ENGINEERING PHYSICS

PHC01

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Topics to be Covered

Harmonic Oscillations - Linear superposition principle, Superposition of two perpendicular oscillations having same and different frequencies and phases, Free, Damped and forced vibrations, Equation of motion, Amplitude resonance, Velocity resonance, Quality factor, sharpness of resonance, etc.

Wave Motion - Wave equation, Longitudinal waves, Transverse waves, Electro-magnetic waves.

Introductory Quantum Mechanics/Wave mechanics - Inadequacy of classical mechanics, Blackbody radiation, Planck's quantum hypothesis, de Broglie's hypothesis, Heisenberg's uncertainty principle and applications, Schrodinger's wave equation and applications to simple problems: Particle in a one-dimensional box, Simple harmonic oscillator, Tunnelling effect.

Interference & Diffraction

Polarisation

Laser and Optical Fiber

Course Outcomes

- **CO1:** To realize and apply the fundamental concepts of physics such as superposition principle, simple harmonic motion to real world problems.
- **CO2:** Learn about the quantum phenomenon of subatomic particles and its applications to the practical field.
- **CO3:** Gain an integrative overview and applications of fundamental optical phenomena such as interference, diffraction and polarization.
- **CO4:** Acquire basic knowledge related to the working mechanism of lasers and signal propagation through optical fibers.

Oscillatory Motion/Vibration

- Introduction
- Periodic motion
- Spring-mass system
- Differential equation of motion
- Simple Harmonic Motion (SHM)
- Energy of SHM
- Pendulum
- Torsional Pendulum

Text Book

1. The Physics of Vibrations and Waves, H. John Pain, Willy and Sons
2. Vibrations and Waves in Physics, Iain G. Main, Cambridge University Press

Introduction

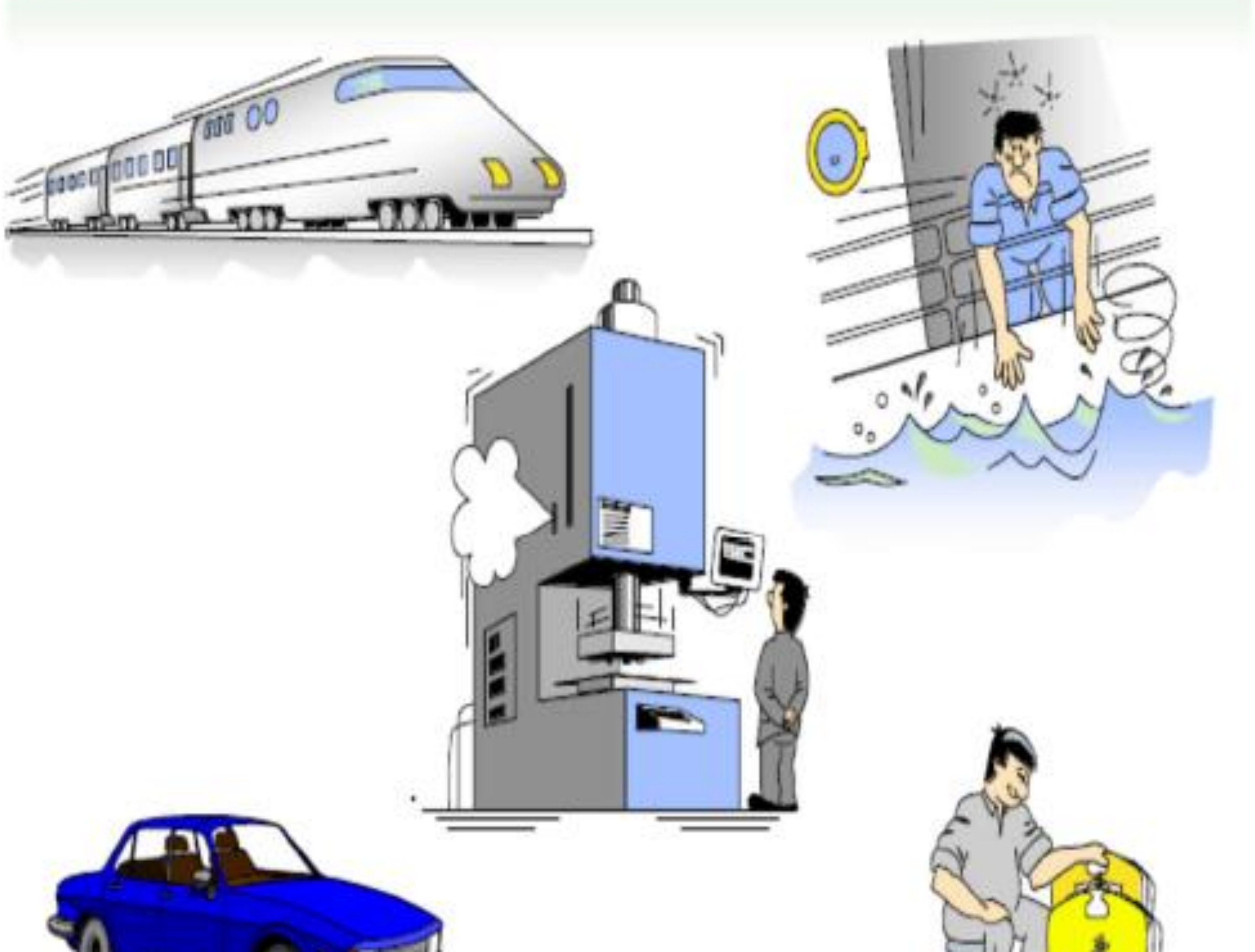
- Any motion that repeats itself after an interval of time is called *vibration or oscillation.*

- The general terminology of “Vibration” is used to describe oscillatory motion of mechanical and structural systems

- The Vibration of a system involves the transfer of its potential energy to kinetic energy and kinetic energy to potential energy, alternately

Vibration

It is also an everyday phenomenon we meet on everyday life



Vibration

Compressor

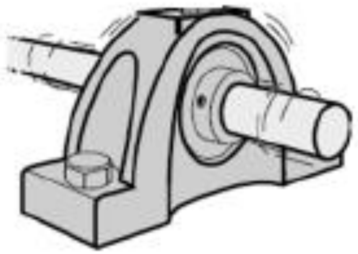
Testing

Ultrasonic cleaning

Harmful effect of vibration

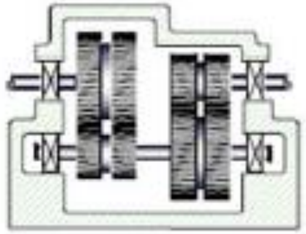
Noise

Destruction



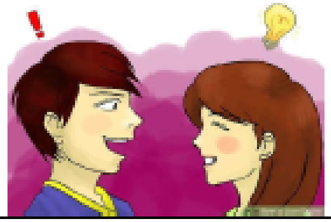
Wear

Fatigue



Vibration in our Lives

- Our heart beats
- Our lungs oscillate
- We hear because our ear drums vibrate
- Vibration makes us snore
- Light waves permit us to see
- Sound waves allow us to hear
- We move because of oscillation of legs
- We can not utter ‘vibration’ without the oscillation of larynges and vocal cords



- Vibrations are an engineering concern in these applications because they may cause a catastrophic failure (complete collapse) of the machine or structure because of excessive stresses and amplitudes (resulting mainly from resonance) or because of material fatigue over a period of time

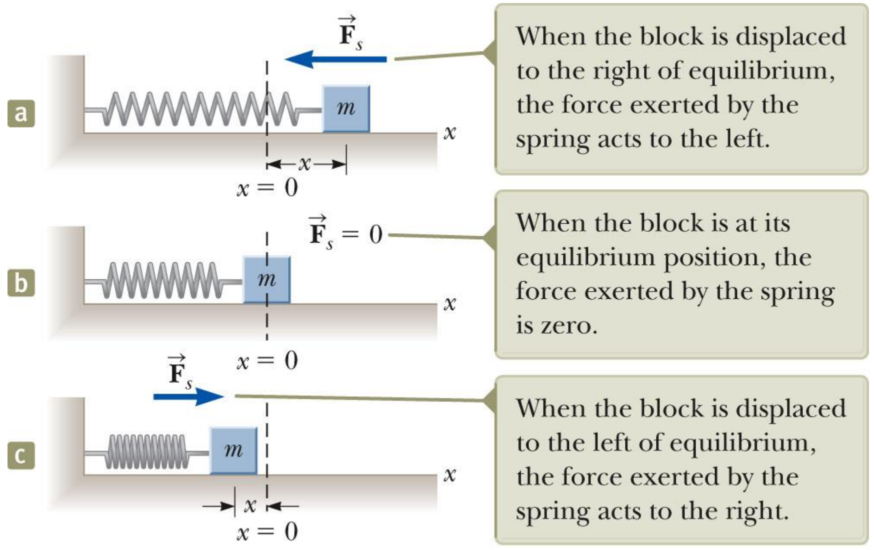
Example: - Failure of Tacoma Narrows Bridge in 1940 due to 42-mile-per-hour wind undergoing a torsional mode resonance
- Vibration of machine components generate annoying noise
- Vibration of string generate pleasing music (already discussed before)

Some historical background

- Historically studies on vibration (acoustics) started long ago (around 4000BC)
- Musicians and philosophers have sought out the rules and laws of sound production, used them in improving musical instruments, and passed them on from generation to generation
- Music had become highly developed and was much appreciated by Chinese, Hindus, Japanese, and, perhaps, the Egyptians.
- These early peoples observed certain definite rules in connection with the art of music, although their knowledge did not reach the level of a science.
- Early applications (by Egyptian) to single or multiple string instruments known as Harps
- Our present system of music is based on ancient Greek civilization.
- The Greek philosopher and mathematician Pythagoras (582-507 B.C.) is considered to be the first person to investigate musical sounds on a scientific basis [later on we will be talking about Mathematical Basis as well]

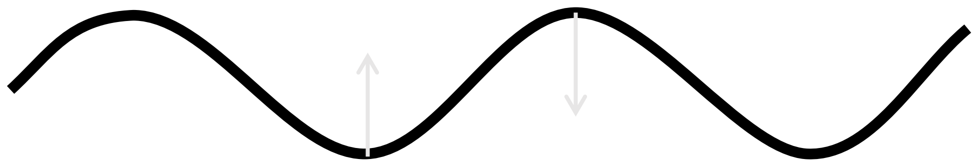
Periodic Motion

- Periodic motion is a motion that regularly returns to a given position after a fixed time interval.
- A particular type of periodic motion is “simple harmonic motion,” which arises when the force acting on an object is proportional to the position of the object about some equilibrium position.
- The motion of an object connected to a spring is a good example.

