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Mathematics - I

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1 INTEGRAL CALCULUS

1. First Mean Value Theorem for Definite Integrals: Let $f(x)$ and $\phi(x)$ be two bounded functions integrable on $a \leq x \leq b$ and let $\phi(x)$ keep the same sign on $[a, b]$, then

$$\int_a^b f(x)\phi(x)dx = \mu \int_a^b \phi(x)dx,$$

where $m \leq \mu \leq M$, m and M being the greatest lower and least upper bounds of $f(x)$ on $[a, b]$.

Note that here $\mu = f(\xi)$ for some $\xi \in [a, b]$.

2. Mean Value Theorem (Simple form): [Particular case of above choosing $\phi(x) = 1$] If $f(x)$ is continuous on $[a, b]$, then at some point ξ in $[a, b]$,

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)dx.$$

3. Second Mean Value Theorem for Definite Integrals: [Bonnet's Form] Let $f(x)$ be bounded monotone non-increasing and never negative on $[a, b]$; and let $\phi(x)$ be bounded and integrable on $[a, b]$. Then there exists a value ξ of x on $[a, b]$, such that

$$\int_a^b f(x)\phi(x)dx = f(a) \int_a^\xi \phi(x)dx; \quad a \leq \xi \leq b.$$

[Weierstrass's Form] Let $f(x)$ be bounded and monotonic on $[a, b]$; and let $\phi(x)$ be bounded and integrable on $[a, b]$. Then there exists at least one value of x, say ξ on $[a, b]$, such that

$$\int_a^b f(x)\phi(x)dx = f(a) \int_a^\xi \phi(x)dx + f(b) \int_\xi^b \phi(x)dx; \quad a \leq \xi \leq b.$$

Example 1. Show that for $k^2 < 1$,

$$\frac{\pi}{6} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \leq \frac{\pi}{6} . \frac{1}{\sqrt{1-\frac{k^2}{4}}} .$$

Solution. Applying first mean-value theorem for integrals which we can do since it satisfies all the conditions. Let $f(x) = \frac{1}{\sqrt{1-k^2x^2}}$ and $\phi(x) = \frac{1}{\sqrt{1-x^2}}$. For $0 \leq \xi \leq \frac{1}{2}$, we get

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{\sqrt{1-k^2\xi^2}} \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} .$$

Now

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_0^{\frac{1}{2}} = \frac{\pi}{6} .$$

Hence

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{6} . \frac{1}{\sqrt{1-k^2\xi^2}} .$$

Putting $\xi = 0$ and $\xi = \frac{1}{2}$, we get

$$\frac{\pi}{6} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \leq \frac{\pi}{6} . \frac{1}{\sqrt{1-\frac{k^2}{4}}} .$$

■

4. Comparison test 1: If $f(x)$ be a non-negative integrable function when $x \geq a$ and $\int_a^B f(x)\mathrm{d}x$ is bounded above for every $B > a$, then $\int_a^\infty f(x)\mathrm{d}x$ will converge; otherwise it will diverge to ∞.
5. Comparison test 2: If $f(x)$ and $g(x)$ be integrable functions when $x \geq a$ such that $0 \leq f(x) \leq g(x)$, then

$$(i) \quad \int_a^\infty f(x)\mathrm{d}x \text{ converges if } \int_a^\infty g(x)\mathrm{d}x \text{ converges}$$

$$(ii) \quad \int_a^\infty g(x)\mathrm{d}x \text{ diverges if } \int_a^\infty f(x)\mathrm{d}x \text{ diverges.}$$

6. Limit test: Let $f(x)$ and $g(x)$ be integrable functions when $x \geq a$ and $g(x)$ be positive. Then if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lambda \neq 0 \quad (finite),$$

the integrals

$F = \int_a^\infty f(x)\mathrm{d}x$ and $G = \int_a^\infty g(x)\mathrm{d}x$ both converge absolutely or both diverge.

