Vector Calculus

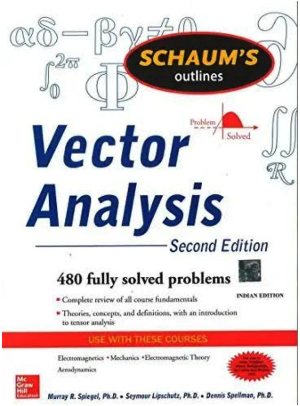
Pinaki Pal

Department of Mathematics
National Institute of Technology Durgapur
West Bengal, India

pinaki.pal@maths.nitdgp.ac.in

Topic Details and Reference

Vector Calculus: Vector valued functions and its differentiability, Line integral, Surface integral, Volume integral, Gradient, Curl, Divergence, Greens theorem in the plane (including vector form), Stokes theorem, Gausss divergence theorem and their applications.



Vector Valued Function

A function whose domain is a subset of $\mathbb{R}^n$ for some natural number n and range is a set of vectors is called a **vector values function**.

Examples

1. $\mathbf{R}(u) = \cos u\hat{i} + \sin^2 u\hat{j} + 2u\hat{k}$
2. $\mathbf{R}(x, y) = x^2y\hat{i} - x^3\hat{j} + 3xy\hat{k}$
3. $\mathbf{R}(x, y, z) = e^x z\hat{i} + 3xy\hat{j} - 4xyz\hat{k}$
4. $\mathbf{R}(t) = \sin 2t\hat{i} + \cos t\hat{j}$
5. $\mathbf{R}(x, y) = xy\hat{i} - xe^y\hat{j}$
6. $\mathbf{R}(x, y, z) = x^2yz\hat{i} - 2\hat{k}$

Vector Differentiation

Ordinary Derivatives of Vectors

Let $\mathbf{R}(u)$ be a vector function of a single scalar variable u. Then the derivative of $\mathbf{R}$ with respect to u is denoted by $\frac{\mathrm{d}\mathbf{R}}{du}$ and defined by

$$\frac{\mathrm{d}\mathbf{R}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{R}(u+\Delta u) - \mathbf{R}(u)}{\Delta u},$$

provided the limit exists. Now if $\mathbf{R}(u) = \mathrm{R}_1(u)\hat{i} + \mathrm{R}_2(u)\hat{j} + \mathrm{R}_3(u)\hat{k}$, it can be easily deduced that

$$\frac{\mathrm{d}\mathbf{R}}{du} = \frac{\mathrm{dR}_1}{du}\hat{i} + \frac{\mathrm{dR}_2}{du}\hat{j} + \frac{\mathrm{dR}_3}{du}\hat{k}.$$

Further, as $\frac{\mathrm{d}\mathbf{R}}{du}$ is a vector function of the scalar variable u, we can differentiate is once again with respect to u. If the derivative exists, it is denoted by $\frac{\mathrm{d}^2\mathbf{R}}{du^2}$. Similarly we can define the higher derivatives of $\mathbf{R}(u)$.

Continuity and Differentiability

A vector function $\mathbf{R}(t) = \mathrm{R}_1(t)\hat{i} + \mathrm{R}_2(t)\hat{j} + \mathrm{R}_3(t)\hat{k}$ is said to be **continuous** at t, if the scalar functions $\mathrm{R}_1(t)$, $\mathrm{R}_2(t)$ and $\mathrm{R}_3(t)$ are continuous at t. A vector function is said to be differentiable of order n if its nth derivative exists. A differentiable function is necessarily continuous but the converse is not true.

Differentiation Formula

If $\mathbf{A}$ and $\mathbf{B}$ are differentiable vector functions of the scalar variable t, ψ is a differentiable scalar function of t, then the following are true.