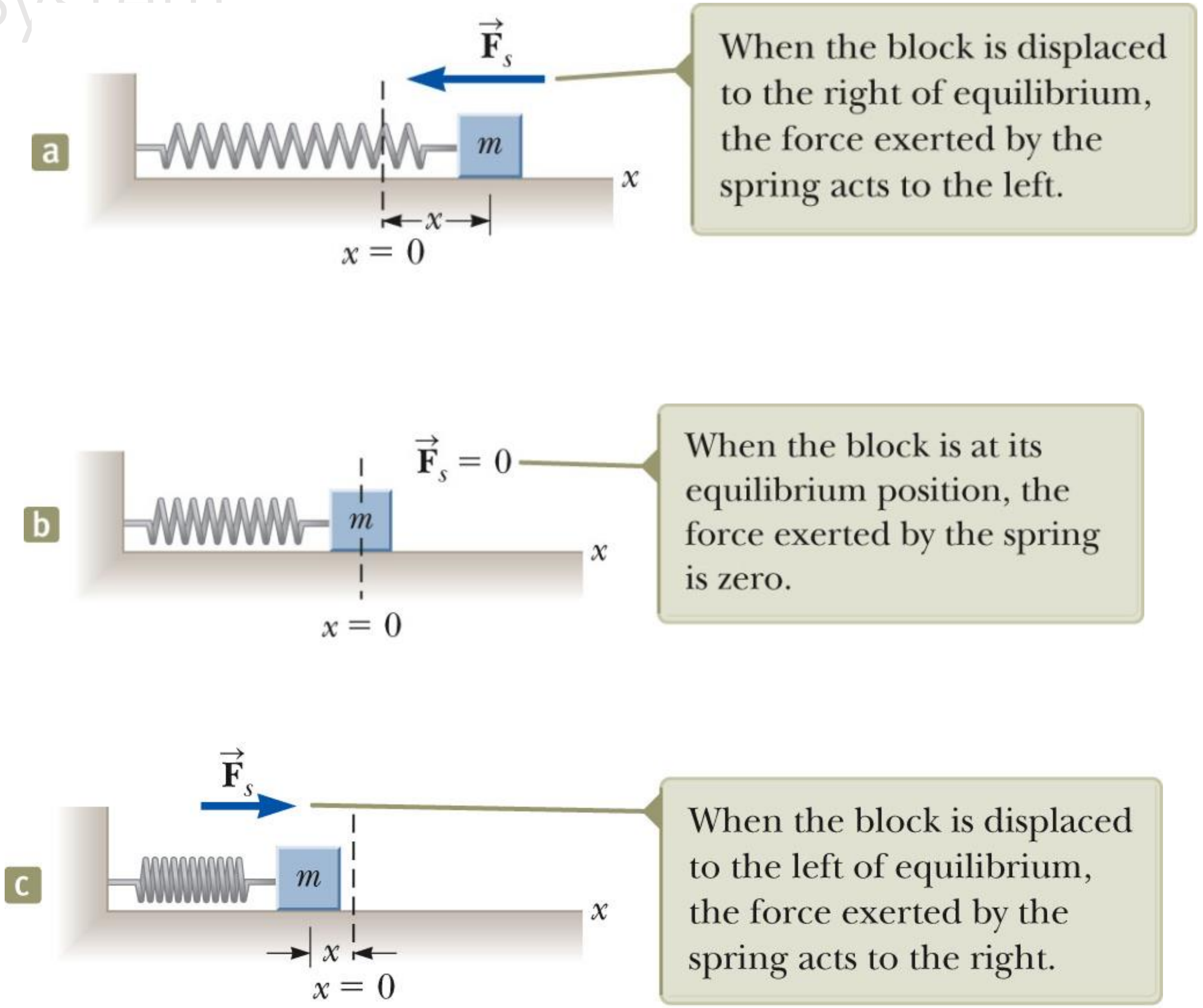


Recall Hooke's Law

- Hooke's Law states $F_s = -kx$
  - F_s is the restoring force.
    - It is always directed toward the equilibrium position.
    - Therefore, it is always opposite the displacement from equilibrium.
  - k is the force (spring) constant.
  - x is the displacement.
- What is the restoring force for a surface water wave?



Restoring Force and the Spring Mass System



Differential Equation of Motion

- Using $F = ma$ for the spring, we have $ma = -kx$
- But recall that acceleration is the second derivative of the position:

$$a = \frac{d^2x}{dt^2}$$

- So this simple force equation is an example of a *differential equation*,

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

- An object moves in simple harmonic motion whenever its acceleration is proportional to its position and has the opposite sign to the displacement from equilibrium.

Acceleration

- Note that the acceleration is NOT constant, unlike our earlier kinematic equations.
- If the block is released from some position $x = A$, then the initial acceleration is $-kA/m$, but as it passes through 0 the acceleration falls to zero.
- It only continues past its equilibrium point because it now has momentum (and kinetic energy) that carries it on past $x = 0$.
- The block continues to $x = -A$, where its acceleration then becomes $+kA/m$.

Analysis Model, Simple Harmonic Motion

- What are the units of k/m, in

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

- They are $1/\mathrm{s}^2$, which we can regard as a frequency-squared, so let's write it as

$$\omega^2 = \frac{k}{m}$$

- Then the equation becomes

$$a = -\omega^2 x$$

- A typical way to solve such a differential equation is to simply search for a function that satisfies the requirement, in this case, that its second derivative yields the negative of itself! The sine and cosine functions meet these requirements.

Let we assumed our trial solution is $x = e^{mt}$

The equation of motion for SHM

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Then

$$\frac{dx}{dt} = me^{mt} \quad \text{and} \quad \frac{d^2x}{dt^2} = m^2e^{mt}$$

Substitute these to equation of motion, we will get $m^2 + \omega^2 = 0$ $\Longrightarrow$ $m = \pm\sqrt{-1}\,\omega = \pm i\omega$

So solutions are $x_1 = e^{i\omega t}$ or $x_2 = e^{-i\omega t}$

Superposition principle : If x_1 *and* x_2 *are two independent solution of a linera differential equation, then their liner combination also will be the solution of the differential equation.*

So the general solution will be the linear combination of x_1 *and* x_2

$$x = C_1x_1 + c_2x_2$$

Since $x_1\ and\ x_2$ are the solution of the differential equation, then

$$\frac{d^2x_1}{dt^2} + \omega^2 x_1 = 0 \qquad (1)$$

$$\frac{d^2x_2}{dt^2} + \omega^2 x_2 = 0 \qquad (2)$$

Now if $$x = C_1 x_1 + c_2 x_2$$

$$\frac{d^2x}{dt^2} + \omega^2 x = C_1\left(\frac{d^2x_1}{dt^2} + \omega^2 x_1\right) + C_2\left(\frac{d^2x_2}{dt^2} + \omega^2 x_2\right) = 0$$

$$x = C_1 e^{i\omega t} + C_2 e^{-i\omega t} = C_1(\cos\omega t + i\; sin\; \omega t) + C_2(\cos\omega t - i\sin\omega t)$$
$$= (C_1 + C_2)\;\text{cos}\omega t - i(c_2 - C1) sin\varphi$$

Let $A\cos\emptyset = (C_1 + C_2)\; and$ $A\; sin\; \emptyset = i(c_2 - C_1)$

So our final solution would be

$$x = A\cos\omega t\cos\emptyset - A\sin\omega t\sin\emptyset = A\cos(\omega t + \emptyset)$$

SHM Graphical Representation

- A solution to the differential equation is

$$x(t) = A\cos(\omega t + \phi)$$

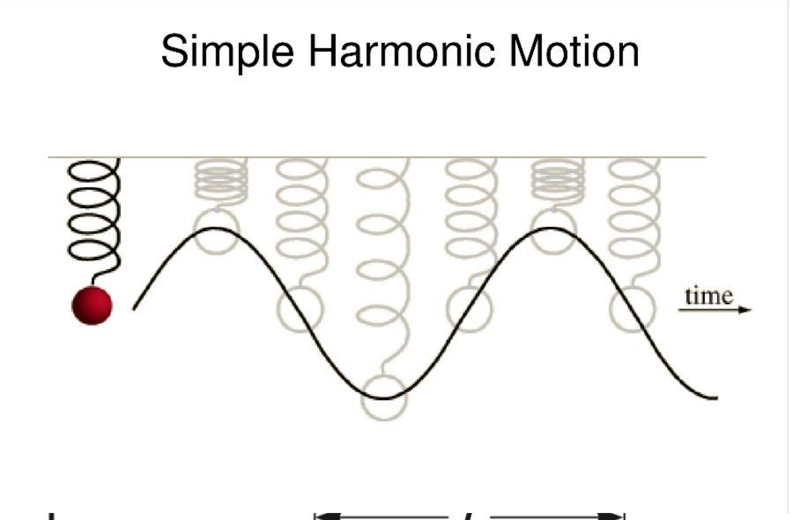
- A, ω, ϕ are all constants:

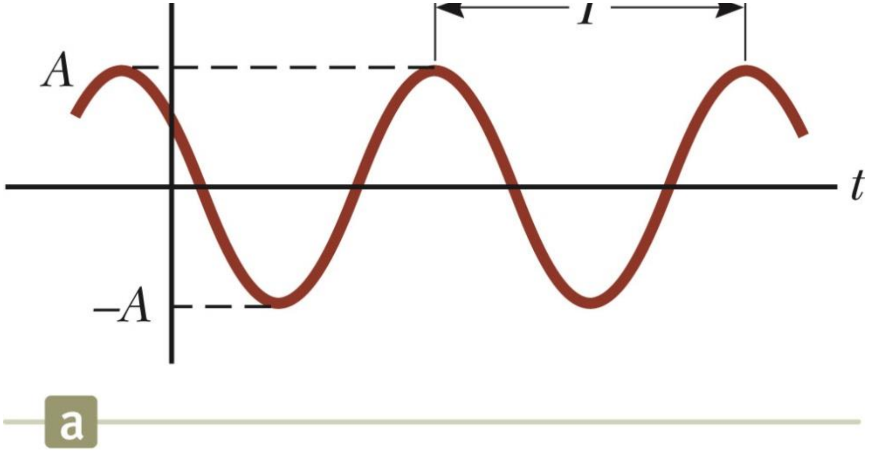
A = amplitude (maximum position in either positive or negative x direction,

ω = angular frequency, $\sqrt{\frac{k}{m}}$

ϕ = phase constant, or initial phase angle.

A and ϕ are determined by initial conditions.





Remember, the period and frequency are:

$$T = \frac{2\pi}{\omega} \quad \left(f = \frac{1}{T} = \frac{\omega}{2\pi} \right)$$

