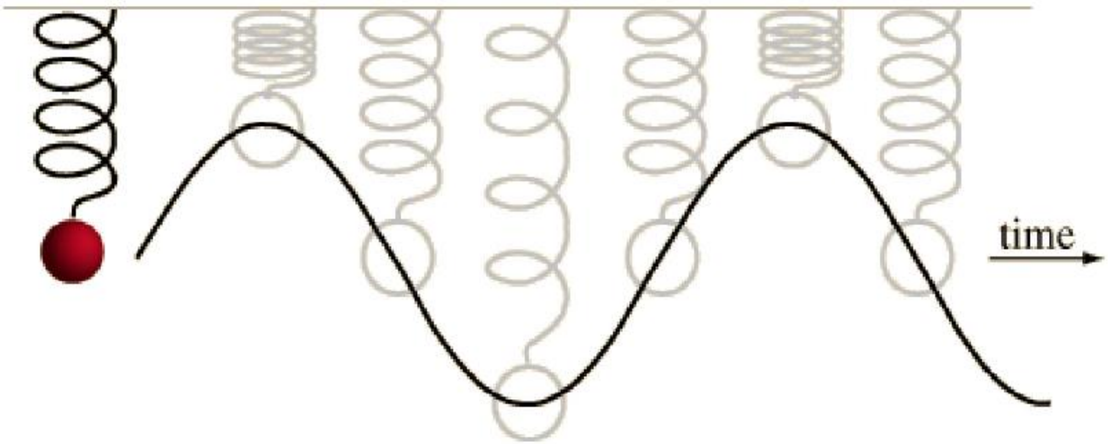
Free Vibration/ Damped Vibration:

The equation of motion for SHM

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$x(t) = A\cos(\omega t + \phi)$$

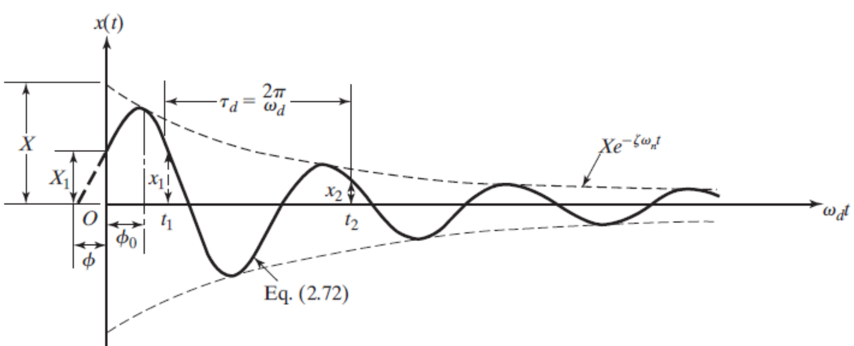
Simple Harmonic Motion



$$F_s = -kx$$

Damped Vibration/Free vibration

Figure 1. In order to counteract dampening forces, this dad needs to keep pushing the swing. (credit: Erik A. Johnson, Flickr)



