Solid Mechanics (MEc 301)

Chapter 4: Concept of Shear Stresses in Beams

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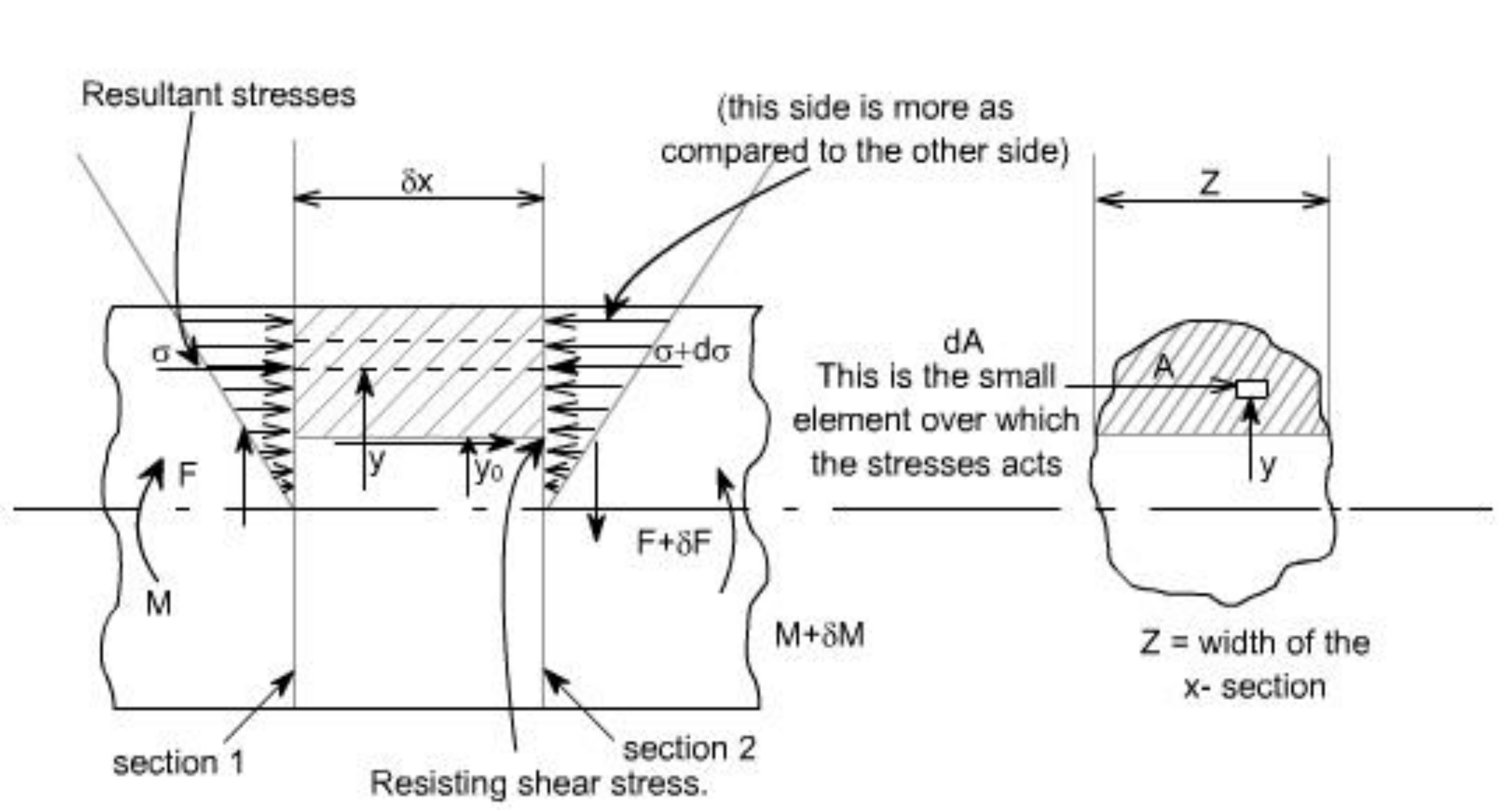
Books:

1. Strength of Materials: Part I, II, S. Timoshenko, CBS Publishers, 1985.
2. Engineering Mechanics of Solids, E. P. Popov, PHI, 1993.
3. Introduction to Solid Mechanics, I. H. Shames and J. M. Pittariesi, PHI, 2003.
4. Strength of Materials, F. L. Singer and A. Pytel, HarperCollins Publishers, 1991

Concept of Shear Stresses in Beams: Bending moment represents the resultant of certain linear distribution of normal stresses τ_x over the cross-section. Similarly, the shear force F_x over any cross-section must be the resultant of a certain distribution of shear stresses.



Assumptions :

- ❖ Stress is uniform across the width (i.e. parallel to the neutral axis)
- ❖ The presence of the shear stress does not affect the distribution of normal bending stresses.

i.e $\tau.z\delta x = \int d\sigma.dA$

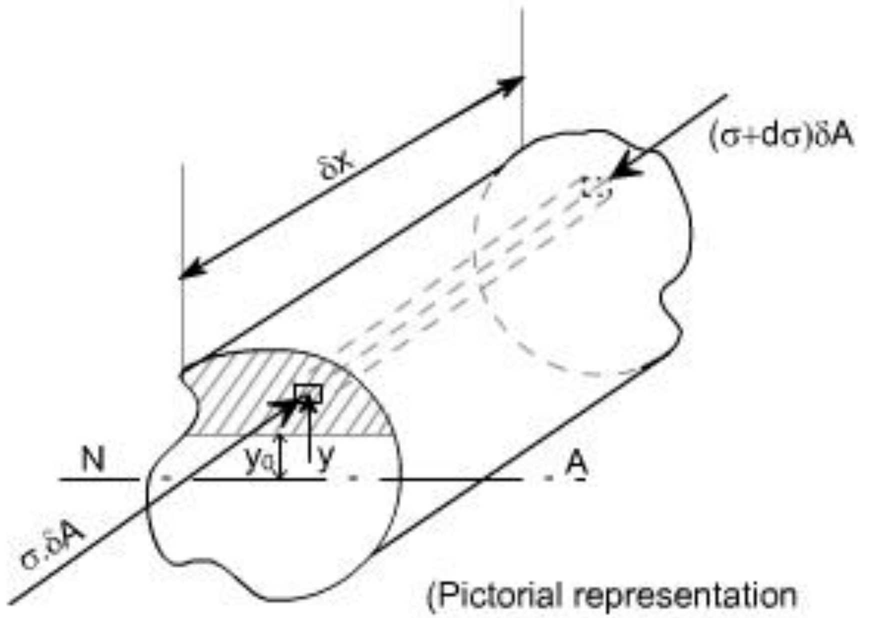
from the bending theory equation

$$\frac{\sigma}{y} = \frac{M}{I}$$

$$\sigma = \frac{M.y}{I}$$

$$\sigma + d\sigma = \frac{(M + \delta M).y}{I}$$

Thus $$d\sigma = \frac{\delta M.y}{I}$$



(Pictorial representation of entire part)

$$d\sigma = \frac{\delta M.y}{I}$$

$$\tau.z\delta x = \int d\sigma.dA$$

$$= \int \frac{\delta M.y.\delta A}{I}$$

$$\tau.z\delta x = \frac{\delta M}{I}\int y.\delta A$$

But $$F = \frac{\delta M}{\delta x}$$

i.e. $$\tau = \frac{F}{I.z}\int y.\delta A$$

But from definition, $\int y.dA = A\bar{y}$

$\int y.dA$ is the first moment of area of the shaded portion and $\bar{y}$ = centroid of the area'A'

Hence

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

Shearing stress distribution in typical cross-sections

Rectangular x-section: Consider a rectangular x-section of dimension b and d

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

for this case, $A = b\left(\frac{d}{2} - y\right)$

While $\bar{y} = \left[\frac{1}{2}\left(\frac{d}{2} - y\right) + y\right]$

i.e $\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$ and $z = b$; $I = \frac{b.d^3}{12}$

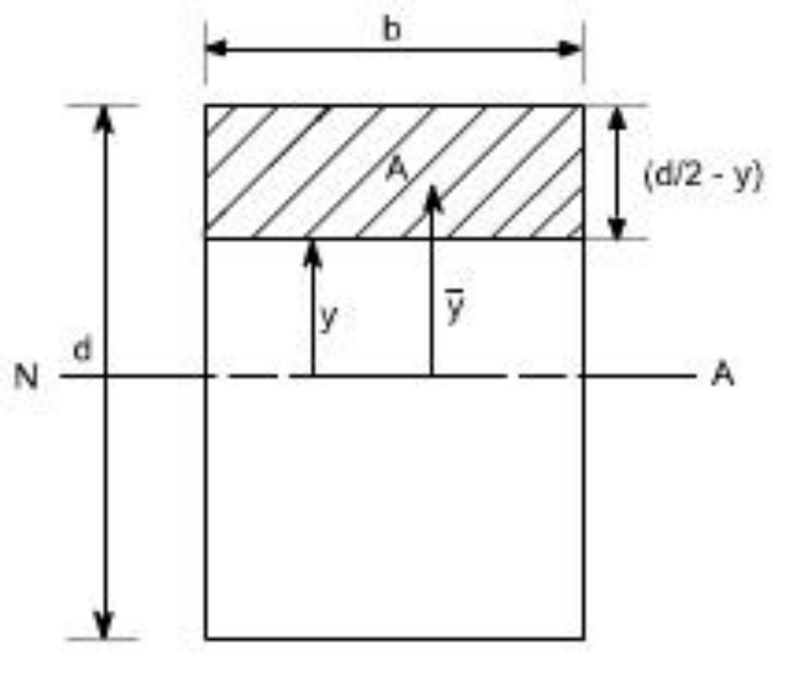
substituting all these values, in the formula

