Linear PDE with Constant Coefficients

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Linear PDE

A linear PDE of order n in the dependent variable z and independent variables x and y can be represented in the form

$$f\left(\frac{\partial^n z}{\partial x^n}, \frac{\partial^n z}{\partial x^{n-1}\partial y}, \ldots, \frac{\partial^n z}{\partial y^n}, \frac{\partial^{n-1} z}{\partial x^{n-1}}, \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y}, \ldots, \frac{\partial^{n-1} z}{\partial y^{n-1}}, \ldots, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, z, x, y\right) = 0, \tag{1}$$

where the powers of the dependent variable and its derivatives are strictly equal to **one**. Moreover, no product of the dependent variable and its derivatives can appear in any term of the equation (1).

Examples

Check whether the following PDEs are linear or not

1. $(x^2+y)\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial y} + z = x^3$ (Linear PDE of order 2)
2. $2\frac{\partial^3 z}{\partial y^3} + (3x-y)\frac{\partial z}{\partial y} + x^2 z = yx$ (Linear PDE of order 3)
3. $\frac{\partial^4 z}{\partial x^4} - 5\frac{\partial z}{\partial y} + 2z = yx$ (Linear PDE of order 4)
4. $\frac{\partial^4 z}{\partial x^4} - 5\left[\frac{\partial z}{\partial y}\right]^2 + 2z = yx$ (Not linear)
5. $\frac{\partial^4 z}{\partial x^4} - 5\frac{\partial z}{\partial y} + 2z^2 = yx + 3$ (Not linear)
6. $\frac{\partial^4 z}{\partial y^4} - 5\frac{\partial z}{\partial y}z = yx^2$ (Not linear)

Linear PDE with constant coefficients

A linear PDE, in which coefficients of the dependent variable and its derivatives are constants is called a linear PDE with constant coefficients.

Examples

Check whether the following equations are linear PDE with constant coefficients

1. $\frac{\partial^4 z}{\partial y^4} - 5\frac{\partial z}{\partial y} = yx^2$ (Yes)
2. $\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial z}{\partial y} = y$ (Yes)
3. $\frac{\partial^3 z}{\partial y^3} - 5\frac{\partial z}{\partial y} = 0$ (Yes)
4. $\frac{\partial^3 z}{\partial y^3} + 2x\frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial z}{\partial y} = 5x^2$ (Linear but not with variable coefficients)
5. $4\frac{\partial^3 z}{\partial y^3} + 3\frac{\partial^3 z}{\partial x \partial y^2} - 5\frac{\partial z}{\partial y} + 7z = x + \sin y$ (Yes)

Linear PDE with constant coefficients can further be divided into two different classes namely homogeneous and nonhomogeneous.

Liner **homogeneous** PDE with constant coefficients

An equation of the form

$$\frac{\partial z^n}{\partial x^n} + a_1 \frac{\partial z^n}{\partial x^{n-1}\partial y} + a_2 \frac{\partial z^n}{\partial x^{n-2}\partial y^2} + \cdots + a_{n-1} \frac{\partial z^n}{\partial x \partial y^{n-1}} + a_n \frac{\partial z^n}{\partial y^n} = f(x, y), \tag{2}$$

where a_1, a_2, ... and a_n are constants and f is a function of the independent variables, is called a linear **homogeneous** PDE of order n with constant coefficients. Note that all the derivatives appeared in the equation are of same order.

Liner **nonhomogeneous** PDE with constant coefficients

A linear PDE with constant coefficients consisting of at least two derivatives of different orders or a term involving the dependent variable is called **nonhomogeneous** linear PDE with constant coefficients.

