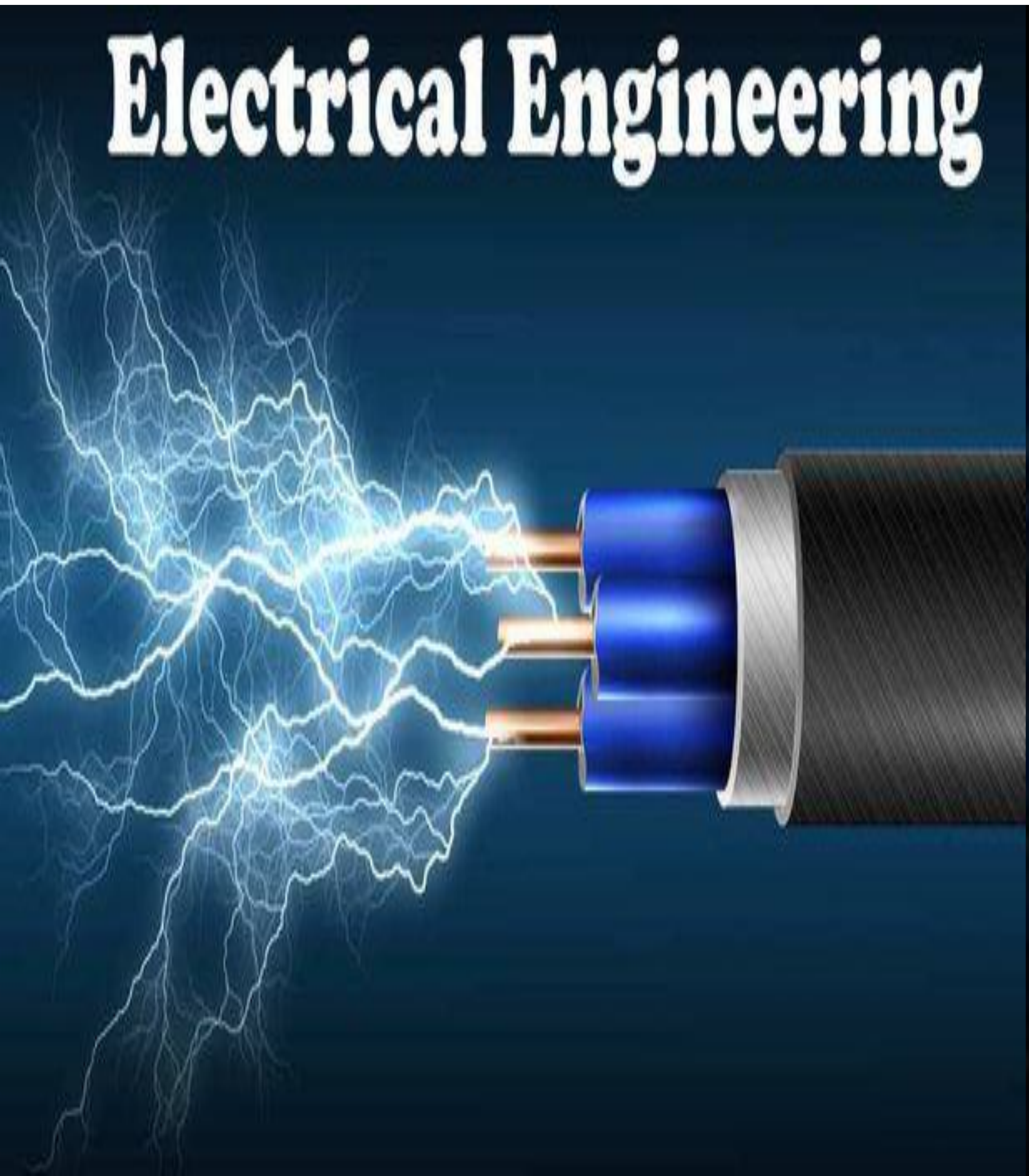
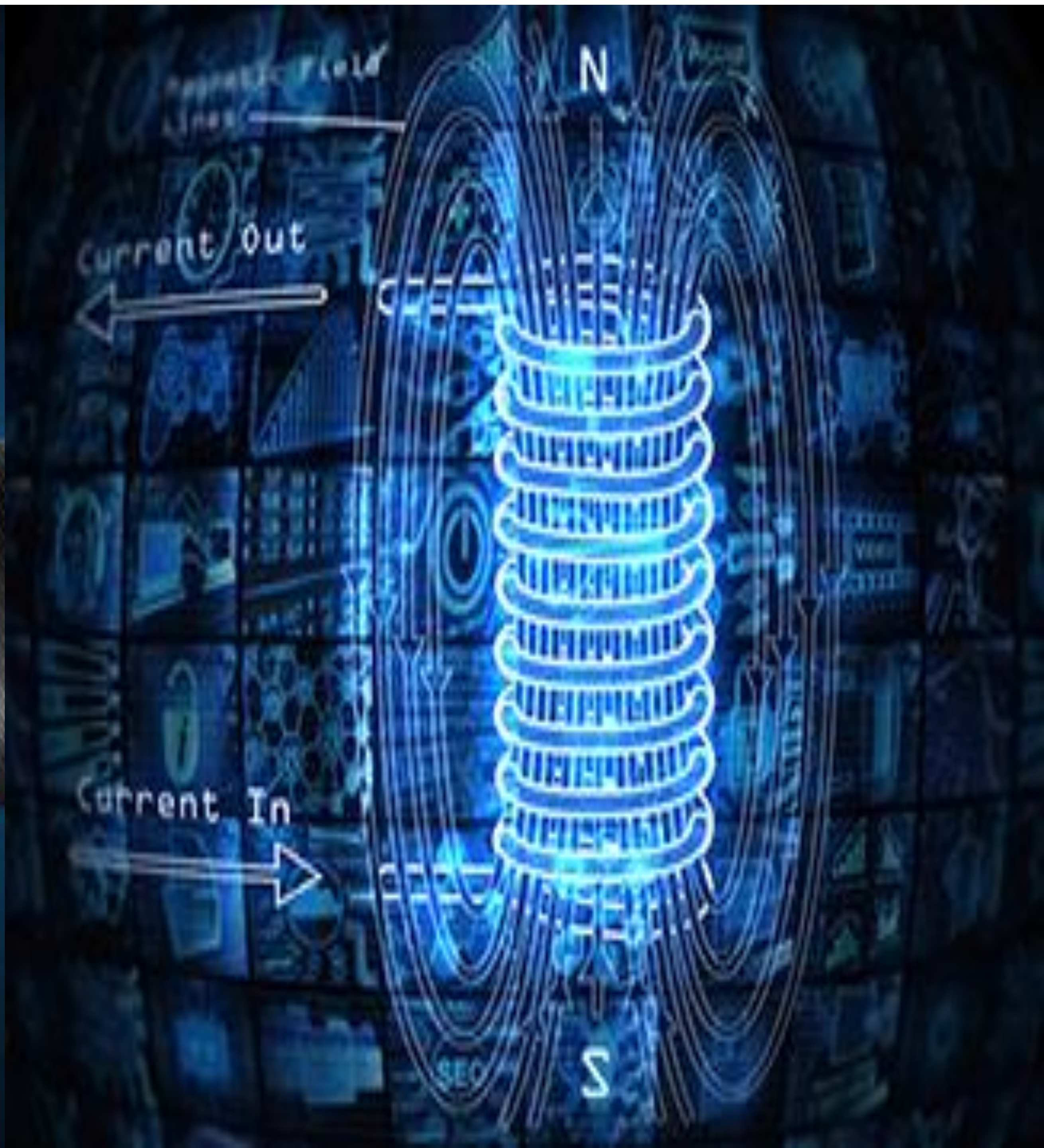