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$= \frac{F.b.\left(\frac{d}{2} - y\right).\frac{1}{2}.\left(\frac{d}{2} + y\right)}{b.\frac{b.d^3}{12}}$$

$$= \frac{\frac{F}{2}.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{\frac{b.d^3}{12}}$$

$$= \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$



Such that $\tau_{max} = \frac{6.F}{b.d^3} \cdot \frac{d^2}{4}$

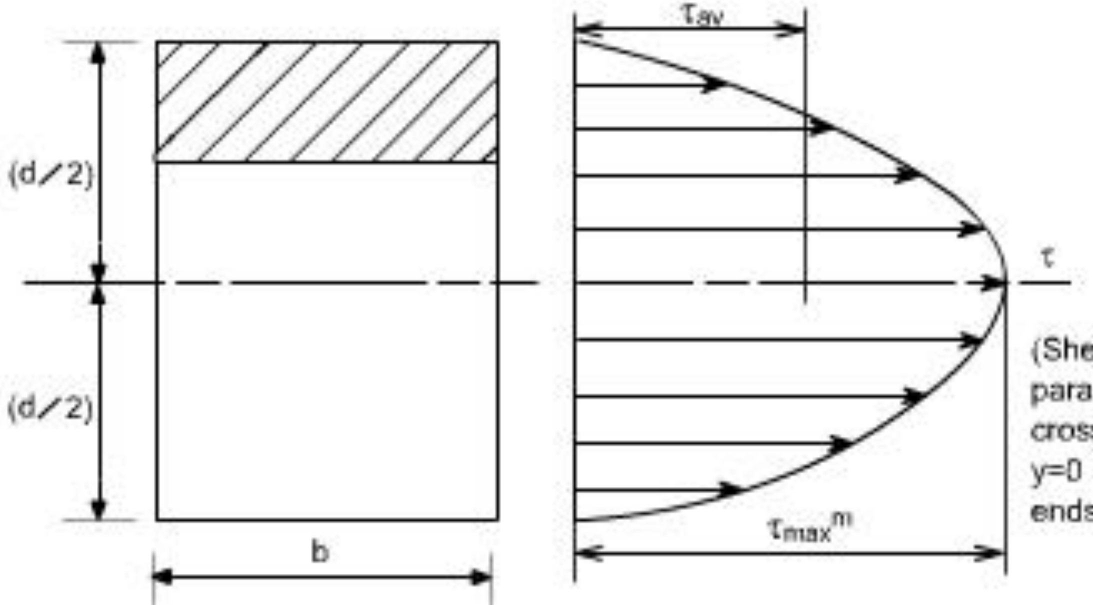
$$= \frac{3.F}{2.b.d}$$

So $\boxed{\tau_{max} = \frac{3.F}{2.b.d}}$ The value of τ_{max} occurs at the neutral axis

The mean shear stress in the beam is defined as

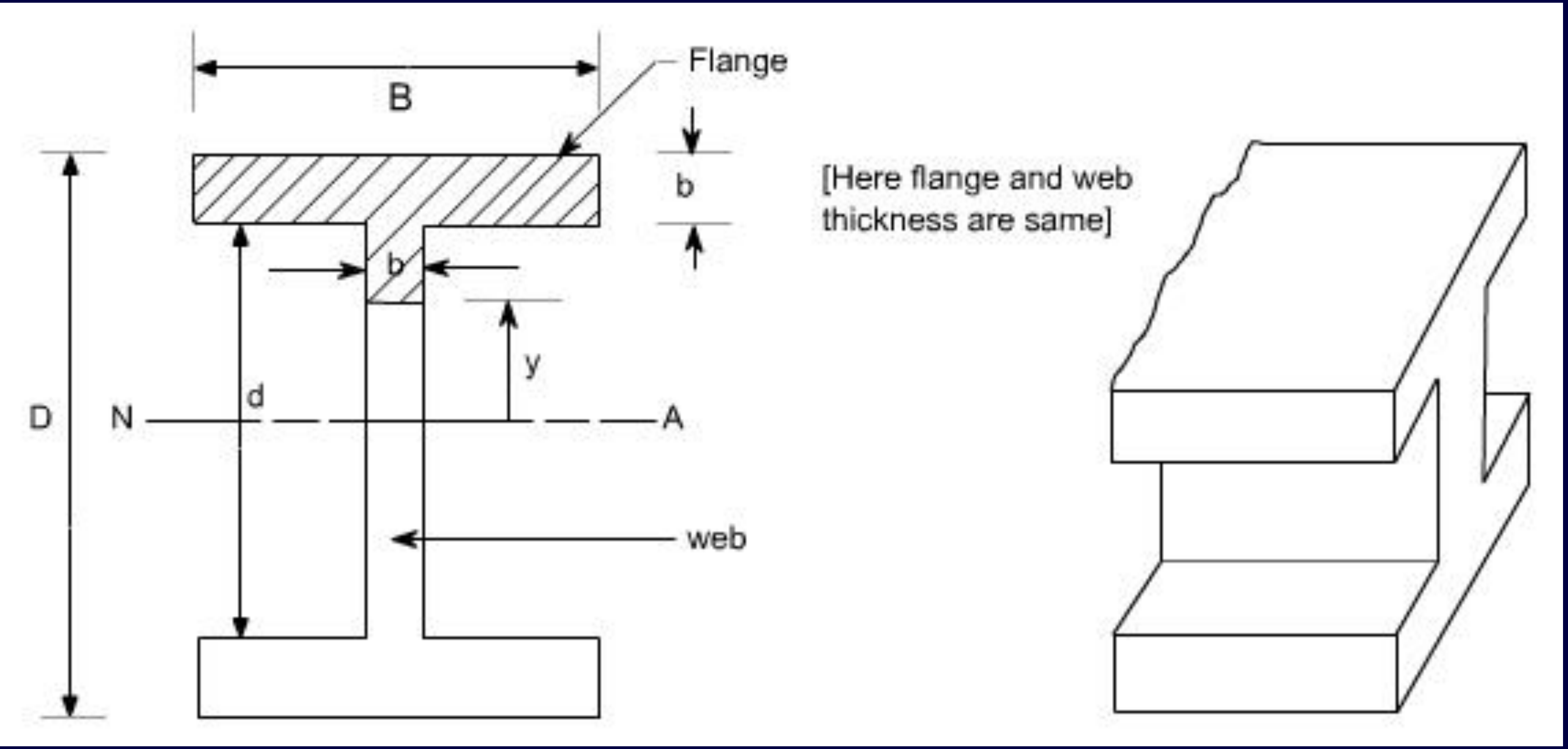
$$\tau_{mean} \text{ or } \tau_{avg} = F/A = F/b.d$$

So $\tau_{max} = 1.5\ \tau_{mean} = 1.5\ \tau_{avg}$



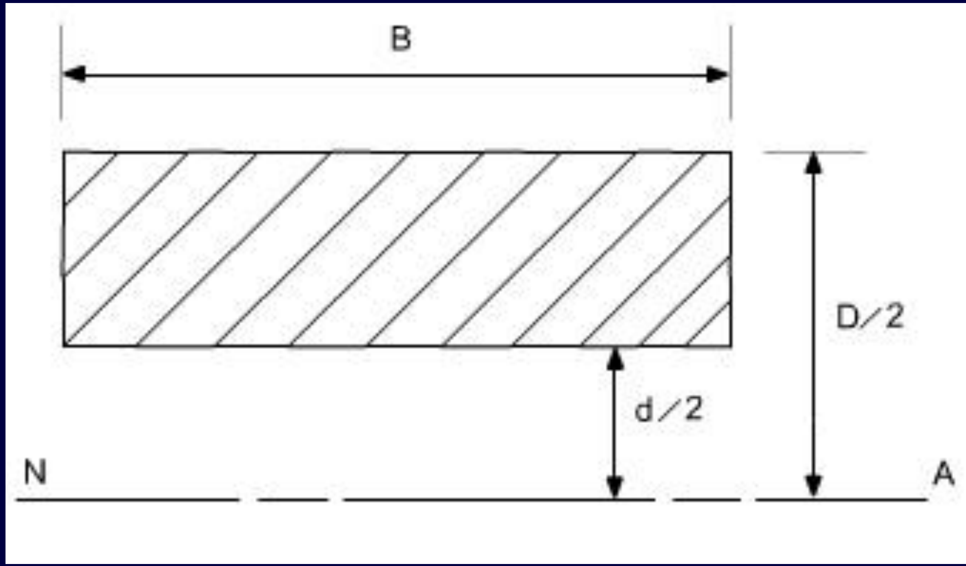
(Shear stress is distributed parabolically over a rectangular cross-section,it is maximum at y=0 and is zero at the extreme ends)

I - section:



$$\tau = \frac{FA\bar{y}}{ZI}$$

Flange area



Area of the flange $= B\left(\frac{D-d}{2}\right)$

Distance of the centroid of the flange from the N.A

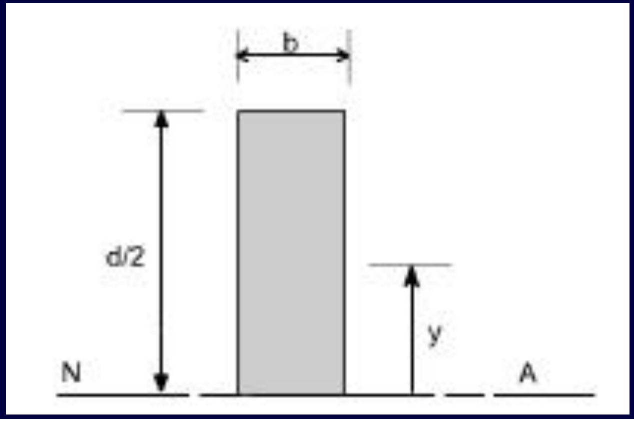
$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$

Web area



Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{Total} = B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

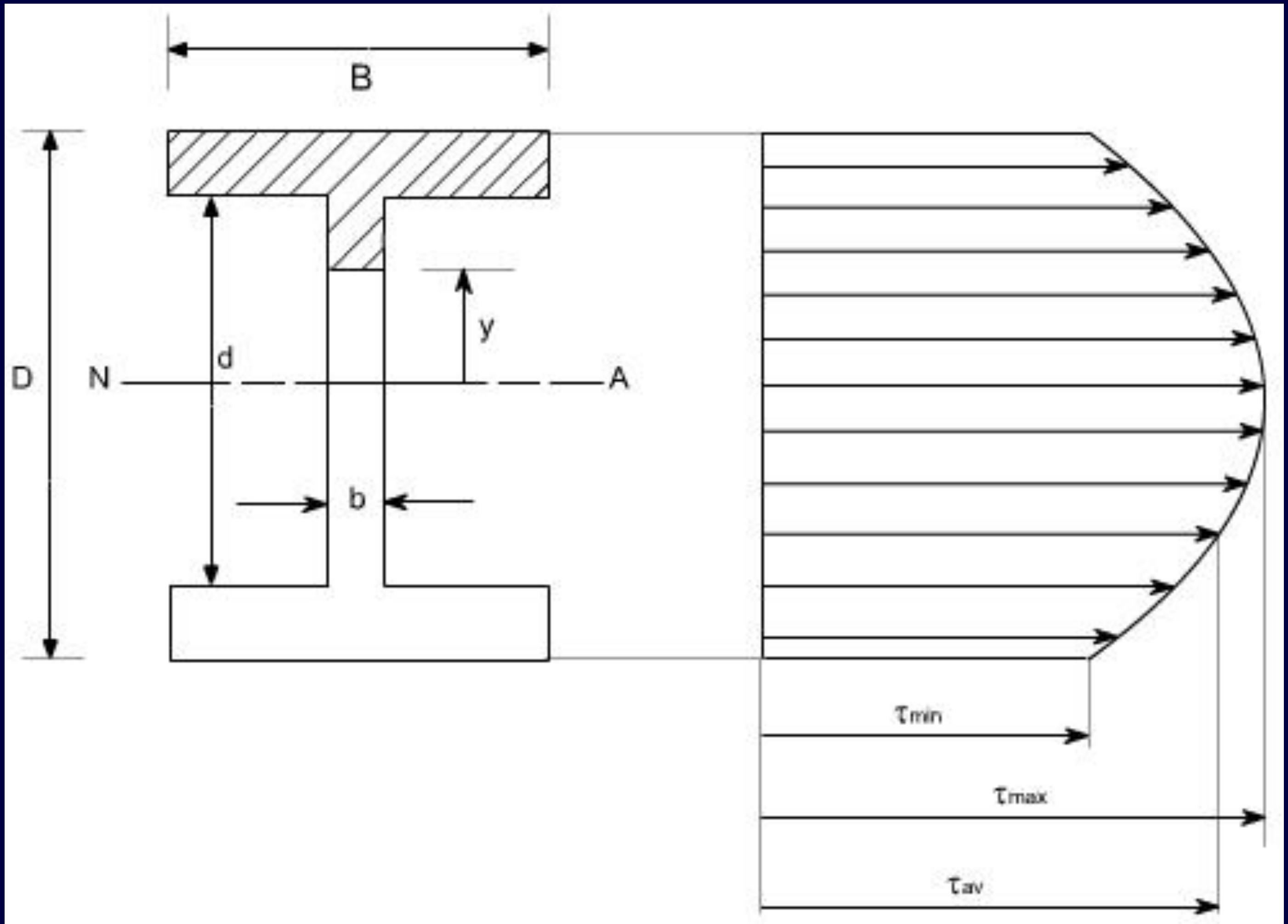
$$A\bar{y}\big|_{Total} = B\left(\frac{D^2-d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