Examples

- $\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} - 3\frac{\partial^2 z}{\partial y^2} = x \sin x$ (Homogeneous)
- $\frac{\partial^3 z}{\partial x^3} + 7\frac{\partial^3 z}{\partial x^2 \partial y} = 0$ (Homogeneous)
- $\frac{\partial^3 z}{\partial x^3} + \frac{\partial^2 z}{\partial x \partial y} = 0$ (Nonhomogeneous)
- $\frac{\partial^4 z}{\partial y^4} + \frac{\partial^4 z}{\partial x^2 \partial y^2} - 3z = 0$ (Nonhomogeneous)
- $\frac{\partial^3 z}{\partial x^3} + \frac{\partial^2 z}{\partial x \partial y} = x$ (Nonhomogeneous)

General solution of the homogeneous linear PDE with constant coefficients

Consider the homogeneous linear PDE

$$\frac{\partial z^n}{\partial x^n} + a_1 \frac{\partial z^n}{\partial x^{n-1}\partial y} + a_2 \frac{\partial z^n}{\partial x^{n-2}\partial y^2} + \cdots + a_{n-1} \frac{\partial z^n}{\partial x \partial y^{n-1}} + a_n \frac{\partial z^n}{\partial y^n} = f(x,y), \tag{3}$$

where a_1, a_2, ... and a_n are constants and f is a function of the independent variables. Introducing the notations $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$ we can write the above equation in the form

$$F(D, D')z = f(x,y), \tag{4}$$

where $f(D, D') = D^n + a_1 D^{n-1} D' + \cdots + a_{n-1} D D'^{n-1} + a_n D'^n$. Now to find the general solution of the PDE (3), we first find the general solution of it when $f(x,y) = 0$. In that case the PDE (3) reduces to

$$F(D, D')z = 0. \tag{5}$$

Let $z = \phi(y + mx)$ be a solution of the PDE (5). Then substituting z in (5) we get

$$
\begin{aligned}
(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n)\phi^n(y + mx) &= 0 \quad (6) \\
\implies m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1} m + a_n &= 0. \quad (7)
\end{aligned}
$$

Now we observe that the equation (7), called the **auxiliary equation** is a polynomial of degree n and let us suppose the not necessarily distinct roots of the polynomial are $m_1, m_2, m_3, \ldots, m_n$. In the following we consider following two possible cases.

- Distinct roots: $m_1, m_2, m_3, \ldots, m_n$ are distinct real or complex numbers. The complete solution of the PDE (5) is then given by

  $$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x),$$

  where ϕ_1, ϕ_2, ... are arbitrary functions.

- Repeated roots: Let a particular root m of the equation (7) is repeated r times. Then the contribution of this root to the complete solution of the PDE (5) will be

$$x^{r-1}\phi_1(y+mx)+x^{r-2}\phi_2(y+mx)+\cdots+\phi_r(y+mx),$$

while the contributions of the distinct roots to complete solution will be same as the previous case. The following examples will illustrate the procedure. Note that the compete solution of the PDE (5) is called the **complementary function (CF)** of the PDE (2).

Examples

Solve the PDEs: (i)$(D^3-6D^2D'+11DD'^2-6D'^2)z=0$,
(ii) $(D^3-3D^2D'+3DD'^2-D'^3)z=0$, (iii) $(D^4-D'^4)z=0$,
(iv) $(D^4-2D^3D'+2DD'^2-D'^4)z=0$.

Solution (i)

Given equation is $(D^3 - 6D^2D' + 11DD'^2 - 6D'^2)z = 0$.
Auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$.
The roots of the auxiliary equation are $m = 1, 2, 3$, which are distinct.
Therefore, complete solutions of the given PDE is

$$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x),$$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Solution (ii)

Given equation is $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$.
Auxiliary equation is $m^3 - 3m^2 + 3m - 1 = 0$.
The roots of the auxiliary equation are $m = 1, 1, 1$, which are repeated.
Therefore, complete solutions of the given PDE is

$$z = x^2\phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + x),$$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Solution (iii)

Given equation is $(D^4 - D'^4)z = 0$.
Auxiliary equation is $m^4 - 1 = 0$.
The roots of the auxiliary equation are $m = 1, -1, i, -i$, which are distinct. Note that there are complex conjugate roots. Therefore, complete solutions of the given PDE is

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix),$$

where ϕ_1, ϕ_2, phi_3 and ϕ_4 are arbitrary functions.

