CHAPTER 16

Three-Dimensional Geometry

16.1 INTRODUCTION

It is well known that the coordinate geometry is the analytic method to study the geometric shape. The students are familiar with two-dimensional geometry and this geometry deals only with the shapes those are drawn on a plane. But, three-dimensional geometry deals with the solid shapes which frequently occur in scientific and engineering problems. In this chapter, we study the very common three-dimensional geometrical objects, such as plane, straight line, sphere, cone and cylinder.

16.2 THREE-DIMENSIONAL COORDINATES

In two dimensions, a point can be represented by two quantities x and y, but in three dimensions, to represent a single point needs three quantities x, y and z. These quantities are measure with respect to a fixed point O called the origin and from three mutually perpendicular lines XOX', YOY' and ZOZ' called *rectangular coordinates axes*, namely x, y and z-axes. OX, OY, OZ whose directions are right-handed are taken as positive directions, opposite directions as negative. The planes XOY, XOZ and ZOX are called xy, yz and zx coordinates planes respectively.

Let P be any point in space (the three-dimensional coordinate system). PN is drawn perpendicular to XOY plane, NM is perpendicular to OX, i.e. MN is parallel to OY. If $OM = x$, $MN = y$, $PN = z$, the coordinates of P are (x, y, z). These coordinates are positive or negative according to their measurement from the origin—along the positive or negative direction of the axes (see Figure 16.1).

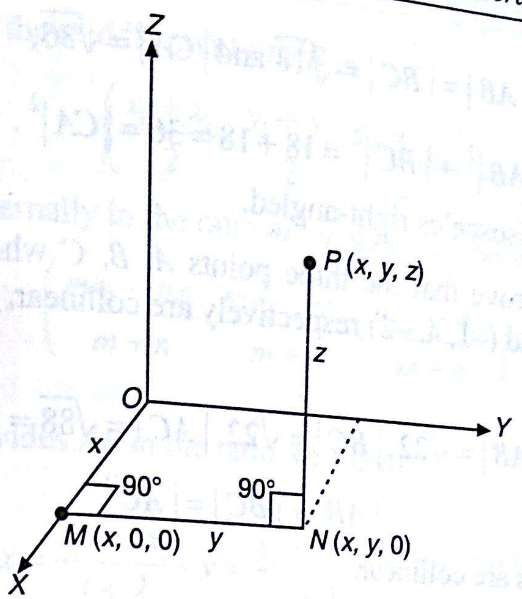

Figure 16.1 Point in space.

The projections of the point $P(x, y, z)$ on the xy, yz and zx-planes are respectively $(x, y, 0)$, $(0, y, z)$ and $(x, 0, z)$.

The three coordinate planes divide the whole of the space into eight parts which are called *octants*. The octant $OXYZ$ in which the point P is situated is called first octant. Any point in this octant has each of its coordinates positive.

16.2.1 Distance between Two Points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space. The distance between these two points is denoted by $|AB|$ and is defined by

$$|AB| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

From this definition, we can test the collinearity between three points in space. If A, B, C be three points in space and if they are collinear then one of the following conditions hold:

(a) $|AB| + |BC| = |AC|$

(b) $|AB| + |AC| = |BC|$

(c) $|AC| + |BC| = |AB|$.

EXAMPLE 16.1 Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form an isosceles right angled triangle.

Solution

Let $A(0, 7, 10)$, $B(-1, 6, 6)$, $C(-4, 9, 6)$ be three points then

$$|AB| = \sqrt{(0+1)^2 + (7-6)^2 + (10-6)^2} = \sqrt{18}.$$

Similarly,

$$|AB| = \sqrt{36} \text{ and } |BC| = \sqrt{18}.$$

Here $$|AB| = |BC| = \sqrt{18} \text{ and } |CA| = \sqrt{36},$$

i.e. $$|AB|^2 + |BC|^2 = 18 + 18 = 36 = |CA|^2.$$

Hence the triangle is isosceles right-angled.

EXAMPLE 16.2 Prove that the three points A, B, C whose coordinates are $(3, -2, 4)$, $(1, 1, 1)$ and $(-1, 4, -2)$ respectively are collinear.

Solution

Here $$|AB| = \sqrt{22}, |BC| = \sqrt{22}, |AC| = \sqrt{88} = 2\sqrt{22}.$$

Hence, $$|AB| + |BC| = |AC|$$

Therefore, the points are collinear.

EXAMPLE 16.3 Find the locus of a point the sum of whose distances from $(4, 0, 0)$ and $(-4, 0, 0)$ is 10.

Solution

Let $P(x, y, z)$ be any point in the space. The distance between P and $(4, 0, 0)$ is $\sqrt{(x-4)^2 + y^2 + z^2}$ and that of between P and $(-4, 0, 0)$ is $\sqrt{(x+4)^2 + y^2 + z^2}$.

Since the sum of distances is 10,

$$\sqrt{(x-4)^2 + y^2 + z^2} + \sqrt{(x+4)^2 + y^2 + z^2} = 10.$$

or $$\sqrt{(x-4)^2 + y^2 + z^2} = 10 - \sqrt{(x+4)^2 + y^2 + z^2}.$$

Squaring both sides, we get

$$(x-4)^2 + y^2 + z^2 = 100 + (x+4)^2 + y^2 + z^2 - 20\sqrt{(x+4)^2 + y^2 + z^2}$$

or $$4x - 25 = -5\sqrt{(x+4)^2 + y^2 + z^2}.$$

Again squaring both sides, we get

$$16x^2 - 200x + 625 = 25(x^2 + y^2 + z^2 + 8x + 16)$$

or $$9x^2 + 25(y^2 + z^2) = 225.$$

16.2.2 Division of the Line Joining Two Points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in space and P be a point which divides the line AB in the ratio $m : n$. Then the coordinates of P is

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right).$$

The coordinates of the middle point of AB is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

If P divides AB externally in the ratio $m:n$, then the coordinates of P are

$$\left(\frac{mx_2-nx_1}{m+n}, \frac{my_2-ny_1}{m+n}, \frac{mz_2-nz_1}{m+n}\right).$$

If $\frac{m}{n}=\lambda$, i.e. P divides AB in the ratio $\lambda:1$ then

$$x=\frac{x_1+\lambda x_2}{1+\lambda},\ y=\frac{y_1+\lambda y_2}{1+\lambda},\ z=\frac{z_1+\lambda z_2}{1+\lambda}.$$

EXAMPLE 16.4 Find the ratio in which the yz-plane divides the join of the points $(-2, 4, 7)$ and $(3, -5, 8)$, and also find the coordinates of the point of intersection of the line with the yz-plane.

Solution

The coordinates of any point on the line joining the two points are

$$\left(\frac{3\lambda-2}{\lambda+1}, \frac{-5\lambda+4}{\lambda+1}, \frac{8\lambda+7}{\lambda+1}\right).$$

If the point is in the yz-plane, then its x-coordinate should be zero.

$$\therefore \quad \frac{3\lambda-2}{\lambda+1}=0 \text{ or } 3\lambda-2=0 \text{ or } \lambda=\frac{2}{3}.$$

Hence the required ratio is $2:3$. The required point is $\left(0, \frac{2}{5}, \frac{37}{5}\right)$.

16.2.3 Area of a Triangle in Space

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of the triangle ABC in space. Then its area is given by

$$\Delta=\sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}$$

where Δ_x, Δ_y and Δ_z are the areas of the projection of the triangle ABC on the yz-, xz- and xy-planes. That is

$$\Delta_x=\frac{1}{2}\{y_1(z_2-z_3)+y_2(z_3-z_1)+y_3(z_1-z_2)\}$$

$$\Delta_y=\frac{1}{2}\{z_1(x_2-x_3)+z_2(x_3-x_1)+z_3(x_1-x_2)\}$$

$$\Delta_z = \frac{1}{2}\{x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)\}.$$

EXAMPLE 16.5 Find the area of the triangle ABC whose vertices are $A(1, 2, 1)$, $B(0, 1, -1)$, $C(2, 1, 1)$.

Solution

Here
$$\Delta_x = \frac{1}{2}\{2(-1-1) + 1(1-1) + 1(1+1)\} = -1,$$

$$\Delta_y = \frac{1}{2}\{1(0-2) + (-1)(2-1) + 1(1-0)\} = -1,$$

$$\Delta_z = \frac{1}{2}\{1(1-1) + 0 + 2(2-1)\} = 1.$$

Therefore, the required area is $\sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2} = \sqrt{3}$.

16.3 DIRECTION COSINES AND DIRECTION RATIOS

If a directed line makes angles α, β, γ with the positive directions of x, y and z axes respectively, then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* (DCs) of this line. There are generally denoted by l, m, n or $\{l, m, n\}$.

The direction cosines of x, y and z axes are respectively $\{1, 0, 0\}$, $\{0, 1, 0\}$ and $\{0, 0, 1\}$.

Let $P(x, y, z)$ be any point with respect to the origin O. Let OP makes angles α, β, γ with the coordinate axes (see Figure 16.2).

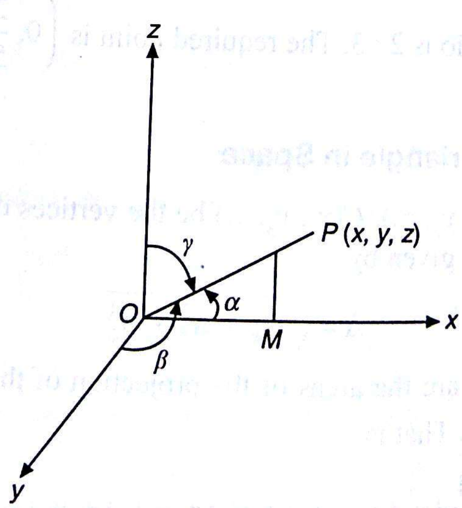

Figure 16.2 Direction cosines.

...should be perpendicular to x-axis

$$\cos\alpha = \frac{|OM|}{|OP|} = \frac{x}{r}$$

...but

$$|OP| = \sqrt{x^2 + y^2 + z^2} = r$$

...ere

Similarly,

$$\cos\beta = \frac{y}{r}, \cos\gamma = \frac{z}{r}.$$

Now,

$$l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2 + y^2 + z^2}{r^2} = 1.$$

Again

$$l = \cos\alpha = \frac{x}{r}, m = \frac{y}{r}, n = \frac{z}{r}.$$

That is,

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r.$$

This implies that x, y, z are proportional to the direction cosines. Thus, any three numbers which are proportional to the direction cosines of a line are called the direction ratios (DRs) of the line. Therefore, if l, m, n are the DCs of a line, then for any r, lr, mr, nr are the DRs of this line.

Again, if a, b, c be the DRs of a line, then its DCs are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

EXAMPLE 16.6 If α, β, γ be the direction angles of a line, then show that (WBUT 2005)

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

Solution

If α, β, γ be the direction angles of a line

then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

i.e.

$$(1-\sin^2\alpha) + (1-\sin^2\beta) + (1-\sin^2\gamma) = 1$$

or

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

16.3.1 Direction Cosine of a Line Joining Two Points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in space then $x_2 - x_1, y_2 - y_1, z_2 - z_1$ be the direction ratios of the line AB and $\frac{x_2 - x_1}{|AB|}, \frac{y_2 - y_1}{|AB|}, \frac{z_2 - z_1}{|AB|}$ are the direction cosines of the line AB.

