CHAPTER 11

Laplace Transforms

11.1 Introduction

The Laplace transform is a very important tool for applied mathematics. This transformation provides an easy and effective means for the solution of many problems arising in engineering. The method of Laplace transform solves differential equations and corresponding initial and boundary value problems. This method reduces the problem of solving a differential equation to that of solving an algebraic equation. This process is made easier by using the table of functions and their transformations. Then, on solving this algebraic equation, one can solve the differential equation.

The partial differential equations can also be solved by using Laplace transforms.

11.2 Definition

Suppose $f(t)$ is a real-valued function defined over the interval $(-\infty, \infty)$ such that $f(t) = 0$ for all $t < 0$.

The Laplace transform of $f(t)$, denoted by $L\{f(t)\}$, is defined as

$$\overline{f}(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt \tag{11.1}$$

Here L is called the **Laplace transformation operator**. The parameter s is either a real or a complex number. In general, the parameter s is taken to be a positive number.

11.3 Sufficient Conditions for Existence of Laplace Transform

Definition 11.3.1 *A function is called* ***sectionally*** *or* ***piecewise continuous*** *in an interval if the interval can be subdivided into a finite number of intervals, in each of which the function is continuous and has finite left and right hand limits.*

Definition 11.3.2 *A function $f(t)$ is said to be of* ***exponential order of*** α *(or briefly as* ***exponential order****) if there exist constant α, M and N such that*

$$|f(t)| \le Me^{\alpha t} \text{ for all } t \ge N$$

Theorem 11.1 *If $f(t)$ is sectionally continuous in every finite interval $0 \le t \le N$, and of exponential order γ for $t > N$, then its Laplace transform $\overline{f}(s)$ exists for all $s > \gamma$.*

Proof. We have

$$\int_0^\infty e^{-st} f(t)dt = \int_0^N e^{-st} f(t)dt + \int_N^\infty e^{-st} f(t)dt \tag{i}$$

Since $f(t)$ is sectionally continuous in every finite interval $0 \le t \le N$, $\int_0^\infty f(t)dt$ exists.

Again

$$\left|\int_N^\infty e^{-st} f(t)dt\right| \le \int_N^\infty |e^{-st} f(t)|dt = \int_N^\infty e^{-st}|f(t)|dt$$

$$\le \int_N^\infty e^{-st} M e^{\gamma t} dt \qquad (\because |f(t)| \le Me^{\gamma t}, \text{ for all } t > N)$$

$$\le M\int_0^\infty e^{-(s-\gamma)t} dt = M\left[\frac{e^{-(s-\gamma)t}}{-(s-\gamma)}\right]_0^\infty = \frac{M}{s-\gamma}, \qquad \text{if } s > \gamma$$

Thus, $\int_0^\infty e^{-st} f(t)dt$ exists if $s > \gamma$.

For example, let $f(t) = \sin \alpha t$. Then

$$\lim_{t\to\infty} |f(t)|e^{-\beta t} = \lim_{t\to\infty} |\sin \alpha t| e^{-\beta t} \le \lim_{t\to\infty} e^{-\beta t} = 0, \beta > 0 \tag{11.2}$$

Therefore, $\sin \alpha t$ is exponential order; also, $\sin \alpha t$ is continuous. Hence the Laplace transform of $\sin \alpha t$ exists.

Again, let $f(t) = e^{t^2}$. For any $\gamma > 0$, we have $e^{-\gamma t}|e^{t^2}| = e^{-\gamma t}e^{t^2} = e^{(t^2-\gamma t)}$; this can be larger and larger for large t.

$\therefore$ $\lim_{t\to\infty} e^{-\gamma t}|e^{t^2}|$ does not exist, and also, the Laplace transform of e^{t^2} does not exist.

The conditions of this theorem are not necessary but sufficient, as $L\{t^{-1/2}\} = \sqrt{\pi/s}$, but $t^{-1/2}$ is not sectionally continuous, since it is finite at $t = 0$.

It may be noted that $L\{f(t)\}$ is a function of s and we denote it by $\overline{f}(s)$, i.e. $L\{f(t)\} = \overline{f}(s)$. This can be written as $f(t) = L^{-1}\{\overline{f}(s)\}$. Then $f(t)$ is called the **inverse Laplace transform** of $\overline{f}(s)$.

11.4 Transformations of Elementary Functions

A table of Laplace transforms of some functions is given below

(i) $L\{k\} = \frac{k}{s}$, k is a constant, $s > 0$ (11.3)

(ii) $L\{t^n\} = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, \ldots$ (11.4)

(iii) $L\{e^{at}\} = \frac{1}{s-a}$, $s > a$ (11.5)

(iv) $L\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$ (11.6)

(v) $L\{\cos at\} = \frac{s}{s^2+a^2}$, $s > 0$ (11.7)

(vi) $L\{\sinh at\} = \frac{a}{s^2-a^2}$, $s > |a|$ (11.8)

(vii) $L\{\cosh at\} = \frac{s}{s^2-a^2}$, $s > |a|$ (11.9)

Proof.

(i) $L\{k\} = \int_0^\infty k e^{-st}dt = \left[-k\frac{e^{-st}}{s}\right]_0^\infty = \frac{k}{s}$ if $s > 0$

(ii) $L\{t^n\} = \int_0^\infty t^n e^{-st}dt = \int_0^\infty e^{-z}(z/s)^n \frac{1}{s}dz$ where $z = st$

$= \frac{1}{s^{n+1}} \int_0^\infty z^n e^{-z} dz = \frac{\Gamma(n+1)}{s^{n+1}}$ if $n > -1$ and $s > 0$

where Γ is the Gamma function.

If n is a positive integer, then $\Gamma(n+1) = n!$ and therefore $L\{t^n\} = \frac{n!}{s^{n+1}}$.

(iii) $L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{e^{-(s-a)}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}$ if $s > a$

(iv) $L\{\sin at\} = \int_0^\infty \sin at\, e^{-st}dt = \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty = \frac{a}{s^2+a^2}$

The proof of (v) is similar to (iv).

(vi) $L\{\sinh at\} = \int_0^\infty e^{-st}\sinh at\, dt = \int_0^\infty e^{-st}\left[\frac{e^{at}-e^{-at}}{2}\right]dt$

$= m\frac{1}{2}\left(\int_0^\infty e^{-(s-a)t}dt - \int_0^\infty e^{-(s+a)t}dt\right) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$

$= \frac{a}{s^2-a^2}$ if $s > |a|$

EXAMPLE 11.4.1 Find the $L\{f(t)\}$, when

(i) $f(t) = \begin{cases} 1, & \text{if } 0 < t < 2 \\ 2, & \text{if } t > 2 \end{cases}$ (WBUT 2002)

(ii) $f(t) = \begin{cases} e^t, & \text{if } 0 < t \leq 1 \\ 0, & \text{if } t > 1 \end{cases}$ (WBUT 2003)

(iii) $f(t) = \begin{cases} 0, & \text{if } 0 < t \leq 1 \\ t, & \text{if } 1 < t \leq 2 \\ 0, & \text{if } t > 2 \end{cases}$ (WBUT 2005)

(iv) $f(t) = \begin{cases} 1, & \text{if } t > \alpha \\ 0, & \text{if } t < \alpha \end{cases}$ (WBUT 2006)

(v) $f(t) = \begin{cases} \sin t, & \text{if } 0 < t < \pi \\ 0, & \text{if } t > \pi. \end{cases}$ (WBUT 2008)

Solution (i)

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^2 f(t)e^{-st}dt + \int_2^\infty f(t)e^{-st}dt \\ &= \int_0^2 1 \cdot e^{-st}dt + \int_2^\infty 2 \cdot e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^2 + 2\left[\frac{e^{-st}}{-s}\right]_2^\infty \\ &= \left[\frac{e^{-2s}}{-s} + \frac{1}{s}\right] + 2\left[0 + \frac{e^{-2s}}{s}\right] = \frac{1}{s}(1 + e^{-2s}) \end{aligned}$$

(ii)

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^1 e^t e^{-st}dt + \int_1^\infty 0 \cdot e^{-st}dt \\ &= \int_0^1 e^{-(s-1)t}dt = \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 = \frac{1}{s-1}[1 - e^{-(s-1)}] \end{aligned}$$

here take $e^{(1-s)t}$ ✗

take $e^{-(s-1)t}$ ✓

(iii)

