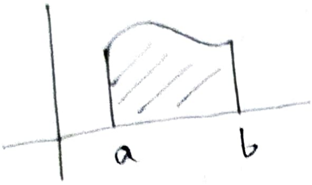
Multiple Integral

Line Integral

$f \geq 0 \quad \int_a^b f(x)\,dx$

= Area of A

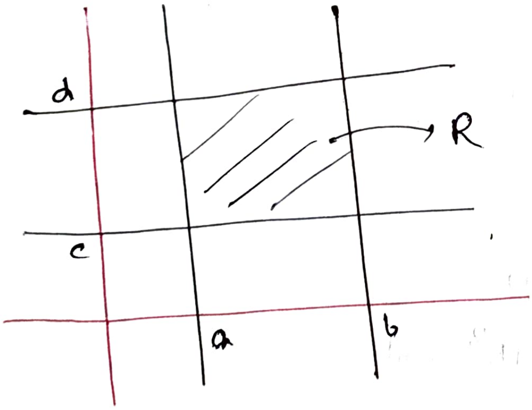


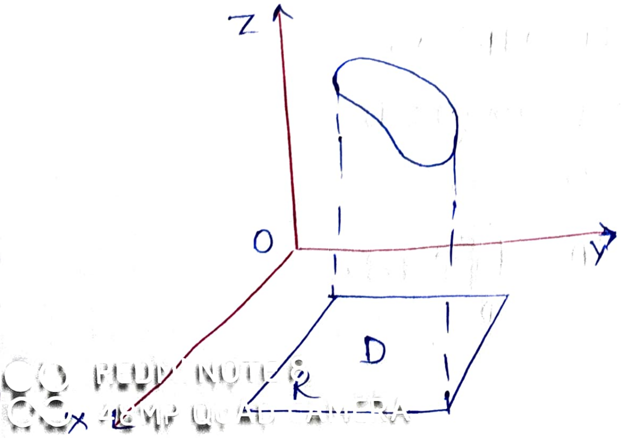
$\int_0^\pi \sin x \, dx$

$F'(x) = f(x)$

$F(x) \Big|_0^\pi$

$F(\pi) = F(0)$



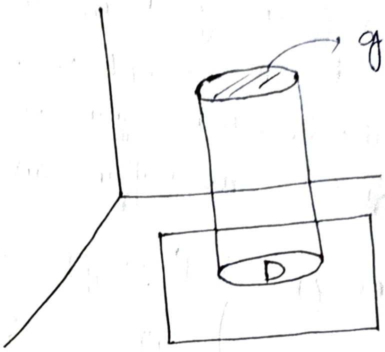


Double Integral

$f : R \to \mathbb{R}$

$R = [a,b] \times [c,d]$

Rectangle (Region of integration)



graph of f

The volume under the function and above D equals to that under f and above R

$f : \mathbb{R} \to \mathbb{R}$

Graph $\subseteq \mathbb{R}^2$ plane

$f : \mathbb{R}^2 \to \mathbb{R}$

Graph $\subseteq \mathbb{R}^3$

Graph $f = \{ (x, f(x)) : x \in A \} \subseteq A \times$

$f : A \to B$

when,

$f(x,y) \geq 0$

$$\iint_R f(x,y)\, dA = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx.$$

$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy.$$

↳ Double And Iterated Integrals.

Method – Repeated Integral

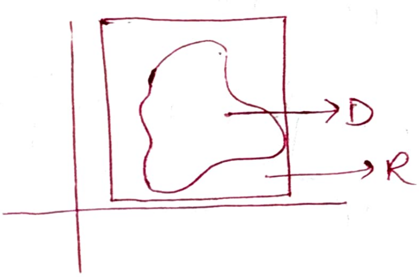
NOTE: We define the double integral only for rectangular regions of integration. However, it is not difficult to extend the concept to more general regions. Let, D be the bounded region, and enclose D in a rectangle R. Define a new function $\bar{f}$ on R as follows.

$$\bar{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \setminus D \end{cases}$$

$$\iint_D f(x,y)\, dx\, dy = \iint_R \bar{f}(x,y)\, dx\, dy$$

$f : R \to \mathbb{R}$ → bounded

$f : D\ (\leq \mathbb{R}^2) \longrightarrow \mathbb{R}$



$f \geq 0$

$R \leq \mathbb{R}^2$ s.t

$R \geq D$

$\tilde{f} : R \to \mathbb{R}^2$ as

$\tilde{f} = f,\ (x,y) \in D$

$= 0$

$$\iint_R \bar{f}\, dA = \iint_D f\, dx\, dy$$

Ex 1 $\iint_D xy^2\,dx\,dy$, when $D : [0,2] \times [0,1]$

$$= \int_0^2 \left\{ \int_0^1 xy^2\,dy \right\} dx$$

$$= \int_0^2 \left[\frac{xy^3}{3} \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \left[\frac{x}{3} - 0 \right] dx = \frac{x^2}{6} \Big|_0^2 = \frac{2}{3}$$

Same answer will come if we exchange the position of dx and dy.

