SUPPLEMENTARY PROBLEMS

37. Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region defined by: (a) $y = \sqrt{x}$, $y = x^2$; (b) $x = 0$, $y = 0$, $x + y = 1$.
Ans. (a) common value = 3/2 (b) common value = 5/3

38. Evaluate $\oint_C (3x + 4y)\,dx + (2x - 3y)\,dy$ where C, a circle of radius two with centre at the origin of the xy plane, is traversed in the positive sense. *Ans.* -8π

39. Work the previous problem for the line integral $\oint_C (x^2 + y^2)\,dx + 3xy^2\,dy$. *Ans.* 12π

40. Evaluate $\oint (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy$ around the boundary of the region defined by $y^2 = 8x$ and $x = 2$ (a) directly, (b) by using Green's theorem. *Ans.* 128/5

41. Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy - y^2)\,dx + (3x^2 - 2xy)\,dy$ along the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.
Ans. $6\pi^2 - 4\pi$

42. Evaluate $\oint (3x^2 + 2y)\,dx - (x + 3\cos y)\,dy$ around the parallelogram having vertices at (0,0), (2,0), (3,1) and (1,1). *Ans.* -6

43. Find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a > 0$, and the x axis.
Ans. $3\pi a^2$

44. Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$, $a > 0$.
Hint: Parametric equations are $x = a\cos^3\theta$, $y = a\sin^3\theta$. *Ans.* $3\pi a^2/8$

45. Show that in polar coordinates (ρ, ϕ) the expression $x\,dy - y\,dx = \rho^2\,d\phi$. Interpret $\frac{1}{2}\int x\,dy - y\,dx$.

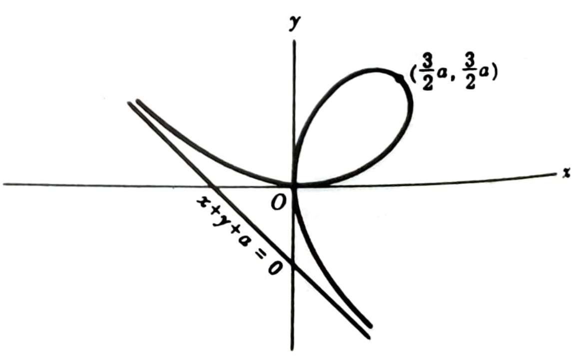
46. Find the area of a loop of the four-leafed rose $\rho = 3\sin 2\phi$. *Ans.* $9\pi/8$

47. Find the area of both loops of the lemniscate $\rho^2 = a^2\cos 2\phi$. *Ans.* a^2

48. Find the area of the loop of the folium of Descartes $x^3 + y^3 = 3axy$, $a > 0$ (see adjoining figure).
Hint: Let $y = tx$ and obtain the parametric equations of the curve. Then use the fact that

$$\text{Area} = \tfrac{1}{2}\oint x\,dy - y\,dx = \tfrac{1}{2}\oint x^2\,d\left(\tfrac{y}{x}\right) = \tfrac{1}{2}\oint x^2\,dt$$

Ans. $3a^2/2$



49. Verify Green's theorem in the plane for $\oint_C (2x - y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. *Ans.* common value = 60π

50. Evaluate $\int_{(1,0)}^{(-1,0)} \frac{-y\,dx + x\,dy}{x^2 + y^2}$ along the following paths:

(*a*) straight line segments from (1,0) to (1,1), then to (−1,1), then to (−1,0).
(*b*) straight line segments from (1,0) to (1,−1), then to (−1,−1), then to (−1,0).

Show that although $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the line integral *is dependent* on the path joining (1,0) to (−1,0) and explain.

Ans. (*a*) π (*b*) $-\pi$

51. By changing variables from (x,y) to (u,v) according to the transformation $x = x(u,v)$, $y = y(u,v)$, show that the area A of a region R bounded by a simple closed curve C is given by

$$A = \iint_R \left| J\left(\frac{x,y}{u,v}\right) \right| du\, dv \qquad \text{where} \qquad J\left(\frac{x,y}{u,v}\right) \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of x and y with respect to u and v. What restrictions should you make ? Illustrate the result where u and v are polar coördinates.

Hint: Use the result $A = \frac{1}{2}\int x\,dy - y\,dx$, transform to u,v coordinates and then use Green's theorem.

52. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F} = 2xy\,\mathbf{i} + yz^2\,\mathbf{j} + xz\,\mathbf{k}$ and S is :

(*a*) the surface of the parallelepiped bounded by $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 1$ and $z = 3$,
(*b*) the surface of the region bounded by $x = 0$, $y = 0$, $y = 3$, $z = 0$ and $x + 2z = 6$.

Ans. (*a*) 30 (*b*) 351/2

53. Verify the divergence theorem or $\mathbf{A} = 2x^2y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$. *Ans.* 180

54. Evaluate $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS$ where (*a*) S is the sphere of radius 2 with centre at (0,0,0), (*b*) S is the surface of the cube bounded by $x = -1$, $y = -1$, $z = -1$, $x = 1$, $y = 1$, $z = 1$, (*c*) S is the surface bounded by the paraboloid $z = 4 - (x^2 + y^2)$ and the xy plane. *Ans.* (*a*) 32π (*b*) 24 (*c*) 24π

55. If S is any closed surface enclosing a volume V and $\mathbf{A} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, prove that $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = (a+b+c)V$.