Electrical Engineering

N
Current Out
Current In
S

LECTURE 12

Transients

- **Steady State and Transient Response**
  - **Steady State**
    - A circuit having constant sources is said to be in steady state if the currents and voltages do not change with time.
    - A circuit having sinusoidal currents and voltages of constant amplitude and constant frequency are also considered to be in a steady state.
  - **Transient State**
    - A circuit is said to be in transient state if the currents and voltages change from one steady state to another steady state.
    - The behaviour of the voltage or current of the circuit containing energy storage elements is changed from one state to another with change in excitation.
    - The time taken for the circuit to change from one state to other state is called the transient time.
    - The application of KVL and KCL to circuits containing energy storage elements results in differential, rather than algebraic equations.

Transients

- **Steady State and Transient Response**
  - **Transient State**
    - The response of the circuit containing storage elements depends upon the nature of the circuit and so, it is called the natural response.
    - Storage elements deliver their energy to the resistances. Hence the response change with time, gets saturated after some time, and is referred to as the transient response.
    - When sources acting on a circuit, the response depends on the nature of the source or sources. This response is called forced response.
  - **Complete Response**
    - The complete response of a circuit consists of two parts: the forced response and the transient response.
    - The complete solution consists of two parts: the complementary function and the particular solution.
    - The complementary function dies out after short interval, and is referred to as the transient response or source free response.
    - The particular solution is the steady state response, or the forced response.

Transients

✓ **DC RESPONSE OF AN R-L CIRCUIT**

Consider a circuit consisting of a resistance and inductance as shown in **Fig. 5.1**.

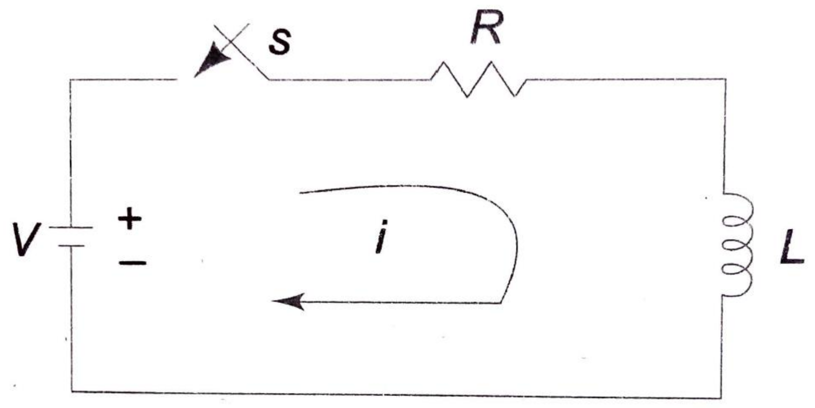


Fig. 5.1

The inductor in the circuit is initially uncharged and is in series with the resistor.

Now, the switch S is closed and the voltage V is applied to the circuit.

Apply Kirchhoff's Voltage Law to the circuit. We get the following differential equation.

$$V = Ri + L\frac{di}{dt}$$

or $$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

Where, V is the constant applied voltage and the current i is to be found out.

Transients

- ✓ **DC RESPONSE OF AN R-L CIRCUIT**

The equation is a linear differential equation of first order. Now, compare it with the following non-homogeneous differential equation

$$\frac{dx}{dt} + Px = K$$

Whose solution is

$$x = e^{-pt} \int K e^{+Pt} dt + c e^{-Pt}$$ where, c is an arbitrary constant.

Similarly, the current equation is

$$i = c e^{-(R/L)t} + e^{-(R/L)t} \int \frac{R}{L} e^{(R/L)t} dt$$

or, $$i = c e^{-(R/L)t} + \frac{V}{R}$$

Transients

✓ DC RESPONSE OF AN R-L CIRCUIT

- **Determination of the value of c by using the initial conditions.**

In the circuit as shown in **Fig. 5.1**, the switch S is closed at $t = 0$.

At $t = 0^-$, i.e. just before closing the switch S, the current in the inductor is zero.

Since the inductor does not allow sudden changes in currents, at $t = 0^+$ just after the switch is closed, the current remains zero.

Thus at $$t = 0, \qquad i = 0$$

So,

$$0 = c + \frac{V}{R}$$

Hence

$$c = -\frac{V}{R}$$

Transients

✓ **DC RESPONSE OF AN R-L CIRCUIT**

$$\therefore \quad i = \frac{V}{R} - \frac{V}{R} e^{-(R/L)t} = \frac{V}{R}\left[1 - e^{-(R/L)t}\right]$$

This equation consists of two parts, the steady state part V/R, and the transient part $(V/R)e^{-(R/L)t}$.

When switch S is closed, the response reaches a steady state value after a time interval as shown in **Fig. 5.2**

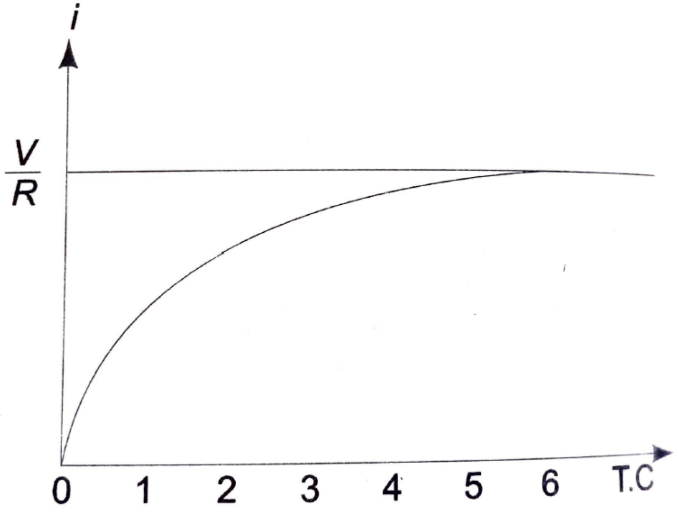


Fig. 5.2

Here the transition period is defined as the time taken for the current to reach its final or steady state value from its initial value.

The quantity L/R is important in describing the curve since L/R is the time required for the current to reach from its initial value of zero to the final value V/R.