7. The μ-test for convergence for Type I: Let $f(x)$ be an integrable function when $x \geq a$. Then $F = \int_a^\infty f(x)\mathrm{dx}$ converges absolutely if

$$\lim_{x\to\infty} x^\mu f(x) = \lambda, \quad \text{for some} \quad \mu > 1,$$

and F diverges if

$$\lim_{x\to\infty} x^\mu f(x) = \lambda(\neq 0), \quad \textit{or} \quad \pm\infty; \quad \text{for some} \quad \mu \leq 1.$$

8. The μ-test for convergence for Type II: Let $f(x)$ be an integrable function in the arbitrary interval $(a+\epsilon, b)$, where $0 < \epsilon < b-a$. Then $F = \int_a^b f(x)\mathrm{dx}$ converges absolutely if

$$\lim_{x\to a+} (x-a)^\mu f(x) = \lambda, \quad \text{for some} \quad 0 < \mu < 1$$

and F diverges if

$$\lim_{x\to a+} (x-a)^\mu f(x) = \lambda(\neq 0), \quad \textit{or} \quad \pm\infty; \quad \text{for some} \quad \mu \geq 1.$$

Example 2. Show that $\int_0^\infty e^{-x^2}\mathrm{dx}$ converges.

Solution. Applying μ-test,

$$\lim_{x\to\infty} x^2 e^{-x^2} = \lim_{x\to\infty} \frac{x^2}{e^{x^2}} = 0$$

for $\mu = 2 > 1$.

So $\int_0^\infty e^{-x^2}\mathrm{dx}$ is convergent. ■

Example 3. Show that $\int_1^\infty e^{-x}x^n\mathrm{dx}$ converges for all values of n.

Solution. Applying μ-test,

$$\lim_{x\to\infty} \frac{x^{n+2}}{e^x} = 0$$

for $\mu = 2 > 1$. Hence $\int_1^\infty e^{-x}x^n\mathrm{dx}$ is convergent for any value of n. ■

Example 4. Show that $\int_1^\infty \frac{\log x}{x^2}\mathrm{dx}$ converges.

Solution. Applying μ-test,

$$\lim_{x\to\infty} x^{\frac{3}{2}} \frac{\log x}{x^{\frac{1}{2}}} = \lim_{x\to\infty} x\log x = 0$$

for $\mu = \frac{3}{2} > 1$. Hence $\int_1^\infty \frac{\log x}{x^2}\mathrm{dx}$ is convergent. ■

Example 5. Show that $\int_1^\infty \frac{x^{\frac{3}{2}}}{3x^2+5}\mathrm{dx}$ is divergent.

Solution. Applying μ-test,

$$\lim_{x\to\infty} x^{\frac{1}{2}} f(x) = \lim_{x\to\infty} x^{\frac{1}{2}} \frac{x^{\frac{3}{2}}}{3x^2+5} = \lim_{x\to\infty} \frac{x^2}{3x^2+5} = \frac{1}{3}$$

for $\mu = \frac{1}{2} < 1$. Hence $\int_1^\infty \frac{x^{\frac{3}{2}}}{3x^2+5}\text{dx}$ is divergent. ■

Example 6. Show that $\int_0^\pi \frac{\sin x}{x^3}\text{dx}$ is diverges.

Solution. By μ-test, since

$$\lim_{x\to 0+} x^2 \frac{\sin x}{x^3} = \lim_{x\to 0+} \frac{\sin x}{x} = 1.$$

Hence $\int_0^\pi \frac{\sin x}{x^3}\text{dx}$ is divergent. ■

9. Gamma Function: Let us discuss the convergence of

$$\int_0^\infty e^{-x} x^{n-1}\text{dx}, \quad n > 0. \tag{1.1}$$

We write, $f(x) = e^{-x}x^{n-1}$, $I_1 = \int_0^1 e^{-x}x^{n-1}\text{dx}$ and $I_2 = \int_1^\infty e^{-x}x^{n-1}\text{dx}$.

The part I_1 is proper when $n \geq 1$ and improper but absolutely convergent when $0 < n < 1$ by the following test.

By second μ-test,

$$\lim_{x\to 0+} x^{1-n} f(x) = \lim_{x\to 0+} x^{1-n} e^{-x} x^{n-1} = \lim_{x\to 0+} e^{-x} = 1,$$

for $0 < \mu = 1 - n < 1$, i.e., for $0 < n < 1$.

The part I_2 also converges absolutely for all values of n by first μ-test,

$$\lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} x^2 e^{-x} x^{n-1} = \lim_{x\to\infty} \frac{x^{n+1}}{e^x} = 0.$$

Thus equation (1.1) converges for $n > 0$. This is called gamma function denoted by $\Gamma(n)$.

