4. State the relations for:
   a. Speed of sound in an arbitrary medium
   b. Speed of sound in a perfect gas
   c. Mach number
5. Discuss the propagation of signal waves from a moving body in a fluid by explaining *zone of action*, *zone of silence*, *Mach cone*, and *Mach angle*. Compare subsonic and supersonic flow in these respects.
6. Write an equation for the stagnation enthalpy (h_t) of a perfect gas in terms of enthalpy (h), Mach number (M), and ratio of specific heats (γ).
7. Write an equation for the stagnation temperature (T_t) of a perfect gas in terms of temperature (T), Mach number (M), and ratio of specific heats (γ).
8. Write an equation for the stagnation pressure (p_t) of a perfect gas in terms of pressure (p), Mach number (M), and ratio of specific heats (γ).
9. (*Optional*) Demonstrate manipulative skills by developing simple relations in terms of Mach number for a perfect gas, such as

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

10. Demonstrate the ability to utilize the concepts above in typical flow problems.

is defined as the velocity of a wave front, created by a disturbance of infinitesamally small amplitude through an elastic medium.

4.3 SONIC VELOCITY AND MACH NUMBER

Explain

We now examine the means by which disturbances pass through an elastic medium. A disturbance at a given point creates a region of compressed molecules that is passed along to its neighboring molecules and in so doing creates a *traveling wave*. Waves come in various *strengths*, which are measured by the amplitude of the disturbance. The speed at which this disturbance is propagated through the medium is called the *wave speed*. This speed not only depends on the type of medium and its thermodynamic state but is also a function of the strength of the wave. The *stronger* the wave is, the faster it moves.

If we are dealing with waves of *large amplitude*, which involve relatively large changes in pressure and density, we call these *shock waves*. These will be studied in detail in Chapter 6. If, on the other hand, we observe waves of *very small amplitude*, their speed is characteristic *only* of the medium and its state. These waves are of vital importance to us since sound waves fall into this category. Furthermore, the presence of an object in a medium can only be felt by the object's sending out or reflecting infinitesimal waves which propagate at the characteristic *sonic velocity*.

Let us hypothesize how we might form an infinitesimal pressure wave and then apply the fundamental concepts to determine the wave velocity. Consider a long constant-area tube filled with fluid and having a piston at one end, as shown in Figure 4.1. The fluid is initially at rest. At a certain instant the piston is given an

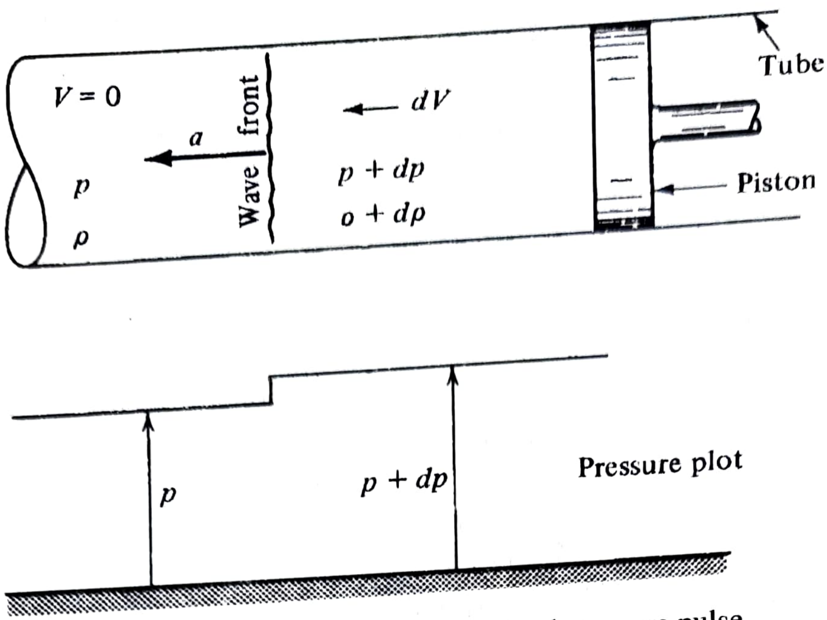
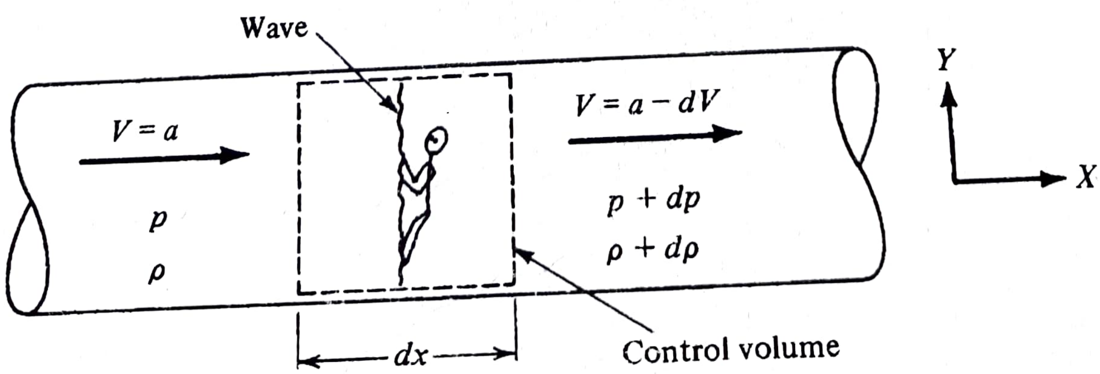


Figure 4.1 Initiation of infinitesimal pressure pulse.

incremental velocity dV to the left. The fluid particles immediately next to the piston are compressed a very small amount as they acquire the velocity of the piston.

As the piston (and these compressed particles) continue to move, the next group of fluid particles is compressed and the *wave front* is observed to propagate through the fluid at the characteristic *sonic velocity* of magnitude a. All particles between the wave front and the piston are moving with velocity dV to the left and have been compressed from ρ to $\rho + d\rho$ and have increased their pressure from p to $p + dp$.