The time constant of a function $\frac{V}{R} e^{-(R/L)t}$ is the time at which the exponent of e is unity, where e is the base of the natural logarithms. The term L/R is called the time constant and is denoted by τ

$$\therefore \quad \tau = \frac{L}{R} \ \text{sec}$$

Transients

- ✓ **DC RESPONSE OF AN R-L CIRCUIT**

∴ The transient part of the solution is

$$i = \frac{V}{R} e^{-(R/L)t} = \frac{V}{R} e^{-\frac{t}{\tau}}$$

At one TC, i.e. at one time constant, the transient term reaches 36.8 percent of its initial value.

$$i(\tau) = -\frac{V}{R} e^{-\frac{t}{\tau}} = -\frac{V}{R} e^{-1} = -0.368\frac{V}{R}$$

Similarly,

$$i(2\tau) = -\frac{V}{R} e^{-2} = -0.135\frac{V}{R}$$

$$i(3\tau) = -\frac{V}{R} e^{-3} = -0.0498\frac{V}{R}$$

$$i(5\tau) = -\frac{V}{R} e^{-5} = -0.0067\frac{V}{R}$$

After 5 TC, the transient part reaches more than 99 percent of its initial value.

Transients

- ✓ **DC RESPONSE OF AN R-L CIRCUIT**

Voltage across the resistor is

$$v_R = Ri = R \times \frac{V}{R}\left[1 - e^{-(R/L)t}\right] = V\left[1 - e^{-(R/L)t}\right]$$

Similarly, the voltage across the inductance is

$$v_L = L\frac{di}{dt} = L \times \frac{V}{R} \times \frac{R}{L} e^{-(R/L)t} = Ve^{-(R/L)t}$$

The response are shown in **Fig. 5.3**

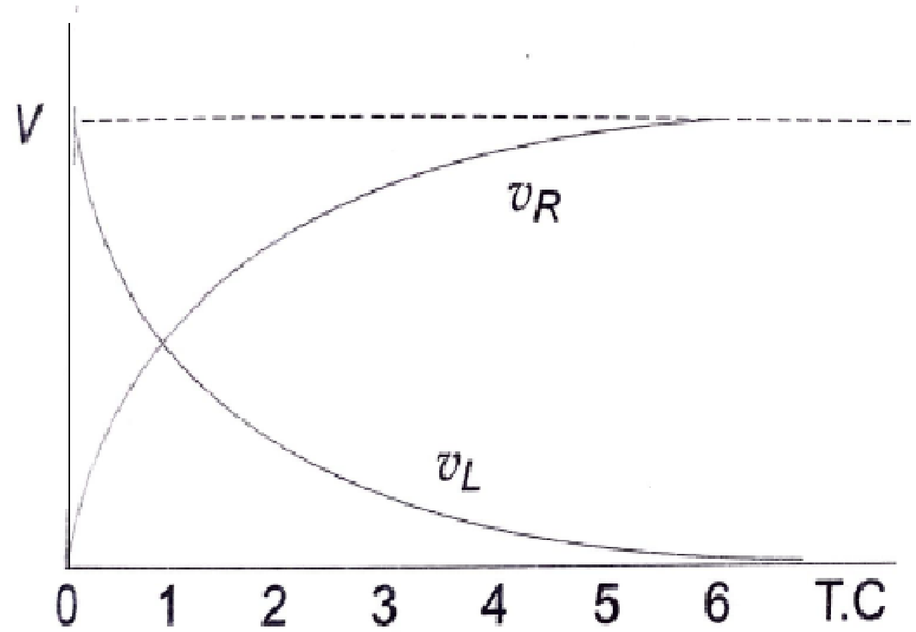


Fig. 5.3

Transients

✓ **DC RESPONSE OF AN R-L CIRCUIT**

Power in the resistor is

$$P_R = v_R i = V\left[1 - e^{-(R/L)t}\right] \times \frac{V}{R}\left[1 - e^{-(R/L)t}\right]$$

$$\therefore P_R = \frac{V^2}{R}\left[1 - 2e^{-(R/L)t} + e^{-(2R/L)t}\right]$$

Power in the inductor is

$$P_L = v_L i = Ve^{-(R/L)t} \times \frac{V}{R}\left[1 - e^{-(R/L)t}\right]$$

$$\therefore \quad P_L = \frac{V^2}{R}\left[e^{-(R/L)t} - e^{-(2R/L)t}\right]$$

The responses are shown in **Fig. 5.4**

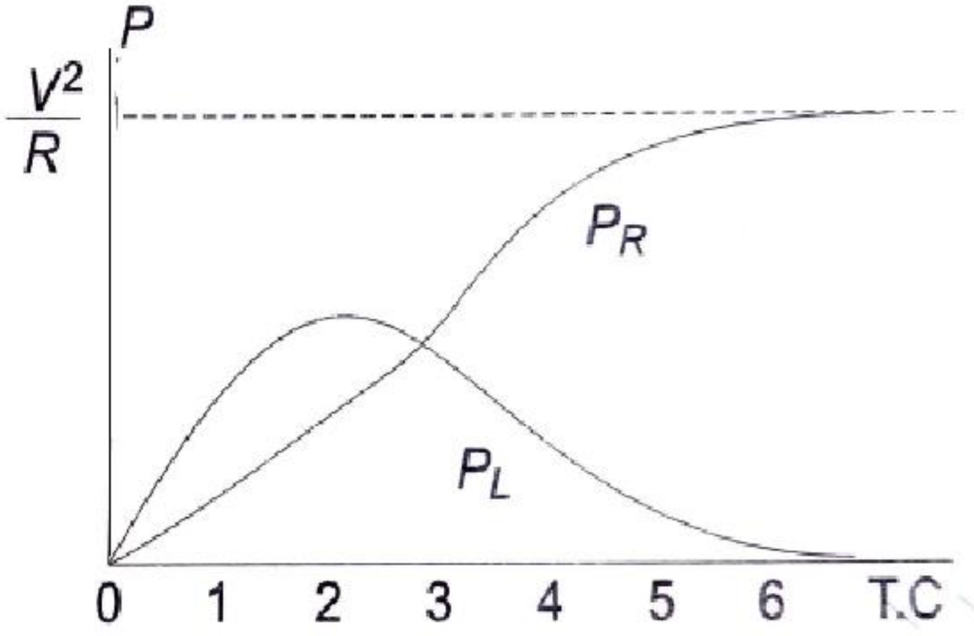


Fig. 5.4

Transients

Example – P5.1

A series RL circuit with R = 30 Ω and L = 15 H has a constant voltage V = 60 volt applied at t = 0 as shown in **Fig. P5.1**. Determine the current

