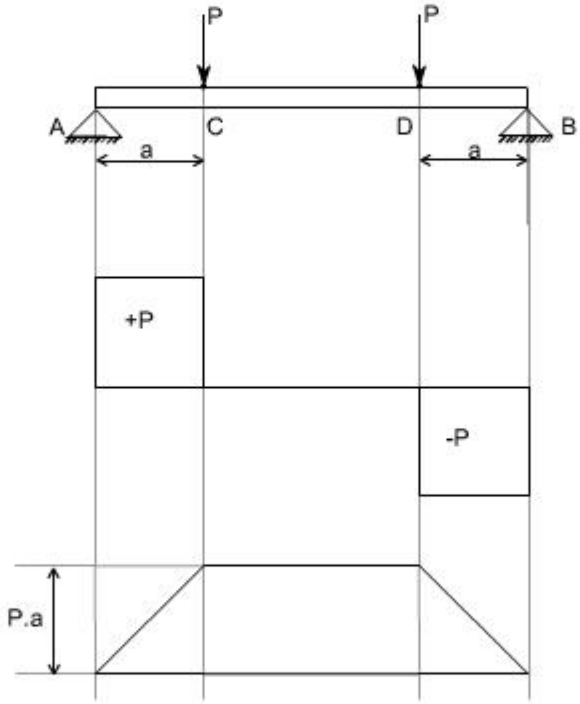
LECTURE 28 and 29: Shearing stress distribution in typical cross-sections

All the theory which has been discussed earlier, while we discussed the bending stresses in beams was for the case of pure bending i.e. constant bending moment acts along the entire length of the beam.



Let us consider the beam AB transversely loaded as shown in the figure above. Together with shear force and bending moment diagrams we note that the middle potion CD of the beam is free from shear force and that its bending moment. M = P.a is uniform between the portion C and D. This condition is called the pure bending condition.

Since shear force and bending moment are related to each other F= dM/dX (eq) therefore if the shear force changes than there will be a change in the bending moment also, and then this won't be the pure bending.

Conclusions : Hence one can conclude from the pure bending theory was that the shear force at each X-section is zero and the normal stresses due to bending are the only ones produced.

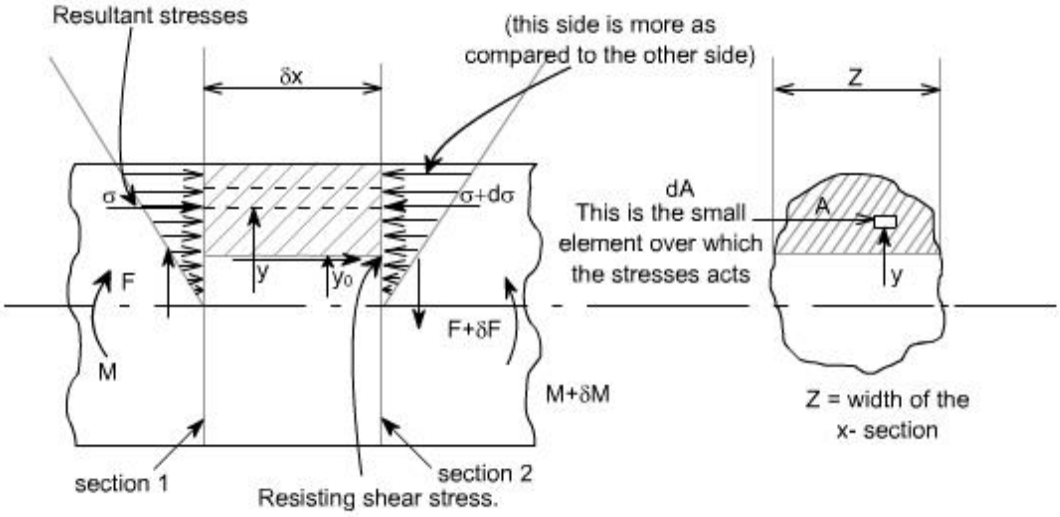
In the case of non-uniform bending of a beam where the bending moment varies from one X-section to another, there is a shearing force on each X-section and shearing stresses are also induced in the material. The deformation associated with those shearing stresses causes warping of the x-section so that the assumption which we assumed while deriving the relation $\frac{\sigma}{y}=\frac{M}{I}=\frac{E}{R}$ that the plane cross-section after bending remains plane is violated. Now due to warping the plane cross=section before bending do not remain plane after bending. This complicates the problem but more elaborate analysis shows that the normal stresses due to bending, as calculated from the equation $\frac{\sigma}{y}=\frac{M}{I}=\frac{E}{R}$.

