Complex Integration

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Complex Integration

Complex Integration:

- Let $C : z(t),\ a \le t \le b$ be a contour and f be any complex function defined on C.
- Let $P : a = t_0 < t_1 < \ldots.. < t_n = b$ be a partition of $[a, b]$.
- Corresponding to the partition P, the curve C is divided in n smaller arcs $\sigma_k = z_{k-1} \to z_k,\ k = 1, 2, \ldots., n$ where $z_k = z(t_k.)$
- Let $\zeta_k = z(s_k),\ t_{k-1} \le s_k \le t_k$ be an arbitrary point in σ_k.
- Let $S_P = \displaystyle\sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$.
  - Let $||P|| := \displaystyle\max_{1 \le k \le n} |t_k - t_{k-1}|$.
  - Choose $n \to \infty$ in such a way that $||P|| \to 0$.

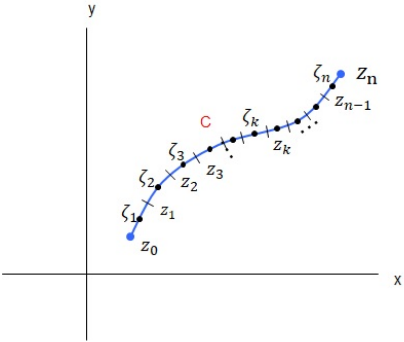


Figure: Complex Integration

Complex Integration

- If $\lim_{||P|| \to 0} S_P$ exists, f is said to be integrable and written as

$$\int_C f(z)dz = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}).$$

- If C is a closed path the integral is denoted by $\oint_C f(z)dz$.

Theorem

- *If f is continuous on a contour C, then f is integrable along C.*
- *If $f = u + iv$ is continuous on a contour $C : z(t) = x(t) + iy(t),\ t \in [a, b]$ then*

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b (udx - vdy) + i\int_a^b (vdx + udy).$$

- *The length of the contour $C : z(t) = x(t) + iy(t),\ t \in [a, b]$ is given by*

$$L(C) := \int_C |dz| = \int_a^b |z'(t)|dt = \int_a^b \sqrt{x'^2(t) + y'^2(t)}\, dt$$

Complex Integration

1. **Linearity:** $\int_C [\alpha f(z) + \beta g(z)]dz = \alpha \int_C f(z)dz + \beta \int_C g(z)dz.$
2. $\int_{-C} f(z)dz = -\int_C f(z)dz.$

   Here if C is the curve joining the points from z_0 to z_1 then $-C$ is the curve joining the points from z_1 to z_0.
3. $\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$
4. **M-L inequality:** If L is the length of the curve C and $M = \max\limits_{t\in[a,b]} |f(z(t))|$ then
$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)||dz| \leq ML.$$

Hints:

$$\begin{aligned}\left|\int_C f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right| \\ &\leq \int_a^b |f(z(t))||z'(t)|dt = \int_C |f(z)||dz| \\ &\leq M\int_a^b |z'(t)|dt = ML.\end{aligned}$$

Examples

Example: Show that $\left|\int_C \frac{dz}{z^2+10}\right| \leq \frac{2\pi}{3}$ where $C : z(t) = 2e^{it}, -\pi \leq t \leq \pi$.

Solution: For $z \in C$, $|z^2+10| \geq 10 - |z|^2 = 10 - |z(t)|^2 = 10 - 4 = 6$ and so

$$\left|\int_C \frac{dz}{z^2+10}\right| = \int_C \frac{|dz|}{|z^2+10|} \leq \frac{1}{6}\int_C |dz| = \frac{1}{6} \times 4\pi = \frac{2\pi}{3}.$$

Theorem (Fundamental theorem of integration)

If a continuous function f has a primitive F in domain D i.e., $F'(z) = f(z)$ for all $z \in D$ then for all paths C in D joining two points z_0 and z_1 in D, we have

$$\int_C f(z)\,dz = F(z_1) - F(z_0).$$

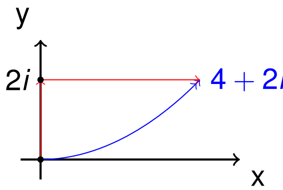
Proof: Let $z(t),\ t \in [a, b]$ be a parameterization of C with $z_0 = z(a), z_1 = z(b)$. Then

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt == \int_a^b F'(z(t))z'(t)\,dt = \int_a^b \frac{dF(z(t))}{dt}\,dt$$

$$= F(z(b)) - F(z(a)) = F(z_1) - F(z_0).$$

Examples

Example: Find $\int_C \bar{z}\, dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by
$(a)\ \ z(t) = t^2 + it \quad (b)$ the line from $z = 0$ to $2i$ and then line from $z = 2i$ to $4 + 2i$.



Solution: (a) For the curve $z(t) = t^2 + it$ the point $z = 0$ and $z = 4 + 2i$ corresponds $t = 0$ and 2 respectively. Therefore

$$\int_C \bar{z}\, dz = \int_0^2 \overline{(t^2 + it)}(2t + i)\, dt = \int_0^2 (2t^3 - it^2 + t)\, dt = 10 - \frac{8}{3}i.$$

(b) Let $C = C_1 + C_2$ where $C_1 : x = 0, 0 \le y \le 2$ and $C_2 : y = 2, 0 \le x \le 4$. Then

$$\int_C \bar{z}\, dz = \int_{C_1} \bar{z}\, dz + \int_{C_2} \bar{z}\, dz = \int_0^2 \overline{(iy)}i\, dy + \int_0^4 \overline{(x + 2i)}\, dx$$

$$= \int_0^2 y\, dy + \int_0^4 (x - 2i)\, dx = 2 + (8 - 8i) = 10 - 8i$$

Examples

Example: Find $\int_C 3z^2\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by
(a) $z(t) = t^2 + it$ (b) the line from $z = 0$ to $2i$ and then line from $z = 2i$ to $4 + 2i$.

Solution: Let $f(z) = 3z^2$ and $F(z) = z^3$. Then $F'(z) = f(z)$. Thus

$$\int_C 3z^2\,dz = F(4+2i) - F(0) = (4+2i)^3$$

where C is any curve joining $z = 0$ and $z = 4 + 2i$.

Cauchy's Theorem

Theorem (Green's Theorem)

Let $M(x, y)$ and $N(x, y)$ be continuous with continuous partial derivative in a simply connected domain R whose boundary is a simple closed contour. Then

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

where C is traversed in the positive sense.

Cauchy's Theorem

Theorem (Cauchy's weak Theorem)

If $f(z)$ is analytic with a continuous derivative in a simply connected domain D, and C is closed contour lying in D, then we have $\int_C f(z)\,dz = 0$.

Proof:

- Let $f(z) = u(x,y) + iv(x,y)$. By C-R equation, we have $u_x = v_y,\ \&\ u_y = -v_x$ for $x, y \in \mathbb{D}$.
- Since $f'(z) = u_x + iv_x$ is continuous, all these partial derivatives are continuous.
- Let C be a simple closed contour.
- Then by Green's Theorem

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx + udy)$$
$$= \iint_R (-v_x - u_y)dxdy + i\iint_R (u_x - v_y)dxdy$$
$$= 0.$$

Cauchy's Theorem

Theorem (Cauchy's Theorem or Cauchy-Goursat Theorem)

If f is analytic in a simply connected domain D and C is any closed contour lying in D, then $\oint_C f(z)dz = 0$.

Remark: The domain bounded by a simple closed contour is always simply connected domain.

Corollary

If f is analytic within and on a simple closed contour C then $\oint_C f(z)dz = 0.$.

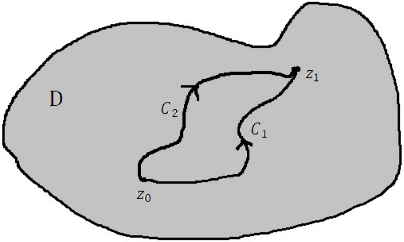
Corollary

Let f be analytic in a simply connected domain D and $a \in D$. Then the function $F(z) = \int_a^z f(\xi)\, d\xi,\ z \in D$ is analytic in D such that $F'(z) = f(z)$ for all $z \in D$.

Cauchy's Theorem

Remark-1: Let f be analytic in a SCD D and z_0 and z_1 be any two points inside D. Then $\int_{z_0}^{z_1} f(z)\,dz$ is independent of the path in D joining the point z_0 and z_1.

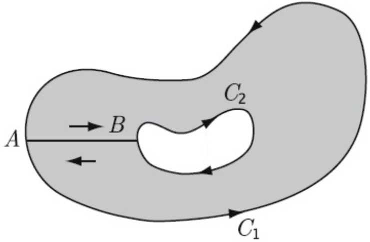
- Let C_1 and C_2 be two distinct curves joining z_0 and z_1.
- Let $C = C_1 + (-C_2)$. Then C is a closed curve lying inside D.
- By Cauchy's theorem $\oint_C f(z)\,dz = 0$
- But $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$
- Thus $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.



