Solid Mechanics (MEC 301)

CHAPTER 3: COLUMNS

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Books:

1. Strength of Materials: Part I, II, S. Timoshenko, CBS Publishers, 1985.
2. Engineering Mechanics of Solids, E. P. Popov, PHI, 1993.
3. Introduction to Solid Mechanics, I. H. Shames and J. M. Pittariesi, PHI, 2003.
4. Strength of Materials, F. L. Singer and A. Pytel, HarperCollins Publishers, 1991

Elastic Stability of Columns

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns: Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded.

Struts: Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons.

- the strut may not be perfectly straight initially.
- the load may not be applied exactly along the axis of the Strut.
- one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

- It is assumed that the deformation to be both progressive with increasing load and simple in form i.e. a member in simple tension/compression becomes progressively longer/shorter but remains straight. Under some circumstances, however, these assumptions may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.
- At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.
- Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions.

The resistance of any member to bending is determined by its flexural rigidity EI

where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k.

There will be two principal moments of inertia, if the least of these is taken then the ratio

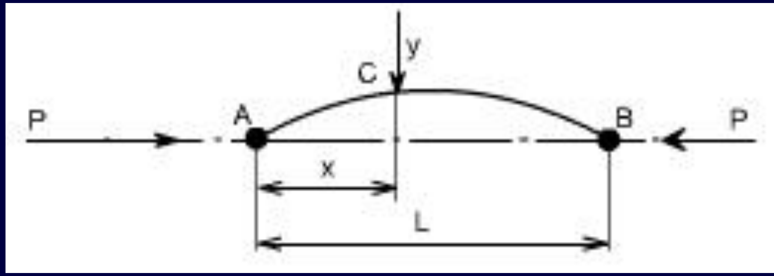
Is called the slenderness ratio. It's numerical value indicates whether the member falls into the class of columns or struts.

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

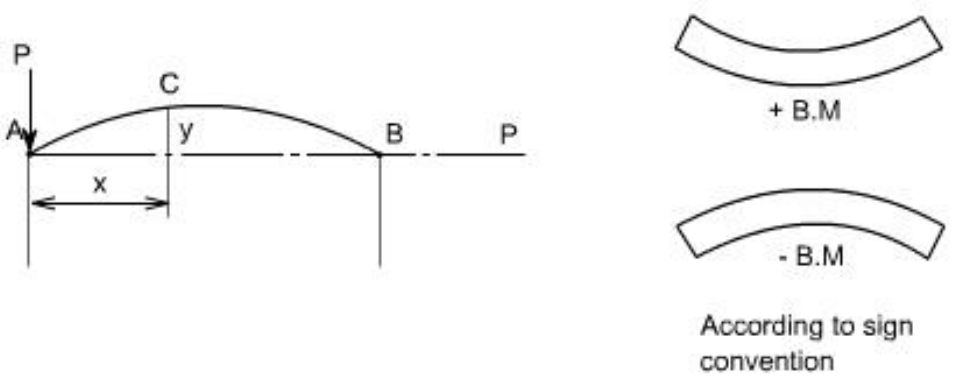
Euler's Theory

Case A: Strut with pinned ends: Consider an axially loaded strut, shown below, and is subjected to an axial load 'P' this load 'P' produces a deflection 'y' at a distance 'x' from one end.

Assume that the ends are either pin jointed or rounded so that there is no moment at either end.



Assumption: The strut is assumed to be initially straight, the end load being applied axially through centroid.



$$B.M|_C = -Py$$

Futher,we know that

$$EI\frac{d^2y}{dx^2} = M$$

$$EI\frac{d^2y}{dx^2} = -P.y = M$$

Thus,

$$EI\frac{d^2y}{dx^2} + Py = 0$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

Let us define a operator D = d/dx

$(D^2 + n^2)\, y = 0$ where $n^2 = P/EI$

$$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x$$

In order to evaluate the constants A and B let us apply the boundary conditions,

(i) at x = 0; y = 0

(ii) at x = L ; y = 0

Applying the first boundary condition yields A = 0.

