NATIONAL INSTITUTE OF TECHNOLOGY DURGAPUR
DEPARTMENT OF MATHEMATICS

Course Materials for Multiple Integrals

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1 MULTIPLE INTEGRALS

Main Content:

1. Double integration, Triple integration in cartesian coordinates and polar coordinates.
2. Change of order of integration and change of variables.

Introduction: We already know that the line integrals $\int_a^b f(x)dx$ is introduced first for functions defined and bounded on finite intervals, and later for unbounded functions and infinite intervals. Now integrals $\iint f$ is generalized the concept of line integrals. The one-dimensional interval $[a,b]$ is replaced by a two-dimensional set R, called the region of integration. First we consider rectangular regions $[a,b]\times[c,d]$; later we consider more general regions with curvilinear boundaries. The integrand is a scalar field f defined and bounded on R. The resulting integral is called a double integral and is denoted by the symbol $\iint_R f\ dx\ dy$. Also, we shall find that most double integrals occurring in practice can be computed by repeated one-dimensional integration i.e. it enables us to evaluate certain double integrals by means of two successive one-dimensional integrations.

Double and Iterated Integrals: If f is a continuous function on a rectangle R,

$$\iint_R f(x,y)\, dA = \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx.$$

If we reverse the roles of x and y, we obtain

$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy.$$

Note: We define the double integral only for rectangular regions of integration. However, it is not difficult to extend the concept to more general regions. Let D be a bounded region, and enclose D in a rectangle R. Let f be defined and bounded on D. Define a new function $\bar{f}$ on R as follows :

$$\bar{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D; \\ 0 & \text{if } (x,y) \in R \setminus D. \end{cases}$$

In other words, extend the definition of f to the whole rectangle R by making the function values equal to 0 outside D. Now ask whether or not the extended function f is integrable on R. If so, we say that f is integrable on D and that, by definition,

$$\iint_D f(x,y)\,dx\,dy = \iint_R \bar{f}(x,y)\,dx\,dy.$$

Example: Compute the integral

$$\iint_D xy^2 dxdy$$

where D is the rectangle defined by $0 \le x \le 2$ and $0 \le y \le 1$.
Solution:

$$\int_0^1 \left(\int_0^2 xy^2 dx\right) dy = \int_0^1 2y^2 dy = \frac{2}{3}.$$

Geometry of the Double Integral: If $f(x,y) \ge 0$, then

$$\iint_R f(x,y)\,dx\,dy$$

is the volume of the region under the graph of f and above the rectangle R in the xy plane.

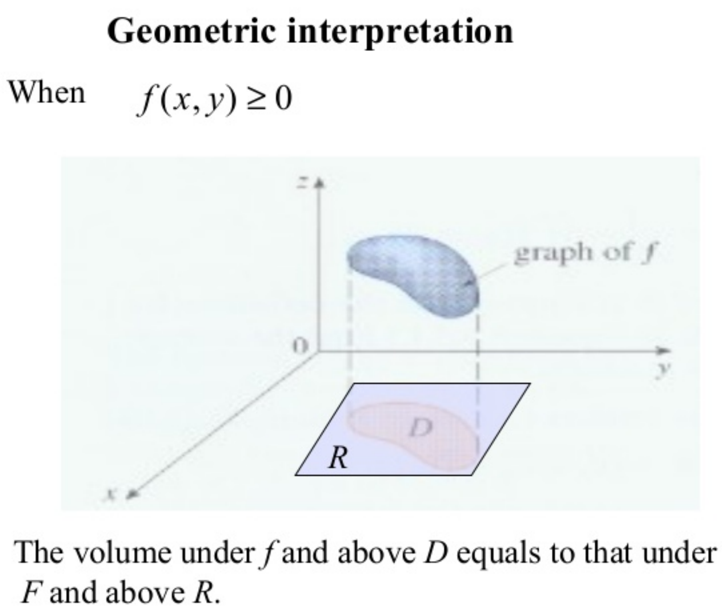


Figure 1.1: Geometry of the Double Integral

Theorem 1.1. ***(Integrability of continuous function)*** *If a function f is continuous on a rectangle R, then f is integrable on R.*

Now we will give a more general result on the existence of integrals on a rectangle R.

Theorem 1.2. ***(Existence of integrals)*** *If a function f is bounded on R and is continuous except possibly discontinuities along a finite number of graphs of continuous functions, then f is integrable on R.*

Properties of the double integrals: The following elementary properties of double integrals on rectangles are very much useful. The proofs are similar to those in one dimensional case, so omitted.

- If $f(x, y)$ be integrable on R, it is integrable on any sub-rectangle of R.
- If f be integrable on R, so are f^+, f^- and $|f|$.
- If f and g be integrable on R, so are $f+g$, $f-g$, fg, cf for some constant c and if $|g| \geq c$ for some constant $c > 0$; so is f/g.
- If f be integrable on R and if R_1 and R_2 be formed from R by cutting it with a line parallel to one of the co-ordinate axes, then

$$\iint_R f(x,y)dxdy = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy.$$

- If f and g be integrable on R and if $f \geq g$ on R, then

$$\iint_R f(x,y)dxdy \geq \iint_R g(x,y)dxdy.$$

- If $f(x, y) = k$ for all $(x, y) \in R$,

$$\iint_R f(x,y)dxdy = k(area\ of\ R).$$

Consider the case, where R is not a rectangle but a region bounded by a simple closed rectifiable curve C which is cut by any line parallel to either axis in at most two points. In this case, we can integrate using type I, type II and type III regions.

