Multiple Integral:

16 December 2021 08:39

Line integral

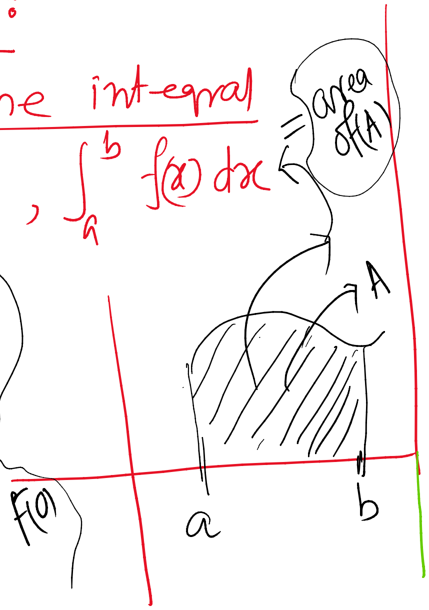
$f \geq 0$, $\int_a^b f(x)\,dx$ = area of (A)

$\int_0^\pi \sin x\,dx$, $f(x)$

$F(x)$, $F'(x) = f(x)$

$F(x)\Big|_0^\pi$

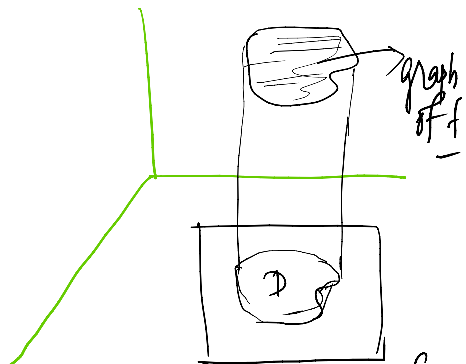
$F(\pi) - F(0)$



Double integral

$f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$, $R = [a,b]\times[c,d]$ $\boxed{f \geq 0}$

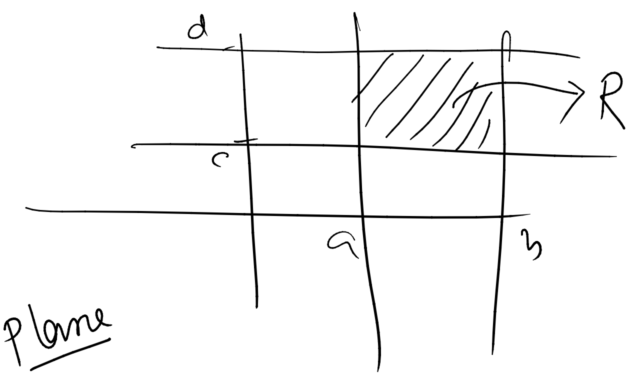
Rectangle (Region of integration)



The volume under f and above D equals

$\mathbb{R}$

$R = [a,b]\times[c,d]$



$f : \mathbb{R} \to \mathbb{R}$

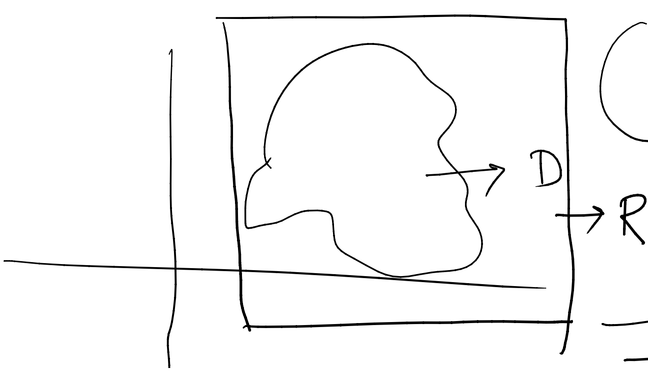
$Grf \subseteq \mathbb{R}^2$ plane

$$\text{Graph } f = \{(x, f(x)) : x \in A\} \subseteq A \times B$$

$f : A \to B$

$f : \mathbb{R}^2 \to \mathbb{R}$

$Grf \subseteq \mathbb{R}^3$



$f : R \to \mathbb{R}$ bounded

$f : D (\subseteq \mathbb{R}^2) \to \mathbb{R}$

$f \geq 0$

$\exists\ R \subset \mathbb{R}^2$ s.t.

