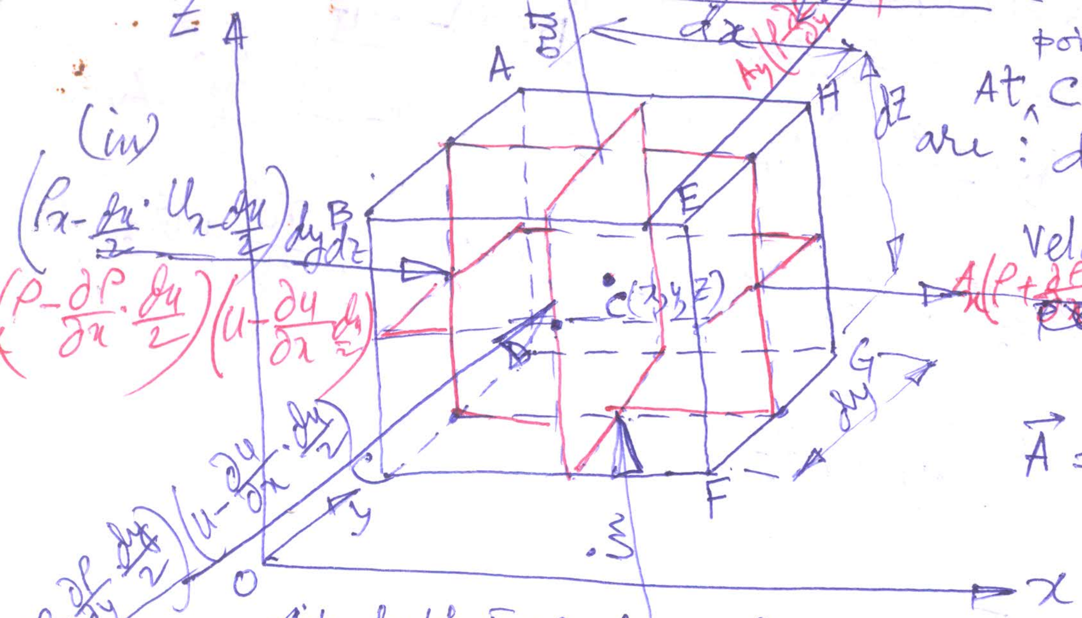
volume approach)



At point C(x, y, z), the properties are: density, ρ,

Velocity $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

At left Face (ABCD)

Density at left face (ABCD) $\rho_{x-\frac{dx}{2}} = \rho - \frac{\partial \rho}{\partial x}\cdot\frac{dx}{2} + \frac{\partial^2 \rho}{\partial x^2}\cdot\frac{1}{2!}\left(-\frac{dx}{2}\right)^2 + \cdots$

(Using Tailor Series expansion) $= \rho - \frac{\partial \rho}{\partial x}\frac{dx}{2}$

Velocity component in x direct^n, ~~in left face~~ $u_{x-\frac{dx}{2}} = u - \frac{\partial u}{\partial x}\frac{dx}{2}$

~~At Right Face (EFGH)~~

mass flow rate (in) $= \rho_{(x-\frac{dx}{2})}\, u_{x-\frac{dx}{2}}\cdot dy\,dz$

$$= \left(\rho - \frac{\partial \rho}{\partial x}\frac{dx}{2}\right)\left(u - \frac{\partial u}{\partial x}\frac{dx}{2}\right)dy\,dz.$$

$$= \rho u\,dy\,dz - \rho\frac{\partial u}{\partial x}\frac{dx}{2}dy\,dz - u\frac{\partial \rho}{\partial x}\frac{dx}{2}dy\,dz + \overset{0}{\cdots}$$

$$= \rho u\,dy\,dz - \left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}dy\,dz.$$

At Right Face (EFGH)

mass flow rate (out) $\rho_{x+\frac{dx}{2}}\cdot u_{x+\frac{dx}{2}}\cdot dy\,dz$

$$= \left(\rho + \frac{\partial \rho}{\partial x}\frac{dx}{2}\right)\left(u + \frac{\partial u}{\partial x}\cdot\frac{dx}{2}\right)dy\,dz$$

$$= \rho u\,dy\,dz + \left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}dy\,dz.$$

Net mass flowrate out $=$ out $-$ in
(along x-direction) $= \rho u\,dy\,dz + \frac{\partial(\rho u)}{\partial x}\frac{dx\,dy\,dz}{2} - \rho u\,dy\,dz + \frac{\partial \rho u}{\partial x}\frac{dx\,dy\,dz}{2}$

$$= \frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz = \frac{\partial}{\partial x}(\rho u)\,d\forall.$$

