LECTURE 38: Energy Methods

Strain Energy: Strain Energy of the member is defined as the internal work done in defoming the body by the action of externally applied forces. This energy in elastic bodies is known as **elastic strain energy**

Strain Energy in uniaxial Loading

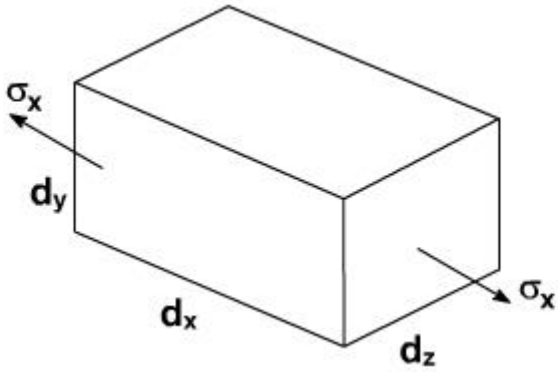


Fig .1

Let as consider an infinitesimal element of dimensions as shown in Fig .1. Let the element be subjected to normal stress σ_x.

The forces acting on the face of this element is σ_x. dy. dz

where dydz = Area of the element due to the application of forces, the element deforms to an amount = $\in_x$ dx

 $\in_x$ = strain in the material in x ? direction

$$= \frac{\text{Change in length}}{\text{Orginal in length}}$$

Assuming the element material to be as linearly elastic the stress is directly proportional to strain (Fig . 2.)

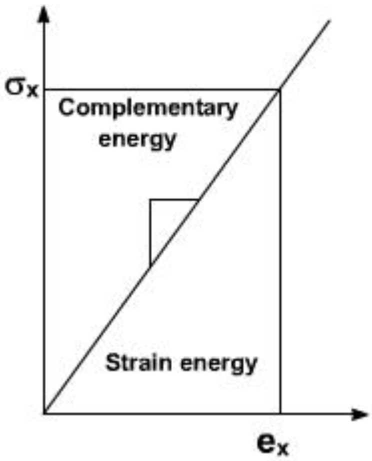


Fig .2

∴ From Fig .2 the force that acts on the element increases linearly from zero until it attains its full value.

Hence average force on the element is equal to ½ σ_x . dy. dz.

∴ Therefore the work done by the above force

Force = average force x deformed length = ½ σ_x. dydz . $\in_x$. dx

For a perfectly elastic body the above work done is the internal strain energy 'du'.

$$\boxed{du = \frac{1}{2}\sigma_x dydz\, \epsilon_x\, dx} \quad .....(2)$$

$$= \frac{1}{2}\sigma_x \epsilon_x\, dxdydz$$

$$\boxed{du = \frac{1}{2}\sigma_x\, \epsilon_x\, dv} \quad ....(3)$$

where dv = dxdydz = Volume of the element

By rearranging the above equation we can write

$$U_o = \boxed{\frac{du}{dv} = \frac{1}{2}\sigma_x\, \epsilon_x} \quad .....(4)$$

The equation (4) represents the strain energy in elastic body per unit volume of the material its strain energy density 'u_o'.

From Hook's Law for elastic bodies, it may be recalled that

$$\boxed{\sigma = E\epsilon}$$

$$\boxed{U_o = \frac{du}{dv} = \frac{\sigma_x^{\ 2}}{2E} = \frac{E\,\epsilon_x^{\ 2}}{2}} \quad ......(5)$$

$$\boxed{U = \int_{Vol} \frac{\sigma_x^{\ 2}}{2E} dv} \quad ....(6)$$

In the case of a rod of uniform cross section subjected at its ends an equal and opposite forces of magnitude P as shown in the Fig .3.

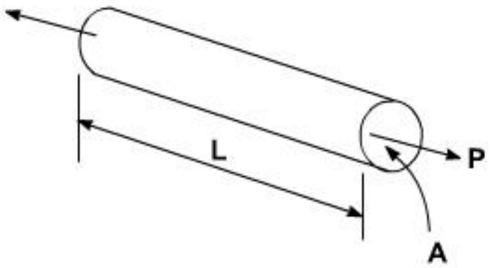


Fig .3

$$U = \int_{Vol}^{L} \frac{\sigma_x^{\ 2}}{2E} dv \qquad \sigma_x = \frac{P}{A}$$

$$U = \int_{0}^{L} \frac{P^2}{2EA^2} Adx \qquad dv = Adx = \text{Element volume}$$

A = Area of the bar.

L = Length of the bar

$$\boxed{U = \frac{P^2 L}{2AE}} \quad .....(7)$$

Modulus of resilience :

Suppose 'σ_x' in strain energy equation is put equal to σ_y i.e. the stress at proportional limit or yield point. The resulting strain energy gives an index of the materials ability to store or absorb energy without permanent deformation

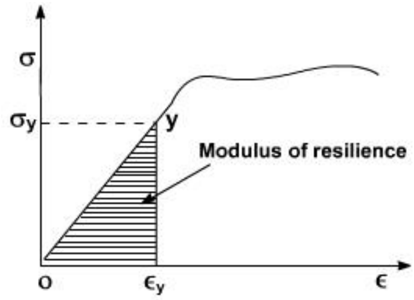


Fig .4

So $$\boxed{U_y = \frac{\sigma_y^{\ 2}}{2E}} \quad .....(8)$$

The quantity resulting from the above equation is called the Modulus of resilience

The modulus of resilience is equal to the area under the straight line portion 'OY' of the stress? strain diagram as shown in Fig .4 and represents the energy per unit volume that the material can absorb without yielding. Hence this is used to differentiate materials for applications where energy must be absorbed by members.

