Complex Analysis

Pinaki Pal

Department of Mathematics
National Institute of Technology Durgapur
West Bengal, India

pinaki.pal@maths.nitdgp.ac.in

Syllabus and books

Syllabus

Functions of complex variable, Limit, Continuity and Derivative; Analytic function; Harmonic function; Complex integration; Cauchy's integral theorem; Cauchy's integral formula; Taylor's theorem, Laurent's theorem (Statement only); Singular points and residues; Cauchy's residue theorem.

Reference

- Engineering Mathematics- Babu Ram
- Engineering Mathematics (Oxford University Press)- S. Pal and S.C. Bhunia
- Advanced Engineering Mathematics- E. Kreyszig

Complex numbers

- A complex number z is an ordered pair (x, y) of real numbers x and y and the set of all complex numbers is denoted by $\mathbb{C}$, i.e.,
$$\mathbb{C} = \{z = (x, y) : x, y \in \mathbb{R}\}.$$
- The sets $\mathbb{C}$ and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ are same as sets, but the algebra in these sets are different.
- For two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, the addition and multiplication are defined as
$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\ z_1 z_2 &= (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \end{aligned}$$
- The complex number $(x, 0)$ is simply denoted by x (indeed, a real number).
- Let $i = (0, 1)$. Then $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$ and so we can write $i = \sqrt{-1}$ (notation).
- Then $z = (x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)(y, 0) = x + iy$.
- The number x and y are called the real and imaginary parts of z and we write $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$

Geometric representation: polar form

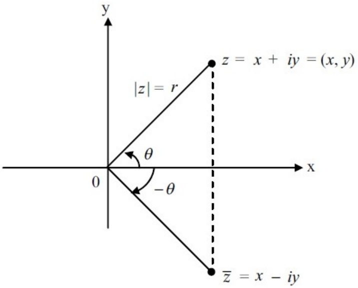

Figure: Argand diagram

- Let $z = x + iy \neq 0$ and $x = r\cos\theta, y = r\sin\theta$. Then

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \qquad \text{(Euler formula)}$$

- The **modulus** (or absolute value) of z is $|z| = r = \sqrt{x^2 + y^2}$.
- The angle θ is called the **amplitude** or **argument** of the complex number z and denoted by $\arg z = \theta$ and we have $\tan\theta = \frac{y}{x}$.

- If α is an argument of an complex number z then $\alpha + 2k\pi, k \in \mathbb{Z}$ is also an argument of the same complex number z.
- Among infinitely many values of θ, the one which lies in $(-\pi, \pi]$ is called the principal argument of z and is denoted by $\operatorname{Arg} z$.
- The **conjugate** of a complex number z is defined by $\overline{z} = x - iy$.

Some properties of complex numbers

1. $|z| = 0 \iff z = 0$
2. $z\overline{z} = |z|^2$
3. $|z| = |\overline{z}|$
4. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
5. $|z_1 z_2| = |z_1||z_2|$
6. $\arg z^n = n \arg z$
7. $|z_1 - z_2|$ represent the distance between the complex numbers z_1 and z_2.
8. $|z - z_0| = r$ represent a circle with center at z_0 and radius r.

Topology of complex plane: Some definition

1. Neighbourhood: Let z_0 be a point in the complex plane. Then the set of all points z such that $|z - z_0| < \delta$ where $\delta > 0$ is called neighbourhood or δ-neighbourhood of z_0.
2. Interior point: A point z_0 is called an interior point of set S if there exists a neighbourhood of z_0 lying wholly in S.
3. Open set: A set S is said to be open if every point of S is an interior point.
4. Limit point: A point z_0 is called a limit point of a point set S if every neighbourhood of z_0 contains at least one point of S other than z_0.
5. Closure: The union of a set S and the set of its limit points is called the closure of S and is denoted by $\overline{S}$.
6. Closed set: A set S is said to be closed if it contains all of its limit points.
7. A set S is said to be closed if and only if its complement S^c is open.
8. Bounded set: A set S is said to be bounded if there exist $M > 0$ such that $|z| \leq M$ for all $z \in M$ i.e., S is contained in some disk of radius M.