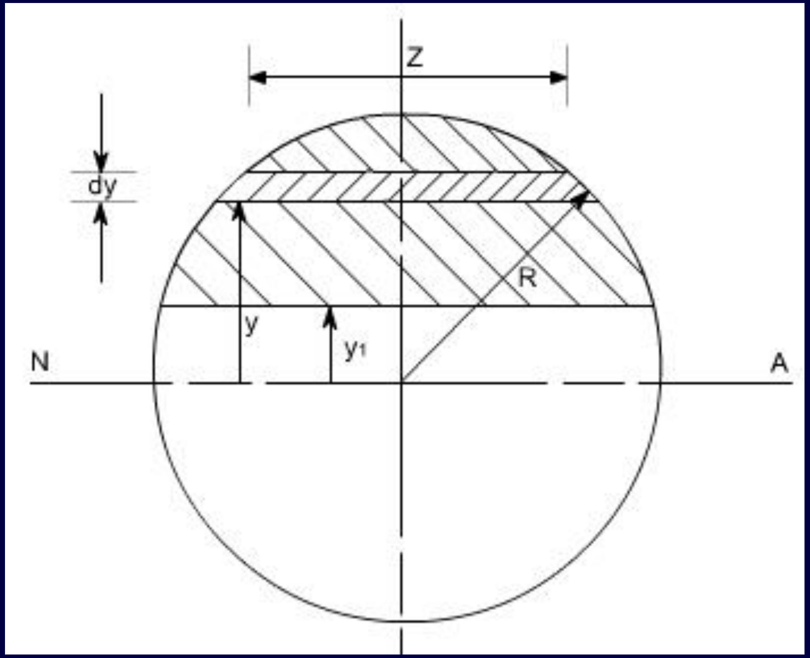
Therefore shear stress,

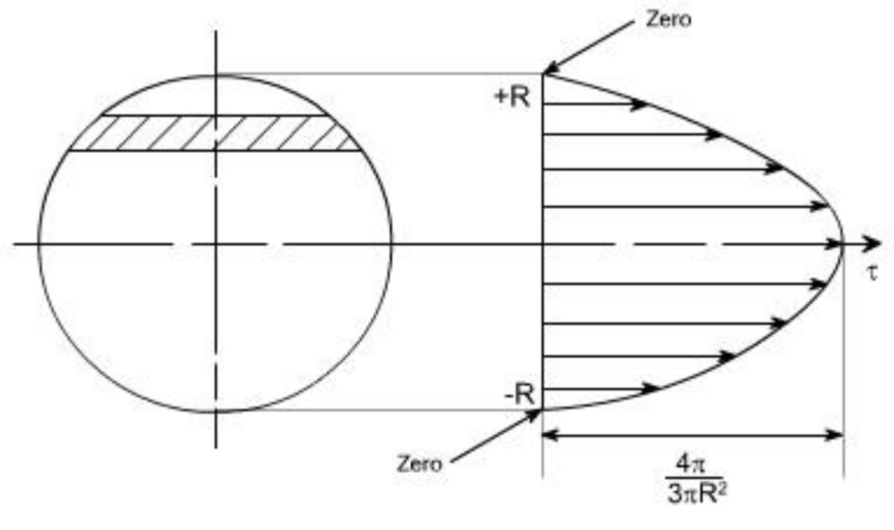
$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2-d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

$$\tau_{max} \text{ at } y = 0 = \frac{F}{8\,b\,I}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

$$\tau_{min} \text{ at } y = d/2 = \frac{F}{8\,b\,I}\left[B\left(D^2 - d^2\right)\right]$$







$$\left(\frac{Z}{2}\right)^2 + y^2 = R^2$$

$$\left(\frac{Z}{2}\right)^2 = R^2 - y^2 \text{ or } \frac{Z}{2} = \sqrt{R^2 - y^2}$$

$$Z = 2\sqrt{R^2 - y^2}$$

$$dA = Z\,dy = 2.\sqrt{R^2 - y^2}\,.dy$$

$I_{N.A.}$ for a circular cross-section $= \dfrac{\pi R^4}{4}$

Hence,

$$\tau = \frac{FA\bar{y}}{ZI} = \frac{F}{\frac{\pi R^4}{4} 2\sqrt{R^2 - y^2}} \int_{y_1}^{R} 2\,y\sqrt{R^2 - y^2}\,dy$$

Where R = radius of the circle.

[The limits have been taken from y_1 to R because we have to find moment of area the shaded portion]

$$= \frac{4F}{\pi R^4 \sqrt{R^2 - y^2}} \int_{y_1}^{R} y\sqrt{R^2 - y^2}\,dy$$

The integration yields the final result to be

$$\tau = \frac{4F\left(R^2 - y_1^2\right)}{3\pi R^4}$$

Again this is a parabolic distribution of shear stress, having a maximum value when $y_1 = 0$

$$\tau_{maxm}|y_1 = 0 = \frac{4F}{3\pi R^2}$$

Obviously at the ends of the diameter the value of $y_1 = \pm R$ thus $\tau = 0$ so this again a parabolic distribution; maximum at the neutral axis

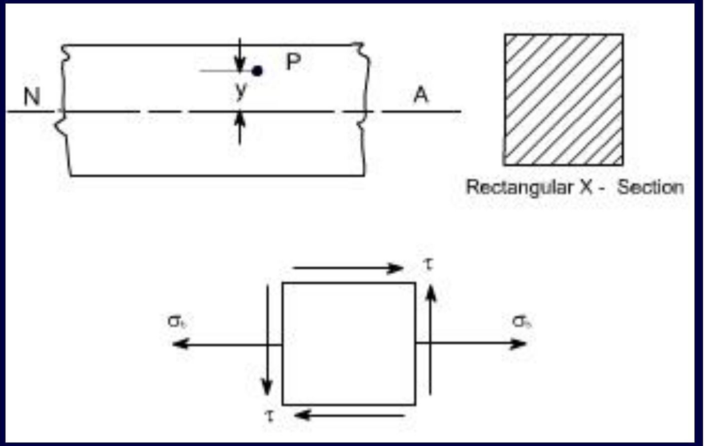
Also

$$\tau_{avg} \text{ or } \tau_{mean} = \frac{F}{A} = \frac{F}{\pi R^2}$$

Hence,

$$\boxed{\tau_{maxm} = \frac{4}{3}\tau_{avg}}$$

Principal Stresses in Beams



$\sigma_b = \dfrac{My}{I}$ for a beam of rectangular cross - section of dimensions b and d; $I = \dfrac{bd^3}{12}$

$$\sigma_b = \frac{12\,My}{bd^3}$$

whereas the value shear stress in the rectangular cross - section is given as

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right]$$

Hence the values of principle stress can be determined from the relations,

$$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}$$

Letting $\sigma_y = 0$; $\sigma_x = \sigma_b$, the values of σ_1 and σ_2 can be computed as

$$\text{Hence } \sigma_1 / \sigma_2 = \frac{1}{2}\left(\frac{12My}{bd^3}\right) \pm \frac{1}{2}\sqrt{\left(\frac{12\,My}{bd^3}\right)^2 + 4\left(\frac{6F}{bd^3}\left(\frac{d^2}{4} - y^2\right)\right)^2}$$

$$\sigma_1, \sigma_2 = \frac{6}{bd^3}\left[My \pm \sqrt{\left\{M^2y^2 + F^2\left(\frac{d^2}{4} - y^2\right)^2\right\}}\right]$$

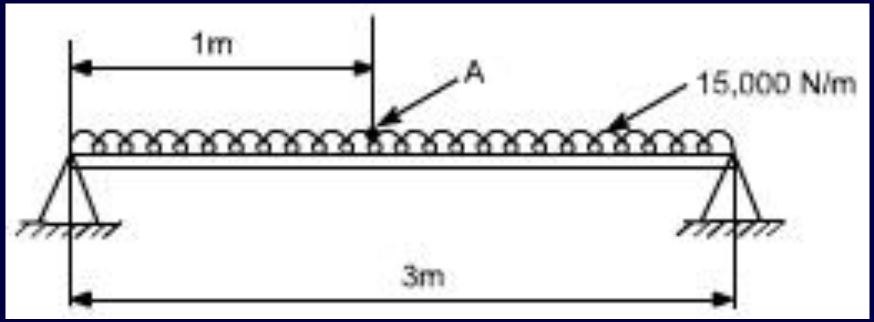
Also,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{putting } \sigma_y = 0$$

we get,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x}$$

Find the principal stress at a point A in a uniform rectangular beam 200 mm deep and 100 mm wide, simply supported at each end over a span of 3 m and carrying a uniformly distributed load of 15,000 N/m.