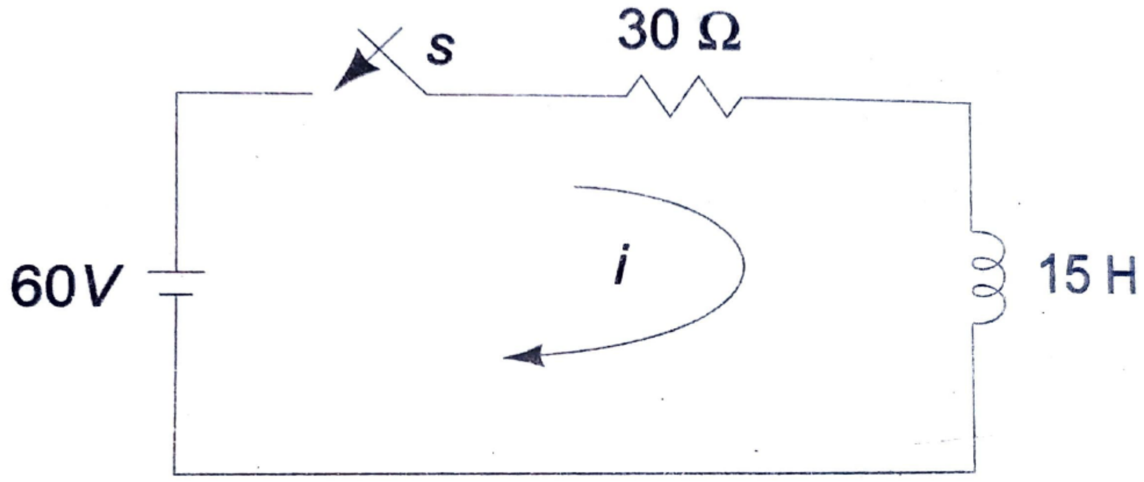


Fig. P5.1

Solution of Example – P5.1

By applying Kirchhoff's voltage law, we get

$$15\frac{di}{dt} + 30i = 60$$

$$\therefore\ \frac{di}{dt} + 2i = 4$$

The general solution for a linear differential equation is

$$i = ce^{-Pt} + e^{-Pt}\int Ke^{Pt}dt$$

Where $P = 2, K = 4$

$$\therefore\ i = ce^{-2t} + e^{-2t}\int 4e^{2t}dt$$

$$\therefore\ i = ce^{-2t} + 2$$

Solution of Example – P5.1

At $t = 0$, the switch S is closed.

Since the inductor never allows sudden changes in currents. At $t = 0^+$ the current in the circuit is zero.

Therefore at $t = 0^+$, $i = 0$

$$\therefore \quad 0 = c + 2 \qquad\qquad \therefore \quad c = -2$$

Substituting the value of c in the current equation, we have

$$i = 2[1 - e^{-2t}]\ \text{A}$$

Voltage across resistor

$$v_R = Ri = 30 \times 2[1 - e^{-2t}] = 60[1 - e^{-2t}]$$

Voltage across inductor

$$v_L = L\frac{di}{dt} = 15 \times \frac{d}{dt}[2(1 - e^{-2t})] = 30 \times 2e^{-2t} = 60e^{-2t}\ V$$

Transients

➢ DC RESPONSE OF AN R-C CIRCUIT

Consider a circuit consisting of resistance and capacitance as shown in **Fig. 5.5**.

The capacitor in the circuit is initially uncharged and is in series with a resistor. The switch S is closed at t = 0 and the voltage V is applied to the circuit.

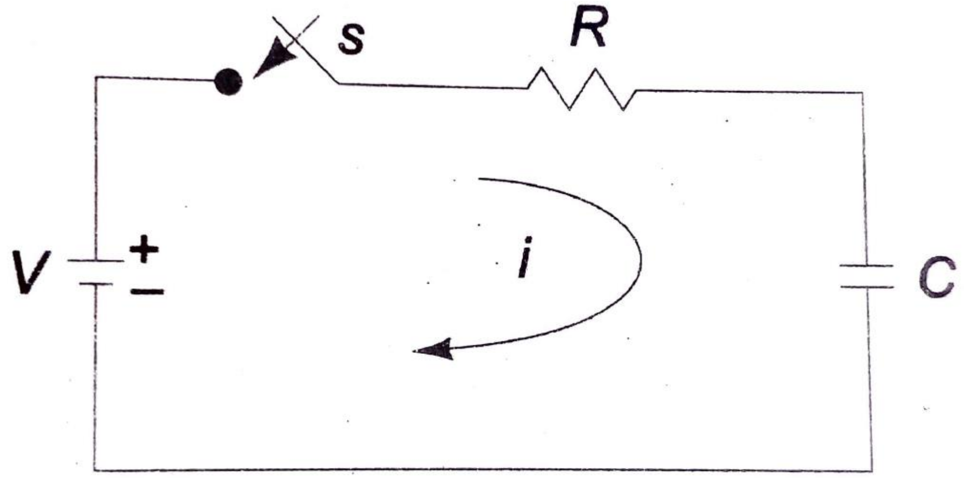


Fig. 5.5

Application of the Kirchhoff's voltage law to the circuit results in the following differential equation.

$$V = Ri + \frac{1}{C}\int i\,dt$$

By differentiating the above equation, we get

$$R\frac{di}{dt} + \frac{1}{C}i = 0$$

or, $$\frac{di}{dt} + \frac{1}{RC}i = 0$$

Transients

- **DC RESPONSE OF AN R-C CIRCUIT**

The equation is a linear differential equation with only the complementary function. The particular solution for the equation is zero.

The solution for this type of differential equation is $i = ce^{-(1/RC)t}$

Determination of the value of c by using the initial conditions.

In the circuit shown in **Fig. 5.5**, switch S is closed at $t = 0$.

Since the capacitor never allows sudden changes in voltage, it will act as a short circuit at $t = 0^+$.

So, the current in the circuit at $t = 0^+$ is V/R

At $t = 0$, the current $i = V/R$ $\therefore \frac{V}{R} = c$

So, the current equation becomes $i = \frac{V}{R} e^{-(1/RC)t}$

When switch S is closed, the response decays with time as shown in **Fig. 5.6**.

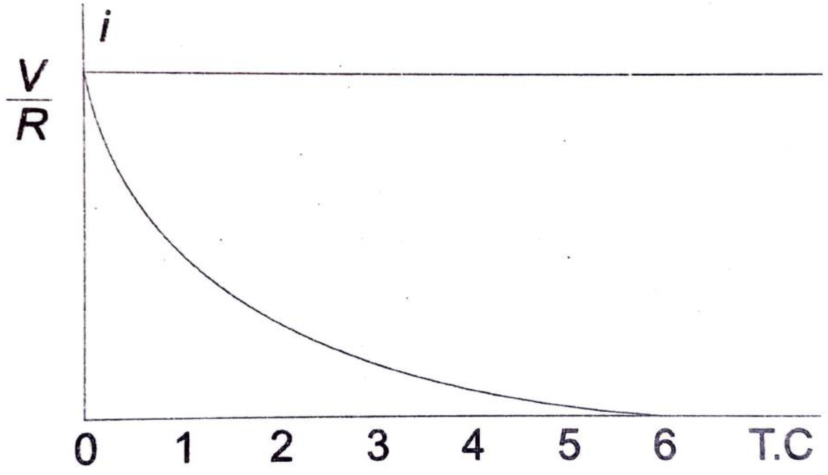


Fig. 5.6.

In the solution, the quantity RC is the time constant, and is denoted by τ, where τ= RC sec

After 5 TC, the curve reaches 99 per cent of its final value.