Hence

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\text{dx}, \quad n > 0.$$

10. Beta Function: Next, let us discuss the convergence of

$$\int_0^1 x^{m-1}(1-x)^{n-1}\text{dx}, \quad m > 0, \quad n > 0. \tag{1.2}$$

This is a proper integral when $m, n \geq 1$ but is improper at the lower limit when $m < 1$, at the upper limit when $n < 1$. We, therefore, split it into two parts $I_1 + I_2$ where

$$I_1 = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\text{dx} \ \text{ and } \ I_2 = \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\text{dx}.$$

We write, $f(x) = x^{m-1}(1-x)^{n-1}$. Now I_1 converges for $0 < m < 1$, diverges when $m \le 0$, by second μ-test

$$\lim_{x\to 0+} x^{1-m} f(x) = \lim_{x\to 0+} x^{1-m} x^{m-1}(1-x)^{n-1} = \lim_{x\to 0+}(1-x)^{n-1} = 1,$$

for $\mu = 1-m$ and for convergence $0 < \mu = 1-m < 1$ that is $0 < m < 1$.

Also

$$\lim_{x\to 0+} x f(x) = \lim_{x\to 0+} x x^{m-1}(1-x)^{n-1} = \lim_{x\to 0+} x^m(1-x)^{n-1} = \begin{cases} 1 & \text{when} \quad m = 0, \\ \infty & \text{when} \quad m < 0. \end{cases}$$

Next if we make the change of variable $x = 1-y$, the second integral reduces to the first with m and n interchanged. Hence we may draw the same conclusion as before with n in place of m. Thus equation (1.2) converges for $m, n > 0$. This is called Beta function denoted by $\beta(m,n)$, or,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x \quad \text{for} \quad m,n > 0.$$

Definition 1 (Gamma function). *The Gamma function denoted by* $\Gamma(n)$ *is defined by*

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt,$$

$n > 0$.

Definition 2 (Beta function). *The Beta function denoted by* $\beta(m,n)$ *is defined by*

$$\beta(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1} dt,$$

$m > 0$, $n > 0$.

Example 7. Show that

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}\mathrm{d}x = (b-a)^{m+n-1}\ B(m,n), \quad m, \quad n > 0.$$

Solution. Put $x = a\cos^2\theta + b\sin^2\theta$, then

$$\begin{aligned} \int_a^b (x-a)^{m-1}(b-x)^{n-1}\mathrm{d}x &= \int_0^{\frac{\pi}{2}} (b-a)^{m+n-1}\sin^{2m-1}\theta\cos^{2n-1}\theta\mathrm{d}\theta \\ &= (b-a)^{m+n-1}B(m,n). \end{aligned}$$

■

Example 8. Show that

$$\int_0^\infty x^{\frac{1}{2}} e^{-x^3}\mathrm{d}x = \frac{\sqrt{\pi}}{3}.$$

Solution. Put $x^3 = z$, then

$$\begin{aligned}\int_0^\infty x^{\frac{1}{2}} e^{-x^3} \mathrm{dx} &= \frac{1}{3}\int_0^\infty z^{-\frac{1}{2}} e^{-z} \mathrm{dz} \\ &= \frac{1}{3}\Gamma(\frac{1}{2}) \\ &= \frac{\sqrt{\pi}}{3}.\end{aligned}$$

