Multiple Integral:

16 December 2021 08:39

Line integral

$f \geq 0$, $\int_a^b f(x)\,dx$ = area of (A)

$\int_0^{\pi} \sin x \, dx$, $\sin x = f(x)$

$F(x)$, $F'(x) = f(x)$

$F(x)\Big|_0^{\pi}$

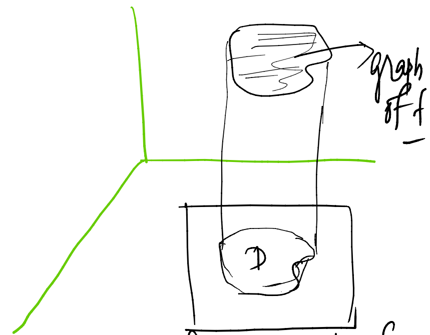
$F(\pi) - F(0)$

Double integral

$f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ $\quad \boxed{f \geq 0}$

$R = [a,b]\times[c,d]$

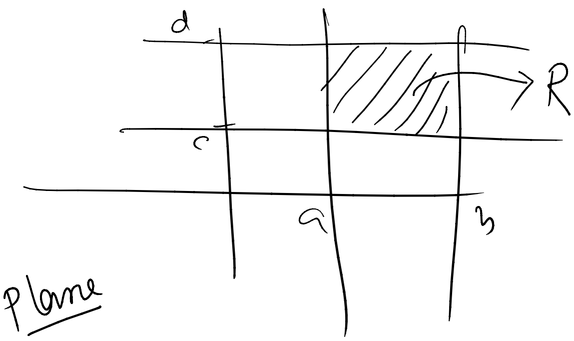
Rectangle (Region of integration)



The volume under f and above D equals

$\mathbb{R}$

$R = [a,b] \times [c,d]$



$f : \mathbb{R} \to \mathbb{R}$

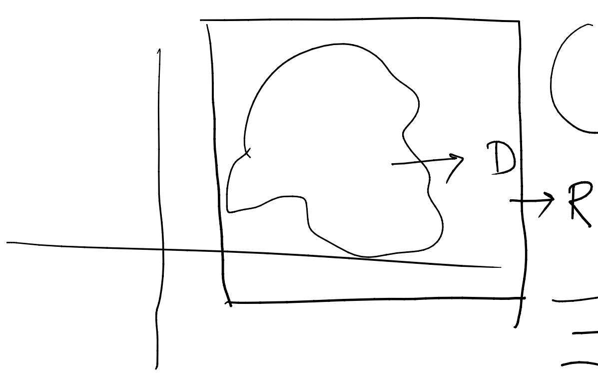
$Grf \subseteq \mathbb{R}^2$ plane

$f : A \to B$

$\text{Graph } f = \{ (x, f(x)) : x \in A \} \subseteq A \times B$

$f : \mathbb{R}^2 \to \mathbb{R}$

$Grf \subseteq \mathbb{R}^3$



$f : R \to \mathbb{R}$ bounded

$f : D \, (\subseteq \mathbb{R}^2) \to \mathbb{R}$

$f \geq 0$

$\exists \; R \subseteq \mathbb{R}^2$ s.t.

$R \supseteq D$

(Note that the volume $\quad \tilde{f} : R \to \mathbb{R}^2$ an

$R \supseteq D$

(Note that the volume contribution of $\tilde{f}$ and f are same)

$\tilde{f} : R \to \mathbb{R}^2$ as

$\tilde{f} = f, \ (x,y) \in D$

$= 0$ oth

$$\iint_R \tilde{f}\, dA = \iint_D f\, dx\,dy$$

Ex1 $\displaystyle\iint_D xy^2\,dx\,dy$, where $D = [0,2]\times[0,1]$

cont.

$$= \int_0^2 \left\{ \int_0^1 xy^2\, dy \right\} dx$$

$$= \int_0^2 \left[x y^3/3 \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \left[\frac{x}{3} - 0 \right] dx = x^2/6 \Big|_0^2$$

$$= 2/3$$

$$= \int_0^1 \left\{ \int_0^2 xy^2\, dx \right\} dy$$

$$= \int_0^1 \frac{x^2y^2}{2} \Big|_{x=0}^{x=2} dy$$

$$= \int_0^1 (2y^2 - 0)\, dy$$

$$= 2/3$$

Remark: $\displaystyle\int_a^b \left\{ \int_c^d f(x,y)\, dx \right\} dy$

$R = [c,d] \times [a,b]$

$\displaystyle\int_a^b \int_c^d f\, dx\, dy \to$ Repeated int.