1. $\frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$
2. $\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{B}$
3. $\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B}$
4. $\frac{d}{dt}(\psi\mathbf{A}) = \psi\frac{d\mathbf{A}}{dt} + \frac{d\psi}{dt}\mathbf{A}$

Partial Derivatives of Vector Functions

If $\mathbf{V}$ is a vector function of more than one scalar variable, say x and y, then the partial derivatives of $\mathbf{V}$ with respect to x and y are defined by

$$\begin{aligned}\frac{\partial \mathbf{V}}{\partial x} &= \lim_{\Delta x \to 0} \frac{\mathbf{V}(x+\Delta x, y) - \mathbf{V}(x,y)}{\Delta x},\\ \frac{\partial \mathbf{V}}{\partial y} &= \lim_{\Delta y \to 0} \frac{\mathbf{V}(x, y+\Delta y) - \mathbf{V}(x,y)}{\Delta y},\end{aligned}$$

provided the limits exist. Note that the idea of continuity and differentiability of the functions of several variables can easily be extended to the vector functions. Moreover, higher order derivatives of the vector functions can also be naturally defined.

Differentiation Formula

1. $\frac{\partial}{\partial x}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{B}$
2. $\frac{\partial}{\partial x}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B}$

Exercise

1. If $\mathbf{R} = e^{-t}\hat{i} + ln(t^2+1)\hat{j} - \tan t\hat{k}$, find (a) $\frac{d\mathbf{R}}{dt}$, (b) $\frac{d^2\mathbf{R}}{dt^2}$, (c) $|\frac{d\mathbf{R}}{dt}|$ and (d) $|\frac{d^2\mathbf{R}}{dt^2}|$ at $t=0$.
2. If $\mathbf{A} = (2x^2y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + x^2\cos y\hat{k}$, find $\frac{\partial \mathbf{A}}{\partial x}$, $\frac{\partial \mathbf{A}}{\partial y}$, $\frac{\partial^2 \mathbf{A}}{\partial x^2}$, $\frac{\partial^2 \mathbf{A}}{\partial y^2}$, $\frac{\partial^2 \mathbf{A}}{\partial x \partial y}$, and $\frac{\partial^2 \mathbf{A}}{\partial y \partial x}$.
3. If $\phi(x,y,z) = xy^2z$ and $\mathbf{A} = xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}$, find $\frac{\partial^3}{\partial x^2 \partial z}(\phi\mathbf{A})$ at the point $(2,-1,1)$.
4. Let $\mathbf{A}(t) = 3t^2\hat{i} - (t+4)\hat{j} + (t^2-2t)\hat{k}$ and $\mathbf{B}(t) = \sin t\hat{i} + 3e^{-t}\hat{j} - 3\cos t\hat{k}$, find $\frac{d^2}{dt^2}(\mathbf{A}\times\mathbf{B})$ at $t=0$.
5. Find $\frac{\partial^2}{\partial x \partial y}(\mathbf{A}\times\mathbf{B})$, where $\mathbf{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$ and $\mathbf{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$.

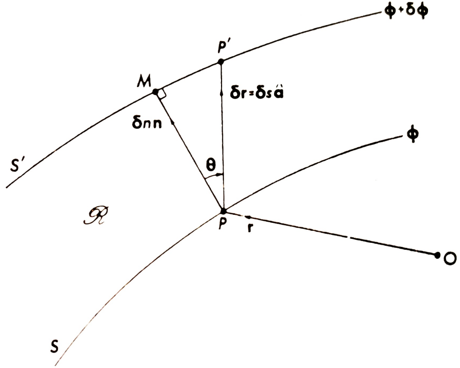
Scalar Field and Level Surfaces

If at each point $P(x, y, z)$ of a region R of space a **scalar function** $\phi(x, y, z)$ is uniquely defined, then the region R together with the function ϕ is said to form a **scalar field**. Examples include temperature in room, gravitation potential etc. For a scalar field ϕ defined over some region of space, the surfaces defined by $\phi = constant$ are called *iso*-ϕ **surfaces** or **level surfaces**. Note that no two level surfaces can intersect, as the function ϕ is uniquely defined at any point in R.

Vector Field

Suppose at each point $P(x, y, z)$ of a region R of space a **vector function** $\mathbf{F}(x, y, z)$ is uniquely defined, then the region R together with the function $\mathbf{F}$ is said to form a **vector field**. Examples include velocity of air in a hall, gravitation force due to earth etc.