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^1 0 \cdot e^{-st}dt + \int_1^2 t \cdot e^{-st}dt + \int_2^\infty 0 \cdot e^{-st}dt \\ &= \int_1^2 te^{-st}dt = \left[\frac{te^{-st}}{-s}\right]_1^2 - \int_1^2 1 \cdot \frac{e^{-st}}{-s}dt \\ &= \frac{1}{s}(e^{-s} - 2e^{-2s}) + \frac{1}{s^2}(e^{-s} - 2e^{-2s}) \end{aligned}$$

(iv)

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^\alpha 1 \cdot e^{-st}dt + \int_\alpha^\infty 0 \cdot e^{-st}dt \\ &= \left[\frac{e^{-st}}{-s}\right]_0^\alpha = \frac{1}{s}[1 - e^{-s\alpha}] \end{aligned}$$

(v)

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^\pi \sin t \cdot e^{-st}dt + \int_\pi^\infty 0 \cdot e^{-st}dt \\ &= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi = \frac{1}{s^2+1}(1 + e^{-s\pi}) \end{aligned}$$

11.5 Properties of Laplace Transforms

11.5.1 Linear Property

Suppose $\overline{f}_1(s)$ and $\overline{f}_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$\begin{aligned} L\{af_1(t) + bf_2(t)\} &= aL\{f_1(t)\} + bL\{f_2(t)\} \\ &= a\overline{f}_1(s) + b\overline{f}_2(s) \end{aligned}$$

where a and b are any constants.

The proof of this result follows directly from the definition.

EXAMPLE 11.5.1 Find $L\{at+b\}$. (WBUT 2005)

Solution $L\{at+b\} = aL\{t\} + L\{b\} = \frac{a}{s^2} + \frac{b}{s}$ [using (11.3) and (11.4)]

11.5.2 Shifting Property

Theorem 11.2 (First shifting theorem). *If $L\{f(t)\} = \overline{f}(s)$, then $L\{e^{at}f(t)\} = \overline{f}(s-a)$.*

Proof. $L\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt = \int_0^\infty f(t)e^{-pt}dt$

where $p = s - a$

$$= \overline{f}(p) = \overline{f}(s-a)$$

Corollary 11.5.1 *If $L\{f(t)\} = \overline{f}(s)$, then $\overline{f}(s+a) = L\{e^{-at}f(t)\}$.*

Using this result, we have the following transformations:

(i) Since $L\{1\} = \frac{1}{s}$

$$L\{e^{at}\} = \frac{1}{s-a} \tag{11.10}$$

(ii) Since $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$L\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}} \tag{11.11}$$

(iii) Since $L\{\sin bt\} = \frac{b}{s^2+b^2}$

$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \tag{11.12}$$

(iv) Since $L\{\cos bt\} = \frac{s}{s^2+b^2}$

$$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \tag{11.13}$$

(v) Since $L\{\sinh bt\} = \frac{b}{s^2-b^2}$

$$L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2} \tag{11.14}$$

(vi) Since $L\{\cosh bt\} = \frac{s}{s^2-b^2}$

$$L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2} \tag{11.15}$$

in every case $s > a$.

EXAMPLE 11.5.2 Find the Laplace transforms of

(i) te^{2t} (WBUT 2006) (ii) $e^t \sin t \cos t$ (WBUT 2006)

(iii) $e^{3t}\cos^2 t$ (iv) $e^{4t}\sin 2t \cos t$.

Solution (i) We know $L\{t\} = \frac{1}{s^2}$. Then by shifting property, $L\{te^{2t}\} = \frac{1}{(s-2)^2}$.

(ii) $L\{\sin t \cos t\} = L\{\sin 2t/2\} = \frac{1}{2}\frac{2}{s^2+4} = \frac{1}{s^2+4}$.

By shifting property, $L\{e^t \sin t \cos t\} = \frac{1}{(s-1)^2+4}$.

(iii) $L\{\cos^2 t\} = L\{(1+\cos 2t)/2\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$.

Then $\{e^{3t}\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s-3} + \frac{s-3}{(s-3)^2+4}\right]$.

(iv) $L\{\sin 2t \cos t\} = \frac{1}{2}L\{\sin 3t + \sin t\} = \frac{1}{2}\left[\frac{3}{s^2+3^2} + \frac{1}{s^2+1}\right]$.

By shifting property, $L\{e^{4t}\sin 2t \cos t\} = \frac{1}{2}\left[\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)^2+1}\right]$.

Theorem 11.3 (Second shifting theorem). *If $L\{f(t)\} = \overline{f}(s)$ and*

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

then $L\{g(t)\} = e^{-as}\overline{f}(s)$.

Proof.

$$\begin{aligned} L\{g(t)\} &= \int_0^\infty e^{-st}g(t)dt = \int_0^a e^{-st}g(t)dt + \int_a^\infty e^{-st}g(t)dt \\ &= \int_0^a e^{-st}\cdot 0dt + \int_a^\infty e^{-st}f(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt \end{aligned}$$

Substituting, $t - a = p$. Then

$$L\{g(t)\} = \int_0^\infty e^{-s(p+a)}f(p)dp = e^{-sa}\int_0^\infty e^{-sp}f(p)dp = e^{-sa}\overline{f}(s)$$

11.5.3 Change of Scale Property

Theorem 11.4 *If* $L\{f(t)\} = \overline{f}(s)$*, then* $L\{f(at)\} = \frac{1}{a}\overline{f}\left(\frac{s}{a}\right)$

Proof.

$$\begin{aligned}
L\{f(at)\} &= \int_0^\infty e^{-st} f(at)dt \\
&= \int_0^\infty e^{-sx/a} f(x)\frac{dx}{a} \qquad \text{where } at = x \\
&= \frac{1}{a}\int_0^\infty e^{-sx/a} f(x)dx = \frac{1}{a}\int_0^\infty e^{-pt} f(t)dt \qquad \text{where } p = \frac{s}{a} \\
&= \frac{1}{a}\overline{f}(p) = \frac{1}{a}\overline{f}\left(\frac{s}{a}\right)
\end{aligned}$$

EXAMPLE 11.5.3 (i) Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$. (WBUT 2007)

(ii) If $L\{f(t)\} = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$, then show that $L\{f(2t)\} = \frac{s^2 - 2s + 4}{4(s+1)^2(s-2)}$

(WBUT 2002)

Solution (i) Let $f(t) = \frac{\sin t}{t}$. Then $L\{f(t)\} = L\left\{\frac{\sin t}{t}\right\} = \overline{f}(s)$. Then

$$L\left\{\frac{\sin at}{t}\right\} = aL\left\{\frac{\sin at}{at}\right\} = aL\{f(at)\} = a\cdot\frac{1}{a}\overline{f}\left(\frac{s}{a}\right) = \overline{f}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right).$$

(ii) Here $\overline{f}(s) = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$.

Therefore

$$L\{f(2t)\} = \frac{1}{2}\overline{f}\left(\frac{s}{2}\right) = \frac{1}{2}\frac{(s/2)^2 - (s/2) + 1}{\{2(s/2)+1\}^2(s/2-1)} = \frac{1}{4}\frac{s^2 - 2s + 4}{(s+1)^2(s-2)}$$

11.5.4 Multiplication by Powers of t

$L\{t^n f(t)\} = -\frac{d}{ds^n} L\{f(t)\}$

Theorem 11.5 *If* $L\{f(t)\} = \overline{f}(s)$*, then* $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{f}(s)$, $n = 1, 2, 3, \ldots$.

(WBUT 2005)

Proof. Let $L\{f(t)\} = \overline{f}(s)$ and $\frac{d^n \overline{f}(s)}{ds^n} = \overline{f}^{(n)}(s)$.

Then $\overline{f}(s) = \int_0^\infty e^{-st} f(t)dt$. Therefore

$$\frac{d\overline{f}(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} f(t)\}dt = -\int_0^\infty te^{-st} f(t)dt$$

or

$$(-1)\frac{d\overline{f}(s)}{ds} = \int_0^\infty e^{-st}\{tf(t)\}dt = L\{tf(t)\}$$

Thus, $(-1)\overline{f}'(s) = L\{tf(t)\}$.

This proves that the theorem is true for $n = 1$.

Let us suppose that the theorem is true for $n = m$, so that

$$(-1)^m\overline{f}^{(m)}(s) = L\{t^m f(t)\} = \int_0^\infty e^{-st}t^m f(t)dt$$

Now

$$(-1)^m\frac{d^{m+1}\overline{f}(s)}{ds^{m+1}} = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st}t^m f(t)\}dt = -\int_0^\infty te^{-st}t^m f(t)dt$$

or

$$(-1)^{m+1}\overline{f}^{(m+1)}(s) = \int_0^\infty e^{-st}\{t^{m+1}f(t)\}dt = L\{t^{m+1}f(t)\}$$

This proves that the theorem is true for $n = m + 1$. Hence by mathematical induction, the theorem is true for $n = 1, 2, 3, \ldots$.