Modulus of Toughness :

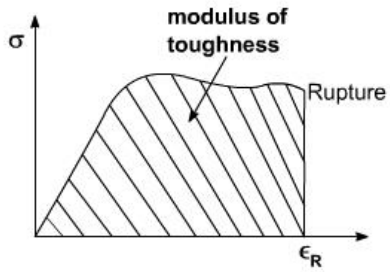


Fig .5

Suppose '$\in$' [strain] in strain energy expression is replaced by $\in_R$ strain at rupture, the resulting strain energy density is called modulus of toughness

$$U = \int_0^{\epsilon} E\,\epsilon_x \, dx = \frac{E\,\epsilon_R^2}{2} dv$$

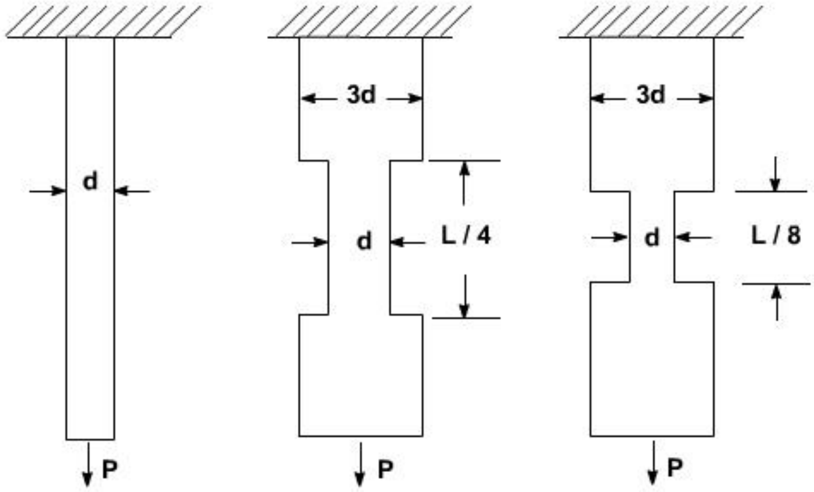
$$\boxed{U = \frac{E\,\epsilon_R^2}{2}} \qquad \ldots\ldots(9)$$

From the stress-strain diagram, the area under the complete curve gives the measure of modules of toughness. It is the materials.

Ability to absorb energy upto fracture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength and that the capacity of a structure to withstand an impact Load depends upon the toughness of the material used.

ILLUSTRATIVE PROBLEMS

1. Three round bars having the same length 'L' but different shapes are shown in fig below. The first bar has a diameter 'd' over its entire length, the second had this diameter over one-fourth of its length, and the third has this diameter over one eighth of its length. All three bars are subjected to the same load P. Compare the amounts of strain energy stored in the bars, assuming the linear elastic behavior.



Solution :

1.The strain Energy of the first bar is expressed as

$$U_1 = \frac{P^2 L}{2EA}$$

2.The strain Energy of the second bar is expressed as

$$U_2 = \frac{P^2 (L/4)}{2EA} + \frac{P^2 (3L/4)}{2E9A} = \frac{P^2 L}{6EA}$$

$$\boxed{U_2 = \frac{U_1}{3}}$$

3.The strain Energy of the third bar is expressed as

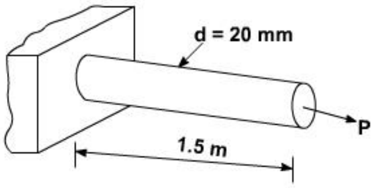
$$U_3 = \frac{P^2 (L/8)}{2EA} + \frac{P^2 (7L/8)}{2E(9A)}$$

$$U_3 = \frac{P^2 L}{9EA}$$

$$\boxed{U_3 = \frac{2U_1}{9}}$$

From the above results it may be observed that the strain energy decreases as the volume of the bar increases.

2. Suppose a rod AB must acquire an elastic strain energy of 13.6 N.m using E = 200 GPa. Determine the required yield strength of steel. If the factor of safety w.r.t. permanent deformation is equal to 5.



Solution : Factor of safety = 5

Therefore, the strain energy of the rod should be u = 5 [13.6] = 68 N.m

Strain Energy density

The volume of the rod is

$$V = AL = \frac{\pi}{4} d^2 L$$
$$= \frac{\pi}{4} 20 \times 1.5 \times 10^3$$
$$= 471 \times 10^3 \ mm^3$$

Yield Strength : As we know that the modulus of resilience is equal to the strain energy density when maximum stress is equal to σ_x .

$$U = \frac{\sigma_y^2}{2E}$$
$$0.144 = \frac{\sigma_y^2}{2 \times (200 \times 10^3)}$$
$$\boxed{\sigma_y = 200 \, \text{Mpa}}$$

It is important to note that, since energy loads are not linearly related to the stress they produce, factor of safety associated with energy loads should be applied to the energy loads and not to the stresses.