$\displaystyle\int_a^b \int_c^d f\, dy\, dx \to$ Repeated Integral

$\displaystyle\iint f\, dx\, dy$

Repeated Integ...

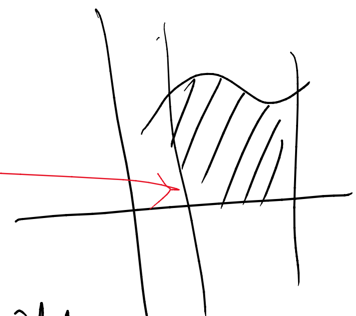
double int. $\begin{cases} \iint f \, dx\, dy \\ \quad \| \\ \iint f \, dy \, dx \end{cases}$

Res- If f is cont. on a rectangle R, then it is integrable on R.

$\mathbb{R}$ Set of all real no.

$R \to$ Rectangle $= [a,b] \times [c,d]$

Note: $\int_a^b f(x)\, dx$, $f \geq 0$



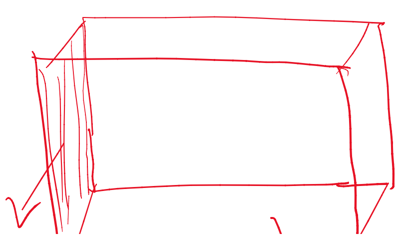
if $f < 0$, we consider $f = -(-f)$ $\to > 0$ and we can apply our definition for $(-f)$

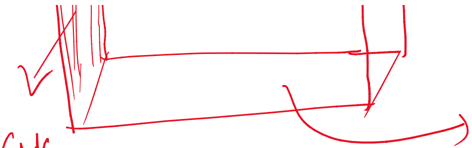
Similarly, if $f(x,y) < 0$ for some $(x,y) \in D$, then we consider $-f(x,y)$ and apply it as

$$f = -(-f)_{>0}.$$

Properties: If $f = k, \forall x,y \in R = [a,b] \times [c,d]$

$$\iint_R f \, dx\, dy = k \text{ area of } (R) = k \cdot (b-a) \cdot (d-c)$$





52 Cards → Volume

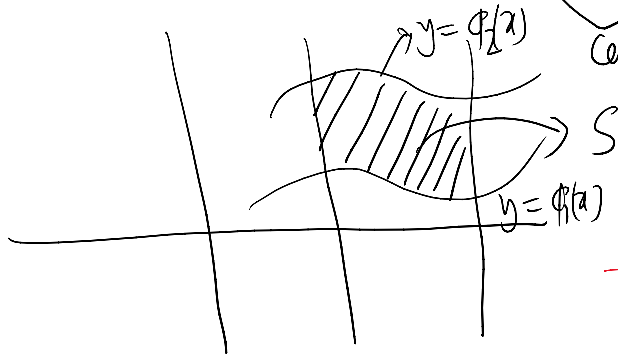
$R \to [a,b]\times[c,d]$

$\hookrightarrow D \subseteq \mathbb{R}^2$

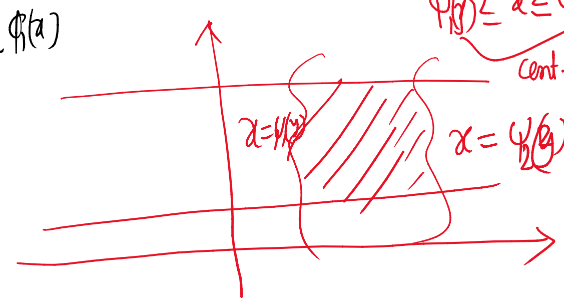
Different types of Regions:

Type-I | Type-II | Type-III

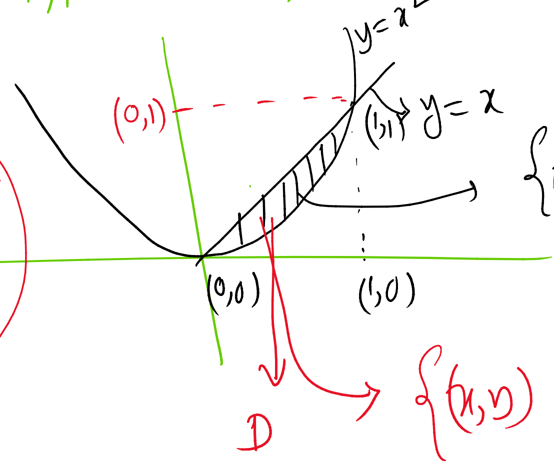
$S = \{(x,y) : a \le x \le b,\ \underbrace{\varphi_1(x) \le y \le \varphi_2(x)}_{\text{Cont. function}}\}$



$T = \{(x,y) \in \mathbb{R}^2 : c \le y \le d,\ \underbrace{\psi_1(y) \le x \le \psi_2(y)}_{\text{Cont. funct.}}\}$



Type III: If it can be written as either type I or type II.



$\{(x,y) : 0 \le x \le 1,\ x^2 \le y \le x\}$

$\{(x,y) : 0 \le y \le 1,\ y \le x \le \sqrt{y}\}$

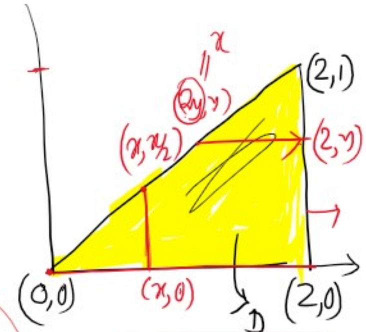
$$\int_0^1 \left\{ \int_{x^2}^{x} \frac{xy^3}{5}\, dy \right\} dx$$

→ Ex. Evaluate $\iint_D \frac{xy^3}{5}\, dx\, dy.$

Ex. Evaluate $\iint_D \frac{x\ y}{5}$ ...

23 | 12 | 21

Ex 1. Evaluate $\iint_D x e^{2xy}\, dx\, dy$ → Cont.



Type-III

$$= \int_0^2 \int_0^{x/2} x e^{2xy}\, dy\, dx$$

or

$$= \int_0^1 \int_{2y}^2 (x e^{2xy})\, dx\, dy$$

→ H.Ex

- Fubini's Theorem: If f is integrable on $R = [a,b]\times[c,d]$, then either of the iterated integrals (if it exists), equals the double integral

$$\iint_R f\, dx\, dy.$$

Cor 1. If f is cont. on R, then both the iterated integrals exist and they are same.

Cor 2. In Fubini's theorem, if both the iterated integrals exist then they are same.

Consider x^y on $[0,1]\times[0,1]$

$$\int_0^1 \int_0^1 x^y\, dx\, dy \quad \text{or} \quad \int_0^1 \int_0^1 x^y\, dy\, dx$$

Note that exist

Note that

$$\int_0^1 \int_0^1 x^y \, dx \, dy$$ exist is easy to calculate and $= \ln 2$.

Other iterated integral

$$\int_0^1 \int_0^1 x^y \, dx \, dy$$ is difficult to find but it will also be same as $\ln 2$.

Remark 1. Both iterated integrals exist but double integral may not exist → this doesn't contradict Fubini's theorem

Example 2. $$\int_0^1 \int_0^1 \frac{y-x}{(2-x-y)^3} \, dy \, dx \neq \int_0^1 \int_0^1 \frac{y-x}{(2-x-y)^3} \, dx \, dy$$ (Try to show it)

f is not integrable on $R = [0,1] \times [0,1]$

Ex 3. $$f = \frac{x^2-y^2}{(x^2+y^2)^2}$$ and note that

$$\frac{x^2-y^2}{(x^2+y^2)^2} = -\frac{\partial^2}{\partial x \partial y} \tan^{-1}(y/x)$$

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} \, dy \, dx = \pi/4$$

and $$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} \, dx \, dy = -\pi/4$$

$f\left(= \frac{x^2-y^2}{\ldots}\right)$ is not integrable on

$f\left(= \dfrac{x^2-y^2}{(x^2+y^2)^2}\right)$ is not integrable on $R=[0,1]\times[0,1]$.