Connected Set

1. Connected Set: A set S is said to be connected if there do not exist two non-empty disjoint open sets A and B such that $S \subseteq A \cup B,\ A \cap S \neq \phi,\ B \cap S \neq \phi$.
2. Domain: An open connected set is called a domain.
3. Any two points in a domain can be joined by a polygonal line that lies in the domain.

Example:

$$A = \{z \in \mathbb{C} : |z| < 1\}, \quad B = \{z \in \mathbb{C} : |z - 2| \leq 1\}, \quad S = A \cup B$$

A is open and connected set. B is closed and connected set. S is neither open nor closed but S is connected.

Complex Functions

Functions of a complex Variable: Let $D \subset \mathbb{C}$. A function f defined on D is a rule that assigns a complex number w to each complex number z in D.

$$w = f(z) = u + iv \Longleftrightarrow w = f(x + iy) = u(x, y) + iv(x, y)$$

If only one value of w corresponds to each value of z, we say that $w = f(z)$ is a single-valued function of z or that $f(z)$ is single valued.

If more than one value of w corresponds to a value of z, then $f(z)$ is called multiple-valued or many-valued function of z.

Example: Let $w = f(z) = z^2$.

Then $f(z) = (x + iy)^2 = (x^2 - y^2) + i2xy$. Here $u(x, y) = x^2 - y^2$ and $v = 2xy$.

Limit

Limit of a functions of a complex variable: Let $f(z)$ be defined and single valued in a deleted nbd of z_0. The function $f(z)$ is said to have the limit l as z approaches z_0 if for given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - l| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

We then write $\lim_{z \to z_0} f(z) = l$. Here the limit is independent of the direction of approach of z to z_0.

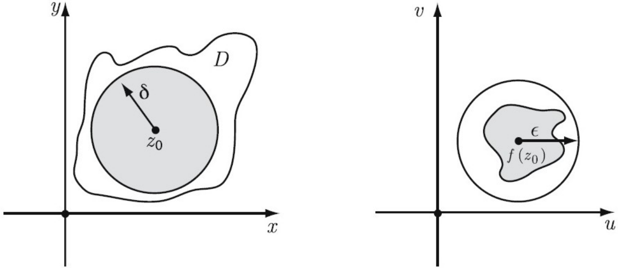

Figure: limit of a function

Limit

Limit in terms of its real and imaginary parts of a complex functions:

Let $f(z) = u(x, y) + iv(x, y)$, $l = l_1 + il_2$ and $z_0 = x_0 + iy_0$. Then

$$\lim_{z \to z_0} f(z) = l \iff \lim_{(x,y)\to(x_0,y_0)} u(x, y) = l_1 \quad \& \lim_{(x,y)\to(x_0,y_0)} v(x, y) = l_2.$$

Example: $\lim_{z\to 1+2i} |z|^2 = \lim_{(x,y)\to(1,2)} (x^2 + y^2) = 5$

Example:

$$\lim_{z\to 1+3i} \frac{z^2 - 3z + 1}{z - 1} = \lim_{z\to 1+3i} \frac{(z - 1)(z - 2)}{z - 1} = \lim_{z\to 1+3i} (z - 2) = -1 + 3i.$$

Example: $\lim_{z\to 0} \frac{\overline{z}}{z}$ does not exist.

Along x-axis ($y = 0$), we have $\lim_{z\to 0} \frac{\overline{z}}{z} = \lim_{x\to 0} \frac{x}{x} = 1.$

Along y-axis ($x = 0$), we have $\lim_{z\to 0} \frac{\overline{z}}{z} = \lim_{y\to 0} \frac{-iy}{iy} = -1.$

Continuity

Continuous functions: Let $f(z)$ be defined and single valued in a nbd of z_0. The function $f(z)$ is said to be continuous at z_0 if for given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

Alternatively, the function $f(z)$ is said to be continuous at z_0 if $\lim_{z \to z_0} f(z)$ exist and is equal to $f(z_0)$.