■

1.1 Problem Set

1. Do the following integrals exist? If exist, find the value:
   a) $\int_0^\infty \frac{1}{1+x^2} dx$ Ans: $\frac{\pi}{2}$
   b) $\int_0^\infty \frac{1}{x^2} dx$ Ans: $\times$
   c) $\int_0^\infty \sin x dx$ Ans: $\times$
   d) $\int_0^\infty e^{-x^2} dx$ Ans: Using β and Γ
   e) $\int_2^\infty \frac{1}{x \log x} dx$ Ans: $\times$
   f) $\int_{-\infty}^\infty x e^{-x^2} dx$ Ans: 0
   g) $\int_0^\infty e^{-ax} \sin bx \ dx$ Ans:
   h) $\int_0^1 \frac{dx}{x}$ Ans: $\times$
   i) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ Ans: $\frac{\pi}{2}$
   j) $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1+\cot^2 x}}$ Ans:
   k) $\int_0^4 \frac{dx}{2x-8}$ Ans: $\times$
   l) $\int_{-1}^1 \frac{dx}{x^2}$ Ans: $\times$
   m) $\int_{-1}^\infty \frac{dx}{x^2}$ Ans: 1
   n) $\int_0^\infty \dfrac{dx}{\sqrt{1+x^2}}$ Ans: $\frac{\pi}{2}$
   o) $\int_{-\infty}^\infty \dfrac{dx}{\sqrt{1+x^2}}$ Ans: π
   p) $\int_0^1 \dfrac{dx}{\sqrt{1-x}}$ Ans: 2
2. Does the integral
$$\int_0^1 \frac{dx}{\sqrt{x}+x^3}$$
converges? Ans: Yes, use comparison test compared with $\frac{1}{\sqrt{x}}$.
3. Examine the convergence of the improper integral
$$\int_0^\infty \frac{x^{p-1}}{1+x} \mathrm{dx},$$
where $p \in \mathbb{R}$.
4. Prove that $\int_{-1}^1 \frac{dx}{x^3}$ exists in Cauchy principal value sense but not in general sense.

5. Prove the following relations ($a > 0$, $m > 0$, $n > 0$):

   a) $\int_0^\infty e^{-at}t^{n-1}dt = \frac{\Gamma(n)}{a^x}$. Hint: Let $at = u$.

   b) $\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt$. Hint: Put $\frac{1}{1+t} = u$.

   c) $\Gamma(1) = 1$, $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n+1) = n!$, n being a fixed positive integer.

   d) $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}dx$. Hint: Let $t = x^2$.

   e) $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$. Hint: Let $t = \sin^2\theta$.

   f) $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. Hint: Start from $\Gamma(m)\Gamma(n)$ and (d).

   g) $\beta(m,n) = \beta(n,m)$.

   h) $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$.

   i) $\beta(\frac{1}{2},\frac{1}{2}) = \pi$. Hint: Put $m = n = \frac{1}{2}$ in (e).

   j) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Hint: Put $m = n = \frac{1}{2}$ in (f) and then use (c) and (i).

6. Express

$$\int_0^1 t^m(1-t^n)^p dt$$

in terms of Beta function and hence evaluate

$$\int_0^1 t^5(1-t^3)^9 dt.$$

⊛ Hint: Let $t^n = u$. Ans: $\frac{1}{330}$.

7. Evaluate:

   a) $\int_0^1 x^3(1-x)^{\frac{1}{2}}dx$ Ans: $\beta\left(4,\frac{3}{2}\right) = \frac{32}{315}$.

   b) $\int_0^1 x(1-x)^7dx$ Ans: $\beta(2,8) = \frac{1}{72}$.

   c) $\int_0^\infty x^2e^{-x^2}dx$ Ans: $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$.

   d) $\int_0^1 t^3(1-t^2)^{\frac{5}{2}}dt$ Ans: $\frac{1}{2}\beta\left(\frac{4}{2},\frac{5}{2}+1\right) = \frac{2}{63}$.

   e) $\int_0^\infty x^4e^{-x}dx$ Ans: $\Gamma(5) = 24$.

   f) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}d\theta$ Ans: $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}$.

   g) $\int_0^1 t^{13}(1-t^7)^7dt$ Ans: $\frac{1}{7}\beta(2,8) = \frac{1}{504}$.

8. Prove that:

   a) $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

   b) $\int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \Gamma(n)\Gamma(1-n)$

   c) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}d\theta = \frac{\pi}{\sqrt{2}}$ using $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \sqrt{2}\pi$

   d) $\int_0^{\frac{\pi}{2}} \sin^7\theta\cos^4\theta d\theta = \frac{1}{120}$

   e) $\int_0^\infty x^n e^{-a^2x^2}\mathrm{dx} = \frac{1}{2a^{n+1}}\Gamma(\frac{n+1}{2})$ Hint: Put $ax = \sqrt{t}$.

   f) $\beta(n,n) = \frac{\sqrt{\pi}\Gamma(n)}{2^{2n-1}\Gamma(n+\frac{1}{2})}$.

   g) $2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2}) = \Gamma(2n)\sqrt{\pi}$. (This is known as **duplication** formula.)

Example 9. Show that $\Gamma(n+1) = n\Gamma(n)$.

Solution.