$\exists\ R \subseteq \mathbb{R}^2$ s.t.

$R \supseteq D$

(Note that the volume contribution of $\tilde{f}$ and f are same)

$\tilde{f} : R \to \mathbb{R}^2$ as

$\tilde{f} = f, \ (x,y) \in D$

$= 0$ oth

$$\iint_R \tilde{f}\, dA = \iint_D f\, dx\, dy$$

Ex 1

$$\iint_D xy^2\, dx\, dy, \quad \text{where } D = [0,2]\times[0,1]$$

cont.

$$= \int_0^2 \left\{ \int_0^1 xy^2\, dy \right\} dx$$

$$= \int_0^2 \left[\frac{xy^3}{3} \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \left[\frac{x}{3} - 0 \right] dx = \frac{x^2}{6}\Big|_0^2$$

$$= 2/3$$

$$= \int_0^1 \left\{ \int_0^2 xy^2\, dx \right\} dy$$

$$= \int_0^1 \frac{x^2y^2}{2}\Big|_{x=0}^{x=2} dy$$

$$= \int_0^1 (2y^2 - 0)\, dy$$

$$= 2/3$$

Remark:

$$\int_a^b \left\{ \int_c^d f(x,y)\, dx \right\} dy$$

$\int_a^b \int_c^d f \, dx \, dy$ → Repeated int.

$R = [c,d] \times [a,b]$

$\int_a^b \int_c^d f \, dy \, dx$ → Repeated Integral

double int. $\left\{ \begin{array}{l} \iint f \, dx \, dy \\ \parallel \\ \iint f \, dy \, dx \end{array} \right.$

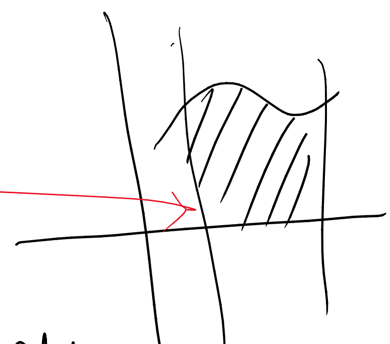
Result – If f is cont. on a rectangle R, then it is integrable on R.

$\mathbb{R}$ Set of all real no.

$R \to$ Rectangle $= [a,b] \times [c,d]$

Note $\Longrightarrow$ $\int_a^b f(x)\, dx$

$f \geq 0$



if $f < 0$, we consider

$f = -(-f)$ > 0 and we can apply our definition for $(-f)$

Similarly, if $f(x,y) < 0$ for some $(x,y) \in D$, then we consider $-f(x,y)$ and apply it as

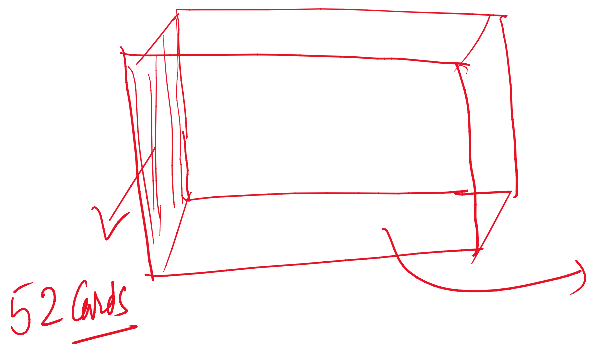
$f = -(-f)$ > 0.

$f = k \;\; \forall\, x,y \in R = [a,b] \times [c,d]$

Properties:

If $f = k, \forall x,y \in R = [a,b]\times[c,d]$

$$\iint_R f \, dx\,dy = k \text{ area of } (R) = k\cdot(b-a)\cdot(d-c)$$



Volume

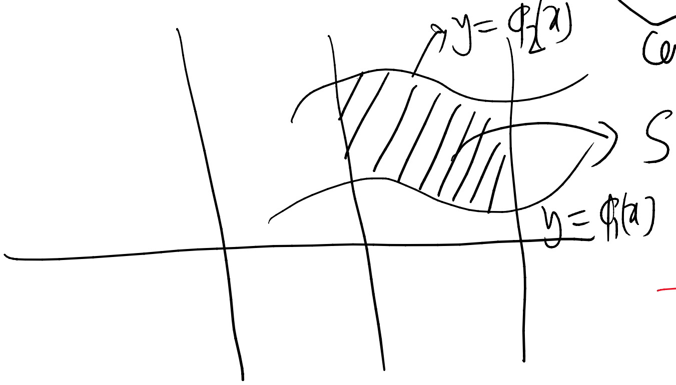
$R \to [a,b]\times[c,d]$

$D \subseteq \mathbb{R}^2$

Different types of Regions:

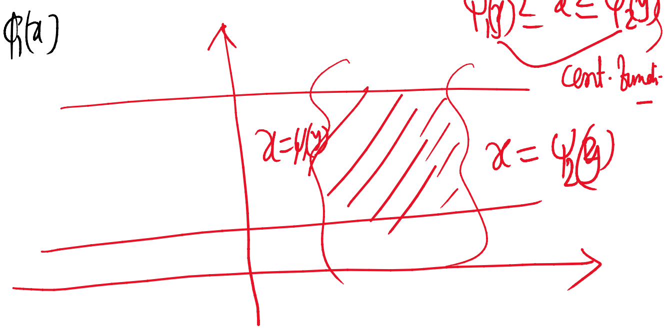
Type-I **Type-II** **Type-III**

$S = \{(x,y) : a \le x \le b, \ \underbrace{\phi_1(x) \le y \le \phi_2(x)}_{\text{Cont. functions}}\}$

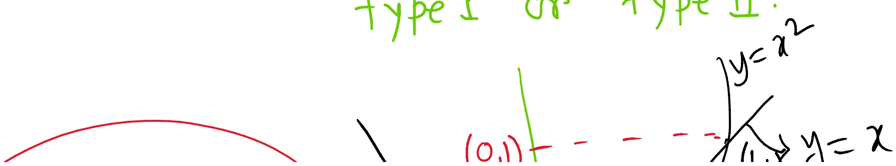


$T = \{(x,y) \in \mathbb{R}^2 : c \le y \le d$

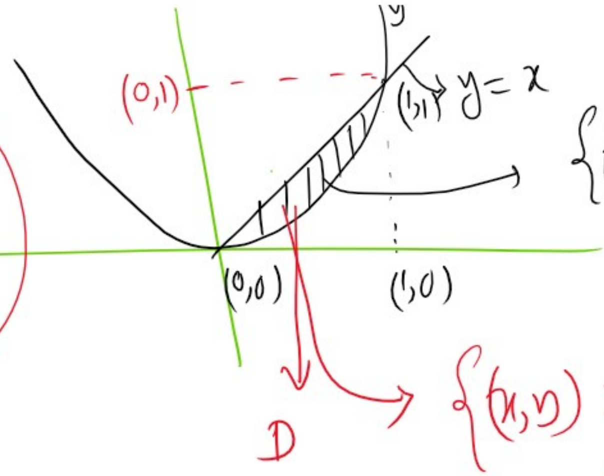
$\underbrace{\psi_1(y) \le x \le \psi_2(y)}_{\text{Cont. funct.}}\}$



Type III: If it can be written as either type I or type II.



$$\int_0^1 \left\{ \int_{x^2}^{x} \frac{xy^3}{5}\, dy \right\} dx$$



$\{(x,y) : 0 \le x \le 1,\ x^2 \le y \le x\}$

$\{(x,y) : 0 \le y \le 1,\ y \le x \le \sqrt{y}\}$

→ Ex. Evaluate $\displaystyle\iint_D \frac{x y^3}{5}\, dx\, dy$.

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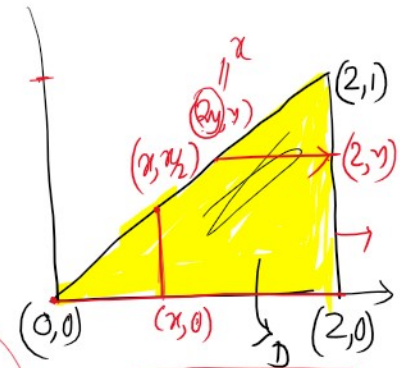
Ex 1. Evaluate $\displaystyle\iint_D x e^{2y}\, dx\, dy$ Cont.