Theorem

A function $f(z) = u(x,y) + iv(x,y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if the functions $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0).

Continuity

Example: Let $f(z) = z^2 + 1 = (x^2 - y^2 + 1) + i2xy$.

Then

$$\lim_{z \to i}(z^2 + 1) = 0 = f(i).$$

Thus $f(z)$ is continuous at $z = i$.

Example: The signum function defined by

$$f(z) = \begin{cases} \frac{|z|}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous in $\mathbb{C} \setminus \{0\}$.

Solution: Let $z_0 \neq 0$. Then

$$\lim_{z \to z_0} f(z)$$

But

$$\lim_{z \to 0} \frac{|z|}{z} = \begin{cases} 1 & \text{when} \quad z = x + i.0 \ \ \& \ \ x \to 0^+ \\ -1 & \text{when} \quad z = x + i.0 \ \ \& \ \ x \to 0^- \end{cases}$$

Thus f is not continuous at $z = 0$.

Differentiability

Differentiable functions: A function $f : D \to \mathbb{C}$ is said to be differentiable at $z_0 \in D$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{or} \quad \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists and it is denoted by $f'(z_0)$.

If f is differentiable at each point of D, we say that f is differentiable in D.

Theorem

If $f : D \to \mathbb{C}$ is differentiable at $z_0 \in D$, then f is continuous at z_0.

Proof:

$$\begin{aligned}\lim_{z \to z_0}(f(z) - f(z_0)) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.(z - z_0) \\ &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \lim_{z \to z_0}(z - z_0) \\ &= f'(z_0).0 = 0.\end{aligned}$$

Thus f is continuous.

Differentiability

Example: Let $f(z) = z^2, z \in \mathbb{C}$. Then

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2zh}{h} = \lim_{h\to 0} (h + 2z) = 2z \end{aligned}$$

Example: The function $f(z) = \overline{z}$ is continuous everywhere but not differentiable at any point.

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h} \to \begin{cases} 1 & \text{if } h \to 0 \text{ along real axis} \\ -1 & \text{if } h \to 0 \text{ along imaginary axis} \end{cases}$$

Thus f is not differentiable at any point. But clearly f is continuous at all points.

Differentiability

Example: Let $f(z) = |z| = \sqrt{x^2 + y^2}, z \in \mathbb{C}$. Then f is continuous on $\mathbb{C}$ but not differentiable at the origin.

Clearly, $\lim\limits_{z \to 0} f(z) - f(0) = 0$. But,

$$\frac{f(z) - f(0)}{z - 0} = \frac{|z|}{z} \to \begin{cases} 1 & \text{if } z = x > 0,\ x \to 0^+ \\ -1 & \text{if } z = x < 0,\ x \to 0^+ \\ -i & \text{if } z = iy,\ y \to 0^+ \\ i & \text{if } z = iy,\ y \to 0^- \end{cases}$$

Thus f is not differentiable at $z = 0$.

Theorem

Let $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ be two differentiable function. Then

(i) $(f \pm g)' = f' \pm g'$;

(ii) $(fg)' = f'g + fg'$;

(iii) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},\ g \neq 0$;

(iv) $[f(g(z))]' = f'(g(z)).g'(z)$;

Theorem

A real valued function of a complex variable either has derivative zero or the derivative does not exist.

Proof: Suppose that $f : D \to \mathbb{R}$ is differentiable at $z_0 \in D$. Then

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exist. If $h \to 0$ along real axis, then $f'(z_0)$ is purely real and if $h \to 0$ along imaginary axis, then $f'(z_0)$ is purely imaginary. This is possible only when $f'(z_0) = 0$.

Example: Show that $f(z) = Re(z)$ is nowhere differentiable.

Hint:

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{Re(h)}{h} = \begin{cases} 1 & \text{for } h = h_1 + i.0 \in \mathbb{R} \backslash \{0\} \\ 0 & \text{for } h = 0 + i.h_2 \in i\,\mathbb{R} \backslash \{0\} \end{cases}$$

Example: Show that the functions $Im(z),\ \bar{z},\ \mathrm{Arg}\,(z)$ is nowhere differentiable.