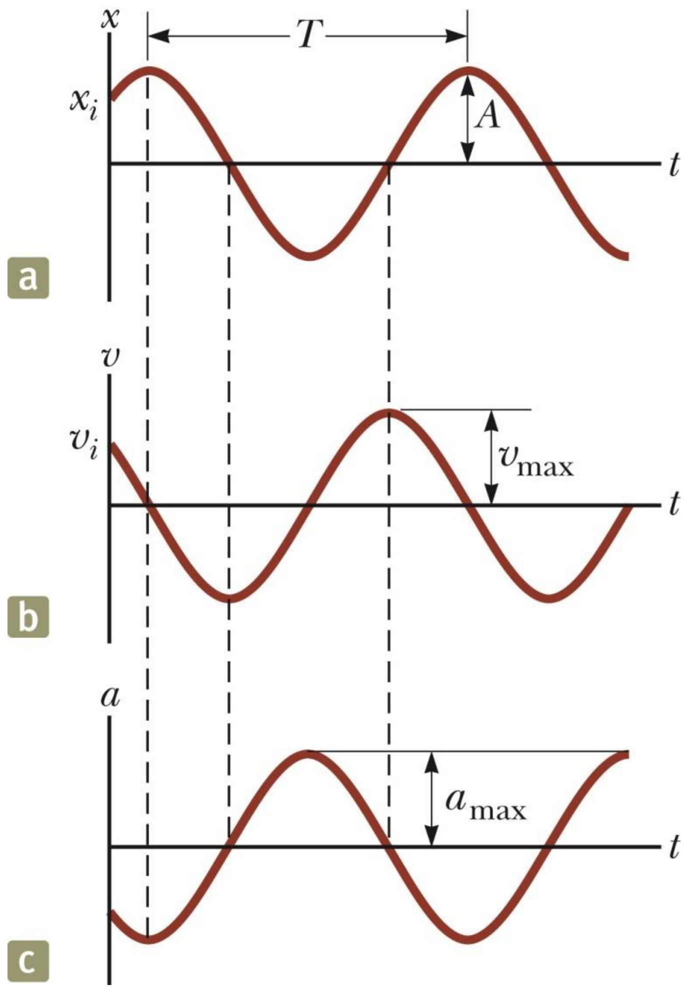
Motion Equations for SHM

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

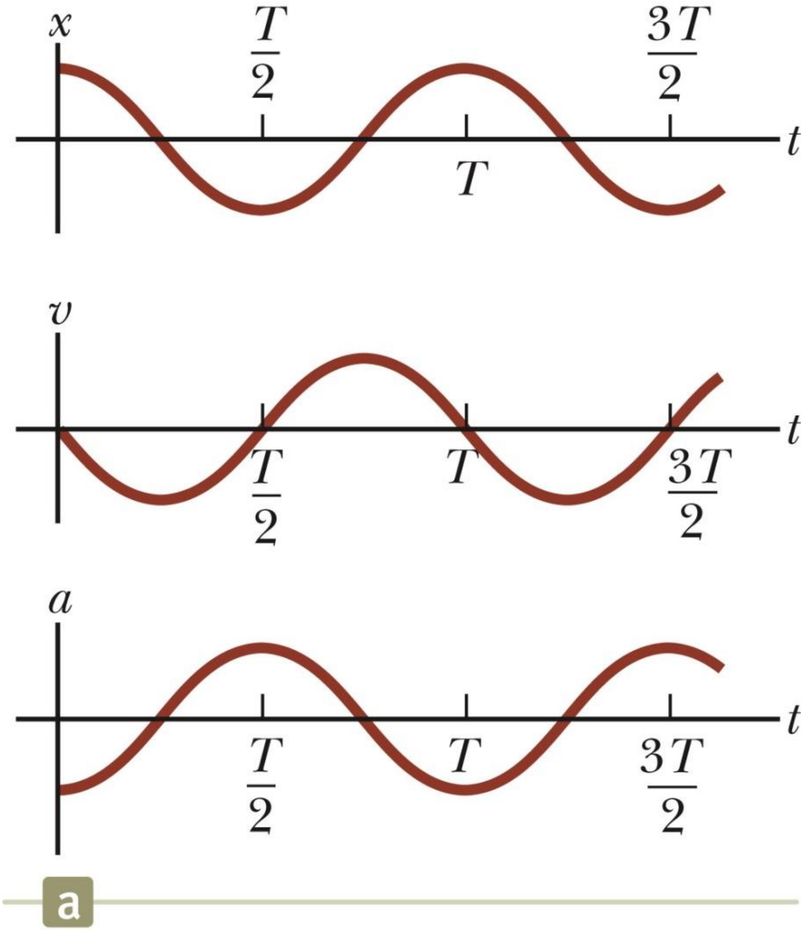
$$a(t) = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$

The velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement.



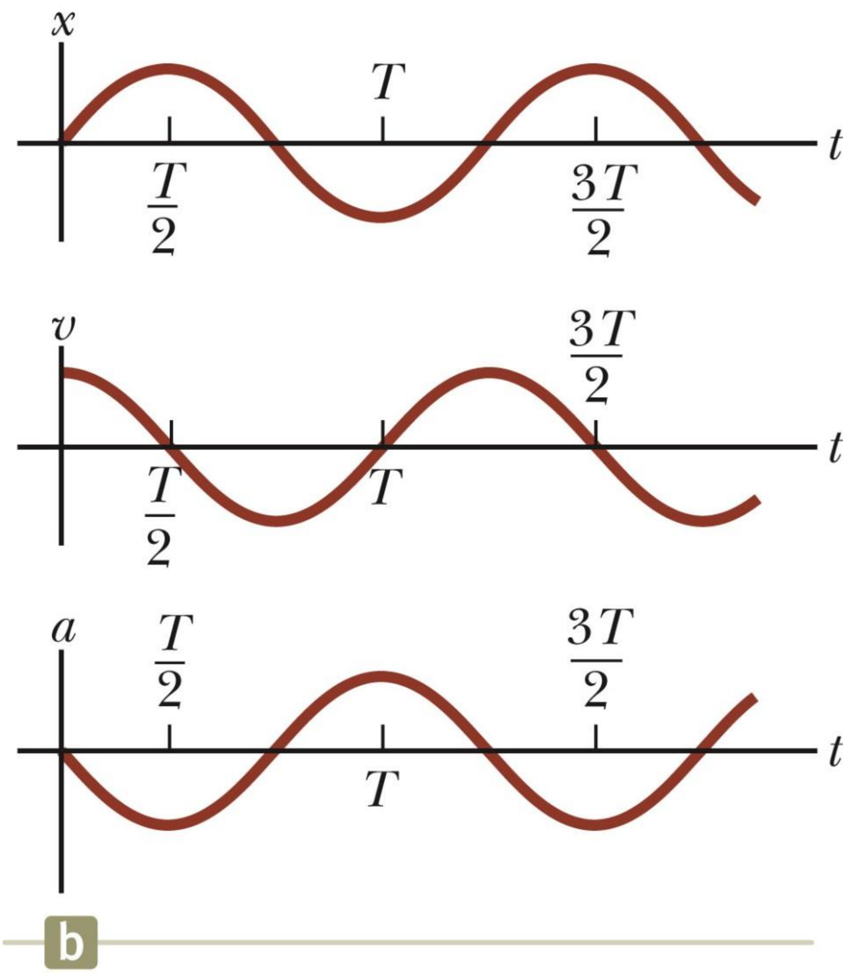
SHM Example 1

- Initial conditions at t = 0 are
  - x (0)= A
  - v (0) = 0
- This means ϕ = 0
- The acceleration reaches extremes of $\pm\,\omega^2 A$ at $\pm\,A$.
- The velocity reaches extremes of $\pm\,\omega A$ at x = 0.



SHM Example 2

- Initial conditions at $t = 0$ are
  - $x(0) = 0$
  - $v(0) = v_i$
- This means $\phi = -\pi/2$
- The graph is shifted one-quarter cycle to the right compared to the graph of $x(0) = A$.



Consider the Energy of SHM Oscillator

- The spring force is a conservative force, so in a frictionless system the energy is constant
- Kinetic energy, as usual, is

$$K = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi)$$

- The spring potential energy, as usual, is

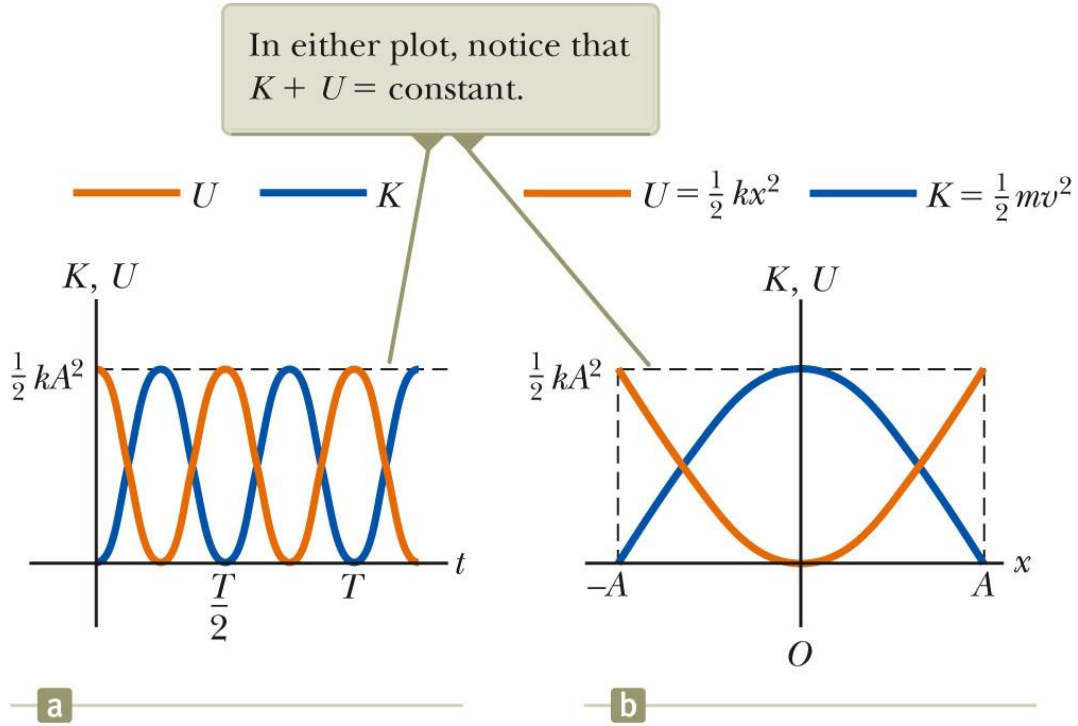
$$U = \tfrac{1}{2} k x^2 = \tfrac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

- Then the total energy is just

$$E = K + U = \tfrac{1}{2} k A^2 \quad \text{(a constant)}$$

Transfer of Energy of SHM

- The total energy is constant at all times, and is $E = \frac{1}{2}kA^2$ (proportional to the square of the amplitude)
- Energy is continuously being transferred between potential energy stored in the spring, and the kinetic energy of the block.



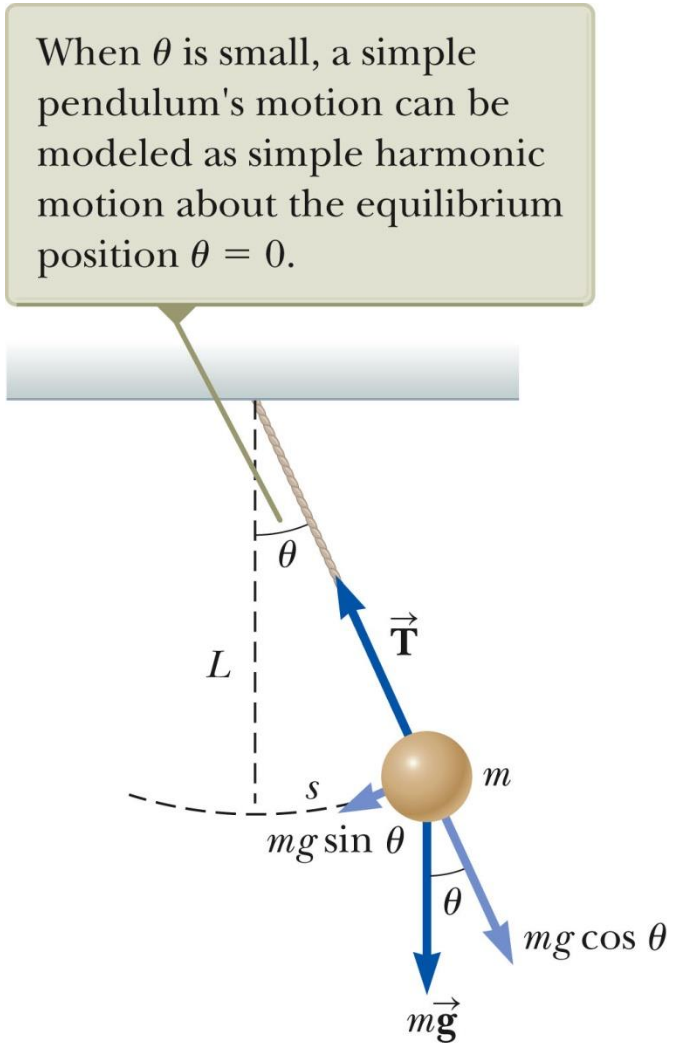
Simple Pendulum

- The forces acting on the bob are the tension and the weight.
- **T** is the force exerted by the string
- m**g** is the gravitational force
- The tangential component of the gravitational force is the restoring force.
- Recall that the tangential acceleration is

$$a_t = r\alpha = L\alpha = L\frac{d^2\theta}{dt^2}$$

- This gives another differential equation

$$m\frac{d^2\theta}{dt^2} = -m\frac{g}{L}\sin\theta \approx -m\frac{g}{L}\theta \quad \text{(for small } \theta\text{)}$$