EXAMPLE 11.5.4 Find the Laplace transforms of (i) $t^2\cos at$, (ii) $te^{-t}\sin 2t$, (iii) $t^2(1-e^t)$.

Solution (i) $L\{\cos at\} = \dfrac{s}{s^2+a^2} = \overline{f}(s)$.

Thus,

$$L\{t^2\cos at\} = (-1)^2\frac{d^2}{ds^2}\overline{f}(s) = \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\} = \frac{d}{ds}\left\{-\frac{s^2-a^2}{(s^2+a^2)^2}\right\} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

(ii) $L\{\sin 2t\} = \dfrac{2}{s^2+4} = \overline{f}(s)$.

Then

$$L\{t\sin 2t\} = (-1)\frac{d}{ds}\overline{f}(s) = -\frac{d}{ds}\left\{\frac{2}{s^2+4}\right\} = \frac{4s}{(s^2+4)^2}$$

Now, by shifting property

$$L\{e^{-t}t\sin 2t\} = \frac{4(s+1)}{[(s+1)^2+4]^2} = \frac{4(s+1)}{(s^2+2s+5)^2}$$

(iii) $L\{1-e^t\} = L\{1\} - L\{e^t\} = \dfrac{1}{s} - \dfrac{1}{s-1} = \overline{f}(s)$. Therefore

$$L\{t^2(1-e^t)\} = (-1)^2\frac{d^2}{ds^2}\overline{f}(s) = \frac{d^2}{ds^2}\left\{\frac{1}{s} - \frac{1}{s-1}\right\}$$

$$= \frac{d}{ds}\left\{-\frac{1}{s^2} + \frac{1}{(s-1)^2}\right\} = \frac{2}{s^3} - \frac{2}{(s-1)^3}$$

$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} f(s)\,ds$

11.5.5 Division by t

Theorem 11.6 *If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(t)/t\} = \int_s^{\infty} \bar{f}(s)ds$, provided the integral exists.*

Proof. Let $L\{f(t)\} = \bar{f}(s)$. Then $\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt$.

Integrating this with respect to s between s and ∞

$$\int_s^{\infty} \bar{f}(s)ds = \int_s^{\infty} ds \int_0^{\infty} e^{-st} f(t)dt$$

Here s and t are independent variables and hence the order of integration in the repeated integral can be interchanged.

Thus

$$\int_s^{\infty} \bar{f}(s)ds = \int_0^{\infty} f(t)dt \int_s^{\infty} e^{-st}ds = \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} ds$$

$$= \int_0^{\infty} \frac{e^{-st}}{t} f(t)dt = \int_0^{\infty} e^{-st}\left\{\frac{f(t)}{t}\right\}dt = L\left\{\frac{f(t)}{t}\right\}$$

EXAMPLE 11.5.5 Find the Laplace transforms of

(i) $\frac{1-e^t}{t}$ (WBUT 2004) (ii) $\frac{\sin at - \sin bt}{t}$ (iii) $\frac{\sin t}{t}$ (WBUT 2003)

Solution (i) $L\{1-e^t\} = L\{1\} - L\{e^t\} = \frac{1}{s} - \frac{1}{s-1} = \bar{f}(s)$ (say). Therefore

$$L\left\{\frac{1-e^t}{t}\right\} = \int_s^{\infty} \left[\frac{1}{s} - \frac{1}{s-1}\right]ds$$ ← same time Integration

$$= \left[\log s - \log(s-1)\right]_s^{\infty} = \left[\log\frac{s}{s-1}\right]_s^{\infty} = \left[\log\frac{1}{1-1/s}\right]_s^{\infty}$$

$$= \log 1 - \log\left(\frac{1}{1-1/s}\right) = \log\left(\frac{s-1}{s}\right)$$

(ii) $L\{\sin at - \sin bt\} = L\{\sin at\} - L\{\sin bt\} = \frac{a}{s^2+a^2} - \frac{b}{s^2+b^2} = \bar{f}(s)$ (say).

Now

$$L\left\{\frac{\sin at - \sin bt}{t}\right\} = \int_s^{\infty} \left[\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2}\right]ds$$

$$= \left[\tan^{-1}\frac{s}{a} - \tan^{-1}\frac{s}{b}\right]_s^{\infty} = \tan^{-1}\frac{s}{b} - \tan^{-1}\frac{s}{a}$$

(iii) $L\{\sin t\} = \frac{1}{s^2+1}$.

Therefore $L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \frac{1}{s^2+1}ds = \left[\tan^{-1} s\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$

11.6 Laplace Transform of Periodic Functions

Let a function $f(t)$ be periodic with period w, so that $f(t+nw) = f(t)$, for $n = 1, 2, 3, \ldots$. Then

$$L\{f(t)\} = \frac{\int_0^w e^{-st} f(t)dt}{1 - e^{-sw}} \tag{11.16}$$

Proof.

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt = \int_0^w e^{-st} f(t)dt + \int_w^{2w} e^{-st} f(t)dt + \cdots \\
&= \sum_{n=0}^{\infty} \int_{nw}^{(n+1)w} e^{-st} f(t)dt = \sum_{n=0}^{\infty} \int_0^w e^{-s(x+nw)} f(x+nw)dx \qquad \text{where } t = x + nw \\
&= \sum_{n=0}^{\infty} \int_0^w e^{-sx} e^{-snw} f(x)dx \quad [\because f(x+nw) = f(x) \text{ for } n = 0, 1, 2, \ldots] \\
&= \sum_{n=0}^{\infty} e^{-snw} \int_0^w e^{-sx} f(x)dx = (1 + e^{-sw} + e^{-2sw} + \cdots) \int_0^w e^{-sx} f(x)dx \\
&= \frac{1}{1 - e^{-sw}} \int_0^w e^{-sx} f(x)dx \qquad \text{as } e^{-sw} < 1 \\
&= \frac{1}{1 - e^{-sw}} \int_0^w e^{-st} f(t)dt
\end{aligned}$$

EXAMPLE 11.6.1 Find the Laplace transform of the periodic function $f(t)$ given by

$$f(t) = \begin{cases} t, & \text{if } 0 < t < c \\ 2c - t, & \text{if } c < t < 2c \end{cases}$$

(WBUT 2003)

Solution Here $f(t)$ is a periodic function with period $2c$. Therefore

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{1 - e^{-2cs}} \int_0^{2c} e^{-st} f(t)dt \\
&= \frac{1}{1 - e^{-2cs}} \left[\int_0^c e^{-st} f(t)dt + \int_c^{2c} e^{-st} f(t)dt \right] \\
&= \frac{1}{1 - e^{-2cs}} \left[\int_0^c te^{-st} dt + \int_c^{2c} (2c - t)e^{-st} dt \right]
\end{aligned}$$

$$= \frac{1}{1-e^{-2cs}}\left\{\left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^c + \left[(2c-t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right]_c^{2c}\right\}$$

$$= \frac{1}{1-e^{-2cs}}\left[\frac{-ce^{-sc}}{s} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2} + \frac{e^{-2cs}}{s^2} + \frac{ce^{-sc}}{s} - \frac{e^{-sc}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2cs}}\frac{(1-e^{-sc})^2}{s^2} = \frac{1}{s^2}\frac{1-e^{-sc}}{1+e^{-sc}}$$

EXAMPLE 11.6.2 Find the Laplace transform of the function

$$f(t) = \begin{cases} \sin wt, & \text{if } 0 < t < \pi/w \\ 0, & \text{if } \pi/w < t < 2\pi/w \end{cases}$$

Solution Here $f(t)$ is a periodic function with period $2\pi/w$. Therefore

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1-e^{-2\pi s/w}}\int_0^{2\pi/w} e^{-st}f(t)dt \\ &= \frac{1}{1-e^{-2\pi s/w}}\left[\int_0^{\pi/w} e^{-st}\sin wt dt + \int_{\pi/w}^{2\pi/w} e^{-st}\cdot 0 dt\right] \\ &= \frac{1}{1-e^{-2\pi s/w}}\left[\frac{e^{-st}(-s\sin wt - w\cos wt)}{s^2+w^2}\right]_0^{\pi/w} \\ &= \frac{we^{-\pi s/w}+w}{(1-e^{-2\pi s/w})(s^2+w^2)} = \frac{w}{(1-e^{-2\pi s/w})(s^2+w^2)} \end{aligned}$$