Cauchy-Riemann equation

Theorem (Necessary condition for derivative)

If $f(z) = u + iv$ is differentiable at z_0, then $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$ exists at z_0 and satisfy the Cauchy-Riemann(C-R) equation at z_0, i.e,

$$f_y(z_0) = if_x(z_0) \quad \textit{or, equivalently} \quad u_x(z_0) = v_y(z_0) \;\; \& \;\; u_y(z_0) = -v_x(z_0).$$

Proof: Let $f'(z_0)$ exists finitely. Then

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists and is independent of the path along which $h = h_1 + ih_2$ approaches to 0. In particular, along x-axis we have

$$\begin{aligned} f'(z_0) &= \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1 + i.0} = f_x(z_0) \\ &= \lim_{h_1 \to 0} \frac{u(x_0 + h_1, y_0) - u(x_0, y_0)}{h_1} + i \lim_{h_1 \to 0} \frac{v(x_0 + h_1, y_0) - v(x_0, y_0)}{h_1} \\ &= u_x(z_0) + iv_x(z_0) \end{aligned} \tag{1}$$

Cauchy-Riemann equation

Again, along y-axis we have

$$
\begin{aligned}
f'(z_0) &= \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{0 + i.h_2} = \frac{1}{i} f_y(z_0) \\
&= \frac{1}{i} \lim_{h_2 \to 0} \frac{u(x_0, y_0 + h_2) - u(x_0, y_0)}{h_2} + i \lim_{h_2 \to 0} \frac{v(x_0, y_0 + h_2) - v(x_0, y_0)}{h_2} \\
&= \frac{1}{i}(u_y(z_0) + iv_y(z_0)) \\
&= v_y(z_0) - iu_y(z_0) \qquad (2)
\end{aligned}
$$

From (1) and (2), we have

$$f_y = if_x \text{ or } \text{ equivalently } u_x = v_y, \; u_y = -v_x.$$

Note 1: If a function $f(z)$ is known to be differentiable then its derivative is given by

$$f'(z) = f_x = -if_y = u_x + iv_x = v_y - iu_y.$$

Note 2: The C-R equations are necessary condition for f to be differentiable at a point. If they are not satisfied at a point then $f'(z)$ does not exist at that point.

If C-R equation hold a point z_0 then f may or may not be differentiable at z_0.

Example: Let $f(z) = \sqrt{|\mathrm{Re}\, z \mathrm{Im}\, z|} = \sqrt{|xy|}$. Then f satisfies the C-R equation at $z = 0$ but $f'(0)$ does not exists.

Solution: Here $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$. Then

$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = 0$$

$$u_y(0,0) = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = 0$$

Similarly, $v_x(0,0) = v_y(0,0) = 0$. Thus, f satisfy C-R equation.

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{(x,y) \to (0,0)} \frac{\sqrt{|xy|}}{x + iy} = \lim_{x \to 0} \frac{\sqrt{m}}{1 + im} \quad (\text{along } \ y = mx) = \frac{\sqrt{m}}{1 + im}$$

which is different for different values of m. Thus $f'(0)$ does not exist

Example: Show that the function

$$f(z) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$$

satisfy C-R equation at $z = 0$ but $f'(0)$ does not exists.

Theorem

Let $f = u + iv$ be differentiable in a domain D. Show that f is constant in D, if one of the following conditions hold

(i) $f'(z) \equiv 0$ *in* D.

(ii) $\mathrm{Re}\, f(z)$ *is constant in* D.

(iii) $\mathrm{Im}\, f(z)$ *is constant in* D.

(iv) $|f(z)|$ *is constant in* D.

Proof: (i) If $f'(z) = f_x = u_x + iv_x = 0$ then $u_x = v_x = 0$ in D. The by C-R equation $u_y = v_y = 0$ in D. Thus u and v both are constant in D and consequently, f is constant in D.