Transients

- **DC RESPONSE OF AN R-C CIRCUIT**

Voltage across the resistor $$v_R = Ri = R \times \frac{V}{R} e^{-(1/RC)t} = Ve^{-(1/RC)t}$$

Similarly, voltage across the capacitor

$$v_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int \frac{V}{R} e^{-(1/RC)t}\,dt$$

$$= -\left[\frac{V}{RC} \times RCe^{-(1/RC)t}\right] + c$$

$$= -Ve^{-(1/RC)t} + c$$

At $t = 0$, voltage across capacitor is zero $\therefore \quad c = V$

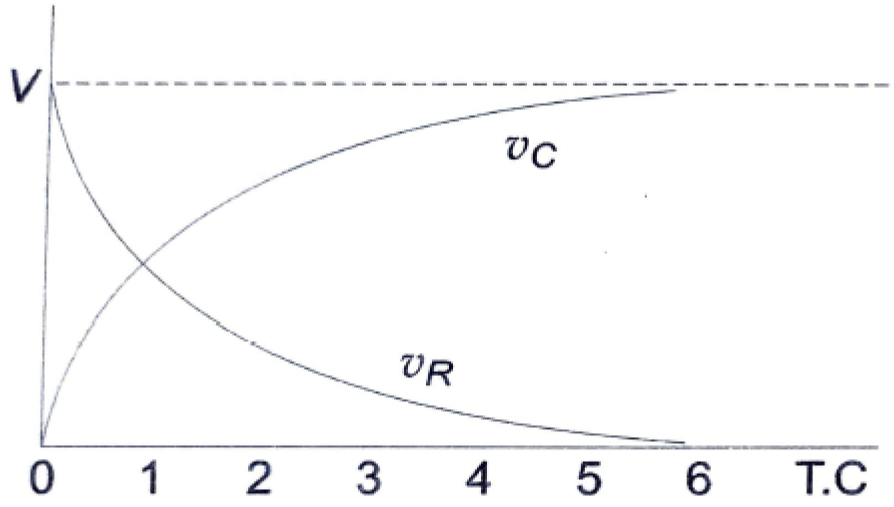


Fig. 5.7

$$\therefore \quad v_c = V\left[1 - e^{-(1/RC)t}\right]$$

The voltage across each element is shown in **Fig. 5.7**

Transients

- **DC RESPONSE OF AN R-C CIRCUIT**

Power in the resistor $P_R = v_R i = Ve^{-(1/RC)t} \times \frac{V}{R} e^{-(1/RC)t} = \frac{V^2}{R} e^{-(2/RC)t}$

Power in the capacitor $P_C = v_C i = V\left[1 - e^{-(1/RC)t}\right] \times \frac{V}{R} e^{-(1/RC)t}$

$$= \frac{V^2}{R}\left[e^{-(1/RC)t} - e^{-(2/RC)t}\right]$$

The responses are shown in **Fig. 5.8**

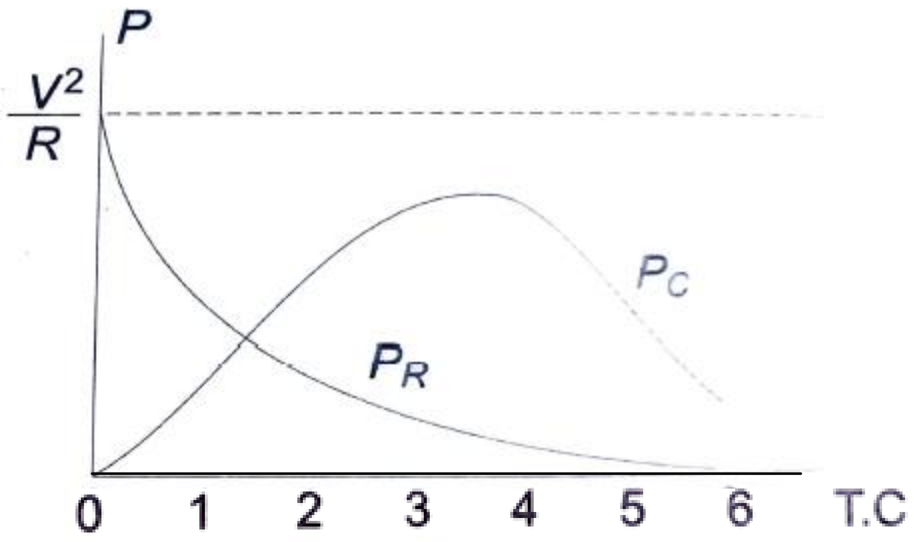


Fig. 5.8

Example – P5.2

A series RC circuit consists of resistor of 10 Ω and capacitor of 0.1 F as shown in **Fig. P5.2.** A constant voltage of 20 V is applied to the circuit at $t = 0$. Obtain the current equation. Determine the voltage across the resistor and the capacitor.

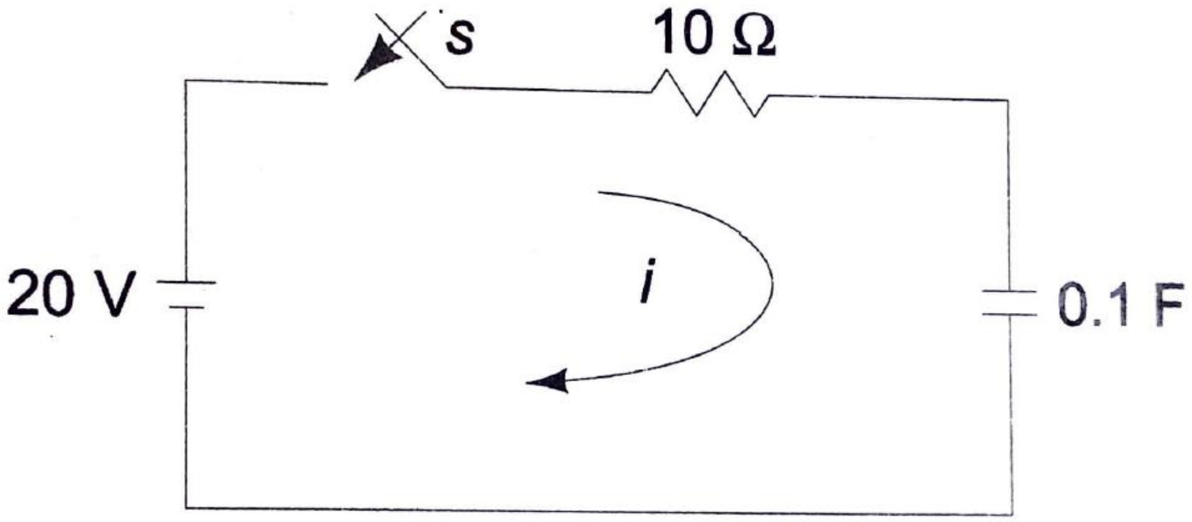


Fig. P5.2.

