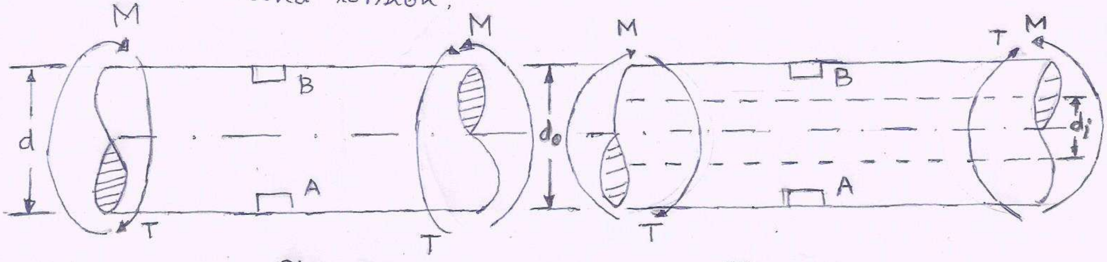
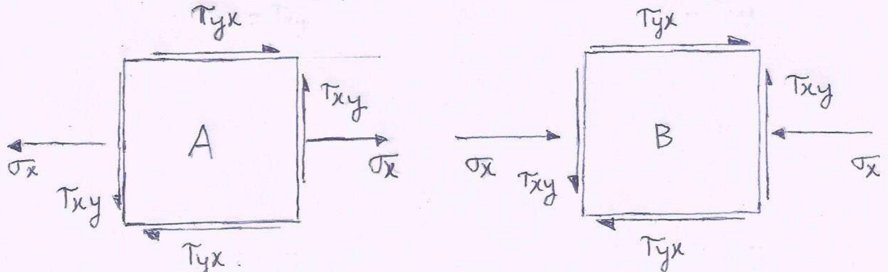
Analysis of shaft under bending moment and torsion.





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Simple Shaft Design - page 1

$$\tau_{yx} = \tau_{xy}$$

$$\sigma_1 = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_2 = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_3 = 0$$

$$\sigma_1 > 0 > \sigma_2 \; ; \quad \tau_{max} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Bending Moment = M and Axial Torsion = T

For solid Shaft:

$$I = \frac{\pi}{64} d^4$$

$$J = \frac{\pi}{32} d^4$$

$$\sigma_b = \frac{M}{I} \cdot \frac{d}{2}$$

$$\sigma_b = \frac{M}{\frac{\pi}{64} d^4} \cdot \frac{d}{2} = \frac{32M}{\pi d^3}$$

$$\tau \text{(Max Torsional stress)} = \frac{T}{J} \cdot \frac{d}{2}$$

$$\tau = \frac{T}{\frac{\pi}{32} d^4} \cdot \frac{d}{2} = \frac{16T}{\pi d^3}$$

N.B.:
M ~~fo~~ from B.M.D
T from Torque Diagram

For hollow shaft:

$$I = \frac{\pi}{64}(d_o^4 - d_i^4)$$

$$J = \frac{\pi}{32}(d_o^4 - d_i^4)$$

$$\sigma_b = \frac{M}{I} \cdot \frac{d_o}{2} = \frac{M}{\frac{\pi}{64}(d_o^4 - d_i^4)} \cdot \frac{d_o}{2}$$

$$= \frac{32M}{\pi d_o^4\left(1 - \frac{d_i^4}{d_o^4}\right)} \cdot d_o$$

$$= \frac{32M}{\pi d_o^3 (1 - k^4)} \quad \text{when } \frac{d_i}{d_o} = k$$

$$\tau \text{(Max Torsional stress)} = \frac{T}{J} \cdot \frac{d_o}{2}$$

$$\tau = \frac{T}{\frac{\pi}{32}(d_o^4 - d_i^4)} \cdot \frac{d_o}{2} = \frac{16T d_o}{\pi d_o^4\left(1 - \frac{d_i^4}{d_o^4}\right)}$$

$$= \frac{16T}{\pi d_o^3 (1 - k^4)} = \frac{16T}{\pi d_o^3 f}$$

where $1 - k^4 = f$

$$\sigma = \frac{32M}{\pi d_o^3 f}$$

Stresses at A

$$\sigma_b = \frac{32M}{\pi d^3}\ ; \ \tau = \frac{16T}{\pi d^3}$$

Stresses at B

$$\sigma_b = -\frac{32M}{\pi d^3}\ ; \ \tau = \frac{16T}{\pi d^3}$$

$$\tau_{max} = \sqrt{\left(\frac{\frac{32M}{\pi d^3}}{2}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \cancel{16T}\left(\frac{16}{\pi d^3}\right)\sqrt{M^2+T^2}$$