The above equation gives the distribution of stresses which are normal to the cross-section that is in x-direction or along the span of the beam are not greatly altered by the presence of these shearing stresses. Thus, it is justifiable to use the theory of pure bending in the case of non uniform bending and it is accepted practice to do so.

Let us study the shear stresses in the beams.

Concept of Shear Stresses in Beams : By the earlier discussion we have seen that the bending moment represents the resultant of certain linear distribution of normal stresses σ_x over the cross-section. Similarly, the shear force F_x over any cross-section must be the resultant of a certain distribution of shear stresses.

Derivation of equation for shearing stress :



Assumptions :

- Stress is uniform across the width (i.e. parallel to the neutral axis)
- The presence of the shear stress does not affect the distribution of normal bending stresses.

It may be noted that the assumption no.2 cannot be rigidly true as the existence of shear stress will cause a distortion of transverse planes, which will no longer remain plane.

In the above figure let us consider the two transverse sections which are at a distance '**δx**' apart. The shearing forces and bending moments being **F, F + δF** and **M, M + δM** respectively. Now due to the shear stress on transverse planes there will be a complementary shear stress on longitudinal planes parallel to the neutral axis.

Let **τ** be the value of the complementary shear stress (and hence the transverse shear stress) at a distance ?**Y'**$_0$ from the neutral axis. **Z** is the width of the x-section at this position

A is area of cross-section cut-off by a line parallel to the neutral axis.

$\bar{y}$ = distance of the centroid of Area from the neutral axis.

Let **σ , σ + dσ** are the normal stresses on an element of area **δA** at the two transverse sections, then there is a difference of longitudinal forces equal to (**dσ . δA)** , and this quantity summed over the area A is in equilibrium with the transverse shear stress **τ** on the longitudinal plane of area z **δx** .

$$\text{i.e } \tau.z\delta x = \int d\sigma.dA$$

from the bending theory equation

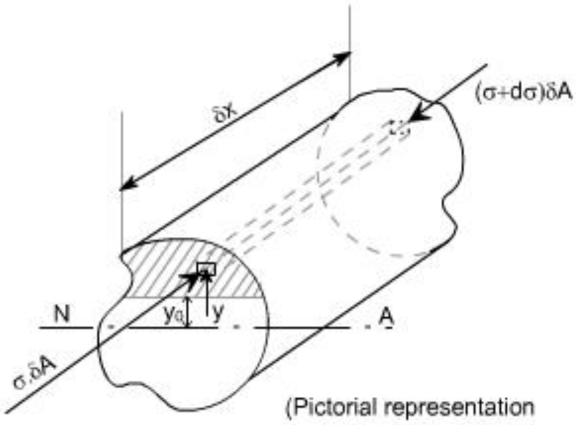
$$\frac{\sigma}{y} = \frac{M}{I}$$

$$\sigma = \frac{M\,.\,y}{I}$$

$$\sigma + d\sigma = \frac{(M + \delta M)\,.\,y}{I}$$

$$\text{Thus} \qquad d\sigma = \frac{\delta M.y}{I}$$

The figure shown below indicates the pictorial representation of the part.



$$d\sigma = \frac{\delta M . y}{I}$$

$$\tau . z\delta x = \int d\sigma . dA$$

$$= \int \frac{\delta M \; . \; y . \delta A}{I}$$

$$\tau . z\delta x = \frac{\delta M}{I} \int y . \delta A$$

But $F = \dfrac{\delta M}{\delta x}$

i.e. $\tau = \dfrac{F}{I . z} \int y . \delta A$

But from definition, $\int y . dA = A\bar{y}$

$\int y . dA$ is the first moment of area of the shaded portion

and $\bar{y}$ = centroid of the area'A'

