CHAPTER 1

Basic Concepts of Differential Equations

1.1 Introduction

The differential equation is one of the most important entities in the study of applied sciences. It occurs in different disciplines of engineering as well as in most areas of science. Studies of differential equations by pure and applied mathematicians, theoretical and applied physicists, chemists, engineers and other scientists have established that there are certain definite methods by which many of the differential equations can be solved. Many interesting methods have been developed so for, however, there remain many unsolved equations, some of which are of great importance. Here some methods are discussed to solve some particular types of differential equations.

1.2 Definition and Terminology

Definition 1.2.1 (Differential equation). *An equation containing independent and dependent variables and the derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a* **differential equation**.

The following are some examples of differential equation:

$$m\frac{d^2x}{dt^2} = -kx \tag{1.1}$$

$$5y = x\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}} \tag{1.2}$$

$$\frac{d^2y}{dx^2} = \frac{W}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{1.3}$$

$$r = \frac{[1+(y')^2]^{\frac{3}{2}}}{y''} \tag{1.4}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.5}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \tag{1.6}$$

Definition 1.2.2 (Ordinary differential equation (ODE)). *A differential equation which involves derivatives with respect to a single independent variable is known as an **ordinary differential equation (ODE)**.*

Equations (1.1) to (1.4) are examples of ordinary differential equations.

Definition 1.2.3 (Partial differential equation (PDE)). *A differential equation which contains two or more independent variables and partial derivatives with respect to them is called a **partial differential equation (PDE)**.*

Equations (1.5) and (1.6) are examples of partial differential equations.

Definition 1.2.4 (Order of differential equation). *The order of a differential equation (ODE or PDE) is the order of the highest derivative in the equation.*

The order of Eqs. (1.2) is one and the order of Eqs. (1.1), (1.3), (1.4), (1.5) and (1.6) is two.

Definition 1.2.5 (Degree of differential equation). *The degree of a differential equation is the degree of the highest-order derivative in the equation, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.*

The degree of Eqs. (1.1), (1.5) and (1.6) is one. The degree of Eqs. (1.2), (1.3) and (1.4) is two as they are written as

$$5y\left(\frac{dy}{dx}\right) = x\left(\frac{dy}{dx}\right)^2 + k$$

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{W}{H}\right)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

and

$$(y''r)^2 = [1 + (y')^2]^3$$

Definition 1.2.6 (Linear and non-linear differential equations). *An nth order differential equation is said to be **linear in** y if it can be written in the form*

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x)$$

*where $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and $f(x)$ are functions of x on some interval. The functions $a_0, a_1, \ldots, a_n$ are called **coefficient functions**.*

A differential equation that is not linear is called **non-linear**, *i.e. the equation other than the above form.*

The equations $y'' - 5y' + 6y = 0$, $x^2y'' - 2xy' + 3e^xy = 0$ and $\cos x\ y'' + 2e^xxy' - 3y + x^5 = 0$ are linear, while $yy'' + y' + 3 = 0$, $y'' + \sin y = 5$ and $yy'' + yy' + xy = 5$ are non-linear.

Note 1.2.1 An ordinary differential equation is linear if the following conditions are satisfied:

- The unknown function and its derivatives occur in the first degree only;
- There are no products involving either the unknown function and its derivatives or two or more derivatives;
- There are no transcendental functions involving in the unknown function or any of its derivatives.

1.3 Solution of Differential Equation

A **solution** of a differential equation is a relation between the dependent and independent variables, not involving the derivatives, such that this relation and its derivatives satisfy the given differential equation.

For example, $y = c_1e^x + c_2e^{-x}$ is the general solution of the differential equation $y'' - y = 0$.

A solution of a differential equation of order n will have n independent arbitrary constants. This solution is called a **general solution**. Any solution obtained by assigning particular values to some or all of the arbitrary constants is called a **particular solution**.

The solutions $y = c_1e^x + 5e^{-x}$ and $y = 2e^x - 3e^{-x}$ are particular solutions of $y'' - y = 0$.

A solution of a differential equation that is not obtained from a general solution by assigning particular values to the arbitrary constants is called a **singular solution**.

For example, $y = 0$ is a singular solution of $y' = 2x\sqrt{y}$.

If the solution of a differential equation can be expressed in the form $y = \phi(x)$, where x and y are respectively independent and dependent variables, then the solution is called an **explicit solution**.

Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation $y'' + a_1(x)y' + a_2(x)y = 0$ are said to be **linearly independent** if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$.

If c_1 and c_2 both are not zero, then the two solutions y_1 and y_2 are said to be **linearly dependent**. If y_1 and y_2 are two solutions, then their linear combination $c_1y_1 + c_2y_2$ is also a solution of that differential equation and this process is called the **principle of superposition**.

1.4 Initial-value and Boundary-value Problems

It is mentioned that an nth order differential equation has n arbitrary constants. To obtain a particular solution, the n conditions are required on dependent variables and its derivatives. There are two well-known methods for specifying these auxiliary conditions – **initial conditions** and **boundary conditions**.

Definition 1.4.1 (Initial-value problem). *If the auxiliary conditions for a given differential equations are assigned to a single value of x, the conditions are called* ***initial conditions***. *The differential equation with its initial conditions is called an* ***initial-value problem*** **(IVP)**.

Definition 1.4.2 (Boundary-value problem). *If the auxiliary conditions for a given differential equations are assigned to two or more values of x, the conditions are called* ***boundary conditions*** *or* ***boundary values***. *The differential equation with its boundary conditions is called a* ***boundary-value problem*** **(BVP)**.

The problem $y'' + 3y = x$, $y(0) = 1$, $y'(0) = 2$ is a second order initial-value problem, since the conditions $y(0) = 1$, $y'(0) = 2$ are specified at a single point, $x = 0$. But the problem $y'' - y' + 2y = 0$, $y(0) = -2$, $y(1) = 2$ is a second order boundary-value problem. Here the boundary conditions are specified at two distinct values of x viz., $x = 0$ and $x = 1$.

1.5 Formation of Differential Equation

A differential equation is formed in two different ways. One is by eliminating arbitrary constants from a relation in the variables (independent and dependent) and constants. The other way is by formulation of the geometrical or physical problem as per some mathematical models.

1.5.1 Differential Equation of a Family of Curves

Suppose there is a differential equation containing n arbitrary constants. Then by differentiating it successfully n times, we get n equations containing n arbitrary constants and derivatives. By eliminating n constants, we get a differential equation of order n.

EXAMPLE 1.5.1

(i) Find the differential equation of the family of curves $y = e^x\ (A\cos x + B\sin x)$, where A and B are arbitrary constants.

(ii) Form a differential equation by eliminating the parameters A and B from the equation $y = A\cos x + B\sin x + x\sin x$.

Solution

(i) Here $y = e^x(A\cos x + B\sin x)$.
Differentiating with respect to x, we get

$$\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$$= y - e^x(A\sin x - B\cos x)$$

Again differentiating, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} - e^x(A\sin x - B\cos x) - e^x(A\cos x + B\sin x)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y$$

or

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

(ii) Differentiating the given equation twice with respect to x, we get

$$y_1 = -A\sin x + B\cos x + \sin x + x\cos x$$

and

$$y_2 = -A\cos x - B\sin x + \cos x + \cos x - x\sin x$$

$$= -(A\cos x + B\sin x + x\sin x) + 2\cos x = -y + 2\cos x$$

or

$$y_2 + y = 2\cos x$$

EXAMPLE 1.5.2

(i) Construct a differential equation by the elimination of the constants a and b from the equation $ax^2 + by^2 = 1$.

(ii) If $v = A + \frac{B}{r}$, where A, B are arbitrary constants, prove that $\dfrac{d^2v}{dr^2} + \dfrac{2}{r}\dfrac{dv}{dr} = 0$.

Solution

(i) Differentiating $ax^2 + by^2 = 1$ with respect to x, $2ax + 2byy_1 = 0$.

or
$$ax + byy_1 = 0 \tag{i}$$

Again differentiating, we get
$$a + by_1^2 + byy_2 = 0$$

or
$$a + b(y_1^2 + yy_2) = 0$$

or
$$a = -b(y_1^2 + yy_2)$$

Putting this value in (i), we get
$$-b(y_1^2 + yy_2)x + byy_1 = 0 \qquad \text{or} \qquad xyy_2 + xy_1^2 - yy_1 = 0$$

(ii) Differentiating $v = A + \frac{B}{r}$ with respect to r twice, we get
$$\frac{dv}{dr} = -\frac{B}{r^2} \qquad \text{and} \qquad \frac{d^2v}{dr^2} = \frac{2B}{r^3}$$

Now
$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = \frac{2B}{r^3} - \frac{2B}{r^3} = 0$$

EXAMPLE 1.5.3

(i) Obtain the differential equation whose general solution is $ax + by + c = 0$, where a, b, c are arbitrary constants.

(ii) Find the differential equation of the curve $y = A\cos(px - B)$, where A and B are the parameters and p is a constant.

Solution

(i) Differentiating $ax + by + c = 0$ with respect to x, we get $a + b\dfrac{dy}{dx} = 0$, or $\dfrac{dy}{dx} = -\dfrac{a}{b}$.

Again differentiating with respect to x, $\frac{d^2y}{dx^2} = 0$.

Thus the required differential equation is $\dfrac{d^2y}{dx^2} = 0$.

(ii) Differentiating $y = A\cos(px - B)$ twice, we get

$$y_1 = -A\sin(px - B)p$$

and

$$y_2 = -Ap^2\cos(px - B) = -p^2\{A\cos(px - B)\} = -p^2 y$$

or

$$y_2 + p^2 y = 0$$

EXAMPLE 1.5.4

(i) Find the differential equation of all the straight lines passing through the point $(-2, 3)$.

(ii) Find the differential equation of all circles having constant radius a.

Solution

(i) The equation of straight lines passing through $(-2, 3)$ are

$$y - 3 = m(x + 2) \tag{i}$$

Differentiating, we get $y_1 = m$. Putting the value of m in (i), we have $y - 3 = y_1(x + 2)$.

(ii) Let the equation of the circle of constant radius a be

$$x^2 + y^2 = a^2$$

Differentiating, we get $2x + 2yy_1 = 0$ or $x + yy_1 = 0$, which is the required differential equation.

EXAMPLE 1.5.5

(i) Find the differential equation of all circles touching the x-axis at the origin.

(ii) Find the differential equation of the system of circles having constant radii and centres lying on the x-axis.

(iii) Find the differential equation of all parabolas having their axes parallel to the y-axis.

(iv) Show that the differential equation of all parabolas with the foci at the origin and axis along the x-axis is given by $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$.

Solution

(i) Since the circles touch the x-axis at the origin, so their centre lies on the y-axis. Let the centre be $(0, a)$.

Then the equation is

$$x^2 + (y - a)^2 = a^2 \qquad \text{or} \qquad x^2 + y^2 - 2ay = 0 \tag{i}$$

Differentiating with respect to x, we get

$$2x + 2yy_1 - 2ay_1 = 0 \qquad \text{or} \qquad a = (x + yy_1)/y_1$$

Putting in (i), we have

$$x^2 + y^2 - 2y(x + yy_1)/y_1 = 0 \qquad \text{or} \qquad y_1(x^2 + y^2) - 2y(x + yy_1) = 0$$

(ii) Let the centre of the circles be $(a, 0)$ with constant radius r. Then the equations are

$$(x - a)^2 + y^2 = r^2 \tag{i}$$

Differentiating with respect to x, we get

$$2(x - a) + 2yy_1 = 0 \qquad \text{or} \qquad (x - a) = -yy_1$$

Putting the value of $(x - a)$ in (i), we get

$$y^2y_1^2 + y^2 = r^2 \qquad \text{or} \qquad y^2(1 + y_1^2) = r^2$$

(iii) The equations of the parabolas whose axes are parallel to y-axis are $y = ax^2 + bx + c$. Differentiating twice, we get $y_1 = 2ax + b$ and $y_2 = 2a$. Again differentiating, we finally have $y_3 = 0$. This is the required differential equation.

(iv) The equation of the parabola whose foci is at the origin and axis is along the x-axis is

$$y = 4a(x + a) \tag{i}$$

where a is the parameter.

Differentiating with respect to x, we get $2yy_1 = 4a$, or $2a = yy_1$.

Putting the value of a in (i), we get

$$y^2 = 2yy_1\left(x + \frac{1}{2}yy_1\right) = 2xyy_1 + (yy_1)^2$$

or $yy_1^2 + 2xy_1 - y = 0$, which is the required differential equation.

EXAMPLE 1.5.6 Find the differential equation of the family of circles touching the x-axis at the origin.

Solution Let the equation of circle passing through the origin be

$$x^2 + y^2 + 2gx + 2fy = 0 \tag{i}$$

If it touches the x-axis then $y = 0$. That is, $\dfrac{dy}{dx} = 0$ at (0,0).

Now, from (i), $2x + 2y\dfrac{dy}{dx} + 2g + 2f\dfrac{dy}{dx} = 0$, or $2g = 0$, or $g = 0$.

Then (i) becomes

$$x^2 + y^2 + 2fy = 0 \tag{ii}$$

Differentiating (ii) with respect to x,

$$2x + 2yy_1 + 2fy_1 = 0, \text{ or } x + yy_1 + fy_1 = 0, \text{ or } f = -(x + yy_1)/y_1$$

Putting this value in (ii), we get

$$x^2 + y^2 - 2y\left(\frac{x + yy_1}{y_1}\right) = 0 \qquad \text{or} \qquad y_1(x^2 + y^2) - 2xy - 2y^2y_1 = 0$$

or

$$y_1(x^2 - y^2) - 2xy = 0 \qquad \text{or} \qquad (x^2 - y^2)\,dy - 2xy\,dx = 0$$

which is the required differential equation.

EXAMPLE 1.5.7 Find the differential equation whose two independent solutions are $\cos x$ and $\sin x$.

Solution The general solution of the required differential equation is

$$y = c_1 \cos x + c_2 \sin x \tag{i}$$

Differentiating (i) with respect to x, we get

$$y' = -c_1 \sin x + c_2 \cos x \tag{ii}$$

Again differentiating (ii) with respect to x we obtain

$$y'' = -c_1 \cos x - c_2 \sin x \tag{iii}$$

Adding (iii) and (i), we get the required differential equation as

$$y'' + y = 0$$

1.5.2 Physical Origins of Differential Equations

Differential equations are used to model different type of problems which appear in engineering, physics, economics and many other fields. These equations are studied in different perspectives. For examples, engineers are tried to model a problem in terms of differential equation, and mathematicians are mainly involved to find out the solution of a differential equation.

EXAMPLE 1.5.8 Suppose a particle of mass m is falling freely under the influence of gravity only. If x is the distance travelled by the particle and if we assume that the upward direction is positive, then by Newton's law the equation is

$$m\frac{d^2x}{dt^2} = -mg \qquad \text{or} \qquad \frac{d^2x}{dt^2} = -g \tag{1.7}$$

The negative sign is used since the weight of the body is a force directed opposite to the positive direction.

EXAMPLE 1.5.9 The human population growth can also be modelled using differential equations. A very common assumption of the Malthusian model is that the rate at which a population grows at a certain time is proportional to the total population at that time. In mathematical form, if $N(t)$ denotes the total population at time t, then this assumption can be expressed as

$$\frac{dN}{dt} \propto N \qquad \text{or} \qquad \frac{dN}{dt} = kN(t) \tag{1.8}$$

where k is the constant of proportionality.

EXAMPLE 1.5.10 We know that Newton's law of cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Let $T(t)$ be the temperature of a body, and T_0 denote the constant temperature of the surrounding medium. Then the rate at which the body cools is $\frac{dT(t)}{dt}$, which is proportional to $T(t) - T_0$ according to Newton's law of cooling. That is

$$\frac{dT(t)}{dt} = \alpha[T(t) - T_0] \tag{1.9}$$

where α is the constant of proportionality.

EXAMPLE 1.5.11 Consider the single-loop series circuit containing an inductor, resistor and capacitor shown in Fig. 1.1. L, C and R denote inductance, capacitance and resistance, and they are all constants.

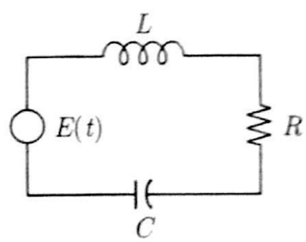

Figure 1.1: Example 1.5.11.

Let $i(t)$ denote the amount of current after the switch is closed, and let the charge on a capacitor at time t be denoted by $q(t)$. Let $E(t)$ denote the impressed voltage on a closed loop. $i(t)$ and $q(t)$ are related by $i = \frac{dq}{dt}$. The voltage drops across an inductor, a resistor and a capacitor are

$$L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

$$iR = R\frac{dq}{dt}$$

and

$$\frac{1}{C}q$$

respectively. According to Kirchhoffs' second law, the impressed voltage $E(t)$ on a closed loop must be equal to the sum of the voltage drops in the loop. Therefore,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \tag{1.10}$$

This is a second order differential equation.

EXERCISES

Section A Multiple Choice Questions

1. The degree and order of the differential equation $\left[1 + \left(\frac{d^3y}{dx^3}\right)^2\right]^{\frac{1}{2}} = 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2$ are

   (a) 2, 3 (b) 2, 2 (c) 3, 3 (d) 3, 2.

2. The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y$ are

   (a) 2, 2 (b) 2, 1 (c) 1, 2 (d) 1, 1. (WBUT 2007)

3. The degree and order of the differential equation

$$\left(\frac{d^2y}{dx^2} + 2\right)^{3/2} = x\frac{dy}{dx}$$

   are

   (a) degree $= \frac{3}{2}$, order $= 2$ (b) degree $= 2$, order $= 3$ (c) degree $= 3$, order $= 2$
   (d) degree $= 2$, order $= 1$. (WBUT 2006)

4. The order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + 5\left(\frac{dy}{dx}\right)^3 + 3y = x^2\log x + \sin^2 x$$

   are

   (a) 2, 2 (b) 2, 3 (c) 3, 2 (d) 3, 3.

5. The order and degree of the differential equation $\frac{d^3y}{dx^3} = \left[y + \left(\frac{d^2y}{dx^2}\right)^3\right]^{1/5}$ are

   (a) 3, 5 (b) 5, 3 (c) 3, 1 (d) 3, $\frac{1}{5}$.

6. The degree and order of the differential equation $\left(\frac{d^2y}{dx^2}\right)^{1/4} + 5\left(\frac{dy}{dx}\right)^{1/3} = 0$ are

   (a) $\frac{1}{4}$, 2 (b) $\frac{1}{3}$, 2 (c) 3, 2 (d) $\frac{1}{4}$, $\frac{1}{3}$.

7. The order of the differential equation whose general solution $x = a\sin(\omega t + b)$, where ω is a constant, is

   (a) 0 (b) 1 (c) 2 (d) 3.

8. The order of the differential equation whose general solution $y = ae^{2x} + be^{-3x} + ce^{2x}$ is

   (a) 0 (b) 1 (c) 2 (d) 3.

9. The general solution of the differential equation $y'' + 9y = 0$ is
   (a) $Ae^{3x} + Be^{-3x}$ (b) $(A + Bx)e^{3x}$ (c) $A\cos 3x + B\sin 3x$
   (d) $(A + Bx)\sin 3x$. (WBUT 2008)

10. The general solution of the ordinary differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$ (where A and B are arbitrary constants) is
    (a) $Ae^{2x} + Be^{-2x}$ (b) $(A + Bx)e^{2x}$ (c) $A\cos 2x + B\sin 2x$
    (d) $(A + Bx)\cos 2x$. (WBUT 2006)

11. The differential equation of the family of curves $y = c_1/x + c_2$ is
    (a) $y'' + \frac{2}{x}y' = 0$ (b) $y'' + 2y' = 0$ (c) $y' + \frac{2}{x}y = 0$ (d) $y'' - y = 0$.

12. The differential equation of the family of straight lines $y = ax + b$ is
    (a) $y' - y = 0$ (b) $y'' + y = 0$ (c) $y' = 0$ (d) $y'' = 0$.

13. The differential equation of the family of straight lines $y = mx$ is
    (a) $y' = 0$ (b) $y'' = 0$ (c) $xy' = y$ (d) $xy = y'$.

14. The differential equation of the family of curves $y = ce^{2x}$ is
    (a) $y'' - 2y = 0$ (b) $y' - 2y = 0$ (c) $y' + 2y = 0$ (d) $y'' + y = 0$.

15. The family of curves $y = cx + c^2 + c^3$ is a solution of a differential equation of degree
    (a) 1 (b) 2 (c) 3 (d) 4.

16. The differential equation whose two independent solutions e^x and e^{-x} is
    (a) $y' - y = 0$ (b) $y'' - y = 0$ (c) $y''' - 2y'' + y' = 0$ (d) $y'' + y = 0$.

17. If the set of independent solutions of a differential equation is $\{e^x, xe^x, x^2e^x\}$, then its order is
    (a) 1 (b) 2 (c) 3 (d) 4.

18. The differential equation $yy'' + xy' + 5y = 0$ is
    (a) linear (b) non-linear.

19. The differential equation $x^2y'' + x^3y' + 3y = \cos x$ is
    (a) linear (b) non-linear.

20. If the rate of change of momentum of a particle of mass m (when particle is moving with velocity v) is proportional to the applied force F, then the corresponding differential equation (for proportional constant k) is
    (a) $m\dfrac{dv}{dt} = kF$ (b) $\dfrac{d}{dt}(mv) = kF$ (c) $\dfrac{d^2}{dt^2}(mv) = kF$ (d) $\dfrac{dF}{dt} = kmv$.

21. $y = e^x - 1$ is a solution of the differential equation
    (a) $y'' - 2y + 1 = 0$ (b) $y' = e^x + 2$ (c) $y' - y = 1$ (d) $y' - 2y = 0$.

Section B Review Questions

1. State whether the following differential equations are linear or non-linear and write the order of each equation:

(a) $(1-x^2)y'' - 16xy' + 2y = \sin x$ (b) $x^2\dfrac{d^3y}{dx^3} - 5\left(\dfrac{dy}{dx}\right)^2 + 2y = 0$

(c) $yy' + 2y = 3 + x^3$ (d) $\dfrac{d^2y}{dx^2} + \dfrac{1}{y} = \cos y$

(e) $\dfrac{dy}{dx} = \left[1 + \left(\dfrac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{2}}$ (f) $\dfrac{d^2r}{dt^2} = -\dfrac{k}{r^2}$

2. Form the differential equations from the following equations:
   (a) $y = e^x(A\cos x + B\sin x)$ (b) $xy = Ae^x + Be^{-x}$
   (c) $c_1y^2 + 4y = 2x^2$ (d) $y^2 = 4c_1(x + c_1)$
   (e) $y = c_1 + c_2\log x$ (f) $r = c_1(1 + \cos\theta)$
   (g) $Ax^2 + By^2 = 1$ (h) $y = a\sin x + b\cos x + x\sin x$
3. Find the differential equation whose three independent solutions are e^x, xe^x, x^2e^x.
4. Find the differential equation of a family of circles passing through the origin.
5. Find the differential equation of a family of circles passing through the origin with their centres on the x-axis.
6. Find the differential equation of a family of circles whose centres are on the y-axis and which touch the x-axis.
7. Find the differential equations of all circles in the XOY plane which have their centres on the x-axis and have given radii.
8. Find the differential equation of all circles having constant radius a.
9. Find the differential equation of all parabolas whose axes are parallel to the y axis.
10. Find the differential equation of a family of parabolas whose vertices and foci are on the x-axis.
11. Show that the differential equation of a general parabola $a^2x^2 + 2abxy + b^2y^2 + 2gx + 2fy + c = 0$ is

$$\frac{d^2}{dx^2}\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right] = 0$$

12. Show that the differential equation corresponding to the family of curves $x^2 + y^2 + 2c_1x + 2c_2y + c_3 = 0$, where c_1, c_2, c_3 are arbitrary constants, is

$$\frac{d^3y}{dx^3}\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2$$

13. The equation to a system of confocal ellipses is $\dfrac{x^2}{a^2+k} + \dfrac{y^2}{b^2+k} = 1$, where k is an arbitrary constant. Find the corresponding differential equation.
14. Obtain the differential equation for the velocity v of a body of mass m falling vertically downward through a medium offering a resistance proportional to the square of the instantaneous velocity.

15. Find the differential equation of all rectangular hyperbolas which have the axes of coordinates as asymptotes.

16. If $y = y_1(x)$ and $y = y_2(x)$ are two independent solutions of the following differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

then show that $y = c_1y_1(x) + c_2y_2(x)$, where c_1, c_2 are arbitrary constants, is also a solution of the given differential equation.

17. A spherical rain drop evaporates at a rate proportional to its surface area. Write a differential equation which gives the formula for its volume V as a function of time.

18. The growth rate of a population of bacteria is directly proportional to the population. The number of bacteria in a culture grow from 100 to 400 in 24 hours. Write down the initial value problem which helps to determine the population after 12 hours.

19. If u and v are two particular solutions of $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x, then show that $y = k(u - v) + v$, where k is any constants, is a solution of the given differential equation.