Cauchy's Theorem

Remark-2: Suppose that $f(z)$ is analytic in a multiply connected domain D and on its boundary C. Then we have $\int_C f(z)\,dz = 0$, where the integration is performed along C in the positive sense.

- Suppose we construct the line segment AB, called a **cross-cut**, which connects the outer boundary C_1 with the inner boundary C_2.
- Then the domain bounded by the contour C_1, the line segment AB, the contour C_2, and the line segment BA (traversed as illustrated in Figure) is simply connected.



- By Cauchy's theorem

$$\oint_{C_1} f(z)\,dz + \int_{AB} f(z)\,dz + \oint_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz = 0.$$

- But $\int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz$.
- Thus $\oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz = 0 \implies \int_C f(z)\,dz = 0.$

Cauchy's Theorem

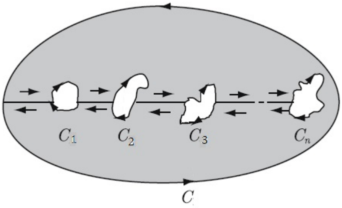
Remark-3: Let $f(z)$ be analytic in a domain D bounded by two simple closed contour C_1 and C_2 and also on C_1 and C_2. Then $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$ where C_1 and C_2 are both traversed counterclockwise.

Theorem (Cauchy's Theorem for multiply connected domains)

Let D be a multiply connected domain bounded externally by a simple closed contour C and internally by n simple closed nonintersecting contours $C_1, C_2, \ldots, C_n$. Let f be analytic on $D \cup C_1 \cup C_2 \cup \ldots \cup C_n$. Then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

where C and $C_1, C_2, \ldots, C_n$ are all traversed counterclockwise.



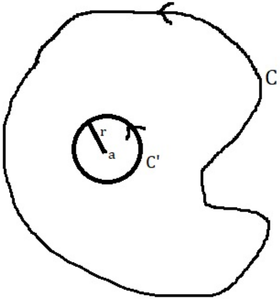
Cauchy's Theorem

Example: Evaluate $\oint_C \frac{dz}{(z-a)^n}$, $n \in \mathbb{Z}$ where C is any closed contour.

Solution:

- If a lies outside C then $1/(z-a)^n$ is analytic inside and on C.
- By Cauchy's theorem $\oint_C \frac{dz}{(z-a)^n} = 0$.

- If a lies inside C then consider a circle C' lying inside C of radius r with center at $z = a$.
- By Cauchy's theorem for multiply connected domain, $\oint_C \frac{dz}{(z-a)^n} = \oint_{C'} \frac{dz}{(z-a)^n}$.
- We know that
$$\oint_{C'} \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \neq 1 \end{cases}$$
- Thus
$$\oint_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \neq 1 \end{cases}$$



Cauchy's Theorem

Example: Evaluate $\oint_C \frac{dz}{z^2+1}$ where C is the circle

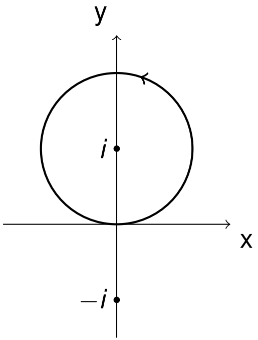
$(a)\ |z-i| = 1 \quad (b)\ |z+i| = 1 \quad (c)\ |z| = 2 \quad (d)\ |z-1| = 1.$

Solution: Let

$$I = \oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_C \frac{dz}{z-i} + \frac{1}{2i}\oint_C \frac{dz}{z+i}$$

(a) Let $C : |z-i| = 1$. In this case, $\frac{1}{z+i}$ is analytic within and on C. Then

$$I = \frac{1}{2i}\oint_C \frac{dz}{z-i} + \frac{1}{2i}\oint_C \frac{dz}{z+i} = \frac{1}{2i} \times 2\pi i + 0 = \pi$$

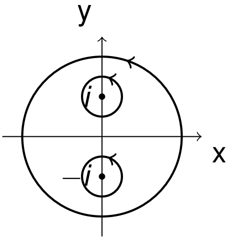


(b) Let $C : |z+i| = 1$. In this case, $\frac{1}{z-i}$ is analytic within and on C. Then

$$I = \frac{1}{2i}\oint_C \frac{dz}{z-i} + \frac{1}{2i}\oint_C \frac{dz}{z+i} = 0 + \frac{1}{2i} \times 2\pi i = \pi$$

Cauchy's Theorem

(c) Let $C : |z| = 2$. In this case, both the point i and $-i$ lies inside C. The curve $C_1 : |z - i| = \frac{1}{2}$ and $C_2 : |z + i| = \frac{1}{2}$ lies inside C. Then

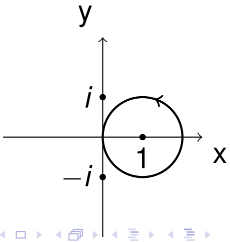


$$I = \frac{1}{2i}\oint_C \frac{dz}{z-i} + \frac{1}{2i}\oint_C \frac{dz}{z+i}$$

$$= \frac{1}{2i}\left(\oint_{C_1} \frac{dz}{z-i} + \oint_{C_2} \frac{dz}{z-i} + \oint_{C_1} \frac{dz}{z+i} + \oint_{C_2} \frac{dz}{z+i}\right)$$

$$= \frac{1}{2i}(2\pi i + 0 + 0 + 2\pi i) = 2\pi.$$

(d) Let $C : |z - 1| = 1$. In this case, both the points i and $-i$ lies outside C and so $\dfrac{1}{z^2+1}$ is analytic within and on C. Thus



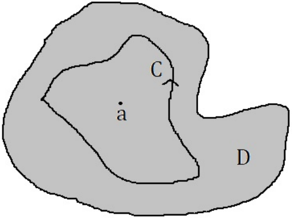
$$I = \oint_C \frac{dz}{z^2+1} = 0$$

Cauchy's Integral Formula

Theorem (Cauchy's Integral Formula)

Let $f(z)$ be analytic in a simply connected domain D. Then for any point $a \in D$ and any simple closed contour C enclosing a, we have

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$



Theorem (Cauchy's Integral Formula for derivatives)

Let $f(z)$ be analytic in a simply connected domain D. Then for any point $a \in D$ and any simple closed contour C enclosing a, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad n = 0, 1, 2, \cdots .$$

Cauchy's Integral Formula

Example: Evaluate $\oint_C \frac{e^z}{z-2}\,dz$ where C is the circle (a) $|z| = 3$ (b) $|z| = 1$.

Solution: The integrand $\frac{e^z}{z-2}$ is not analytic at $z = 2$.

(a) Let $C : |z| = 3$ and so $z = 2$ lies inside C. If $f(z) = e^z$ then $f(z)$ is analytic within and on C. Then by Cauchy's integral formula

$$\oint_C \frac{e^z}{z-2}\,dz = \oint_C \frac{f(z)}{z-2}\,dz = 2\pi i \times f(2) = 2\pi i e^2$$

(b) Let $C : |z| = 1$ and so $z = 2$ lies outside C. If $f(z) = \frac{e^z}{z-2}$ then $f(z)$ is analytic within and on C. Then by Cauchy's integral theorem

$$\oint_C \frac{e^z}{z-2}\,dz = \oint_C f(z)\,dz = 0.$$

Cauchy's Integral Formula

Example: Evaluate $\oint_C \frac{\tan z}{z^2-1}\,dz$ where C is the circle $|z| = \frac{3}{2}$.

Solution: The integrand $\frac{\tan z}{z^2-1}$ is not analytic at

$z = i, -i, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots.$

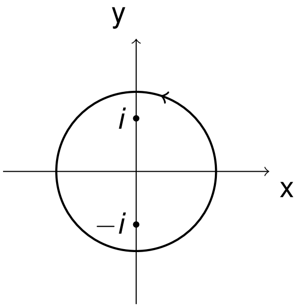
The points $z = i, -i$ lies inside $C : |z| = \frac{3}{2}$. Then by Cauchy's integral formula

$$\oint_C \frac{\tan z}{z^2-1}\,dz = \oint_C \frac{\tan z}{(z-1)(z+1)}\,dz$$

$$= \frac{1}{2}\oint_C \frac{\tan z}{(z-1)}\,dz - \frac{1}{2}\oint_C \frac{\tan z}{(z+1)}\,dz$$

$$= \frac{1}{2} \times 2\pi i \tan 1 - \frac{1}{2} \times 2\pi i \tan(-1)$$

$$= 2\pi i \tan 1$$



Cauchy's Integral Formula

Example: Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}\, dz$ where C is the circle $|z| = 3$.