Region of Type I, Type II, and Type III: First we consider sets of points S in the xy-plane described as follows: $S = \{(x, y) : a \leq x \leq b \ and\ \phi_1(x) \leq y \leq \phi_2(x)\}$; where $\phi_1(x)$ and $\phi_2(x)$ are functions continuous on a closed interval $[a, b]$ and satisfying $\phi_1 \leq \phi_2$, which we call a region of Type I. In a similar manner we can define region of Type II. Type III can be describe both as region of type I and as region of type II region.

Theorem 1.3. ***(Iterated Integrals for elementary region)*** *Let D be a region of Type I, between the graphs of ϕ_1 and ϕ_2. Assume that f is defined and bounded on the interior D and that f is continuous on D. Then the double integral $\iint_D f$ exists and can be evaluated by repeated one-dimensional integration,*

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right] dx.$$

If D is region of type 2,

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy.$$

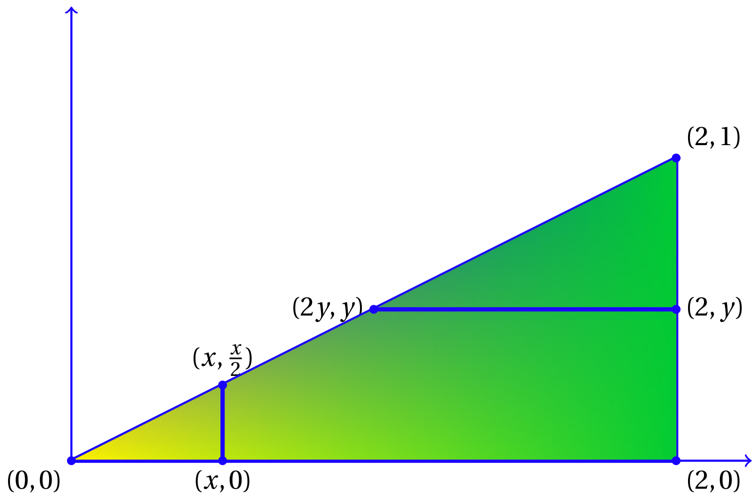


Figure 1.2: Find $\iint_D (xy^2)\,dx\,dy$.

Example: Find

$$\iint_D (xy^2)\,dx\,dy,$$

where D is the shaded region in Figure 1.2.

Solution: $\iint_D xy^2 dxdy = \int_0^2 \left(\int_0^{\frac{x}{2}} xy^2 dy\right) dx = \int_0^1 \left(\int_{2y}^2 xy^2 dx\right) dy = \frac{4}{15}$.

Fubini's Theorem: If f is integrable on $R = [a,b] \times [c,d]$, then either of the iterated integrals

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \text{ or } \int_a^b \int_c^d f(x,y)\,dy\,dx,$$

if it exists, equals the double integral

$$\iint_R f(x,y)\,dx\,dy.$$

Remark 1.1. *If f is continuous on $R = [a,b] \times [c,d]$, then both the iterated integrals*

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \text{ or } \int_a^b \int_c^d f(x,y)\,dy\,dx,$$

exist, equals the double integral

$$\iint_R f(x,y)\,dx\,dy.$$

Remark 1.2. *In Fubini's theorem, if both the iterated integrals exist then they are same. For example, if we take the integrable function x^y on the interval $R = [0,1] \times [0,1]$ then both the iterated integral $\int_0^1 \int_0^1 x^y\,dy\,dx$ and $\int_0^1 \int_0^1 x^y\,dx\,dy$ do exist. But the iterated integral*

$$\int_0^1 \int_0^1 x^y\,dx\,dy$$

can be easily calculated and is equal to $\log 2$; where as the other iterated integral can not be calculated easily (it is equal to $\ln 2$). So, by Fubini's theorem, we can easily calculate double integral.

Remark 1.3. *There may be the case, where both the iterated integrals of $f(x,y)$ in R exist but double integral does not exist. But this does not contradict the above theorem (Fubini's Theorem). As for example, both the iterated integrals*

$$\int_0^1 \int_0^1 \frac{y-x}{(2-x-y)^3}\,dy\,dx \quad \text{and} \quad \int_0^1 \int_0^1 \frac{y-x}{(2-x-y)^3}\,dx\,dy$$

exist but not same (Try to show!). From Fubini's Theorem, it can be concluded that the double integral of $\frac{y-x}{(2-x-y)^3}$ *over* $[0,1]\times[0,1]$ *does not exist.*
If we take the integrand function $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ *and note that*

$$\frac{x^2-y^2}{(x^2+y^2)^2} = -\frac{\partial^2}{\partial x \partial y}\arctan(y/x).$$

Now we use the above fact to evaluate the following two repeated integrals as

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \frac{\pi}{4}$$

and

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy = -\frac{\pi}{4}.$$

However, the corresponding double integral does not converge absolutely (in other words the integral of the absolute value is not finite):

$$\iint_{[0,1\times[0,1]} \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|\,dy\,dx = \infty.$$