Angle between two lines: Let $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$ be the direction ratios of two lines then the angle θ between them is

$$\theta = \cos^{-1} \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

If $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$ be the DCs of two lines then angle θ is given by

$$\theta = \cos^{-1}(l_1 l_2 + m_1 m_2 + n_1 n_2).$$

The above two expressions for θ can also be written as

$$\theta = \sin^{-1} \frac{\{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2\}^{1/2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and $$\theta = \sin^{-1}\{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2\}^{1/2}.$$

Conditions for perpendicularity

If the lines are perpendicular to each other, then $\theta = \pi/2$, and hence $\cos\theta = 0$.

Therefore, $a_1a_2 + b_1b_2 + c_1c_2 = 0$ or $l_1l_2 + m_1m_2 + n_1n_2 = 0.$

These are the conditions for perpendicularity.

Conditions for parallel lines: If the lines are parallel, then $\theta = 0$, i.e. $\sin\theta = 0$.

Therefore, $(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1) = 0.$

This implies,

$$a_1b_2 - a_2b_1 = 0, \quad b_1c_2 - b_2c_1 = 0 \text{ and } c_1a_2 - c_2a_1 = 0$$

or $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2},$$

i.e. the DRs are proportional.

In terms of DCs the result is similar, i.e.,

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

These are the conditions for parallel lines.

EXAMPLE 16.7 Find the direction cosines of the line which are equally inclined to the axes.

Solution

If the line makes angles α, β, γ with the axes, then

$$\cos\alpha = \cos\beta = \cos\gamma.$$

Therefore, $$\alpha = \beta = \gamma.$$

If the edges
of a parallelo

Hence,
$$\frac{l}{1}=\frac{m}{1}=\frac{n}{1}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+1}}=\frac{1}{\sqrt{3}}.$$

Thus, the DCs of the line are $\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$.

EXAMPLE 16.8 A, B, C and D are points of $(\alpha, 3, -1)$, $(3, 5, -3)$, $(1, 2, 3)$ and $(3, 5, 7)$ respectively. If AB is perpendicular to CD then find the value of α.

(WBUT 2006)

Solution

The DRs of the lines AB and CD are respectively $\{\alpha - 3, 3 - 5, -1 + 3\}$ and $\{1 - 3, 2 - 5, 3 - 7\}$, i.e. $\{\alpha - 3, -2, 2\}$ and $\{-2, -3, -4\}$.

If AB and CD are perpendicular, then
$$(\alpha-3)\times(-2)+(-2)\times(-3)+2\times(-4)=0$$
or
$$2\alpha-4=0 \quad\text{or}\quad \alpha=2.$$

EXAMPLE 16.9 If the edges of a rectangular parallelopiped be a, b, c, show that the angles between the four diagonals are given by
$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right).$$

Solution

Let $ABC\ LMN\ PO$ be the rectangular parallelopiped shown in Figure 16.3.

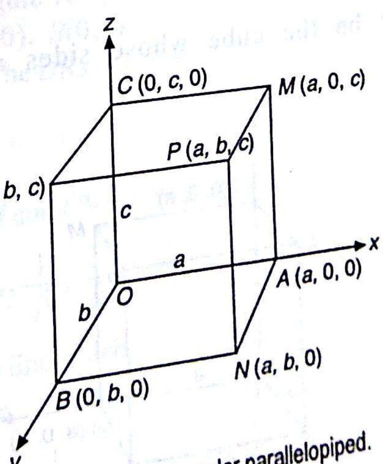

Figure 16.3 A rectangular parallelopiped.

Its four diagonals are AL, BM, CN and OP. The corner points are $O(0, 0, 0)$ $P(a, b, c)$, $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$, $L(0, b, c)$, $M(a, 0, c)$ and $N(a, b, 0)$.

The DRs of OP are $a - 0, b - 0, c - 0$ or a, b, c.

Therefore, DCs are

$$\frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}, \text{ where } \Sigma a^2 = a^2 + b^2 + c^2.$$

The DRs of AL are $0 - a, b - 0, c - 0$ or $-a, b, c$ and hence DCs are

$$\frac{-a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}.$$

Similarly, the DCs of BM and CN are

$$\frac{a}{\sqrt{\Sigma a^2}}, \frac{-b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}} \text{ and } \frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{-c}{\sqrt{\Sigma a^2}} \text{ respectively.}$$

If θ be the angle between OP and AL then

$$\cos\theta = \frac{a \times (-a) + b \times b + c \times c}{\sqrt{a^2 + b^2 + c^2}\sqrt{a^2 + b^2 + c^2}} = \frac{-a^2 + b^2 + c^2}{a^2 + b^2 + c^2}.$$

Similarly, we can determine the angle between other pairs and hence the angles between the six pairs are given by

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right).$$

EXAMPLE 16.10 Prove that the angle between two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

Solution

Let $OABCLMNP$ be the cube whose sides are of length a (see Figure 16.4).

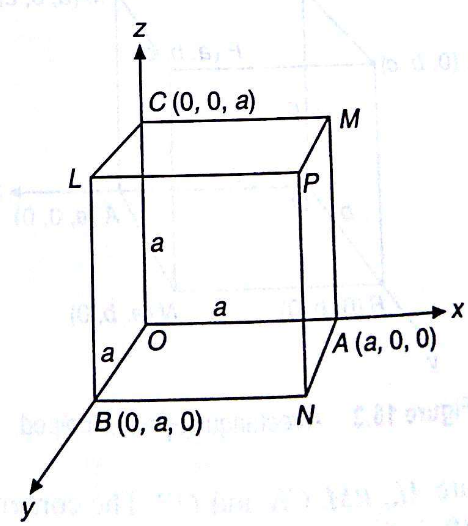

Figure 16.4 A cube.

The corner points of the cube are $O(0, 0, 0)$, $P(a, a, a)$, $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$, $L(0, a, a)$, $M(a, 0, a)$ and $N(a, a, 0)$.

The DRs of OP are $a-0, a-0, a-0$ or a, a, a.

Therefore, the DCs of OP are $\dfrac{a}{\sqrt{a^2+a^2+a^2}}, \dfrac{a}{\sqrt{a^2+a^2+a^2}}, \dfrac{a}{\sqrt{a^2+a^2+a^2}}$

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

Similarly, DCs of AL are $-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

Hence the required angle between the diagonals is

$$\cos^{-1}\left(\frac{\frac{1}{\sqrt{3}}\times\left(-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}}}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}\right)=\cos^{-1}\frac{1}{3}.$$

EXAMPLE 16.11 A line angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}.$$

Solution

Let $OABC\ LMNP$ (see Figure 16.4) be a cube of side a. The corners of the cube are $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$, $L(0, a, a)$, $M(a, 0, a)$, $N(a, a, 0)$ and $P(a, a, a)$. The DRs of OP are $a-0, a-0, a-0$ and hence DCs are

$\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

Similarly, DCs of AL, BM and CN are respectively

$$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}.$$

Let l, m, n be the DCs of a line which is inclined at angles $\alpha, \beta, \gamma, \delta$ respectively to the four diagonals, then

$$\cos\alpha = l\cdot\frac{1}{\sqrt{3}}+m\cdot\frac{1}{\sqrt{3}}+n\cdot\frac{1}{\sqrt{3}}=\frac{l+m+n}{\sqrt{3}}.$$

Similarly, $\cos\beta=\dfrac{-l+m+n}{\sqrt{3}}, \cos\gamma=\dfrac{l-m+n}{\sqrt{3}}, \cos\delta=\dfrac{l+m-n}{\sqrt{3}}.$

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Hence $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2\ (l+m-n)^2]$$

$$= \frac{1}{3}\cdot 4(l^2+m^2+n^2) = \frac{4}{3}.$$

EXAMPLE 16.12 Find the acute angle between the lines whose DCs are given by the relations

$$l+m+n=0 \text{ and } l^2+m^2-n^2=0.$$

Solution

Putting $n = -(l+m)$ to $l^2+m^2-n^2=0$. We get

$$l^2+m^2-(l+m)^2=0 \text{ or } 2lm=0.$$

When $\quad l=0$ then $m+n=0$ or $m=-n$.

$$\therefore \quad \frac{l}{0}=\frac{m}{1}=\frac{n}{-1}.$$

When $\quad m=0$ then $l+n=0$ or $l=-n$.

$$\therefore \quad \frac{l}{1}=\frac{m}{0}=\frac{n}{-1}.$$

Hence the DRs of the lines are 0, 1, –1 and 1, 0, –1.
The angle θ between the lines is

$$\cos\theta = \frac{0\cdot1+1\cdot0+(-1)(-1)}{\sqrt{0+1+1}\sqrt{0+1+1}} = \frac{1}{2}.$$

Hence, $\quad \theta = \dfrac{\pi}{3}.$

EXAMPLE 16.13 Find the direction cosines of the straight line passing through the points (1, 2, 4) and (3, 1, 3).

Solution

The DRs of the line joining the points (1, 2, 4) and (3, 1, 3) are $1-3, 2-1, 4-3$, or –2, 1, 1.

The DCs are $\dfrac{-2}{\sqrt{4+1+1}}, \dfrac{1}{\sqrt{4+1+1}}, \dfrac{-1}{\sqrt{4+1+1}}$, i.e. $\dfrac{-2}{\sqrt6}, \dfrac{1}{\sqrt6}, \dfrac{1}{\sqrt6}$.

16.4 PLANE

A *plane* is a surface such that if any two points are taken on it, the straight line joining them lies wholly in the surface, i.e., every point on the line joining the two points will be on the plane.

The most general equation of a plane is $ax + by + cz + d = 0$, where a, b, c, d are constants and a, b, c are not all zero. If the plane passes through the origin then its equation is $ax + by + cz = 0$. It can be shown that a, b, c represent the DRs of the plane and is directed perpendicular to the plane.

Equation of the plane passing through a point: Let $ax + by + cz + d = 0$ be the plane and it passes through the point (x_1, y_1, z_1). Then $ax_1 + by_1 + cz_1 + d = 0$. Subtracting these two equations, we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is the required equation of the plane.

Intercept form of a plane: Let the plane cuts the coordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ then its equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. This form is known as intercept form.

Normal form of a plane: If l, m, n are the DCs of a plane and p be the perpendicular distance from the origin to the plane, then $lx + my + nz = p$ represents the normal form of a plane.

Let $ax + by + cz + d = 0$ be the equation of a plane we divide this equation by $\sqrt{a^2 + b^2 + c^2}$, we get

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}x + \frac{b}{\sqrt{a^2 + b^2 + c^2}}y + \frac{c}{\sqrt{a^2 + b^2 + c^2}}z = -\frac{d}{\sqrt{a^2 + b^2 + c^2}}.$$

The perperpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is $\frac{d}{\sqrt{a^2 + b^2 + c^2}}$. Also, the sum of squares of the coefficients of x, y, z is 1. Hence the above equation is the normal form of the plane.

