Axiomatic Definition of Probability:

Probability can be defined as a set function P(E) which assigns to every event E a number known as the probability of E subject to the following axioms:

(i) The probability of an event E is greater than or equal to zero,

(ii) The probability of a certain event is always one,

(iii) If two events are disjoint, the probability that either of the two events happens is the sum of their probabilities that each happens.

Theorems on Probability

(I) Theorem of total Probability:

If the events $A_1, A_2, \dots, A_n$ are mutually exclusive, then probability of occurrence of either A_1 or A_2 or $\dots A_n$ is given by the sum of their probabilities

i.e. $P(A_1 or A_2 or \ \dots A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$

(This is also known as addition theorem)

(II) Probability of occurrence of at least one of the two events A and B (which may not be mutually exclusive) is given by $P(A+B) = P(A) + P(B) - P(AB)$

Note: Probability of occurrence of at least one of the three events A, B, C (which are not mutually exclusive) is that

$$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$$

In general

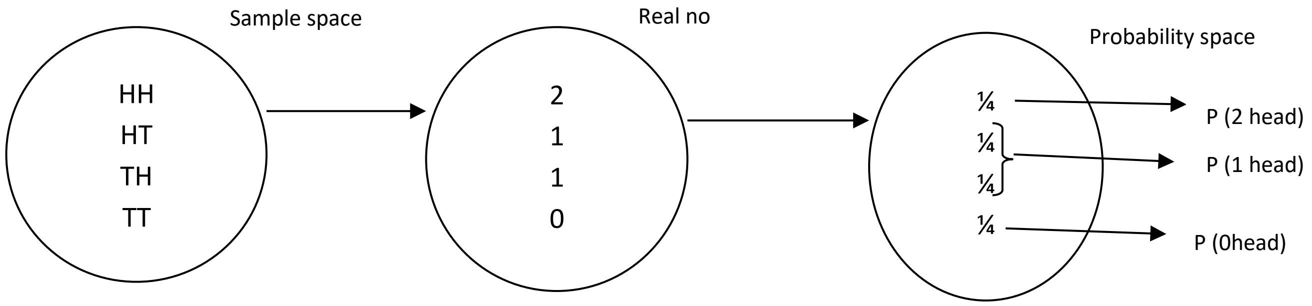
(III) If the events $A_1, A_2, \dots, A_n$ are not mutually exclusive then the probability of the happening of at least one of the 'n' events is given by

$$P(A_1 + A_2 + \ \ldots + A_n)$$
$$= \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1 \\ i<j}}^{n} P\left(A_{ij}\right) + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} P\left(A_{ijk}\right) - \cdots$$
$$+ (-1)^{n-1} P(A_1 A_2 \ \ldots A_n)$$

where A_{ij} means $A_i A_j$ for all *I,j*

Lecture 3

Random variables (RV):



Let X=No. of heads

X(HH)=2

X(HT)=1

X(TH)=1

X(TT)=0

Thus RV Can be defined as follows:

Random Variable is a function that associates a non (-ve) real number with each element of the sample space.

Discrete RV : A RV is said to be discrete RV if its set of possible outcomes is countable.

Continuous RV: When a RV takes on values on a continuous scale, it is called a continuous RV.

In the practical problems, continuous RVs represent measured data, such as possible heights, weights, temperature, distance, or life periods.

Whereas discrete RV represent count data, such as the no. of defectives in a sample of k items or no. of high way fatalities per year in a given state etc.

Frequently, it is convenient to represent all the probabilities of a RV X by a formula. Such a formula would necessarily be a function of the numerical values x that we shall denote by f(x), g(x), r(x) and so forth. Therefore, we write f(x)=P(X=x) i.e. f(3)=P(X=3).

Definition: The set of ordered pairs (x, f(x)) is a probability distribution of discrete RV X if for each possible outcomes x,

1. $f(x) \geq 0 \quad \forall x$
2. $\sum f(x) = 1$
3. $P(X = x) = f(x)$.

Ex.1 Check whether the following can serve a probability distribution

(a) f(x)=(x-2)/2 for x=1,2,3,4.

(b) f(x)=x^2/25 for x=0,1,2,3,4.

<u>Solution:</u>

(a) f(x)=(x-2)/2.

Therefore, f(1) = (1-2)/2 =-1/2, which is not possible as $f(x) \geq 0 \quad \forall x$.

Thus this function f(x) cannot serve as a probability distribution.

(b) f (x)=x^2/25 for x=0,1,2,3,4.

Here, f(0)=0, f(1)=25, f(2)=4/25, f(3)=9/25, f(4)=16/25

Therefore, each f(x)>=0.

And $\sum f(x) = \frac{0}{25} + \frac{1}{25} + \frac{4}{25} + \frac{16}{25} = \frac{6}{5} \geq 1$ which is not possible as$\sum f(x) = 1$.

Therefore the function defined in (b) cannot serve as a probability distribution.

2. A shipment of 8 similar micro computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the no. of defectives and also find a formula for the probability distribution for the no. of defectives.

Solution: Let X be the RV whose values x are the possible numbers of defective computers purchased by the school so x can be any nos. 0, 1, 2.

Therefore, f(0)=P(X=0)= $\frac{^3C_0 \, ^5C_2}{^8C_2} = \frac{\frac{5!}{2!3!}}{\frac{8!}{2!6!}} = 5/14$

f(1)=P(X=1)= $\frac{^3C_1 \, ^5C_1}{^8C_2} = 15/28$

and f(2)=P(X=2)= $\frac{^3C_2 \, ^5C_0}{^8C_2} = 3/28$

Therefore, probability distribution of the RV X is

X	0	1	2
f(x)=P(X=x)	10/28	15/28	3/28

Therefore, formula for probability distribution for no. of defectives is

f(x)=P(X=x)= $\frac{^3C_x \, ^5C_{2-x}}{^8C_2}$, $0 \leq x \leq 2$

Cumulative distribution:

In many problems we wish to compute the prob. that the observed value of a RV X will be less than or equal to some real no. x.