Solution: The integrand $\frac{e^{2z}}{(z+1)^4}$ is not analytic at $z = -1$ and the point $z = -1$ lies inside C.
Let $f(z) = e^{2z}$. Then $f(z)$ is analytic within and on C.

Then by Cauchy's integral formula for derivatives we have

$$f^{(3)}(-1) = \frac{3!}{2\pi i} \oint_C \frac{f(z)}{(z+1)^4}\, dz = \frac{3}{\pi i} \oint_C \frac{e^{2z}}{(z+1)^4}\, dz.$$

Now $f'(z) = 2e^{2z},\ f''(z) = 4e^{2z},\ f'''(z) = 8e^{2z}$.

$$\oint_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{\pi i}{3} f^{(3)}(-1) = \frac{8\pi i}{3e^2}.$$

Cauchy's Integral Formula

Example: Evaluate $\oint_C \frac{e^z}{(z+1)^2(z-2)}\,dz$ where C is the circle $|z-1| = 3$.

Solution: The integrand is not analytic at $z = -1, 2$ and these points lies inside C.

By partial fraction

$$\frac{1}{(z+1)^2(z-2)} = \frac{1/9}{(z-2)} - \frac{1/9}{(z+1)} - \frac{1/3}{(z+1)^2}.$$

Let $f(z) = e^z$. Then $f(z)$ is analytic within and on C.

Then by Cauchy's integral formula for derivatives we have

$$\oint_C \frac{e^z}{(z+1)^2(z-2)}\,dz = \frac{1}{9}\oint_C \frac{e^z}{(z-2)}\,dz - \frac{1}{9}\oint_C \frac{e^z}{(z+1)}\,dz - \frac{1}{3}\oint_C \frac{e^z}{(z+1)^2}\,dz$$

$$= \frac{1}{9} \times 2\pi i f(2) - \frac{1}{9} \times 2\pi i f(-1) - \frac{1}{3} \times 2\pi i f'(-1)$$

$$= \frac{2\pi i}{9}(e^2 - e^{-1} - 3e^{-1}) = \frac{2\pi i}{9}(e^2 - 4e^{-1})$$

Cauchy's Integral Formula

Example: Evaluate $\oint_C \frac{z+4}{z^2+2z+5}\,dz$ where C is the circle $|z+1-i|=2$.

Solution: The integrand is not analytic at $z=-1\pm 2i$. Note that the point $-1+2i$ lies inside C and the point $-1-2i$ lies outside C.

Let $f(z)=\frac{z+4}{z+1+2i}$. Then $f(z)$ is analytic within and on C.

Then by Cauchy's integral formula we have

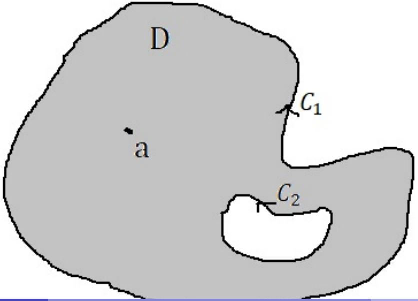
$$\begin{aligned}
\oint_C \frac{z+4}{z^2+2z+5}\,dz &= \oint_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz \\
&= \oint_C \frac{f(z)}{(z+1-2i)}\,dz = 2\pi i\; f(-1+2i) \\
&= 2\pi i\frac{-1+2i+4}{-1+2i+1+2i} = \frac{\pi}{2}(3+2i).
\end{aligned}$$

Cauchy's Integral Formula

Theorem (Cauchy integral formula for multiply connected domain)

Let D be a multiply connected domain bounded by two simple closed contour C_1 and C_2 (C_2 lying wholly within C_1) and $f(z)$ is analytic in $D \cup C_1 \cup C_2$. If a is any interior point of D, then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz - \frac{n!}{2\pi i}\oint_{C_2} \frac{f(z)}{(z-a)^{n+1}}dz, \quad n = 0,\ 1,\ 2, \ldots.$$



Theorem

Let f be analytic in a domain D. Then the derivatives of all orders of f exist and are analytic in D.

Converse of Cauchy's Theorem:

Theorem (Morera's Theorem)

Let $f(z)$ be continuous in a domain D with the property that $\oint_C f(z)\,dz = 0$ for every simple closed contour C. Then $f(z)$ is analytic.

Theorem (Cauchy Inequality)

Let $f(z)$ be analytic in the open disk $\Delta(a,r)=\{z\in\mathbb{C}:|z-a|<r\}$ and $|f(z)|\leq M$ for all $z\in\partial\Delta(a,r)$. Then

$$|f^{(n)}(a)|\leq\frac{Mn!}{r^n},\quad n\in\mathbb{N}.$$

Proof: By Cauchy integral formula for derivatives, we have

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint_C\frac{f(z)}{(z-a)^{n+1}}dz,\qquad C:|z-a|=r$$

Then

$$|f^{(n)}(a)|\leq\frac{n!}{2\pi}\oint_C\frac{|f(z)|}{|z-a|^{n+1}}|dz|\leq\frac{Mn!}{2\pi r^{n+1}}\oint_C|dz|=\frac{Mn!}{2\pi r^{n+1}}2\pi r=\frac{Mn!}{r^n}.$$

Theorem (Liouville's Theorem)

Any bounded entire function is constant.

Proof:

- Let $f(z)$ be a bounded entire function.
- Then there exists a positive constant M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.
- Let a be any point of the complex plane and C be the circumference of the circle $|z - a| = R$.
- Then, by Cauchy's inequality, we have $|f'(a)| \leq \dfrac{M}{R}$.
- Since $f(z)$ is an entire function, R may be taken arbitrarily large and, therefore, $|f'(a)| \leq \frac{M}{R} \to 0$ as $R \to \infty$.
- Thus $f'(a) = 0$. Since a is arbitrary, $f'(z) = 0$ for all $z \in \mathbb{C}$.
- Thus f is constant.

Example:

- The function $\sin z$ is an entire function and it is not bounded.
- The function $\cos z$ is an entire function and it is not bounded.

Power Series

- A series of the form
$$\sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$
is known as the power series about the point $z = a$ where a_n and a are fixed complex numbers and z is a complex variable.
- The power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ is called absolutely convergent if $\sum_{n=0}^{\infty} |a_n||z-a|^n$ is convergent.
- For every power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ there exist a real number R such that for every z in $|z-a| < R$, the series is absolutely convergent and for every z in $|z-a| > R$, the series is divergent.
- The number R is called the radius of convergence of the power series and the circle $|z-a| = R$ is called the circle of convergence of the power series.
- No general statement can be made about the convergence of a power series on the circle of convergence.

Radius of convergence of a power series:

If R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ then

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \quad \text{(Ratio Test)}$$

or,

$$\frac{1}{R} = \lim_{n\to\infty}|a_n|^{1/n} \quad \text{(Root Test)}$$

Power Series

Example: Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Solution: Here $a_n = \frac{1}{n!}$ and $a = 0$. Then

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Thus the radius of convergence is $R = \infty$ and the series converges for all $z \in \mathbb{C}$.

Example: Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n}{4n^2+1} z^n$.

Solution: Here $a_n = \frac{n}{4n^2+1}$ and $a = 0$. Then

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{4(n+1)^2+1} \times \frac{4n^2+1}{n} = 1.$$

Thus the radius of convergence is $R = 1$ and the series converges for

Power Series

Example: Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(3+i)^{-n}z^n$.

Solution: Here $a_n = (3+i)^{-n}$ and $a = 0$. Then

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{3+i}\right| = \frac{1}{\sqrt{10}}.$$

Thus the radius of convergence is $R = \sqrt{10}$ and the series converges for $|z| < \sqrt{10}$ and diverges for $|z| > \sqrt{10}$.

Power Series

Example: Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(3+i)^{-n}z^{3n}$.

Solution: Let $w = z^3$. Then the series becomes $\sum_{n=0}^{\infty} a_n w^n$ where $a_n = (3+i)^{-n}$. Then

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{3+i}\right| = \frac{1}{\sqrt{10}}.$$

Thus the series $\sum_{n=0}^{\infty} a_n w^n$ convergence is converges for $|w| < \sqrt{10}$ and diverges for $|w| > \sqrt{10}$.

Hence the power series $\sum_{n=0}^{\infty}(3+i)^{-n}z^{3n}$ convergence is converges for $|z| < 10^{1/6}$ and diverges for $|z| > 10^{1/6}$. Thus the radius of convergence of the given series is $R = 10^{1/6}$.

Power Series

Theorem

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ and it converges to the function $f(z)$ in $|z-a| < R$. The $f(z)$ is analytic in $|z-a| < R$, i.e., a power series represents an analytic function inside its circle of convergence.

