Chapter 14

Improper Integral, Gamma and Beta Functions

In the definite integral $\int_a^b f(x)\,dx$ it is assumed that
(i) the limits a and b both are finite, and
(ii) the integrand $f(x)$ is bounded within $a \le x \le b$.

If a definite integral satisfies these two conditions, then the integral is called **proper integral**.

But, if a or b or both are infinite or $f(x)$ is not finite in $a \le x \le b$, then the integral is called **improper integral** or **infinite integral** or **generalised integral**.

If the integral is proper and integrable, then it has a finite value. But the value of improper integral may be finite or infinite. If the value of the integral is finite, then it is said to be **convergent**, otherwise the integral is said to be **divergent**.

The improper integrals are of two types, viz., first type or first kind and second type or second kind.

Gamma and beta functions are two improper integrals and they are used to solve a large number of problems involved in integration.

14.1 First Type Improper Integrals

The integrals $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{-\infty}^{\infty} f(x)\,dx$ are called **first type** improper integrals. The values of these integrals are evaluated as follows.

(i) Let $f(x)$ be bounded and integrable in $a \le x \le B$ for every $B > a$. Then $\int_a^{\infty} f(x)\,dx$ is said to be **converge** or **exist** if $\lim\limits_{B\to\infty} \int_a^B f(x)\,dx$ exists. Thus

$$\int_a^{\infty} f(x)\,dx = \lim_{B\to\infty} \int_a^B f(x)\,dx \tag{14.1}$$

If $\lim\limits_{B\to\infty} \int_a^B f(x)\,dx$ does not exist, then the integral $\int_a^{\infty} f(x)\,dx$ is said to be a diverge provided $\lim_{B\to\infty}\int_a^B f(x)\,dx$ tends to be an infinity with a fixed sign. If the improper integral $\int f(x)\,dx$ is neither a converge nor diverge, then it is called **oscillatory**.

The improper integral $\int_{-\infty}^{b} f(x)\,dx$ can be evaluated as

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{A\to-\infty}\int_A^b f(x)\,dx$$

provided the limit exists and $f(x)$ is bounded and integrable in $A \le x \le b$.

The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is broken up into two integrals of the previous forms,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx \qquad (14.2)$$

where a is any point.

If $f(x)$ be bounded and integrable in $A \le x \le a$ for every $A < a$, and in $a \le x \le B$ for every $B > a$ and $\lim_{A\to-\infty}\int_A^a f(x)\,dx$ and $\lim_{B\to\infty}\int_a^B f(x)\,dx$ for $A < a < B$ exist finitely, then the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to be convergent. In this case, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{A\to-\infty}\int_A^a f(x)\,dx + \lim_{B\to\infty}\int_a^B f(x)\,dx \qquad (14.3)$$

EXAMPLE 14.1.1 Evaluate $\int_1^{\infty} \frac{1}{x^2}\,dx$.

Solution Here the upper limit is infinite and the integrand $1/x^2$ is bounded in $1 \le x \le B$, for every $B \ge 1$. Now

$$\lim_{B\to\infty}\int_1^B \frac{1}{x^2}\,dx = \lim_{B\to\infty}\left[-\frac{1}{x}\right]_1^B = \lim_{B\to\infty}\left[-\frac{1}{B}+1\right] = 1$$

Hence the integral $\int_1^{\infty}\frac{1}{x^2}\,dx$ is convergent and $\int_1^{\infty}\frac{1}{x^2}\,dx = 1$.

EXAMPLE 14.1.2 Evaluate $\int_1^{\infty}\frac{1}{x}\,dx$.

Solution This is first type improper integral since the upper limit is ∞. Here also the integrand $1/x$ is bounded in every $1 \le B$.

Now,

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{B\to\infty} \int_1^B \frac{1}{x}\,dx = \lim_{B\to\infty} \Big[\log x\Big]_1^B$$

$$= \lim_{B\to\infty} [\log B - 0] = \infty$$

Since $\lim\limits_{B\to\infty} \int_1^B \frac{1}{x}\,dx$ does not exist, therefore $\int_1^{\infty} \frac{1}{x}\,dx$ also does not exist.

EXAMPLE 14.1.3 Find the value of $\int_a^{\infty} \cos x\,dx$, if it exists.

Solution Here $\int_a^{\infty} \cos x\,dx = \lim\limits_{B\to\infty} \int_a^B \cos x\,dx = \lim\limits_{B\to\infty} \Big[\sin x\Big]_a^B = \lim\limits_{B\to\infty} (\sin B - \sin a)$.

Since the value of $\lim\limits_{B\to\infty} \sin B$ is not fixed (its value lies between -1 and 1), $\lim\limits_{B\to\infty} (\sin B - \sin a)$ oscillates finitely.

Hence $\int_a^{\infty} \cos x\,dx$ is oscillatory.

EXAMPLE 14.1.4 Examine for convergence of the following integrals:

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

(ii) $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

(iv) $\displaystyle\int_2^{\infty} \frac{dx}{x\,\log x}$.

Solution

(i) Here both the limits are infinite. We choose a point 0 within the interval of integratior Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{A\to-\infty} \int_A^0 \frac{dx}{1+x^2} + \lim_{B\to\infty} \int_0^B \frac{dx}{1+x^2}$$

$$= \lim_{A\to-\infty} \left[\tan^{-1} 0 - \tan^{-1} A\right] + \lim_{B\to\infty} \left[\tan^{-1} B - \tan^{-1} 0\right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Thus the integral converges.

(ii) In this problem,

$$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{B\to\infty}\int_0^{B} xe^{-x^2}\,dx$$

$$= \lim_{B\to\infty}\left[\frac{1}{2}\int_0^{B^2} e^{-z}\,dz\right]$$

[where $x^2 = z,\ 2x\,dx = dz$]

$$= \frac{1}{2}\lim_{B\to\infty}\left[-e^{-z}\right]_0^{B^2} = \frac{1}{2}\lim_{B\to\infty}\left[1-e^{-B^2}\right] = \frac{1}{2}$$

Hence the integral converges.

(iii) We divide the interval at the point $x = 0$. That is,

$$I = \int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2} = \int_{-\infty}^{0}\frac{dx}{(1+x^2)^2} + \int_0^{\infty}\frac{dx}{(1+x^2)^2}$$

$$= 2\int_0^{\infty}\frac{dx}{(1+x^2)^2} = 2\lim_{B\to\infty}\int_0^{B}\frac{dx}{(1+x^2)^2}$$

Substituting $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$.

When $x \to 0, \theta \to 0$ and when $x \to B$ then $\theta \to \tan^{-1} B$. Therefore,

$$I = 2\lim_{B\to\infty}\int_0^{\tan^{-1}B}\frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2} = 2\lim_{B\to\infty}\int_0^{\tan^{-1}B}\cos^2\theta\,d\theta$$

$$= \lim_{B\to\infty}\int_0^{\tan^{-1}B}(1+\cos 2\theta)\,d\theta = \lim_{B\to\infty}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\tan^{-1}B}$$

$$= \lim_{B\to\infty}\left[\tan^{-1}B+\frac{B}{1+B^2}\right] = \frac{\pi}{2}$$

This integral also converges.

(iv) Here

$$\int_2^{\infty}\frac{dx}{x\log x} = \lim_{B\to\infty}\int_2^{B}\frac{dx}{x\log x}$$

[Putting $\log x = z$, $\frac{1}{x}\,dx = dz$.

When $x \to 2, z \to \log 2$ and when $x \to B, z \to \log B$]

$$= \lim_{B\to\infty}\int_{\log 2}^{\log B}\frac{dz}{z} = \lim_{B\to\infty}\left[\log z\right]_{\log 2}^{\log B}$$

$$= \lim_{B\to\infty}\left[\log\log B - \log\log 2\right]$$

But $\lim_{B\to\infty} \log B \to \infty$ and hence $\lim_{B\to\infty} \log\log B \to \infty$.

Therefore, $\lim_{B\to\infty} [\log\log B - \log\log 2]$ does not exist.

Hence $\displaystyle\int_2^\infty \frac{dx}{x\,\log x}$ diverges.

14.2 Second Type Improper Integrals

In this type, both the lower and upper limits of the integral are finite, but the integrand $f(x)$ is infinite for some point within $a \le x \le b$. $f(x)$ may be infinite

(i) at the lower limit a,

(ii) at the upper limit b, and

(iii) at a point c in $a < c < b$.

(i) If $f(x)$ has an infinite discontinuity only at the lower limit a, then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b f(x)\,dx, \qquad 0 < \varepsilon < b - a$$

If this limit exists, then we say that $\displaystyle\int_a^b f(x)\,dx$ converges, otherwise it diverges.

(ii) If $f(x)$ has an infinite discontinuity at the upper limit only, then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+} \int_a^{b-\varepsilon} f(x)\,dx, \qquad 0 < \varepsilon < b - a$$

(iii) If $f(x)$ has an infinite discontinuity at the point $x = c$ where $a < c < b$, then $\displaystyle\int_a^b f(x)\,dx$ can be evaluated as

$$\lim_{\varepsilon_1\to 0+} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2\to 0+} \int_{c+\varepsilon_2}^b f(x)\,dx$$

provided all these limits exist. If either of the limits does not exist, we say that the integral does not exist.

If we take $\varepsilon_1 = \varepsilon_2 = \varepsilon$ then the integral

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+} \left\{ \int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right\}$$

This is called the **Cauchy principal value** of the integral. Sometimes it may happen that the Cauchy principal of the integral exists when the integral as per general definition does not exist.

EXAMPLE 14.2.1 Evaluate $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$, if it exists.

Solution The integrand $1/x^2$ has the an infinite discontinuity at $x = 0$. So, it is an improper integral of second type.

Now,

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon\to 0+}\int_{0+\varepsilon}^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon\to 0+}\left[-\frac{1}{x}\right]_\varepsilon^1$$

$$= \lim_{\varepsilon\to 0+}\left[-1+\frac{1}{\varepsilon}\right] = \infty$$

Hence $\lim\limits_{\varepsilon\to 0+}\int_{0+\varepsilon}^1 \frac{1}{x^2}\,dx$ does not exist and consequently $\int_0^1 \frac{1}{x^2}\,dx$ does not exist.

EXAMPLE 14.2.2 Evaluate $\int_{-1}^1 \frac{1}{x^3}\,dx$.

Solution $1/x^3$ has the infinite discontinuity at $x = 0$, so it is an improper integral of second type.

