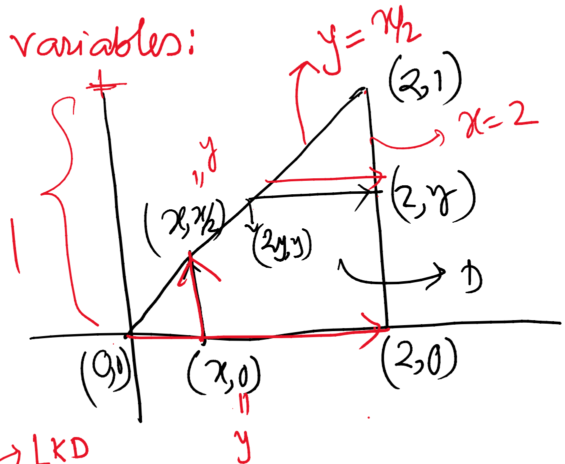
06 January 2022 08:06

Change of variables:

Recap:



$$\iint_D (xy^2)\,dx\,dy$$

$$= \int_0^2 \int_0^{x/2} xy^2\,dy\,dx$$

$$= \int_0^1 \int_{x=2y}^{x=2} xy^2\,dx\,dy = 4/15$$

MCQ → No negative marking

LKD

C-D | Question will be same for all

9 → 15 marks

11-01-22

9 to 9:30 AM

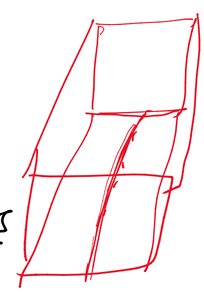
Question: Let $f : D \to \mathbb{R}^+$, $D \subseteq \mathbb{R}^2$, then $\iint_D f\,dx\,dy$ will always exist. (T|F) F

Existence Th. Suppose f is bounded on B and cont. except for discont. along a finite number of graphs of cont. functions

$\Downarrow$

f is integrable

i.e. $\iint_B f\,dx\,dy$ exists



$\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx$$

Let f is bounded on $[a,b]$ and the set of points of discontinuities D is a zero set.

$\Downarrow$ f is integrable

Q: $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ using double integration

$$\int_{-\infty}^{0} e^{-x^2}\,dx + \int_{0}^{\infty} e^{-x^2}\,dx \rightarrow \text{Gamma function}$$

Cylindrical Co-ordinate:

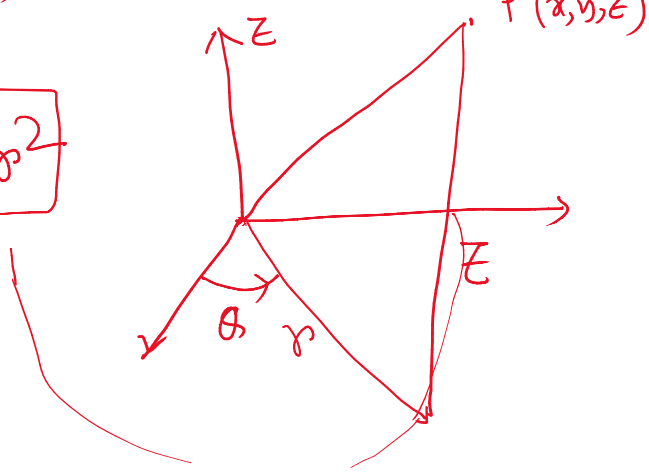
(r, θ, z) of a point

(x, y, z)

$\downarrow$

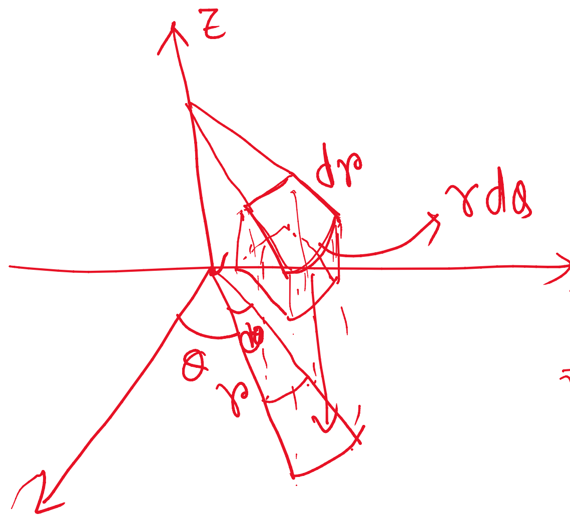
(r, θ, z) : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $r \geq 0$, $0 \leq \theta \leq 2\pi$

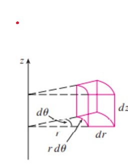
$\boxed{x^2 + y^2 = r^2}$



$(x, y, z) \rightarrow (r, \theta, z)$

Infinitesimal volume element:





$\boxed{r\,dr\,d\theta}$

x

Triple integration: If W is a region in space,

$$\iiint_W f(x,y,z)\,\underline{dx\,dy\,dz} = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

Where W^* is the corresponding region in (r,θ,z) space.