Taylor Series

Theorem (Taylor's theorem)

Let $f(z)$ be analytic in a domain D whose boundary is C. Then for all $z \in D$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad |z-a| < \delta$$

where $a_n = \dfrac{f^{(n)}(a)}{n!}$ are called Taylor's coefficients of $f(z)$ and δ is the distance from a to the nearest point of C.

Remark:

- The infinite series is called the Taylor's series and

$$a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \oint_{|z-a|=\delta} \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

- If $a = 0$ then the Taylor's series is called Maclaurin's series.

Taylor Series

When D is a disk then Taylor's theorem can be written as:

Taylor's theorem: Let $f(z)$ be analytic in the disk $|z-a|<R$. Then for all z in the disk

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n, \quad |z-a|<R.$$

The Taylor series $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n$ converges to $f(z)$ in the disk $|z-a|<R$.

Remark:

- Any function $f(z)$ which is analytic at a point z_0 must have a Taylor's series about z_0 valid in some nbd of z_0.
- If f is an entire function then the radius of convergence can be chosen arbitrary large, i.e., the region of validity of the Taylor's series becomes $|z-z_0|<\infty$.

Taylor Series

Example: Find the Taylor series of $f(z) = \dfrac{1}{1-z}$ about $z = 0$.

Solution: Clearly,

$$f'(z) = \frac{1}{(1-z)^2}, \quad f''(z) = \frac{1.2}{(1-z)^3}, \quad f'''(z) = \frac{1.2.3}{(1-z)^4}, \ldots, f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$$

Thus $f^{(n)}(0) = n!$ and so the Taylor series of $f(z)$ about $z = 0$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^n = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots .$$

If R is the radius of convergence of the Taylor series then

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{1} = 1.$$

Example: The Taylor series of $f(z) = \dfrac{1}{1+z}$ about $z = 0$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^n = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1.$$

Taylor Series

Example: Find the Taylor series of $f(z) = \dfrac{z+2}{1-z^2}$ about $z = 0$.

Solution: Here

$$f(z) = \frac{z+2}{1-z^2} = \frac{3}{2}\frac{1}{1-z} + \frac{1}{2}\frac{1}{1+z}$$

$$= \frac{3}{2}\left(\sum_{n=0}^{\infty} z^n\right) + \frac{1}{2}\left(\sum_{n=0}^{\infty}(-1)^n z^n\right), \quad |z| < 1$$

$$= 2 + z + 2z^2 + z^3 + \cdots, \quad |z| < 1.$$

Taylor Series

Example: Find the Taylor series of $f(z) = \dfrac{1}{(z-2)(z-3)}$ about $z = 0$.

Solution:

$$f(z) = \frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2} = -\frac{1}{3}\frac{1}{1-z/3} + \frac{1}{2}\frac{1}{1-z/2}$$

$$= -\frac{1}{3}\left(\sum_{n=0}^{\infty}(z/3)^n\right) + \frac{1}{2}\left(\sum_{n=0}^{\infty}(z/2)^n\right)$$

where the first series is valid in $|z| < 3$ and the second series is valid in $|z| < 2$. Thus both series are valid in $|z| < 2$. Hence

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right)z^n, \quad |z| < 2.$$

Laurent Series

Theorem (Laurent's theorem)

Let $f(z)$ be analytic in the annular region (annulus) $D : R_1 < |z-a| < R_2$. Then for each $z \in D$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

with

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad \text{and} \quad b_n = a_{-n}$$

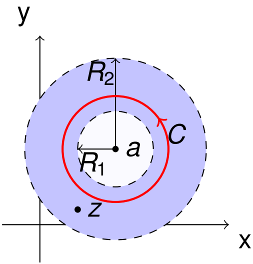
where C is any simple closed contour lying in D that makes a complete counterclockwise revolution about a.

Alternatively,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$



Laurent Series

Remark:

- Suppose $f(z)$ is analytic inside the disk $|z-a| < R_1$. Then by Cauchy's theorem

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{-n+1}}\, dz = 0$$

In this case, the Laurent series reduce to the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Laurent Series

Example: Find the Taylor's/Laurent's series of $f(z) = \dfrac{1}{1-z}$ in

(i) $|z| < 1$ (ii) $|z| > 1$.

Solution: (i) Let $|z| < 1$. Then

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

(ii) Let $1 < |z| < \infty$. Then

$$f(z) = \frac{1}{1-z} = \frac{1}{-z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, \quad \left|\frac{1}{z}\right| < 1$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}, \quad 1 < |z| < \infty.$$

Example: Find the Laurent's series of $f(z) = \dfrac{1}{1+z}$ in $|z| > 1$.

Solution:

$$f(z) = \frac{1}{1+z} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}}, \quad 1 < |z| < \infty.$$

Laurent Series

Example: Find the Taylor's/Laurent's series of $f(z) = \dfrac{1}{(z+1)(z+3)}$ in

(i) $|z| < 1$ (ii) $1 < |z| < 3$ (iii) $|z| > 3$ (iv) $0 < |z+1| < 2$

Solution: We have

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

(i) Let $|z| < 1$. Then

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} = \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \cdots.$$

This is a Taylor's series valid for $|z| < 1$.

Laurent Series

(ii) Let $1 < |z| < 3$. Then

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= \left(\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots\right) + \left(-\frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots\right).$$

This is a Laurent's series valid for $1 < |z| < 3$.
(iii) Let $|z| > 3$. Then

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right)$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots.$$

This is a Laurent's series valid for $|z| > 3$.

Laurent Series

(iv) Let $0 < |z+1| < 2$. We substitute $u = z+1$. Then $0 < |u| < 2$ and so

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} = \frac{1}{2u} - \frac{1}{2(u+2)} = \frac{1}{2u} - \frac{1}{4}\left(1 + \frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left(1 - \frac{(z+1)}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + \cdots\right).$$

This is a Laurent's series valid for $0 < |z+1| < 2$.

Laurent Series

Example: Find the Taylor's/Laurent's series of $f(z) = \dfrac{z}{z^2 - 3z + 2}$ in

(i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$.

Solution: We have

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{z}{(z-1)(z-2)} = \frac{1}{1-z} - \frac{2}{2-z}.$$

(i) Let $|z| < 1$. Then

$$f(z) = \frac{1}{1-z} - \frac{2}{2-z} = (1-z)^{-1} - \left(1 - \frac{z}{2}\right)^{-1} = \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n.$$

(ii) Let $1 < |z| < 2$. Then

$$f(z) = \frac{1}{1-z} - \frac{2}{2-z} = -\frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \left(1 - \frac{z}{2}\right)^{-1} = -\frac{1}{z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n.$$

(iii) Let $|z| > 2$. Then

$$f(z) = \frac{1}{1-z} - \frac{2}{2-z} = -\frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1 - \frac{2}{z}\right)^{-1} = -\frac{1}{z}\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \frac{2}{z}\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n.$$

Laurent Series

Example: Suppose that $f(z)$ is an entire function and that $|f(z)| \leq M|z|^k$ as $|z| \to \infty$ for some $k > 0$. Then $f(z)$ is a polynomial of degree at most k.

Solution: Since $f(z)$ is entire function, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad z \in \mathbb{C}.$$

By Cauchy's inequality, on the circle $|z| = R$ (R is a very large number), we have

$$|a_n| = \left| \frac{f^{(n)}(0)}{n!} \right| \leq \frac{MR^k}{R^n} = \frac{M}{R^{n-k}}$$

Letting $R \to \infty$, we see that $a_n = 0$ whenever $n > k$. Hence, $f(z)$ is a polynomial of degree at most k.

Zeros of analytic function

Definition

Let $f(z)$ be analytic at z_0. Then z_0 is called a zero of $f(z)$ if $f(z_0) = 0$.

Theorem (The Fundamental Theorem of Algebra)

Every nonconstant polynomial has at least one zero in $\mathbb{C}$.

Theorem

Every polynomial of degree n has exactly n (not necessarily distinct) zeros in $\mathbb{C}$.

Theorem (Zeros are isolated)

Suppose $f(z)$ is analytic at a point $z = z_0$. Then either $f(z) \equiv 0$ in some neighborhood of z_0, or there exists a real number r such that $f(z) \neq 0$ in the punctured disk $0 < |z - z_0| < r$.

Zeros of analytic function

Definition

Let $f(z)$ be analytic at z_0. Then $f(z)$ has a Taylor series expansion in a nbd of z_0 as follows

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < \delta$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$.

If $a_0 = a_1 = a_2 = \cdots = a_{m-1} = 0$ but $a_m \neq 0$ then $f(z)$ can be written as

$$f(z) = (z - z_0)^m \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} = (z - z_0)^m \phi(z),$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$ and we say that $f(z)$ has a zero of order m at $z = z_0$.