(ii) If $\mathrm{Re}\, f(z) = u = c$ then $u_x = u_y = 0$. By C-R equation $v_x = v_y = 0$. and so $f'(z) = 0$ in D. Thus f is constant in D.
(iv) Let $|f(z)| = k$, a constant. Then

$$\begin{aligned} u^2 + v^2 = k^2 &\implies uu_x + vv_x = 0 \quad \text{and} \quad uu_y + vv_y = 0 \\ &\implies (u^2 + v^2)({u_x}^2 + u_y^2) = 0 \qquad \text{[squaring and adding]} \\ &\implies k^2|f'(z)|^2 = 0 \end{aligned}$$

Analytic Function

- **Analytic Function:** A function $f : D \to \mathbb{C}$ is said to be analytic at a point $z_0 \in D$ if it is differentiable at every point of some neighbourhood of z_0.
- Alternative terms for analytic functions are **regular function** or **holomorphic function**.
- The function f is said to be analytic on D if it analytic at every point of D.
- A function which is analytic at every point in the complex plane is called entire function.

Example: Show that the function $f(z) = \bar{z} = x - iy$ is nowhere analytic.

Solution:

- Here $u(x, y) = x$ and $v(x, y) = -y$.
- Then $u_x = 1 \ \& \ u_y = 0$ and $v_x = 0 \ \& \ v_y = -1$.
- Thus $f(z)$ does not satisfy the CR equation at any point (alternatively, $f_{\bar{z}} = 1 \neq 0$).
- Thus f is not differentiable at any point and so f is not analytic at any point.

Analytic Function

Example: The function $f(z) = |z|^2 = z\bar{z}$ is differentiable only at the origin and hence nowhere analytic.

Solution: Here $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Then $u_x = 2x \;\&\; u_y = 2y$ and $v_x = 0 \;\&\; v_y = 0$. Thus $f(z)$ satisfy the CR equation only at the origin (alternatively, $f_{\bar{z}} = z$). Thus f is not differentiable at z if $z \neq 0$ and so f is not analytic at any point. You can check that f is differentiable at origin.

Example:

1. Any polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ is entire function.
2. The function $\sin z$, $\cos z$, e^z are entire function.
3. The function $f(z) = \frac{z}{1-z}$ is analytic in $\mathbb{C}\backslash\{1\}$.
4. The functions $f_(z) = Rez = \frac{z+\bar{z}}{2}$, $f_2(z) = Imz = \frac{z-\bar{z}}{2i}$, $f_3(z) = e^{\bar{z}}$ are nowhere differentiable/analytic.
5. The function $\mathrm{Log}\, z = \log|z| + i\mathrm{Arg}\, z$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$.
6. We can not talk about the analyticity of the function $\log z = \log|z| + i \arg z$ as it is a multi-valued function.

Harmonic Function

Harmonic Function: A function $\phi : \Omega \to \mathbb{R}$ is said to be harmonic in an open set Ω if it has continuous partial derivatives of second order and satisfies the Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \phi_{xx} + \phi_{yy} = 0.$$

Theorem

Both the real and imaginary parts of an analytic function are harmonic.

Proof: If $f = u + iv$ is analytic then $f'(z) = u_x + iv_x$. By C-R equation, $u_x = v_y, \ \& \ u_y = -v_x$. Thus $u_{xx} = v_{xy}, \ u_{yy} = -v_{yx}$. Therefore $u_{xx} + u_{yy} = 0$.

Example: The functions $u(x, y) = x, \ v(x, y) = -y$ both are harmonic in $\mathbb{C}$. But $f = u + iv = \bar{z}$ is not analytic at any point of $\mathbb{C}$.

Harmonic Function

Theorem

Let u be a harmonic function in a simply connected domain (to be discussed latter). Then there exist another harmonic function v such that $f = u + iv$ is analytic. (The function v is called the harmonic conjugate of u.)

