CHAPTER 9

Jacobians

9.1 DEFINITION

If $u(x, y)$ and $v(x, y)$ are two functions of two variables x and y then the function determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \text{ or } \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the *Jacobian* of u and v with respect to x and y. The Jacobian of u, v with respect to x and y is denoted by $\frac{\partial(u, v)}{\partial(x, y)}$. The Jacobian is also denoted by $J(u, v)$.

The Jacobian is used to evaluate multiple integral when transformation is required. Also, one can test the dependence of functional relations, using the concept of Jacobian.

EXAMPLE 9.1 If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x, y)}{\partial(r, \theta)}$. (WBUT 2007)

Solution

By definition $$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

EXAMPLE 9.2 If $x = a\cos\theta\cosh\phi$, $y = a\sin\theta\sinh\phi$, prove that

$$\frac{\partial(x, y)}{\partial(\phi, \theta)} = \frac{1}{2}a^2(\cosh 2\phi - \cos 2\theta).$$

Solution

We have $$\frac{\partial(x, y)}{\partial(\phi, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\[2ex] \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta\sinh\phi & -a\sin\theta\cosh\phi \\ a\sin\theta\cosh\phi & a\cos\theta\sinh\phi \end{vmatrix}$$

$$= a^2(\cos^2\theta\sinh^2\phi + \sin^2\theta\cosh^2\phi)$$

$$= a^2\left[\cos^2\theta(\cosh^2\phi - 1) + (1 - \cos^2\theta)\cosh^2\phi\right]$$

$$= a^2\left[\cosh^2\phi - \cos^2\theta\right]$$

$$= a^2\left[\frac{1}{2}(1 + \cosh 2\phi) - \frac{1}{2}(1 + \cos 2\theta)\right]$$

$$= \frac{a^2}{2}\left[\cosh 2\phi - \cos 2\theta\right].$$

9.2 PROPERTIES OF JACOBIANS

Theorem 9.1 If u_1, u_2 are functions of the variables y_1, y_2 and y_1, y_2 are the functions of x_1, x_2, then

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \times \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}.$$

Proof: By chain rule of partial derivative

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_2}$$

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$

and $$\frac{\partial u_2}{\partial x_2} = \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_2}.$$

Now, right hand side

$$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \times \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} & \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_2} \\ \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

[row by column multiplication]

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix} = \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}.$$

Hence proved.

In general, if $u_1, u_2, ..., u_n$ are functions of the set of the variables $y_1, y_2, ..., y_n$ and $y_1, y_2, ..., y_n$ are themselves functions of $x_1, x_2, ..., x_n$, then

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(u_1, u_2, ..., u_n)}{\partial(y_1, y_2, ..., y_n)} \times \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}.$$

Theorem 9.2 If J be the Jacobian of the system u, v with regard to x, y and J' the Jacobian of x, y with regard to u, v, then $JJ' = 1$, i.e.

$$\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1.$$

Proof: Let $u = f_1(x, y), v = f_2(x, y)$, then differentiating these w.r.t. u and v partially, we get

$$1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}, \qquad 0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}$$

$$0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}, \qquad 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}.$$

Now,
$$JJ' = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)}$$

$$=\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$=\begin{vmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial v}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial v} \\ \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial v}+\dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}=1.$$

Thus, $JJ'=1$. From this relation, we can write

$$\frac{\partial(u, v)}{\partial(x, y)}=\frac{1}{\dfrac{\partial(x, y)}{\partial(u, v)}}.$$

In general, $$\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)}\times\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)}=1.$$

Theorem 9.3 (Jacobian of implicit functions). If 4 variables u_1, u_2 and x_1, x_2 are connected implicitly by two independent relations

$$f_1(u_1, u_2, x_1, x_2)=0$$

$$f_2(u_1, u_2, x_1, x_2)=0$$

then $$\frac{\partial(f_1, f_2)}{\partial(u_1, u_2)}\times\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}=\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}.$$

Proof: Differentiating $f_1(u_1, u_2, x_1, x_2)=0$ w.r.t. x_1, x_2 we get

$$\frac{\partial f_1}{\partial x_1}+\frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_1}=0 \text{ or } \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_1}=-\frac{\partial f_1}{\partial x_1}$$

$$\frac{\partial f_1}{\partial x_2}+\frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_2}=0 \text{ or } \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_2}=-\frac{\partial f_1}{\partial x_2}.$$