Answers

Section A Multiple Choice Questions

1. (a)	2. (b)	3. (c)	4. (a)	5. (a)	6. (c)	7. (c)	8. (d)	9. (c)
10. (c)	11. (a)	12. (d)	13. (c)	14. (b)	15. (a)	16. (b)	17. (c)	18. (b)
19. (a)	20. (b)	21. (c)						

Section B Review Questions

1. (a) linear, 2 (b) linear, 3 (c) non-linear, 1

   (d) non-linear, 2 (e) linear, 2 (f) linear, 2

2. (a) $y'' - 2y' + 2y = 0$

   (b) $xy'' + 2y' - xy = 0$

   (c) $(x^2 - y)y' = xy$

   (d) $y[1 - y'^2] = 2xy'$

   (e) $xy'' + y' = 0$

   (f) $(1 + \cos\theta)\frac{dr}{d\theta} + r\sin\theta = 0$

   (g) $x[yy'' + y'^2] = yy'$

   (h) $y'' + y = 2\cos x$

3. $y''' - 3y'' + 3y' - y = 0$

4. $xy' - y = 0$

5. $2xyy' = y^2 - x^2$
6. $(x^2 - y^2)y' = 2xy$
7. $\pm\sqrt{(a^2 - y^2)} + yy' = 0$, a is the given radius
8. $x + yy' = 0$
9. $y''' = 0$
10. $yy'' + (y')^2 = 0$
13. $(x^2 - y^2) + xy(p - \frac{1}{p}) = a^2 - b^2$, where $p = \frac{dy}{dx}$
14. $\frac{dv}{dt} + k\frac{v^2}{m} = g$
15. $xy' + y = 0$
17. $\frac{dV}{dt} = -cV^{2/3}$, c is constant
18. $\frac{dN}{dt} = kt$, $N(0) = 100$ (assumed)

Chapter 2

Differential Equations of First Order

2.1 Introduction

In the previous chapter, we have seen how a differential equation is constructed from geometrical problems as well as from real-life problems. It is also observed that the form of a differential equation is not simple; it may be linear or non-linear, and the degree and order are also one or more than one. It is not possible to solve a differential equation by a fixed method. The solution procedure of a first order and first degree differential equation is relatively easier than that of other differential equations. In this chapter, the differential equations of first order and first degree are considered.

2.2 Equations of First Order and of First Degree

The first order and first degree differential equation is of the form

$$\frac{dy}{dx} = f(x, y) \tag{2.1}$$

which is sometimes written as

$$M(x, y)dx + N(x, y)dy = 0 \tag{2.2}$$

There is no common method to solve the equation of the form (2.1) or (2.2). However, some special methods are applied to the following types of equations

(a) Equations in which variables are separable

(b) Homogeneous equations

(c) Linear equations

(d) Exact equations

The equations of the types (c) and (d) will only be discussed here. Other types of equations are not within the scope of this book.

2.3 Linear Equations

A differential equation is said to be **linear** if the dependent variable y and its differential coefficients occur only in the first degree and are not multiplied together.

Thus the standard form of a linear differential equation of first order is of the form

$$\frac{dy}{dx} + Py = Q \tag{2.3}$$

where P and Q are functions of x only. This equation is also known as **Leibnitz's linear equation**. This type of equation can be solved by making the left hand side a perfect differential after multiplying a suitable function $I(x)$, called the **integrating factor (I.F.)**.

Multiplying both sides of (2.3) by $I(x)$, so that

$$I(x)\frac{dy}{dx} + I(x)Py = I(x)Q \tag{2.4}$$

Let us assume that

$$I\frac{dy}{dx} + IPy = \frac{d}{dx}(Iy)$$

Then

$$I\frac{dy}{dx} + IPy = I\frac{dy}{dx} + y\frac{dI}{dx}; \text{ this gives } \frac{dI}{dx} = IP$$

That is, $I = e^{\int P dx}$.

Thus Eq. (2.4) becomes

$$e^{\int Pdx}\frac{dy}{dx} + e^{\int Pdx}Py = e^{\int Pdx}Q$$

That is,

$$\frac{d}{dx}(ye^{\int Pdx}) = e^{\int Pdx}Q$$

After integration it becomes

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c \qquad \text{or} \qquad y = e^{-\int Pdx}\int e^{\int Pdx}Q\,dx + ce^{-\int Pdx} \tag{2.5}$$

which is the required solution.

Note 2.3.1 The factor $e^{\int Pdx}$ is called the **integrating factor**, which makes the LHS an exact differential.

Note 2.3.2 Sometimes a differential equation becomes linear if we take x as the dependent variable and y as the independent variable, i.e. it can be written as

$$\frac{dx}{dy} + P_1 x = Q_1 \tag{2.6}$$

where P_1 and Q_1 are constants or functions of y only. In this case the I.F. $= e^{\int P_1 dy}$ and the solution is given by

$$xe^{\int P_1\,dy} = \int Q_1\,e^{\int P_1\,dy}dy + c$$

Steps to solve linear differential equation:

Step 1: Express the first order and first degree differential equation of the form (2.3).

Step 2: Identify $P(x)$ and compute I.F. $I(x) = e^{\int P\,dx}$.

Step 3: Multiply Eq. (2.3) by $I(x)$.

Step 4: The solution is $yI(x) = \int Q\,I(x)\,dx + c$.

EXAMPLE 2.3.1 Solve $(1-x^2)\dfrac{dy}{dx} + 2xy = x(1-x^2)^{\frac{1}{2}}$.

Solution The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x\sqrt{1-x^2}}{1-x^2}$$

Here

$$P = \frac{2x}{1-x^2} \text{ and } Q = \frac{x\sqrt{1-x^2}}{1-x^2}$$

Therefore, the integrating factor is

$$e^{\int P\,dx} = e^{\int \frac{2x}{1-x^2}\,dx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

Hence, the solution of the given equation is

$$ye^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$$

or

$$y\frac{1}{1-x^2} = \int \frac{x\sqrt{1-x^2}}{1-x^2}\,\frac{1}{1-x^2}\,dx + c$$

We substitute, $z = 1-x^2$ to the RHS. Then $dz = -2xdx$.

$$\therefore \quad \frac{y}{1-x^2} = -\int \frac{1}{2}\,\frac{\sqrt{z}}{z^2}\,dz + c = \frac{1}{\sqrt{z}} + c = \frac{1}{\sqrt{1-x^2}} + c$$

or $y = \sqrt{1-x^2} + c(1-x^2)$, where c is arbitrary constant.

EXAMPLE 2.3.2 Solve $(1+x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1}x}$.

Solution The given equation can be written as

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$$

The integrating factor (I.F.) is $e^{\int \frac{1}{1+x^2}\,dx} = e^{\tan^{-1}x}$.
Multiplying both sides by I.F., we get

$$\frac{d}{dx}\left(ye^{\tan^{-1}x}\right) = \frac{e^{\tan^{-1}x}}{1+x^2}e^{\tan^{-1}x}$$

Integrating

$$ye^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2} e^{\tan^{-1}x}\, dx + c$$

Substituting $e^{\tan^{-1}x} = z$, we get

$$\frac{e^{\tan^{-1}x}}{1+x^2}\, dx = dz$$

Thus

$$ye^{\tan^{-1}x} = \int z\, dz = \frac{z^2}{2} + c \qquad \text{or} \qquad 2ye^{\tan^{-1}x} = e^{2\tan^{-1}x} + 2c$$

where c is arbitrary constant.

EXAMPLE 2.3.3 Solve $(1+x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$.

Solution The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

$$\text{I.F.} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$$

Multiplying both sides by I.F., we get

$$\frac{d}{dx}\{y(1+x^2)\} = 4x^2$$

Integrating

$$y(1+x^2) = \int 4x^2\, dx + c \qquad \text{or} \qquad y(1+x^2) = \frac{4}{3}x^3 + c$$

EXAMPLE 2.3.4 Solve $y^2 + \left(x - \dfrac{1}{y}\right)\dfrac{dy}{dx} = 0$.

Solution The given equation can be written as

$$y^2\frac{dx}{dy} + x = \frac{1}{y} \qquad \text{or} \qquad \frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3}$$

This is a linear equation in x and I.F. $= e^{\int \frac{1}{y^2}dy} = e^{-\frac{1}{y}}$
Multiplying the above equation by I.F. and integrating, we obtain

$$xe^{-\frac{1}{y}} = \int e^{-\frac{1}{y}}\frac{1}{y^3}\, dy = -\int z\, e^{-z}\, dz \qquad (\text{where } z = 1/y)$$

$$= -\left[-ze^{-z} + \int e^{-z}\, dz\right] = ze^{-z} + e^{-z} + c$$

$$= e^{-1/y}\left(\frac{1}{y} + 1\right) + c$$

or

$$x = \left(\frac{1}{y} + 1\right) + ce^{1/y}$$

EXAMPLE 2.3.5 Solve $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$.

Solution The given equation can be written as

$$\frac{dx}{dy} + \frac{1}{1 + y^2}x = \frac{\tan^{-1} y}{1 + y^2}$$

This is a linear equation in x and I.F. $= e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$.
Multiplying the given equation by I.F. and integrating, we get

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1 + y^2} e^{\tan^{-1} y}\, dy$$

Substituting $\tan^{-1} y = z$. Then $\frac{1}{1+y^2}\,dy = dz$.
Therefore

$$\begin{aligned} xe^{\tan^{-1} y} &= \int z e^z\, dz = (z - 1)e^z + c \\ &= (\tan^{-1} y - 1)e^{\tan^{-1} y} + c \end{aligned}$$

or

$$x = (\tan^{-1} y - 1) + ce^{-\tan^{-1} y}$$

2.3.1 Bernoulli's Equation

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n \tag{2.7}$$

where P and Q are constants or functions of x only and n is constant other than 0 and 1, is called **Bernoulli's equation.**

This equation can be reduced to the linear form by suitable substitution.

Divide (2.7) by y^n, so that $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$.

Let $y^{1-n} = z$, so that $(1 - n)y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

Thus Eq. (2.7) reduces to

$$\frac{1}{-n + 1}\frac{dz}{dx} + Pz = Q \qquad \text{or} \qquad \frac{dz}{dx} + (1 - n)Pz = (1 - n)Q$$

which is a linear equation in z and can be solved easily.

EXAMPLE 2.3.6 Solve $x\frac{dy}{dx} + y = y^2 \log x$. (WBUT 2008)

Solution This equation can be written as

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x}$$

Substituting $\frac{1}{y} = z$. Then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$.
Then the above equation reduces to

$$-\frac{dz}{dx} + \frac{1}{x}z = \frac{\log x}{x} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x}z = -\frac{\log x}{x}$$

This is a linear equation in z and I.F. $= e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$.
Multiplying the above equation by I.F. and integrating, we obtain

$$z\left(\frac{1}{x}\right) = \int -\frac{\log x}{x^2}\,dx = \frac{1}{x}\log x - \int \frac{1}{x}\frac{1}{x}\,dx \quad \text{(integration by parts)}$$
$$= \frac{1}{x}\log x + \frac{1}{x} + c$$

or

$$z = \log x + 1 + cx = \log(xe) + cx$$

or

$$\frac{1}{y} = cx + \log(ex)$$

or

$$y[cx + \log(ex)] = 1$$

2.3.2 Equations Reducible to Linear Form

If the equation is of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \tag{2.8}$$

where P, Q are functions of x, then it can be reduced to the linear form by substituting $f(y) = z$.

EXAMPLE 2.3.7 Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ (WBUT 2007)

Solution This equation can be written as

$$\cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$$

Substituting $\sin y = z$. Then $\cos y\frac{dy}{dx} = \frac{dz}{dx}$.

Then the above equation reduces to

$$\frac{dz}{dx} - \frac{1}{1+x}z = (1+x)e^x$$

which is linear equation in z and I.F. $= e^{-\int \frac{1}{1+x}\,dx} = \dfrac{1}{1+x}$.
Multiplying the above equation by I.F. and integrating, we get

$$z\frac{1}{1+x} = \int e^x\,dx = e^x + c$$

That is, $\sin y = (1+x)(e^x + c)$ is the required solution.

EXAMPLE 2.3.8 Solve $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

Solution Dividing this equation by $\cos^2 y$

$$\sec^2 y\frac{dy}{dx} + 2x\frac{\sin y\cos y}{\cos^2 y} = x^3$$

or

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

Substituting, $\tan y = z$, so that $\sec^2 y\dfrac{dy}{dx} = \dfrac{dz}{dx}$.
Then the above equation reduces to

$$\frac{dz}{dx} + 2xz = x^3$$

which is linear equation in z and I.F. $= e^{\int 2x\,dx} = e^{x^2}$.
Multiplying this linear equation by I.F. and integrating, we obtain

$$ze^{x^2} = \int x^3 e^{x^2}\,dx + c = \frac{1}{2}(x^2-1)e^{x^2} + c$$

or

$$z = \frac{1}{2}(x^2-1) + ce^{-x^2} \quad \text{or} \quad \tan y = \frac{1}{2}(x^2-1) + ce^{-x^2}$$

EXAMPLE 2.3.9 Solve $\dfrac{dy}{dx} + \dfrac{y\log y}{x} = \dfrac{y(\log y)^2}{x^2}$. (WBUT 2007)

Solution Dividing by $y(\log y)^2$, the given equation reduces to

$$\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x(\log y)} = \frac{1}{x^2}$$

Substituting $\dfrac{1}{\log y} = z$. Then $-\dfrac{1}{y(\log y)^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.
Therefore, the above equation becomes

$$\frac{dz}{dx} - \frac{1}{x}z = -\frac{1}{x^2}$$

This is linear equation in z and I.F. $= e^{\int -\frac{1}{x}dx} = \frac{1}{x}$.
Multiplying the above equation by I.F. and integrating we obtain

$$z\frac{1}{x} = \int -\frac{1}{x^3}\,dx = \frac{1}{2}\frac{1}{x^2} + c$$

or

$$\frac{1}{x\log y} = \frac{1}{2x^2} + c \qquad \text{or} \qquad (1+2cx^2)\log y = 2x$$

EXAMPLE 2.3.10 Solve $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$.

Solution The given equation can be written as

$$x^2y\frac{dy}{dx} = xy^2 - e^{1/x^3} \qquad \text{or} \qquad y\frac{dy}{dx} - \frac{1}{x}y^2 = -\frac{1}{x^2}e^{1/x^3}$$

Substituting $y^2 = z$. Then $2y\dfrac{dy}{dx} = \dfrac{dz}{dx}$. Therefore

$$\frac{dz}{dx} - \frac{2}{x}z = -\frac{2}{x^2}e^{1/x^3}$$

which is a linear equation in z and I.F. $= e^{-\int 2/x\,dx} = \dfrac{1}{x^2}$.
Multiplying by I.F. and integrating

$$z\frac{1}{x^2} = \int -\frac{2}{x^4}e^{1/x^3}\,dx$$

Substituting $1/x^3 = u$ to the RHS.
Then $-\dfrac{3}{x^4}dx = du$. Therefore

$$z\frac{1}{x^2} = \frac{2}{3}\int e^u\,du = \frac{2}{3}e^u + c \qquad \text{or} \qquad 3y^2 = 2x^2e^{1/x^3} + cx^2$$

2.3.3 Applications of First Order Linear Equations

EXAMPLE 2.3.11 Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y = 2(e^x - x - 1)$.

Solution Given that $\dfrac{dy}{dx} = y + 2x$ or $\dfrac{dy}{dx} - y = 2x$. This is a linear equation and I.F. $= e^{-\int dx} = e^{-x}$.
Therefore, its solution is $ye^{-x} = \int 2xe^{-x}\,dx = -2e^{-x}(x+1) + c$
It passes through (0,0), so that $c = 2$.
Hence the required curve is $y = 2(e^x - x - 1)$.

EXAMPLE 2.3.12 Show that the curve in which the portion of the tangent included between the coordinate axes is bisected by the point of contact is a rectangular hyperbola.

Solution Let (x, y) be a point on the curve. The equation of tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

or

$$Y - Xy_1 = y - xy_1$$

or

$$\frac{Y}{y - xy_1} + \frac{X}{-(y - xy_1)/y_1} = 1$$

The points of intersection between the tangent and coordinates axes are respectively $(\frac{y-xy_1}{-y_1}, 0)$ and $(0, y - xy_1)$.

Since (x, y) is the middle point of the line segment joining these two points, therefore, $(y - xy_1)/(-y_1) = 2x$ and $y - xy_1 = 2y$.
Dividing, we get

$$2y/(-y_1) = 2x, \qquad \text{or} \qquad y = -x\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{y} = -\frac{dx}{x}$$

Integrating, $\log y = -\log x + \log c$, or $xy = c$, which represents a rectangular hyperbola.

EXAMPLE 2.3.13 A particle P moves so that its component velocities parallel to the axes of x and y are respectively $-ky$ and kx, where k is a constant other than zero. Find the path of the particle if it passes through the point (3, 4).

Solution The differential equations of the path are

$$\frac{dx}{dt} = -ky \qquad \text{and} \qquad \frac{dy}{dt} = kx$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = -\frac{x}{y}$$

That is, $ydy + xdx = 0$. Integrating, we get $x^2 + y^2 = c$.
If it passes through the point (3, 4), therefore $c = 9 + 16 = 25$.
Hence the required solution is $x^2 + y^2 = 25$.

EXAMPLE 2.3.14 Radium decomposes at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years, how much will remain at the end of 100 years?

Solution Let x be the amount of radium at time t. Then the differential equation is $\frac{dx}{dt} = -kx$, k is proportional constant, i.e. $\frac{dx}{x} = -k\,dt$. This gives $\log x = -kt + \log c$,
or

$$x = ce^{-kt} \tag{i}$$

Let A be the amount of radium when $t = 0$.
$\therefore$ $A = c$, (i) becomes $x = Ae^{-kt}$.

Given that 5% of the original amount disappears in 50 years, i.e. 95% of the original amount present in 50 years. Therefore

$$0.95A = Ae^{-50k} \quad \text{or} \quad e^{-50k} = 0.95 \tag{ii}$$

When $t = 100$, then let $x = X$.
$\therefore \quad X = Ae^{-100k}$, or $X = A(0.95)^2 = 0.9025A$.
Hence 90.25% of the original amount will remain at the end of 100 years.

2.4 Exact Differential Equations

A differential equation of the form $Mdx + Ndy = 0$, where M and N are functions of x and y, is said to be **exact** if its left hand member is the exact differential of some function $u(x, y)$. i.e. $du \equiv Mdx + Ndy = 0$ and the solution is $u(x, y) = c$.

Theorem 2.1 *The necessary and sufficient condition for the differential equation* $Mdx + Ndy = 0$ *to be exact is that* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Proof. The condition is necessary:
Let $Mdx + Ndy = 0$ is exact. Then by definition

$$Mdx + Ndy = du \tag{2.9}$$

where u is a function of x and y.
Again, by Chain rule

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{2.10}$$

Therefore

$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Equating the coefficients of dx and dy, we get $M = \dfrac{\partial u}{\partial x}$ and $N = \dfrac{\partial u}{\partial y}$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Hence $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, assuming $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

The condition is sufficient:

Let $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. We have to show that $Mdx + Ndy = 0$ is exact.

Let $P = \int M\,dx$, where y is taken as constant while performing the integration. Then $\dfrac{\partial P}{\partial x} = M$, so that $\dfrac{\partial^2 P}{\partial y \partial x} = \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ by assumption.

That is, $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial P}{\partial y}\right)$.

Integrating both sides with respect to x (taking y as constant), we get

$$N = \frac{\partial P}{\partial y} + \phi(y)$$

where $\phi(y)$ is a function of y only.

Thus

$$\begin{aligned} M dx + N dy &= \frac{\partial P}{\partial x} dx + \left[\frac{\partial P}{\partial y} + \phi(y)\right] dy \\ &= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy\right) + \phi(y) dy \\ &= dP + d\psi(y) \qquad \text{where } d\psi(y) = \phi(y) dy \\ &= d[P + \psi(y)] \end{aligned} \tag{2.11}$$

which shows that $M dx + N dy = 0$ is exact.

Note 2.4.1 From Eq. (2.11), $M dx + N dy = 0$ becomes $d[P + \psi(y)] = 0$. Integrating, $P + \psi(y) = c$, or $P + \int \phi(y) dy = c$.

But

$$P = \underset{y \text{ is constant}}{\int M\,dx} \qquad \text{and} \qquad \phi(y) = \text{terms of } N \text{ not containing } x.$$

Thus the solution of $M dx + N dy = 0$ is given by

$$\underset{y \text{ is constant}}{\int M\,dx} \quad + \quad \underset{\text{terms free from } x}{\int N\,dy} = c$$

That is, the solution of an exact differential equation can be obtained by performing the following steps:

Step 1: Integrate M with respect to x, taking y as constant.

Step 2: Integrate the terms of N which do not contain x, with respect to y

Step 3: Add the two expressions obtained in Steps 1 and 2 and equate the result to an arbitrary constant.

EXAMPLE 2.4.1 Solve $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$.

Solution The given equation can be written as

$$(ax+hy+g)dx + (hx+by+f)dy = 0$$

Here $M = ax+hy+g$ and $N = hx+by+f$

Now, $\frac{\partial M}{\partial y} = h = \frac{\partial N}{\partial x}$, i.e. the given equation is exact.

$$\underset{y \text{ is constant}}{\int M dx} = \int (ax+hy+g)dx = a\frac{x^2}{2} + hxy + gx$$

$$\underset{\text{terms free from } x}{\int N dy} = \int (by+f)dy = b\frac{y^2}{2} + fy.$$

Hence the required solution is

$$a\frac{x^2}{2} + hxy + gx + b\frac{y^2}{2} + fy = \frac{k}{2}$$

or

$$ax^2 + by^2 + 2hxy + 2gx + 2fy = k, \ k \text{ is arbitrary constant}$$

EXAMPLE 2.4.2 Solve $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$.

Solution Here $M = y^2e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2$.

Now,

$$\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 2xye^{xy^2} = e^{xy^2}(2y + 2xy^3)$$

and

$$\frac{\partial N}{\partial x} = y(e^{xy^2} + xe^{xy^2}y^2) = e^{xy^2}(2y + 2xy^3)$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, i.e. the equation is exact.

Its solution is

$$\underset{y \text{ is constant}}{\int (y^2e^{xy^2} + 4x^3)\, dx} + \underset{\text{terms free from } x}{\int (2xye^{xy^2} - 3y^2)\, dy} = c$$

or

$$e^{xy^2} + x^4 + \int (-3y^2)\, dy = c$$

or

$$e^{xy^2} + x^4 - y^3 = c$$

EXAMPLE 2.4.3 Solve $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$.

Solution Here $M = 1 + e^{x/y}$, $N = e^{x/y}(1 - x/y)$ and $\dfrac{\partial M}{\partial y} = \dfrac{1}{y}e^{x/y} = \dfrac{\partial N}{\partial x}$.
Therefore, the given equation is exact and the solution is given by

$$\underset{y \text{ is constant}}{\int M dx} \quad + \quad \underset{\text{terms free from } x}{\int N dy} = c$$

$$\underset{y \text{ is constant}}{\int (1 + e^{x/y} dx + 0)} = c \qquad \text{or} \qquad x + ye^{x/y} = c$$

2.4.1 Equations Reducible to Exact Form

If the equation $Mdx + Ndy = 0$ is not exact, then it can be made exact by multiplying a suitable function of x and y. Such a function is called an **integrating factor.** But, there is no general method to find an integrating factor. Here some methods are discussed to find an I.F.

Rule I. If the differential equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $\dfrac{1}{Mx + Ny}$ is an integrating factor.

Proof. Since the equation $M\,dx + N\,dy = 0$ is homogenous, so M and N are homogeneous. Let M and N be the homogeneous functions of x and y of degree n. Then by Euler's theorem

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = nM \quad \text{and} \quad x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN \tag{i}$$

Multiplying the given equation by $\frac{1}{Mx+Ny}$, we get

$$\frac{M}{Mx + Ny}\,dx + \frac{N}{Mx + Ny}\,dy = 0 \quad \text{or} \quad M'\,dx + N'\,dy = 0 \tag{ii}$$

where $M' = \dfrac{M}{Mx + Ny}$ and $N' = \dfrac{N}{Mx + Ny}$.

Now

$$\frac{\partial M'}{\partial y} = \frac{\frac{\partial M}{\partial y}(Mx + Ny) - M\left(x\frac{\partial M}{\partial y} + y\frac{\partial N}{\partial y} + N\right)}{(Mx + Ny)^2}$$

$$= \frac{Ny\frac{\partial M}{\partial y} - My\frac{\partial N}{\partial y} - MN}{(Mx + Ny)^2}$$

$$\frac{\partial N'}{\partial x} = \frac{\frac{\partial N}{\partial y}(Mx + Ny) - N\left(M + x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x}\right)}{(Mx + Ny)^2}$$

$$= \frac{Mx\frac{\partial N}{\partial x} - Nx\frac{\partial M}{\partial x} - MN}{(Mx + Ny)^2}$$

Equation (ii) will be exact if $\dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x}$

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i.e. if

$$Ny\frac{\partial M}{\partial y} - My\frac{\partial N}{\partial y} - MN = Mx\frac{\partial N}{\partial x} - Nx\frac{\partial M}{\partial x} - MN$$

or, if

$$N\left(x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}\right) = M\left(x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y}\right)$$

or, if $N \cdot nM = M \cdot nN$, which is true.

Hence $\dfrac{1}{Mx + Ny}$ is an integrating factor.

EXAMPLE 2.4.4 Solve the equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$. (WBUT 2008)

Solution Here $M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$, both are homogenous and $Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$.

$\therefore$ $\frac{1}{Mx+Ny}$ is an integrating factor.

Thus

$$\frac{x^2y - 2xy^2}{x^2y^2}\,dx - \frac{x^3 - 3x^2y}{x^2y^2}\,dy = 0$$

or

$$\frac{x - 2y}{xy}\,dx - \frac{x - 3y}{y^2}\,dy = 0 \text{ is exact}$$

Now

$$\underset{\text{taken } y \text{ constant}}{\int (1/y - 2/x)\,dx} - \underset{\text{terms free from } x}{\int (x/y^2 - 3/y)\,dy} = 0$$

or

$$\frac{x}{y} - 2\log x + 3\log y = c$$

or

$$\frac{x}{y} + \log(y^3/x^2) = c \qquad c \text{ is arbitrary constant}$$

EXAMPLE 2.4.5 Show that $\{x(x^2 - y^2)\}^{-1}$ is an integrating factor of the differential equation $(x^2 + y^2)dx - 2xydy = 0$ and hence solve the equation.