Solution (iv)

Given equation is $(D^4 - 2D^3D' + 2DD'^2 - D'^4)z = 0$.
Auxiliary equation is $m^4 - 2m^3 + 2m - 1 = 0$.
The roots of the auxiliary equation are $m = 1, 1, 1, -1$, which are distinct.
Therefore, complete solutions of the given PDE is

$$z = x^2\phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + x) + \phi_4(y - x),$$

where ϕ_1, ϕ_2, phi_3 and ϕ_4 are arbitrary functions.

Now we consider the the PDE

$$F(D, D')z = f(x, y),$$

again and define the **Particular Integral (PI)** of as follows.

Particular Integral

It is a function u of x and y denoted by the symbol

$$\frac{1}{F(D, D')} f(x, y).$$

The function u is such that the action of the operator $F(D, D')$ on u returns the function $f(x, y)$. i.e.

$$F(D, D')u = f(x, y).$$

Clearly $u(x, y)$ is a particular function which satisfy the PDE homogeneous PDE (4). Note that **PI** does not involve any arbitrary function or constant.

General Method

Meaning of $\frac{1}{D}f(x,y)$ and $\frac{1}{D'}f(x,y)$

Let

$$\begin{aligned}\frac{1}{D}f(x,y) &= X(x,y), \implies \frac{\partial X}{\partial x} = f(x,y)\\ \text{Integrating, } X &= \int f(x,y)dx \text{ (Partil Integration w.r.t. x)}\end{aligned}$$

Similarly $\frac{1}{D'}f(x,y)$ means partial integration of $f(x,y)$ w.r.t. y.

.

Now let $m_1, m_2, \ldots, m_n$ are the roots of the auxiliary equation. Then

$$F(D,D') = (D - m_1 D')(D - m_2 D') \ldots (D - m_n D'),$$

which implies

$$\begin{aligned}\frac{1}{F(D,D')}f(x,y) &= \frac{1}{(D-m_1D')(D-m_2D')\dots(D-m_nD')}f(x,y)\\ &= \frac{1}{(D-m_1D')(D-m_2D')\dots(D-m_{n-1}D')}X_1, \quad (8)\end{aligned}$$

where $X_1 = \frac{1}{(D-m_nD')}f(x,y)$.
Therefore,

$$\begin{aligned}(D-m_nD')X_1 &= f(x,y),\\ \text{Or, } \frac{\partial X_1}{\partial x} - m_n\frac{\partial X_1}{\partial y} &= f(x,y). \quad (9)\end{aligned}$$

Now the equation (9) is a Lagrange's equation whose subsidiary equation is given by

$$\frac{dx}{1} = \frac{dy}{-m_n} = \frac{dX_1}{f(x,y)}. \quad (10)$$

From the first two terms of the simultaneous differential equations (10) we get

$$y = c - m_n x,$$

where c is an arbitrary constant. Now from the first and third terms of equation (10), we get

$$X_1 = \int f(x, c - m_n x) dx$$

and after integration we have to replace c by $y + m_n x$.
Therefore, to find the particular integral

$$\frac{1}{F(D, D')} f(x, y)$$

we need to repeat the above procedure n number of times. Although the method is general but often it is lengthy. In the following we discuss some short methods for finding particular integral.

Short Methods

I: $f(x, y) = x^m y^n$, m and n are some positive integers

In this case the the function $\frac{1}{F(D,D')}$ is either expanded in an infinite series of ascending powers of D or D' and action of the resulting infinite series operator on $f(x, y)$ is determined to find the **PI**.

Examples

Find particular integrals of the PDEs (i) $(D^2 - a^2 D'^2)z = x(a = constant)$, (ii) $(D^2 + 3DD' + 2D'^2)z = x + y$, (iii) $(D^2 - DD' - 6D'^2)z = xy$

Solution (i)

Here $f(x,y) = x$.

$$
\begin{aligned}
\therefore \mathrm{PI} &= \frac{1}{D^2 - a^2 D'^2} x \\
&= -\frac{1}{a^2 D'^2}\left[1 - \frac{D^2}{a^2 D'^2}\right]^{-1} x \\
&= -\frac{1}{a^2 D'^2}\left[1 + \frac{D^2}{a^2 D'^2} + \dots\right] x \\
&= -\frac{1}{a^2 D'^2} x = -\frac{1}{a^2} \cdot \frac{1}{D'^2} x \\
&= -\frac{xy^2}{2a^2}.
\end{aligned}
$$