11.7 Laplace Transforms of Derivatives

Theorem 11.7 *If $L\{f(t)\} = \overline{f}(s)$ the $L\{f'(t)\} = s\overline{f}(s) - f(0)$, provided $f(t)$ is continuous for $0 \le t \le N$ and of exponential order γ for $t > N$, $s > \gamma$ where $f'(t)$ is sectionally continuous for $0 \le t \le N$.*

Proof.

$$\begin{aligned} L\{f'(t)\} &= \int_0^\infty e^{-st}f'(t)dt = \lim_{A\to\infty}\int_0^A e^{-st}f'(t)dt \\ &= \lim_{A\to\infty}\left\{\left[e^{-st}f(t)\right]_0^A + s\int_0^A e^{-st}f(t)dt\right\} \\ &= \lim_{A\to\infty} e^{-sA}f(A) - f(0) + s\overline{f}(s) \\ &= s\overline{f}(s) - f(0), \qquad \text{since } \lim_{A\to\infty} e^{-sA}f(A) = 0 \end{aligned}$$

Corollary 11.7.1 *If $f(t)$ fails to be continuous at $t = 0$ but $\lim_{t\to 0+} f(t) = f(0+)$ exists, then $L\{f'(t)\} = s\overline{f}(s) - f(0+)$.*

Theorem 11.8 *If $L\{f(t)\} = \bar{f}(s)$, then $L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$ provided $f(t)$ and $f'(t)$ are continuous for $0 \le t \le N$ and of exponential order for $t > N$, where $f''(t)$ is sectionally continuous for $0 \le t \le N$.*

Proof. $L\{f''(t)\} = sL\{f'(t)\} - f'(0) = s\{s\bar{f}(s) - f(0)\} - f'(0) = s^2\bar{f}(s) - sf(0) - f'(0)$.
In general

$$\begin{aligned} L\{f^{(n)}(t)\} &= s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots \\ &\quad - s^1 f^{(n-2)}(0) - f^{(n-1)}(0) \end{aligned}$$

provided $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous for $0 \le t \le N$ and of exponential order for $0 \le t \le N$.

11.8 Laplace Transform of Integral

Theorem 11.9 *If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$.*

Proof. Let

$$L\{f(t)\} = \bar{f}(s) \qquad \text{and} \qquad g(t) = \int_0^t f(u)du \tag{i}$$

From (i), it is clear that $g(0) = 0$ and $g'(t) = \frac{d}{dt}\left(\int_0^t f(u)du\right) = f(t)$.
Now

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

or

$$L\{f(t)\} = sL\{g(t)\}$$

i.e.

$$L\{g(t)\} = \frac{1}{s}\bar{f}(s)$$

Hence $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$.

EXAMPLE 11.8.1 Find the Laplace transforms of (i) $\int_0^t \frac{\sin t}{t}dt$, (ii) $\int_0^t e^{-t}\cos t\, dt$.

Solution (i) We know $L\{\sin t\} = \frac{1}{1+s^2}$. Therefore

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{1+s^2}ds = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

Hence $L\left\{\int_0^t \frac{\sin t}{t}dt\right\} = \frac{1}{s}\cot^{-1} s$.

(ii) We know $L\{\cos t\} = \dfrac{s}{s^2+1}$.

By shifting property

$$L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$$

Hence $L\left\{\displaystyle\int_0^t e^{-t}\cos t\,dt\right\} = \dfrac{1}{s}\,\dfrac{s+1}{s^2+2s+2}$.

11.9 Evaluation of Integrals by Laplace Transforms

EXAMPLE 11.9.1 Evaluate (i) $\displaystyle\int_0^\infty \frac{\sin at}{t}dt$, (ii) $\displaystyle\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt$, (iii) $\displaystyle\int_0^\infty \frac{e^{-t}\sin^2 t}{t}dt$.

Solution (i) $L\{\sin at\} = \dfrac{a}{s^2+a^2} = \bar{f}(s)$ (say).

Then

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{a}{s^2+a^2}ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

Thus

$$\int_0^\infty e^{-st}\frac{\sin at}{t}dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

Now

$$\lim_{s\to 0}\tan^{-1}\frac{s}{a} = \begin{cases} 0, & \text{if } a>0 \\ \pi, & \text{if } a<0 \end{cases}$$

Thus, taking limit as $s \to 0$, we get

hen $\tan^{-1} 0$, $\frac{\pi}{2} - 0 = \frac{\pi}{2}$

hen $\tan^{-1}\pi$, $\frac{\pi}{2} - \pi = -\frac{\pi}{2}$

$$\int_0^\infty \frac{\sin at}{t}dt = \begin{cases} \dfrac{\pi}{2}, & \text{if } a>0 \\ -\dfrac{\pi}{2}, & \text{if } a<0 \end{cases}$$

(ii) We know $L\left\{e^{-at}-e^{-bt}\right\} = \dfrac{1}{s+a} - \dfrac{1}{s+b} = \bar{f}(s)$ say.

Then

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right]ds$$

$$= \left[\log\frac{s+a}{s+b}\right]_s^\infty = \left[\log\frac{1+a/s}{1+b/s}\right]_s^\infty = \log 1 - \log\left(\frac{1+a/s}{1+b/s}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

That is, $\displaystyle\int_0^\infty e^{-st}\frac{e^{-at}-e^{-bt}}{t}dt = \log\left(\frac{s+b}{s+a}\right)$.

Taking limit as $s \to 0$, we get

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log\left(\frac{b}{a}\right)$$

(iii) We know that $L\{\sin^2 t\} = \frac{1}{2} L\{1 - \cos 2t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$.

Then

$$\begin{aligned} L\left\{\frac{\sin^2 t}{t}\right\} &= \int_s^\infty \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty \\ &= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty = -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right) \end{aligned}$$

That is

$$\int_0^\infty e^{-st} \frac{\sin^2 t}{t} dt = \frac{1}{4}\left(\frac{s^2+4}{s^2}\right)$$

Now, taking limit as $s \to 1$, we get

$$\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{\log 5}{4}$$

11.10 Laplace Transforms of Bessel's Functions

We know from the definition of Bessel function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Therefore

$$\begin{aligned} L\{J_0(x)\} &= \frac{1}{s} - \frac{1}{2^2} \cdot \frac{2!}{s^3} + \frac{1}{2^2 \cdot 4^2} \cdot \frac{4!}{s^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{6!}{s^7} + \cdots \\ &= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{s^4}\right) + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{s^6}\right) + \cdots\right] \\ &= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{1+s^2}} \end{aligned}$$

Again, we know that $J_0'(x) = -J_1(x)$.

Thus, $L\{J_1(x)\} = -L\{J_0'(x)\} = -[sL\{J_0(x)\} - J_0(0)] = 1 - \frac{s}{\sqrt{1+s^2}}$.

EXAMPLE 11.10.1 Find the Laplace transforms of the following functions

(i) $e^{-at} J_0(bt)$ (ii) $tJ_1(t)$ (iii) $\int_0^\infty te^{-3t} J_0(4t) dt$.

Solution (i) We know that $L\{J_0(t)\} = \dfrac{1}{\sqrt{1+s^2}}$.

Then by change of scale property

$$L\{J_0(bt)\} = \frac{1}{b}\frac{1}{\sqrt{1+(s/b)^2}} = \frac{1}{\sqrt{s^2+b^2}}$$

Now by shifting property, $L\{e^{-at}J_0(bt)\} = \dfrac{1}{\sqrt{(s-a)^2+b^2}}$.

(ii) We know that $L\{J_1(t)\} = 1 - \dfrac{s}{\sqrt{1+s^2}}$.

Therefore

$$L\{tJ_1(t)\} = (-1)\frac{d}{ds}\left\{1 - \frac{s}{\sqrt{1+s^2}}\right\} = \frac{1}{(1+s^2)^{3/2}}$$

(iii) Again, $L\{J_0(t)\} = \dfrac{1}{\sqrt{1+s^2}}$. Therefore

$$L\{J_0(4t)\} = \frac{1}{4}\frac{1}{\sqrt{1+(s/4)^2}} = \frac{1}{\sqrt{s^2+16}}$$

Also

$$L\{tJ_0(4t)\} = (-1)\frac{d}{ds}\left\{\frac{1}{\sqrt{s^2+16}}\right\} = \frac{s}{(s^2+16)^{3/2}}$$

That is

$$\int_0^\infty e^{-st}tJ_0(4t)dt = \frac{s}{(s^2+16)^{3/2}}$$

Now, taking limit as $s \to 3$, we get

$$\int_0^\infty e^{-3t}tJ_0(4t)dt = \frac{3}{(9+16)^{3/2}} = \frac{3}{125}$$

11.11 Unit Step Function and Its Laplace Transformation

The **unit step function** is generally denoted by u and is defined as

$$u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \tag{11.17}$$

This function has a unit jump at any point a. The unit step function is also called the **Heaviside function**. It is shown in Fig. 11.1.