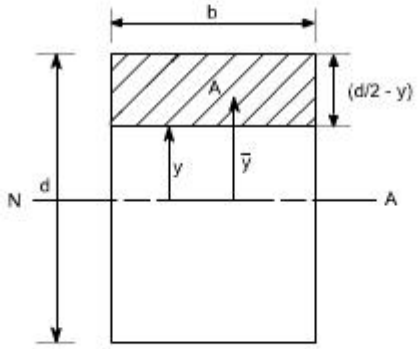
Hence

So substituting $$\tau = \frac{F . A . \bar{y}}{I . z}$$

Where 'z' is the actual width of the section at the position where ? τ ' is being calculated and I is the total moment of inertia about the neutral axis.

Let us consider few examples to determine the sheer stress distribution in a given X- sections

Rectangular x-section: Consider a rectangular x-section of dimension b and d



A is the area of the x-section cut off by a line parallel to the neutral axis. $\bar{y}$ is the distance of the centroid of A from the neutral axis

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$\text{for this case, } A = b\left(\frac{d}{2} - y\right)$$

$$\text{While} \quad \bar{y} = \left[\frac{1}{2}\left(\frac{d}{2} - y\right) + y\right]$$

$$\text{i.e} \quad \bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right) \text{ and } z = b\,;\, I = \frac{b.d^3}{12}$$

substituting all these values, in the formula

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$= \frac{F.b.\left(\frac{d}{2} - y\right).\frac{1}{2}.\left(\frac{d}{2} + y\right)}{b.\frac{b.d^3}{12}}$$

$$= \frac{\frac{F}{2}.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{\frac{b.d^3}{12}}$$

$$= \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$

This shows that there is a parabolic distribution of shear stress with y.
The maximum value of shear stress would obviously beat the location y = 0.

$$\text{Such that} \quad \tau_{max} = \frac{6.F}{b.d^3}.\frac{d^2}{4}$$

$$= \frac{3.F}{2.b.d}$$

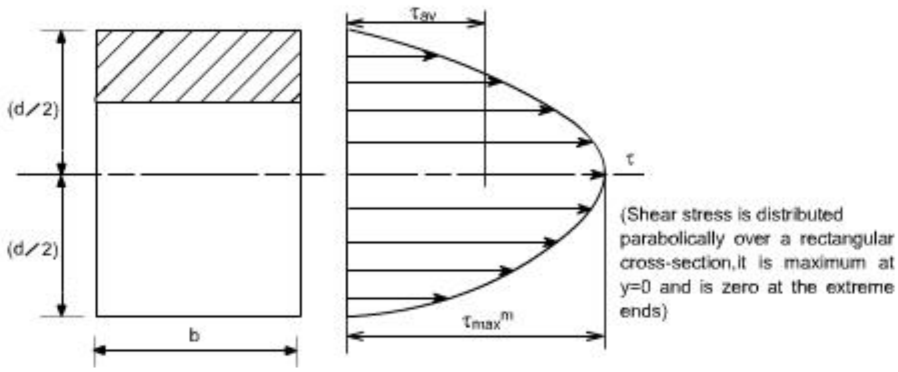
$$\text{So} \quad \boxed{\tau_{max} = \frac{3.F}{2.b.d}} \text{ The value of } \tau_{max} \text{ occurs at the neutral axis}$$