We next recognize that this is a difficult situation to analyze. Why? Because it is *unsteady* flow! [As you observe any given point in the tube, the properties change with time (e.g., pressure changes from p to $p + dp$ as the wave front passes).] This difficulty can easily be solved by superimposing on the entire flow field a constant velocity to the right of magnitude a. *This procedure changes the frame of reference to the wave front as it now appears as a stationary wave.* An alternative way of achieving this result is to jump on the wave front. Figure 4.2 shows the problem that we now



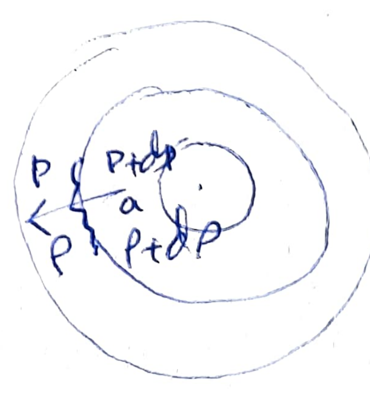


Figure 4.2 Steady-flow picture corresponding to Figure 4.1.

have. *Note that changing the reference frame in this manner does not in any way alter the actual (static) thermodynamic properties of the fluid, although it will affect the stagnation conditions.* Since the wave front is extremely thin, we can use a control volume of infinitesimal thickness.

Continuity

For steady one-dimensional flow, we have

$$\dot{m} = \rho A V = \text{const} \tag{2.30}$$

But $A = \text{const}$; thus

$$\rho V = \text{const} \tag{4.1}$$

Application of this to our problem yields

$$\rho a = (\rho + d\rho)(a - dV)$$

Expanding gives us

$$\cancel{\rho a} = \cancel{\rho a} - \rho\, dV + a\, d\rho - \overbrace{\cancel{d\rho\, dV}}^{\text{HOT}}$$

Neglecting the higher-order term and solving for dV, we have

$$dV = \frac{a\, d\rho}{\rho} \tag{4.2}$$

Momentum

Since the control volume has infinitesimal thickness, we can neglect any shear stresses along the walls. We shall write the x-component of the momentum equation, taking forces and velocity as positive if to the right. For steady one-dimensional flow we may write

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{\text{out}_x} - V_{\text{in}_x}) \tag{3.46}$$

$$\cancel{p}A - (\cancel{p} + dp)A = \frac{\rho A a}{g_c}\,[(\cancel{a} - dV) - \cancel{a}]$$

$$A\, dp = \frac{\rho A a}{g_c}\, dV$$

Canceling the area and solving for dV, we have

$$dV = \frac{g_c\, dp}{\rho a} \tag{4.3}$$

Equations (4.2) and (4.3) may now be combined to eliminate dV, with the result

$$a^2 = g_c \frac{dp}{d\rho} \tag{4.4}$$

However, the derivative $dp/d\rho$ is not unique. It depends entirely on the process. Thus it should really be written as a *partial* derivative with the appropriate subscript. But what subscript? What kind of a process are we dealing with?

Remember, we are analyzing an infinitesimal disturbance. For this case we can assume negligible losses and heat transfer as the wave passes through the fluid. Thus the process is both reversible and adiabatic, which means that it is isentropic. (Why?) After we have studied shock waves, we shall prove that very weak shock waves (i.e., small disturbances) approach an isentropic process in the limit. Therefore, equation (4.4) should properly be written as

$$\boxed{a^2 = g_c \left(\frac{\partial p}{\partial \rho}\right)_s} \tag{4.5}$$

This can be expressed in an alternative form by introducing the *bulk* or *volume modulus of elasticity* E_v. This is a relation between volume or density changes that occurs as a result of pressure fluctuations and is defined as

$$E_v \equiv -v\left(\frac{\partial p}{\partial v}\right)_s \equiv \rho\left(\frac{\partial p}{\partial \rho}\right)_s \tag{4.6}$$

Thus

$$\boxed{a^2 = g_c \left(\frac{E_v}{\rho}\right)} \tag{4.7}$$

Equations (4.5) and (4.7) are equivalent general relations for sonic velocity through *any* medium. The bulk modulus is normally used in connection with liquids and solids. Table 4.1 gives some typical values of this modulus, the exact value depending on the temperature and pressure of the medium. For solids it also depends on the type of loading. The reciprocal of the bulk modulus is called the *compressibility*. What is the sonic velocity in a truly incompressible fluid? [*Hint:* What is the value of $(\partial p/\partial \rho)_s$?]

Equation (4.5) is normally used for gases and this can be greatly simplified for the case of a gas that obeys the perfect gas law. For an isentropic process, we know that

Table 4.1 Bulk Modulus Values for Common Media

Medium	Bulk Modulus (psi)
Oil	185,000–270,000
Water	300,000–400,000
Mercury	approx. 4,000,000
Steel	approx. 30,000,000

$$pv^{\gamma} = \text{const} \quad \text{or} \quad p = \rho^{\gamma}\ \text{const} \tag{4.8}$$

Thus

$$\frac{\partial P}{\partial \rho} = \gamma \rho^{\gamma-1}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \rho^{\gamma-1}\ \text{const}$$

But from (4.8), the constant $= p/\rho^{\gamma}$. Therefore, Assuming the gas obeys perfect gas law: $(P = \rho R T)$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \rho^{\gamma-1} \frac{p}{\rho^{\gamma}} = \gamma \frac{p}{\rho} = \gamma R T$$

and from (4.5)

$$\boxed{a^2 = \gamma g_c R T} \tag{4.9}$$

or

$$\boxed{a = \sqrt{\gamma g_c R T}} \tag{4.10}$$

Notice that for perfect gases, sonic velocity is a function of the individual gas and temperature *only*.

Example 4.1 Compute the sonic velocity in air at 70°F.

$$a^2 = \gamma g_c R T = (1.4)(32.2)(53.3)(460 + 70)$$

$$a = 1128 \text{ ft/sec}$$

Example 4.2 Sonic velocity through carbon dioxide is 275 m/s. What is the temperature in Kelvin?

$$a^2 = \gamma g_c R T$$

$$(275)^2 = (1.29)(1)(189)(T)$$

$$T = 310.2 \text{ K}$$

(23)

Always keep in mind that in general, sonic velocity is a property of the fluid and varies with the state of the fluid. *Only* for gases that can be treated as perfect is the sonic velocity a function of temperature alone.