The unit step function is a typical engineering function made to measure for engineering applications.

The unit step function can also be written as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

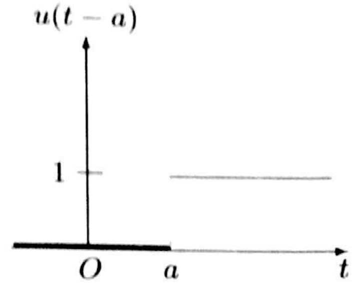

Figure 11.1: Unit step function $u(t-a)$.

The function

$$f(t) = \begin{cases} f_1(t), & t < a \\ f_2(t), & t \geq a \end{cases}$$

can be expressed by a unit step function as

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a)$$

Because, when $t \geq a$, then $u(t-a) = 1$. In this case, $f(t) = f_2(t)$, and when $t < a$, then $u(t-a) = 0$. Hence $f(t) = f_1(t)$.

Similarly, if

$$f(t) = \begin{cases} f_1(t), & t < a_1 \\ f_2(t), & a_1 < t < a_2 \\ f_3(t), & a_2 < t \end{cases}$$

then $f(t)$ can be expressed as

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a_1) + \{f_3(t) - f_2(t)\}u(t-a_2)$$

Theorem 11.10 *If $u(t-a)$ is a unit step function then $L\{u(t-a)\} = \dfrac{e^{-as}}{s}$.*

Proof.

$$\begin{aligned} L\{u(t-a)\} &= \int_0^\infty e^{-st}\, u(t-a)dt = \int_0^a e^{-st} \cdot 0dt + \int_a^\infty e^{-st} \cdot 1dt \\ &= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-st}}{s} \end{aligned}$$

Theorem 11.11 *Let $L\{f(t)\} = \overline{f}(s)$ and $u(t-a)$ be a unit step function. Then $L\{f(t-a)u(t-a)\} = e^{-st}\overline{f}(s) = e^{-as}L\{f(t)\}$.*

Proof.

$$
\begin{aligned}
& L\{f(t-a)\,u(t-a)\} \\
&= \int_0^{\infty} e^{-st} f(t-a)\,u(t-a)dt \\
&= \int_0^{a} e^{-st} f(t-a)\times 0\,dt + \int_a^{\infty} e^{-st} f(t-a)\,dt \\
&= \int_a^{\infty} e^{-st} f(t-a)dt = \int_0^{\infty} e^{-s(x+a)} f(x)dx \qquad \text{where } x = t-a \\
&= e^{-sa}\int_0^{\infty} e^{-sx} f(x)dx = e^{-sa}L\{f(t)\} = e^{-sa}\bar{f}(s)
\end{aligned}
$$

EXAMPLE 11.11.1 Express the function

$$f(t) = \begin{cases} e^{-t}, & 0 < t < 2 \\ 0, & t \geq 2 \end{cases}$$

in terms of unit step function and hence find $L\{f(t)\}$. (WBUT 2004)

Solution The function $f(t)$ can be written in terms of unit step function as $f(t) = e^{-t} + (0 - e^{-t})u(t-3)$, where $u(t-3)$ is the unit step function.
Now

$$L\{f(t)\} = L\{e^{-t}\} - L\{e^{-t}u(t-3)\}$$

We know that

$$L\{e^{-t}\} = \frac{1}{s+1} \quad \text{and} \quad L\{u(t-3)\} = \frac{e^{-st}}{s}$$

Therefore

$$L\{e^{-t}u(t-3)\} = \frac{e^{-3(s+1)}}{s+1}$$

Hence

$$L\{f(t)\} = \frac{1}{s+1} - \frac{e^{-3(s+1)}}{s+1} = \frac{1}{s+1}\left[1 - e^{-3(s+1)}\right]$$

EXAMPLE 11.11.2 Express

$$f(t) = \begin{cases} 2, & \text{if } 0 < t < \pi \\ 0, & \text{if } \pi < t < 2\pi \\ \sin t, & \text{if } t > 2\pi \end{cases}$$

n terms of unit step function and hence evaluate $L\{f(t)\}$.

olution The function $f(t)$ in terms of step function is

$$
\begin{aligned}
f(t) &= 2 + (0-2)u(t-\pi) + (\sin t - 0)u(t-2\pi) \\
&= 2 - 2\,u(t-\pi) + \sin t\; u(t-2\pi)
\end{aligned}
$$

The function $\sin t\, u(t-2\pi)$ is equal to $\sin(t-2\pi)\, u(t-2\pi)$ because of periodicity. Thus

$$L\{\sin(t-2\pi)\cdot u(t-2\pi)\} = e^{-2\pi s} L\{\sin t\} = \frac{e^{-2\pi s}}{s^2+1}$$

Hence

$$\begin{aligned} L\{f(t)\} &= L\{2\} - L\{2u(t-\pi)\} + L\{\sin t\, u(t-2\pi)\} \\ &= \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1} \end{aligned}$$

11.12 Additional Worked-Out Examples

EXAMPLE 11.12.1 Find the Laplace transforms of

(i) $e^{-2t}(2\cos 3t - 3\sin 4t)$ (ii) $e^{5t}\sin 2t \cos t$
(iii) $e^{3t} + 5t^2 - 2\sin 4t + 3\cos 2t$ (iv) $\cos(at+b)$.

Solution (i) We know $L\{\cos 3t\} = \dfrac{s}{s^2+9}$ and $L\{\sin 4t\} = \dfrac{4}{s^2+16}$.

Therefore

$$L\{2\cos 3t - 3\sin 4t\} = \frac{2s}{s^2+9} - \frac{12}{s^2+16}$$

Hence by shifting property

$$\begin{aligned} L\{e^{-2t}(2\cos 3t - 3\sin 4t)\} &= \frac{2(s+2)}{(s+2)^2+9} - \frac{12}{(s+2)^2+16} \\ &= \frac{2(s+2)}{s^2+2s+13} - \frac{12}{s^2+4s+20} \end{aligned}$$

(ii) $L\{\sin 2t\ \cos t\} = \dfrac{1}{2}L\{\sin 3t + \sin t\} = \dfrac{1}{2}\left\{\dfrac{3}{s^2+9} + \dfrac{1}{s^2+1}\right\}$

By shifting property

$$L\{e^{5t}\sin 2t\ \cos t\} = \frac{1}{2}\left\{\frac{3}{(s-5)^2+9} + \frac{1}{(s-5)^2+1}\right\}$$

(iii) We have

$$\begin{aligned} & L\{e^{3t} + 5t^2 - 2\sin 4t + 3\cos 2t\} \\ = \ & L\{e^{3t}\} + 5L\{t^2\} - 2L\{\sin 4t\} + 3L\{\cos 2t\} \\ = \ & \frac{1}{s-3} + 5\frac{2}{s^3} - 2\frac{4}{s^2+16} + 3\frac{s}{s^2+4} \\ = \ & \frac{1}{s-3} + \frac{10}{s^3} - \frac{8}{s^2+16} + \frac{3s}{s^2+4} \end{aligned}$$

(iv) Since $\cos(at+b) = \cos at \cos b - \sin at \sin b$

$$L\{\cos(at+b)\} = \cos b \; L\{\cos at\} - \sin b \; L\{\sin at\}$$

$$= \cos b \frac{s}{s^2+a^2} - \sin b \frac{a}{s^2+a^2} = \frac{1}{s^2+a^2}(s\cos b - a\sin b)$$

EXAMPLE 11.12.2 If $L\{f(t)\} = \bar{f}(s)$, show that

$$L\{\sinh at \; f(t)\} = \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)]$$

and

$$L\{\cosh at \; f(t)\} = \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$$

Hence find the value of $L\{\sinh 2t \; \cos 4t\}$.

