CHAPTER 5

Simultaneous Linear Differential Equations with Constant Coefficients

5.1 Introduction

In many engineering problems, it is observed that the number of dependent variables may be more than one, but the independent variable is one. In such cases the problem can be formulated as a system of differential equations.

To solve a system of differential equations, the simplest technique is construct an equation of higher order in one dependent variable by eliminating the other dependant variables. The elimination can be done in many different ways.

5.2 Solution of First Order Simultaneous Equations

EXAMPLE 5.2.1 Solve the equations: $\dfrac{dx}{dt} + y = e^t$, $\dfrac{dy}{dt} - x = e^{-t}$. (WBUT 2003)

Solution Given:

$$\frac{dx}{dt} + y = e^t \tag{i}$$

and

$$\frac{dy}{dt} - x = e^{-t} \tag{ii}$$

From (i)

$$y = -\frac{dx}{dt} + e^t \tag{iii}$$

Substituting y in (ii), we get $-\dfrac{d^2x}{dt^2} + e^t - x = e^{-t}$, or $\dfrac{d^2x}{dt^2} + x = e^t - e^{-t}$.

Let $x = ce^{mt}$ be a trial solution of $\dfrac{d^2x}{dt^2} + x = 0$.

$\therefore$ A.E. is $m^2 + 1 = 0$ or $m = \pm i$.

$\therefore$ C.F. is $c_1 \cos t + c_2 \sin t$.

$$\text{P.I.} = \frac{1}{D^2+1}(e^t - e^{-t}) = \frac{1}{D^2+1}e^t - \frac{1}{D^2+1}e^{-t} = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$$

Thus

$$x = c_1 \cos t + c_2 \sin t + \frac{1}{2}(e^t - e^{-1})$$

Now, from (iii)

$$\begin{aligned} y &= -\frac{d}{dt}[c_1 \cos t + c_2 \sin t + \frac{1}{2}(e^t - e^{-t})] + e^t \\ &= c_1 \sin t - c_2 \cos t + \frac{1}{2}(e^t + e^{-t}) + e^t \\ &= c_1 \sin t - c_2 \cos t + \frac{1}{2}(3e^t + e^{-t}) \end{aligned}$$

Hence the required solution is

$$x = c_1 \cos t + c_2 \sin t + \frac{1}{2}(e^t - e^{-t}) \text{ and } y = c_1 \sin t - c_2 \cos t + \frac{1}{2}(3e^t + e^{-t})$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 5.2.2 Solve: $\dfrac{dx}{dt} - \dfrac{dy}{dt} + 2y = \cos 2t$ and $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x = \sin 2t$.

Solution The given equations can be written as

$$Dx - (D-2)y = \cos 2t \tag{i}$$

and

$$(D-2)x + Dy = \sin 2t \tag{ii}$$

Operating (i) and (ii) by D and $D-2$ respectively

$$D^2 x - D(D-2)y = -2\sin 2t$$

and

$$(D-2)^2 x + D(D-2)y = 2\cos 2t - 2\sin 2t$$

Adding these two equations, we get

$$(2D^2 - 4D + 4)x = 2\cos 2t - 4\sin 2t \qquad \text{or} \qquad (D^2 - 2D + 2)x = \cos 2t - 2\sin 2t$$

Let $x = ce^{mt}$ be a trial solution of $(D^2 - 2D + 2)x = 0$.

$\therefore$ A.E. is $m^2 - 2m + 2 = 0$ or $m = 1 \pm i$.

