Inverse Laplace Transforms

12.1 Introduction

It is seen that the Laplace transform converts ordinary derivatives to algebraic functions. To solve a differential equation, we convert it into an algebraic equation, and then after simplification, we apply the inverse Laplace transform to obtain the value of the dependent variable. Thus the inverse Laplace transform is very essential to solve a differential equation.

12.2 Definition and Some Standard Results

If $L\{f(t)\} = \overline{f}(s)$, then we say that $f(t)$ is the **inverse Laplace transform** of $\overline{f}(s)$ and symbolically

$$L^{-1}\{\overline{f}(s)\} = f(t)$$

For example, $L\{1\} = \frac{1}{s}$, thus $L^{-1}\left\{\frac{1}{s}\right\} = 1$.

Some Standard Results

(i) Since $L\{1\} = \frac{1}{s}$

$$L^{-1}\left\{\frac{1}{s}\right\} = 1 \tag{12.1}$$

(ii) Since $L\{e^{at}\} = \frac{1}{s-a}$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{12.2}$$

(iii) Since $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} \quad n = 1, 2, \ldots \tag{12.3}$$

(iv) Since $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at \tag{12.4}$$

(v) Since $L\{\cos at\} = \frac{s}{s^2+a^2}$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at \tag{12.5}$$

(vi) Since $L\{\sinh at\} = \frac{a}{s^2-a^2}$

$$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at \tag{12.6}$$

(vii) Since $L\{\cosh at\} = \frac{s}{s^2-a^2}$

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at \tag{12.7}$$

(viii)

$$L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}t^{n-1}}{(n-1)!} \tag{12.8}$$

(ix)

$$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt \tag{12.9}$$

(x)

$$L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt \tag{12.10}$$

The inverse Laplace transform may not be unique. Lerch's theorem gives the condition for uniqueness of the inverse Laplace transform.

Theorem 12.1 (Lerch's theorem) *If the function $f(t)$ is sectionally continuous in $[0, N]$ for each positive integer N and if there exists a real constant $M > 0$ such that for all $t > N$, $|f(t)| < Me^{\alpha t}$ for some α, then $L^{-1}\{\overline{f}(s)\} = f(t)$ is unique.*

12.3 Properties of Inverse Laplace Transform

12.3.1 Linear Property

Theorem 12.2 *If $L\{f_1(t)\} = \overline{f_1}(s)$ and $L\{f_2(t)\} = \overline{f_2}(s)$, then $L^{-1}\{c_1\overline{f_1}(s) + c_2\overline{f_2}(s)\} = c_1L^{-1}\{\overline{f_1}(s)\} + c_2L^{-1}\{\overline{f_2}(s)\}$, where c_1 and c_2 are arbitrary constants.*

The proof of this theorem is trivial.

For example

$$\begin{aligned} L^{-1}\left\{\frac{1}{s^2}+\frac{4}{s-5}-\frac{s}{s^2+9}\right\} &= L^{-1}\left\{\frac{1}{s^2}\right\}+L^{-1}\left\{\frac{4}{s-5}\right\}-L^{-1}\left\{\frac{s}{s^2+9}\right\} \\ &= t+4e^{5t}-\cos 3t \end{aligned}$$

12.3.2 Shifting Property

Theorem 12.3 (First shifting theorem) *If* $L\{f(t)\} = \overline{f}(s)$, *then* $L^{-1}\{\overline{f}(s-a)\} = e^{at}f(t)$ $= e^{at}L^{-1}\{\overline{f}(s).\}$

This result follows from the relation $L\{e^{at}f(t)\} = \overline{f}(s-a)$.

For example, $L^{-1}\left\{\dfrac{1}{s^2}\right\} = t$, therefore $L^{-1}\left\{\dfrac{1}{(s-1)^2}\right\} = e^t t$. (WBUT 2005)

EXAMPLE 12.3.1 Find $L^{-1}\left\{\dfrac{s+1}{s^2+s+1}\right\}$. (WBUT 2003)

Solution We have

$$\begin{aligned}
L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} &= L^{-1}\left\{\frac{s+1/2+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} \\
&= L^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} + L^{-1}\left\{\frac{1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} \\
&= e^{-t/2}L^{-1}\left\{\frac{s}{s^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{2}\frac{2}{\sqrt{3}}L^{-1}\left\{\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} \\
&= e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}\right)t + \frac{1}{\sqrt{3}}e^{-t/2}L^{-1}\left\{\frac{\sqrt{3}/2}{s^2+(\sqrt{3}/2)^2}\right\} \\
&= e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}\right)t + \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}\right)t
\end{aligned}$$

Theorem 12.4 (Second shifting theorem) *If* $L^{-1}\{\overline{f}(s)\} = f(t)$, *then*

$$L^{-1}\{e^{-as}\overline{f}(s)\} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

Proof. Let us define a function $g(t)$ as

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

Since $L^{-1}\{\overline{f}(s)\} = f(t)$, $L\{f(t)\} = \overline{f}(s)$.
Now by second shifting property of Laplace transform, $L\{g(t)\} = e^{-as}\overline{f}(s)$.
Therefore, $L^{-1}\{e^{-as}\overline{f}(s)\} = g(t)$, i.e.

$$L^{-1}\{e^{-as}\overline{f}(s)\} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

EXAMPLE 12.3.2 Find (i) $L^{-1}\left\{\dfrac{2e^{-2s}}{s^2+9}\right\}$ (ii) $L^{-1}\left\{\dfrac{3se^{-\pi s}}{s^2+16}\right\}$.