We break up the interval at the point $x = 0$ as

$$\int_{-1}^0 \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$$

Now

$$\int_{-1}^1 \frac{1}{x^3}\,dx = \lim_{\varepsilon_1\to 0+}\int_{-1}^{0-\varepsilon_1} \frac{1}{x^3}\,dx + \lim_{\varepsilon_2\to 0+}\int_{0+\varepsilon_2}^1 \frac{1}{x^3}\,dx$$

$$= \lim_{\varepsilon_1\to 0+}\left[-\frac{1}{2x^2}\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2\to 0+}\left[-\frac{1}{2x^2}\right]_{\varepsilon_2}^1$$

$$= \lim_{\varepsilon_1\to 0+}\left[\frac{1}{2}-\frac{1}{2\varepsilon_1^2}\right] + \lim_{\varepsilon_2\to 0+}\left[-\frac{1}{2}+\frac{1}{2\varepsilon_2^2}\right]$$

Since $\lim\limits_{\varepsilon_1\to 0+}\frac{1}{2\varepsilon_1^2}$ and $\lim\limits_{\varepsilon_2\to 0+}\frac{1}{2\varepsilon_2^2}$ do not exist, therefore $\int_{-1}^1 \frac{1}{x^3}\,dx$ does not exist.

But, according to Cauchy principal value the integral exists, as shown below

$$\lim_{\varepsilon\to 0+}\left[\int_{-1}^{0-\varepsilon} \frac{1}{x^3}\,dx + \int_{0+\varepsilon}^1 \frac{1}{x^3}\,dx\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\left(\frac{1}{2}-\frac{1}{2\varepsilon^2}\right)+\left(-\frac{1}{2}+\frac{1}{2\varepsilon^2}\right)\right]$$

$$= \lim_{\varepsilon\to 0+}(0) = 0$$

Thus for Cauchy principal value sense $\int_{-1}^1 \frac{1}{x^2}\,dx = -2$, and for general sense it is divergent.

EXAMPLE 14.2.3 Evaluate $\int_0^1 \frac{dx}{(1-x)^{1/3}}$.

Solution Here $x = 1$ is the point of infinite discontinuity, so it is also a second type improper integral.

Now,

$$\int_0^1 \frac{dx}{(1-x)^{1/3}} = \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{dx}{(1-x)^{1/3}} = \lim_{\varepsilon\to 0+} \left[\frac{(1-x)^{2/3}}{2/3}\right]_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} \frac{3}{2}\left[\varepsilon^{2/3} - 1\right] = -\frac{3}{2}$$

Thus $\lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{dx}{(1-x)^{1/3}}$ exists and $\int_0^1 \frac{dx}{(1-x)^{1/3}} = -\frac{3}{2}$.

EXAMPLE 14.2.4 Test for convergence.

(i) $\int_{1/2}^1 \frac{dx}{x \log x}$ (ii) $\int_1^4 \frac{dx}{x-1}$ (iii) $\int_0^\pi \frac{dx}{\sin x}$.

Solution

(i) Here $x = 1$ is the point of infinite discontinuity. Therefore,

$$\int_{1/2}^1 \frac{dx}{x \log x} = \lim_{\varepsilon\to 0+} \int_{1/2}^{1-\varepsilon} \frac{dx}{x \log x}$$

[Putting $\log x = z.\ \frac{1}{x}\, dx = dz$.

When $x \to 1/2, z \to \log(1/2)$ and when $x \to 1-\varepsilon, z \to \log(1-\varepsilon)$.]

$$= \lim_{\varepsilon\to 0+} \int_{\log(1/2)}^{\log(1-\varepsilon)} \frac{dz}{z} = \lim_{\varepsilon\to 0+} \left[\log z\right]_{\log(1/2)}^{\log(1-\varepsilon)}$$

$$= \lim_{\varepsilon\to 0+} \left[\log\log(1-\varepsilon) - \log\log(1/2)\right].$$

Here $\lim_{\varepsilon\to 0+} \log\log(1-\varepsilon) = \log 0 \to -\infty$ and $\log\log(1/2) = \log(-\log 2)$, does not exist.

Hence the limit does not exist and consequently the integral diverges.

(ii) Here also $x = 1$ is the point of infinite discontinuity.

Therefore,

$$\int_1^4 \frac{dx}{x-1} = \lim_{\varepsilon\to 0+} \int_{1+\varepsilon}^4 \frac{dx}{x-1}$$

$$= \lim_{\varepsilon\to 0+} \left[\log(x-1)\right]_{1+\varepsilon}^4$$

$$= \lim_{\varepsilon\to 0+} [\log 3 - \log \varepsilon] \to \infty$$

Hence the integral does not converge.

(iii) Here 0 and π both are points of infinite discontinuity. Thus we write

$$\begin{aligned}
\int_0^{\infty} \frac{dx}{\sin x} &= \int_0^{\pi/2} \frac{dx}{\sin x} + \int_{\pi/2}^{\pi} \frac{dx}{\sin x} \\
&= \lim_{\varepsilon_1 \to 0+} \int_{\varepsilon_1}^{\pi/2} \frac{dx}{\sin x} + \lim_{\varepsilon_2 \to 0+} \int_{\pi/2}^{\pi-\varepsilon_2} \frac{dx}{\sin x} \\
&= \lim_{\varepsilon_1 \to 0+} \Big[\log|\tan(x/2)|\Big]_{\varepsilon_1}^{\pi/2} + \lim_{\varepsilon_2 \to 0+} \Big[\log|\tan(x/2)|\Big]_{\pi/2}^{\pi-\varepsilon_2} \\
&= \lim_{\varepsilon_1 \to 0+} \Big[\log|\tan(\pi/4)| - \log|\tan(\varepsilon_1/2)|\Big] \\
&\quad + \lim_{\varepsilon_2 \to 0+} \Big[\log|\tan\{(\pi-\varepsilon_2)/2\}| - \log|\tan(\pi/4)|\Big] \\
&= \lim_{\varepsilon_1 \to 0+} \Big[0 - \log|\tan(\varepsilon_1/2)|\Big] + \lim_{\varepsilon_2 \to 0+} \Big[\log|\tan\{(\pi-\varepsilon_2)/2\}| - 0\Big] \\
&= \infty
\end{aligned}$$

Hence the integral does not converge.

In the previous problems, whether an integral converges or diverges, is tested by computing its value. But, without computing the actual value of the integration one can determine the convergence of the integral. Several methods are available for testing the convergence of an integral, which are stated below.

14.3 Test for First Type Improper Integrals

It is mentioned earlier that the integral $\int_a^{\infty} f(x)\, dx$ evaluated as

$$\int_a^{\infty} f(x)\, dx = \lim_{B \to \infty} \int_a^{B} f(x)\, dx$$

where $f(x)$ is bounded and integrable in $[a, B]$ for every $B \geq a$.

Theorem 14.1 *A necessary and sufficient condition for the convergence of* $\int_a^{\infty} f(x)\, dx$, *where* $f(x)$ *is positive in* $[a, B]$ *is that there exists a positive number* M, *independent of* B, *such that*

$$\int_a^{B} f(x)\, dx < M$$

for every $B \geq a$, *i.e.* $\int_a^{B} f(x)\, dx$ *is bounded above.*

The integral $\int_a^\infty f(x)\,dx$ *is said to be convergent if* $\lim_{B\to\infty}\int_a^B f(x)\,dx$ *tends to a finite limit.*

If $\int_a^B f(x)\,dx$ *is not bounded above then* $\lim_{B\to\infty}\int_a^B f(x)\,dx$ *tends to* ∞, *i.e. the integral diverges.*

For example, let us consider the integral $\int_1^\infty \frac{1}{x^2}\,dx$. The integral $1/x^2$ is positive in $[1, B]$ for any $B \geq 1$. Also,

$$\int_1^B \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^B = 1 - \frac{1}{B} < 1 \text{ for any } B \geq 1$$

Thus the integral $\int_1^\infty \frac{1}{x^2}\,dx$ satisfies the necessary and sufficient conditions of the above theorem. Hence the integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

14.3.1 A Standard Integral

The improper integral

$$\int_a^\infty \frac{dx}{x^n},\ a > 0 \tag{14.4}$$

converges if $n > 1$ and diverges if $n \leq 1$.

Proof. We have

$$\begin{aligned}
\int_a^\infty \frac{dx}{x^n} &= \lim_{B\to\infty}\int_a^B \frac{dx}{x^n} \\
&= \lim_{B\to\infty}\left[\frac{1}{1-n}\,\frac{1}{x^{n-1}}\right]_a^B = \lim_{B\to\infty}\left[\frac{1}{1-n}\left\{\frac{1}{B^{n-1}} - \frac{1}{a^{n-1}}\right\}\right], n \neq 1 \\
&= \begin{cases} \dfrac{1}{n-1}\,\dfrac{1}{a^{n-1}}, & \text{if } n-1 > 0 \\ \infty, & \text{if } n-1 < 0. \end{cases}
\end{aligned}$$

Again, if $n = 1$,

$$\begin{aligned}
\lim_{B\to\infty}\int_a^B \frac{dx}{x} &= \lim_{B\to\infty}\left[\log B - \log a\right], n = 1 \\
&= \infty
\end{aligned}$$

That is,

$$\int_a^\infty \frac{dx}{x^n} = \begin{cases} \frac{1}{n-1}\,\frac{1}{a^{n-1}}, & \text{if } n > 1 \\ \infty, & \text{if } n \leq 1 \end{cases}$$

Hence $\int_a^\infty \frac{dx}{x^n}$ converges if $n > 1$ and diverges if $n \leq 1$.

Comparison test I (comparison of two integrals)

Theorem 14.2 *If $f(x)$ and $g(x)$ are two positive integrable functions and $f(x) \le g(x)$ on $[a, B]$ for any $B \ge a$, then*

(i) $\int_a^\infty f(x)\,dx$ *converges, if* $\int_a^\infty g(x)\,dx$ *converges, and*

(ii) $\int_a^\infty g(x)\,dx$ *diverges, if* $\int_a^\infty f(x)\,dx$ *diverges.*

EXAMPLE 14.3.1 Examine the convergence of $\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$.

Solution It is easy to verify that

$$0 \le \frac{\cos^2 x}{x^2} \le \frac{1}{x^2} \text{ for } 1 \le x.$$

Also, $\int_1^\infty \frac{dx}{x^2}$ is convergent $(n = 2 > 1)$. Hence by comparison test $\int_1^\infty \frac{dx}{x^2}$ converges.

Comparison test II (limit form)

Theorem 14.3 *Let $f(x)$ and $g(x)$ be two positive integrable functions when $x \ge a$ and $g(x)$ be positive. Let*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$$

(i) If $l \ne 0$, then the integrals $\int_a^\infty f(x)\,dx$ *and* $\int_a^\infty g(x)\,dx$ *both converge and diverge together.*

(ii) If $l = 0$ and $\int_a^\infty g(x)\,dx$ *converges, then* $\int_a^\infty f(x)\,dx$ *converges.*

(iii) If $l = \pm\infty$ and $\int_a^\infty g(x)\,dx$ *diverges, then* $\int_a^\infty f(x)\,dx$ *diverges.*

EXAMPLE 14.3.2 Test for convergence.