Mach Number

We define the *Mach number* as

$$\boxed{M \equiv \frac{V}{a}} \tag{4.11}$$

where

$V \equiv$ the velocity of the medium

$a \equiv$ sonic velocity through the medium

Subsonic $M<1$
Sonic $M=1$
Supersonic $M>1$
Hypersonic $(M \geq 5)$

It is important to realize that both V and a are computed *locally* for conditions that actually exist at the same point. If the velocity at one point in a flow system is twice that at another point, we *cannot* say that the Mach number has doubled. We must seek further information on the sonic velocity, which has probably also changed. (What property would we be interested in if the fluid were a perfect gas?)

If the velocity is less than the local speed of sound, M is less than 1 and the flow is called *subsonic*. If the velocity is greater than the local speed of sound, M is greater than 1 and the flow is called *supersonic*. We shall soon see that the Mach number is the most important parameter in the analysis of compressible flows.

4.4 WAVE PROPAGATION

Let us examine a point disturbance that is at rest in a fluid. *Infinitesimal* pressure pulses are continually being emitted and thus they travel through the medium at *sonic* velocity in the form of spherical wave fronts. To simplify matters we shall keep track of only those pulses that are emitted every second. At the end of 3 seconds the picture will appear as shown in Figure 4.3. Note that the wave fronts are concentric.

Now consider a similar problem in which the disturbance is no longer stationary. Assume that it is moving at a speed less than sonic velocity, say $a/2$. Figure 4.4 shows such a situation at the end of 3 seconds. Note that the wave fronts are no longer concentric. Furthermore, the wave that was emitted at $t = 0$ is always in front of the disturbance itself. *Therefore, any person, object, or fluid particle located upstream will feel the wave fronts pass by and know that the disturbance is coming.*

Next, let the disturbance move at exactly sonic velocity. Figure 4.5 shows this case and you will note that all wave fronts coalesce on the left side and move along with the disturbance. After a long period of time this wave front would approximate a plane indicated by the dashed line. In this case, no region upstream is forewarned of the disturbance as the disturbance arrives at the same time as the wave front.

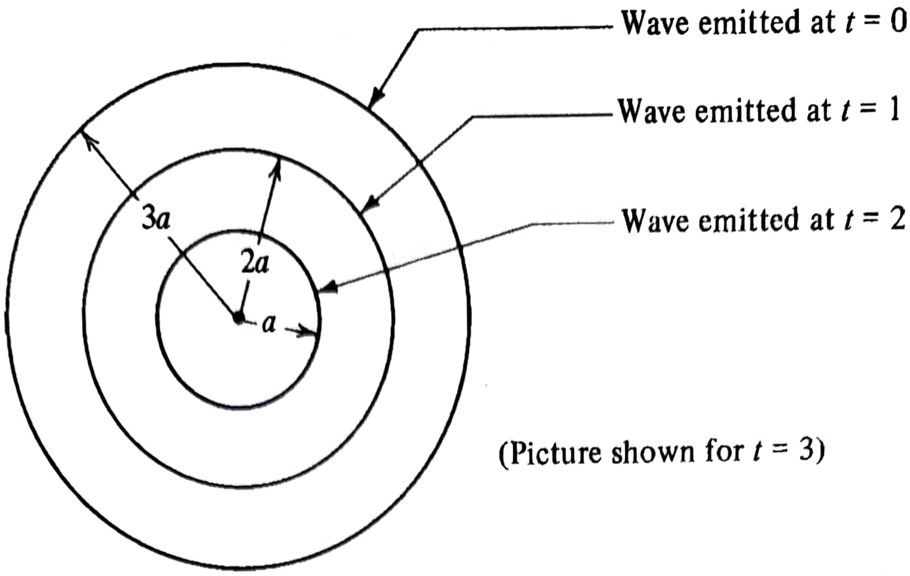


Figure 4.3 Wave fronts from a stationary disturbance. V = 0 , M = 0

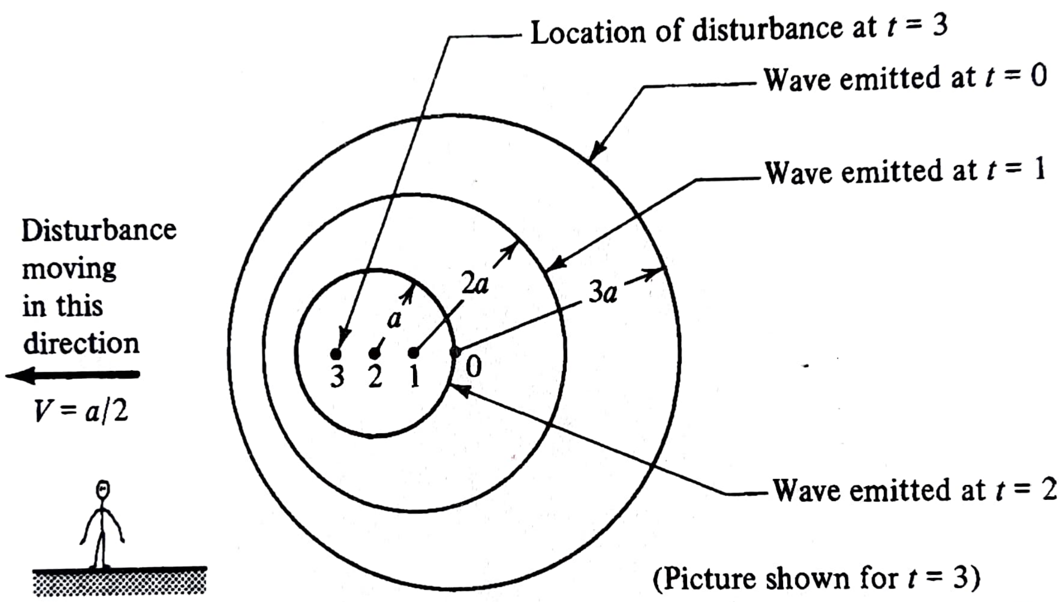


Figure 4.4 Wave fronts from subsonic disturbance. $V = a/2$, $M = \frac{1}{2}$

The only other case to consider is that of a disturbance moving at velocities greater than the speed of sound. Figure 4.6 shows a point disturbance moving at Mach number = 2 (twice sonic velocity). The wave fronts have coalesced to form a cone with the disturbance at the apex. This is called a *Mach cone*. The region inside the cone is called the *zone of action* since it feels the presence of the waves. The outer region is called the *zone of silence*, as *this entire region is unaware of the disturbance*. The surface of the Mach cone is sometimes referred to as a *Mach wave*; the half-angle at the apex is called the *Mach angle* and is given the symbol μ. It should be easy to see that