Strain Energy in Bending :

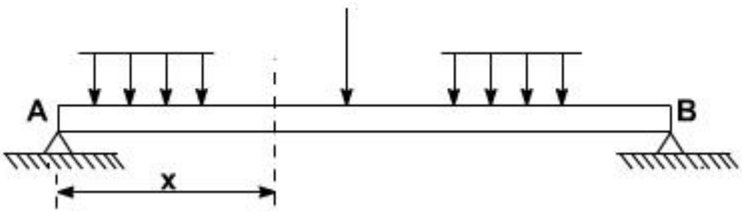


Fig .6

Consider a beam AB subjected to a given loading as shown in figure.

Let M = The value of bending Moment at a distance x from end A.

From the simple bending theory, the normal stress due to bending alone is expressed as.

$$\boxed{\sigma = \frac{MY}{I}}$$

Substituting the above relation in the expression of strain energy

i.e. $U = \int \frac{\sigma^2}{2E} dv$

$$= \int \frac{M^2.y^2}{2EI^2} dv \qquad .....(10)$$

Substituting dv = dxdA

Where dA = elemental cross-sectional area

$\frac{M^2.y^2}{2EI^2} \rightarrow$ is a function of x alone

Now substitiuting for dy in the expression of U.

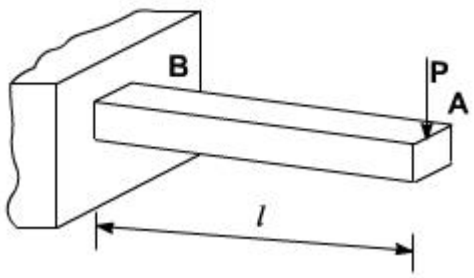
$$U = \int_0^L \frac{M^2}{2EI^2}\left(\int y^2 dA\right) dx \qquad .....(11)$$

We know $\int y^2 dA$ represents the moment of inertia 'I' of the cross-section about its neutral axis.

$$\boxed{U = \int_0^L \frac{M^2}{2EI} dx} \qquad .....(12)$$

ILLUSTRATIVE PROBLEMS

1. Determine the strain energy of a prismatic cantilever beam as shown in the figure by taking into account only the effect of the normal stresses.



Solution : The bending moment at a distance x from end A is defined as

$$\boxed{M = -Px}$$

Substituting the above value of M in the expression of strain energy we may write

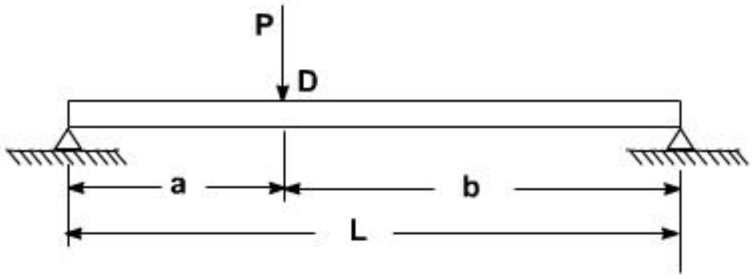
$$U = \int_0^L \frac{P^2x^2}{2EI} dx$$

$$\boxed{U = \int_0^L \frac{P^2L^3}{EI}}$$

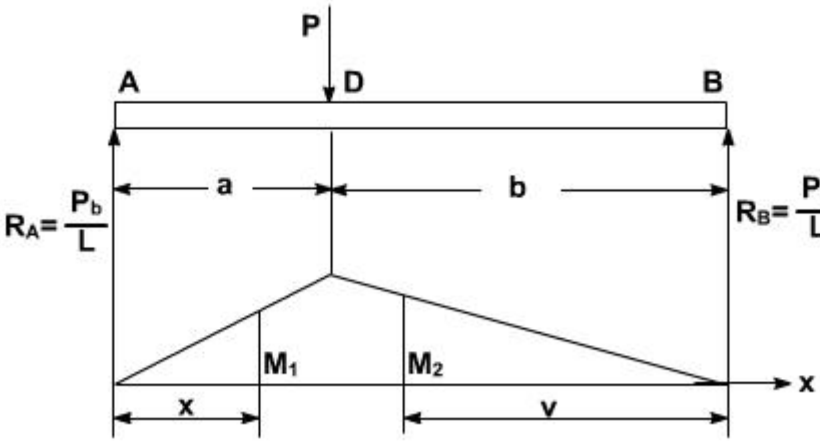
Problem 2 :

a. Determine the expression for strain energy of the prismatic beam AB for the loading as shown in figure below. Take into account only the effect of normal stresses due to bending.
b. Evaluate the strain energy for the following values of the beam