(i) $\int_1^\infty \frac{x^2}{x^4+1}\,dx$ (ii) $\int_1^\infty \frac{\log x}{x^2}\,dx$ (iii) $\int_1^\infty \frac{x^{3/2}}{x^2+2}\,dx$.

Solution

(i) Let $f(x) = \frac{x^2}{x^4+1}$ and $g(x) = \frac{1}{x^2}$.

Now, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^4}{x^4+1} = \lim_{x\to\infty} \frac{1}{1+1/x^4} = 1 \ne 0$

Also, $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx$ converges (since $n = 2 > 1$).

Therefore, by comparison test $\int_1^\infty \frac{x^2}{x^4+1}\,dx$ converges.

(ii) Let $f(x) = \dfrac{\log x}{x^2}$ and $g(x) = \dfrac{1}{x^{3/2}}$.

Now,

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{\log x}{x^2} \cdot x^{3/2} \\
&= \lim_{x\to\infty} \frac{\log x}{x^{1/2}} \quad \left(\frac{\infty}{\infty}\text{form}\right) \\
&= \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0
\end{aligned}$$

Also, $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges (since $n = 3/2 > 1$), hence by comparison test $\displaystyle\int_1^\infty \frac{\log x}{x^2}\,dx$ converges.

(iii) Let $f(x) = \dfrac{x^{3/2}}{x^2+2}$ and $g(x) = \dfrac{1}{x}$

Then, $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{5/2}}{x^2+2} = \lim_{x\to\infty} \frac{\sqrt{x}}{1+2/x^2} = \infty$

Also, $\displaystyle\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x}\,dx$ diverges (since $n = 1$)

Hence, by comparison test $\displaystyle\int_1^\infty \frac{x^{3/2}}{x^2+2}\,dx$ diverges.

Alternate.

Let $g(x) = \dfrac{1}{\sqrt{x}}$. Then

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x^2+2} = 1 \neq 0$ and $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges (since $n = 1/2 < 1$). Hence $\displaystyle\int_1^\infty f(x)\,dx$ ~~co~~diverges.

EXAMPLE 14.3.3 Examine the convergence of the following integrals.

(i) $\displaystyle\int_1^\infty e^{-x}x^n\,dx$ (ii) $\displaystyle\int_{e^2}^\infty \frac{dx}{x\log\log x}$

Solution

(i) Let $f(x) = e^{-x}x^n$ and $g(x) = \dfrac{1}{x^2}$.

Now,

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{e^{-x}\cdot x^n}{1/x^2} = \lim_{x\to\infty} \frac{x^{n+1}}{e^x} \quad \left(\frac{\infty}{\infty}\text{ form}\right) \\
&= \lim_{x\to\infty} \frac{(n+1)x^n}{e^x} \quad \left(\frac{\infty}{\infty}\text{ form}\right)
\end{aligned}$$

and so on, finally $\to 0$ for all n.

Also, $\int_1^{\infty} \frac{1}{x^2}\, dx$ is convergent.

Hence, by comparison test $\int_1^{\infty} e^{-x}x^n\, dx$ is convergent for all n.

Substituting $\log x = z$. Then $\frac{1}{x}\, dx = dz$. When $x \to e^2$ then $z \to 2$ and when $x \to \infty, z \to \infty$. Therefore,

$$\int_{e^2}^{\infty} \frac{1}{x \log\log x}\, dx = \int_2^{\infty} \frac{dz}{\log z}$$

Now, let $f(z) = \frac{1}{\log z}$ and $g(z) = \frac{1}{z}$.

Then

$$\lim_{z\to\infty} \frac{f(z)}{g(z)} = \lim_{z\to\infty} \frac{z}{\log z} \left(\frac{\infty}{\infty} \text{ form}\right) = \lim_{z\to\infty} \frac{1}{1/z} \to \infty$$

Also, $\int_{e^2}^{\infty} \frac{1}{z}\, dz$ is divergent and hence $\int_2^{\infty} \frac{dz}{\log z}$, i.e. $\int_{e^2}^{\infty} \frac{1}{x \log\log x}\, dx$ is divergent.

14.3.2 Absolute and Conditional Convergent

The improper integral $\int_a^{\infty} f(x)\, dx$ is said to be **absolute convergent** if $\int_a^{\infty} |f(x)|\, dx$ is convergent.

It can easily be verified that, if $\int_a^{\infty} |f(x)|\, dx$ exists, then obviously $\int_a^{\infty} f(x)\, dx$ exists. But, the converse is not true. This leads to another type of convergence, called conditional convergent

If the integral $\int_a^{\infty} f(x)\, dx$ is convergent, but $\int_a^{\infty} |f(x)|\, dx$ does not converge, then the integral $\int_a^{\infty} f(x)\, dx$ is called **conditionally convergent**.

That is, if the integral $\int_a^{\infty} f(x)\, dx$ absolutely convergent, then both the integrals $\int_a^{\infty} f(x)\, dx$ and $\int_a^{\infty} |f(x)|\, dx$ convergent, but for conditionally convergent, only the integral $\int_a^{\infty} f(x)\, dx$ is convergent and $\int_a^{\infty} |f(x)|\, dx$ is dievergent.

EXAMPLE 14.3.4 Show that $\int_1^{\infty} \frac{\cos x}{x^p}\, dx$ converges absolutely if $p > 1$.

Solution Here $\left|\frac{\cos x}{x^p}\right| = \frac{|\cos x|}{x^p} \leq \frac{1}{x^p}$ for all $x \geq 1$.

Again, $\int_1^{\infty} \frac{dx}{x^p}$ converges iff $p > 1$.

Therefore, $\int_1^{\infty} \left|\frac{\cos x}{x^p}\right| dx$ converges if $p > 1$.

Hence $\int_1^{\infty} \frac{\cos x}{x^p} dx$ converges absolutely if $p > 1$.

EXAMPLE 14.3.5 If $\phi(x)$ be bounded when $0 < a \le x$ and $p > 1$, show that $\int_a^{\infty} \frac{|\phi(x)|}{x^p} dx$ converges.

Solution Since $\phi(x)$ is bounded for $0 < a \le x$, then there exists a positive number M such that $|\phi(x)| \le M$, for $0 < a \le x < \infty$.

Thus
$$\frac{|\phi(x)|}{x^p} \le \frac{M}{x^p}$$

Now,
$$\int_a^{\infty} \frac{|\phi(x)|}{x^p} dx \le \int_a^{\infty} \frac{M}{x^p} dx = M \int_a^{\infty} \frac{1}{x^p} dx$$

Since $\int_a^{\infty} \frac{1}{x^p} dx$ exists for $p > 1$, therefore $\int_a^{\infty} \frac{|\phi(x)|}{x^p} dx$ exists for $p > 1$, i.e. the given integral is convergent for $p > 1$.

14.4 Test for Second Type Improper Integrals

14.4.1 A Standard Integral

The integral
$$\int_a^b \frac{dx}{(x-a)^p} \tag{14.5}$$

exists, if $p < 1$ and does not exist, if $p \ge 1$.

Proof. Here a is the point of infinite discontinuity. Let $p \ne 1$. Then

$$\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^p} = \lim_{\varepsilon \to 0+} \left[\frac{1}{1-p}(x-a)^{1-p}\right]_{a+\varepsilon}^b$$
$$= \lim_{\varepsilon \to 0+} \left[\frac{1}{1-p}(b-a)^{1-p} - \varepsilon^{1-p}\right]$$
$$= \begin{cases} \frac{1}{1-p}(b-a)^{1-p}, & \text{if } 1-p > 0 \\ -\infty, & \text{if } 1-p < 0 \end{cases}$$

when $p = 1$ then

$$\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0+} \Big[\log(x-a)\Big]_{a+\varepsilon}^b$$
$$= \lim_{\varepsilon \to 0+} [\log(b-a) - \log \varepsilon] = -\infty$$

Thus $\int_a^b \frac{dx}{(x-a)^p}$ exists if $p < 1$ and does not exist if $p \geq 1$.

Similarly, the integral $\int_a^b \frac{dx}{(b-x)^p}$ exists if $p < 1$ and does not exist if $p \geq 1$.

Comparison test III (comparison of two integrals)

Theorem 14.4 *Let a be the only point of infinite discontinuity. If $f(x)$ and $g(x)$ be integrable functions in $a < x \leq b$ such that $0 \leq f(x) \leq g(x)$, then*

(i) $\int_a^b f(x)\,dx$ *converges, if* $\int_a^b g(x)\,dx$ *converges, and*

(ii) $\int_a^b g(x)\,dx$ *diverges, if* $\int_a^b f(x)\,dx$ *diverges.*

EXAMPLE 14.4.1 Show that $\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$ convergence.

Solution Here $x = 0$ is the only point of infinite discontinuity.

Now,

$$\frac{\sin x}{x^{3/2}} = \frac{1}{x^{1/2}}\,\frac{\sin x}{x}.$$

The function $\frac{\sin x}{x}$ is bounded and $\frac{\sin x}{x} \leq 1$. Therefore, $\frac{\sin x}{x^{3/2}} \leq \frac{1}{x^{1/2}}$.

Since $\int_0^1 \frac{dx}{x^{1/2}}$ is convergent, then $\int_0^1 \frac{\sin x}{x^{3/2}}\,dx \left(\leq \int_0^1 \frac{1}{x^{1/2}}\,dx\right)$ is also convergent.

Comparison test IV (limit form)

Theorem 14.5 *Let a be the only point of infinite discontinuity. If $f(x)$ and $g(x)$ are two positive integrable functions in $[a, b]$ such that*

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = l$$

(i) If $l \neq 0$ and finite, then the integrals $\int_a^b f(x)\,dx$ *and* $\int_a^b g(x)\,dx$ *both converge and diverge together at a.*

(ii) If $l = 0$ and $\int_a^b g(x)\,dx$ *converges, then* $\int_a^b f(x)\,dx$ *converges at a.*

(iii) If $l = \pm\infty$ and $\int_a^b g(x)\,dx$ *diverges, then* $\int_a^b f(x)\,dx$ *diverges at a.*

EXAMPLE 14.4.2 Test the convergence.

(i) $\int_0^1 e^{-x} x^{n-1}\,dx$ (ii) $\int_0^{\pi/2} \frac{x^p}{\sin x}\,dx$ (iii) $\int_0^1 \frac{x^n}{1+x}\,dx$

Solution

(i) Here $x = 0$ is the point of infinite discontinuity when $n - 1 < 0$.

Let $f(x) = e^{-x}x^{n-1}$ and $g(x) = \dfrac{1}{x^{1-n}}$.

Now
$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^{n-1}e^{-x}}{1/x^{1-n}} = \lim_{x\to 0+} e^{-x} = 1$$

Also, $\displaystyle\int_0^1 \frac{1}{x^{1-n}}\,dx$ converges if $1 - n < 1$, or $n > 0$

Therefore, by comparison test $\displaystyle\int_0^1 e^{-x}x^{n-1}\,dx$ converges for $n > 0$.