$$\boxed{\sin \mu = \frac{a}{V} = \frac{1}{M}} \tag{4.12}$$

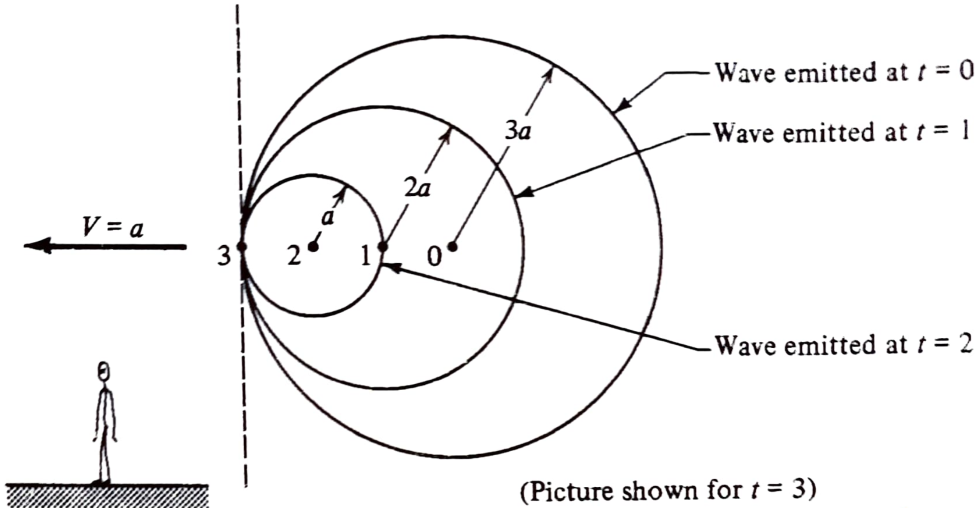


Figure 4.5 Wave fronts from sonic disturbance. M = 1, V = a

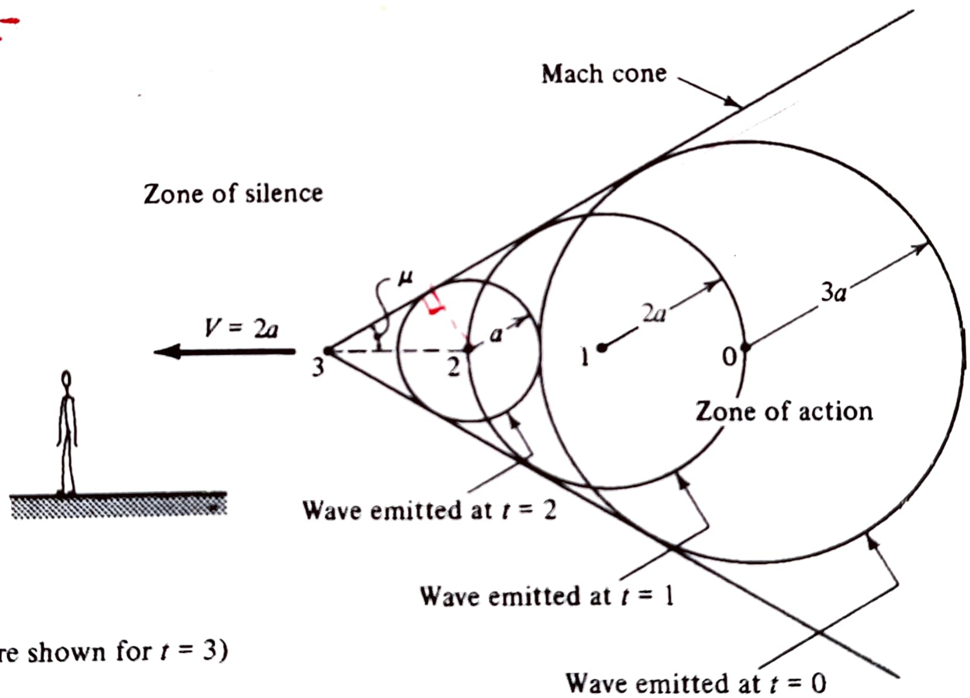


Figure 4.6 Wave fronts from supersonic disturbance. V = 2a, M = 2

In this section we have discovered one of the most significant differences between subsonic and supersonic flow fields. In the subsonic case the fluid can "sense" the presence of an object and smoothly adjust its flow around the object. In supersonic flow this is not possible, and thus flow adjustments occur rather abruptly in the form of shock or expansion waves. We study these in great detail in Chapters 6 through 8.

Reference states
- Stagnation state
- Critical state

Stagnation states & static states:-

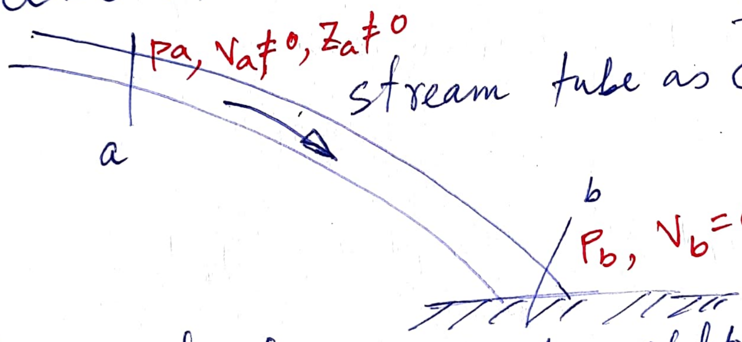
Static properties are those that would be measured if the instrument is ~~you~~ moved with fluid. The word static is usually omitted.