The Vector Gradient



The above figure shows two *iso*-ϕ surfaces S and S' of the scalar field defined by $\phi(x, y, z)$ passing through two nearby points P and P' respectively.

The Vector Gradient ...

The function assumes the values ϕ and $\phi + \delta\phi$ respectively on the surfaces S and S'.
Let us suppose

$$\vec{OP} = \mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ and } \vec{OP'} = (x+\delta x)\hat{i} + (y+\delta y)\hat{j} + (z+\delta z)\hat{k},$$

so that $\vec{PP'} = \delta\mathbf{r} = \delta s\hat{\mathbf{a}}$, where $\delta s = |\delta\mathbf{r}| = PP'$. Further let PM be drawn perpendicularly to S' to meet it at M such that

$$\vec{PM} = \delta n\hat{n}, \text{ where } PM = \delta n.$$

When S and S' are taken to coincide, $\hat{n}$ defines the unit normal at P to S in the direction from S to S'.
Suppose $\angle MPP' = \theta$ so that $\delta n = \delta s \cos\theta$. Then the rate of change of ϕ with respect to s as we go from P on S to P' on S' is

$$\frac{\delta\phi}{\delta s} = \frac{\delta\phi}{\delta n}\frac{\delta n}{\delta s} = \cos\theta\frac{\delta\phi}{\delta n}.$$

The Vector Gradient . . .

Hence as $P' \to P$, along $P'P$,

$$\frac{\partial \phi}{\partial s} = \cos\theta \frac{\partial \phi}{\partial n}$$

. Note that $max.|\frac{\partial \phi}{\partial s} = |\frac{\partial \phi}{\partial n}|$, at a chosen point P on S. This maximum rate of change takes place when $\theta = 0$ or π. Thus, the vector quantity $\frac{\partial \phi}{\partial n}\hat{\mathbf{n}}$ completely specifies the maximum rate of change of ϕ together with its direction and is called the **gradient** of ϕ at P. It is denoted either by $grad\phi$ i.e.

$$grad\phi = \nabla\phi = \frac{\partial \phi}{\partial n}\hat{\mathbf{n}}$$

and it can be proved that

$$grad\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}.$$

Directional Derivative

Therefore the quantity $\frac{\partial \phi}{\partial s}$ is given by

$$\frac{\partial \phi}{\partial s} = \cos\theta \frac{\partial \phi}{\partial n} = (\hat{a} \cdot \hat{n}) \frac{\partial \phi}{\partial n} = \hat{a} \cdot \nabla \phi$$

and is called the **directional derivative** of ϕ along $\hat{a}$. The operators $\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ and $\hat{a} \cdot \nabla$ are called the **vector differential operator** and **directional differentiator** respectively. The operator ∇ is also known as **nabla**.

Exercise

1. Find the gradient of the function $\phi(x, y, z) = 2x^2 - 3xy + 9z - 2$ at the point $(1, 0, 0)$ and differentiate it in the direction specified by $2\hat{i} - 2\hat{j} + k$.
2. Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.
3. In what direction from the point $(2, 1, -1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum? What is the magnitude of this maximum?
4. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.
5. Find the unit outward drawn normal to the surface $(x - 1)^2 + y^2 + (z + 2)^2 = 9$ at the point $(3, 1, -4)$.
6. Find the equation of the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(2, -1, 5)$.

Vector Integration

Integration with respect to a scalar variable

If a vector function $\mathbf{R}(u)$ of a scalar variable u be such that

$$\mathbf{a} = \frac{d}{du}[\mathbf{R(u)}],$$

then the indefinite integration of $\mathbf{a}(u)$ is defined by

$$\int \mathbf{a}(u)du = \int \frac{d}{du}[\mathbf{R(u)}]du = \mathbf{R}(u) + \mathbf{C}$$

in which $\mathbf{C}$ is an arbitrary constant vector independent of u. Now if $\mathbf{a}(u) = a_1(u)\hat{i} + a_2(u)\hat{j} + a_3(u)\hat{k}$, then

$$\int \mathbf{a}(u)du = \hat{i}\int a_1(u)du + \hat{j}\int a_2(u)du + \hat{k}\int a_3(u)du.$$