$$= \frac{16\,T_e}{\pi d^3}$$

$$[\tau] = \frac{\tau_{yield}}{F.S.}$$

T_e = Equivalent Torque $= \sqrt{M^2+T^2}$

$$[\tau] \geq \tau_{max}$$

$$[\tau] \geq \frac{16\,T_e}{\pi d^3}\ ; \ d^3 \geq \frac{16\,T_e}{\pi[\tau]}$$

$$d \geq \sqrt[3]{\frac{16\,T_e}{\pi[\tau]}}$$

Stresses at A

$$\sigma_b = \frac{32M}{\pi d_o^3 f}\ ; \ \tau = \frac{16T}{\pi d_o^3 f}$$

Stresses at B

$$\sigma_b = -\frac{32M}{\pi d_o^3 f}\ ; \ \tau = \frac{16T}{\pi d_o^3 f}$$

$$\tau_{max} = \sqrt{\left(\frac{\frac{32M}{\pi d_o^3 f}}{2}\right)^2 + \left(\frac{16T}{\pi d_o^3 f}\right)^2}$$

$$= \sqrt{\left(\frac{16M}{\pi d_o^3 f}\right)^2 + \left(\frac{16T}{\pi d_o^3 f}\right)^2}$$

$$= \left(\frac{16}{\pi d_o^3 f}\right)\sqrt{M^2+T^2} = \frac{16\,T_e}{\pi d_o^3 f}$$

$$[\tau] = \frac{\tau_{yield}}{F.S.}$$

T_e = Equivalent Torque $= \sqrt{M^2+T^2}$

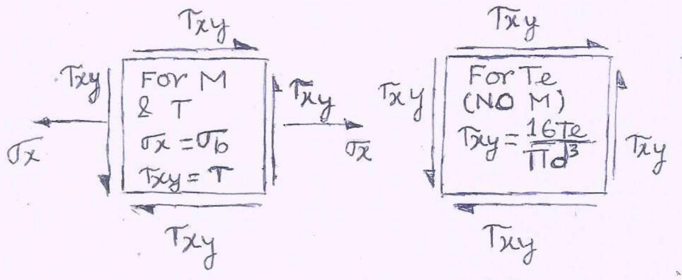
$$[\tau] \geq \tau_{max}$$

$$[\tau] \geq \frac{16\,T_e}{\pi d_o^3 f}\ ; \ d_o^3 \geq \frac{16\,T_e}{\pi[\tau] f}$$

$$d_o \geq \sqrt[3]{\frac{16\,T_e}{\pi[\tau] f}}$$

$f = 1 - k^4$ and $k = \frac{d_i}{d_o}$

for solid shaft, $d_i = 0$, $k = 0$; $f = 1$; $d_o = d$.



The effects of shock and fatigue are incorporated with the introduction of k_b and k_t into the expression for T_e.

$$T_e = \sqrt{(k_b M)^2 + (k_t T)^2}$$

where k_b = Combined shock and fatigue factor for bending moment

k_t = Combined shock and fatigue factor for torsional moment

(Table 9.2, page – 332, Fourth Edition)

From A.S.M.E. code,

$$[\tau] = \min(0.3\,\sigma_{yield},\ 0.18\,\sigma_{ut})$$

Additional reduction of 25% for keyway and shoulders on shaft.

Shaft:

A shaft is a rotating or stationary member, usually of circular cross-section, having mounted upon it such elements as gears, pulleys, flywheels, cranks, sprockets and other power-transmitting elements. The shafts are relatively long and may be subjected to bending, tension, compression or torsional loads, acting singly or in combination with one another. When they are combined, one may expect to find both static and fatigue strengths to be important design consideration, since a single shaft may be subjected to static stresses, completely reversed stresses and repeated stresses, all acting at the same time.

a) Transmission shaft – It is used to transmit power between the power source and the machines absorbing power. It is generally subjected to torque, bending moment and axial load in combination e.g. line shaft, counter shaft, head shaft and all factory shafts.
b) Machine shaft – It is integral part of a machine. It is used to transfer motion and power within the machine e.g. crank shaft, gear shaft.
c) Axle – It is a stationary or rotating shaft. It does not carry any torsional load. It is subjected to bending moment due to transverse load only. It is used to support rotating parts e.g. axles of automobiles.
d) Spindle – It is a short rotating shafts used to impart motion either to cutting tool or work piece e.g. lathe spindle, drill press spindle.