Frequency of Simple Pendulum

- The equation for θ is the same form as for the spring, with solution

$$\theta(t) = \theta_{\max} \cos(\omega t + \phi)$$

where now the angular frequency is

$$\omega = \sqrt{\frac{g}{L}} \quad \left(\text{so the period is T} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \right)$$

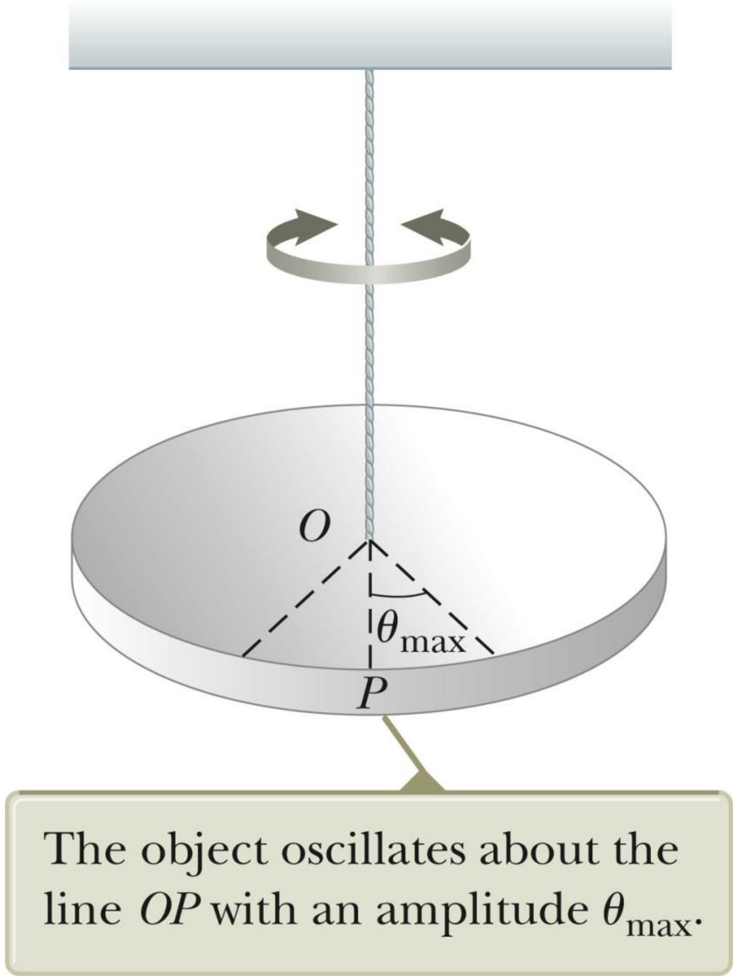
Summary: the period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity. The period is independent of mass.

Torsional Pendulum

- Assume a rigid object is suspended from a wire attached at its top to a fixed support.
- The twisted wire exerts a restoring torque on the object that is proportional to its angular position.
- The restoring torque is $\tau = -\kappa\,\theta$.
    - κ is the *torsion constant* of the support wire.
- Newton's Second Law gives

$$\tau = I\alpha \rightarrow -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$



Importance of the Superposition Principle

Let us consider the example of motion of a simple pendulum. For small oscillation the equation of motion is $\frac{d^2\psi}{dt^2} = -\omega^2\psi$.
Under a given set of initial condition(at t=0), the solution be ψ_1, given by
ψ_1,=$A_1\cos(\omega\ t + \phi_1)$
where A_1 and $\phi_1$are amplitude and phase constant respectively. $\omega = \sqrt{\frac{k}{m}}$
angular frequency does not depend on initial conditions.
Under another set of initial conditions, the displacement ψ_2 is given by
ψ_2,=$A_2\cos(\omega\ t + \phi_2)$
A_2 and $\phi_2$are amplitude and phase constant respectively.

If we have third set of initial condition as follows: If we superpose the initial conditions corresponding to ψ_1 and ψ_2.
Then, from the superposition principle, the new motion described by ψ_3 is simply superposition of ψ_1 and ψ_2
$\psi_3 = \psi_1 + \psi_2$

Superposition of

Oscillations Having Equal Frequencies

1. Suppose we have two SHMs of equal frequencies but of different amplitudes and phase constants acting on a particle(or a system) along x axis.
2. The displacements x_1 and x_2 of the two harmonic motions, of the same angular frequency ω, are given by

$$x_1 = A_1 \cos(\omega t + \phi_1) \tag{1}$$
$$x_2 = A_2 \cos(\omega t + \phi_2) \tag{2}$$

   where A_1 and A_2 are the amplitudes and ϕ_1 and ϕ_2 are the phase constants of two SHMs.
3. The resultant displacement x of the two harmonic oscillations using superpostion principle is the sum of individual displacement x_1 and x_2 and is written as:

$$x = x_1 + x_2 \tag{3}$$
$$x = A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2) \tag{4}$$

Using

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

we can have

$$x = (A_1\cos\phi_1 + A_2\cos\phi_2)\cos\omega t - (A_1\sin\phi_1 + A_2\sin\phi_2)\sin\omega t \quad (5)$$

Now let suppose:

$$A_1\cos\phi_1 + A_2\cos\phi_2 = A\cos\delta \quad (6)$$

$$A_1\sin\phi_1 + A_2\sin\phi_2 = A\sin\delta \quad (7)$$

Using 6 and 7 in 5 we obtain resultant displacement as:

$$x = A\cos\delta\cos\omega t - \sin\delta\sin\omega t \tag{8}$$

$$x = A(\cos\omega t + \delta) \tag{9}$$

where A and δ are the constants to be determined.
Equation 9 shows that the resulting motion is simple harmonic with an angular frequency ω the same as that of individual SHMs. The resulting motion has an Apmlitude A and a phase constant δ.
We can obtain A by squaring and adding Eqs. 6 and 7.

$$A^2(\cos^2\delta + \sin^2\delta) = (A_1\cos\phi_1 + A_2\cos\phi_2)^2 + (A_1\sin\phi_1 + A_2\sin\phi_2)^2 \tag{10}$$

$$\begin{aligned} A^2 &= A_1^2\cos^2\phi_1 + A_2^2\cos^2\phi_2 + 2A_1A_2\cos\phi_1\cos\phi_2 \\ &+ A_1^2\sin^2\phi_1 + A_2^2\sin^2\phi_2 + 2A_1A_2\sin\phi_1\sin\phi_2 \end{aligned} \tag{11}$$

Using trignometric identity $\cos^2\delta + \sin^2\delta = 1$ and $cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2 = \cos(\phi_2 - \phi_1)$. Collecting coefficients of A_1 and A_2 we obatin resultant amplitude as:

$$\begin{aligned} A^2 &= A_1^2 + A_2^2 + 2A_1A_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) && (12) \\ A^2 &= A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1) && (13) \end{aligned}$$

Phase constant δ of resultant motion can obtained by dividing Eqs. 6 and 7.

$$\frac{A\sin\delta}{A\cos\delta} = \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2} \qquad (14)$$

$$\tan\delta = \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2} \qquad (15)$$

Thus we conclude that the resultant effect of two collinear SHMs of equal freqencies is a simple harmonic motion of the same frequency but with modified amplitude and phase constant obtained in Eq. 12 and 58 respectively.

Oscillations Having Different Frequencies

1. Let us consider two harmonic oscillations of different amplitudes A_1 and A_2 and different frequencies ω_1 and ω_2.
2. For Simplicity, we assume that two oscillations have the same phase constant which we take to be Zero in this case.
3. The two SHMs can be written as:

$$x_1 = A_1 \cos \omega_1 t \tag{16}$$
$$x_2 = A_2 \cos \omega_2 t \tag{17}$$

From superposition principle, the resulting oscillation is given by

$$x = x_1 + x_2 \tag{18}$$

$$x = x_1 + x_2 \tag{19}$$
$$= A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \tag{20}$$

Let us define average frequency ω_a and modulation ω_m as:

$$\omega_a = \frac{\omega_1 + \omega_2}{2} \tag{21}$$

$$\omega_m = \frac{\omega_2 - \omega_1}{2} \tag{22}$$

where $\omega_2 > \omega_1$
Adding and subtracting Eqs. 21 and 22 we obtain

$$\omega_1 = \omega_a - \omega_m \tag{23}$$

$$\omega_2 = \omega_a + \omega_m \tag{24}$$

Substituting ω_1 and ω_2 in Eq. 19

$$x = A_1 \cos(\omega_a - \omega_m)t + A_2 \cos(\omega_a + \omega_m)t \tag{25}$$

Using trignometric identity $\cos(\omega_a - \omega_m)$ and $\cos(\omega_a + \omega_m)$

$$\begin{aligned} x &= A_1(\cos\omega_a t \cos\omega_m t + \sin\omega_a t \sin\omega_m t) \\ &+ A_2(\cos\omega_a t \cos\omega_m t - \sin\omega_a t \sin\omega_m t) \end{aligned} \tag{26}$$

Collecting coefficients of A_1 and A_2 we obtain:

$$x = (A_1 + A_2)\cos\omega_m t \cos\omega_a t + (A_1 - A_2)\sin\omega_m t \sin\omega_a t \tag{27}$$

Again, let us suppose

$$(A_1 + A_2)\cos\omega_m t = A_m \cos\delta_m \tag{28}$$

$$(A_1 - A_2)\sin\omega_m t = A_m \sin\delta_m \tag{29}$$

Finally, we have

$$x = A_m \cos\delta_m \cos\omega_a t + A_m \sin\delta_m \sin\omega_a t \tag{30}$$

A_m and δ_m are the resultant amplitude and phase constant.