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty e^{-x} x^{n+1-1} dx \\
&= \int_0^\infty x^n e^{-x} dx \\
&= \lim_{B\to\infty} \int_0^B x^n e^{-x} dx \\
&= \lim_{B\to\infty} \left\{ x^n \int_0^B e^{-x} dx - \int_0^B n x^{n-1} \int_0^B e^{-x} dx \right\} \\
&= \lim_{B\to\infty} \left\{ \left[-x^n e^{-x} \right]_0^B \right\} + \lim_{B\to\infty} \left\{ n \int_0^B x^{n-1} e^{-x} dx \right\} \\
&= 0 + n \int_0^\infty e^{-x} x^{n-1} dx \\
&= n\Gamma(n).
\end{aligned}$$

■

Example 10. Show that

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx.$$

Solution. From definition, we know that

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0.$$

Let $x = t^2$. Then $dx = 2tdt$ and

$$\begin{aligned}
\Gamma(n) &= \int_0^\infty e^{-t^2} t^{2n-2} \cdot 2tdt \\
&= 2\int_0^\infty e^{-t^2} t^{2n-1} dt \\
&= 2\int_0^\infty e^{-x^2} x^{2n-1} dx.
\end{aligned}$$

■

Example 11. Prove that

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta.$$

Solution. From definition, we know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,$$

Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta d\theta$ and

$$\begin{aligned}\beta(m,n) &= \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta d\theta\\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta.\end{aligned}$$

■

Example 12. Show that

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Solution. We know that

$$\Gamma(m) = 2\int_0^\infty e^{-x^2}x^{2m-1}dx.$$

So

$$\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}dy.$$

Therefore

$$\begin{aligned}\Gamma(m)\Gamma(n) &= 2\int_0^\infty e^{-x^2}x^{2m-1}dx\cdot 2\int_0^\infty e^{-y^2}y^{2n-1}dy\\ &= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}dxdy\end{aligned}$$

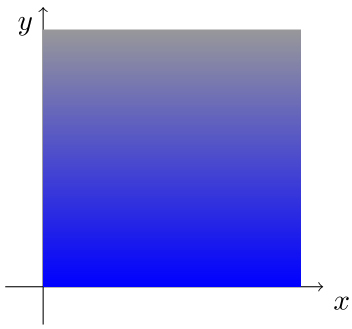

Figure 1.1: Change of limit

Let $x = r\cos\theta$ and $y = r\sin\theta$. So $dxdy = rdrd\theta$. Then from Figure 1.1, we get

$$\begin{aligned}\Gamma(m)\Gamma(n) &= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^\infty e^{-r^2}r^{2m-1}\cos^{2m-1}\theta\cdot r^{2n-1}\sin^{2n-1}\theta\cdot rdrd\theta\\ &= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^\infty e^{-r^2}r^{2(m+n)-1}\cos^{2m-1}\theta\sin^{2n-1}\theta drd\theta\\ &= 2\int_{r=0}^\infty e^{-r^2}r^{2(m+n)-1}dr\cdot 2\int_{\theta=0}^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta\\ &= \Gamma(m+n)\cdot\beta(m,n).\end{aligned}$$

$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. ■

Example 13. Express

$$\int_0^1 t^m (1-t^n)^p \, dt$$

in terms of Beta function and hence evaluate

$$\int_0^1 t^5(1-t^3)^9 dt.$$

Solution. The given expression is the following

$$\int_0^1 t^m(1-t^n)^p dt \tag{1.3}$$

Let $t^n = u$. Then $nt^{n-1}dt = du \implies dt = \frac{1}{n}u^{-\frac{n-1}{n}}du$ and $t = u^{\frac{1}{n}}$.
From Equation (1.3), we get

$$\begin{aligned}\int_0^1 t^m(1-t^n)^p dt &= \int_0^1 u^{\frac{m}{n}}(1-u)^p \cdot \frac{1}{n}u^{-\frac{n-1}{n}}du \\ &= \frac{1}{n}\int_0^1 u^{\frac{m}{n}-\frac{n-1}{n}}(1-u)^p \, du \\ &= \frac{1}{n}\int_0^1 u^{\frac{m+1}{n}-1}(1-u)^{p+1-1} \, du \\ &= \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right).\end{aligned}$$

