Lecture3: Charpit's Method

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Charpit's method

Consider a first order PDE of the form

$$f(x, y, z, p, q) = 0, \tag{1}$$

where f is a given function and the objective is to find it's solution. To achieve this we first note that

$$dz = pdx + qdy \tag{2}$$

and try to find another relation of the form

$$g(x, y, z, p, q) = 0, \tag{3}$$

such that the relation (2) becomes integrable when the expressions of p and q derived from (1) and (3) are substituted in (2). The integral of (2) will then satisfy (1), because p and q are obtained from (1).

Differentiating the equations (1) and (3) partially with respect to x and y respectively we get,

$$\begin{aligned}
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} &= 0, \quad (4)\\
\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z} p + \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial x} &= 0, \quad (5)\\
\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} &= 0, \quad (6)\\
\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} q + \frac{\partial g}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial y} &= 0. \quad (7)
\end{aligned}$$

Eliminating $\frac{\partial p}{\partial x}$ from (4) and (5) we get

$$\left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial p}\right] + p\left[\frac{\partial f}{\partial z}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial p}\right] + \frac{\partial q}{\partial x}\left[\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right] = 0. \quad (8)$$

Similarly eliminating $\frac{\partial q}{\partial y}$ from the equations (6) and (7) we get

$$\left[\frac{\partial f}{\partial y}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial q}\right] + q\left[\frac{\partial f}{\partial z}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial q}\right] + \frac{\partial p}{\partial y}\left[\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\right] = 0. \quad (9)$$

and note that $\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y}$.

Now adding the equations (8) and (9) we get

$$\begin{aligned}\left[\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right]\frac{\partial g}{\partial p} \quad &+ \quad \left[\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right]\frac{\partial g}{\partial q} + \left[-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right]\frac{\partial g}{\partial z} \\ &+ \quad \left[-\frac{\partial f}{\partial p}\right]\frac{\partial g}{\partial x} + \left[-\frac{\partial f}{\partial p}\right]\frac{\partial g}{\partial x} = 0. \qquad (10)\end{aligned}$$

Now following the similar type of calculation as was done for deriving Lagrange's subsidiary equations we find the following simultaneous ordinary differential equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}, \qquad (11)$$

which is known as **Charpit's auxiliary equations**.

Any integral of equations (11) will involve p, q or both. Now from a suitable integral of (11) and equation (1) one can find p and q which are then substituted in equation (2). The resulting equation is then integrated to find the complete integral of equation (1).

Example 1

Use Charpit's method to solve the equation $(p^2 + q^2)y = qz$.

Solution

Given

$$f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0.$$

$\therefore$ Charpit's auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\implies \frac{dp}{-qp} = \frac{dq}{p^2 + q^2 - q^2} = \frac{dz}{-p(2py) - q(2qy - z)} = \frac{dx}{-2py} = \frac{dy}{-2qy + z}$$

From the first two terms we get

$$pdp + qdp = 0.$$

Integrating we get the

$$p^2 + q^2 = c^2,$$

where c is an arbitrary constant.
Putting this in the given equation we get,

$$q = \frac{c^2 y}{z}.$$

Therefore,

$$p^2 = c^2 - \frac{c^4 y^2}{z^2}.$$

Now $dz = pdx + qdy = \frac{c}{z}\sqrt{z^2 - c^2 y^2}dx + \frac{c^2 y}{z}dy.$

$$\text{Or, } \frac{zdz - c^2 ydy}{\sqrt{z^2 - c^2 y^2}} = cdx$$

.

Integrating we get,

$$\sqrt{z^2 - c^2y^2} = cx + d,$$

where c and d are arbitrary constants.

Therefore the complete integral of the given PDE is

$$z^2 = (xc + d)^2 + c^2y^2.$$

Example 2

Use Charpit's method to solve the equation $pq = 1$.

Solution

$$\text{Here} \qquad f(x, y, z, p, q) = f(p, q) = pq - 1 = 0$$

$$\implies \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = q, \frac{\partial f}{\partial q} = p.$$

$\therefore$ The Charpit's auxiliary equations are given by

$$\begin{aligned}
\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} &= \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \\
\implies \frac{dp}{0} &= \frac{dq}{0} = \frac{dz}{2pq} = \frac{dx}{q} = \frac{dy}{p}
\end{aligned}$$

$$\Rightarrow \frac{dx}{q} = \frac{dy}{p} = \frac{dz}{2pq} = \frac{dp}{0} = \frac{dq}{0}$$

$$\Rightarrow dp = 0$$

$$\Rightarrow p = constant = a, \text{(say)}$$

Now from the given relation

$$pq = 1 \Rightarrow q = \frac{1}{p} = \frac{1}{a}$$

$$\text{Again,} \quad dz = pdx + qdy = adx + \frac{1}{a}dy$$

$$\text{Integrating,} \quad z = ax + \frac{y}{a} + b,$$

which is the complete integral of the given PDE.