Solution Here $M = x^2 + y^2$ and $N = -2xy$, both are homogeneous.
Also, $Mx + Ny = x(x^2 - y^2) \neq 0$. Hence $\dfrac{1}{Mx + Ny} = \dfrac{1}{x(x^2 - y^2)}$ is an integrating factor.
Divide the given equation by I.F. we get

$$\frac{x^2 + y^2}{x(x^2 - y^2)}dx - \frac{2xy}{x(x^2 - y^2)}dy = 0$$

or

$$\left(\frac{2x}{x^2 - y^2} - \frac{1}{x}\right)dx - \frac{2y}{x^2 - y^2}dy = 0$$

Integrating

$$\underbrace{\int\left(\frac{2x}{x^2-y^2}-\frac{1}{x}\right)dx}_{y \text{ is constant}}-\underbrace{\int\frac{2y}{x^2-y^2}dy}_{\text{terms free from } x}=0$$

or

$$\log(x^2-y^2)-\log x-0=\log c$$

or

$$\log\left(\frac{x^2-y^2}{x}\right)=\log c \qquad \text{or} \qquad x^2-y^2=cx$$

which is the required solution.

Rule II. If the equation $Mdx+Ndy=0$ is not exact but is of the form

$$yf_1(xy)dx+xf_2(xy)dy=0 \tag{2.12}$$

then $\dfrac{1}{Mx-Ny}$ is an integrating factor provided $Mx-Ny\neq 0$.
The proof is similar to Rule I.

EXAMPLE 2.4.6 Solve $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$.

Solution The given equation is of the form

$$yf_1(xy)dx+xf_2(xy)dy=0$$

Now

$$Mx-Ny=xy(xy+2x^2y^2)-xy(xy-x^2y^2)=3x^3y^3\neq 0$$

$$\therefore \qquad \text{I.F.}=\frac{1}{Mx-Ny}=\frac{1}{3x^3y^3}$$

Multiplying the given equation by I.F.

$$\frac{y(xy+2x^2y^2)}{3x^3y^3}dx+\frac{x(xy-x^2y^2)}{3x^3y^3}dy=0$$

or

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0$$

Now

$$\frac{\partial M}{\partial y}=-\frac{1}{3x^2y^2}=\frac{\partial N}{\partial x}$$

Hence the given equation becomes exact and its solution is

$$\underbrace{\int\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx}_{y \text{ is constant}}+\underbrace{\int\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy}_{\text{terms free from } x}=0$$

or

$$-\frac{1}{3xy}+\frac{2}{3}\log x+\int-\frac{1}{3y}\,dy=c$$

or

$$-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=c \qquad \text{or} \qquad \log\left(\frac{x^2}{y}\right)-\frac{1}{xy}=3c$$

EXAMPLE 2.4.7 Solve $(x^3y^3 + x^2y^2 + xy + 1)ydx + (x^3y^3 - x^2y^2 - xy + 1)xdy = 0.$

Solution Here the equation is of the form $yf_1(xy)dx + xf_2(xy)dy = 0.$

Now, $Mx - Ny = 2x^2y^2(xy + 1).$

$$\text{I.F. is } \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2(xy+1)}$$

Multiplying the given equation by I.F.

$$\frac{(xy+1)(x^2y^2+1)}{2x^2y^2(xy+1)}ydx + \frac{(xy+1)(x^2y^2-2xy+1)}{2x^2y^2(xy+1)}xdy = 0$$

or

$$\frac{x^2y^2+1}{2x^2y^2}ydx + \frac{x^2y^2-2xy+1}{2x^2y^2}xdy = 0$$

or

$$\frac{1}{2}\left(y + \frac{1}{x^2y}\right)dx + \frac{1}{2}\left(x - \frac{2}{y} + \frac{1}{xy^2}\right)dy = 0$$

Also

$$\frac{\partial M}{\partial y} = \frac{1}{2}\left(1 - \frac{1}{x^2y^2}\right) = \frac{\partial N}{\partial x}$$

Thus the equation is now exact and its solution is

$$\underset{y \text{ is constant}}{\int \frac{1}{2}\left(y + \frac{1}{x^2y}\right)dx} + \underset{\text{terms free from } x}{\frac{1}{2}\int -\frac{2}{y}dy} = 0$$

or

$$\frac{1}{2}\left(xy - \frac{1}{xy}\right) - \log y = c$$

or

$$xy - \frac{1}{xy} - \log y^2 = 2c$$

Rule III. If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then the integrating factor of $Mdx + Ndy = 0$ is $e^{\int f(x)\,dx}$.

EXAMPLE 2.4.8 Solve $(x^3 + xy^4)dx + 2y^3dy = 0.$

Solution Here $M = x^3 + xy^4$ and $N = 2y^3$.

Now, $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{2y^3}(4xy^3 - 0) = 2x = f(x)$, say.

Then

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

Multiplying the given equation by I.F., we get $e^{x^2}(x^3 + xy^4)dx + 2e^{x^2}y^3dy = 0.$

Now,

$$\frac{\partial}{\partial y}\left[e^{x^2}(x^3+xy^4)\right] = 4xe^{x^2}y^3 = \frac{\partial}{\partial x}(2e^{x^2}y^3)$$

Thus the above equation is exact and its solution is

$$\underset{y \text{ is constant}}{\int e^{x^2}(x^3+xy^4)dx} + \underset{\text{terms free from } x}{\int 2e^{x^2}y^3 dy} = 0$$

That is

$$\frac{1}{2}\int e^z(z+y^4)dz + 0 = c \quad \text{where } z = x^2$$

or

$$\frac{1}{2}\{(z-1)+y^4\}e^z = c \qquad \text{or} \qquad (x^2-1+y^4)e^{x^2} = 2c$$

EXAMPLE 2.4.9 Solve $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$.

Solution Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$.

Now, $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{-x^2y}(2xy+2xy) = -\dfrac{4}{x} = f(x)$, say.

$\therefore$ I.F. $= e^{\int f(x)\,dx} = e^{-\int (4/x)\,dx} = \dfrac{1}{x^4}$

Multiplying the given equation by I.F.

$$\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}dy = 0 \qquad \text{which is obviously exact}$$

Its solution is

$$\underset{y \text{ is constant}}{\int \left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx} - \underset{\text{terms free from } x}{\int \frac{y}{x^2}dx} = 0$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3}\int e^{1/x^3}\left(-\frac{3}{x^4}\right)dx + 0 = c \qquad \text{or} \qquad -\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = c$$

which is the required solution.

Rule IV. If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is a function of y only, say $g(y)$, then an integrating factor of $M\,dx + N\,dy = 0$ is $e^{\int g(y)\,dy}$.

EXAMPLE 2.4.10 Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$

Solution Here $M = 3x^2y^4 + 2xy$ and $N = 2x^3y^3 - x^2$.

Now

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x, \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

and

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{xy(3xy^3 + 2)} = \frac{-2x(3xy^3 + 2)}{xy(3xy^3 + 2)} = -\frac{2}{y} = g(y) \quad \text{say}$$

Therefore

$$\text{I.F.} = e^{\int -\frac{2}{y}\,dy} = e^{\log \frac{1}{y^2}} = \frac{1}{y^2}$$

Multiplying the given equation by I.F., we get

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0 \qquad \text{which is obviously exact}$$

Integrating

$$\underset{y \text{ is constant}}{\int \left(3x^2y^2 + \frac{2x}{y}\right)dx} + \underset{\text{terms free from } x}{\int \left(2x^3y - \frac{x^2}{y^2}\right)dy} = 0$$

or

$$x^3y^2 + \frac{x^2}{y} + 0 = c$$

or

$$x^3y^3 + x^2 = cy$$

EXAMPLE 2.4.11 Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$

Solution Here $M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$ and also

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \qquad \frac{\partial N}{\partial x} = y^3 - 4$$

Now

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y^4 + 2y}(y^3 - 4 - 4y^3 - 2) = -\frac{3}{y} = g(y) \qquad \text{say}$$

$$\therefore \quad \text{I.F.} = e^{\int -\frac{3}{y}\,dy} = e^{-3\log y} = \frac{1}{y^3}$$

Multiplying the given equation by I.F. we get

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

which is obviously exact.

Integrating

$$\underset{y \text{ is constant}}{\int \left(y + \frac{2}{y^2}\right)dx} + \underset{\text{terms free from } x}{\int \left(x + 2y - \frac{4x}{y^3}\right)dy} = 0$$

or

$$\left(y + \frac{2}{y^2}\right)x + \int 2y\,dy = c \qquad \text{or} \qquad \left(y + \frac{2}{y^2}\right)x + y^2 = c$$

Rule V. If the equation $M\,dx + N\,dy = 0$ is of the form

$$x^a y^b(my\,dx + nx\,dy) + x^c y^d(py\,dx + qx\,dy) = 0 \tag{2.13}$$

where a, b, c, d, m, n, p and q are constants and $x^h y^k$ is an integrating factor, where h and k are constants and can be determined by applying that condition, then after multiplication by $x^h y^k$ the given equation becomes exact.

EXAMPLE 2.4.12 Solve $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$.

Solution The given equation can be written as

$$(2y + 6xy^2)dx + (3x + 8x^2y)dy = 0$$

Assume that the I.F. of this equation is of the form $x^h y^k$. Multiplying the above equation by $x^h y^k$, we get

$$(2x^h y^{k+1} + 6x^{h+1}y^{k+2})dx + (3x^{h+1}y^k + 8x^{h+2}y^{k+1})dy = 0$$

Let

$$M = 2x^h y^{k+1} + 6x^{h+1}y^{k+2}, N = 3x^{h+1}y^k + 8x^{h+2}y^{k+1}$$

and

$$\frac{\partial M}{\partial y} = 2(k+1)x^h y^k + 6(k+2)x^{h+1}y^{k+1}, \frac{\partial N}{\partial x} = 3(h+1)x^h y^k + 8(h+2)x^{h+1}y^{k+1}$$

If the equation $M\,dx + N\,dy = 0$ is exact, then we must have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, i.e.

$$2(k+1)x^h y^k + 6(k+2)x^{h+1}y^{k+1} = 3(h+1)x^h y^k + 8(h+2)x^{h+1}y^{k+1}$$

Equating the coefficients of $x^h y^k$ and $x^{h+1}y^{k+1}$, we get

$$2(k+1) = 3(h+1) \text{ and } 6(k+2) = 8(h+2)$$

The solution of these equations is $h = 1$, $k = 2$.

Thus the integrating factor is $x^h y^k = xy^2$.

Multiplying the given equation by xy^2, we get

$$(2xy^3 + 6x^2y^4)dx + (3x^2y^2 + 8x^3y^3)dy = 0$$

and obviously it is an exact equation. Integrating

$$\underbrace{\int(2xy^3 + 6x^2y^4)dx}_{y \text{ is constant}} + \underbrace{\int(3x^2y^2 + 8x^3y^3)dy}_{\text{terms free from } x} = 0$$

or

$$x^2y^3 + 2x^3y^4 + 0 = c$$

or

$$x^2y^3 + 2x^3y^4 = c$$

EXAMPLE 2.4.13 Solve $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$.

Solution Multiplying both sides by x^hy^k, we get

$$(x^hy^{k+1}\,dx + 3x^{h+1}y^k\,dy) + x^{h+2}y^{k+1}(2x^hy^{k+1}\,dx - x^{h+1}y^k\,dy) = 0$$

or

$$(x^hy^{k+1} + 2x^{2h+2}y^{2k+2})\,dx + (3x^{h+1}y^k - x^{2h+3}y^{2k+1})\,dy = 0$$

Let

$$M = x^hy^{k+1} + 2x^{2h+2}y^{2k+2}$$

and

$$N = 3x^{h+1}y^k - x^{2h+3}y^{2k+1}$$

Then

$$\frac{\partial M}{\partial y} = x^h(k+1)y^k + 2x^{2h+2}(2k+2)y^{2k+1}$$

and

$$\frac{\partial N}{\partial x} = 3(h+1)x^hy^k - (2h+3)x^{2h+2}y^{2k+1}$$

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $k+1 = 3(h+1)$ and $2(2k+2) = -(2h+3)$.

Solving, we get $h = -11/7, k = -19/7$.

Therefore, (i) becomes

$$(x^{-11/7}y^{-12/7} + 2x^{3/7}y^{-5/7})\,dx - (x^{10/7}y^{-12/7} - 3x^{-4/7}y^{-19/7})\,dy = 0$$

Integrating, we get

$$\int(x^{-11/7}y^{-12/7} + 2x^{3/7}y^{-5/7})\,dx - \int(x^{10/7}y^{-12/7} - 3x^{-4/7}y^{-19/7})\,dy = 0$$

or

$$y^{-12/7}\frac{x^{-4/7}}{-4/7} + 2\frac{x^{10/7}}{10/7}y^{-5/7} = c \text{ or } -\frac{7}{4}y^{-12/7}x^{-4/7} + \frac{7}{5}x^{10/7}y^{-5/7} = c$$

EXERCISES

Section A Multiple Choice Questions

Linear Equations

1. An integrating factor of $\frac{dy}{dt} + y = 1$ is (WBUT 2008)
   (a) e^t (b) e/t (c) et (d) t/e.
2. Integrating factor of $\frac{dy}{dx} + 2xy = 5$ is
   (a) e^x (b) e^{x^2} (c) e^{x^3} (d) e^{2x}.
3. The I.F. of $(1+x)\frac{dy}{dx} - y = e^{3x}(1+x)^2$ is
   (a) $1+x$ (b) $1+x^2$ (c) $(1+x)^{-1}$ (d) $(1+x)^{-2}$.
4. The I.F. of $y(\log y)dx + (x - \log y)dy = 0$ is
   (a) y (b) $\log x$ (c) e^{xy} (d) $\log y$.
5. The I.F. of the differential equation $(x + 2y^3)\frac{dy}{dx} = y$ is
   (a) y (b) $1/y$ (c) x (d) $1/x$.
6. The I.F. of the differential equation $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ is
   (a) x (b) y (c) $1/x$ (d) $1/y$.
7. The I.F. of $\frac{dy}{dx} + 2y = 3$ is
   (a) e^x (b) e^{x^2} (c) e^{-2x} (d) e^{2x}.
8. The value of $I(x)$ for which the LHS of the equation $I(x)\frac{dy}{dx} + I(x)y\tan x = x^2$ becomes exact is
   (a) $\cos x$ (b) $\sin x$ (c) $\operatorname{cosec} x$ (d) $\sec x$.
9. The solution of $\frac{dy}{dx} + y = 1$ is
   (a) $y = 1 + ce^{-x}$ (b) $\log y + x = cx$ (c) $y = e^x + c$ (d) $ye^x = c$.
10. The solution of $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}$ is
    (a) $xy = c$ (b) $xy = x + c$ (c) $\log(xy) = c$ (d) $e^{xy} = x + c$.

Exact Equations

11. The solution of the differential equation $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$ is
    (a) $ax^2 + by^2 + c = 0$ (b) $ax^2 + 2hxy + by^2 = 0$ (c) $ax^2 + hxy + gx = 0$
    (d) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.
12. The equation $(k + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$ is exact if k is
    (a) 0 (b) 1 (c) 2 (d) 3.
13. The I.F. of the equation $a(x\,dy + 2y\,dx) = xy\,dy$ is
    (a) $1/(xy)$ (b) $1/x$ (c) $1/y$ (d) xy.

14. The I.F. of $x^2y\,dx - (x^3 + y^3)dy = 0$ is
(a) $-1/y$ (b) $-1/y^2$ (c) $-1/y^3$ (d) $-1/y^4$.
15. The I.F. of $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ is
(a) $1/(xy)$ (b) xy (c) x^2y^2 (d) $1/x^2y^2$.
16. The I.F. of $(x^2 + y^2)\,dx - 2xy\,dy = 0$ is
(a) $1/x^2$ (b) $1/x$ (c) $1/y^2$ (d) $1/y$.
17. The I.F. of $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ is
(a) y (b) $1/y$ (c) $1/x$ (d) x.
18. If x^hy^k is the integrating factor of $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$, then the values of h and k are respectively
(a) 2, 1 (b) $-1/2$, $-1/2$ (c) $-5/2$, $-1/2$ (d) -2, -1.
19. The solution of $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$ is
(a) $x^2y + xy - x\tan y + \tan y = c$ (b) $x^2y + xy + x^3 - x^2\tan^2 y = c$
(c) $x^2 + y^2 = cxy\tan x\tan y$ (d) $xy^2 + x^2y + \tan x + \tan y = c$.
20. The solution of $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$ is
(a) $x^4 + y^4 = c$ (b) $x^4 + y^4 + 6x^2y^2 = c$ (c) $x^4 + 3x^2y^2 = c$
(d) $x^3 + 3xy^2 + 3x^2y^2 + y^4 = c$.
21. The differential equation $(xe^{kxy} + 2y)\dfrac{dy}{dx} + ye^{xy} = 0$ is exact for c equal to
(a) 0 (b) 1 (c) 2 (d) 3.

Section B Review Questions

Linear Equations

Solve the following differential equations:

1. $(1 + x)\frac{dy}{dx} - xy = 1$
2. $y\,dx - x\,dy + \log x\,dx = 0$
3. $(1 - x^2)\frac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$
4. $\cos^2 x\frac{dy}{dx} + y = \tan x$
5. $\frac{dy}{dx} - \frac{2y}{x+1} = (x + 1)^3$
6. $\frac{dy}{dx} + 2xy = 2e^{-x^2}$
7. $(x + 2y^3)\frac{dy}{dx} = y$
8. $\sin 2x\left(\frac{dy}{dx}\right) - y = \tan x$
9. $\frac{dx}{dy} + \frac{xy}{1-y^2} - y\sqrt{x} = 0$
10. $x\log x\frac{dy}{dx} + y = \log x^2$

11. $ye^y dx = (y^3 + 2xe^y)dy$

12. $e^{-y}\sec^2 y dy = dx + x\,dy$

13. $xy(1 + xy^2)\frac{dy}{dx} = 1$

14. $\frac{dy}{dx} + x\sin^2 y = x^3\cos^2 y$

15. $x\frac{dy}{dx} + y = x^3y^6$

16. $y(2xy + e^x)dx - e^x dy = 0$

17. $\frac{dy}{dx} = y\tan x - y^2\sec x$

18. $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$

19. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x \sec y$

20. $\tan y\frac{dy}{dx} + \tan x = \cos y\cos^2 x$

21. $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

22. $(x + y + 1)\frac{dy}{dx} = 1$

23. $\frac{dy}{dx} - y\cot x = \operatorname{cosec} x$

24. $\frac{dy}{dx} + \frac{n}{x}y = \frac{a}{x^n}$

25. $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$

26. $(1 + y^2)dx = (\tan^{-1} y - x)dy$

27. $(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$

28. $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$

29. $x\log x\frac{dy}{dx} + y = 2\log x$

30. $dx + x\,dy = e^{-y}\sec^2 y dy$

31. $\frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$

32. $\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{\sin^2 x}{1+x^3}$

33. $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

34. $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y\sin y$

35. $\frac{dy}{dx} + y\cos x = y^n\sin 2x$

36. $\sin y\frac{dy}{dx} = \cos y(1 - x\cos y)$

37. Solve $\dfrac{dy}{dx} + 2y\tan x = \sin x$, given that $y = 0$ when $x = \pi/3$.

38. Solve $x\dfrac{dy}{dx} = -y + e^x$, given that $y = 2$ when $x = 1$.

Exact equations

Solve the following differential equations:

39. $x\,dy + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$

40. $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$

41. $(\sin x\cos y + e^{2x})dx + (\cos x\sin y + \tan y)dy = 0$

42. $\{y(1 + 1/x) + \cos y\}dx + (x + \log x - x\sin y)dy = 0$

43. $(1 + 2xy\cos x^2 - 2xy)dx + (\sin x^2 - x^2)dy = 0$

44. $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

45. $(x^2 + y^2 - a^2)x\,dx + (x^2 - y^2 - b^2)y\,dy = 0$

46. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

47. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$

48. $ye^{xy}\,dx + (xe^{xy} + 2y)\,dy = 0$

49. $y\,dx - x\,dy + (1 + x^2)\,dx + x^2\sin y\,dy = 0$

50. $y(axy + e^x)\,dx - e^x\,dy = 0$

51. $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$

52. $y\sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$

53. $(\sec x\tan x\tan y - e^x)\,dx + \sec x\sec^2 y\,dy = 0$

54. $y(2xy + e^x)\,dx = e^x\,dy$

55. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

56. $x^2y\,dx - (x^3 + y^3)\,dy = 0$

57. $(x^3 + xy^4)\,dx + 2y^3\,dy = 0$

58. $(xy + 2x^2y^2)y\,dx + (xy - x^2y^2)x\,dy = 0$

59. $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

60. $(1+xy)y\,dx + (1-xy)x\,dy = 0$

61. $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$

62. $(x^4y^4 + x^2y^2 + xy)y\,dx + (x^4y^4 - x^2y^2 + xy)x\,dy = 0$

63. $(x^2+y^2)\,dx - 2xy\,dy = 0$

64. $(y + \frac{1}{3}y^3 + \frac{1}{2}x^2)\,dx + \frac{1}{4}(x + xy^2)\,dy = 0$

65. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

66. $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$

67. $(xy^3 + y)\,dx + 2(xy^2 + x + y^4)\,dy = 0$

68. $(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0$

69. $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$

70. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

71. $(3x + 2y^2)y\,dx + 2x(2x + 3y^2)\,dy = 0$

72. Prove that $\frac{1}{(x+y+1)^4}$ is an integrating factor of

$$(2xy - y^2 - y)dx + (2xy - x^2 - x)dy = 0$$

and hence solve it.

73. If $x^\alpha y^\beta$ is an integrating factor of $2y\,dx - 3xy^2\,dx - x\,dy = 0$, find α and β and use it to solve the equation.

74. Show that a constant k can be found so that $(x+y)^k$ is an integrating factor of $(4x^2 + 2xy + 6y)\,dx + (2x^2 + 9y + 3x)\,dy = 0$ and hence solve the equation.

Answers

Section A Multiple Choice Questions

1. (a) 2. (b) 3. (c) 4. (d) 5. (b) 6. (c) 7. (d) 8. (d) 9. (a)
10. (b) 11. (d) 12. (b) 13. (a) 14. (d) 15. (d) 16. (a) 17. (a) 18. (c)
19. (a) 20. (b) 21. (b)

Section B Review Questions

1. $y(1+x) = x + ce^x$
2. $y = cx - (1 + \log x)$
3. $y = \sqrt{1-x^2} + c(1-x^2)$
4. $y = ce^{-\tan x} + \tan x - 1$

5. $y = \frac{1}{2}(x+1)^4 + c(x+1)^2$

6. $ye^{x^2} = 2x + c$

7. $x = y^3 + cy$

8. $y = (c+1)\sqrt{\tan x}$

10. $y = \log x + \frac{c}{\log x}$

11. $x/y^2 = c - e^{-y}$

12. $xe^y = c + \tan y$

13. $1/x = (2 - y^2) + ce^{-y^2/2}$

14. $2\tan y = (x^2 - 1) + 2ce^{-x^2}$

15. $(5/2 + cx^2)x^3y^5 = 1$

16. $e^x = y(c - x^2)$

17. $\sec x = y(c + \tan x)$

18. $y^2 = x^2 + cx - 1$

19. $\sin y = (1+x)(e^x + c)$

20. $\cos y = \cos x(\sin x + c)$

21. $\sqrt{x} = \sqrt{y}(\log\sqrt{y} + c)$

22. $x = ce^y - (y+2)$

23. $y\mathrm{cosec}\, x = -\cot x + c$

24. $yx^n = ax + c$

25. $2ye^{\tan^{-1}x} = e^{2\tan^{-1}x} + 2c$

26. $x = \tan^{-1}y - 1 + ce^{\tan^{-1}y}$

27. $xe^{\tan^{-1}y} = \tan^{-1}y + c$

28. $y = \tan^{-1}x - 1 + ce^{\tan^{-1}x}$

29. $y\log x = (\log x)^2 + c$

30. $xe^y = \tan y + c$

31. $y = z + ce^{-z}$, where $z = \frac{x}{\sqrt{(1-x^2)}}$

32. $y(1+x^3) = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

33. $y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3}x^{3/2} + c$

34. $2x = \sin y(1 - 2cx^2)$

35. $\frac{1}{y^{n-1}} = 2\sin x - \frac{2}{1-n} + ce^{(n-1)\sin x}$

36. $\sec y\, e^{-x} = e^{-x}(1+x) + c$

37. $y = \cos x - 2\cos^2 x$

38. $y = \frac{1}{x}e^x + \frac{2-e}{x}$

39. $x^2 - 2\tan^{-1}(x/y) + y^2 = 2c$

40. $x + ye^{x/y} = c$

41. $\frac{1}{2}e^{2x} - \cos x\cos y + \log\sec y = c$

42. $(x + \log x)y + x\cos y = c$

43. $x + y\sin x^2 - x^2 y = c$

44. $y\sin x + (\sin y + y)x = c$

45. $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$

46. $x^3 - 6x^2y - 6xy^2 + y^3 = c$

47. $\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$

48. $e^{xy} + y^2 = c$

49. $x^2 - y - 1 - x\cos y = cx$

50. $\frac{ax^2}{2} + e^{x/y} = c$

51. $x = cye^{\frac{1}{xy}}$

52. $3y\cos 2x + 6y + 2y^3 = c$

53. $e^x = \sec x\tan y + c$

54. $e^x + x^2y = cy$

55. $x/y - 2\log x + 3\log y = c$

56. $x^3 = 3y^3(\log y - c)$

57. $(x^2 - 1 + y^4)e^{x^2} = 2c$

58. $2\log x - \log y = \frac{1}{xy} + c$

59. $xy + \log(\frac{x}{y}) - \frac{1}{xy} = c$

60. $\log(x/y) - \frac{1}{xy} = c$

61. $x = cy\cos xy$

62. $\frac{1}{2}x^2y^2 - \frac{1}{xy} + \log(x/y) = c$

63. $x - y^2/x = c$

64. $x^6 + 3x^4y + x^4y^3 = c$

65. $x^3y^3 + x^2 = cy$

66. $e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3} + \frac{y^2}{6} - \frac{y}{18} + \frac{1}{108}\right) = c$

67. $3x^2y^4 + 6xy^3 + 2y^6 = c$

68. $5x^{-36/13}y^{24/13} - 12x^{-10/13}y^{-15/13} = c$

69. $x^2y^3 + 2x^3y^4 = c$

70. $4x^{1/2}y^{1/2} + \frac{2}{3}x^{-3/2}y^{3/2} = c$

71. $x^2y^4(x + y^2) = c$

72. $xy = c(x + y + 1)^3$

73. $\alpha = 1,\ \beta = -2,\ x^2 - x^3y = cy$

74. $k = 1,\ x^4 + 2x^3y + x^2y^2 + 3x^2y + 6xy^2 + 3y^3 = c$

Chapter 3

Differential Equations of First Order and Higher Degree

3.1 Introduction

The most general form of a differential equation of the first order and of higher degree, say, nth degree is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0 \tag{3.1}$$

where $p = \frac{dy}{dx}$ and $P_1, P_2, \ldots, P_n$ are functions of x and y. This equation can also be written as $F(x, y, p) = 0$. But, this equation cannot be solved in its general form. In this chapter, the following three special types of such equations are discussed:

- Equations solvable for p
- Equations solvable for y
- Equations solvable for x

3.2 Equations Solvable for p

Suppose the differential Eq. (3.1) can be solved for p and is of the form

$$(p - f_1)(p - f_2) \cdots (p - f_n) = 0 \tag{3.2}$$

where each f is a function of x and y.

