Mathematics - I

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1 FUNCTIONS OF SINGLE VARIABLE

1.1 Expansion of Functions

Theorem 1 (Extreme Value Theorem (EVT)). *If a real-valued function $f(x)$ is continuous in the closed and bounded interval $[a, b]$, then $f(x)$ must attain a maximum and a minimum, each at least once. That is, there exist numbers c and d in $[a, b]$ such that $f(c) \geq f(x) \geq f(d)$ for all $x \in [a, b]$.*

Theorem 2 (Theorem on Local Extrema). *If $f(c)$ is a local extremum, then either $f(x)$ is not differentiable at c or $f'(c) = 0$. That is, at a local max or min f either has no tangent, or $f(x)$ has a horizontal tangent there.*

Theorem 3 (Rolle's Theorem). *Let $a < b$ and suppose $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$, and $f(a) = f(b)$. Then there exists a $c \in (a, b)$ such that $f'(c) = 0$. That is, under these hypotheses, f has a horizontal tangent somewhere in between a and b.*

Proof. We seek a c in (a, b) with $f'(c) = 0$. That is, we wish to show that f has a horizontal tangent somewhere between a and b.

Since f is continuous on the closed interval $[a, b]$, the Extreme Value Theorem says that f has a maximum value $f(M)$ and a minimum value $f(m)$ on the closed interval $[a, b]$. Either $f(M) = f(m)$ or $f(M) \neq f(m)$.

Case 1. We suppose the maximum value $f(M) = f(m)$, the minimum value. So all values of f on $[a, b]$ are equal, and f is constant on $[a, b]$. Then $f'(x) = 0$ for all x in (a, b). So one may take c to be anything in (a, b); for example, $c = \frac{a+b}{2}$ would suffice.

Case 2. Now we suppose $f(M) \neq f(m)$. So at least one of $f(M)$ and $f(m)$ is not equal to the value $f(a) = f(b)$.

Case 2.a We first consider the case where the maximum value $f(M) \neq f(a) = f(b)$. So M is neither a nor b. But M is in $[a, b]$ and not at the end points. So M must be in the open interval (a, b). We have the maximum value $f(M) \geq f(x)$ for all x in the closed interval $[a, b]$ which contains the open interval (a, b). So we also have $f(M) \geq f(x)$ for every x in the open interval (a, b), Since M is also in the open interval (a, b), this means by definition that $f(M)$ is a local maximum.

Since M is in the open interval (a, b), by hypothesis we have that f is differentiable at M. Now by the Theorem on Local Extrema, we have that f has a horizontal tangent at M; that is, we have that $f'(M) = 0$. So we take $c = M$, and we are done with this case.

Case 2.b We now consider the case where the minimum value $f(m) \neq f(a) = f(b)$. (This case is very similar to the previous case.)

So m is neither a nor b. But m is in $[a, b]$ and not at the endpoints. So m must be in the open interval (a, b). We have the minimum value $f(m) \leq f(x)$ for all x in the closed interval $[a, b]$ which contains the open interval (a, b). Thus $f(m) \leq f(x)$ for every x in the open interval (a, b), Since m is also in the open interval (a, b), this means by definition that $f(m)$ is a local minimum.

Since m is in the open interval (a, b), by hypothesis we have that f is differentiable at m. Now by the Theorem on Local Extrema, we have that f has a horizontal tangent at m; that is, we have that $f'(m) = 0$. So we take $c = m$, and we are done with this case. $\square$

Theorem 4 (Langrange's Mean Value Theorem "MVT"). *Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b), and continuous on $[a, b]$ where $a < b$. Then there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. The equation of the secant through $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \tfrac{f(b)-f(a)}{(b-a)}(x - a)$$

which we can rewrite as

$$y = \tfrac{f(b)-f(a)}{(b-a)}(x - a) + f(a).$$

Let

$$g(x) = f(x) - \left[\tfrac{f(b)-f(a)}{(b-a)}(x - a) + f(a)\right]$$

Note that $g(a) = g(b) = 0$. Also, $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b) since f is. So by Rolle's Theorem there exists c in (a, b) such that $g'(c) = 0$.

But $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, so $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$.

Therefore, $f'(c) = \frac{f(b)-f(a)}{b-a}$ and the proof is complete. $\square$

Theorem 5 (Cauchy's Mean Value Theorem). *Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

Proof. Hint: Define $h(x) = f(x) - rg(x)$, where r is fixed in such a way that $h(a) = h(b)$, namely

$$r = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Then use Rolle's Theorem on $h(x)$. $\square$

Theorem 6 (Generalization for determinants)**.** *Assume that $f(x)$, $g(x)$, and $h(x)$ are differentiable functions on (a, b) that are continuous on $[a, b]$. Define*

$$D(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

There exists $c \in (a, b)$ such that $D'(c) = 0$. Notice that

$$D'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

and if we place $h(x) = 1$, we get Cauchy's mean value theorem. If we place $h(x) = 1$ and $g(x) = x$ we get Lagrange's mean value theorem.

Proof. Each of $D(a)$ and $D(b)$ are determinants with two identical rows, hence $D(a) = D(b) = 0$. The Rolle's theorem implies that there exists $c \in (a, b)$ such that $D'(c) = 0$. □

Taylor Polynomial for $f(x)$ about a: Suppose $f(x)$ and its derivatives $f'(x)$, $f''(x)$, $\ldots$, $f^{(n)}(x)$ exist at a. We define the n^{th} Taylor polynomial for $f(x)$ about a by the formula.

$$T_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

i.e.

$$T_n(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{n!},$$

where $f^{(0)}(x) = f(x)$ and $f^{(k)}(x)$ denote the k^{th} derivative of $f(x)$.

Theorem 7 (Taylor's Theorem)**.** *Let $n \in \mathbb{N}$. Suppose a function $f(x)$ satisfies the following conditions:*

1. *$f(x)$ and its first $n-1$ derivatives are continuous in a closed interval $[a, b]$.*
2. *$f^{(n-1)}(x)$ is differentiable in the open interval (a, b).*

Then, for any given positive integer p, there exists at least one point c in the open interval (a, b) such that

$$f(b) = f(a) + (b-a)f^{'}(a) + \frac{(b-a)^2}{2!}f^{''}(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-c)^{n-p}(b-a)^p}{(n-1)!p}f^{(n)}(c).$$

Proof. Hint: Let us consider the function $\phi(x)$ defined by

$$\phi(x) = f(b) - f(x) - (b-x)f^{'}(x) - \frac{(b-x)^2}{2!}f^{''}(x) + \cdots + \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) - r(b-x)^p,$$

where r is a constant to be chosen appropriately. □

Aliter: Taking $b - a = h$ and $c = a + \theta h$, where $\theta = \dfrac{c-a}{b-a}$, we find that

$$(b-a)^{n-p}(b-a)^p = \{(b-a) - (c-a)\}^{n-p}(b-a)^p = (b-a)^{n-p}\{1 - \frac{c-a}{b-a}\}^{n-p}(b-a)^p$$

$$= (b-a)^n\{1 - \frac{c-a}{b-a}\}^{n-p} = h^n(1-\theta)^{n-p}.$$

Accordingly, the above expression may be rewritten in the following alternative form:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \tag{1.1}$$

where

$$R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)!p}f^{(n)}(a+\theta h). \tag{1.2}$$

Here R_n is known as the remainder after n terms in the Taylor's theorem. The definition of θ indicates that $0 < \theta < 1$.