(ii) Here $x = 0$ is the only point of infinite discontinuity.

Let $f(x) = \dfrac{x^p}{\sin x}$ and $g(x) = \dfrac{1}{x^{1-p}}$.

Now
$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^p}{\sin x}\cdot x^{1-p} = \lim_{x\to 0+} \frac{x}{\sin x} = 1$$

Again, $\displaystyle\int_0^{\pi/2} \frac{1}{x^{1-p}}\,dx$ converges only if $1 - p < 1$, or $p > 0$

Hence by comparison test $\displaystyle\int_0^{\pi/2} \frac{x^p}{\sin x}\,dx$ is convergent only when $p > 0$.

(iii) Here also $x = 0$ is the point of infinite discontinuity.

Let $f(x) = \dfrac{x^n}{1+x}$ and $g(x) = \dfrac{1}{x^{1-n}}$.

Now
$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^n}{1+x}\cdot x^{1-n} = \lim_{x\to 0+} \frac{x}{1+x} = 0$$

Also, $\displaystyle\int_0^1 \frac{1}{x^{1-n}}\,dx$ is convergent when $1 - n < 1$ or $n > 0$

Hence by comparison test $\displaystyle\int_0^1 \frac{x^n}{1+x}\,dx$ converges only when $n > 0$.

EXAMPLE 14.4.3 Show that $I = \displaystyle\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ converges whereas $J = \displaystyle\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ diverges.

Solution *For the integral I.*

In this case $x = 0$ is the point of discontinuity.

Let $f(x) = \dfrac{\log x}{\sqrt{x}}$ and $g(x) = \dfrac{1}{x^{3/4}}$

Now, $\displaystyle\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{\log x}{\sqrt{x}}\cdot x^{3/4} = \lim_{x\to 0+} \frac{\log x}{x^{-1/4}} = \lim_{x\to 0+} (-4x^{1/4}) = 0$

But, $\int_0^1 \frac{1}{x^{3/4}}\,dx$ is convergent $(p = 3/4 < 1)$.

Hence by comparison test $I = \int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ converges.

For the integral J.

Here $x = 1$ is the only point of infinite discontinuity.

Let $f(x) = \frac{\sqrt{x}}{\log x}$ and $g(x) = \frac{1}{x-1}$.

Now

$$\begin{aligned}
\lim_{x\to 1+} \frac{f(x)}{g(x)} &= \lim_{x\to 1+} \frac{\sqrt{x}}{\log x}\cdot(x-1) \\
&= \lim_{x\to 1+} \frac{x^{3/2}-x^{1/2}}{\log x} \quad \left(\frac{0}{0} \text{ form}\right) \\
&= \lim_{x\to 1+} \frac{\frac{3}{2}x^{1/2}-\frac{1}{2}x^{-1/2}}{1/x} = \lim_{x\to 1+}\left(\frac{3}{2}x^{3/2}-\frac{1}{2}x^{1/2}\right) \\
&= 1
\end{aligned}$$

But, $\int_1^2 \frac{1}{x-1}\,dx$ is divergent $(p = 1)$.

Hence by comparison test $I = \int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ diverges.

14.4.2 Absolute Convergent

The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f(x)|\,dx$ is convergent.

EXAMPLE 14.4.4 Show that $\int_0^1 \frac{\sin x}{x^p}\,dx,\; p > 0$ converges absolutely for $p < 1$.

Solution Since $p > 0$, $x = 0$ is the only point of infinite discontinuity.

Now, $\left|\frac{\sin x}{x^p}\right| = \frac{|\sin x|}{x^p} < \frac{1}{x^p}$, in $0 < x \le 1$

Also, $\int_0^1 \frac{1}{x^p}\,dx$ converges only if $p < 1$

Hence by comparison test, $\int_0^1 \left|\frac{\sin x}{x^p}\right| dx$ converges and hence $\int_0^1 \frac{\sin x}{x^p}\,dx$ converges absolutely only if $p < 1$.

14.5 Gamma Function

Definition 14.5.1 *The gamma function is denoted by $\Gamma(n)$ and is defined by*

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx, \qquad n > 0 \tag{14.6}$$

This is a first type improper integral if $n - 1 \geq 0$ and if $n - 1 < 0$, then it is also a second type improper integral, because in this case, $x = 0$ is a point of discontinuity. If $n \leq 0$, then the integral does not converge, i.e. the integral does not give any finite value.

The gamma function satisfies several important properties, they are studied here.

Property 14.5.1 $\Gamma(n+1) = n\Gamma(n), n > 0$.

Proof. We have

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty e^{-x} x^n dx, n+1 > 0 \\
&= \lim_{\substack{B\to\infty \\ \epsilon\to 0+}} \left\{ \left[x^n \frac{e^{-x}}{-1} \right]_\epsilon^B - \int_\epsilon^B n x^{n-1} \frac{e^{-x}}{-1} dx \right\} \\
&= 0 + n \lim_{\substack{B\to\infty \\ \epsilon\to 0+}} \int_\epsilon^B x^{n-1} e^{-x}\, dx \\
&= n\Gamma(n)
\end{aligned}$$

$\therefore \Gamma(n+1) = n\,\Gamma(n)$.

Property 14.5.2 *If $n > 0$ is an integer, then $\Gamma(n+1) = n!$.*

Proof. If n being an integer, then

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \\
&= n(n-1)\Gamma(n-1) \\
&= n(n-1)(n-2)\Gamma(n-2) \\
&\quad \ldots\ldots\ldots\ldots\ldots \\
&\quad \ldots\ldots\ldots\ldots\ldots \\
&= n(n-1)(n-2)\ldots 1 \cdot \Gamma(1)
\end{aligned}$$

Again, $\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$

Hence $\Gamma(n+1) = n!$.

Some initial values of gamma function are:

$$\Gamma(1) = 0! = 1, \qquad \Gamma(2) = 1! = 1, \qquad \Gamma(3) = 2! = 2,$$

$$\Gamma(4) = 3! = 6, \qquad \Gamma(5) = 4! = 24, \qquad \Gamma(6) = 5! = 120.$$

Note that $\Gamma(1)$ and $\Gamma(2)$ have same value. It may be remembered that $\Gamma(0)$ does not exist. Also, $\Gamma(-n), n > 0$ does not exist.

Property 14.5.3 *For any $a > 0$,*

$$\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}, n > 0$$

Proof. To prove this result substituting $ax = y$. Then $a\, dx = dy$. Also, when $x \to 0$ then $y \to 0$ and when $x \to \infty$ then $y \to \infty$.

Thus

$$\begin{aligned} \int_0^\infty e^{-ax} x^{n-1}\, dx &= \int_0^\infty e^{-y} \frac{y^{n-1}}{a^{n-1}} \frac{dy}{a} \\ &= \frac{1}{a^n} \int_0^\infty e^{-y} y^{n-1}\, dy = \frac{\Gamma(n)}{a^n} \end{aligned}$$

EXAMPLE 14.5.1 Find the value of $\int_0^\infty e^{-3x} x^6\, dx$.

Solution By the Property 14.5.3,

$$\int_0^\infty e^{-3x} x^6\, dx = \int_0^\infty e^{-3x} x^{7-1}\, dx = \frac{\Gamma(7)}{3^7} = \frac{6!}{3^7} = \frac{80}{243}$$

14.6 Beta Function

Definition 14.6.1 *The beta function is denoted by $B(m, n)$ and is defined by*

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \qquad m, n > 0. \tag{14.7}$$

It is a second type improper integral. If $m - 1 < 0$, then $x = 0$ is the point of infinite discontinuity and when $n - 1 < 0$, then $x = 1$ is the point of infinite discontinuity. It can be shown that the integral has a finite value if $m > 0$ and $n > 0$.

The beta function satisfies many interesting results which are discussed below.

Property 14.6.1 *$B(m,n) = B(n,m)$, i.e. beta function is commutative.*

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m,n > 0.$

Putting $x = 1 - y$, $dx = -dy$ and when $x \to 1$, $y \to 0$ and when $x \to 0$, $y \to 1$.

$$\therefore B(m,n) = \int_1^0 (1-y)^{m-1}y^{n-1}(-dy) = \int_0^1 y^{n-1}(1-y)^{m-1}dx = B(n,m).$$

Property 14.6.2 $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$

Proof. Substituting $x = \frac{1}{1+y}$, $dx = -\frac{1}{(1+y)^2}dy$. Again, $x + xy = 1$, $y = \frac{1-x}{x}$. When $x \to 1$, $y \to 0$ and when $x \to 0$, $y \to \infty$.

Therefore

$$\begin{aligned} B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}dx \\ &= \int_0^\infty \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy. \end{aligned}$$

Thus $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx.$

Since $B(m,n) = B(n,m)$,

$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}}\,dx$ [interchanging m and n]

Therefore, $\quad B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+m}}\,dx$

Property 14.6.3

$$B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

Proof. Let $B(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}\,dz$. Putting $z = \frac{1}{1+y}$, $dz = -\frac{1}{(1+y)^2}\,dy$. Again, $z + zy = 1$ or, $y = \frac{1-z}{z}$, when $z \to 1$, $y \to 0$, and when $z \to 0$, $y \to \infty$.

Now,

$$
\begin{aligned}
B(m,n) &= \int_0^1 z^{m-1}(1-z)^{n-1}dz \\
&= \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{-1}{(1+y)^2}dy \\
&= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy \\
&= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} + \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy
\end{aligned}
$$

[Again substituting $y = \frac{1}{x}$ in second integral.

$dy = -\frac{1}{x^2}\,dx$ and when $y \to 1, x \to 1;\ y \to \infty, x \to 0.$]

$$
\begin{aligned}
&= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_1^0 \frac{\left(\frac{1}{x}\right)^{n-1}}{\left(1+\frac{1}{x}\right)^{m+n}}\left(-\frac{1}{x^2}\right)dx \\
&= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{m+n}}{x^{n-1}(1+x)^{m+n}}\frac{1}{x^2}dx \\
&= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx \\
&= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx.
\end{aligned}
$$

Property 14.6.4

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta, \qquad m,n>0$$

Proof. We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$. Substituting $x = \sin^2\theta$, $dx = \sin 2\theta\,d\theta$. When $x \to 1$, $\theta \to \frac{\pi}{2}$ and when $x \to 0$, $\theta \to 0$.