Solution of Example – P5.2

By applying Kirchhoff's law, we get $$10i + \frac{1}{0.1}\int i dt = 20$$

Differentiating with respect to t we get

$$10\frac{di}{dt} + \frac{i}{0.1} = 0$$

$$\therefore \quad \frac{di}{dt} + i = 0$$

The solution for the above equation is $i = ce^{-t}$

At $t = 0,$ switch S is closed. Since the capacitor does not allow sudden changes in voltage, the current in the circuit is $i = V/R = 20/10 = 2\ A$.

At $t = 0,\ i = 2\ A$

$\therefore$ The current equation $i = 2e^{-t}$

Voltage across the resistor is $v_R = i \times R = 2e^{-t} \times 10 = 20e^{-t}\ V$

Voltage across the capacitor is $v_C = V\left[1 - e^{-(1/RC)t}\right] = 20[1 - e^{-t}]\ V$

Transients

➢ DC RESPONSE OF R-L-C CIRCUIT

Consider a circuit consisting of resistance, inductance and capacitance as shown in **Fig. 5.9**.

The capacitor and inductor are initially uncharged, and are in series with a resistor.

The switch S is closed at $t = 0$, and determine the complete solution for the current.

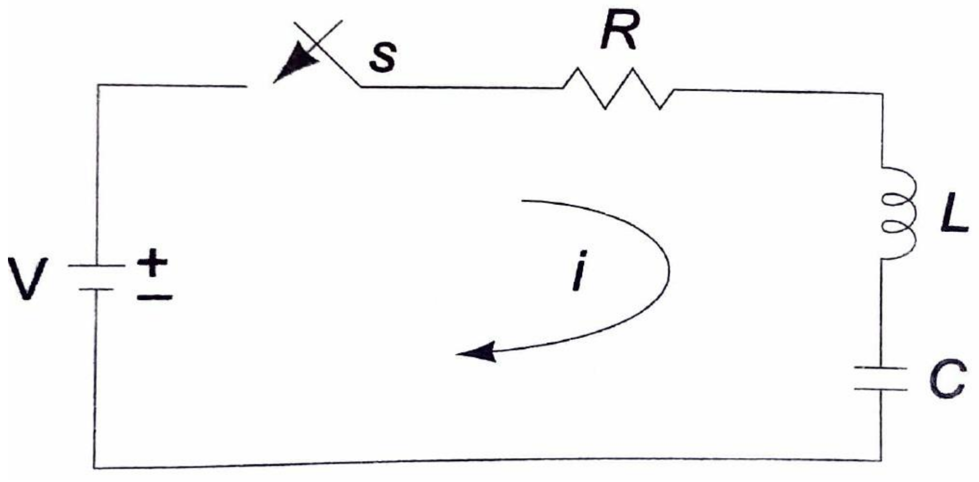


Fig. 5.9

Application of Kirchhoff's voltage law to the circuit results in the following differential equation.

$$V = Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt$$

By differentiating the above equation, we have

$$0 = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i$$

or, $$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

➢ **DC RESPONSE OF R-L-C CIRCUIT**

The equation is a second order linear differential equation, with only complementary function. The particular solution for the equation is zero.

Characteristic equation for the above differential equation $\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) = 0$

The roots of the equation are $D_1, D_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

By assuming $K_1 = -\frac{R}{2L}$ and $K_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

$\therefore \quad D_1 = K_1 + K_2$ and $D_2 = K_1 - K_2$

Now, the current of R-L-C circuit for different cases are determined as follows:

Transients

- **DC RESPONSE OF R-L-C CIRCUIT**

Case 1: When $R = 0$

The roots are complex conjugate, thus the characteristic equation is written as

$$[D - jK_2][D + jK_2]i = 0$$

The solution for the equation is

$$i = c_1 e^{(jK_2)t} + c_2 e^{(-jK_2)t}$$

$$= c_1 e^{jK_2 t} + c_2 e^{-jK_2 t}$$

$$= c_1(\cos K_2 t + j \sin K_2 t) + c_2(\cos K_2 t - j \sin K_2 t)$$

$$= (c_1 + c_2) \cos K_2 t + j(c_1 - c_2) \sin K_2 t$$

$$= A \cos K_2 t + B \sin K_2 t \quad [\because A = c_1 + c_2, \quad B = c_1 - c_2]$$

$$= I_m \sin(K_2 t + \emptyset) \quad \left[\because I_m = \sqrt{A^2 + B^2}, \ \ \emptyset = \tan^{-1}(A/B)\right]$$

Transients

- **DC RESPONSE OF R-L-C CIRCUIT**

The current, i as shown in **Fig. 5.10** is **undamped** in nature.

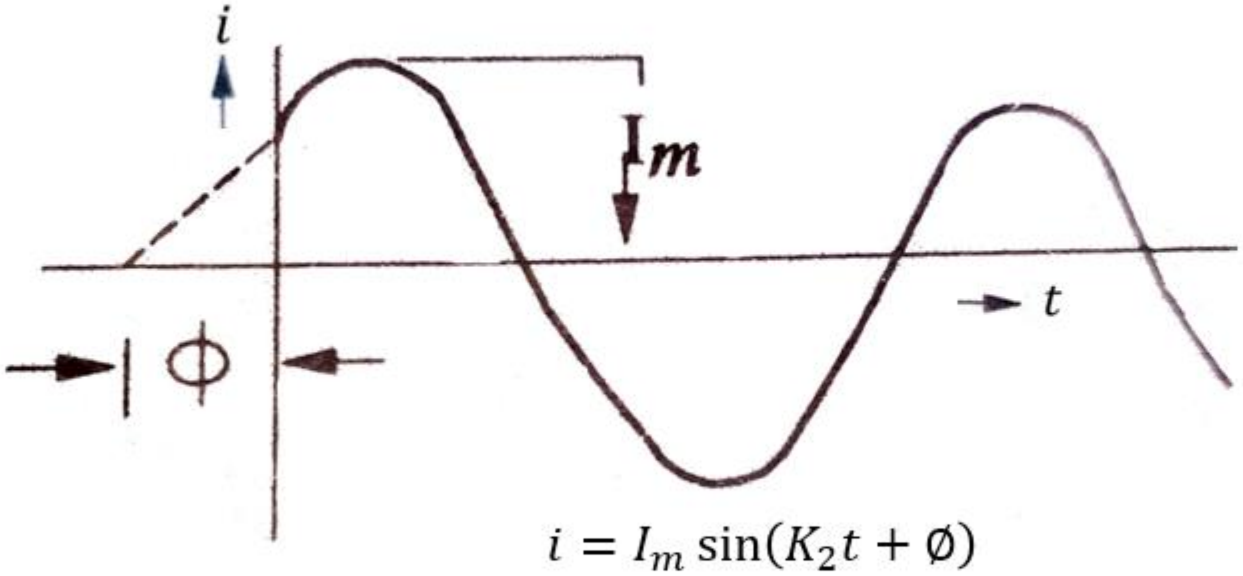


Fig. 5.10: Undamped Current Waveform

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

Transients

- **DC RESPONSE OF R-L-C CIRCUIT**

Case 2: when $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$

The roots are real and unequal. So, the characteristic equation is written as

$$[D-(K_1+K_2)][D-(K_1-K_2)]i = 0$$

The solution for the above equation is

$$i = c_1 e^{(K_1+K_2)t} + c_2 e^{(K_1-K_2)t}$$

$$= e^{K_1 t}(c_1 e^{K_2 t} + c_2 e^{-K_2 t})$$

$$= e^{K_1 t}[c_1(\cosh K_2 t + \sinh K_2 t) + c_2(\cosh K_2 t - \sinh K_2 t)]$$

$$= e^{K_1 t}[(c_1 + c_2)\cosh K_2 t + (c_1 - c_2)\sinh K_2 t]$$

$$= e^{K_1 t}[A\cosh K_2 t + B\sinh K_2 t] \qquad [\because A = c_1 + c_2, \quad B = c_1 - c_2]$$

- **DC RESPONSE OF R-L-C CIRCUIT**

The current, i as shown in **Fig. 5.11** is **overdamped** in nature.