Equations of the coordinates planes: The planes XOY, YOZ, ZOX are called the coordinates planes the equations of these planes are respectively $z = 0, x = 0, y = 0$.

Equations of planes parallel to the axes: The equations of the planes parallel to x, y and z axes are respectively, $by + cz + d = 0$, $ax + cz + d = 0$ and $ax + by + d = 0$.

16.4.1 Distance from a Point to the Plane

Let $ax + by + cz + d = 0$ be the equation of the plane and $A(x_1, y_1, z_1)$ be the given point. Then the perpendicular distance from A to the plane is

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|.$$

EXAMPLE 16.14 Find the distance between the planes

$$x - 2y + z = 4 \quad \text{and} \quad 2x - 4y + 2z = 5.$$

Solution

The distance d_1 from (0, 0, 0) to the plane $x - 2y + z = 4$ is

$$d_1 = \frac{-4}{\sqrt{1+4+1}} = \frac{-4}{\sqrt{6}}.$$

Again, the distance d_2 from (0, 0, 0) to the plane $2x - 4y + 2z = 5$ is

$$d_2 = \frac{-5}{\sqrt{4+16+4}} = \frac{-5}{2\sqrt{6}}.$$

Hence the required distance between the planes is

$$|d_1 - d_2| = \left|\frac{-4}{\sqrt{6}} + \frac{5}{2\sqrt{6}}\right| = \frac{3}{2\sqrt{6}}.$$

16.4.2 Angle between Two Planes

The angle between planes is the angle between the normals to the plane drawn through a point. Let

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0$$

be the equations of the planes.

The DRs of the normals to the planes are a_1, b_1, c_1 and a_2, b_2, c_2.

Hence the angle θ between the planes is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Condition for perpendicular planes: If the planes are perpendicular then

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Condition for parallel planes: If the above planes are parallel, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

EXAMPLE 16.15 Determine the value of h for which the planes

$$3x - 2y + hz - 1 = 0 \text{ and } x + hy + 5z + 2 = 0$$

may be perpendicular to each other.

Solution

The DRs of the planes are 1, h, 5 and 3, −2, h. If they are perpendicular, then

$$1 \times 3 + h \times (-2) + 5 \times h = 0 \quad \text{or} \quad 3h + 3 = 0$$

or

$$h = -1.$$

16.4.3 Planes Bisecting the Angles between Two Planes

Let $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ be the equations of two planes. If $P(\alpha, \beta, \gamma)$ be any point on either of the two bisecting planes, then P is equidistant from the two planes. Therefore,

$$\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2\alpha+b_2\beta+c_2\gamma+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Hence the equations of the planes are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.$$

To distinguish the two bisecting planes follow the rule:

First write the equations of the planes with the same sign of the constants. Then calculate the angle between the planes. If $\cos\theta<0$, i.e. $a_1a_2+b_1b_2+c_1c_2<0$, then the origin is within the acute angle between the planes. If $a_1a_2+b_1b_2+c_1c_2>0$, then the origin is within the obtuse angle between the planes.

16.4.4 Worked-Out Examples on Plane

EXAMPLE 16.16 Show that the four points (0, –1, 0), (2, 1, –1), (1, 1, 1) and (3, 3, 0) are coplanar and obtain the equation of the plane.

Solution

Let the equation of the plane passes through the point (0, –1, 0) be

$$a(x-0)+b(y+1)+c(z-0)=0. \tag{1}$$

Since it pass through the points (2, 1, –1) and (1, 1, 1).

$$2a+2b-c=0 \tag{2}$$

and

$$a+2b+c=0. \tag{3}$$

Solving equations (2) and (3), we get

$$\frac{a}{2+2}=\frac{b}{-1-2}=\frac{c}{4-2} \quad \text{or} \quad \frac{a}{4}=\frac{b}{-3}=\frac{c}{2}=k \text{ (say)}$$

i.e.

$$a=4k, b=-3k, c=2k.$$

Putting these values in equation (1), we get

$$4x-3y+2z=3. \tag{4}$$

Again,

$$\text{LHS}=4\times3-3\times3+2\times0=3=\text{RHS}$$

i.e. (3, 3, 0) satisfies equation (4). Hence the given four points are coplanar and the required plane is $4x-3y+2z=3$.

EXAMPLE 16.17 Find the equation of the plane which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ and which is perpendicular to the plane $5x+3y+6z+8=0$.

Solution

Let the equation of the plane be

$$x + 2y + 3z - 4 + \lambda\,(2x + y - z + 5) = 0$$

or $$x\,(1 + 2\lambda) + y\,(2 + \lambda) + z\,(3 - \lambda) + 5\lambda - 4 = 0. \qquad (1)$$

Since it is perpendicular to the plane

$$5x + 3y + 6z + 8 = 0$$

therefore, $$5\,(1 + 2\lambda) + 3\,(2 + \lambda) + 6\,(3 - \lambda) = 0$$

or $$7\lambda + 29 = 0 \quad \text{or} \quad \lambda = -\frac{29}{7}.$$

Hence the required equation of the plane is

$$x\left(1 - \frac{58}{7}\right) + y\left(2 - \frac{29}{7}\right) + z\left(3 + \frac{29}{7}\right) - \frac{145}{7} - 4 = 0$$

or $$51x + 15y - 50z + 173 = 0.$$

EXAMPLE 16.18 Find the equation of the plane passing through the points (2, 3, –4), (1, –1, 3) and perpendicular to the plane $2x + 6y + 9z = 9$.

Solution

Let the equation of the plane passing through the point (2, 3, –4) be

$$a\,(x - 2) + b\,(y - 3) + c\,(z - 4) = 0. \qquad (1)$$

Since it passes through the point (1, –1, 3),

$$-a - 4b + 7c = 0. \qquad (2)$$

Also, (1) is perpendicular to the plane

$$2x + 6y + 9z = 9$$

$$\therefore \qquad 2a + 6b + 9c = 0. \qquad (3)$$

Solving equations (2) and (3) we get

$$a = 78k,\ b = -23k,\ c = -2k.$$

Putting these values in equation (1) we get the required equation of the plane

$$78\,(x - 2) - 23\,(y - 3) - 2\,(z + 4) = 0$$

or $$78x - 23y - 2z + 217 = 0.$$

EXAMPLE 16.19 Find the equation of the plane which passes through the point (1, 2, 1) and parallel to the plane $x + 2y - 3z = 5$.

Solution

The plane parallel to the given plane is

$$x + 2y - 3z = k.$$

Since it passes through the point (1, 2, 1),

$$1 + 2 \times 2 - 3 \times 1 = k \quad \text{or} \quad k = 2.$$

Hence the required equation of the plane is $x + 2y - 3z = 2$.

EXAMPLE 16.20 Find the equation of the plane passing through the point (2, 5, –8) and perpendicular to each of the planes $2x - 3y + 4z + 7 = 0$ and $4x + y - 2z + 16 = 0$.

Solution

Let the equation of the plane passing through the point (2, 5, –8) be

$$a(x-2)+b(y-5)+c(z+8)=0.$$

Since it is perpendicular to both the given planes,

$$2a-3b+4c=0 \quad \text{and} \quad 4a+b-2c=0.$$

Solving these equations, we get

$$\frac{a}{1}=\frac{b}{10}=\frac{c}{7}.$$

Hence the required equation of the plane is

$$(x-2)+10(y-5)+7(z+8)=0$$

or

$$x+10y+7z+4=0.$$

EXAMPLE 16.21 A variable plane passes through a fixed point (a, b, c) and meets the axes of reference in A, B and C. Show that the locus of the points of intersection of the planes through A, B and C parallel to the coordinate planes is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1. \qquad \text{(WBUT 2005)}$$

Solution

Let the equation of the plane be

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1. \qquad (1)$$

Since it passes through the point (a, b, c),

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1.$$

It meets the axes at $A(\alpha, 0, 0)$, $B(0, \beta, 0)$, $C(0, 0, \gamma)$.

Also planes through A, B and C and parallel to the coordinates planes are

$$a=\alpha,\ y=\beta,\ z=\gamma.$$

The locus of the point of intersection is obtained by eliminating the variables α, β, γ.

Putting the value of α, β, γ in equation (1) we get the required locus as

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

EXAMPLE 16.22 A variable plane at a constant distance p from the origin meets the axes in A, B and C. Through A, B, C, planes are drawn parallel to the coordinates planes. Show that the locus of their point of intersection is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$$ (WBUT 2006)

Solution

Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

It meets the axes in $A\,(a, 0, 0)$, $B\,(0, b, 0)$, $C\,(0, 0, c)$. The distance from origin to the plane is p (given).

$$\therefore \qquad \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p \quad \text{or} \quad \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}. \qquad (1)$$

Planes passing through A, B, C and parallel to the coordinates planes are $x=a$, $y=b$, $z=c$.

Putting the values of a, b, c in equation (1), we get the required locus as

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$$

EXAMPLE 16.23 A variable plane is at a constant distance $3p$ from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the triangle ABC is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$$

Solution

Let the coordinates of A, B and C be respectively $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Then the equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{x}{c}=1.$$

Its distance from origin is

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=3p$$

or

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{9p^2}. \qquad (1)$$

If (α, β, γ) be the centroid of the triangle ABC, then

$$\alpha = \frac{a+0+0}{3} = \frac{a}{3}, \beta = \frac{0+b+0}{3} = \frac{b}{3}, \gamma = \frac{0+0+c}{3} = \frac{c}{3}.$$

Putting the values of $a = 3\alpha, b = 3\beta, c = 3\gamma$ to equation (1), we get

$$\frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\gamma^2} = \frac{1}{9p^2}.$$

Hence the required locus is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}.$$

EXAMPLE 16.24 A variable plane is at a constant distance p from the origin and meets the axes in A, B, and C. Show that the locus of the centroid of the tetrahedron $OABC$ is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}.$$

Solution

Let the coordinates of A, B, C be $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

Then the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Since its distance from origin is p,

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = p \quad \text{or} \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \tag{1}$$

Let (α, β, γ) be the centroid of the tetrahedron $OABC$. Therefore,

$$\alpha = \frac{0+a+0+0}{4} = \frac{a}{4}, \beta = \frac{0+0+b+0}{4} = \frac{b}{4}, \gamma = \frac{0+0+0+c}{4} = \frac{c}{4}.$$

Putting the values of $a = 4\alpha, b = 4\beta, c = 4\gamma$ to equation (1), we get,

$$\frac{1}{16\alpha^2} + \frac{1}{16\beta^2} + \frac{1}{16\gamma^2} = \frac{1}{p^2}.$$

Hence the required locus is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}$.

EXAMPLE 16.25 Find the equations of the planes bisecting the angle between the planes $x + 2y + 2z = 9$ and $4x - 3y + 12z + 13 = 0$ and distinguish them.