Equating each factor to zero and we get n equations of first order and first degree. Let their solutions be

$$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, \ldots, \phi_n(x, y, c_n) = 0$$

All possible solutions of Eq. (3.2) will then be included in the relation

$$\phi_1(x, y, c_1)\phi_2(x, y, c_2) \cdots \phi_n(x, y, c_n) = 0 \tag{3.3}$$

It may be observed that the general solution contains n arbitrary constants, whereas we expect only one constant, as the equation is only of the first order.

See that the generality will still be maintained if all the constants $c_1, c_2, \dots, c_n$ be made the same c. Hence the complete solution of (3.2) is

$$\phi_1(x, y, c)\phi_2(x, y, c)\cdots\phi_n(x, y, c) = 0 \tag{3.4}$$

where c is arbitrary constant.

EXAMPLE 3.2.1 Solve $p^2 + px + py + xy = 0$.

Solution The equation may be written as $(p+x)(p+y) = 0$. Now, $p+x = 0$ gives $\dfrac{dy}{dx} + x = 0$ or $2y = -x^2 + c_1$ and $p + y = 0$ gives $\dfrac{dy}{dx} + y = 0$ or $\log y + x = c_2$.

Thus the complete solution is $(2y + x^2 + c)(x + \log y + c) = 0$.

EXAMPLE 3.2.2 Solve $p^2 - p(e^x + e^{-x}) + 1 = 0$.

Solution The given equation can be written as $(p - e^x)(p - e^{-x}) = 0$. Now, $p - e^x = 0$ gives $\dfrac{dy}{dx} = e^x$, or $y = e^x + c_1$ and $p - e^{-x} = 0$ gives $\dfrac{dy}{dx} = e^{-x}$, or $y = -e^{-x} + c_2$.

Hence the general solution is $(y - e^x - c)(y + e^{-x} - c) = 0$, where c is arbitrary constant.

EXAMPLE 3.2.3 Solve $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$, i.e. $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$. (WBUT 2006)

Solution This equation can be written as

$$p^2 + p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$$

or

$$\left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0$$

When $p + \dfrac{y}{x} = 0$, then $\dfrac{dy}{dx} + \dfrac{y}{x} = 0$, or $\dfrac{dy}{y} + \dfrac{dx}{x} = 0$, or $\log(xy) = \log c_1$, or $xy = c_1$.

When $p - \dfrac{x}{y} = 0$, then $\dfrac{dy}{dx} - \dfrac{x}{y} = 0$, or $ydy - xdx = 0$ or, $y^2 - x^2 = c_2$.

Hence the solution is $(xy - c)(x^2 - y^2 - c) = 0$, where c is arbitrary constant.

EXAMPLE 3.2.4 Solve $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$.

Solution From the given equation

$$p = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2} = \frac{y \pm \sqrt{(x^2 - y^2)}}{x}$$

This is a homogeneous equation. Substituting $y = vx$. Then $p = \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Thus $v + x\dfrac{dv}{dx} = \dfrac{v \pm \sqrt{1 - v^2}}{1}$, or $x\dfrac{dv}{dx} = \pm\sqrt{1 - v^2}$ or, $\dfrac{dv}{\sqrt{1 - v^2}} = \pm\dfrac{dx}{x}$.

Integrating, we get $\sin^{-1} v = \pm\log x \pm \log c = \pm\log cx$.

That is, $\sin^{-1}(y/x) = \pm\log(cx)$ is the general solution.

3.3 Equations Solvable for y

If the equation is solvable for y then it can be written as

$$y = f(x, p) \tag{3.5}$$

Differentiating with respect to x, gives

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right) \tag{3.6}$$

which is a differential equation in two variables x and p. Let its solution be

$$F(x, p, c) = 0 \tag{3.7}$$

By eliminating p between Eqs. (3.5) and (3.7), we get the required solution. If the elimination of p is not possible, then we solve (3.5) and (3.7) for x and y and obtain

$$x = F_1(p, c) \qquad y = F_2(p, c) \tag{3.8}$$

as required solution, where p is taken as parameter.

EXAMPLE 3.3.1 Solve $y = 2px + p^4x^2$.

Solution Differentiating with respect to x,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 4p^3x^2\frac{dp}{dx} + 2p^4x$$

or

$$\frac{dp}{dx}(2x + 4p^3x^2) + (p + 2p^4x) = 0$$

or

$$\left(p + 2x\frac{dp}{dx}\right)(1 + 2p^3x) = 0$$

Now, $p + 2x\dfrac{dp}{dx} = 0$, or $2\dfrac{dp}{p} + \dfrac{dx}{x} = 0$.

Integrating, $\log p^2 + \log x = \log c$, or $p^2x = c$, or $p^2 = c/x$.

To eliminate p, substituting $p^2 = c/x$ to the given equation, i.e. we obtain $y = 2px + c^2$, or $(y - c^2)^2 = 4p^2x^2 = 4x^2\dfrac{c}{x}$, or $(y - c^2)^2 = 4cx$, which is the required solution.

EXAMPLE 3.3.2 Solve $y = 2px + p^2$.

Solution Given

$$y = 2px + p^2 \tag{i}$$

Differentiating with respect to x,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or

$$p = 2p + (2x + 2p)\frac{dp}{dx}$$

or

$$2(x + p)\frac{dp}{dx} + p = 0$$

or

$$\frac{dx}{dp} + \frac{2x}{p} = -2$$

which is a linear differential equation in x and I.F. is $e^{\int (2/p)dp} = p^2$.

Multiplying the above equation by p^2, we get $\frac{d}{dp}(xp^2) = -2p^2$.

Integrating, $$xp^2 = -\frac{2}{3}p^3 + \frac{c}{3}, \text{ or } x = -\frac{2}{3}p + \frac{c}{3p^2} \qquad \text{(ii)}$$

The Eqs. (i) and (ii) taken together, with parameter p, constitute the general solution.

EXAMPLE 3.3.3 Solve $y = \sin p - p\cos p$.

Solution Differentiating with respect to x, we get

$$p = \frac{dy}{dx} = (\cos p - \cos p + p\sin p)\frac{dp}{dx} \qquad \text{or} \qquad \sin p\, dp = dx$$

Integrating, $\cos p = x + c$. Therefore, $\sin p = \sqrt{1 - (x + c)^2}$.

Eliminating p, we get $y = \sqrt{1 - (x + c)^2} - (x + c)\cos^{-1}(x + c)$, which is the required solution.

EXAMPLE 3.3.4 Solve $y = (p + p^2)x + p^{-1}$. (WBUT 2003)

Solution Differentiating with respect to x, we get

$$p = (p + p^2) + (1 + 2p)x\frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx}$$

or

$$p^2 + \left\{(1 + 2p)x - \frac{1}{p^2}\right\}\frac{dp}{dx} = 0$$

or

$$\frac{dx}{dp} + \frac{1 + 2p}{p^2}x = \frac{1}{p^4}$$

This is a linear equation in p and its I.F. is

$$e^{\int (1+2p)/p^2 dp} = p^2 e^{-1/p}$$

Multiplying the above equation by I.F. and integrating, we get

$$xp^2 e^{-1/p} = \int \frac{1}{p^2}e^{-1/p}dp = -\int e^{-1/p}d(1/p) = e^{-1/p} + c$$

or

$$x = \frac{1}{p^2} + \frac{c}{p^2}e^{1/p} \qquad \text{(i)}$$

Thus the given equation and (i) gives the required solution, where p is taken as parameter.

3.4 Equations Solvable for x

When the equation is solvable for x, then it can be expressed as

$$x = f(y, p) \tag{3.9}$$

Then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dp} = \phi\left(y, p, \frac{dp}{dy}\right)$$

It becomes an equation containing two variables y and p. Let its solution be

$$F(y, p, c) = 0 \tag{3.10}$$

By eliminating p between (3.9) and (3.10), the required solution is obtained. If the elimination is not possible, (3.9) and (3.10) may be expressed in terms of p, and p may be considered the parameter.

EXAMPLE 3.4.1 Solve $(1 + p^2)y - 2px = 0$.

Solution The given equation can be written as $2x = yp + y/p$.

Differentiating with respect to y, we get

$$\frac{2}{p} = p + y\frac{dp}{dx} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

or

$$\frac{1}{p} - p = \left(y - \frac{y}{p^2}\right)\frac{dp}{dy}$$

or

$$\left(\frac{1 - p^2}{p}\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Thus

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

or

$$\frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating, $\log y + \log p = \log c$, or $yp = c$, or $p = c/y$.

Substituting $p = c/y$ to the given equation, we get $2x = c + y^2/c$, or $y^2 + c^2 - 2cx = 0$, which is the required solution.

EXAMPLE 3.4.2 Solve $x - y\left(\frac{dy}{dx}\right) = 3\left(\frac{dy}{dx}\right)^2$.

Solution The given equation is

$$x - yp = 3p^2 \qquad \text{or} \qquad x = yp + 3p^2 \tag{i}$$

Differentiating with respect to y,

$$\frac{dx}{dy} = p + y\frac{dp}{dy} + 6p\frac{dp}{dy}$$

or

$$\frac{1}{p} = p + (y + 6p)\frac{dp}{dy}$$

or

$$\left(\frac{1}{p} - p\right)\frac{dy}{dp} = y + 6p$$

or

$$\frac{dy}{dp} - y\frac{p}{1-p^2} = \frac{6p^2}{1-p^2} \tag{ii}$$

This is a linear equation and its I.F.

$$= e^{-\int \frac{p}{1-p^2}dp} = e^{\frac{1}{2}\int \frac{-2p}{1-p^2}dp} = e^{\frac{1}{2}\log(1-p^2)} = \sqrt{1-p^2}$$

Multiplying (ii) by $\sqrt{1-p^2}$ and integrating

$$y\sqrt{1-p^2} = \int \frac{6p^2}{\sqrt{1-p^2}}dp = -6\int \sqrt{1-p^2}dp + 6\int \frac{1}{\sqrt{1-p^2}}dp$$

$$= -6\left[\frac{p}{2}\sqrt{1-p^2} + \frac{1}{2}\sin^{-1}p\right] + 6\sin^{-1}p + c$$

$$= -3p\sqrt{1-p^2} + 3\sin^{-1}p + c$$

Therefore

$$y = -3p + \frac{3}{\sqrt{1-p^2}}\sin^{-1}p + \frac{c}{\sqrt{1-p^2}} \tag{iii}$$

Equations (i) and (iii) give the required solution where p is treated as parameter.

3.5 Clairaut's Equation

A differential equation of the form

$$y = px + f(p) \tag{3.11}$$

is called **Clairaut's equation**. The general solution of this equation can be determined easily.

Differentiating with respect to x, we have

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or

$$\frac{dp}{dx}\left\{x + f'(p)\right\} = 0$$

Therefore, either $\dfrac{dp}{dx} = 0$ or $x + f'(p) = 0$.

The equation $\dfrac{dp}{dx} = 0$ gives $p = c$.

if $\dfrac{dp}{dx} = 0$, then $p = c$ and singular sol is $y = cx + f(c)$

Substituting $p = c$ in (3.11), we get

$$y = cx + f(c) \tag{3.12}$$

as the general solution.

Thus, the solution of Clairaut's equation is obtained on replacing p by c.

By eliminating p between $x + f'(p) = 0$ and (3.11) we get an equation without any constant. This is also a solution of the given equation and this solution is known as a **singular solution** of (3.11). This solution gives the envelope of the family of the straight line (3.12).

EXAMPLE 3.5.1 Find the general solution of $p = \cos(y - px)$. (WBUT 2007)

Solution The given equation can be written as $y = px + \cos^{-1} p$, which is the Clairaut's equation.

Differentiating with respect to x, we get

$$p = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1 - p^2}}\frac{dp}{dx}$$

or

$$\left(x - \frac{1}{\sqrt{1 - p^2}}\right)\frac{dp}{dx} = 0$$

Now, $\dfrac{dp}{dx} = 0$ gives $p = c$.

Therefore, the general solutions is $y = cx + \cos^{-1} c$, where c is arbitrary constant.

EXAMPLE 3.5.2 Solve the differential equation $y = px + \sqrt{a^2p^2 + b^2}$. Also, find its singular solution. (WBUT 2005)

Solution The given equation is $y = px + \sqrt{a^2p^2 + b^2}$. (i)

Differentiating (i) with respect to x, we get

$$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{a^2p^2 + b^2}}a^2p\frac{dp}{dx} \quad \text{or} \quad \frac{dp}{dx}\left(x + \frac{a^2p}{\sqrt{a^2p^2 + b^2}}\right) = 0$$

Either $\dfrac{dp}{dx} = 0$ or $x + \dfrac{a^2p}{\sqrt{a^2p^2 + b^2}} = 0$.

When $\dfrac{dp}{dx} = 0$, then $p = c$, a constant.

Then the general solution is

$$y = cx + \sqrt{a^2c^2 + b^2} \qquad \text{[obtain by replacing } p \text{ by } c \text{ in (i)]}$$

When $x + \dfrac{a^2p}{\sqrt{a^2p^2 + b^2}} = 0$, then

$$x = -\frac{a^2p}{\sqrt{a^2p^2 + b^2}} \qquad \text{(ii)}$$

From (i)

$$y = -\frac{a^2p^2}{\sqrt{a^2p^2 + b^2}} + \sqrt{a^2p^2 + b^2} = \frac{b^2}{\sqrt{a^2p^2 + b^2}} \qquad \text{(iii)}$$

Squaring and adding (ii) and (iii)

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{(-ap)^2 + b^2}{a^2p^2 + b^2} = 1 \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the required singular solution.

EXAMPLE 3.5.3 Solve $y = 2px + y^2p^3$.

Solution Multiplying the given equation by y. Then $y^2 = 2pxy + y^3p^3$.

Substituting $y^2 = v$ and $2y\dfrac{dy}{dx} = \dfrac{dv}{dx}$, i.e. $2yp = P$, where $P = \dfrac{dv}{dx}$.

Therefore, the above equation reduces to $v = xP + \dfrac{1}{8}P^3$.

This is Clairaut's form and its general solution is

$$v = xc + \frac{c^3}{8} \qquad \text{or} \qquad y^2 = xc + \frac{c^3}{8}$$

EXAMPLE 3.5.4 Find the general solution of $(px - y)(x - py) = 2p$.

Solution Putting $x^2 = u$ and $y^2 = v$ and differentiating, we get $2x\,dx = du$ and $2y\,dy = dv$

$\therefore \dfrac{y\,dy}{x\,dx} = \dfrac{dv}{du}$ or, $\dfrac{yp}{x} = q$ where $q = \dfrac{dv}{du}$. Therefore, $p = \dfrac{xq}{y}$.

Putting this value in $(px - y)(x - py) = 2p$, we get

$$\left(\frac{x^2q}{y} - y\right)(x - xq) = \frac{2xq}{y}$$

or

$$(x^2q - y^2)x(1 - q) = 2xq$$

or

$$(uq - v)(1 - q) = 2q$$

or

$$v = uq - \frac{2q}{1 - q}$$

This is the Clairaut's form.

The general solution is

$$v = uc - \frac{2c}{1-c} \qquad \text{or} \qquad y^2 = cx^2 - \frac{2c}{1-c}$$

where c is any arbitrary constant.

EXAMPLE 3.5.5 Reduce the differential equation

$$(x^2 + y^2 - 1)\frac{dy}{dx} = xy\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$$

into Clairaut's form and find its general solution.

Solution The given equation is

$$(x^2 + y^2 - 1)p = xy(1 + p^2) \tag{i}$$

Substituting $x^2 = u$ and $y^2 = v$, $2xdx = du$ and $2ydy = dv$.

Now, $\dfrac{dy}{dx}\dfrac{y}{x} = \dfrac{dv}{du}$, or $p\dfrac{y}{x} = q$, where $p = \dfrac{dy}{dx}$ and $q = \dfrac{dv}{du}$.

Using $p = xq/y$, (i) becomes

$$(u + v - 1)\frac{xq}{y} = xy\left(1 + \frac{x^2q^2}{y^2}\right)$$

or

$$(u + v - 1)q = y^2 + x^2q^2 = v + uq^2$$

or

$$v(q-1) + uq(1-q) = q$$

or

$$v = uq + \frac{q}{q-1}$$

This is Clairaut's form and its general solution is

$$v = uc + \frac{c}{c-1} \qquad \text{or} \qquad y^2 = cx^2 + \frac{c}{c-1}$$

where c is arbitrary constant.

EXAMPLE 3.5.6 Reduce the equation $x^2\left(\dfrac{dy}{dx}\right)^2 + y(2x+y)\dfrac{dy}{dx} + y^2 = 0$ to Clairaut's form by the substitution $y = u, xy = v$. Hence find its general solution.

Solution Here $y = u$ and $xy = v$, or $x = v/u$.

$\therefore dy = du$ and $dx = \dfrac{udv - vdu}{u^2}$.

Thus, $\dfrac{dy}{dx} = \dfrac{u^2du}{udv - vdu} = \dfrac{u}{p - v/u}$ where $p = \dfrac{dv}{du}$.

Using this value, the given equation reduces to

$$\frac{v^2}{u^2}\frac{u^2}{(p-v/u)^2}+v\Big(2\frac{v}{u}+u\Big)\frac{u}{p-v/u}+u^2=0$$

After simplification this equation reduces to $v=up+p^2$.

This equation is clearly of Clairaut's form and its general solution is $v=uc+c^2$ or, $xy=cy+c^2$, c is arbitrary constant.

EXAMPLE 3.5.7 Reduce the equation $(px^2+y^2)(px+y)=(p+1)^2$ where $p=\frac{dy}{dx}$, to Clairaut's form by the substitution $xy=u$ and $x+y=v$ and hence find its singular solutions.

Solution Putting $u=xy, v=x+y$.

Differentiating with respect to x, we get $\dfrac{du}{dx}=y+xp$, $\dfrac{dv}{dx}=1+p$.

Now

$$\frac{dv}{du}=\frac{1+p}{y+xp}$$

or

$$q=\frac{1+p}{y+xp}$$

where $q=\frac{dv}{du}$, or

$$q(y+xp)=1+p \quad\text{or}\quad p=\frac{1-yq}{xq-1}$$

Now

$$px^2+y^2=\frac{x^2-x^2yq}{xq-1}+y^2=\frac{x^2-y^2-xy(x-y)q}{xq-1}$$

$$=\frac{(x-y)\{(x+y)-xyq\}}{xq-1}=\frac{(x-y)(v-uq)}{xq-1}$$

Putting this value into the given equation, we get

$$\frac{(x-y)(v-uq)}{xq-1}\cdot\frac{1+p}{q}=(1+p)^2$$

or

$$\frac{(x-y)(v-uq)}{xq-1}=q(1+p)=q\Big\{1+\frac{1-yq}{xq-1}\Big\}=\frac{q^2(x-y)}{xq-1}$$

or $v-uq=q^2$, or $v=uq+q^2$, which is Clairaut's form and its general solution is $v=cu+c^2$, or $x+y=cxy+c^2$, where c is arbitrary constant.

To find singular solution, differentiate $v=uq+q^2$ with respect to u.

$$\frac{dv}{du}=q+u\frac{dq}{du}+2q\frac{dq}{du}$$

or

$$\frac{dq}{du}(u+2q)=0$$

Now, $u+2q=0$ gives $q=-u/2$.

Substituting $q = -u/2$ to $v = uq + q^2$, we get the singular solution as $v = -u^2/2 + u^2/4$, or $4v + u^2 = 0$, or $4(x + y) + (xy)^2 = 0$.

(A singular solution may also be obtained by equating the c-discriminant to the zero of the general solution)

EXERCISES

Section A Multiple Choice Questions

1. The general solution of the equation $p^2 - 5p + 6 = 0$ is
   (a) $(y - 2x + c)(y - 3x + c) = 0$ (b) $(y - 2x + c_1)(y - 3x + c_2) = 0$
   (c) $(y - 2x + c)^2(y - 3x + c)^2 = 0$ (d) $p = 2, p = 3$.
2. The general solution of the equation $\left(\frac{dy}{dx}\right)^2 - (x + y)\frac{dy}{dx} + xy = 0$ is
   (a) $(y - x^2)(y - \log x + c) = 0$ (b) $(2y - x^2 + c)(\log y - x + c) = 0$
   (c) $(2y - x^2)(y - \log x) = 0$ (d) $(2y - x^2 + c_1)(\log y - x - c_2) = 0$.
3. The general solution of $x^2p^2 + xyp - 6y^2 = 0$ is
   (a) $(y - x^2)(xy - c) = 0$ (b) $(y - cx^2)(yx^3 - c) = 0$
   (c) $(y - x^2 - c)(yx^2 - c) = 0$ (d) $(y - x^2)(xy^2 - c) = 0$.
4. The general solution of $y = x^4p^2 - px$ is
   (a) $(x^2 - c)(y^2 - c) = 0$ (b) $(xy - c)(x^2 + y^2 - c) = 0$
   (c) $xy + c = c^2x$ (d) none of these.
5. The general solution of $x = y + a \log p$ is
   (a) $y = c - a\log(1 - p)$ (b) $(y - x)(y - x^2 - c) = 0$
   (c) $x = c - a\log(1 - p)$ (d) $x^2 + y^2 = cp$.
6. The general solution of $(xp - y)(p - xy) = 0$ is
   (a) $(xy - y^2 - c)(y - x^2y - c) = 0$ (b) $(xy - c)(x^2 - y^2 - c) = 0$
   (c) $(y - cx)(2y - x^2 - c) = 0$ (d) $(y - xc)(2\log y - x^2 - c) = 0$.
7. The general solution of $y = x\frac{dy}{dx} + \sqrt{a\left(\frac{dy}{dx}\right)^2 + 5}$ is
   (a) $y = xc + ac$ (b) $y = x^2 + ay^2$
   (c) $y^2 = x^2 + \sqrt{ac^2 + 5}$ (d) $y = xc + \sqrt{ac^2 + 5}$.
8. The general solution of $(y - px)(p - 1) = p$ is
   (a) $(y - cx)(c - 1) = c$ (b) $(y^2 - x^2 - c)(y - x) = y$
   (c) $(y - cx)(c - 1) = cx$ (d) $(y^2 - x^2 - c_1)(y^2 - x^2 - c_2) = 0$.
9. The general solution of $p = \log(y - xp)$ is
   (a) $y = \log(y - cx)$ (b) $c = \log(y - cx)$
   (c) $e^c = y^2 - x^2$ (d) $y = \log(y^2 - x^2 - c)$.
10. The general solution of $(y - px)^2 = a^2p^2 + b^2$ is
    (a) $(y^2 - cx)(x^2 - cy) = 0$ (b) $(y - x)^2 = a^2c^2 + b^2$
    (c) $(y - cx)^2 = a^2c^2 + b^2$ (d) none of these.
11. The singular solution of $y = px + e^p$ is
    (a) $y = x(\log x + 1)$ (b) $y = cx + e^c$
    (c) $y = x^2 + e^c$ (d) $y^2 = x(\log x + y)$.

12. The solution of the equation $(p-xy)(p-x^2)(p-y^2)=0$ is
(a) $(xy-c)(x^2-c)(y^2-c)=0$ (b) $(xy-x^2)(y-cx)(x-cy)=0$
(c) $(x+1/y+c)(y-x^3/3-c)(\log y-x^2/2-c)=0$ (d) none of these.

Section B Review Questions

Find the general solution of the following differential equations:

1. $xy\left(\frac{dy}{dx}\right)^2+(x^2+y^2)\frac{dy}{dx}+xy=0$
2. $y(y-2)p^2-(y-2x+xy)p+x=0$
3. $p^2+2py\cot x=y^2$
4. $yp^2+(x-y)p-x=0$
5. $p^3+2xp^2-y^2p^2-2xy^2p=0$
6. $p(p+y)=x(x+y)$
7. $y=x[p+\sqrt{1+p^2}]$
8. $(p+y+x)(xp+y+x)(p+2x)=0$
9. $xy^2(p^2+2)=2py^3+x^3$
10. $x^2p^3+y(1+x^2y)p^2+y^3p=0$
11. $y=yp^2+2px$
12. $y=2px+p^n$
13. $y=x+a\tan^{-1}p$
14. $y-2px-\tan^{-1}(xp^2)=0$
15. $y+px=x^4p^2$
16. $x^2p^4+2xp-y=0$
17. $y=2px+y^2p^3$
18. $p^3-4xyp+8y^2=0$
19. $p=\tan(x-\frac{p}{1+p^2})$
20. $y=(1+p)x+ap^2$
21. $e^{p-y}=p^2-1$
22. $y=2px+f(xp^2)$

23. $p^3 - 4xyp + 8y^2 = 0$

24. $y = p\tan p + \log\cos p$

25. $y = \log(p^3 + p)$

26. $x(1 + p^2) = 1$

27. $(y - 1)p - xp^2 + 2 = 0$

28. $(y - px)(p - 1) = p$ (WBUT 2009)

29. $(x - a)p^2 + (x - y)p - y = 0$

30. $p = \log(px - y)$

31. $\sin px\ \cos y = \cos px\ \sin y + p$

32. Solve $x^2p^2 + yp(2x+y) + y^2 = 0$ by reducing it to Clairaut's form by using the substitution $y = u$ and $xy = v$.