Writing F(x) = P(X≤x) for every real no. x, we define F(x) to be the cumulative distribution of the RV X and F(x) is called distribution function.

Defn: The cumulative distribution F(x) of a discrete RV X with probability distribution f(x) is given by $F(x) = P(X \le x) = \sum_{t \le x} f(t),\ -\infty < x < \infty$.

Ex. 1. An unbiased coin is thrown three times. If the RV X denotes the no. of heads obtained, find the cumulative distribution function (cdf) of X.

Solun: Here the sample space contains the following events.

S = {HHH, HHT, HTH, THH, HTT,THT, TTH,TTT}.

Let X represent the no. of heads.

The prob. distribution of X is

x	0	1	2	3	Total
P(X=x)=f(x)	1/8	3/8	3/8	1/8	1

From this table we can find the prob. of obtaining 0 or less head, 1 or less head, 2 or less heads, 3 or less heads, as follows.

F(0)=P(X≤0)=P(X=0)=1/8.

F(1)=P(X≤1)=P(X=0)+ P(X=1)=1/8+3/8=4/8=1/2.

F(2)=P(X≤2)=P(X=0)+ P(X=1)+P(X=2)=1/8+3/8+3/8=7/8..

F(3)=P(X≤3)=P(X=0)+ P(X=1)+P(X=2)+ P(X=3)=1/8+3/8+3/8+1/8=1.

Thus we have

X	0	1	2	3
F(x)	1/8	½	7/8	1

In general if x denotes any arbitrary real no. the prob. that the no of heads (X) is x or less can be given as

F(x)=0 when x<0.

Because we can not have less than 0 heads.

F(x)=1/8 when 0≤ x<1, (denoted by F(0)=P(X≤0))

e.g. F(0.7)=1/8, because the prob. of having 0.7 or less head is the same as that of having 0 head.

Again, F(x)=1/8+3/8=4/8=1/2 when 1≤x<2, (denoted by F(1)=P(X≤1))

e.g. F(1.34)=1/2 because the prob. of having 1.34 or less heads is the same as that of having 0 or 1 head.

Similarly, $F(x)=1/8+3/8+3/8=7/8$ when $2\le x<3$, (denoted by $F(2)=P(X\le 2)$)

$F(x)=1/8+3/8+3/8+1/8=1$ when $3\le x$, (denoted by $F(3)=P(X\le 3)$)

e.g. $F(8.75)=1$ because the prob. having 8.75 heads or less is the same as that of having 0,1,2,3 heads. The cumulative distribution function (cdf) of X is thus given as follows.

$$F(x) = \begin{cases} 0 & \text{when } x<0 \\ 1/8, & 0\le x<1 \\ 1/2, & 1\le x<2 \\ 7/8, & 2\le x<3 \\ 1, & 3\le x \end{cases}$$

<u>Continuous Probability Distribution:</u>

Probability Density function f(x) for a Continuous RV is defined over the set of real nos. R as follows:

1. $f(x)\ge 0 \quad \forall x \in R$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a<X<b)= \int_a^b f(x)dx$

Ex. 1 Suppose that the error in the reaction temperature in ^0C for a controlled laboratory experiment is a continuous RV X having the probability density function

$$f(x) = \begin{cases} x^2/3, & -1<x<2, \\ 0 & \text{, elsewhere} \end{cases}$$

(a) Verify the condition (2) of above defn.

(b) Find $P(0<X\le 1)$.

Soln: (a) Here $f(x) = x^2/3, \quad -1<x<2,$

Therefore, $\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{2} \frac{x^2}{3} dx = [\frac{x^3}{9}]_{-1}^{2} = 1$

(b) Find $P(0<X\le 1)= \int_0^1 \frac{x^2}{3} dx = [\frac{x^3}{9}]_0^1 = \frac{1}{9}$

<u>Cumulative Distribution</u> F(x) of a continuous RV X with density function f(x) is given by

$F(x) = P(X\le x) = \int_{-\infty}^{x} f(t)dt \quad for -\infty < x < \infty$

According to this defn. we can write the result $P(a<X<b) = F(b) - F(a)$.

Because $F(b)= P(X\le b)$

And $F(a) = P(X\le a)$

Therefore, $F(b)-F(a) = P(X\le b) - P(X\le a)$

$= \int_{-\infty}^{b} f(t)dt - \int_{-\infty}^{a} f(t)dt$ from property of definite integration.

$=\int_a^b f(t)dt$

$=P(a<X<b)$

Properties of Continuous Distribution function:

1. $F(+\infty) = 1, F(-\infty) = 0$
2. $F(x)$ is a non-decreasing fun. of x.
3. If $F(x_0) = 0$, then $F(x) = 0$ for $x \le x_0$.
4. $P(X>x)=1-P(X\le x)=1-F(x)$
5. The function F(x) is continuous from the right i.e. $F(x^+) = F(x)$
6. $P(x_1<X\le x_2)= F(x_2)-F(x_1)$
7. $P(X=x)=F(x)- F(x^-)$
8. $P\{x_1\le X\le x_2\}=F(x_2)-F(x_1^-)$

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6. $P(x_1<X\le x_2)=P(x\le x_2)-P(x\le x_1)= F(x_2)-F(x_1)$
7. $P(X=x)=P(X\le x)-P(X\le x^-)=F(x)- F(x^-)$
8. $P\{x_1\le X\le x_2\}=F(x_2)-F(x_1^-)$