Remark 1.4. *But, it may so happen that, one of the iterated integral does exist but other iterated integral and the double integral do not exist.*

$$f(x,y) = \begin{cases} \frac{1}{2}, y = \text{ rational} \\ x, y = \text{ irrational} \end{cases}$$

and $R = [0,1]\times[0,1]$.
Now, $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \frac{1}{2}$; *where as* $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ *and* $\iint_R f(x,y)dxdy$ *do not exist. Note that* $f(x,y)$ *is not continuous.*

Remark 1.5. *Let* $f : (-1,1]\times(-1,1] \to \mathbb{R}$ *be defined as*

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{otherwise}. \end{cases}$$

Then it is quite surprising to note that both the iterated integrals exist and same (equals to 0) but the double integral of $f(x,y)$ *does not exist as the function is unbounded near the point origin. [Hint: For evaluating the iterated integrals: anti-derivative of* $\frac{xy}{(x^2+y^2)^2}$ *is* $-\frac{x}{2(x^2+y^2)}$ *for fixed* x*.]*

Example 1.1. *Use Fubini's theorem to evaluate*

$$\int_0^\pi \int_0^1 \cos x \sin(y^2)\,dy\,dx.$$

We see the the given iterated integral is difficult to evaluate it, however, the integrand function is continuous and hence integrable. Therefore we can use Fubini's theorem to evaluate

$$\int_0^{\pi}\int_0^1 \cos x \sin(y^2)\, dy dx$$

in terms of other iterated integral

$$\begin{aligned}&\int_0^1\int_0^{\pi} \cos x \sin(y^2)\, dx dy\\ &= \int_0^1 \sin(y^2)\sin x|_{x=0}^{\pi} dy\\ &= \int_0^1 \sin(y^2)(0-0)\, dy = 0.\end{aligned}$$

Exercise 1.1. *Is*

$$\int_{-1}^2\int_0^6 x^2\sin(x-y)\, dx\, dy = \int_0^6\int_{-1}^2 x^2\sin(x-y)\, dy\, dx$$

correct? Justify your answer. [Hint. Correct]

Exercise 1.2. *Evaluate*

$$\iint_R y \sin xy\, dx\, dy$$

where $R = [1,2]\times[0.\pi]$. [Hint. Both the iterated integral exist and

$$\int_0^{\pi}\int_1^2 y\sin xy\, dx\, dy = \int_0^{\pi} (-\cos xy)_{x=1}^{x=2}\, dy = 0.$$

However, the other iterated integral is tough.]

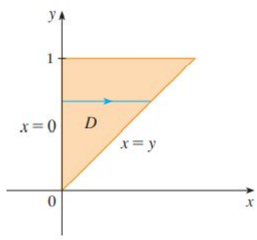
Example 1.2. *In order to evaluate*

$$\int_0^1\int_x^1 \sin(y^2)\, dy\, dx$$

we have be cautious. If we try to evaluate the integral as it stands, we are faced with the task of first evaluating. But it's impossible to do so in finite terms since $\int \sin(y^2)\, dy$ is not an elementary function. So we must change the order of integration i.e. we have to calculate other iterated integral. As the integrand function, is continuous both the iterated integrals will exist and same. We can write the given region $D = \{(x,y) : 0 \le x \le 1, x \le y \le 1\}$ as as Type-II region

$$D = \{(x,y) : 0 \le y \le 1, 0 \le x \le y\}.$$

Now



$$\int_0^1\int_x^1 \sin(y^2)\, dy\, dx = \int_0^1\int_0^y \sin(y^2)\, dx\, dy = \ldots = \frac{1}{2}(1-\cos 1).$$

Mean Value Theorem for Double Integrals: Suppose that $f : D \to \mathbb{R}$ is continuous and D is an elementary region. Then for some point (x_0, y_0) in D we have

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f(x, y)\, dA,$$

where $A(D)$ denotes the area of D.

Triple and Iterated Integrals: Let $f(x, y, z)$ be integrable on the box $B := [a, b] \times [c, d] \times [e, f]$. Then any iterated integral that exists is equal to the triple integral; that is,

$$\iiint_B f(x, y, z)\, dx\, dy\, dz = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz = \int_e^f \int_a^b \int_c^d f(x, y, z)\, dy\, dx\, dz,$$

and so on. (There are six possible orders altogether.)

Theorem 1.4. ***(Existence of integrals)*** *If a function f is bounded on B and is continuous except possibly discontinuities along a finite number of graphs of continuous functions [such as $x = \alpha(y, z)$, $y = \beta(x, z)$, or $z = \gamma(x, y)$], then f is integrable on B.*

Note 1. *Triple integral can be defined over a general bounded region E in three-dimensional space (a solid) by much the same procedure that we used for double integrals.*

The developments of triple integrals are analogous to that of double integrals. Students are referred to see the text book [1] for details.