The mean shear stress in the beam is defined as

$$\tau_{mean} \text{ or } \tau_{avg} = F/A = F/b.d$$

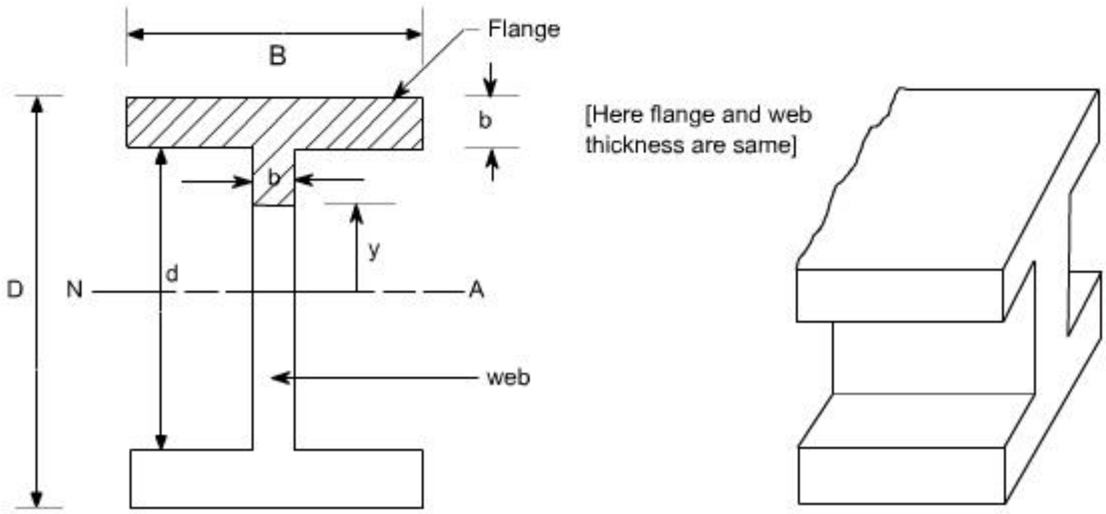
$$\text{So} \quad \tau_{max} = 1.5\,\tau_{mean} = 1.5\,\tau_{avg}$$

Therefore the shear stress distribution is shown as below.



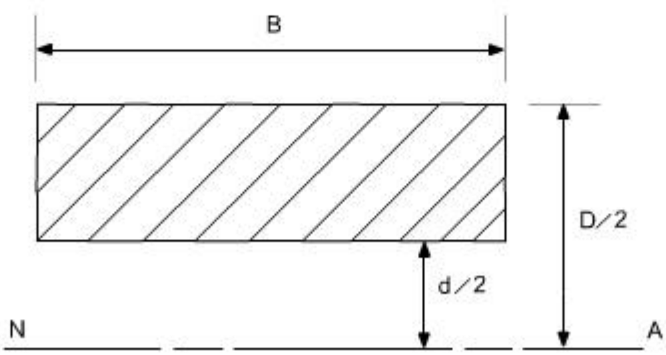
It may be noted that the shear stress is distributed parabolically over a rectangular cross-section, it is maximum at y = 0 and is zero at the extreme ends.

I - section : Consider an I - section of the dimension shown below.



The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area



Flange area

$$\text{Area of the flange} = B\left(\frac{D-d}{2}\right)$$

Distance of the centroid of the flange from the N.A

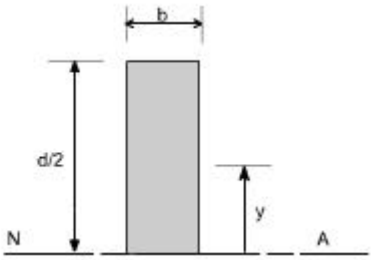
$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\big|_{\text{Flange}} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$

Web Area



Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{\text{web}} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{\text{Total}} = B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

$$A\bar{y}\big|_{\text{Total}} = B\left(\frac{D^2-d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2-d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

To get the maximum and minimum values of τ substitute in the above relation.

y = 0 at N. A. And y = d/2 at the tip.

