Free Damped Oscillations

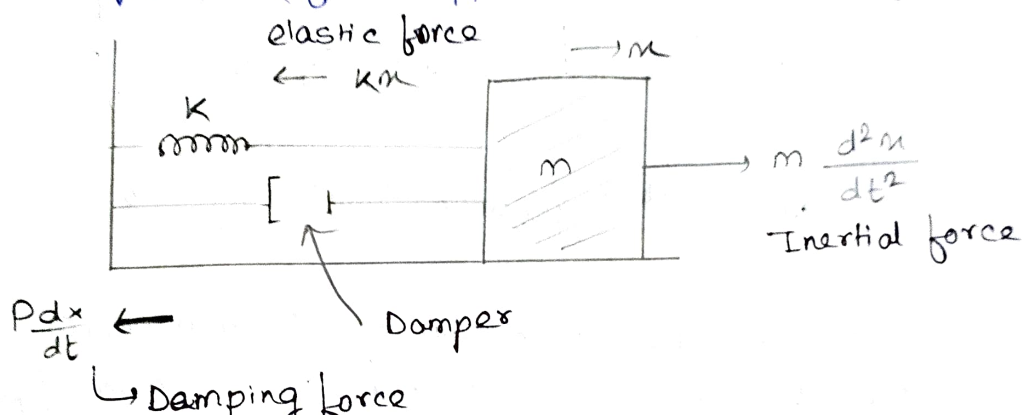
gradually decreases the energy and the amplitude

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

$$x = A\sin(\omega_0 t + \phi)$$

$$F = -Pv$$

$F/v = -P$ (force opposes the ossilallations)



(i) $-Kx$ --- ②

(ii) $-P\,dx/dt$ -- ③ proportional to instantaneous velocity of particle

(iii) $m\,d^2x/dt^2$ --- ④

$$m\,d^2x/dt^2 = -Kx - P\,dx/dt \quad \text{------ ⑤}$$

$$\frac{d^2x}{dt^2} + P/m\,dx/dt + K/m\,x = 0$$

$$\boxed{\frac{d^2x}{dt^2} + \gamma\,dx/dt + \omega_0^2 x = 0 \quad \text{------ ⑥}}$$

$\gamma = P/m \qquad \omega_0^2 = K/m$

$$\frac{L/T}{T} \quad \gamma\,L/T = \omega_0^2 L$$

$T^{-1} = \gamma \qquad T^{-1} = \omega_0$

General Solution:

$x = Ae^{\alpha t}$ ----> ⑦

$\frac{dx}{dt} = \alpha A e^{\alpha t}$

$$\frac{d^2x}{dt^2} = \alpha^2 A e^{\alpha t}$$

$$(\alpha^2 + \gamma\alpha + \omega_0^2) A e^{\alpha t} = 0 \quad (10)$$

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0 \quad (11)$$

roots are

$$\alpha_1 = -\gamma/2 + 1/2 (\gamma^2 - 4\omega_0^2)^{1/2} \quad \ldots (12)$$

$$\alpha_2 = -\gamma/2 - 1/2 (\gamma^2 - 4\omega_0^2)^{1/2} \quad \ldots (13)$$

$$x_1 = A_1 e^{\alpha_1 t} = A_1 \exp\left[-\gamma/2 + 1/2 (\gamma^2 - 4\omega_0^2)^{1/2}\right] t \quad \ldots (14)$$

$$x_2 = A_2 e^{\alpha_2 t} = A_2 \exp\left[-\gamma/2 - 1/2 (\gamma^2 - 4\omega_0^2)^{1/2}\right] t \quad \ldots (15)$$

$x = Ae^{\alpha t}$ general solution of (6).

$x = x_1 + x_2$ also a solution of equation no. (6) becoz it is a linear equation so we can apply superposition principle

$$x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$
$$= A_1 \exp\left[-\gamma/2 + 1/2 (\gamma^2 - 4\omega_0^2)^{1/2}\right] t + A_2 \exp\left[-\gamma/2 + 1/2 (\gamma^2 - 4\omega_0^2)^{1/2}\right] t \quad (16)$$

$\gamma > 2\omega_0$, $\gamma = 2\omega_0$, $\gamma < 2\omega_0$

i) case I $\gamma > 2\omega_0$ Large Damping / Overdamped motion

$$\left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2} = q \quad \ldots (17)$$

$$x_2 + x_1 = A_1 \exp\left[-\gamma/2 + q\right] t + A_2 \exp(-\gamma/2 - q) t.$$

$$x = \exp(-\gamma/2) t \left[A_1 e^{qt} + A_2 e^{-qt}\right] \quad \ldots (18)$$