Solution (i) We know $L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin 3t.$

Thus by second shifting property

$$L^{-1}\left\{\frac{2e^{-2s}}{s^2+9}\right\} = \begin{cases} \frac{2}{3}\sin 3(t-2), & t>2 \\ 0, & t<2 \end{cases}$$

(ii) Since $L^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t.$

Thus

$$L^{-1}\left\{\frac{3se^{-\pi s}}{s^2+16}\right\} = \begin{cases} 3\cos 4(t-\pi), & t>\pi \\ 0, & t<\pi \end{cases}$$

12.3.3 Change of Scale Property

Theorem 12.5 *If $L^{-1}\{\overline{f}(s)\} = f(t)$, then $L^{-1}\{\overline{f}(as)\} = \frac{1}{a}f(t/a)$, a is a constant.* (WBUT 2002)

Proof. Let us consider

$$L\left\{\frac{1}{a}f(t/a)\right\} = \frac{1}{a}L\{f(t/a)\} = \frac{1}{a}\cdot\frac{1}{1/a}\overline{f}\left(\frac{s}{1/a}\right) = \overline{f}(as)$$

(By change of scale property of Laplace transform)

Thus, $L^{-1}\{\overline{f}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right).$

For example, $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at.$ So $L^{-1}\left\{\frac{3s}{9s^2+a^2}\right\} = \frac{1}{3}\cos\frac{at}{3}.$

12.4 On Derivatives

Theorem 12.6 *Let $L\{f(t)\} = \overline{f}(s)$. Then $L^{-1}\left\{\overline{f}'(s)\right\} = -tL^{-1}\{\overline{f}(s)\} = -tf(t)$.*

Proof. $L\{tf(t)\} = -\frac{d}{ds}\overline{f}(s) = -\overline{f}'(s)$, that is, $-L^{-1}\left\{\overline{f}'(s)\right\} = tf(t)$, or $L^{-1}\left\{\overline{f}'(s)\right\} = -tf(t)$.

In general, since

$$L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}\left\{\overline{f}(s)\right\}$$

$$L^{-1}\left\{\overline{f}^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

where $\overline{f}^{(n)}(s) = \frac{d^n}{ds^n}\left\{\overline{f}(s)\right\}$

EXAMPLE 12.4.1 Find (i) $L^{-1}\left\{\log\frac{s(s^2+9)}{s^2+16}\right\}$ (ii) $L^{-1}\{\cot^{-1}\frac{s}{2}\}$.

Solution Let $\bar{f}(s) = \log \dfrac{s(s^2+9)}{s^2+16} = \log s + \log(s^2+9) - \log(s^2+16)$,

$\bar{f}'(s) = \dfrac{1}{s} - \dfrac{2s}{s^2+9} - \dfrac{2s}{s^2+16}$.

Therefore

$$\begin{aligned} L^{-1}\{\bar{f}'(s)\} &= L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{2s}{s^2+9}\right\} - L^{-1}\left\{\frac{2s}{s^2+16}\right\} \\ &= 1 - 2\cos 3t - 2\cos 4t \end{aligned}$$

or

$$-tf(t) = 1 - 2\cos 3t - 2\cos 4t$$

or

$$f(t) = \frac{2\cos 3t + 2\cos 4t - 1}{t} = L^{-1}\left\{\frac{s(s^2+9)}{s^2+16}\right\}$$

(ii) Let $\bar{f}(s) = \cot^{-1}\left(\frac{s}{2}\right)$

Therefore,

$$\bar{f}'(s) = -\frac{1}{1+(s/2)^2}\cdot\frac{1}{2} = -\frac{2}{4+s^2}$$

Now

$$L^{-1}\{\bar{f}'(s)\} = -L^{-1}\left\{\frac{2}{s^2+2^2}\right\} = -\sin 2t$$

That is, $-tf(t) = -\sin 2t$, or $f(t) = \dfrac{\sin 2t}{t}$, or $L^{-1}\{\bar{f}(s)\} = \cot^{-1}\left(\dfrac{s}{2}\right) = \dfrac{\sin 2t}{t}$.

12.5 Multiplication by s^n

Theorem 12.7 *If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0) = 0$, then $L^{-1}\{s\bar{f}(s)\} = f'(t)$.*

Proof. From the relation $L\{f'(t)\} = s\bar{f}(s) - f(0)$, we have $L^{-1}\{s\bar{f}(s)\} = f'(t)$ as $f(0) = 0$.

In general, $L^{-1}\{s^n\bar{f}(s)\} = f^{(n)}(t)$, provided $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$.

EXAMPLE 12.5.1 Find (i) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$, (ii) $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)^2}\right\}$, (iii) $L^{-1}\left\{\dfrac{s^2}{(s+1)^5}\right\}$

(WBUT 2006)

Solution (i) Let $f(t) = \sin at$. Then $L\{f(t)\} = L\{\sin at\} = \dfrac{a}{s^2+a^2}$.

Now, $L\{t\sin at\} = -\dfrac{d}{ds}\left\{\dfrac{a}{s^2+a^2}\right\} = \dfrac{2as}{(s^2+a^2)^2}$, or $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a}t\sin at$.

(ii) Let $\phi(t) = \dfrac{1}{2a}t\sin at$. Then $\phi(0) = 0$.

Now, from the formula $L^{-1}\{s\bar{f}(s)\} = f'(t)$

$$L^{-1}\left\{s\cdot\frac{s}{(s^2+a^2)^2}\right\} = \phi'(t) = \frac{1}{2a}(\sin at + at\cos at)$$

(iii) We know that $L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$.

Therefore, $L^{-1}\left\{\frac{1}{(s+1)^5}\right\} = e^{-t}\frac{t^4}{4!} = f(t)$, say, by first shifting property.

Now by the formula, $L^{-1}\{s\bar{f}(s)\} = f'(t)$, as $f(0) = 0$, we have

$$L^{-1}\left\{\frac{s}{(s+1)^5}\right\} = \frac{d}{dt}\left\{e^{-t}\frac{t^4}{4!}\right\} = e^{-t}\frac{t^3}{4!}(4-t) = \phi(t) \qquad \text{(say)}$$

Also, $\phi(0) = 0$.

Again, using the same formula

$$L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\} = \frac{d}{dt}\left\{e^{-t}\frac{t^3}{4!}(4-t)\right\} = e^{-t}\frac{t^2}{4!}(t^2 - 8t + 12)$$

12.6 Division by s

Theorem 12.8 *If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)dt$.*

Proof. Let $F(t) = \int_0^t f(u)du$. Thus, $F'(t) = f(t)$ and $F(0) = 0$.

Then

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

or

$$L\{F'(t)\} = sL\{F(t)\}$$

or

$$L\{F(t)\} = \frac{1}{s}L\{F'(t)\} = \frac{1}{s}L\{f(t)\} = \frac{\bar{f}(s)}{s}$$

That is, $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = F(t) = \int_0^t f(u)du$.

EXAMPLE 12.6.1 Find (i) $\left\{\frac{1}{(s^2+a^2)^2}\right\}$ (ii) $\left\{\frac{1}{s^2(s^2+1)}\right\}$.