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

Thus either $B = 0$, or $\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$

if B=0,that y0 for all values of x hence the strut has not buckled yet.Therefore,the solution required is

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \text{ or } \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or } \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \text{ or } P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the least value of P which will cause the strut to buckle, and it is called the **Euler Crippling Load** P_e from which w obtain.

$$\boxed{P_e = \frac{\pi^2 EI}{L^2}}$$

It may be noted that the value of I used in this expression is the least moment of inertia

It should be noted that the other solutions exists for the equation

$$\sin\left(l\sqrt{\frac{P}{EI}} \right) = 0 \qquad \text{i.e. } \sin\ nL = 0$$

The interpretation of the above analysis is that for all the values of the load P, other than those which make $\sin nL = 0$; the strut will remain perfectly straight since $y = B \sin nL = 0$

For the particular value of

$$P_e = \frac{\pi^2 EI}{L^2}$$

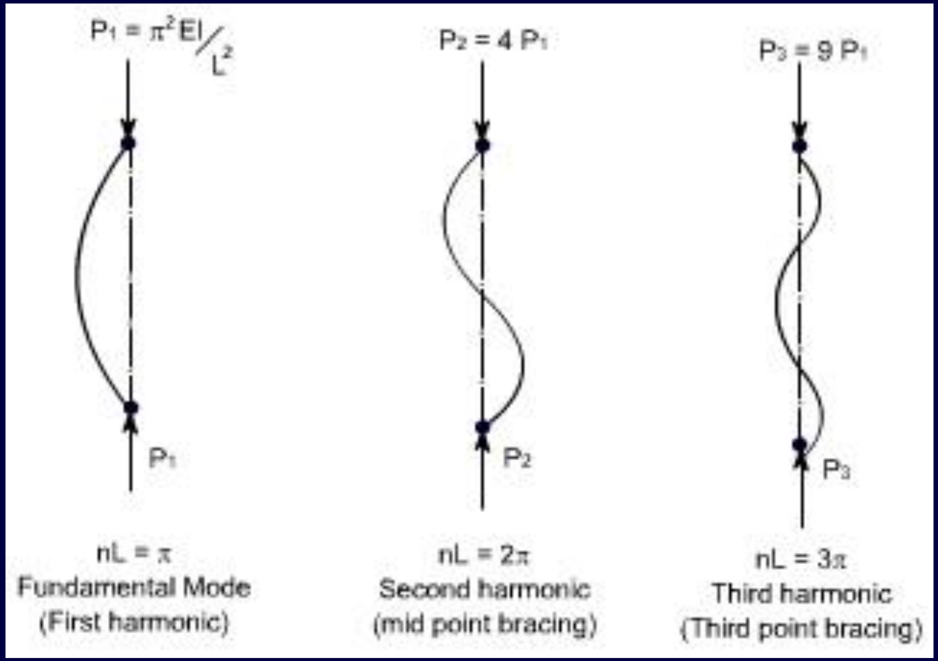
$$\sin nL = 0 \quad \text{or } nL = \pi$$

$$\text{Therefore} \quad n = \frac{\pi}{L}$$

$$\text{Hence } y = B \sin nx = B \sin \frac{\pi x}{L}$$

Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that 'L' remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

- Further it should be noted that the deflection is not proportional to load, and this applies to all strut problems; like wise it will be found that the maximum stress is not proportional to load.
- The solution chosen of $nL = \pi$ is just one particular solution; the solutions $nL = 2\pi$, 3π, 5π etc are equally valid mathematically and they do, infact, produce values of 'P_e' which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of P_e, each corresponding with a different mode of buckling.
- The value selected above is so called the fundamental mode value and is the lowest critical load producing the single bow buckling condition.
- The solution $nL = 2\pi$ produces buckling in two half waves, 3π in three half-waves etc.

