Solid Mechanics (MEC 301)

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Books:

1. Strength of Materials: Part I, II, S. Timoshenko, CBS Publishers, 1985.
2. Engineering Mechanics of Solids, E. P. Popov, PHI, 1993.
3. Introduction to Solid Mechanics, I. H. Shames and J. M. Pittariesi, PHI, 2003.
4. Strength of Materials, F. L. Singer and A. Pytel, HarperCollins Publishers, 1991
5. NPTEL

Syllabus

- Introduction to stress and strains, Generalized Hooke's Law, Relationship among different elastic coefficients. 4
- Analysis of bi-axial stress and Mohr's Circle. 6
- Theory of Bending, Shearing Forces and Bending Moments in beams, SF and BM Diagrams. 6
- Bending Stresses in Beams, Flexural rigidity, Section Modulus, Shear Flow, Shear Centre. 6
- Deflection of Beams: Double-Integration method, Area-Moment method; Propped cantilever and Fixed beams. 6
- Statically indeterminate beam problems. 4
- Torsion of Circular shafts. 4
- Combined Loading and Theories of Failure. 4
- Columns: Buckling of columns, Euler's formula for stability of column. 6
- Stresses in Thin Cylinder 2
- Strain Energy methods – Castigliano's Theorem. **4**

LECTURE 1: INTRODUCTION AND REVIEW

Preamble: Engineering science is usually subdivided into number of topics such as

- Solid Mechanics
- Fluid Mechanics
- Heat Transfer
- Properties of materials and soon Although there are close links between them in terms of the physical principles involved and methods of analysis employed.

The solid mechanics as a subject may be defined as a branch of applied mechanics that deals with behaviors of solid bodies subjected to various types of loadings. This is usually subdivided into further two streams i.e, Mechanics of rigid bodies or simply Mechanics and Mechanics of deformable solids.

The mechanics of deformable solids which is branch of applied mechanics is known by several names i.e. strength of materials, mechanics of materials etc.

Mechanics of rigid bodies: The mechanics of rigid bodies is primarily concerned with the static and dynamic behavior under external forces of engineering components and systems which are treated as infinitely strong and undeformable. Primarily we deal here with the forces and motions associated with particles and rigid bodies.

Mechanics of deformable solids:

Mechanics of solids: The mechanics of deformable solids is more concerned with the internal forces and associated changes in the geometry of the components involved.

Of particular importance are the properties of the materials used, the strength of which will determine whether the components fail by breaking in service, and the stiffness of which will determine whether the amount of deformation they suffer is acceptable.

Therefore, the subject of mechanics of materials or strength of materials is central to the whole activity of engineering design.

Usually the objectives in analysis here will be the determination of the stresses, strains, and deflections produced by loads. Theoretical analyses and experimental results have an equal roles in this field.

Analysis of stress and strain

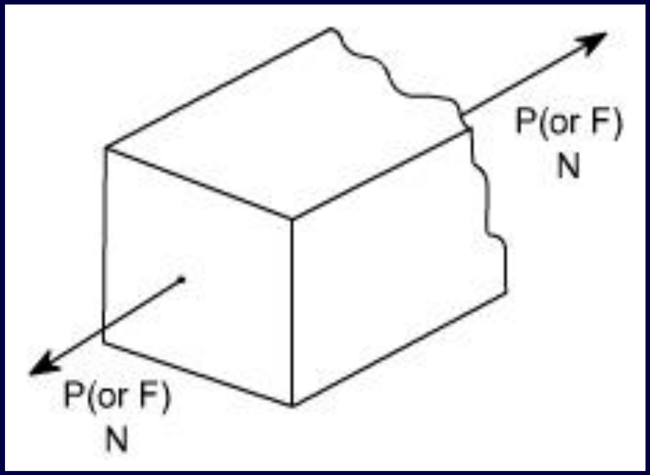
Concept of stress

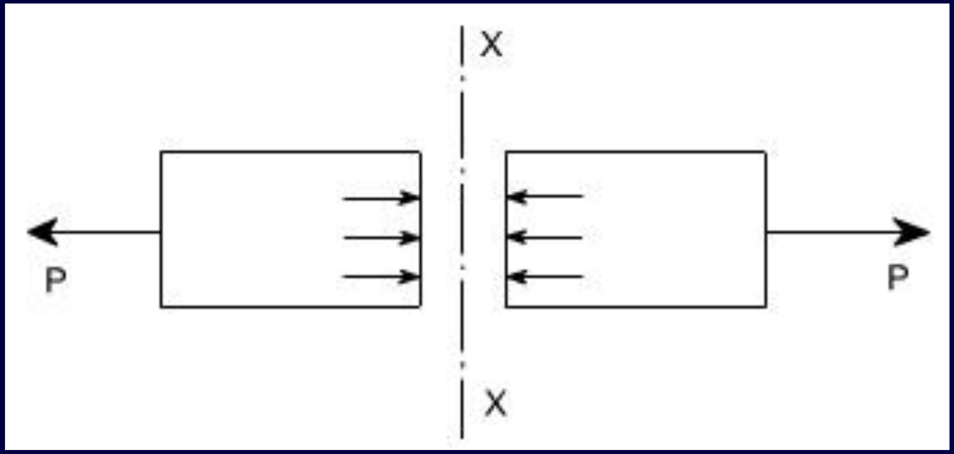
- Main problem of engineering mechanics of material is the investigation of the internal resistance of the body, i.e., the nature of forces set up within a body to balance the effect of the externally applied forces.
- The externally applied forces are termed as loads. These externally applied forces may be due to any one of the reason.

- due to service conditions,
- due to environment in which the component works,
- through contact with other members,
- due to fluid pressures,
- due to gravity or inertia forces.

- In mechanics of deformable solids, externally applied forces acts on a body and body suffers a deformation. From equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particles of material due to cohesion.
- These internal forces give rise to a concept of stress. Therefore, let us define a term stress

Stress





- Let us consider a rectangular bar of some cross sectional area and subjected to some load or force (in Newtons)
- Let us imagine that the same rectangular bar is assumed to be cut into two halves at section XX. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been shown
- Now stress is defined as the force intensity or force per unit area. Here we use a symbol sigma to represent the stress.

$$\sigma = \frac{P}{A}$$

- Where A is the area of the XX section

- Here we are using an assumption that the total force or total load carried by the rectangular bar is uniformly distributed over its cross section.
- But the stress distributions may be far from uniform, with local regions of high stress known as stress concentrations.
- If the force carried by a component is not uniformly distributed over its cross sectional area, A, we must consider a small area, dA which carries a small load dF, of the total force F, Then definition of stress is

$$\boxed{\sigma = \frac{\delta F}{\delta A}}$$

Units : The basic units of stress in S.I units are N / m2 (or Pa)

MPa = 10^6 Pa ; GPa = 10^9 Pa ; KPa = 10^3 Pa

Some times N / mm2 units are also used, because this is an equivalent to MPa. While US customary unit is pound per square inch psi.

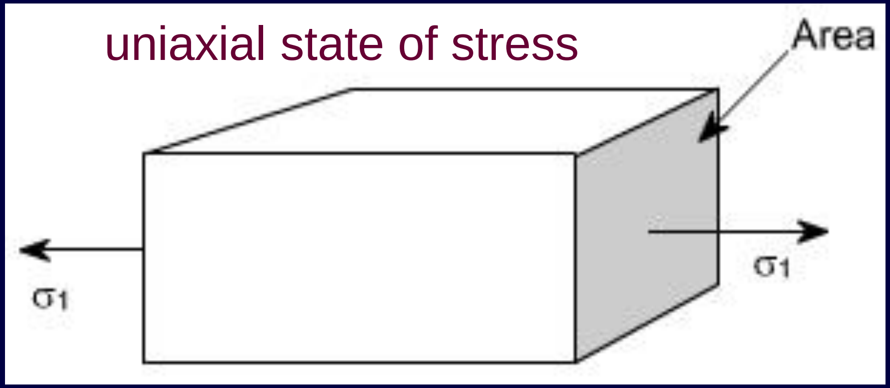
TYPES OF STRESSES: only two basic stresses exists :

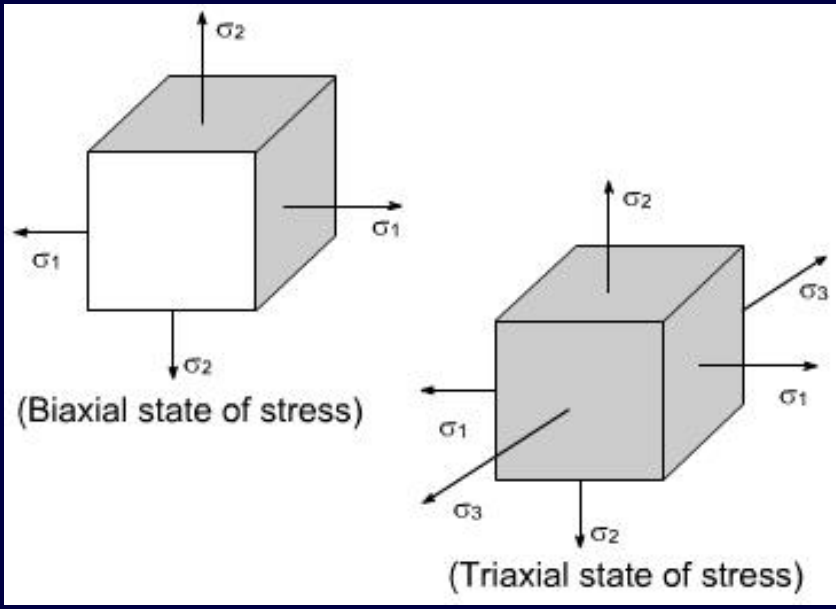
(1) normal stress

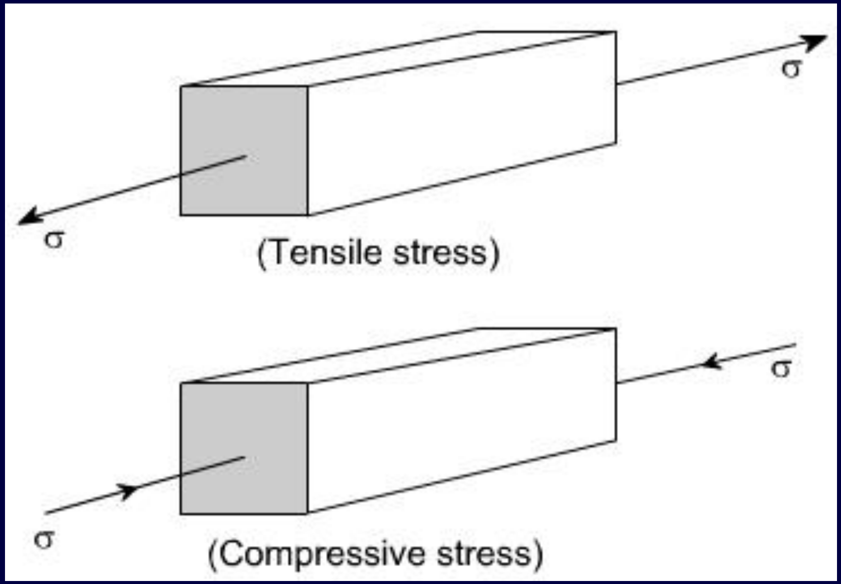
(2) shear stress.

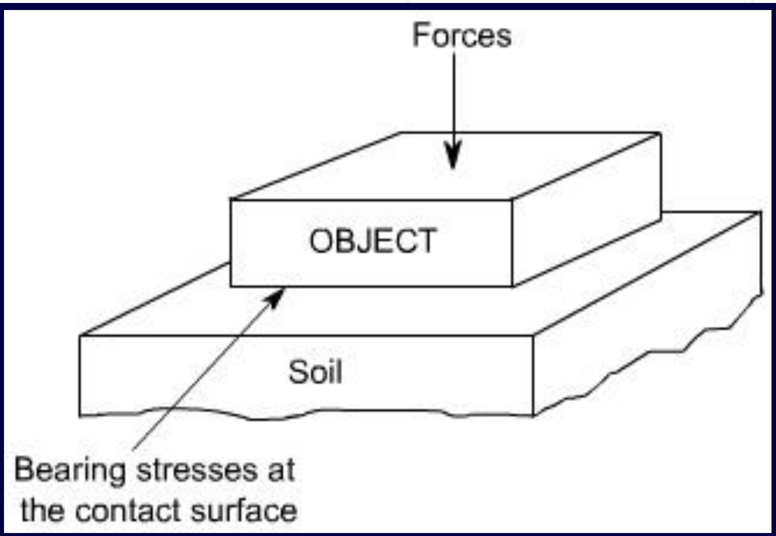
Other stresses either are similar to these basic stresses or are a combination of these e.g. bending stress is a combination of tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress.

Normal stresses : If the stresses are normal to the areas concerned, then these are termed as normal stresses. The normal stresses are generally denoted by a Greek letter (σ)



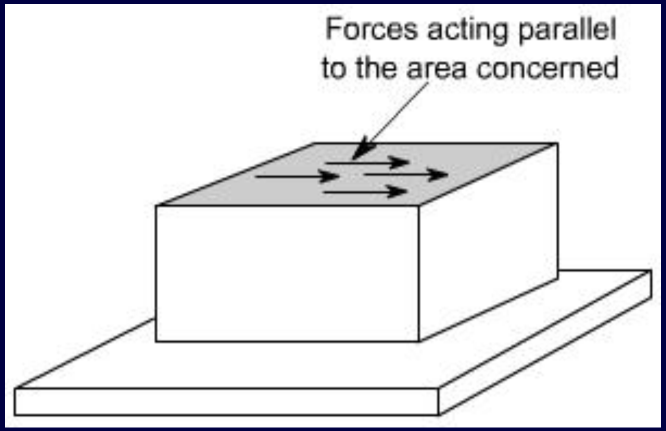






Bearing Stress : When one object presses against another, it is referred to a bearing stress (They are in fact the compressive stresses).