Solution

The equations of the planes bisecting the angle between the given planes are

$$\frac{x+2y+2z-9}{\sqrt{1+4+4}}=\pm\frac{4x-3y+12z+13}{\sqrt{16+9+144}}$$

or $$25x+17y+62z-78=0$$

and $$x+35y-10z-156=0.$$

Acute or obtuse

The angle between the planes $x+2y+2z=9$ and the bisecting plane $x+35y-10z-156=0$ is given by

$$\cos\theta=\frac{1\times 1+2\times 35+2\times(-10)}{\sqrt{1+35^2+10^2}\sqrt{1+2^2+2^2}}=\frac{17}{36}.$$

$\therefore$ $$\tan\theta=\frac{\sqrt{1007}}{17}>1.$$

Thus, $$\theta>45^\circ.$$

Hence the plane $x-35y-10z-156=0$ bisects the obtuse angle between the planes and hence the other plane $25x+17y+62z-78=0$ bisects the acute angle.

With respect to origin: We write the equations of the planes so that the constant terms in both are positive, i.e.

$$-x-2y-2z+9=0 \quad \text{and} \quad 4x-3y+12z+13=0.$$

Then the plane $$\frac{-x-2y-2z+9}{\sqrt{1^2+2^2+2^2}}=+\frac{4x-3y+12z+13}{\sqrt{4^2+3^3+12^2}}$$

or $$25x+17y+62z-78=0$$

bisects the angle between the planes that contains the origin.

16.5 STRAIGHT LINE

Let (x_1, y_1, z_1) be a given point on the line and l, m, n be the DCs or DRs of the line. Let (x, y, z) be any point on the line, then the DCs are proportional to $x-x_1$, $y-y_1$, $z-z_1$.

Hence the equations of the line passing through the point (x_1, y_1, z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

This equation is called the *standard form* or *canonical form* or *symmetric form* of a line.

In particular, the equations of straight line passing through the origin are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

It may be noted that the direction of l, m, n is parallel to the line.

Equations of the straight line passing through two points: Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in space. Then the equations of the line AB are

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

As $x_1 - x_2, y_1 - y_2, z_1 - z_2$ represents DRs of the line.

Parametric form: If the line passes through the point (x_1, y_1, z_1) and DRs or DCs are l, m, n then its equations are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (say)}.$$

These equation can be written as

$$x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr.$$

This form of a straight line is called parametric form and r is called the parameter.

Plane intercept form: If two planes intersect, then the intersection between them generates a straight line. Thus the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ represent a straight line.

Equation of coordinates axes: We know any point on x-axis is of the form $(a, 0, 0)$. Thus, the equation of x-axis is $y = 0, z = 0$.

Similarly, the equations of y-axis and z-axis are respectively $z = 0, x = 0$ and $x = 0, y = 0$.

16.5.1 Perpendicular Distance of a Point from a Straight Line

Let $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ be the equation of the line and $P(\alpha, \beta, \gamma)$ be the given point. Then the perpendicular distance from P to the line is given by

$$[(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - \{(x_1 - \alpha)l + (y - \beta)m + (z_1 - \gamma)n\}^2]^{1/2}.$$

16.5.2 Condition of Coplanarity of Two Straight Lines

Let $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ be two lines. If these two lines are coplanar then the plane should pass through the points (x_1, y_1, z_1) and (x_2, y_2, z_2). Therefore, the equation of the plane can be taken as $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$. Since it passes through the point (x_2, y_2, z_2),

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \quad (16.1)$$

Also, the DRs of the lines are perpendicular to the *DRs* of the plane, hence

$$al_1 + bm_1 + cn_1 = 0 \quad (16.2)$$

and

$$al_2 + bm_2 + cn_2 = 0. \quad (16.3)$$

Eliminating *a, b, c* between equations (16.1), (16.2) and (16.3), we get the required conditions of the co-planarity

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

The equation of the plane where the straight lines lie (co-planar) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

EXAMPLE 16.26 Show that the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

are co-planar. Find also the equation of the plane.

Solution

Here $\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$

$$= 1(16-15) - 1(12-10) + (9-8) = 0.$$

Hence the given lines are co-planar.

The equation of the plane is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

or $(x-1)(15-16) - (y-2)(10-12) + (z-3)(8-9) = 0$

or $x - 2y + z = 0.$

EXAMPLE 16.27 Prove that the following lines:

$$x + 2y - 5z + 9 = 0 = 3x - y + 2z - 5$$

$$2x + 3y - z - 3 = 0 = 4x - 5y + z + 3$$

are coplanar and also find the equation of the plane containing them.

Solution

Let l, m, n be the DRs of the first line, then

$$l + 2m - 5n = 0 \quad \text{and} \quad 3l - m + 2n = 0.$$

Solving these equations, we get $\frac{l}{1} = \frac{m}{17} = \frac{n}{7}$.

Let $(0, \beta, \gamma)$ be any point on the first line, then

$$2\beta - 5\gamma + 9 = 0 \text{ and } -\beta + 2\gamma - 5 = 0.$$

The solution of these equations is $\beta = -7, \gamma = -1$.

Therefore, $(0, -7, -1)$ is a point on the first line. Hence the symmetrical form of the first line is

$$\frac{x}{1} = \frac{y+7}{17} = \frac{z+1}{7}.$$

Similarly, if l', m', n' be the DRs of the second line, then

$$2l' + 3m' - n' = 0 \text{ and } 4l' - 5m' + n' = 0.$$

Solving we get, $\frac{l'}{1} = \frac{m'}{3} = \frac{n'}{11}$.

The point $(0, 0, -3)$ lie on the second line. Hence the equation of the second line in symmetrical form is

$$\frac{x}{1} = \frac{y}{3} = \frac{z+3}{11}.$$

Test of coplanarity: Now, $\begin{vmatrix} 0-0 & -7-0 & -1+3 \\ 1 & 17 & 7 \\ 1 & 3 & 11 \end{vmatrix} = 0.$

Hence the lines are coplanar.

Equation of the plane: The equation of the required plane is

$$\begin{vmatrix} x & y+7 & z+1 \\ 1 & 17 & 7 \\ 1 & 3 & 11 \end{vmatrix} = 0$$

or $$x(187-21) - y(11-7) + (z+3)(3-17) = 0$$

or $$83x - 2y - 7z - 51 = 0.$$

16.5.3 Shortest Distance between Two Skew Lines

The non-parallel non-intersecting lines are called *skew lines* (see Figure 16.5).

Let the equations of the skew lines LA and MB be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} = r_1 \text{ (say)}$$

and $$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}=r_2 \text{ (say)}.$$

Let LM be the shortest distance. Therefore, LM is perpendicular to both the lines.

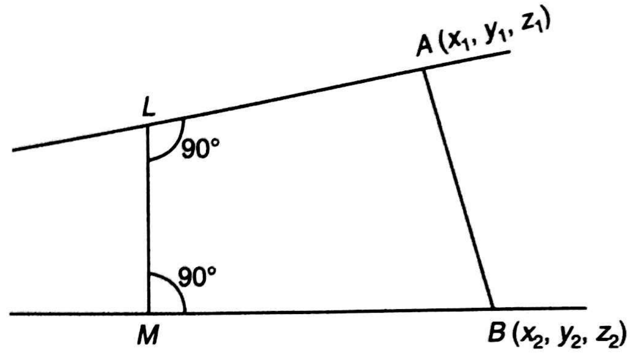

Figure 16.5 Skew lines.

Any point on first and second lines are respectively

$$(x_1+l_1r_1, y_1+m_1r_1, z_1+n_1r_1) \text{ and } (x_2+l_2r_2, y_2+m_2r_2, z_2+n_2r_2).$$

For particular values of r_1 and r_2 these points will be L and M. So, the DRs of the line LM are

$$x_1-x_2+l_1r_1-l_2r_2,\ y_1-y_2+m_1r_1-m_2r_2,\ z_1-z_2+n_1r_1-n_2r_2.$$

Since LM is perpendicular to both LA and MB, then

$$l_1(x_1-x_2+l_1r_1-l_2r_2)+m_1(y_1-y_2+m_1r_1-m_2r_2)+n_1(z_1-z_2+n_1r_1-n_2r_2)=0$$

and $$l_2(x_1-x_2+l_1r_1-l_2r_2)+m_2(y_1-y_2+m_1r_1-m_2r_2)+n_2(z_1-z_2+n_1r_1-n_2r_2)=0.$$

There are linear equations of r_1 and r_2 and solving these two equations, we get the values of r_1 and r_2 and hence, we get the coordinates of L, M. Then the distance between L and M is the required shortest distance and the equation of LM is the equation of shortest distance line.

The following example illustrates the method completely.

EXAMPLE 16.28 Find the shortest distance and its equation between the lines

$$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5},\quad \frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}.$$

Solution

Given $$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}=r_1 \text{ (say)}$$

and $$\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}=r_2 \text{ (say)}.$$

Any points on the first and the second lines are respectively,

$$L(2r_1 + 3, -7r_1 - 15, 5r_1 + 9) \text{ and } M(2r_2 - 1, r_2 + 1, -3r_2 + 9).$$

The DRs of the line LM are

$$2r_1 - 2r_2 + 4, -7r_1 - r_2 - 16, 5r_1 + 3r_2.$$

If LM is the shortest distance then LM is perpendicular to both the lines, and hence

$$2(2r_1 - 2r_2 + 4) - 7(-7r_1 - r_2 - 16) + 5(5r_1 + 3r_2) = 0$$

and

$$2(2r_1 - 2r_2 + 4) + (-7r_1 - r_2 - 16) - 3(5r_1 + 3r_2) = 0.$$

That is,

$$13r_1 + 3r_2 + 20 = 0 \text{ and } 9r_1 + 7r_2 + 4 = 0.$$

Solving these equations, we get

$$r_1 = -2, \quad r_2 = 2.$$

Hence the coordinates of L and M are respectively

$$(-1, -1, -1) \text{ and } (3, 3, 3).$$

Therefore, the required shortest distance is

$$\sqrt{(-1-3)^2 + (-1-3)^2 + (-1-3)^2} = 4\sqrt{3}$$

and the equation of shortest distance line is

$$\frac{x+1}{-1-3} = \frac{y+1}{-1-3} = \frac{z+1}{-1-3}$$

or

$$\frac{x+1}{1} = \frac{y+1}{1} = \frac{z+1}{1}$$

or

$$x = y = z.$$

EXAMPLE 16.29 Find the magnitude of the shortest distance between the lines

$$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}$$

and

$$5x - 2y - 3z + 6 = 0, x - 3y + 2z - 3 = 0.$$

Solution

Any plane through the second line is

$$(5x - 2y - 3z + 6) + \lambda(x - 3y + 2z - 3) = 0$$

or

$$(5 + \lambda)x + (-2 - 3\lambda)y + (-3 + 2\lambda)z + (6 - 3\lambda) = 0.$$

If it is parallel to the first line, then

$$4(5 + \lambda) + 3(-2 - 3\lambda) + 2(-3 + 2\lambda) = 0 \text{ or } \lambda = 8.$$

Hence the equation of the plane containing the second line and parallel to the first line is

$$13x - 26y + 13z - 18 = 0. \tag{1}$$

Let (0, −1, 2) be a point on the first line. Then the required shortest distance between the given lines is equal to the shortest distance from the point (0, −1, −2) to the plane of equation (1), which is

$$\frac{26 + 26 - 18}{\sqrt{13^2 + 26^2 + 13^2}} = \frac{34}{13\sqrt{6}} = \frac{17}{39}\sqrt{6}.$$

16.5.4 Worked-Out Example on Straight Lines

EXAMPLE 16.30 Find the value of k, so that the lines

$$\frac{x+4}{k} = \frac{y+6}{5} = \frac{z-1}{-2} \text{ and } 3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$$

may intersect. What is the point of intersection ?