1.1.1 Remainder in Cauchy's and Lagrange's forms

According to the statement of the theorem, expression (1.1) and Taylor's theorem holds for any positive integer p.

For $p = 1$, the expression (1.2) becomes

$$R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(a+\theta h). \tag{1.3}$$

This R_n is called the remainder in Cauchy's form. Also, expression (1.1) with R_n given by (1.3) is called the Taylor's theorem with remainder in Cauchy's form.

For $p = n$, the expression (1.2) becomes

$$R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h). \tag{1.4}$$

This R_n is called the remainder in Lagrange's form. Also, expression (1.1) with R_n given by (1.4) is called the Taylor's theorem with remainder in Lagrange's form.

1.1.2 Taylor's series

Taking $a + h = x$ in expression (1.1) we obtain

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n(1-\theta)^{n-p}}{(n-1)!p}f^{(n)}(a+\theta[x-a]). \tag{1.5}$$

This expression gives the expansion of $f(x)$ in powers of $(x-a)$ and the expansion contains $n+1$ terms. Let us denote the sum of the first n terms by $S_n(x)$ and the last term by $R_n(x)$, i.e.,

$$S_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) \tag{1.6}$$

$$= f(a) + \sum_{r=1}^{n-1}\frac{(x-a)^r}{r!}f^{(r)}(a) \tag{1.7}$$

and

$$R_n(x) = \frac{(x-a)^n(1-\theta)^{n-p}}{(n-1)!p}f^{(n)}(a+\theta[x-a]). \tag{1.8}$$

Then the expression (1.5) may be put in form

$$f(x) = S_n(x) + R_n(x) \tag{1.9}$$

Now, suppose that $f(x)$ possesses derivatives of all orders and that $R_n(x)$ tends to zero as $n \to \infty$. Then taking the limit as $n \to \infty$ on both sides of (1.9), we get

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a) \tag{1.10}$$

The right hand side of (1.10) is an infinite series in ascending powers of $x - a$. This series is called **Taylor's series** for the function $f(x)$ about the point a. It is also referred to as the **Taylor's expansion** of $f(x)$ in power series about $x = a$.

For $a = 0$, expression (1.10) becomes

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} f^{(n)}(0) \tag{1.11}$$

The series on the right hand of this expression is the Taylor's expansion of $f(x)$ in power series about $x = 0$. This expansion is usually called **Maclurin's expansion** of $f(x)$.

Example 1. Prove, using the Taylor's Theorem, that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$, $0 < \theta < 1, \quad x > 0$.
Deduce that $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}, \qquad x > 0$.

Solution. For $n = p = 3$, the Taylor's theorem yields

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a+\theta h), \quad 0 < \theta < 1. \tag{1.12}$$

Now taking $f(x) = \log x \;\; x > 0$, we find that $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$,, $f'''(x) = \frac{2}{x^3}$.

Then , expression (1.12) becomes

$$\log(a+x) = \log a + x \cdot \frac{1}{a} + \frac{x^2}{2!}(-\frac{1}{a^2}) + \frac{x^3}{3!} \cdot \frac{2}{(a+\theta x)^3}, \quad 0 < \theta < 1.$$

For $a = 1$, this reduces to

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdot \frac{1}{(1+\theta x)^3}, \quad 0 < \theta < 1. \tag{1.13}$$

Which is the first of the required results.

Next, we note that, since $x > 0$ and $\theta > 0$, $(1+\theta x)^3 > 1$, or $\frac{1}{(1+\theta x)^3} < 1$. Consequently, (1.13) yields

$$\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}.$$

∎

Example 2. Use Taylor's Theorem to prove that $1 - \frac{1}{2}x^2 \leq \cos x$ for all $x \in \mathbb{R}$.

Solution. Use $f(x) := \cos x$ and $x_0 = 0$ in **Taylor's Theorem**, to obtain

$$\cos x = 1 - \frac{1}{2}x^2 + R_2(x),$$

where for some c between 0 and x we have

$$R_2(x) = \frac{f'''(c)}{3!}x^3 = \frac{\sin c}{6}x^3.$$

if $0 \leq x \leq \pi$, then $0 \leq c < \pi$; since c and x^3 are both positive, we have $R_2(x) \geq 0$. Also, if $-\pi \leq x \leq 0$, then $-\pi \leq c \leq 0$; since $\sin c$ and x^3 are both negative, we again have $R_2(x) \geq 0$.Therefore, we see that $1 - \frac{1}{2}x^2 \leq \cos x$ for $\mid x \mid\leq \pi$. If $\mid x \mid\geq \pi$, then we have $1 - \frac{1}{2}x^2 < -3 \leq \cos x$ and the inequality is trivially valid. Hence, the inequality holds for all $x \in \mathbb{R}$. ∎

Example 3. For any $k \in \mathbb{N}$, and for all $x > 0$, we have

$$x - \frac{1}{2}x^2 + \cdots - \frac{1}{2k}x^{2k} < \ln(1+x) < x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Solution. Using the fact that the derivative of $\ln(1+x)$ is $\frac{1}{(1+x)}$ for $x > 0$, we see that the nth Taylor polynomial for $\ln(1+x)$ with $x_0 = 0$ is

$$S_n(x) = x - \frac{1}{2}x^2 + \cdots + (-1)^{n-1}\frac{1}{n}x^n$$

and the remainder is given by

$$R_n(x) = \frac{(-1)^n c^{n+1}}{n+1}x^{n+1}$$

for some c satisfying $0 < c < x$. Thus for any $x > 0$, if $n = 2k$ is even, then we have $R_{2k}(x) > 0$; and if $n = 2k+1$ is odd, then we have $R_{2k+1}(x) < 0$. The stated inequality then follows immediately. ∎

Example 4. Use **Talyor's Theorem** with $n = 2$ to approximate $\sqrt[3]{1+x}$, $x > -1$.

Solution. We take the function $f(x) := (1+x)^{1/3}$, the point $x_0 = 0$, and $n = 2$. Since $f'(x) = \frac{1}{3}(1+x)^{-2/3}$, and $f''(x) = \frac{1}{3}(\frac{-2}{3})(1+x)^{-5/3}$, we have $f'(0) = \frac{1}{3}$ and $f''(0) = -\frac{2}{9}$. Thus we obtain

$$f(x) = S_2(x) + R_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + R_2(x),$$

where $R_2(x) = \frac{1}{3!}f'''(c)x^3 = \frac{5}{81}(1+c)^{-8/3}x^3$ for some point c between 0 and x.

For example, if we let $x = 0.3$, we get the approximation $S_2(0.3) = 1.09$ for $\sqrt[3]{1.3}$. Moreover, since $c > 0$ in this case, then $(1+c)^{-8/3} < 1$ and so the error is at most

$$R_2(0.3) \leq \frac{5}{81}(\frac{3}{10})^3 = \frac{1}{600} < 0.17 * (10)^{-2}.$$

Hence, we have $\mid \sqrt[3]{1.3} - 1.09 \mid< 0.5 * 10^{-2}$, so that two decimal place accuracy is assured. ∎

Example 5. Prove that the equation $x^7 + x^5 + x^3 + 1 = 0$ has exactly one real solution. You should use Rolle's Theorem at some point in the proof.