Shear stresses : Where the cross sectional area of a block of material is subject to a distribution of forces which are parallel, rather than normal, to the area concerned. Such forces are associated with a shearing of the material, and are referred to as shear forces.

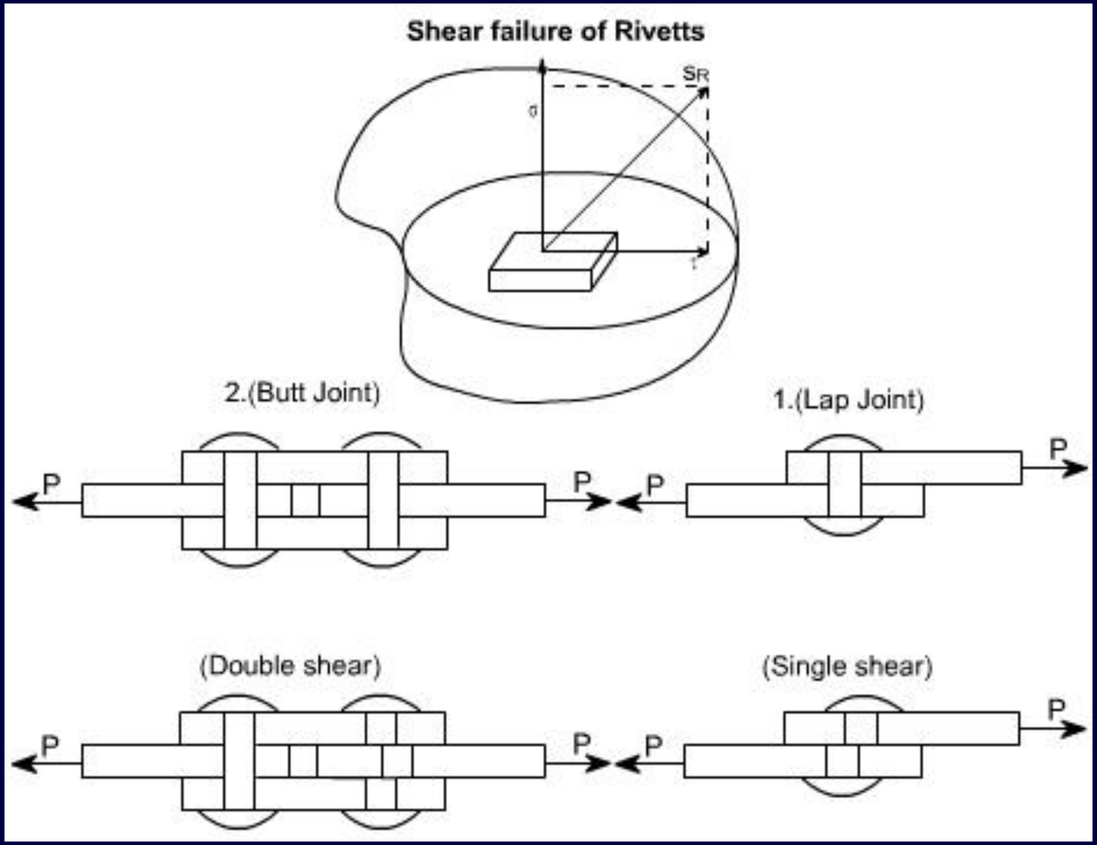


The resulting force intensities are known as shear stresses, the mean shear stress being equal to

$$\tau = \frac{P}{A}$$

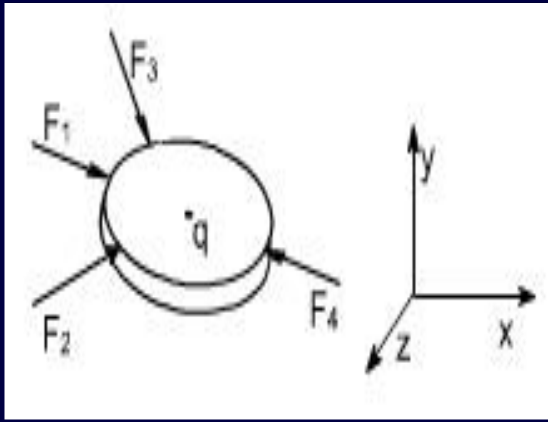
where P is the total force and A the area over which it acts.

The stress (resultant stress) at any point in a body is basically resolved into two components σ and τ one acts perpendicular and other parallel to the area concerned, as it is clearly defined in the following figure.

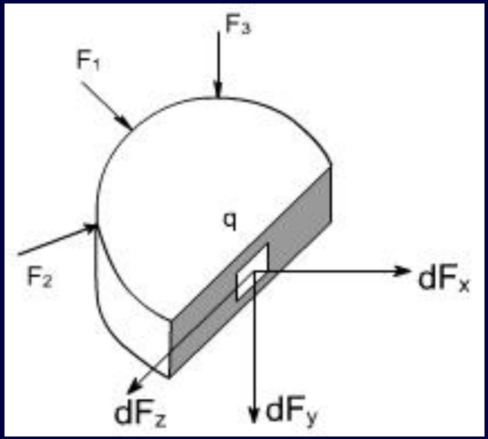


Lecture 2: ANALYSIS OF STERSSES

General State of stress at a point : Stress at a point in a material body has been defined as a force per unit area. But this definition is some what ambiguous since it depends upon what area we consider at that point. Let us, consider a point ‘q' in the interior of the body



Let us pass a cutting plane through a point 'q' perpendicular to the x - axis as shown below

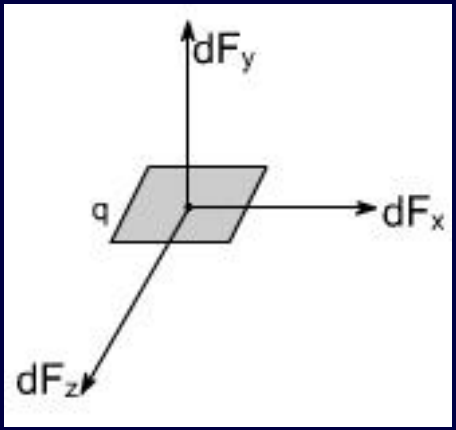


The corresponding force components can be shown like this

$$dF_x = \sigma_{xx} . da_x$$
$$dF_y = \tau_{xy} . da_x$$
$$dF_z = \tau_{xz} . da_x$$

where da_x is the area surrounding the point 'q' when the cutting plane perpendicular is to x - axis.

In a similar way it can be assumed that the cutting plane is passed through the point 'q' perpendicular to the y - axis. The corresponding force components are shown below



The corresponding force components may be written as

$$dF_x = \tau_{yx} . da_y$$

$$dF_y = \sigma_{yy} . da_y$$

$$dF_z = \tau_{yz} . da_y$$

where da_y is the area surrounding the point 'q' when the cutting plane perpendicular to y - axis.

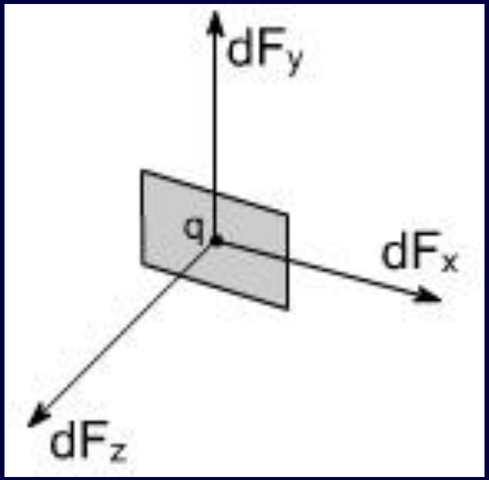
In the last it can be considered that the cutting plane is passed through the point 'q' perpendicular to the z - axis.

The corresponding force components may be written as

$$dF_x = \tau_{zx} . da_z$$

$$dF_y = \sigma_{zy} . da_z$$

$$dF_z = \tau_{zz} . da_z$$

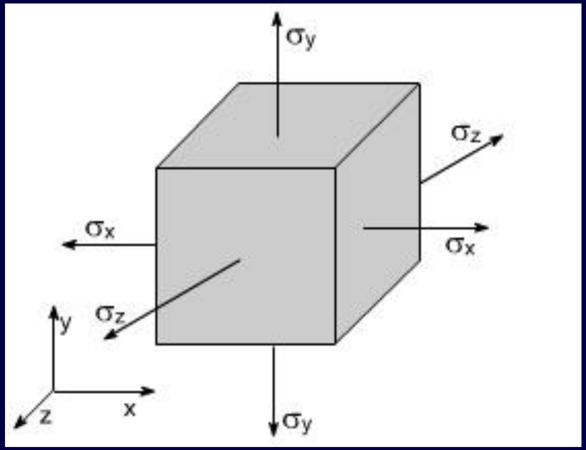


where da_z is the area surrounding the point 'q' when the cutting plane perpendicular to z - axis.

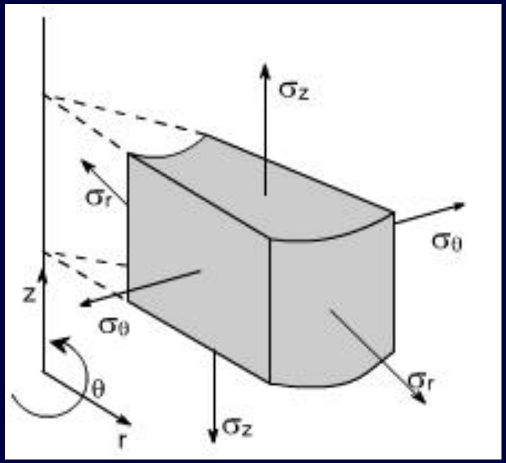
Salient Features

- It is amply clear that there is nothing like stress at a point 'q'
- There exist a situation where it is a combination of state of stress at a point q.
- Thus, it becomes imperative to understand the term state of stress at a point 'q'.
- Therefore, it becomes easy to express a state of stress by the scheme as discussed earlier, where the stresses on the three mutually perpendicular planes are labeled in the manner as shown earlier. the state of stress as depicted earlier is called the general or a tri-axial state of stress that can exist at any interior point of a loaded body.
- Before defining the general state of stress at a point. Let us make ourselves conversant with the notations for the stresses.
- We have already chosen to distinguish between normal and shear stress with the help of symbols σ and τ .

Cartesian - co-ordinate system:



Cylindrical - co-ordinate system:



Sign convention : The tensile forces are termed as (+ve) while the compressive forces are termed as negative (-ve).

Normal Stress:

- tensile +ve
- compressive -ve

First subscript : it indicates the direction of the normal to the surface.
Second subscript : it indicates the direction of the stress.

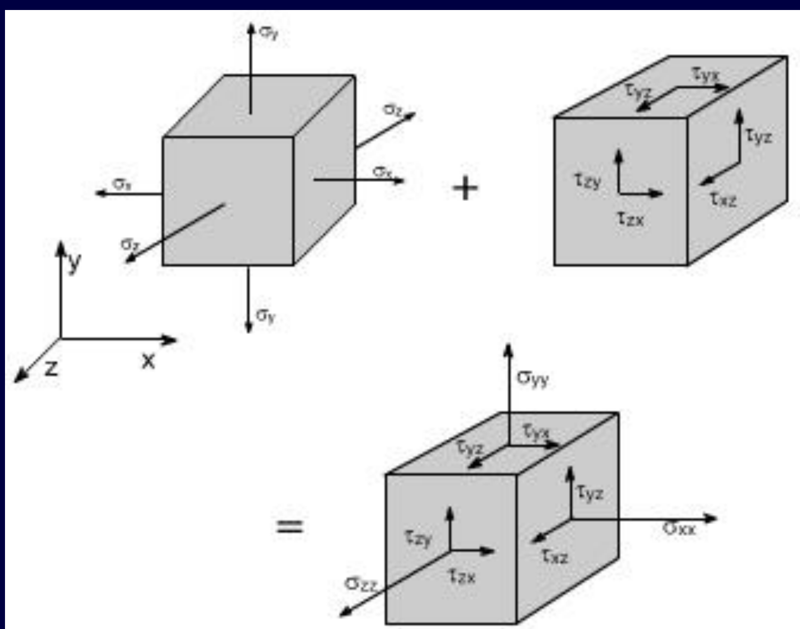
It may be noted that in the case of normal stresses the double script notation may be dispensed with as the direction of the normal stress and the direction of normal to the surface of the element on which it acts is the same. Therefore, a single subscript notation as used is sufficient to define the normal stresses.

Shear Stresses

Single subscript notation is not practical, because such stresses are in direction parallel to the surfaces on which they act. We therefore have two directions to specify, that of normal to the surface and the stress itself.