Example 1.3. *We like to focus on elementary region E in case of triple integration like double integral. Let us take*

$$E = \{(x, y, z) : a \le x \le b, g_1(x) \le y \le g_2(x), \phi_1(x, y) \le z \le \phi_2(x, y)\}$$

where g_1, g_2, ϕ_1, ϕ_2 are continuous functions of x and x, y respectively. In this case

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1}^{g_2} \int_{\phi_1(x,y)}^{\phi_2(x,y)} dz\, dy\, dx.$$

On the other hand, there can be another type region as

$$E = \{(x, y, z) : c \le y \le d, h_1(y) \le x \le h_2(y), \phi_1(x, y) \le z \le \phi_2(x, y)\}$$

where h_1, h_2, ϕ_1, ϕ_2 are continuous functions of y and x, y respectively. In this case

$$\iiint_E f(x, y, z)\, dv = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} dz\, dx\, dy.$$

Example 1.4. *Express this as an elementary region: The region cut out of the ball $x^2 + y^2 + z^2 \le 4$ by the cylinder $2x^2 + z^2 = 1$; that is the region inside the cylinder and the ball.*

Example 1.5. *Consider*

$$\iiint_E z\, dV,$$

where V is the solid tetrahedron bounded by the four planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$. In order to understand the region as an elementary region, it's wise to draw two diagrams: one of the solid region (see Figure 5) and one of its projection D on the xy-plane (see Figure 6). The lower boundary of

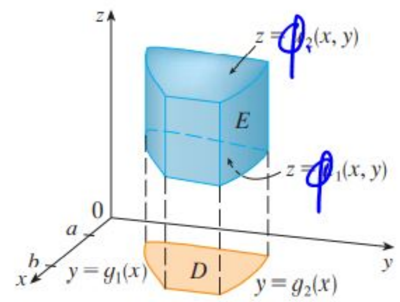


FIGURE 3
A type 1 solid region

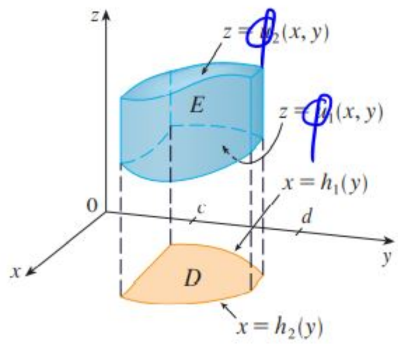


FIGURE 4
Another type 1 solid region

Figure 1.3: Elementary regions in space

the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ or $z = 1 - x - y$, so we take $\phi_1 = 0$ and $\phi_2 = 1 - x - y$. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ in the xy-plane. So the projection of E is the triangular region shown in Figure 6, and we have

$$E\{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}.$$

Now we can evaluate the given integral easily

$$\iiint_E z dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = \dots = \frac{1}{24}.$$

Change of Variables: For double integrals the rule for changing of variables is more complicated than that of in single integration. Suppose we have

$$\iint_D f(x, y)\,dx\,dy$$

and want to change the variables to u and v given by $x = x(u, v)$, $y = y(u, v)$. The change of variables formula is

$$\iint_D f(x, y)\,dx\,dy = \iint_{D*} f(x(u, v), y(u, v))|J|\,du\,dv$$

where J is the Jacobian, given by $J = x_u y_v - y_u x_v$ and D^* is the new region of integration, in the (u, v) plane.

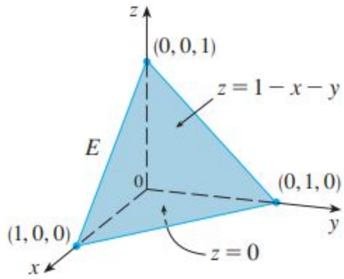


FIGURE 5

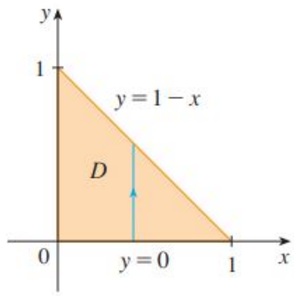


FIGURE 6

Example: Let us find

$$\iint_D (x+y)^2 \, dx \, dy$$

where D is the parallelogram bounded by the lines $x+y=0, x+y=1, 2x-y=0$ and $2x-y=3$. In this example, the boundary curves of D can suggest what substitution to use. So let us try $u=x+y, v=2x-y$. In these new variables the region D is described by $0\le u\le 1, 0\le v\le 3$. After calculation of the Jacobian J, we get $J=-\frac{1}{3}$ and then the given integral can be found easily and answer is $\frac{1}{3}$.
Example: Let us find

$$\iint_D (x+y) \, dx \, dy$$

where D is the trapezoidal region with vertices given by $(0,0), (5,0), (\frac{5}{2},\frac{5}{2})$, and $(\frac{5}{2},-\frac{5}{2})$ using the transformation $x=2u+3v, y=2u-3v$.
In these new variables the region D is described by $0\le u\le \frac{5}{4}, 0\le v\le \frac{5}{6}$. After calculation of the Jacobian J, we get $J=-12$ and then the given integral can be found easily and answer is $\frac{125}{4}$.
Example: Let us find

$$\iint_D \left(\frac{x-y}{x+y+2}\right)^2 dx \, dy$$

where D is the square region with vertices given by $(1,0), (-1,0), (0,1)$, and $(0,-1)$.
Clearly, the region is bounded by the lines

$$x+y=\pm 1, x-y=\pm 1.$$

In this example, the boundary curves of D can suggest what substitution to use. So let us try $u=x+y, v=x-y$. After calculation of the Jacobian J, we get $J=-\frac{1}{2}$ and then the given integral can be found easily and answer is $\frac{2}{9}$.
Example: Let us find

$$\iint_D (x^2-y^2) \, dx \, dy$$

where D is the square region with vertices given by $(0,0),(1,-1),(1,1)$, and $(2,0)$.
In this example, the boundary curves of D can suggest what substitution to use. So let us try $x = u+v, y = u-v$. After calculation of the Jacobian J, we get $J = -2$ and then the given integral can be found easily and answer is 2.
Changing coordinates in triple integrals: Here the coordinate change will involve three functions