$A_1 + A_2 = C$, $A_1 - A_2 = D$

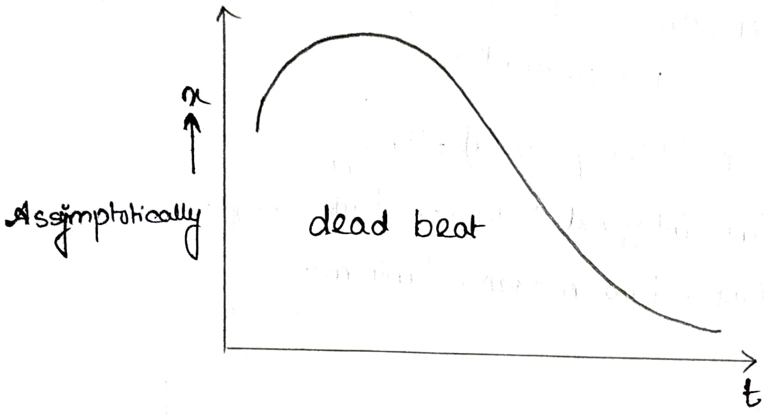
$$\frac{C+D}{2} = A_1 \qquad \frac{C-D}{2} = A_2 \quad \ldots (19)$$

$$x = \exp(-\gamma/2)t \left[\frac{C+D}{2} e^{qt} + \frac{C-D}{2} e^{-qt}\right]$$

$$x = e^{-\gamma t/2}\left[C\left(\frac{e^{qt}+e^{-qt}}{2}\right) + D\left(\frac{e^{qt}-e^{-qt}}{2}\right)\right]$$

$$x = e^{-\gamma t/2}\left[C\cosh qt + D\sinh qt\right] \quad \text{--- (20)}$$

it is representing a motion which is non-oscillatory, and aperiodic



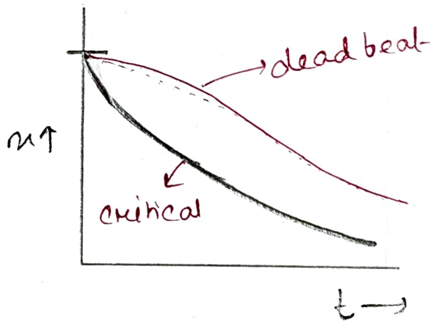
$e^{-\gamma t/2}$

(ii) case II $\gamma = 2\omega_0$ Critical Damping

$$x = (A_1 + A_2)e^{-\gamma t/2}$$

$$x = Be^{-\gamma t/2}$$

$\alpha_1 = \alpha_2$



(iii) case III $\gamma < 2\omega_0$ weakly damped / Under damped motion

$$\left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2} = i\omega$$

$$\omega = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}, \quad i = \sqrt{-1}$$

$$x = e^{-\gamma t/2}\left[A_1 \exp(i\omega t) + A_2 \exp(-i\omega t)\right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

(24)

$x = e^{-\gamma t/2} (A_1 + A_2) \cos \omega t + i (A_1 - A_2) \sin \omega t$ ---- (26)

$x = e^{-\gamma t/2} A \cos \delta \cos \omega t + i \, A \sin \delta \sin \omega t$ ---- (27)

Oscillaratory

$$\boxed{x = A e^{-\gamma t/2} \cos (\omega t - \delta)} \quad \text{---- (28)}$$

$$\omega = \omega_0 \left(1 - \frac{\gamma^2}{4\omega_0^2}\right)^{1/2} \quad \text{---- (29)}$$

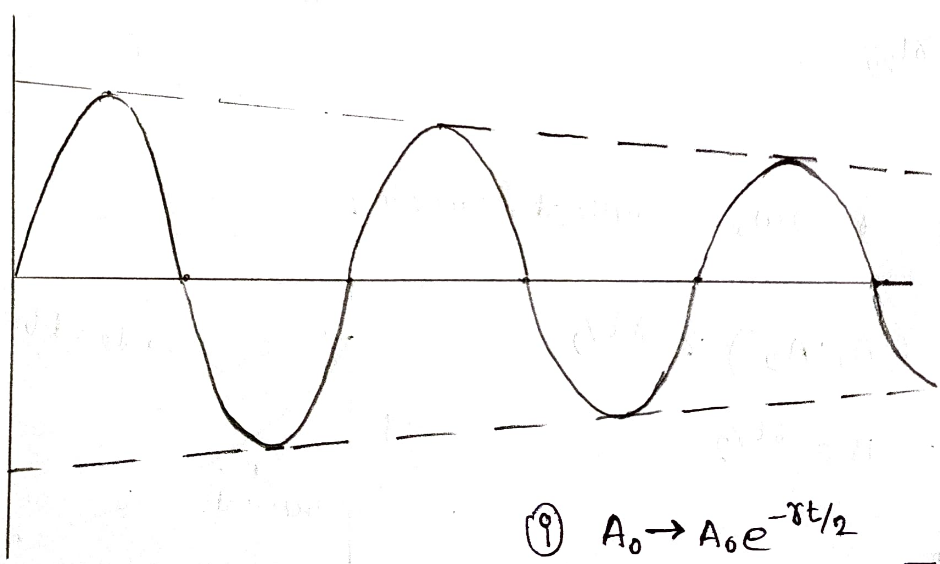
→ not a SHM, Amp decreases

→ non periodic

→ it has a T (time period) $= 2\pi/\omega$

time interval between two zero's

$T/2$ → between two maxima, minima



(i) $A_0 \rightarrow A_0 e^{-\gamma t/2}$

(ii) $\omega_0 \rightarrow \omega_0 \left[1 - \gamma^2/4\omega_0^2\right]^{1/2}$