$$S.F|_{x=1m} = 7{,}500\ N$$

$$B.M|_{x=1m} = 15{,}000\ N.m$$

$$\sigma_x = \frac{My}{I}$$

$$= \frac{15{,}000 \times 5 \times 10^{-2} \times 12}{10 \times 10^{-12} \times \left(20 \times 10^{-2}\right)^3}$$

$$\sigma_x = 11.25\ MN/m^2$$

For the compution of shear stresses

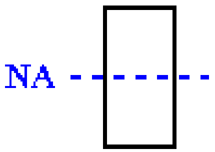
$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right] \qquad \text{putting } y = 50\ mm,\ d = 200\ mm$$

$$F = 7500\ N$$

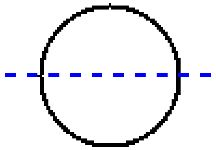
$$\tau = 0.422\ MN/m^2$$

$\sigma_1 = 11.27\ MN/m^2$ and

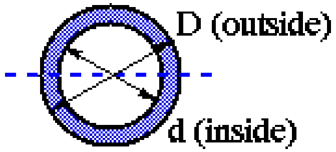
$\sigma_2 = -\,0.025\ MN/m^2$



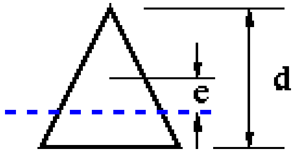
$\tau = \dfrac{3V}{2A}$ $\qquad e = 0$



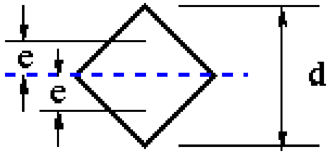
$\tau = \dfrac{4V}{3A}$ $\qquad e = 0$



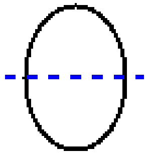
$\tau = \dfrac{4V}{3A}\left(1 + \dfrac{Dd}{D^2 + d^2}\right)$ $\qquad e = 0$



$\tau = \dfrac{3V}{2A}$ $\qquad e = \dfrac{d}{8}$

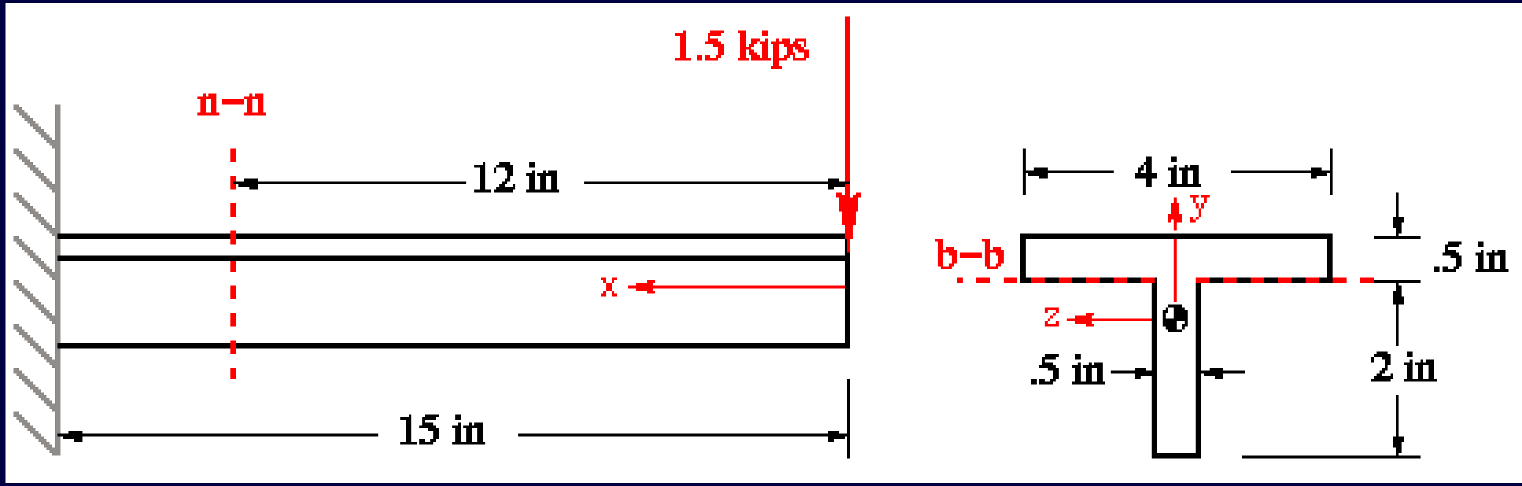


$\tau = \dfrac{9V}{8A}$ $\qquad e = \dfrac{d}{8}$



$\tau = \dfrac{4V}{3A}$ $\qquad e = 0$

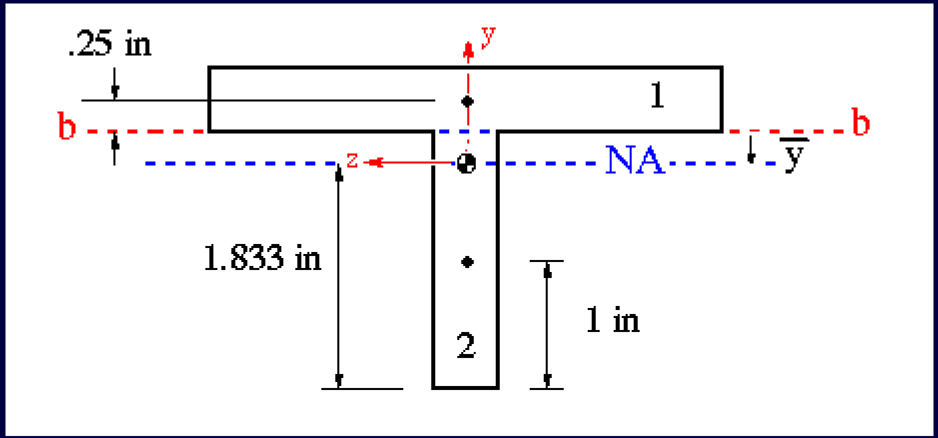
Problem 2



$$\bar{y} = \frac{\Sigma A y}{\Sigma A}$$

$$= \frac{4(.5)(-.25) + 2(.5)(1)}{4(.5) + 2(.5)}$$

$$\bar{y} = .1667 \ in$$

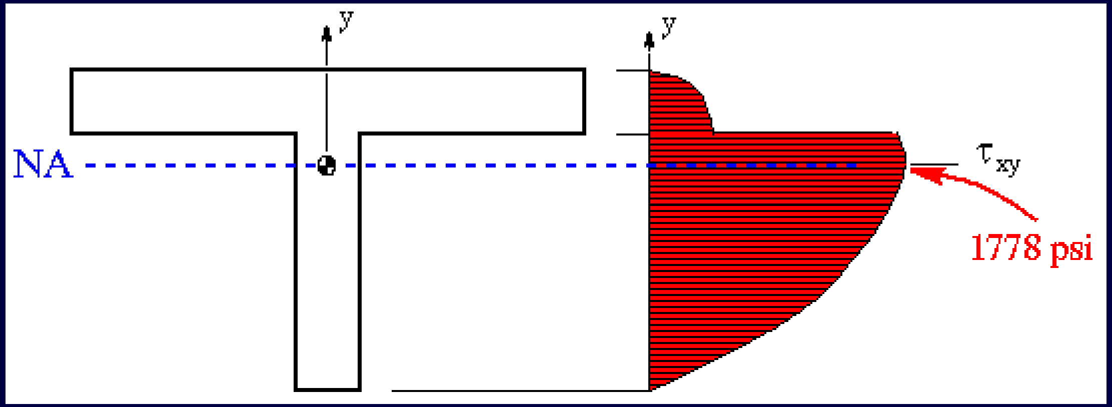


$$I_z = \frac{1}{12}(4)(.5)^3 + 4(.5)(.25+.1667)^2$$
$$+ \frac{1}{12}(.5)(2)^3 + 2(.5)(1-.1667)^2$$
$$I_z = 1.417\ in^4$$

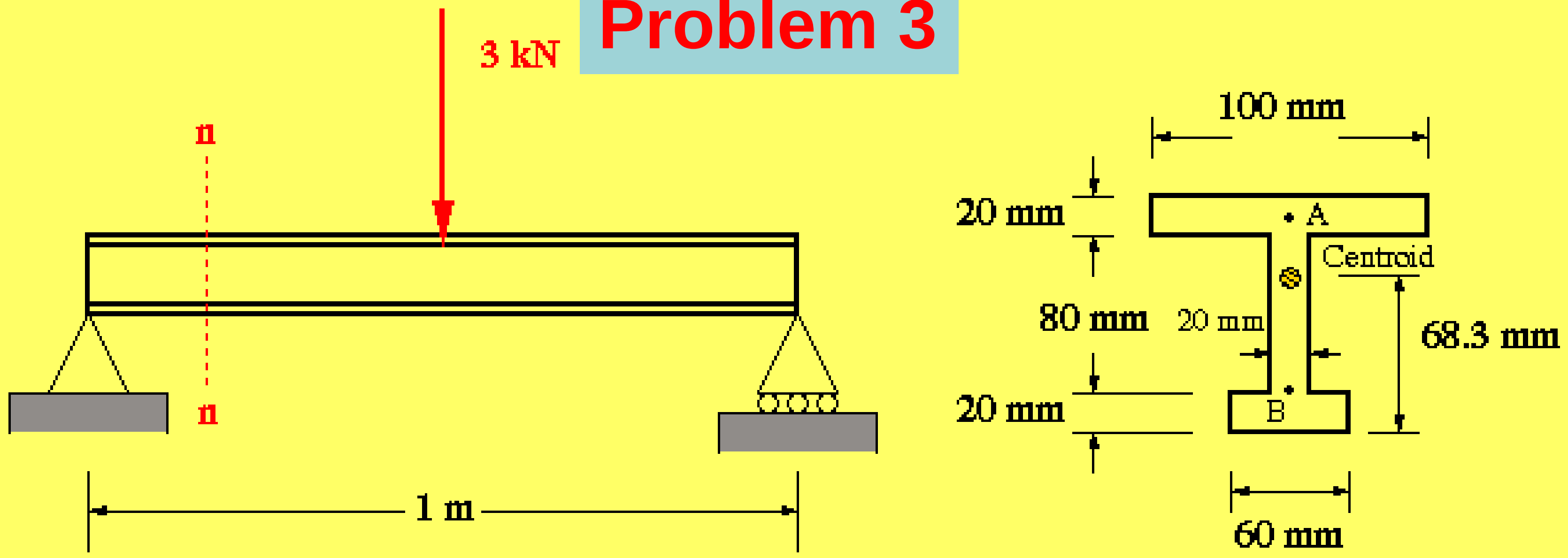
$$M_z = Px$$
$$= 1500(12)$$
$$= 18{,}000\ lb\text{-}in$$

$$\sigma_{x_{top}} = -\frac{M_z y_{top}}{I_z}$$
$$= -\frac{18{,}000(1.833)}{1.417}$$
$$\sigma_{x_{top}} = -23{,}300\ psi$$