Stagnation state is a reference state which would exist if the fluid were brought to rest i.e. zero velocity & zero potential, isentropically. ($Q = 0$ & $ds = 0$)
i.e. reversible adiabatic process
adiabatic reversible

i.e. $Q = 0$ & $s = \text{const}$.

1st law of thermo

stream tube as C.V. $\frac{\delta Q}{dt} = \frac{\delta W}{dt} + \frac{dE}{dt}$



The fluid at state 'a' is brought to state 'b' isentropically when $V_b = 0$ & $Z_b = 0$.
velo potential

Energy equ^n:-
0 (adiabatic) 0 0 no shaft work

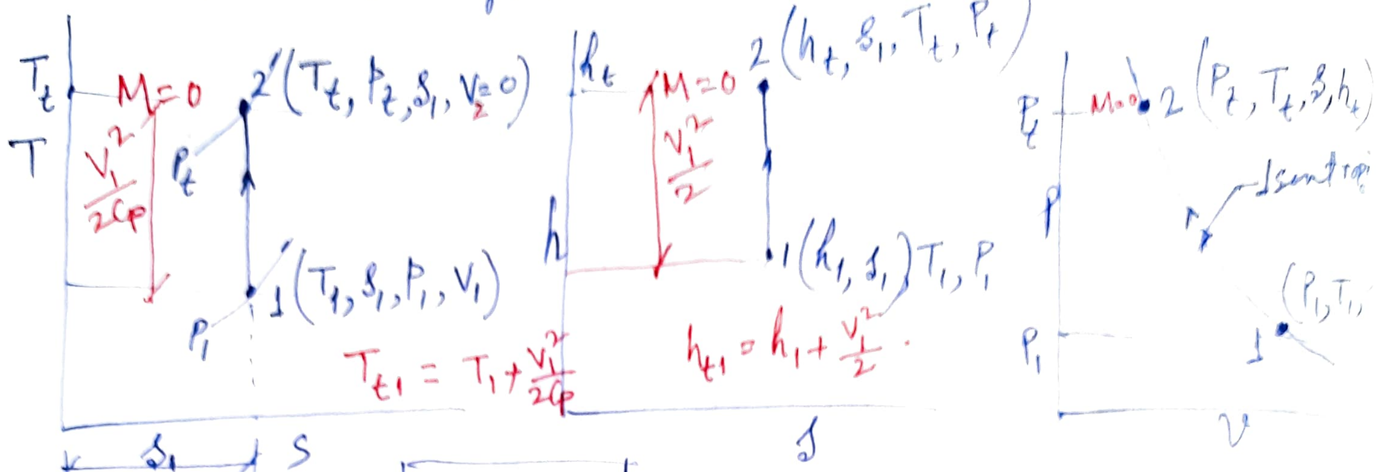
$$h_a + \frac{V_a^2}{2} + gZ_a + \cancel{q} = h_b + \cancel{\frac{V_b^2}{2}} + \cancel{gZ_b} + \cancel{w_s}$$

$$\therefore h_b = h_a + \frac{V_a^2}{2} + gZ_a$$

$$\therefore \boxed{h_{t_1} = h_1 + \frac{V_1^2}{2} + gZ_1} \approx \boxed{h_{t_1} = h_1 + \frac{V_1^2}{2}}$$

where h_t = stagnation enthalpy or Total enthalpy correspondi

h_t = Stagnation enthalpy or total enthalpy corresponding to any state having static properties shown on rhs. This is depicted in T-s, h-s & P-v diagram.



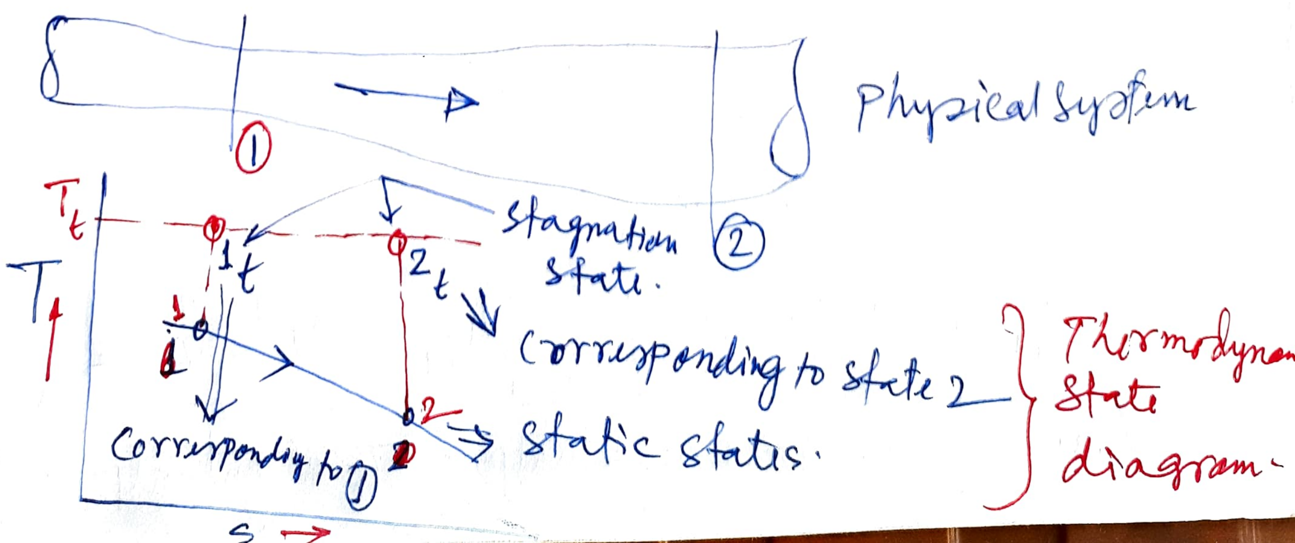
$$\boxed{h = C_pT}$$

From the above diagram state point '2' is the stagnation state corresponding to the static state point '1' of any system.