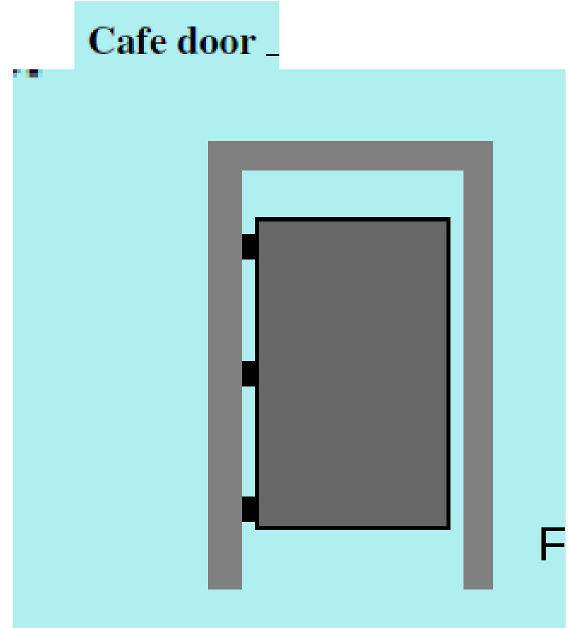


Figure. 2

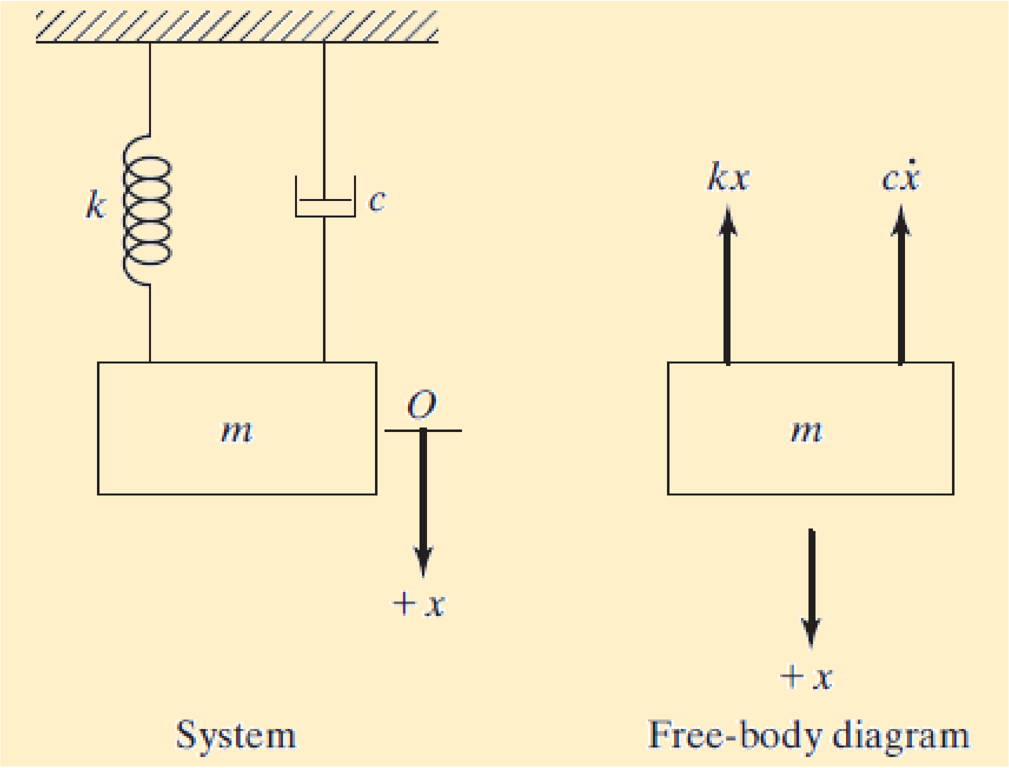
A cafe door on three hinges with damper in the lower hinge

Figure 3. Door Closure: over damped vibration

Damped Vibration

- Single degree of freedom system with viscous damping:
- Degree of Freedom : **The minimum number of independent coordinates required to determine completely the position of all parts of a system at any** instant of time defines the degree of freedom of the system.
- System free body diagram

C is the damping constant or coefficient



Mathematical model (governing equation of motion):

$$m\ddot{x} = -kx - C\dot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$c\dot{x} = F_d$ is the damping force.

It is proportional to velocity

Solution:

Assume $x(t) = B\,e^{St}$, substitute into the equation of motion:

$$ms^2 + cs + k = 0 \quad \Rightarrow \quad s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Define $C = 2\zeta m\omega_n$, then the roots of the solution can be written as:

$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$, where ζ is known as the *damping ratio*.