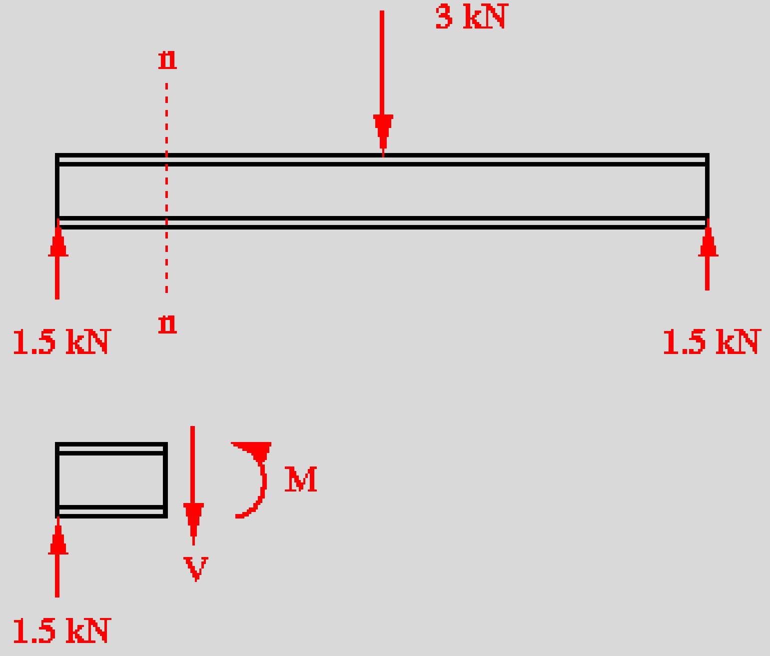
$$\tau_{xy_{max}} = \frac{V_y Q_z}{I_z t}$$
$$= \frac{1500(.84)}{1.417(.5)}$$
$$\tau_{xy_{max}} = 1778\ psi$$



Problem 3



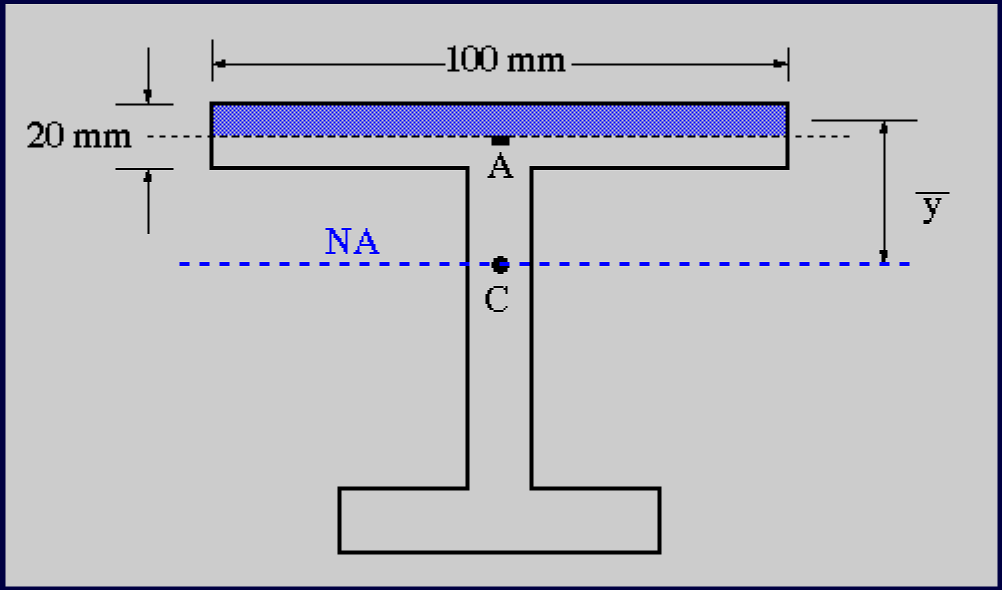
$I = 8.63 \times 10^{-6}\ m^4$



$$\sum F_y = 0$$

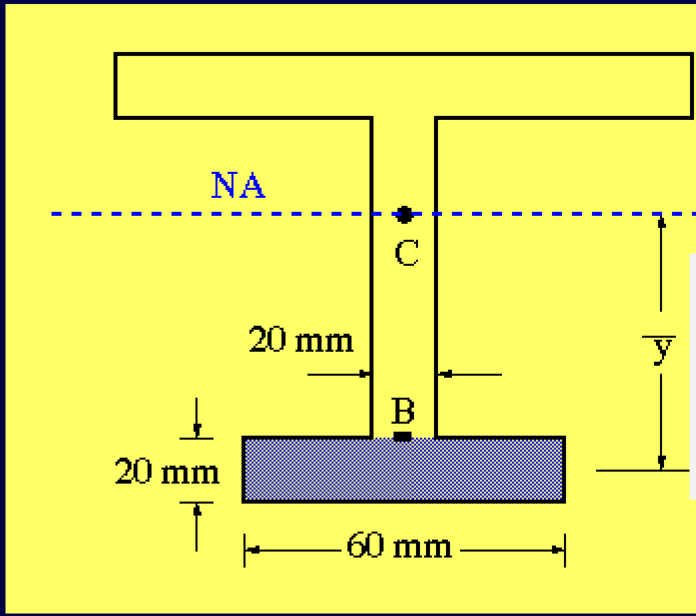
$$1.5 - V = 0$$

$$V = 1.5kN$$



$$Q = A\bar{y}$$
$$= (.1)(.01)(.0517 - .005)$$
$$Q = 4.67x10^{-5}\ m^3$$

$$\tau_{ave} = \frac{VQ}{It}$$
$$= \frac{(1500)(4.67x10^{-5})}{(8.63x10^{-6})(.1)}$$
$$\tau_{ave} = 81.17\ kPa$$



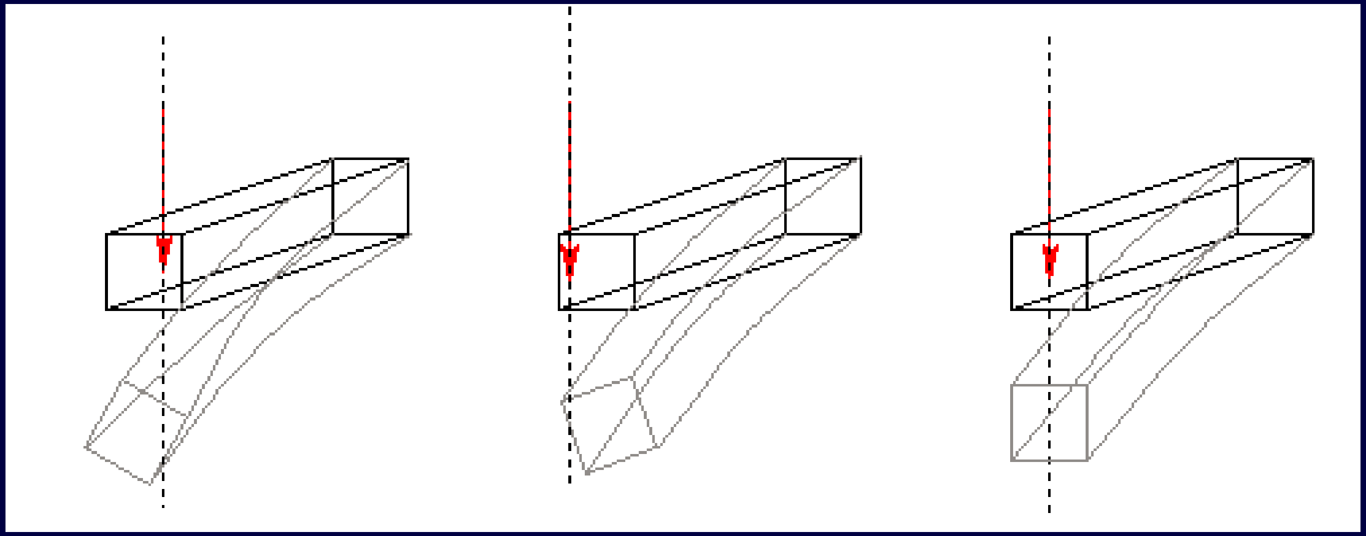
$$Q = A\bar{y}$$
$$= (.06)(.02)(.0683 - .02/2)$$
$$Q = 7x10^{-7}\ m^3$$

$$\tau_{ave} = \frac{VQ}{It}$$
$$= \frac{(1500)(7x10^{-7})}{(8.63x10^{-6})(.02)}$$
$$\tau_{ave} = 608\ kPa$$

Shear Centre

Consider the figure below showing a cantilever beam with a transverse force at the tip.

- Under the action of this load, the beam may twist as it bends.
- It is the line of action of the lateral force that is responsible for this bend-twist coupling.
- If the line of action of the force passes through the **Shear Center** of the beam section, then the beam will only bend without any twist.
- Otherwise, twist will accompany bending.

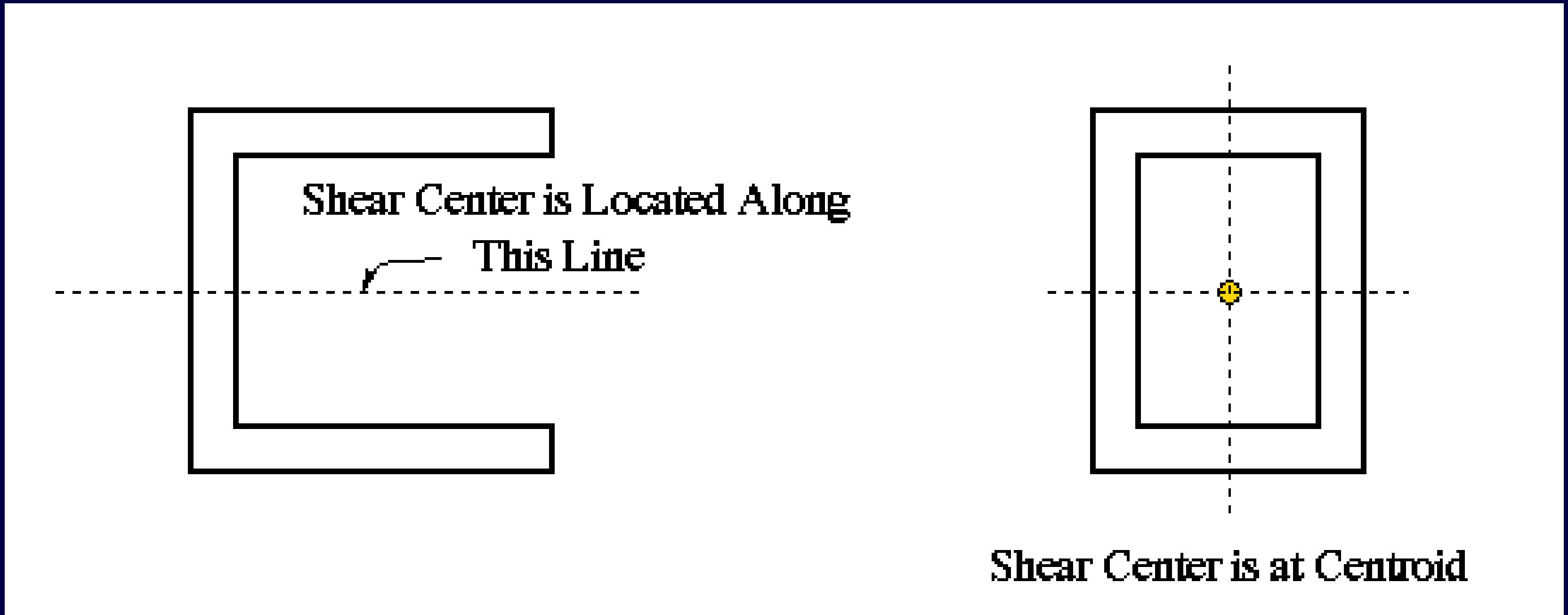


Shear center is defined as the point on the beam section where load is applied and no twisting is produced