Solution (i) From previous example, we have

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$$

Therefore

$$\begin{aligned}
L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} &= L^{-1}\left\{\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \frac{t\sin at}{2a}dt \\
&= \frac{1}{2a}\left\{\left[t\frac{-\cos at}{a}\right]_0^t - \int_0^t 1\cdot\frac{-\cos at}{a}dt\right\} \\
&= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right] = \frac{1}{2a^3}[\sin at - at\cos at]
\end{aligned}$$

(ii) We know $L^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$.

Therefore

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u du = -\left[\cos u\right]_0^t = 1 - \cos t$$

Again, applying the same rule

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos u)du = \left[u - \sin u\right]_0^t = t - \sin t$$

12.7 Inverse Laplace Transform of Integrals

Theorem 12.9 *If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $\left\{\int_s^\infty \bar{f}(u)du\right\} = \dfrac{f(t)}{t}$.*
This result follows from the Laplace transform of $f(t)/t$, as

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(u)du, \text{ i.e. } L^{-1}\left\{\int_s^\infty \bar{f}(u)du\right\} = \frac{f(t)}{t}$$

EXAMPLE 12.7.1 Find the inverse Laplace transform of $\dfrac{s}{(s^2+a^2)^2}$.

Solution Let $f(t) = L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$.

Now

$$\begin{aligned} L\left\{\frac{f(t)}{t}\right\} &= \int_s^\infty \frac{s}{(s^2+a^2)^2}ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+a^2)^2}ds \\ &= -\frac{1}{2}\left[\frac{1}{s^2+a^2}\right]_s^\infty = \frac{1}{2}\frac{1}{s^2+a^2} \end{aligned}$$

Thus, $\dfrac{f(t)}{t} = \dfrac{1}{2}L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{2a}\sin at$.

Hence, $f(t) = \dfrac{1}{2a}t\sin at$.

12.8 Convolution Theorem

Theorem 12.10 (Convolution theorem) *If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$, then $L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)du = F * G$. ($F * G$ is called the* ***convolution*** *of two functions F and G.)*

Proof. Let $\phi(t) = \int_0^t f(u)g(t-u)du$. Then $L\{\phi(t)\} = \int_0^\infty e^{-st}\int_0^t f(u)g(t-u)du\,dt$.
Now, we change the order of the integration. The domain is shown in Fig. 12.1.

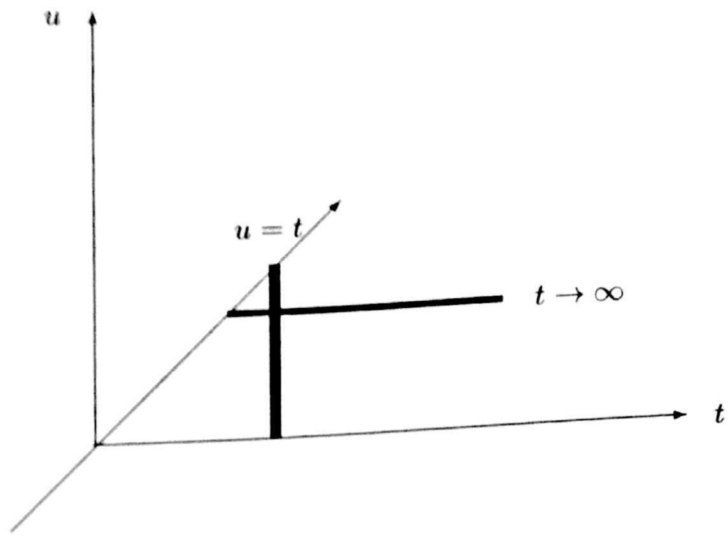

Figure 12.1: Change of order of integration.

Therefore

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u)\, dt\, du$$

(Substituting $t - u = v$. Then $dt = dv$.)

$$\begin{aligned}
&= \int_0^\infty \int_0^\infty e^{-(u+v)s} f(u) g(v)\, dv\, du \\
&= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-vs} g(v)\, dv \right\} du \\
&= \int_0^\infty e^{-su} f(u) \overline{g}(s)\, du = \overline{g}(s) \int_0^\infty e^{-su} f(u)\, du \\
&= \overline{g}(s) \overline{f}(s)
\end{aligned}$$

Therefore, $L^{-1}\{\overline{g}(s)\overline{f}(s)\} = \phi(t) = \int_0^t f(u) g(t-u)\, du.$

EXAMPLE 12.8.1 Use convolution theorem to evaluate

(i) $L^{-1}\left\{ \dfrac{s^2}{(s^2+a^2)(s^2+b^2)} \right\}$ (ii) $L^{-1}\left\{ \dfrac{s}{(s^2+a^2)^2} \right\}$.

Solution (i) Since $f(t) = L^{-1}\left\{ \dfrac{s}{s^2+a^2} \right\} = \cos at$ and $g(t) = L^{-1}\left\{ \dfrac{s}{s^2+b^2} \right\} = \cos bt$.

Then by convolution theorem, we get

$$\begin{aligned}
L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\} &= \int_0^t \cos au \cos b(t-u)du \\
&= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\}+\cos\{(a+b)u-bt\}]du \\
&= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{a-b}+\frac{\sin\{(a+b)u-bt\}}{a+b}\right]_0^t \\
&= \frac{1}{2}\left[\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right] \\
&= \frac{a\sin at-b\sin bt}{a^2-b^2}
\end{aligned}$$

(ii) Let $f(t)=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at$ and $g(t)=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at.$

Therefore, by convolution theorem

$$\begin{aligned}
L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} &= \int_0^t f(u)g(t-u)du \\
&= \int_0^t \frac{1}{a}\sin au\cos a(t-u)du = \frac{1}{2a}\int_0^t [\sin at-\sin(2au-at)]\,dt \\
&= \frac{1}{2a}\left[u\sin at+\frac{1}{2a}\cos(2au-at)\right]_0^t = \frac{1}{2a}t\sin at
\end{aligned}$$

Hence, $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{1}{2a}t\sin at.$

EXAMPLE 12.8.2 Use convolution theorem to find the values of

(i) $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$ (WBUT 2002)

(ii) $L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}$ (WBUT 2004)

(iii) $L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\}.$ (WBUT 2008)

Solution (i) Let $f(t)=L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$ and $g(t)=L^{-1}\left\{\frac{1}{s^2+9}\right\}=\frac{1}{3}\sin 3t.$