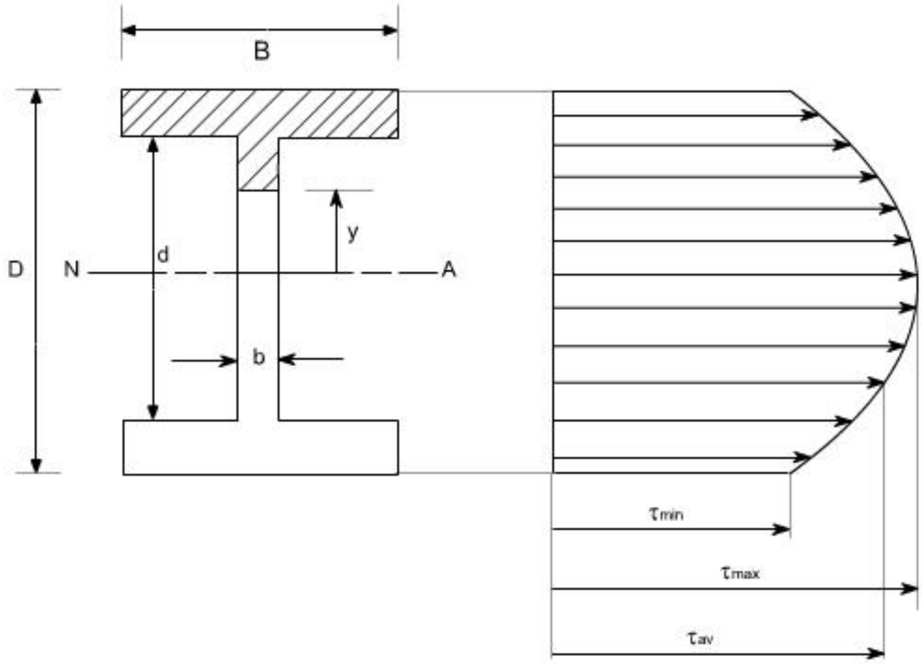
The maximum shear stress is at the neutral axis. i.e. for the condition y = 0 at N. A.

Hence, τ_{max} at $y = 0 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$ ....(2)

The minimum stress occur at the top of the web, the term bd 2 goes off and shear stress is given by the following expression

$$\tau_{min} \text{ at } y = d/2 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right)\right] \qquad ....(3)$$

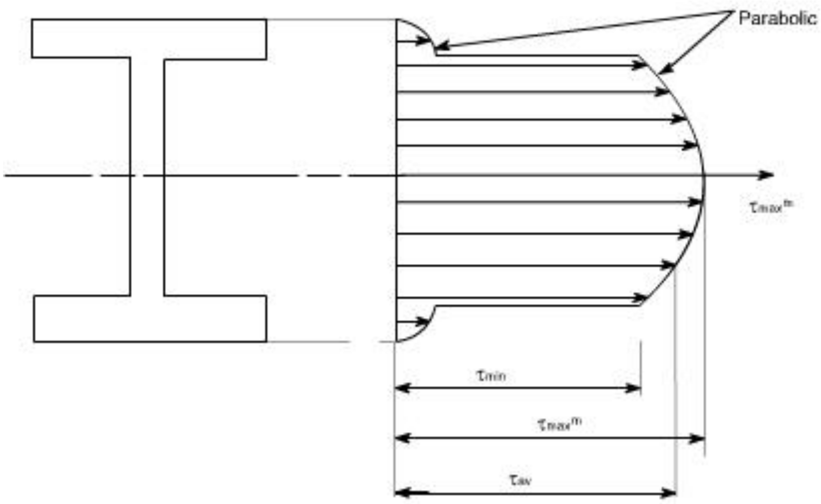
The distribution of shear stress may be drawn as below, which clearly indicates a parabolic distribution



$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

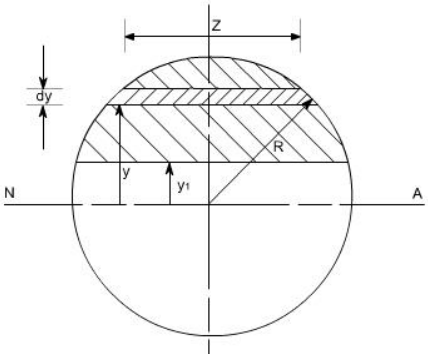
Note: from the above distribution we can see that the shear stress at the flanges is not zero, but it has some value, this can be analyzed from equation (1). At the flange tip or flange or web interface y = d/2.Obviously than this will have some constant value and than onwards this will have parabolic distribution.

In practice it is usually found that most of shearing stress usually about 95% is carried by the web, and hence the shear stress in the flange is negligible however if we have the concrete analysis i.e. if we analyze the shearing stress in the flange i.e. writing down the expression for shear stress for flange and web separately, we will have this type of variation.