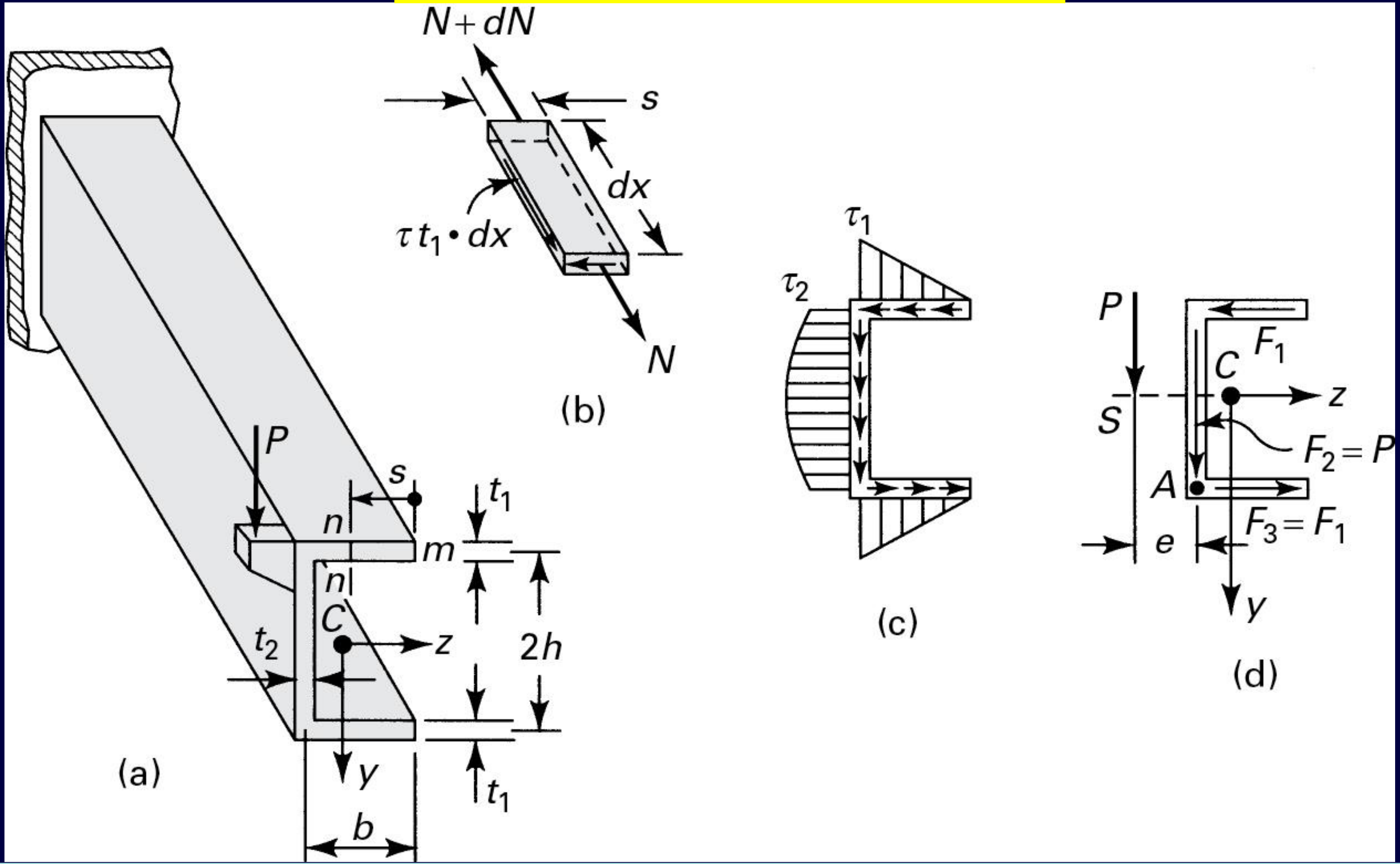
Shear Centre

The two following points facilitate the determination of the shear center location.

1. The shear center always falls on a cross-sectional axis of symmetry.
2. If the cross section contains two axes of symmetry, then the shear center is located at their intersection. Notice that this is the only case where shear center and centroid coincide.



Shear Centre in C-Sections



The shearing stress in the upper flange at any section *nn* will be found first. This section is located a distance *s* from the free edge *m*, as shown in the figure. At *m*, the shearing stress is zero. The first moment of area st_1 about the *z* axis is $Q_z = st_1h$. The shear stress at *nn*,

$$\tau_{xz} = \frac{V_y Q_z}{I_z b} = P\frac{sh}{I_z} \qquad \text{(a)}$$

Here the normal force $N = t_1 s \sigma_x$, owing to the bending of the beam, increases with dx by dN. Hence, the x equilibrium of the element requires that $\tau t_1 \cdot dx$ must be directed as shown in the figure. This flange force is directed to the left, because the shear forces must intersect at the corner of the element.

$$\tau_1 = P\frac{bh}{I_z} \qquad \text{(b)}$$

$$\tau_2 = P\frac{bt_1 h}{t_2 I_z} \qquad \text{(c)}$$

$$F_1 = \tfrac{1}{2}\tau_1 b t_1 = P\frac{b^2 h t_1}{2I_z} \qquad \text{(d)}$$

Symmetry of the section dictates that $F_1 = F_3$. We will assume that the web force F_2 = P, since the vertical shearing force transmitted by the flange is negligibly small. The shearing force components acting in the section must be statically equivalent to the resultant shear load P. Thus, the principle of the moments for the system of forces applied at A, yields $M_x = pe = 2F_1 h$. Substituting F_1 from Eq. (d) into this expression, we obtain

$$e = \frac{b^2 h^2 t_1}{I_z}$$

$$I_z = \tfrac{2}{3} t_2 h^3 + 2bt_1 h^2$$

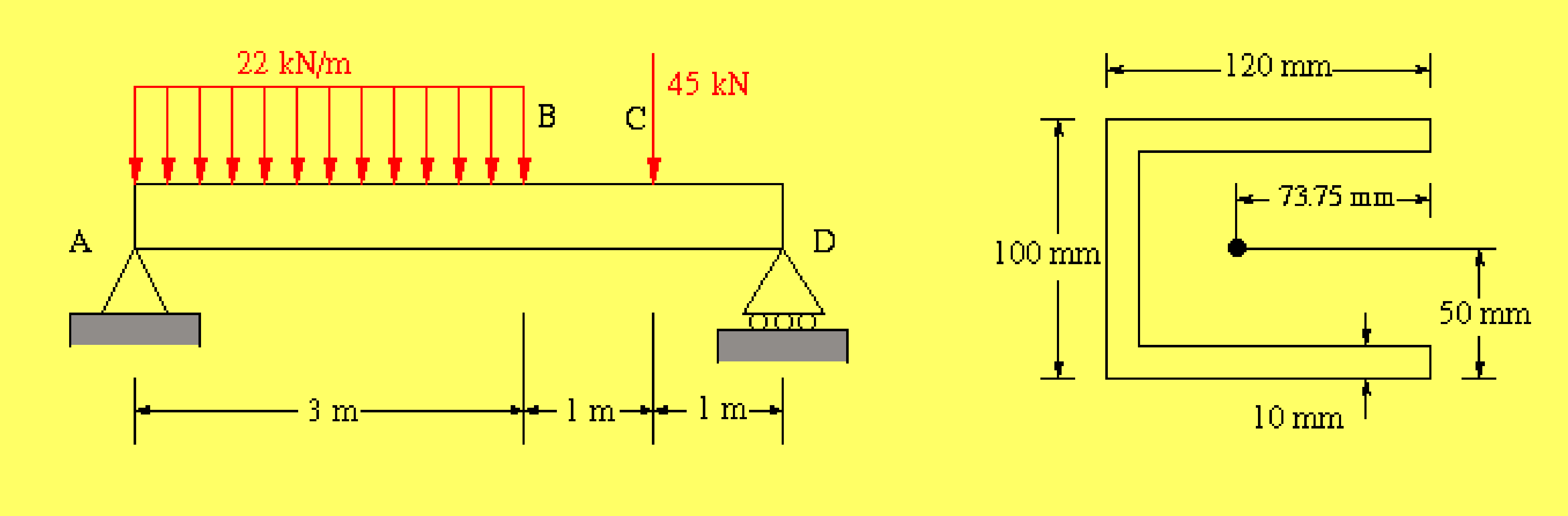
$$e = \tfrac{3}{2}\frac{b^2 t_1}{ht_2 + 3bt_1} \qquad \text{(e)}$$

Problem 4

For the beam and loading shown, determine:

- the location and magnitude of the maximum transverse shear force 'Vmax',
- the shear flow 'q' distribution due the 'Vmax',
- the 'x' coordinate of the shear center measured from the centroid,
- the maximum shear stress and its location on the cross section.

Stresses induced by the load do not exceed the elastic limits of the material.



$$\sum M_A = 0$$
$$0 = 22(3)(1.5) + 45(4) - R_D(5)$$
$$R_D = 55.8\ kN$$

$$\sum F_y = 0$$
$$0 = R_A - 22(3) - 45 + 55.8$$
$$R_A = 55.2\ kN$$

The transverse shear can be found by using mechanics of materials

$$V = -\int w(x)\ dx$$

A to B

$$V = -\int (22)\ dx + 55.2$$
$$= 55.2 - 22x \quad (linear\ variation)$$
$$V_A = 55.2\ kN$$
$$V_B = -10.8\ kN$$