Ex. $$\iiint (z^2x^2+z^2y^2)\,dx\,dy\,dz$$

$W=\{(x,y,z): x^2+y^2\le 1,\ -1\le z\le 1\}$

Using cylindrical transformation,

$W^*=\{(r,\theta,z): 0\le r\le 1,\ 0\le\theta\le 2\pi,\ -1\le z\le 1\}$

$$= \int_{-1}^{1}\int_0^{2\pi}\int_0^{1} z^2 r^2 \boxed{r\,dr\,d\theta\,dz}$$

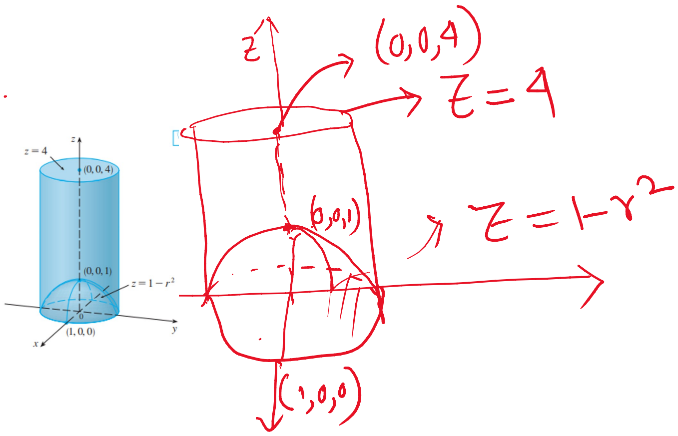
$$= \int_{-1}^{1}\int_0^{2\pi} z^2\cdot \frac{r^4}{4}\Big|_{r=0}^{1} d\theta\,dz = \int_{-1}^{1}\int_0^{2\pi} z^2/4\,d\theta\,dz$$

$$= 2\pi\cdot\frac{1}{12}\left(z^3\Big|_{-1}^{1}\right)$$

$$= \pi/3.$$

Ex. A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any pt

$z = 1 - x^2 - y^2$. The density at any pt is proportional to its distance from the axis of the cylinder. Find the mass of E.



In cylindrical system

$$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ \underline{1 - r^2} \le z \le 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, so the density function

$$f(x,y,z) = k\underline{\sqrt{x^2+y^2}} = kr, \quad \underline{k>0} \text{ const.}$$

So, mass of E

$$= \iiint_E k\underline{\sqrt{x^2+y^2}}\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^{4} kr \cdot r \, dz\, dr\, d\theta$$

(do it!) $$= \frac{12\pi k}{5}$$

Ex. Evaluate $\underline{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2}} (x^2+y^2)\, dz\, dy\, dx$

$\sqrt{4-x^2}$ $\quad$ $\sqrt{x^2+y^2}$

Using cylindrical transformation E, we get $\quad \{(x,y,z): -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ \sqrt{x^2+y^2} \le z \le 2\}$

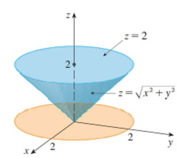
$$E^* = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ r \le z \le 2\}$$

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2 \cdot r\, dz\, dr\, d\theta$$

Check it

$$= \frac{16\pi}{5}$$

$\boxed{r^2 = x^2+y^2}$



Figure

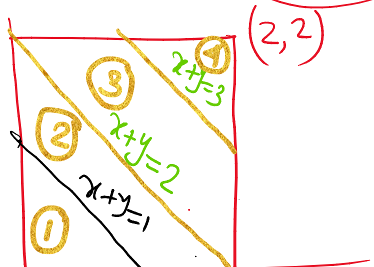
I

20-01-22 $\quad$ Spherical co-ordinate :-

$f = k \quad \iint_D f\, dx\,dy = k \cdot A(D)$

$\iint_D (1)\, dx\,dy = 1 \cdot \boxed{\text{Area of } (D)}$

$\iint_{[0,2]\times[0,2]} f(x+y)\, dx\,dy$, $\quad$ f(t) greatest int. $\le t$



Whenever x, y ∈ ①②③④

$f(x,y) = 0, 1, 2,$ and 3 resp.