33. Use the transformation $x^2 = u$ and $y^2 = v$ solve the equation

$$axyp^2 + (x^2 - ay^2 - b)p - xy = 0$$

34. By the substitution $x^2 = u, y^2 = v$, reduce the equation $x^2 + y^2 - (p + p^{-1})xy = c^2$ to Clairaut's form and find the general integral and the singular solution.

35. Solve the equation $(px + y)^2 = py^2$ using the transformation $u = y, v = xy$.

36. Solve the equation $y^2(y - xp) = x^4p^2$ using the substitution $x = 1/u, y = 1/v$.

Answers

Section A Multiple Choice Questions

1. (a) 2. (b) 3. (b) 4. (c) 5. (a) 6. (d) 7. (d) 8. (a) 9. (b)
10. (c) 11. (a) 12. (c)

Section B Review Questions

1. $(x^2 + y^2 - c)(xy - c) = 0$
2. $(y^2 - 2y - x - c)(y^2 - x^2 - c) = 0$
3. $y(1 \pm \cos x) = c$
4. $(x - y + c)(x^2 + y^2 + c) = 0$
5. $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$
6. $(2y - x^2 + c)(y + x + ce^{-x} - 1) = 0$
7. $x^2 + y^2 = cx$
8. $(1 - x - y - ce^{-x})(2xy + x^2 - c)(y + x^2 - c) = 0$

9. $(y^2 - x^2 - c)(y^2 - cx^4 - x^2) = 0$
10. $(y - c)(ye^{1/x} - c)(xy + cy - 1) = 0$
11. $cxy + 2x\sqrt{1 + cx} = 0$
12. $y = \frac{2c}{p} + \frac{1-n}{1+n}p^n$
13. $x = c + \frac{a}{2}\left[\log \frac{p-1}{\sqrt{1+p^2}} - \tan^{-1} p\right]$
14. $y = 2\sqrt{cx} + \tan^{-1} c$
15. $xy = c^2x + c$
16. $y = 2\sqrt{cx} + c^2$
17. $y^2 = 2cx + c^3$
18. $y = c(x - c)^2$
19. $y + (1 + p^2)^{-1} = c$
20. $x = 2a(1 - p) + ce^{-p}$
21. $p(p + 1) = c(p - 1)e^x$
22. $y = 2c\sqrt{x} + f(c^2)$
23. $y = c(x - c)^2$
24. $x = \tan p + c$
25. $x + (1/p) + c = 2\tan^{-1} p$
26. $y - c = \sqrt{y^2 - x^2} - \tan^{-1} \sqrt{(1 - x)/x}$
27. $y = cx + 1 - 2/c$
28. $y = cx + c/(c - 1)$
29. $y = cx - ac^2/(c + 1)$
30. $y = cx - e^c$
31. $y = cx - \sin^{-1} c$
32. $xy = cy + e^2$
33. $y^2 = cx^2 - bc/(ac + 1)$
34. $y^2 = Ax^2 + \{c^2A/(A - 1)\}$
35. $xy = cy - c^2$
36. $x = yc + xyc^2$

CHAPTER 4

Linear Differential Equations of Second and Higher Order

4.1 Introduction

An nth order differential equation is called **linear** if it is of **first degree** in the dependent variable y and its derivatives $y', y'', \ldots, y^{(n)}$. That is, the general form of an nth order linear differential equation is

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = g(x) \tag{4.1}$$

where $a_1, a_2, \ldots, a_n$ and $g(x)$ are functions of x.

If $g(x) \neq 0$, then the equation is called a **non-homogeneous linear equation**. If $g(x) = 0$, then the equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{4.2}$$

is called a **homogeneous linear equation**.

Two solutions y_1 and y_2 of (4.2) are called **dependent** if $c_1 y_1 + c_2 y_2 = 0$ for $c_1 \neq 0$ and $c_2 \neq 0$, i.e. if the ratio $\frac{y_1}{y_2} = -\frac{c_2}{c_1} =$ a constant. Otherwise, the two solutions are called **linearly independent**.

In general, a set of solutions $y_1, y_2, \ldots, y_n$ of an nth order differential equation is said to be linearly independent if $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$ implies that $c_i = 0$ for all $i = 1, 2, \ldots, n$.

The dependence of a set of functions can be tested by computing the value of the **Wronskian**, which is defined below.

Wronskian

Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two functions. Then $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$ is called the **Wronskian** of two functions y_1 and y_2 and it is denoted by $W(y_1, y_2)$.

If y_1 and y_2 are linearly dependent, then the Wronskian is identically 0, for if $y_1 = c y_2$, c is a constant, then $y_1' = c y_2'$ and hence

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = y_1 c y_1' - y_1' c y_1 = 0$$

The Wronskian of n functions $y_1, y_2, \dots, y_n$ is

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

It can be shown by the previous method that if $y_1, y_2, \dots, y_n$ are dependent, then

$$W(y_1, y_2, \dots, y_n) = 0$$

Theorem 4.1 *If y_1 and y_2 are two linearly independently solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{4.3}$$

on $[a, b]$ in which $p(x)$ and $q(x)$ are continuous, then their Wronskian is not zero at any point on $[a, b]$.

It can be shown that every nth order linear homogeneous differential equation, i.e. the equation of the form (4.2) has n independent solutions.

Theorem 4.2 *If y_1, y_2 are two solutions of the homogeneous Eq. (4.2), then $u = c_1y_1 + c_2y_2$ is also its solution.*

Proof. Let $y = y_1$ and $y = y_2$ be two solutions of the homogeneous Eq. (4.2). Then

$$\frac{d^n y_1}{dx^n} + a_1\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_1}{dx} + a_n y_1 = 0 \tag{4.4}$$

and

$$\frac{d^n y_2}{dx^n} + a_1\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_2}{dx} + a_n y_2 = 0 \tag{4.5}$$

Let $u = c_1y_1 + c_2y_2$.

Therefore

$$\begin{aligned}
&\frac{d^n u}{dx^n} + a_1\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_{n-1}\frac{du}{dx} + a_n u \\
&= \frac{d^n(c_1y_1 + c_2y_2)}{dx^n} + a_1\frac{d^{n-1}(c_1y_1 + c_2y_2)}{dx^{n-1}} + \cdots + a_{n-1}\frac{d(c_1y_1 + c_2y_2)}{dx} + a_n(c_1y_1 + c_2y_2) \\
&= c_1\left(\frac{d^n y_1}{dx^n} + a_1\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_1}{dx} + a_n y_1\right) \\
&\quad + c_2\left(\frac{d^n y_2}{dx^n} + a_1\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_2}{dx} + a_n y_2\right) \\
&= c_1.0 + c_2.0 \qquad \text{[using (4.4) and (4.5)]} \\
&= 0
\end{aligned}$$

Hence the result.

In general, if $y_1, y_2, \ldots, y_n$ are n independent solutions of (4.2), then $c_1y_1 + c_2y_2 + \cdots + c_ny_n(= u)$ where $c_1, c_2, \ldots, c_n$ are arbitrary constants, is the complete solution of (4.2). That is

$$\frac{d^nu}{dx^n} + a_1\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_{n-1}\frac{du}{dx} + a_nu = 0 \tag{4.6}$$

If $y = v$ be any particular solution of (4.1), then

$$\frac{d^nv}{dx^n} + a_1\frac{d^{n-1}v}{dx^{n-1}} + \cdots + a_{n-1}\frac{dv}{dx} + a_nv = g(x) \tag{4.7}$$

Adding (4.6) and (4.7), we have

$$\frac{d^n(u+v)}{dx^n} + a_1\frac{d^{n-1}(u+v)}{dx^{n-1}} + \cdots + a_{n-1}\frac{d(u+v)}{dx} + a_n(u+v) = g(x) \tag{4.8}$$

This shows that $y = u + v$ is also a solution of (4.1), and this solution is the complete solution of (4.1).

The part u is called the **complementary function (C.F.)** and the part v is called the **particular integral (P.I.)** of (4.1). Thus the complete solution is

$$y = u + v = \text{ C.F. + P.I.} \tag{4.9}$$

Introducing the operators D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, etc. We write the Eq. (4.1) in the form

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = g(x)$$

The expression $D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$ is called the **linear differential operator of order** n and is denoted by $F(D)$.

4.2 Complementary Function

Let us consider Eq. (4.1), i.e.

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = 0 \tag{4.10}$$

where the coefficients $a_1, a_2, \ldots, a_n$ are constants.

Let $y = ce^{mx}$, where c is arbitrary constant, be a possible solution of (4.10). Then $y' = Dy = mce^{mx}, y'' = D^2y = m^2ce^{mx}, \ldots, y^{(n)} = D^ny = m^nce^{mx}$. Substituting these values, (4.10) reduces to

$$cm^ne^{mx} + a_1cm^{n-1}e^{mx} + a_2cm^{n-2}e^{mx} + \cdots + a_{n-1}cme^{mx} + a_nce^{mx} = 0$$

Since $ce^{mx} \neq 0$ for all m and x, we have

$$m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n = 0 \tag{4.11}$$

This equation is known as the **auxiliary equation (A.E.)** of (4.10) and this equation can easily be obtained from (4.11) by replacing y' with m, y'' with $m^2, \ldots, y^{(n)}$ by m^n.

Equation (4.11) is an algebraic equation in m of degree n and has exactly n roots. Let $m_1, m_2, \ldots, m_n$ be its roots. These roots may be distinct, equal or complex.

Case I. *Let all the roots $m_1, m_2, \ldots, m_n$ be real and distinct.*

In this case, the Eq. (4.10) can be expressed as

$$(D - m_1)(D - m_2)\cdots(D - m_n)y = 0 \tag{4.12}$$

This equation is satisfied by $(D - m_n)y = 0$, that is, $\frac{dy}{dx} - m_n y = 0$, or $\frac{dy}{y} = m_n dx$. Integrating, we get $y = c_n e^{m_n x}$, where c_n is arbitrary constant.

Since the factor of the above equation can be taken in any order, $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$, etc. are also the solutions. Hence the complete solution of (4.10) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \tag{4.13}$$

Case II. *Let two roots m_1, m_2 be equal (i.e. $m_1 = m_2$).*

In this case, the solution (4.13) reduces to

$$\begin{aligned} y &= (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\ &= c e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \end{aligned}$$

which contains $(n - 1)$ arbitrary constants, so it is not the complete solution.

The part of (4.12) corresponding to the roots m_1, m_1 is $(D - m_1)(D - m_1) = 0$. Let $(D - m_1)y = z$. Then $(D - m_1)z = 0$, i.e. $z = c_1 e^{m_1 x}$ (as in the previous case).

Therefore, $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$, which is linear in y and I.F. is $e^{-m_1 x}$.

Thus, $\frac{d}{dx}(y e^{-m_1 x}) = c_1$. Integrating, $y e^{-m_1 x} = c_1 x + c_2$, or $y = (c_1 x + c_2)e^{m_1 x}$.

Thus the complete solution of (4.10) is

$$y = (c_1 x + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

In case of k equal roots, the complete solution is

$$y = (c_1 x + c_2 x^2 + \cdots + c_k x^k)e^{m_1 x} + c_{k+1} e^{m_{k+1} x} + \cdots + c_n e^{m_n x}$$

Case III. *Let a pair of roots be complex.*

Let the roots be $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Then the part of the solution is

$$\begin{aligned}
y &= c_1 e^{m_1 x} + c_2 e^{m_2 x} \\
&= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\
&= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) \\
&= e^{\alpha x}\{c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\} \\
&= e^{\alpha x}\{(c_1+c_2)\cos\beta x + (c_1 i - c_2 i)\sin\beta x\} \\
&= e^{\alpha x}(A\cos\beta x + B\sin\beta x)
\end{aligned}$$

where $A = c_1 + c_2$ and $B = (c_1 - c_2)i$ are arbitrary constants.

Thus the complete solution of (4.10) is

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

If two pair of complex roots be equal, i.e. $m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta$, then the part of complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x]$$

EXAMPLE 4.2.1 Find the complete solution of $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0.$

Solution Let $y = ce^{mx}$ be a solution. Then $y' = cme^{mx}$ and $y'' = cm^2e^{mx}$. The given equation reduces to

$$cm^2e^{mx} - cme^{mx} - 6ce^{mx} = 0$$

or

$$m^2 - m - 6 = 0$$

or

$$(m-3)(m+2) = 0$$

or

$$m = 3, -2$$

Thus the complete solution is $y = c_1e^{3x} + c_2e^{-2x}$.

EXAMPLE 4.2.2 Solve $(D^3 - 5D^2 + 8D - 4)y = 0.$

Solution Let $y = ce^{mx}$ be a solution. Then *A.E.* is

$$m^3 - 5m^2 + 8m - 4 = 0$$

or

$$(m-1)(m^2 - 4m + 4) = 0$$

or

$$(m-1)(m-2)^2 = 0$$

or

$$m = 1, 2, 2$$

Therefore, the complete solution is $y = c_1 e^x + (c_2 x + c_3)e^{2x}$

EXAMPLE 4.2.3 Solve $D^2y + 9y = 0$. (WBUT 200

Solution Let $y = ce^{mx}$ be a solution. Then $A.E.$ is

$$m^2 + 9 = 0$$

or

$$m = \pm 3i = 0 \pm 3i$$

Therefore, the complete solution is

$$y = e^{0x}(c_1 \cos 3x + c_2 \sin 3x) = c_1 \cos 3x + c_2 \sin 3x$$

EXAMPLE 4.2.4 Solve $(D^2 + 1)^3 y = 0$.

Solution In this case, the $A.E.$ is $(m^2 + 1)^3 = 0$, or $m = \pm i, \pm i, \pm i$.
Therefore, the complete solution is

$$y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$$

4.3 Particular Integral (P.I.)

To obtain a P.I. of $F(D)y = g(x)$, we apply the operator $[F(D)]^{-1}$ or $\frac{1}{F(D)}$, called the **inverse operator**, we have

$$\frac{1}{F(D)}F(D)y = y = \frac{1}{F(D)}g(x)$$

Suppose that $F(D) = (D - m_1)(D - m_2)\cdots(D - m_n)$ has no repeated factor. So, $\frac{1}{F(D)}$ can be expressed as

$$\frac{1}{F(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \cdots + \frac{A_n}{D - m_n} = \sum_{i=1}^{n} \frac{A_i}{D - m_i}$$

Then

$$y = \frac{1}{F(D)}g(x) = \sum_{i=1}^{n} \frac{A_i}{D - m_i}g(x) \tag{4.14}$$

Let $\dfrac{A_i}{D - m_i}g(x) = z$, i.e. $(D - m_i)z = A_i g(x)$, or $\dfrac{dz}{dx} - m_i z = A_i g(x)$.

This is a linear differential equation and its I.F. is $e^{-m_i x}$. Multiplying this equation by I.F. and integrating, we get

$$ze^{-m_i x} = \int A_i g(x) e^{-m_i x} dx \qquad \text{or} \qquad z = e^{m_i x} \int A_i g(x) e^{-m_i x} dx$$

Equation (4.14) reduces to

$$y = \sum_{i=1}^{n} A_i e^{m_i x} \int g(x) e^{-m_i x} dx \tag{4.15}$$

We note that to find the P.I. the arbitrary constant which arises in integration need not be written down. The appropriate number of constants are included in the C.F.

Inverse Operators

Case I. $\frac{1}{D} g(x) = \int g(x) dx$

Let $\frac{1}{D} g(x) = z$. That is, $Dz = g(x)$, or $\frac{dz}{dx} = g(x)$ or $z = \int g(x) dx$.
Hence

$$\frac{1}{D} g(x) = \int g(x) dx \tag{4.16}$$

Case II. $\frac{1}{D-a} g(x) = e^{ax} \int g(x) e^{-ax} dx$

Let

$$\frac{1}{D-a} g(x) = z \qquad \text{or} \qquad \frac{dz}{dx} - az = g(x)$$

This is a linear equation and I.F. $= e^{-ax}$.

Its solution is

$$ze^{-ax} = \int g(x) e^{-ax} dx \qquad \text{or} \qquad z = e^{ax} \int g(x) e^{-ax} dx$$

Hence

$$\frac{1}{D-a} g(x) = e^{ax} \int g(x) e^{-ax} dx$$

4.3.1 Rules for Finding Particular Integral

In general, the P.I. of the Eq. (4.10) is

$$\text{P.I.} = \frac{1}{D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n} g(x) = \frac{1}{F(D)} g(x)$$

Case I. Let $g(x) = e^{ax}$.
Then

$$F(D)e^{ax} = (D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n)e^{ax}$$
$$= a^n e^{ax} + a_1 a^{n-1} e^{ax} + \cdots + a_{n-1} a e^{ax} + a_n e^{ax}$$
$$= (a^n + a_1 a^{n-1} + a_2 a^{n-2} + \cdots + a_{n-1}a + a_n)e^{ax}$$
$$= F(a)e^{ax}$$

Operating both side by $\frac{1}{F(D)}$, we get

$$e^{ax} = \frac{1}{F(D)} F(a) e^{ax}$$

i.e.

$$\frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}, \qquad \text{provided } F(a) \neq 0$$

If $F(a) = 0$, then there should be a factor $(D - a)$ in $F(D)$. Let $F(D) = (D - a)F_1(D)$, s and $F_1(a) \neq 0$.

Then

$$\frac{1}{F(D)} e^{ax} = \frac{1}{(D-a)F_1(D)} e^{ax} = \frac{1}{D-a} \frac{1}{F_1(a)} e^{ax}$$
$$= \frac{1}{F_1(a)} \frac{1}{D-a} e^{ax} = \frac{1}{F_1(a)} e^{ax} \int dx$$
$$= \frac{1}{F_1(a)} x e^{ax}$$

i.e.

$$\frac{1}{F(D)} e^{ax} = x \frac{1}{F'(a)} e^{ax}$$

$$\left[\text{as } F'(D) = F_1(D) + (D-a)F_1'(D), \text{ or } F'(a) = F_1(a)\right] \text{ provi}$$
$$F'(a) \neq 0$$

Again, if $F'(a) = 0$, then by the same method, $\frac{1}{F(D)} e^{ax} = x^2 \frac{1}{F''(a)} e^{ax}$, provided $F''(a) \neq$ (

EXAMPLE 4.3.1 Find the particular integral of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 9e^x$.

Solution The given equation is $(D^2 - 3D + 2)y = 9e^x$.
The P.I. is

$$\frac{1}{D^2 - 3D + 2} 9e^x = \frac{1}{(D-1)(D-2)} 9e^x = 9 \frac{1}{D-1} \frac{1}{D-2} e^x$$
$$= 9 \frac{1}{D-1} \frac{1}{1-2} e^x$$
$$= -9 \frac{1}{D-1} e^x = -9xe^x$$

Case II. Let $g(x) = \sin ax$ or $\cos ax$.

We know that

$$D(\sin ax) = a\cos ax \qquad D^2(\sin ax) = -a^2 \sin ax$$
$$D^3(\sin ax) = -a^3\cos ax \qquad D^4(\sin ax) = a^4 \sin ax$$
$$D^5(\sin ax) = a^5\cos ax \qquad D^6(\sin ax) = -a^6 \sin ax$$

Thus we observed that

$$D^2(\sin ax) = -a^2 \sin ax \qquad (D^2)^2(\sin ax) = (-a^2)^2 \sin ax$$
$$(D^2)^3(\sin ax) = (-a^2)^3 \sin ax \qquad (D^2)^4(\sin ax) = (-a^2)^4 \sin ax$$

and so on, and

$$(D^2)^n(\sin ax) = (-a^2)^n \sin ax$$

Hence $F(D^2)(\sin ax) = F(-a^2)\sin ax$.

That is

$$\frac{1}{F(D^2)}F(D^2)\sin ax = \frac{1}{F(D^2)}F(-a^2)\sin ax$$

or

$$\sin ax = F(-a^2)\frac{1}{F(D^2)}\sin ax$$

Thus

$$\frac{1}{F(D^2)}\sin ax = \frac{\sin ax}{F(-a^2)} \tag{4.17}$$

Similarly

$$\frac{1}{F(D^2)}\cos ax = \frac{\cos ax}{F(-a^2)} \tag{4.18}$$

In both cases, $F(-a^2) \neq 0$. In general,

$$\frac{1}{F(D^2)}\sin(ax+b) = \frac{1}{F(-a^2)}\sin(ax+b)$$

and

$$\frac{1}{F(D^2)}\cos(ax+b) = \frac{1}{F(-a^2)}\cos(ax+b)$$

provided $F(-a^2) \neq 0$.

If $F(-a^2) = 0$, then the above formula is not applicable. We proceed as follows: Since $\cos ax + i\sin ax = e^{iax}$, so the imaginary part (I.P.) of e^{iax} is equal to $\sin ax$. Therefore

$$\begin{aligned}\frac{1}{F(D^2)}\sin ax &= \text{I.P. of } \frac{1}{F(D^2)}e^{iax}\\ &= \text{I.P. of } x\frac{1}{F'(D^2)}e^{iax} \qquad [\because F(-a^2) = 0 \text{ as } D^2 = -a^2]\end{aligned}$$

Thus

$$\frac{1}{F(D^2)}\sin ax = x\frac{1}{F'(-a^2)}\sin ax \qquad \text{provided } F'(-a^2) \neq 0 \tag{4.1}$$

Again, if $F'(-a^2) = 0$, then

$$\frac{1}{F(D^2)}\sin ax = x^2\frac{1}{F''(-a^2)}\sin ax \qquad \text{provided } F''(-a^2) \neq 0 \tag{4.2}$$

EXAMPLE 4.3.2 Find the P.I. of $\dfrac{d^3y}{dx^3} + 9\dfrac{dy}{dx} = \cos 3x$.

Solution This equation can be written as $(D^3 + 9D)y = \cos 3x$. Therefore

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D(D^2+9)}\cos 3x \qquad [\text{Here } D^2 = -9, F(D^2) = 0] \\
&= x\frac{1}{3D^2+9}\cos 3x \qquad [\text{Using the rule (4.19)}] \\
&= x\frac{1}{3(-9)+9}\cos 3x \qquad (\text{Putting } D^2 = -9) \\
&= -\frac{x}{18}\cos 3x
\end{aligned}$$

Case III. Let $g(x) = x^m$. Here

$$\text{P.I.} = \frac{1}{F(D)}x^m = [F(D)]^{-1}x^m$$

From $F(D)$, take out the lowest degree term and the remaining factor be of the form $[1 \pm F_1(D)]^{-1}$. Then expand $[1 \pm F_1(D)]^{-1}$ in ascending power of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)th$ and higher order derivatives of x^m is zero we need not consider terms beyond D^m.

EXAMPLE 4.3.3 Find the P.I. of $(2D^2 + 5D + 2)y = 5 + 2x$.

Solution

$$\begin{aligned}
\text{P.I.} &= \frac{1}{2D^2+5D+2}(5+2x) \\
&= \frac{1}{2}\frac{1}{1+\frac{5D+2D^2}{2}}(5+2x) \\
&= \frac{1}{2}\left(1+\frac{5D+2D^2}{2}\right)^{-1}(5+2x)
\end{aligned}$$

$$= \frac{1}{2}\left(1 - \frac{5D + 2D^2}{2} + \cdots\right)(5 + 2x)$$

$$= \frac{1}{2}\left[5 + 2x - \frac{5}{2}(2)\right]$$

$$= \frac{1}{2} \cdot 2x = x$$

Case IV. When $g(x) = e^{ax}f(x)$, $f(x)$ is a function of x.

Let v is a function of x, then

$$D(e^{ax}v) = e^{ax}Dv + ae^{ax}v = e^{ax}(D + a)v$$

$$D^2(e^{ax}v) = e^{ax}D^2v + 2ae^{ax}Dv + a^2e^{ax}v = e^{ax}(D + a)^2v$$

In this way

$$D^n(e^{ax}v) = e^{ax}(D + a)^n v$$

Therefore,

$$F(D)(e^{ax}v) = e^{ax}F(D + a)v$$

Operating both sides by $\dfrac{1}{F(D)}$

$$\frac{1}{F(D)}F(D)(e^{ax}v) = \frac{1}{F(D)}[e^{ax}F(D + a)v]$$

or

$$e^{ax}v = \frac{1}{F(D)}[e^{ax}F(D + a)v]$$

Let $F(D + a)v = f(x)$. That is, $v = \dfrac{1}{F(D + a)}f(x)$.

Thus, $e^{ax}\dfrac{1}{F(D + a)}f(x) = \dfrac{1}{F(D)}[e^{ax}f(x)]$.

Finally

$$\frac{1}{F(D)}[e^{ax}f(x)] = e^{ax}\frac{1}{F(D + a)}f(x) \tag{4.21}$$

EXAMPLE 4.3.4 Solve $(D^2 - 2D)y = e^x \sin x$. (WBUT 2007)

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 2D)y = 0$. Therefore, A.E. is

$$m^2 - 2m = 0 \qquad \text{or} \qquad m = 0, 2$$

Therefore C.F. is $c_1 + c_2 e^{2x}$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D(D-2)} e^{x} \sin x = e^{x} \frac{1}{(D+1)(D+1-2)} \sin x \\
&= e^{x} \frac{1}{(D^2-1)} \sin x = e^{x} \frac{1}{(-1^2-1)} \sin x \\
&= -\frac{1}{2} e^{x} \sin x
\end{aligned}$$

Hence the complete solution is $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^{x} \sin x$.

4.4 Additional Worked-Out Examples

EXAMPLE 4.4.1 Solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x + \sin x$.

Solution Let the equation be $(D^2 + D - 2)y = x + \sin x$ and $y = ce^{mx}$ be a solution of $(D^2 + D - 2)y = 0$.