Now,

$$
\begin{aligned}
B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^{\frac{\pi}{2}} \sin^{2(m-1)}\theta(1-\sin^2\theta)^{n-1}\sin 2\theta\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2+1}\theta\cos^{2n-2+1}\theta\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.
\end{aligned}
$$

Property 14.6.5 *For any $p > -1, q > -1$,*

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\Big(\frac{p+1}{2}, \frac{q+1}{2}\Big).$$

Proof. Substituting $\sin^2\theta = x$, $\sin 2\theta \, d\theta = dx$, when $\theta \to \frac{\pi}{2}$, $x \to 1$ and when $\theta \to 0$, $x \to 0$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta &= \int_0^{\frac{\pi}{2}} (\sin\theta)^{p-1}(\cos\theta)^{q-1} \sin\theta\cos\theta \, d\theta \\
&= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{p-1}{2}} (\cos^2\theta)^{\frac{q-1}{2}} \sin\theta\cos\theta \, d\theta \\
&= \int_0^1 x^{\frac{p-1}{2}} (1-x)^{\frac{q-1}{2}} \frac{1}{2} dx \\
&= \frac{1}{2}\int_0^1 x^{\frac{p-1}{2}} (1-x)^{\frac{q-1}{2}} dx \\
&= \frac{1}{2} B\Big(\frac{p-1}{2} + 1, \frac{q-1}{2} + 1\Big) \\
&= \frac{1}{2} B\Big(\frac{p+1}{2}, \frac{q+1}{2}\Big).
\end{aligned}$$

Property 14.6.6 $B(\frac{1}{2}, \frac{1}{2}) = \pi$.

Proof. Substituting $p = q = 0$ to the above property.

Then $\displaystyle\int_0^{\pi/2} \cos^0\theta \, \sin^0\theta \, d\theta = \frac{1}{2} B\Big(\frac{0+1}{2}, \frac{0+1}{2}\Big)$

This gives $\dfrac{\pi}{2} = \dfrac{1}{2} B\Big(\dfrac{1}{2}, \dfrac{1}{2}\Big)$ or, $B\Big(\dfrac{1}{2}, \dfrac{1}{2}\Big) = \pi$.

Property 14.6.7 (Relation between beta and gamma functions)

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Proof. From Property 14.6.4 and the definition of the gamma function

$$B(m, n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta \tag{i}$$

$$\text{and } \Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t} \, dt = 2\int_0^{\infty} r^{2m-1} e^{-r^2} \, dr \text{ where } t = r^2. \tag{ii}$$

Let us consider three regions (Fig. 14.1).

E_1 :	first quadrant of the circle $x^2 + y^2 = R^2$	: OABCO
E :	the square $0 \le x \le R$ and $0 \le y \le R$	: OADCO
E_2 :	first quadrant of the circle $x^2 + y^2 = 2R^2$	: OFGDHO

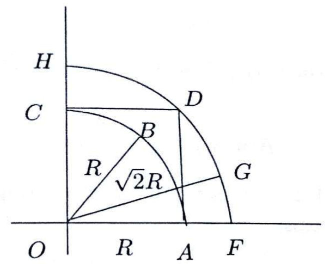

Figure 14.1: The regions E_1, E, E_2.

From (ii) it is observed that the integral

$$4\iint_E x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}\,dx\,dy$$

tends to $\Gamma(m)\Gamma(n)$ as $R \to \infty$.

The positive quadrant (E_1) of the circle $x^2 + y^2 = R^2$ is a part of the square E which, again is a part of the positive quadrant (E_2) of the $x^2 + y^2 = 2R^2$. Thus $E_1 \subseteq E \subseteq E_2$.

The integrand being positive, we have

$$\begin{aligned}4\iint_{E_1} x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy &\leq 4\iint_E x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy\\ &\leq 4\iint_{E_2} x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy.\end{aligned}$$

Substituting $x = r\cos\theta, y = r\sin\theta$ to the first integral. Therefore,

$$\begin{aligned}&4\iint_{E_1} x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy\\ =\ & 4\int_0^{\pi/2} \cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta \times \int_0^R e^{-r^2}r^{2m+2n-1}\,dr\\ =\ & 2B(m,n)\int_0^R e^{-r^2}r^{2m+2n-1}\,dr.\end{aligned}$$

Similarly, $4\iint_{E_2} x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy = 2B(m,n)\int_0^{\sqrt{2}R} e^{-r^2}r^{2m+2n-1}\,dr.$

Therefore,

$$\begin{aligned}2B(m,n)\int_0^R e^{-r^2}r^{2m+2n-1}\,dr &\leq 4\iint_E x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy\\ &\leq 2B(m,n)\int_0^{\sqrt{2}R} e^{-r^2}r^{2m+2n-1}\,dr.\end{aligned}$$

Taking $R \to \infty$, we get

$$B(m,n)\Gamma(m+n) \le \Gamma(m)\Gamma(n) \le B(m,n)\Gamma(m+n)$$

or

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

It is easy to observe that $B(1,1) = 1$ and $B(0,n)$, $B(m,0)$ do not exist, for any m, n. Also, $B(-m,-n)$; $m, n > 0$ does not exist.

Property 14.6.8 $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof. Since $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$ and by Property 14.6.7,

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi.$$

But, $\Gamma(1) = 1$, therefore $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Thus, $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

Combining Properties 14.6.5 and 14.6.7, we obtain the following result.

Property 14.6.9 *Since* $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \int_0^{\pi/2} \cos^p\theta \sin^q\theta \, d\theta$, *thus*

$$\int_0^{\pi/2} \cos^p\theta \sin^q\theta \, d\theta = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

for any $p > -1, q > -1$

EXAMPLE 14.6.1 Find the value of the following integrals.

(i) $\int_0^{\pi/2} \sin^5 x \, dx$ (ii) $\int_0^{\pi/2} \cos^6 x \, dx$

(iii) $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$ (iv) $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$.

Solution

(i)

$$\begin{aligned} \int_0^{\pi/2} \sin^5 x \, dx &= \int_0^{\pi/2} \sin^5 x \cos^0 x \, dx \\ &= \frac{1}{2}\frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{5+0+2}{2}\right)} \end{aligned}$$

[by Property 14.6.9]

$$= \frac{1}{2}\frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{1}{2}\frac{2! \times \Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)} = \frac{8}{15}$$

(ii)

$$\int_0^{\pi/2} \cos^6 x\, dx = \int_0^{\pi/2} \cos^6 x \sin^0 x\, dx$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{6+0+2}{2}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = \frac{1}{2}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\times\Gamma\left(\frac{1}{2}\right)}{3!} = \frac{15}{16}\cdot\frac{\pi}{6} = \frac{5\pi}{32}.$$

(iii)

$$\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx = \frac{1}{2}\frac{\Gamma\left(5/2\right)\Gamma\left(7/2\right)}{\Gamma\left(12/2\right)}$$

$$= \frac{1}{2}\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(6)} = \frac{45\pi}{64\times 5!} = \frac{3\pi}{512}.$$

(iv)

$$\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx = \frac{1}{2}\frac{\Gamma\left(5/2\right)\Gamma\left(6/2\right)}{\Gamma\left(11/2\right)}$$

$$= \frac{1}{2}\frac{\Gamma(5/2)\cdot\Gamma(3)}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma(5/2)} = \frac{8}{315}$$

Property 14.6.10 (Duplicating formula) *For any $m > 0$,*

$$2^{2m-1}\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2m)$$

Proof. By the property of beta function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta. \qquad \text{(i)}$$

Substituting $n = m$. Therefore,

$$B(m,m) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2m-1}\theta\, d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}(2\theta)\, d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\, d\phi \qquad \text{[where } 2\theta = \phi]$$

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$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} \phi \, d\phi \tag{ii}$$

Again, substituting $n = 1/2$ in (i) we get

$$\frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \, d\theta$$

or
$$\frac{1}{2^{2m-1}} \frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} \theta \, d\theta \tag{iii}$$

From (ii) and (iii) for $m > 0$,

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = \frac{\sqrt{\pi}\Gamma(m)}{2^{2m-1}\Gamma(m+1/2)}$$

or
$$2^{2m-1}\Gamma(m)\Gamma(m+1/2) = \sqrt{\pi}\Gamma(2m), \qquad m > 0$$

From this relation, one can determine the value of $\Gamma(1/2)$ by putting $m = 1/2$.

Property 14.6.11

$$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}, \qquad 0 < m < 1$$

Proof. From the property of beta function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx.$$

Putting $n = 1 - m, 0 < m < 1$. Therefore,

$$B(m, 1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(m+1-m)} = \Gamma(m)\Gamma(1-m) = \int_0^\infty \frac{x^{m-1}}{1+x} \, dx$$

$$= \int_0^1 \frac{x^{m-1}}{1+x} \, dx + \int_1^\infty \frac{x^{m-1}}{1+x} \, dx$$

[It can be proved that both the integrals convergent for $0 < m < 1$]

Putting $x = 1/y$ in the second integral

$$= \int_0^1 \frac{x^{m-1}}{1+x} \, dx + \int_0^1 \frac{y^{-m}}{1+y} \, dy = \int_0^1 \frac{x^{m-1}}{1+x} \, dx + \int_0^1 \frac{x^{-m}}{1+x} \, dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{-m}}{1+x} \, dx$$

$$= \int_0^1 (x^{m-1} + x^{-m})\left(1 - \frac{x}{1+x}\right) dx$$

$$= \int_0^1 (x^{m-1} + x^{-m}) \, dx - \int_0^1 \frac{x^m + x^{1-m}}{1+x} \, dx$$

Now

$$\int_0^1 (x^{m-1} + x^{-m})\, dx = \left[\frac{x^m}{m} + \frac{x^{1-m}}{1-m}\right]_0^1 = \frac{1}{m} + \frac{1}{1-m}$$

and

$$\int_0^1 \frac{x^m + x^{1-m}}{1+x}\, dx$$

$$= \int_0^1 (x^m + x^{1-m})(1 - x + x^2 - x^3 + x^4 + \cdots)\, dx$$

$$= \left[\frac{x^{m+1}}{m+1} + \frac{x^{2-m}}{2-m} - \frac{x^{m+2}}{m+2} - \frac{x^{3-m}}{3-m} + \cdots\right]_0^1$$

Thus

$$\Gamma(m)\Gamma(1-m) = \frac{1}{m} + \frac{1}{1-m} - \frac{1}{m+1} - \frac{1}{2-m} + \frac{1}{m+2} + \frac{1}{3-m} + \cdots$$

$$= \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{k+m} + \frac{1}{k+1-m}\right) = \pi \operatorname{cosec} m\pi = \frac{\pi}{\sin m\pi}.$$

14.7 Additional Worked-Out Examples

EXAMPLE 14.7.1 Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}$.

Solution

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x \cos^{-\frac{1}{2}} x\, dx$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{2}\frac{\pi}{\sin 45^o} = \frac{1}{\sqrt{2}}\pi$$

EXAMPLE 14.7.2 Find the value of $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Solution Here the integral is even function. Thus, $\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-x^2}\, dx$.

Now, substituting $x^2 = z$. Then $2x\, dx = dz$. When $x \to 0, z \to 0$ and when $x \to \infty$, $z \to \infty$.

Therefore,

$$\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-z}\cdot\frac{1}{2\sqrt{z}}\,dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z}z^{1/2-1}\,dz = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Thus, $\int_{-\infty}^\infty e^{-x^2}\,dx = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$

EXAMPLE 14.7.3 Evaluate $\int_0^\infty a^{-x^2}\,dx, a > 1.$

Solution We know $a^{-x^2} = e^{\log a^{-x^2}} = e^{-x^2\log a} = e^{-bx^2}$, where $b = \log a$.