This distribution is known as the ?top ? hat? distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

Shear stress distribution in beams of circular cross-section: Let us find the shear stress distribution in beams of circular cross-section. In a beam of circular cross-section, the value of Z width depends on y.



Using the expression for the determination of shear stresses for any arbitrary shape or a arbitrary section.

$$\tau = \frac{FA\,\bar{y}}{Z\,I} = \frac{FA\int y\,dA}{Z\,I}$$

Where $\int y\,dA$ is the area moment of the shaded portion or the first moment of area.

Here in this case 'dA' is to be found out using the Pythagoras theorem

$$\left(\frac{Z}{2}\right)^2 + y^2 = R^2$$

$$\left(\frac{Z}{2}\right)^2 = R^2 - y^2 \text{ or } \frac{Z}{2} = \sqrt{R^2 - y^2}$$

$$Z = 2\sqrt{R^2 - y^2}$$

$$dA = Z\,dy = 2.\sqrt{R^2 - y^2}\,.dy$$

$$I_{N.A}\text{ for a circular cross-section} = \frac{\pi R^4}{4}$$

Hence,

$$\tau = \frac{FA\,\bar{y}}{Z\,I} = \frac{F}{\frac{\pi R^4}{4}2\sqrt{R^2 - y^2}}\int_{y_1}^{R} 2\,y\sqrt{R^2 - y^2}\,dy$$

Where R = radius of the circle.

[The limits have been taken from y_1 to R because we have to find moment of area the shaded portion]

$$= \frac{4F}{\pi R^4\sqrt{R^2 - y^2}}\int_{y_1}^{R} y\sqrt{R^2 - y^2}\,dy$$

The integration yields the final result to be

$$\tau = \frac{4F\left(R^2 - y_1^2\right)}{3\pi R^4}$$

Again this is a parabolic distribution of shear stress, having a maximum value when $y_1 = 0$

$$\tau_{maxm}|_{y_1} = 0 = \frac{4F}{3\pi R^2}$$

Obviously at the ends of the diameter the value of $y_1 = \pm R$ thus $\tau = 0$ so this again a parabolic distribution; maximum at the neutral axis

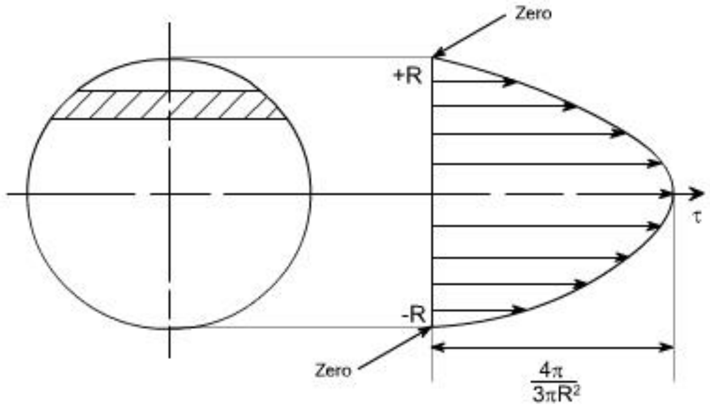
Also

$$\tau_{avg} \text{ or } \tau_{mean} = \frac{F}{A} = \frac{F}{\pi R^2}$$

Hence,

$$\boxed{\tau_{maxm} = \frac{4}{3}\tau_{avg}}$$

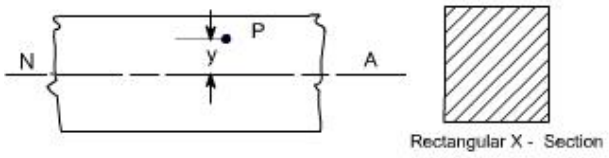
The distribution of shear stresses is shown below, which indicates a parabolic distribution

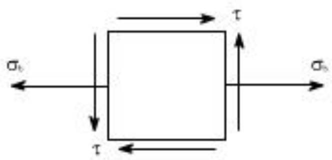


Principal Stresses in Beams

It becomes clear that the bending stress in beam σ_x is not a principal stress, since at any distance y from the neutral axis; there is a shear stress τ (or τ_{xy} we are assuming a plane stress situation)

In general the state of stress at a distance y from the neutral axis will be as follows.