$\boxed{\text{Type-III}}$

$$= \int_0^2 \int_0^{x/2} x e^{2y}\, dy\, dx$$

or

$$= \int_0^1 \int_{2y}^{2} \left(x \cdot e^{2y} \right) dx\, dy$$

→ H.Ex

- Fubini's Theorem: If f is integrable on $R = [a,b] \times [c,d]$, then either of the iterated integrals (if it exists), equals the double integral

$$\iint_R f\, dx\, dy.$$

$\iint\limits_R$

Cor 1. If f is cont. on R, then both the iterated integrals exist and they are same.

Cor 2. In Fubini's theorem, if both the ~~int~~ iterated integrals ~~exist~~ then they are same.

Consider x^y on $[0,1]\times[0,1]$

$$\int_0^1\int_0^1 x^y\,dx\,dy \quad \text{or} \quad \int_0^1\int_0^1 x^y\,dy\,dx$$

exist

Note that
$$\int_0^1\int_0^1 x^y\,dx\,dy$$
is easy to calculate and $= \ln 2$.

Other iterated integral
$$\int_0^1\int_0^1 x^y\,dx\,dy$$
is difficult to find but it will also be same as $\ln 2$.

Remark 1. Both iterated integrals exist but double integral may not exist $\Big] \to$ this doesn't contradict Fubini's theorem

Example 2. $\int^1\int^1 \frac{y-x}{\quad}{}_{3}\,dx\,dy \;\Big|\; \int^1\int^1 \frac{y-x}{}$

Example 2.

$$\int_0^1 \int_0^1 \frac{y-x}{(2-x-y)^3}\,dy\,dx \neq \int_0^1 \int_0^1 \frac{y-x}{(2-x-y)^3}\,dx\,dy$$

(Try to show it)

f is not integrable on $R=[0,1]\times[0,1]$

Ex3. $$f = \frac{x^2-y^2}{(x^2+y^2)^2}$$ and note that

$$\frac{x^2-y^2}{(x^2+y^2)^2} = -\frac{\partial^2}{\partial x \partial y}\tan^{-1}(y/x)$$

$$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \pi/4$$

$\neq$

and $$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy = -\pi/4$$

$f\left(= \dfrac{x^2-y^2}{(x^2+y^2)^2}\right)$ is not integrable on $R=[0,1]\times[0,1]$.

Remark 2. One iterated integral exists but not the other and also double integral does not exist.

Construct a supportive example:—

$f(x,y) = \begin{cases} 1/2, & y \to \text{rational} \\ \ldots \end{cases}$ on $R=[0,1]\times[0,1]$

$$f(x,y) = \begin{cases} \frac{1}{2}, & y \to \text{rational} \\ x, & y = \text{irrational} \end{cases} \quad \text{on } R = [0,1] \times [0,1]$$

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = \frac{1}{2} \qquad \left| \qquad \int_0^1 \left[\int_0^1 f\, dy \right] dx \right.$$

$$\iint_R f\, dx\, dy$$

don't exist

Question: Can you construct an example where both the ~~int~~ iterated integrals exist and same but double integral fails to exist?

$\boxed{\text{Yes, it is possible.}}$

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

∞ near $(0,0)$

$R = [-1,1] \times [-1,1]$

Note f is discont at only $(0,0)$ but it is unbounded near the origin.

$$\int_{-1}^{1} \int_{-1}^{1} f\, dx\, dy$$

$$-\frac{x}{2(x^2+y^2)} \quad \text{for fixed } x$$

$$\int_{-1}^{1} \int_{-1}^{1} f\, dy\, dx = -\int_{-1}^{1} \frac{x}{2(x^2+y^2)}\, dx = 0$$

odd

$$\frac{y}{2(x^2+y^2)} \qquad \int_{-1}^{1} (\,)\, dy = 0 \quad \text{odd}$$

Similarly for the other iterated integral $= 0$

Hence double integral does not exist.

Hence double integral does not exist.

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CR.

F.S.

M.I

I.C

Jan'22

CA-I → 15
CA → II → 25
~~Mid~~ End term Exam → 60

- CA-I (15 marks, mainly MCQ type Questions) $\xrightarrow{\text{On-line}}$ $\boxed{10/1/22 \text{ to } 15/1/22}$
- CA-II (Mid-term Exam) 25 marks (On-line) [14/2/22 to 19/2/22]
- End term Exam (60 marks - on-line) → 21/3/22 - 26/3/22

$\boxed{\text{Total 100 marks}}$