Hence C.F. is $e^t(c_1 \cos t + c_2 \sin t)$.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(\cos 2t - 2\sin 2t) = \frac{1}{-2D-2}(\cos 2t - 2\sin 2t) \quad (\because D^2 = -4)$$

$$= -\frac{1}{2}\frac{D-1}{D^2-1}(\cos 2t - 2\sin 2t)$$

$$= -\frac{1}{2}\frac{D-1}{-4-1}(\cos 2t - 2\sin 2t)$$

$$= \frac{1}{10}(-2\sin 2t - 4\cos 2t - \cos 2t + 2\sin 2t)$$

$$= -\frac{1}{2}\cos 2t$$

$\therefore\ x = e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\cos 2t$. Adding given equations

$$2\frac{dx}{dt} - 2x + 2y = \cos 2t + \sin 2t$$

Therefore

$$y = \frac{1}{2}(\cos 2t + \sin 2t) - \frac{dx}{dt} + x$$

$$= \frac{1}{2}(\cos 2t + \sin 2t) - [e^t(c_1 \cos t + c_2 \sin t) + e^t(-c_1 \sin t + c_2 \cos t) + \sin 2t]$$

$$+ e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\cos 2t$$

$$= e^t(c_1 \sin t - c_2 \cos t) - \frac{1}{2}\sin 2t$$

Hence the general solution is

$$x = e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\cos 2t \quad \text{and} \quad y = e^t(c_1 \sin t - c_2 \cos t) - \frac{1}{2}\sin 2t$$

EXAMPLE 5.2.3 Solve: $\frac{dx}{dt} + 4x + 3y = t$, $\frac{dy}{dt} + 2x + 5y = e^t$.

Solution The given equations are

$$(D+4)x + 3y = t \tag{i}$$

and

$$2x + (D+5)y = e^t \tag{ii}$$

Operating (i) by $(D+5)$, multiplying (ii) by 3 and subtracting, we get

$$(D^2 + 9D + 14)x = 1 + 5t - 3e^t$$

Let $x = ce^{mt}$ be a trial solution of $(D^2 + 9D + 14)x = 0$.

$\therefore$ A.E. is $m^2 + 9m + 14 = 0$ or $m = -2, -7$.

$\therefore$ C.F. is $c_1e^{-2t} + c_2e^{-7t}$.

$$\text{P.I.} = \frac{1}{D^2 + 9D + 14}(1 + 5t - 3e^t)$$

$$= \frac{1}{D^2 + 9D + 14}(1 + 5t) - \frac{1}{D^2 + 9D + 14}3e^t$$

$$= \frac{1}{14}\left(1 + \frac{D^2 + 9D}{14}\right)^{-1}(1 + 5t) - \frac{3}{1 + 9 + 14}e^t$$

$$= \frac{1}{14}\left(1 - \frac{D^2 + 9D}{14} - \cdots\right)(1 + 5t) - \frac{1}{8}e^t$$

$$= \frac{1}{14}\left(1 + 5t - \frac{45}{14}\right) - \frac{1}{8}e^t$$

$$= \frac{1}{14}\left(5t - \frac{31}{14}\right) - \frac{1}{8}e^t$$

Thus $x = c_1e^{-2t} + c_2e^{-7t} + \frac{1}{14}(5t - \frac{31}{14}) - \frac{1}{8}e^t$.

From the first equation

$$y = \frac{1}{3}\left(t - 4x - \frac{dx}{dt}\right)$$

$$= \frac{1}{3}\left[t - 4c_1e^{-2t} - 4c_2e^{-7t} - \frac{2}{7}\left(5t - \frac{31}{14}\right)\right.$$

$$\left. + \frac{1}{2}e^t + 2c_1e^{-2t} + 7c_2e^{-7t} - \frac{5}{14} + \frac{1}{8}e^t\right]$$

$$= \frac{1}{3}\left(- 2c_1e^{-2t} + 3c_2e^{-7t} + \frac{5}{8}e^t - \frac{3}{7}t + \frac{27}{98}\right)$$

Hence the general solution is

$$x = c_1e^{-2t} + c_2e^{-7t} + \frac{1}{14}\left(5t - \frac{31}{14}\right) - \frac{1}{8}e^t$$

and

$$y = \frac{1}{3}\left(- 2c_1e^{-2t} + 3c_2e^{-7t} + \frac{5}{8}e^t - \frac{3}{7}t + \frac{27}{98}\right)$$

where c_1 and c_2 are arbitrary constraints.

EXAMPLE 5.2.4 Solve: $\dfrac{dx}{dt} = 2y, \dfrac{dy}{dt} = 2z, \dfrac{dz}{dt} = 2x$.