Remark: $\int_a^b \left\{ \int_c^d f(x,y)\,dx \right\} dy$

$R = [c,d] \times [a,b]$

(they are diff) $\int_a^b \int_c^d f\,dx\,dy \neq \int_a^b \int_c^d f\,dy\,dx$

Ref: if f is continuous on a rectangle R, then it is integral on R (not real no.)

$\mathbb{R} \to$ set of all real no.

NOTE: $\int_a^b f(x)\,dx$

$\downarrow$

$f \geq 0$

if $f < 0$, we consider $f = -(-f) \to > 0$

and we can apply our defination for $-f$

Similarly: if $f(x,y) < 0$ for some $(x,y) \in D$ then we consider $-f(x,y)$ and apply it as

$f = -(-f) \to > 0$

Properties :

- If $f(x,y)$ be integrable on R, it is integrable on any sub-rectangle of R. $\to R$
- If f be integrable on R, so are f^+, f^- and $|f|$.
- If f and g be integrable on R, so are $f+g$, $f-g$, fg, cf for some constant c and if $|g| \geq c$ for some constant $c > 0$; so is f/g.
- If f be integrable on R and if R_1 and R_2 be formed from R by cutting it with a line parallel to one of the co-ordinate axes then,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

- If f and g be integrable on R and if $f \geq g$ on R, then

$$\iint_R f(x,y)\,dx\,dy \geq \iint_R g(x,y)\,dx\,dy$$

- If $f(x,y) = k$ for all $(x,y) \in R$.

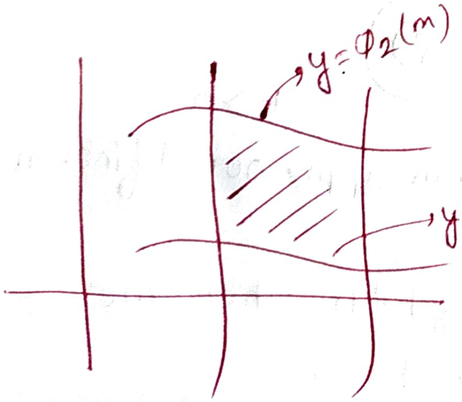
$$\iint_R f(x,y)\,dx\,dy = k\ (\text{area of } R)$$
$$= k\,(b-a)\,(d-c)$$

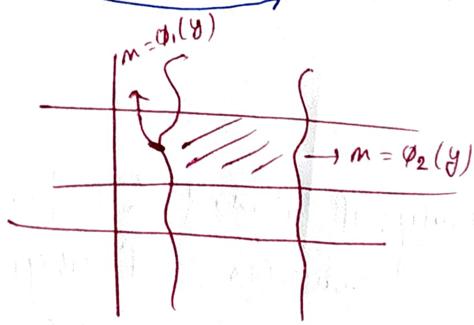
Type-I

Type-II

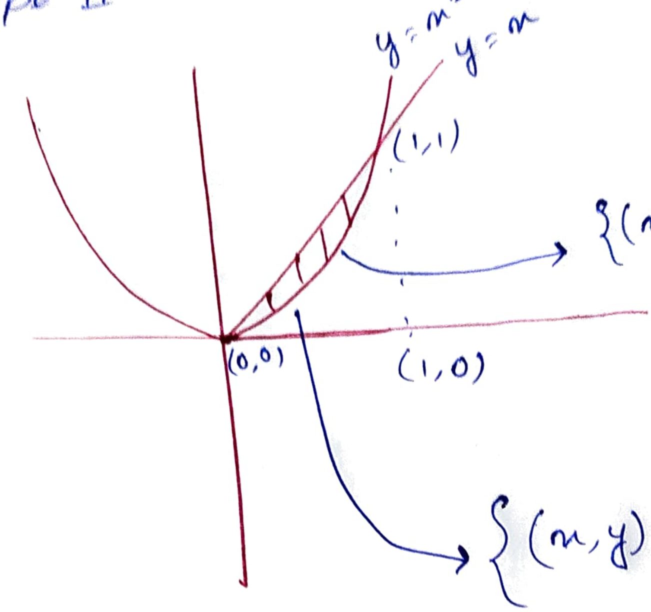
$S = \{(x,y) : a \leq x \leq b,\ \underbrace{\phi_1(x) \leq y \leq \phi_2(x)}_{\text{constant function}}$

$T = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d,\ \underbrace{\phi_1(y) \leq x \leq \phi_2(y)}$





Type III : If it can be written as either type I and type II



$\{(x,y) : 0 \le x \le 1,\ x^2 \le y \le x\}$ Type I

$\{(x,y) : 0 \le y \le 1,\ y \le x \le \sqrt{y}\}$ Type II

hence we can say that it is type III region.

H.W

Ex 2 Evaluate $\cancel{\iint_D x e^y \, dx\, dy}$

$$\iint_D xy^3/_5 \, dx\, dy$$

$$= \int \frac{xy^4}{20} \, dx$$

$$= y^4/_{20} \, \frac{x^2}{2}$$

$$= \frac{x^2 y^4}{40} \quad \text{(Ans.)}$$