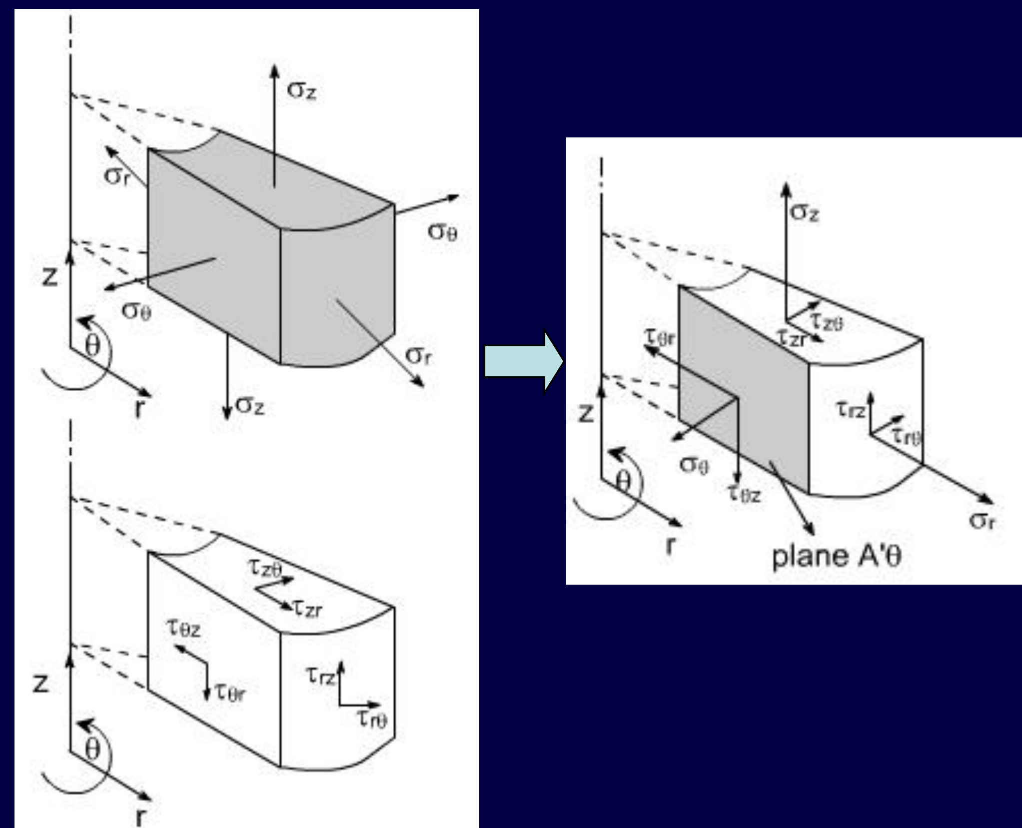
Cartesian - co-ordinate system:

τ_{xy} , τ_{yx} , τ_{yz}, τ_{zy}, τ_{zx}, τ_{xz}



Cylindrical - co-ordinate system:

τ_{rq}, τ_{qr}, τ_{qz}, τ_{zq},τ_{zr}, τ_{rz}



State of stress at a point

By state of stress at a point, we mean an information which is required at that point such that it remains under equilibrium. or simply a general state of stress at a point involves all the normal stress components, together with all the shear stress components as shown in earlier figures.

Therefore, we need nine components, to define the state of stress at a point

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \sigma_y & \tau_{yx} & \tau_{yz} \\ \sigma_z & \tau_{zx} & \tau_{zy} \end{bmatrix}$$

we apply the conditions of equilibrium which are as follows:

$$\sum F_x = 0 \; ; \; \sum M_x = 0$$
$$\sum F_y = 0 \; ; \; \sum M_y = 0$$
$$\sum F_z = 0 \; ; \; \sum M_z = 0$$

Then we get $\tau_{xy} = \tau_{yx}; \quad \tau_{yz} = \tau_{zy}; \quad \tau_{zx} = \tau_{xy}$

Then we will need only six components to specify the state of stress at a point i.e,

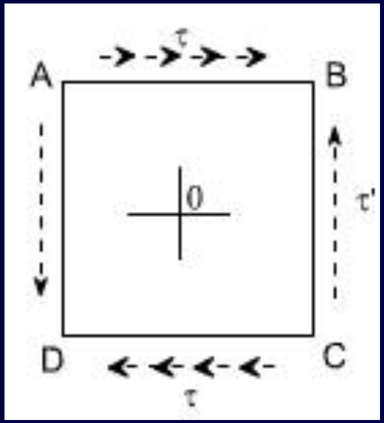
$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$$

Shear stresses:

- tending to turn the element C.W +ve.
- tending to turn the element C.C.W -ve.

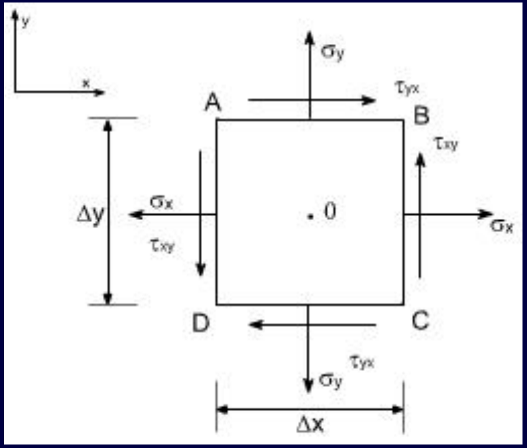
Complementary shear stresses

The existence of shear stresses on any two sides of the element induces complementary shear stresses on the other two sides of the element to maintain equilibrium.



- On planes AB and CD, the shear stress τ acts.
- To maintain the static equilibrium of this element, on planes AD and BC, τ' should act,
- we shall see that τ' which is known as the complementary shear stress would come out to equal and opposite to the τ.

Let us prove this thing for a general case as discussed below:



The figure shows a small rectangular element with sides of length Δx, Δy parallel to x and y directions. Its thickness normal to the plane of paper is Δz in z direction. All nine normal and shear stress components may act on the element, only those in x and y directions are shown.

Now forces on top and bottom surfaces produce a couple which must be balanced by the forces on left and right hand faces

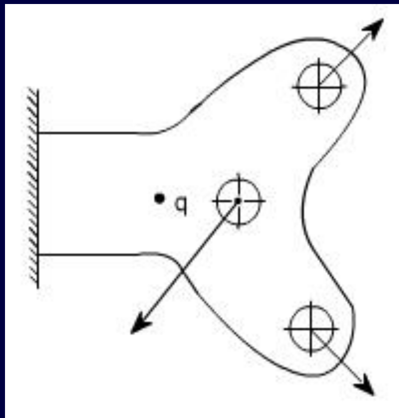
$$\tau_{yx} \,.\, \Delta x \,.\, \Delta z \,.\, \Delta y = \tau_{xy} \,.\, \Delta x \,.\, \Delta z \,.\, \Delta y$$

$$\Rightarrow \quad \tau_{yx} = \tau_{xy}$$

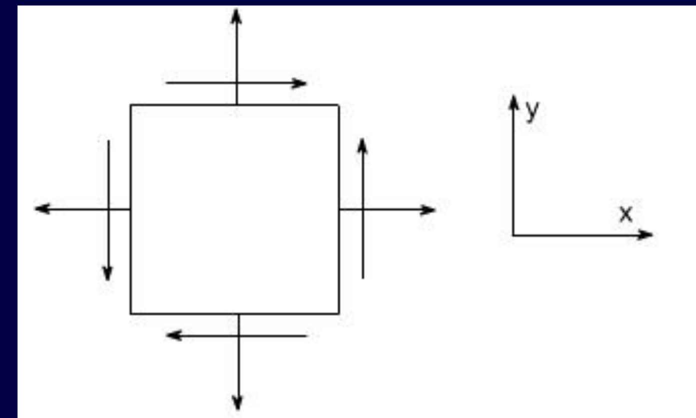
Similarly, $\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$

The complementary shear stresses are equal in magnitude

LECTURE 3: ANALYSIS OF STRESSES

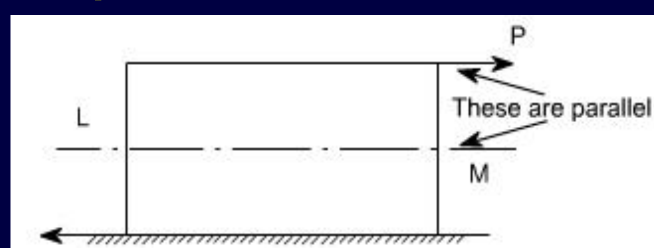


Consider a point 'q' in some sort of structural member. Assuming that at point ‘q‘, a plane state of stress exist. i.e., the state of stress is to describe by a parameters σ_x, σ_y and $\tau_{xy.}$ These stresses could be indicated on the two dimensional diagram as shown here:

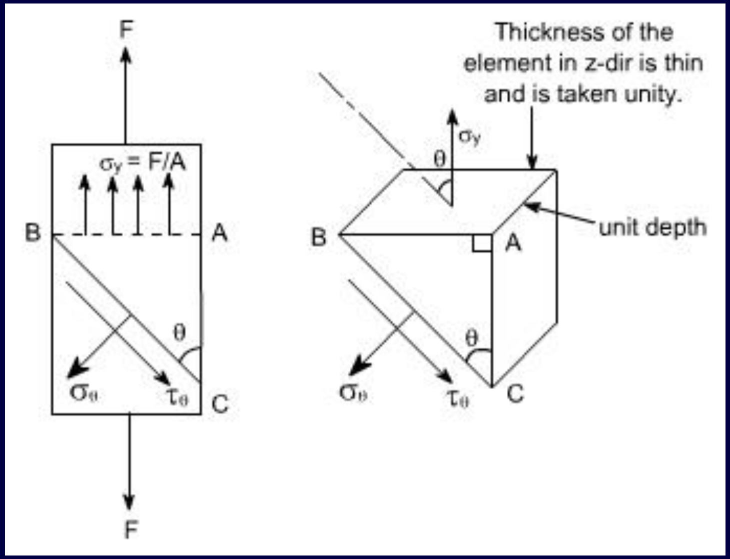


This is a common way of representing the stresses. It must be realized that the material is unaware of what we have called the x and y axes. i.e. the material has to resist the loads irrespective less of how we wish to name them or whether they are horizontal, vertical or otherwise further more, the material will fail when the stresses exceed beyond a permissible value.

Thus, a fundamental problem in engineering design is to determine the maximum normal stress or maximum shear stress at any particular point in a body. There is no reason to believe a priori that σ_x, σ_y and τ_{xy} are the maximum value. Rather the maximum stresses may associates themselves with some other planes located at ‘q'. Thus, it becomes imperative to determine the values of σ_q and τ_q. In order to achieve this let us consider the following.



Stresses on oblique plane



Till now we have dealt with either pure normal direct stress or pure shear stress. In many instances, however both direct and shear stresses acts and the resultant stress across any section will be neither normal nor tangential to the plane.

A plane state of stress is a 2-D state of stress in a sense that the stress components in one direction are all zero i.e $\sigma_z = \tau_{yz} = \tau_{zx} = 0$

Let the block be of unit depth now considering the equilibrium of forces on the triangle portion ABC. Resolving forces perpendicular to BC, gives

$\sigma_\theta.BC.1 = \sigma_y \sin\theta \,.\, AB \,.\, 1$

but $AB/BC = \sin\theta$ or $AB = BC \sin\theta$

$\sigma_\theta.BC.1 = \sigma_y \sin\theta.\ BC \sin\theta \,.\, 1$

⇨ $\sigma_\theta = \sigma_y \sin^2\theta = ½\, \sigma_y\,(1- \cos 2\theta)$

Now resolving the forces parallel to BC

$\tau_\theta.BC.1 = \sigma_y \cos\theta \,.\, AB \sin\theta \,.\, 1$

again $AB = BC \cos\theta$

$\tau_\theta.BC.1 = \sigma_y \cos\theta \,.\, BC \sin\theta \,.\, 1$ or $\tau_\theta = \sigma_y \sin\theta \cos\theta$

⇨ $\tau_\theta = ½\, \sigma_y \sin 2\theta$

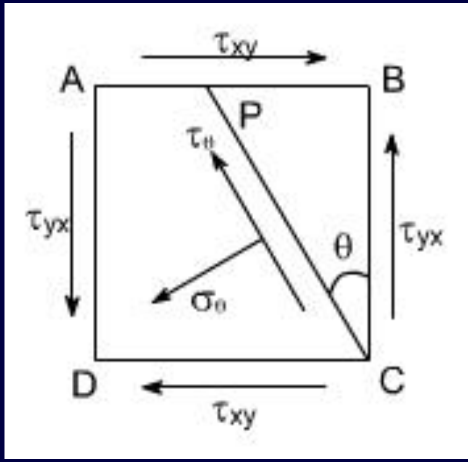
By examining the equations (1) and (2), the following conclusions may be drawn

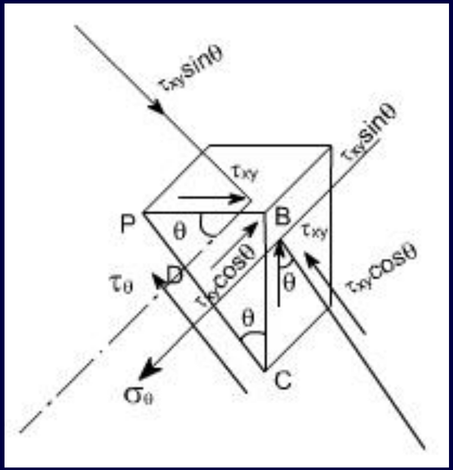
- The value of direct stress σ_θ is maximum and is equal to σ_y when $\theta = 90^0$.
- The shear stress τ_θ has a maximum value of $0.5\sigma_y$ when $\theta = 45^0$
- The stresses σ_θ and τ_θ are not simply the resolution of σ_y

Material Subjected to Pure Shear

Shear stresses have been applied to the sides AB and DC

Consider the equilibrium of portion of PBC





Assuming unit depth and resolving normal to PC or in the direction of σ_θ

$\sigma_\theta.PC.1 = \tau_{xy}.\ PB.\ \cos\theta.1 + \tau_{xy}.\ BC.\ \sin\theta.1$

$= \tau_{xy}.\ PB.\ \cos\theta + \tau_{xy}.\ BC.\ \sin\theta$

Now $PB/PC = \sin\theta$ $BC/PC = \cos\theta$

$\sigma_\theta.PC.1 = \tau_{xy}.\ \cos\theta\ \sin\theta\ PC + \tau_{xy}.\ \cos\theta.\sin\theta\ PC$

$\sigma_\theta = 2\ \tau_{xy} \sin\theta\ \cos\theta = \tau_{xy}.\sin 2\theta$ ………….. (1)

Now resolving forces parallel to PC or in the direction τ_θ.

then τ_{xy} PC . 1 = τ_{xy} . PB $\sin\theta$ - τ_{xy} . BC $\cos\theta$

-ve sign has been put because this component is in the same direction as that of τ_θ.