Solution (ii)

$$
\begin{aligned}
PI &= \frac{1}{D^2 + 3DD' + 2D'^2}(x+y) \\
&= \frac{1}{D^2}\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(x+y) \\
&= \frac{1}{D^2}\left[1 - \frac{3D'}{D} + \dots\right] \\
&= \frac{1}{D^2}\left[x + y - \frac{3}{D}1\right] \\
&= \frac{1}{D^2}(y + x - 3x) = \frac{1}{D^2}(y - 2x) \\
&= \frac{x^2y}{2} - \frac{x^3}{3}.
\end{aligned}
$$

Solution (iii)

$$
\begin{aligned}
PI &= \frac{1}{D^2 - DD' - 6D'^2} xy \\
&= \frac{1}{D^2}\left[1 - \frac{D}{D'} - 6\frac{D'^2}{D^2}\right]^{-1} xy \\
&= \frac{1}{D^2}\left[1 + \frac{D}{D'} + \dots\right] xy \\
&= \frac{1}{D^2}\left[xy + \frac{x^2}{2}\right] \\
&= \frac{1}{6}x^3y + \frac{x^4}{24}
\end{aligned}
$$

Lecture 5: 01.10.2020

II: $f(x,y) = \phi(ax+by)$

Under this case there are two possibilities.

- (a) $F(a,b) \neq 0$

$$PI = \frac{1}{F(D,D')}\phi(ax+by) = \frac{1}{F(a,b)} \iint \ldots n \text{ times } \phi(t)dtdt \ldots dt,$$

  where n is the order of the PDE and $t = ax+by$.

- (b) $F(D,D') = (bD - aD')^r \psi(D,D')$ such that $\psi(a,b) \neq 0$.

$$PI = \frac{1}{F(D,D')}\phi(ax+by) = \frac{x^r}{r!b^r}\frac{1}{\psi(a,b)} \iint \ldots (n-r) \text{ times } \psi(t)dtdt \ldots dt.$$

Examples

Find the particular integrals for the following PDEs.
(i) $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$, (ii) $(D^2 + 3DD' + 2D'^2)z = x + y$,
(iii) $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x+y)$.

Solution (i)

Here

$$F(D, D') = D^2 + 2DD' + D'^2,$$

$f(x, y) = e^{2x+3y}$ is of the form $\phi(ax + by)$ with $a = 2$ and $b = 3$ and

$$F(a, b) = F(2, 3) = 2^2 + 2 \times 2 \times 3 + 3^2 = 25 \neq 0.$$

Therefore,

$$\begin{aligned} PI &= \frac{1}{D^2 + 2DD' + D'^2} e^{2x+3y} \\ &= \frac{1}{2^2 + 2 \times 2 \times 3 + 3^2} \int \left(\int e^t dt \right) dt \\ &= \frac{1}{25} e^t = \frac{1}{25} e^{2x+3y} \end{aligned}$$

Solution (ii)

Here $F(D, D) = D^2 + 3DD' + 2D'^2$, $f(x, y) = x + y$ is of the form $\phi(ax + by)$ with $a = b = 1$ and $F(1, 1) = 6 \neq 0$. Therefore,

$$
\begin{aligned}
PI &= \frac{1}{D^2 + 3DD' + 2D'^2}(x + y) \\
&= \frac{1}{1^2 + 3 \times 1 \times 1 + 2 \times 1^2} \int (\int tdt)dt \\
&= \frac{1}{6}\frac{t^3}{6} = \frac{(x + y)^3}{36}.
\end{aligned}
$$

Solution (iii)

Here $F(D, D) = D^3 - 4D^2D' + 4DD'^2$, $f(x, y) = \cos(2x + y)$ is of the form $\phi(ax + by)$ with $a = 2, b = 1$ and $F(2, 1) = 0$.
Now we note that $F(D, D') = D(D - 2D')^2$. So the particular integral is given by

$$
\begin{aligned}
PI &= \frac{1}{D^3 - 4D^2D' + 4DD'^2} \cos(2x + y) \\
&= \frac{1}{D(D - 2D')^2} \cos(2x + y) = \frac{x^2}{2!1^2} \frac{1}{D} \cos(2x + y) \\
&= \frac{x^2}{2} \times \frac{1}{2} \int \cos t dt = \frac{x^2}{4} \sin t \\
&= \frac{x^2}{4} \sin(2x + y).
\end{aligned}
$$