B to C

$$V_{B-C} = -10.8\ kN \quad (constant)$$

@ C

$$V_C = -10.8 - 45$$
$$= -55.8\ kN$$

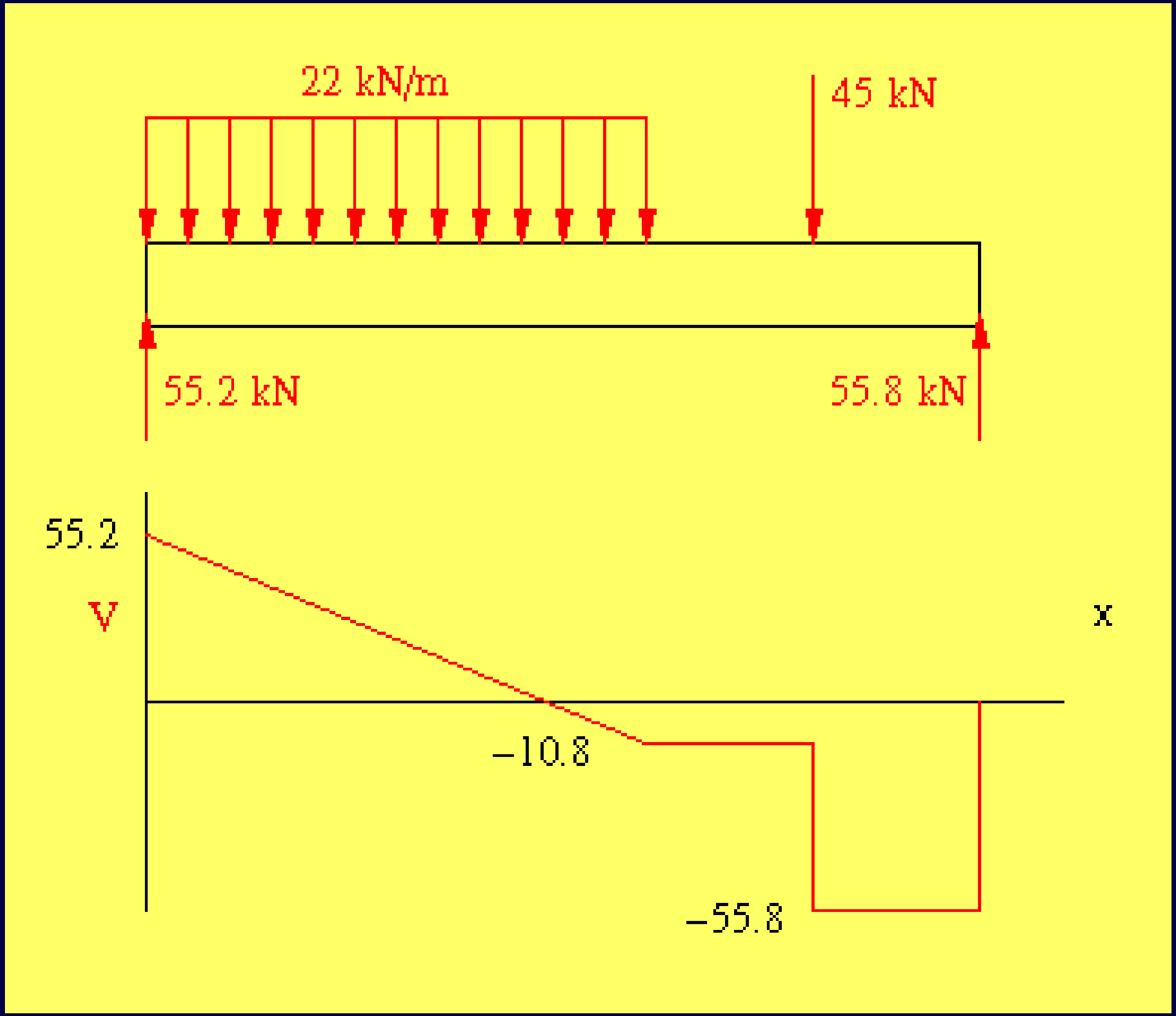
C to D

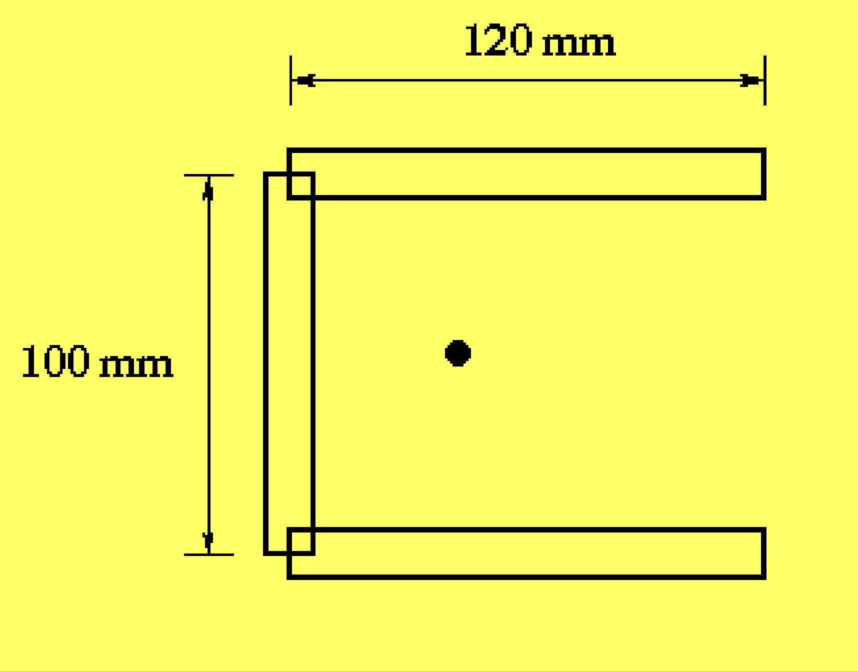
$$V_{C-D} = -55.8\ kN \quad (constant)$$

@ D

$$V_D = -55.8 + 55.8$$
$$= 0\ kN$$

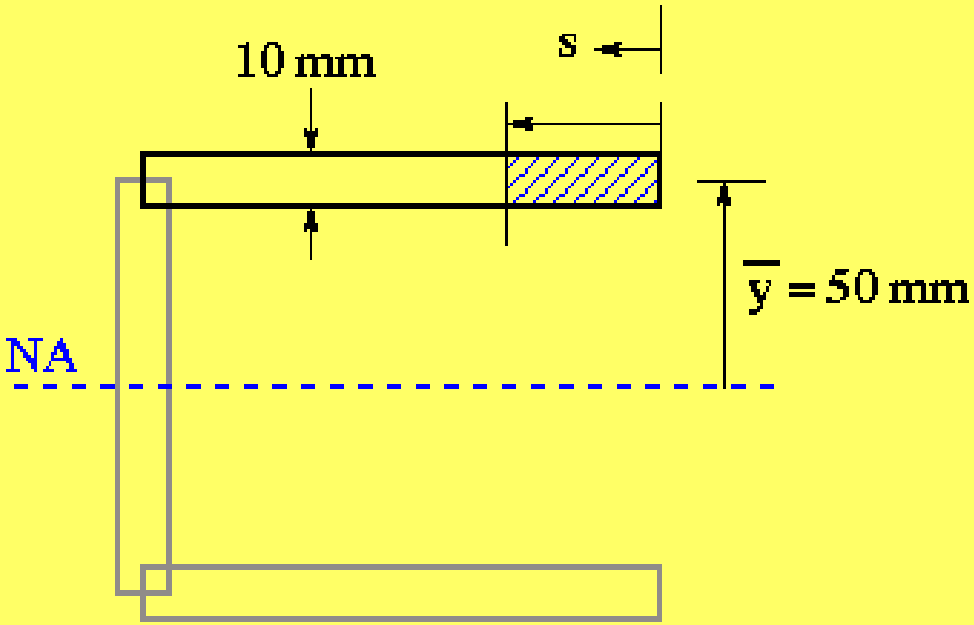
Shear Force Diagram





$$I = \frac{1}{12}(.01)(.1)^3 + 2\left[\frac{1}{12}(.12)(.01)^3 + (.12)(.01)(.05)^2)\right]$$