Comparing with Equation (1.3), we get $m=5$, $n=3$ and $p=9$. So

$$\begin{aligned}\int_0^1 t^5(1-t^3)^9 dt &= \frac{1}{3}\beta\left(\frac{5+1}{3}, 9+1\right) \\ &= \frac{1}{3}\beta(2,10) \\ &= \frac{1}{3}\frac{\Gamma(2)\Gamma(10)}{\Gamma(12)} = \frac{1}{330}.\end{aligned}$$

■

1.2 APPLICATIONS OF DEFINITE INTEGRAL

Rule 1: Let the equation of the curve in rectangular cartesian coordinates be $y = f(x)$. We assume it to be continuous on the finite interval $[a,b]$ where $b > a$ and the value of $f(x)$ are all positive throughout the range. Then area of the given region

$$A = \int_a^b f(x)dx.$$

Rule 2: Let the equation of the curve in rectangular cartesian coordinates be $x = \phi(y)$. We assume it to be continuous on the finite interval $[c, d]$ where $d > c$ and the value of $\phi(y)$ are all positive throughout the range. Then area of the given region

$$A = \int_c^d \phi(y)dy.$$

Rule 3: If the equation of a curve in polar coordinates be $r = f(\theta)$, the area included between the curve and two radii vectors $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta.$$

1.3 Problem Set

1. Find the area of the region bounded by the upper half of the circle $x^2 + y^2 = 25$, the x-axis and the ordinates $x = -3$ and $x = 4$. Ans: $12 + \frac{25}{4}\pi$ sq unit.
2. Find the area of the circle $r = 2a\sin\theta$. Ans: πa^2 square unit.
3. Find the area of the cardioide $r = a(1 - \cos\theta)$. Ans: $\frac{3\pi}{2}a^2$ square unit.
4. Obtain the area common to the two circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$. Ans: $(\pi - 1)a^2$ sq unit.
5. Find the area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes. Ans.
6. Show that the area included between the cardiodes $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$ is $\frac{a^2(3\pi-8)}{2}$.
7. Find the common area to the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 4ax$. Ans.

Rule 4: The length of the curve $y = f(x)$ between two points $x = x_1$ and $x = x_2$ is given by

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Rule 5: The length of the curve $x = \phi(y)$ between two points $y = y_1$ and $y = y_2$ is given by

$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Rule 6: The length of the curve $r = f(\theta)$ between the points (r_1, θ_1) and (r_2, θ_2) is given by

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Rule 7: The length of the curve $\theta = f(r)$ between the points (r_1, θ_1) and (r_2, θ_2) is given by

$$s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr.$$

Note: We look at the arc length of the curve given by, $r = f(\theta), \quad \alpha \le \theta \le \beta$ where we also assume that the curve is traced out exactly once.

First write the curve in terms of a set of parametric equations,

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta$$

and we can now use the parametric formula for finding the arc length.

We need the following derivatives for these computations.

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta, \quad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

We need the following for our ds.

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\cos\theta - f(\theta)\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + f(\theta)\cos\theta\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2.$$

The *arc* length formula for polar coordinates is then,

$$L = \int ds = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

1.4 Problem Set

1. Find the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the points of intersection of the parabola and the straight line $3y = 8x$. Ans: $\frac{9}{16}a^{\frac{7}{2}}$ unit.
2. Determine the length of any arc of the parabola $y^2 = 4ax$, the arc being measured from the vertex.
3. Show that the length of the arc of that part of the cardioide $r = a(1 + \cos\theta)$ which lies on the side of the line $4r = 3a\sec\theta$ remote from the pole is equal to 4a.
4. Find the length of the arc of the parabola $r = a\sec^2\frac{\theta}{2}$.
5. Find the length of the loop of the curve $9ay^2 = (x - 2a)(x - 5a)^2$. Ans: $4\sqrt{3a}$ unit.
6. Find the length of the cardiode $r = a(1 - \cos\theta)$ lying inside the circle $r = a\cos\theta$. Ans: $4a(2 - \sqrt{3})$ unit.
7. Find the perimeter of the cardiode $r = a(1 + \cos\theta)$. Ans: $8a$ unit.
8. Find the entire length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. Ans: $6a$ unit.
9. Find the perimeter of the loop of the curve $3ay^2 = x(x - a)^2$. Ans: $\frac{4}{\sqrt{3}}a$ unit.