Materials: Commonly adopted materials are mild steel. In addition, different types of alloy steels are also used for shafts, depending on situation and types of application.

Deflection restraints: $\delta = L/1200$, where δ and L are maximum transverse deflection of the shaft and length of shaft between bearing supports respectively.

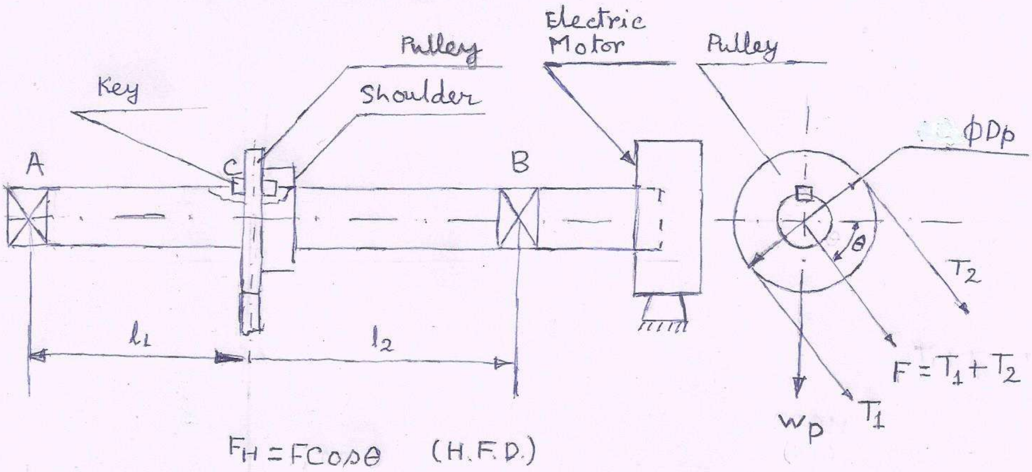
Twist restraint: Maximum twist is 2.5-3.5 degree per meter of shaft length for line shaft. It is with 0.25 degree per meter of machine shaft.

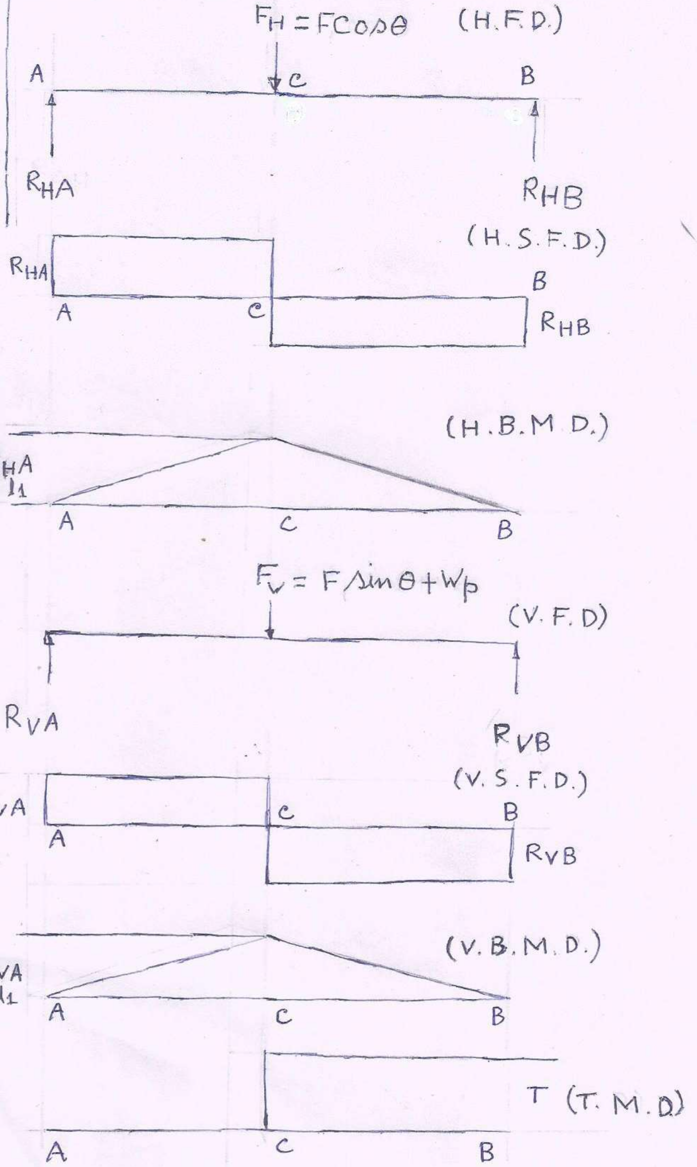
Problem:

A solid horizontal steel shaft is to run in self-aligning bearings A and B, 2500mm apart. The shaft is driven from the right of right-hand bearing with an electric motor. The power is supplied to a machine from the shaft through a 500-mm diameter pulley, mounted on the shaft through a rectangular key at 1200mm from the left-hand bearing. The belt tensions are 1800N and 840N and they are parallel to each other and at 64^{O} to the horizontal. The weight of the pulley is 500N. Determine the diameter of the shaft.
Use the following data:

Material = FeE 200;   Factor of safety = 2.5;
Combined shock and fatigue factor for bending = 1.6;
Combined shock and fatigue factor for torsion = 1.2;
Allowable tensile stress for key material = 120MPa

AKM
Shaft Problem - page 1





F → Force
H → Horizontal
V → Vertical
S → Shear
B → Bending
T → Twisting
M → Moment
D → Diagram

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Shaft Problem – page 2

$l_1 = 1200\,mm$
$l_2 = 1300\,mm$
$T_1 = 1800N$
$T_2 = 840N$
$W_p = 500N$
$D_p = 500\,mm$
$\theta = 64^\circ$
Material = Fe E 200
F.S. = 2.5
$K_m = 1.6$
$K_t = 1.2$

From Design Data Book,
$\sigma_{yield} = 200 MPa$.
$\tau_{yield} = \frac{200}{2} = 100 MPa$
$[\tau] = \frac{100}{2.5} = 40 MPa$

Torque calculation:
$T = (1800-840) \times \frac{D_P}{2}$
$= 960 \times \frac{500}{2} = 240000\,N\text{-}mm$

C is most critical point
$M = 1932629.473\,N\text{-}mm$.

$F = T_1 + T_2 = 1800N + 840 = 2640N$
$F_H = F\cos\theta = 2640\cos 64^\circ = 1157.299828\,N$
$F_V = F\sin\theta + W_P = 2640 \sin 64^\circ + 500$
$= 2372.816282 + 500 = 2872.816282\,N$

$R_{HA} = \frac{2640 \times 1300}{1200+1300} \cos 64^\circ = \frac{1157.299828 \times 1300}{2500}$
$= 601.7959103N$

$R_{HB} = \frac{1157.299828 \times 1200}{1200+1300} = 555.5039172N$

$R_{VA} = \frac{2872.816282 \times 1300}{1200+1300} = 1493.864466N$

$R_{VB} = \frac{2872.816282 \times 1200}{1200+1300} = 1378.951814N$.

$M_{HC} = R_{HA} \times l_1 = 601.7959103 \times 1200$
$= 722155.0924\,N\text{-}mm$

$M_{VC} = R_{VA} \times l_1 = 1493.864466 \times 1200$
$= 1792637.36\,N\text{-}mm$

$M_C = \sqrt{M_{HC}^2 + M_{VC}^2}$
$= \sqrt{(722155.0924)^2 + (1792637.36)^2}$
$= 1932629.473\,N\text{-}mm$

$T_e = \sqrt{(K_m M)^2 + (K_t T)^2} = \sqrt{(1.6 \times 1932629.473)^2 + (1.2 \times 240000)^2}$
$= 3105589.977\,N\text{-}mm$

$d \geq \sqrt[3]{\frac{16\,T_e}{\pi[\tau]}}$; $d \geq \sqrt[3]{\frac{16 \times 3105589.977}{\pi \times 40}}$; $d \geq \sqrt[3]{395415.9969}$

$d \geq 73.3980877\,mm$; $d \geq 73.3980877 \times 1.1$: $d \geq 80.73789647\,mm$

We take, $d = 82\,mm$ For commercial shaft, we take, $d = 85\,mm$ or $90\,mm$ (as available)

N.B.: ~~They~~ Keyway is located at point C where Te is maximum and is critical. So the value of allowable stress is further reduced by 25% which equivalent to multiplying with 1.1 i.e. increasing the diameter by 10%. $\sqrt[3]{\frac{1}{0.75}} = 1.100642416 \approx 1.1$.