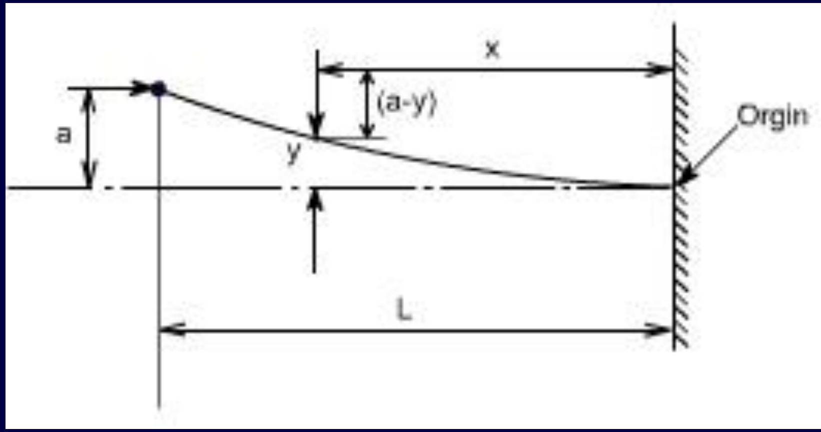


$$L\sqrt{\frac{P}{EI}} = \pi \text{ or } P_1 = \frac{\pi^2 EI}{L^2}$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 2\pi \text{ or } P_2 = \frac{4\pi^2 EI}{L^2} = 4P_1$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 3\pi \text{ or } P_3 = \frac{9\pi^2 EI}{L^2} = 9P_1$$

Case b: One end fixed and the other free:



$B.M|_C = P(a-y)$

Hence, the differential equation becomes,

$$EI\frac{d^2y}{dx^2} = P(a-y)$$

On rearranging we get

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

Let $\frac{P}{EI} = n^2$

$y_{gen} = A\cos(nx) + \sin(nx) + P.I$

P.I is a particular value of y which satisfies the differential equation, Hence yP.I = a

Therefore the complete solution becomes Y = A cos(nx) + B sin(nx) + a

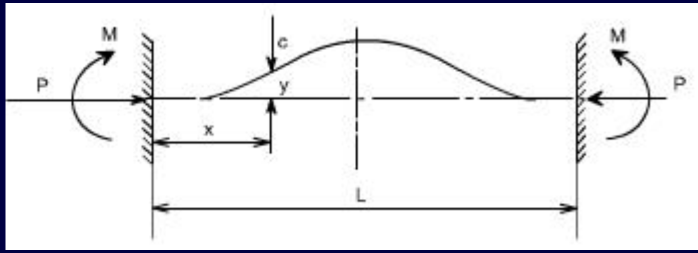
Now imposing the boundary conditions to evaluate the constants A and B
(i) at x = 0; y = 0 ; This yields A = -a and (ii) at x = 0; dy/dx = 0 ; This yields B = 0
Hence, y = -a cos(nx) + a; Further, at x = L; y = a, Therefore a = - a cos(nx) + a or 0 = cos(nL)

$$nL = \frac{\pi}{2}$$

$\sqrt{\frac{P}{EI}}\,L = \frac{\pi}{2}$, Therefore, the Euler's crippling load is given as

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case 3: Strut with fixed ends



$$EI\frac{d^2y}{dx^2} = M - Py$$

$$\text{or } \frac{d^2y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$$

$n^2 = \frac{P}{EI}$, Therefore in the operator from, the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{M}{EI}$$

$$y_{general} = y_{complementary} + y_{particular\,integral}$$

$$y|_{P.I} = \frac{M}{n^2EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B\,Cosnx + A\,Sinnx + \frac{M}{P}$$