$$u = u(x,y,z), v = v(x,y,z), w = w(x,y,z)$$

but the general principles remain the same. Jacobian will be

$$J = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}.$$

Example 1.6. *In order to evaluate*

$$\iint_D \frac{(x-y)^2}{x^2+y^2}\, dx\, dy$$

where $D = \{(x,y) : x^2+y^2 \le 1\}$, *we can use the polar coordinate transformation as*

$$x = r\cos\theta, y = r\sin\theta$$

where the revised region D^* *in* $r\theta$*-plane is* $[0,1]\times[0,2\pi]$. *Hence, we can find the above integral as*

$$\int_0^1 \int_0^{2\pi} \frac{r^2(\cos\theta - \sin\theta)^2}{r^2}.r\, d\theta\, dr = \int_0^1 r\, dr \int_0^{2\pi} (1-\sin(2\theta))\, d\theta$$

$$= \frac{1}{2}.(2\pi - 0) = \pi.$$

Alternatively, we can do this in terms of the elementary regions as

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - \frac{2xy}{x^2+y^2})\, dy\, dx = \pi - \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2xy}{x^2+y^2}\, dy\, dx = \pi - 0 = \pi.$$

Note that the function $f(x,y) = \frac{2xy}{x^2+y^2}$ *is an odd function (in the variable* y*) keeping* x *as constant.*

Caution: In the above integral, it can't be done by taking 4 times of

$$\int_0^1 \int_0^{\frac{\pi}{2}} \frac{r^2(\cos\theta - \sin\theta)^2}{r^2}.r\, d\theta\, dr$$

as the integrand function is not symmetric over all the four quadrants.
Cylindrical Coordinates: The cylindrical coordinates (r,θ,z) of a point (x,y,z) are defined by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, where $r \ge 0$ and $0 \le \theta < 2\pi$.

Example 1.7. *Evaluate*

$$\iiint_W (z^2x^2 + z^2y^2)\, dx\, dy\, dz,$$

where W *is the cylindrical region defined by* $x^2+y^2 \le 1, -1 \le z \le 1$, *we use cylindrical transformation. The transformed region is:* $0 \le r \le 1, 0 \le \theta \le 2\pi, -1 \le z \le 1$. *Answer is* $\frac{\pi}{3}$.

Example 1.8. *Evaluate*

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)^2\,dz\,dy\,dx$$

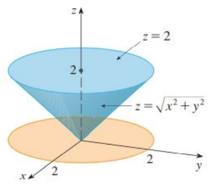
In this case we will use cylindrical transformation to evaluate this integral easily. This region has a much simpler description in cylindrical coordinates:

$$E = \{(r,\theta,z) : 0 \le r \le 2, 0 \le \theta \le 2\pi, r \le z \le 2\}.$$

So, it will be

$$\int_0^{2\pi}\int_0^2\int_r^2 r^4 r\,dz\,dr\,d\theta.$$

Now it is easy to calculate.



Spherical coordinates: The spherical coordinates of (x, y, z) are defined as follows:

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

where

$$\rho \ge 0, \;\; 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

We can easily calculate the Jacobian of the transformation as $J = \rho^2\sin\phi$.

Example 1.9. *Evaluate*

$$\iiint_W e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dx\,dy\,dz$$

where $W = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$. *We use spherical coordinate transformation*

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

where

$$\rho \ge 0, \;\; 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

The new region W^* *is described as*

$$0 \le \rho \le 1, 0 \le \theta < 2\pi, 0 \le \phi \le \pi.$$

Therefore,

$$\int_0^1\int_0^{2\pi}\int_0^{\pi} e^{\rho^3}\rho^2\sin\phi\,d\phi\,d\theta\,d\rho = \frac{4\pi}{3}(e-1).$$

Example 1.10. *Evaluate*

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx.$$

In this case from the region of iteration it is clear that we can use spherical coordinate transformation

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

The new region W^* is described as

$$0 \le \rho \le 1, 0 \le \theta < \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}.$$

Therefore,

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\rho^2}} \rho^2 \sin\phi d\phi d\theta d\rho = \ldots = \frac{\pi}{2} \int_0^1 \frac{\rho^2}{\sqrt{1-\rho^2}} d\rho = \frac{\pi}{2} \Big[\int_0^1 \frac{1}{\sqrt{1-\rho^2}} d\rho - \int_0^{\rho} \sqrt{1-\rho^2}\, d\rho \Big] = \ldots = \frac{\pi^2}{8}.$$

Example 1.11. *Use spherical coordinates or otherwise to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$. In order to find the volume, we first try to draw the picture as Figure 1.4. Note that the sphere passes through the origin and has center $(0,0,1/2)$. We*

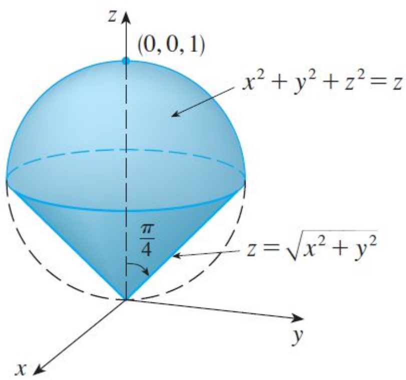


Figure 1.4: Find the volume of the shaded portion

write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho \cos\phi \;\; or \;\; \rho = \cos\phi.$$