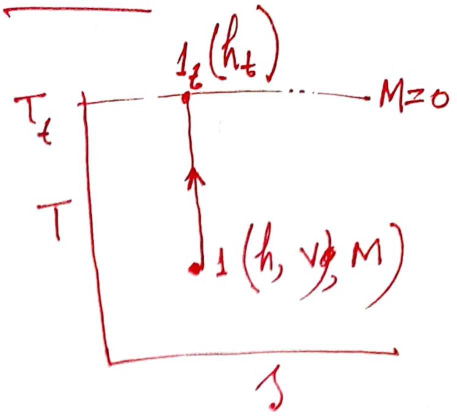
General form of energy equ^n can be written as

$$\underbrace{h_1 + \frac{V_1^2}{2} + gz_1}_{h_{t1}} + q = \underbrace{h_2 + \frac{V_2^2}{2} + gz_2}_{h_{t2}} + w_s$$

$$\therefore \boxed{h_{t1} + q = h_{t2} + w_s}$$



Stagnation Relation :—



$$h_t = h + \frac{V^2}{2} \quad \cdots (1)$$

$$M = \frac{V}{a} \quad \therefore \quad V^2 = M^2 a^2$$

$$V^2 = M^2 \gamma R T =$$

$$\cancel{C_p = \frac{\gamma R}{\gamma - 1}}$$

$$C_p = \frac{\gamma R}{\gamma - 1}, \quad \cancel{V^2 = M^2 \cdot \frac{\gamma R}{\gamma-1} (\gamma - 1) T}$$

$$=$$

From (1) we have :-

$$h_t = h + \frac{V^2}{2}$$

$$= C_p T + \frac{M^2 \gamma R T}{2}$$

$$= h + \frac{M^2}{2} \cdot \left(\frac{\gamma R}{\gamma - 1}\right) \cdot (\gamma - 1) \cdot T$$

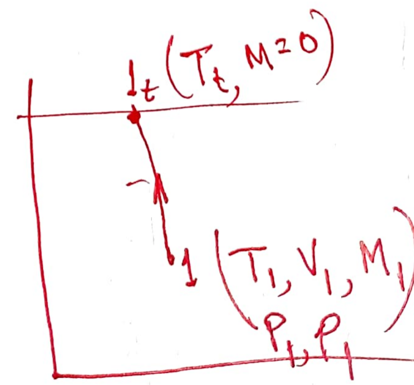
$$= h + \frac{M^2}{2} (\gamma - 1) C_p T$$

$$= h + \frac{M^2}{2} (\gamma - 1) \cdot h \quad [\because h = C_p T]$$

$$\boxed{h_t = h \left(1 + \frac{\gamma - 1}{2} M^2\right)}$$

Putting $h_t = C_p T_t$ & $h = C_p T$ we get :-

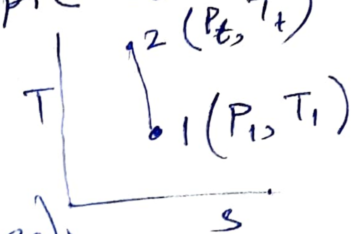
$$\boxed{T_t = T \left(1 + \frac{\gamma - 1}{2} M^2\right)}$$



$$\boxed{\frac{T_t}{T_1} = 1 + \frac{\gamma - 1}{2} M_1^2}$$

29

As stagnation process is isentropic so:



$P v^\gamma = \text{constant}$. (isentropic relatⁿ)

$P v = RT$ (equⁿ of state assuming perfect gas).

$$v = \frac{RT}{P}.$$

$$\therefore \quad P \cdot \left(\frac{RT}{P}\right)^\gamma = \text{const.}$$

$$P^{1-\gamma} \cdot R^\gamma T^\gamma = \text{Const.}$$

$$P_1^{1-\gamma} \cancel{R^\gamma} T_1^\gamma = P_t^{1-\gamma} \cancel{R^\gamma} T_t^\gamma.$$

$$\text{a,} \quad P_t^{\gamma-1} T_1^\gamma = P_1^{\gamma-1} T_t^\gamma$$

$$\left(\frac{P_t}{P_1}\right)^{\gamma-1} = \left(\frac{T_t}{T_1}\right)^\gamma$$

$$\therefore \quad \frac{P_t}{P_1} = \left(\frac{T_t}{T_1}\right)^{\gamma/(\gamma-1)}$$

$$\boxed{\frac{P_t}{P_1} = \left(1 + \frac{\gamma-1}{2} M_1^2\right)^{\gamma/\gamma-1}}$$

Again: $\dfrac{P}{\rho^\gamma} = \text{Constant}.$

$$\frac{P_1}{\rho_1^\gamma} = \frac{P_t}{\rho_t^\gamma}.$$

$$\frac{\rho_t}{\rho_1} = \left(\frac{P_t}{P_1}\right)^{1/\gamma}$$

$$\therefore \quad \boxed{\frac{\rho_t}{\rho_1} = \left(1 + \frac{\gamma-1}{2} M_1^2\right)^{1/(\gamma-1)}}$$

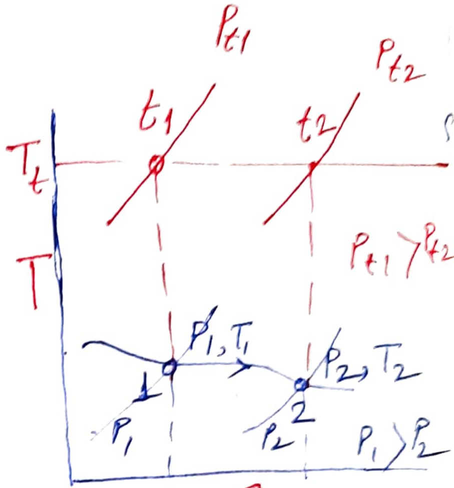
Example: For a steady 1-d flow of a perfect gas with no heat transfer & no shaft (external) work; the energy equ^n :-

$$h_{t1} + \cancel{q}^{\,0} = h_{t2} + \cancel{w_s}^{\,0}$$

$$h_{t1} = h_{t2} = \text{constant.}$$

$$\therefore \boxed{h_t = \text{const.}}$$

$$\therefore \boxed{T_t = \text{constant}}$$



$$S_2 - S_1 = \Delta S_{1-2} = c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1}$$

process from (1) to (2) is some arbitrary process but adiabatic with no ext. work

$$\Delta S_{t1-t_2} = \Delta S_{1-2} = c_p \ln\frac{T_{t2}}{T_{t1}} - R\ln\frac{P_{t2}}{P_{t1}}$$