$$x = A_m \cos(\omega_a t - \delta_m) \tag{31}$$

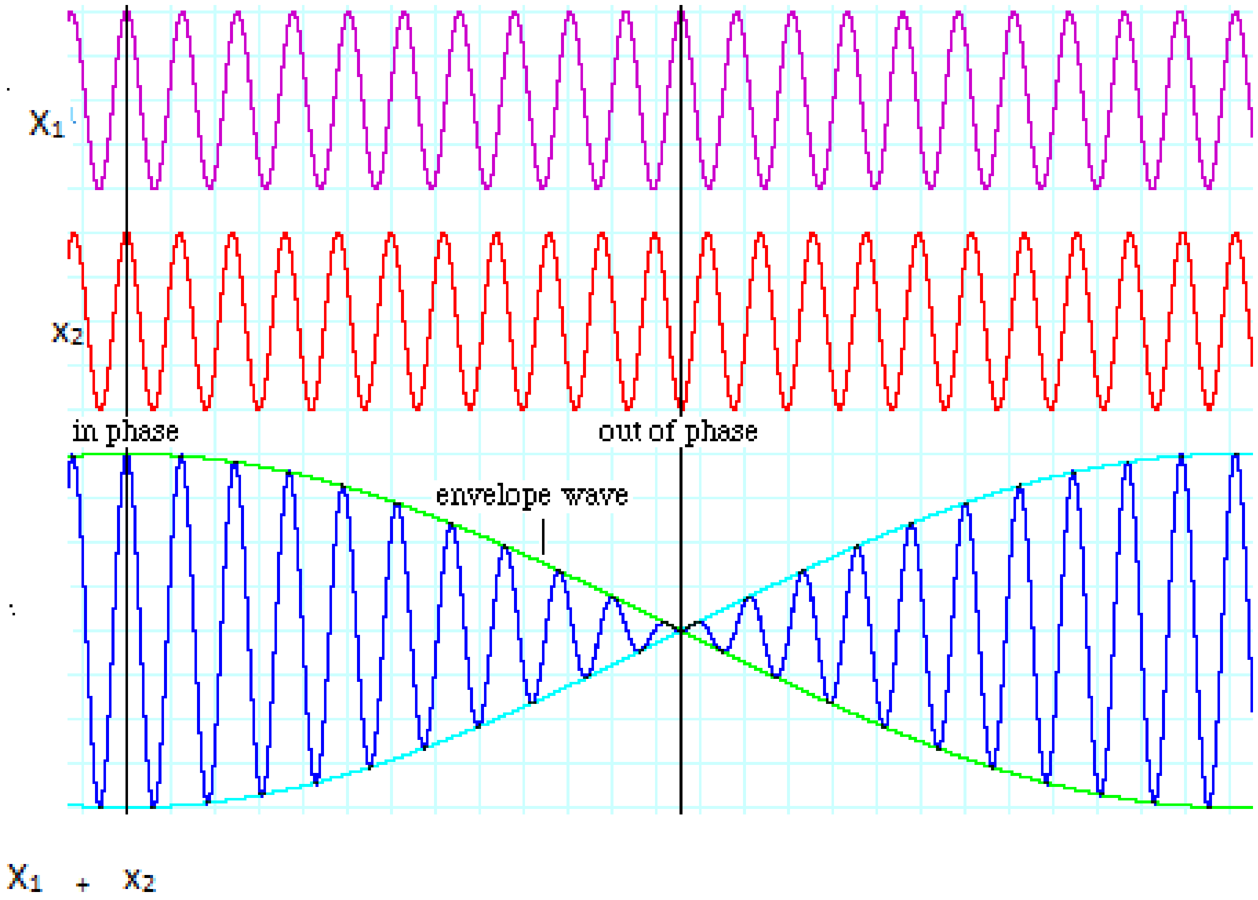
Now obtain A_m and δ_m using Eqs. 28 and 29 as done in the case of oscilations having same frequencies :

$$A_m^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(2\omega_m t) \tag{32}$$

$$\tan\delta_m = \frac{(A_1 - A_2)\sin\omega_m t}{(A_1 + A_2)\cos\omega_m t} \tag{33}$$

From the expressions obtained for A_m and δ_m is time dependent and hence the resultant motion is not having SHM behaviour.

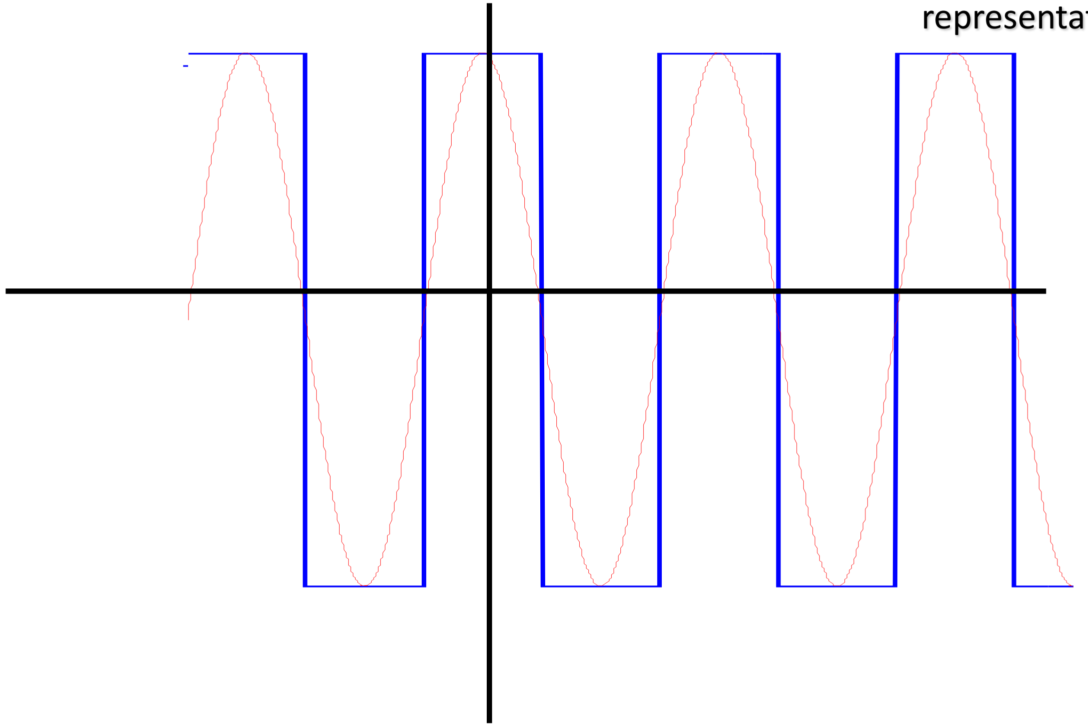
This oscillation can, at best, be described as periodic with an angular frequency of ω_a, average of two component frequencies.


X1
X2
in phase
out of phase
envelope wave
X1 + X2

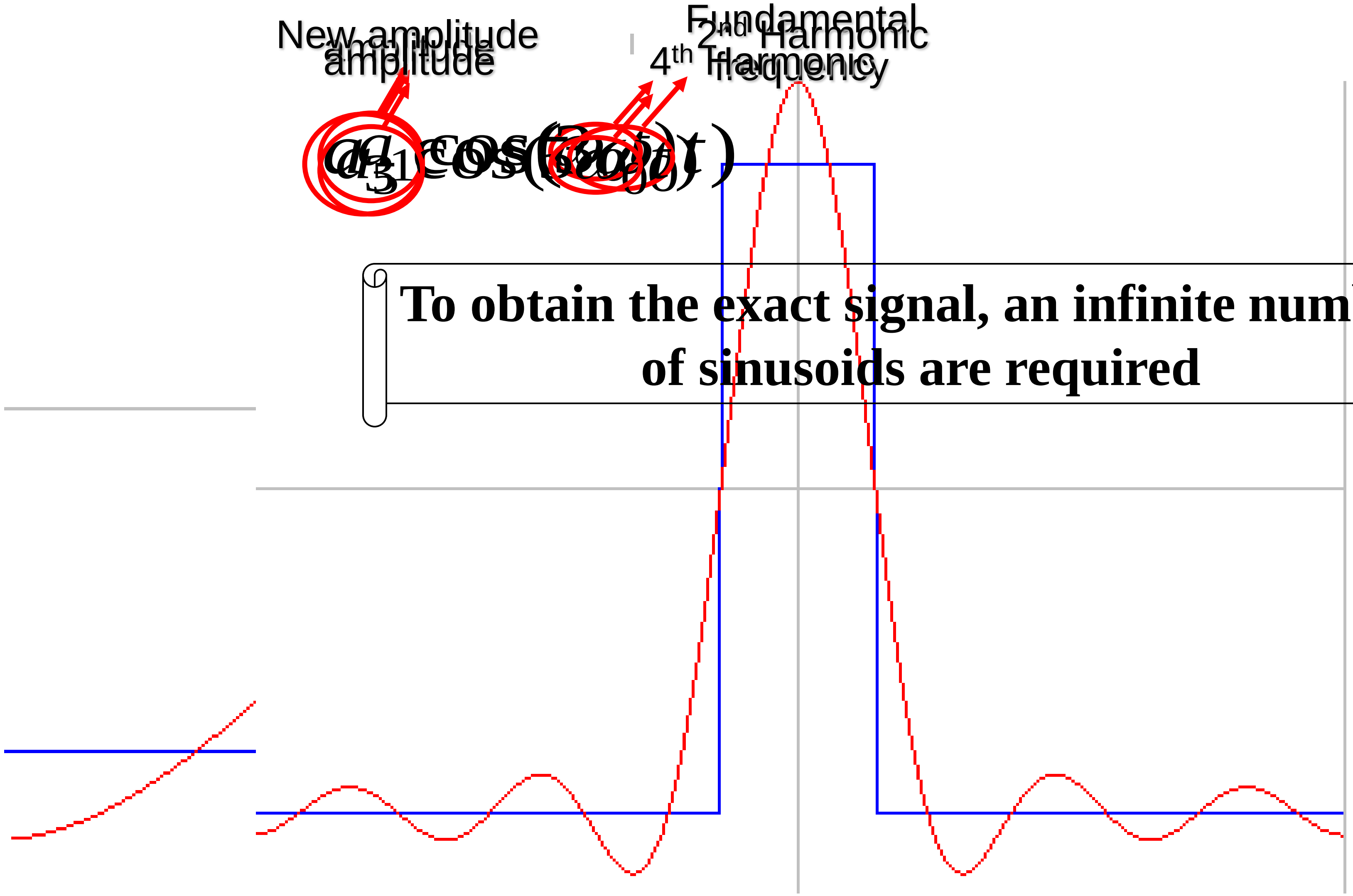
Visualization

Can you represent this simple function using sinusoids?

Single sinusoid representation

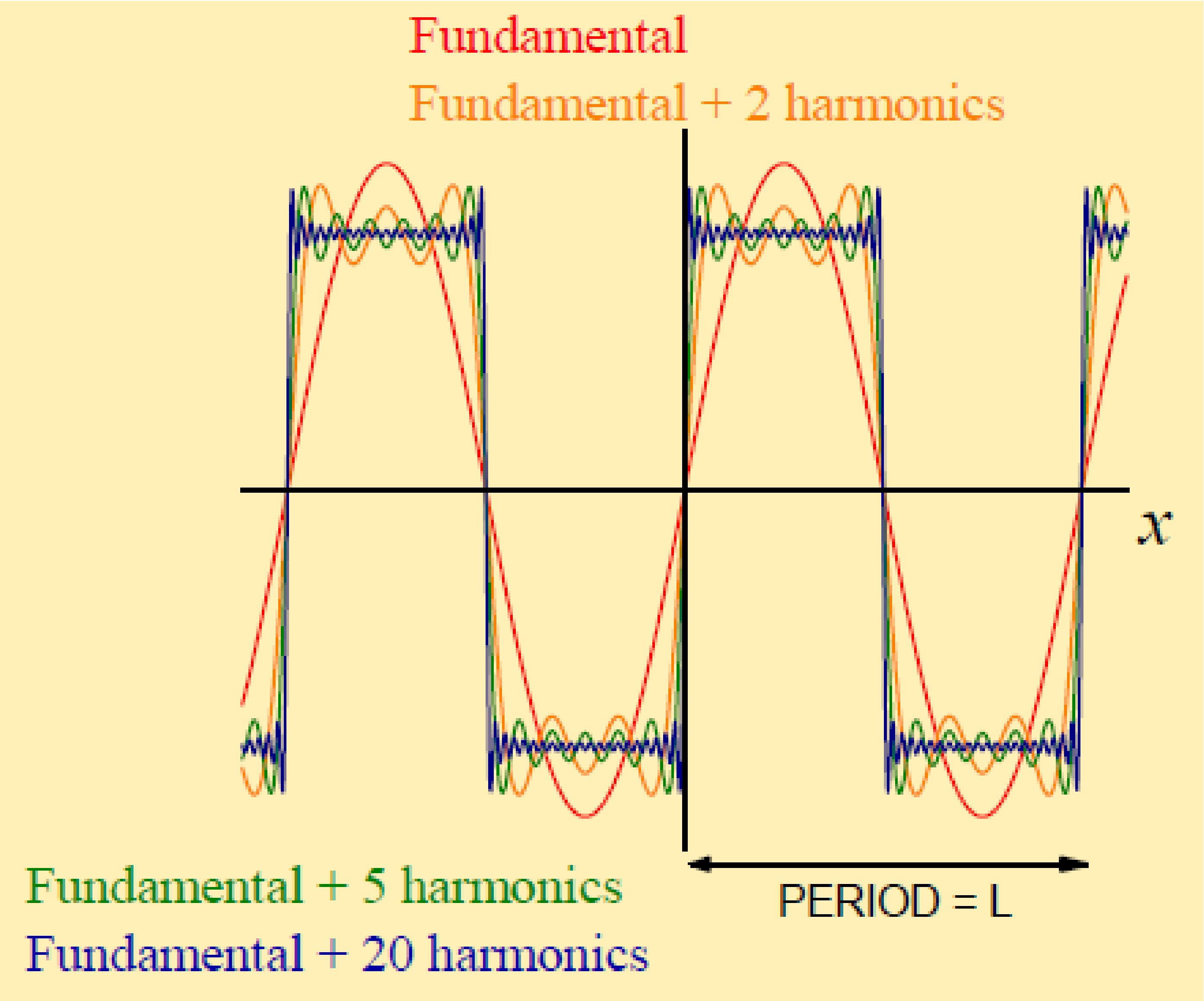


Visualization



To obtain the exact signal, an infinite number of sinusoids are required

Fourier Theorem



Superposition of two perpendicular Harmonic oscillations:

Analytical Method

i) Vibration having Equal frequencies

Suppose that a particle moves under the simultaneous influence of two simple harmonic oscillations of equal frequency, one along x-axis, the other along the perpendicular y axis.

what is its subsequent motion?