$\zeta = \dfrac{C}{C_c}$, where C_c is the *critical damping coefficient*

$C_c = 2m\omega_n$. In the face of the value of the square root shown above three possible solutions are exist. These solutions are as follows:

Solution # 1: under-damped vibration:

When ζ<1

The roots $S_{1,2}$ of the characteristic equation can now be written as:

$$s_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\omega_n$$

The solution for this case becomes:

$$x(t) = e^{-\zeta\omega_n t}\left\{B_1\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + B_2\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

OR $$x(t) = Xe^{-\xi\omega_n t}\left\{\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi\right)\right\}$$

Define $\omega_{\mathrm{d}} = \sqrt{1-\zeta^2}\,\omega_n$ as the *damped natural frequency*, then

$$x(t) = Ae^{-\zeta\omega_n t}\left\{\cos\left(\omega_{\mathrm{d}} t - \phi\right)\right\}$$

The solution (above) for the under-damped vibration case (ζ<1) consists of a harmonic motion of frequency ω_d and an amplitude

$$Ae^{-\zeta\omega_n t}$$

Note that:

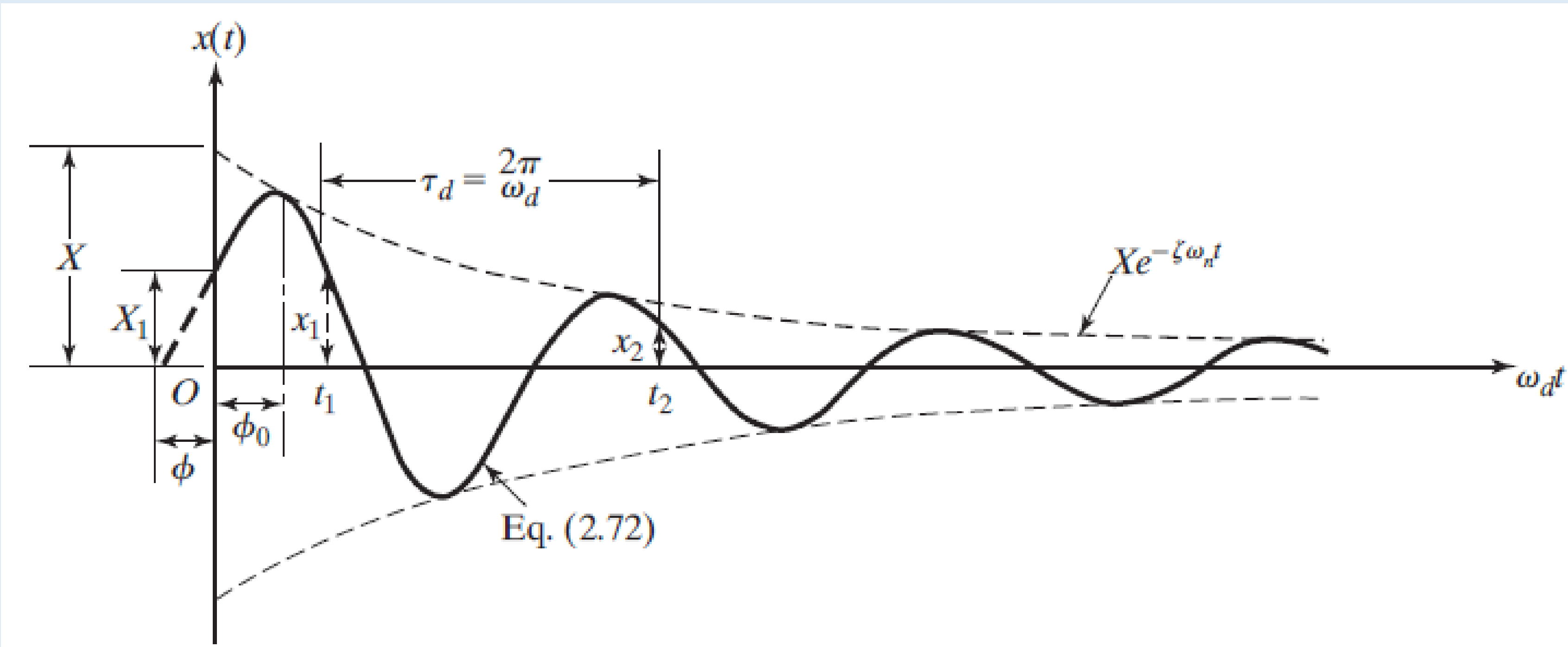
$$A = \sqrt{B_1^2 + B_2^2} \quad \text{and } \phi = \tan^{-1}\left[\frac{B_2}{B_1}\right]$$