Note: From the above discussion, it is understood that the calculated diameter of the shaft depends on the computed values of T_e and $[\tau]$. With the appropriately selected values of K_m and K_t, the value of T_e depends on the values of T and M. It is clear that values of T and M varies from location to location on the same shaft. The values of allowable shear stress also not same along the shaft. The diameter of the shaft is to be calculated at all critical locations and design decision is to be taken based on engineering judgement.

Type of Loading	C_m	C_t
Stationary Shaft :		
Load applied gradually	1.0	1.0
Load applied suddenly	1.5-2.0	1.5-2.0
Rotating Shaft:		
Load applied gradually	1.5	1.0
Steady Loads	1.5	1.0
Loads applied gradually with minor shocks	1.5-2.0	1.0-1.5
Loads applied suddenly with heavy shocks	2.0-3.0	1.5-3.0

Note: In some books, C_m and C_t are denoted as K_m and K_t respectively.

Calculation of average torque T_{av} from rated power [kW] of motor with rpm n:
$(2\pi n \times T_{av})/60 = 1000 \times [kW]$;
$T_{av} = (60 \times 1000) / (2\pi n) \times [kW]$ N-m = (9549.296586/n) N-m ≈ (9550/n) × [kW] N-m
$T_{av} = (9550/n) \times 10^3 \times [kW]$ N-mm.
Design Torque $T_d = T_{av} \times C_s$.

Note:

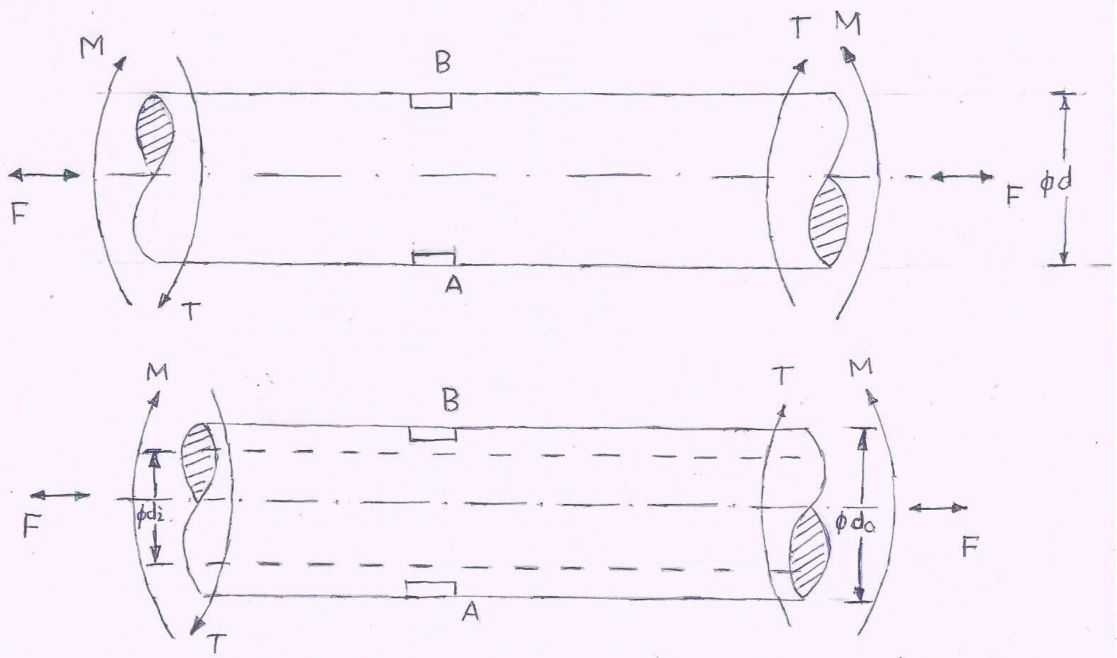
The torque T in the formula $T_e = \sqrt{(K_m M)^2 + (K_t T)^2}$ is to be considered as design torque T_d. If T be calculated from any data which is an averge based time like rated power etc, it reflects averge value of torque. But the design calculation is to based on maximum instantane value T_{max} of T.

Service factor $C_s = \frac{T_{max}}{T_{av}}$ where T_{av} = Average value of T

$$T_{max} = T_{av} \times C_s$$

$$T_d = T_{max}$$

C_s is selected on the basis of available literature and experience. C_s value for different combination of driving machine and driven machines are different. It also depends on operating condition.