The displacement may be written as

$$x = A_1 \cos \omega t \quad \text{--- (1)}$$

$$y = A_2 \cos(\omega t + \delta) \quad \text{--- (2)}$$

A_1 and A_2 are the amplitudes of x & y oscillation

Assuming that the phase constant of the x oscillation is zero and that of y is δ, so that δ is the phase difference between them.

Using the trignometric identity in eq. (2) we obtain:

$$\frac{y}{A_2} = \cos \omega t \cos \delta - \sin \omega t \sin \delta \quad \text{--- (3)}$$

from (1) we can have

$$\frac{x}{A_1} = \cos\omega t \quad , \quad \sin\omega t = \left(1 - \frac{x^2}{A_1^2}\right)^{1/2}$$

Use $\cos\omega t$ and $\sin\omega t$ in (3)

$$\frac{y}{A_2} = \frac{x}{A_1}\cos\delta - \left(1 - \frac{x^2}{A_1^2}\right)^{1/2}\sin\delta$$

$$\left(1 - \frac{x^2}{A_1^2}\right)^{1/2}\sin\delta = \frac{x}{A_1}\cos\delta - \frac{y}{A_2} \quad - (4)$$

Squaring eq (4) on both sides

$$\left(\frac{x}{A_1}\cos\delta - \frac{y}{A_2}\right)^2 = \left(1 - \frac{x^2}{A_1^2}\right)\sin^2\delta$$

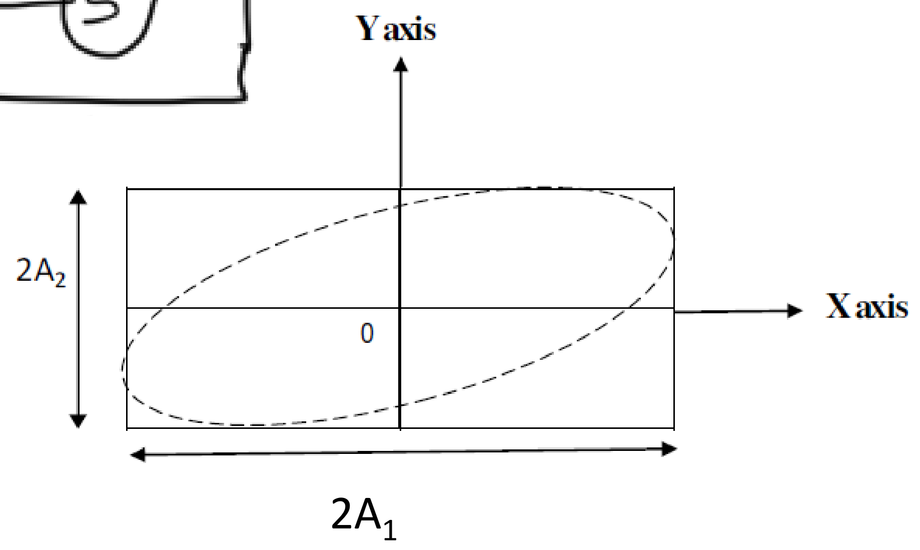
$$\frac{x^2}{A_1^2}\cos^2\delta + \frac{y^2}{A_2^2} - \frac{2xy}{A_1A_2}\cos\delta = \sin^2\delta - \frac{x^2}{A_1^2}\sin^2\delta$$

Rearranging above equation :

$$\boxed{\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1A_2}\cos\delta = \sin^2\delta \qquad (5)}$$

This is a general expression of ellipse confined inside a rectangle of sides 2/
The major axis of the ellipse makes an angle to with the x-axis is

$$\tan 2\varphi = \frac{2A_1A_2}{A_1^2 - A_2^2}\cos\delta$$



the path followed by the particle, which is subjected to the two rectangular SHMs of equal frequencies, is, in general, an ellipse.

Let us consider a few special cases:

i) $\delta = 0$, Eq. 5 reduces to

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$

or $\left(y - \frac{A_2}{A_1} x\right)^2 = 0$

$$\boxed{y = \frac{A_2}{A_1} x}$$

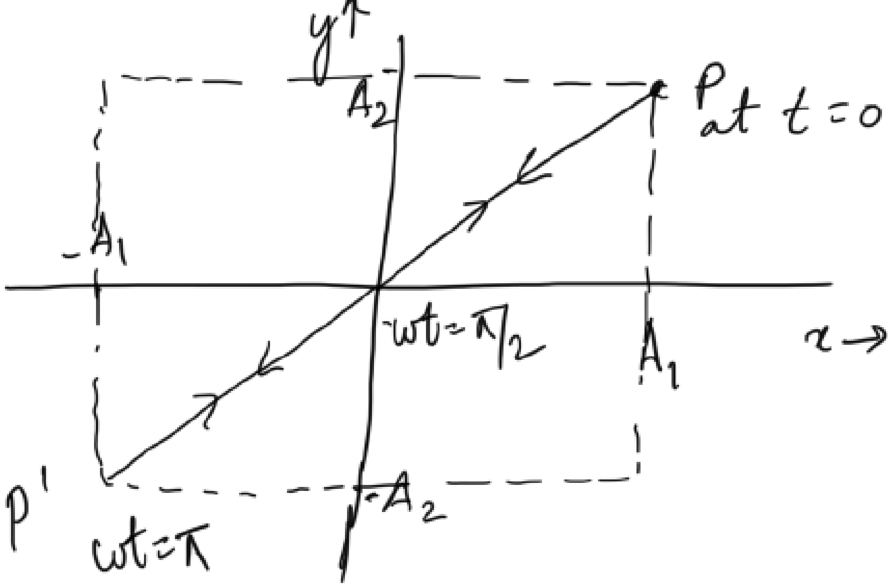
a pair of coincident straight lines, having positive slope A_2/A_1 and passing through origin

The directions of motion can be easily determined from Eq. (1) and (2) by subsituting $\delta = 0$

$$x = A_1 \cos \omega t$$
$$y = A_2 \cos(\omega t + \delta) \rightarrow 0$$

$$\boxed{\text{at } t = 0 \quad x = A_1, \; y = A_2}$$

particle is at P



This in optics known as linearly polarized vibration.

(ii) $\delta = \pi/2$ Eq (5) reduces to

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

which is the equation of an ellipse whose principal axes lie along the x and y axes. The particle moves on elliptical path.

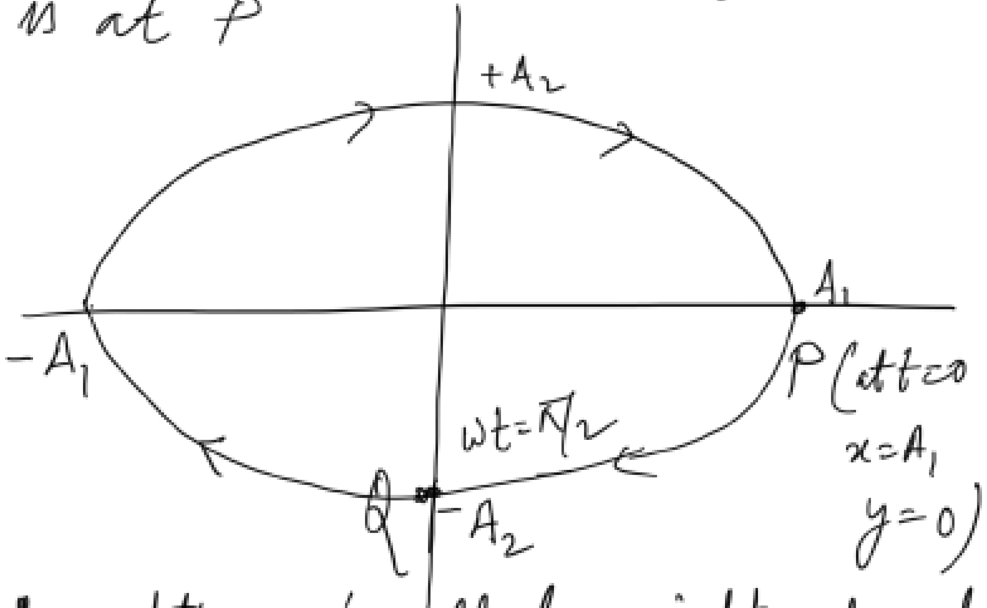
The direction of its motion can be determined from eq. (1) & (2) by setting $\delta = \pi/2$

$$x = A_1 \cos \omega t$$

$$y = A_2 \cos(\omega t + \pi/2)$$

$$y = -A_2 \sin \omega t$$

at $t = 0$, $x = A_1$, $y = 0$ particle is at P



In optics is called right-handed elliptically polarized vibration

if $A_1 = A_2 = A$, the ellipse degenerates into a circle

$$\boxed{x^2 + y^2 = A^2}$$

Two SHM having slightly different frequencies at right angles to each other :

$$x = A_1 \cos \omega t \qquad y = A_2 \cos\{(\omega - \Delta\omega)t + \delta\}$$

$$= A_2 \cos[\omega t + (\delta - \Delta\omega t)]$$

Then $y = A_2 \cos(\omega t + \alpha)$ $\qquad \alpha = (\delta - \Delta\omega t)$

So Equation for trajectory of the motion is

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} \cos\alpha = \sin^2\alpha$$