Solution. Let $y = y(x) = x^7 + x^5 + x^3 + 1$. $y(0) = 1$, $y(-1) = -2$. By EVT, there exists at least one real root in $x \in (-1, 0)$ such that $y(x) = 0$. Now I claim that there is EXACTLY one such real root, by using the method of contradiction.

Suppose not, there exists at least 2 real roots x_1, x_2 such that $y(x_1) = 0$, $y(x_2) = 0$. Since y is differentiable, by Rolle's theorem, there exists a number $a \in (x_1, x_2)$ such that $y'(a) = 0$. However, $y'(x) = 7x^6 + 5x^4 + 3x^2 > 0$ for all $x \neq 0$. ∎

Example 6. Use Taylor's theorem to prove that

$$1 + \frac{x}{2} - \frac{x^3}{8} < (1+x)^{\frac{1}{2}} < 1 + \frac{x}{2} \quad if \quad x > 0.$$

Solution. Let $f(x) = (1+x)^{\frac{1}{2}}$, $x \geq 0$.

Then $f'(x) = \frac{1}{2(1+x)^{\frac{1}{2}}}$, $f''(x) = -\frac{1}{4(1+x)^{\frac{3}{2}}}$, $f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}}$.

By Taylor's theorem with Lagrange's from of remainder, for any $x > 0$,

$f(x) = f(0) + xf'(x) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(c)$, for some $c \in (0, x)$.

or, $(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16(1+c)^{\frac{5}{2}}}$.

Therefore for $x > 0$, $(1+x)^{\frac{1}{2}} > 1 + \frac{x}{2} - \frac{x^2}{8}$, Since $\frac{x^3}{16(1+c)^{\frac{5}{2}}} > 0$.

By Taylor's theorem with Lagrange's form of remainder, for any $x > 0$,

$f(x) = f(0) + xf'(x) + \frac{x^2}{2}f''(d)$, for some $d \in (0, x)$.

Therefore for $x > 0$, $(1+x)^{\frac{1}{2}} < 1 + \frac{x}{2}$, Since $\frac{x^2}{8(1+d)^{\frac{3}{2}}} > 0$. ∎

Example 7. Let $c \in R$ and a real function f be such f'' is continuous on some neighbourhood of c. Prove that

$$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2}.$$

Solution. Let f'' be continuous on $(c - \delta, c + \delta)$ for some $\delta > 0$.

By Taylor's theorem with Lagrange's form of remainder, for any h satisfying $0 < h < \delta$

$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2}f''(c + \theta h)$, $0 < \theta < 1$

and $f(c-h) = f(c) - hf'(c) + \frac{h^2}{2}f''(c - \theta' h)$, $0 < \theta' < 1$.

Therefore $f(c+h) + f(c-h) - 2f(c) = \frac{h^2}{2}[f''(c + \theta h) + f''(c - \theta' h)]$

or, $\dfrac{f(c+h) - 2f(c) + f(c-h)}{h^2} = \frac{1}{2}[f''(c + \theta h) + f''(c - \theta' h)]$

Since f'' is continuous at c,

$$\begin{aligned}\lim_{h\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2} &= \lim_{h\to 0}\frac{1}{2}[f''(c+\theta h)+f''(c-\theta' h)]\\ &= f''(c).\end{aligned}$$

■

1.2 Problem Set

1. Use Rolle's theorem to prove that the equation $5x^3-2x^2+x-6=0$ can not have more than one real root.
2. Prove that the equation $x^{13}+7x^3-5=0$ has exactly one real root.
3. If f is differentiable on an interval I, and $f'(x)=0$ for all $x\in I$, then prove that f is constant on I.
4. If f and g are differentiable on an interval I, $f'(x)=g'(x)$ for all $x\in I$, then prove that there exists a constant $C\in R$ such that $f(x)=g(x)+C$ for all $x\in I$.
5. Suppose f is differentiable on an interval I. If $f'(x)\geq 0$ for all $x\in I$, then prove that f is monotone increasing on I.
6. In each of the following, give an example of a function that fits the given conditions and for which the conclusion of Rolle's theorem does not hold:
   a) f is continuous on $[a,b]$, and $f(a)=f(b)$.
   b) f is differentiable on (a,b) and $f(a)=f(b)$.
   c) f is differentiable on (a,b) and continuous on $[a,b]$.
7. Verify Rolle's theorem of (i) $f(x)=\cos x$ in $[-\frac{\pi}{2},\frac{\pi}{2}]$,
   (ii) $f(x)=|x|$, $1\leq x\leq 1$.
8. Find the 4^{th} Taylor polynomial of the function $f(x)=\sin x$ about 0.
9. Find the 4^{th} Taylor polynomial of $f(x)=3+5x^2-4x^3+x^4$ about 0.
10. Find the 4^{th} Taylor polynomial of $f(x)=3+5x^2-4x^3+x^4$ about 1.
11. Find the 4^{th} Taylor polynomial of the function $f(x)=e^x$ about 0.
12. Use Mean Value Theorem to show that $\sqrt[3]{28}$ lies between $3+\frac{1}{28}$ and $3+\frac{1}{27}$.
13. Use Taylor's Theorem to prove that for all $x>0$,

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}<e^x<1+x+\frac{x^2}{2!}+\frac{x^3}{3!}e^x.$$

14. If $f(x)$ and $g(x)$ are differentiable functions for $0\leq x\leq 1$ such that $f(0)=5$, $g(0)=1$, $f(1)=8$ and $g(1)=2$, then show that there exists c satisfying $0<c<1$ and $f'(c)=3g'(c)$.
15. Prove that between any two real roots of $e^x\cos x+1=0$ there exists at least one real root of $e^x\sin x+1=0$.
16. A functions f is differentiable on $[0,2]$ and $f(0)=0, f(1)=2, f(2)=1$. Prove that $f'(c)=0$ for some $c\in(0,2)$.
17. Find the Maclaurin's expansion of the following functions:
    (i) $\tan x$ (ii) $e^{\sin x}$ (iii) $\sin^{-1}x$.
18. Expand $5x^2+7x+3$ in power of $(x-2)$.

19. $f(x)$ is continuous and differentiable on $[6, 15]$. Suppose $f(6) = -2$ and $f'(x) \leq 10$. What is the largest possible value of $f(15)$?

20. $f(x)$ is continuous and differentiable everywhere. $f(x)$ has two real roots. Show that $f'(x)$ has one real root.

21. $f(x)$ is differentialbe on $[a, b]$, where $ab > 0$. Show that

$$\frac{1}{a-b}\begin{bmatrix} a & b \\ f(a) & f(b) \end{bmatrix} = f(c) - cf'(c),$$

for some $c \in (a, b)$. Hint: $F(x) = \dfrac{f(x)}{x}$ and $G(x) = \dfrac{1}{x}$

22. Let f and g be functions, continuous on $[a, b]$, differentiable on (a, b) and let $f(a) = f(b) = 0$. Prove that there is a point $c \in (a, b)$ such that $g'(c)f(c) + f'(c) = 0$. Hint: Let $h(x) = e^{g(x)} \cdot f(x)$.