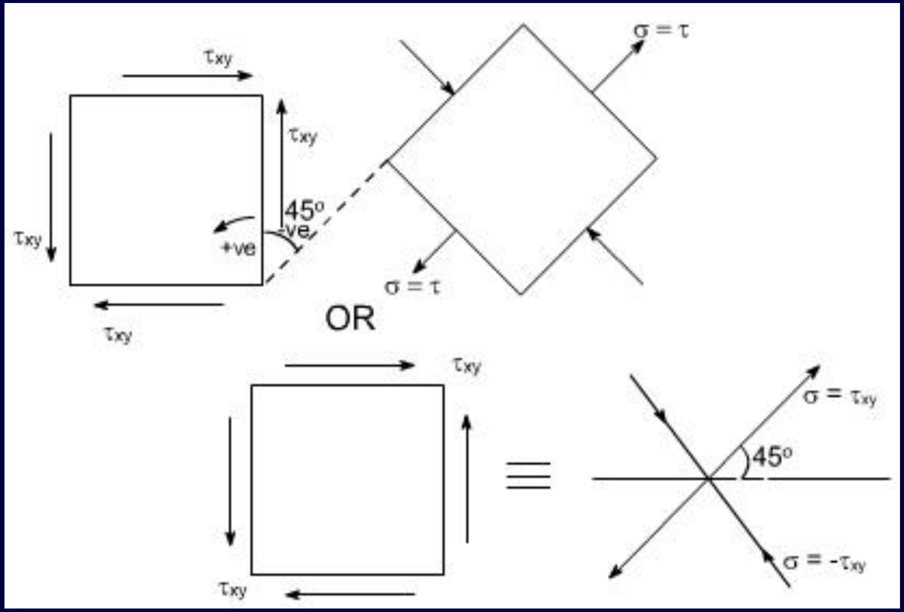
again converting the various quantities in terms of PC we have

$\tau_{xy}PC . 1 = \tau_{xy} . PB.\sin^2\theta - \tau_{xy} . PC\cos^2\theta$

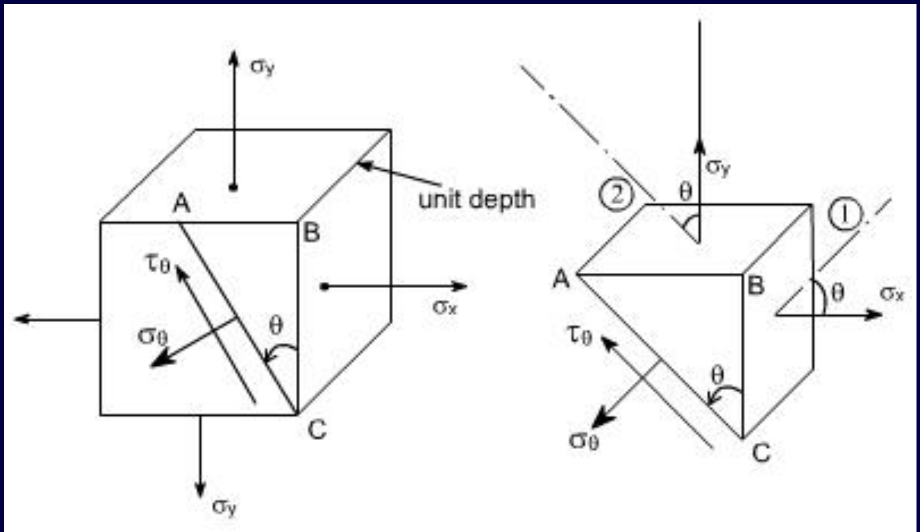
$\tau_\theta = -[\ \tau_{xy} (\cos^2\theta - \sin^2\theta)\ ] = -\tau_{xy}\cos 2\theta$ …………………….(2)

Salient Features

- The negative sign means that the sense of τ_θ is opposite to that of assumed one. Let us examine the equations (1) and (2) respectively
- Equation (1) ($\sigma_\theta = \tau_{xy}$ sin2θ), represents that maximum value of $\sigma_\theta = \tau_{xy}$ when $\theta = 45^0$.
- Equation (2) ($\tau_\theta = -\tau_{xy}\cos 2\theta$), which indicates that the maximum value of τ_θ is τ_{xy} when $\theta = 0^0$ or 90^0. it has a value zero when $\theta = 45^0$.
- From equation (1) it may be noticed that the normal component σ_θ has maximum and minimum values of $+\tau_{xy}$ (tension) and $-\tau_{xy}$ (compression) on plane at $\pm 45^0$ to the applied shear and on these planes the tangential component τ_θ is zero.
- Hence the system of pure shear stresses produces an equivalent direct stress system, one set compressive and one tensile each located at 45^0 to the original shear directions as depicted in the figure below:



Material subjected to two mutually perpendicular direct stresses



For equilibrium of the portion ABC, resolving perpendicular to AC

$$\sigma_\theta .\ AC.1 = \sigma_y \sin\theta .\ AB.1 + \sigma_x \cos\theta .\ BC.1$$

Converting AB and BC in terms of AC so that AC cancels out from the sides On rearranging the various terms we get

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta \ \text{..............} \ (3)$$

Now resolving parallel to AC: $\tau_\theta .AC.1 = -\sigma_y .\cos\theta .AB.1 + \sigma_x .BC.\sin\theta .1$

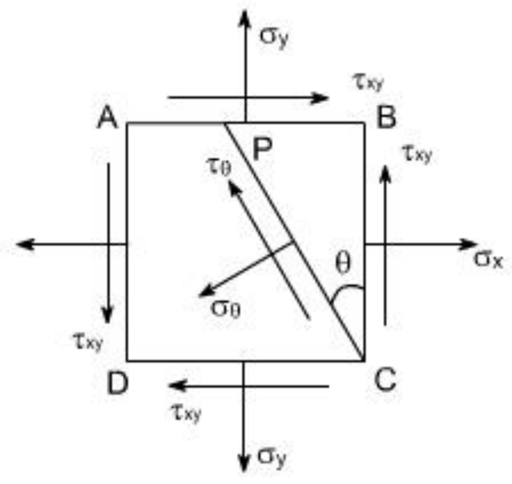
Again converting the various quantities in terms of AC so that the AC cancels out from the two sides.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta \ \text{............................} \ (4)$$

- The maximum direct stress would be equal to σ_x or σ_y which ever is the greater, when $\theta = 0^0$ or 90^0
- The maximum shear stress in the plane of the applied stresses occurs when $\theta = 45^0$

$$\tau_{max} = \frac{(\sigma_x - \sigma_y)}{2}$$

LECTURE 4: Material subjected to combined direct and shear stresses



By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

- Material subjected to pure state of stress shear. In this case the various formulas deserved are as follows

$$\sigma_\theta = \tau_{yx} \sin 2\theta$$

$$\tau_\theta = -\tau_{yx} \cos 2\theta$$

- Material subjected to two mutually perpendicular direct stresses. In this case the various formula's derived are as follows.

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

To get the required equations for the case under consideration, let us add the respective equations for the above two cases such that

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{yx} \sin 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{yx} \cos 2\theta$$

- ➢ These are the equilibrium equations for stresses at a point. They do not depend on material proportions and are equally valid for elastic and inelastic behavior
- ➢ This eqn. gives two values of 2θ that differ by 180^0. Hence the planes on which maximum and minimum normal stresses occur ate 90^0 apart.

For σ_θ to be a maximum or minimum $\dfrac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$

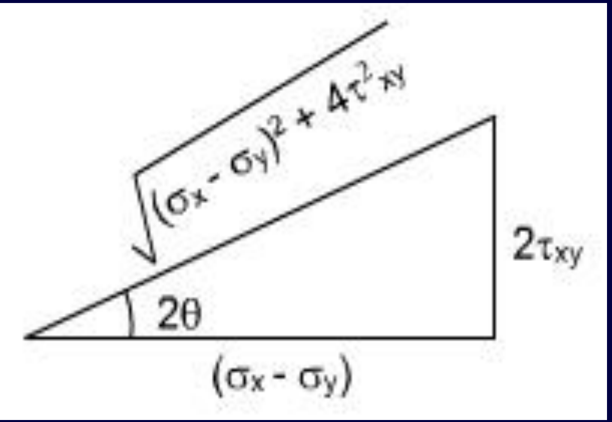
$$= 0$$

i.e. $-(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$

$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$

Thus, $$\boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$$

From the triangle it may be determined



$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

Substituting the values of cos2θ and sin2 θ in equation (5) we get

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cdot\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} + \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\cdot\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} + \frac{1}{2}\frac{4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

or

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\cdot\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\cdot\frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Hence we get the two values of σ_θ, which are designated σ_1 as σ_2 and respectively,therefore

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\cdot\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

The σ_1 and σ_2 are termed as the principle stresses of the system.

Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (6) we see that

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}\cdot(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = 0$$

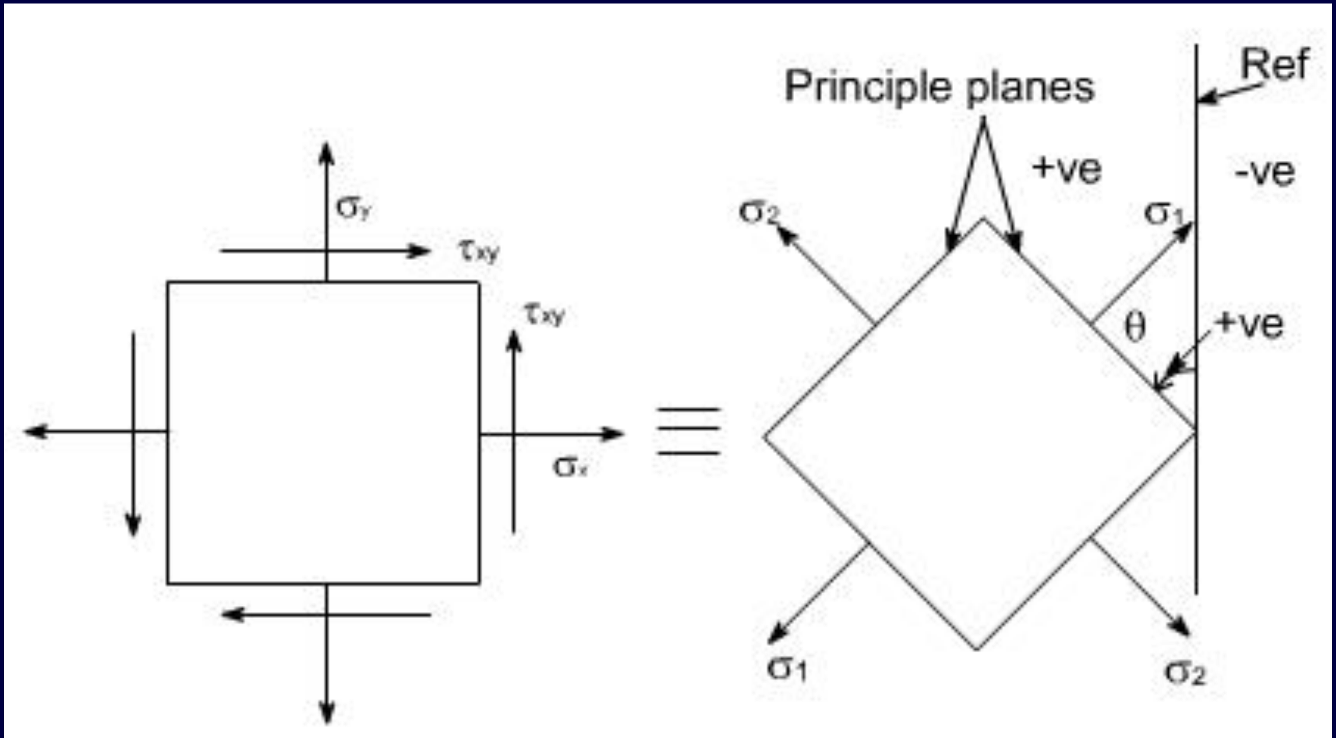
This shows that the values of shear stress is zero on the principal planes.

Maximum and minimum values of normal stresses occur on planes of zero shearing stress. The maximum and minimum normal stresses are called the principal stresses, and the planes on which they act are called principal plane the solution of equation

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

will yield two values of 2θ separated by 180^0 i.e. two values of θ separated by 90^0 . Thus the two principal stresses occur on mutually perpendicular planes termed principal planes.