Boundry conditions relevant to this case are at $x=0; y=0$

$$B = -\frac{M}{P}$$

Also at $x = 0; \frac{dy}{dx} = 0$ hence

$A=0$

Therefore,

$$y = -\frac{M}{P}\,Cos\,nx + \frac{M}{P}$$

$$y = \frac{M}{P}\,(1 - Cos\,nx)$$

Futher, it may be noted that at $x = L; y = 0$

$$\text{Then } 0 = \frac{M}{P}\,(1 - Cos\,nL)$$

Thus, either $\frac{M}{P} = 0$ or $(1 - Cos\,nL) = 0$

obviously, $(1 - Cos\,nL) = 0$

$cos\,nL = 1$

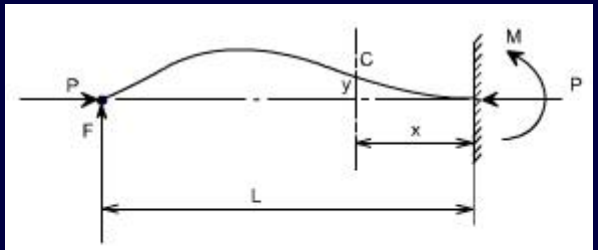
Hence the least solution would be

$nL = 2\pi$

$\sqrt{\frac{P}{EI}}\,L = 2\pi$, Thus, the buckling load or crippling load is

$$P_e = \frac{4\pi^2.EI}{L^2}$$

Case 4 : One end fixed, the other pinned



$$EI\frac{d^2y}{dx^2} = -Py + F(L-x)$$

$$EI\frac{d^2y}{dx^2} + Py = F(L-x)$$

Hence

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L-x)$$

In the operator form the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{F}{EI}(L-x)$$