23. Using Mean Value Theorem, show that

    a) $\frac{x-1}{x} < \log x < x - 1$ for $x > 1$.

    b) $e^x \geq 1 + x$ for $x \in \mathbb{R}$.

    c) $1 - \frac{x^2}{2!} < \cos x$, for $x \neq 0$.

    d) $x - \frac{x^3}{3!} < \sin x$, for $x > 0$.

    e) $x > \sin x$ for $0 < x < \frac{\pi}{2}$.

    f) $\frac{x}{1+x} < \log(1 + x) < x$ for all $x > 0$.

24. (Using the MVT) Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Consequently, $\forall x \in \mathbb{R}$, $|\sin x| \leq |x|$.

25. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = a$ and $f(b) = b$. Show that there is $c \in (a, b)$ such that $f'(c) = 1$. Further, show that there are distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

Example 8. Use MVT to show that $\sqrt[3]{28}$ lies between $3 + \frac{1}{28}$ and $3 + \frac{1}{27}$ i.e.,

$$3 + \frac{1}{28} < \sqrt[3]{28} < 3 + \frac{1}{27}.$$

Solution. Let f be the real function defined by $f(x) = \sqrt[3]{x}$, then f is continuous and differentiable on $[27, 28]$. So by the MVT, there exists $c \in (27, 28)$ such that

$$f(28) = f(27) + (28 - 27)f'(c).$$

$$\implies \sqrt[3]{28} = 3 + \frac{1}{3c^{\frac{2}{3}}}$$

Since $27 < c < 28$ and f is strictly increasing $[0, \infty]$, it follows that

$$3 + \frac{1}{3.(28)^{\frac{2}{3}}} < \sqrt[3]{28} < 3 + \frac{1}{3.(27)^{\frac{2}{3}}} \tag{1.14}$$

The right hand inequality of (1.14) gives

$$\sqrt[3]{28} < 3 + \frac{1}{3.3^{3}.\frac{2}{3}} \qquad i.e. \qquad \sqrt[3]{28} < 3 + \frac{1}{27} \tag{1.15}$$

Using left hand inequality of (1.14) gives

$$3 + \frac{1}{3.(28)^{\frac{2}{3}}} < \sqrt[3]{28} \tag{1.16}$$

Now $27^{\frac{1}{3}} < 28^{\frac{1}{3}}$ i.e. $3 < 28^{\frac{1}{3}} \implies 3.(28)^{\frac{2}{3}} < (28)^{\frac{1}{3}}.(28)^{\frac{2}{3}}$ i.e.

$$3.(28)^{\frac{2}{3}} < 28 \tag{1.17}$$

From (1.16) and (1.17) (for left) and (1.15), we get $3 + \frac{1}{28} < \sqrt[3]{28} < 3 + \frac{1}{27}$. ■

2 FUNCTIONS OF SEVERAL VARIABLES

2.1 Functions, Limit and Continuity

Definition 1. *A function $f(x, y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$, and we write*

$$\lim_{(x,y)\to(a,b)} f(x, y) = l.$$

Definition 2. *A function $f(x, y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to the positive number ϵ there exists another positive number $\delta = \delta(\epsilon)$ such that $|f(x, y) - l| < \epsilon$, whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$.*

Definition 3. *If a function $f(x, y)$ has distinct limits as (x, y) approaches a point (a, b) along two distinct paths, the limit $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.*

Definition 4. *A function $f(x, y)$ is said to be continuous at the point (a, b) if (i) $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and is equal to l, say, and (ii) $f(a, b) = l$.*

Example 1. Show that the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & if \ (x, y) \neq (0, 0), \\ 0 & if \ (x, y) = (0, 0) \end{cases}$$

is continuous at the origin.

Solution. Let $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore |\frac{xy}{\sqrt{x^2+y^2}} - 0| = |r\cos\theta\sin\theta| \leq r = \sqrt{x^2 + y^2} < \epsilon$, if $x^2 < \frac{\epsilon^2}{2}$ and $y^2 < \frac{\epsilon^2}{2}$.

Thus $|\frac{xy}{\sqrt{x^2+y^2}} - 0| < \epsilon$ whenever $0 < \sqrt{x^2 + y^2} < \epsilon$.

So for any $\delta > 0$ there exists an $\epsilon = \delta$ such that
$|\frac{xy}{\sqrt{x^2+y^2}} - 0| < \epsilon$ whenever $0 < \sqrt{x^2 + y^2} < \delta$.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0.$$

Hence, $f(x, y)$ is continuous at $(0, 0)$. ■

Example 2. Show that the function $f(x,y)$ defined by

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{when } (x,y) \neq (0,0), \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

Solution. Let $x = r\cos\theta$, $y = r\sin\theta$.
$\therefore |xy\frac{x^2-y^2}{x^2+y^2}| = r^2|\cos\theta\sin\theta\cos 2\theta| < r^2 = x^2+y^2 < \epsilon$, if $x^2 < \frac{\epsilon}{2}$ and $y^2 < \frac{\epsilon}{2}$.

$\therefore |xy\frac{x^2-y^2}{x^2+y^2} - 0| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \sqrt{\epsilon}$

So for any $\delta > 0$ there exists an $\epsilon = \delta^2$ such that
$|xy\frac{x^2-y^2}{x^2+y^2} - 0| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$.

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Hence, $f(x,y)$ is continuous at $(0,0)$. ■

Example 3. Prove that the function

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y} & \text{when } x \neq y, \\ 0 & \text{when } x = y, \end{cases}$$

is not continuous at $(0,0)$.

Solution. Putting $y = x - mx^3$, then as $x \to 0$, $y \to 0$. Now

$$f(x, x-mx^3) = \frac{x^3 + (x-mx^3)^3}{mx^3} = \frac{1+(1-mx^2)^3}{m}.$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{1+(1-mx^2)^3}{m} = \frac{2}{m}.$$

So that the limit is different for different choices of m. Therefore the limit does not exist. ■

2.2 Problem Set

1. Show that

$$\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0.$$

2. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist for

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & if\ \ x^4+y^2 \neq 0, \\ 0 & if\ \ x = y = 0. \end{cases}$$

3. Show that the function $f(x,y)$ defined by

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{when } (x,y) \neq (0,0), \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

4. Show that the following function is continuous at the origin.

$$f(x,y) = \begin{cases} \frac{x^3y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

5. Given that $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$, where, $x^2+y^2 \neq 0$, is it possible to assign a value for $f(0,0)$ such that $f(x,y)$ is continuous at $(0,0)$? Why?

2.3 Partial Differentiation

Differentiation of homogeneous functions

A function $f(x,y)$ of two independent variables x and y is called a **homogeneous** function of degree n if the function can be put in the form $x^n\phi(\frac{y}{x})$, or $y^n\phi(\frac{x}{y})$.