56. If $\mathbf{H} = \text{curl}\,\mathbf{A}$, prove that $\iint_S \mathbf{H}\cdot\mathbf{n}\, dS = 0$ for any closed surface S.

57. If $\mathbf{n}$ is the unit outward drawn normal to any closed surface of area S, show that $\iiint_V \text{div}\,\mathbf{n}\, dV = S$.

58. Prove $\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS$.

59. Prove $\iint_S r^5\,\mathbf{n}\, dS = \iiint_V 5r^3\,\mathbf{r}\, dV$.

60. Prove $\iint_S \mathbf{n}\, dS = \mathbf{0}$ for any closed surface S.

61. Show that Green's second identity can be written $\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S (\phi\frac{d\psi}{dn} - \psi\frac{d\phi}{dn})\,dS$

62. Prove $\iint_S \mathbf{r}\times d\mathbf{S} = \mathbf{0}$ for any closed surface S.

63. Verify Stokes' theorem for $\mathbf{A} = (y-z+2)\mathbf{i} + (yz+4)\mathbf{j} - xz\,\mathbf{k}$, where S is the surface of the cube $x=0$, $y=0$, $z=0$, $x=2$, $y=2$, $z=2$ above the xy plane. *Ans.* common value $= -4$

64. Verify Stokes' theorem for $\mathbf{F} = xz\,\mathbf{i} - y\,\mathbf{j} + x^2y\,\mathbf{k}$, where S is the surface of the region bounded by $x=0$, $y=0$, $z=0$, $2x+y+2z = 8$ which is not included in the xz plane. *Ans.* common value $= 32/3$

65. Evaluate $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$, where $\mathbf{A} = (x^2+y-4)\mathbf{i} + 3xy\,\mathbf{j} + (2xz+z^2)\mathbf{k}$ and S is the surface of (*a*) the hemisphere $x^2+y^2+z^2 = 16$ above the xy plane, (*b*) the paraboloid $z = 4-(x^2+y^2)$ above the xy plane.
Ans. (*a*) -16π, (*b*) -4π

66. If $\mathbf{A} = 2yz\,\mathbf{i} - (x+3y-2)\mathbf{j} + (x^2+z)\mathbf{k}$, evaluate $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$ over the surface of intersection of the cylinders $x^2+y^2 = a^2$, $x^2+z^2 = a^2$ which is included in the first octant. *Ans.* $-\frac{a^2}{12}(3\pi+8a)$

67. A vector $\mathbf{B}$ is always normal to a **given** closed surface S. Show that $\iiint_V \operatorname{curl}\mathbf{B}\,dV = \mathbf{0}$, where V is the region bounded by S.

68. If $\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \mathbf{H}\cdot d\mathbf{S}$, where S is any surface bounded by the curve C, show that $\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}$.

69. Prove $\oint_C \phi\,d\mathbf{r} = \iint_S d\mathbf{S}\times\nabla\phi$.

70. Use the operator equivalence of Solved Problem 25 to arrive at (*a*) $\nabla\phi$, (*b*) $\nabla\cdot\mathbf{A}$, (*c*) $\nabla\times\mathbf{A}$ in rectangular coordinates.

71. Prove $\iiint_V \nabla\phi\cdot\mathbf{A}\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS - \iiint_V \phi\nabla\cdot\mathbf{A}\,dV$.

72. Let $\mathbf{r}$ be the position vector of any point relative to an origin O. Suppose ϕ has continuous derivatives of order two, at least, and let S be a closed surface bounding a volume V. Denote ϕ at O by ϕ_o. Show that

$$\iint_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right]\cdot d\mathbf{S} = \iiint_V \frac{\nabla^2\phi}{r}\,dV + \alpha$$

where $\alpha = 0$ or $4\pi\phi_o$ according as O is outside or inside S.

73. The potential $\phi(P)$ at a point $P(x,y,z)$ due to a system of charges (or masses) $q_1, q_2, \ldots, q_n$ having position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ with respect to P is given by

$$\phi = \sum_{m=1}^{n} \frac{q_m}{r_m}$$

Prove *Gauss' law*

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = 4\pi Q$$

where $\mathbf{E} = -\nabla\phi$ is the electric field intensity, S is a surface enclosing all the charges and $Q = \sum_{m=1}^{n} q_m$ is the total charge within S.

74. If a region V bounded by a surface S has a continuous charge (or mass) distribution of density ρ, the potential $\phi(P)$ at a point P is defined by $\phi = \iiint_V \frac{\rho\,dV}{r}$. Deduce the following under suitable assumptions:

(*a*) $\iint_S \mathbf{E}\cdot d\mathbf{S} = 4\pi\iiint_V \rho\,dV$, where $\mathbf{E} = -\nabla\phi$.

(*b*) $\nabla^2\phi = -4\pi\rho$ (Poisson's equation) at all points P where charges exist, and $\nabla^2\phi = 0$ (Laplace's equation) where no charges exist.