Then

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right\} &= \int_0^t f(u)g(t-u)du \\
&= \int_0^t \sin u\,\frac{1}{3}\sin 3(t-u)du = \frac{1}{6}\int_0^t [\cos(4u-3t)-\cos(3t-2u)]du \\
&= \frac{1}{6}\left[\frac{\sin(4u-3t)}{4}-\frac{\sin(3t-2u)}{-2}\right]_0^t \\
&= \frac{1}{6}\left[\frac{1}{4}(\sin t-\sin 3t)+\frac{1}{2}(\sin t-\sin 3t)\right] \\
&= \frac{1}{8}(\sin t-\sin 3t)
\end{aligned}
$$

(ii) Let $f(t) = L^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$ and $g(t) = L^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$.
Therefore, by convolution theorem

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{s-2}\cdot\frac{1}{s^2+1}\right\} &= \int_0^t \sin u\, e^{2(t-u)}du \\
&= e^{2t}\int_0^t \sin u\, e^{-2u}du = e^{2t}\frac{1}{5}\left[e^{-2u}\cos u-2e^{-2u}\sin u\right]_0^t \\
&= \frac{1}{5}(e^{2t}-2\sin t-\cos t)
\end{aligned}
$$

(iii) Let

$$
\begin{aligned}
f(t) &= L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} \\
&= e^{-t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}e^{-t}\sin 2t
\end{aligned}
$$

By convolution theorem

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\} &= L^{-1}\left\{\frac{1}{s^2+2s+5}\cdot\frac{1}{s^2+2s+5}\right\} \\
&= \int_0^t f(u)\, f(t-u)du \\
&= \int_0^t \frac{e^{-2u}\sin 2u}{2}\cdot\frac{e^{-(t-u)}\sin 2(t-u)}{2}du \\
&= \frac{1}{4}\int_0^t e^{-t}\sin 2u\ \sin(2t-2u)du
\end{aligned}
$$

$$= \frac{1}{8}\int_0^t e^{-t}[\cos(4u-2t) - \cos 2t]du = \frac{e^{-t}}{8}\left[\frac{\sin(4u-2t)}{4} - \cos 2t \cdot u\right]_0^t$$

$$= \frac{e^{-t}}{8}\left[\frac{\sin 2t + \sin 2t}{4} - t\cos 2t\right] = \frac{e^{-t}}{8}\left[\frac{1}{2}\sin 2t - t\cos 2t\right]$$

EXAMPLE 12.8.3 Use convolution theorem to prove that

$$\int_0^t \sin u \ \cos(t-u)du = \frac{t}{2}\sin t$$

(WBUT 2007)

Solution By convolution theorem

$$\int_0^t f(u)g(t-u)du = L^{-1}\left\{\overline{f}(s)\overline{g}(s)\right\}$$

Comparing, we get $f(t) = \sin t$ and $g(t) = \cos t$.

Therefore, $\overline{f}(s) = L\{\sin t\} = \dfrac{1}{s^2+1}$ and $\overline{g}(s) = L\{\cos t\} = \dfrac{s}{s^2+1}$.

Thus, $\displaystyle\int_0^t \sin u \ \cos(t-u)du = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$.

Again, $\overline{f}'(s) = \dfrac{-2s}{(s^2+1)^2}$.

We know that $L^{-1}\left\{\overline{f}'(s)\right\} = -tf(t)$, that is $L^{-1}\left\{\dfrac{-2s}{(s^2+1)^2}\right\} = -t\sin t$.

or $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{t\sin t}{2}$.

Hence, $\displaystyle\int_0^t \sin u\cos(t-u)du = \frac{t\sin t}{2}$.

12.9 Method of Partial Fractions

In this method the transformed expression is divided in such a way that each term has some known standard Laplace transformation.

EXAMPLE 12.9.1 Find the Laplace inverse transforms of

(i) $\dfrac{4s+5}{(s-4)^2(s+3)}$ (WBUT 2003)

(ii) $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$ (WBUT 2006)

(iii) $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

(iv) $\dfrac{s}{s^4+4a^4}$.

Solution (i) Let

$$\frac{4s+5}{(s-4)^2(s+3)} = \frac{A}{s-4} + \frac{B}{(s-4)^2} + \frac{C}{(s+3)}$$

or

$$4s+5 = A(s-4)(s+3) + B(s+3) + C(s-4)^2$$

Substituting $s = 4,\ -3$ and 0 respectively

$$21 = 7B \quad \text{or} \quad B = 3$$
$$-7 = 49C \quad \text{or} \quad C = -\frac{1}{7}$$
$$5 = -12A + 3B + 16C$$

This gives $A = \frac{1}{7}$. Therefore

$$\frac{4s+5}{(s-4)^2(s+3)} = \frac{1}{7}\frac{1}{s-4} + \frac{3}{(s-4)^2} - \frac{1}{7}\frac{1}{(s+3)}$$

Hence

$$\begin{aligned} L^{-1}\left\{\frac{4s+5}{(s-4)^2(s+3)}\right\} &= \frac{1}{7}L^{-1}\left\{\frac{1}{s-4}\right\} + 3L^{-1}\left\{\frac{1}{(s-4)^2}\right\} - \frac{1}{7}L^{-1}\left\{\frac{1}{(s+3)}\right\} \\ &= \frac{1}{7}e^{4t} + 3te^{4t} - \frac{1}{7}e^{-3t} \end{aligned}$$

(ii) $\dfrac{1}{(s^2+a^2)(s^2+b^2)} = \dfrac{1}{b^2-a^2}\left[\dfrac{1}{s^2+a^2} - \dfrac{1}{s^2+b^2}\right].$

Therefore

$$\begin{aligned} L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} &= \frac{1}{b^2-a^2}\left[L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - L^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right] \\ &= \frac{1}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right] \end{aligned}$$

(iii) Denominator $s^3 - 6s^2 + 11s - 6 = (s-1)(s-2)(s-3)$.

Let

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

or

$$2s^2 - 6s + 5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

Substituting $s = 1, 2, 3$, we get $A = 1/2$, $B = -1$, $C = 5/2$.

Therefore

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2}\frac{1}{s-3}$$

Hence

$$L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$$
$$= \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$$

(iv) The denominator $s^4+4a^4 = (s^2+2as+2a^2)(s^2-2as+2a^2)$.