Theorem 8 (Euler's theorem). *If u is a homogeneous function of x and y with degree n, then*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu. \tag{2.1}$$

Proof. Since u is a homogeneous function of degree n, we can put u in the form $u = x^n\phi(\frac{y}{x})$. This gives

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi(\frac{y}{x}) + x^n\phi'(\frac{y}{x})(-y/x^2) = nx^{n-1}\phi(\frac{y}{x}) - x^{n-2}y\phi'(\frac{y}{x}),$$

$$\text{and} \quad \frac{\partial u}{\partial y} = x^n\phi'(\frac{y}{x})(1/x) = x^{n-1}\phi'(\frac{y}{x}).$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\phi(\frac{y}{x}) - x^{n-1}y\phi'(\frac{y}{x}) + x^{n-1}y\phi'(\frac{y}{x}) = nx^n\phi(\frac{y}{x}) = nu.$$

□

Example 4. If u is a homogeneous function of x and y with degree n, then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u. \tag{2.2}$$

Solution. Differentiating (2.1) partially with respect to x, we obtain

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

$$\text{or} \qquad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x} \tag{2.3}$$

Similarly, differentiating (2.1) partially with respect to y, we get

$$y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial y \partial x} = (n-1)\frac{\partial u}{\partial y} \tag{2.4}$$

Multiplying (2.3) by x and (2.4) by y and adding we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = n(n-1)u.$$

■

Example 5. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u(1 - 4\sin^2 u).$$

Solution. $\tan u = \frac{x^3+y^3}{x-y}$ is a homogeneous function in x and y of degree 2.
By Euler's Theorem

$$x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2\tan u \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

Differentiate with respect to x and y, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (2\cos 2u - 1)\frac{\partial u}{\partial x} \tag{2.5}$$

Similarly, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = (2\cos 2u - 1)\frac{\partial u}{\partial y}. \tag{2.6}$$

Adding above equations, we get

$$\begin{aligned} x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= (2\cos 2u - 1)(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}) \\ &= \sin 2u(1 - 4\sin^2 u). \end{aligned}$$

■

Example 6. If $u = \frac{(x^2+y^2)^n}{2n(2n-1)} + xf(\frac{y}{x}) + g(\frac{y}{x})$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (x^2+y^2)^n.$$

Proof. Let $u = U + V + W$, where $U = \frac{(x^2+y^2)^n}{2n(2n-1)}$, $V = xf(\frac{y}{x})$ and $W = g(\frac{y}{x})$.
Differentiate with respect to x and y, we get

$$x^2\frac{\partial^2 U}{\partial x^2} + 2xy\frac{\partial^2 U}{\partial x \partial y} + y^2\frac{\partial^2 U}{\partial y^2} = (x^2+y^2)^n.$$

Since V is a homogeneous function of x and y of degree 1, by Euler's Theorem

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = V.$$

Differentiate with respect to x and y, we have

$$x^2\frac{\partial^2 V}{\partial x^2} + 2xy\frac{\partial^2 V}{\partial x \partial y} + y^2\frac{\partial^2 V.}{\partial y^2} = 0.$$

Similarly, W is a homogeneous function of x and y of degree 0,

$$x^2\frac{\partial^2 W}{\partial x^2} + 2xy\frac{\partial^2 W}{\partial x \partial y} + y^2\frac{\partial^2 W}{\partial y^2} = 0.$$

■

Example 7. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 = 2(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}).$$

Solution. Given that $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$.
Differentiating partially with respect to x, we get

$$\frac{2x}{a^2+u} - \{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\}\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{P}\frac{2x}{a^2+u},$$

where $P = \sum \frac{x^2}{(a^2+u)^2}$.
Similarly,

$$\frac{\partial u}{\partial y} = \frac{1}{P}\frac{2x}{a^2+u} \quad , \quad \frac{\partial u}{\partial z} = \frac{1}{P}\frac{2x}{a^2+u}.$$

Now

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 = \frac{4}{P} = 2(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}).$$

■

Example 8. Show that the expression $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ can be resolved into linear factors if $\det \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} = 0$.

Solution. Let $u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is resolvable into two linear factors $v = l_1x + m_1y + n_1z$ and $w + l_2x + m_2y + n_2z$. Thus u, v and w are connected by $u = vw$. The Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$, and

$$\det\begin{pmatrix} 2(ax+gz+hy) & 2(by+fz+hx) & 2(cz+gx+fy) \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{pmatrix} = 0.$$

Since x, y, z are independent variables, their coefficients in the above determinant must separately vanish.

Hence

$$\begin{aligned} a(m_1n_2 - m_2n_1) + h(l_2n_1 - l_1n_2) + g(l_1m_2 - l_2m_1) &= 0 \\ h(m_1n_2 - m_2n_1) + b(l_2n_1 - l_1n_2) + f(l_1m_2 - l_2m_1) &= 0 \\ g(m_1n_2 - m_2n_1) + f(l_2n_1 - l_1n_2) + c(l_1m_2 - l_2m_1) &= 0. \end{aligned}$$

Eliminating $(m_1n_2 - m_2n_1)$, $(l_2n_1 - l_1n_2)$, $(l_1m_2 - l_2m_1)$, we get $\det\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} = 0.$ ■

Example 9. If $f(0) = 0$, $f'(x) = \dfrac{1}{1+x^2}$, prove, without using the method of integration that $f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$.

Solution. Let $u = f(x) + f(y)$, and $v = \frac{x+y}{1-xy}$.

The Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \end{pmatrix} = 0.$$

Hence u, v are connected by a functional relation $u = \phi(v)$, i.e., $f(x) + f(y) = \phi\left(\dfrac{x+y}{1-xy}\right)$.

Putting $y = 0$, using $f(0) = 0$, we get $f(x)\phi(x)$. Thus $f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$. ■

2.4 Implicit Function

If $F(x,y)$ be a function of two variables and $y = \phi(x)$ be a function of x such that for every x for which $\phi(x)$ is defined, $F(x, \phi(x)) = 0$, then we say $y = \phi(x)$ is an implicit function defined by $F(x,y) = 0$.

For example $y = 3 - \frac{3}{2}x$ is an implicit function defined by $3x + 2y - 6$.

Theorem 9 (Existence Theorem)**.** *Let $F(x,y)$ be a function of two variables x and y and let (x_0, y_0) be a point in its domain of definition such that*

1. *$F(x_0, y_0) = 0$;*
2. *F_x and F_y are continuous in a certain neighborhood of (x_0, y_0);*
3. *$F_y(x_0, y_0) \neq 0$.*

Then there exists a rectangle:

$$x_0 - h \le x \le x_0 + h; \quad y_0 - k \le y \le y_0 + k.$$

centered at (x_0, y_0) *such that for every value of* x *in the interval* I: $x_0 - h \le x \le x_0 + h$ *the functional equation* $F(x, y) = 0$ *determines one and only one value* $y = \phi(x)$ *which lies in the interval* $y_0 - k \le y \le y_0 + k$.

Example 10. Examine the existence of unique implicit function for $xy \sin x + \cos y$ at $(0, \frac{\pi}{2})$.