$$\Delta S = c_p \ln\frac{T_t}{T_t} - R\ln\frac{P_{t2}}{P_{t1}}$$

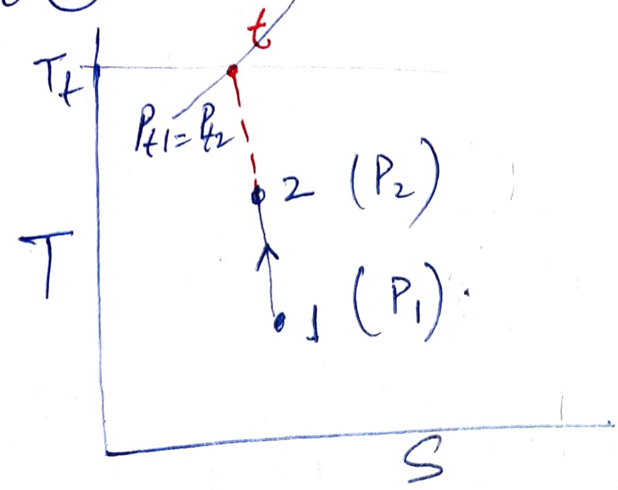
$$\ln\frac{P_{t2}}{P_{t1}} = 0 - \frac{\Delta S}{R}$$

$$\boxed{\frac{P_{t2}}{P_{t1}} = e^{-\frac{\Delta S}{R}}} \Rightarrow \text{For 1-2 is adiabatic process.}$$

$$T\,ds = du + p\,dv$$
$$ds = c_v\frac{dT}{T} + R\frac{dv}{v}$$
$$T\,ds = dh - v\,dp$$
$$ds = c_p\frac{dT}{T} - R\frac{dp}{P}$$

If the process from (1) to (2) is isentropic then

$\Delta S = 0$ & $P_{t2} = P_{t1}$



The * (Critical) Reference Concept

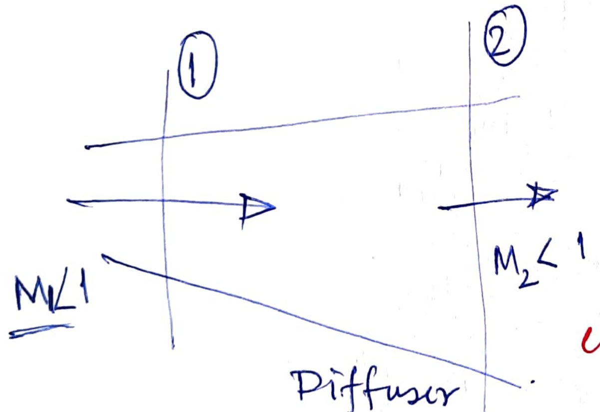
Stagnation state is not a feasible reference state while dealing with area changes (Why?)

The * (critical) state is defined as that thermodynamic state which would exist if the fluid reached a Mach no. of unity by some particular process

① Isentropic * reference state :- Using Isentropic process. (Flow through varying area duct).

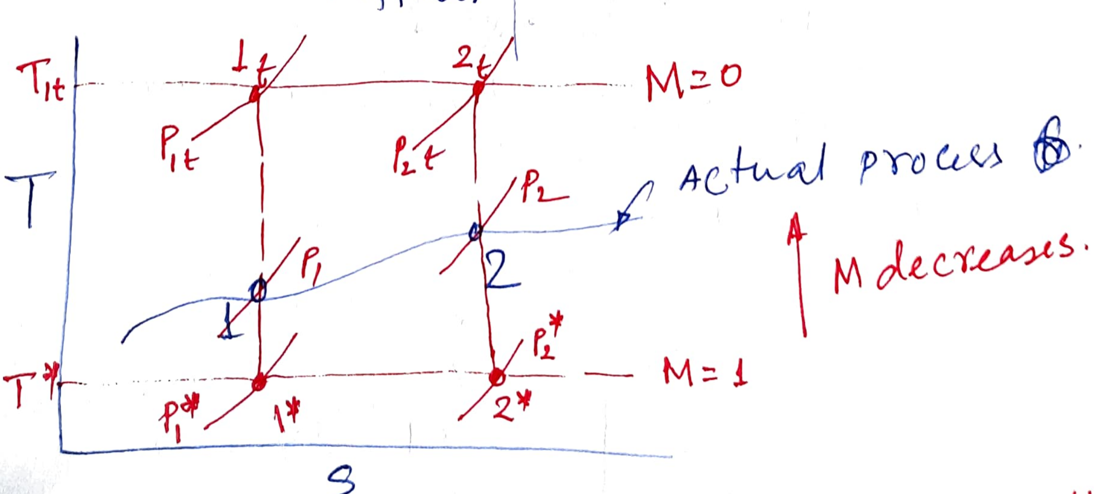
② Rayleigh * reference state :- Using Rayleigh flow (No friction, const. Area but heat transfer)

③ Fanno * reference state :- Using Fanno flow (No heat transfer, const Area but friction considered)



$M_1 > M_2$

with $q=0$ & $w_s=0$



Isentropic * reference state exists with change in Area.

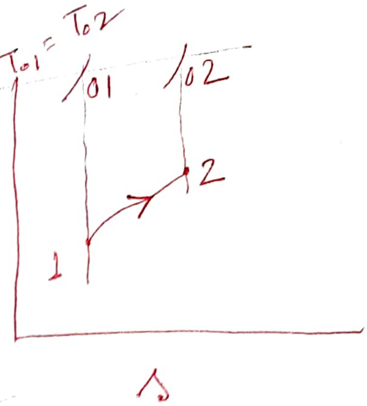
Adiabatic energy equ^n :—

$h_{01} = h_{02}$

or, $h_1 + \frac{V_1^2}{2} = h_{0_1} = h_{02} = h_2 + \frac{V_2^2}{2}$ = constant.