is the phase angle. A and ϕ can be determined from the inatial conditions of the motion.

The amplitde for this case is exponantially decaying with time as shown.

Frequency of damped vibration:

$$\omega_d = \sqrt{1-\zeta^2}\ \omega_n$$



Under-damped vibration

Critical damping (c_c) (ζ=1)

The critical damping c_c is defined as the value of the damping constant c for which the radical in s-equation becomes zero:

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \Rightarrow c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

Solution # 2: critically damped vibration (ζ=1):

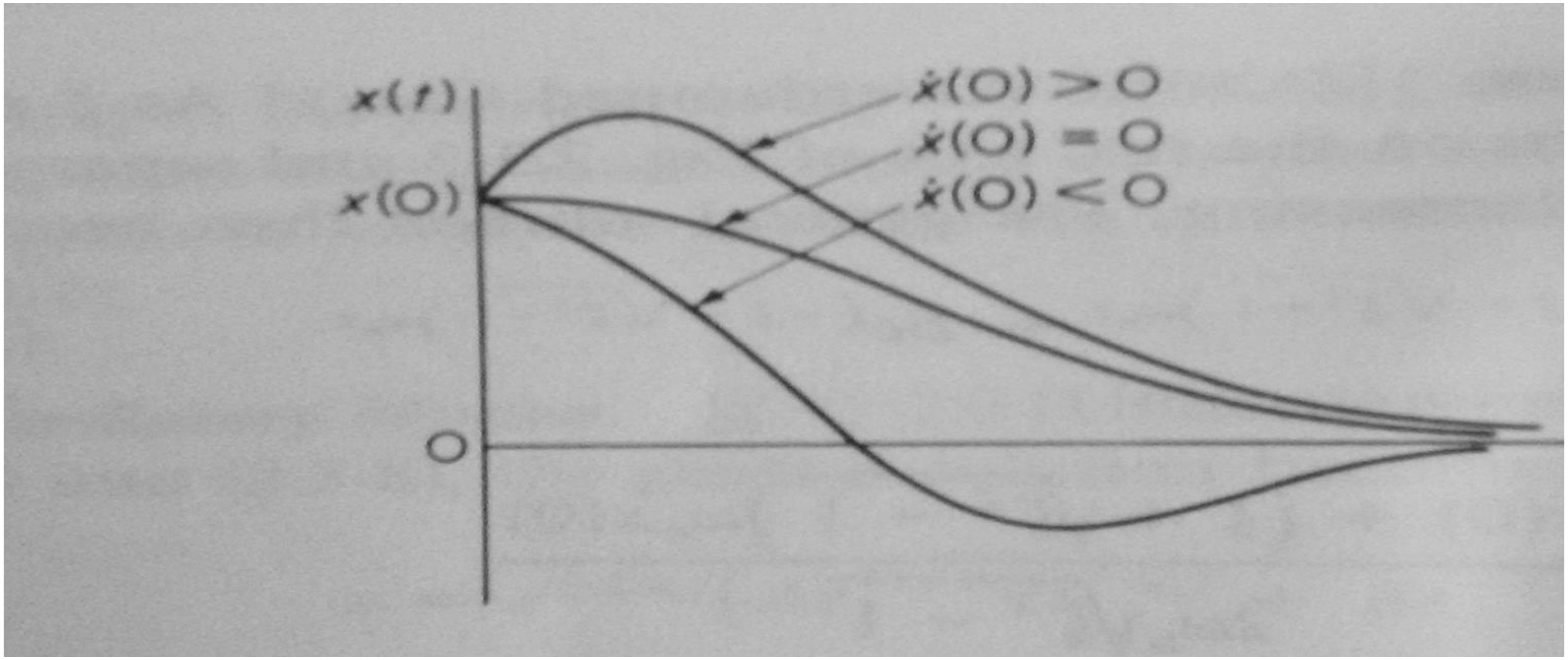
For this case the roots of the characteristic equation become:

$$s_{1,2} = -\omega_n$$

Therefor the solution can be written as:

$$x(t) = e^{-\omega_n t}\{B_1 + B_2 t\}$$

B_1 and B_2 are constants that can be determined from inatial conditions. The motion is no longer harmonic as shown in the figure.



-The system returns to the equilibrium position in short time
-The shape of the curve depends on initial conditions as shown
-The moving parts of many electrical meters and instruments are critically damped to avoid overshoot and oscillations .

Solution # 3: over-damped vibration (ζ>1)

For this case the roots of the characteristic equation become:

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

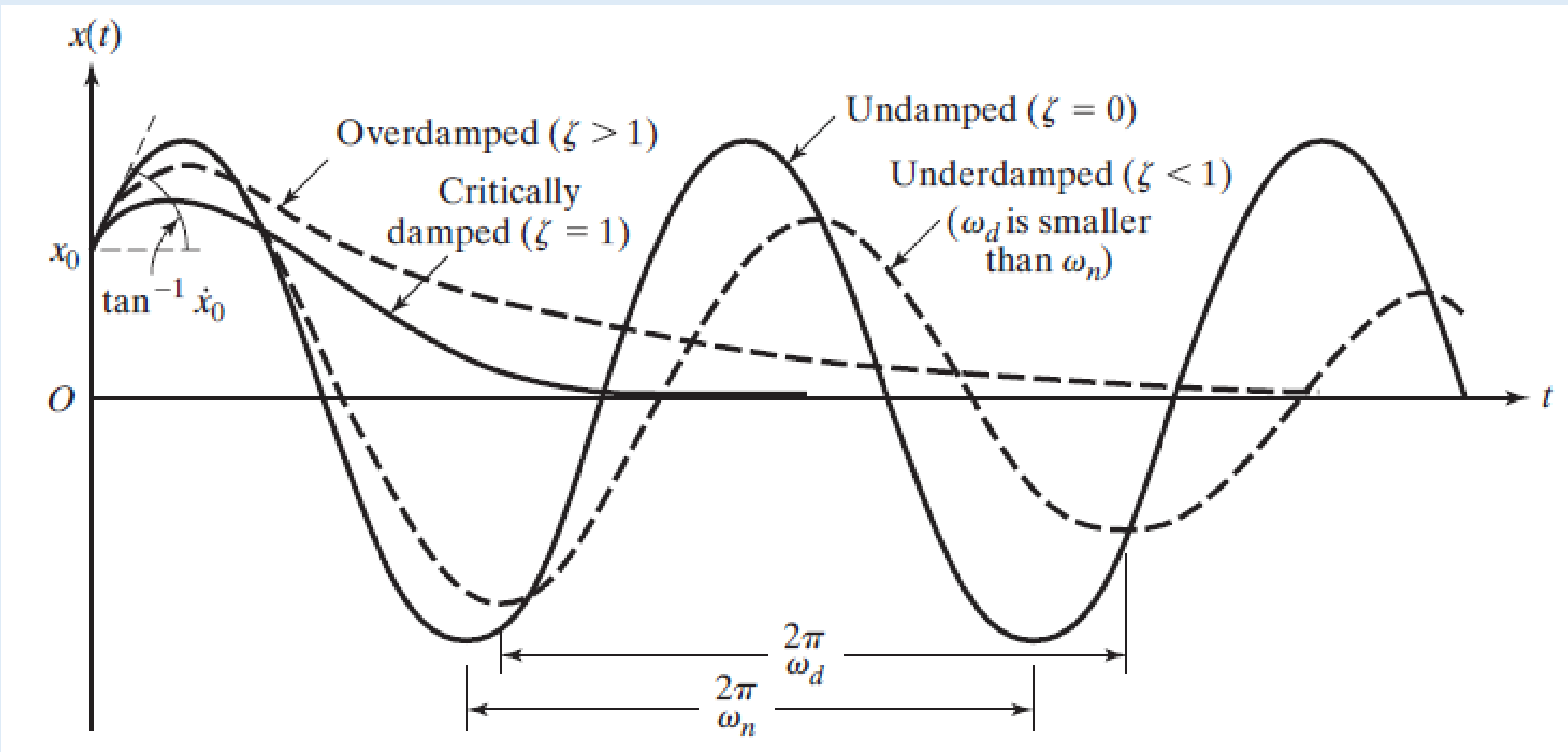
Therefore, the solution can be written as,