Solution. Let $F(x, y) = xy \sin x + \cos y$, then $F_x = y \sin x + xy \cos x$ and $F_y = x \sin x - \sin y$. Observe that $F(0, \frac{\pi}{2}) = 0$, F_x and F_y are always continuous and $F_y(0, \frac{\pi}{2}) = -1 \neq 0$. Thus all the conditions of implicit Function Theorem are satisfied at $(0, \frac{\pi}{2})$. ■

2.5 Problem Set

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when $u = \tan^{-1} \frac{x^2 - y^2}{x^2 + y^2}$.
2. If $U = \sqrt{xy}$, find the value of $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$.
3. If $x^x y^y z^z = k$, where x and y are independent variable, show that $z_{xy} = -(x \log(ex))^{-1}$, at $x = y = z$.
4. If $\theta = t^n\, e^{-\frac{r^2}{4t}}$, find the value of n for which $\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial \theta}{\partial r}) = \frac{\partial \theta}{\partial t}$.
5. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that (i) $u_x + u_y + u_z = \frac{3}{x+y+z}$ (ii) $u_{xx} + u_{yy} + u_{zz} = -\frac{3}{(x+y+z)^2}$ (iii) $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u = -\frac{9}{(x+y+z)^2}$.
6. If $z = f(x + ay) + \phi(x - ay)$ show that $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$.
7. State and prove Euler's theorem (Case of two variables).
8. Verify Euler's theorem for the function $u = f(x, y) = ax^2 + 2hxy + by^2$.
9. Let $u = \sin^{-1} \sqrt{\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}}$, show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u)$.
10. If $u = x\phi(\frac{y}{x}) + y\psi(\frac{y}{x})$, prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$.
11. If $u = x\ \phi(x + y) + y\ \psi(x + y)$ prove that $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$.
12. If $f(x, y) = \tan^{-1} \frac{x^3 + y^3}{x - y}$, using Euler's theorem, show that $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = (2 \cos 2f - 1) \sin 2f$.
13. If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, prove that $xu_x + yu_y = \sin 2u$.
14. If $u = f(x^2 + 2yz, y^2 + 2xz)$, prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.
15. If $u = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, prove that $u_{xx} + u_{yy} + u_{zz} = \frac{2}{u}$.
16. If $z = u^3 + v^3$, where $u = \sin xy$ and $v = y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
17. If $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, where u and v are functions of x and y, prove that $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$, where $x = r\cos\theta$, $y = r\sin\theta$. [Hint: $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$.]

2.6 Total differential and total derivative

For a function $f = f(x, y)$ of two independent variables x and y the total differential (or the exact differential or just the differential) df is defined by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{2.7}$$

Further, if x and y are themselves functions of an independent variable t, that is if $f = f(x, y)$ where $x = x(t)$ and $y = y(t)$, then the total derivative of f is given by the formula

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{2.8}$$

Similarly, the df of a function $f = f(x, y, z)$ is defined by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz. \tag{2.9}$$

Further, if x, y, z are themselves functions of t, then the total derivative of f is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}. \tag{2.10}$$

The total differential and the the total derivative of a function of n independent variables are defined similarly.

2.7 Differentiation of composite functions

Suppose f is a function of the variables x and y, and x and y are themselves functions of two other variables u and v. Then $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are computed by using the following formulas:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}. \tag{2.11}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \tag{2.12}$$

Similarly, if f is a function of u and v, where u and v are themselves functions of x and y, then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}. \tag{2.13}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}. \tag{2.14}$$

The second and higher order partial derivatives of f can be obtained by repeated application of the above formulas. Also, the formulas can be extended to functions of three and more variables. The formulas (2.11)-(2.14) are called the **Chain rules** for partial differentiation.

2.8 Jacobians

Suppose u and v are functions of two independent variables x and y. Then the determinant

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \tag{2.15}$$

is called the **Jacobian** of u and v with respect to x and y. It is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\frac{(u,v)}{(x,y)}$. Similarly, if u, v, w are functions of three independent variables x, y, z, then the **Jacobian** of u, v, w with respect to x, y, z is defined as follows:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J\frac{(u,v,w)}{(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}. \tag{2.16}$$

The Jacobian of n independent variables is defined in an analogous way.

One of the fundamental properties of the Jacobian is that two functions u and v, which depend on two independent variables x and y, are themselves independent of one another iff $J = \frac{\partial(u,v)}{\partial(x,y)} \neq 0$. If this condition is satisfied, then we can express x and y in terms of u and v explicitly. Consequently, we can define the Jacobian of x and y with respect to u and v as follows:

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}. \tag{2.17}$$

It can be proved that $JJ' = 1$. In view of this result, J' is called the inverse of J.

2.8.1 Jacobians of functions of functions

Suppose u and v are functions of two independent variables s and t, s and t are themselves functions of two independent variables x and y. Then, by using the chain rule of partial differentiation, we can prove that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(x,y)}.$$

The above chain rule can be extended in a natural way to Jacobian of n variables , $n \geq 3$.

2.9 Problem Set

1. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$.
2. If $x = a\cos\theta\cosh\phi$, $y = a\sin\theta\sinh\phi$, find $\frac{\partial(x,y)}{\partial(\phi,\theta)} = \frac{1}{2}a^2(\cosh 2\phi - \cos 2\theta)$.
3. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{y^2-x^2}{uv(u-v)}$.
4. If $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = -r^2\sin\phi$.
5. If $x = r + s + t$, $y = st + tr + rs$, $z = rst$, show that $\frac{\partial(x,y,z)}{\partial(r,s,t)} = (r-s)(r-t)(s-t)$. Also, find the value of $\frac{\partial(r,s,t)}{\partial(x,y,z)}$.
6. If $x + y = u$ and $x = uv$ then show that $\frac{\partial(x,y)}{\partial(u,v)} = -u$. Also, find the value of $\frac{\partial(u,v)}{\partial(x,y)}$.
7. If $u = x(1-r^2)^{-\frac{1}{2}}$ and $v = y(1-r^2)^{-\frac{1}{2}}$ where $r^2 = x^2 + y^2$, find the value of $\frac{\partial(u,v)}{\partial(x,y)}$.
8. Show that the functions:

$$u = \frac{x+y}{1-xy}, \quad v = \tan^{-1}x + \tan^{-1}y$$

are not independent and find the relationship between them.

9. Show that the functions: $u = x + y + z$, $v = xy + yz + zx$, $w = x^3 + y^3 + z^3 - 3xyz$ are not independent but they are related by $u^3 = 3uv + w$.
10. If $u = x^2 + y^2$, $v = x^2 - y^2$ and $x = r\theta$, $y = r + \theta$ then find the value of the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$. Ans: $8r\theta(r^2 - \theta^2)$.
11. If $u = x + 2y + z$, $v = x - 2y + 3z$, and $w = 2xy - xz + 4yz - 2z^2$, then find the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. Is there any relation between u, v, and w? If yes, then what is the relation between them?

2.10 Taylor's and Maclaurin's Series

We proved the Taylor's theorem for a function of a single variable and used it to expand the function in power series. Here, we consider the corresponding theorem that hold for function of two independent variables.