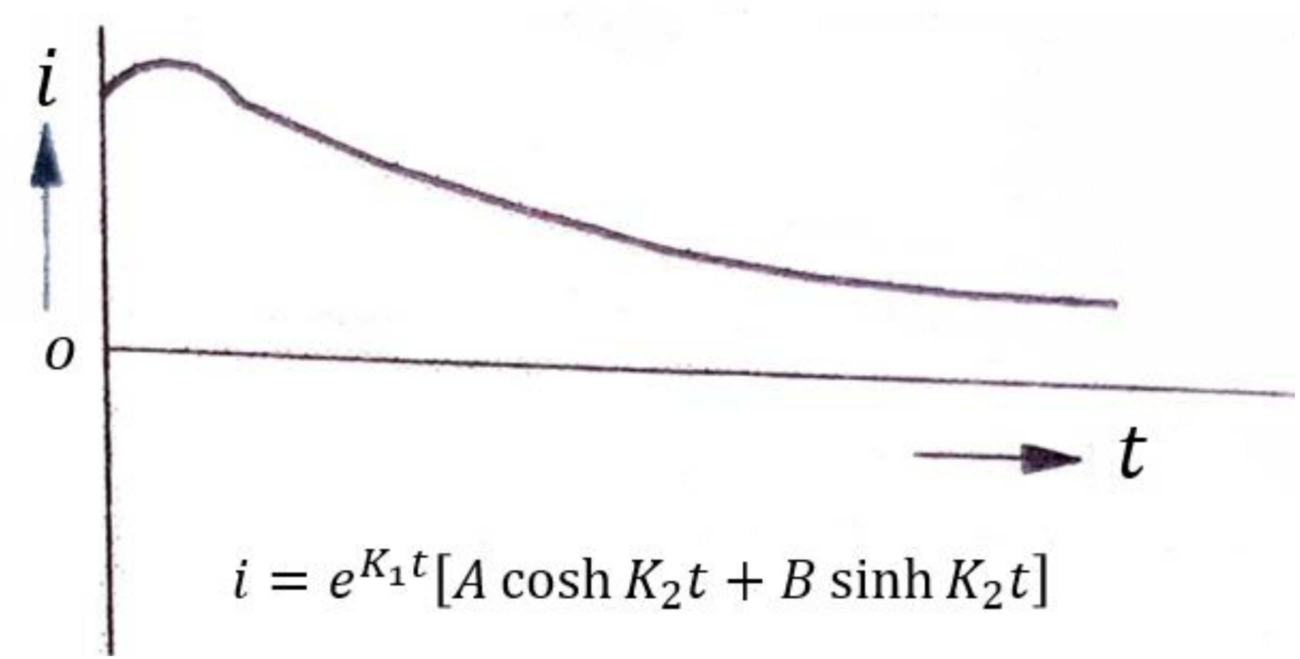


Fig. 5.11: Overdamped Current Waveform

Transients

- **DC RESPONSE OF R-L-C CIRCUIT**

Case 3: when $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$

The roots are complex conjugate, thus the characteristic equation is written as

$[D - (K_1 + jK_2)][D - (K_1 - jK_2)]i = 0$

The solution for the above equation is

$$i = c_1 e^{(K_1+jK_2)t} + c_2 e^{(K_1-jK_2)t}$$

$$= e^{K_1 t}\left(c_1 e^{jK_2 t} + c_2 e^{-jK_2 t}\right)$$

$$= e^{K_1 t}[c_1(\cos K_2 t + j\sin K_2 t) + c_2(\cos K_2 t - j\sin K_2 t)]$$

$$= e^{K_1 t}[(c_1 + c_2)\cos K_2 t + j(c_1 - c_2)\sin K_2 t]$$

$$= e^{K_1 t}\ [A\cos K_2 t + B\sin K_2 t] \qquad [\because A = c_1 + c_2, \quad B = c_1 - c_2]$$

$$i = I_m e^{K_1 t}\sin(K_2 t + \emptyset) \qquad \left[\because I_m = \sqrt{A^2 + B^2}, \ \ \emptyset = \tan^{-1}(A/B)\right]$$

Transients

- **DC RESPONSE OF R-L-C CIRCUIT**

The current, i as shown in **Fig. 5.12** is **underdamped** in nature.