Let

$$\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

or

$$s = (As+B)(s^2-2as+2a^2) + (Cs+D)(s^2+2as+2a^2)$$

Equating coefficients of s^3, s^2, s, we get respectively

$$0 = A+C$$
$$0 = -2aA+B+2aC+D$$
$$1 = 2a^2A-2aB+2a^2C+2aD$$

Substituting $s=0$, we obtain

$$0 = 2a^2B+2a^2D$$

The solution of these equations is

$$A=C=0,\ B=-\frac{1}{4a},\ D=\frac{1}{4a}$$

Therefore

$$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = -\frac{1}{4a}L^{-1}\left\{\frac{1}{s^2+2as+2a^2}\right\} + L^{-1}\left\{\frac{1}{s^2-2as+2a^2}\right\}$$
$$= -\frac{1}{4a}L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\} + \frac{1}{4a}L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$
$$= -\frac{1}{4a}e^{-at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} + \frac{1}{4a}e^{at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$
$$= -\frac{1}{4a}e^{-at}\frac{1}{a}\sin at + \frac{1}{4a}e^{at}\frac{1}{a}\sin at$$
$$= \frac{1}{2a^2}\sin at\frac{e^{at}-e^{-at}}{2}$$
$$= \frac{1}{2a^2}\sin at \sinh at$$

12.10 Additional Worked-Out Examples

EXAMPLE 12.10.1 Evaluate $L^{-1}\left\{\dfrac{se^{-4s}}{s^2-2s+5}\right\}$.

Solution We have

$$\begin{aligned} L^{-1}\left\{\frac{s}{s^2-2s+5}\right\} &= L^{-1}\left\{\frac{s-1+1}{(s-1)^2+4}\right\} = L^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} \\ &= e^t L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + e^t L^{-1}\left\{\frac{1}{s^2+2^2}\right\} \\ &= e^t\cos 2t + e^t\frac{1}{2}\sin 2t = e^t(\cos 2t + \frac{1}{2}\sin 2t) \end{aligned}$$

Hence by second shifting property

$$L^{-1}\left\{\frac{se^{-4s}}{s^2-2s+5}\right\} = \begin{cases} e^{(t-4)}[\cos 2(t-4) + \frac{1}{2}\sin 2(t-4)], & t > 4 \\ 0, & t < 4 \end{cases}$$

EXAMPLE 12.10.2 Evaluate $L^{-1}\{\tan^{-1}(s+3)\}$.

Solution Let $\overline{f}(s) = \tan^{-1}(s+3)$.

Therefore, $\overline{f}'(s) = \dfrac{1}{1+(s+3)^2}$, or

$$L^{-1}\left\{\overline{f}'(s)\right\} = L^{-1}\left\{\frac{1}{1+(s+3)^2}\right\} = e^{-3t}L^{-1}\left\{\frac{1}{1+s^2}\right\} = e^{-3t}\sin t$$

That is, $-tL^{-1}\{\overline{f}(s)\} = e^{-3t}\sin t$, or

$$L^{-1}\{\overline{f}(s)\} = L^{-1}\{\tan^{-1}(s+3)\} = -\frac{e^{-3t}\sin t}{t}$$

EXAMPLE 12.10.3 Find the inverse Laplace transform of $s\log\dfrac{s-1}{s+1}$.

Solution Let $\overline{f}(s) = \log\dfrac{s-1}{s+1} = \log(s-1) - \log(s+1)$.

Therefore, $\overline{f}'(s) = \dfrac{1}{s-1} - \dfrac{1}{s+1}$.

Then

$$L^{-1}\left\{\overline{f}'(s)\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

or

$$-tf(t) = e^t - e^{-t} \quad \text{or} \quad f(t) = \frac{e^{-t}-e^t}{t}$$

Hence

$$L^{-1}\left\{s\log\frac{s-1}{s+1}\right\} = \frac{d}{dt}\{f(t)\} = -\frac{(e^{t}+e^{-t})t+(e^{-t}-e^{t})}{t^2}$$

$$= \frac{2(\sinh t - t\cosh t)}{t^2}$$

EXAMPLE 12.10.4 Evaluate $L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\}$.

Solution Let $\overline{f}(s) = \log\left(1+\frac{1}{s^2}\right)$.

Therefore, $\overline{f}'(s) = -\dfrac{2}{s(s^2+1)}$.

$$L^{-1}\left\{\overline{f}'(s)\right\} = -2L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = -2L^{-1}\left\{\frac{1}{s}-\frac{s}{s^2+1}\right\}$$

or

$$-tL^{-1}\left\{\overline{f}(s)\right\} = -2(1-\cos t)$$

or

$$L^{-1}\left\{\overline{f}(s)\right\} = \frac{2(1-\cos t)}{t}$$

Hence

$$L^{-1}\left\{\frac{1}{s}\overline{f}(s)\right\} = L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\} = \int_0^t \frac{2(1-\cos t)}{t}dt$$

EXAMPLE 12.10.5 Use convolution theorem to evaluate $L^{-1}\left\{\dfrac{s}{(s^2+9)^2}\right\}$.

(WBUT 2007)

Solution Let $\overline{f}(s) = \dfrac{1}{s^2+9}$ and $\overline{g}(s) = \dfrac{s}{s^2+9}$.

Then $f(t) = L^{-1}\left\{\dfrac{1}{s^2+9}\right\} = \dfrac{1}{3}\sin 3t$ and $g(t) = L^{-1}\left\{\dfrac{s}{s^2+9}\right\} = \cos 3t$.

Now by convolution theorem

$$L^{-1}\left\{\overline{f}(s)\overline{g}(s)\right\} = \int_0^t f(u)g(t-u)du$$

That is

$$
\begin{aligned}
L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} &= \int_0^t \frac{1}{3}\sin 3u \cdot \cos 3(t-u)du \\
&= \frac{1}{3}\int_0^t [\sin 3t - \sin(6u-3t)]\, du \\
&= \frac{1}{3}\left[u \sin 3t + \frac{1}{6}\cos(6u-3t)\right]_0^t \\
&= \frac{1}{6} t \sin 3t
\end{aligned}
$$

EXAMPLE 12.10.6 Find $L^{-1}\left\{\frac{1}{s}e^{1/s}\right\}$.

Solution We know that $e^{1/s} = 1 + \frac{1}{s} + \frac{1}{2!}\frac{1}{s^2} + \frac{1}{3!}\frac{1}{s^3} + \cdots$.

Therefore

$$\frac{1}{s}e^{1/s} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2!}\frac{1}{s^3} + \frac{1}{3!}\frac{1}{s^4} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{s^{n+1}}$$

Hence

$$L^{-1}\left\{\frac{1}{s}e^{1/s}\right\} = \sum_{n=0}^{\infty}\frac{1}{n!}L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{(n!)^2}$$

EXAMPLE 12.10.7 Use convolution theorem to determine

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\}$$

Solution Let $f(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $g(t) = L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{\sin 3t}{3}$.