$$x(t) = B_1 e^{-\left(\varsigma + \sqrt{\varsigma^2 - 1}\right)\omega_n t} + B_2 e^{-\left(\varsigma - \sqrt{\varsigma^2 - 1}\right)\omega_n t}$$

B_1 and B_2 are constants to be determined from knowing the initial conditions of the motion.

Classification	Physical Meaning
Overdamped	The door closes slowly without oscillations.
Critically damped	The door closes without oscillations, in the least amount of time.
Underdamped	The door oscillates through the jamb position many times with decreasing amplitudes.

Graphical representation of the motions of the damped systems



Single DoF free vibration system

Relaxation time

This is defined as the time for the amplitude to decay to e− 1 ≈ 0.368 of its original value.

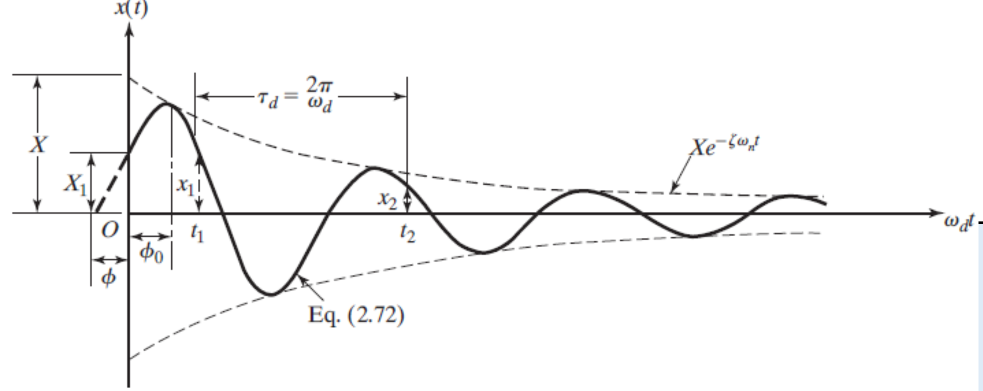
Amplitude decay as $Ae^{-\zeta\omega_n t}$

$$Ae^{-1} = Ae^{-\zeta\,\omega_n\tau_r} \Longrightarrow \tau_r = \frac{1}{\zeta\,\omega_n} = \frac{c}{2m}$$

Logarithmic decrement

The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes.