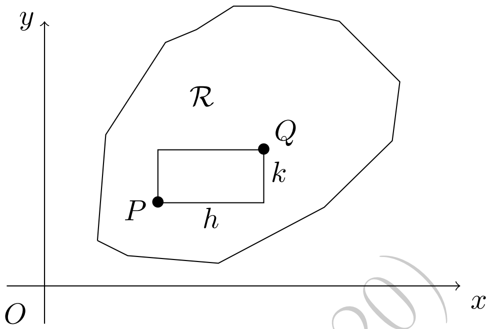

Figure 2.1: Taylor's Theorem

Consider a function $f = f(x, y)$ of two independent real variables x and y, defined over a region $\mathcal{R}$ in the xy-plane. Let $P(a, b)$ and $Q(a + h, b + k)$ be two neighbouring points in $\mathcal{R}$. Then the value of the function f at the point Q is expressed in terms of the value of f at the point P through the following results:

$$f(a+h, b+k) = f(a, b)+\frac{1}{1!}\Delta f(a, b)+\frac{1}{2!}\Delta^2 f(a, b)+\cdots+\frac{1}{(n-1)!}\Delta^{n-1} f(a, b)+\frac{1}{n!}\Delta^n f(a+\theta h, b+\theta k). \tag{2.18}$$

Here Δ is a partial differential operator defined by

$$\Delta = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y},$$

so that

$$\Delta f = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y},$$

$$\Delta^2 f = \Delta(\Delta f) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\Delta f,$$

$$\Delta^3 f = \Delta(\Delta^2 f) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\Delta^2 f,$$

and so on, and θ is a real constant such that $0 < \theta < 1$.

The expression (2.18) is a generation to functions of two variables with Lagrange's form of remainder. This expression is known as the **Taylor's theorem** for the function $f = f(x, y)$. This theorem holds whenever

1. for $f(x,y)$ and its partial derivatives of order $n-1$ are continuous for $a \leq x \leq a+h$ and $b \leq y \leq b+k$, and
2. the nth order partial derivatives of f exist for $a < x < a+h$ and $b < y < b+k$.

Write x for $a+h$ and y for $b+k$ in (2.18), so that $h = x-a$ and $k = y-b$, we obtain the following alternative form of the Taylor's theorem for $f(x,y)$:

$$f(x,y) = f(a,b) + \sum_{r=1}^{n-1} \frac{1}{r!}\Delta^r f(a,b) + \frac{1}{n!}\Delta^n f(a+\theta[x-a], b+\theta[y-b]) \tag{2.19}$$

Here,

$$\Delta = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}; \quad h = x-a, \quad k = y-b \tag{2.20}$$

2.10.1 Taylor Series for $f(x,y)$

Let us rewrite expression (2.19) as

$$f(x,y) = S_n(x,y) + R_n(x,y), \tag{2.21}$$

where

$$S_n(x,y) = f(a,b) + \sum_{r=1}^{n-1} \frac{1}{r!}\Delta^r f(a,b) \tag{2.22}$$

and

$$R_n(x,y) = \frac{1}{n!}\Delta^n f(a+\theta[x-a], b+\theta[y-b]) \tag{2.23}$$

which is the remainder after n terms.

We suppose that $f(x,y)$ possesses partial derivatives of all orders and that $R_n(x,y) \to 0$ as $n \to \infty$. Then taking limits as $n \to \infty$ on both sides of (2.21), we get

$$f(x,y) = \lim_{n\to\infty} S_n(x,y) = f(a,b) + \sum_{n=1}^{\infty} \frac{1}{n!}\Delta^n f(a,b). \tag{2.24}$$

The right hand side of the above expression is an infinite series in ascending powers of $h = x-a$ and $k = y-b$. This is referred to as the Taylor's series (or the Taylor's expansion) of $f(x,y)$ about (or near) the point (a,b).

2.10.2 Maclaurin's Expansion for $f(x,y)$

For $a = 0$, $b = 0$, expression (2.24) becomes

$$f(x,y) = f(0,0) + \sum_{n=1}^{\infty} \frac{1}{n!}\Delta^n f(0,0). \tag{2.25}$$

Here,

$$\Delta = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \quad \text{since} \quad h = x, \quad k = y \tag{2.26}$$

Expression (2.25) gives the Taylor's expansion of $f(x,y)$ about (or near) the point $(0,0)$. This expression is an infinite series in power of x and y, and is known the Maclaurin's expansion of $f(x,y)$.

Note: The following expressions are useful in computational work:

$$\Delta f = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}, \tag{2.27}$$

$$\Delta^2 f = \Delta(\Delta f) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) = h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}, \tag{2.28}$$

$$\begin{aligned}\Delta^3 f = \Delta(\Delta^2 f) &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)\\ &= h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\end{aligned} \tag{2.29}$$

and so on.

Example 11. Obtain the Taylor's expansion of $f(x, y) = x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$.

Solution. Since we have to expand $f(x, y)$ in powers of $x-1$ and $y-2$, we take $a = 1$, $b = -2$. Then $h = x - a = x - 1$ and $k = y - b = y + 2$.

We find that

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 3$$

$$\frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^3 f}{\partial x^3} = 0, \quad \frac{\partial^3 f}{\partial x^2 \partial y} = 2, \quad \frac{\partial^3 f}{\partial x \partial y^2} = 0, \quad \frac{\partial^3 f}{\partial y^3} = 0$$

Partial derivatives of all higher-order are zero. Therefore,

$$\Delta f(x, y) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = 2hxy + k(x^2 + 3)$$

$$\Delta^2 f(x, y) = 2h^2y + 4hkx \;\text{ and }\; \Delta^3 f(x, y) = 6h^2k,$$

$\Delta^n f(x, y) = 0$, for $n \geq 4$.

These yield

$$\Delta f(a, b) = \Delta f(1, -2) = -4h + 4k$$

$$\Delta^2 f(a, b) = \Delta^2 f(1, -2) = -4h^2 + 4hk \;\text{ and }\; \Delta^3 f(a, b) = \Delta^4 f(1, -2) = 6h^2k,$$

$\Delta^n f(a, b) = \Delta^n f(1, -2) = 0$, for $n \geq 4$. Also $f(a, b) = f(1, -2) = -10$. Hence, for the given $f(x, y)$, the Taylor's expansion, namely

$$f(x, y) = f(a, b) + \sum_{n=1}^{\infty} \frac{1}{n!}\Delta^n f(a, b).$$

is $x^2y + 3y - 2 = -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2(y + 2)$, on recalling that $h = x - 1$ and $k = y + 2$. ■

Example 12. Expand $f(x, y) = \frac{y^2}{x^3}$ about the point $(1, -1)$ upto and including the second degree terms.