Solution We have $\dfrac{dx}{dt} = 2y$.

$$\therefore\ \frac{d^2x}{dt^2} = 2\frac{dy}{dt} = 4z \text{ and } \frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 8x$$

Let $x = ce^{mt}$ be a trial. solution of $\dfrac{d^3x}{dt^3} - 8x = 0$.

$\therefore$ A.E. is $m^3 - 8 = 0$, or $(m-2)(m^2+2m+4) = 0$, or $m = 2, -1 \pm i\sqrt{3}$.

$\therefore$ C.F. is $c_1e^{2t} + e^{-t}(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t)$.

Hence $x = c_1e^{2t} + e^{-t}(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t)$.

Now

$$y = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}\left[2c_1e^{2t} - e^{-t}(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t) + e^{-t}(-\sqrt{3}c_2\sin\sqrt{3}t + \sqrt{3}c_3\cos\sqrt{3}t)\right]$$

$$= c_1e^{2t} + \frac{e^{-t}}{2}\left[(-c_2+\sqrt{3}c_3)\cos\sqrt{3}t + (-c_3-\sqrt{3}c_2)\sin\sqrt{3}t\right]$$

$$z = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}\left[2c_1e^{2t} - \frac{e^{-t}}{2}\{(-c_2+\sqrt{3}c_3)\cos\sqrt{3}t - (c_3+\sqrt{3}c_2)\sin\sqrt{3}t\}\right.$$

$$\left.+\frac{e^{-t}}{2}\{-\sqrt{3}(-c_2+\sqrt{3}c_3)\sin\sqrt{3}t - \sqrt{3}(c_3+\sqrt{3}c_2)\cos\sqrt{3}t\}\right]$$

$$= c_1e^{2t} - \frac{e^{-t}}{2}\left[(\sqrt{3}c_3+c_2)\cos\sqrt{3}t + (c_3-\sqrt{3}c_2)\sin\sqrt{3}t\right]$$

Hence the general solution is

$$x = c_1e^{2t} + e^{-t}(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t)$$

$$y = c_1e^{2t} + \frac{e^{-t}}{2}\left[\left(-c_2+\sqrt{3}c_3\right)\cos\sqrt{3}t - \left(c_3+\sqrt{3}c_2\right)\sin\sqrt{3}t\right]$$

$$z = c_1e^{2t} - \frac{e^{-t}}{2}\left[\left(\sqrt{3}c_3+c_2\right)\cos\sqrt{3}t + \left(c_3-\sqrt{3}c_2\right)\sin\sqrt{3}t\right]$$

where c_1, c_2, c_3 are arbitrary constants.

EXAMPLE 5.2.5 Solve: $\dfrac{dx}{dt} + \dfrac{2}{t}(x-y) = 1,\ \dfrac{dy}{dt} + \dfrac{1}{t}(x+5y) = t.$

Solution We have

$$\frac{dx}{dt} + \frac{2}{t}(x-y) = 1 \qquad \text{(i)}$$

$$\frac{dy}{dt} + \frac{1}{t}(x+5y) = t \qquad \text{(ii)}$$

Differentiating (i) with respect to t, we get

$$\frac{d^2x}{dt^2} - \frac{2}{t^2}(x-y) + \frac{2}{t}\left(\frac{dx}{dt} - \frac{dy}{dt}\right) = 0$$

or

$$\frac{d^2x}{dt^2} - \frac{2}{t^2}(x-y) + \frac{2}{t}\left(\frac{dx}{dt} - t + \frac{1}{t}(x+5y)\right) = 0$$

or

$$\frac{d^2x}{dt^2}+\frac{2}{t}\frac{dx}{dt}-2+\frac{2}{t^2}(x+5y-x+y)=0$$

or

$$\frac{d^2x}{dt^2}+\frac{2}{t}\frac{dx}{dt}-2+\frac{12}{t^2}y=0$$

From (i), $x-y=\frac{t}{2}\left(1-\frac{dx}{dt}\right)$, or $y=x-\frac{t}{2}\left(1-\frac{dx}{dt}\right)$.