Similarly, from $f_2(u_1, u_2, x_1, x_2)=0$, we get

$$\frac{\partial f_2}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial f_2}{\partial u_2}\frac{\partial u_2}{\partial x_1}=-\frac{\partial f_2}{\partial x_1}$$

and
$$\frac{\partial f_2}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial f_2}{\partial u_2}\frac{\partial u_2}{\partial x_2}=-\frac{\partial f_2}{\partial x_2}.$$

Now,
$$\frac{\partial(f_1, f_2)}{\partial(u_1, u_2)}+\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}$$

$$=\begin{vmatrix}\dfrac{\partial f_1}{\partial u_1} & \dfrac{\partial f_1}{\partial u_2}\\ \dfrac{\partial f_2}{\partial u_1} & \dfrac{\partial f_2}{\partial u_2}\end{vmatrix}\times\begin{vmatrix}\dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2}\\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2}\end{vmatrix}$$

$$=\begin{vmatrix}\dfrac{\partial f_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial f_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial f_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial f_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_2}\\ \dfrac{\partial f_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial f_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial f_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial f_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_2}\end{vmatrix}$$

$$=\begin{vmatrix}\dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2}\\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2}\end{vmatrix}=\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}.$$

9.3 WORKED-OUT EXAMPLES

EXAMPLE 9.3 If $f(u, v)=3uv^2, g(u, v)=u^2-v^2$, find the Jacobian $\dfrac{\partial(f, g)}{\partial(u, v)}$. (WBUT 2004)

Solution

$$\frac{\partial f}{\partial u}=3v^2, \frac{\partial f}{\partial v}=6uv, \frac{\partial g}{\partial u}=2u, \frac{\partial g}{\partial v}=-2v.$$

Therefore,
$$\frac{\partial(f, g)}{\partial(u, v)}=\begin{vmatrix}\dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v}\\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v}\end{vmatrix}=\begin{vmatrix}3v^2 & 6uv\\ 2u & -2v\end{vmatrix}$$

$$=-6v^3-12u^2v=-6v(v^2+2u^2).$$

EXAMPLE 9.4 If $f(x, y)=\dfrac{x+y}{1-xy}$ and $g(x, y)=\tan^{-1}x+\tan^{-1}y$ find the Jacobian $\dfrac{\partial(f, g)}{\partial(x, y)}$.

Solution

$$\frac{\partial f}{\partial x}=\frac{1.(1-xy)-(x+y)(-y)}{(1-xy)^2}=\frac{1+y^2}{(1-xy)^2}.$$

Similarly,

$$\frac{\partial f}{\partial y}=\frac{1+x^2}{(1-xy)^2},\ \frac{\partial g}{\partial x}=\frac{1}{1+x^2}\text{ and }\frac{\partial g}{\partial y}=\frac{1}{1+y^2}.$$

$$\therefore\quad \frac{\partial(f,g)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y}\\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y}\end{vmatrix}=\begin{vmatrix}\dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2}\\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2}\end{vmatrix}$$

$$=\frac{1}{(1-xy^2)}-\frac{1}{(1-xy^2)}=0.$$

EXAMPLE 9.5 If $x+y=u$ and $x=uv$, then show that $\dfrac{\partial(x,y)}{\partial(u,v)}=-u$. Also, find the value of $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Solution

Here $x=uv$, $y=u-x=u-uv=u(1-v)$.

Therefore, $$\frac{\partial x}{\partial u}=v,\ \frac{\partial x}{\partial v}=u,\ \frac{\partial y}{\partial u}=1-v,\ \frac{\partial y}{\partial v}=-u.$$

$$\therefore\quad \frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}v & u\\ 1-v & -u\end{vmatrix}$$

$$=-u.$$

We know, $$\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=1.$$

$$\therefore\quad \frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}}=\frac{1}{-u}=-\frac{1}{u}.$$

EXAMPLE 9.6 If $u^3+v^3=x+y$, and $u^2+v^2=x^3+y^3$, show that $\dfrac{\partial(u,v)}{\partial(x,y)}=\dfrac{1}{2}\dfrac{y^2-x^2}{uv(u-v)}$.