P = 208 KN ; L = 3.6 m = 3600 mm; A = 0.9 m = 90mm ; b = 2.7m = 2700 mm ; E = 200 GPa ; I = 104 x 10^8 mm^4



Solution:

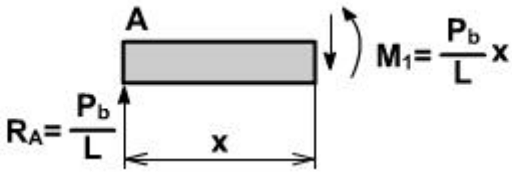


a. Bending Moment: Using the free-body diagram of the entire beam, we may determine the values of reactions as follows:

$$R_A = P_b / L \quad R_B = P_a / L$$

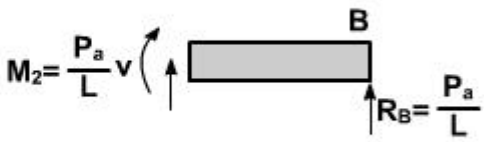
For Portion AD of the beam, the bending moment is

$$\boxed{M_1 = \frac{P_b}{L} x}$$



For Portion DB, the bending moment at a distance v from end B is

$$\boxed{M_2 = \frac{P_a}{L} x}$$



Strain Energy: Since strain energy is a scalar quantity, we may add the strain energy of portion AD to that of DB to obtain the total strain energy of the beam.

$$U = U_{AD} + U_{DB}$$

$$= \int_0^a \frac{M_1^2}{2EI} dx + \int_0^b \frac{M_2^2}{2EI} dv$$

$$= \frac{1}{2EI} \int_0^a \left(\frac{P_b}{L} x \right)^2 dx + \frac{1}{2EI} \int_0^b \left(\frac{P_a}{L} v \right)^2 dx$$

$$= \frac{1}{2EI} \frac{P^2}{L^2} \left(\frac{b^2 a^3}{3} + \frac{a^2 b^3}{3} \right)$$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL^2} (a+b)}$$

$$\text{Since}(a+b) = L$$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL}}$$

b. Substituting the values of P, a, b, E, I, and L in the expression above.

$$U = \frac{(200 \times 10^3)^2 \times (900)^2 \times (2700)^2}{6\,(200 \times 10^3) \times (104 \times 10^6) \times (3600)} = 5.27 \times 10^7 \ \text{KN.m}$$

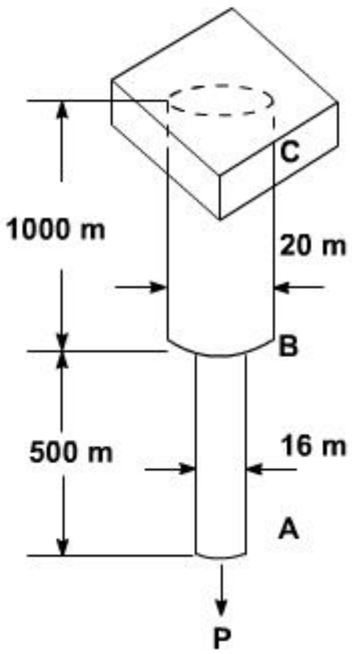
Problem 3) Determine the modulus of resilience for each of the following materials.

- Stainless steel. E = 190 GPa ; σ_y = 260MPa
- Malleable cast iron E = 165GPa ; σ_y = 230MPa
- Titanium E = 115GPa σ_y = 830MPa
- Magnesium E = 45GPa σ_y = 200MPa

4) For the given Loading arrangement on the rod ABC determine

(a). The strain energy of the steel rod ABC when P = 40 KN.

(b). The corresponding strain energy density in portions AB and BC of the rod.



LECTURE 39: Complementary Strain Energy

Consider the stress strain diagram as shown Fig 39.1. The area enclosed by the inclined line and the vertical axis is called the complementary strain energy. For a linearly elastic materials the complementary strain energy and elastic strain energy are the same.

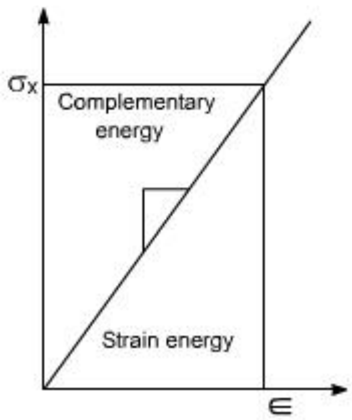


Fig 39.1

Let us consider elastic non linear prismatic bar subjected to an axial load. The resulting stress strain plot is as shown.