The above equation becomes

$$\frac{d^2x}{dt^2}+\frac{2}{t}\frac{dx}{dt}-2+\frac{12}{t^2}\left(x-\frac{t}{2}+\frac{t}{2}\frac{dx}{dt}\right)=0$$

or

$$\frac{d^2x}{dt^2}+\frac{8}{t}\frac{dx}{dt}+\frac{12x}{t^2}=2+\frac{6}{t}$$

or

$$t^2\frac{d^2x}{dt^2}+8t\frac{dx}{dt}+12x=t^2(2+6/t)=2t^2+6t$$

Putting $\log t=z$, or $t=e^z$, we have

$$t\frac{dx}{dt}=Dx \qquad \text{and} \qquad t^2\frac{d^2x}{dt^2}=D(D-1)x$$

where $D\equiv\frac{d}{dz}$

The above equation becomes

$$D(D-1)x+8Dx+12x=2e^{2z}+6e^z \qquad \text{or} \qquad (D^2+7D+12)x=2e^{2z}+6e^z$$

Let $x=ce^{mz}$ be a trial solution.

$\therefore$ A.E. is $m^2+7m+12=0$, or $m=-3,-4$.

Thus C.F. is $c_1e^{-3z}+c_2e^{-4z}$.

$$\text{P.I.}=\frac{1}{D^2+7D+12}(2e^{2z}+6e^z)$$

$$=\frac{1}{4+14+12}2e^{2z}+\frac{1}{1+7+12}6e^z=\frac{e^{2z}}{15}+\frac{3}{10}e^z.$$

The general solution is

$$x=c_1t^{-3}+c_2t^{-4}+\frac{t^2}{15}+\frac{3}{10}t \text{ and}$$

$$y=x-\frac{t}{2}\left(1-\frac{dx}{dt}\right)$$

$$=x-\frac{t}{2}+\frac{t}{2}\left(-3c_1t^{-4}-4c_2t^{-5}+\frac{2t}{15}+\frac{3}{10}\right)$$

where c_1, c_2 are arbitrary constants.

5.3 Solution of Second Order Simultaneous Equations

EXAMPLE 5.3.1 Solve: $2\dfrac{d^2x}{dt^2} + 3\dfrac{dy}{dt} = 4,\ \ 2\dfrac{d^2y}{dt^2} - 3\dfrac{dx}{dt} = 0$, given $x(0) = y(0) = x'(0) = y'(0) = 0$.

Solution Differentiating first equation with respect to t, we get

$$2\frac{d^3x}{dt^3} + 3\frac{d^2y}{dt^2} = 0 \quad \text{or} \quad 2\frac{d^3x}{dt^3} + 3 \cdot \frac{3}{2}\frac{dx}{dt} = 0 \qquad \text{(using second equation)}$$

or

$$(4D^3 + 9D)x = 0$$

Let $x = ce^{mt}$ be a trial solution.

$\therefore$ A.E. is $4m^3 + 9m = 0$ or $m(4m^2 + 9) = 0$ or $m = 0, \pm\frac{3}{2}i$.

$\therefore$ $x = c_1 + c_2 \cos\dfrac{3}{2}t + c_3 \sin\dfrac{3}{2}t$.

Integrating first equation with respect to t, we get $2\dfrac{dx}{dt} + 3y = 4t + c_4$.

That is

$$\begin{aligned} y &= \frac{1}{3}\left(4t + c_4 - 2\frac{dx}{dt}\right) \\ &= \frac{1}{3}\left[4t + c_4 - 2\left\{-\frac{3}{2}c_2 \sin\frac{3}{2}t + \frac{3}{2}c_3 \cos\frac{3}{2}t\right\}\right] \\ &= \frac{4}{3}t + c_2 \sin\frac{3}{2}t - c_3 \cos\frac{3}{2}t + \frac{1}{3}c_4 \end{aligned}$$