Substituting $bx^2 = z$. Then $2bx\,dx = dz$ or $dx = \frac{1}{2bx}\,dz = \frac{1}{2b}\sqrt{\frac{b}{z}}\,dz = \frac{1}{2\sqrt{bz}}\,dz.$

Hence

$$\int_0^\infty a^{-x^2}\,dx = \int_0^\infty \frac{1}{2\sqrt{bz}}e^{-z}\,dz$$

$$= \frac{1}{2\sqrt{b}}\int_0^\infty e^{-z}z^{1/2-1}\,dz$$

$$= \frac{1}{2\sqrt{b}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{b}} = \frac{\sqrt{\pi}}{2\sqrt{\log a}}$$

EXAMPLE 14.7.4 Show that $\int_0^\infty e^{-x^4}dx \times \int_0^\infty e^{-x^4}.x^2dx = \frac{\pi}{8\sqrt{2}}.$

Solution Let $x^4 = z$, $4x^3dx = dz$, or $dx = \frac{dz}{4x^3}$, $x = z^{\frac{1}{4}}$.

Let

$$I_1 = \int_0^\infty e^{-x^4}dx$$

$$= \int_0^\infty e^{-z}\frac{dz}{4z^{\frac{3}{4}}}dz$$

$$= \int_0^\infty e^{-z}z^{\frac{1}{4}-1}dz = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

and

$$I_2 = \int_0^\infty e^{-x^4}\cdot x^2dx$$

$$= \frac{1}{4}\int_0^\infty e^{-z}z^{-\frac{1}{4}}dz$$

$$= \frac{1}{4}\int_0^\infty e^{-z}z^{\frac{3}{4}-1}dz = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\therefore I = I_1 \times I_2 = \frac{1}{16}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{16}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{16}\pi\sqrt{2} = \frac{\pi}{8\sqrt{2}}.$$

EXAMPLE 14.7.5 Show that $\int_0^\infty e^{-x^2} x^{\alpha^2} dx = \frac{1}{2}\Gamma(\frac{\alpha^2+1}{2})$, $\alpha > 1$.

Solution Putting $x^2 = z$, $2x\, dx = dz$.

$$\begin{aligned}\int_0^\infty e^{-x^2} x^{\alpha^2} dx &= \frac{1}{2}\int_0^\infty e^{-z}(\sqrt{z})^{\alpha^2}\frac{dz}{2\sqrt{z}} = \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{\alpha^2-1}{2}} dz \\ &= \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{\alpha^2+1}{2}-1} dz = \frac{1}{2}\Gamma\left(\frac{\alpha^2+1}{2}\right),\ \alpha > 1\end{aligned}$$

EXAMPLE 14.7.6 Prove that $\displaystyle\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}} dx = \frac{3\pi}{128}$.

Solution

$$\begin{aligned}\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}} dx &= \int_0^1 x^{\frac{5}{2}-1}(1-x)^{\frac{5}{2}-1} dx \\ &= \frac{\Gamma(\frac{5}{2})\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}+\frac{5}{2})} = \frac{\{\Gamma(\frac{5}{2})\}^2}{\Gamma(5)} = \frac{\{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\}^2}{4\cdot 3\cdot 2} = \frac{3\pi}{128}\end{aligned}$$

Since $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$ and $\Gamma(5) = 4!$

EXAMPLE 14.7.7 Prove that $\displaystyle\int_a^b (x-a)^3(b-x)^2 dx = \frac{(b-a)^6}{60}$.

Solution Substituting $x = a+(b-a)t$, $dx = (b-a)dt$, $b-x = b-a-(b-a)t = (b-a)(1-t)$. When $x \to a$, $t \to 0$ and when $x \to b$, $t \to 1$.

$$\begin{aligned}\int_a^b (x-a)^3(b-x)^2 dx dx &= \int_0^1 \{t(b-a)\}^3\{(b-a)(1-t)\}^2(b-a)dt \\ &= \int_0^1 (b-a)^6 t^3(1-t)^2 dt \\ &= (b-a)^6\int_0^1 t^{4-1}(1-t)^{3-1} dt \\ &= (b-a)^6 B(4,3) \\ &= (b-a)^6\frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} = (b-a)^6\frac{3\cdot 2\cdot\Gamma(3)}{6\cdot 5\cdot 4\cdot 3\Gamma(3)} \\ &= \frac{(b-a)^6}{60}\end{aligned}$$

EXAMPLE 14.7.8 Show that $\int_0^1 \frac{dx}{\sqrt{1-x^4}} dx = \frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}$.

Solution Substituting $1 - x^4 = y$, $x^4 = 1 - y$, $4x^3 dx = -dy$, $dx = -\frac{1}{4}\frac{dy}{(1-y)^{\frac{3}{4}}}$.
When $x \to 0$, $y \to 1$; when $x \to 1$, $y \to 0$.

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt{1-x^4}} dx &= \int_1^0 \frac{1}{\sqrt{y}} \frac{1}{4} \frac{-dy}{(1-y)^{\frac{3}{4}}} \\
&= \int_0^1 \frac{1}{4} y^{-\frac{1}{2}} (1-y)^{-\frac{3}{4}}\, dy \\
&= \frac{1}{4}\int_0^1 y^{\frac{1}{2}-1}(1-y)^{\frac{1}{4}-1} dy \\
&= \frac{1}{4} B\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}
\end{aligned}$$

EXAMPLE 14.7.9 Show that $\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}} dx = \frac{\pi}{3}$.

Solution Let $x^6 = z$, $6x^5 dx = dz$

$$\begin{aligned}
\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}} dx &= \int_0^1 \frac{1}{6} \frac{z^{-\frac{5}{6}}}{(1-z)^{\frac{1}{6}}} dz = \frac{1}{6}\int_0^1 z^{\frac{1}{6}-1}(1-z)^{\frac{5}{6}-1} dz \\
&= \frac{1}{6} B\left(\frac{1}{6}, \frac{5}{6}\right) = \frac{1}{6}\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma(1)} = \frac{1}{6}\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right) \\
&= \frac{1}{6}\frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{3}
\end{aligned}$$

EXAMPLE 14.7.10 Prove that $\int_0^{\pi/2} \sin^p x\, dx \times \int_0^{\pi/2} \sin^{p+1} x\, dx = \frac{\pi}{2(p+1)}$.

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sin^p x\, dx \times \int_0^{\pi/2} \sin^{p+1} x\, dx &= \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} \times \frac{1}{2}\frac{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+3}{2}\right)} \\
&= \frac{1}{4}\frac{\pi\Gamma\left(\frac{p+1}{2}\right)}{\frac{p+1}{2}\Gamma\left(\frac{p+1}{2}\right)} = \frac{\pi}{2(p+1)}.
\end{aligned}$$

EXAMPLE 14.7.11 Find the value of $\int_{-1}^{1}(1+x)^p(1-x)^q\,dx$.

Solution Putting $1+x=2y$, or $x=2y-1$. Therefore, $dx=2dy$. When $x\to -1, y\to 0$ and when $x\to 1$ then $y\to 1$.

$$\begin{aligned}\int_{-1}^{1}(1+x)^p(1-x)^q\,dx &= \int_0^1 (2y)^p 2^q(1-y)^q 2\,dy\\ &= 2^{p+q+1}\int_0^1 y^{(p+1)-1}(1-y)^{q+1-1}\,dy\\ &= 2^{p+q+1}B(p+1,q+1)\\ &= 2^{p+q+1}\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\end{aligned}$$

provided $p>-1, q>-1$.

EXAMPLE 14.7.12 Prove that $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{1\cdot 2\cdot 3\ldots(m-1)}{n(n+1)\ldots(n+m-1)}$, where m is a positive integer and n is any positive quantity.

Solution We have

$$\begin{aligned}B(m,n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{\Gamma(m)\Gamma(n)}{(m+n-1)\Gamma(m+n-1)}\\ &= \frac{\Gamma(m)\Gamma(n)}{(m+n-1)(m+n-2)\Gamma(m+n-2)}\\ &= \frac{\Gamma(m)\Gamma(n)}{(m+n-1)(m+n-2)(m+n-3)\Gamma(m+n-3)}\\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= \frac{\Gamma(m)\Gamma(n)}{(m+n-1)(m+n-2)\ldots n\cdot\Gamma(n)}\\ &= \frac{1\cdot 2\cdot 3\ldots(m-1)}{n(n+1)\ldots(n+m-1)} \quad [\because\ m \text{ is a positive integer}]\end{aligned}$$

EXAMPLE 14.7.13 Prove that

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = (b-a)^{m+n-1}B(m,n),\quad m,n>0$$

Solution Putting $x = a + (b-a)t$, $dx = (b-a)dt$, $b - x = b - a - (b-a)t = (b-a)(1-t)$.
When $x \to a$, $t \to 0$ and when $x \to b$, $t \to 1$.

$$\begin{aligned}
\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx &= \int_0^1 (b-a)^{m-1}t^{m-1}(b-a)^{n-1}(1-t)^{n-1}(b-a)dt \\
&= \int_0^1 (b-a)^{m+n-1}t^{m-1}(1-t)^{n-1}dt \\
&= (b-a)^{m+n-1}B(m,n)
\end{aligned}$$

EXAMPLE 14.7.14 Show that $\displaystyle\int_0^1 x^{m-1}(1-x^2)^{n-1}\,dx = \frac{1}{2}B\left(\frac{m}{2},n\right)$, $m,n>0$.

Solution Putting $x^2 = z$, $2x\,dx = dz$

$$\begin{aligned}
\int_0^1 x^{m-1}(1-x^2)^{n-1}dx &= \frac{1}{2}\int_0^1 z^{\frac{m-1}{2}-\frac{1}{2}}(1-z)^{n-1}dz \\
&= \frac{1}{2}\int_0^1 z^{\frac{m}{2}-1}(1-z)^{n-1}dz \\
&= \frac{1}{2}\left(\frac{m}{2},n\right),\ m,n>0
\end{aligned}$$

EXAMPLE 14.7.15 Prove that $\displaystyle\int_0^1 x^p(1-x^q)^n\,dx = \frac{1}{q}B\left(\frac{p+1}{q},n+1\right)$.

Solution Substituting $x^q = z$. Then $qx^{q-1}\,dx = dz$.

$$\begin{aligned}
\int_0^1 x^p(1-x^q)^n\,dx &= \int_0^1 z^{\frac{p}{q}-\frac{q-1}{q}}(1-z)^{n+1-1}\,dz = \frac{1}{q}\int_0^1 z^{(p-q+1)/q}(1-z)^n\,dz \\
&= \frac{1}{q}\int_0^1 z^{(p+1)/q-1}(1-z)^{n+1-1}\,dz \\
&= \frac{1}{q}B\left(\frac{p+1}{q},n+1\right)
\end{aligned}$$

EXAMPLE 14.7.16 Prove that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\cdots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$.