$$y_{particular} = \frac{F}{n^2EI}(L-x) \text{ or } y = \frac{F}{P}(L-x)$$

The full solution is therefore

$$y = A\,Cos\,mx + B\,Sin\,nx + \frac{F}{P}(L-x)$$

The boundry conditions relevants to the problem are at x=0;y=0

$$\text{Hence } A = -\frac{FL}{P}$$

$$\text{Also at } x = 0; \frac{dy}{dx} = 0$$

$$\text{Hence } B = \frac{F}{nP}$$

$$\text{or } y = -\frac{FL}{P}Cos\,nx + \frac{F}{nP}Sin\,nx + \frac{F}{P}(L-x)$$

$$y = \frac{F}{nP}\left[Sin\,nx - nL\,Cos\,nx + n(L-x)\right]$$

Also when x = L ; y = 0

Therefore, nL Cos nL = Sin nL or tan nL = nL

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is nL = 4.49radian

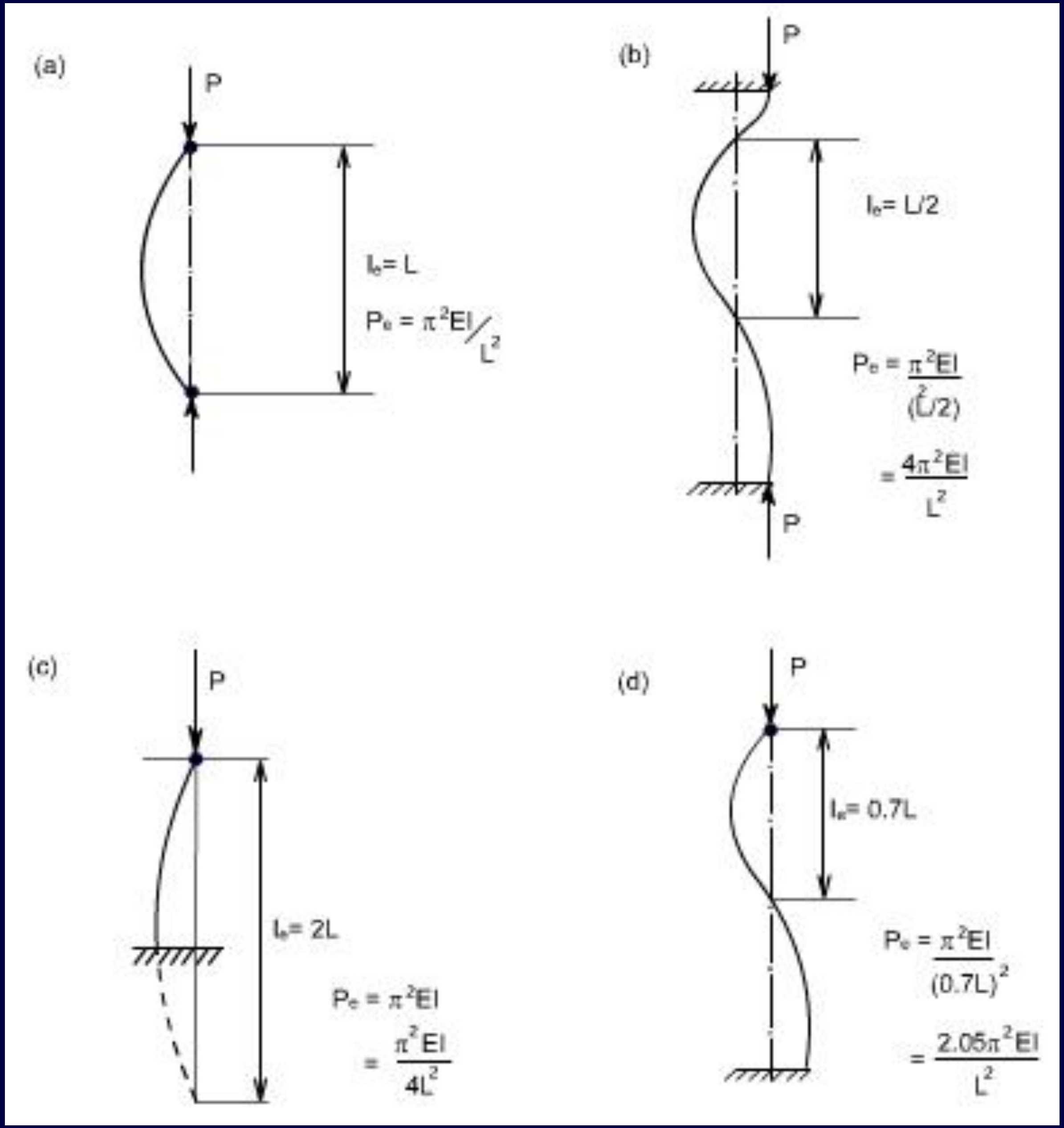
$$\text{or } \sqrt{\frac{P}{EI}}\,L = 4.49$$

$$\frac{P_e}{EI}L^2 = 20.2$$

$$P_e = \frac{2.05\pi^2\,EI}{L^2}$$

Equivalent Strut Length:

$$P_e = \frac{\pi^2 EI}{L_e^2}$$

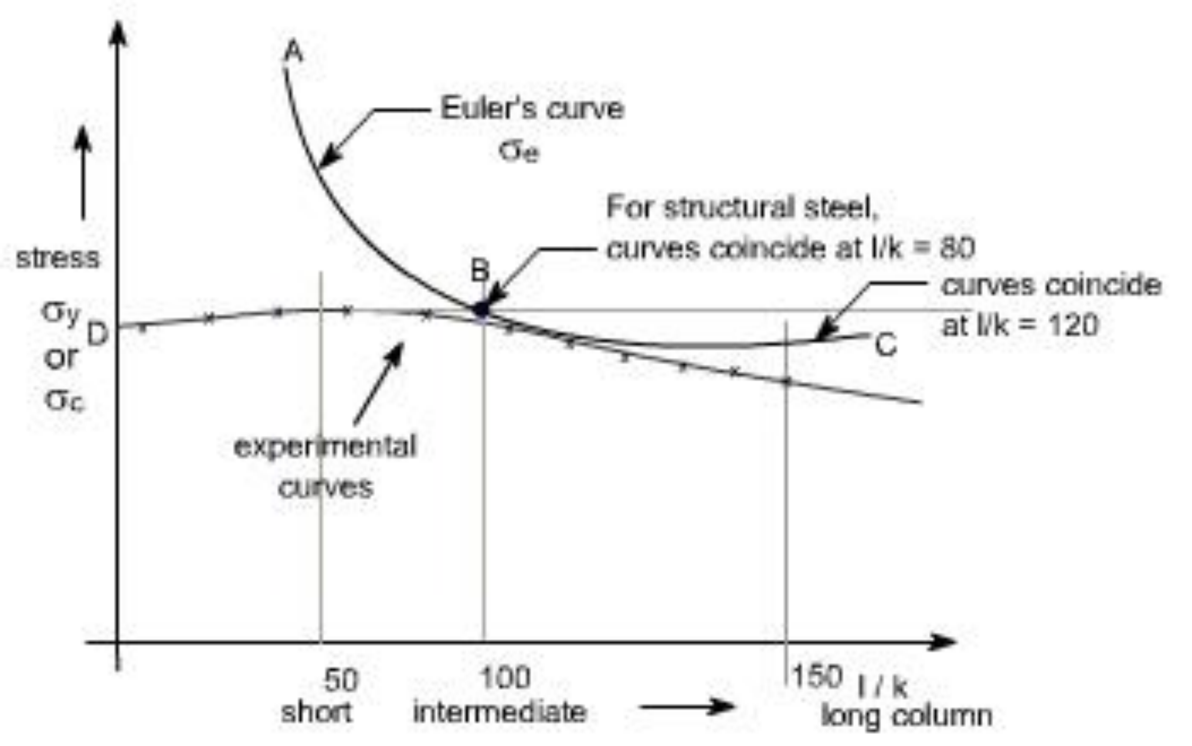


Comparison of Euler Theory with Experiment results

$$\text{Euler's stress, } \sigma_e = \frac{P_e}{A} = \frac{\pi^2 EI}{Al^2}$$

$$\text{But, } I = Ak^2$$

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$



(a) Straight line formulae:

$$P = \sigma_y A\left[1 - n\left(\frac{l}{k}\right)\right]$$

(b) Johnson parabolic formulae: $P = \sigma_y A\left[1 - b\left(\frac{l}{k}\right)^2\right]$

(c) Rankine Gordon Formulae : $\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$

where P_e = Euler crippling load

P_c = Crushing load or Yield point load in Compression

P_R = Actual load to cause failure or Rankine load

(c) Rankine Gordon Formulae :

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

where P_e = Euler crippling load

P_c = Crushing load or Yield point load in Compression

P_R = Actual load to cause failure or Rankine load

- Since the Rankine formulae is a combination of the Euler and crushing load for a strut.
- For a very short strut P_e is very large hence $1/P_e$ would be large so that $1/P_e$ can be neglected.
- Thus $P_R = P_c$, for very large struts, P_e is very small so $1/P_e$ would be large and $1/P_c$ can be neglected ,hence $P_R = P_e$
- The Rankine formulae is therefore valid for extreme values of L/k. It is also found to be fairly accurate for the intermediate values in the range under consideration.

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_E} = \frac{P_C + P_E}{P_C P_E}$$

$$P_R = \frac{P_C P_E}{P_C + P_E} = \frac{P_C}{1 + {P_C}/{P_E}} = \frac{\sigma_C \bullet A}{1 + {\sigma_C \bullet A \bullet L^2}/{\pi^2 EI}} = \frac{\sigma_C \bullet A}{1 + {\sigma_C \bullet A \bullet L^2}/{\pi^2 EAk^2}}$$

$$P_R = \frac{\sigma_C \bullet A}{1 + a\left(\frac{L}{k}\right)^2}$$

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l}{k}\right)^2}$$

where σ_c is the crushing stress

a is the Rankine's constant $(\sigma_c/\pi^2 E)$

➢Rewriting the formula in terms of stresses and equivalent lengths for different end conditions, we have

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2}$$

Material	**σ_y or σ_c MN/m^2**	**Value of a**	
		Pinned ends	**Fixed ends**
Low carbon steel	315	1/7500	1/30000
Cast Iron	540	1/1600	1/6400
Timber	35	1/3000	1/12000

note a = 4 x (a for fixed ends)