So, $h + \frac{V^2}{2} = h_0$ = const. [for any state bet^n 1-2]

$C_pT + \frac{V^2}{2} = C_pT_0$ —— (1)



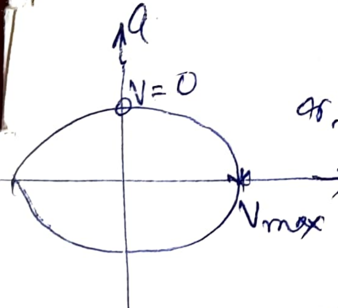
$$\frac{\gamma RT}{\gamma-1} + \frac{V^2}{2} = \frac{\gamma R T_0}{\gamma-1}$$

$$\frac{a^2}{\gamma-1} + \frac{V^2}{2} = \frac{a_0^2}{\gamma-1} \quad \text{(A)}$$

Since all are kinetic form so this is called kinetic form of energy equ^n.

Now if all the thermal energy (C_pT) is ~~extracted~~ converted to K.E. (i.e. $\frac{V^2}{2}$), then K.E. becomes maximum, $\frac{V_{max}^2}{2}$. When the whole thermal energy is extracted, the ~~[illegible]~~ temperature of the fluid becomes zero and hence velocity of sound i.e. ($\sqrt{\gamma RT}$) in the medium becomes also zero. Thus, ~~[illegible]~~ putting $a=0$ & $V=V_{max}$, we get,

$$\frac{a^2}{\gamma-1} + \frac{V^2}{2} = \frac{a_0^2}{\gamma-1}$$



or, $\frac{0^2}{\gamma-1} + \frac{V_{max}^2}{2} = \text{const} = \frac{a_0^2}{\gamma-1} = h_0$.

$\therefore \frac{V_{max}^2}{2} = h_0 \Rightarrow V_{max} = \sqrt{2C_pT_0}$.

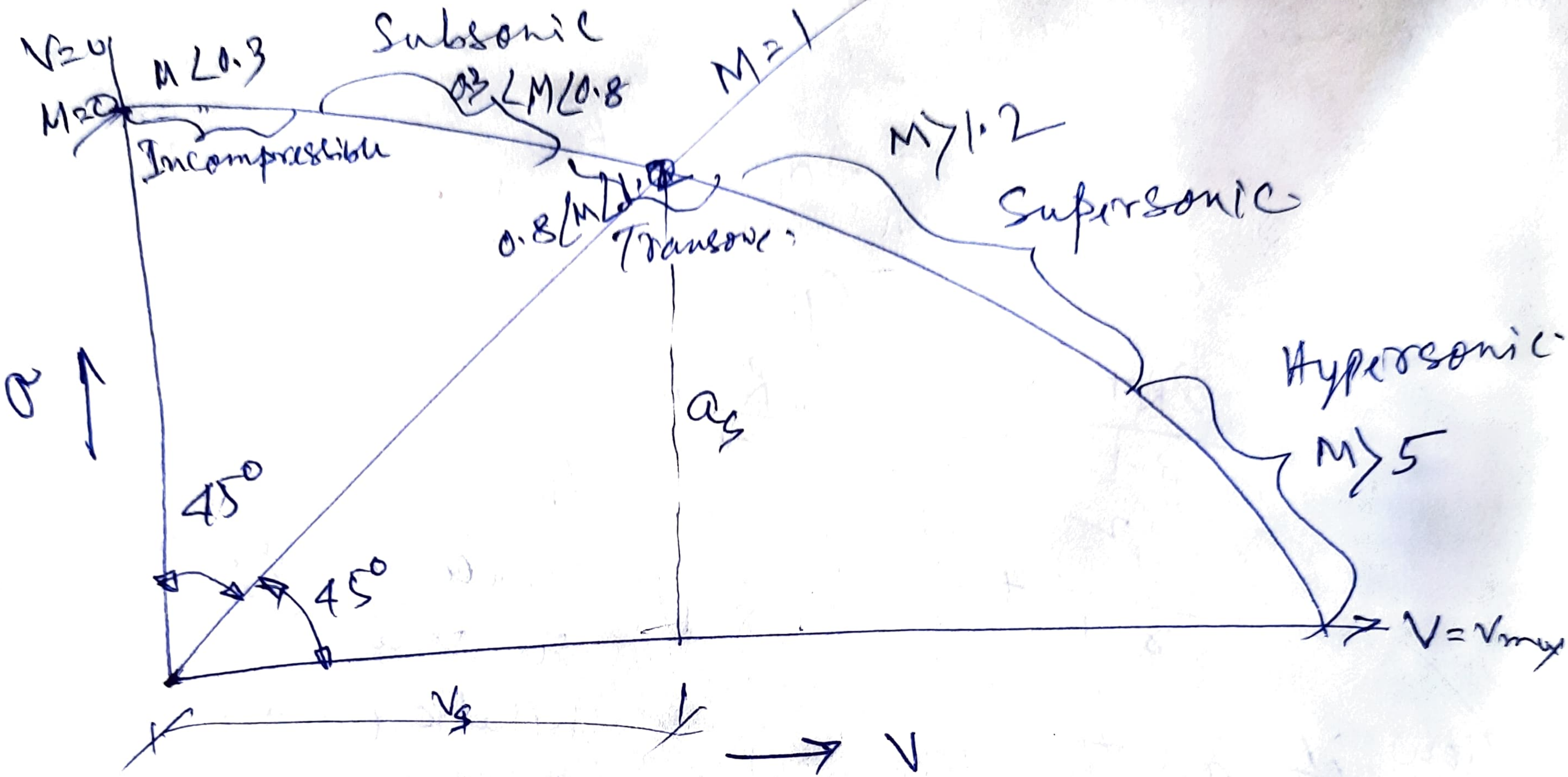
Now, Equation (A) can be written as —

$$\frac{a^2}{\gamma-1} + \frac{V^2}{2} = \left(\frac{V_{max}^2}{2}\right) = \frac{a_0^2}{\gamma-1} = h_0 = \text{Const}$$

--- (B)

Since Eq^n (B) is the equation of ellipse with a & V as the coordinates and only first quad is obtained as both a & V are positive, thus the first quadrant of ellipse is called Prandtl velocity ellipse

$M = \frac{V}{a}$



V=0
M=0
M<0.3
Incompressible
Subsonic
0.3<M<0.8
M=1
0.8<M<1.2
Transonic
M>1.2
Supersonic
Hypersonic
M>5
45°
45°
a_s
V_s
V
V=Vmax