Singularity

- **Singular points:** A point z_0 is called a singular point of a function $f(z)$ if $f(z)$ is not analytic at z_0 (may even be undefined at z_0) but is analytic at some point in every neighborhood of z_0. The function $f(z)$ is said to have a singularity at z_0.
- **Isolated singular point:** The singular point z_0 is called isolated singular point of $f(z)$ if there exist a nbd of z_0 containing no other singular point of $f(z)$.
- **Non-isolated singular point:** The singular point z_0 is called a non-isolated singular point of $f(z)$ if every neighborhood of z_0 contains at least one singularity of $f(z)$ other than z_0.

Example: Consider the function $f(z) = |z|^2 = z\bar{z}$. This function is nowhere analytic. It has no singular point.

Example: Consider the function $f(z) = \tan z = \dfrac{\sin z}{\cos z}$. Singular points of $f(z)$ are given by

$$\cos z = 0 \implies z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots .$$

All these singular points are isolated.

Singularity

Example: Consider the function $f(z) = \tan\frac{1}{z} = \dfrac{\sin\frac{1}{z}}{\cos\frac{1}{z}}$.

Singular points of $f(z)$ are given by

$$\cos\frac{1}{z} = 0 \implies \frac{1}{z} = (2n+1)\frac{\pi}{2} \implies z = \frac{2}{(2n+1)\pi}, \quad n \in \mathbb{Z}.$$

All these singular points are isolated.

Note that $f(z)$ is not defined at $z = 0$ but $f(z)$ is analytic in some nbd of $z = 0$. Thus $z = 0$ is also a singular points of $f(z)$.

Since $\lim\limits_{n\to\infty} \dfrac{2}{(2n+1)\pi} = 0$, every nbd of $z = 0$ contains many other singular point of $f(z)$.

Thus $z = 0$ is a non-isolated singular point of $f(z)$.

Singularity

Isolated singularity of a function $f(z)$ at $z = z_0$ can be further classified.

Let $z = z_0$ be an isolated singularity of an analytic function $f(z)$. Then in a deleted nbd of $z = z_0$, $f(z)$ has a Laurent series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}, \quad 0 < |z - z_0| < \delta.$$

The term $\sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$ is called the principal part of $f(z)$ at $z = z_0$. Now there are three possibilities.

1. Removable singularity:

- If the principal part in Laurent expansion of $f(z)$ does not contain any term, that is $b_n = 0$ for all $n \in \mathbb{N}$ then z_0 is called a removable singularity of $f(z)$.
- If a function $f(z)$ is not defined at z_0 but $\lim_{z \to z_0} f(z)$ exist then z_0 is called a removable singularity of $f(z)$.
- In this case, if we define $f(z)$ at z_0 as equal to $\lim_{z \to z_0} f(z)$ then $f(z)$ will be analytic at z_0.

2. Pole:

- If the principal part has only a finite number of terms, that is $b_n = 0$ for all $n > m$ for some $m \in \mathbb{N}$ and $b_m \neq 0$ then z_0 is called a pole of order m of $f(z)$.
- If $m = 1$ then z_0 is called a simple pole. If $m = 2$ then z_0 is called a double pole.
- In this case, $f(z)$ has the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{m} b_n (z - z_0)^{-n}.$$

- If z_0 is an isolated singularity and we can find a positive integer m such that

$$\lim_{z \to z_0} (z - z_0)^m f(z) = A \neq 0$$

  then z_0 is called a pole of order m of $f(z)$.
- An isolated singularity z_0 of $f(z)$ is a pole of order m iff $f(z)$ can be expressed as $f(z) = \dfrac{\phi(z)}{(z - z_0)^m}$ where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.
- An isolated singularity z_0 of $f(z)$ is a pole of $f(z)$ iff $\lim\limits_{z \to z_0} f(z) = \infty$.

3. Essential Singularity:

- If the principal part in Laurent expansion of $f(z)$ contains an infinite number of terms, then z_0 is called an isolated essential singularity of $f(z)$.
- In this case, $f(z)$ has the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}.$$

- If z_0 is a isolated singularity and there exist no positive integer m such that

$$\lim_{z \to z_0} (z - z_0)^m f(z) = A$$

  is called an essential singularity of $f(z)$.
- If z_0 is a isolated singularity of $f(z)$ and $\lim\limits_{z \to z_0} f(z)$ does not exist in $\mathbb{C}_\infty$ then z_0 is called an essential singularity of $f(z)$.

Remark: A point z_0 is a pole of order m of $f(z)$ iff z_0 is a zero of order m of $1/f(z)$.

Singularity

Example: Consider the function $f(z) = \dfrac{\sin z}{z}$.

Note that $f(0)$ is not defined but $\lim_{z \to 0} \dfrac{\sin z}{z} = 1$. Then $z = 0$ is a removable singularity of $f(z)$.

Example: Consider the function $f(z) = \dfrac{1}{z(z-2)^5} + \dfrac{3}{(z-2)^2}$.

$f(z)$ has isolated singularity at $z = 0, 2$. Now,

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \left(\frac{1}{(z-2)^5} + \frac{3z}{(z-2)^2} \right) = -\frac{1}{32}.$$

$$\lim_{z \to 0} (z-2)^5 f(z) = \lim_{z \to 2} \left(\frac{1}{z} + 3z(z-2)^3 \right) = \frac{1}{2}.$$

Thus $z = 0$ is a simple pole and $z = 2$ is a pole of order 5 of $f(z)$.

Example: The function $f(z) = e^{\frac{1}{z}}$ has essential singularity at $z = 0$ because

$$f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots.$$

Singularity

Example: Find the singularities of the function $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ and classify them.

Solution: Singularities of $f(z)$ are given by

$$\cos z = 0 \implies z = z_n := (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{N}.$$

Since $\lim\limits_{z \to z_n} f(z)$ does not exist, there are no removable singularity.

$$\begin{aligned}\lim_{z \to z_n} (z - z_n) f(z) &= \lim_{z \to z_n} \frac{z - z_n}{\cot z} \quad (\frac{0}{0} \text{ form}) \\ &= \lim_{z \to z_n} \frac{1}{-\sec^2 z} = -1\end{aligned}$$

Thus all the singularities of $f(z)$ are simple pole.

Singularity

Example: Find the nature of the singularities of $f(z) = \dfrac{1}{z(e^z - 1)}$.

Solution: Singularities of $f(z)$ are given by

$$z = 0 \;\;\&\;\; e^z = 1 \implies z = 0, 2n\pi i, \quad n \in \mathbb{Z}.$$

Thus $z = 0$ is a double pole. All other poles are simple.

Example: Find the nature of the singularities of $f(z) = z \sin \dfrac{1}{z}$.

Solution: The only singularity of $f(z)$ is at $z = 0$. Note that

$$f(z) = z \sin \frac{1}{z} = z \left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots \right) = 1 - \frac{1}{3!z^2} + \frac{1}{5!z^4} - \cdots$$

Since the series does not terminate, $z = 0$ is an essential singularity of $f(z)$.

Singularity

Isolated singularity at ∞: A function $f(z)$ has an isolated singularity at $z = \infty$ if and only if $f(1/z)$ has an isolated singularity at $z = 0$.

Moreover, we make the definition that the singularity of $f(z)$ at $z = \infty$ is removable, a pole, or essential according as the singularity of $f(1/z)$ at $z = 0$ is removable, a pole, or essential.

Example:

1. The function $f(z) = z^2 + 1$ has a pole of order 2 at $z = \infty$ because $f(1/z) = (1/z^2) + 1$ has a pole of order 2 at $z = 0$.
2. The function $f(z) = e^z$ has an isolated essential singularity at $z = \infty$ because $f(1/z) = e^{1/z}$ has an isolated essential singularity at $z = 0$.
3. Let $f(z) = \dfrac{1}{z(z^2+4)}$. Then $f(1/z) = \frac{z^3}{1+4z^2}$ which is analytic at $z = 0$. Thus $f(z)$ is analytic at $z = \infty$.

Meromorphic function: A function $f(z)$ is said to be meromorphic if it is analytic in the finite complex plane $\mathbb{C}$ except possibly at a finite number of poles.

Residue

Residue at a finite point: We recall that if $f(z)$ has an isolated singularity at z_0, then in a deleted nbd of $z = z_0$, $f(z)$ has a Laurent series expansion of the

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \quad 0 < |z - z_0| < \delta.$$

with

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad \text{and} \quad b_n = a_{-n}$$

where C is any simple closed contour lying in the nbd of z_0 that makes a complete counterclockwise revolution about z_0.