Remark 2. It may happen that one of the iterated integrals exists but not the other and also the double integral does not exist.

Construct a supportive example:—

$$f(x,y) = \begin{cases} 1/2, & y \to \text{rational} \\ x, & y = \text{irrational} \end{cases} \quad \text{on } R=[0,1]\times[0,1]$$

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = \frac{1}{2} \qquad \Bigg| \qquad \int_0^1\left[\int_0^1 f\,dy\right]dx$$

$\iint_R f\,dx\,dy$ do not exist

Question: Can you construct an example where both the ~~int~~ iterated integrals exist and same but double integral fails to exist?

$\boxed{\text{Yes, it is possible.}}$

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2}, & (x,y)\neq(0,0) \\ 0 & \text{otherwise} \end{cases}$$

0, 0, ∞ near $(0,0)$

$R=[-1,1]\times[-1,1]$

Note f is discont at only $(0,0)$ but it is unbounded near the origin.

only (0,0) but it is unbounded near the origin.

$R = [-1,1] \times [-1,1]$

$$\int_{-1}^{1}\int_{-1}^{1} f \, dx\, dy \qquad - \frac{x}{2(x^2+y^2)} \text{ for fixed } x$$

$$\int_{-1}^{1}\int_{-1}^{1} (f\, dy)\, dx = -\int_{-1}^{1} \frac{x}{2(x^2+y^2)}\, dx = 0 \quad \text{odd}$$

$$\frac{y}{2(x^2+y^2)} \qquad \int_{-1}^{1} (\;)\, dy \quad \text{odd} \quad 0$$

Similarly for the other iterated integral $= 0$

Hence double integral does not exist.

$$V = \{(x,y,f(x,y)) : (x,y) \in D\} \qquad \iint_D f\, dx\, dy \qquad f \geq 0$$

30-12-21

Question: Can we use double integral to evaluate the area of some region D?

Ans: Yes, if we integrate the const. function $f(x,y)=1$ over a region D, we get the area of D:

$$\iint_D 1\, dA = \text{Area}(D) \quad (A(D))$$

Note that for a solid cylinder with base D and height 1 has volume $A(D)\cdot 1 = A(D)$, but we can also write its as $\iint_D 1 \cdot dA$.

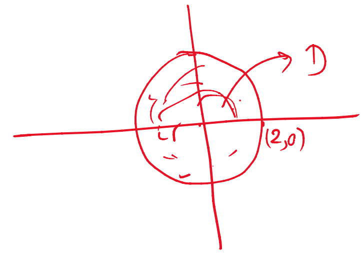
Note: If $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$m\, A(D) \leq \iint_D f(x,y)\, dA \leq M\, A(D)$$

Question: Can you estimate $\iint e^{\sin x \sin y}\, dx\, dy$,

Question: Can you estimate $\iint_D e^{\sin x \sin y}\, dx\, dy$,

where $D = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 4\}$.



$\cos x$

$-1 \le \sin x \le 1$

So, $-1 \le \sin x \cos x \le 1$

$$\frac{1}{e} \le \underbrace{e^{\sin x \cos x}}_{f} \le e$$

$m \quad M$

$\frac{1}{e} \le f \le e$

$A(D) = \pi 2^2 = 4\pi$

Now using the above note, We have,

$$\boxed{\frac{1}{e}\cdot 4\pi \le \iint_D f\, dx\, dy \le e \cdot 4\pi}$$

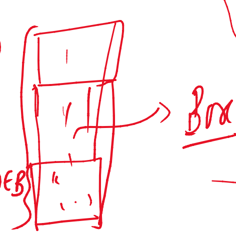
Triple Integration: $\displaystyle\iiint_B f(x,y,z)\, \overset{dV}{\underline{dx\, dy\, dz}}$