Solution

Let $\frac{x+4}{k} = \frac{y+6}{5} = \frac{z-1}{-2} = r$ (say). Any point on the line is

$$(kr - 4, 5r - 6, -2r + 1).$$

If the lines intersect then this point lies on the second line. Therefore,

$$3(kr - 4) - 2(5r - 6) + (-2r + 1) + 5 = 0$$

and $$2(kr - 4) + 3(5r - 6) + 4(-2r + 1) - 4 = 0$$

or $$kr - 4r + 2 = 0 \text{ and } 2kr + 7r - 26 = 0.$$

Solving these equations, we get

$$r = 2, \quad k = 3.$$

Therefore, the required value of k is 3 and the point of intersection is (2, 4, −3).

EXAMPLE 16.31 Find the equations of the image of the line

$$\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{4}$$

in the plane $3x + y - 4z + 21 = 0$.

Solution

The find point of intersection between the line and the plane (see Figure 16.6).

Let $$\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{4} = r \text{ (say)}.$$

Any point on this line is

$$(2r + 2, 3r + 3, 4r + 4).$$

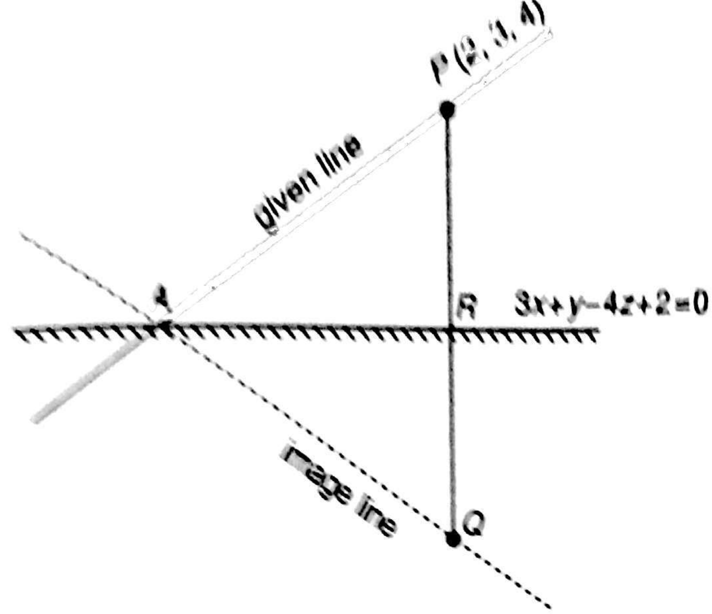

Figure 16.6 AQ is the image of AP.

If it lies on the plane, then

$$3(2r+2)+(3r+3)-4(4r+4)+21=0$$

or

$$r=2.$$

Hence the point of intersection is A (6, 9, 12).

To find the image of the point P(2, 3, 4): Here P (2, 3, 4) is a point on the given line. Let Q (a, b, c) be the image of P. Let R be the point of intersection between the given plane and the line PQ.

$$PR = RQ, \text{ i.e. } R \text{ is the middle point of } PQ.$$

∴ Therefore, the coordinate of R is $\left(\frac{a+2}{2}, \frac{b+3}{2}, \frac{c+4}{2}\right)$. Again, PQ is perpendicular to the given plane and passing through the point P, so its equation is

$$\frac{x-2}{3}=\frac{y-3}{1}=\frac{z-4}{-4}=r_1 \text{ (say)}.$$

Any point on this line (PQ) is $(3r_1+2, r_1+3, -4r_1+4)$.

If it lies on the plane, $3(3r_1+2)+(r_1+3)-4(-4r_1+4)+21=0$.

$$r_1=-\frac{7}{13}$$

i.e.

Therefore, the coordinates of R are $\left(\frac{5}{13}, \frac{32}{13}, \frac{80}{13}\right)$

$$\frac{a+2}{2}=\frac{5}{13}, \frac{b+3}{2}=\frac{32}{13}, \frac{c+4}{2}=\frac{80}{13}$$

Thus,

$$a=-\frac{16}{13},\ b=\frac{25}{13},\ c=\frac{108}{13}.$$

or

Hence the image of P is $Q\left(-\frac{16}{13},\frac{25}{13},\frac{108}{13}\right)$.

The image of the line AP is the line AQ and its equation is

$$\frac{x-6}{6+\frac{16}{13}}=\frac{y-9}{9-\frac{25}{13}}=\frac{z-12}{12-\frac{108}{13}}$$

or

$$\frac{x-6}{47}=\frac{y-9}{46}=\frac{z-12}{24}.$$

EXAMPLE 16.32 Prove that the equation of the perpendicular from the point (3, –1, 11) to the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is $\frac{x-3}{1}=\frac{y+1}{-6}=\frac{z-11}{4}$ and that the coordinates of its foot are (2, 5, 7).

Solution

Any point on the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r$ is $A(2r, 3r+2, 4r+3)$. Let B be the point whose coordinates are (3, –1, 11) (see Figure 16.7).

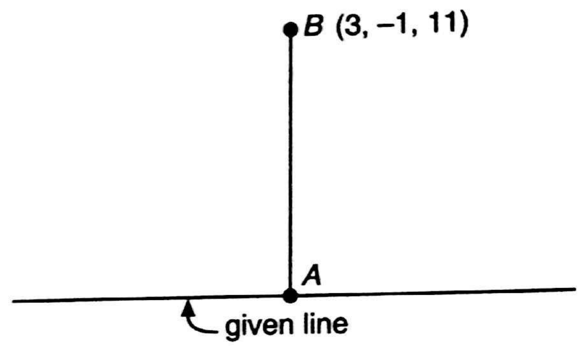

Figure 16.7

The DRs of AB are

$$2r-3, 3r+3, 4r-8.$$

If AB is perpendicular to the line, then

$$2(2r-3)+3(3r+3)+4(4r-8)=0$$

or $\quad r=1.$

Therefore, the coordinates of A are (2, 5, 7). Hence the equation of the line perpendicular to AB is

$$\frac{x-3}{3-2}=\frac{y+1}{-1-5}=\frac{z-11}{11-7}\quad\text{or}\quad\frac{x-3}{1}=\frac{y+1}{-6}=\frac{z-11}{4}$$

and the coordinates of foot of the perpendicular are (2, 5, 7).

...AMPLE 16.33 Obtain the equation of the plane through the straight line $...-4y+5z-10=0,\ 2x+2y-3z-4=0$ and parallel to the line $x=2y=3z$. (WBUT 2006)

...olution

...t the equation of the plane passing through the line $3x-4y+5z-10=0,\ 2x+...y-3z-4=0$ be

$$3x-4y+5z-10+\lambda(2x+2y-3z-4)=0$$

...r $$(3+2\lambda)x+(-4+2\lambda)y+(5-3\lambda)z-10-4\lambda=0.$$

If this plane is parallel to the line $x=2y=3z$

...r $$\frac{x}{1}=\frac{y}{1/2}=\frac{z}{1/3}\ \text{then}$$

$$1(3+2\lambda)+\frac{1}{2}(-4+2\lambda)+\frac{1}{3}(5-3\lambda)=0$$

...r $$\lambda=-\frac{4}{3}.$$

Hence the required equation of the plane is

$$\left(3-\frac{8}{3}\right)x+\left(-4-\frac{8}{3}\right)y+(5+4)z-10+\frac{16}{3}=0$$

or $$x-20y+27z-14=0.$$

EXAMPLE 16.34 A straight line with direction ratios $2, 7, -5$ is drawn to intersect the lines $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$ and $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}$.

Find the coordinates of the points of intersection and length intercepted on it. (WBUT 2005)

Solution

Let AB be the line whose DRs are $2, 7, -5$ and it meets the given lines at

$$(3r_1+5,\ -r_1+7,\ r_1-2) \text{ and } (-3r_2-3,\ 2r_2+3,\ 4r_2+6).$$

Then the DRs of the line AB are $-3r_2-3r_1-8,\ 2r_2+r_1-4,\ 4r_2-r_1+8$.

These are proportional to $2, 7, -5$.

$$\therefore \quad \frac{-3r_2-3r_1-8}{2}=\frac{2r_2+r_1-4}{7}=\frac{4r_2-r_1+8}{-5}.$$

Form the first and the second ratio, we have

$$-21r_2-21r_1-56=4r_2+2r_1-8$$

or $$23r_1+25r_2+48=0$$

$$10r_2-5r_1+20=28r_2-7r_1+56$$

or $$r_1 - 19\, r_2 - 18 = 0.$$

Solving these equations, we get $r_1 = r_2 = -1$.

Thus the points of intersection are (2, 8, –3) and (0, 1, 2).

The length of intercept between the lines is

$$\sqrt{(2-0)^2 + (8-1)^2 + (-3-2)^2} = \sqrt{78}.$$

EXAMPLE 16.35 Find the equation of the projection of the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4}$$

on the plane $x + 3y + z + 5 = 0$.

Solution

The projection of a line on a plane is the line of intersection of the given plane and the plane passing through the given line and perpendicular to the given plane.

The given line and the plane are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4} \tag{1}$$

and $$x + 3y + z + 5 = 0. \tag{2}$$

The equation of a plane through the line of equation (1) is

$$a\,(x-1) + b\,(y-2) + c\,(z-4) = 0 \tag{3}$$

where $$2a + 3b + 4c = 0 \tag{4}$$

If the plane of equation (3) perpendicular to the plane of equation (2), then

$$a + 3b + c = 0 \tag{5}$$

Solving equations (4) and (5), we get

$$\frac{a}{9} = \frac{b}{2} = \frac{c}{3}$$

Thus the equation of the plane containing the line of equation (1) and perpendicular to the plane of equation (2) is

$$-9\,(x-1) + 2\,(y-2) + 3\,(z-4) = 0$$

or $$9x - 2y - 3z + 7 = 0$$

Hence the equations of the projection are

$$x + 3y + z + 5 = 0, 9x - 2y - 3z + 7 = 0.$$

EXAMPLE 16.36 Show that the equation to the plane containing the line

$$\frac{y}{b} + \frac{z}{c} = 1,\ x = 0$$

and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1,\ y = 0$ is $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$

and if $2d$ is the shortest distance prove that

$$\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

Solution

Let the equation of the plane containing first line be

$$\frac{y}{b}+\frac{z}{c}-1+\lambda x=0. \tag{1}$$

It will be parallel to the second line, if it is parallel to the plane

$$\frac{x}{a}-\frac{z}{c}-1+\mu y=0 \tag{2}$$

through the second line.