Complete solution

Consider the homogeneous linear PDE of the form

$$F(D, D')z = f(x, y),$$

with constant coefficients. Then we know how to find the **complementary function (CF)** and **particular integral (PI)** of the PDE. Now if we know the CF and PI of the PDE, then the **complete solution** of the PDE is given by

$$z = CF + PI.$$

Examples

Solve the PDEs

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$
2. $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial^2 x \partial y} = 2e^{2x} + 3x^2 y$
3. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$

Solution (i)

Auxiliary Equation is

$$m^2 - m = 0 \text{ and the roots of it are } m = 0, 1.$$

Therefore complementary function is $z = \phi_1(y) + \phi_2(y + x)$.
Now

$$\begin{aligned} PI &= \frac{1}{D^2 - DD'} \cos x \cos 2y = \frac{1}{2} \frac{1}{D^2 - DD'} [\cos(x + 2y) + \cos(x - 2y)] \\ &= \frac{1}{2} \left[\frac{1}{1-2} \iint \cos t dt dt + \frac{1}{1+2} \iint \cos t_1 dt_1 dt_1 \right] \\ &= \frac{1}{2} \left[\cos t - \frac{1}{3} \cos t_1 \right] \\ &= \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y) \end{aligned}$$

Hence the complete solution is give by,

$$z = \phi_1(y) + \phi_2(y + x) + \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y).$$

Solution (ii)

Auxiliary Equation is $m^3 - 2m^2 = 0$ and the roots of it are $m = 0, 0, 2$.
Therefore complementary function is $z = x\phi_1(y) + \phi_2(y) + \phi_3(y + 2x)$.
Now

$$
\begin{aligned}
PI &= \frac{1}{D^3 - D^2D'}(2e^2x + 3x^2y) = 2\frac{1}{D^3 - D^2D'}e^{2x} + 3\frac{1}{D^3 - D^2D'}x^2y \\
&= 2\frac{1}{2^3 - 2^2 \times 0}e^{2x} + \frac{3}{D^3}\frac{1}{1 - \frac{D'}{D}}x^2y \\
&= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left[1 - \frac{D'}{D}\right]^{-1}x^2y \\
&= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left[1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \dots\right]x^2y \\
&= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left[x^2y + \frac{1}{D}x^2\right] = \frac{1}{4}e^{2x} + \frac{3}{D^3}\left[x^2y + \frac{x^3}{3}\right] \\
&= \frac{1}{4}e^{2x} + \frac{x^5y}{20} + \frac{x^6}{60}
\end{aligned}
$$

So the complete solution is given by

$$z = x\phi_1(y) + \phi_2(y) + \phi_3(y + 2x + \frac{1}{4}e^{2x} + \frac{x^5 y}{20} + \frac{x^6}{60}.$$

Solution (iii)

Auxiliary Equation is $m^2 + m - 6 = 0$ and the roots of it are $m = 2, -3$.
Therefore complementary function is $z = \phi_1(y + 2x) + \phi_2(y - 3x)$.
Now

$$\begin{aligned}
PI &= \frac{1}{D^2 + DD' - 6D'^2} y \cos x = \frac{1}{(D + 3D')(D - 2D')} y \cos x \\
&= \frac{1}{(D + 3D')} \int (c - 2x) \cos x dx \;\; [\because y = c - 2x] \\
&= \frac{1}{(D + 3D')} [c \sin x - 2x \sin x - \cos x] \\
&= \frac{1}{(D + 3D')} [(y + 2x) \sin x - 2x \sin x - \cos x]
\end{aligned}$$

$$
\begin{aligned}
\therefore \; PI &= \frac{1}{(D+3D')}[y\sin x - \cos x] \\
&= \int [(c+3x)\sin x - \cos x]\,dx \;\; [\because y = c+3x] \\
&= -c\cos x - \sin x - 3x\cos x + 3\sin x \\
&= -(y-3x)\cos x - \sin x - 3x\cos x + 3\sin x \\
&= -y\cos x + 2\sin x
\end{aligned}
$$

Hence the complete solution is given by

$$
z = \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + 2\sin x.
$$

Lecture 6: 08.10.2020

Classification of Second Order PDE

We consider the most general second order linear PDE of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \qquad (11)$$

where A, B, C, D, E, F, G are functions of the independent variables x and y.