Now

$$x' = -\frac{3}{2}c_2 \sin\frac{3}{2}t + \frac{3}{2}c_3 \cos\frac{3}{2}t \qquad \text{and} \qquad y' = \frac{4}{3} + \frac{3}{2}c_2 \cos\frac{3}{2}t + \frac{3}{2}c_3 \sin\frac{3}{2}t$$

Using the initial condition $t = 0,\ x = 0,\ y = 0,\ x' = 0,\ y' = 0$, we obtain

$$0 = c_1 + c_2 \qquad 0 = -c_3 + \frac{1}{3}c_4 \qquad 0 = \frac{3}{2}c_3 \qquad 0 = \frac{4}{3} + \frac{3}{2}c_2$$

Solution of these equations is $c_3 = 0,\ c_4 = 0,\ c_1 = -c_2 = \dfrac{8}{9}$.

Hence the required solution is

$$x = \frac{8}{9}\left(1 - \cos\frac{3}{2}t\right) \qquad y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3}{2}t$$

EXAMPLE 5.3.2 Solve: $\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = 2x + 2t,\ \ \dfrac{dx}{dt} + 4\dfrac{dy}{dt} = 3y$.

Solution We have

$$\frac{d^2x}{dt^2} - \frac{dy}{dt} = 2x + 2t \qquad \text{(i)}$$

and

$$\frac{dx}{dt} + 4\frac{dy}{dt} = 3y$$

Differentiating (ii) with respect to t, we get

$$\frac{d^2x}{dt^2} + 4\frac{d^2y}{dt^2} = 3\frac{dy}{dt}$$

Subtracting (i) from (iii), we get

$$4\frac{d^2y}{dt^2} + \frac{dy}{dt} = 3\frac{dy}{dt} - 2x - 2t \qquad \text{or} \qquad 2\frac{d^2y}{dt^2} - \frac{dy}{dt} = -x - t$$

Again differentiating it with respect to t

$$2\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} = -\frac{dx}{dt} - 1 = -\left(3y - 4\frac{dy}{dt}\right) - 1$$

or

$$2\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = -1$$

Let $y = ce^{mt}$ be a trial solution.

$\therefore$ A.E. is $2m^3 - m^2 - 4m + 3 = 0$, or $(m-1)^2(2m+3) = 0$, or $m = 1, 1, -3/2$.

$\therefore$ C.F. is $(c_1 + c_2t)e^t + c_3e^{-3t/2}$.

$$\text{P.I.} = \frac{1}{(D-1)^2(2D+3)}(-1) = \frac{1}{3}(1-D)^{-1}\left(1 + \frac{2}{3}D\right)^{-1}(-1) = -\frac{1}{3}$$

$\therefore \quad y = (c_1 + c_2t)e^t + c_3e^{-3t/2} - \frac{1}{3}$.

Now

$$\frac{dy}{dt} = c_2e^t + (c_1 + c_2t)e^t - \frac{3}{2}c_3e^{-3t/2}$$

and hence

$$\frac{d^2y}{dt^2} = 2c_2e^t + (c_1 + c_2t)e^t + \frac{9}{4}c_3e^{-3t/2}$$

Therefore, from (iv)

$$\begin{aligned} x = -2\frac{d^2y}{dt^2} + \frac{dy}{dt} - t &= -2\left[2c_2e^t + (c_1 + c_2t)e^t + \frac{9}{4}c_3e^{-3t/2}\right] \\ &+\left[c_2e^t + (c_1 + c_2t)e^t - \frac{3}{2}c_3e^{-3t/2}\right] - t \\ &= -3c_2e^t - (c_1 + c_2t)e^t - 6c_3e^{-3t/2} - t \end{aligned}$$

Equations (v) and (vi) gives the solution, where c_1, c_2, and c_3 are constants.

5.4 Using Eigenvalues

In previous sections, a system of differential equations has been solved by converting it into a higher order differential equation. But it is well known that the solution of higher order equation is a very difficult task. Here an efficient method is used to solve a system of differential equations. Let us consider a system of differential equation of the form

$$\begin{aligned}
\dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\
\dot{y}_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{y}_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n
\end{aligned}$$

where $y_1, y_2, y_3, \ldots, y_n$ are dependent variables. The solution of this system of equations can be done by finding eigenvalues and eigenvectors of the associated system.