Solution

Here the variables u, v and x, y are implicitly connected by two relations, viz.

$$f_1 \equiv u^3 + v^3 - x - y = 0$$

$$f_2 \equiv u^2 + v^2 - x^3 - y^3 = 0.$$

Now,
$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(f_1, f_2)}{\partial(x, y)} \div \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} \div \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} \div \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= (3y^2 - 3x^2) \div (6u^2v - 6uv^2)$$

$$= \frac{1}{2}\,\frac{y^2 - x^2}{uv\,(u - v)}.$$

EXAMPLE 9.7 The roots of the equation $(\lambda - x)^2 + (\lambda - y^2) = 0$ in λ are u, v, prove that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{y - x}{u - v}.$$

Solution

The equation $(\lambda - x)^2 + (\lambda - y)^2 = 0$ can be written as

$$\lambda^2 - \lambda(x + y) + \frac{1}{2}(x^2 + y^2) = 0.$$

If u, v are the roots, then

$$u + v = -\text{ coefficient of } \lambda / \text{coefficient of } \lambda^2$$

$$= x + y$$

$$uv = \text{constant term/coefficient of } \lambda^2$$

$$= \frac{1}{2}(x^2 + y^2).$$

These relations can be written as

$$f_1 \equiv u + v - x - y = 0 \text{ and } f_2 \equiv uv - \frac{1}{2}(x^2 + y^2).$$

Now,
$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(f_1, f_2)}{\partial(x, y)} \div \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} \div \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 \\ -x & -y \end{vmatrix} \div \begin{vmatrix} 1 & 1 \\ v & u \end{vmatrix}$$

$$= (y - x) \div (u - v) = \frac{y - x}{u - v}.$$

EXAMPLE 9.8 If $u = x(1 - r^2)^{-1/2}$ and $v = y(1 - r^2)^{-1/2}$, where $r^2 = x^2 + y^2$, find the value of $\dfrac{\partial(u, v)}{\partial(x, y)}$.

Solution

Given $\quad u = x(1 - x^2)^{-1/2}$

i.e. $\quad u^2(1 - r^2) = x^2$

or $\quad x^2 - u^2(1 - x^2 - y^2) = 0.$

Similarly, $\quad y^2 - v^2(1 - x^2 - y^2) = 0$

Let $\quad f_1 \equiv x^2 - u^2(1 - x^2 - y^2) = 0$

and $\quad f_2 \equiv y^2 - v^2(1 - x^2 - y^2) = 0.$

Now,
$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x + 2xu^2 & 2yu^2 \\ 2xv^2 & 2y + 2yv^2 \end{vmatrix}$$

$$= 4xy \begin{vmatrix} 1 + u^2 & u^2 \\ v^2 & 1 + v^2 \end{vmatrix}$$

$$= 4xy(1 + u^2 + v^2),$$

and
$$\frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -2u(1-x^2-y^2) & 0 \\ 0 & -2v(1-x^2-y^2) \end{vmatrix} = 4uv(1-x^2-y^2)^2.$$

$$\therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(f_1,f_2)}{\partial(x,y)} \div \frac{\partial(f_1,f_2)}{\partial(u,v)}$$

$$= \frac{4xy(1+u^2+v^2)}{4uv(1-x^2-y^2)^2} = \frac{xy(1+u^2+v^2)}{uv(1-r^2)^2}.$$

Now,
$$uv = xy(1-r^2)^{-1}$$

and
$$1+u^2+v^2 = 1+x^2(1-r^2)^{-1}+y^2(1-r^2)^{-1}$$

$$= 1+\frac{x^2+y^2}{1-r^2} = 1+\frac{r^2}{1-r^2} = \frac{1}{1-r^2}.$$

Hence,
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{xy(1-r^2)^{-1}}{xy(1-r^2)^{-1}(1-r^2)^2} = (1-r^2)^{-2}.$$

EXAMPLE 9.9 If $u = x^2+y^2$, $v = x^2-y^2$ and $x = r\theta$, $y = r+\theta$ then find the value of the Jacobian $\dfrac{\partial(u,v)}{\partial(r,\theta)}$.

Solution

We know,
$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \theta & r \\ 1 & 1 \end{vmatrix}$$

$$= (-4xy-4xy) \times (\theta - r)$$

$$= 8x(r-\theta).$$