Therefore the two dimensional complex stress system can now be reduced to the equivalent system of principal stresses.



Let us recall that for the case of a material subjected to direct stresses the value of maximum shear stresses

$$\tau_{max^m} = \frac{1}{2}(\sigma_x - \sigma_y) \text{ at } \quad \theta = 45^0$$,Thus, for a 2-dimensional state of stress, subjected to principle stresses

$$\tau_{max^m} = \frac{1}{2}(\sigma_1 - \sigma_2)$$, on substituting the values if σ_1 and σ_2, we get

$$\tau_{max^m} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Alternatively this expression can also be obtained by differentiating the expression for τ_θ with respect to θ i.e.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta.2 + \tau_{xy}\sin 2\theta.2$$

$$= 0$$

or $(\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$

$$\tan 2\theta_s = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Recalling that

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Thus,

$$\boxed{\tan 2\theta_p . \tan 2\theta_s = 1}$$

- Therefore, it can be concluded that the equation (2) is a negative reciprocal of equation (1) hence the roots for the double angle of equation (2) are 90^0 away from the corresponding angle of equation (1).
- This means that the angles that angles that locate the plane of maximum or minimum shearing stresses form angles of 45^0 with the planes of principal stresses.
- Further, by making the triangle we get

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

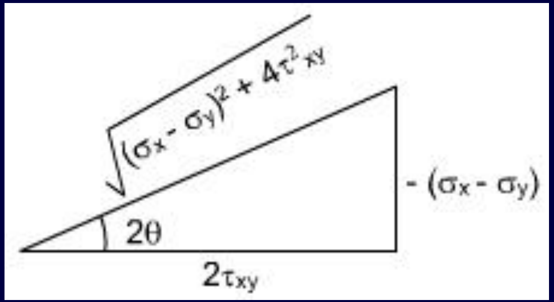
Therefore by substituting the values of $\cos 2\theta$ and $\sin 2\theta$ we have

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2} . -\frac{(\sigma_x - \sigma_y).(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

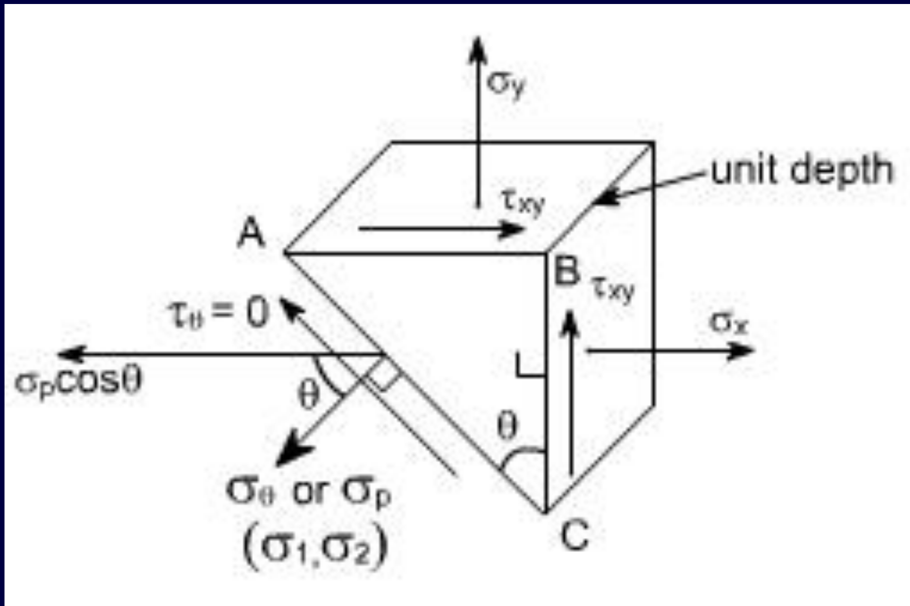
$$= -\frac{1}{2} . \frac{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = \pm\frac{1}{2} . \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$



- ✓ Because of root the difference in sign convention arises from the point of view of locating the planes on which shear stress act. From physical point of view these sign have no meaning.
- ✓ The largest stress regard less of sign is always know as maximum shear stress.

Principal plane inclination in terms of associated principal stress



Consider once again the equilibrium of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses σ_p acts, and the shear stress is zero.

$$\sigma_x . + \tau_{xy} \frac{AB}{BC} = \sigma_p . \cos\theta . \frac{AC}{BC}$$

or

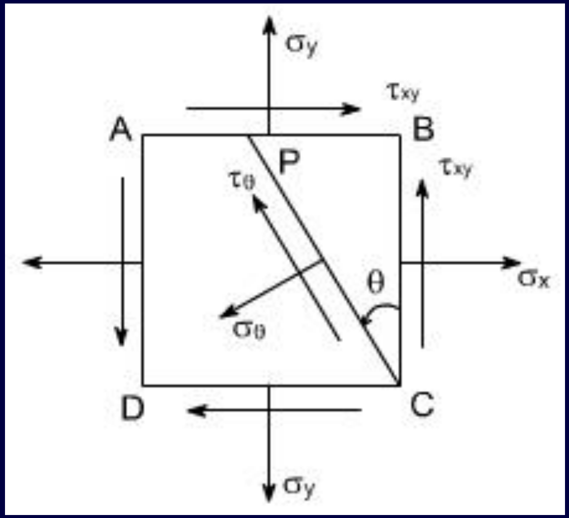
$$\sigma_x + \tau_{xy} \tan\theta = \sigma_p$$

Thus

$$\boxed{\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}}$$

LECTURE 5: GRAPHICAL SOLUTION / MOHR'S STRESS CIRCLE

The Mohr's stress circle is used to find out graphically the direct stress σ and sheer stress τ on any plane inclined at θ to the plane on which σ_x acts. The direction of θ here is taken in anticlockwise direction from the BC.



STEPS:

1. Label the Block ABCD.
2. Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)
3. Plot the stresses on two adjacent faces e.g. AB and BC, using the following sign convention.

Direct stresses - tensile -- positive; compressive -- negative

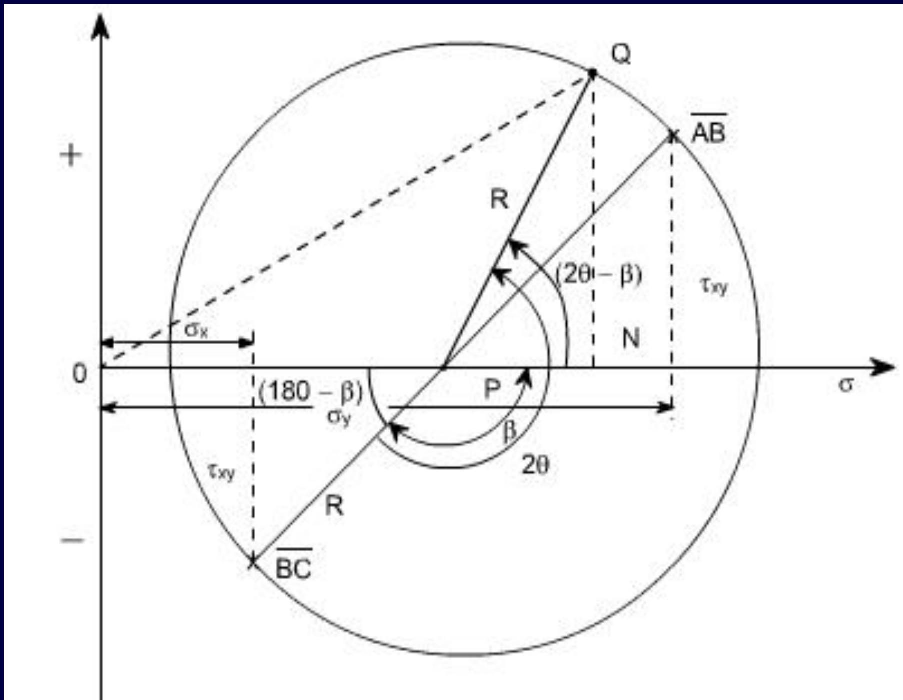
Shear stresses - tending to turn block clockwise, positive

tending to turn block counter clockwise, negative

This gives two points on the graph which may than be labeled as respectively to denote stresses on these planes.

4. Join AB and BC
5. The point P where this line cuts the σ axis is then the center of Mohr's stress circle and the line joining AB and BC is diameter. Therefore the circle can now be drawn.

Now every point on the circle then represents a state of stress on some plane through C.



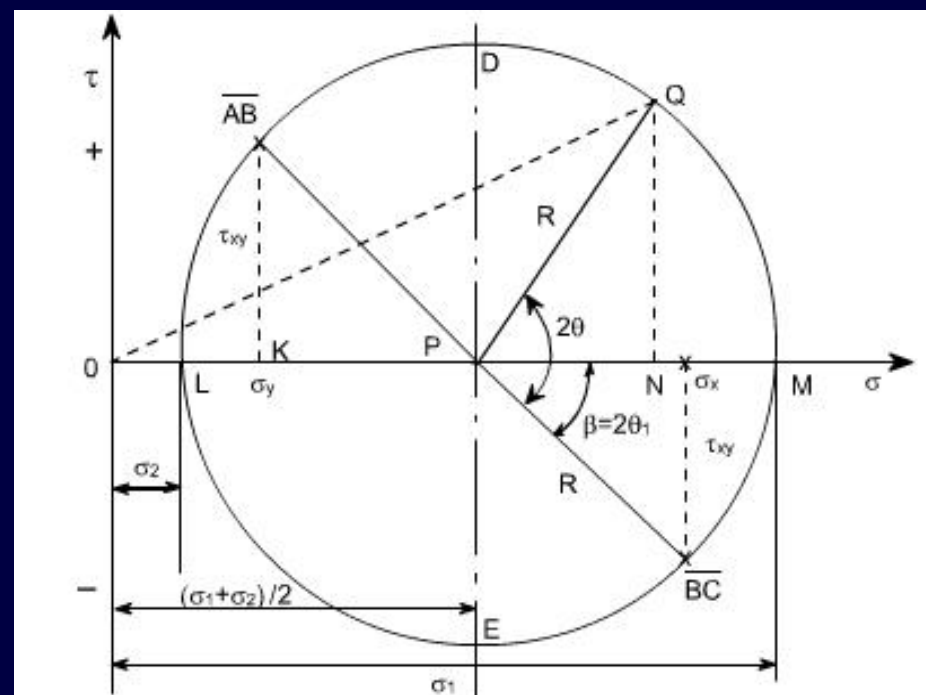
Proof: Consider any point Q on the circumference of the circle, such that PQ makes an angle 2θ with BC, and drop a perpendicular from Q to meet the σ axis at N. Then OQ represents the resultant stress on the plane an angle θ to BC. Here we have assumed that $\sigma_x > \sigma_y$. Now let us find out the coordinates of point Q.

These are ON and QN. From the figure drawn earlier ON = OP + PN and OP = OK + KP

$$OP = \sigma_y + 1/2\,(\sigma_x - \sigma_y) = (\sigma_x + \sigma_y)/2$$

$$PN = R\cos(2\theta - \beta)$$

$$R\cos\beta = \frac{(\sigma_x - \sigma_y)}{2};\ R\sin\beta = \tau_{xy}$$



$$ON = (\sigma_x + \sigma_y)/2 + 1/2\,(\sigma_x + \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$QM = R\sin(2\theta - \beta) = R\sin 2\theta\cos\beta - R\cos 2\theta\sin\beta$$

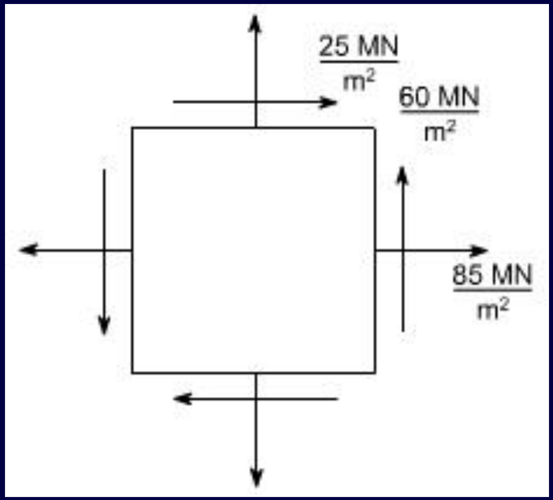
$$= 1/2\,(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

These are same as derived analytically

Salient Features

1. The direct stress is maximum when Q is at M and at this point obviously the sheer stress is zero, hence by definition OM is the length representing the maximum principal stresses σ_1 and $2\theta_1$ gives the angle of the plane θ_1 from BC. Similar OL is the other principal stress and is represented by σ_2
2. The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle.
3. From the above point the maximum sheer stress i.e. the Radius of the Mohr's stress circle would be $(\sigma_x - \sigma_y)/2$
4. As already defined the principal planes are the planes on which the shear components are zero. Therefore, we may conclude that on principal plane the sheer stress is zero.
5. Since the resultant of two stress at 90^0 can be found from the parallelogram of vectors as shown in the diagram. Thus, the resultant stress on the plane at q to BC is given by OQ on Mohr's Circle.
6. The graphical method of solution for a complex stress problems using Mohr's circle is a very powerful technique, since all the information relating to any plane within the stressed element is contained in the single construction. It thus, provides a convenient and rapid means of solution. Which is less prone to arithmetical errors and is highly recommended.