The above equation can be written as

$$\dot{Y} = AY \tag{5.1}$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The solution of the above equation may be of the form

$$Y(t) = Xe^{\lambda t} \tag{5.2}$$

where X is a nonzero constant vector and λ is a constant scalar, and both are unknown.

Substituting (5.2) in (5.1), we get

$$\lambda X e^{\lambda t} = AXe^{\lambda t}$$

from which we have

$$AX = \lambda X \qquad \text{i.e.} \qquad (A - \lambda I)X = 0$$

Thus λ is the eigenvalue of A and X is the corresponding eigenvector. Hence we can determine $Y(t)$ using (5.2).

EXAMPLE 5.4.1 Solve: $\dfrac{dx}{dt} = x + 2y, \quad \dfrac{dy}{dt} = x.$

Solution The given equation can be written as

$$\dot{Y}(t) = AY(t)$$

where $Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

The eigenvalues λ of the matrix A are given by $\begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0$, or $\lambda = 2, -1$.

For $\lambda_1 = 2$ let the eigenvector be $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

That is $-x_1 + 2x_2 = 0$, or $x_1 - 2x_2 = 0$. This equation gives $x_1 = k_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -1$, let the eigenvector be $X_2 = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$.

Then $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $x_1' + x_2' = 0$.

This gives $x_2 = k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Hence the required solution is

$$Y(t) = k_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \quad \text{or} \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2k_1 e^{2t} + k_2 e^{-t} \\ k_1 e^{2t} - k_2 e^{-t} \end{bmatrix}$$

Thus

$$x(t) = 2k_1 e^{2t} + k_2 e^{-t} \quad \text{and} \quad y(t) = k_1 e^{2t} - k_2 e^{-t}$$

EXAMPLE 5.4.2 Solve: $\dfrac{dx}{dt} - 7x + y = 0$, $\dfrac{dy}{dt} - 2x - 5y = 0$.

(WBUT 2005, 2007)

Solution The given equations are

$$\frac{dx}{dt} = 7x - y \quad \text{and} \quad \frac{dy}{dt} = 2x + 5y$$

and these can be written as

$$\dot{Y} = AY$$

where $Y = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 7 & -1 \\ 2 & 5 \end{bmatrix}$.

The eigenvalues λ of the matrix A are given by

$$\begin{vmatrix} 7-\lambda & -1 \\ 2 & 5-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 12\lambda + 37 = 0 \quad \text{or} \quad \lambda = 6 \pm i$$

Let $\lambda_1 = 6 + i$ and $\lambda_2 = 6 - i$.

For $\lambda_1 = 6 + i$, let the eigenvector be $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then

$$\begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$(1-i)x_1 - x_2 = 0 \qquad \text{or} \qquad 2x_1 - (1+i)x_2 = 0$$

The solution of these equations is $x_1 = k_1$, $x_2 = (1-i)k_1$.

This gives $\quad x_1 = k_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$.

The eigenvector corresponding to $\lambda_2 = 6 - i$ is obtained by replacing i by $-i$.

This gives $\quad x_2 = k_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$.

Hence the required solution is

$$Y(t) = k_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(6+i)t} + k_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(6-i)t}$$

Thus

$$\begin{aligned} x &= k_1 e^{(6+i)t} + k_2 e^{(6-i)t} \\ &= e^{6t}(k_1 e^{it} + k_2 e^{-it}) \\ &= e^{6t}[k_1(\cos t + i\sin t) + k_2(\cos t - i\sin t)] \\ &= e^{6t}[(k_1 + k_2)\cos t + (k_1 - k_2)i\sin t] \\ &= e^{6t}(c_1\cos t + c_2\sin t) \end{aligned}$$

where

$$c_1 = k_1 + k_2 \qquad \text{and} \qquad c_2 = (k_1 - k_2)i$$

and

$$\begin{aligned} y &= k_1(1-i)e^{(6+i)t} + k_2(1+i)e^{(6-i)t} \\ &= e^{6t}[k_1(1-i)(\cos t + i\sin t) + k_2(1+i)(\cos t - i\sin t)] \\ &= e^{6t}[\{(k_1 + k_2) - (k_1 - k_2)i\}\cos t + \{(k_1 + k_2) + (k_1 - k_2)i\}\sin t] \\ &= e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t] \end{aligned}$$

where c_1, c_2 are arbitrary constants.