In case case of definite integration, we have

$$\int_a^b \mathbf{a}(u)du = \int_a^b \frac{d}{du}[\mathbf{R(u)}]du = \mathbf{R}(b) - \mathbf{R}(a).$$

Examples

1. If $\mathbf{a(u)} = u^2\hat{i} + (u-1)\hat{j} - 3u\hat{k}$, then find $\int \mathbf{a}(u)du$ and $\int_2^3 \mathbf{a}(u)du$.
2. Evaluate $\int_1^3 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt$, where $\mathbf{r} = 5t^2\hat{i} - t^3\hat{j} + 2t\hat{k}$.
3. The acceleration of a particle at time t is given by

$$\mathbf{a}(t) = \sin 3t\hat{i} - 5\cos t\hat{j} + 10t\hat{k}.$$

   If $\mathbf{v}$ and $\mathbf{r}$ respectively denote the velocity and displacement of the particle at time $t = 0$, then find $\mathbf{v}$ and $\mathbf{r}$.
4. If $\mathbf{A}(t) = t\hat{i} - t^2\hat{j} + (t-1)k$ and $\mathbf{B}(t) = 3t^2\hat{i} - 2t\hat{j} + 4\hat{k}$, evaluate (a) $\int_0^3 \mathbf{A} \cdot \mathbf{B}dt$ and (b) $\int_0^3 \mathbf{A} \times \mathbf{B}dt$.

Line Integrals

Any integral which is evaluated along a curve is called a **line integral.**

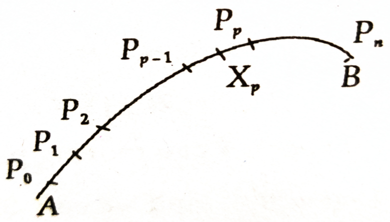
Definition

Let us consider a space curve $\mathbb{C}$, joining the points A and B. The position vector of a point (x, y, z) on it is given by $\mathbf{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$. The values of u corresponding to the terminal points A and B are u_1 and u_2 respectively. Let

$$\vec{\mathbf{A}}(x, y, z) = A_1(x, y, z)\hat{i} + A_2(x, y, z)\hat{j} + A_3(x, y, z)\hat{k}$$

be a vector function of position defined and continuous along $\mathbb{C}$.

The curve $\mathbb{C}$ is shown below is divided into n parts by the $n+1$ points $P_0, P_1, \ldots P_{p-1}, P_p, \ldots, P_n$. The line segment $P_{p-1}P_p$ is given by the vector $\vec{\delta r} = \delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k}$.



Let (x, y, z) be any point X_p on the arc $\widehat{P_{p-1}P_p}$, then the line integral of $\vec{\mathbf{A}}$ over $\mathbb{C}$ is defined by

$$\int_{\mathbb{C}} \vec{\mathbf{A}} \cdot \vec{dr} = lim_{\delta r \to 0} \sum_{p=1}^{n} \vec{\mathbf{A}} \cdot \vec{\delta r},$$

provided the limit in the right hand side exists. In particular, if $\mathbb{C}$ is a closed curve, then the above line integral is called the **circulation** of $\vec{\mathbf{A}}$ around $\mathbb{C}$.

The line integral $\int_{\mathbb{C}} \vec{\mathbf{A}} \cdot \vec{dr}$ can also be represented as

$$\int_{\mathbb{C}} \vec{\mathbf{A}} \cdot \vec{dr} = \int_A^B \vec{\mathbf{A}} \cdot \vec{dr} = \int_A^B A_1 dx + A_2 dy + A_3 dz.$$