We call the function

$$B(x,y)^2 - 4A(x,y)C(x,y)$$

as discriminant of the equation (11).

Classification

At a point (x_0, y_0) in its domain, the PDE (11) is called

- Hyperbolic if $B(x_0,y_0)^2 - 4A(x_0,y_0)C(x_0,y_0) > 0$.
- Parabolic if $B(x_0,y_0)^2 - 4A(x_0,y_0)C(x_0,y_0) = 0$.
- Elliptic if $B(x_0,y_0)^2 - 4A(x_0,y_0)C(x_0,y_0) < 0$.

Cannonical Forms

Consider the most general of the independent variables x and y of the PDE (11) to new variables ξ and η given by

$$\xi = \xi(x,y), \;\; \eta = \eta(x,y)$$

such that the functions are continuously differentiable and the Jacobian

$$J(x,y) = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0,$$

in the domain of definition of the PDE. Using the chain rule we now have the following

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}.
\end{aligned}$$

Substituting all in (11) we get

$$\bar{A}\frac{\partial^2 u}{\partial x^2} + \bar{B}\frac{\partial^2 u}{\partial x \partial y} + \bar{C}\frac{\partial^2 u}{\partial y^2} + \bar{D}\frac{\partial u}{\partial x} + \bar{E}\frac{\partial u}{\partial y} + \bar{F}u = \bar{G}, \tag{12}$$

where

$$\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\bar{B} &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
\bar{F} &= F, \bar{G} = G.
\end{aligned}$$

Now we can verify that

$$\bar{B}^2 - 4\bar{A}\bar{C} = (\xi_x\eta_y - \xi_y\eta_x)^2(B^2 - 4AC)$$

and the transformation discussed above does not alter the type of the PDE. Finally it can be shown that following some simple steps, one can determine specific forms of ξ and η, so that the PDE (11) takes the following three cannonical forms

1. $u_{\xi\xi} - u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ or $u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ in the hyperbolic case.
2. $u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ or $u_{\xi\xi} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ in the parabolic case.
3. $u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ in the elliptic case.
   In the following slide we discuss each cases separately.

Lecture 7: 09.10.2020

Cannonical Forms for Hyperbolic Equations

Since the discriminant $\bar{B}^2 - 4\bar{A}\bar{C} > 0$ for hyperbolic PDE, we choose ξ and η in such a way that

$$\bar{A} = 0 \text{ and } \bar{C} = 0.$$

Thus we have

$$A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \text{ and } A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0.$$

Rewriting the above equations we get

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \text{ and } A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0 = 0.$$

Solving the above equations we get

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \text{ and } \frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}. \tag{13}$$

Now along the curves $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$ we know

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} \text{ and } \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$$

respectively.

Now solving the differential equations

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} \text{ and } \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} \tag{14}$$

we can determine the surfaces $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$ which will give the coordinates ξ and η that will ensure $\bar{A} = \bar{C} = 0$. Using these coordinates we will be able to determine the desired normal form of the hyperbolic PDE. Note that the equations (14) are called **characteristic equations**.