EXERCISES

Section A Multiple Choice Questions

1. The solution of the equations $y' = z,\ z' = -y$ is
   (a) $y = c_1 \cos x + c_2 \sin x,\ z = -c_1 \sin x + c_2 \cos x$
   (b) $y = c_1 e^x + c_2 e^{-x};\ z = c_1 e^x - c_2 e^{-x}$
   (c) $y = e^x(c_1 \cos x + c_2 \sin x),\ z = e^{-x}(c_1 \cos x + c_2 \sin x)$
   (d) none of these.
2. The solution of the equations $\frac{dy}{dx} = z + 2x,\ \frac{dz}{dx} = y + 2x$ is given by
   (a) $y = xe^{-z}$ (b) $y = z + ce^{-x}$ (c) $y + z = \log(xc)$ (d) $x + y + z = c$
3. The solution of $\dfrac{dx}{dt} = x,\ \dfrac{dy}{dt} + \dfrac{dx}{dt} = 0$ is
   (a) $x = xt + c_1,\ y = x + c_2$ (b) $x + y = c$ (c) $x = ce^x$
   (d) $x = c_1 e^t,\ y + c_1 e^t = c_2$.
4. $x = c_1 e^t + c_2 e^{-t},\ y = -c_1 e^t + c_2 e^{-t} + c_3$ is solution of the equations
   (a) $\frac{dy}{dt} = -x,\ \frac{d^2x}{dt^2} + \frac{dy}{dt} = 0$ (b) $\frac{dy}{dt} = -x,\ \frac{dx}{dt} = -y$
   (c) $\frac{dx}{dt} + \frac{dy}{dt} = e^{-t},\ \frac{dx}{dt} = t$ (d) none of these.
5. If $\frac{dy}{dt} = 2x + 3y,\ \frac{dx}{dt} = -x - 2y$. Then the solution of these equations are
   (a) $x + y = ce^t$ (b) $x - y = ce^{-t}$ (c) $x = c_1 e^t + c_2 e^{-t},\ y = -x$
   (d) none of these.

Section B Review Questions

Solve the following system of equations:

1. $\frac{dx}{dt} = -\omega y,\ \frac{dy}{dt} = \omega x$. Also show that the point (x, y) lies on a circle .
2. $4\frac{dx}{dt} - \frac{dy}{dt} + 3x = \sin t,\ \frac{dx}{dt} + y = \cos t$.
3. $\frac{dx}{dt} - y = t,\ \frac{dx}{dt} + x = t^2$.
4. $x' + y = \sin t,\ y' + x = \cos t,\ x(0) = 2$ *and* $y(0) = 0$.
5. $\frac{dx}{dt} + x - y = 0,\ \frac{dy}{dt} + 2x + 5y = 0$.
6. $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0,\ \frac{dy}{dt} + 5x + 3y = 0$.
7. $\frac{dx}{dt} + 2x - 3y = t,\ \frac{dy}{dt} - 3x + 2y = e^{2t}$.
8. $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t,\ \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$.
9. $\frac{dy}{dx} = 3z - y,\ \frac{dz}{dx} = 4z - 2y$.
10. $5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x},\ \frac{dy}{dx} + 8y - 3z = 5e^{-x}$.
11. $\frac{dy}{dx} + y - z = e^x,\ \frac{dz}{dx} - y + z = e^x$.