Then by convolution theorem

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{s^2+1}\frac{1}{s^2+9}\right\} &= \int_0^t \sin u \, \frac{1}{3}\sin 3(t-u)du \\
&= \frac{1}{6}\int_0^t [\cos(4u-3t) - \cos(3t-2u)]du \\
&= \frac{1}{6}\left[\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right]_0^t \\
&= \frac{1}{6}\left[\frac{1}{4}(\sin t - \sin 3t) + \frac{1}{2}(\sin t - \sin 3t)\right] = \frac{1}{8}[\sin t - \sin 3t]
\end{aligned}
$$

Now we consider

$$f_1(t) = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{8}(\sin t - \sin 3t)$$

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and
$$f_2(t) = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

Again, by convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\} = L^{-1}\{\overline{f_1}(s)\overline{f_2}(s)\}$$
$$= \int_0^t f_1(u)f_2(t-u)du$$
$$= \int_0^t \frac{1}{8}(\sin u - \sin 3u)\cos 2(t-u)du$$
$$= \frac{1}{8}\int_0^t [\sin u \cos 2(t-u) - \sin 3u \cos 2(t-u)]\, du$$
$$= \frac{1}{8}\int_0^t \left[\frac{1}{2}\{\sin(2t-u) - \sin(3u-2t)\} - \frac{1}{2}\{\sin(u+2t) - \sin(5u-2t)\}\right] du$$
$$= \frac{1}{16}\left\{\left[\cos(2t-u) + \frac{1}{3}\cos(3u-2t)\right]_0^t - \left[-\cos(u+2t) + \frac{1}{5}\cos(5u-2t)\right]_0^t\right\}$$
$$= \frac{1}{16}\left\{\frac{4}{3}(\cos t - \cos 2t) + \frac{4}{5}(\cos 3t - \cos 2t)\right\}$$
$$= \frac{1}{12}\cos t - \frac{2}{15}\cos 2t + \frac{1}{20}\cos 3t$$

EXAMPLE 12.10.8 Find $L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}$.

Solution Let $\overline{f}(s) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right) = \log(s^2+b^2) - \log(s^2+a^2)$.

Therefore,
$$\overline{f}'(s) = \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$$

Thus, $L^{-1}\left\{\overline{f}'(s)\right\} = 2\cos bt - 2\cos at$, or $-tf(t) = 2\cos bt - 2\cos at$, or $f(t) = 2\left(\frac{\cos at - \cos bt}{t}\right)$.

Hence
$$L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\} = \frac{\cos at - \cos bt}{t}$$

EXERCISES

Section A Multiple Choice Questions

1. The inverse Laplace transform of $1/s^2$ is

   (a) 1 (b) t (c) $\frac{t^2}{2!}$ (d) $2t^2$.

2. The value of $L^{-1}\left\{\frac{2}{s-3}\right\}$ is

   (a) $2e^{3t}$ (b) $3e^{2t}$ (c) $2e^{-3t}$ (d) e^{-3t}.

3. The value of $L^{-1}\left\{\frac{1}{s}+\frac{4}{s-3}\right\}$ is

   (a) $\frac{1}{t}+\frac{4}{t-3}$ (b) $1+4e^{-3t}$ (c) $1+4e^{3t}$ (d) $1+e^{3t}$.

4. $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$ is equal to

   (a) $\frac{1}{3}(\sin t-\sin 2t)$ (b) $\frac{1}{3}(\cos t-\cos 2t)$ (c) $\cos t-\cos 2t$
   (d) $\sin t-\sin 2t$.

5. The inverse Laplace transform of $\frac{6}{(s-1)^4}$ is

   (a) $\frac{5}{t^4}$ (b) $5e^{-t}$ (c) $e^t t^3$ (d) $6t^3e^{3t}$.

6. If $L^{-1}\{\overline{f}(s)\}=f(t)$ then $L^{-1}\left\{\overline{f}'(s)\right\}$ is equal to

   (a) $tf(t)$ (b) $-tf(t)$ (c) $f'(t)$ (d) $-t^2f'(t)$.

7. If $f(t)=\frac{t+2}{t^2+2t+5}$ then $L^{-1}\{\overline{f}(2s)\}$ is equal to

   (a) $\frac{t+4}{t^2+4t+20}$ (b) $\frac{t}{t^2+2t+1}$ (c) $\frac{1}{4}\frac{t+4}{t^2+4t+20}$ (d) $\frac{2(t+2)}{t^2+2t+5}$.

8. $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}$ is

   (a) $\sinh at$ (b) $\frac{1}{a}\sinh at$ (c) $\cosh at$ (d) $\frac{1}{a}\cosh at$.

9. $L^{-1}\left\{\frac{7}{(s-2)(s+5)}\right\}$ equal to

   (a) $e^{-2t}-e^{5t}$ (b) $e^{2t}-e^{-5t}$ (c) $7(e^{2t}-e^{-5t})$ (d) $\frac{1}{7}(e^t-e^{-5t})$.

10. The value of $L^{-1}\left\{\frac{4}{s^2-16}+\frac{3s}{s^2-9}\right\}$ is

    (a) $4\sin 4t+3\cos 3t$ (b) $\sinh 4t+3\cosh 3t$ (c) $4\sinh 4t+\cosh 3t$
    (d) $\sin 4t+3\cos 3t$.

11. $L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$ is equal to (WBUT 2006)

    (a) $\frac{1}{2a}t\sin at$ (b) $\frac{1}{a}t\sin at$ (c) $\frac{1}{a^2}\sin at$ (d) $\cos at$

12. $L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$ is equal to

(a) $\frac{1}{t^2}$ (b) e^{2t} (c) te^{2t} (d) $\frac{e^{2t}}{t}$.

13. $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$ is equal to

(a) $\frac{1}{t^n}$ (b) t^n (c) t^{n+1} (d) $\frac{t^n}{n!}$.

14. $L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$ is

(a) $e^{2t}\sin 3t$ (b) $\frac{1}{3}e^{2t}\sin 3t$ (c) $e^{2t}\cos 3t$ (d) $\frac{1}{3}e^{2t}\cos 3t$.

15. The inverse Laplace transform of $\frac{s-2}{(s-2)^2+16}$ is

(a) $e^{2t}\cos 4t$ (b) $e^{4t}\cos 4t$ (c) $e^{2t}\sin 4t$ (d) $\sin 4t$.