Examples

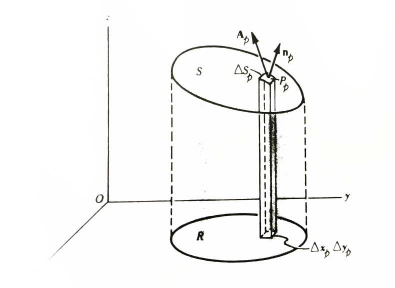
1. Evaluate $\int_{\mathbb{C}} \vec{\mathbf{A}} \cdot \vec{dr}$, where $\vec{\mathbf{A}} = x^2y^2\hat{i} + y\hat{j}$ and the curve $\mathbb{C}$ is $y^2 = 4x$ in the xy-plane from $(0,0)$ to $(4,4)$.
2. If $\vec{\mathbf{A}} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_{\mathbb{C}} \vec{\mathbf{A}} \cdot \vec{dr}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths $\mathbb{C}$:
   - (a) $x = t$, $y = t^2$ and $z = t^3$.
   - (b) the straight lines from $(0,0,0)$ to $(1,0,0)$, then to $(1,1,0)$ and then to $(1,1,1)$.
   - (c) the straight line joining $(0,0,0)$ to $(1,1,1)$.
3. Evaluate $\int_{\mathbb{C}} \vec{\mathbf{A}} \cdot \vec{dr}$, where $\vec{\mathbf{A}} = (x - 3y)\hat{i} + (y - 2x)\hat{j}$ and $\mathbb{C}$ is the curve in the xy-plane, $x = 2\cos t$, $y = 3\sin t$ from $t = 0$ to π.

Surface Integrals

Any integral evaluated over a surface is called a surface integral. In the following definition of a surface integral is given.

Surface integral of a vector field (Normal flux over a surface)

Let S be a surface in space as shown below and $\mathbf{A}(x, y, z)$ be a vector function defined on it.



Divide the area S into N parts of elements ΔS_p, where $p = 1, 2, 3, \ldots, N$. Choose any point $P_p(x_p, y_p, z_p)$ within ΔS_p. Let $\mathbf{n}_p$ be positive unit normal to ΔS_p at P_p. Then form the sum

$$\sum_{p=1}^{N} \mathbf{A}(x_p, y_p, z_p) \cdot \mathbf{n}_p \Delta S_p,$$

Now take the limit of the sum as $N \to \infty$ in such a way that each $\Delta S_p \to 0$. Then if this limit exists, it is called a surface integral of $\mathbf{A}$ and it is denoted by

$$\iint_S \mathbf{A} \cdot \mathbf{n} dS.$$

Suppose that the surface S has one to one project R on the xy-plane (see the figure above) then it can be shown that

$$\iint_S \mathbf{A} \cdot \mathbf{n} dS = \iint_R \mathbf{A} \cdot \mathbf{n} \frac{dxdy}{|\mathbf{n} \cdot k|}.$$

If the said project is not one to one we can evaluate the integral by projecting the surface S on other suitable co-ordinate plane.

Surface integral of a scalar field

Surface integral of a scalar field ϕ over the surface S is denoted by $\iint_S \phi dS$ and defined by

$$\iint_S \phi dS = \lim_{N\to\infty} \sum_{p=1}^{N} \phi(x_p, y_p, z_p)\Delta S_p,$$

provided the limit exists, where limit of the sum as $N \to \infty$ is taken in such a way that each $\Delta S_p \to 0$.
It can shown that

$$\iint_S \phi dS = \iint_R \phi \frac{dxdy}{|\mathbf{n} \cdot k|},$$

where R is the projection of S on the xy-plane.

Now if $z = f(x, y)$ defined the surface S, then

$$|\mathbf{n} \cdot k| = |\frac{\nabla f}{|\nabla f|} \cdot k| = \frac{1}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1}}.$$

Hence the surface integral $\iint_S \phi dS$ can be evaluated using the double integral as

$$\iint_S \phi dS = \iint_R \phi \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1}\, dx\, dy.$$

Examples

1. Evaluate $\iint_S \mathbf{A} \cdot \mathbf{n} dS$, where $\mathbf{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.
2. If $\mathbf{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} dS$, where S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.
3. For the function $\phi(x, y, z) = x^2 + 2y + z - 1$, show that

$$\int_S \phi dS = \frac{3}{4},$$

the region S consisting of the part of the plane $2x + 2y + z = 2$ lying in the first octant.