$$= 6.8533x10^{-6}\ m^4$$



Top Flange

$$q = \frac{V}{I}\bar{y}A$$

$$= \frac{55.8}{6.8533x10^{-6}}(.05)(.01)s$$

$$q = 4071.2s$$

$$0 \le s \le .12\ m$$

$$q_0 = 0\ \frac{kN}{m}$$

$$q_{.12} = 488.5\ \frac{kN}{m}$$

Web

$$q = 488.5 + \frac{55.8}{6.8533x10^{-6}}\left(.05 - \frac{w}{2}\right)(.01)w$$

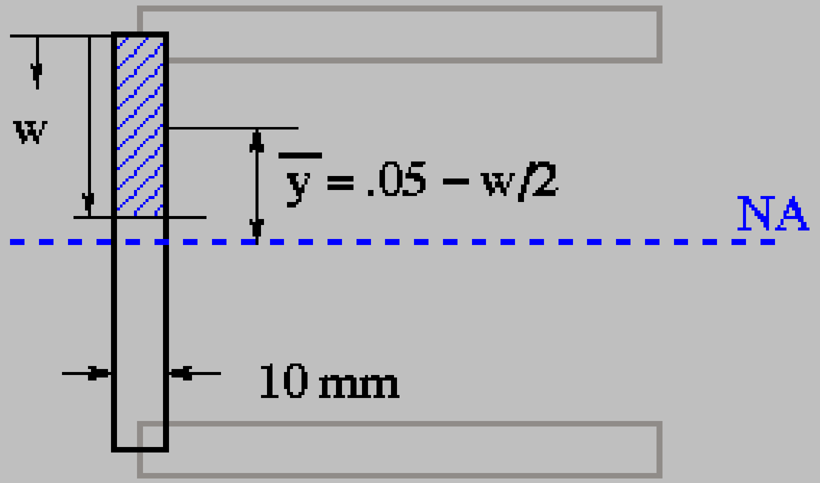
$$= 488.5 + 4071.2w - 40{,}712w^2$$

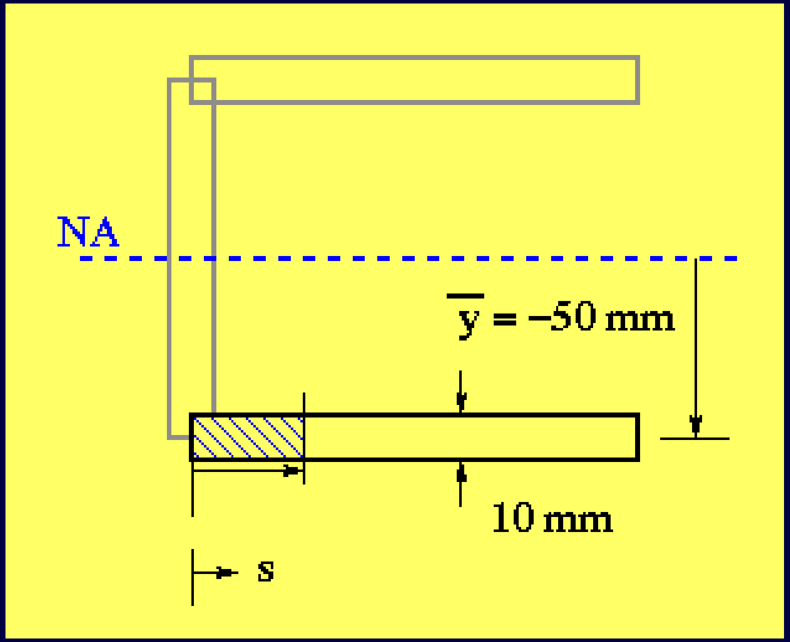
$$0 \le w \le .1\ m$$

$$q_o = 488.5\ \frac{kN}{m}$$

$$q_{NA} = 590.3\ \frac{kN}{m}$$

$$q_{.1} = 499.5\ \frac{kN}{m}$$





Bottom Flange

$$q = \frac{V}{I} A\bar{y}$$

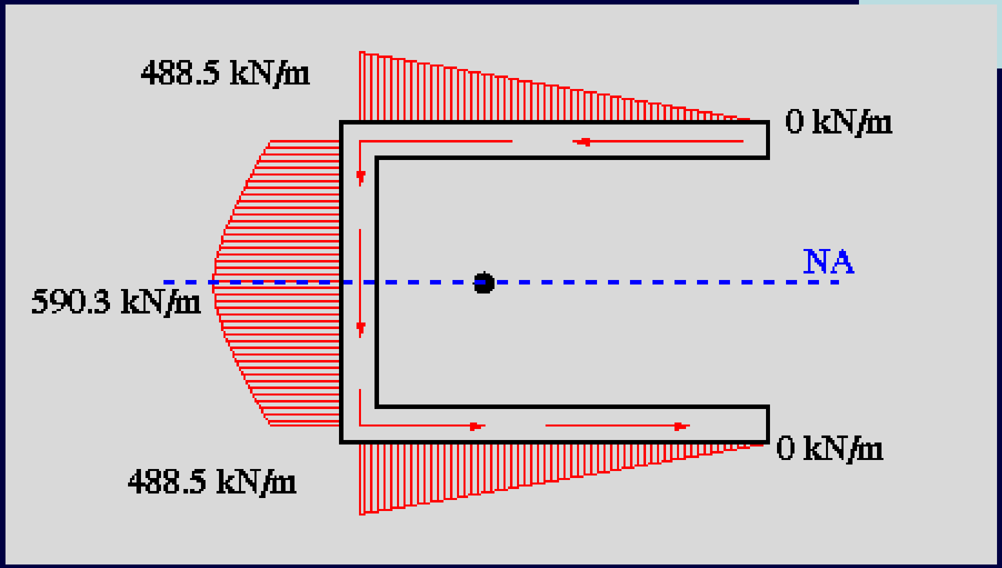
$$= 488.5 + \frac{55.8}{6.8533x10^{-6}}(-.05)(.01)s$$

$$q = 488.5 - 4071.2s$$

$$0 \le s \le .12\ m$$

$$q_0 = 488.5\ \frac{kN}{m}$$

$$q_{.12} = 0\ \frac{kN}{m}$$



Top Flange

$$F_{Top} = \int_0^{.12} 4071.2s \; ds$$

$$= 29.31 \; kN$$

Web

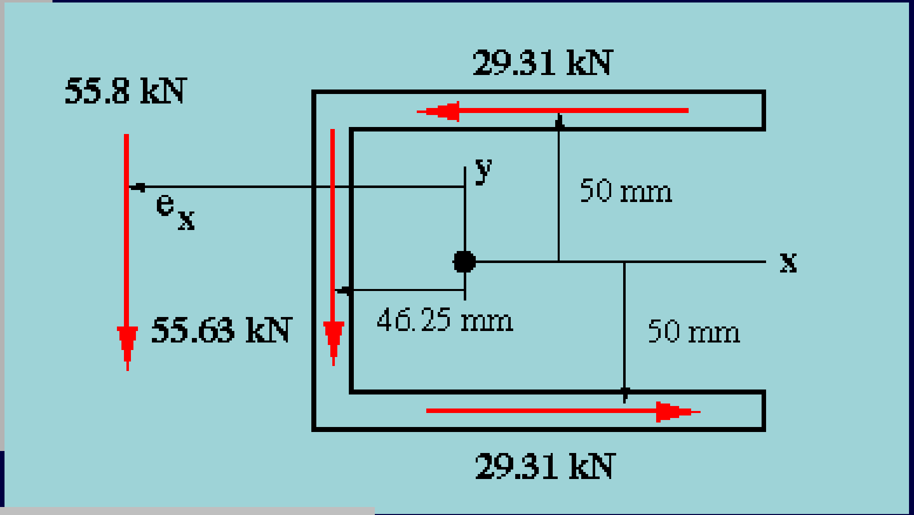
$$F_{Web} = \int_0^{.1} (488.5 + 4071.2w - 40{,}712w^2) \; dw$$

$$= 55.64 \; kN$$

Top Flange

$$F_{Top} = \int_{.12}^{0} (-4071.2s) \; ds$$

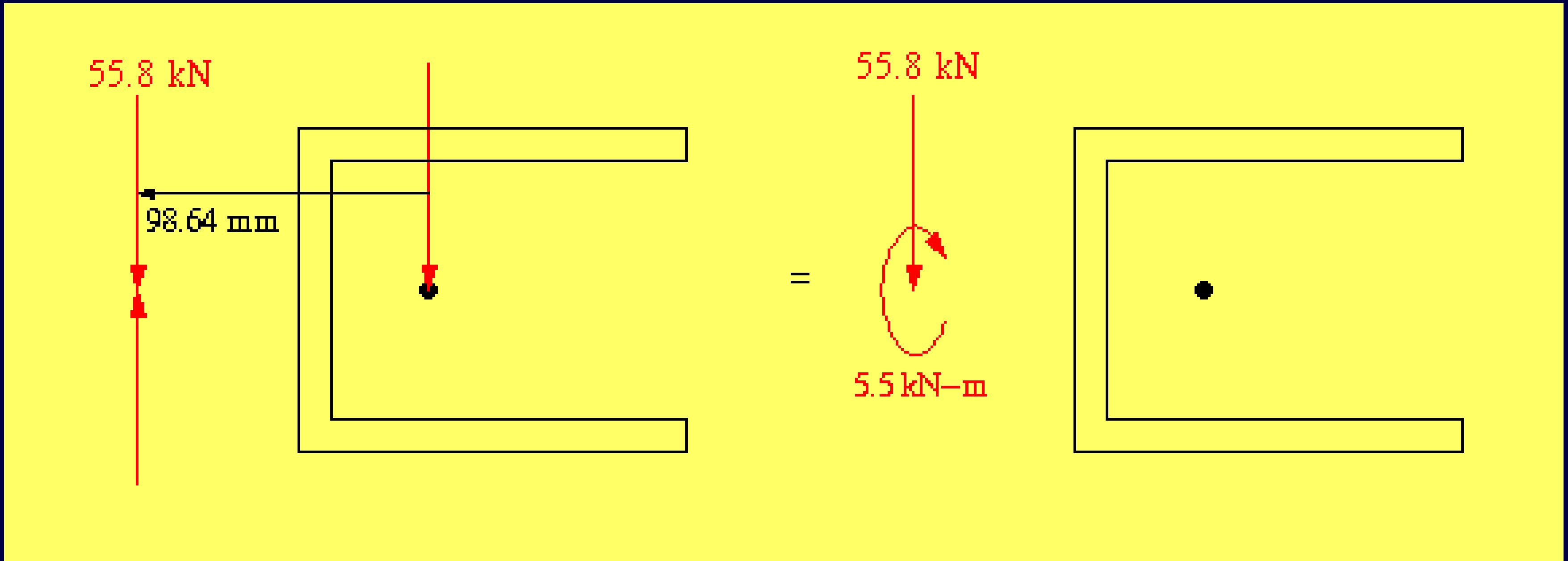
$$= 29.31 \; kN$$



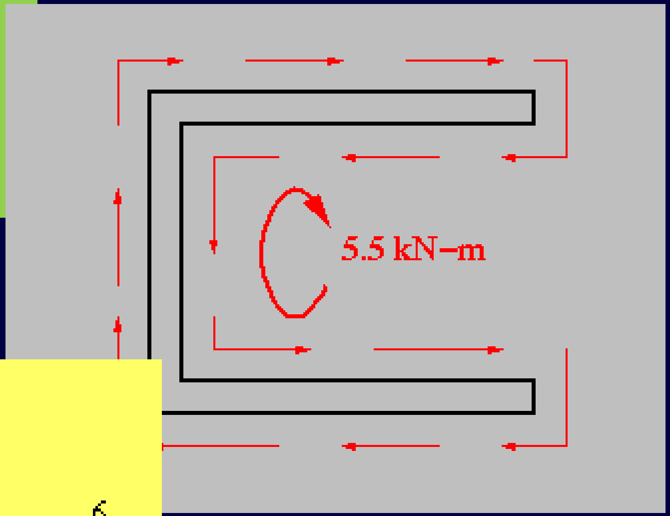
$$\sum M_o \Big|_{Componants} = \sum M_o \Big|_{Result}$$

$$29.31(.05) + 29.31(.05) + 55.64(.04625) = -(55.8)e_x$$

$$e_x = .09864 \; m$$



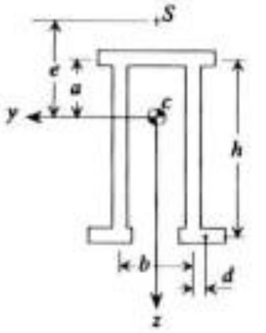
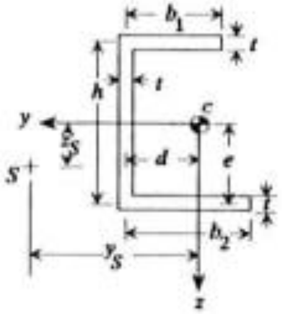
$$T = 55.8(.09864)$$
$$= 5.5 \ kN\text{–}m$$



$$\tau_{max} = \tau_{\text{max}}\big|_V + \tau_{max}\big|_T$$
$$\tau_{\text{max}} = 590.3x10^3 + 485.3x10^6$$
$$= 485.89 \ MPa$$

$$\frac{a}{b} = \frac{.12+.12+.1}{.01}$$
$$= 34 > 10 \quad OK$$

$$\tau_{max} = \frac{3T}{at^2}$$
$$= \frac{3(5.5)}{(.12+.12+.1)(.02)^2}$$
$$\tau_{max} = 485.3 \ MPa$$

Shape	Location of Shear Center (S), e	Warping Constant, Γ
4. 	$-a\left(1+\frac{b^2A}{4I_z}\right)+2h\frac{I_F}{I_z}$ where A = total area I_F = moment of inertia of each lower flange with respect to web axis I_z, I_y = moments of inertia with respect to z, y axes Ref. 2.7	$\frac{b^2}{4}\left[I_y+a^2A\left(1-\frac{b^2A}{4I_z}\right)\right]$ $+2h^2I_F-2bdh^2A_F+b^2ahA\frac{I_F}{I_z}-4h^2\frac{I_F^2}{I_z}$ where A_F = area of each lower flange d = distance of centroid of lower flange from the web axis Ref. 2.7
5. Channel with unequal flanges 	$z_S = e - \frac{b_1^2ht}{6(I_yI_z - I_{yz}^2)}$ $\times\left[-3I_{yz}(h-e)+I_y(2b_1-3d)\right]$ $y_S = d + \frac{b_1^2ht}{6(I_yI_z - I_{yz}^2)}$ $\times\left[-I_{yz}(2b_1-3d)+3I_z(h-e)\right]$ $e = \frac{h^2+2b_1h}{2h(b_1+b_2)}, \quad d = \frac{b_1^2+b_2^2}{2h(b_1+b_2)}$ Ref. 2.9	Use computer program