Solution We have

$$L\{\sinh at \; f(t)\} = L\{\frac{1}{2}(e^{at} - e^{-at})f(t)\}$$

$$= \frac{1}{2}[L\{e^{at}f(t)\} - L\{e^{-at}f(t)\}]$$

$$= \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)] \qquad \text{(by shifting property)}$$

Similarly

$$L\{\cosh at \; f(t)\} = L\left\{\frac{1}{2}(e^{at} + e^{-at})f(t)\right\} = \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$$

Also

$$L\{\sinh 2t \; \cos 4t\} = \frac{1}{2}[\bar{f}(s-2) - \bar{f}(s+2)] \qquad \text{and} \qquad L\{\cos 4t\} = \frac{s}{s^2+16} = \bar{f}(s)$$

Hence

$$L\{\sinh 2t \; \cos 4t\} = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+16} - \frac{s+2}{(s+2)^2+16}\right]$$

EXAMPLE 11.12.3 If $L\{f(t)\} = \dfrac{e^{-2s}}{s}$, then find the value of $L\{e^{-3t}f(4t)\}$.

Solution Since $L\{f(t)\} = \dfrac{e^{-2s}}{s}$, then by change of scale property

$$L\{f(4t)\} = \frac{1}{4}\frac{e^{-2(s/4)}}{(s/4)} = \frac{e^{-s/2}}{s}$$

Now, by shifting property

$$L\{e^{-3t}f(4t)\} = \frac{e^{-(s+3)/2}}{s+3}$$

EXAMPLE 11.12.4 Find the Laplace transform of $t^3 \sin t$ and hence find the value of $\int_0^{\infty} t^3 e^{-t} \sin t \, dt$.

Solution We know $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Therefore

$$L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right) = -\frac{d^2}{ds^2}\left(\frac{-2s}{(1+s^2)^2}\right)$$

$$= \frac{d}{ds}\left\{\frac{2-6s^2}{(1+s^2)^3}\right\} = \frac{24(s^2-1)s}{(1+s^2)^4}$$

Putting $s = 1$, we have $\int_0^{\infty} t^3 e^{-t} \sin t \, dt = 0.$

EXAMPLE 11.12.5 Evaluate $\int_0^{\infty} \int_0^t \dfrac{e^{-t} \sin u}{u} \, du \, dt.$

Solution We know $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Therefore

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \frac{1}{s^2+1} ds = \left[\tan^{-1} s\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

Again, $L\left\{\int_0^t \dfrac{\sin u}{u} \, du\right\} = \dfrac{1}{s} \cot^{-1} s.$

Now, by shifting property, $L\left\{e^{-t} \int_0^t \dfrac{\sin u}{u} \, du\right\} = \dfrac{1}{s+1} \cot^{-1}(s+1).$

That is

$$\int_0^{\infty} e^{-st}\left\{e^{-t} \int_0^t \frac{\sin u}{u} du\right\} dt = \frac{1}{s+1} \cot^{-1}(s+1)$$

or

$$\int_0^{\infty} e^{-st} \int_0^t \frac{e^{-t} \sin u}{u} \, du \, dt = \frac{1}{s+1} \cot^{-1}(s+1)$$

Putting $s = 0$, we get

$$\int_0^{\infty} \int_0^t \frac{e^{-t} \sin u}{u} \, du \, dt = \cot^{-1}(1) = \frac{\pi}{4}$$

EXAMPLE 11.12.6 Evaluate

(i) $L\left\{t \int_0^t \dfrac{e^{-t} \sin t}{t} dt\right\}$ (ii) $L\left\{\int_0^t \int_0^t \int_0^t (t \cos t) dt \, dt \, dt\right\}$.

Solution (i) We know $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Thus,

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}ds = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$\therefore \quad L\left\{\int_0^t \frac{\sin t}{t}dt\right\} = \frac{1}{s}\cot^{-1} s.$$

Now, by shifting property

$$L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t}dt\right)\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

(ii) $L\{\cos t\} = \dfrac{s}{s^2+1}$.

Therefore

$$L\{t\cos t\} = -\frac{d}{ds}\left\{\frac{s}{s^2+1}\right\} = \frac{s^2-1}{(s^2+1)^2}$$

Hence

$$L\left\{\int_0^t\int_0^t\int_0^t (t\cos t)dt\,dt\,dt\right\} = \frac{1}{s^3}L\{t\cos t\} = \frac{1}{s^3}\frac{s^2-1}{(s^2+1)^2}$$

EXAMPLE 11.12.7 Find the Laplace transform of $f(t) = |t-1| + |t+1|, t \geq 0$.

Solution This function is written as

$$f(t) = \begin{cases} -(t-1)+(t+1) = 2, & \text{when } 0 \leq t \leq 1 \\ (t-1)+(t+1) = 2t, & \text{when } t > 1 \end{cases}$$

Therefore

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st}f(t)dt = \int_0^1 e^{-st}\cdot 2dt + \int_1^\infty 2te^{-st}dt \\ &= \left[\frac{2e^{-st}}{-s}\right]_0^1 + \left[2t\frac{e^{-st}}{-s}\right]_1^\infty - \int_1^\infty 2\frac{e^{-st}}{-s}dt \\ &= \left[\frac{2e^{-s}}{-s} + \frac{2}{s}\right] + \frac{2e^{-s}}{s} - \left[\frac{2e^{-st}}{s^2}\right]_1^\infty \\ &= \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right) \end{aligned}$$

EXERCISES

Section A Multiple Choice Questions

1. The Laplace transform of 5 is

   (a) 5 (b) $5s$ (c) $5/s$ (d) s^5.

2. The Laplace transform of t^3 is

   (a) $\frac{1}{3}$ (b) $\frac{3}{s}$ (c) $\frac{1}{s^4}$ (d) $\frac{6}{s^4}$. $\frac{6}{s^4}$

3. The Laplace transform of e^{-4t} is

   (a) $\frac{1}{s+4}$ (b) $\frac{1}{s-4}$ (c) $(s-4)$ (d) e^{-4s}.

4. $L\{\sin 3t\}$ is

   (a) $\frac{3}{s^2+9}$ (b) $\frac{s}{s^2+9}$ (c) $\frac{s}{s^2-9}$ (d) $\frac{3}{s^2-9}$.

5. If $L\{f(t)\} = 1/s^2$ then $L\{e^{2t}f(t)\}$ is

   (a) $\frac{e^{2t}}{s^2}$ (b) $\frac{e^{2s}}{s^2}$ (c) $\frac{1}{(s-2)^2}$ (d) $\frac{1}{(s+2)^2}$.

6. $L\{te^{2t}\}$ is equal to

   (a) $\frac{1}{s-2}$ (b) $2(s-2)^2$ (c) $\frac{1}{(s-2)^2}$ (d) $\frac{2!}{s^2}$. $\frac{2!}{(s-2)^2}$ (WBUT 2006)

7. The Laplace transform of the function $\cos at$ is

   (a) $\frac{s}{s^2-a^2}$ (b) $\frac{a}{s^2+a^2}$ (c) $\frac{s}{s^2+a^2}$ (d) $\frac{1}{s^2-a^2}$. (WBUT 2007)

8. $L\{e^{2t}\sin 3t\}$ is equal to

   (a) $\frac{3}{(s-2)^2+9}$ (b) $\frac{3}{(s+2)^2+9}$ (c) $\frac{s}{(s-2)^2+9}$ (d) $\frac{s}{(s-2)^2+1}$.

9. $L\{t\sin t\}$ is equal to

   (a) $\frac{s}{(1+s^2)^2}$ (b) $\frac{1}{s}\frac{1}{(s^2+1)^2}$ (c) $-\frac{2s}{(1+s^2)^2}$ (d) $-\frac{2}{(1+s^2)^2}$.

10. If $L\{f(t)\} = \frac{s^2+1}{s^2-1}$, then $L\{f(2t)\}$ is

    (a) $\frac{s^2+4}{s^2-4}$ (b) $\frac{1}{2}\frac{s^2+4}{s^2-4}$ (c) $\frac{1}{2}\frac{s^2+1}{s^2-1}$ (d) $\frac{1}{2}\frac{s^2+2}{s^2-2}$.

11. If $L\{f(t)\} = \frac{1}{s}$ and $f(0) = 1$, then $L\{f'(t)\}$ is

    (a) 0 (b) s (c) $\frac{1}{s}$ (d) $s-1$.

12. If $L\{f(t)\} = \tan^{-1}(1/s)$ then $L\{tf(t)\}$ is

    (a) $\tan^{-1}\left(\frac{1}{s}\right)$ (b) $\frac{1}{s^2+1}$ (c) $\frac{1}{s+1}$ (d) $\tan^{-1}\left(\frac{2}{\pi s}\right)$. (WBUT 2008)

13. If $f(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$, then $L\{f(t)\}$ is equal to

    (a) $\frac{1}{s}(1-e^{-2s})$ (b) $\frac{1}{s}(1+e^{-2s})$ (c) $\frac{1}{s}$ (d) $\frac{e^{-2s}}{s}$.