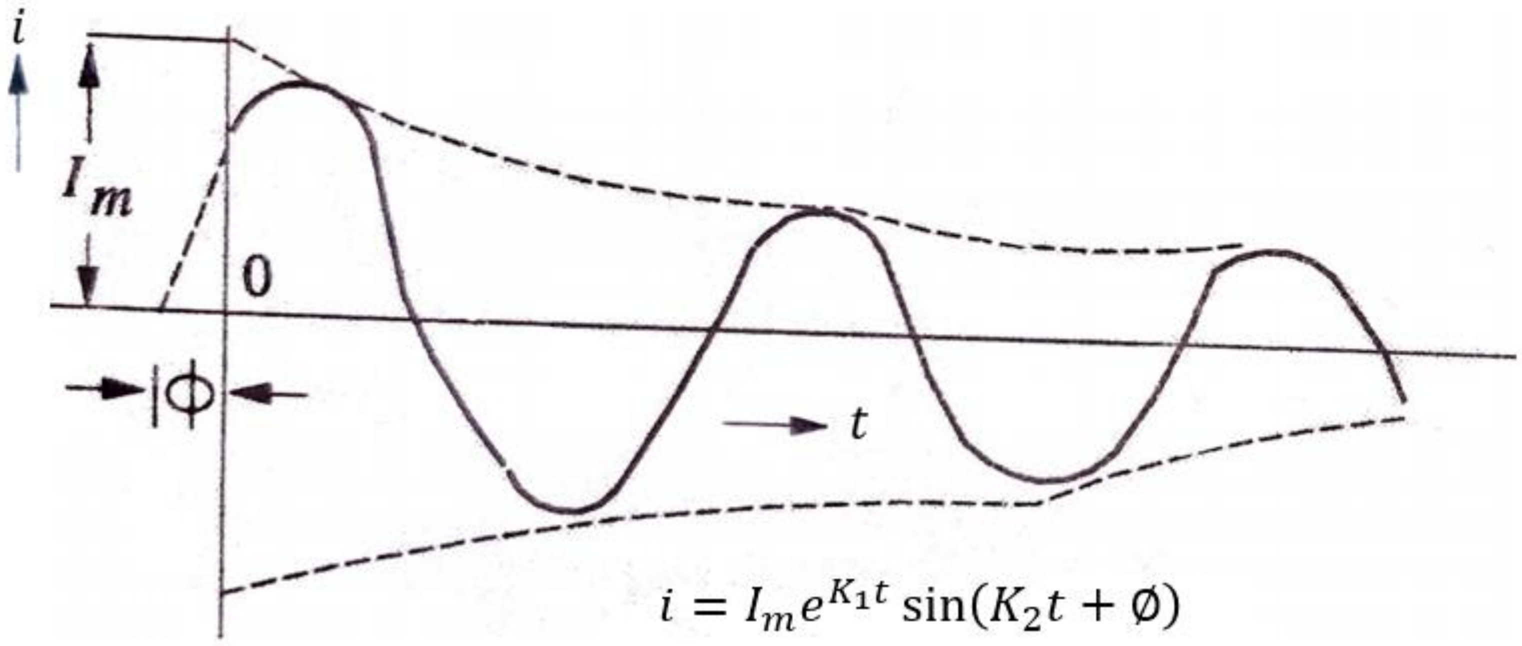


Fig. 5.12: Underdamped Current Waveform

$$\therefore f = \frac{1}{2\pi}\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

= Natural frequency of the circuit.

Transients

➢ **DC RESPONSE OF R-L-C CIRCUIT**

Case 4: when $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$

The roots are equal, and thus the characteristic equation is written as

$$(D - K_1)(D - K_1)i = 0$$

The solution for the equation is

$$i = e^{K_1 t}(c_1 + c_2 t)$$

The current, i as shown in **Fig. 5.13** is **critically damped** in nature.

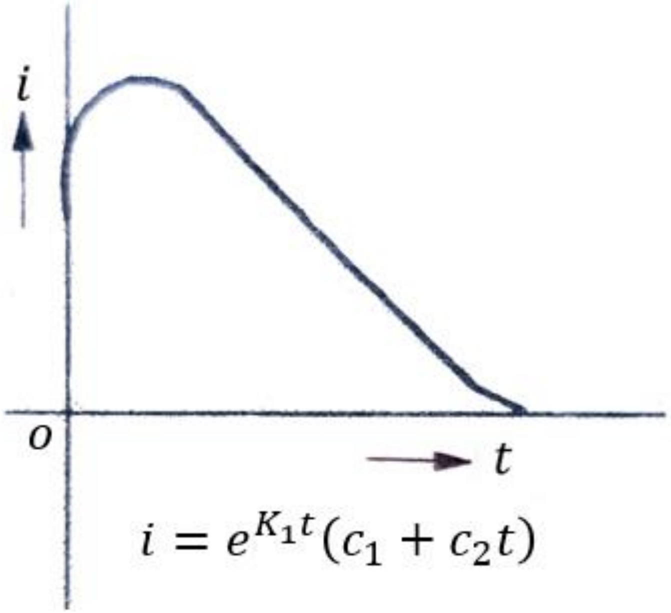


Fig. 5.13: Critically Damped Current Waveform

Example – P5.3

The circuit shown in **Fig. P5.3** consists of resistance, inductance and capacitance in series with a 100 V constant source when the switch is closed at t = 0. Find the current transient.

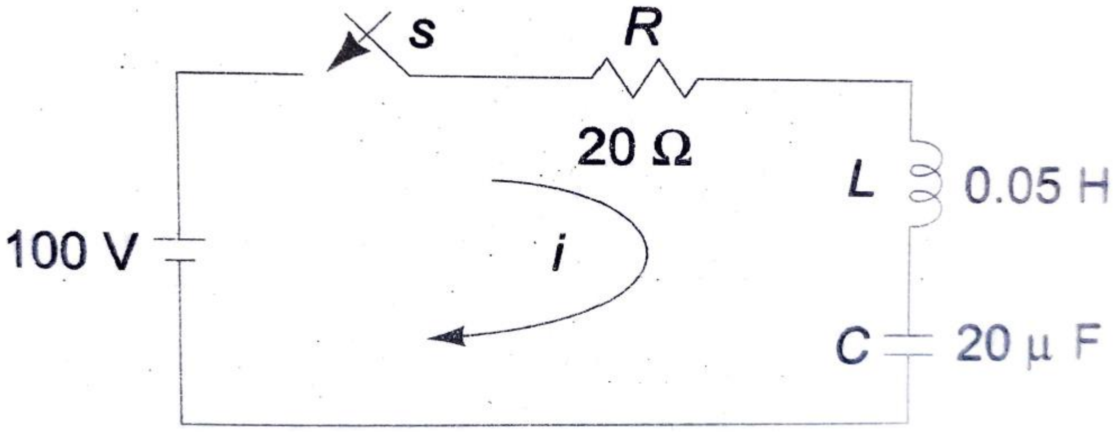


Fig. P5.3

Solution of Example – P5.3

At t = 0, switch S is closed when the 100 V source is applied to the circuit and results in the following differential equation.

$$100 = 20i + 0.05\frac{di}{dt} + \frac{1}{20\times 10^{-6}}\int i\,dt$$

Differentiating, we get

$$0.05\frac{d^2i}{dt^2} + 20\frac{di}{dt} + \frac{1}{20\times 10^{-6}}i = 0$$

$$\frac{d^2i}{dt^2} + 400\frac{di}{dt} + 10^6\,i = 0$$

$$(D^2 + 400D + 10^6)i = 0$$

$$D_1, D_2 = -\frac{400}{2} \pm \sqrt{\left(\frac{400}{2}\right)^2 - 10^6}$$

$$= -200 \pm \sqrt{(200)^2 - 10^6}$$

$$D_1 = -200 + j979.8$$

$$D_2 = -200 - j979.8$$

Therefore the current

$$i = e^{+K_1 t}[c_1\cos K_2 t + c_2\sin K_2 t]$$

Transients

Solution of Example – P5.3

At $t = 0$, the current flowing through the circuit is zero

$$i = 0 = (1)[c_1 \cos 0 + c_2 \sin 0] \qquad \therefore \;\; c_1 = 0$$

$$\therefore \; i = e^{-200t} c_2 \sin 979.8t \;\; A$$

Differentiating, we get $\dfrac{di}{dt} = c_2[e^{-200t} 979.8 \cos 979.8t + e^{-200t} \sin 979.8t]$

At $t = 0$, the voltage across inductor is 100 V

$$\therefore \;\; L\frac{di}{dt} = 100 \qquad \text{or,} \qquad \frac{di}{dt} = 2000$$

At $t = 0$ $\qquad \dfrac{di}{dt} = 2000 = c_2 979.8 \cos 0$

$$\therefore \;\; c_2 = \frac{2000}{979.8} = 2.04$$

The current equation is $\quad i = e^{-200t}(2.04 \sin 979.8t) \;\; A$


Thank you