The coefficient b_1 is called the residue of $f(z)$ at z_0 and is denoted by $\mathrm{Res}\,[f(z); z_0]$. Thus

$$\mathrm{Res}\,[f(z); z_0] = b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Residue

Residue at ∞: If $f(z)$ has an isolated singularity at ∞, then in a deleted nbd of ∞, $f(z)$ has a Laurent series expansion of the

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}, \quad \delta < |z| < \infty.$$

Let C be any simple closed contour lying in the nbd of ∞ that makes a complete clockwise revolution about ∞. Then

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\sum_{n=0}^{\infty} a_n \oint_C z^n\,dz + \frac{1}{2\pi i}\sum_{n=1}^{\infty} b_n \oint_C z^{-n}\,dz = -b_1.$$

Therefore, we define the residue of $f(z)$ at $z = \infty$ as

$$\begin{aligned}\mathrm{Res}\,[f(z);\infty] &= \frac{1}{2\pi i}\oint_C f(z)\,dz \\ &= -b_1 = -(\text{coefficient of } 1/z \text{ in the Laurent series expansion of } f(z)).\end{aligned}$$

Remark:

$$\mathrm{Res}\,[f(z);\infty] = -\mathrm{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right].$$

Residue

Theorem (Residue at a pole)

If f(z) has a pole of order m at $z = z_0$, then

$$\mathrm{Res}\,[f(z); z_0] = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)].$$

In particular, if $f(z)$ has a simple pole at z_0, then

$$\mathrm{Res}\,[f(z); z_0] = \lim_{z\to z_0}(z-z_0)f(z).$$

Theorem (Residue at a pole)

Let $f(z)$ and $g(z)$ be analytic at z_0. If $g(z)$ has a simple pole at z_0 and $f(z_0) \neq 0$, then

$$\mathrm{Res}\left[\frac{f(z)}{g(z)}; z_0\right] = \frac{f(z_0)}{g'(z_0)}.$$

Remark: Let z_0 is a essential singularity of $f(z)$. To find $\mathrm{Res}\,[f(z); z_0]$,

Residue

Example: Find the singularities in the complex plane and the residue at those singular points of the function $\dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$

Solution: The function $f(z)$ has a pole of order 2 at $z = -1$ and simple poles at $z = \pm 2i$. Therefore

$$\operatorname{Res}[f(z); -1] = \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d}{dz}[(z+1)^2 f(z)] = \lim_{z \to -1} \frac{d}{dz} \frac{z^2 - 2z}{(z^2+4)}$$

$$= \lim_{z \to -1} \frac{(z^2+4)(2z-2) - (z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}.$$

$$\operatorname{Res}[f(z); 2i] = \lim_{z \to 2i} (z - 2i) f(z) = \lim_{z \to 2i} \frac{z^2 - 2z}{(z+1)^2(z+2i)} = \frac{7+i}{25}.$$

$$\operatorname{Res}[f(z); -2i] = \lim_{z \to -2i} (z + 2i) f(z) = \lim_{z \to -2i} \frac{z^2 - 2z}{(z+1)^2(z-2i)} = \frac{7-i}{25}.$$

Residue

Example: Find the singularities in the complex plane and the residue at those singular points of the function $\dfrac{e^{z^2}}{(z-i)^3}$.

Solution: The function $f(z)$ has a pole of order 3 at $z = i$. Therefore

$$\mathrm{Res}\,[f(z); i] = \frac{1}{(3-1)!} \lim_{z \to i} \frac{d^2}{dz^2}[(z-i)^3 f(z)] = \frac{1}{2} \lim_{z \to i} \frac{d^2}{dz^2} e^{z^2}$$

$$= \frac{1}{2} \lim_{z \to i} (2z^2 e^{z^2} + e^{z^2}) = -\frac{1}{e}.$$

Example: Find the singularities in the complex plane and the residue at those singular points of the functions $f(z) = \dfrac{\sin z - z}{z^3}$.

Solution: The function $f(z)$ has pole of order 3 at $z = 0$. Therefore, Therefore

$$\mathrm{Res}\,[f(z); 0] = \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2}[z^3 f(z)] = \frac{1}{2} \lim_{z \to 0} (\sin z - z) = 0$$

Residue

Example: Find the singularities in the complex plane and the residue at those singular points of the function $f(z) = e^{1/z}$ and evaluate $\oint_{|z|=1} e^{1/z}\, dz$.

Solution: The function

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots$$

has an isolated essential singularity at $z = 0$. Thus $\mathrm{Res}\,[f(z); 0] = b_1 = 1$ and so

$$\oint_{|z|=1} e^{1/z}\, dz = 2\pi i \times \mathrm{Res}\,[f(z); 0] = 2\pi i.$$

Example: Find the singularities in the complex plane and the residue at those singular points of the function $f(z) = \sin\frac{1}{z^2}$ and evaluate $\oint_{|z|=1} \sin\left(\frac{1}{z^2}\right) dz$.

Solution: The function

$$f(z) = \sin\frac{1}{z^2} = \frac{1}{z^2} - \frac{1}{3!}\frac{1}{z^6} + \cdots$$

has an isolated essential singularity at $z = 0$. Thus $\mathrm{Res}\,[f(z); 0] = b_1 = 0$ and so

$$\oint_{|z|=1} \sin\frac{1}{z^2}\, dz = 2\pi i \times \mathrm{Res}\,[f(z); 0] = 0.$$

Residue

Example: Find the singularities in the extended complex plane and the residue at those singular points of the function $f(z) = \dfrac{z^n}{1+z}$.

Solution: The function $f(z)$ has an simple pole at $z = -1$. Now

$$f(1/z) = \frac{1}{z^{n-1}(1+z)}$$

which shows that $z = \infty$ is a pole of order $n - 1$ of $f(z)$. Now

$$\mathrm{Res}\,[f(z); -1] = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1} z^n = (-1)^n$$

We know that

$$\mathrm{Res}\,[f(z); \infty] = -\mathrm{Res}\,\left[\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right].$$

If $g(z) = \dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) = \dfrac{1}{z^{n+1}(1+z)}$ then $z = 0$ is a pole of order $n+1$ of $g(z)$. Thus

$$\mathrm{Res}\,[f(z); \infty] = -\mathrm{Res}\,[g(z); 0] = -\frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}[z^{n+1}g(z)] = -\frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}(1+z)^{-1}$$

$$= (-1)^{n+1}\lim_{z\to 0}(1+z)^{-(n+1)} = (-1)^{n+1}.$$

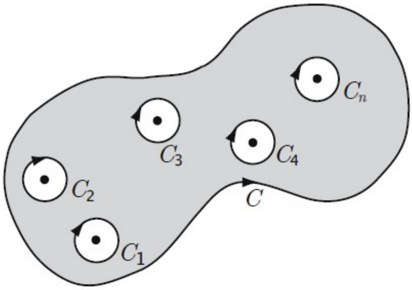
Remark: In the previous example, $\mathrm{Res}\,[f(z); -1] + \mathrm{Res}\,[f(z); \infty] = 0$.

Residue

Theorem (Cauchy's Residue Theorem)

Let $f(z)$ be analytic inside and on a simple closed contour C except for isolated singularities at $z_1, z_2, z_3, \ldots, z_n$ inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,[f(z); z_k]$$



Theorem (Residue Theorem for $\mathbb{C}_\infty$)

Suppose f(z) is analytic in $\mathbb{C}_\infty$ except for isolated singularities at $z_1, z_2, z_3, \ldots, z_n, \infty$. Then the sum of its residues (including the point at infinity) is zero. That is,

$$\mathrm{Res}\,[f(z); \infty] + \sum_{k=1}^{n} \mathrm{Res}\,[f(z); z_k] = 0$$

Residue

Example: Evaluate $\displaystyle\int_{|z|=2} \frac{dz}{z^2(z^2-1)}$.

Solution: The function $f(z) = \dfrac{1}{z^2(z^2-1)}$ has double pole at $z = 0$ and simple pole at $z = \pm 1$. Note that all the singular points of $f(z)$ lies inside $|z| = 2$. Now

$$\mathrm{Res}\,[f(z);0] = \lim_{z\to 0} \frac{d}{dz}[(z-0)^2 f(z)] = \lim_{z\to 0} \frac{d}{dz}\frac{1}{z^2-1} = \lim_{z\to 0} \frac{-2z}{(z^2-1)^2} =$$

$$\mathrm{Res}\,[f(z);1] = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1} \frac{1}{z^2(z+1)} = \frac{1}{2}$$

$$\mathrm{Res}\,[f(z);-1] = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1} \frac{1}{z^2(z-1)} = -\frac{1}{2}.$$

Then by Cauchy's Residue theorem

$$\int_{|z|=2} \frac{dz}{z^2(z^2-1)} = 2\pi i(\mathrm{Res}\,[f(z);0] + \mathrm{Res}\,[f(z);1] + \mathrm{Res}\,[f(z);-1]) = 0.$$

Residue

Example: Evaluate $\displaystyle\int_{|z-1|=1/2} \frac{dz}{z^2(z^2-1)}$.