Shaft under axial load F in addition to axial twisting moment T and bending moment M

Design of shaft subjected to axial tension or ~~com~~ axial compression in addition to Bending moment M and twisting moment T

Let the axial tension or compression is caused by an axial load F.

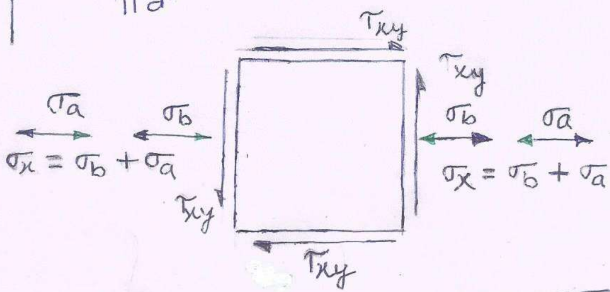
$$\sigma_a = \frac{F}{\frac{\pi}{4} d^2} = \frac{4Fd}{\pi d^3}$$

$$\sigma_a = \frac{32 Fd}{8 \pi d^3}$$

$$\sigma = \sigma_b + \sigma_a = \frac{32M}{\pi d^3} + \frac{32 Fd}{8\pi d^3}$$

$$= \frac{32}{\pi d^3}\left(M + \frac{Fd}{8}\right)$$

$$= \frac{32 M'}{\pi d^3} \quad \text{where } M' = M + \frac{Fd}{8}$$



$$T_e = \sqrt{\left[K_m M + \frac{\alpha F d_o (1+k^2)}{8}\right]^2 + (K_t T)^2}$$

For solid shaft $k = 0$ & $d_o = d$

If the shaft be slender and be subjected to compressive axial Force F, the force F is multiplied a factor α to account for column effect. F will be αF.

$$M' = M + \frac{\alpha F d}{8}$$

$\alpha = 1$ for axial tensile load

Radius of gyration (r) for circular shaft

$Ar^2 = I$; $r = \sqrt{\frac{I}{A}}$

$r = \frac{d}{4}$ for solid shaft

$= \frac{d_o}{4}\sqrt{1+k^2}$ for hollow shaft

critical slenderness ratio $= \sqrt{\frac{2\pi^2 E}{\sigma_{yield}}}$

$$\sigma_a = \frac{F}{\frac{\pi}{4}(d_o^2 - d_i^2)}$$

$$= \frac{4 F d_o}{\pi d_o^3 \left(1 - \frac{d_i^2}{d_o^2}\right)}$$

$$= \frac{32 F d_o}{8\pi d_o^3 (1-k^2)}$$

$$= \frac{32 F d_o (1+k^2)}{8\pi d_o^3 (1-k^4)}$$

$$= \frac{32 F d_o (1+k^2)}{8\pi d_o^3 f}$$

$$\sigma = \sigma_b + \sigma_a$$

$$= \frac{32 M}{\pi d_o^3 f} + \frac{32 F d_o (1+k^2)}{8\pi d_o^3 f}$$

$$= \frac{32}{\pi d_o^3 f}\left[M + \frac{F d_o (1+k^2)}{8}\right]$$

$$= \frac{32 M'}{\pi d_o^3 f}$$

where $M' = \left[M + \frac{F d_o (1+k^2)}{8}\right]$

$$M' = M + \frac{\alpha F d_o (1+k^2)}{8}$$

expression for α : $\alpha = 1$ for tensile load

$$\alpha = \frac{1}{1 - 0.0044\left(\frac{L}{r}\right)} \quad \text{for short column effect}$$

$$= \frac{\sigma_{yield}\left(\frac{L}{r}\right)^2}{C \pi^2 E}$$

where E = Young's Modulus

$C = 1.0$ for pinned-pinned ends

$= 2.25$ for fixed ends

$= 1.6$ for ends that are partially restrained as bearings

Problem:

Q.1. A solid shaft and a hollow shaft are to be of equal strength. The hollow shaft is to be 10% larger than the solid shaft. What will be the ratio of the weight of the hollow shaft to that of the solid shaft? Both the shafts are to be made of the same material.

Q.2. A Hollow shaft, 500mm outside diameter and 300mm inside diameter, is supported in two bearings 6m apart. The shaft is driven by a flexible coupling at one end and drives a ship's propeller at 100rpm. The maximum trust on the propeller is 500kN when the shaft is transmitting 6000kW. Determine the factor of safety. Give your comment on the result obtained by you.