- The harmonic conjugate v is unique, upto an addition of a real constant.
- Indeed, if v_1 is another harmonic conjugate, then $F = u + iv_1$ is also analytic in Ω and so $F - f = i(v_1 - v)$ becomes analytic in Ω. But then $\mathrm{Re}\,(F - f) = 0$ and so $F - f = c$ (constant).

Harmonic Function

Example: Show that $u(x, y) = 4xy - x^3 + 3xy^2$ harmonic and find v such that $f = u + iv$ is analytic.

- Here
  $u_x = 4y - 3x^2 + 3y^2, \quad u_y = 4x + 6xy, \quad u_{xx} = -6x, \quad u_{yy} = 6x.$
- So, $u_{xx} + u_{yy} = 0$ and therefore u is harmonic in $\mathbb{C}$.
- Now

$$u_x = 4y - 3x^2 + 3y^2 = v_y$$
$$\implies v = \int v_y dy + \phi(x) = 2y^2 - 3x^2y + y^3 + \phi(x)$$
$$\implies v_x = -6xy + \phi'(x) = -u_y = -4x - 6xy$$
$$\implies \phi'(x) = -4x$$
$$\implies \phi(x) = -2x^2 + k \quad k \text{ is a real constant}$$

Therefore $v = 2y^2 - 3x^2y + y^3 - 2x^2 + k$ and hence $f = u + iv$ is analytic.

To find the function: Let $f = u + iv$ is the corresponding analytic function. Then by C-R equation

$$\begin{aligned} f'(z) &= u_x + iv_x = u_x - iu_y = (4y - 3x^2 + 3y^2) - i(4x + 6xy) \\ &= -3(x^2 - y^2 + 2ixy) - 4i(x + iy) = -3z^2 - 4iz. \end{aligned}$$

Thus $f(z) = -z^3 - 2iz^2 + c$.

Harmonic Function

Example: Find the analytic function $f = u + iv$ given that $u(x, y) = x^3 - 3xy^2$.

- Here $u_x = 3x^2 - 3y^2, \quad u_y = -6xy, \quad u_{xx} = 6x, \quad u_{yy} = -6x$.
- So, $u_{xx} + u_{yy} = 0$ and therefore u is harmonic in $\mathbb{C}$.
- Now

$$
\begin{aligned}
u_x &= 3x^2 - 3y^2 = v_y \\
\implies v &= \int v_y dy + \phi(x) = 3x^2y - y^3 + \phi(x) \\
\implies v_x &= 6xy + \phi'(x) = -u_y = 6xy \\
\implies \phi'(x) &= 0 \\
\implies \phi(x) &= k \quad k \text{ is a real constant}
\end{aligned}
$$

Therefore $v = 3x^2y - y^3 + k$. Hence

$$f = u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3 + k) = (x + iy)^3 + ik = z^3 + c.$$

The Extended complex plane and Stereographic projection

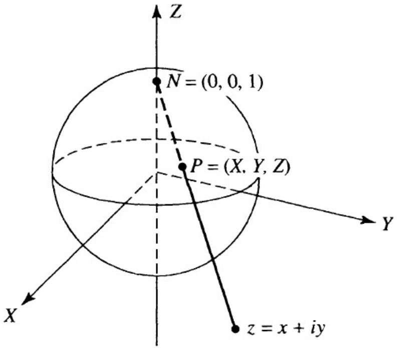

Figure: Stereographic projection

- The extended complex plane is the complex plane together with the point at infinity.
- The extended complex plane is denoted by $\mathbb{C}_\infty$ so that $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.
- One way to visualize the extended complex plane is the Stereographic projection.
- We consider a sphere of radius 1 centered at the origin $(0, 0, 0)$ in $\mathbb{R}^3$.
- We identify the complex number $z = x + iy$ by $(x, y, 0)$ in $\mathbb{R}^3$.

- If $P(X, Y, Z)$ is any point on the unit sphere other than the north pole $N(0, 0, 1)$ then the straight line joining P and N meets the complex plane at exact one point, namely at $z = x + iy$ or $(x, y, 0)$.
- Thus to each point on sphere (except the north pole $N(0, 0, 1)$) there correspond one and only one point on the complex plane and conversely.
- For completeness, we say that the north pole $N(0, 0, 1)$ correspond to the point at infinity (∞).