ILLUSRATIVE PROBLEMS



σ_1 and σ_2

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$= \frac{1}{2}(85 + 25) \pm \frac{1}{2}\sqrt{(85 + 25)^2 + (4x60^2)}$$

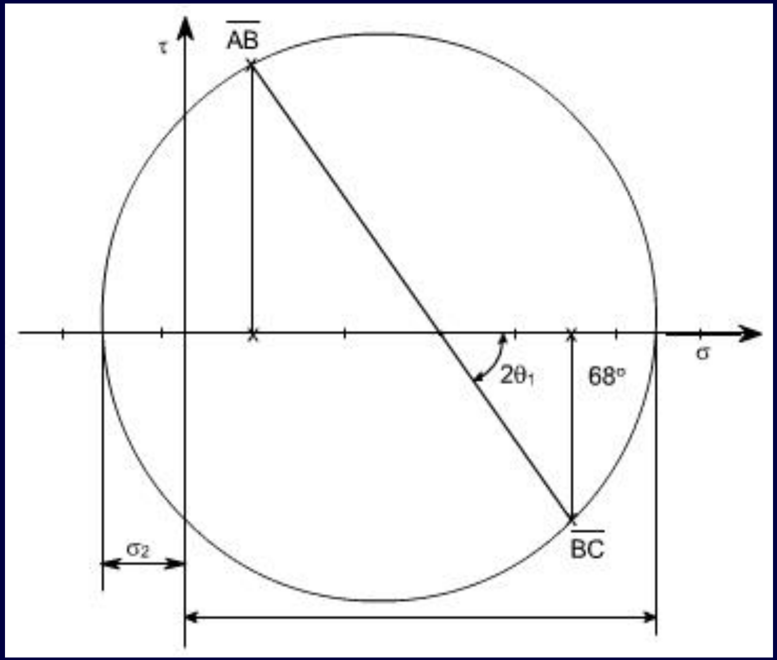
$$= 55 \pm \frac{1}{2}.60\sqrt{5} = 55 \pm 67$$

$$\Rightarrow \sigma_1 = 122\ MN/m^2$$

$$\sigma_2 = -12\ MN/m^2 (\text{compressive})$$

$$\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

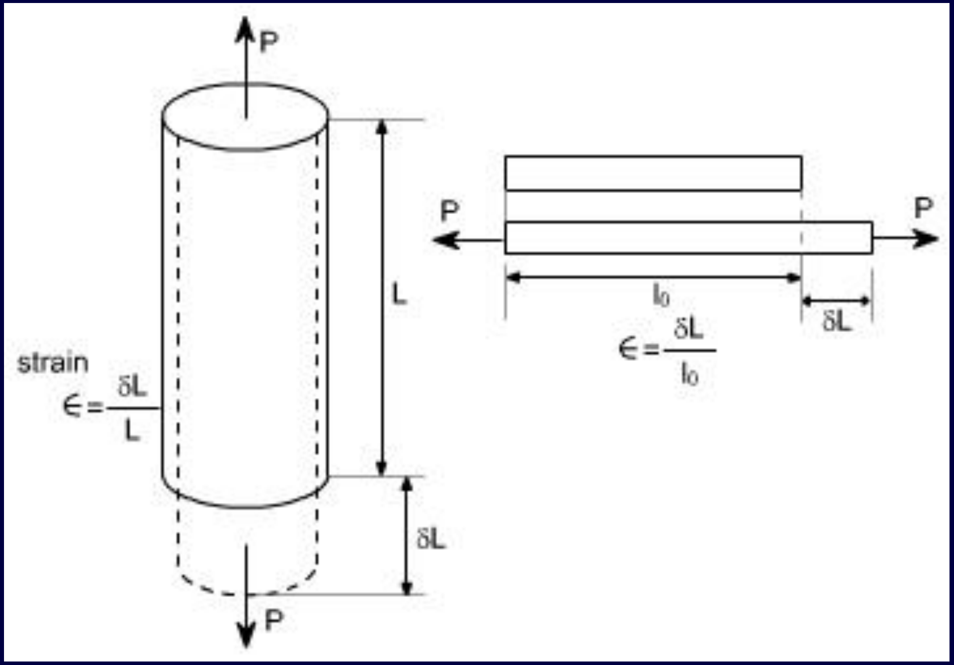
$$\Rightarrow \theta_1 = 31^0 71' \text{ and } \theta_2 = 121^0 71'$$



LECTURE 6: ANALYSIS OF STRAINS

Linear strain or normal strain or the longitudinal strain:

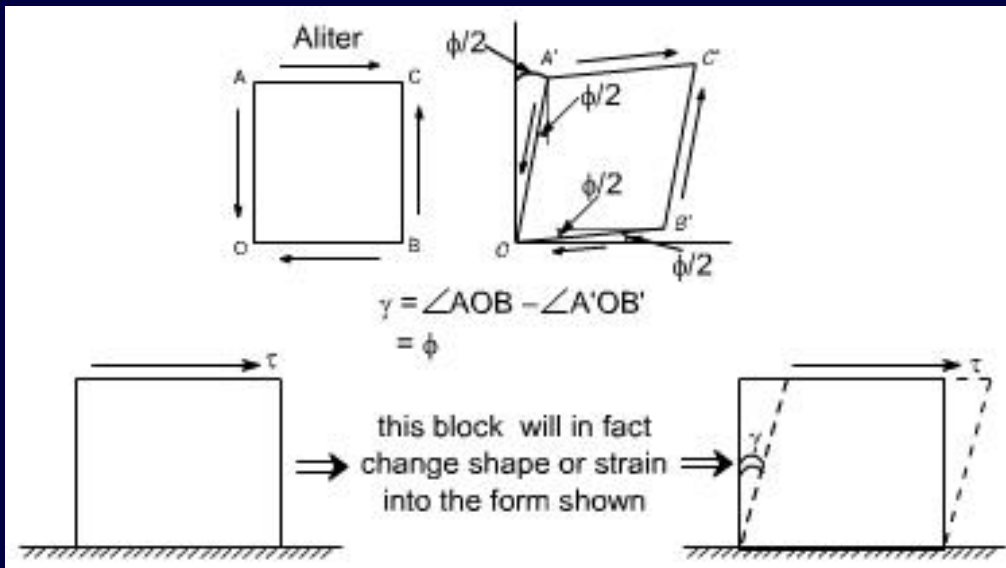
$$\text{strain}(\varepsilon) = \frac{\text{change in length}}{\text{orginal length}} = \frac{\delta L}{L}$$



Sign convention for strain: Tensile strains are positive whereas compressive strains are negative.

NEXT SLIDE -- Shear Strain

- The tangent of the angle through which two adjacent sides rotate relative to their initial position is termed shear strain. In many cases the angle is very small and the angle it self is used, (in radians), instead of tangent, so that $\gamma = \angle AOB - \angle A'OB' = \phi$



Hook's Law: A material is said to be elastic if it returns to its original, unloaded dimensions when load is removed.

$$\boxed{\frac{\text{stress}}{\text{strain}} = \text{constant}}$$

This constant is given by the symbol E and is termed as the modulus of elasticity or Young's modulus of elasticity

$$E = \frac{\text{strain}}{\text{stress}} = \frac{\sigma}{\epsilon}$$

$$= \frac{P/A}{\delta L/L}$$

$$\boxed{E = \frac{PL}{A\delta L}}$$

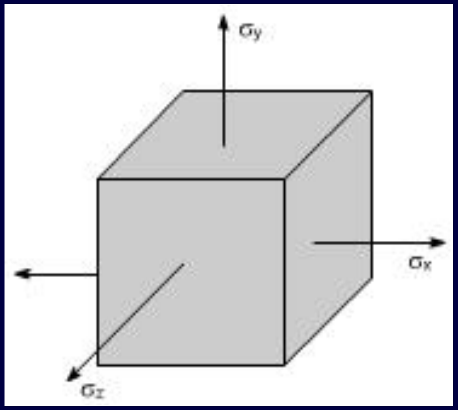
Poisson's ratio: If a bar is subjected to a longitudinal stress there will be a strain in this direction equal to σ/E . There will also be a strain in all directions at right angles to σ. The final shape being shown by the dotted lines.

It has been observed that for an elastic materials, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to longitudinal strain is known as the poison's ratio .

Poison's ratio (μ) = -lateral strain /longitudinal strain

For most engineering materials the value of m his between 0.25 and 0.33.

Three dimensional state of strain: Consider an element subjected to three mutually perpendicular tensile stresses σ_x , σ_y and σ_z as shown in the figure below.



$$\varepsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_z}{E}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E}$$

The negative sign indicating that if σ_y and σ_z are positive i.e. tensile, these they tend to reduce the strain in x direction.

Principal strains in terms of stress: In the absence of shear stresses on the faces of the elements let us say that σ_x, σ_y and σ_z are in fact the principal stress. The resulting strain in the three directions would be the principal strains.

i.e. We will have the following

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\sigma_2 - \mu\sigma_3\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\sigma_1 - \mu\sigma_3\right]$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\sigma_1 - \mu\sigma_2\right]$$

Two dimensional state of strain: stress in third direction is zero ($\sigma_z = 0$ or $\sigma_3 = 0$). Although the strains will be there in this direction due to the stresses σ_1 , σ_2

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\sigma_2\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\sigma_1\right]$$

$$\varepsilon_3 = \frac{1}{E}\left[-\mu\sigma_1 - \mu\sigma_2\right]$$

Hence a strain can exist without a stress in that direction

i.e if $\sigma_3 = 0$; $\varepsilon_3 = \frac{1}{E}\left[-\mu\sigma_1 - \mu\sigma_2\right]$

Also

$\varepsilon_1 .E = \sigma_1 - \mu\sigma_2$

$\varepsilon_2 .E = \sigma_2 - \mu\sigma_1$

so the solution of above two equations yields

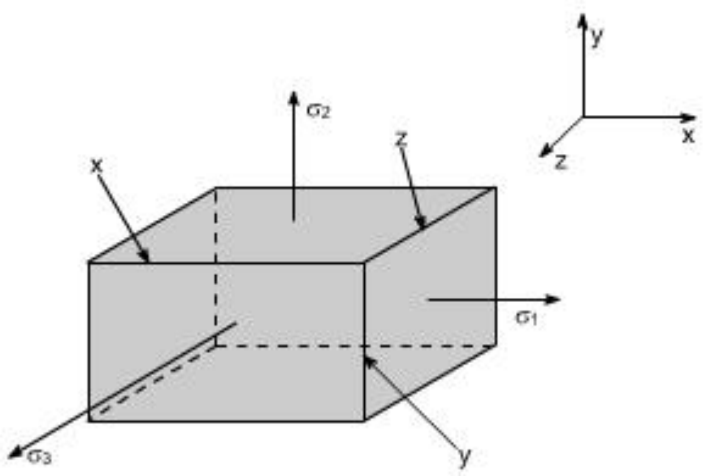
$$\boxed{\begin{aligned}\sigma_1 &= \frac{E}{(1-\mu^2)}\left[\varepsilon_1 + \mu\varepsilon_2\right]\\ \sigma_2 &= \frac{E}{(1-\mu^2)}\left[\varepsilon_2 + \mu\varepsilon_1\right]\end{aligned}}$$

Volumetric Strain

$$\text{Volumetric strain} = \frac{\text{Increase in volume}}{\text{Original volume}}$$

$$= \frac{x(1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3)z - xyz}{xyz}$$

$$= (1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3) - 1 \cong \epsilon_1 + \epsilon_2 + \epsilon_3 \quad \left[\text{Neglecting the products of } \epsilon\text{'s}\right]$$



$$\epsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

Futher Volumetric strain $= \epsilon_1 + \epsilon_2 + \epsilon_3$

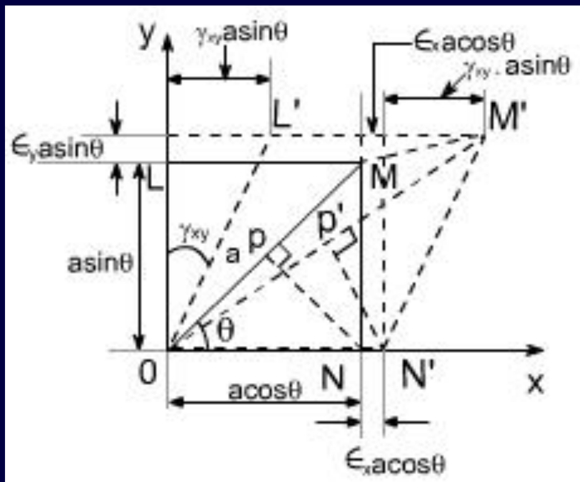
$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{E} - \frac{2\mu(\sigma_1 + \sigma_2 + \sigma_3)}{E}$$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}$$

hence the

$$\boxed{\text{Volumetric strain} = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}}$$

Strains on an oblique plane ---- (a) Linear strain



Thus the movement of M parallel to OM , which since the strains are small is practically coincident with MM'. and this would be the summation of portions (1) and (2) respectively and is equal to

$$= (\epsilon_y\ a\sin\theta)\sin\theta + (\epsilon_x\ a\cos\theta + \gamma_{xy} a\sin\theta)\cos\theta$$

$$= a\left[\epsilon_y \sin\theta.\sin\theta + \epsilon_x \cos\theta.\cos\theta + \gamma_{xy} \sin\theta.\cos\theta\right]$$

hence the strain along OM

$$= \frac{\text{extension}}{\text{original length}}$$

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \gamma_{xy} \sin\theta.\cos\theta + \epsilon_y \sin^2\theta$$