At some point 'P' in the beam, the value of bending stresses is given as

$$\sigma_b = \frac{My}{I} \text{ for a beam of rectangular cross - section of dimensions b and d; } I = \frac{bd^3}{12}$$

$$\sigma_b = \frac{12\,My}{bd^3}$$

whereas the value shear stress in the rectangular cross - section is given as

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right]$$

Hence the values of principle stress can be determined from the relations,

$$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}$$

Letting $\sigma_y = 0$; $\sigma_x = \sigma_b$,the values of σ_1 and σ_2 can be computed as

$$\text{Hence } \sigma_1 / \sigma_2 = \frac{1}{2}\left(\frac{12My}{bd^3}\right) \pm \frac{1}{2}\sqrt{\left(\frac{12\,My}{bd^3}\right)^2 + 4\left(\frac{6F}{bd^3}\left(\frac{d^2}{4} - y^2\right)\right)^2}$$

$$\sigma_1, \sigma_2 = \frac{6}{bd^3}\left[My \pm \sqrt{\left\{M^2y^2 + F^2\left(\frac{d^2}{4} - y^2\right)^2\right\}}\right]$$

Also,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{putting } \sigma_y = 0$$

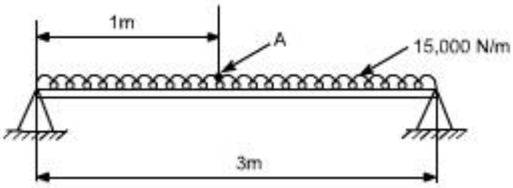
we get,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x}$$

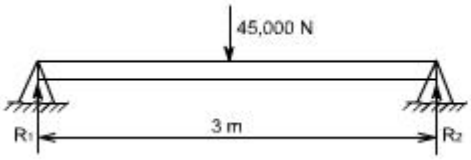
After substituting the appropriate values in the above expression we may get the inclination of the principal planes.

Illustrative examples: Let us study some illustrative examples, pertaining to determination of principal stresses in a beam

1. Find the principal stress at a point A in a uniform rectangular beam 200 mm deep and 100 mm wide, simply supported at each end over a span of 3 m and carrying a uniformly distributed load of 15,000 N/m.



Solution: The reaction can be determined by symmetry



$R_1 = R_2 = 22{,}500$ N

consider any cross-section X-X located at a distance x from the left end.

Hence,

S. F $_{\text{at XX}}$ =22,500 ? 15,000 x

B.M $_{\text{at XX}}$ = 22,500 x 15,000 x (x/2) = 22,500 x ? 15,000 . x^2 / 2

Therefore,

S. F $_{\text{at x}}$ = 1 m = 7,500 N

B. M $_{\text{at x}}$ = 1 m = 15,000 N

$$S.F|_{x=1m} = 7{,}500\,\text{N}$$

$$B.M|_{x=1m} = 15{,}000\,\text{N.m}$$

$$\sigma_x = \frac{My}{I}$$

$$= \frac{15{,}000 \times 5 \times 10^{-2} \times 12}{10 \times 10^{-12} \times \left(20 \times 10^{-2}\right)^3}$$

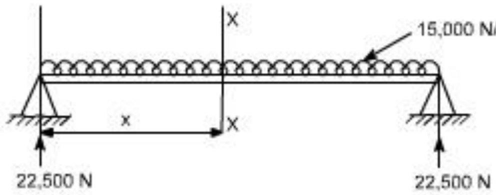
$$\sigma_x = 11.25\,\text{MN/m}^2$$

For the compution of shear stresses

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right] \qquad \text{putting } y = 50\,\text{mm}, d = 200\,\text{mm}$$