16. $L^{-1}\left\{\frac{1}{(s-2)(s+2)}\right\}$ is equal to

(a) $e^{2t}+e^{-2t}$ (b) $\cosh 2t$ (c) $2\cosh 2t$ (d) $2\sinh 2t$.

17. $L^{-1}\left\{\frac{5s^2-3s+4}{s^3}\right\}$ is equal to

(a) $1-3t+2t^2$ (b) $5-3t+2t^2$ (c) $5t^2-3t+4$ (d) $5-3t^2+4t^3$.

18. $L^{-1}\{\log(5+s)\}$ is

(a) e^{-5t} (b) $-\frac{e^{-5t}}{t}$ (c) $-e^{-5t}$ (d) $-te^{-5t}$.

19. $L^{-1}\left\{\int_s^\infty \overline{f}(u)du\right\}$ is equal to

(a) $\frac{1}{t}L^{-1}\{\overline{f}(s)\}$ (b) $\frac{1}{t^2}L^{-1}\{\overline{f}(s)\}$ (c) $L^{-1}\{\overline{f}(s)\}$ (d) $tL^{-1}\{\overline{f}(s)\}$.

20. If $L^{-1}\{\overline{f}(s)\}=f(t)$, then $L^{-1}\left\{\frac{1}{s^2}\overline{f}(s)\right\}$ is equal to

(a) $-\frac{d^2}{ds^2}\overline{f}(s)$ (b) $\int_0^t\int_0^p f(u)\,du\,dp$ (c) $\int_0^t\int_0^p \overline{f}(s)\,ds\,dp$
(d) $\int_0^t\int_0^p \frac{1}{t^2}f(t)\,dt\,dp$.

21. $L^{-1}\left\{\frac{1}{(s+a)^n}\right\}$ is

(a) $\frac{1}{t^n}$ (b) $\frac{a^n}{t^n}$ (c) $e^{-at}\frac{t^{n-1}}{(n-1)!}$ (d) $e^{-at}\frac{t^n}{n!}$.

Section B Review Questions

1. Prove that $L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -tL^{-1}\left\{\bar{f}(s)\right\}$.
2. Prove that $L^{-1}\left\{\bar{f}(s-a)\right\} = e^{at}L^{-1}\left\{\bar{f}(s)\right\}$.
3. Show that $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)dt$ where $L^{-1}\left\{\bar{f}(s)\right\} = f(t)$.

Find the inverse Laplace transforms of the following

4. $\dfrac{6s-4}{s^2-4s+20}$
5. $\dfrac{4s+12}{s^2+8s+16}$
6. $\dfrac{s^2+2s+8}{s^3}$
7. $\dfrac{s+2}{s^2-4s+13}$
8. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$
9. $\dfrac{4s+5}{(s-1)^2(s+2)}$
10. $\dfrac{6e^{-2s}}{s^2+9}$
11. $\dfrac{(s+1)e^{-2\pi s}}{s^2+s+1}$
12. $\dfrac{s}{(s^2+9)^2}$
13. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$
14. $\dfrac{s}{s^4+64}$
15. $\dfrac{3s+2}{s^2-s-2}$
16. $\dfrac{1-7s}{(s-3)(s-1)(s+2)}$
17. $\dfrac{s}{(s^2-1)^2}$
18. $\dfrac{s^3}{s^4-a^4}$
19. $\dfrac{s}{s^4+s^2+1}$
20. $\dfrac{3s+12}{4s^2+12s+9}$
21. $\dfrac{1}{s^2-6s+10}$

22. $\dfrac{3s+1}{(s+1)^4}$

23. $\dfrac{e^{-4s}}{s^2}$

24. $\dfrac{s+1}{(s^2+2s+2)^2}$

25. $s\log\dfrac{s}{\sqrt{1+s^2}}+\cot^{-1}s$

26. $\log\dfrac{s+a}{s+b}$

27. $\tan^{-1}\dfrac{2}{s^2}$

28. $\tan^{-1}\left(\dfrac{s-2}{3}\right)$

29. $\log(1-\dfrac{1}{s^2})$

30. $\log\sqrt{\dfrac{s-1}{s+1}}$

31. $\log\dfrac{s^2+1}{s(s+1)}$

32. $\dfrac{1}{s(s+2)^3}$

33. $\dfrac{1}{s}\log\dfrac{s+2}{s+1}$

34. $\dfrac{s+1}{s^2+s+1}$

35. $\dfrac{1}{(s+1)(s^2+1)}$ (WBUT 20

36. $\dfrac{4s+5}{(s-4)^2(s+3)}$

37. $\dfrac{1}{(s^2+1)^3}$

38. $\dfrac{s}{(s^2+4)^2}$

39. $\dfrac{s^2}{(s^2+4)(s^2+9)}$

40. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

41. $\log\dfrac{1+s}{s}$

42. $\log\sqrt{\dfrac{s^2+b^2}{s^2+a^2}}$

43. $\log \dfrac{s^2+1}{(s-1)^2}$

44. $3\log \dfrac{s+1}{(s+2)(s+4)}$

45. $\dfrac{6}{(s+2)(s-4)}$

46. $\dfrac{s^2+9s-9}{s^3-9s}$

47. $\dfrac{2s^3}{s^4-81}$

48. $\dfrac{s^4+3(s+1)^3}{s^4(s+1)^3}$

49. $\dfrac{s^3+6s^2+14s}{(s+2)^4}$

50. $\dfrac{s+1}{s(s-2)(s+3)}$

51. $\dfrac{s}{(s^2+\pi^2)^2}$

52. $\dfrac{w}{s^2(s^2+w^2)}$

53. $\dfrac{s}{s^4+4a^4}$

54. $\dfrac{s^2}{s^4+4a^4}$

55. $\dfrac{s^3}{s^4+4a^4}$

56. Show that

(a) $L^{-1}\left\{\dfrac{1}{s}\sin\dfrac{1}{s}\right\} = t - \dfrac{t^3}{(3!)^2} + \dfrac{t^5}{(5!)^2} - \dfrac{t^7}{(7!)^2} + \cdots$

(b) $L^{-1}\left\{\dfrac{1}{s}\cos\dfrac{1}{s}\right\} = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + \cdots$