Similarly net mass flow rate (out) along y-direction $= (out)_y - (in)_y$.
In - Along Front Face (BCFE) & Rear Face (ADGH) - out

$$= \left(\rho + \frac{\partial \rho}{\partial y}\cdot\frac{dy}{2}\right)\left(v + \frac{\partial v}{\partial y}\cdot\frac{dy}{2}\right)A_y - \left(\rho - \frac{\partial \rho}{\partial y}\frac{dy}{2}\right)\left(v - \frac{\partial v}{\partial y}\cdot\frac{dy}{2}\right)\cdot A_y.$$

$$= \cancel{\rho v A_y} + \left(\rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right)\frac{dy}{2}A_y - \cancel{\rho v A_y} + \left(\rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right)\cdot\frac{dy}{2}A_y$$

$$= \frac{\partial}{\partial y}(\rho v)\cdot dy\, dx\, dz$$

$$= \frac{\partial}{\partial y}(\rho v)\, d\forall.$$

Net mass flow rate out along Z-direct^n $= (out)_z - (in)_z$

[In - ~~Along~~ Through Bottom (CDGF) & out - Through top (ABEH)]

$$= \left(\rho + \frac{\partial \rho}{\partial z}\frac{dz}{2}\right)\left(w + \frac{\partial w}{\partial z}\cdot\frac{dz}{2}\right)A_z - \left(\rho - \frac{\partial \rho}{\partial z}\frac{dz}{2}\right)\left(w - \frac{\partial w}{\partial z}\frac{dz}{2}\right)A_z$$

$$= \cancel{\rho w A_z} + \left(\rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}A_z - \cancel{\rho w A_z} + \left(\rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}\cdot A_z.$$

$$= \frac{\partial}{\partial z}(\rho w)\cdot dx\, dy\, dz$$

$$= \frac{\partial}{\partial z}(\rho w)\, d\forall.$$

~~Rate~~ Time rate of change in mass within CV $= \frac{\partial(\rho\, d\forall)}{\partial t} = \frac{\partial \rho}{\partial t}d\forall.$

From conservation of mass statement, $\frac{dm}{dt} = 0$

or, Net mass flow rate out through control surfaces + ~~Change in ma~~ Time rate of change in mass within control volume $= 0$

or, $\frac{\partial}{\partial x}(\rho u) + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + \frac{\partial \rho}{\partial t} = 0$ ... Continuity

$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{V} = 0$ or, $\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = 0$ $\longrightarrow$ Substantial Derivative.

$= \partial\rho/\partial t + \rho\vec{\nabla}\cdot\vec{V} + (\vec{V}\cdot\vec{\nabla})\rho$

$$\frac{\partial}{\partial r}(\rho v_r r\, d\theta\, dz)\, dr = \left[r\frac{\partial}{\partial r}(\rho v_r\, d\theta\, dz) + \rho v_r\, d\theta\, dz\frac{\partial r}{\partial r}\right] dr$$

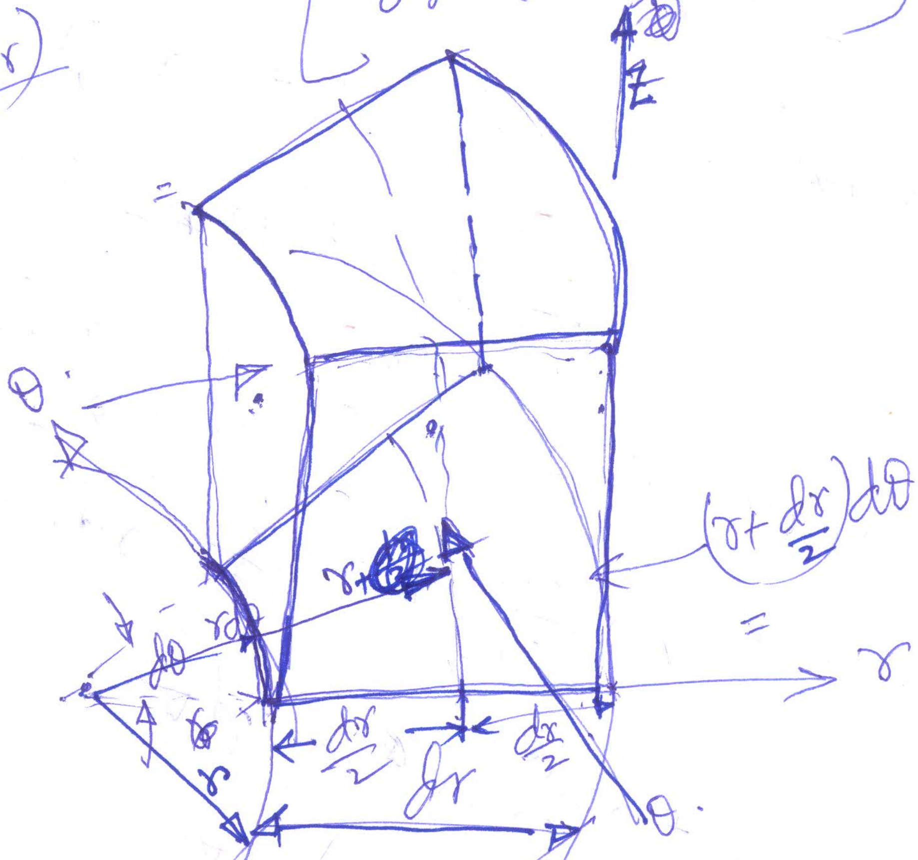
$$= r\frac{\partial}{\partial r}(\rho v_r)\, d\theta\, dz\, dr + \rho v_r\, d\theta\, dz\, dr.$$