The equation of the cone can be written as

$$\rho \cos\phi = \rho \sin\phi$$

and it will give $\phi = \pi/4$. Therefore the region in spherical coordinate system becomes

$$W^* = \{(\rho,\theta,\phi) : 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{4}, 0 \le \rho \le \cos\phi\}.$$

Hence the required volume is

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = .. = \frac{\pi}{8}.$$

Exercises

Problems on Double integrals:

1. Evaluate the double integrals by repeated integration, given that each integral exists.
   a) $\iint_D xy(x+y)dxdy,\ where\ D=[0,1]\times[0,1].$ Ans: $\frac{1}{3}$
   b) $\iint_D (x^3+3x^2y+y^3)dxdy,\ where\ D=[0,1]\times[0,1].$ Ans: 1
   c) $\iint_D sin^2xsin^2ydxdy,\ where\ D=[0,\pi]\times[0,\pi].$ Ans: $\frac{\pi^2}{4}$
   d) $\iint_D |cos(x+y)|dxdy,\ where\ D=[0,\pi]\times[0,\pi].$ Ans: 2π
   [Hint. We need to subdivide the square of integration D into parts where $\cos(x+y)$ has a single sign, i.e., we will divide it into four parts as Figure 1.5. Therefore, we can write the

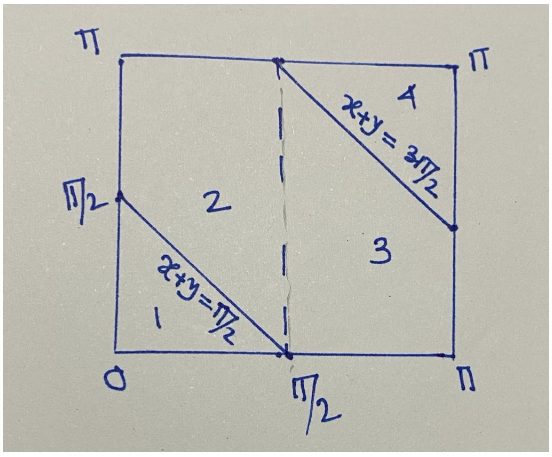


Figure 1.5: Subdivided region

given integral as

$$\begin{aligned}\iint_D |cos(x+y)|dxdy &= I_1+I_2+I_3+I_4\\ &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}-x}\cos(x+y)\,dydx - \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}-x}^{\pi}\cos(x+y)\,dydx\\ &\quad -\int_{\frac{\pi}{2}}^{\pi}\int_0^{\frac{3\pi}{2}-x}\cos(x+y)\,dydx + \int_{\frac{\pi}{2}}^{\pi}\int_{\frac{3\pi}{2}-x}^{\pi}\cos(x+y)\,dydx\\ &= (\frac{\pi}{2}-1)+(\frac{\pi}{2}+1)+(\frac{\pi}{2}+1)+(\frac{\pi}{2}-1)=2\pi.\end{aligned}$$

]
   e) $\iint_D f(x+y)dxdy,\ where\ D=[0,2]\times[0,2]$ and $f(t)$ denotes the greatest integer $\le t$. Ans: 6

2. Find the integral of $xe^x\sin(\frac{1}{2}\pi y)$ over the rectangle $[0,2]\times[-1,0]$. Ans: $\frac{-2(1+e^2)}{\pi}$

3. Evaluate $\iint_D (x^3+y^2x)\ dx\ dy$; where D is the region bounded under the graph $y=x^2$; from $x=0\ to\ x=2$. Ans: $\frac{64}{3}$

4. Find the volume under the graph of the mapping $f(x,y) = 1+2x+3y$, over the rectangle $[1,2] \times [0,1]$. Ans: $\frac{11}{2}$

5. Evaluate $\iint_D sec(x^2+y^2)\ dx\ dy$; where D is the region bounded by $x^2+y^2 \le 1$. Ans: $\pi.\ln(\sec 1 + \tan 1)$

6. By changing the order of integration evaluate the following integrals:

   a) $\int_0^{2a}\int_{\frac{y^2}{4a}}^{3a-y} (x^2+y^2)\,dx\,dy.$ Ans: $\frac{314a^4}{35}$

   b) $\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\,dx.$ Ans: 0.667

   c) $\int_0^1\int_{x^2}^{2-x} (x+y)\,dy\,dx.$ Ans: $\frac{89}{60}$

   d) $\int_1^4\int_1^{\sqrt{x}} (x^2+y^2)\,dy\,dx.$ Ans: $\frac{1934}{105}$

   e) $\int_0^1\int_{1-y}^1 (x+y^2)\,dx\,dy.$ Ans: $\frac{7}{12}$

   f) $\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{2}} \frac{\sin y}{y}\,dy\,dx$ Ans: 1

7. Compute
$$\iint_R (x^2+y)\,dA,$$
where R the square is $[0,1]\times[0,1]$. A consequence of the reduction to iterated integrals is that interchanging the order of the integration in the iterated integrals does not change the answer. Verify this. Ans: $\frac{5}{6}$