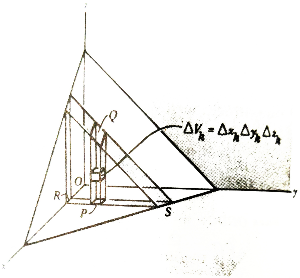
4. Find the surface area of the portion of the plane bounded by (a) $x = 0$, $y = 0$, $x = 1$ and $y = 1$; (b) $x = 0$, $y = 0$ and $x^2 + y^2 = 16$.

Volume Integrals

Consider a closed surface in space enclosing a volume V. Then the symbols

$$\iiint_V \mathbf{A} dV \text{ and } \iiint_V \phi dV$$

denote the volume integrals of the vector function $\mathbf{A}(x, y, z)$ and the scalar function $\phi(x, y, z)$ respectively and defined as follows. Subdivide the region V into N parts having volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k,\ k = 1, 2, \ldots, N$ (see the figure below). Let (x_k, y_k, z_k) be a point within the cube and define $\phi_k = \phi(x_k, y_k, z_k)$ and $\mathbf{A_k} = \mathbf{A}(x_k, y_k, z_k)$.



Then the volume integrals $\iiint_V \mathbf{A} dV$ and $\iiint_V \phi dV$ are defined by

$$\iiint_V \mathbf{A} dV = \lim_{N\to\infty} \sum_{k=1}^{N} \mathbf{A_k} \Delta V_k \text{ and } \iiint_V \phi dV = \lim_{N\to\infty} \sum_{k=1}^{N} \phi_k \Delta V_k,$$

provided the limits exist.

Example

1. Show that $\iiint \phi dV = \frac{1}{4}$, where $\phi = 15xy$ and V denotes the closed region bounded by the planes
$$x + 4y + z = 2, x = 0, y = 0, z = 0.$$
2. If $\mathbf{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4z\hat{k}$, evaluate $\iiint_V \mathbf{F} dV$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.
3. Find $\iiint_V \mathbf{A} dV$, where $\mathbf{A} = x\hat{i} + y\hat{j} + z\hat{k}$ and V is the region bounded by the surfaces $x = 0, x = 2, y = 0, y = 6, z = x^2, z = 4$.

Green's Theorem in the Plane

If R be a closed region of the xy-plane bounded by a simple closed regular curve $\mathbb{C}$,, and P and Q are continuous functions of x and y having continuous derivatives in R, then

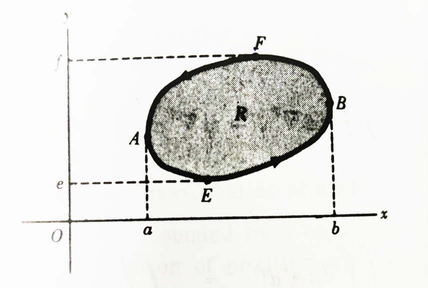
$$\int_{\mathbb{C}} P\,dx + Q\,dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy,$$

where the line integral is taken in positive sense.

Proof

Let the equations of the curves AEB and AFB be $y = Y_1(x)$ and $y = Y_2(x)$ respectively. Then

$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b} \left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial P}{\partial y}\,dy\right] dx$$



$$
\begin{aligned}
\iint_R \frac{\partial P}{\partial y}\, dx\, dy &= \int_{x=a}^{b} P(x, y)\big|_{y=Y_1(x)}^{y=Y_2(x)}\, dx \\
&= \int_{x=a}^{b} \left[P(x, Y_2) - P(x, Y_1)\right]\, dx \\
&= -\int_a^b P(x, Y_1)\, dx - \int_b^a P(x, Y_2)\, dx \\
&= -\int_{\mathbb{C}} P\, dx \qquad (1)
\end{aligned}
$$