Logarithmic decrement:



$$\frac{x_1(t)}{x_2(t)} = \frac{Ae^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi)}{Ae^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi)}$$

But

$$t_2 = t_1 + \tau_d \Rightarrow \tau_d = \frac{2\pi}{\omega_d}$$

$$\cos(\omega_d t_2 - \phi) = \cos(2\pi + \omega_d t_1 - \phi) = \cos(\omega_d t_1 - \phi)$$

So

$$\frac{x_1(t)}{x_2(t)} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d} = \frac{x_2(t)}{x_3(t)}$$

Assume: δ is the logarithmic decrement

$$\delta = \ln\frac{x_1(t)}{x_2(t)} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Logarithmic decrement: is dimensionless

Logarithmic decrement

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

For small damping ; ζ <<1

$$\delta \approx 2\pi\zeta$$

Generally, when the amplitude after a number of cycles "n" is known, logarithmic decrement can be obtained as follows:

$$\delta = \frac{1}{n}\ln\frac{x_1(t)}{x_n(t)} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

where x_1(t) is the first known amplitude, x_n(t) is the other known amplitude after n cycles of the decayed motion.

Energy Relations:

- Let x is the displacement of the particles execution damped vibration at a time t.

Then potential energy : $E_p = \int_0^x kxdx = \frac{1}{2}kx^2$

The instantaneous kinetic energy is

$$E_k = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

The total Energy $E = E_k + E_p = \frac{1}{2}kx^2 + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$

The instantaneous dissipation of energy E due to resisting force $c\frac{dx}{dt}$ is the product of this force and the velocity. Thus

$$-\frac{dE}{dt} = c\left(\frac{dx}{dt}\right)^2 \quad or, -\left[kx\frac{dx}{dt} + m.\frac{dx}{dt}.\frac{d^2x}{dt^2}\right] = c\left(\frac{dx}{dt}\right)^2$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Starting from the displacement x_1 on one side of swing the particle after t has the maximum displacement x_2 (say) on the other side

Since both are the maximum position, the energy is entirely potential energy

Thus, the loss of energy in one time period is

$$\Delta E = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 = \frac{1}{2}kx_1^2\left(1 - \frac{x_2^2}{x_1^2}\right)$$

$$\frac{x_2}{x_1} = e^{-\varsigma\omega_n\tau_d}$$

$$or, \frac{\Delta E}{E} = (1 - e^{-2\varsigma\omega_n\tau_d}) = 2\varsigma\omega_n\tau_d$$

If $\varsigma\omega_n\tau_d$ Is very small

Quality factor Q or figure of merit is defined as

$$Q = 2\pi\frac{maximum\ energy\ stored\ per\ cycle}{energy\ lost\ or\ dissipated\ per\ cycle} = 2\pi\frac{E}{\Delta E} = \frac{\pi}{\varsigma\omega_n\tau_d}$$

Single DoF free vibration system

Example

An under-damped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig.(a)).When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig.(b).