Example

Find the cannonical of the PDE

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0.$$

Solution

Comparing the given equation with the general form we get

$$A = 3, B = 10, C = 3, D = E = F = G = 0.$$

Discriminant, $B^2 - 4AC = 100 - 36 = 64 > 0$. Hence the PDE is of hyperbolic type for all (x, y). Characteristic equations are given by

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{1}{3} \text{ and } \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 3.$$

Solving the characteristic equations we get

$$y - 3x = c_1 \text{ and} y - \frac{x}{3} = c_2.$$

Now the new coefficients of the PDE are given by

$$\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\
\bar{B} &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y = -\frac{64}{3} \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = 0 \\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0 \\
\bar{F} &= F = 0, \bar{G} = G = 0.
\end{aligned}$$

Hence the cannonical form of the given PDE is

$$\frac{64}{3}u_{\xi\eta} = 0 \text{ or } u_{\xi\eta} = 0.$$

Integrating the cannonical equation partially with respect to ξ we get

$$u_\eta(\xi,\eta) = f_1(\eta),$$

where f_1 is some arbitrary function of η.

Integrating once again with respect to η we get

$$u_{\xi\eta} = f(\xi) + \int f_1(\eta)d\eta = f(\xi) + g(\eta),$$

where f and g both are arbitrary functions of ξ and η. Now the solution of the PDE in terms of the independent variables x and y is given by

$$u(x, y) = f(y - 3x) + g(y - \frac{x}{3}).$$

Example

Reduce the PDE

$$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0$$

to a cannonical form.

Solution

Comparing the given equation with the general form we get

$$A = 1, B = -2\sin x, C = -\cos^2 x, D = 0, E = -\cos x, F = G = 0.$$

Discriminant, $B^2 - 4AC = 4(\sin^2 x + \cos^2 x) = 4 > 0$. Hence the PDE is of hyperbolic type for all (x, y). Characteristic equations are given by

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\sin x - 1 \text{ and } \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 1 - \sin x.$$

Solving the characteristic equations we get

Now the new coefficients of the PDE are given by

$$
\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0\\
\bar{B} &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y = -4\\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0\\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = 0\\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0\\
\bar{F} &= F = 0, \bar{G} = G = 0.
\end{aligned}
$$

Hence the cannonical form of the given PDE is

$$-4u_{\xi\eta} = 0 \text{ or } u_{\xi\eta} = 0.$$

Integrating the cannonical equation partially with respect to ξ we get

$$u_\eta(\xi, \eta) = f_1(\eta),$$

where f_1 is some arbitrary function of η.

Integrating once again with respect to η we get

$$u_{\xi\eta} = f(\xi) + \int f_1(\eta)d\eta = f(\xi) + g(\eta),$$

where f and g both are arbitrary functions of ξ and η. Now the solution of the PDE in terms of the independent variables x and y is given by

$$u(x, y) = f(y + x - \cos x) + g(y - x - \cos x).$$

Lecture 8: 29.10.2020

Cannonical forms for Prabolic Equations

In this case the Discriminant $\bar{B}^2 - 4\bar{A}\bar{C} = 0$, which can be true if $\bar{B} = 0$ and $\bar{A} = 0$ or $\bar{C} = 0$. Now $\bar{A} = 0$ implies

$$\begin{aligned} \bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\ \implies & A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \end{aligned}$$

Which further implies

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = -\frac{B}{2A}.$$

Now to find the curve which satisfy the above equation we write the following characteristic equation

$$\frac{dy}{dx} = \frac{B}{2A}.$$

Solving the above equation we get $\xi(x, y) = c$.
Now we can show that $\bar{A} = 0$ implies $\bar{B} = 0$.
We know that

$$\begin{aligned} \bar{B} &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\ &= 2A\xi_x\eta_x + 2\sqrt{AC}\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\ &= 2(\sqrt{A}\xi_x + \sqrt{C}\xi_y)(\sqrt{A}\eta_x + \sqrt{C}\eta_y) \end{aligned}$$

Here $\frac{\xi_x}{\xi_y} = -\frac{B}{2A} = -\frac{2\sqrt{AC}}{2A} = -\frac{\sqrt{C}}{\sqrt{A}} \implies \sqrt{A}\xi_x + \sqrt{C}\xi_y = 0$.
Therefore we get $\bar{B} = 0$. So for the above choice of $\xi(x, y)$, we have $\bar{A} = \bar{B} = 0$. Which further implies $\bar{B}^2 - 4\bar{A}\bar{C} = 0$.
Hence, in this case we can choose the function η as per our wish only with the restriction