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho v_r r)$$

$$\frac{1}{r}\left[\rho v_r\frac{\partial r}{\partial r} + r\frac{\partial(\rho v_r)}{\partial r}\right] = \underbrace{r\, d\theta\, dr\, dz}_{d\forall}\frac{\partial}{\partial r}(\rho v_r) + \frac{1}{r}\rho v_r\, d\forall$$

$$= \frac{\rho v_r}{r} + \frac{\partial(\rho v_r)}{\partial r}$$

$$= \left[\frac{\partial}{\partial r}(\rho v_r) + \frac{\rho v_r}{r}\right] d\forall.$$



$\dot{m} = \rho v_r\, r\, d\theta\, dz$

out along r $= \frac{\partial}{\partial r}(\rho v_r\, r\, d\theta\, dz)\, dr$ $d\forall$

$\rho v_r\, r\, d\theta\, dz$ $\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)$

ρ at $r+dr$.

$$\rho_{r-\frac{dr}{2}} = \rho - \frac{\partial\rho}{\partial r}\cdot\frac{dr}{2}, \quad v_r\Big|_{r-\frac{dr}{2}} = v_r - \frac{\partial v_r}{\partial r}\cdot\frac{dr}{2}.$$

$$\dot{m} = \left(\rho - \frac{\partial\rho}{\partial r}\cdot\frac{dr}{2}\right)\left(v_r - \frac{\partial v_r}{\partial r}\cdot\frac{dr}{2}\right)\left(r - \frac{dr}{2}\right) d\theta\, dz.$$

$$\text{out} = \left(\rho + \frac{\partial\rho}{\partial r}\frac{dr}{2}\right)\left(v_r + \frac{\partial v_r}{\partial r}\frac{dr}{2}\right)\left(r + \frac{dr}{2}\right) d\theta\, dz$$

Net out along r-direction :-

$$\left(\rho+\frac{\partial\rho}{\partial r}\cdot\frac{dr}{2}\right)\left(V_r+\frac{\partial V_r}{\partial r}\cdot\frac{dr}{2}\right)\left(r+\frac{dr}{2}\right)d\theta\, dz-\left(\rho-\frac{\partial\rho}{\partial r}\frac{dr}{2}\right)\left(V_r-\frac{\partial V_r}{\partial r}\frac{dr}{2}\right)\left(r-\frac{dr}{2}\right)d\theta\, dz$$

$$= \cancel{\rho V_r r\, d\theta\, dz + \rho V_r \frac{dr}{2}d\theta\, dz + \rho\frac{\partial V_r}{\partial r}\frac{dr}{2}\, r\, d\theta\, dz}$$

$$\left(\rho V_r \pm \rho\frac{\partial V_r}{\partial r}\cdot\frac{dr}{2} \pm V_r\frac{\partial\rho}{\partial r}\frac{dr}{2} + \cancel{\ }^{0}\right)\left(r\pm\frac{dr}{2}\right)d\theta\, dz$$

$$= \rho V_r r\, d\theta\, dz + \rho V_r \frac{dr}{2}d\theta\, dz + \rho r\frac{\partial V_r}{\partial r}\frac{dr}{2}d\theta\, dz + \cancel{\ }^{0} + r V_r\frac{\partial\rho}{\partial r}\frac{dr}{2}d\theta\, dz + \cancel{\ }^{0}$$

$$-\left(\rho V_r r\, d\theta\, dz - \rho V_r\frac{dr}{2}d\theta\, dz - \rho r\frac{\partial V_r}{\partial r}\cdot\frac{dr}{2}d\theta\, dz - r V_r\frac{\partial\rho}{\partial r}\frac{dr}{2}d\theta\, dz\right)$$

$$\rho V_r\, dr\, d\theta\, dz + \rho r\frac{\partial V_r}{\partial r}dr\, d\theta\, dz + r V_r\frac{\partial\rho}{\partial r}dr\, d\theta\, dz$$

$$\frac{1}{r}\left(\rho V_r + \rho r\frac{\partial V_r}{\partial r} + r V_r\frac{\partial\rho}{\partial r}\right) r\, dr\, d\theta\, dz$$

$$\frac{1}{r}\left(\rho V_r + r\frac{\partial}{\partial r}(\rho \cancel{r} V_r)\right) d\forall$$

$$= \frac{1}{r}\left(\rho V_r + r\frac{\partial}{\partial r}(\rho V_r)\right) d\forall$$

$$= \left(\frac{\rho V_r}{r} + \frac{\partial}{\partial r}(\rho V_r)\right) d\forall$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(\rho V_r r)\, d\forall$$

Similarly θ direct^n $\Rightarrow$ $\frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta)\, d\forall$

Along direct^n z $\Rightarrow$ $\frac{\partial}{\partial z}(\rho V_z)\, d\forall$

mass within CV $\Rightarrow$ $\frac{\partial}{\partial t}(\rho\, d\forall) = \frac{\partial\rho}{\partial t}d\forall$

$$d\forall\,\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho V_r r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

$$\therefore \frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho V_r r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$$