Since these two planes are parallel, therefore,

$$\frac{\lambda}{1/a}=\frac{1/b}{\mu}=\frac{1/c}{-1/c} \text{ or } \lambda=\frac{1}{a} \text{ and } \mu=-\frac{1}{b}.$$

Substituting these values of λ in equation (1), we get

$$-\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0 \text{ or } \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0, \tag{3}$$

which is the required equation of the plane.

A point on the second line is $(a, 0, 0)$. The shortest distance is the distance of the plane of equation (3) from the point $(a, 0, 0)$.

$$\therefore \quad 2d=\frac{2}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \text{ or } \frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

EXAMPLE 16.37 Find the distance of the point (3, 8, 2) from the line

$$\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}$$

measured parallel to the plane $3x+2y-2z+15=0$.

Solution

Let P be the given point (3, 8, 2) and Q be the point on the given line whose coordinates are $(2r+1, 4r+3, 3r+2)$. The DRs of PQ are

$$2r+1-3, 4r+3-8, 3r+2-2 \text{ or } 2r-2, 4r-5, 3r$$

Now, PQ is parallel to the plane

$$3x+2y-2z+15=0$$

and hence it is perpendicular to normal 3, 2, –2.

$$\therefore \quad 3(2r-2)+2(4r-5)-2\times 3r=0 \text{ or } r=2.$$

Hence the point Q is (5, 11, 8) and the required distance is

$$PQ = \sqrt{(5-3)^2 + (11-8)^2 + (8-2)^2} = 7.$$

EXAMPLE 16.38 Prove that the straight lines

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}, \frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma}, \frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

will lie in one plane if

$$\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$$

Solution

Given lines lie on a plane. Let L, M, N be the DRs of such plane. Then L, M, N are perpendicular to the DRs of the straight lines. Hence

$$L\alpha + M\beta + N\gamma = 0 \tag{1}$$

$$La\alpha + Mb\beta + Nc\gamma = 0 \tag{2}$$

$$Ll + Mm + Nn = 0 \tag{3}$$

Solving equations (1) and (3), we get

$$\frac{L}{\beta n - m\gamma} = \frac{M}{l\gamma - n\alpha} = \frac{N}{\alpha m - l\beta} = \lambda \text{ (say)}.$$

Substituting the values of L, M, N to equation (2), we get

$$a\alpha(\beta n - m\gamma) + b\beta(l\gamma - n\alpha) + c\gamma(\alpha m - l\beta) = 0$$

or
$$l(b-c)\beta\gamma + m(c-a)\alpha\gamma + n(a-b)\alpha\beta = 0$$

or
$$\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$$

EXAMPLE 16.39 A variable line intersects the lines $x = b, y + c = 0; y = c, z + a = 0; z = a, x + b = 0$. Show that the locus of the line is

$$axy + byz + czx + abc = 0.$$

Solution

Any line intersecting the first two lines is

$$x - b + k_1(y + c) = 0, \; y - c + k_2(z + a) = 0. \tag{1}$$

It meets the third line. Therefore, putting $z = a$ and $x = -b$, we get

$$-2b + k_1(y + c) = 0 \text{ and } y - c + 2ak_2 = 0.$$

Eliminating y, we get

$$ak_1k_2 - ck_1 + b = 0. \tag{2}$$

Putting the value of λ_1 and λ_2 from equations (1) in (2), we get

$$a\,\frac{x-?}{y-?}\;\frac{y-?}{z-?}+?\,\frac{x-?}{y-?}+? = 0$$ [illegible]

$$axy + \ldots = 0$$ [illegible]

or

EXAMPLE 16.40 Show that the surface generated by a straight line which intersects the lines $y = 0$, $z = c$; $x = 0$, $z = -c$ and the curve $z = 0$, $xy + c^2 = 0$ is $z^2 - c^2 = xy$.

Solution

Any line intersecting the first two lines is

$$y + \lambda_1 (z - c) = 0 \text{ and } x + \lambda_2 (z + c) = 0. \tag{1}$$

It meets the curve $z = 0$, $xy + c^2 = 0$.

Putting $z = 0$ in equation (1), we get

$$y - \lambda_1 c = 0 \text{ and } x + \lambda_2 c = 0 \text{ i.e. } xy = -\lambda_1 \lambda_2 c^2$$

$$[\because xy + c^2 = 0]$$

$$-c^2 = -\lambda_1 \lambda_2 c^2 \tag{2}$$

or

$$\lambda_1 \lambda_2 = 1.$$

or

Putting the values of λ_1, λ_2 from equation (1) to (2), we get

$$\frac{y}{z-c} \times \frac{x}{z+c} = 1 \quad \text{or} \quad xy = z^2 - c^2,$$

which is the required surface.

EXAMPLE 16.41 Prove that the locus of the point which is equidistant from the lines $y = mx$, $z = c$ and $y = -mx$, $z = -c$ is the surface $mxy + (1 + m^2)\, cz = 0$.

Solution

Let (α, β, γ) be the variable point. The gives equations of the given lines can be written as

$$\frac{x}{1} = \frac{y}{m} = \frac{z-c}{0} \text{ and } \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0}.$$

The distance from (α, β, γ) to the straight lines are

$$\left\{\alpha^2 + \beta^2 + (\gamma - c)^2 - \frac{(\alpha + m\beta + 0)^2}{1^2 + m^2}\right\}^{1/2}$$

$$\left\{\alpha^2 + \beta^2 + (\gamma + c)^2 - \frac{(\alpha - m\beta + 0)^2}{1^2 + m^2}\right\}^{1/2}.$$

and

Since these are equal,

$$\alpha^2+\beta^2+(\gamma-c)^2-\frac{(\alpha+m\beta)^2}{1+m^2}=\alpha^2+\beta^2+(\gamma+c)^2-\frac{(\alpha-m\beta)^2}{1+m^2}$$

or

$$4\gamma c=\frac{(\alpha-m\beta)^2-(\alpha+m\beta)^2}{1+m^2}$$

or

$$\gamma c\,(1+m^2)=-\,m\alpha\beta.$$

Hence the required locus of (α,β,γ) is

$$mxy+(1+m^2)\,cz=0.$$

16.6 QUADRIC SURFACES

A surface defined in space by an equation of the second degree in x, y, z is called a *quadric surface* or a quadric or conicoid. The general equation of a conicoid is $ax^2 + by^2 + cz^2 + 2gzx + 2fyz + 2hxy + 2ux + 2vy + 2wz + d = 0$. For different values of its coefficients this equation represents different quadric, viz. sphere, cone, cylinder, ellipsoid, hyperboloid, paraboloid, etc. In this, section we discuss the quadric sphere, right circular cone and right circular cylinder. The equations of some standard conicoids are give below:

Ellipsoid : $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Hyperboloid of one sheet : $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

Hyperboloid of two sheets : $\quad -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Elliptic paraboloid : $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$

Hyperbolic paraboloid : $\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}.$

16.7 SPHERE

A sphere is the locus of a point which moves in a space so that it is always at a constant distance from a fixed point. The fixed point is called the *centre* of the sphere and the distance is called the *radius* of the sphere.

Equation of a sphere

General form: The very general equation of a sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

It may be noted that the coefficient of x^2, y^2, z^2 are same, there is no term of the form yz, zx, xy. If $u^2 + v^2 + w^2 - d > 0$ then the above quadric represents a sphere. The centre and radius of this sphere are respectively $(-u, -v, -w)$ and $\sqrt{(u^2 + v^2 + w^2 - d)}$.

Centre and radius being given: If (a, b, c) be the coordinates of centre and r be the radius, then the equation of the sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Diameter form: If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the end points of a diameter then the equation of the sphere is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

and its centre is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$.

We have seen that a straight line in space can be drawn uniquely if two points are known, a plane can be drawn if three points are given. Similarly, if four non-coplanar points are given then we can drawn a sphere uniquely.

In the following example, we illustrate how a sphere can be obtained if four non-coplanar points are given.

EXAMPLE 16.42 Find the equation of the sphere through the four points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

Solution

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2zw + d = 0.$$

Since it passes through the points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$,

$$d = 0$$

$$a^2 + 2ua = 0 \text{ or } 2u = -a$$

$$b^2 + 2vb = 0 \text{ or } 2v = -b$$

$$c^2 + 2wc = 0 \text{ or } 2w = -c.$$

Hence the required equation of the sphere is

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

16.7.1 Plane Section of a Sphere

If the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cuts by the plane $ax + by + cz + d' = 0$ then the section generates a circle in space. In Figure 16.8, O is the centre of the sphere, C is the centre of the circle PQ, OP is the radius of the sphere and CP is the radius of the circle. CO is perpendicular to the plane. It may be noted that the radius of the circle is $CP = \sqrt{OP^2 - OC^2}$. The equation of a circle is the combination of the sphere and the plane, i.e.,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,\ ax + by + cz + d' = 0$$

together represent a circle.

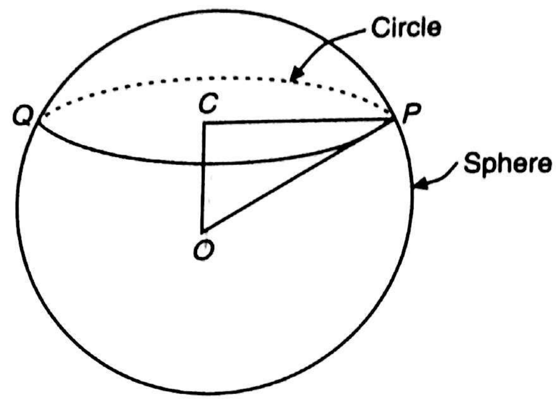

Figure 16.8 Circle in space.

If the circle passes through the centre of the sphere, then it is called *great circle.*

If $S = 0$, $P = 0$ ($S = 0$ is a sphere and $P = 0$ is a plane) represents a circle, then the equation of a sphere passing through this circle is $S + \lambda P = 0$, for a suitable value of λ.

EXAMPLE 16.43 Find the centre and radius of the circle

$$x^2 + y^2 + z^2 - 2x + 6y + 4z - 35 = 0, \quad x - 2y - 2z + 7 = 0.$$

Solution

The centre and the radius of the sphere are respectively $(1, -3, -2)$ and $\sqrt{1 + 9 + 4 + 35} = 7$.

The equation of the line perpendicular to the plane and passing through the centre of the sphere are

$$\frac{x-1}{1} = \frac{y+3}{-2} = \frac{z+2}{-2}.$$

Any point on the line is $(r + 1, -2r - 3, -2r - 2)$. If this point lies on the plane, then

$$1 \cdot (r+1) + 2 \cdot (2r+3) + 2\,(2r+2) + 7 = 0 \text{ or } 9r = -18 \text{ or } r = -2.$$

Therefore, the centre of the circle is $(-1, 1, 2)$.