8. Find the volume of the tetrahedron bounded by the planes $x=0$, $z=0$, $y=0$, and $-x+y+z=1$. Ans: $\frac{1}{6}$

9. Find the volume under the graph of $f(x,y) = x^2+y^2$ between the planes $x=0$, $x=3$, $y=-1$, and $y=1$. Ans: 20

10. Find the average value of $f(x,y) = x^2 + sin2y + 1$ on D, where $D = [-3,1]\times[0,\pi]$. Ans: $\frac{40\pi}{3}$

11. Compute the volume of the solid below the graph $z = x^2 + y$ and lying above the rectangle $R = [0,1]\times[1,2]$. Ans: $\frac{11}{6}$

12. Evaluate the double integral
$$\iint_R (1-\frac{x^2}{a^2}-\frac{y^2}{b^2})\,dx\,dy,$$
where $R = \{(x,y) :\ x\ge 0,\ y\ge 0,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$. Ans: $\frac{\pi ab}{8}$

13. Find
$$\iint_D (1-\sin\pi x)y\,dx\,dy,$$
where D is bounded by $y=0$, $y=x$, $x=1$.

14. A lumberjack cuts out a wedge-shaped piece of a cylindrical tree of radius r obtained by making two saw cuts to the tree's centre, one horizontally and one at an angle θ. Compute the volume of the wedge W using Cavalieri's principle (see Figure-1)

    [Hint. First we like to find the area of the cross section A_x as $\frac{1}{2}b.h$. On the other hand, from the Figure 1.6, we get, $\tan\theta = \frac{h}{b}$ and $r^2 = x^2+b^2$. Therefore, $A_x = \frac{1}{2}b^2\tan\theta = \frac{1}{2}(r^2-x^2)\tan\theta$.

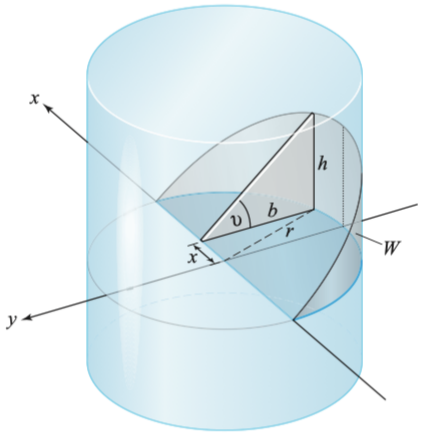


Figure 1.6: Find the volume of W

Now by Cavalieri's theorem, we can find the volume (V) of the wedge W as

$$V = \int_{-r}^{r} A_x \, dx = \frac{1}{2} \tan\theta \int_{-r}^{r} (r^2 - x^2) \, dx = \frac{2}{3} r^3 \tan\theta.$$

]

15. Evaluate

$$\iint_D (x^3 y + \cos x) \, dx \, dy,$$

where D is the triangle defined by $0 \le x \le \frac{\pi}{2}$, $0 \le y \le x$. Ans:1.82

a)

$$\int_0^1 \int_{x^2}^{x} (x+y)^2 \, dy \, dx.$$

Ans: $\frac{71}{420}$.

b)

$$\int_0^{\pi} \int_{\sin x}^{3\sin x} x(1+y) \, dy \, dx.$$

Ans: $\pi(2+\pi)$

c)

$$\int_{-1}^{1} \int_{y^{\frac{2}{3}}}^{(2-y)^2} (\frac{3}{2}\sqrt{x} - 2y) \, dx \, dy.$$

d)

$$\int_0^2 \int_{-\frac{2}{3}(\sqrt{4-x^2})}^{\frac{2}{3}(\sqrt{4-x^2})} (\frac{3}{2}\sqrt{x} - 2y) \, dy \, dx.$$

16. Compute the surface integral of x over the triangle in the space having the vertices $(1,1,1)$, $(2,1,1)$ and $(2,0,4)$.

Problems on triple integrals:

1. Verify the volume formula for the ball of radius 1:

$$\iiint_W dx\,dy\,dz = \frac{4}{3}\pi,$$

where W is the set of (x, y, z) with $x^2 + y^2 + z^2 \le 1$. [Hint. You can evaluate this triple integration by using elementary region. Another way: using spherical coordinate transformation. Check which one is easier.]

2. Evaluate

$$\iiint_W (x^2 + y^2 + z^2)\,dx\,dy\,dz,$$

where W is the region bounded by $x + y + z = a$ (where $a > 0$), $x = 0$, $y = 0$, and $z = 0$.

3. Determine the volume of the region that lies behind the plane $x + y + z = 8$ and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$. [Hint. First try to express the region as an elementary region. Limits for y, z, x are $0 \le y \le 4, \frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}; 0 \le x \le 8 - y - z$. Ans: $\frac{49}{5}$.]

4. Let W be the region bounded by the planes $x = 0$, $y = 0$, and $z = 2$, and the surface $z = x^2 + y^2$ and lying in the quadrant $x \ge 0$, $y \ge 0$. Compute

$$\iiint_W x\,dx\,dy\,dz$$

and sketch the region. Ans: $\frac{8\sqrt{2}}{15}$.

5. Evaluate

$$\int_0^1 \int_1^2 \int_2^3 \cos[\pi(x + y + z)]\,dx\,dy\,dz.$$

Ans: 0

6. Evaluate

$$\iiint_B e^{-xy}\,dx\,dy\,dz,$$

where $B := [0, 1] \times [0, 1] \times [0, 1]$.

