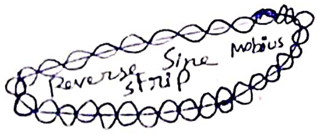
$2\pi i \sin\frac{\pi}{2})$.

Q) $\int_{|z|=2} \frac{1}{z^3+1}\, dz \rightarrow$ H.W.



25/08/2022

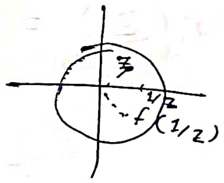
Complex Integrals

Dialation

I) $f(z) = az$, $a \in \mathbb{C}$

$a = |a| e^{i\alpha}$ [Both magnification and Rotation = Dialation]

- $f(z) = az + b$ [Linear ~~operation~~ transformation]
- $f(z) = \frac{1}{z} \rightarrow$ Inversion. [Reflection about unit circle as well as with the real axis] (+ve x axis)



- $f(z) = \dfrac{az+b}{cz+d}$

$ad - bc \neq 0$

If $ad - bc = 0 \Rightarrow$ the $f(z)$ is a constant function

Proof

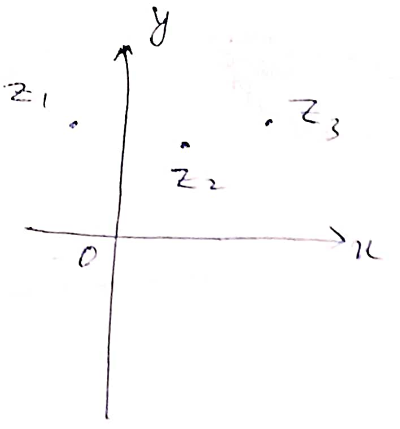
$ad = bc$

$\Rightarrow \frac{b}{a} = \frac{d}{c} = k \Rightarrow b = ak$ and $d = ck$

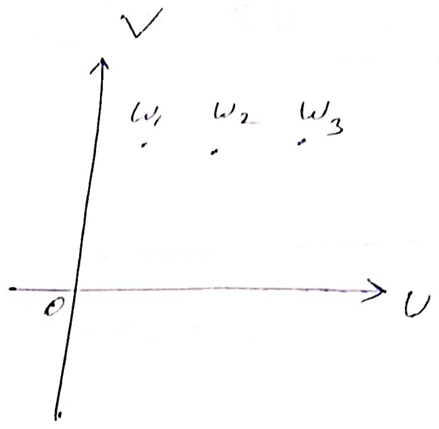
$\Rightarrow \dfrac{az + ak}{cz + ck} \Rightarrow \frac{a}{c}\left(\frac{z+k}{z+k}\right)$

$= \frac{a}{c} =$ constant

Bilinear transformation



Z-plane



w-plane

$$f(z_i) = w_i \quad [i = 1, 2, 3]$$

We need 3 points as even though we have 4 unknowns i.e. a, b, c and d

We can convert it as

$$f(z) = \frac{az+b}{cz+d}$$

$$= \frac{(a/b)z + 1}{(c/b)z + d/b} \rightarrow 3 \text{ unknowns}$$

Now, $f(z_1) = w_1 = \dfrac{az_1+b}{cz_1+d}$

$$\Rightarrow W - W_1 = \left(\frac{az+b}{cz+d}\right) - \left(\frac{az_1+b}{cz_1+d}\right)$$

We know, $f(z_2) = w_2 = \dfrac{az_2+b}{cz_2+d}$

$$f(z) = W_3 = \frac{az_3+b}{cz_3+d}$$

$$\Rightarrow W - W_3 = \left(\frac{az+b}{cz+d}\right) - \left(\frac{az_3+b}{cz_3+d}\right)$$

$$\Rightarrow \frac{(W-W_1)(W_2-W_3)}{(W_1-W_2)(W-W_3)} = \frac{(z-z_1)}{z_1-z_2} \cdot \frac{(z_2-z_3)}{z-z_3}$$

∴ We can express W in terms of Z.