$\therefore$ A.E. is $m^2 + m - 2 = 0$, or $m = -2, 1$

$\therefore$ C.F. is $c_1 e^{-2x} + c_2 e^{x}$.

$$\begin{aligned}
\text{P.I. for } x &= \frac{1}{D^2 + D - 2} x = -\frac{1}{2}\left(1 - \frac{D^2 + D}{2}\right)^{-1} x \\
&= -\frac{1}{2}\left(1 + \frac{D^2 + D}{2} + \cdots\right) x = -\frac{1}{2}\left(x + \frac{1}{2}\right) \\
&= -\frac{1}{4}(2x + 1)
\end{aligned}$$

$$\begin{aligned}
\text{P.I. for } \sin x &= \frac{1}{D^2 + D - 2} \sin x \\
&= \frac{1}{-1 + D - 2} \sin x \qquad (\text{Putting } D^2 = -1) \\
&= \frac{D+3}{D^2 - 9} \sin x = \frac{1}{-1-9}[D(\sin x) + 3\sin x] \\
&= -\frac{1}{10}(\cos x + 3\sin x)
\end{aligned}$$

Therefore, the complete solution is $y = c_1 e^{-2x} + c_2 e^{x} - \frac{1}{10}(\cos x + 3\sin x)$.

EXAMPLE 4.4.2 Solve $(D^2 - 4D + 3)y = e^{2x} \sin x$.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 4D + 3)y = 0$.

Then A.E. is $m^2 - 4m + 3 = 0$ or $m = 3, 1$. The C.F. is $c_1e^x + c_2e^{3x}$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 4D + 3}e^{2x}\sin x \\
&= e^{2x}\frac{1}{(D+2)^2 - 4(D+2) + 3}\sin x \\
&= e^{2x}\frac{1}{D^2 - 1}\sin x \\
&= e^{2x}\frac{1}{-1-1}\sin x \qquad (\text{Putting } D^2 = -a^2 = -1) \\
&= -\frac{1}{2}e^{2x}\sin x
\end{aligned}$$

Therefore, the general solution is $y = c_1e^x + c_2e^{3x} - \frac{1}{2}e^{2x}\sin x$.

EXAMPLE 4.4.3 Solve $(D^2 - 2D + 1)y = xe^x \sin x$.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 2D + 1)y = 0$. Then A.E. is $m^2 - 2m + 1 = 0$, or $m = 1, 1$. Therefore, C.F. is $(c_1 + c_2x)e^x$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D-1)^2}xe^x\sin x = e^x\frac{1}{(D+1-1)^2}x\sin x \\
&= e^x\frac{1}{D^2}x\sin x = e^x\frac{1}{D}\frac{1}{D}x\sin x \\
&= e^x\frac{1}{D}\int x\sin x\,dx = e^x\frac{1}{D}(-x\cos x + \sin x) \\
&= e^x\Big[-\int x\cos x\,dx + \int \sin x\,dx\Big] \\
&= e^x\Big[-x\sin x + \int \sin x\,dx + \int \sin x\,dx\Big] \\
&= e^x(-x\sin x - 2\cos x)
\end{aligned}$$

Therefore, the general solution is

$$y = (c_1 + c_2x)e^x - e^x(x\sin x + 2\cos x)$$

EXAMPLE 4.4.4 Solve $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x$.

Solution The given equation is $(D^3 - 2D^2 + D)y = e^{2x} + x$.
Let $y = ce^{mx} (\neq 0)$ be a trial solution of $(D^3 - 2D^2 + D)y = 0$.
The A.E. is $m^3 - 2m^2 + m = 0$, or $m = 0, 1, 1$.
$\therefore$ C.F. is $c_1 + (c_2 + c_3x)e^x$, where c_1, c_2, c_3 are arbitrary constants.

$$\frac{e^{2x}}{D(D-1)^2} = \frac{\frac{1}{2}e^{2x}}{D(2-1)^2} = \frac{1}{2}\frac{e^{2x}}{D} = \frac{e^{2x}}{2}$$

P.I. of $e^{2x} + x$

$$= \frac{1}{D^3 - 2D^2 + D}(e^{2x} + x) = \frac{1}{2}e^{2x} + \frac{1}{D(1 + D^2 - 2D)}x$$

$$= \frac{e^{2x}}{2} + \frac{1}{D}(1 + D^2 - 2D)^{-1}x = \frac{e^{2x}}{2} + \frac{1}{D}(1 - D^2 + 2D - \cdots)x$$

$$= \frac{e^{2x}}{2} + \frac{1}{D}(x - 0 + 2) = \frac{e^{2x}}{2} + \frac{x^2}{2} + 2x$$

Therefore, the general solution is

$$y = c_1 + (c_2 + c_3 x)e^x + \frac{e^{2x}}{2} + \frac{x^2}{2} + 2x$$

where c_1, c_2, c_3 are arbitrary constants.

EXAMPLE 4.4.5 Solve $\dfrac{d^2x}{dt^2} + 2x = t^2e^{3t} + e^t \cos 2t$.

Solution The given equation is $(D^2 + 2)x = t^2e^{3t} + e^t\cos 2t$, where $D \equiv \frac{d}{dt}$.

Let $x = ce^{mt}$ be a trial solution.

$\therefore$ A.E. is $m^2 + 2 = 0$, or $m = \pm\sqrt{2}i$.

$\therefore$ C.F. is $c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t$.

$$\text{P.I.} = \frac{1}{D^2 + 2}(t^2e^{3t} + e^t\cos 2t) = e^{3t}\frac{1}{(D+3)^2 + 2}t^2 + e^t\frac{1}{(D+1)^2 + 2}\cos 2t$$

$$= e^{3t}\frac{1}{D^2 + 6D + 11}t^2 + e^t\frac{1}{D^2 + 2D + 3}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(1 + \frac{D^2 + 6D}{11}\right)^{-1}t^2 + e^t\frac{1}{-4 + 2D + 3}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(1 - \frac{D^2 + 6D}{11} + \left(\frac{D^2 + 6D}{11}\right)^2 + \cdots\right)t^2 + e^t\frac{1}{2D - 1}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(t^2 - \frac{12t}{11} + \frac{50}{121}\right) + e^t\frac{2D + 1}{4D^2 - 1}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(t^2 - \frac{12t}{11} + \frac{50}{121}\right) + e^t\frac{2D + 1}{4(-4) - 1}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(t^2 - \frac{12t}{11} + \frac{50}{121}\right) - \frac{e^t}{17}(-4\sin 2t + \cos 2t)$$

Therefore, the general solution is

$$x = c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t + \frac{e^{3t}}{11}\left(t^2 - \frac{12t}{11} + \frac{50}{121}\right) - \frac{e^t}{17}(-4\sin 2t + \cos 2t)$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.4.6 Solve $(D^2+4)y = x\sin^2 x$. (WBUT 2008)

Solution Let $y = ce^{mx}$ be a solution of $(D^2+4)y = 0$

$\therefore$ A.E. is $m^2+4=0$, or $m = \pm 2i$.

Thus, C.F. is $c_1\cos 2x + c_2 \sin 2x$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+4}x\sin^2 x = \frac{1}{D^2+4}\cdot\frac{1}{2}x(1-\cos 2x)\\
&= \frac{1}{2}\cdot\frac{1}{D^2+4}x - \frac{1}{2}\cdot\frac{1}{D^2+4}x\cos 2x\\
&= \frac{1}{8}\left(1+\frac{D^2}{4}\right)^{-1}x - \frac{1}{2}\text{ R.P. of }\frac{1}{D^2+4}xe^{2ix} \qquad (\because \cos x = \text{Real part of } e^{2ix})\\
&= \frac{1}{8}\left(1-\frac{D^2}{4}+\cdots\right)x - \frac{1}{2}\text{ R.P. of }e^{2ix}\frac{1}{(D+2i)^2+4}x\\
&= \frac{1}{8}x - \frac{1}{2}\text{ R.P. of }e^{2ix}\frac{1}{D^2+4Di}x\\
&= \frac{1}{8}x - \frac{1}{2}\text{ R.P. of }e^{2ix}\frac{1}{4Di}\left(1+\frac{D}{4i}\right)^{-1}x\\
&= \frac{1}{8}x - \frac{1}{8}\text{ R.P. of }e^{2ix}\frac{1}{Di}\left(x-\frac{1}{4i}\right)\\
&= \frac{1}{8}x - \frac{1}{8}\text{ R.P. of }e^{2ix}\frac{1}{i}\left(\frac{x^2}{2}-\frac{x}{4i}\right)\\
&= \frac{1}{8}x - \frac{1}{8}\text{ R.P. of }(\cos 2x + i\sin 2x)\left(-\frac{ix^2}{2}+\frac{x}{4}\right)\\
&= \frac{1}{8}x - \frac{1}{8}\left(\frac{x}{4}\cos 2x + \frac{x^2}{2}\sin 2x\right)\\
&= \frac{1}{8}x - \frac{1}{32}(x\cos 2x + 2x^2\sin 2x)
\end{aligned}$$

Therefore, the general solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{8}x - \frac{1}{32}(x\cos 2x + 2x^2\sin 2x)$$

EXAMPLE 4.4.7 Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x\cos x$.

Solution Let $y = ce^{mx}$ be a trial solution.

$\therefore$ A.E. is $m^2-2m+2=0$, or $m = \dfrac{2\pm\sqrt{4i}}{2} = 1\pm i$.

$\therefore$ C.F. is $(c_1 \cos x + c_2 \sin x)e^x$, where c_1, c_2 are arbitrary constants.

$$\text{P.I} = \frac{1}{D^2 - 2D + 2} e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} \cos x$$

$$= e^x \frac{1}{D^2+1} \cos x = e^x \cdot \text{ R.P. of } \frac{1}{D^2+1} e^{ix}$$

$$= e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{(D+i)^2+1} 1 = e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{D^2 + 2Di} 1$$

$$= e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{2Di}\left(1 + \frac{D}{2i}\right)^{-1} 1 = e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{2Di} 1$$

$$= e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{2i} x = xe^x \cdot \text{ R.P. of } (\cos x + i \sin x) \frac{1}{2i}$$

$$= xe^x \cdot \text{ R.P. of } \frac{1}{2}(-i\cos x + \sin x) = \frac{1}{2} xe^x \sin x$$

Therefore, the general solution is $y = (c_1 \cos x + c_2 \sin x)e^x + \frac{1}{2} xe^x \sin x$, where c_1, c_2 are arbitrary constants.

EXAMPLE 4.4.8 Solve $\dfrac{d^4 y}{dx^4} + 2\dfrac{d^2 y}{dx^2} + y = x^2 \cos x$.

Solution We have $(D^4 + 2D^2 + 1)y = x^2 \cos x$, or $(D^2+1)^2 y = x^2 \cos x$.

Let $y = ce^{mx}$ be a trial solution.

$\therefore$ A.E. is $(m^2+1)^2 = 0$, or $m^2 + 1 = 0$ (twice), or $m = \pm i$ (twice).

$\therefore$ C.F. is $(c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x$, where c_1, c_2, c_3 and c_4 are arbitrary constants

P.I. $\dfrac{1}{(D^2+1)^2} x^2 \cos x$

$$= \text{R.P. of } \frac{1}{(D^2+1)^2} x^2 e^{ix} = \text{R.P. of } e^{ix} \frac{1}{\{(D+i)^2+1\}^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{(D^2 + 2Di)^2} x^2 = \text{R.P. of } e^{ix} \frac{1}{-4D^2\left(1 + \frac{D}{2i}\right)^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{-4D^2}\left(1 - \frac{2D}{2i} - \frac{3D^2}{4} - \cdots\right) x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{-4D^2}\left(x^2 + i \cdot 2x - \frac{3}{4} \cdot 2\right)$$

$$= \text{R.P. of } e^{ix} \frac{1}{-4D}\left(\frac{x^3}{3} + i \cdot x^2 - \frac{3}{2} x\right)$$

$$= \text{R.P. of } e^{ix}\frac{1}{-4}\left(\frac{x^4}{12} + i\cdot\frac{x^3}{3} - \frac{3x^2}{4}\right)$$

$$= -\frac{1}{48}\text{R.P. of } (\cos x + i\sin x)(x^4 + 4x^3 i - 9x^2)$$

$$= -\frac{1}{48}\{(x^4 - 9x^2)\cos x - 4x^3\sin x\}$$

Therefore, the general solution is

$$y = (c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x - \frac{1}{48}\{(x^4 - 9x^2)\cos x - 4x^3\sin x\}$$

EXAMPLE 4.4.9 Solve $\dfrac{d^2y}{dx^2} + 2n\cos\alpha\dfrac{dy}{dx} + n^2y = a\cos nx$, where n, α are constants.

Solution The given equation is $(D^2 + 2n\cos\alpha D + n^2)y = a\cos nx$.
Let $y = ce^{mx}$ be a trial solution. Then A.E. is $m^2 + 2n\cos\alpha m + n^2 = 0$, or

$$m = \frac{-2n\cos\alpha \pm \sqrt{4n^2\cos^2\alpha - 4n^2}}{2} = -n\cos\alpha \pm ni\sin\alpha$$

$\therefore$ C.F. is $[c_1\cos\{(n\sin\alpha)x\} + c_2\sin\{(n\sin\alpha)x\}]e^{-nx\cos\alpha}$.

P.I. is

$$\frac{1}{D^2 + 2n\cos\alpha D + n^2}a\cos nx$$

$$= \frac{(D^2 + n^2) - 2n\cos\alpha D}{(D^2 + n^2)^2 - 4n^2\cos^2\alpha D^2}a\cos nx$$

$$= \frac{(D^2 + n^2) - 2n\cos\alpha D}{-4n^2\cos^2\alpha(-n^2)}a\cos nx$$

$$= a\frac{-n^2\cos nx + n^2\cos nx - 2n\cos\alpha(-n\sin nx)}{4n^4\cos^2\alpha}$$

$$= \frac{a\sin nx}{2n^2\cos\alpha}$$

Therefore, the general solution is

$$y = [c_1\cos\{(n\sin\alpha)x\} + c_2\sin\{(n\sin\alpha)x\}]e^{-nx\cos\alpha} + \frac{a\sin nx}{2n^2\cos\alpha}$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 4.4.10 Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 3x^2e^{2x}\sin 2x$.

Solution Let $y = ce^{mx}$ be a trial solution of $(D^2 - 4D + 4)y = 0$.
$\therefore$ A.E. is $m^2 - 4m + 4 = 0$, or $(m-2)^2 = 0$, or $m = 2, 2$.

Then C.F. is $(c_1 + c_2x)e^{2x}$.

$$\text{P.I.} = \frac{1}{(D-2)^2}3x^2e^{2x}\sin 2x = e^{2x}\frac{1}{(D+2-2)^2}3x^2\sin 2x$$

$$= e^{2x}\frac{1}{D^2}3x^2\sin 2x = \text{I.P. of } e^{2x}\frac{1}{D^2}3x^2e^{2ix}$$

$$= \text{I.P. of } 3e^{2x}e^{2ix}\frac{1}{(D+2i)^2}x^2$$

$$= \text{I.P. of } 3e^{2x}e^{2ix}\frac{-1}{4}\left(1+\frac{D}{2i}\right)^{-2}x^2$$

$$= \text{I.P. of } 3e^{2x}e^{2ix}\frac{-1}{4}\left(1-\frac{D}{i}-\frac{3D^2}{4}+\cdots\right)x^2$$

$$= \text{I.P. of } 3e^{2x}e^{2ix}\frac{-1}{4}\left(x^2+i\cdot 2x-\frac{3}{4}2\right)$$

$$= \text{I.P. of } 3e^{2x}(\cos 2x + i\sin 2x)\frac{-1}{4}\left\{(x^2-3/2)+i\cdot 2x\right\}$$

$$= -\frac{3}{4}e^{2x}\{2x\cos 2x + (x^2-3/2)\sin 2x\}$$

Therefore, the general solution is

$$y = (c_1+c_2x)e^{2x} - \frac{3}{4}e^{2x}\{2x\cos 2x + (x^2-3/2)\sin 2x\}$$

EXAMPLE 4.4.11 Solve $\dfrac{d^2y}{dx^2} + a^2y = \sec ax + x\sin ax$.

Solution Let $y = ce^{mx}$ be a trial solution $(D^2+a^2)y = 0$.
A.E. is $m^2 + a^2 = 0$, or $m = \pm ai$.
C.F. is $c_1\cos ax + c_2\sin ax$.
P.I. of $x\sin ax$

$$= \frac{1}{D^2+a^2}x\sin ax = \text{I.P. of } \frac{1}{D^2+a^2}xe^{iax}$$

$$= \text{I.P. of } e^{iax}\frac{1}{(D+ia)^2+a^2}x$$

$$= \text{I.P. of } e^{iax}\frac{1}{D^2+2iaD}x$$

$$= \text{I.P. of } e^{iax}\frac{1}{2iaD}\left(1+\frac{D}{2ia}\right)^{-1}x$$

$$= \text{I.P. of } e^{iax}\frac{1}{2iaD}\left(1-\frac{D}{2ia}+\cdots\right)x$$

$$= \text{I.P. of } e^{iax}\frac{1}{2iaD}\left(x - \frac{1}{2ia}\right)$$

$$= \text{I.P. of } e^{iax}\frac{-i}{2a}\left(\frac{x^2}{2} + \frac{xi}{2a}\right)$$

$$= \text{I.P. of } (\cos ax + i\sin ax)\left(-\frac{x^2 i}{4a} + \frac{x}{4a^2}\right)$$

$$= -\frac{x^2}{4a}\cos ax + \frac{x}{4a^2}\sin ax$$

P.I. of $\sec ax$

$$= \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D+ia)(D-ia)}\sec ax = \frac{1}{2ia}\left(\frac{1}{D-ia} - \frac{1}{D+ia}\right)\sec ax$$

Now

$$\frac{1}{D-ia}\sec ax$$

$$= \frac{1}{D-ia}e^{iax}\frac{e^{-iax}}{\cos ax} = e^{iax}\frac{1}{D+ia-ia}\frac{e^{-iax}}{\cos ax}$$

$$= e^{iax}\frac{1}{D}\frac{\cos ax - i\sin ax}{\cos ax}$$

$$= e^{iax}\int (1 - i\tan ax)\, dx = e^{iax}(x - \frac{i}{a}\log\sec ax)$$

Similarly, $\dfrac{1}{D+ia}\sec ax = e^{-iax}\left(x + \dfrac{i}{a}\log\sec ax\right)$.

Hence P.I. of $\sec ax$

$$= \frac{1}{2ia}\left[e^{iax}\left(x - \frac{i}{a}\log\sec ax\right) - e^{-iax}\left(x + \frac{i}{a}\log\sec ax\right)\right]$$

$$= \frac{x}{a}\frac{e^{iax} - e^{-iax}}{2i} - \frac{1}{a^2}\log\sec ax\,\frac{e^{iax} + e^{-iax}}{2}$$

$$= \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax\log\sec ax$$

Therefore, the general solution is

$$y = c_1\cos ax + c_2\sin ax - \frac{x^2}{4a}\cos ax + \frac{x}{4a^2}\sin ax + \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax\log\sec ax$$

EXAMPLE 4.4.12 Solve $(D^2 - 3D + 2)y = e^x$, if $y = 3$ and $Dy = 3$ when $x = 0$.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 3D + 2)y = 0$.

Then A.E. is $m^2 - 3m + 2 = 0$, or $m = 1, 2$ and C.F. is $c_1e^x + c_2e^{2x}$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 3D + 2}e^x = \frac{1}{(D-1)(D-2)}e^x \\
&= \frac{1}{D-1}e^x\frac{1}{1-2} = -\frac{1}{D-1}e^x \\
&= -e^x\frac{1}{D+1-1}(1) = -e^x x
\end{aligned}$$

Therefore, the general solution is $y = c_1e^x + c_2e^{2x} - xe^x$.
We are given that when $x = 0$, then $y = 3$ and $Dy = 3$.
$\therefore \quad 3 = c_1 + c_2$ and $3 = c_1 + 2c_2 - 1$ (as $Dy = c_1e^x + 2c_2e^{2x} - e^x - xe^x$).
The solution of these equations is $c_1 = 2, c_2 = 1$.
Hence the required solution is $y = 2e^x + e^{2x} - xe^x$.

EXAMPLE 4.4.13 Solve $\dfrac{d^2y}{dx^2} - y = 1$, given that $y = 0$ when $x = 0$ and y approaches a finite limit when x approaches $-\infty$.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 1)y = 0$.
$\therefore$ A.E. is $m^2 - 1 = 0$, or $m = \pm 1$.
C.F. is $c_1e^x + c_2e^{-x}$. Now

$$\text{P.I.} = \frac{1}{D^2 - 1}(1) = -(1 - D^2)^{-1}(1) = -1$$

Thus, the general solution is $y = c_1e^x + c_2e^{-x} - 1$.
When $x = 0$, then $y = 0$, which gives $c_1 + c_2 - 1 = 0$
Since when x approaches $-\infty$ and y approaches a finite limit, then c_2 should be equal to zero otherwise y becomes infinite.
$\therefore \quad c_1 = 1$.
Hence the required solution is $y = e^x - 1$.

4.5 Cauchy–Euler and Legendre Differential Equations

In this section, two special types of ordinary differential equations with variable coefficients are considered. Both types of equations can be transformed to linear equations with constant coefficients by a suitable transformation of the independent variable. These reduced equations can then be solved by the methods discussed earlier.

4.5.1 Cauchy–Euler Equations

The equation of the form

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_ny = f(x) \qquad (4.2$$

where $a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is known as the **Cauchy–Euler equation** and it is homogeneous in the sense that the coefficient of the nth derivative is x^n.

For this type of equation the suitable transformation is $x = e^z$ or $z = \log x$.

Therefore
$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = \frac{1}{x}Dy \qquad \text{where } D \equiv \frac{d}{dz}$$

That is, $x\dfrac{dy}{dx} = Dy$.

Again
$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} = \frac{d}{dz}\left(\frac{1}{x}Dy\right)\frac{1}{x}$$
$$= \frac{1}{x}\left(-\frac{1}{x^2}\frac{dx}{dz}Dy + \frac{1}{x}D^2y\right)$$
$$= \frac{1}{x^2}(D^2y - Dy)$$

Thus, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$.

Similarly, $x^3\dfrac{d^3y}{dx^3} = D(D-1)(D-2)y$, etc.

By these substitution, Eq. (4.22) becomes an equation in y with z as the independent variable with constant coefficients.

EXAMPLE 4.5.1 Solve $x^3\dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} - y = x^2$.

Solution Substitute $z = \log x$, or $x = e^z$

Now
$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x} \qquad \text{or} \qquad x\frac{dy}{dx} = \frac{dy}{dz} = Dy$$

where $\equiv \dfrac{d}{dz}$

Differentiating with respect to x, we get
$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dz^2}\frac{dz}{dx} = \frac{1}{x}\frac{d^2y}{dz^2}$$

or
$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - x\frac{dy}{dx} = D^2y - Dy = D(D-1)y$$

Similarly, $x^3\dfrac{d^3y}{dx^3} = D(D-1)(D-2)y$.

Using these results, the given equation becomes
$$\{D(D-1)(D-2) + D - 1\}y = e^{2z} \qquad \text{or} \qquad (D^3 - 3D^2 + 3D - 1)y = e^{2z}$$

Let $y = ce^{mz}$ be a trial solution of $(D^3 - 3D^2 + 3D - 1)y = 0$.

Then A.E. is $m^3 - 3m^2 + 3m - 1 = 0$, or $(m-1)^3 = 0$, or $m = 1, 1, 1$.

$\therefore$ C.F. is $(c_1 + c_2 z + c_3 z^2)e^z$.

$$\text{P.I.} = \frac{1}{(D-1)^3}e^{2z} = \frac{e^{2z}}{(2-1)^3} = e^{2z}$$

Therefore, the general solution is

$$y = (c_1 + c_2 z + c_3 z^2)e^z + e^{2z} = \{c_1 + c_2 \cdot \log x + c_3(\log x)^2\}x + x^2$$

where c_1, c_2, c_3 are arbitrary constants.

EXAMPLE 4.5.2 Solve $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$.

Solution Substitute $\log x = z$, or $x = e^z$. Therefore

$$x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y, \quad x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

The given equation becomes

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$$

or

$$(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

Let $y = ce^{mz}$ be a trial solution of $(D^3 - D^2 + 2)y = 0$.

Thus the A.E. is $m^3 - m^2 + 2 = 0$, or $(m+1)(m^2 - 2m + 2) = 0$, or $m = -1, 1 \pm i$.

C.F. is $c_1e^{-z} + (c_2 \cos z + c_3 \sin z)e^z$.

P.I. is

$$\begin{aligned}
&\frac{1}{D^3 - D^2 + 2}10(e^z + e^{-z}) \\
&= 10\frac{1}{2}e^z + 10e^{-z}\frac{1}{(D-1)^3 - (D-1)^2 + 2}1 \\
&= 5e^z + 10e^{-z}\frac{1}{D^3 - 4D^2 + 5D}1 \\
&= 5e^z + 10e^{-z}\frac{1}{5D}\left(1 + \frac{D^2 - 4D}{5}\right)^{-1}1 \\
&= 5e^z + 2e^{-z}\frac{1}{D}1 = 5e^z + 2e^{-z}z
\end{aligned}$$

Therefore, the general solution is

$$\begin{aligned}
y &= c_1e^{-z} + (c_2 \cos z + c_3 \sin z)e^z + 5e^z + 2e^{-z}z \\
&= \frac{c_1}{x} + \{c_2 \cos(\log x) + c_3 \sin(\log x)\}x + 5x + \frac{2\log x}{x}
\end{aligned}$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.5.3 Solve $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x^2e^{2x}$.

Solution Put $\log x=z$, or $x=e^z$. Therefore

$$x\frac{dy}{dx}=Dy,\ \ x^2\frac{d^2y}{dx^2}=D(D-1)y$$

where $D\equiv\dfrac{d}{dz}$.