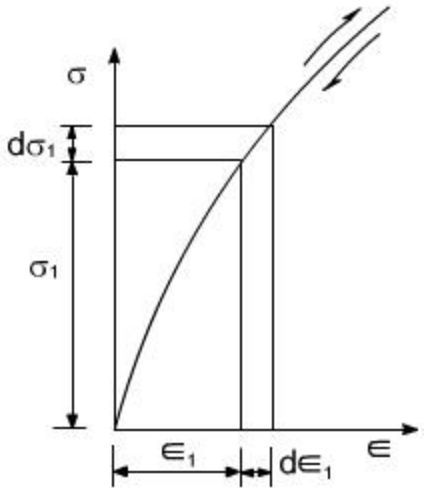


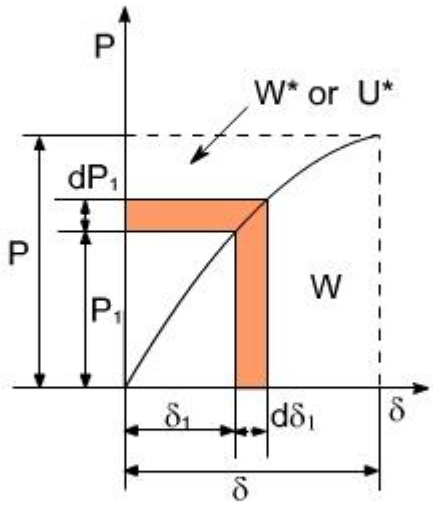
Fig 39. 2

The new term complementary work is defined as follows

$$W^* = \int_0^P \delta_1 dP_1$$

we also know

$$\boxed{W^* + W = P\delta}$$



So In geometric sense the work "W* is the complement of the work 'W' because it completes rectangle as shown in the above figure

Complementary Energy

$$U^* = W^* = \int_0^P \delta_1 dP$$

Likewise the complementary energy density u* is obtained by considering a volume element subjected to the stress σ_1 and $\in_1$, in a manner analogous to that used in defining the strain energy density. Thus

$$U^* = \int_0^\sigma \in_1 \, d\sigma 1$$

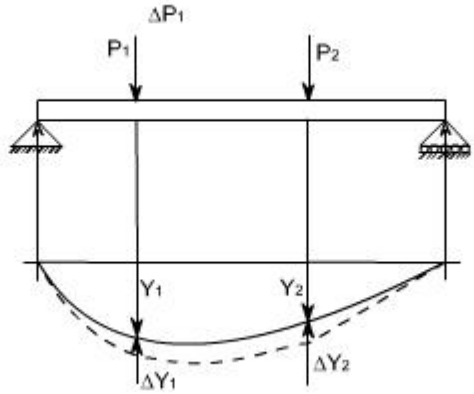
The complementary energy density is equal to the area between the stress strain curve and the stress axis. The total complementary energy of the bar may be obtained from u* by integration

$$U^* = \int dv$$

Sometimes the complementary energy is also called the stress energy. Complementary Energy is expressed in terms of the load and that the strain energy is expressed in terms of the displacement.

Castigliano's Theorem : Strain energy techniques are frequently used to analyze the deflection of beam and structures. Castigliano's theorem were developed by the Italian engineer Alberto castigliano in the year 1873, these theorems are applicable to any structure for which the force deformation relations are linear

Castigliano's Therom :



Consider a loaded beam as shown in figure

Let the two Loads P_1 and P_2 produce deflections Y_1 and Y_2 respectively strain energy in the beam is equal to the work done by the forces.

$$\boxed{U = \frac{1}{2}P_1Y_1 + \frac{1}{2}P_2Y_2} \qquad .....(1)$$

Let the Load P_1 be increased by an amount ΔP_1.

Let ΔP_1 and ΔP_2 be the corresponding changes in deflection due to change in load to ΔP_1.

Now the increase in strain energy $\Delta U = \frac{1}{2}\Delta P_1\Delta Y_1 + P_1\Delta Y_1 + P_2\Delta Y_2 \qquad ....(2)$

Suppose the increment in load is applied first followed by P_1 and P_2 then the resulting strain energy is

$$\boxed{U+\Delta U = \frac{1}{2}\Delta P_1\Delta Y_1 + \Delta P_1Y_1 + P_2\Delta Y_2\frac{1}{2}P_1Y_1 + \frac{1}{2}P_2Y_2} \qquad ......(3)$$

Since the resultant strain energy is independent of order loading,

Combing equation 1, 2 and 3. One can obtain

$$\boxed{\Delta P_1Y_1 = P_1\Delta Y_1 + P_2\Delta Y_2} \qquad .....(4)$$

equations (2) and (4) can be combined to obtain

$$\boxed{\frac{\Delta U}{\Delta P_1} = y_1 + \frac{1}{2}\Delta Y_1} \qquad .....(5)$$

or upon taking the limit as ΔP_1 approaches zero [Partial derivative are used because the starin energy is a function of both P_1 and P_2]

$$\boxed{\frac{\partial U}{\partial P} = Y_1} \qquad ....(6)$$

For a general case there may be number of loads, therefore, the equation (6) can be written as

$$\boxed{\frac{\partial U}{\partial P_i} = Y_i} \qquad ....(7)$$

The above equation is castigation's theorem:

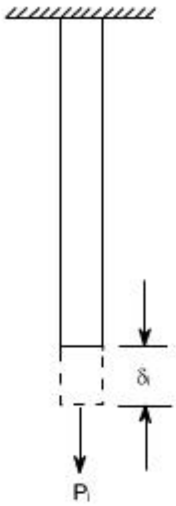
The statement of this theorem can be put forth as follows; if the strain energy of a linearly elastic structure is expressed in terms of the system of external loads. The partial derivative of strain energy with respect to a concentrated external load is the deflection of the structure at the point of application and in the direction of that load.