Lissajous Figure

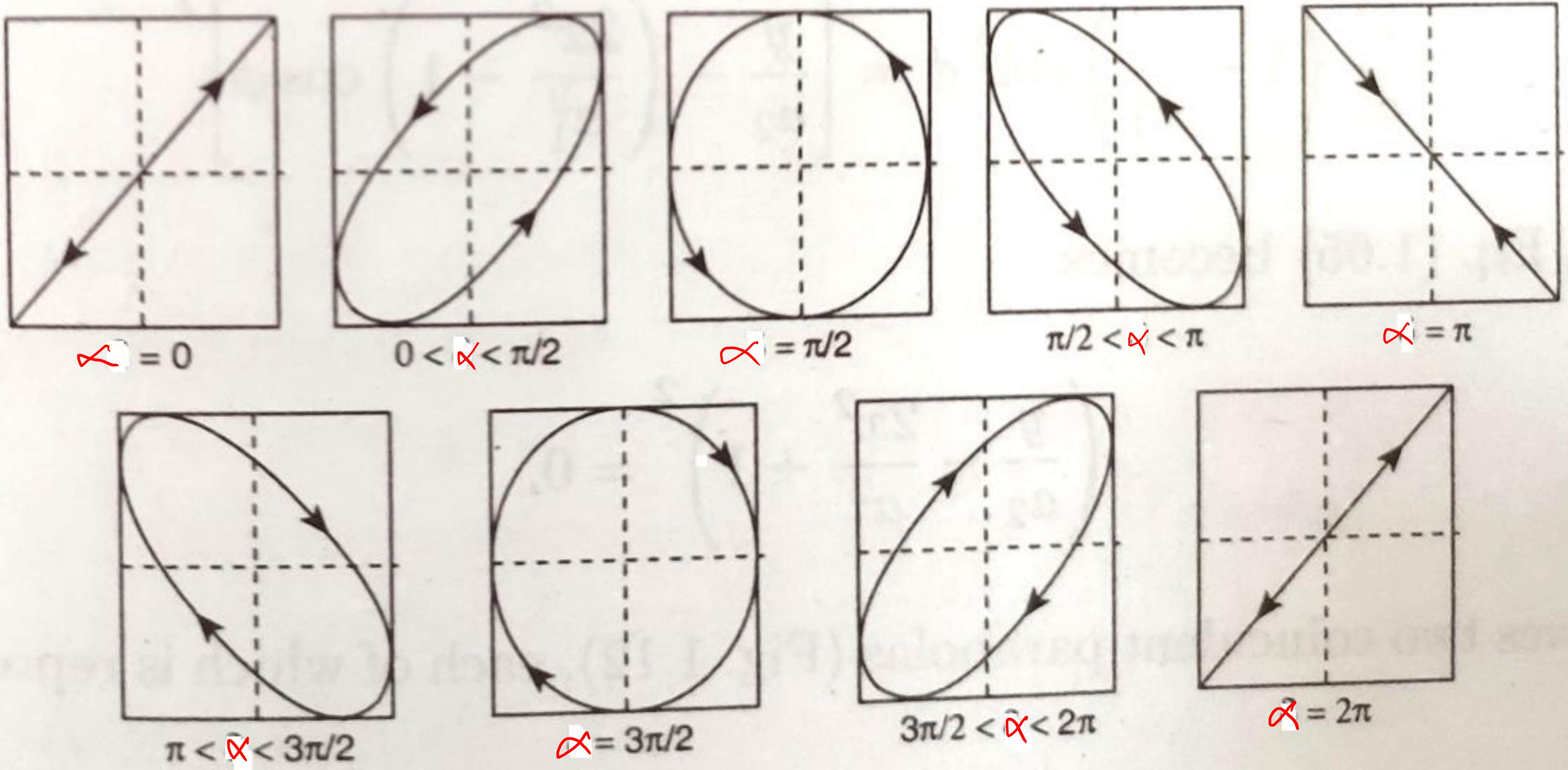


Fig. 1.11 Sequence of paths as δ increases from 0 to 2π

(2) Vibrations having different frequencies (Lissajous Figures)

When the frequencies of the two perpendicular simple harmonic vibrations are not equal. The pattern which are traced from the resulting motion are called Lissajous figures.

i) Frequency in the ratio 1:2

Let us first consider the case when frequency ω_2 of y oscillations twice the frequency of ω_1 of x

i.e $\omega_1 = \omega$ and $\omega_2 = 2\omega$.

The two SHMs are then given by

$$x = A_1 \cos \omega t \quad \text{—} (1)$$

$$y = A_2 \cos(2\omega t + \delta) \quad \text{—} (2)$$

where A_1 and A_2 are amplitudes and δ is the phase difference between them.

Using Analytical Method

$$\frac{y}{A_2} = \cos(2\omega t + \delta)$$

$$\frac{y}{A_2} = \cos 2\omega t \cos\delta - \sin 2\omega t \sin\delta$$

$$\frac{y}{A_2} = (2\cos^2\omega t - 1)\cos\delta - 2\sin\omega t \cos\omega t \sin\delta$$

from (1) we can have

$$\frac{x}{A_1} = \cos\omega t, \quad \sin\omega t = \left(1 - \frac{x^2}{A_1^2}\right)^{1/2}$$

$$\frac{y}{A_2} = \left(2\frac{x^2}{A_1^2} - 1\right)\cos\delta - \frac{2x}{A_1}\left(1 - \frac{x^2}{A_1^2}\right)^{1/2} \cdot \sin\delta$$

$$\left(\frac{y}{A_2} + \cos\delta\right) - \frac{2x^2}{A_1^2}\cos\delta = -\frac{2x}{A_1}\left(1 - \frac{x^2}{A_1^2}\right)^{1/2}\sin\delta$$

Squaring above equation on both sides

$$\left(\frac{y}{A_2} + \cos\delta\right)^2 + \frac{4x^4}{A_1^4}\cos^2\delta - \frac{4x^2}{A_1^2}\cos\delta\left(\frac{y}{A_2} + \cos\delta\right)$$

$$= \frac{4x^2}{A_1^2}\left(1 - \frac{x^2}{A_1^2}\right)\sin^2\delta \quad (3)$$

Rearranging Eq (3) we obtain:

$$\left(\frac{y}{A_2} + \cos\delta\right)^2 + \frac{4x^4}{A_1^4} - \frac{4x^2}{A_1^2}\cos^2\delta - \frac{4x^2}{A_1^2}\sin^2\delta - \frac{4x^2}{A_1^2}\cos\delta\,\frac{y}{A_2} = 0$$

$$\left(\frac{y}{A_2} + \cos\delta\right)^2 + \frac{4x^4}{A_1^4} - \frac{4x^2}{A_1^2} - \frac{4x^2}{A_1^2}\cdot\frac{y}{A_2}\cos\delta = 0$$

$$\boxed{\left(\frac{y}{A_2} + \cos\delta\right)^2 + \frac{4x^2}{A_1^2}\left[\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\cos\delta\right] = 0} \rightarrow (4)$$

In general this equation represents a closed curve having two loops.

For special cases

$\delta = 0$ in eq (4)

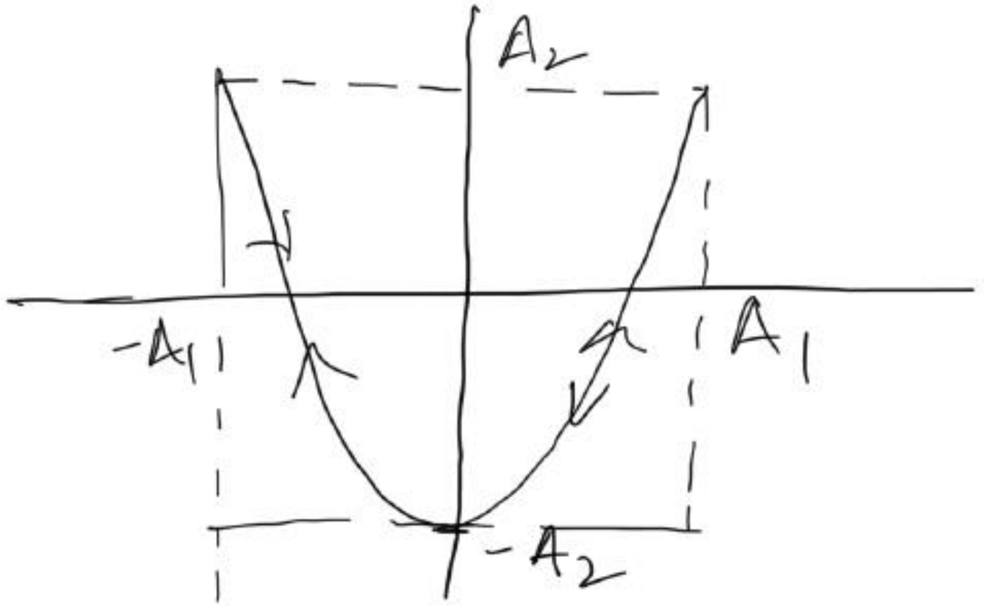
$$\left(\frac{y}{A_2} + 1\right)^2 + \frac{4x^3}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\right) = 0$$

$$\left(\frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2}\right)^2 = 0$$

This represents two coincident parabolas with their vertices at $(0, -A_2)$, the equation of each parabola being

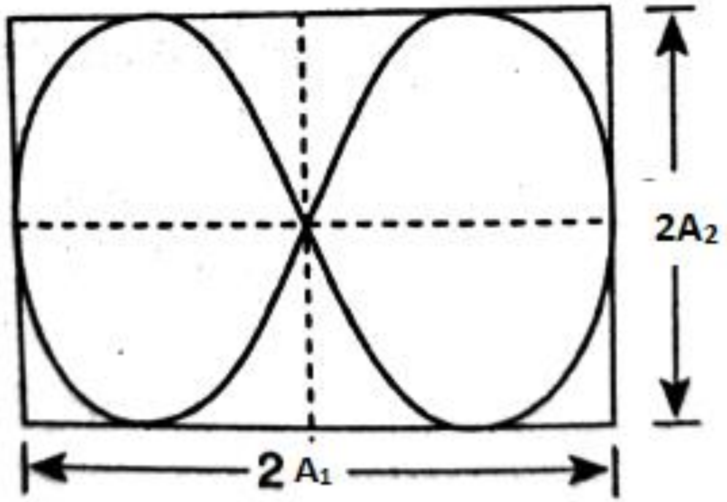
$$\frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2} = 0$$

$$x^2 = \frac{A_1^2}{2A_2}(y + A_2)$$



If $\delta = \frac{\pi}{2}$, the equation (4) reduced to

$$\frac{y^2}{A_2^2} + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1\right) = 0$$



Superposition of two SHM with frequency ratio 1:3 at right angle:

So $x = A_1 \cos \omega t$ $\qquad$ $y = A_2 \cos(3\omega t + \delta)$

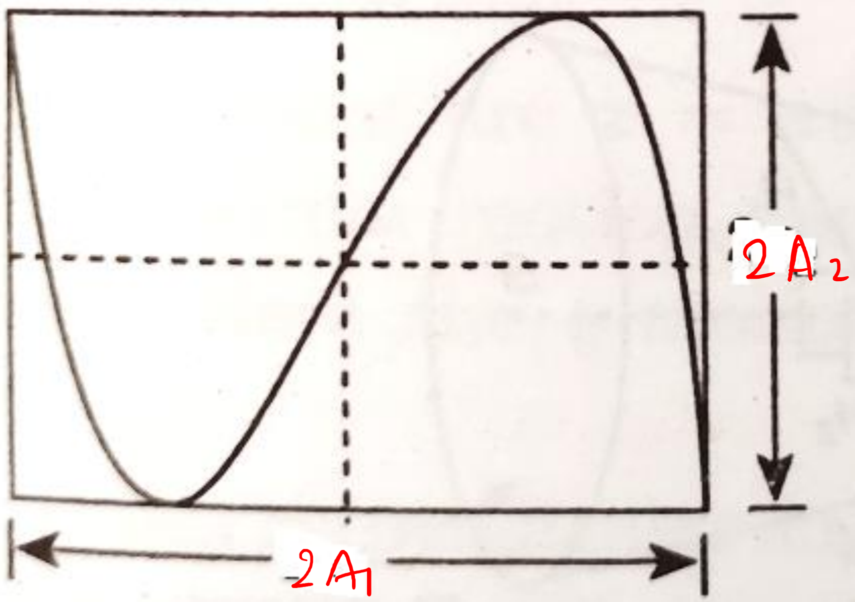
$$y/A_2 = \cos 3\omega t \cos\delta - \sin 3\omega t \sin\delta$$