The given equation becomes

$$[D(D-1)+D-1]y=e^{2z}\cdot e^{2e^z}$$

or

$$(D^2-1)y=e^{2z}\cdot e^{2e^z}$$

Let $y=ce^{mz}$ be a trial solution of $(D^2-1)y=0$.

$\therefore$ A.E. is $m^2-1=0$, or $m=\pm 1$.

C.F. is $c_1e^z+c_2e^{-z}$.

$$\text{P.I.}=\frac{1}{D^2-1}e^{2z}\cdot e^{2e^z}=e^{2z}\frac{1}{(D+2)^2-1}e^{2e^z}=e^{2z}\frac{1}{D^2+4D+3}e^{2e^z}$$

$$=e^{2z}\frac{1}{(D+3)(D+1)}e^{2e^z}$$

$$=e^{2z}\frac{1}{D+3}u,\ \text{ where } u=\frac{1}{D+1}e^{2e^z}$$

That is, $(D+1)u=e^{2e^z}$, or $\dfrac{du}{dz}+u=e^{2e^z}$, which is a linear differential equation and its I.F. is e^z.

Multiplying the above equation by e^z, we get $\dfrac{d}{dz}(ye^z)=e^ze^{2e^z}$.

Integrating, $ue^z=\int e^ze^{2e^z}\,dz$.

Putting $e^z=t$. Then $e^z\,dz=dt$.

$\therefore\ \ ue^z=\displaystyle\int e^{2t}\,dt=\frac{e^{2t}}{2}=\frac{e^{2e^z}}{2}$, or $u=\dfrac{e^{-z}e^{2e^z}}{2}$.

$\therefore\ \ $ P.I. is $e^{2z}\dfrac{1}{D+3}u=e^{2z}v$, where $v=\dfrac{1}{D+3}u$, or $\dfrac{dv}{dz}+3v=u=\dfrac{e^{-z}e^{2e^z}}{2}$, which is linear in v and its I.F. is e^{3z}.

Multiplying the above equation by e^{3z} and integrating, it becomes $ve^{3z}=\displaystyle\int\frac{e^{2z}e^{2e^z}}{2}\,dz$.

Putting $e^z=t$, $e^z\,dz=dt$.

$$ve^{3z}=\frac{1}{2}\int te^{2t}\,dt=\frac{1}{2}\left[t\frac{e^{2t}}{2}-\int\frac{e^{2t}}{2}\,dt\right]$$

$$=\frac{1}{4}\left[te^{2t}-\frac{e^{2t}}{2}\right]=\frac{1}{8}(2e^z-1)e^{2e^z}$$

or

$$v = \frac{1}{8}(2e^z - 1)e^{2e^z} \cdot e^{-3z}$$

$\therefore$ P.I. is $e^{2z}v = e^{2z}\frac{1}{8}(2e^z - 1)e^{2e^z} \cdot e^{-3z} = \frac{1}{8}e^{-z}(2e^z - 1)e^{2e^z}$.

Therefore, the general solution is

$$y = c_1 e^z + c_2 e^{-z} + \frac{1}{8}e^{-z}(2e^z - 1)e^{2e^z}$$
$$= c_1 x + \frac{c_2}{x} + \frac{1}{8}\left(2 - \frac{1}{x}\right)e^{2x}$$

where c_1, c_2 are arbitrary constants.

4.5.2 Legendre Equations

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + c_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + c_{n-1}(ax+b)\frac{dy}{dx} + c_n y = f(x) \qquad (4.23)$$

where $c_1, c_2, \ldots, c_{n-1}, c_n, a, b$ are constants, is called **Legendre's equation**.

For this type of problem the suitable transformation is $ax + b = e^z$ or $z = \log(ax+b)$.

Then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{a}{ax+b}Dy$, where $D \equiv \frac{d}{dz}$.

That is, $(ax+b)\frac{dy}{dx} = aDy$.

Also

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} = \frac{d}{dz}\left(\frac{a}{ax+b}Dy\right)\frac{dz}{dx}$$
$$= -\frac{a^2}{(ax+b)^2}Dy + \frac{a}{ax+b}D^2y\frac{dz}{dx}$$
$$= \frac{a^2}{(ax+b)^2}(D^2y - Dy)$$

Thus, $(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y$.

Similarly, $(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$ and so on.

By these substitutions too the given equation reduces to a linear equation with constant coefficients whose independent variable is z.

EXAMPLE 4.5.4 Solve

$$(2+x)^2\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} + 4y = \sin\{2\log(2+x)\}$$

Solution. Let $z = \log(2+x)$, or $2 + x = e^z$. Therefore

or
$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{2+x}$$

$$(2+x)\frac{dy}{dx} = \frac{dy}{dz}$$

Again, on differentiating we get

$$(2+x)\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dz^2}\frac{dz}{dx} = \frac{1}{2+x}\frac{d^2y}{dz^2}$$

or
$$(2+x)^2\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

Substituting these values into the given equation, we get

$$\frac{d^2y}{dz^2} + 4y = \sin 2z$$

Let $y = e^{mz}$ be the trial solution.

The A.E. is $m^2 + 4 = 0$, or $m = \pm 2i$.

The C.F. is $c_1 \cos 2z + c_2 \sin 2z$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+4}\sin 2z = \text{I.P. of } \frac{1}{D^2+4}e^{2iz} \\
&= \text{I.P. of } e^{2iz}\frac{1}{(D+2i)^2+4}1 = \text{I.P. of } e^{2iz}\frac{1}{D^2+4Di}1 \\
&= \text{I.P. of } e^{2iz}\frac{1}{4Di}\left(1+\frac{D}{4i}\right)^{-1}1 = \text{I.P. of } e^{2iz}\frac{1}{4Di}\left(1-\frac{D}{4i}+\cdots\right)1 \\
&= \text{I.P. of } e^{2iz}\frac{1}{4Di}(1) = \text{I.P. of } e^{2iz}\frac{1}{4i}z \\
&= \text{I.P. of } \frac{z}{4}(\cos 2z + i\sin 2z)(-i) = -\frac{z}{4}\cos 2z
\end{aligned}$$

Hence the general solution is

$$\begin{aligned}
y &= c_1\cos 2z + c_2 \sin 2z - \frac{z}{4}\cos 2z \\
&= c_1\cos\{2\log(2+x)\} + c_2\sin\{2\log(2+x)\} - \frac{\log(2+x)}{4}\cos\{2\log(2+x)\}
\end{aligned}$$

where c_1, c_2 are arbitrary constants.

EXAMPLE [illegible] Solve $(1+2x)^2\frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$

Solution Let $\log(1+2x) = z$ or $e^z = 1+2x$.

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{2}{1+2x}\frac{dy}{dz}$ or $(1+2x)\frac{dy}{dx} = 2Dy$, where $D = \frac{d}{dz}$.

Again differentiating with respect to x, we get

$$(1+2x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2\frac{d^2y}{dz^2}\frac{dz}{dx} = 2\frac{2}{1+2x}\frac{d^2y}{dz^2}$$

or

$$(1+2x)^2\frac{d^2y}{dx^2} + 2(1+2x)\frac{dy}{dx} = 4\frac{d^2y}{dz^2}$$

or

$$(1+2x)^2\frac{d^2y}{dx^2} = 4\frac{d^2y}{dz^2} - 2\cdot 2Dy = 4D(D-1)y$$

Thus the given equation reduces to

$$\{4D(D-1) - 6\cdot 2D + 16\}y = 8e^{2z} \qquad \text{or} \qquad (D^2 - 4D + 4)y = 2e^{2z}$$

The A.E. is $m^2 - 4m + 4 = 0$, or $m = 2, 2$. Thus C.F. is $(c_1 + c_2 z)e^{2z}$.

$$\text{P.I.} = \frac{1}{(D-2)^2}2e^{2z} = 2e^{2z}\frac{1}{(D+2-2)^2}1 = 2e^{2z}\frac{1}{D^2}1 = z^2e^{2z}$$

Therefore, the general solution is

$$y = (c_1 + c_2 z)e^{2z} + z^2e^{2z} = \{c_1 + c_2\log(1+2x)\}(1+2x)^2 + (1+2x)^2\{\log(1+2x)\}^2$$

4.6 Method of Variation of Parameters

Let us consider the second order linear differential equation of the form

$$y'' + p(x)y' + q(x)y = r(x)$$

Again, we suppose that $y = y_1(x)$ and $y = y_2(x)$ be two independent solutions of

$$y'' + p(x)y' + q(x)y = 0$$

are known.

Then C.F. is $c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are arbitrary constants.

The method of variation of parameters actually determines the particular solution based on the C.F. by taking c_1 and c_2 as functions of x. That is, we assume that

$$y = Ay_1(x) + By_2(x)$$

is the general solution of (4.24), where A and B are functions of x. Differentiating, we get

$$y' = Ay_1' + By_2' + A'y_1 + B'y_2$$

We choose A and B in such a way that

$$A'y_1 + B'y_2 = 0 \tag{4.27}$$

Then y' reduces to $y' = Ay_1' + By_2'$.

Again we differentiate and obtain

$$y'' = Ay_1'' + By_2'' + A'y_1' + B'y_2'$$

The values of y, y' and y'' are substituted in (4.24). Therefore

$$Ay_1'' + By_2'' + A'y_1' + B'y_2' + p(x)[Ay_1' + By_2'] + q(x)[Ay_1 + By_2] = r(x) \tag{4.28}$$

This gives

$$A[y_1'' + p(x)y_1' + q(x)y_1] + B[y_2'' + p(x)y_2' + q(x)y_2] + A'y_1' + B'y_2' = r(x) \tag{4.29}$$

In view of the fact that y_1 and y_2 are the solutions of (4.25), then the first two terms of above equation become zero; hence

$$A'y_1' + B'y_2' = r(x) \tag{4.30}$$

The solutions of (4.27) and (4.30) for A' and B' is

$$A' = -\frac{r(x)y_2(x)}{W(x)} \quad \text{and} \quad B' = \frac{r(x)y_1(x)}{W(x)}$$

where $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$

Thus

$$A = -\int \frac{r(x)y_2(x)}{W(x)}dx + c_1 \qquad \text{and} \qquad B = \int \frac{r(x)y_1(x)}{W(x)}dx + c_2$$

Hence the required general solution is

$$y = -y_1 \int \frac{r(x)y_2(x)}{W(x)}dx + y_2 \int \frac{r(x)y_1(x)}{W(x)}dx + c_1y_1 + c_2y_2$$

EXAMPLE 4.6.1 Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 9e^x \qquad \text{(WBUT 2004)}$$

Solution Let $y = ce^{mx}$ be a solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$.

The A.E. is $m^2 - 3m + 2 = 0$, or $m = 1, 2$.

$\therefore$ C.F. is $c_1e^x + c_2e^{2x}$.

Let the general solution be $y = Ae^x + Be^{2x}$, where A and B are functions of x.

Let $y_1 = e^x$ and $y_2 = e^{2x}$. Then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$

The values of A and B are given by

$$A = -\int \frac{y_2(x)r(x)}{W}\,dx + c_1 \qquad \text{and} \qquad B = \int \frac{y_1(x)r(x)}{W}\,dx + c_2$$

That is

$$A = -\int \frac{e^{2x} \cdot 9e^x}{e^{3x}}\,dx + c_1 = -\int 9dx + c_1 = -9x + c_1$$

and

$$B = \int \frac{e^x \cdot 9e^x}{e^{3x}}\,dx + c_2 = \int 9e^{-x}dx + c_2 = -9e^{-x} + c_2$$

Hence the general solution is

$$y = (-9x + c_1)e^x + (-9e^{-x} + c_2)e^{2x} = -9e^x(x+1) + c_1e^x + c_2e^{2x}$$

EXAMPLE 4.6.2 Apply the method of variation of parameter to solve the equation

$$\frac{d^2y}{dx^2} + y = \sec^3 x \cdot \tan x \qquad \text{(WBUT 2007)}$$

Solution Let $y = ce^{mx}$ be a solution of $(D^2 + 1)y = 0$.

$\therefore$ A.E. is $m^2 + 1 = 0$, or $m = \pm i$.

$\therefore$ C.F. is $c_1 \cos x + c_2 \sin x$.

Let the general solution be $y = A\cos x + B\sin x$, where A and B are functions of x.

Let $y_1 = \cos x$ and $y_2 = \sin x$. Then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

The values of A and B are given by

$$A = -\int \frac{y_2(x)r(x)}{W}\,dx + c_1 = -\int \frac{\sin x \cdot \sec^3 x \tan x}{1}\,dx + c_1$$

$$= -\int \frac{\sin^2 x}{\cos^4 x}dx + c_1 = -\int \tan^2 x \sec^2 x\,dx + c_1$$

(Putting $\tan x = z, \sec^2 x\,dx = dz$)

$$= -\int z^2\,dz + c_1 = -\frac{z^3}{3} + c_1 = -\frac{1}{3}\tan^3 x + c_1$$

and

$$B = \int \frac{y_1(x)r(x)}{W}\,dx + c_2$$

$$= -\int \frac{\cos x \cdot \sec^3 x \tan x}{1}\,dx + c_2$$

$$= \int \tan x \sec^2 x\,dx + c_2 = \int \tan x\,d(\tan x)\,dx + c_2$$

$$= \frac{1}{2}\tan^2 x + c_2$$

Hence the required solution is

$$y = -\frac{1}{3}\tan^3 x \cos x + c_1 \cos x + \frac{1}{2}\tan^2 x \sin x + c_2 \sin x$$

$$= \frac{1}{6}\tan^2 x \sin x + c_1 \cos x + c_2 \sin x$$

EXAMPLE 4.6.3 Solve $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$. The method of variation of parameters may be used.

Solution Let $y = ce^{mx}$ be a trial solution of

$$\frac{d^2y}{dx^2} + a^2y = 0 \tag{i}$$

$\therefore$ A.E. is $m^2 + a^2 = 0$, or $m = \pm ia$.

$\therefore$ $y = \cos ax$ and $y = \sin ax$ are two independent solutions of (i). *

Let $y = A\cos ax + B\sin ax$, where A and B are functions of x, be the general solution of the given differential equation.

Then

$$\frac{dy}{dx} = A'\cos ax - Aa\sin ax + B'\sin ax + Ba\cos ax$$

We choose A and B in such a way that

$$A'\cos ax + B'\sin ax = 0 \tag{ii}$$

Therefore

$$\frac{dy}{dx} = -Aa\sin ax + Ba\cos ax$$

Again differentiating

$$\frac{d^2y}{dx^2} = -A'a\sin ax - Aa^2\cos ax + B'a\cos ax - Ba^2\sin ax$$

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Using the values of y and $\dfrac{d^2y}{dx^2}$, the given equation becomes

$$-Aa' \sin ax + B'a \cos ax = \sec ax \tag{iii}$$

The solution of (ii) and (iii) is $A' = -\dfrac{1}{a} \tan ax$ and $B' = \dfrac{1}{a}$.

Integrating, $A = \dfrac{1}{a^2} \log \cos ax + c_1$ and $B = \dfrac{x}{a} + c_2$.

Hence

$$\begin{aligned} y &= A \cos ax + B \sin ax \\ &= \left(\frac{1}{a^2} \log \cos ax + c_1\right) \cos ax + \left(\frac{x}{a} + c_2\right) \sin ax \\ &= c_1 \cos ax + c_2 \sin ax + \frac{\cos ax}{a^2} \log \cos ax + \frac{x}{a} \sin ax \end{aligned}$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.6.4 Solve the equation $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = \dfrac{e^{2x}}{x^2}$ by the method of variation of parameters.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 4D + 4)y = 0$.

$\therefore$ A.E. is $m^2 - 4m + 4 = 0$, or $m = 2, 2$.

$\therefore$ C.F. is $(c_1 + c_2x)e^{2x}$.

Let the general solution be $y = (A + Bx)e^{2x}$, where A, B are functions of x.

$$\frac{dy}{dx} = 2Ae^{2x} + A'e^{2x} + Be^{2x} + 2Bxe^{2x} + B'xe^{2x}$$

We choose A, B such that

$$A' + B'x = 0 \tag{i}$$

Now, $\dfrac{dy}{dx} = 2Ae^{2x} + Be^{2x} + 2Bxe^{2x}$.

Again, differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 2A'e^{2x} + 2Be^{2x} + B'e^{2x} + 2Be^{2x} + 2B'xe^{2x} + 4Bxe^{2x}$$

Putting these values into the given equation, we obtain

$$(4A + 2A' + 4B + B' + 2B'x + 4Bx)e^{2x} - 4(2A + B + 2Bx)e^{2x} + 4(A + Bx)e^{2x} = \frac{e^{2x}}{x^2}$$

or

$$2A' + B' + 2B'x = \frac{1}{x^2} \tag{ii}$$

Solving (i) and (ii) we obtain $A' = -1/x$, $B' = 1/x^2$. Therefore

$$A = -\log x + c_3, \; B = -\frac{1}{x} + c_4$$

Hence the general solution is $y = (-\log x + c_3 - 1 + c_4x)e^{2x}$, where c_3, c_4 are arbitrary constants.

EXAMPLE 4.6.5 Solve the equation $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by using the method of variation of parameters.

Solution Let $y = ce^{mx}$ be a trial solution of $(D^2+1)y = 0$.

$\therefore$ A.E. is $m^2 + 1 = 0$, or $m = \pm i$.

C.F. is $c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are arbitrary constants.

Let $y = A\cos x + B\sin x$ be the general solution of the given equation, where A and B are functions of x.

Now, $\frac{dy}{dx} = A' \cos x - A\sin x + B' \sin x + B\cos x$.

We choose A and B in such a way that

$$A' \cos x + B' \sin x = 0 \tag{i}$$

Then $\frac{dy}{dx} = -A\sin x + B\cos x$.

Again, on differentiating we get

$$\frac{d^2y}{dx^2} = -A\cos x - A' \sin x - B\sin x + B' \cos x$$

On putting the values of y and $\frac{d^2y}{dx^2}$ into the given equation, it reduces to

$$-A\cos x - A' \sin x - B\sin x + B' \cos x + A\cos x + B\sin x = \operatorname{cosec} x$$

or

$$-A' \sin x + B' \cos x = \operatorname{cosec} x \tag{ii}$$

The solution of (i) and (ii) is $A' = -1, B' = \cot x$.

$\therefore$ $A = -x + c_3$ and $B = \log\sin x + c_4$.

Hence the general solution is $y = (-x+c_3)\cos x + (\log\sin x + c_4)\sin x = c_3\cos x + c_4 \sin x - x\cos x + \sin x \log\sin x$, where c_3, c_4 are arbitrary constants.

EXAMPLE 4.6.6 Find the general solution of $(1+x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1+x)^2$ by the method of variation of parameters. It is given that $y = x$ and $y = e^{-x}$ are two independent solutions of the corresponding homogeneous equation.

Solution Let $y = x$ and $y = e^{-x}$ be two solutions. To test their independence, we consider the Wronskian

$$W(x) = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = -xe^{-x} - e^{-x} \neq 0$$

Thus the two solutions x and e^{-x} are independent.

Let $y = Ax + Be^{-x}$, where A and B are functions of x, be the general solution.

$\frac{dy}{dx} = A'x + A + B'e^{-x} - Be^{-x}$.

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We choose A, B in such a way that

$$A'x + B'e^{-x} = 0 \quad \text{(i)}$$

$$\therefore \quad \frac{dy}{dx} = A - Be^{-x}.$$

Again, on differentiating we get $\dfrac{d^2y}{dx^2} = A' - B'e^{-x} + Be^{-x}$.

Putting the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the given equation, we get

$$(1+x)(A' - B'e^{-x} + Be^{-x}) + x(A - Be^{-x}) - (Ax + Be^{-x}) = (1+x)^2$$

or

$$(1+x)(A' - B'e^{-x}) = (1+x)^2 \qquad \text{or} \qquad A' - B'e^{-x} = 1 + x \quad \text{(ii)}$$

The solution of (i) and (ii) is $A' = 1$ and $B' = -xe^{-x}$.

Integrating, we get $A = x + c_1$ and $B = -\int xe^x\,dx = -xe^x + \int e^x\,dx = -xe^x + e^x + c_2$.

Hence the general solution is

$$y = (x + c_1)x + (-xe^x + e^x + c_2)e^{-x} = c_1x + c_2e^{-x} + x^2 + (1 - x)$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.6.7 Solve $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 0$, given that $y = x^3$ is a solution.

Solution Since $y = x^3$ is a solution, let the solution of the given equation be $y = ux^3$, where u is a function of x.

Therefore, $\dfrac{dy}{dx} = 3x^2u + \dfrac{du}{dx}x^3$ and $\dfrac{d^2y}{dx^2} = 6xu + 6x^2\dfrac{du}{dx} + x^3\dfrac{d^2u}{dx^2}$.

By these substitutions, the given equation reduces to

$$x^2\left[x^3\frac{d^2u}{dx^2} + 6x^2\frac{du}{dx} + 6xu\right] + x\left[3x^2u + x^3\frac{du}{dx}\right] - 9x^3u = 0$$

or

$$x^5\frac{d^2u}{dx^2} + 7x^4\frac{du}{dx} + 6x^3u + 3x^3u - 9x^3u = 0$$

or

$$x^4\left[x\frac{d^2u}{dx^2} + 7\frac{du}{dx}\right] = 0$$

or

$$x\frac{d^2u}{dx^2} + 7\frac{du}{dx} = 0$$

Let $\dfrac{du}{dx} = p$. Then the above equation becomes

$$x\frac{dp}{dx} + 7p = 0 \quad \text{or} \quad \frac{dp}{p} = -7\frac{dx}{x}$$

Integrating, we get $\log p = -7\log x + \log c_1$, where c_1 is arbitrary constant, or $p = c_1/x^7$. That is, $\dfrac{du}{dx} = \dfrac{c_1}{x^7}$, or $du = \dfrac{c_1}{x^7}dx$.

Now, integrating it, we get $u = -\dfrac{c_1}{6x^6} + c_2$, where c_2 is arbitrary constant.

Hence the solution is $y = ux^3 = -\dfrac{c_1}{6x^3} + c_2x^3$.

EXAMPLE 4.6.8 If $y = y_1$ and $y = y_2$ be two solutions of $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = 0$, then show that $y_1\dfrac{dy_2}{dx} - y_2\dfrac{dy_1}{dx} = c\cdot e^{-\int P dx}$ where c is a constant.

Solution Given that $y = y_1$ is a solution, let $y = vy_1$. Therefore,

$$\frac{dy}{dx} = v\frac{dy_1}{dx} + y_1\frac{dv}{dx}$$

and

$$\frac{d^2y}{dx^2} = v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2}$$

Putting these values to the given equation, we have

$$\left(v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2}\right) + P\left(v\frac{dy_1}{dx} + y_1\frac{dv}{dx}\right) + Qvy_1 = 0$$

or

$$v\left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1\right) + y_1\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{dv}{dx}\frac{dy_1}{dx} = 0 \qquad \text{(i)}$$

Since $y = y_1$ is a solution of the given equation

$$\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1 = 0$$

Therefore, (i) becomes

$$y_1\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{dv}{dx}\frac{dy_1}{dx} = 0$$

or

$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right) = 0$$

Now we put, $\dfrac{dv}{dx} = p$, so that $\dfrac{d^2v}{dx^2} = \dfrac{dp}{dx}$. Therefore,

$$\frac{dp}{dx} + p\left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right) = 0, \quad \text{or} \quad \frac{dp}{p} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)dx = 0$$

Integrating, we get

$$\log p + \int P\,dx + \int \frac{2}{y_1}\,dy_1 = 0$$

or

$$\log p + \int P\,dx + 2\log y_1 = \log c \qquad \text{or} \qquad \log\left(\frac{py_1^2}{c}\right) = -\int P\,dx$$

or

$$py_1^2 = ce^{-\int P\,dx} \qquad \text{or} \qquad \frac{dv}{dx}y_1^2 = ce^{-\int P\,dx}$$

Now, $y = vy_1$. But $y = y_2$ is also a solution.

$\therefore\ y_2 = vy_1$, or $v = \frac{y_2}{y_1}$. $\quad$ So $\dfrac{dv}{dx} = \dfrac{y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}}{y_1^2}$.

Hence from (ii), $y_1\dfrac{dy_2}{dx} - y_2\dfrac{dy_1}{dx} = ce^{-\int P\,dx}$.

EXERCISES

Section A Multiple Choice Questions

1. The general solution of the differential equation $D^2y + 9y = 0$ is
   (a) $Ae^{3x} + Be^{-3x}$ (b) $(A + Bx)e^{3x}$ (c) $A\cos 3x + B\sin 3x$
   (d) $(A + Bx)\sin 3x$.
   (WBUT 20
2. The general solution of $(D^2 - 1)^2y = 0$ is
   (a) $c_1e^{x} + c_2e^{-x}$ (b) $c_1e^{x} + c_2e^{x} + c_3e^{-x} + c_4e^{-x}$
   (c) $(c_1 + c_2x)e^{x} + (c_3 + c_4x)e^{-x}$ (d) $(c_1\cos x + c_2\sin x)^2$.
3. The general solution of $\dfrac{d^2y}{dx^2} + 4y = 0$ is
   (a) $Ae^{2x} + Be^{-2x}$ (b) $(A + Bx)e^{2x}$ (c) $A\cos 2x + B\sin 2x$
   (d) $(A + Bx)\cos 2x$.
   (WBUT 20
4. The general solution of $(D^2 + 5D + 6)y = 0$ is
   (a) $c_1e^{-3x} + c_2e^{-2x}$ (b) $(c_1 + c_2x)e^{-2x}$ (c) $c_1\cos 3x + c_2\sin 3x$
   (d) $e^{-3x} + e^{-2x}$.
5. The C.F. of the differential equation $(D^2 + 16)y = \cos x$ is
   (a) $c_1e^{4x} + c_2e^{-4x}$ (b) $c_1e^{4x} + c_2e^{-4x} + \frac{1}{15}\cos x$
   (c) $c_1\cos 4x + c_2\sin 4x + \frac{1}{15}\cos x$ (d) $c_1\cos 4x + c_2\sin 4x$.
6. A particular solution of $(D^2 - 1)y = 0$ when $x = 0, y = 0$ and $x = 0, Dy = 1$ is
   (a) $c_1e^{x} + c_2e^{-x}$ (b) $\frac{1}{2}(e^{x} - e^{-x})$ (c) $c_1\cos x + c_2\sin x$ (d) $\frac{1}{2}(e^{x} + e^{-x})$
7. The P.I. of $(D^2 - 2D + 4)y = e^{2x}$ is
   (a) e^{2x} (b) $\frac{1}{2}e^{2x}$ (c) $\frac{x^2}{2}$ (d) $\frac{1}{4}e^{2x}$.
8. The P.I. of $(D^2 + 4)y = \sin 3x$ is
   (a) $\frac{1}{5}\sin 3x$ (b) $-\frac{1}{5}\sin 3x$ (c) $\frac{1}{5}\cos 3x$ (d) $-\frac{1}{5}\cos 3x$.
9. The value of $\frac{1}{D^3}x$ is
   (a) $x^2/2$ (b) $x^3/6$ (c) $x^4/24$ (d) 0.