Rule 8: For rotation of the curve $y = f(x)$ between $x = a$ and $x = b$ about x-axis, then

$$V = \pi\int_a^b y^2 dx \quad \text{and} \quad S = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Rule 9: For rotation of the curve $x = \phi(y)$ between $y = a$ and $y = b$ about y-axis, then

$$V = \pi\int_a^b x^2 dy \quad \text{and} \quad S = 2\pi\int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Rule 10: A sectorial element bounded by the radii vectors $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$ revolves about the initial line, then the volume of the solid of revolution is given by

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \sin\theta d\theta \quad \text{and} \quad S = 2\pi \int_{\alpha}^{\beta} r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

1.5 Problem Set

1. The curve $r = a(1+\cos\theta)$ revolves about the initial line. Find the volume and the surface area of the figure formed. Ans: $V = \frac{8\pi a^3}{3}$ cubic unit, $S = \frac{32\pi a^2}{5}$ sq. unit.
2. Find the volume of the solid generated by the revolution about the y-axis of the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$. Ans:
3. Find the surface area of the sphere generated by the circle
$$x^2 + y^2 = 16$$
about x-axis. Ans: $S = 64\pi$, $V = \frac{256\pi}{3}$
4. Find the area of the surface formed by the revolution of the curve $6xy = y^4 + 3$ about the axis of y, from $y = 1$ to $y = 4$. Ans:
5. A parabolic reflector of an automobile headlight is 12 cm in diameter and 4 cm deep. Find the cost of plating of the front portion of the reflection if the cost of plating is Rs. 50 per sq.cm. Ans: Rs 7700.00
6. Find the volume of the solid obtained by revolving the loop of the curve $a^2y^2 = x^2(2a-x)(x-a)$ about the x-axis. Ans: $\frac{23}{60}\pi a^3$ Cubic unit.
7. Find the volume of the solid obtained by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis. Ans: $\frac{4}{3}\pi ab^2$ Cubic unit.
8. Find the volume of the solid obtained by revolving the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis. Ans: $\frac{32\pi a^3}{105}$ Cubic unit.
9. Find the surface area of the solid generated by revolving the arc of the parabola $y^2 = 4ax$ bounded by its latus rectum about the x-axis. Ans: $8a^2\pi\frac{2\sqrt{2}-1}{3}$ Cubic unit.
10. Find the area of the surface of the solid formed by revolving the curve $r = 2a\cos\theta$ about the initial line. Ans: $4\pi a^2$ Cubic unit.

Hard Problem:

1. Find the volume of the solid obtained by revolution of the cissoid $y^2(a-x) = x^3$ about its asymptote. Ans: $\pi^2\frac{a^3}{4}$ cubic unit.
2. The axes of symmetry of 2-inch right circular cylinders intersect at right angles. What volume do the cylinders have in common? Ans: $(\frac{4\pi r^3}{3})(\frac{4}{\pi})$ cubic unit.

Example 14. Prove that a sectorial element bounded by the radii vectors $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$ revolves about the initial line, then the volume of the solid of revolution is given by
$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \sin\theta d\theta.$$

Solution.

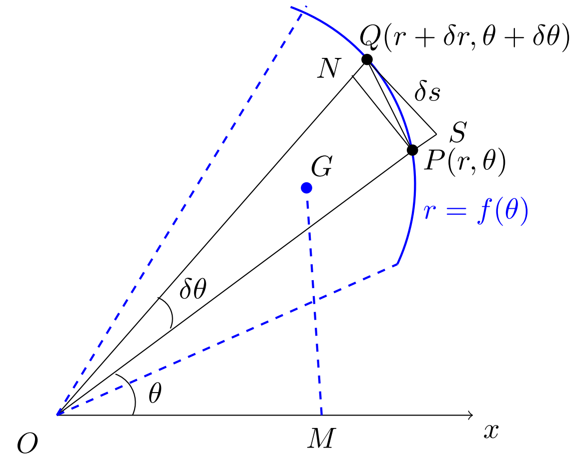

Figure 1.2: Polar curve relation

Let $OP = OR = r$ and $OS = OQ = r + \delta r$. Let area of sector $POQ = \delta V$.

Area of sector $= \frac{1}{2} \times (radius)^2 \times (anlge\ \ beteen\ \ radii)$.