Solution: The function $f(z) = \dfrac{1}{z^2(z^2-1)}$ has double pole at $z = 0$ and simple pole at $z = \pm 1$. Note that only the pole $z = 1$ of $f(z)$ lies inside $|z-1| = 1/2$. Now

$$\mathrm{Res}\,[f(z); 1] = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{1}{z^2(z+1)} = \frac{1}{2}$$

Then by Cauchy's Residue theorem

$$\int_{|z|=2} \frac{dz}{z^2(z^2-1)}\,dz = 2\pi i \times \mathrm{Res}\,[f(z); 1] = \pi i.$$

Residue

Example: Evaluate $\displaystyle\int_{|z-3|=1} \frac{1-\cos 2(z-3)}{(z-3)^3}\, dz.$

Solution: First we note that

$$
\begin{aligned}
f(z) &= \frac{1-\cos 2(z-3)}{(z-3)^3} \\
&= \frac{1}{(z-3)^3}\left[1 - 1 + \frac{4(z-3)^2}{2!} - \frac{16(z-3)^4}{4!} + \cdots\right] \\
&= \frac{2}{(z-3)} - \frac{16(z-3)}{4!} + \cdots .
\end{aligned}
$$

Thus $f(z)$ has a simple pole at $z = 3$.

The Laurent's series is in the power of $z-3$. The coefficient of $\frac{1}{(z-3)}$ is 2. Hence, $\operatorname{Res}[f(z); 3] = 2$.

Then by Cauchy's Residue theorem

$$
\int_{|z-3|=1} \frac{1-\cos 2(z-3)}{(z-3)^3}\, dz = 2\pi i \times \operatorname{Res}[f(z); 3] = 4\pi i.
$$

Residue

Example: Evaluate $\displaystyle\int_{|z|=2} \tan z \, dz$.

Solution: The singularity of the function $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ are given by

$$\cos z = 0 \implies z = (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z}.$$

The singular points $z_n = (2n+1)\frac{\pi}{2}$ are simple poles of $f(z)$.

Note that only the pole $z_0 = \frac{\pi}{2}$ and $z_{-1} = -\frac{\pi}{2}$ of $f(z)$ lies inside $|z| = 2$. Now

$$\mathrm{Res}\,[f(z); \frac{\pi}{2}] = \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \tan z = \lim_{z \to \frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right)}{\cot z} = \lim_{z \to \frac{\pi}{2}} \frac{1}{\csc^2 z} = 1$$

$$\mathrm{Res}\,[f(z); -\frac{\pi}{2}] = \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) \tan z = \lim_{z \to -\frac{\pi}{2}} \frac{\left(z + \frac{\pi}{2}\right)}{\cot z} = \lim_{z \to -\frac{\pi}{2}} \frac{1}{\csc^2 z} = 1.$$

Then by Cauchy's Residue theorem

$$\int_{|z|=2} \tan z \, dz = 2\pi i \left(\mathrm{Res}\,[f(z); \frac{\pi}{2}] + \mathrm{Res}\,[f(z); -\frac{\pi}{2}]\right) = 4\pi i.$$

Residue

Example: Evaluate $\displaystyle\int_{|z|=2} \frac{dz}{(z^n-1)(z-3)}$.

Solution: The function $f(z) = \dfrac{1}{(z^n-1)(z-3)}$ has simple pole at $z=3$ and at $z = z_k = e^{2k\pi i/n}$, $k = 0, 1, 2, \ldots, n-1$.

Note that the poles $z = z_k$, $k = 0, 1, 2, \ldots, n-1$ of $f(z)$ lies inside $|z| = 2$. Then by Cauchy's Residue theorem

$$\int_{|z|=2} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,[f(z); z_k] = -2\pi i \left(\mathrm{Res}\,[f(z); 3] + \mathrm{Res}\,[f(z); \infty]\right).$$

Now

$$\mathrm{Res}\,[f(z); 3] = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 1} \frac{1}{z^n-1} = \frac{1}{3^n-1},$$

$$\mathrm{Res}\,[f(z); \infty] = -\mathrm{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right] = \mathrm{Res}\left[\frac{z^{n-1}}{(1-3z)(1-z^n)}; 0\right] = 0.$$

Therefore,

$$\int \frac{dz}{(z^n-1)(z-3)}\,dz = \frac{2\pi i}{1-3^n}.$$

Problem set

Example: Discuss continuity of the function

$$f(z) = \begin{cases} \frac{\mathrm{Re}\, z}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Solution: Let $z_0 \neq 0$. Then

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{\mathrm{Re}\, z}{z} = \frac{\mathrm{Re}\, z_0}{z_0} = f(z_0).$$

But

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\mathrm{Re}\, z}{z} = \begin{cases} 1, & \text{if } z = x \to 0 \\ 0, & \text{if } z = iy \to 0. \end{cases}$$

Thus $f(z)$ is continuous at all complex number except at the origin.

Problem set

Example: Find the limit $\lim_{z\to 0} f(z)$ if exist where $f(z) = \dfrac{xy}{x^2+y^2} + 2xi$.

Solution: If $f(z) = u(x,y) + iv(x,y)$ then $u(x,y) = \dfrac{xy}{x^2+y^2}$ and $v(x,y) = 2x$. Now

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} u(x,y) &= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \\
&= \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2+m^2x^2} \quad \text{(along } y = mx\text{)} \\
&= \frac{m}{1+m^2}.
\end{aligned}$$

Since $\lim_{(x,y)\to(0,0)} u(x,y)$ does not exists, $\lim_{z\to 0} f(z)$ does not exists.

Problem set

Example: Determine where the following functions satisfy the Cauchy Riemann equations and where the functions are differentiable.

(i) $f(z) = \overline{z}^2$ (ii) $f(z) = z\mathrm{Re}\, z$ (iii) $f(z) = 2z + 4\overline{z} + 5$.

Solution: (i) We have

$$f(z) = \overline{z}^2 \implies f_{\overline{z}}(z) = 2\overline{z}.$$

Thus $f(z)$ satisfy the C-R equation only at the origin. Hence $f(z)$ is not differentiable at all non zero points.

At $z = 0$, we have to check it separately by definition. (Home work!)

(ii) We have

$$f(z) = z\mathrm{Re}\, z = \frac{z(z+\overline{z})}{2} \implies f_{\overline{z}}(z) = z/2.$$

Thus $f(z)$ satisfy the C-R equation only at the origin. Hence $f(z)$ is not differentiable at all non zero points.

At $z = 0$, we have to check it separately by definition. (Home work!)

$$f(z) = 2z + 4\overline{z} + 5 \implies f_{\overline{z}}(z) = 4.$$

Thus $f(z)$ does not satisfy the C-R equation at any points of the complex plane. Hence $f(z)$ is not differentiable at at any points of the complex plane.

Problem set

Example: If $f(z)$ is continuous at a point z_0, show that $\overline{f(z)}$ is also continuous at z_0. Is the same true for differentiability at z_0?

Solution: Let $f(z) = u(x, y) + iv(x, y)$. If $f(z)$ is continuous at a point z_0 then $u(x, y)$ and $v(x, y)$ is continuous at a point z_0. Thus $\overline{f(z)} = u(x, y) - iv(x, y)$ is also continuous at z_0.

The same is not true for differentiability at z_0. For example, let $f(z) = z$. Then $f(z)$ is differentiable at $z = 0$ but $\overline{f(z)} = \overline{z}$ is not differentiable at $z = 0$.

Problem set

Example: Find the values of the constants a, b, c so that the following functions becomes entire function:

(i) $f(z) = x + ay - i(bx + cy)$ (ii) $f(z) = a(x^2 + y^2) + ibxy + c$.

Solution: (i) If $f(z) = x + ay - i(bx + cy)$ is entire then it must satisfy the C-R equation at all points. Here $u(x, y) = x + ay$ and $v(x, y) = -(bx + cy)$. Then

$$\begin{aligned} u_x = v_y &\implies 1 = -c \implies c = -1 \\ u_y = -v_x &\implies a = b. \end{aligned}$$

(ii) If $f(z) = a(x^2 + y^2) + ibxy + c$ is entire then it must satisfy the C-R equation at all points. Here $u(x, y) = a(x^2 + y^2) + c$ and $v(x, y) = bxy$. Then

$$\begin{aligned} u_x = v_y &\implies 2ax = bx \implies 2a = b \\ u_y = -v_x &\implies 2ay = -by \implies 2a = -b. \end{aligned}$$

Thus $a = b = 0$.