Use the following data:

Material = 40C8
Combined shock and fatigue factor for bending = 1.5
Combined shock and fatigue factor for torsion = 1.0
Acceleration due to gravity = 9.8m/sec^2.
Density of material = 8.1 gms/cc.

A Key is a machine element which is used for connecting two machine parts for preventing relative motion of rotation with respect to each other. In many application, the Key prevents the lengthwise relative motion also. The connected parts act as a single unit. The Key joint consists of shaft, hub and key.

Primary function of the Key is to transmit the torque from the shaft to hub and vice versa.

A groove called a Keyseat or Keyway is usually cut into the shaft and the hub of the part to be connected.

The commonly adopted forms of ~~P~~ Keys may be divided into four classes.

Keys

- Saddle Key
  - Flat Key
- Tangent Keys
  - Kennedy Key
- Sunk Key
  - i) Rectangular or square Keys
  - ii) Gib head Keys
  - iii) Feather Keys and splined Shafts
  - iv) woodruff Keys
- Round Keys
  - Taper pins

Taper Key is uniform in width but tappred in height. The standardtaper is 1 in 100. The bollom surface ~~of~~ the Key is straight and the top surface is given a taper. The laper is provided for the following reasons.

i) when the Key is inserted in the Keyways of the shaft and hub and pressed by means of hammer, it becomes tight due to wedge action. This ensures the tightness of joint in operating conditions and prevents loosening of the parts.

ii) Due to taper, it is easy to remove the Key and dismantle the joint.

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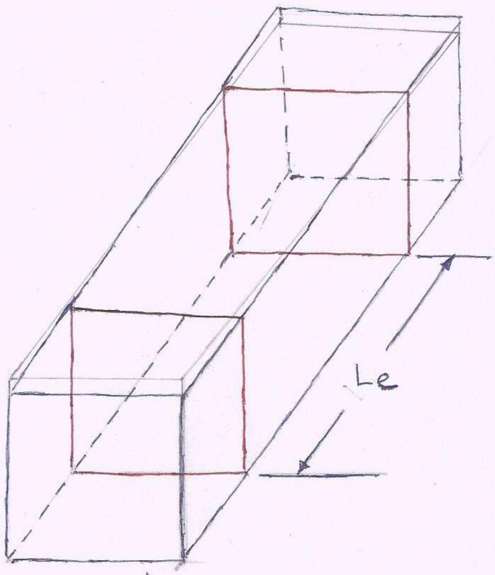
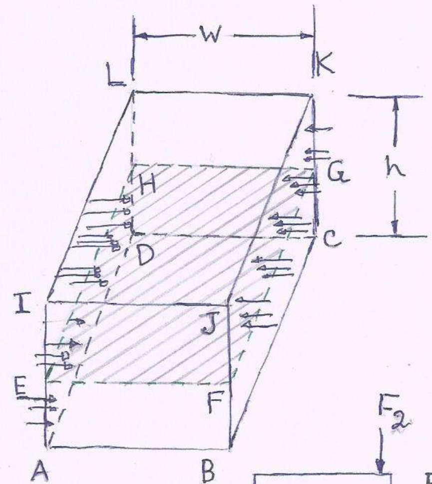
Key – page 1

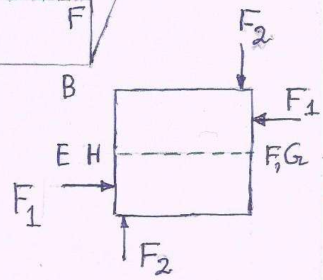
Tangential force for Key $= F_t$; $F_t \times \frac{d_s}{2} = T_d$; $F_t = \frac{2T_d}{d_s}$