To remember

W W₁ W₂ W₃

$$w - w_1 \qquad w_2 - w_3$$

$$w_1 - w_2 \qquad w - w_3$$

Similar for Z

This result is true for extended plane (∞ is included)

Ex If $z_3 = \infty$

then In the RHS we can take limit

$$\frac{z - z_1}{z_1 - z_2} \cdot \frac{z_2 - z_3}{z - z_3} = \frac{(z - z_1) \cdot \frac{z_2 - z_3}{z_3} = -1}{(z_1 - z_2) \cdot \frac{z - z_3}{z_3} = -1}$$

$$= \frac{z - z_1}{z_1 - z_2}$$

• We can also write

$$\frac{w - w_1}{w_1 - w_2} \cdot \frac{w_2 - w_3}{w_3 - w} = \frac{z - z_1}{z_1 - z_2} \cdot \frac{z_2 - z_3}{z_3 - z}$$

Q) $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ Find the Bilinear transformation

$$\Rightarrow \frac{w - w_1 \times \frac{w_2 - w_3}{w_3} = -1}{w_1 - w_2 \times \frac{w_3 - w}{w_3} = 1}$$

$$\Rightarrow \frac{w_1 - w}{w_1 - w_2} = \frac{-w}{-1} = w = \frac{z - z_1}{z_1 - z_2} \cdot \frac{z_2 - z_3}{z_3 - z}$$

(Ans)
This is the bilinear ~~transform~~ transformation

Cross Ratio → Remains invariant under valid Bilinear transformation

$z_1, z_2, z_3, z_4 \xrightarrow{f} w_1, w_2, w_3, w_4$

$$\Rightarrow \frac{(z_1 - z_2)\cdot(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)}$$

Now, the cross ratio of ws will be the same as that of zs

Q) Determine the bilinear transformation which transforms the points $i, 2, -2 \longrightarrow i, 1, -1$

Ans) Let us suppose the point z on the z-plane is mapped to the point w on the w plane by the bilinear transformation

$\therefore$ Cross ratio of $(z, i, 2, -2)$ = cross ratio of $(w, i, 1, -1)$

$$\Rightarrow \frac{(z-i)}{(i-2)} \cdot \frac{(2+2)}{-2-z} = \frac{w-i}{i-1} \cdot \frac{(1+1)}{-1-w}$$

$$\Rightarrow \frac{z-i}{(i-2)} \cdot \frac{\cancel{4}\ 2}{(-2-z)} = \frac{(w-i)(2)}{(i-1)(-1-w)}$$

$$\Rightarrow \frac{2(z-i)}{(i-2)(z+2)} = \frac{w-i}{(i-1)(w+1)}$$ ⇒ Find w (that is the bilinear transformation)

Q) $z_1 = \infty$, $z_2 = i$, $z_3 = 0$

$w_1 = 0$, $w_2 = i$, $w_3 = \infty$

$$\Rightarrow \frac{(w-w_1)\cdot(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)\cancel{(w_2-w_3)}(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{w-w_1 \cdot \frac{w_2-w_3}{w_3}}{\frac{w_1-w_2}{} \cdot \frac{w_3-w}{w_3}} = \frac{\frac{z-z_1}{z_1}(z_2-z_3)}{\frac{z_1-z_2}{z_1}\cdot(z_3-z)}$$

$$\Rightarrow \frac{(w-w_1)(-1)}{(w_1-w_2)\,1} = \frac{-1(z_2-z_3)}{1(z_3-z)}$$

$$\Rightarrow \frac{(w-0)}{i} = \frac{+1(i-0)}{(0+z)}$$

$$\Rightarrow w = \frac{-1}{z} \quad \text{(Answer)}$$

Q) $z_1 = 1$, $z_2 = i$, $z_3 = -1$

$w_1 = 0$, $w_2 = 1$, $w_3 = \infty$

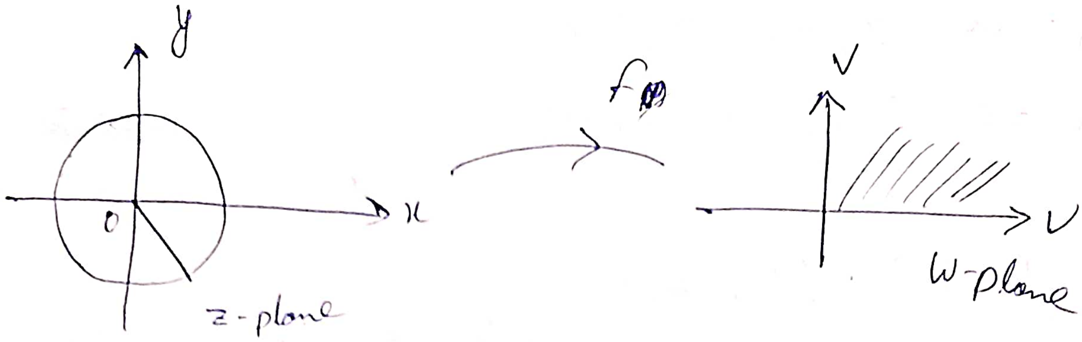
(Show that under the bilinear transformation as shown, $|z|<1$ is mapped to $\operatorname{Im} w > 0$)