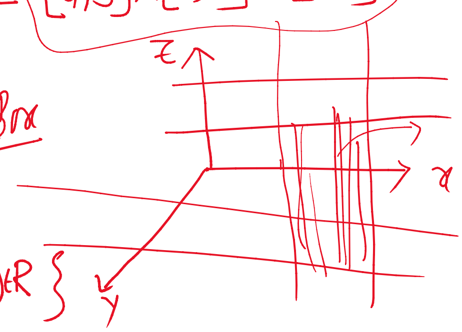
$B = [a,b]\times[c,d]\times[p,q]$

$f \ge 0$

$f : B \to \mathbb{R}^+$

$\text{Gr} f = \{(x,y,z,w) :$ $\quad f(x,y,z) = w$, $(x,y,z)\in B$





$R = [a,b]\times[c,d]$

$f : R \to \mathbb{R}^+$ $\quad \text{Gr} f = \{(x,y,f(x,y)) : (x,y)\in R\}$

Can we visulise this? $\quad \subseteq \mathbb{R}^4$

$$\iiint_B f\, dV = \text{Vol}(W), \text{ where}$$

$W = \{(x,y,z,w)\ a\le x\le b,\ c\le y\le d,$

$\iint f\, dx\, dy$

$\iint_D f\,dx\,dy$ B

$\iint_D f\,dy\,dx$

$\int_{-1}^{1}\int_0^{\pi} f\,dx\,dy$

$\int_{-1}^{1}\int_0^{\pi} f\,dy\,dx$

$W = \{(x,y,z,w) \mid a \le x \le b,\ c \le y \le d,\ p \le z \le q,\ 0 \le f(x,y,z) \le w\}$

$$\underbrace{\int_a^b}_{3} \underbrace{\int_c^d}_{2} \underbrace{\int_p^q}_{1} f(x,y,z)\,dz\,dy\,dx$$

Total no of iterated integrals is 6

$\iiint_B f\,dx\,dy\,dz$

Cont. So it is integrable

<u>Example 2.</u> $\iiint_B xyz^2\,dx\,dy\,dz$

$B = [0,1]\times[-1,2]\times[0,3]$

$$= \int_0^3 \int_{-1}^2 \left(\int_0^1 xyz^2\,dx\right) dy\,dz$$

$$= \int_0^3 \left(\int_{-1}^2 \frac{x^2yz^2}{2}\Big|_{x=0}^{1} dy\right) dz$$

$$= \int_0^3 \frac{y^2z^2}{4}\Big|_{y=-1}^{2} dz = \int_0^3 \frac{3z^2}{4}\,dz$$

$$= \frac{z^3}{4}\Big|_0^3 = \frac{27}{4}.$$

[Try to find other iterated integrals.]

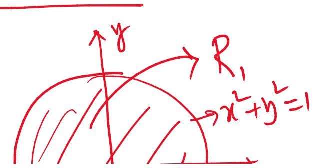
$\int_0^{\pi} x\sin(x^2)\,dx$

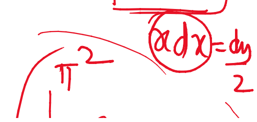
$\boxed{y = x^2}$

$x\,dx = \frac{dy}{2}$

→ Change of variables:

Target:





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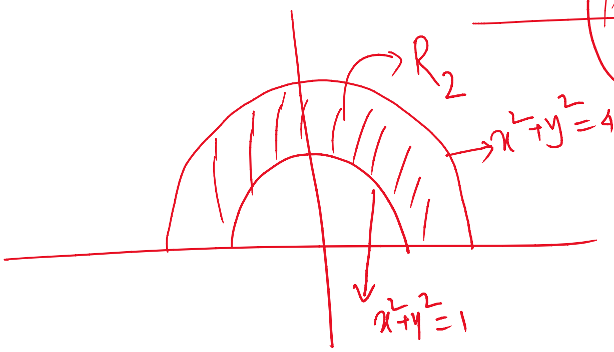
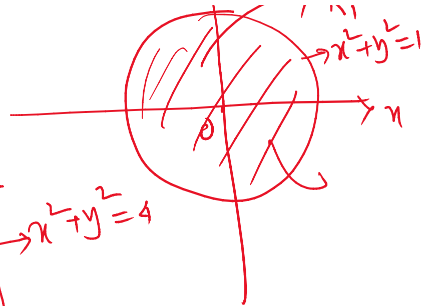