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0.$$

Example

Find a cannonical form of the PDE

$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} = e^x.$$

Solution

Here $A = x^2,\ B = -2xy,\ C = y^2,\ D = E = F = G = 0.$
Discriminant

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0,$$

so the PDE is of parabolic type for all x and y.
The characteristic equation is

$$\frac{dy}{dx} = \frac{B}{2A} = -\frac{y}{x}.$$

Solving we get

$$xy = c,$$

and we choose

$$\xi = xy.$$

Now the other coordinate can be chosen as per our wish and here we choose $\eta = y$. For such choice of ξ and η we have

$$\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\
\bar{B} &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y = 0 \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = y^2 \\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = -2xy \\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0 \\
\bar{F} &= F = 0, \bar{G} = G = e^x.
\end{aligned}$$

Hence the transformed PDE is given by

$$y^2 u_{\eta\eta} - 2xyu_\xi = e^x.$$

In terms of ξ and η it can be written as

$$\eta^2 u_{\eta\eta} = 2\xi u_\xi + e^{\frac{\xi}{\eta}} \implies u_{\eta\eta} = \frac{2\xi}{\eta^2} u_\xi + \frac{1}{\eta^2} e^{\frac{\xi}{\eta}},$$

which is the required cannonical form.

Canonical form for Elliptic PDE ($B^2 - 4AC < 0$)

The characteristic equations are taken to be

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} \text{ and } \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A},$$

solving which we get the complex conjugate co-ordinates α and β. Then to get the cannonical form we need to take the transformation

$$\xi = \frac{1}{2}(\alpha + \beta), \ \eta = \frac{i}{2}(\beta - \alpha).$$

Using the above transformation we'll get the **canonical form of the elliptic equations** as

$$u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta).$$

Example

Reduce the PDE $u_{xx} + x^2 u_{yy} = 0$ to a canonical form.

Solution

Compairing the given PDE with the standard form we get We may write $A = 1,\ B = 0,\ C = x^2,\ D = E = F = G = 0$ and hence $B^2 - 4AC = -4x^2 < 0$. Thus, the given PDE is **elliptic** in nature. The characteristic equations are given by

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -ix \text{ and } \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = ix.$$

Solving the characteristic equations we get $y + i\frac{x^2}{2} = C_1$ and $y - i\frac{x^2}{2} = C_2$. The complex characteristic curves are given by

$$\alpha = y + i\frac{x^2}{2},\ \ \beta = y - i\frac{x^2}{2}.$$

Now to get a canonical form, we use the transformation given by

$$\xi = \frac{1}{2}(\alpha + \beta), \ \eta = \frac{i}{2}(\beta - \alpha),$$

which imply

$$\xi = y, \ \eta = \frac{x^2}{2}.$$

For the above co-ordinates we get

$$\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = x^2, \\
\bar{B} &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y = 0, \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = x^2, \\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y = 1, \\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y = 0, \\
\bar{F} &= F = 0, \bar{G} = G = 0.
\end{aligned}$$

Therefore, the required cannonical form is given by

$$x^2 u_{\xi\xi} + x^2 u_{\eta\eta} + u_\xi = 0 \implies u_{\xi\xi} + u_{\eta\eta} = -\frac{1}{2\eta} u_\xi.$$

Exercise

Reduce the following PDEs to cannonical forms

1. $u_{xx} + xu_{yy} = 0, \quad x \neq 0.$
2. $u_{xx} + 2u_{xy} + \sin^2(x)u_{yy} + u_y = 0.$
3. $\sin^2(x)u_{xx} + \sin 22u_{xy} + \cos^2(x)u_{yy} + u_y = x.$
4. $u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0.$
5. $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x.$
6. $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y.$
7. $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0.$
8. $u_{xx} - 2\sin x\ u_{xy} - \cos^2 x\ u_{yy} - \cos x\ u_y = 0.$
9. $u_{xx} + 2xu_{yy} = 0.$
10. $y^2u_{xx} - x^2u_{yy} = 0,\ x > 0,\ y > 0.$
11. $u_{xx} + 2u_{xy} + u_{yy} = 0.$
12. $e^xu_{xx} + e^yu_{yy} = u.$