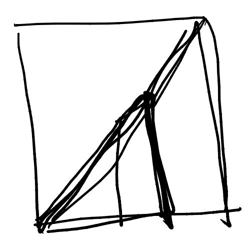
$$= A(1)\cdot 0 + A(2)\cdot 1 + A(3)\cdot 2 + A(4)\cdot 3$$

$$= \tfrac{1}{2}\cdot 0 + \left(2-\tfrac{1}{2}\right)\cdot 1 + \left(2-\tfrac{1}{2}\right)\cdot 2 + \tfrac{1}{2}\cdot 3$$

$$= 6$$

→ Ex. $\iiint_W e^{(x^2+y^2+z^2)^{3/2}}\, dV$

$W = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$



$W = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$

We can solve via
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$$

(extremely awkward to evaluate it)

Alternatively, we like to use spherical co-ordinate transf. to evaluate this

$$W^* = \{(\rho,\theta,\phi) : 0 \leq \rho \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$$

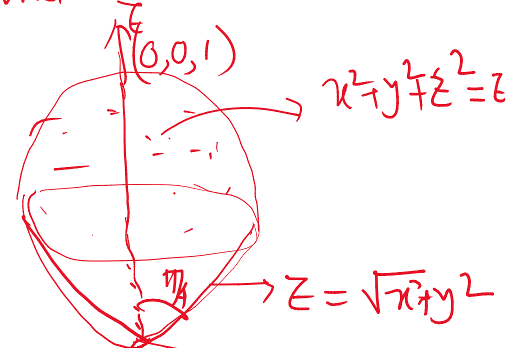
$$\iiint_W f\, dv = \iiint_{W^*} e^{(\rho^2)^{3/2}} (\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi$$

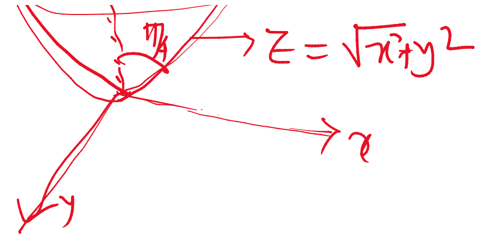
$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{\rho^3} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^{1} e^{\rho^3}\rho^2\, d\rho$$

$$\cdots = \frac{4}{3}\pi(e-1) \quad \text{(Ans)}$$

Ex 2. Volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.





$\underline{\text{Sol}^n}$ $\Longrightarrow$ Sphere passes through the origin and has the centre $(0,0,\tfrac{1}{2})$.

$$\rho^2 = \rho\cos\phi, \quad \underline{\rho = \cos\phi}$$

- eq^n of the cone

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos\theta + \rho^2\sin^2\phi\sin^2\theta}$$

$$= \rho\sin\phi$$

$$\sin\phi = \cos\phi, \quad \phi = \pi/4$$

Therefore, the solid E in spherical co-ordinate system is

$$E = \{(\rho,\theta,\phi) : 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4,\ 0 \le \rho \le \cos\phi\}$$

$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\cos\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$\cdots = \pi/8$$

$\underline{\text{General Case}}$: Change of variables:

$$\iint_D f(x,y)\,dx\,dy$$

$x \sim u$
$y \sim v$

$x = x(u,v)$
$y = y(u,v)$

$x = x(u,v)$
$y = y(u,v)$

$$\iint_D f\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v))\,|J|\,du\,dv$$

$$J = \boxed{x_u y_v - y_u x_v} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$x_u = \frac{\partial x}{\partial u}, y_v = \frac{\partial y}{\partial v}$

D^* is the new region in u-v plane.

Ex. $\iint_D (x+y)^2\,dx\,dy$

$D \to$ parallelogram $x+y=0$, $x+y=1$ | $2x-y=0$, $2x-y=3$

$\boxed{u = x+y \text{ and } v = 2x-y} \Rightarrow x = \frac{u+v}{3}, y = \frac{2u-v}{3}$

$D^* = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 3\}$

$$J = x_u y_v - x_v y_u = -\frac{1}{9} - \frac{2}{9} = -\frac{3}{9}$$
$$= -\frac{1}{3}.$$

Using the above transformation, we get

$$\iint_D (x+y)^2\,dx\,dy = \iint_{D^*} u^2 \left|-\frac{1}{3}\right| du\,dv$$
$$= \frac{1}{3}\int_0^1 u^2\,du \int_0^3 dv$$
$$= \frac{1}{3} \cdot \left[\frac{u^3}{3}\right]_0^1 \cdot 3$$

$$= \frac{1}{3}.$$

Triple integration:

$$u = u(x,y,z),\ v = v(x,y,z),\ w = w(x,y,z)$$

$$J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(u,v,w)\,|J|\,du\,dv\,dw$$

Home task: $$\iiint_W \frac{dx\,dy\,dz}{\sqrt{1+x^2+y^2+z^2}}$$

$$W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}.$$