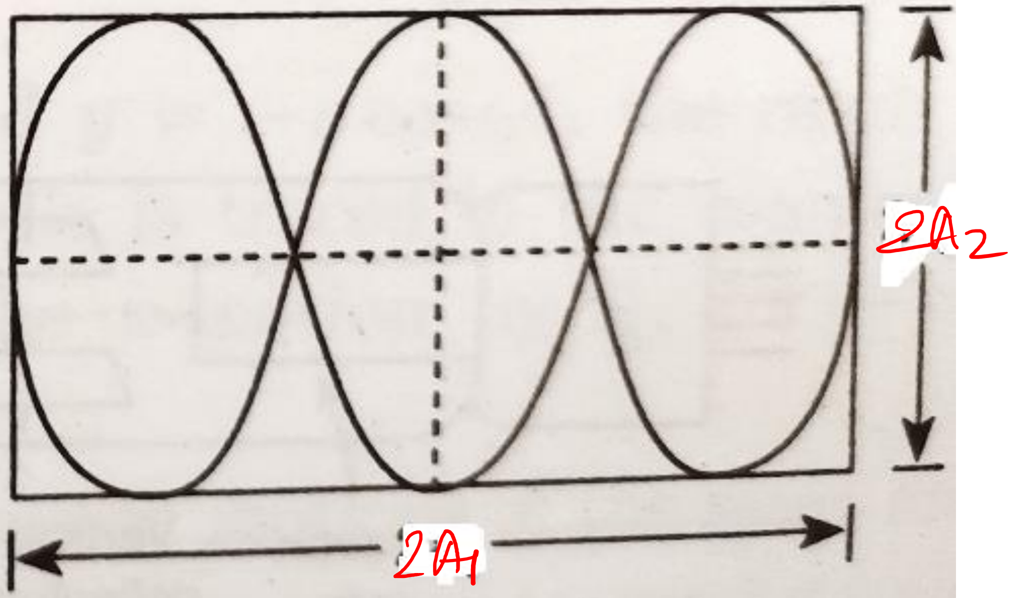
$$= (4\cos^3\omega t - 3\cos\omega t)\cos\delta - (3\sin\omega t - 4\sin^3\omega t)\sin\delta$$

then putting $\cos\omega t = x/A_1$ and $\sin\omega t = \sqrt{1 - \frac{x^2}{A_1^2}}$

$$= \left(4\frac{x^3}{A_1^3} - 3\frac{x}{A_1}\right)\cos\delta - \left[3\left(1 - \frac{x^2}{A_1^2}\right)^{1/2} - 4\left(1 - \frac{x^2}{A_1^2}\right)^{3/2}\right]\sin\delta$$

$$\Rightarrow \left(1 - \frac{x^2}{A_1^2}\right)\left(4\frac{x^2}{A_1^2} - 1\right)^2 \sin^2\delta = \left[y/A_2 - 4\left(\frac{x^3}{A_1^3} - \frac{3x}{A_1}\right)\cos\delta\right]^2$$



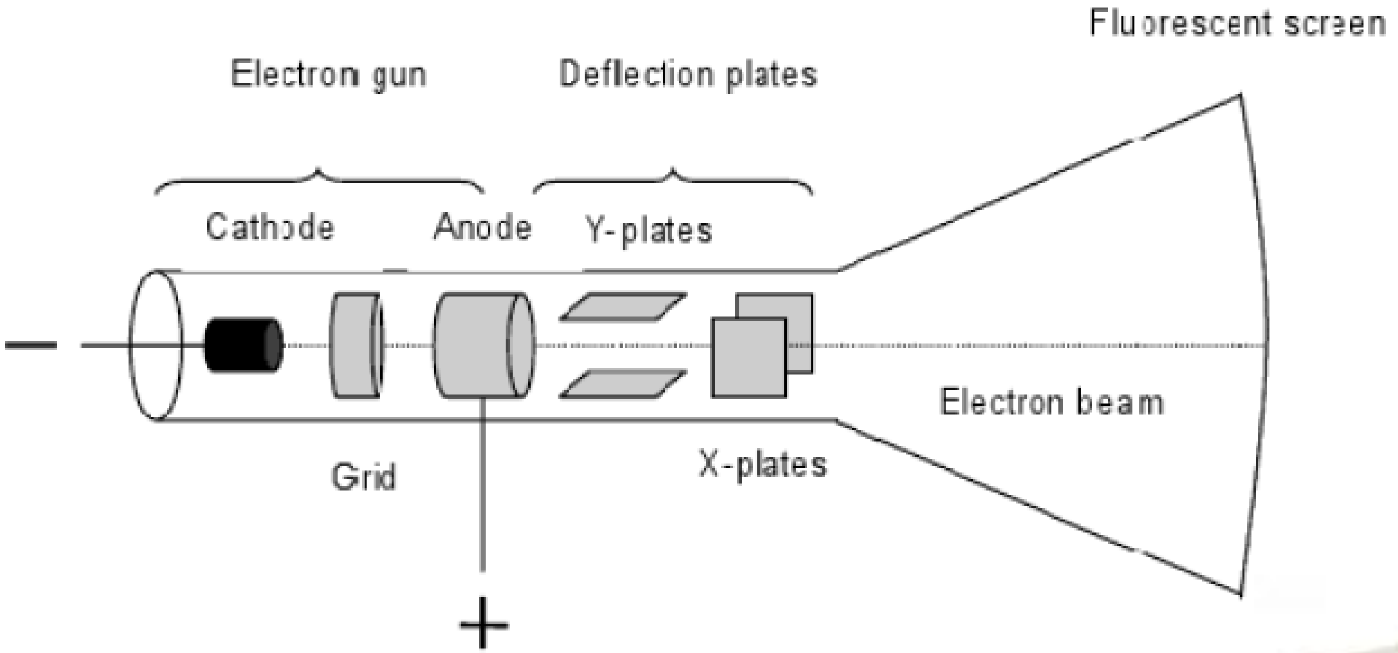


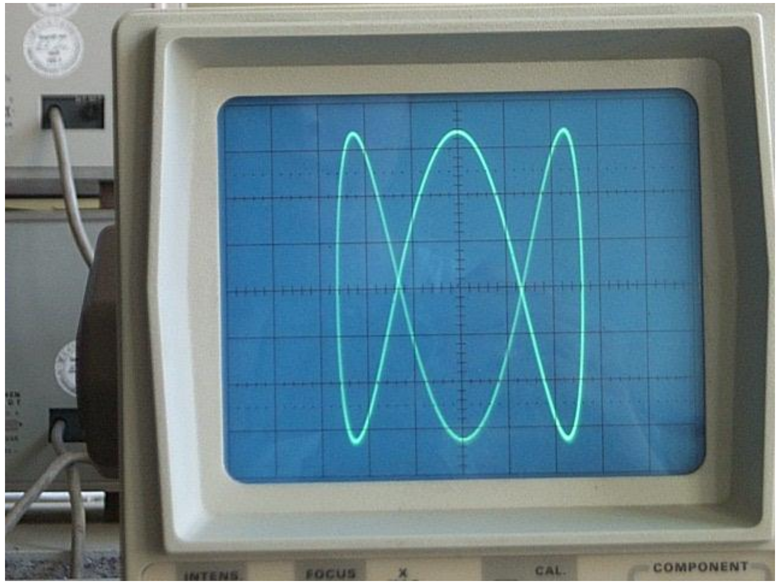
$\delta = 0$

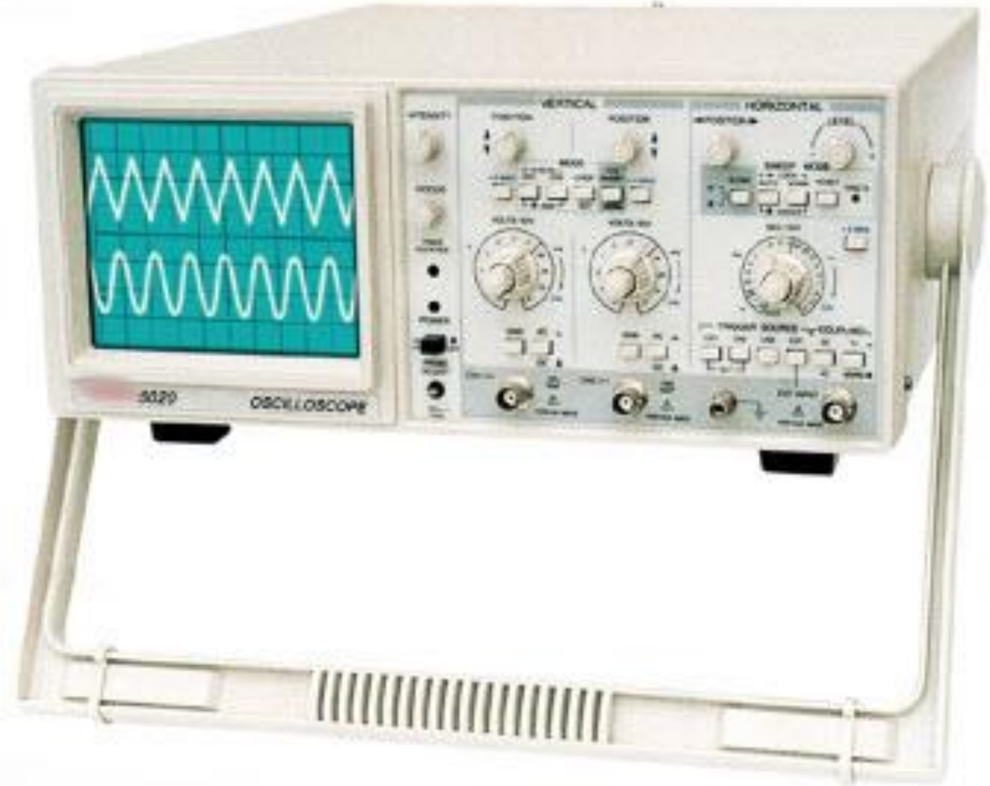
$$\left(\frac{y}{A_2} + \frac{3x_1}{A_1} - \frac{4x^3}{A_1}\right)^2 = 0$$

$\delta = \frac{\pi}{2}$

$$\left(1 - \frac{x^2}{A_1^2}\right)\left(\frac{4x^2}{A_1} - 1\right)^2 - \frac{y^2}{A_2^2} = 0$$


Fluorescent screen
Electron gun
Deflection plates
Cathode
Anode
Y-plates
−
Electron beam
Grid
X-plates
+


INTENS
FOCUS
X
CAL.
COMPONENT


VERTICAL
HORIZONTAL
OSCILLOSCOPE

Jules Antoine Lissajous

Lissajous was a French physicist and mathematician who was born on March 4,1822 at Versaillas, France. Lissajous graduated from the Ecole Normale Superieure in 1847 in Paris. He is known for many innovations but perhaps the best known is a special apparatus – the Lissajous apparatus--- that creates the variety of figures which are now named after him as the Lissajous figures. This apparatus made him realize, in a way, his idea of making 'sound waves visible'!. This apparatus uses two tuning forks with mirrors attached to them , mounted in mutually perpendicular directions. By making the light reflected from the mirror of the first tuning fork , to fall on the mirror of the second fork and then getting the light reflected from this on the screen, one can see the resulting Lissajous figures on the screen. It is possible to vary the frequency ratio and the phase difference between the two oscillations and see the effects of these on the patterns of the Lissajous figures obtained.

Lissajous apparatus led to the invention of other apparatus such as the harmonograph. It is interesting to note that many of the designs, used by fabric and cloth designers, can be correlated with Lissajous figures corresponding to different mutual orientations and to different frequency, phase and amplitude ratios of the two superimposed simple harmonic motions.