10. $\frac{1}{D^2}\cos x$ is equal to
(a) $-\sin x$ (b) $-\cos x$ (c) $\cos x$ (d) $\sin x$.

11. The value of $\frac{1}{D^2-1}4xe^x$ is
(a) $e^x(x^2+x)$ (b) x^2e^x (c) x^3e^x (d) $e^x(x^2-x)$.

12. $\frac{1}{D+2}e^{-2x}\sin 3x$ is equal to
(a) $-\frac{1}{3}e^{-2x}\cos 3x$ (b) $e^{-2x}\cos 3x$ (c) $-\frac{1}{3}e^{-2x}\sin 3x$ (d) $\frac{1}{3}e^{-2x}\sin 3x$.

13. $\frac{1}{D-1}x^2$ is equal to
(a) x^2+2x+2 (b) $-(x^2+2x+2)$ (c) $2x-x^2$ (d) $-(2x-x^2)$.

14. $\frac{1}{D^2+4}\sin 2x$ is equal to
(a) $\frac{x}{4}\cos 2x$ (b) $-\frac{x}{4}\cos 2x$ (c) $\frac{x}{4}\sin 2x$ (d) $x\sin 2x$.

15. The P.I. of $(D^4-8D)y=x^2$ is
(a) $-\frac{x^4}{32}$ (b) $\frac{x^4}{32}$ (c) $\frac{x^3}{8}$ (d) $-\frac{1}{8}x^3$.

16. Using the transformation $x=e^z$ the equation $x\dfrac{dy}{dx}+y=(\log x)^2$ reduces to
(a) $\dfrac{d^2y}{dz^2}+y=z$ (b) $2\dfrac{d^2y}{dz^2}+2y=z^2$ (c) $\dfrac{d^2y}{dz^2}+y=z^2$
(d) none of these.

17. Using the substitution $z=\log x$ the equation $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+4y=2x^2$ reduces to
(a) $\dfrac{d^2y}{dz^2}-4\dfrac{dy}{dz}+4y=2e^{2z}$ (b) $\dfrac{d^2y}{dz^2}+4y=2e^{2z}$ (c) $\dfrac{d^2y}{dz^2}-3\dfrac{dy}{dz}+4y=2e^{2z}$
(d) none of these.

18. The C.F. of the equation $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-4y=x^2$ is
(a) $c_1e^{-x}+c_2e^{2x}$ (b) $c_1x^2+c_2/x$ (c) $c_1\cos 2x+c_2\sin 2x$
(d) none of these.

19. The C.F. of the equation $x^2\dfrac{d^2y}{dx^2}+4y=\sin(\log x)$ is
(a) $c_1\cos 2x+c_2\sin 2x$ (b) $c_1e^{2x}+c_2e^{-2x}$
(c) $\{c_1\cos(\frac{\sqrt{3}}{2}\log x)+c_2\sin(\frac{\sqrt{3}}{2}\log x)\}\sqrt{x}$ (d) $c_1\cos\sqrt{3}x+\sin\sqrt{3}x$.

20. The C.F. of the equation $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=x^2$ is
(a) $(c_1+c_2x)e^{-x}$ (b) $(c_1+c_2x)/x$ (c) $(c_1+c_2\log x)/x$ (d) $c_1e^x+c_2\frac{1}{x}$.

21. The C.F. of $x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+4y=x^4$ is
(a) $(c_1+c_2\log x)/x^2$ (b) $c_1e^x+c_2e^{-2x}$ (c) $c_1x+c_2x^{-1}$
(d) none of these.

22. The P.I. of $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=x$ is
(a) $x^2/2$ (b) $x/2$ (c) $x^3/2$ (d) $x^3/4$.

23. The general solution of $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = \log x$ is

(a) $c_1 + c_2 x + \frac{x^3}{6}$ (b) $c_1 + c_2 \log x + \frac{(\log x)^3}{6}$ (c) $c_1 + c_2 \log x$ (d) [illegible]

24. Using the transformation $z = \log(x+a)$, the equation $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx}$ [illegible]
is transformed to

(a) $\frac{d^2y}{dz^2} - 5\frac{dy}{dz} + 6y = 0$ (b) $\frac{d^2y}{dz^2} - 6\frac{dy}{dz} = 0$ (c) $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 6y = 0$
(d) none of these.

25. By the transformation $z = \log(5+2x)$ the equation $(5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx}$ [illegible]
is transformed to

(a) $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 2y = 0$ (b) $\frac{d^2y}{dz^2} - 6\frac{dy}{dz} + 8y = 0$ (c) $\frac{d^2y}{dz^2} + 4y = 0$
(d) none of these.

Section B Review Questions

Solve:

1. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x}$
2. $(D^3 + D^2 - D - 1)y = \cosh x$
3. $(D^2 - D - 2)y = 4x$
4. $(D^4 - 8D)y = x^2 + e^{2x}$
5. $(D^3 - 3D^2 + 3D - 1)y = xe^x + e^x$
6. $(D^5 - D)y = e^x + \sin x - x$
7. $(D^2 - 2D + 1)y = e^{-2x}\sin 2x$
8. $(D^3 - D^2 + 3D + 5)y = e^x \cos 3x$
9. $(D^2 + D + 1)y = (1 - e^x)^2$
10. $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$
11. $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$
12. $(D^2 - 4)y = x \sinh x$
13. $(D^2 - 1)y = x \sin 3x + \cos x$
14. $(D^2 - 2D + 1)y = xe^x \sin x$
15. $(D^4 + 2D^2 + 1)y = x^2 \cos x$
16. $(D^3 + 2D^2 + D)y = e^{-x} + \sin 2x$

17. $(D^2+1)^2y = x^4 + 2\sin x \cos 3x$

18. $(D^4-1)y = e^x \cos x$

19. $(D^2+2)y = x^2e^{3x} + e^x \cos 2x$

20. $(D^4-1)y = \cos x \cosh x$

21. $(D^2+4)y = x \sin x$

22. $(D^3+2D^2+D)y = x^2e^{2x} + \sin^2 x$

23. $(D^2-1)y = x \sin x + (1+x^2)e^x$

24. $(D^3+3D+2)y = e^{e^x}$

25. $(D^4-1)y = x^2 \sin x$

26. $(D^2+a^2)y = \sec ax$

27. $(D^2+4)y = \sec 2x$

28. $(D^3-3D^2+4D-2)y = e^x + \cos x$

29. $(D^2-4D+3)y = e^x \cos 2x + \cos 3x$

30. $(D^2-2D+2)y = e^x \sin 2x$

31. $(D^4+2D^3+3D^2+2D+1)y = xe^x$

32. $(D^3-1)y = x \sin x$

33. $(D^2-1)y = e^{2x} \sin x + e^{x/2} \sin \frac{x\sqrt{3}}{2}$

34. $(D^4+D^3+D^2)y = x^2(a+bx)$

35. $(D^2+a^2)y = \cos ax$

36. $(D^2-5D+6)y = e^x \cos x$ (WBUT 2002)

37. $(D^2+4)y = x \sin^2 x$ (WBUT 2008)

38. $(D^2-2D+1)y = xe^x$

39. $(D^2+a^2)y = \tan ax$

40. $(D^4+D^2+16)y = 16x^2 + 256$

41. $(D^2-1)y = e^{-x} + \cos x + x^3 + e^x \cos x$

42. $(D^2-4D+4)y = x^2 + e^x + \sin 2x$

43. $(D^5 - D)y = 12e^x + 8\sin x - 2x$

44. $(D^2 + 1)y = \cos x + xe^{2x} + e^x \sin x$

45. $(D^2 + 2D + 1)y = e^x + x^2 - \sin x$

46. $(D^2 - 4D + 1)y = 73\sin 2x + x + 13e^{-x/2}$

47. $(D^2 - 4D + 3)y = e^{2x}\sin 3x$

48. $(D^2 - 5D + 6)y = x(x + e^x)$

49. $(D^2 + 4)(D^2 + 1)y = \cos 2x + \sin x$

50. $(D^3 - 3D - 2)y = 540x^3 e^{-x}$

51. $(D^2 + a^2)y = x\cos ax$

52. $(D^4 - 1)y = x\sin x$

53. If $\dfrac{d^2x}{dt^2} + \dfrac{g}{b}(x - a) = 0$ and $x = a'$ and $\frac{dx}{dt} = 0$ when $t = 0$, show that

$$x = a + (a' - a)\cos\sqrt{\frac{g}{b}}t$$

54. Solve the equation $\dfrac{d^2x}{dt^2} + 2n\cos\alpha\dfrac{dx}{dt} + n^2x = a\cos nt$, given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

55. Solve $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y + 37\sin 3x = 0$, given that $y = 3$, $Dy = 0$ when $x = 0$.

56. Find the value of u which satisfies the equation $\frac{d^2u}{d\theta^2} + u = 2k\cos\theta$ and also the following conditions:

(i) u has the same value when $\theta = \pm\pi/2$

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} u\,d\theta = 0$.

57. Show that the transformation $x = \sinh z$ transforms $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 4y$ to $\dfrac{d^2y}{dz^2} = 4y$ and hence solve it.

58. Solve $(D^4 - n^4)y = 0$ completely. If $Dy = y = 0$ when $x = 0$ and $x = l$, then prove that $y = c_1(\cos nx - \cosh nx) + c_2(\sin nx - \sinh nx)$ and $\cos nl \cosh nl = 1$.

59. Solve the equation $\dfrac{d^2x}{dt^2} + 20\dfrac{dx}{dt} + 64x = 0$, given that $x = 1/3$ and $\frac{dx}{dt} = 0$ at $t = 0$.

60. Solve the equation $\dfrac{1}{5}\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 2x = 5\cos 4t$, given that $x(0) = \frac{1}{2}$ and $\left(\dfrac{dx}{dt}\right)_{t=0} = 0$.

61. $x^3\dfrac{d^3y}{dx^3} + 6x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} - 4y = 0$

62. $x^2\dfrac{d^2y}{dx^2} + y = 3x^2$

63. $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x^4$

64. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - y = 3x^3\cos(\log x)$

65. $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$

66. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$

67. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 2y = (x+1)^2$

68. $x\dfrac{d^2y}{dx^2} - \dfrac{2y}{x} = x + \dfrac{1}{x^2}$

69. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$

70. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

71. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\log x$

72. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10(x + 1/x)$

73. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3\log x$

74. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^m$

75. $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2y}{x^3} = 1$

76. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\log x \cdot \sin(\log x) + 1}{x}$

77. $\dfrac{d^2V}{dr^2} + \dfrac{1}{r}\dfrac{dV}{dr} = 4\pi\rho, \rho = \text{constant}$

78. $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \log x)^2$

79. $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$

80. $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 10$

81. $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$

82. $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + x + 1$

83. $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

84. $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\log(1+x)$

85. $[(1+2x)^2D^2 - 6(1+2x)D + 16]y = 8(1+2x)^2$

86. $(x+1)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} = (2x+3)(2x+4)$

87. $(x+3)^2\frac{d^2y}{dx^2} - 4(x+3)\frac{dy}{dx} + 6y = \log(x+3)$

88. $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Using the method of variation of parameter, solve the following equations:

89. $y'' + 3y' + 2y = 12e^x$

90. $y'' + 2y' + y = x^2e^{-x}$

91. $y'' + y = 4x\sin x$

92. $y'' - 2y' + y = x^2\log x$

93. $y'' - y = \frac{2}{1+e^x}$

94. $y'' - 3y' + 2y = \frac{e^x}{1+e^x}$

95. $y'' + 2y' + y = \frac{1}{e^x x^2}$

96. $y'' + 4y = 4\tan 2x$ (WBUT 20

97. $y'' - 3y' + 2y = 9e^x$ (WBUT 20

98. $y'' + 9y = \sec 3x$ (WBUT 2005, 20

99. $y'' + y = \operatorname{cosec} x$

100. $y'' - 2y' + y = e^x/x$

101. $y'' - 2y' + 2y = e^x\tan x$

102. $y'' - 5y' + 6y = e^{3x} + \sin x$

103. $(D^2 - 2D)y = e^x \sin x$

104. $y'' + 4y = 4\sec^2 2x$ (WBUT 2006)

105. $y'' + y = \sec^3 x \tan x$ (WBUT 2007)

106. Solve $x^2y'' - xy' + y = 0$, given that $y_1 = x$ is a solution.

107. Given that $y = x$ is a solution of $x^2y'' + xy' - y = 0$, find the general solution of $x^2y'' + xy' - y = x$.

Answers

Section A Multiple Choice Questions

1. (c)	2. (c)	3. (c)	4. (a)	5. (d)	6. (b)	7. (d)	8. (b)	9. (c)
10. (b)	11. (d)	12. (a)	13. (b)	14. (b)	15. (a)	16. (c)	17. (a)	18. (b)
19. (c)	20. (c)	21. (a)	22. (b)	23. (b)	24. (a)	25. (a)		

Section B Review Questions

1. $c_1 + (c_2 + c_3x)e^{-x} + \frac{1}{18}e^{2x}$
2. $c_1e^x + (c_2 + c_3x)e^{-x} + \frac{1}{8}xe^x - \frac{1}{8}x^2e^{-x}$
3. $c_1e^{-x} + c_2e^{2x} + 1 - 2x$
4. $c_1 + c_2e^{2x} + e^{-x}(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x) - \frac{x^3}{24} + \frac{xe^{2x}}{24}$
5. $(c_1 + c_2x + c_3x^2)e^x + \frac{1}{24}e^x(4x^3 + x^4)$
6. $c_1 + c_2e^x + c_3e^{-x} + c_4\cos x + c_5 \sin x + \frac{1}{4}xe^x + \frac{1}{8}x^2\cos x - \frac{1}{2}x^2$
7. $c_1e^{-2x} + c_2e^{-3x} - \frac{1}{10}e^{-2x}(\cos 2x + 2\sin 2x)$
8. $c_1e^{-x} + e^x(c_2\cos 2x + c_3 \sin 2x) - \frac{1}{65}e^x(3\sin 3x + 2\cos 3x)$
9. $e^{-x/2}(c_1 \cos\frac{\sqrt{3}x}{2} + c_2 \sin\frac{\sqrt{3}x}{2}) + 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$
10. $(c_1 + c_2x)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$
11. $c_1e^x + c_2e^{2x} + e^{3x}(\frac{x}{2} - \frac{3}{4}) + \frac{1}{20}(3\cos 2x - \sin 2x)$
12. $c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$
13. $c_1e^x + c_2e^{-x} - \frac{1}{50}(5x\sin 3x + 3\cos 3x + 25\cos x)$
14. $(c_1 + c_2x)e^x - e^x(x\sin x + 2\cos x)$
15. $(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x + \frac{1}{48}(4x^3\sin x - x^2(x^2 - 9)\cos x)$
16. $c_1 + (c_2 + c_3x)e^x - \frac{x^2}{2}e^{-x} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$
17. $(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x + x^4 - 24x^2 + 72 + \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x$
18. $c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{1}{5}e^x\cos x$
19. $c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{e^{3x}}{11}(x^2 - \frac{12}{11}x + \frac{50}{121}) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$

20. $c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{1}{6}\cos x\cosh x$

21. $c_1\cos 2x + c_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)$

22. $c_1 + (c_2 + c_3x)e^{-x} + \frac{e^{2x}}{18}(x^2 - \frac{7x}{8} + \frac{11}{6}) + \frac{1}{100}(3\sin 2x + 4\cos 2x)$

23. $c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{xe^x}{12}(2x^2 - 3x + 9)$

24. $c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} + e^{-2x}e^{e^x}$

25. $c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x + \frac{1}{24}(2x^3 - 15x)\cos x - \frac{3x^2}{8}\sin x$

26. $c_1\cos ax + c_2\sin ax + \frac{1}{a^2}\cos ax\log\cos ax + \frac{x}{a}\sin ax$

27. $c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\cos 2x\log\cos 2x + \frac{x}{2}\sin 2x$

28. $c_1e^x + (c_2\cos x + c_3\sin x)e^x + xe^x + \frac{1}{10}(\cos x + 3\sin x)$

29. $c_1e^x + c_2e^{3x} - \frac{e^x}{8}(\cos 2x + \sin 2x) - \frac{1}{30}(2\cos 3x + \sin 3x)$

30. $e^x(c_1\cos x + c_2\sin x) - \frac{1}{3}e^x\sin 2x$

31. $e^{-x/2}(c_1 + c_2x)\cos\frac{\sqrt{3}x}{2} + (c_3 + c_4x)\sin\frac{\sqrt{3}x}{2} + \frac{e^x}{9}(x - 2)$

32. $c_1e^x + e^{-\frac{x}{2}}\{c_2x\cos\frac{\sqrt{3}x}{2} + c_3x\sin\frac{\sqrt{3}x}{2}\} + \frac{1}{2}(x\cos x - x\sin x - 3\cos x)$

33. $c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2 - 3x + 9)$

34. $c_1 + c_2x + e^{-x/2}(c_3\sin\frac{\sqrt{3}x}{2} + c_4\cos\frac{\sqrt{3}x}{2}) + 3bx^2 - \frac{1}{3}cx^3$

35. $c_1\cos ax + c_2\sin ax + \frac{x}{2a}\sin ax$

36. $c_1e^{2x} + c_2e^{3x} + \frac{e^x}{10}(\cos x - 3\sin x)$

37. $c_1\cos 2x + c_2\sin 2x + \frac{1}{2}\{\frac{x}{4} - \frac{1}{8}\sin 2x(x^2 - \frac{1}{8}) + \frac{1}{2}x\cos 2x\}$

38. $(c_1 + c_2x)e^x + \frac{1}{6}e^xx^3$

39. $c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax\log\tan(\frac{\pi}{4} + \frac{ax}{2})$

40. $c_1e^{-\frac{\sqrt{7}}{2}x}\cos(\frac{3x}{2} + c_2) + c_3e^{\frac{\sqrt{7}}{2}x}\sin(\frac{3x}{2} + c_4) + x^2 + \frac{127}{8}$

41. $c_1\cos x + c_2\sin x + \frac{1}{2}e^{-x} + \frac{1}{2}x\sin x + x^3 - 6x + \frac{1}{5}e^x(2\sin x + \cos x)$

42. $(c_1 + c_2x)e^{2x} + \frac{1}{4}(x^2 + 2x + \frac{3}{2})e^x + \frac{1}{8}\cos 2x$

43. $c_1 + c_2e^x + c_3e^{-x} + c_4\cos x + c_5\sin x + 3xe^x + 2x\sin x + x^2$

44. $c_1\cos x + c_2\sin x + \frac{1}{2}x\sin x + \frac{1}{25}e^x(5x - 4) - \frac{1}{5}e^x(2\cos x - \sin x)$

45. $(c_1 + c_2x)e^{-x} + \frac{1}{4}e^x + x^2 - 4x + 6 - \frac{1}{2}\cos x$

46. $c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x} + 8\cos 2x - 3\sin 2x + x + 4 + 4e^{-x/2}$

47. $c_1e^x + c_2e^{3x} - \frac{1}{10}e^{2x}\sin 3x$

48. $c_1e^{2x} + c_2e^{3x} + \frac{1}{108}(18x^2 + 30x + 19) + \frac{e^x}{4}(2x + 3)$

49. $(c_1\cos 2x + c_2\sin 2x) + (c_3\cos x + c_4\sin x) - \frac{1}{12}x\sin 2x - \frac{x}{6}\cos x$

50. $c_1e^{2x} + (c_2 + c_3x)e^{-x} - e^{-x}(20x^2 - 20x^3 - 15x^4 - 9x^5)$

51. $c_1\cos ax + c_2\sin ax + \frac{1}{4a^2}(ax^2\sin ax + x\cos ax)$

52. $c_1e^x + c_2e^{-x} + (c_3\cos x + c_4\sin x) + \frac{1}{3}(x^2\cos x - 3x\sin x)$

54. $x = e^{-(n\cos\alpha)t}\left[-\frac{a}{n^2\sin 2\alpha}\sin(n\sin\alpha)t\right] + \frac{a}{2n^2\cos\alpha}\sin nt$

55. $3e^{-x}\cos 3x + 6\cos 3x - \sin 3x$
56. $u = k(\theta\sin\theta - \cos\theta)$
61. $\frac{c_1 + c_2\log x}{x^2} + c_3 x$
62. $\sqrt{x}(c_1\cos\frac{\sqrt{3}}{2}\log x + c_2\sin\frac{\sqrt{3}}{2}\log x) + x^2$
63. $\frac{c_1 + c_2\log x}{x^2} + \frac{x^4}{36}$
64. $c_1 x^2 + c_2 x^3 + x^5$
65. $(c_1 + c_2\log x)x + c_3 x^{-1} + \frac{1}{4x}\log x$
66. $(c_1\cos\log x + c_2\sin\log x)x + x\log x$
67. $c_1 x + \frac{c_2}{x^2} + \frac{1}{4}x^2 + \frac{1}{3}x\log x - \frac{1}{2}$
68. $(c_1 + c_2\log x)x^2 + \frac{1}{4} + 2x + \frac{1}{2}x^2(\log x)^2$
69. $c_1 x^4 + c_2 x^{-1} + \frac{x^4}{5}\log x$
70. $2(\log x)^3 + c_1\log x + c_2$
71. $\frac{c_1}{x} + c_2 x^4 - \frac{x^2}{6} - \frac{\log x}{2} + \frac{3}{8}$
72. $c_1 x^{-1} + [c_2\cos(\log x) + c_3\sin(\log x)]\,x + 5x + \frac{10\log x}{x}$
73. $c_1 x^3 + c_2 x^{-4} + \frac{x^3}{98}\log x(7\log x - 2)$
74. $c_1 x + c_2 x^{-1} + \frac{x^m}{(m^2-1)}$
75. $c_1 x^2 + x^{\frac{5}{2}}(c_2 x^{\frac{\sqrt{21}}{2}} + c_3 x^{-\frac{\sqrt{21}}{2}}) - \frac{x^3}{5}$
76. $x^2(c_1 x^{\sqrt{3}} + c_2 x^{\sqrt{-3}}) + \frac{1}{6x} + \frac{\log x}{61x}(5\sin\log x + 6\cos\log x) + \frac{2}{3721x}(27\sin\log x + 191\cos\log x)$
77. $(c_1 + c_2\log r) + \pi\rho r^2$
78. $(c_1 + c_2\log x)\cos(\log x) + (c_3 + c_4\log x)\sin(\log x) + (\log x)^2 + 2\log x - 3$
79. $[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)]x + \frac{x}{2}\sin\log x$
80. $\frac{1}{3x}(5x^2 + c_1 x + c_2)$
81. $c_1\cos\log(1+x) + c_2\sin\log(1+x) + 2\log(1+x)\sin\log(1+x)$
82. $c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2)]$
83. $c_1(2x+3)^{(3+\sqrt{57})/4} + c_2(2x+3)^{(3-\sqrt{57})/4} - \frac{3}{14}(2x+3) + \frac{3}{4}$
84. $c_1\cos\log(1+x) + c_2\sin\log(1+x) - \cos x\log(1+x)$
85. $(1+2x)^2[\{\log(1+2x)\}^2 + c_1\log(1+2x) + c_2]$
86. $c_1 + c_2\log(x+1) + \{\log(x+1)\}^2 + x^2 + 8x$
87. $c_1 x^2 + c_2 x^3 + \frac{1}{6}(\log x + \frac{5}{6})$
88. $c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2) + 1]$
89. $c_1 e^{-2x} + c_2 e^{-x} + 2e^x$
90. $(c_1 + c_2 x)e^{-x} + \frac{x^4 e^{-x}}{12}$
91. $c_1\cos x + c_2\sin x - x^2\cos x + x\sin x$
92. $(c_1 + c_2 x)e^x + x^2 e^x(\frac{1}{2}\log x - \frac{3}{4})$

93. $e^x \log(1+e^{-x}) - 1 - e^{-x}\log(1+e^x) + c_1 e^x + c_2 e^{-x}$

94. $e^x \log(1+e^{-x}) + e^{2x}\{\log(1+e^{-x}) - (1+e^{-x})\} + c_1 e^x + c_2 e^{2x}$

95. $(c_1 + c_2 x)e^{-x} - e^{-x}\log x - e^{-x}$

96. $c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

97. $c_1 e^{2x} + c_2 e^x - 9xe^x - 9e^x$

98. $c_1 \cos 3x + c_2 \sin 3x + \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x \log \cos 3x$

99. $c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log \sin x$

100. $(c_1 + c_2 x)e^x + xe^x \log x$

101. $(c_1 \cos x + c_2 \sin x)e^x - e^x \cos x \log(\sec x + \tan x)$

102. $c_1 e^{2x} + c_2 e^{3x} + xe^{3x} + \frac{1}{10}(\sin x + \cos x)$

103. $c_1 + c_2 e^{2x} - \frac{1}{2}e^x \sin x$

104. $c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log \sec 2x + \tan 2x$

105. $c_1 \cos x + c_2 \sin x + \frac{1}{6}\tan^2 x \sin x$

106. $c_1 x + c_2 x \log x$

107. $\frac{c_1}{x} + c_2 x + \frac{x}{2}\log x$