In a similar fashion, castigliano's theorem can also be valid for applied moments and resulting rotations of the structure

$$\boxed{\frac{\partial U}{\partial M_i} = \theta_i} \qquad ....(8)$$

Where M_i = applied moment and θ_i = resulting rotation

Castigliano's First Theorem :



In similar fashion as discussed in previous section suppose the displacement of the structure are changed by a small amount $d\delta_i$. While all other displacements are held constant the increase in strain energy can be expressed as

$$\boxed{dU = \frac{\partial U}{\partial \delta_i} d\delta_i} \qquad ....(9)$$

Where $\partial U / \delta_i \rightarrow$ is the rate of change of the starin energy w.r.t δ_i.

It may be seen that, when the displacement δ_i is increased by the small amount $d\delta$; workdone by the corresponding force only since other displacements are not changed.

The work which is equal to $P_i d\delta_i$ is equal to increase in strain energy stored in the structure

$$\boxed{du = P_i d\delta_i}$$

By rearranging the above expression, the Castigliano's first theorem becomes

$$\boxed{P_i = \frac{dU}{d\delta_i}}$$

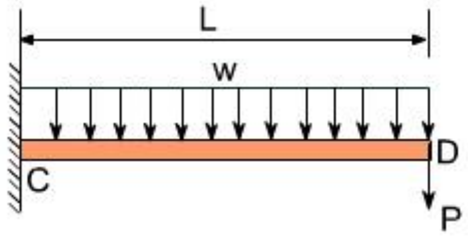
The above relation states that the partial derivative of strain energy w.r.t. any displacement δ_i is equal to the corresponding force P_i provided that the strain is expressed as a function of the displacements.

LECTURE 40: ILLUSTRATIVE PROBLEMS

Using Castigliano's Theorem :

1. The cantilever beam CD supports a uniformly distributed Load w. and a concentrated load P as shown in figure below. Suppose

L = 3m; w = 6KN/m ; P = 6KN and E. I = 5 MN m^2 determine the deflection at D



The deflection 'Y_0' at the point D Where load 'P' is applied is obtained from the relation

$$\boxed{\delta_i = \frac{\partial U}{\partial P_i}}$$

Since P is acting vertical and directed downward δ ; represents a vertical deflection and is positions downward.

$$\boxed{\delta_0 = \frac{\partial U}{\partial P_i} = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial P}\,dx} \quad .....(1)$$

The bending moment M at a distance x from D

$$\boxed{M = -\left(Px + \frac{1}{2}wx^2\right)} \quad ......(2)$$

And its derivative with respect to 'P' is

$$\boxed{\frac{\partial M}{\partial P} = -x} \quad ....(3)$$

Substituting for M and ∂ M/ ∂ P into equation (1)

$$Y_D = \frac{1}{EI}\int_0^l \left(Px + \frac{1}{2}wx^2\right)$$

$$\boxed{Y_D = \frac{1}{EI}\left(\frac{PL^3}{3} + \frac{wL^2}{8}\right)}$$

Substituting the values of P,L,w and EI

$$Y_D = \frac{1}{5\times10^6}\left(\frac{8\times3^3\times10^3}{3} + \frac{6\times10^3\times3^4}{8}\right)$$

$$Y_D = 26.55\times10^{-3}\,\text{m}$$

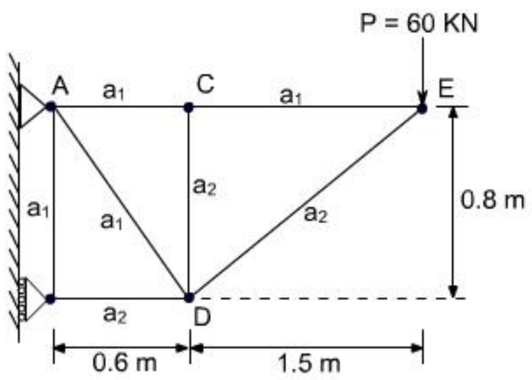
$$\boxed{Y_D = 26.55\,\text{mm}}$$

$$\delta_0 = \frac{1}{EI}\int_0^L -\left(Px + \frac{1}{2}wx^2\right)(-x)dx$$

$$= \frac{1}{EI}\int_0^L \left(\frac{1}{2}wx^3\right)dx \quad \text{Since P is zero}$$

$$= \frac{wL^4}{8EI}$$

2.

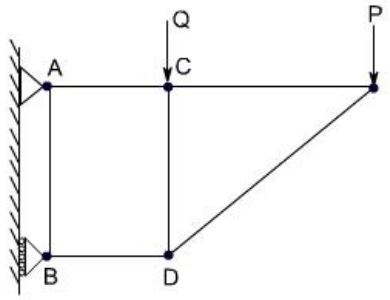


Areas

a_1 = 500 mm^2 ; a_2 = 1000 mm^2

For the truss as shown in the figure above, Determine the vertical deflection at the joint C.