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta.\cos\theta$$

Recalling $\cos^2\theta - \sin^2\theta = \cos 2\theta$

or $2\cos^2\theta - 1 = \cos 2\theta$

$$\cos^2\theta = \left[\frac{1+\cos 2\theta}{2}\right]$$

$$\sin^2\theta = \left[\frac{1-\sin 2\theta}{2}\right]$$

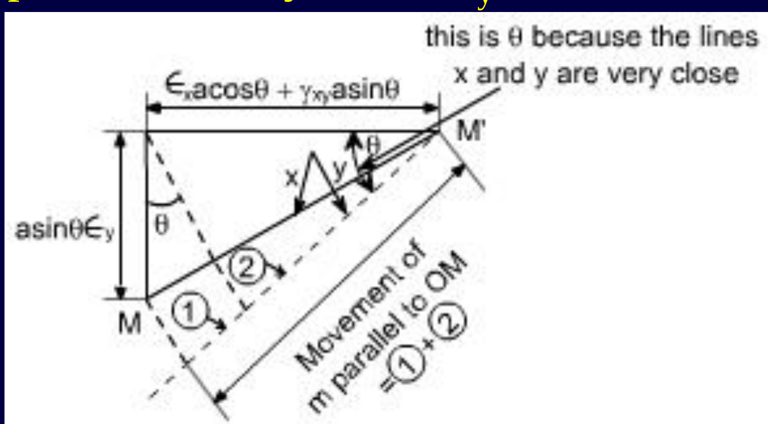
hence

$$\epsilon_\theta = \epsilon_x \left[\frac{1+\cos 2\theta}{2}\right] + \epsilon_y \left[\frac{1-\sin 2\theta}{2}\right] + \gamma_{xy} a \sin\theta.\cos\theta$$

$$= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\epsilon_\theta = \left\{\frac{\epsilon_x + \epsilon_y}{2}\right\} + \left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

If M moves to M', then the movement of M parallel to x axis is €$_x$acosθ + γ_{xy}sinθ and the movement parallel to the y axis is €$_y$asinθ



Shear strain: shear stain in the direction OM = displacement of point P at the foot of the perpendicular from N to OM; and the following expression can be derived as

$$\frac{1}{2}\gamma_\theta = -\left[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta\right]$$

In the above expression ½ is there so as to keep the consistency with the stress relations. Further -ve sign in the expression occurs so as to keep the consistency of sign convention, because OM' moves clockwise with respect to OM it is considered to be negative strain.

The other relevant expressions are the following :

Principalplanes :

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

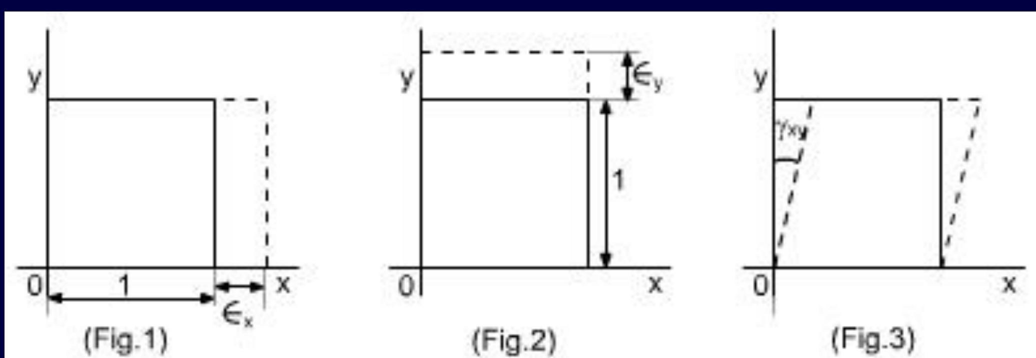
Principalstrains :

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Maximumshearstrains :

$$\frac{\gamma_{max}}{2} = \pm\sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Plane Strain: In xy plane three strain components may exist as can be seen from the following figures:



Plane strain condition is defined only by the components , $€_y$, γ_{xy}: $€_z = 0$; $\gamma_{xz} = 0$; $\gamma_{yz} = 0$

Plane stress is not the stress system associated with plane strain. The plane strain condition is associated with 3-D stress system and plane stress is associated with 3-D strain system.

Principal Strain and Mohr's Circle for Strain

$$\epsilon_\theta = \left\{\frac{\epsilon_x + \epsilon_y}{2}\right\} + \left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \qquad (1)$$

$$\frac{1}{2}\gamma_\theta = -\left[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta\right] \qquad (2)$$

Re writing the equation (1) as below :

$$\left[\epsilon_\theta - \left(\frac{\epsilon_x + \epsilon_y}{2}\right)\right] = \left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \qquad (3)$$

squaring and adding equations (2) and (3)

$$\left[\epsilon_\theta - \left(\frac{\epsilon_x + \epsilon_y}{2}\right)\right]^2 + \left\{\frac{1}{2}\gamma_\theta\right\}^2 = \left[\left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta\right]^2 + \left[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta\right]^2$$

$$\left[\epsilon_\theta - \left(\frac{\epsilon_x + \epsilon_y}{2}\right)\right]^2 + \left\{\frac{1}{2}\gamma_\theta\right\}^2 = \left(\frac{\epsilon_x + \epsilon_y}{2}\right)^2 + \frac{\gamma^2_{xy}}{4}$$

Now as we know that

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\epsilon_1 + \epsilon_2 = \epsilon_x + \epsilon_y$$

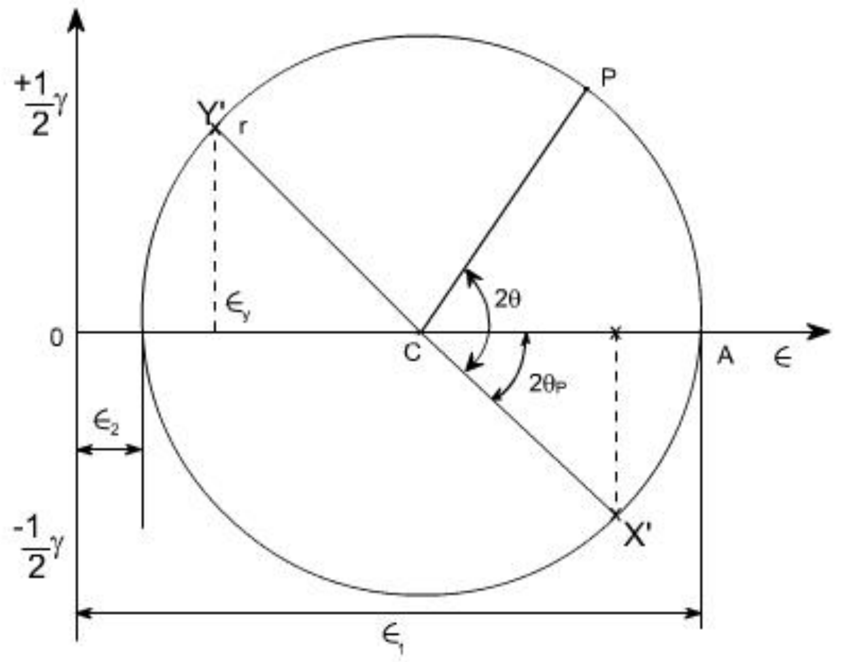
$$\left(\frac{\epsilon_1 - \epsilon_2}{2}\right)^2 = \left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \frac{\gamma^2_{xy}}{4}$$

Therefore the equation gets transformed to

$$\left[\epsilon_\theta - \left(\frac{\epsilon_1 + \epsilon_2}{2}\right)\right]^2 + \left[\frac{\gamma_\theta}{2}\right]^2 = \left(\frac{\epsilon_1 - \epsilon_2}{2}\right)^2 \qquad (4)$$

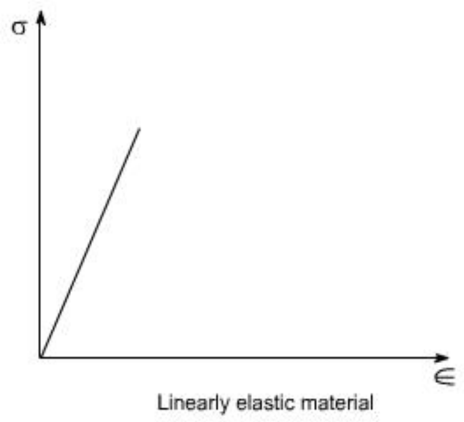
If we plot equation (4) we obtain a circle of radius $\left(\frac{\epsilon_1 - \epsilon_2}{2}\right)$ with center at $\left(\frac{\epsilon_1 + \epsilon_2}{2}, 0\right)$

For the strains on an oblique plane we have two equations which are identical in form with the equation defining the direct stress on any inclined plane θ .

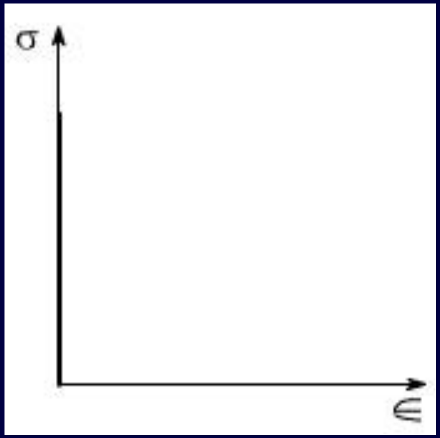


LECTURE 7: STRESS - STRAIN RELATIONS

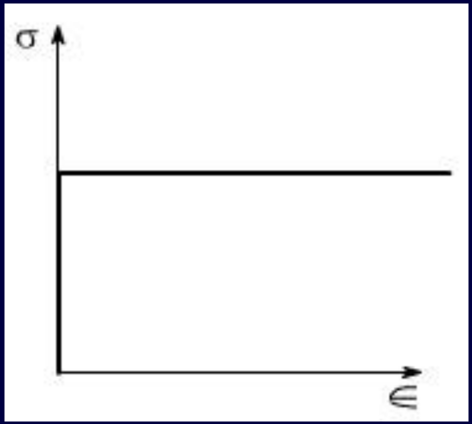
(i) Linear elastic material



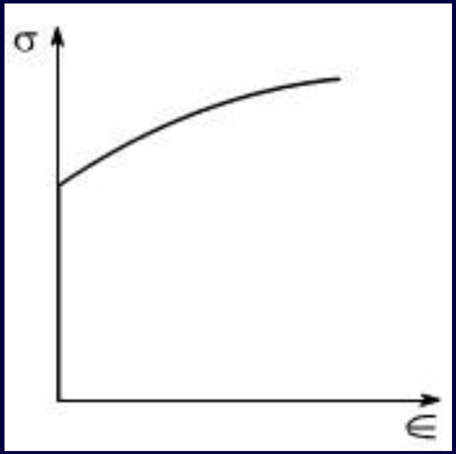
(ii) Rigid Materials



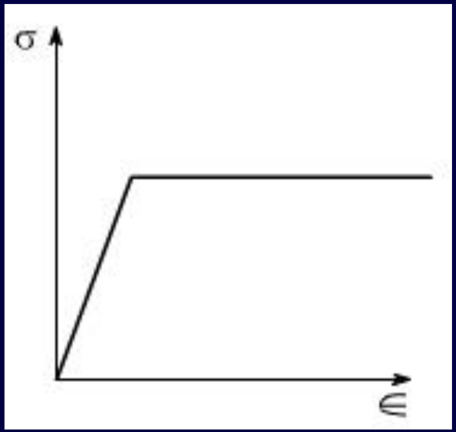
(iii) Perfectly plastic (non-strain hardening):



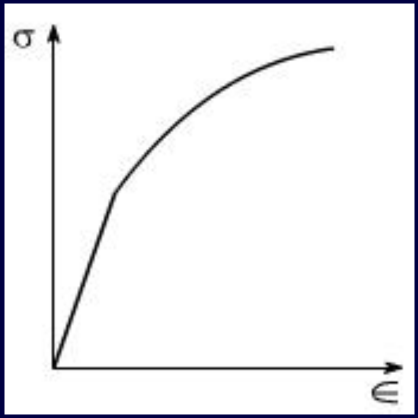
(iv) Rigid Plastic material (strain hardening)



(v) Elastic Perfectly Plastic material



(vi) Elastic Plastic material

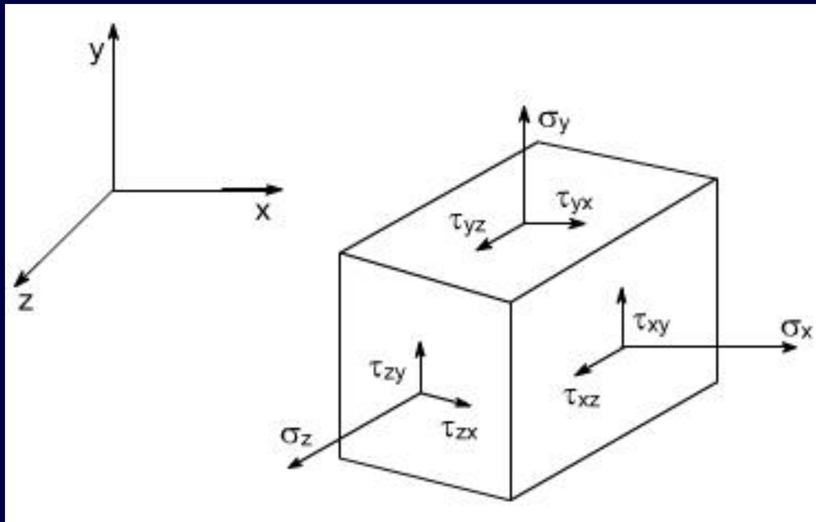


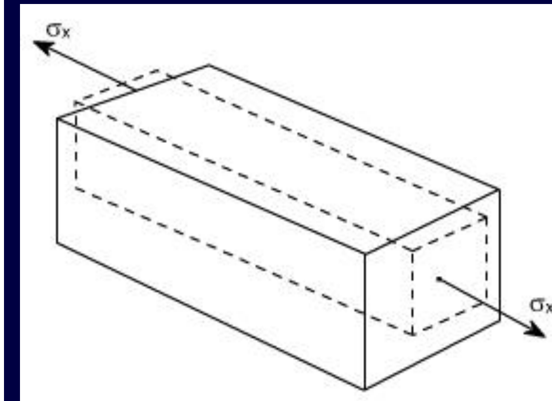
ISOTROPIC: If the response of the material is independent of the orientation of the load axis of the sample, then we say that the material is isotropic or in other words we can say that isotropy of a material in a characteristics, which gives us the information that the properties are the same in the three orthogonal directions x y z, on the other hand if the response is dependent on orientation it is known as an-isotropic.