7. Evaluate

$$\iiint_B (2x + 3y + z)\,dx\,dy\,dz,$$

where $B := [0, 2] \times [-1, 1] \times [0, 1]$. Ans: 10

8. Evaluate

$$\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz\,dy\,dx.$$

Ans: $\frac{1}{6}$

9. Evaluate

$$\iiint_W z\,dx\,dy\,dz,$$

where W is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = 1$, with $x \ge 0$, $y \ge 0$. item Evaluate

$$\int_{-\infty}^{\infty} e^{-10x^2}\,dx.$$

10. Evaluate

$$\iiint_S \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz$$

where S is the solid bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, where $b>a>0$. Ans: $2\pi\left(\frac{1+a^2}{e^{a^2}}-\frac{1+b^2}{e^{b^2}}\right)$.

11. Find the volume inside the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Ans: $\frac{4\pi}{3}abc$.

Miscellaneous:

1. If D is a plane defined by $1\le x\le 2$, $0\le y\le 1$ (measured in centimetres), and the mass density is $\rho(x,y)=ye^{xy}$ grams per square centimetre, integrate ρ over D to find the mass of the plane. Ans: $\frac{e^2}{2}-e+\frac{1}{2}=1.4762$ grams.

2. Write

$$\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}\,dy\,dx$$

as an integral over a region. Sketch the region and show that it is over type 1 and 2. Reverse the order of the integration and evaluate. Ans: $\frac{2}{3}$.

3. Evaluate the integral $\iint_D (x+y)\,dx\,dy$ where $D=\{(x,y):1\le x^2+y^2\le 4, x\le 0\}$. Draw the region. [Hint. Here suggested coordinate transformation is 'polar coordinate' transformation. The revised region is:

$$D^*=\left\{(r,\theta):1\le r\le 2,\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}\right\}$$

and the answer is $-\frac{14}{3}$.]

4. Find

$$\iint_D y(1-\cos(\frac{\pi x}{4}))\,dx\,dy,$$

where D is bounded by $x=0$, $y^2=x$, $y=2$. Ans: $4-\frac{16}{\pi^2}$

5. Evaluate $\iint_D \log(x^2+y^2)\ dx\,dy$, where D is the region in the first quadrant lying between the circles $x^2+y^2=1$ and $x^2+y^2=4$. Ans: $\frac{\pi}{2}(4\ln 2-3/2)$.

6. Evaluate

$$\iint_D (x^2+y^2)^{\frac{3}{2}}\,dx\,dy,$$

where D is the disk $x^2+y^2\le 4$.

7. Find the volume of the intersection of the ellipsoid $x^2+2y^2+2z^2\le 10$ and the cylinder $y^2+z^2\le 1$.

8. Evaluate

$$\iiint_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

where S is the solid bounded by the spheres $x^2+y^2+z^2\le a^2$ and $x^2+y^2+z^2\le b^2$, where $a>b>0$.

9. Evaluate the integral $\iint_D e^{\frac{x+y}{x-y}}\, dx dy$, where D is the trapezoidal region with vertices $(1,0),(2,0),(0,-2)$, and $(0,-1)$. [Hint. Since it isn't easy to integrate, we make a change of variables suggested by the form of this function:

$$u = x + y, v = x - y.$$

It's inverse transformation is

$$x = \frac{1}{2}(u+v), y = \frac{1}{2}(u-v)$$

and Jacobian of the transformation is

$$J = -\frac{1}{2}$$

and the revised region D^* will also be a trapezoidal region. It is

$$D^* = \{u, v) : 1 \le v \le 2, -v \le u \le v\}.$$

Therefore, the given integral becomes:

$$\iint_D e^{\frac{x+y}{x-y}}\, dx dy = \iint_{D^*} e^{\frac{u}{v}} |J|\, du dv = \int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \frac{1}{2}\, du dv = .. = \frac{3}{4}(e - e^{-1}).$$

10. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2 + y^2 \le 4$, $2 \le z \le 3$.

11. Use a double integral to find the area of the region enclosed by the lemniscate $r^2 = 4\cos(2\theta)$ [**Hint.** We know the area of the region $D \subset \mathbb{R}^2$ is given by the double integration

$$\iint_D 1.\, dx dy.$$

If we want to convert it into polar coordinates, we transform x, y into r, θ and accordingly D as D^* in polar coordinates. Or, if we have to find the area enclosed by some closed polar curve, we have to find

$$\iint_{D^*} r.\, dr\, d\theta.$$

In this case, we will find:

$$4 \int_0^{\frac{\pi}{4}} \int_0^{2\sqrt{\cos 2\theta}} r.\, dr\, d\theta$$

and the answer is 4].

12. Change the variable

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} xyz\, dz\, dx\, dy$$

in cylindrical coordinates.

13. Change the variable

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} xyz\, dz\, dx\, dy$$

in spherical coordinates.

14. Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$. Ans: $\frac{2a^3}{3}(\pi - 4/3)$.

15. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ by the cylinder $x^2 + y^2 = 3y$. Ans: $18\pi - 36$.

16. Use a triple integral to find the volume of the tetrahedron bounded by the planes $x = 2y$, $z = 0$, $x = 0$, and $x + 2y + z = 2$. Hint: $V = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\,dy\,dx = \ldots = \frac{1}{3}$.

Acknowledgement. *This part of the lecture note has been prepared and modified with the help of my PhD student Supriti Laha. Further, we use the following books as references.*

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[2] J. Stwart, Multivariable Calculus, Thomson Brooks/Cole, 2005.