14. If $L\{f(t)\} = \frac{1}{s^2}$, then $L\{f(t)/t\}$ is equal to

    (a) $1/s^2$ (b) $1/s^3$ (c) $1/s$ (d) s.

15. $L\{\sin at + \cos at\}$ is

(a) $\dfrac{2a}{s^2+a^2}$ (b) $\dfrac{a+s}{s^2+a^2}$ (c) $\dfrac{2s}{s^2+a^2}$ (d) $\dfrac{a+s}{s^2-a^2}$.

16. $L\left\{\frac{t\sin t}{e^t}\right\}$ is

(a) $\dfrac{s+1}{s^2+2s+2}$ (b) $\dfrac{s+1}{(s^2+2s+2)^2}$ (c) $\dfrac{2(s+1)}{(s^2+2s+2)^2}$ (d) $\dfrac{s+2}{(s^2+2s+2)^2}$.

17. $\displaystyle\int_0^\infty e^{-t}\cos 2t\,dt$ is

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{3}$ (c) $\dfrac{2}{5}$ (d) $\dfrac{1}{5}$.

18. $\displaystyle\int_0^\infty t\sin 2t\,dt$ is

(a) 0 (b) 1 (c) -1 (d) 4.

19. If $f(t) = \begin{cases} 1, & t > 2 \\ 0, & t < 2 \end{cases}$, then $L\{f(t)\}$ is equal to

(a) $\dfrac{e^{-2s}}{s}$ (b) $\dfrac{e^{2s}}{s}$ (c) $\dfrac{e^{s}}{s}$ (d) $\dfrac{e^{-2s}}{5^s}$.

20. If $L\{f(t)\} = e^{-s}$, then $L\left\{\dfrac{f(t)}{t}\right\}$ is

(a) e^s (b) e^{-s} (c) $\dfrac{e^{-s}}{s}$ (d) $\dfrac{e^s}{s}$.

21. $\displaystyle\int_0^\infty t^2 e^{2t}\,dt$ is equal to

(a) $\dfrac{1}{4}$ (b) $\dfrac{-1}{4}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{8}$.

22. If $L\{f(t)\} = \cot^{-1} s$, then $L\left\{\displaystyle\int_0^t f(t)dt\right\}$ is equal to

(a) $\cot^{-1} s$ (b) $-\dfrac{1}{1+s^2}$ (c) $\dfrac{1}{s}\cot^{-1} s$ (d) $s\cot^{-1} s$.

23. $L\{1/t\}$ is equal to

(a) $\dfrac{1}{s}$ (b) s (c) 1 (d) none of these.

Section B Review Questions

1. Prove that the function e^{t^3} does not satisfy the sufficiency condition for existence of Laplace transform.

2. Find the Laplace transform of the following functions:
   (a) $3t^2 + 4e^{2t} + 5$
   (b) $a_0t^n + a_1t^{n-1} + a_2t^{n-2} + \cdots + a_{n-1}t + a_n$.

3. Find the value of $L\{4t^3 - 3\sin 4t + 3e^{-2t}\}$.

4. Evaluate $L\{(t^2-2)^2\}$.

5. Evaluate $L\{e^{-4t}\sin 2t\ \cos t\}$.

6. Evaluate $L\{\sin^3 2t\}$.

7. Evaluate $L\{3\sin 3t\ \sin 4t\}$.

8. Find the Laplace transform of $2\cosh 3t + 4\sinh 2t$.

9. Find the Laplace transform of $4e^{2t}\cos 3t$.

10. Find the values of $L\{t\sin at\}$ and $L\{t\cos at\}$.

11. Evaluate $L\{(\sin t - \cos t)^2\}$.

12. Evaluate $L\{\cosh at - \cos at\}$.

13. Evaluate $L\{\sin 2t\ \cos 3t\}$.

14. Evaluate $L\{\sin^5 t\}$.

15. Evaluate $L\{(1 + te^{-t})^3\}$.

16. Evaluate $L\{\sinh 3t\ \cos^2 t\}$.

17. Find the Laplace transform of $f(t) = |t-1| + |t-2|, t \geq 0$.

18. Find the Laplace transform of $f(t)$, where

$$f(t) = \begin{cases} \sin(t - \pi/3), & \text{if } t > \pi/3 \\ 0, & \text{if } t < \pi/3 \end{cases}$$

(WBUT 2004)

19. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} \sin t, & \text{if } 0 < t < \pi \\ 0, & \text{if } t > \pi \end{cases}$$

20. Find $L\{f(t)\}$, if

$$f(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & 0 < t < 1 \end{cases}$$

21. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} 1, & \text{if } 0 < t < 2 \\ 2, & \text{if } t > 2 \end{cases}$$

(WBUT 2002)

22. Find the Laplace transform of $f(t)$, where

$$f(t) = \begin{cases} \cos(t - 2\pi/3), & \text{if } t > 2\pi/3 \\ 0, & \text{if } t < 2\pi/3 \end{cases}$$

23. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

24. Find the Laplace transform of $f(t)$, where

$$f(t) = \begin{cases} 1, & 0 \le t \le a/2 \\ -1, & a/2 < t \le a \end{cases}$$

25. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$$

26. Find the Laplace transform of the triangular wave function of period $2c$ given by

$$f(t) = \begin{cases} t, & 0 \le t \le c \\ 2c - t, & c < t < 2c \end{cases}$$

27. Find the Laplace transform of the square wave function of period w defined by

$$f(t) = \begin{cases} 1, & 0 < t < w/2 \\ -1, & w/2 < t < w \end{cases}$$

28. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$$

29. Find $L\{f(t)\}$ where $f(t)$ is a periodic function of period $2c$ defined by

$$f(t) = \begin{cases} t/c, & 0 < t < c \\ (2c - t)/c, & c < t < 2c \end{cases}$$

Draw the graph of the function.

30. Find $L\{f(t)\}$ where

$$f(t) = \begin{cases} 2\sin 3t, & 0 < t < \pi/3 \\ 0, & \pi/3 < t < 2\pi/3 \end{cases}$$

where $f(t) = f(t + 2\pi/3)$.

31. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} 5\sin 3(t - \pi/4), & t > \pi/4 \\ 0, & t < \pi/4 \end{cases}$$

32. Find (a) $L\{J_0(ax)\}$ and (b) $L\{e^{-at}J_0(bt)\}$.

33. If

$$f(t) = \begin{cases} 0, & 0 \le t \le 1 \\ t^2, & t > 1 \end{cases}$$

find $L\{f(t)\}$ and $L\{f'(t)\}$.

34. Find $L\{f(t)\}$ and $L\{f'(t)\}$, where

$$f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 2t, & t > 2 \end{cases}$$

35. Find $L\{\sin 2t\}$ and hence find $L\left\{\dfrac{\sin 2t}{t}\right\}$.

36. Find the value of $L\left\{\dfrac{e^{at} - \cos bt}{t}\right\}$.

37. Evaluate $L\left\{\dfrac{e^{-2t} - e^{-4t}}{t}\right\}$.

38. Evaluate $L\{(\cos 4t - \cos 5t)/t\}$.

39. Evaluate $L\{(1 - \cos t)/t^2\}$.

40. Find the Laplace transform of $1 - \cos 3t$ and hence evaluate $L\left\{\dfrac{1 - \cos 3t}{t}\right\}$.

41. Evaluate $L\left\{2^t + \dfrac{\cos 2t - \cos 3t}{t}\right\}$.

42. Evaluate $L\left\{\dfrac{\cos at - \cos bt}{t}\right\}$.

43. Express

$$f(t) = \begin{cases} 0, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$$

as unit step function and hence evaluate $L\{f(t)\}$.

44. Find the Laplace transform of
(a) $(t-1)\,u(t-1)$ (b) $t^2\,u(t-1)$ (c) $4u(t-\pi)\,\cos t$.

45. Express the following function in terms of unit step function

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

Also, find $L\{f(t)\}$. (WBUT 200

46. Express the function

$$f(t) = \begin{cases} e^t, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$$

in terms of unit step function and hence find $L\{f(t)\}$. (WBUT 2004)

47. Express

$$f(t) = \begin{cases} 2t, & 0 < t < \pi \\ 1, & t > \pi \end{cases}$$

in terms of unit step function and hence find $L\{f(t)\}$.

48. Express

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$$

in terms of unit step function and hence evaluate $L\{f(t)\}$.