G is centre of gravity whose position is $\frac{2}{3} \times (radiusvector)$.

Volume of solid generated by revolution of sector POR

$$\begin{aligned}
&= 2\pi(area\ \ of\ \ sector) \times (perpendicular\ \ distance\ \ of\ \ sector)\\
&= 2\pi(\frac{1}{2}r^2\delta\theta) \times (\frac{2}{3}r\sin\theta)\\
&= \frac{2}{3}\pi r^3 \sin\theta\delta\theta.
\end{aligned}$$

Volume of solid generated by revolution of sector SOQ

$$\begin{aligned}
&= 2\pi(\frac{1}{2}(r+\delta r)^2\delta\theta) \times (\frac{2}{3}(r+\delta r)\sin\theta)\\
&= \frac{2}{3}\pi(r+\delta r)^3 \sin\theta\delta\theta.
\end{aligned}$$

Volume of sector $POR <$ Volume of sector $POQ <$ Volume of sector SOQ, this implies

$$\frac{2}{3}\pi r^3 \sin\theta\delta\theta < \delta V < \frac{2}{3}\pi(r+\delta r)^3 \sin\theta\delta\theta$$

as $P \to Q$, $\delta\theta \to 0$, $\delta r \to 0$.

$$\begin{aligned}
\lim_{\delta\theta\to 0} \frac{\delta V}{\delta\theta} &= \frac{2}{3}\pi r^3 \sin\theta\\
\implies \frac{dV}{d\theta} &= \frac{2}{3}\pi r^3 \sin\theta.
\end{aligned}$$

Integrating both sides, we get

$$\int dV = \int_\alpha^\beta \frac{2}{3}\pi r^3 \sin\theta d\theta$$

$$\implies V = \frac{2}{3}\pi \int_\alpha^\beta r^3 \sin\theta d\theta.$$

$\blacksquare$

1. $\int \sqrt{a^2+x^2}\mathrm{dx} = \dfrac{x\sqrt{a^2+x^2}}{2} + \dfrac{a^2}{2}\sinh^{-1}\frac{x}{a} + C.$
2. $\int \sqrt{x^2-a^2}\mathrm{dx} = \dfrac{x\sqrt{x^2-a^2}}{2} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{x}{a} + C.$
3. $\int \sqrt{a^2-x^2}\mathrm{dx}d = \dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + C.$
4. $\int \dfrac{\mathrm{dx}}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C.$
5. $\int \sin^{-1}x\mathrm{dx} = x\sin^{-1}x + \sqrt{1-x^2} + C$
6. $\cosh x = \dfrac{e^x - e^{-x}}{2}$, $\sinh x = \dfrac{e^x+e^{-x}}{2}$, $\dfrac{d}{dx}(\cosh x) = \sinh x$, $\dfrac{d}{dx}(\sinh x) = \cosh x$, $\cosh^2 x - \sinh^2 x = 1$.
7. $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \qquad \lim_{n\to\infty}(1+\frac{1}{n})^n = e \quad \text{and} \quad \lim_{n\to\infty}(1+\frac{x}{n})^n = e^x.$$
8. The equation of a **cardiod** is $r = a(1+\cos\theta)$ and shape of the equation is in Figure 2.1. Similarly the equation of another **cardiod** is $r = a(1-\cos\theta)$ and shape of the equation is in Figure 2.2. The pole of the cardiod is the origin.

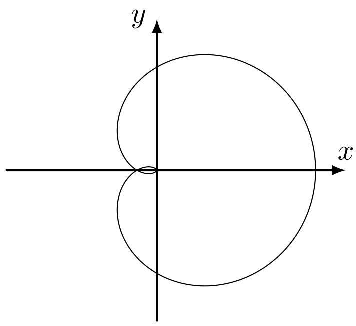

Figure 2.1: cardiod: $r = a(1+\cos\theta)$

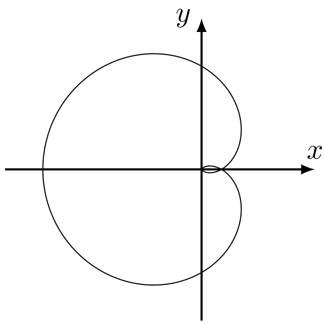

Figure 2.2: cardiod: $r = a(1-\cos\theta)$

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