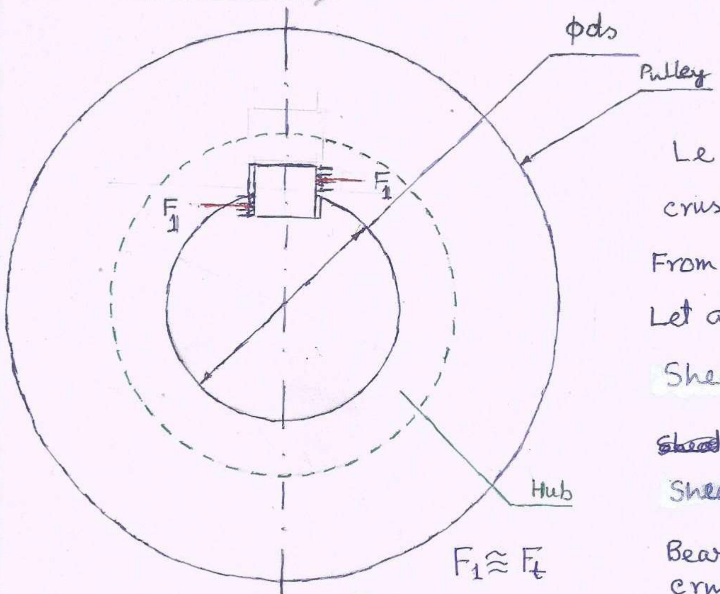
$T_d = T_{av} \times C_s$; C_s = Service factor $= \frac{T_{max}}{T_{av}}$; $T_d = T_{av} \times C_s$

T_d = Design torque ; d_s = Diameter of shaft









$F_1 \approxeq F_t$

Le = Effective Length of Key

crushing area are FGKJ and ADHE

From shearing

Let area EFGH = A

Shear stress $= \frac{F_1}{A}$

~~Shear~~ $A = W \times Le$

Shear stress $= \frac{F_1}{W \times Le}$

Bearing stress is also called crushing stress. From crushing,

Bearing stress $= \frac{F_1}{\frac{h}{2} \times Le}$

$$\frac{F_1}{W \times Le} \leqslant [\tau] ; \quad Le \geqslant \frac{F_1}{W \times [\tau]} \quad \cdots\cdots (1)$$

$[\sigma_{crush}]$ = Allowable crushing stress for key

$[\tau]$ = Allowable shear stress for Key

Le = Maximum value from two equations

$$\frac{F_1}{\frac{h}{2} \times Le} \leqslant [\sigma_{crush}] ; \quad Le \geqslant \frac{F_1}{\frac{h}{2} \times [\sigma_{crush}]} \quad \cdots\cdots (2)$$

Torque = 240000 N-mm.

Dia of shaft = 82 mm. $F_t = \frac{2T}{d_s} = \frac{2 \times 240000}{82} = 5853.658537 \text{ N}$

$F_t = 5854 \text{ N}$

Dia of shaft = 85 mm. $F_t = \frac{2T}{d_s} = \frac{2 \times 240000}{85} = 5647.058824 \text{ N}$

$F_t = 5648 \text{ N}$

Dia of shaft = 90 mm. $F_t = \frac{2T}{d_s} = \frac{2 \times 240000}{90} = 5333.333333 \text{ N}$

$F_t = 5334 \text{ N}.$

$[\tau] = \frac{120}{2} = 60 \text{ MPa}$

$[\sigma_{crush}] = 150 \text{ MPa}$

For $d_s = 82$ mm or 85 mm, $w = 22$ mm & $h = 14$ mm.

$F_t = 5854$ N (For $d_s = 82$ mm) $L_e \geqslant \frac{5854}{22 \times 60}$; $L_e \geqslant 4.434848485$ mm.

$L_e \geqslant \frac{5854}{\frac{14}{2} \times 150}$; $L_e \geqslant \frac{5854}{7 \times 150}$; $L_e \geqslant 5.575238095$ mm

$L_e = 5.575238095$ mm.

For $d_s = 85$ mm ; $w = 22$ mm & $h = 14$ mm

$F_t = 5648$ N (For $d_s = 85$ mm) $L_e \geqslant \frac{5648}{22 \times 60}$; $L_e \geqslant 4.278787879$ mm.

$L_e \geqslant \frac{5648}{\frac{14}{2} \times 150}$; $L_e \geqslant \frac{5648}{7 \times 150}$; $L_e \geqslant 5.379047619$ mm.

~~Le = 537~~ $L_e = 5.379047619$ mm

For $d_s = 90$ mm ; $w = 25$ mm & $h = 14$ mm

$F_t = 5334$ N $L_e \geqslant \frac{5334}{25 \times 60}$; $L_e \geqslant 3.556$ mm.

$L_e \geqslant \frac{5334}{\frac{14}{2} \times 150}$; $L_e \geqslant \frac{5334}{7 \times 150}$; $L_e \geqslant 5.08$ mm

$L_e = 5.08$ mm

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Key Problem - page 1