Use convolution theorem, evaluate

57. $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$

58. $L^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$

59. $L^{-1}\left\{\dfrac{1}{(s^2+a^2)^2}\right\}$

60. $L^{-1}\left\{\dfrac{1}{(s+1)(s+9)^2}\right\}$

61. $L^{-1}\left\{\dfrac{s}{(s^2+9)^2}\right\}$

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62. $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$ (WBUT 2005, 2006)

63. $L^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+9)}\right\}$ (WBUT 2002

64. $L^{-1}\left\{\dfrac{1}{(s^2+a^2)(s^2+b^2)}\right\}$

65. $L^{-1}\left\{\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right\}$

66. $L^{-1}\left\{\dfrac{1}{(s^2+2s+5)^2}\right\}$ (WBUT 200

67. $L^{-1}\left\{\dfrac{1}{(s-2)(s^2+1)}\right\}$ (WBUT 200

68. $L^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$

69. $L^{-1}\left\{\dfrac{1}{s^2(s^2+1)^2}\right\}$

70. Show that

$$L^{-1}\left\{\frac{1}{s}\log\frac{s^2+a^2}{s^2+b^2}\right\} = 2\int_0^t \frac{\cos bu - \cos au}{u}du$$

71. Prove that

$$\int_0^t\int_0^t\int_0^t f(t)\,dt\,dt\,dt = \int_0^t \frac{(t-u)^3}{2}f(u)du$$

72. Using convolution theorem prove that

$$\int_0^t e^u e^{2(t-u)}du = e^{2t} - e^t$$

73. Using convolution theorem, verify that

$$\int_0^t \sin u\cos(t-u)du = \frac{1}{2}t\sin t$$

Answers

Section A Multiple Choice Questions

1. (b) 2. (a) 3. (c) 4. (a) 5. (c) 6. (b) 7. (a) 8. (c) 9. (b)
10. (b) 11. (d) 12. (c) 13. (b) 14. (b) 15. (a) 16. (c) 17. (b) 18. (b)
19. (a) 20. (b) 21. (c)

Section B Review Questions

4 $2e^{2t}(3\cos 4t + \sin 4t)$

5 $4(1-t)e^{-4t}$

6 $1 + 2t + 8t^2$

7. $e^{2t}(\cos 3t + \frac{4}{3}\sin 3t)$

8. $\frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$

9. $\frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}$

10. $f(t) = 2\sin 3(t-2), t > 0, f(t) = 0, t < 3$

11. $e^{-(t-2\pi)/2}\{\cos\frac{\sqrt{3}}{2}(t-2\pi) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-2\pi)\}, t > 2\pi$

12. $\frac{1}{6}t\sin 3t$

13. $e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$

14. $\frac{1}{8}\sin 2t\ \sinh 2t$

15. $\frac{1}{3}8e^{2t} + \frac{1}{3}e^{-t}$

16. $e^t + e^{-2t} - 2e^{3t}$

17. $\frac{1}{2}t\sinh t$

18. $\frac{1}{2}[\cos at + \cosh at]$

19. $\frac{2}{\sqrt{3}}\sinh(\frac{t}{2})\ \sin(\sqrt{3}\frac{t}{2})$

20. $\frac{1}{8}e^{-3t/2}(6-5t)$

21. $e^{3t}\sin t$

22. $e^{-t}(\frac{3t^2}{2} - \frac{t^3}{3})$

23. $t - 4, t > 4$

24. $\frac{1}{2}te^{-t}\sin t$

25. $\frac{1-\cos t}{t^2}$

26. $-e^{-at} + e^{bt}$

27. $2\sinh t\ \sin t$

28. $-\frac{1}{t}e^{2t}\sin 3t$

29. $\frac{2}{t}(1 - \cosh t)$

30. $-\frac{1}{t}\sinh t$

31. $\frac{1}{t}(1+e^{-t}-2\cos t)$

32. $1-e^{-t}(\frac{t^2}{2}+t+1)$

33. $\int_0^t \frac{e^{-u}-e^{-2u}}{u}\,du$

34. $e^{-t/2}(\cos\frac{\sqrt{3}t}{2}+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2})$

35. $\frac{1}{2}(\sin t-\cos t+e^{-t})$

36. $-\frac{1}{7}e^{-3t}+\frac{1}{7}e^{4t}+3te^{4t}$

37. $\frac{1}{8}\{(3-t^2)\sin t-2t\sin t\}$

38. $\frac{1}{4}t\sin 2t$

39. $\frac{1}{5}(-2\sin 2t+3\sin 3t)$

40. $\frac{a\sin at-b\sin bt}{a^2-b^2}$

41. $\frac{1-e^t}{t}$

42. $\frac{1}{t}(\cos at-\cos bt)$

43. $\frac{2}{t}(e^t-\cos t)$

44. $3(e^{-t}-e^{-2t}-e^{-4t})$

45. $e^{4t}-e^{-2t}$

46. $1+3\sinh 3t$

47. $\cos 3t+\cosh 3t$

48. $\frac{1}{2}(t^2e^{-t}+t^3)$

49. $e^{-2t}(1+t^2-2t^3)$

50. $-\frac{1}{6}+\frac{3}{10}e^{2t}-\frac{2}{15}e^{-3t}$

51. $\frac{t\sin\pi t}{2\pi}$

52. $\frac{\cos t-\sin wt}{w^2}$

53. $\frac{1}{2a^2}\sinh at\ \sin at$

54. $\frac{1}{2a}(\cosh at\ \sin at+\sinh at\ \cos at)$

55. $\cosh at\ \cos at$

57. $\frac{e^{-bt}-e^{-at}}{a-b}$

58. $\frac{1}{4}(1-\cos 2t)$

59. $\frac{1}{2a^3}(\sin at - at\cos at)$

60. $\frac{e^{-t}}{64}[1-e^{-8t}(1+8t)]$

61. $\frac{1}{6}t\sin t$

62. $\frac{1}{2}t\sin t$

63. $\frac{1}{24}(3\sin t - \sin 3t)$

64. $\frac{1}{b^2-a^2}(\frac{\sin at}{a} - \frac{\sin bt}{b})$

65. $\frac{1}{b^2-a^2}(\cos at - \cos bt)$

66. $\frac{1}{16}e^{-t}\sin 2t - \frac{1}{8}te^{-t}\cos 2t$

67. $\frac{1}{5}(e^{2t} - 2\sin t - \cos t)$

68. $\frac{t^2}{2} + \cos t - 1$

69. $t(e^{-t}+1) + 2(e^{-t}-1)$