Now, the distance between C and O i.e. $(1, -3, -2)$ and $(-1, 1, 2)$

$$\sqrt{(1+1)^2 + (-3-1)^2 + (-2-2)^2} = 6.$$

Therefore, the radius of the circle is $\sqrt{7^2 - 6^2} = \sqrt{13}.$

EXAMPLE 16.44 Show that the equation to the sphere through the circle

$$x^2 + y^2 + z^2 = 9, \ 2x + 3y + 4z = 5$$

and the point (1, 2, 3) is $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$.

Solution

Let the equation of the sphere be

$$x^2 + y^2 + z^2 - 9 + \lambda(2x + 3y + 4z - 5) = 0.$$

Since it passes through the point (1, 2, 3),

$$1 + 4 + 9 - 9 + \lambda(2 + 6 + 12 - 5) = 0 \text{ or } \lambda = -\frac{1}{3}.$$

Hence the required equation of the sphere is

$$x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

or
$$3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0.$$

EXAMPLE 16.45 Show that the equation of the sphere for which the circle

$$x^2 + y^2 + z^2 + 7y - 2z + 2 = 0, \ 2x + 3y + 4z - 8 = 0$$

is a great circle is $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$.

Solution

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0$$

or
$$x^2 + y^2 + z^2 + 2\lambda x + (7 + 3\lambda)y + (-2 + 4\lambda)z + (2 - 8\lambda) = 0.$$

Its centre is $\left(-\lambda, -\frac{(7 + 3\lambda)}{2}, (1 - 2\lambda)\right)$.

If the circle is a great circle, this centre lies on the given plane

∴
$$-2\lambda - \frac{3}{2}(7 + 3\lambda) + 4(1 - 2\lambda) - 8 = 0 \text{ or } \lambda = -1.$$

Hence the required equation of the sphere be

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0.$$

EXAMPLE 16.46 A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the locus of the centroid of the triangle ABC is the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

Solution

Let the coordinates of A, B and C be $(a, 0, 0)$, $(0, b, 0)$ $(0, 0, c)$ and the fourth point be $(0, 0, 0)$. The equation of the sphere through these points is

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

Its radius is
$$\sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}} = k$$

or
$$a^2 + b^2 + c^2 = 4k^2. \tag{1}$$

Let (α, β, γ) be the coordinates of the centroid of the triangle ABC then

$$\alpha = \frac{a+0+0}{3} = \frac{a}{3}, \beta = \frac{0+b+0}{3} = \frac{b}{3}, \gamma = \frac{0+0+c}{3} = \frac{c}{3}.$$

Putting the values of a, b, c in equation (1), we get

$$9(\alpha^2 + \beta^2 + \gamma^2) = 4k^2$$

Hence the required locus is $9(x^2 + y^2 + z^2) = 4k^2$.

EXAMPLE 16.47 A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is

$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2.$$

Solution

Let the equation of the plane passing through A, B, C be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{1}$$

where the coordinates of A, B, C are respectively $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. The equation of the sphere $OABC$ is

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

If (α, β, γ) be the centre then,

$$\alpha = \frac{a}{2}, \beta = \frac{b}{2}, \gamma = \frac{c}{2}.$$

Since the plane of equation (1) passes through (p, q, r),

$$\frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1.$$

Putting the values of $a = 2\alpha$, $b = 2\beta$, $c = 2\gamma$, we get

$$\frac{p}{2\alpha} + \frac{q}{2\beta} + \frac{r}{2\gamma} = 1.$$

Hence the required locus is $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2.$

EXAMPLE 16.48 Obtain the equations of the tangent planes to the sphere

$$x^2+y^2+z^2+6x-2z+1=0$$

which pass through the line $3(16-x)=3z=2y+30$.

Solution

The equations of the line can be written as

$$3(16-x)=3z \text{ or } x+z-16=0 \text{ and } 3z-2y-30=0$$

Any plane through this line is

$$x+z-16+\lambda(3z-2y-30)=0$$

$$\text{or}\quad x-2\lambda y+(1+3\lambda)z-(16+30\lambda)=0 \qquad (1)$$

If it is a tangent plane then perpendicular from centre $(-3, 0, 1)$ of the sphere should be equal to the radius $\sqrt{9+0+1-1}=3$.

Therefore,
$$\frac{-3+(1+3\lambda)-(16+30\lambda)}{\sqrt{1+4\lambda^2+(1+3\lambda)^2}}=\pm 3$$

$$\text{or}\quad 9(2+3\lambda)=3\sqrt{(2+6\lambda+13\lambda^2)}$$

$$\text{or}\quad 9(4+12\lambda+9\lambda^2)=2+6\lambda+13\lambda^2$$

$$\text{or}\quad 2\lambda^2+3\lambda+1=0 \text{ or } \lambda=-1, -\frac{1}{2}.$$

Hence the equations of the planes are

$$2x+2y-z-2=0 \text{ and } x+2y-2z+14=0,$$

obtained by putting $\lambda=-1, -\frac{1}{2}$.

EXAMPLE 16.49 Find the value of k for which the plane $x+y+z=k$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$.

Solution

The centre and radius of the sphere are respectively $(1, 1, 1)$ and $\sqrt{1+1+1+6}=3=r$ (say).

The perpendicular distance d from $(1, 1, 1)$ to the plane is $\dfrac{1+1+1-k}{\sqrt{3}}$.

If the plane touches the sphere then $d=r$, i.e.

$$\frac{3-k}{\sqrt{3}}=3 \text{ or } k=3-3\sqrt{3}=3(1-\sqrt{3}),$$

which is the required value of k.

16.8 RIGHT CIRCULAR CYLINDER

A *cylinder* is a surface generated by a variable straight line parallel to a fixed straight line called axis and satisfying one or more conditions, i.e. intersecting a given curve or touching a given surface is called a cylinder (see Figure 16.9).

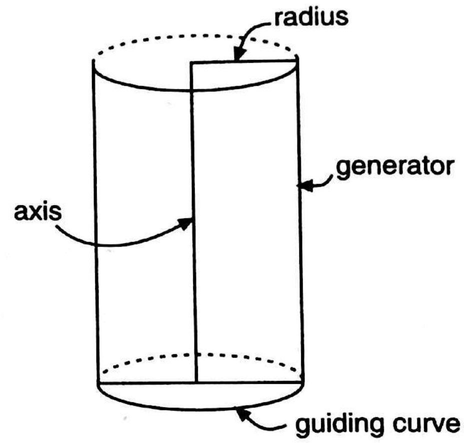

Figure 16.9 A right circular cylinder.

The given curve is called the *guiding curve* or *base* or *directrix* and the variable line is called as *generator* (see Figure 16.9). If the guiding curve is a circle and the fixed line is perpendicular to the plane of the circle through the centre of the circle, then the cylinder is called *right circular cylinder* and the fixed line is called the *axis* of the cylinder. The distance between the axis and any generator is the *radius* of the right circular cylinder and it is equal to the radius of the guiding circle.

EXAMPLE 16.50 Find the equation of the cylinder whose guiding curve is $x^2 + y^2 = 9, z = 1$ and the axis is $\frac{x}{2} = \frac{y}{3} = \frac{z}{-1}$.

Solution

Let (α, β, γ) be any point on the cylinder. The generator through this point is

$$\frac{x-\alpha}{2} = \frac{y-\beta}{3} = \frac{z-\gamma}{-1} \text{ (since axis and generator are parallel).}$$

It cuts the guiding curve at $z = 1$, then

$$\frac{x-\alpha}{2} = \frac{y-\beta}{3} = \frac{1-\gamma}{-1}$$

or
$$x = \alpha + 2\gamma - 2,\ y = \beta + 3\gamma - 3,\ z = 1.$$

This point lies on the curve $x^2 + y^2 = 9$.

$\therefore$
$$(\alpha + 2\gamma - 2)^2 + (\beta + 3\gamma - 3)^2 = 9.$$

Hence the locus of (α, β, γ), i.e. the equation of the cylinder is

$$(x + 2z - 2)^2 + (y + 3z - 3)^2 = 9.$$

EXAMPLE 16.51 Find the equation of the right circular cylinder of radius 3 whose axis passes through the point (1, 3, 4) and has 1, −2, 3 as its direction ratios.

Solution

The equations of the axis is

$$\frac{x-1}{1}=\frac{y-3}{-2}=\frac{z-4}{3}.$$

Let $P(x, y, z)$ be any point on the cylinder. Then perpendicular distance from P to the axis is

$$\left\{(x-1)^2+(y-3)^2+(z-4)^2-\frac{[1\cdot(x-1)-2(y-3)+3(z-4)]^2}{1^2+(-2)^2+3^2}\right\}^{1/2}$$

which is equal to the radius of the cylinder.

Thus, $$(x-1)^2+(y-3)^2+(z-4)^2-\frac{(x-2y+3z-7)^2}{14}=3^2$$

or $$13x^2+10y^2+5z^2+4xy+12yz-6xz-14x-112y-70z+189=0.$$

This is the required equation of the cylinder.

EXAMPLE 16.52 Find the equation of the right circular cylinder whose guiding curve is $x^2+y^2+z^2=9,\ x-y+z=3.$

Solution

The equation of the sphere is

$$x^2+y^2+z^2=9 \qquad (1)$$

whose centre is (0, 0, 0) and radius is 3.

The given plane is

$$x-y+z=3. \qquad (2)$$

The axes of the cylinder is normal to the plane equation (2) and passes through the centre of the sphere (see Figure 16.10).

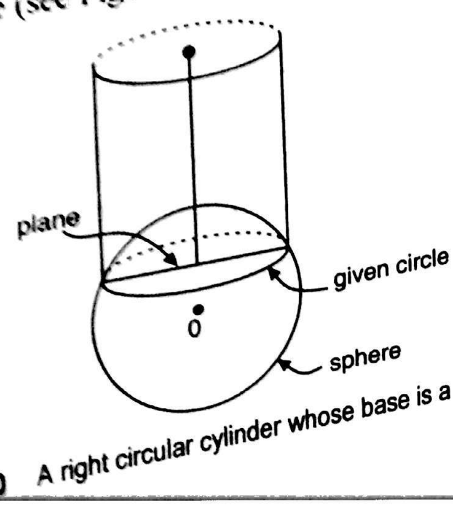

Figure 16.10 A right circular cylinder whose base is a circle in space.

Therefore, the equation of the axes of the cylinder are

$$\frac{x-0}{1}=\frac{y-0}{-1}=\frac{z-0}{1}.$$

Length of the perpendicular from the centre of the sphere on the plane equation (2) is

$$\left|\frac{0+0+0-3}{\sqrt{1+1+1}}\right|=\sqrt{3}.$$

Therefore, the radius of the circle is $\sqrt{3^2-3}=\sqrt{6}$. This is the radius of the cylinder. Let (x, y, z) be any point on the cylinder. Then the perpendicular distance from this point to the axis of the cylinder is

$$\left[x^2+y^2+z^2-\frac{(1\cdot x-1\cdot y+1\cdot z)^2}{1+1+1}\right]^{1/2}$$

which is equal to the radius of the cylinder $\sqrt{6}$.

$$\therefore \qquad x^2+y^2+z^2-\frac{(x-y+z)^2}{3}=6$$

or

$$x^2+y^2+z^2-xz+xy+yz-9=0,$$

which is the required equation of the cylinder.