Similarly, let the equations of the curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

$$\begin{aligned}
\iint_R \frac{\partial Q}{\partial x}\, dx\, dy &= \int_{y=e}^{f} \left[\int_{x=X_1(y)}^{X_2(y)} \frac{\partial Q}{\partial x}\, dx \right] dy = \int_{y=e}^{f} Q(x,y)\Big|_{x=X_1(y)}^{x=X_2(y)} dy \\
&= \int_{y=e}^{f} \left[Q(X_2, y) - P(X_1, y) \right] dy \\
&= \int_{f}^{e} P(X_1, y)\, dy + \int_{e}^{f} P(X_2, y)\, dy \\
&= \int_{\mathbb{C}} Q\, dy. \qquad (2)
\end{aligned}$$

Adding the equations (1) and (2) we get

$$\int_{\mathbb{C}} P\, dx + Q\, dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx\, dy.$$

Vector form of Green's theorem

In vector form Green's theorem can be written as

$$\int_{\mathbb{C}} \mathbf{A} \cdot \mathbf{dr} = \iint_{R} (\nabla \times \mathbf{A}) \cdot \hat{k}\, dR,$$

where $\mathbf{A} = M\hat{i} + N\hat{j}$, $\mathbf{dr} = dx\hat{i} + dy\hat{j}$ and $dR = dx\,dy$.

Examples

1. Verify Green's theorem in the plane for $\int_{\mathbb{C}} (xy + y^2)\, dx + x^2\, dy$, where $\mathbb{C}$ is the closed curve of the region bounded by $y = x$ and $y = x^2$. [Ans: Both sides = -$\frac{1}{20}$].

Examples

1. Verify Green's theorem in the plane for $\int_\Gamma x^2\,dx + xy\,dy$, where Γ is the square in the xy-plane bounded by $x = 0$, $x = a, y = 0$ and $y = q$ $(a > 0)$ described in the positive sense. [Ans: Both sides $= \frac{a^3}{2}$
2. Evaluate $\int_\Gamma (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy$ by Green's theorem, where Γ is the circle $x^2 + y^2 = 1$ in the xy-plane described in the positive sense. [Ans: 0]
3. Verify Green's theorem in the plane for $\int_{\mathbb{C}} (2xy - x^2)\,dx + (x + y^2)\,dy$, where $\mathbb{C}$ is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$. [Ans: Both sides $= \frac{1}{30}$].
4. Show that the area bounded by a simple closed curve Γ is given by $\frac{1}{2}\int_\Gamma x\,dy - y\,dx$. Hence obtain the area of the ellipse $x = a\cos t$ and $y = b\sin t$. [Ans: πab]

Normal flux of a vector field; Divergence

Let $\mathbf{F}$ be a vector function uniquely defined at each point P of a region R and let δS be a surface element at P having unit normal specified by $\mathbf{n}$. Then $\delta\mathbf{S} = \delta S\mathbf{n}$ is called the vector area of the element associated with the direction $\mathbf{n}$.

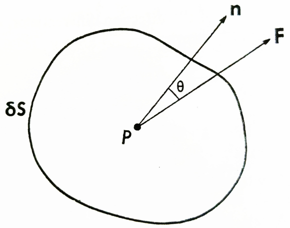


FIG. 1.13

If $\mathbf{F}$ at P makes an angle θ with $\mathbf{n}$, then we define the normal flux of $\mathbf{F}$ across δS in the sense of $\mathbf{n}$ to be

$$(F\cos\theta)\delta S = \mathbf{F}\cdot\mathbf{n}\delta S = \mathbf{F}\cdot\delta \mathrm{S}.$$

Over a finite surface S (Open or Closed) total normal flux would be

$$\int_S F\cos\theta dS = \int_S \mathbf{F}\cdot\mathbf{n} ds = \int_S \mathbf{F}\cdot\mathbf{dS}.$$

Now suppose Δv denotes the element of volume of space containing a point P and enclosed by a closed surface ΔS. Let $\mathbf{n}$ be the unit normal at any surface element δS of ΔS ($\delta S \ll \Delta S$) drawn outwards from Δv. Then the total outward directed normal flux of $\mathbf{F}$ over ΔS is

$$\int_{\Delta S} \mathbf{n}\cdot\mathbf{F} dS = \int_{\Delta S} \mathbf{F}\cdot\mathbf{dS}.$$