~~$\Rightarrow z - z_1 (z-1)$~~

$$\Rightarrow \frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-0)(-1)}{(0-1)\,1}$$

$$\frac{z-1}{i(z+1)}$$

$$\Rightarrow \frac{(z-1)(1+i)}{(i-1)(1+z)} = w$$



Now

Express z in terms of w

$$\Rightarrow z = \frac{iw+1}{-iw+1}$$

Now, $|z| < 1$

$$\Rightarrow |z|^2 - 1 < 0$$

$$\Rightarrow z\bar{z} - 1 = \frac{iw+1}{-iw+1} \times \frac{-i\bar{w}+1}{i\bar{w}+1} - 1$$

→ As w is a complex number

$$\frac{(iw+1)(-i\bar{w}+1) - (-iw+1)(i\bar{w}+1)}{(-iw+1)(i\bar{w}+1)}$$

$$\Rightarrow \frac{+w\bar{w} + iw - i\bar{w} + 1 - (w\bar{w} - iw + i\bar{w} + 1)}{(-iw+1)(i\bar{w}+1)}$$

$$= \frac{\cancel{w\bar{w}} - \cancel{w\bar{w}} + iw + iw - i\bar{w} - i\bar{w} \cancel{+1} \cancel{-1}}{|1-iw|^2}$$

$$= \frac{2i(w-\bar{w})}{|1-iw|^2}$$

$$= \frac{-2 \times 2\, Im(w)}{|1-iw|^2} < 0$$

$\Rightarrow \operatorname{Im}(w) > 0$

Q) $z_1 = i,\ z_2 = 0,\ z_3 = -i$

$w_1 = 0,\ w_2 = -1,\ w_3 = \infty$

Now,
$$\frac{(z-i)(0+i)}{(i-0)(-i-z)} = \frac{(w-0)(-1)}{(0+1)\ 1}$$

$$\Rightarrow \frac{(z-i)(i)}{i(-1)(i+z)} = \frac{-w}{1}$$

$$\Rightarrow \frac{z-i}{z+i} = w$$

Check

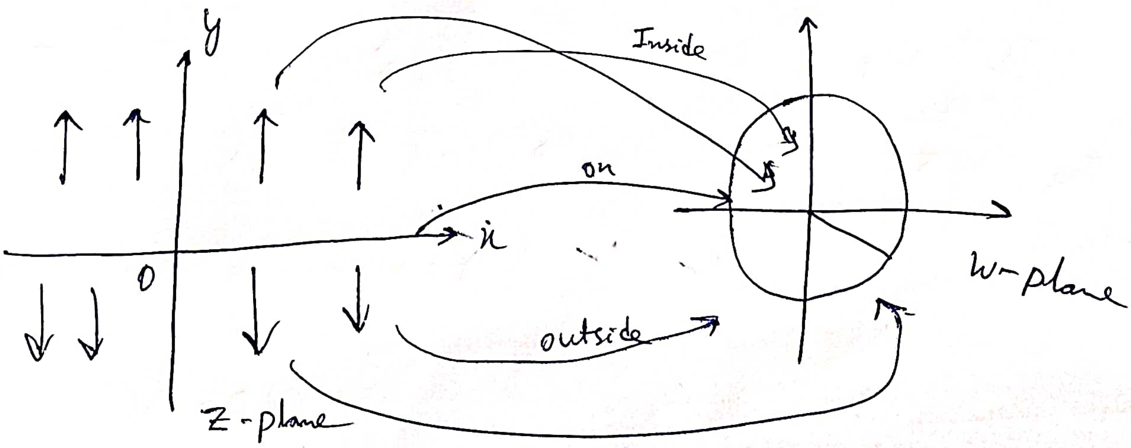
I) $\operatorname{Im}(z) = 0 \longrightarrow |w| = 1$

II) $\operatorname{Im}(z) > 0 \longrightarrow |w| < 1$

III) $\operatorname{Im}(z) < 0 \longrightarrow |w| > 1$

Diagram

1)



Now, to express z in terms of w

$z - i = wz + iw$

$\Rightarrow z(1-w) = iw + i$

$\Rightarrow z = \dfrac{i(w+1)}{1-w}$

Now $\quad w\bar{w} - 1 = |w|^2 - 1 = \dfrac{-4\,Im(z)}{|z+i|^2}$

$\therefore$ In I) $Im(z) = 0$

$\Rightarrow |w|^2 - 1 = 0$

$\Rightarrow |w| = 1$ (We will not take -ve Solutions)

II) If $Im(z) > 0$, $\quad |w|^2 - 1 < 0$

$\Rightarrow |w| < 1$

III) If $Im(z) < 0$, $\quad |w|^2 - 1 > 0$

$\Rightarrow |w| > 1$