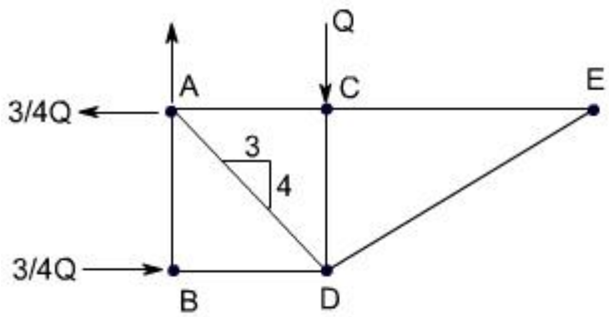
Solution: Since no vertical load is applied at Joint C. we may introduce dummy load Q. as shown below



Using castigliano's theorem and denoting by the force F_i in a given member i caused by the combined loading of P and Q. we have

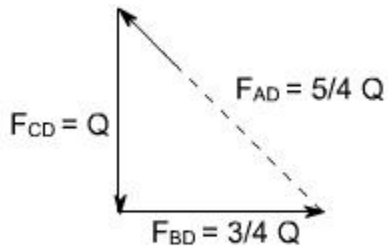
$$\boxed{Y_c = \Sigma\left(\frac{F_iL_i}{A_iE}\right)\frac{\partial F}{\partial Q} = \frac{1}{E}\Sigma\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F}{\partial Q}} \qquad ......(1)$$

Free body diagram : The free body diagram is as shown below



Force in Members: Considering in sequence, the equilibrium of joints E, C, B and D, we may determine the force in each member caused by load Q.

Joint E: $F_{CE} = F_{DE} = 0$
Joint C: $F_{AC} = 0$; $F_{CD} = -Q$
Joint B: $F_{AB} = 0$; $F_{BD} = -3/4Q$



The total force in each member under the combined action of Q and P is

Member	F_i	$\partial F_i / \partial Q$	L_i ,m	A_i ,m^2	$\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F_iL}{\partial A_i}$
AB	0	0	0.8	5000×10^{-6}	0
AC	+15P/8	0	0.6	5000×10^{-6}	0
AD	+5P/4+5O/4	5/4	1.0	5000×10^{-6}	3125P+3125Q
BD	-21P/8-3Q/4	-3/4	0.6		1181P+338Q

CD	-Q	-1	0.8	$1000x10^{-6}$	+800Q
CE	15P/8	0	1.5	$1000x10^{-6}$	0
DE	-17P/8	0	1.7	$500x10^{-6}$	0
				$1000x10^{-6}$	

P = 60 KN

$$\Sigma\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F}{\partial Q} = 4306P + 4263Q \qquad \text{.......(2)}$$

Sub-(2) in (1)

Deflection of C.

$$Y_c = \Sigma\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F}{\partial Q}$$

$$= \frac{1}{E}(4306P + 4263Q)$$

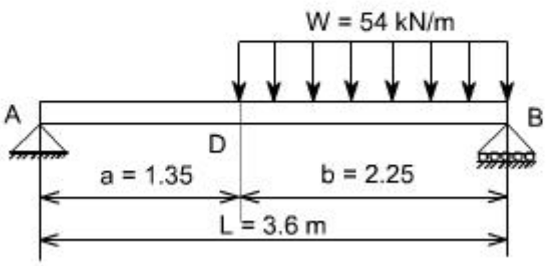
Since the load Q is not the part of loading therefore putting Q = 0

$$Y_C = \frac{1}{73x10^9}[4306]x\left[60x10^3\right]$$

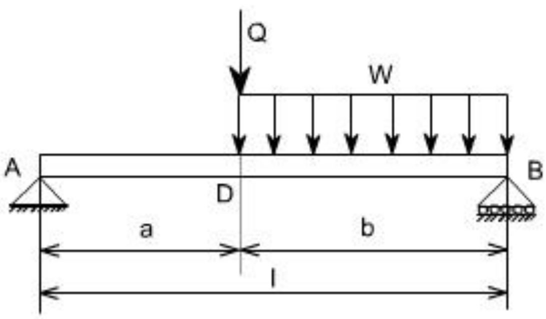
$$Y_C = 3.539x10^{-3}\ m$$

$$\boxed{Y_C = 3.539\ mm}$$

3. For the beam and loading shown, determine the deflection at point D. Take E = 200Gpa, I = $28.9x10^6$ mm^4



Solution: Castigliano's Theorem : Since the given loading does not include a vertical load at point D, we introduce the dummy load Q as shown below. Using Castigliano's Theorem and noting that E.I is constant, we write.