12. $\frac{d^2x}{dt^2} - 3x - 4y = 0,\ \frac{d^2y}{dt^2} + x + y = 0.$

13. $\frac{d^2x}{dt^2} - 3x - y = e^t,\ \frac{dy}{dt} - 2x = 0.$

14. $2\frac{d^2x}{dt^2} - \frac{dy}{dt} - 4x = 2t,\ 2\frac{dx}{dt} + 4\frac{dy}{dt} - 3y = 0.$

15. $2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4,\ 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0,\ x(0) = y(0) = x'(0) = y'(0) = 0.$

16. $2x'' + 3y' = 4,\ 2y'' - 3x' = 0,\ x(0) = y(0) = x'(0) = y'(0) = 0.$

17. The coordinates x and y of the position of a particle at time t satisfy the differential equations $y' + 2x = \sin 2t$ and $x' - 2y = \cos 2t$. If $x(0) = 1$ and $y(0) = 0$, show that the path of the particle is given by $4x^2 + 4xy + 5y^2 = 4$.

Answers

Section A Multiple Choice Questions

1. (a) 2. (b) 3. (d) 4. (a) 5. (a)

Section B Review Questions

1. $x = -c_1 \sin wt + c_2 \cos \omega t,\ y = c_1 \cos \omega t + c_2 \sin \omega t$
2. $x = c_1e^{-t} + c_2e^{-3t},\ y = c_1e^{-t} + c_2e^{-3t} + \cos t$
3. $x = c_1 \cos t + c_2 \sin t + t^2 - 1,\ y = -c_1 \sin t + c_2 \cos t + t$
4. $x = e^t + e^{-t},\ y = \sin t - e^t + e^{-t}$
5. $y = c_1e^{(-3+\sqrt{2})t} + c_2e^{(-3-\sqrt{2})t},\ x = -(1 + \frac{1}{\sqrt{2}})c_1e^{(-3+\sqrt{2})t} + (-1 + \frac{1}{\sqrt{2}})c_2e^{(-3-\sqrt{2})t}$
6. $x = c_1 \cos t + c_2 \sin t,\ y = -\frac{1}{2}(c_1 + 3c_2)\sin t + \frac{1}{2}(c_2 - 3c_1)\cos t$
7. $x = c_1e^t + c_2e^{-5t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{36}{25},\ y = c_1e^t - c_2e^{-5t} + \frac{4}{7}e^{2t} - \frac{3}{5}t - \frac{12}{25}$
8. $x = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + 3\cos t,\ y = (1+\sqrt{2})c_1e^{\sqrt{2}t} + (1-\sqrt{2})c_2e^{-\sqrt{2}t} - 5\sin t$
9. $y = -\frac{1}{2}c_1e^{5x} + 3c_2e^{-2x},\ z = c_1e^{5x} - c_2e^{-2x}$
10. $y = 2e^{-x} + c_1e^x + c_2e^{-2x},\ z = 3e^{-x} + c_3e^x + c_4e^{-2x}$
11. $y = e^x + c_1 + c_2e^{-2x},\ z = e^x + c_1 - c_2e^{-2x}$
12. $x = (c_1 + c_2t)e^t + (c_3 + c_4t)e^{-t},\ y = \frac{1}{2}e^t(c_2 - c_1 - c_2t) - \frac{1}{2}e^{-t}(c_3 + c_4 + c_4t)$
13. $y = (c_1 + c_2t)e^{-t} + c_3e^{2t} - \frac{1}{2}e^t,\ x = \frac{1}{2}[e^{-t}(c_2 - c_1 - c_2t) + 2c_3e^{2t} - \frac{1}{2}e^t]$
14. $x = (c_1 + c_2t)e^t + c_3e^{-\frac{3t}{2}} - \frac{1}{2}t,\ y = -2(c_1 + c_2t - 3c_2)e^t - \frac{1}{3}c_3e^{-\frac{3t}{2}} - \frac{1}{3}$
15. $x = \frac{8}{9}(1 - \cos\frac{3t}{2}),\ y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3t}{2}$
16. $x = \frac{8}{9}(1 - \cos\frac{3t}{2}),\ y = \frac{4}{3} - \frac{8}{9}\sin\frac{3t}{2}$