Analyticity at point at infinity

- Any nbd of the point of infinity is the set of all complex number (including ∞) lies in $|z| > M$ where $M > 0$.
- A function $f(z)$ is continuous/differentiable/analytic at $z = \infty$ iff the function $f(1/z)$ is continuous/differentiable/analytic at $z = 0$ respectively.

Curves

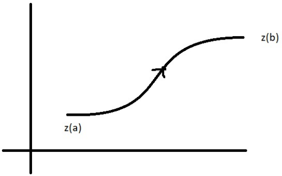

Figure: Curve

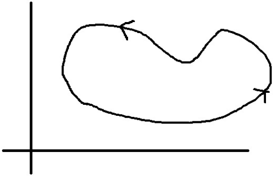

Figure: Simple closed curve

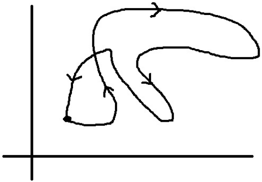

Figure: Not simple closed curve

- **Curve:** A continuous curve or simply curve or arc in $\mathbb{C}$ is a continuous mapping $\gamma : [a, b] \to \mathbb{C}$ and is defined parametrically by $\gamma : z(t) = x(t) + iy(t), t \in [a, b]$ where $x(t)$ and $y(t)$ are continuous real valued functions on $[a, b]$.
- A curve may have more than one parametrization. For example, $z_1(t) = t, t \in [0, 1]$ and $z_2(t) = t^2, t \in [0, 1]$ represent the curve.
- For the parameterized curve $\gamma : [a, b] \to \mathbb{C}$, the point $\gamma(a)$ is called the initial point and $\gamma(b)$ is called the terminal point of γ.
- If $\gamma(a) = \gamma(b)$ then it is called a closed curve.
- The curve γ is called simple or Jordan arc if $\gamma(t)$ is one one (injective) with possible exception that $\gamma(a) = \gamma(b)$.
- A simple closed curve is called a Jordan curve. A domain D bounded by a Jordan curve is called a Jordan domain.

Curves

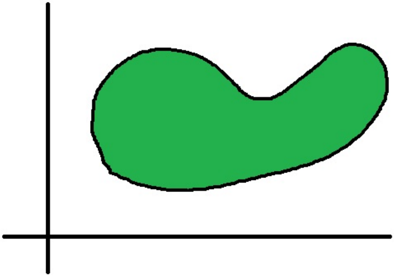

Figure: Simply connected domain

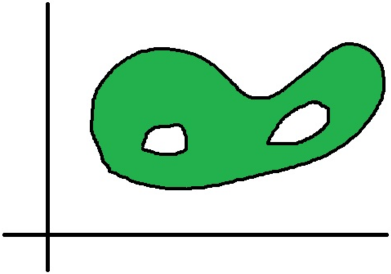

Figure: Multiply connected domain

- A domain D is called simply connected if each simple closed curve contained in D contains only points of D inside.
- A domain D is called simply connected if each simple closed curve contained in D can be contracted to a point without leaving D.
- A domain that is not simply connected is called multiply connected.

Curves

- The boundary C of a domain is said to have positive orientation, or to be traversed in the positive direction if a person walking on C always has the domain to his left.
- A curve $z(t) = x(t) + iy(t), t \in [a, b]$ is said to be smooth or regular or continuously differentiable on $[a, b]$ or C^1 curve if $z(t)$ and $z'(t)$ are continuous.
- A curve $C : z = z(t)$ is called piecewise smooth curve is there exists a subdivision $a = t_0 < t_1 < t_2 < .... < t_n = b$ of $[a, b]$ such that $z(t)$ is a smooth curve on $[t_{j-1}, t_j]$ for $j = 1, 2, ..., n$.
- A contour is just a piecewise smooth curve.