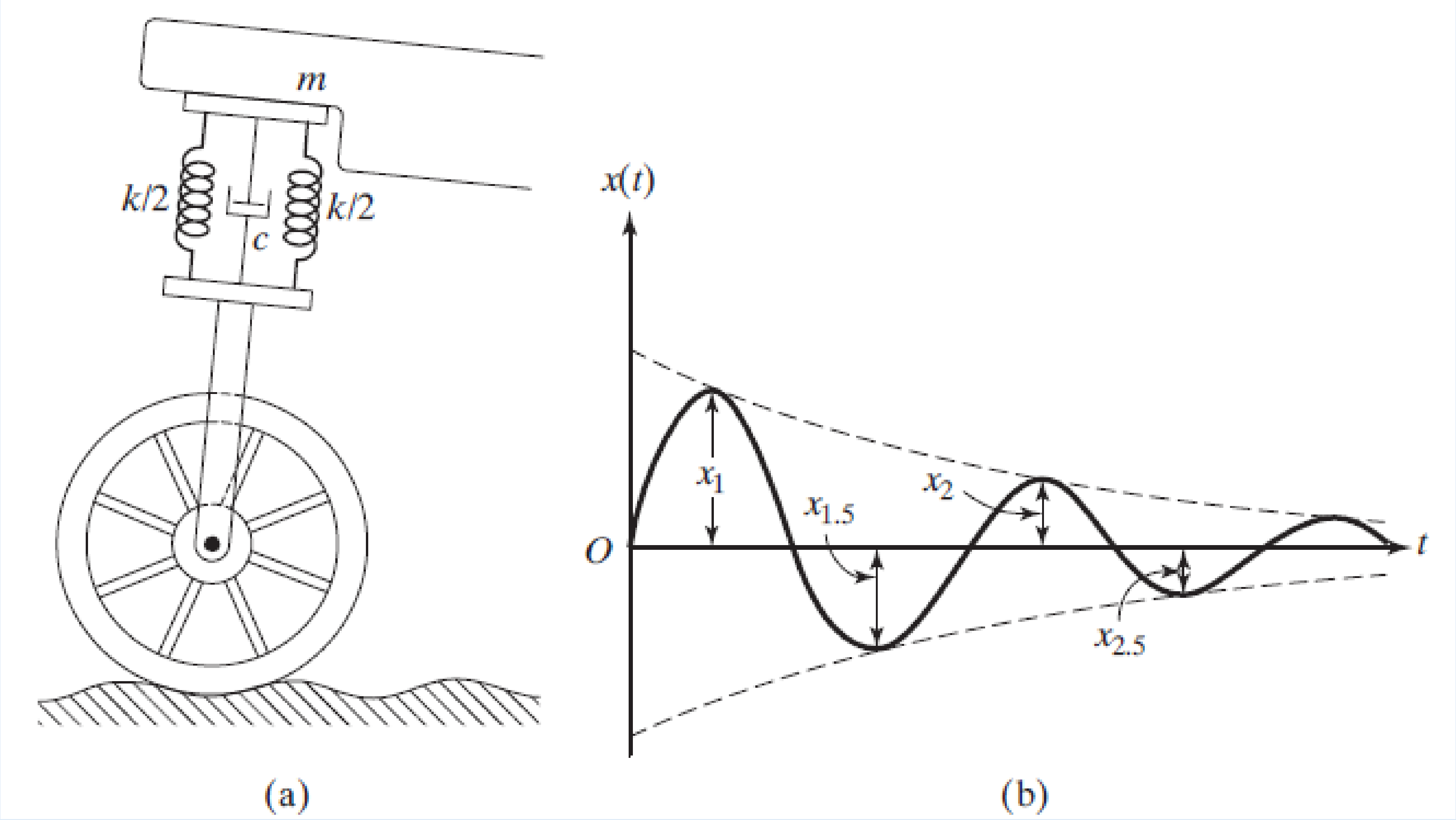
Requirements:

1. Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2s and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e. $x_{1.5} = x_1/4$).

Single DoF free vibration system

Example

Note that this system is under-damped system



Single DoF free vibration system

Solution:

Finding k and c

$$\text{Here, } n = 0.5 \text{ and } \frac{x_1}{x_n} = 4, \text{ then the logarithmi c decrement is :}$$

$$\delta = \frac{1}{n}\ln\frac{x_1}{x_n} = \frac{1}{0.5}\ln(4) = 2.7726 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow \zeta = 0.4037$$

$$\tau_d = 2 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow \omega_n = \frac{2\pi}{2\sqrt{1-0.4037^2}} = 3.4338\, rad/s$$

Single DoF free vibration system

The critical damping can be found as:

$$c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54 \text{ N.sec/m}$$

Damping coefficient C and stiffness K can be found as:

$$c = \zeta\, c_c = (0.4037)(1373.54) = 554.4981 \text{ N.sec/m.......(Ans.)}$$

$$k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652 \text{ N/m........(Ans.)}$$

Single DoF free vibration system

End of chapter