Solution

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\cdots\Gamma\left(\frac{8}{9}\right)$$

$$= \left\{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right)\right\}\left\{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{7}{9}\right)\right\}\left\{\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{6}{9}\right)\right\}\left\{\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\right\}$$

$$= \left\{\Gamma\left(\frac{1}{9}\right)\Gamma\left(1-\frac{1}{9}\right)\right\}\left\{\Gamma\left(\frac{2}{9}\right)\Gamma\left(1-\frac{2}{9}\right)\right\}\left\{\Gamma\left(\frac{3}{9}\right)\Gamma\left(1-\frac{3}{9}\right)\right\}\left\{\Gamma\left(\frac{4}{9}\right)\Gamma\left(1-\frac{4}{9}\right)\right\}$$

$$= \frac{\pi}{\sin\frac{\pi}{9}} \times \frac{\pi}{\sin\frac{2\pi}{9}} \times \frac{\pi}{\sin\frac{3\pi}{9}} \times \frac{\pi}{\sin\frac{4\pi}{9}}$$

$$= \frac{2}{\sqrt{3}}\pi^4 \frac{1}{\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{4\pi}{9}} = \frac{2^3}{\sqrt{3}}\pi^4 \frac{1}{2\sin\frac{\pi}{9}\left(\cos\frac{2\pi}{9} - \cos\frac{2\pi}{3}\right)}\ .$$

$$= \frac{2^3\pi^4}{\sqrt{3}} \frac{1}{2\sin\frac{\pi}{9}\left(\cos\frac{2\pi}{9} + \frac{1}{2}\right)}$$

$$= \frac{2^3\pi^4}{\sqrt{3}} \frac{1}{2\sin\frac{\pi}{9}\cos\frac{2\pi}{9} + \sin\frac{\pi}{9}}$$

$$= \frac{2^3\pi^4}{\sqrt{3}} \frac{1}{\sin\frac{3\pi}{9} - \sin\frac{\pi}{9} + \sin\frac{\pi}{9}}$$

$$= \frac{2^3\pi^4}{\sqrt{3}} \frac{2}{\sqrt{3}} = \frac{16}{3}\pi^4$$

EXAMPLE 14.7.17 Show that $\displaystyle\int_0^p x^m(p^q - x^q)^n\, dx = \frac{p^{nq+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$, if $p > 0, q > 0, m+1 > 0, n+1 > 0$.

Solution Putting $p^q - x^q = zp^q$. Therefore, $-\dfrac{qx^{q-1}}{p^q}\, dx = dz$, i.e. $x = (1-z)^{1/q} \cdot p$.

$$\begin{aligned}
\int_0^p x^m p^{nq}\left(1 - \frac{x^q}{p^q}\right)^n dx &= -\int_1^0 (1-z)^{(m+1-q)/q} p^{nq+q} z^n\, dz \\
&= \int_0^1 \frac{p^{nq+m+1}}{q} z^n (1-z)^{(m+1)/q-1}\, dz \\
&= \frac{p^{nq+m+1}}{q}\int_0^1 z^{n+1-1}(1-z)^{(m+1)/q-1}\, dz \\
&= \frac{p^{nq+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right),
\end{aligned}$$

if $(m+1)/q > 0, n+1 > 0$. That is, if $n+1 > 0, m+1 > 0, q > 0$. Since p is the upper limit of the integral, so it must be positive as lower limit is 0.

EXAMPLE 14.7.18 Show that

$$B(m,n)B(m+n,l) = B(n,l)B(n+l,m) = B(l,m)B(l+m,n).$$

Solution We have

$$\begin{aligned} B(m,n)B(m+n,l) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\cdot\frac{\Gamma(m+n)\Gamma(l)}{\Gamma(m+n+l)} \\ &= \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)} \\ &= \frac{\Gamma(n)\Gamma(l)}{\Gamma(n+l)}\cdot\frac{\Gamma(n+l)\Gamma(m)}{\Gamma(n+l+m)} \\ &= B(n,l)B(n+l,m) \end{aligned}$$

Again

$$\begin{aligned} B(n,l)B(n+l,m) &= \frac{\Gamma(n)\Gamma(l)}{\Gamma(n+l)}\cdot\frac{\Gamma(n+l)\Gamma(m)}{\Gamma(n+l+m)} \\ &= \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)} \\ &= \frac{\Gamma(l)\Gamma(m)}{\Gamma(m+l)}\cdot\frac{\Gamma(m+l)\Gamma(n)}{\Gamma(m+l+n)} \\ &= B(l,m)B(m+l,n) \end{aligned}$$

EXAMPLE 14.7.19 Show that $B(m,n) = B(m+1,n) + B(m,n+1)$.

Solution We have

$$\begin{aligned} B(m,n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \\ &= \frac{\Gamma(m)\Gamma(n)\cdot(m+n)}{\Gamma(m+n)\cdot(m+n)} \\ &= \frac{m\Gamma(m)\Gamma(n) + n\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} \\ &= \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} + \frac{\Gamma(m)\cdot n\Gamma(n)}{(m+n)\Gamma(m+n)} \\ &= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} \\ &= B(m+1,n) + B(m,n+1) \end{aligned}$$

EXAMPLE 14.7.20 Show that $\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q}$.

Solution

$$\begin{aligned}
\frac{B(p,q+1)}{q} &= \frac{\Gamma(p)\Gamma(q+1)}{q\cdot\Gamma(p+q+1)} \\
&= \frac{\Gamma(p)\cdot q\Gamma(q)}{q\cdot\Gamma(p+q+1)} \\
&= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} \\
&= \frac{p\Gamma(p)\Gamma(q)}{p\Gamma(p+q+1)} = \frac{B(p+1,q)}{p}
\end{aligned}$$

Again

$$\begin{aligned}
\frac{B(p+1,q)}{p} &= \frac{\Gamma(p+1)\Gamma(q)}{p\cdot\Gamma(p+q+1)} \\
&= \frac{p\Gamma(p)\Gamma(q)}{p\Gamma(p+q+1)} \\
&= \frac{\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)} = \frac{B(p,q)}{p+q}
\end{aligned}$$

EXAMPLE 14.7.21 Show that $\displaystyle\int_0^\infty \frac{x^{m-1}dx}{(a+bx)^{m+n}} = \frac{1}{a^n\cdot b^m}B(m,n)$.

Solution Substituting $bx = at$, $bdx = adt$. When $x\to\infty$, $t\to\infty$; when $x\to 0$, $t\to 0$.

$$\begin{aligned}
\int_0^\infty \frac{x^{m-1}dx}{(a+bx)^{m+n}}dx &= \int_0^\infty \frac{\left(\frac{a}{b}t\right)^{m-1}}{(a+at)^{m+n}} \times \frac{a}{b}dt = \int_0^\infty \frac{\left(\frac{a}{b}\right)^{m-1+1} t^{m-1}}{a^{m+n}(1+t)^{m+n}}dt \\
&= \int_0^\infty \frac{a^m}{b^m a^m a^n}\frac{t^{m-1}}{(1+t)^{m+n}}dt = \frac{1}{a^n b^m}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt \\
&= \frac{1}{a^n\cdot b^m}B(m,n),\ m,n>0
\end{aligned}$$

EXAMPLE 14.7.22 Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{a^m b^n\Gamma(m+n)}$.

Solution

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{\cos^{2m+2n}\theta(b + a\tan^2\theta)^{m+n}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^{2m-1}\theta\sec^2\theta\, d\theta}{(b + a\tan^2\theta)^{m+n}} = \int_0^{\frac{\pi}{2}} \frac{\tan^{2m-2}\theta\sec^2\theta\tan\theta\, d\theta}{(b + a\tan^2\theta)^{m+n}}$$

[Putting $a\tan^2\theta = bt$. $\therefore$ $2a\tan\theta.\sec^2\theta\, d\theta = b\, dt$.

When $\theta \to 0, t \to 0$ and when $\theta \to \pi/2, t \to \infty$.]

$$= \int_0^\infty \frac{\left(\frac{bt}{a}\right)^{m-1}\frac{b\,dt}{2a}}{(b+bt)^{m+n}}dt = \frac{1}{2}\int_0^\infty \frac{b^{m-1+1}t^{m-1}}{a^{m-1+1}b^{m+n}(1+t)^{m+n}}\,dt$$

$$= \frac{1}{2a^m b^n}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$= \frac{1}{2a^m b^n}B(m,n) = \frac{1}{2a^m b^n}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad \left(\because\; B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx\right)$$

EXAMPLE 14.7.23 Show that $\displaystyle\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(b+cx)^{l+m}}\,dx = \frac{B(l,m)}{(b+c)^l b^m}$.

Solution Substituting $x = \dfrac{1}{1+y}$, $dx = -\dfrac{1}{(1+y)^2}\,dy$. When $x \to 1, y \to 0$ and when $x \to 0, y \to \infty$.

$$\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(b+cx)^{l+m}}\,dx = \int_\infty^0 \frac{\left(\frac{1}{1+y}\right)^{l-1}\left(\frac{y}{1+y}\right)^{m-1}}{\left(b + c\frac{1}{1+y}\right)^{l+m}}\cdot\frac{-1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{m-1}(1+y)^{l+m}}{(1+y)^{l-1+m-1+2}(b+by+c)^{l+m}}\,dy$$

$$= \int_0^\infty \frac{y^{m-1}\,dy}{(b+c)^{l+m}\left(1+\frac{b}{b+c}y\right)^{l+m}}$$

$$= \int_0^\infty \frac{\left(\frac{b+c}{c}\right)^{m-1} z^{m-1}\frac{b+c}{b}\,dz}{(b+c)^{l+m}(1+z)^{l+m}}$$

$$= \frac{(b+c)^m}{b^m(b+c)^{l+m}}\int_0^\infty \frac{z^{m-1}}{(1+z)^{l+m}}\,dz$$

$$= \frac{1}{(b+c)^l b^m}B(m,l)$$

EXAMPLE 14.7.24 Prove that $\displaystyle\int_0^{\pi} \frac{\sin^{n-1} x\, dx}{(a+b\cos x)^n} = \frac{2^{n-1}\cdot B(n/2, n/2)}{(a^2-b^2)^{n/2}}$

Solution From trigonometry

$$\begin{aligned}
a + b\cos x &= a\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + b\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) \\
&= (a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2} \\
&= \left\{(a+b) + (a-b)\tan^2\frac{x}{2}\right\}\cos^2\frac{x}{2} \\
&= \left(1 + \frac{a-b}{a+b}\tan^2\frac{x}{2}\right)(a+b)\cos^2\frac{x}{2}
\end{aligned}$$