Example 3

Using Charpit's method, find the complete integral of the equation

$$2(z + xp + yq) = yp^2.$$

Solution

Here $f(x, y, z, p, q) = 2(z + xp + yq) - yp^2$.

$$\implies \frac{\partial f}{\partial x} = 2p, \frac{\partial f}{\partial y} = 2q - p^2, \frac{\partial f}{\partial z} = 2, \frac{\partial f}{\partial p} = 2x - 2py, \frac{\partial f}{\partial q} = 2y.$$

Now Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2p + 2p} = \frac{dq}{2q - p^2 + 2q} = \frac{dz}{-p(2x - 2yp) - 2qy} = \frac{dx}{-2x + 2yp} = \frac{dy}{-2y}.$$

Considering the first and fifth terms of the above equations we get

$$\frac{dp}{p} + 2\frac{dy}{y} = 0.$$

Integrating, we get

$$p = \frac{c}{y^2}.$$

Substituting p in the given equation, we get

$$q = -\frac{z}{y} - \frac{cx}{y^3} + \frac{c^2}{2y^4}.$$

Now

$$dz = pdx + qdy = \frac{c}{y^2}dx - \frac{z}{y}dy - \frac{cx}{y^3}dy + \frac{c^2}{2y^4}dy$$

$$\implies ydz + zdy = c\left(\frac{ydx - xdy}{y^2}\right) + \frac{c^2}{2y^3}dy.$$

Integrating we get,

$$yz = \frac{cx}{y} - \frac{c^2}{4y^2} + d,$$

where c and d are arbitrary constants. Therefore the complete integral of the PDE is

$$4y^3z = 4dy^2 + 4cxy - c^2.$$

Two special cases

(i) PDEs involving p and q i.e. $f(p,q)=0$.

For such a case, the Charpit's auxiliary equations leads to

$$\frac{dp}{0}=\frac{dq}{0}=\frac{dz}{pf_p+qf_q}=\frac{dx}{f_p}=\frac{dy}{f_q}.$$

$$\Rightarrow p=a, \text{constant}$$

From the given equation $f(p,q)=0$, one may get $q=Q(a)$.
Now, putting $p=a$, $q=Q(a)$ in the equation $dz=pdx+qdy$ we get $dz=adx+Q(a)dy$, which leads to the complete integral

$$z=ax+Q(a)y+b,$$

where a, b are arbitrary constants.

Exercise

Find the complete integrals of the following PDE using Charpit's method

- $p^2q^2 = 1$
- $pq^3 = 5$
- $p^5q^9 = k$
- $f(p, q) = p + q - pq$
- $\sqrt{p} + \sqrt{q} = 1$
- $p^2 + q^2 = npq$, n is a real constant.
- $3p^2 - 2q^2 = 4pq$
- $p^2 + q^2 = n^2$, n is a real constant.

(ii) PDEs not involving independent variables i.e. $f(z, p, q) = 0$

For such a case, the Charpit's auxiliary equations can be written as

$$-\frac{dp}{pf_z} = -\frac{dq}{qf_z} = \frac{dz}{pf_p + qf_q} = \frac{dx}{f_p} = \frac{dy}{f_q}$$

From the first two terns we get $p = aq$, where a is an arbitrary constant. Substituting $p = aq$ in the given equation we get

$$q = Q(a, z) \implies p = aQ(a, z)$$

Now substituting p and q into

$$\begin{aligned} dz &= pdx + qdy \\ \Rightarrow dz = aQ(a,z)dx + Q(a,z)dy \Rightarrow \frac{dz}{Q(a,z)} &= adx + dy \\ \Rightarrow \int \frac{dz}{Q(a,z)} &= ax + y + b. \end{aligned}$$

The last equation gives the complete integral.

Exercise

Find the complete integrals of the following PDE using Charpit's method

- $p^2 = z^2(1 - pq)$
- $p^3 + q^3 = 27z$
- $(p^2 + q^2)y - qz = 0$
- $x^2p^2 + y^2q^2 - 4 = 0$
- $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$
- $2(z + xp + yq) = yp^2$
- $p^2x + q^2y = z$
- $2x(z^2q^2 + 1) = pz$
- $xpq + yq^2 = 1$
- $z = px + qy + log\, pq$
- $z^2(1 + p^2 + q^2) = 1$
- $z(p^2 + q^2) + px + qy = 0$
- $z + xp - x^2yq^2 - x^3pq = 0$
- $2z + p^2 + qy + 2y^2 = 0$
- $pxy + pq + qy = yz$
- $z(p^2 + q^2) + px + qy = 0$