Thus the mean outward normal flux of $\mathbf{F}$ per unit volume is

$$\frac{1}{\Delta v}\int_{\Delta S}\mathbf{n}\cdot\mathbf{F}dS.$$

Keeping P fixed, let $\Delta v \to 0$ in such a way that ΔS shrinks to the point P. Then if the limit

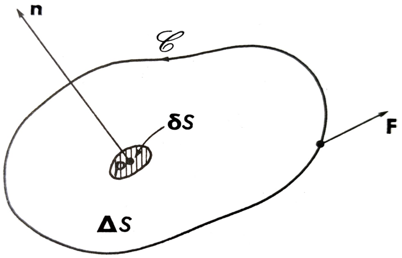
$$\lim_{\Delta v\to 0}\left[\frac{1}{\Delta v}\int_{\Delta S}\mathbf{n}\cdot\mathbf{F}dS\right]$$

exists uniquely, it measures the outward flux of $\mathbf{F}$ per unit volume of P. It is a scalar quantity and it can be shown that its value is independent of the shape of Δv. Such a limit is called the **divergence** of $\mathbf{F}$ at P and written as $div\mathbf{F}$. If $\mathbf{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then it can be shown that

$$div\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Curl of a vector function

Suppose $\mathbb{C}$ denotes a closed curve containing a small element of area ΔS and that $\mathbf{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector function defined at each point of $\mathbb{C}$ and ΔS.



Let P be a point in ΔS and suppose $\delta S(\ll \Delta S)$ is a small element of ΔS and containing P. Further suppose the sense of description of $\mathbb{C}$ is as shown in the figure. At P draw the unit positive normal $\mathbf{n}$ to δS. Then the total line integral or circulation of $\mathbf{F}$ around $\mathbb{C}$ is denoted by

$$\int_{\mathbb{C}} \mathbf{F} \cdot \mathbf{dr}.$$

Hence the mean circulation per unit area of ΔS is $\frac{1}{\Delta S} \int_{\mathbb{C}} \mathbf{F} \cdot \mathbf{dr}$.

Keeping P fixed, let $\Delta S \to 0$ in such a way that the curve $\mathbb{C}$ approaches the point P. Then the limiting value of the mean circulation per unit area at P, if it exists it can be shown to be independent of the shape of the curve $\mathbb{C}$ for a given normal direction $\mathbf{n}$. This limit is a scalar but depends on the normal $\mathbf{n}$ to ΔS. We then define a vector $curl\mathbf{F}$ so that

$$\mathbf{n} \cdot curl\mathbf{F} = \lim_{\Delta S \to 0} \left[\frac{1}{\Delta S} \int_{\mathbb{C}} \mathbf{F} \cdot \mathbf{dr} \right].$$

It can be proved that

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Gauss's divergence theorem

Let V be a region of space enclosed by a closed surface S and $\mathbf{F}$ be a continuously differentiable vector function uniquely defined on V as well as on S. Then

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{n} \cdot \mathbf{F}\, dS,$$

where $\mathbf{n}$ denote the unit outward normal to the surface S.

Example

Evaluate $\iint_S \mathbf{n} \cdot \mathbf{F}\, dS$, where $\mathbf{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by
$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. [Ans: $\frac{3}{2}$]

Stokes' Theorem

If $\mathbb{C}$ is closed curve, not necessarily in a plane, containing an area S and if the continuously differentiable vector function $\mathbf{F}$ is uniquely defined on S as well as on C, then

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr} = \iint_S \mathbf{n} \cdot (\nabla \times \mathbf{F} = \iint_S \mathbf{n} \cdot curl\mathbf{F},$$

where $\mathbf{n}$ is the unit positive normal to a point on S with respect to the sense of description of $\mathbb{C}$.

Example

Verify Stokes' theorem for $\mathbf{A} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. [Ans: Each side $= \pi$.]

Some Vector Identities

If ϕ is a uniform differentiable scalar function, and **F** and **G** be two uniform vector functions, then

1. $\nabla \times \nabla\phi = 0$.
2. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.
3. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$.
4. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$.
5. $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F})$.
6. $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G})$.