Let

$$\begin{aligned}
I = \int_0^{\pi} \frac{\sin^{n-1} x\, dx}{(a+b\cos x)^n} &= \int_0^{\pi} \frac{(2\sin\frac{x}{2}\cos\frac{x}{2})^{n-1}\, dx}{\left(1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)^n (a+b)^n\cos^{2n}\frac{x}{2}} \\
&= \int_0^{\pi} \frac{2^{n-1}(\sin\frac{x}{2}\cos\frac{x}{2})^{n-1}\sec^{2n}\frac{x}{2}\, dx}{\left(1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)^n (a+b)^n} \\
&= \int_0^{\pi} \frac{2^{n-1}\tan^{n-1}\frac{x}{2}\sec^2\frac{x}{2}\, dx}{\left(1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)^n (a+b)^n}
\end{aligned}$$

Putting $\dfrac{a-b}{a+b}\tan^2\dfrac{x}{2} = z$, $\quad\therefore \dfrac{a-b}{a+b}2\cdot\dfrac{1}{2}\tan\dfrac{x}{2}\sec^2\dfrac{x}{2}\, dx = dz$

When $x \to 0$ then $z \to 0$ and when $x \to \pi$ then $z \to \infty$ also $\tan\dfrac{x}{2} = \sqrt{\dfrac{a+b}{a-b}}\sqrt{z}.$

Therefore

$$\begin{aligned}
I &= \int_0^{\infty} \frac{2^{n-1}\left(\frac{a+b}{a-b}\right)^{\frac{n-2}{2}+1} z^{(n-2)/2}\, dz}{(a+b)^n(1+z)^n} \\
&= 2^{n-1}\left(\frac{a+b}{a-b}\right)^{n/2}\frac{1}{(a+b)^n}\int_0^{\infty}\frac{z^{n/2-1}\, dz}{(1+z)^n} \\
&= 2^{n-1}\frac{1}{\{(a-b)(a+b)\}^{n/2}}\int_0^{\infty}\frac{z^{n/2-1}\, dz}{(1+z)^{n/2+n/2}} \\
&= \frac{2^{n-1}}{(a^2-b^2)^{n/2}}B\left(\frac{n}{2}, \frac{n}{2}\right)
\end{aligned}$$

EXAMPLE 14.7.25 Show that $\int_0^1 \log\Gamma(x)\,dx = \frac{1}{2}\log(2\pi)$ where $\int_0^{\pi/2} \log\sin x\,dx = \frac{\pi}{2}\log\frac{1}{2}$.

Solution Let $I = \int_0^1 \log\Gamma(x)\,dx$.

Substituting $x = 1 - y$. Therefore, $dx = -dy$.

Therefore

$$\begin{aligned}
I &= \int_0^1 \log\Gamma(1-y)\,dy = \int_0^1 \log\Gamma(1-x)\,dx \\
\therefore\ 2I &= \int_0^1 \log\Gamma(x)\,dx + \int_0^1 \log\Gamma(1-x)\,dx \\
&= \int_0^1 \log\{\Gamma(x)\Gamma(1-x)\}\,dx \\
&= \int_0^1 \log\frac{\pi}{\sin\pi x}\,dx \\
&= \int_0^1 \log\pi\,dx - \int_0^1 \log\sin\pi x\,dx \\
&= \log\pi - \frac{1}{\pi}\int_0^{\pi} \log\sin z\,dz = \log\pi - \frac{2}{\pi}\times\int_0^{\pi/2} \log\sin z\,dz \quad [\text{ where } \pi x = z] \\
&= \log\pi - \frac{2}{\pi}\cdot\frac{\pi}{2}\log\frac{1}{2} = \log(2\pi)
\end{aligned}$$

Hence $I = \frac{1}{2}\log(2\pi)$.

EXAMPLE 14.7.26 Show that

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx = \frac{1}{9^{1/3}}B(2/3,1/3) = \frac{1}{9^{1/3}}\frac{2\pi}{\sqrt{3}}$$

Solution Putting $\frac{x}{1-x} = \frac{at}{1-t}$, where a is a suitable constant, to be chosen latter.

That is, $x = \frac{at}{1-t+at}$, $dx = \frac{a}{(1-t+at)^2}\,dt$.

When $x \to 0, t \to 0$ and when $x \to 1, t \to 1$.

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx$$

$$= \int_0^1 \left\{\frac{at}{(1-t+at)}\right\}^{-1/3}\left\{\frac{1-t}{1-t+at}\right\}^{-2/3}\left\{\frac{1-t+3at}{1-t+at)}\right\}^{-1}\cdot\frac{a}{(1-t+at)^2}\,dt$$

$$= \int_0^1 a^{2/3}t^{-1/3}(1-t)^{-2/3}(1-t+3at)^{-1}\,dt$$

If we choose $a = 1/3$, the above integral becomes beta function. Therefore, the given integral is equal to

$$\int_0^1 (1/3)^{2/3}t^{-1/3}(1-t)^{-2/3}\,dt$$

$$= \int_0^1 \frac{1}{9^{1/3}}t^{2/3-1}(1-t)^{1/3-1}\,dt = \frac{1}{9^{1/3}}B(2/3,1/3)$$

$$= \frac{1}{9^{1/3}}\frac{\Gamma(2/3)\Gamma(1/3)}{\Gamma(1)} = \frac{1}{9^{1/3}}\cdot\frac{\pi}{\sin\pi/3}$$

$$= \frac{1}{9^{1/3}}\cdot\frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}\cdot 9^{1/3}}$$

EXAMPLE 14.7.27 Find the values of

$$\int_0^\infty e^{-ax}x^{m-1}\cos bx\,dx \text{ and } \int_0^\infty e^{-ax}x^{m-1}\sin bx\,dx,$$

$m > 0$ in terms of gamma function. Hence or otherwise show that

$$\int_0^\infty x^{m-1}\cos bx\,dx = \frac{\Gamma(m)\cos\left(\frac{m\pi}{2}\right)}{b^m}$$

$$\text{and } \int_0^\infty x^{m-1}\cos bx\,dx = \frac{\Gamma(m)\sin\left(\frac{m\pi}{2}\right)}{b^m}$$

Solution Let $I_1 = \int_0^\infty e^{-ax}x^{m-1}\cos bx\,dx$ and $I_2 = \int_0^\infty e^{-ax}x^{m-1}\sin bx\,dx$.

Now, let

$$I = I_1 + iI_2 = \int_0^\infty e^{-ax}x^{m-1}(\cos bx + i\sin bx)\,dx, \text{ where } i = \sqrt{-1},$$

$$= \int_0^\infty e^{-ax}x^{m-1}e^{ibx}\,dx = \int_0^\infty e^{-(a-ib)x}x^{m-1}\,dx$$

Putting $(a-ib)x = z$. Then $(a-ib)dx = dz$. When $x \to 0, z \to 0$ and when $x \to \infty, z \to \infty$. Therefore,

$$\begin{aligned}
I &= \int_0^\infty e^{-z} \frac{z^{m-1}}{(a-ib)^{m-1}} \cdot \frac{1}{a-ib}\, dz \\
&= \frac{1}{(a-ib)^m} \int_0^\infty e^{-z} z^{m-1}\, dz = \frac{\Gamma(m)}{(a-ib)^m} \\
&= \frac{\Gamma(m)\,(a+ib)^m}{(a^2+b^2)^m}
\end{aligned}$$

To separate real and imaginary parts of $(a+ib)^m$, let $a = r\cos\theta, b = r\sin\theta$, where $a^2+b^2 = r^2$ and $\theta = \tan^{-1}\frac{b}{a}$. Therefore, $(a+ib)^m = (r\cos\theta + ir\sin\theta)^m = r^m(\cos m\theta + i\sin m\theta)$.

Thus,

$$\begin{aligned}
I_1 + iI_2 &= \frac{\Gamma(m)(a^2+b^2)^{m/2}(\cos m\theta + i\sin m\theta)}{(a^2+b^2)^m} \\
&= \frac{\Gamma(m)(\cos m\theta + i\sin m\theta)}{(a^2+b^2)^{m/2}}
\end{aligned}$$

Separating real and imaginary parts, we get

$$\begin{aligned}
I_1 &= \int_0^\infty e^{-ax} x^{m-1} \cos bx\, dx = \frac{\cos m\theta\, \Gamma(m)}{(a^2+b^2)^{m/2}} \\
\text{and } I_2 &= \int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx = \frac{\sin m\theta\, \Gamma(m)}{(a^2+b^2)^{m/2}}
\end{aligned}$$

where $\theta = \tan^{-1}\dfrac{b}{a}$.

Second part: Substituting $a = 0$. Then $\theta = \pi/2$.

Therefore, from above integrals

$$\int_0^\infty x^{m-1} \cos bx\, dx = \frac{\Gamma(m)\cos\left(\frac{m\pi}{2}\right)}{b^m}$$

$$\text{and } \int_0^\infty x^{m-1} \cos bx\, dx = \frac{\Gamma(m)\sin\left(\frac{m\pi}{2}\right)}{b^m}$$

EXERCISES

Section A Multiple Choice Questions

1. If $\int_a^\infty f(x)\,dx$ converges then its value is

   (a) finite (b) infinite (c) oscillates (d) none of these

2. If $\int_a^\infty f(x)\,dx$ diverges then its value is

   (a) finite (b) infinite (c) oscillates (d) none of these

3. $\int_1^\infty \frac{dx}{x^p}$ converges only if

   (a) $p > 1$ (b) $p \leq 1$ (c) $p = 1$ (d) for all p

4. $\int_1^3 \frac{dx}{(x-1)^p}$ converges only if

   (a) $p < 1$ (b) $p \geq 1$ (c) $p = 1$ (d) for all p

5. The Cauchy principal value of $\int_{-1}^1 \frac{dx}{x^2}$ is

   (a) 0 (b) 2 (c) -2 (d) 3

6. The integral $\int_{-1}^1 \frac{dx}{x^2}$

   (a) converges (b) diverges

7. Let $f(x)$ be a function such that $f(x) \leq \frac{1}{x^2}$ for all $x \geq 1$. Then the integral $\int_1^\infty f(x)\,dx$

   (a) converges (b) diverges

8. If $|\phi(x)| < M$ for all x in $1 \leq x < \infty$, then $\int_1^\infty \frac{|\phi(x)|}{x^2}\,dx$ is

   (a) converges (b) diverges

9. If $f(x) \geq \frac{1}{\sqrt{x}}$ for all $x \geq a, a > 0$. Then $\int_a^\infty f(x)\,dx$

   (a) converges (b) diverges

10. If $\lim\limits_{x\to\infty} \frac{f(x)}{1/x^2}$ is a non-zero finite number, then $\int_a^\infty f(x)\,dx, a > 0$

    (a) converges (b) diverges

11. If $0 \leq f(x) \leq \frac{1}{\sqrt{x}}$ for all x in $[0, 1]$. Then the integral $\int_0^1 f(x)\,dx$

    (a) converges (b) diverges