$$F = 7500\,\text{N}$$

$$\tau = 0.422\,\text{MN/m}^2$$

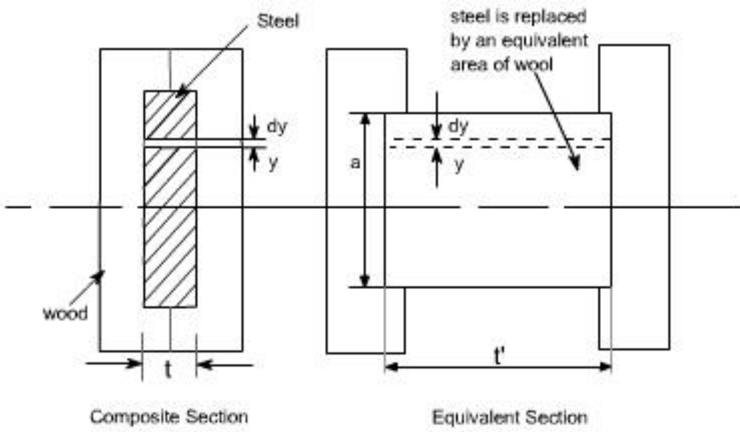


Now substituting these values in the principal stress equation,

We get $\sigma_1 = 11.27$ MN/m^2 and $\sigma_2 = -0.025$ MN/m^2

Bending Of Composite or Fletched Beams: A composite beam is defined as the one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together two timber joists and a reinforcing steel plate, then it is termed as a fletched beam.

The bending theory is valid when a constant value of Young's modulus applies across a section it cannot be used directly to solve the composite-beam problems where two different materials, and therefore different values of E, exists. The method of solution in such a case is to replace one of the materials by an equivalent section of the other.



Consider, a beam as shown in figure in which a steel plate is held centrally in an appropriate recess/pocket between two blocks of wood .Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength. i.e. the moment at any section must be the same in the equivalent section as in the original section so that the force at any given dy in the equivalent beam must be equal to that at the strip it replaces.

$$\sigma.t = \sigma'.t' \text{ or } \boxed{\frac{\sigma}{\sigma'} = \frac{t'}{t}}$$

$$\text{recalling } \sigma = E.\varepsilon$$

Thus

$$\varepsilon E t = \varepsilon' E' t'$$

Again, for true similarity the strains must be equal,

$$\varepsilon = \varepsilon' \text{ or } E t = E' t' \text{ or } \boxed{\frac{E}{E'} = \frac{t'}{t}}$$

$$\text{Thus, } \boxed{t' = \frac{E}{E'}.t}$$

Hence to replace a steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E'.

The equivalent section is then one of the same materials throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows by utilizing the given relations.

$$\frac{\sigma}{\sigma'} = \frac{t'}{t}$$

$$\frac{\sigma}{\sigma'} = \frac{E}{E'}$$

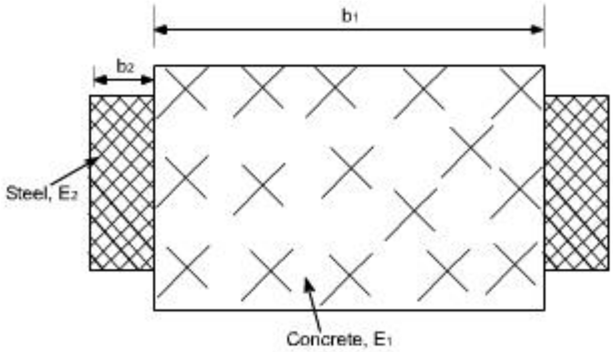
Stress in steel = modular ratio x stress in equivalent wood

The above procedure of course is not limited to the two materials treated above but applies well for any material combination. The wood and steel flitched beam was nearly chosen as a just for the sake of convenience.

Assumption

In order to analyze the behavior of composite beams, we first make the assumption that the materials are bonded rigidly together so that there can be no relative axial movement between them. This means that all the assumptions, which were valid for homogenous beams are valid except the one assumption that is no longer valid is that the Young's Modulus is the same throughout the beam.

The composite beams need not be made up of horizontal layers of materials as in the earlier example. For instance, a beam might have stiffening plates as shown in the figure below.



Again, the equivalent beam of the main beam material can be formed by scaling the breadth of the plate material in proportion to modular ratio. Bearing in mind that the strain at any level is same in both materials, the bending stresses in them are in proportion to the Young's modulus.