EXAMPLE 16.53 Find the equation of the cylinder whose generator touches the following two spheres $x^2+y^2+z^2=36$ and $(x-3)^2+(y-1)^2+z^2=36$.

Solution

Since radii of both the spheres are equal, the generators touch both the sphere (see Figure 16.11). So the radius of the cylinder is 6, and equal for both the spheres.

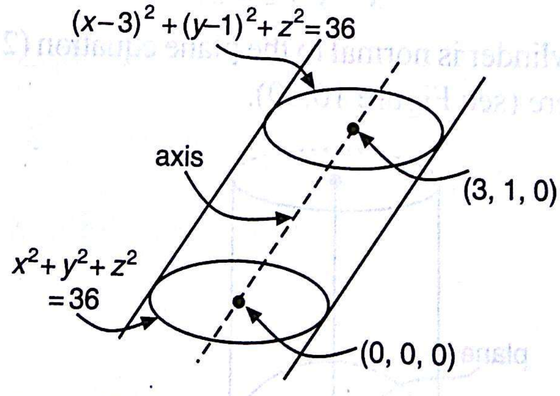

Figure 16.11 Cylinder touches two spheres.

The centres of the spheres are (0, 0, 0) and (3, 1, 0), and the axis of the cylinder passes through the centres. Thus, the equation of the axis is

$$\frac{x}{3}=\frac{y}{1}=\frac{z}{0}.$$

... any point on the generator. The perpendicular distance from P to the axis is

$$\left[x^2+y^2+z^2-\frac{(3x+y+z\cdot 0)^2}{9+1+0}\right]^{1/2}$$

and this is equal to the radius of the cylinder.

$$\therefore \qquad x^2+y^2+z^2-\frac{(3x+y)^2}{10}=36$$

or

$$x^2+9y^2+10z^2-6xy-360=0.$$

This is the required equation of the cylinder.

16.9 RIGHT CIRCULAR CONE

A *cone* is a surface generated by a straight line passing through a fixed point and intersecting a curve or touching a given surface (see Figure 16.12).

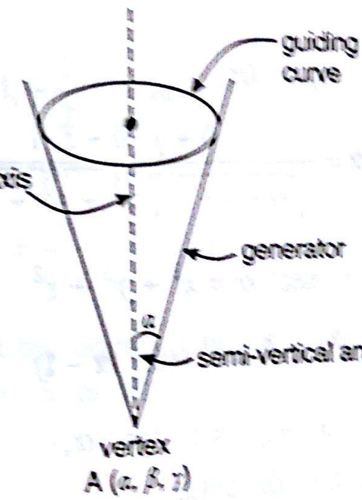

Figure 16.12 A right circular cone.

The fixed point is known as the vertex and the given curve is called the guiding curve or *base* or *directrix*. Any line lying on the cone is called its generator.

A *right circular cone* is the surface generated by a line passing through a fixed point called the *vertex* and which is inclined at a constant angle α to a fixed line through the vertex.

The constant angle is called the *semi-vertical angle* of the cone and the fixed line is called the *axis* of the cone.

It may be noted that the equation of a cone with its vertex as origin is homogeneous in x, y, z and conversely.

EXAMPLE 16.54 Find the equation to a right circular cone whose vertex is origin and axis is z-axis and semi-vertical angle α.

Solution

The equation of the axis is $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$, since the DCs of z-axis are 0, 0, 1 (see Figure 16.13).

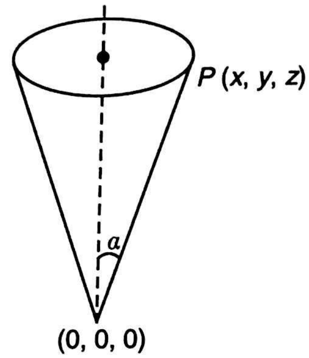

Figure 16.13

Let $P(x, y, z)$ be any point on the cone. The DRs of the generator OP are x, y, z. The semi-vertical angle of the cone is α, which is the angle between OP and the axis.

DRs of OP is $\{x-0, y-0, z-0\}$.

$\therefore$
$$\cos\alpha = \frac{x\cdot 0 + y\cdot 0 + z\cdot 1}{\sqrt{0+0+1}\sqrt{x^2+y^2+z^2}}$$

or
$$z^2\sec^2\alpha = x^2+y^2+z^2$$

or
$$x^2+y^2 = z^2(\sec^2\alpha - 1)$$

or
$$x^2+y^2 = z^2\tan^2\alpha,$$

which is the required equation of the cone.

EXAMPLE 16.55 Find the equations of the lines of intersection of the plane $3x+4y+z=0$ and the cone $15x^2-32y^2-7z^2=0$.

Solution

Let the equation of the generating line in which the plane cuts the cone be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

The line lies on the cone and also on the plane.

$\therefore$
$$3l+4m+n=0$$

and
$$15l^2-32m^2-7n^2=0.$$

$$15l^2 - 32m^2 - 7(3l + 4m)^2 = 0$$

or
$$2l^2 + 7ml + 6m^2 = 0$$

$$(2l + 3m)(l + 2m) = 0$$

or
$$m = -\frac{2}{3}l, -\frac{1}{2}l.$$

When
$$m = -\frac{2}{3}l,$$

then
$$n = -3l - 4m = -\left(3l - \frac{8}{3}l\right) = -\frac{l}{3}$$

Therefore, the DRs are $1, -\frac{2}{3}, -\frac{1}{3}$ or $3, -2, -1$.

Again, when $m = -\frac{1}{2}l$, $n = -(3l - 2l) = -l$.

In this case, the DRs are $1, -\frac{1}{2}, -1$ or $2, -1, -2$.

Hence the equations of the required generators are

$$\frac{x}{3} = \frac{y}{-2} = \frac{z}{-1} \text{ and } \frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}.$$

EXAMPLE 16.56 Find the equation of the right circular cone which passes through the line $2x = 3y = -5z$ and has the line $x = y = z$ as its axis.

Solution

The given line is $2x = 3y = -5z$

or
$$\frac{x}{1/2} = \frac{y}{1/3} = \frac{z}{-1/5}. \quad (1)$$

The equation of the axis is $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$. (2)

Let the angle between the lines represented by equations (1) and (2) be θ, then

$$\cos\theta = \frac{\frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1 + \left(-\frac{1}{5}\right)\cdot 1}{\sqrt{1+1+1}\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{25}}} = \frac{1}{\sqrt{3}}. \quad (3)$$

Let (x, y, z) be a point on the surface of the cone.

Therefore, the DRs of the generator are $x-0, y-0, z-0$.

$$\therefore \qquad \cos\theta = \frac{x\cdot 1 + y\cdot 1 + z\cdot 1}{\sqrt{3}\sqrt{x^2+y^2+z^2}}. \qquad (4)$$

From equations (3) and (4),

$$\frac{1}{\sqrt{3}} = \frac{x+y+z}{\sqrt{3}\sqrt{(x^2+y^2+z^2)}}$$

or $$(x^2+y^2+z^2) = (x+y+z)^2$$

or $$xy + yz + zx = 0$$

which is the required equation of the cone.

EXAMPLE 16.57 Find the equation of the right circular cone whose vertex lies in the yz-plane, axis is the line $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z+1}{-1}$ and which passes through the point $\left(1, 1, -\frac{1}{2}\right)$.

Solution

Let $(0, \alpha, \beta)$ be the vertex on the yz-plane.

Let

$$\frac{x-0}{l} = \frac{y-\alpha}{m} = \frac{z-\beta}{n} \qquad (1)$$

be any generator of the cone.

Since axis passes through the point $(0, \alpha, \beta)$.

$$\therefore \qquad \frac{0-2}{2} = \frac{\alpha+1}{-2} = \frac{\beta+1}{-1}$$

or $$\alpha = 1 \text{ and } \beta = 0.$$

Therefore, the vertex is (0, 1, 0) and equation (1) becomes

$$\frac{x}{l} = \frac{y-1}{m} = \frac{z-0}{n}. \qquad (2)$$

Direction ratios of the line joining (0, 1, 0) and $\left(1, 1, -\frac{1}{2}\right)$ are $1, 0, -\frac{1}{2}$.

The semi-vertical angle is

$$\cos\theta = \frac{2\cdot 1 + (-2)\cdot 0 + (-1)\cdot\left(-\frac{1}{2}\right)}{\sqrt{4+4+1}\sqrt{1+0+\frac{1}{4}}} = \frac{\sqrt{5}}{3}.$$

Also, the angle between generator and the axis is

$$\cos\theta = \frac{2l - 2m - n}{\sqrt{9}\sqrt{l^2 + m^2 + n^2}} = \frac{2l - 2m - n}{3\sqrt{l^2 + m^2 + n^2}}$$

$\therefore$
$$\frac{\sqrt{5}}{3} = \frac{2l - 2m - n}{3\sqrt{l^2 + m^2 + n^2}}$$

or
$$(2l - 2m - n)^2 = 5\,(l^2 + m^2 + n^2). \qquad (3)$$

Eliminating l, m, n between equation (2) and (3) we get the required cone

$$\{2x - 2\,(y-1) - z\}^2 = 5\,\{x^2 + (y-1)^2 + z^2\}$$

or
$$x^2 + y^2 + 4z^2 + 8xy + 4xz - 4yz - 8x - 2y + 4z + 1 = 0.$$

EXERCISES

Short Answer Questions

(Section A)

1. The direction cosines of the straight line passing through the points (1, 2, 4) and (3, 1, 3) are ......$\frac{2}{\sqrt6}, \frac{-1}{\sqrt6}, \frac{-1}{\sqrt6}$
2. A directed straight line makes angles 60°, 45° with the axes of x and y respectively. The angle it makes with the z-axis is .....60.....
3. The coordinates of the middle point of the line joining the points (2, 3, 4) and (–2, 1, 4) are .(0, 2, 4).....
4. The square of the distance between the points (0, 1, 2) and (1, 0, 1) is ....
5. The equation of the plane which passes through the points (1, –1, 2) and is parallel to the plane $2x + 3y + 4z + 1 = 0$, is ....
6. The equation of the plane passing through the point (4, –3, 5) and containing the y-axis is ....
7. The equation of the plane passing through the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) is ....
8. The distance of the point (1, 2, –3) from the plane $5x - 3y + z + 5 = 0$ is ....
9. The distance between the planes $x + 2y - 3z + 5 = 0$ and $x + 2y - 3z - 7 = 0$ is ....
10. The angle between the planes $x - y + 2z = 9$ and $2x + y + z = 7$ is ....
11. The equation of the plane through the point (1, 2, –3) and normal to the straight line joining the points (–1, 3, 4) and (5, 2, –1) is ....