Examples of an-isotropic materials, whose properties are different in different directions are

- ❖ Wood
- ❖ Fibre reinforced plastic
- ❖ Reinforced concrete

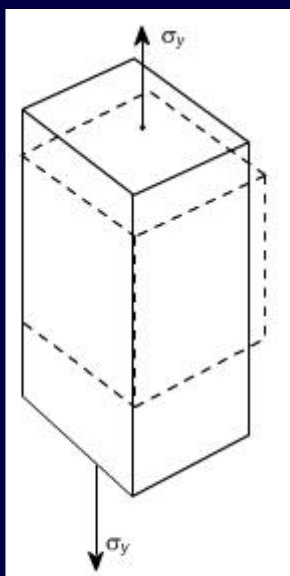
HOMOGENIUS: A material is homogenous if it has the same composition throughout body Hence the elastic properties are the same at every point in the body. However, the properties need not to be the same in all the direction for the material to be homogenous. Isotropic materials have the same elastic properties in all the directions. Therefore, the material must be both homogenous and isotropic in order to have the lateral strains to be same at every point in a particular component.





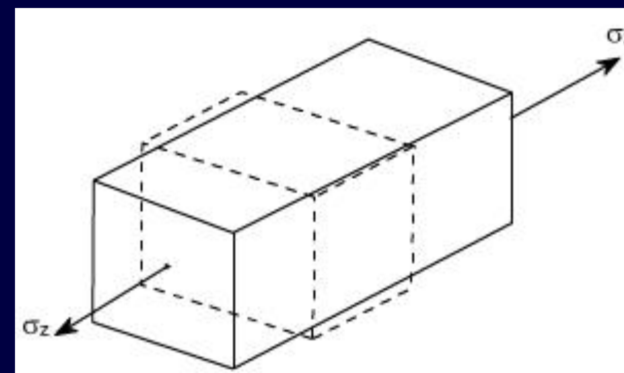
$$\epsilon_x = \frac{\sigma_x}{E} \text{ , } \epsilon_y = -\mu\epsilon_x \text{ ; } \epsilon_z = -\mu\epsilon_x$$

$$\epsilon_x = \frac{\sigma_x}{E} \text{ ; } \epsilon_y = -\mu\frac{\sigma_x}{E} \text{ ; } \epsilon_z = -\mu\frac{\sigma_x}{E}$$



$$\epsilon_y = \frac{\sigma_y}{E} \text{ , } \epsilon_x = -\mu\epsilon_y \text{ ; } \epsilon_z = -\mu\epsilon_y$$

$$\epsilon_y = \frac{\sigma_y}{E} \text{ ; } \epsilon_x = -\mu\frac{\sigma_y}{E} \text{ ; } \epsilon_z = -\mu\frac{\sigma_y}{E}$$



$$\epsilon_z = \frac{\sigma_z}{E} \text{ , } \epsilon_y = -\mu\epsilon_z \text{ ; } \epsilon_x = -\mu\epsilon_z$$

$$\epsilon_z = \frac{\sigma_z}{E} \text{ ; } \epsilon_y = -\mu\frac{\sigma_z}{E} \text{ ; } \epsilon_x = -\mu\frac{\sigma_z}{E}$$

Thus the total strain in any one direction is

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z) \qquad (1)$$

In a similar manner, the total strain in the y and z directions become

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu}{E}(\sigma_x + \sigma_z) \qquad (2)$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\mu}{E}(\sigma_x + \sigma_y) \qquad (3)$$

In the following analysis shear stresses were not considered. It can be shown that for an isotropic material's a shear stress will produce only its corresponding shear strain and will not influence the axial strain. Thus, we can write Hook's law for the individual shear strains and shear stresses in the following manner.

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \text{......................(4)}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \quad \text{......................(5)}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \quad \text{......................(6)}$$

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E}$$

$$\epsilon_z = -\mu\frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} \text{ and } \tau_{xy} = \frac{\gamma_{xy}}{G}$$

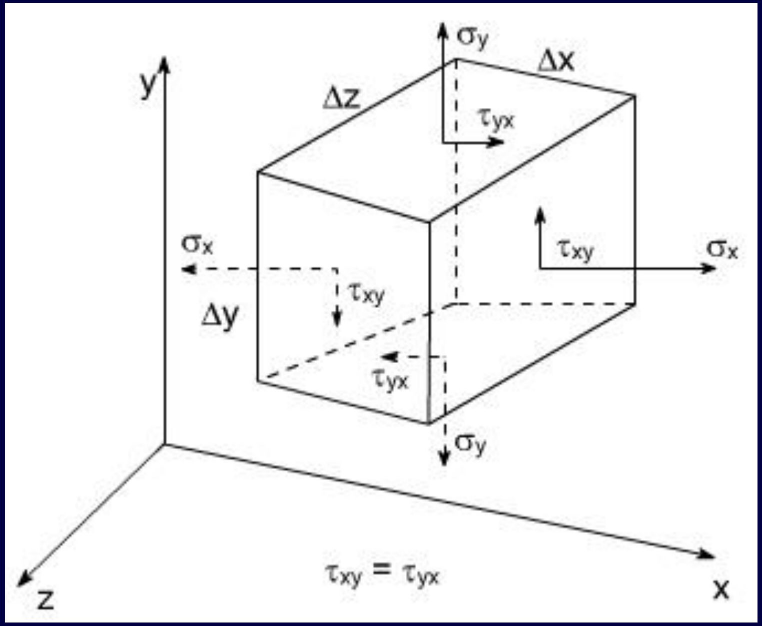
Their inverse relations can be also determined and are given as

$$\sigma_x = \frac{E}{(1-\mu^2)}(\epsilon_x + \mu\epsilon_y)$$

$$\sigma_y = \frac{E}{(1-\mu^2)}(\epsilon_y + \mu\epsilon_x)$$

$$\tau_{xy} = G.\gamma_{xy}$$

PLANE STRESS

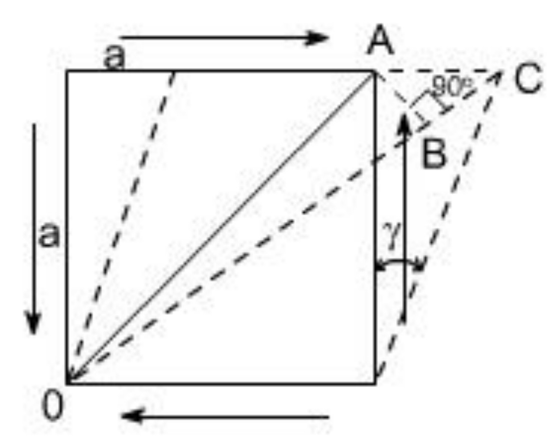


$$\boxed{\sigma_z = \tau_{zx} = \tau_{zy} = 0}$$

The Equations (1) through (6) are known as Generalized Hook's law and are the constitutive equations for the linear elastic isotropic materials. When these equations are used as written, the strains can be completely determined from known values of the stresses. To engineers the plane stress situation is of much relevance (i.e. $\sigma_z = \tau_{xz} = \tau_{yz} = 0$), Thus then the above set of equations reduces to

RELATION AMONG ELASTIC CONSTANTS

E = Young's Modulus of Rigidity = Stress / strain
G = Shear Modulus or Modulus of rigidity= Shear stress / Shear strain
μ = Possion's ratio = - lateral strain / longitudinal strain
K = Bulk Modulus of elasticity = Volumetric stress / Volumetric strain
where Volumetric strain = sum of linear stress in x, y and z direction.
Volumetric stress = stress which cause the change in volume.



Relation between E, G and μ :

Consider a cube of material of side 'a' subjected to the action of the shear and complementary shear stresses as shown in the figure and producing the strained shape as shown in the figure below.

Assuming that the strains are small and the angle ACB may be taken as 45^0.

strain on diagonal $= \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$

$= \frac{\tau}{E} - \mu\frac{(-\tau)}{E}$

$= \frac{\tau}{E}(1 + \mu)$

equating the two strains one may get

$$\frac{\tau}{2G} = \frac{\tau}{E}(1 + \mu)$$

or $\boxed{E = 2G(1 + \mu)}$

Thus, strain on diagonal OA $= \frac{BC}{OA}$

$= \frac{AC\cos 45^0}{OA}$

$OA = \frac{a}{\sin 45^0} = a.\sqrt{2}$

hence strain $= \frac{AC}{a\sqrt{2}}.\frac{1}{\sqrt{2}}$

$= \frac{AC}{2a}$

but AC = aγ

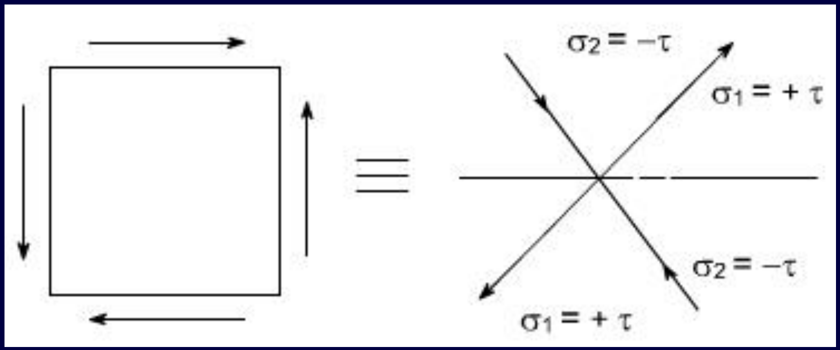
where γ = shear strain

Thus, the strain on diagonal $= \frac{a\gamma}{2a} = \frac{\gamma}{2}$

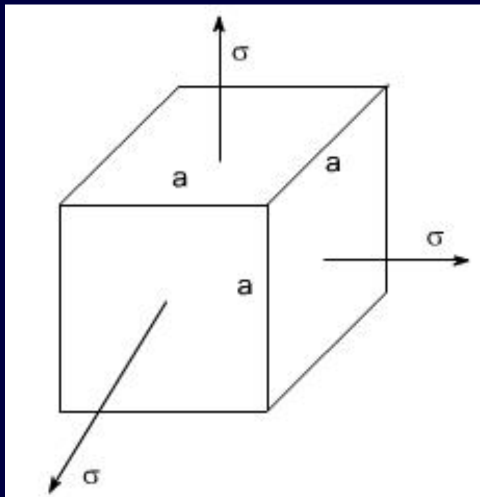
From the definition

$G = \frac{\tau}{\gamma}$ or $\gamma = \frac{\tau}{G}$

thus, the strain on diagonal $= \frac{\gamma}{2} = \frac{\tau}{2G}$



Relation between E, K and μ: Consider a cube subjected to three equal stresses s as shown in the figure below



The total strain in one direction or along one edge due to the application of hydrostatic stress or volumetric stress s is given as

$$= \frac{\sigma}{E} - \gamma\frac{\sigma}{E} - \gamma\frac{\sigma}{E}$$

$$= \frac{\sigma}{E}(1 - 2\gamma)$$

volumetre strain = 3.linear strain

volumetre strain = $\epsilon_x + \epsilon_y + \epsilon_z$

or thus, $\epsilon_x = \epsilon_y = \epsilon_z$

$$\text{volumetric strain} = 3\frac{\sigma}{E}(1 - 2\gamma)$$

By definition

$$\text{Bulk Modulus of Elasticity } (K) = \frac{\text{Volumetric stress}(\sigma)}{\text{Volumetric strain}}$$

or

$$\text{Volumetric strain} = \frac{\sigma}{k}$$

Equating the two strains we get

$$\frac{\sigma}{k} = 3.\frac{\sigma}{E}(1 - 2\gamma)$$

$$\boxed{E = 3K(1 - 2\gamma)}$$

Relation between E, K and G

$$E = 2G(1+\mu) \text{ and } E = 3K(1-2\mu)$$

$$\therefore E = \frac{9GK}{(3K+G)}$$

It may be noted that under condition of simple tension and simple shear, all real materials tend to experience displacements in the directions of the applied forces and Under hydrostatic loading they tend to increase in volume. In other words the value of the elastic constants E, G and K cannot be negative

$$E = 2G(1+\mu) \text{ and } E = 3K(1-2\mu)$$

$$\text{Yields } -1 \le \mu \le 0.5$$

In actual practice no real material has value of Poisson's ratio negative . Thus, the value of u cannot be greater than 0.5, if however μ> 0.5 than $€_v$ = -ve, which is physically unlikely because when the material is stretched its volume would always increase.