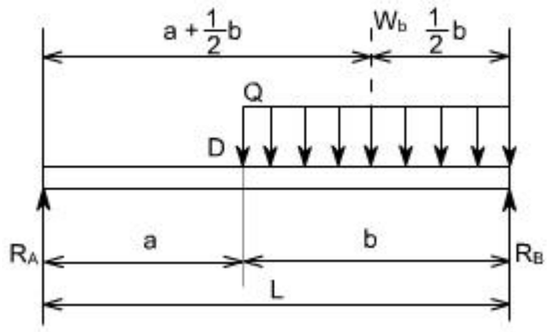


$$Y_D = \int \frac{M}{EI}\left(\frac{\partial M}{\partial Q}\right)dx$$

$$\boxed{Y_D = \frac{1}{EI}\int M\left(\frac{\partial M}{\partial Q}\right)dx} \qquad \ldots\ldots(1)$$

The integration is performed separately for portion AD and DB

Reactions

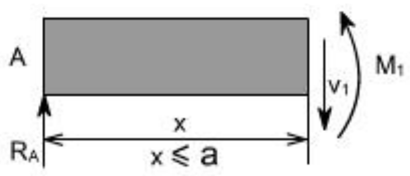


Using F.B.D of the entire beam

$$R_A = \frac{wb^2}{2L} + Q\frac{b}{L}\uparrow$$

$$R_B = \frac{wb(a+1/2b)^2}{L} + Q\frac{a}{L}\uparrow$$

Portion AD of Beam :



From Using the F.B.D.we find

$$M_1 = R_A x = \left(\frac{wb^2}{2L} + Q\frac{b}{L}\right)x$$

$$\frac{\partial M_1}{\partial Q} = \frac{bx}{L}$$

Substituting into equation (1) and integrating from A to D. gives

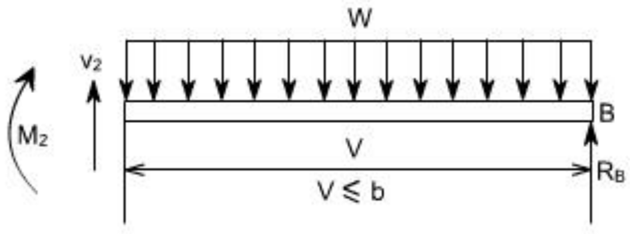
$$\frac{1}{EI}\int M_1\left(\frac{\partial M_1}{\partial Q}\right)dx = \frac{1}{EI}\int_0^a R_A x\left(\frac{bx}{L}\right)dx$$

$$= \frac{R_A a^3 b}{3EIL}$$

Substituting for R_A and then set the dummy load 'q' equal to zero

$$\frac{1}{EI}\int M_1\left(\frac{\partial M_1}{\partial Q}\right)dx = \frac{wa^3b^3}{6EIL^2} \qquad \text{......(2)}$$

Portion DB of Beam : From Using the F.B.D shown below we find the bending moment at a distance V from end B is



$$M_2 = R_2 V - \frac{wV^2}{2}$$

$$= \left(\frac{wb(a+1/2b)^2}{L} + Q\frac{a}{L}\right)V - \frac{wV^2}{2}$$

$$\boxed{\frac{\partial M_2}{\partial Q} = +\frac{aV}{L}}$$

Substituting into equation (1) and integrating from point 'b' where V = 0, to point 'D' where V = b, we write.

$$\frac{1}{EI}\int M_2 \frac{\partial M_2}{\partial Q}dv = \frac{1}{EI}\int_0^b\left(R_B V - \frac{wV^2}{2}\right)\left(\frac{aV}{L}\right)dV$$

$$= \frac{R_B ab^3}{3EIL} - \frac{wab^4}{8EIL}$$

Substituting for R_B and setting $Q = 0$

$$\frac{1}{EI}\int M_2 \frac{\partial M_2}{\partial Q}dv = \int \frac{wb(a+1/2b)}{L}\int \frac{ab^3}{3EIL} - \frac{wab^4}{8EIL}$$

$$= \frac{5a^2b^4 + ab^5}{24EIL^2}w \qquad \text{.......(3)}$$

Deflection at point D:

Recalling eq (1), (2) and (3) we have

$$Y_D = \frac{wab^3}{24EIL^2}\left[4a^2 + 5ab + b^2\right]$$

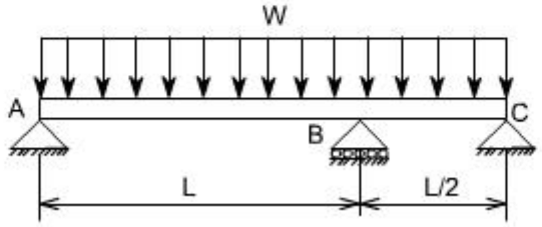
$$= \frac{wab^3}{24EIL}\left[4a + b\right]\left[a + b\right]$$

$$\boxed{Y_D = \frac{wab^3}{24EIL}\left[4a + b\right] \quad \ldots\ldots..(4)}$$

Substituting the values of w,a,b,E,I and L we obtain

$$\boxed{Y_D = 12.72 \text{ mm} \downarrow}$$

4. For the uniform loaded beam with following supports. Determine the reactions at the supports



Solution:

Ans.

$$R_A = \frac{13}{32}WL$$

$$R_B = \frac{33}{32}WL$$

$$R_C = \frac{WL}{16}$$