49. Evaluate $L\left\{e^{-2t}\int_0^t \frac{\sin t}{t}\,dt\right\}$.

50. Evaluate $L\left\{e^{2t}\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$.

51. Evaluate $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$.

52. Find the value of $L\left\{\int_0^t e^{-2t}\cos 3t\,dt\right\}$.

53. Evaluate $L\left\{\cosh t\int_0^t u\cosh u\,du\right\}$.

54. Find the Laplace transform of $\int_0^t \frac{e^{2t}\sin t}{t}\,dt$.

55. Find the Laplace transform of $\frac{\sin at}{t}$. Hence show that $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$. (WBUT 200

56. Evaluate (a) $\int_0^\infty \frac{e^{-2t}-e^{-t}}{t}\,dt$, (b) $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$.

57. Using the Laplace transform, prove that $\int_0^\infty te^{-3t}\sin t\,dt = \frac{3}{50}$.

58. Find the value of $\int_0^\infty \frac{\cos at - \cos bt}{t}\,dt$.

59. Show that $\int_0^\infty \frac{\sin^3 t}{t}\, dt = \frac{\pi}{4}$.

60. Prove that $\int_0^\infty te^{-2t} \sin 3t\, dt = \frac{12}{169}$.

61. Use the Laplace transform to prove that $\int_0^\infty \frac{e^{-\sqrt{2}t} \sinh t \, \sin t}{t}\, dt = \frac{\pi}{8}$.

62. Show that $\int_0^\infty \frac{e^{-t} \sin^2 t}{t}\, dt = \frac{1}{4}\log 5$.

63. Prove that $\int_0^\infty \frac{1 - \cos t}{t^2}\, dt = \frac{\pi}{4}$.

64. Show that $\int_0^\infty te^{-3t} J_0(4t)\, dt = \frac{3}{125}$.

65. Prove that $L\left\{ \int_0^t \int_0^t \int_0^t t \, \sin t \, dt \right\} = \frac{2}{s^2(s^2+1)^2}$.

Answers

Section A Multiple Choice Questions

1. (c) 2. (d) 3. (a) 4. (a) 5. (c) 6. (c) 7. (c) 8. (a) 9. (c)
10. (b) 11. (a) 12. (b) 13. (a) 14. (c) 15. (b) 16. (c) 17. (d) 18. (a)
19. (b) 20. (b) 21. (a) 22. (c) 23. (d)

Section B Review Questions

2. (a) $\frac{6}{s^3} + \frac{4}{s-2} + \frac{5}{s}$ (b) $a_0\frac{n!}{s^{n+1}} + a_1\frac{(n-1)!}{s^n} + \cdots + a_{n-1}\frac{1}{s^2} + a_n\frac{1}{s}$

3. $\frac{24}{s^4} - \frac{12}{s^2+16} + \frac{3}{s+2}$

4. $\frac{24}{s^5} - \frac{8}{s^3} + \frac{4}{s}$

5. $\frac{1}{2}\left\{\frac{3}{(s+4)^2+9} + \frac{1}{(s+4)^2+1}\right\}$

6. $\frac{48}{(s^2+4)(s^2+36)}$

7. $\frac{3s}{2}\left[\frac{1}{s^2+1} - \frac{1}{s^2+40}\right]$

8. $\frac{2s}{s^2-9} + \frac{8}{s^2-4}$

9. $\frac{4(s-2)}{s^2-2s+13}$

10. $\frac{2as}{(s^2+a^2)^2}$ and $\frac{s^2-a^2}{(s^2+a^2)^2}$

11. $\dfrac{s^2 - 2s + 4}{s(s^2 + 4)}$

12. $\dfrac{2a^2 s}{s^4 - a^4}$

13. $\dfrac{2(s^2 - 5)}{(s^2 + 1)(s^2 + 25)}$

14. $\dfrac{5}{4}\left[\dfrac{1}{s^2 + 1} - \dfrac{3/2}{s^2 + 9} + \dfrac{1/2}{s^2 + 25}\right]$

15. $\dfrac{1}{s} + \dfrac{3}{(s + 1)^2} + \dfrac{6}{(s + 2)^3} + \dfrac{6}{(s + 3)^4}$

16. $\dfrac{3}{2}\left[\dfrac{1}{s^2 - 9} + \dfrac{s^2 - 13}{s^4 - 10s^2 + 169}\right]$

17. $\dfrac{1}{s^2}(2e^{-2s} - 2e^{-s} + 3s + 2)$

18. $\dfrac{1}{1 + s^2} e^{-\pi s/3}$

19. $\dfrac{1 + e^{-\pi s}}{1 + s^2}$

20. $\dfrac{2e^{-s}}{s^3}$

21. $\dfrac{1}{s}(1 + e^{-2s})$

22. $e^{-2\pi s/3} \dfrac{s}{1 + s^2}$

23. $\dfrac{1}{1 - s}[e^{1-s} - 1]$

24. $\dfrac{1}{s}[1 + e^{-as} - 2e^{-as/2}]$

25. $\dfrac{2}{s^3} - \dfrac{e^{-2s}}{s^3}(2 + 3s + 3s^2) + \dfrac{e^{-3s}}{s^2}(5s - 1)$

26. $\dfrac{1}{s^2} \sinh \dfrac{cs}{2}$

27. $\dfrac{1}{s} \tanh \dfrac{sw}{4}$

28. $\dfrac{(1 - e^{-\pi s})(1 + \pi s)}{(1 + e^{-\pi s})s^2}$

29. $\dfrac{1}{cs^2} \tanh \dfrac{cs}{2}$

30. $\dfrac{12}{(s^2 + 9)(1 - e^{-2\pi s/3})}$

31. $\dfrac{15e^{-\pi s/4}}{s^2 + 9}$

32. (a) $\dfrac{1}{\sqrt{s^2+a^2}}$ (b) $\dfrac{1}{\sqrt{(s-a)^2+b^2}}$

33. $\dfrac{e^{-s}}{s^2}(s+2-2e^{s})$ and $\dfrac{e^{-s}}{s^2}(s+2-2e^{-s})$

34. $\dfrac{e^{-2s}}{s^2}(2s+1+e^{2s})$ and $\dfrac{e^{-2s}}{s}(2s+1+e^{2s})$

35. $\dfrac{2}{s^2+4}, \cot^{-1}\dfrac{s}{2}$

36. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{(s-a)^2}\right)$

37. $\log\left(\dfrac{s+4}{s+2}\right)$

38. $\dfrac{1}{2}\log\left(\dfrac{s^2+25}{s^2+16}\right)$

39. $\cot^{-1}s-\dfrac{1}{2}s\log(1+\dfrac{1}{s^2})$

40. $\dfrac{1}{s}-\dfrac{s}{s^2+9}, \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2}\right)$

41. $\dfrac{1}{s-\log 2}+\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)$

42. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$

43. $\left(\dfrac{1}{s}+\dfrac{1}{s^2}\right)e^{-s}-\left(\dfrac{1}{s^2}+\dfrac{2}{s}\right)e^{-2s}$

44. (a) $\frac{e^{-s}}{s^2}$, (b) $e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$, (c) $-4e^{\pi s}s/(1+s^2)$

45. $\dfrac{1}{s^2}(1-2e^{-2s})-\dfrac{1}{s}$

46. $\dfrac{1}{s+1}-\dfrac{e^{-2(s+1)}}{s+1}$

47. $\dfrac{2}{s^2}+\left(\dfrac{1-2\pi}{s}-\dfrac{2}{s^2}\right)e^{-as}$

48. $\dfrac{1}{s^2+1}-\dfrac{2e^{-\pi s}}{s^2+4}+\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{3e^{-2\pi s}}{s^2+9}-\dfrac{2e^{-2\pi s}}{s^2+4}$

49. $\dfrac{1}{s+2}\cot^{-1}(s+2)$

50. $\dfrac{1}{s-2}\cot^{-1}(s-1)$

51. $\dfrac{s+(s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$

52. $\dfrac{1}{s}\dfrac{s+2}{s^2+2s+11}$

53. $\dfrac{1}{2}\left[\dfrac{s^2-2s+2}{(s-1)(s^2-2s)^2}+\dfrac{s^2+2s+2}{(s+1)(s^2+2s)^2}\right]$

54. $\frac{1}{s}\cot^{-1}(s-2)$

55. $\frac{\pi}{2}$ if $a > 0$, $\frac{-\pi}{2}$ if $a < 0$, 0 if $a = 0$

56. (a) $\log\frac{1}{2}$, (b) $\log\frac{b}{a}$

58. $\log\frac{b}{a}$.