$$\int_0^{\pi^2} \sin y \, dy \qquad (x\,dx) = \frac{dy}{2}$$

$$\iint_D f(x,y)\, dx\, dy$$

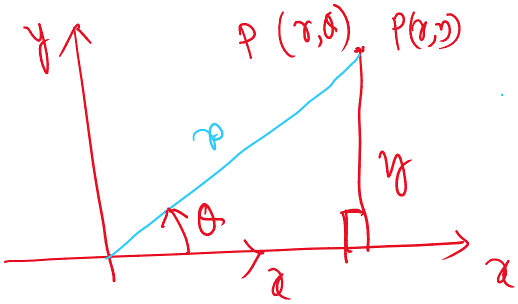
$$\begin{cases} R_1 = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\} \text{ Type-I or Type-II} \\ R_2 = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\} \text{ "} \end{cases}$$

$r^2 = x^2 + y^2$

$\Downarrow$

$x = r\cos\theta,$

$y = r\sin\theta$

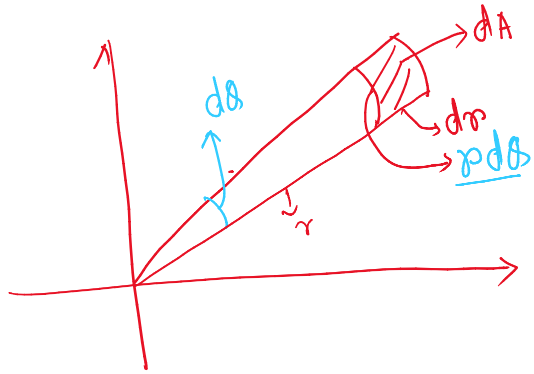


If f is cont. on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

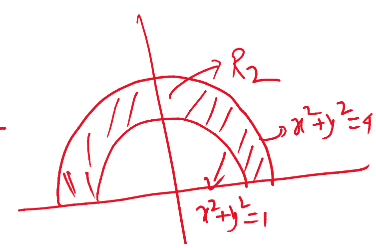
$$\iint_R f(x,y)\, dx\, dy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, \boxed{r\, dr\, d\theta}$$

$dx\,dy$

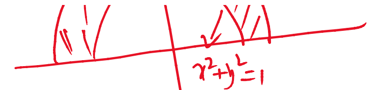
$dx\,dy \to r\,dr\,d\theta$



Ex 3. Evaluate $\iint_{R_2} (3x + 4y^2)\, dA$



Ex 3. Evaluate $\iint_{R_2} (3x + 4y^2)$



$$= \int_0^{\pi}\int_1^2 \left(3r\cos\theta + 4r^2\sin^2\theta\right)\boxed{r\,dr\,d\theta}$$

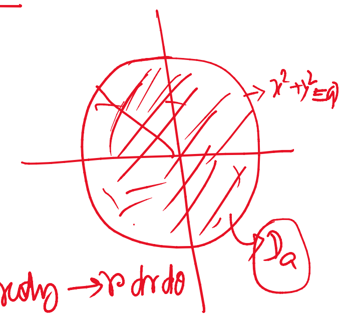
$$\Downarrow$$

$$= \frac{15\pi}{2} \quad \text{(Check!)}$$

Ex 4. $\Longrightarrow$ $\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pi}$ (Gaussian integral) $\qquad r^2 = x^2+y^2$

$$I = \iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy$$

$D_a = \{(x,y) : x^2+y^2 \leq a^2\}$



Put $x = r\cos\theta$, $y = r\sin\theta$, $r \geq 0$ $\qquad dx\,dy \to r\,dr\,d\theta$

$0 \leq r \leq a \qquad 0 \leq \theta \leq 2\pi$

$$I = \int_0^a \int_0^{2\pi} e^{-r^2} r\,d\theta\,dr = 2\pi \int_0^a r e^{-r^2}\,dr$$

$$= \pi\left(1 - e^{-a^2}\right) \text{ ----- (1)}$$

$(-\infty,\infty) \times (-\infty,\infty)$

Letting $a \to \infty$, then (1) will become

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \lim_{a\to\infty} \pi\left(1-e^{-a^2}\right)$$

$$= \pi$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-y^2}\,dy \int_{-\infty}^{\infty} e^{-x^2}\,dx = \pi$$

$$\Rightarrow \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

$$\Rightarrow \left(\int_{-\infty} \quad \right) \Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

— o —