Problem set

Example: Let $f(z) = u + iv$ be analytic. If $\dfrac{\partial u}{\partial x} = u_1(x, y)$ and $\dfrac{\partial u}{\partial y} = u_2(x, y)$ then show that

$$f(z) = \int [u_1(z, 0) - iu_2(z, 0)]\, dz.$$

Solution: We know that

$$f'(z) = u_x + iv_x = u_x - iu_y = u_1(x, y) - iu_2(x, y).$$

Substituting $y = 0$, we get

$$f'(x) = u_1(x, 0) - iu_2(x, 0).$$

Replacing x by z, we get

$$f'(z) = u_1(z, 0) - iu_2(z, 0) \implies \int [u_1(z, 0) - iu_2(z, 0)]\, dz.$$

Example: Let $f(z) = u + iv$ be analytic. If $\dfrac{\partial v}{\partial y} = v_1(x, y)$ and $\dfrac{\partial v}{\partial x} = v_2(x, y)$ then show that

$$f(z) = \int [v_1(z, 0) + iv_2(z, 0)]\, dz.$$

Remark: This method of constructing an analytic function is called Milne-Thomson's method.

Problem set

Example: If $u = e^{-x}(x\sin y - y\cos y)$ then find analytic function $f(z) = u + iv$ in terms of z.

Solution: We have,

$$u_x = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y = u_1(x,y),$$
$$u_y = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y = u_2(x,y).$$

Therefore, by Milne's method

$$f(z) = \int [u_1(z,0) - iu_2(z,0)]\,dz = \int [0 - i(ze^{-z} - e^{-z})]\,dz = ize^{-z} + c.$$

Problem set

Example: If $u - v = (x-y)(x^2+4xy+y^2)$ then find analytic function $f(z) = u + iv$ in terms of z.

Solution: If $f(z) = u + iv$ then $if(z) = -v + iu$. Thus

$$(1+i)f(z) = (u-v) + i(u+v) = U + iV = F(z)$$

is analytic function. Here

$$U = u - v = (x-y)(x^2+4xy+y^2).$$

Hence,

$$U_x = 3x^2 + 6xy - 3y^2 = \phi_1(x,y), \quad U_y = 3x^2 - 6xy - 3y^2 = \phi_2(x,y).$$

Therefore, by Milne's method

$$F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz = \int [3z^2 - i3z^2]\, dz = (1-i)z^3 + c$$

Thus

$$F(z) = (1+i)f(z) = (1-i)z^3 + c \implies f(z) = \frac{1-i}{1+i}z^3 + c' = -iz^3 + c'$$

Problem set

Example: Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n^k}{a^n} z^n$.

Solution: Here $a_n = \frac{n^k}{a^n}$.

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^k a^n}{n^k a^{n+1}} = \frac{1}{a} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^k = \frac{1}{a}.$$

Thus the radius of convergence of the power series is $R = a$.

Example: Find the radius of convergence of the power series $\sum_{n=1}^{\infty} n^{1/n}(z+i)^n$.

Solution: Here $a_n = n^{1/n}$.

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(n^{1/n}\right)^{1/n} = 1^0 = 1.$$

Thus the radius of convergence of the power series is $R = 1$.

Problem set

Example: Find the radius of convergence of the power series
$$\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$$

Solution: Here $a_n = \dfrac{n!}{n^n}$.
$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!\,n^n}{n!(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$
Thus the radius of convergence of the power series is $R = e$.

Example: Find the radius of convergence of the power series
$$\sum_{n=1}^{\infty} \frac{3^n}{n^2+4n} z^{2n}.$$

Solution: Let $w = z^2$. Then the given series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2+4n} w^n$.

Here $a_n = \frac{3^n}{n^2+4n}$.

Problem set

Example: Find $\int_C \overline{z}\, dz$ along the following curves:

$(i)\ \ z(t) = e^{2it},\ \ t \in [-\pi, \pi] \qquad (ii)\ \ z(t) = t + it,\ \ t \in [0, 2]$.

Solution: (i) If $C : z(t) = e^{2it},\ \ t \in [-\pi, \pi]$ then

$$\int_C \overline{z}\, dz = \int_{-\pi}^{\pi} \overline{z(t)} z'(t)\, dt = \int_{-\pi}^{\pi} e^{-2it}(2i)e^{2it}\, dt = 2i \int_{-\pi}^{\pi} dt = 4\pi i.$$

(ii) If $C : z(t) = t + it,\ \ t \in [0, 2]$ then

$$\int_C \overline{z}\, dz = \int_0^2 \overline{z(t)} z'(t)\, dt$$

$$= \int_0^2 (t - it)(1 + i)\, dt = (1 + i)(1 - i) \int_0^2 t\, dt = 4.$$

Problem set

Example: Show that $\left|\int_{|z|=1} \frac{2z+1}{5+z^2}\,dz\right| \leq \frac{3\pi}{2}$.

Solution: Let $f(z) = \frac{2z+1}{5+z^2}$. On $|z| = 1$, we have

$$|f(z)| \leq \frac{2|z|+1}{5-|z|^2} \leq \frac{3}{4}.$$

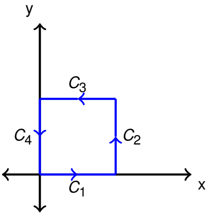
Then by ML-formula

$$\left|\int_{|z|=1} \frac{2z+1}{5+z^2}\,dz\right| \leq \int_{|z|=1} \left|\frac{2z+1}{5+z^2}\right| |dz| \leq \frac{3}{4}\int_{|z|=1} |dz| = \frac{3}{4} \times 2\pi = \frac{3\pi}{2}.$$

Problem set

Example: Evaluate $\int_C |z|^2\,dz$ along the square with vertices $0, 1, 1+i, i,$

Solution: Let $C = C_1 + C_2 + C_3 + C_4$ where $C_1 : z(t) = t,\ t \in [0,1]$, $C_2 : z(t) = 1 + it,\ t \in [0,1]$, $C_3 : z(t) = (1-t) + i,\ t \in [0,1]$, $C_4 : z(t) = i(1-t),\ t \in [0,1]$.



$$\int_C |z|^2\,dz = \int_{C_1} |z|^2\,dz + \int_{C_2} |z|^2\,dz + \int_{C_3} |z|^2\,dz + \int_{C_4} |z|^2\,dz$$

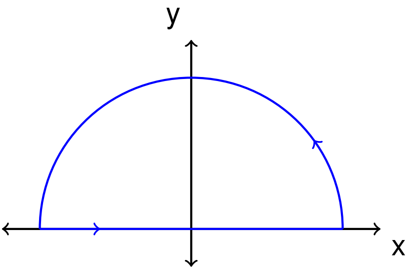
$$= \int_0^1 t^2\,dt + \int_0^1 |1+it|^2(i)\,dt + \int_0^1 |(1-t)+i|^2(-1)\,dt + \int_0^1 |i(1-t)|^2(-i)\,dt$$

$$= \int_0^1 t^2\,dt + i\int_0^1 (1-t^2+2it)\,dt - \int_0^1 (-2t+t^2+2i(1-t))\,dt - i\int_0^1 (1-t)^2\,dt$$

$$= \dots.$$

Problem set

Example: Evaluate $\int_C z|z|\,dz$ along the upper semicircle $|z| = R$ from R to $-R$ and the line segment $[-R, R]$.



Solution: Let $C = C_1 + C_2$ where
$C_1 : z(t) = Re^{it},\ t \in [0, \pi]$,
$C_2 : z(t) = t,\ t \in [-R, R]$.

$$\begin{aligned}
\int_C z|z|\,dz &= \int_{C_1} z|z|\,dz + \int_{C_2} z|z|\,dz \\
&= \int_0^{\pi} Re^{it}|Re^{it}|(iRe^{it})\,dt + \int_{-R}^{R} t|t|\,dt \\
&= iR^3\int_0^{\pi} e^{2it}\,dt + \int_{-R}^{0}(-t^2)\,dt + \int_0^{R}(t^2)\,dt \\
&= \ldots.
\end{aligned}$$

Problem set

Example: Evaluate $\int_C 4z^3\,dz$ along the following curves:

$(i)\;\; z(t) = t^2 + it,\;\; t \in [0,2] \qquad (ii)\;\; z(t) = t + it,\;\; t \in [0,2]$.

Solution: (i) Since the function $f(z) = 4z^3$ is analytic in $\mathbb{C}$, $\int_C 4z^3\,dz$ is independent of the path.

The initial and terminal points of the curve C are $z_0 = z(0) = 0$ and $z_1 = z(2) = 4 + 2i$ respectively.

Note that $F(z) = z^4$ is the anti-derivative of $f(z)$. Thus

$$\int_C 4z^3\,dz = F(z_1) - F(z_0) = (4 + 2i)^4$$

(ii) The initial and terminal points of the curve C are $z_0 = z(0) = 0$ and $z_1 = z(2) = 2 + 2i$ respectively. Thus

$$\int_C 4z^3\,dz = \Big[z^4\Big]_{z_0}^{z_1} = (2 + 2i)^4$$