Solution. Here $f(x,y) = x^{-3}y^2$, $a = 1$, $b = -1$. Therefore, $h = x - a = x - 1$, $k = y - b = y + 1$. We find that

$$\frac{\partial f}{\partial x} = -3x^{-4}y^2, \quad \frac{\partial f}{\partial y} = 2x^{-3}y$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^{-5}y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = -6x^{-4}y, \quad \frac{\partial^2 f}{\partial y^2} = 2x^{-3}$$

Since the expansion is required upto second degree terms, we need not find third and higher-order partial derivatives. Now,

$$\Delta f(x,y) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \frac{-3hy^2}{x^4} + \frac{2ky}{x^3}$$

$$\Delta^2 f(x,y) = \frac{12h^2y^2}{x^5} - \frac{12hky}{x^4} + \frac{2k^2}{x^3}$$

Therefore, $f(a,b) = f(1,-1) = 1$, $\Delta f(a,b) = \Delta f(1,-1) = -3h - 2k$,

$\Delta^2 f(a,b) = \Delta^2 f(1,-1) = 12h^2 + 12hk + 2k^2$

Now, the Taylor's expansion upto second-degree terms, namely

$$f(x,y) = f(a,b) + \sum_{n=1}^{2} \frac{1}{n!}\Delta^n f(a,b)$$

gives

$$\tfrac{y^2}{x^3} = 1 - 3(x-1) - 2(y+1) + 6(x-1)^2 + 6(x-1)(y-1) + (y+1)^2. \qquad \blacksquare$$

2.11 Problem Set

1. Expand the function $f(x,y) = e^{(x+y^2)}$ about the point $(1,1)$. Ans: $e^{(x+y^2)} = e^2[1 + 2\{(x-1)+(y-1)\} + \{3(x-1)^2 + \cdots + 4(x-1)(y-1) + 3(y-1)^2\} + \cdots$.
2. Expand the function $f(x,y) = e^x \cos y$ about the point $(1, \frac{\pi}{4})$ upto second degree terms, by using the Taylor's theorem.
3. Obtain the expansion of $f(x,y) = \tan^{-1}\frac{y}{x}$ about the point $(1,1)$ upto second degree terms, by using the Taylor's theorem. Hence find an approximate value of $f(x,y)$ at $(1.1, 0.9)$. Ans: $f(1.1, 0.9) = 0.6904$.
4. Expand $\sin x \cos y$ in powers of x and y as far as terms of third degree. Ans: $\sin x \cos y = x - \frac{1}{6}(x^3 + 3xy^2)$
5. Expand the function e^{xy} in powers of x and y upto second-degree terms. Ans: $e^{xy} = 1 + xy + \frac{1}{2}(x^2y^2)$

3 SEQUENCE AND SERIES

Test 1 (Integral Test). *A positive term series $f(1) + f(2) + \cdots + f(n) + \cdots$, where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^\infty f(x)\mathrm{d}x$ is finite or infinite.*

Theorem 10. *The series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.*

Test 2 (Comparison Test).

- If two positive term series $\sum u_n$ and $\sum v_n$ be such that (i) $\sum v_n$ converges, (ii) $u_n \leq v_n$ for all values of n, then $\sum u_n$ also converges.
- If two positive term series $\sum u_n$ and $\sum v_n$ be such that (i) $\sum v_n$ diverges, (ii) $u_n \geq v_n$ for all values of n, then $\sum u_n$ also diverges.
- If two positive term series $\sum u_n$ and $\sum v_n$ be such that

  $$\lim_{n\to\infty} \frac{u_n}{v_n} = \text{finite} \quad (\neq 0),$$

  then $\sum u_n$ and $\sum v_n$ both converge or diverge.

Test 3 (D'Alembert's Ratio Test). *In positive term series $\sum u_n$, if*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda,$$

then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$. (for $\lambda = 1$ there is no conclusion.)

Test 4 (Cauchy's Root Test). *In positive term series $\sum u_n$, if*

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lambda,$$

then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$. (for $\lambda = 1$ there is no conclusion.)

Test 5 (Cauchy Condensation Test). *Suppose $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if*

$$\sum_{n=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$$

converges.

Definition 5. *A series in which the terms are alternately positive or negative is called an **alternating series**.*

Definition 6. *If the series of arbitrary terms $u_1 + u_2 + u_3 + u_4 + \cdots + u_n + \cdots$ be such that the series $|u_1| + |u_2| + |u_3| + |u_4| + \cdots + |u_n| + \cdots$ is convergent, then the series $\sum u_n$ is said to be **absolutely convergent**.*
*If $\sum |u_n|$ is divergent but $\sum u_n$ is convergent, then $\sum u_n$ is said to be **conditionally convergent**.*

Test 6 (Leibnitz's Rule). *An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots.$ converges if (i) each term is numerically less than its preceding term, and (ii) $\lim_{n\to\infty} u_n = 0$. ($\lim_{n\to\infty} u_n \neq 0$, the given series is oscillatory.)*

Example 1. Test the convergence of the series $\sum u_n$, where $u_n = (n^4+1)^{\frac{1}{2}} - (n^4-1)^{\frac{1}{2}}$.

Solution. Let $u_n = \frac{2}{(n^4+1)^{\frac{1}{2}} + (n^4-1)^{\frac{1}{2}}}$, $v_n = \frac{1}{n^2}$.

Then

$$\lim_{n\to 0} \frac{u_n}{v_n} = \lim_{n\to 0} \frac{2n^2}{(n^4+1)^{\frac{1}{2}} + (n^4-1)^{\frac{1}{2}}} = 1.$$

Since $\sum v_n$ is convergent, $\sum u_n$ is convergent by comparison test. ∎

Example 2. Test the convergence of the series $\sum u_n$, where $u_n = \frac{n^n}{n!}$.

Solution. $u_n = \dfrac{n^n}{n!}$, $\dfrac{u_{n+1}}{u_n} = \left(\dfrac{n+1}{n}\right)^n$ and $\lim_{n\to 0} \dfrac{u_{n+1}}{u_n} = \lim_{n\to 0}\left(\dfrac{n+1}{n}\right)^n = e > 1$.

$\sum u_n$ is divergent by D'Alembert's ratio test. Therefore the given test is divergent. ■

Example 3. Test the convergence of the series $\sum u_n$, where $u_n = \dfrac{1}{2^{n+(-1)^n}}$.

Solution. Here $u_n = \dfrac{1}{2^{n+(-1)^n}}$ and $\lim_{n\to 0} u_n^{\frac{1}{n}} = \lim_{n\to 0} \dfrac{1}{(2^{n+(-1)^n})^{\frac{1}{n}}} = \dfrac{1}{2}$.

Therefore the series is convergent by Cauchy's root test. ■

Example 4. Test the convergence of the series $\sum u_n$, where $u_n = \dfrac{(-1)^{n+1}}{n}$.

Solution. Since $\{u_n\}$ is a monotone decreasing sequence of positive real numbers and $\lim_{n\to 0} u_n = \lim_{n\to 0} \frac{1}{n} = 0$.

u_n is convergent by Leibnitz's test. ■

Example 5. Test the convergence of the series $\sum u_n$, where $u_n = \dfrac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}$.

Solution. Since $\{u_n\}$ is a monotone decreasing sequence of positive real numbers, as

$$\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} > \frac{1\cdot 3\cdots(2n+1)}{2\cdot 4\cdots(2n+2)}$$

and

$$\lim_{n\to 0} u_n = \lim_{n\to 0} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} = 0.$$

u_n is convergent by Leibnitz's test. ■

3.1 Problem Set

1. Test for convergence of the following series:
   a) $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \cdots$. Ans: Divergent: Int, Comp.
   b) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$. Ans: Convergent.
   c) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$. Ans: Convergent.
   d) $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \cdots$. Ans: ?
   e) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots + \frac{x^n}{n^2+1} + \cdots$. Ans: ?
   f) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$. Ans: Convergent.
   g) $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n}$. Ans: ?

h) $\sum\limits_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\sqrt{n}}$. Ans: ?

i) $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (where $p > 0$). Ans: ?

2. Investigate for convergence of the series:

a) $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}x + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}x^2 + \cdots \infty$, $(x \neq 0)$. Ans: Convt if $x < 1$, Div if $x \geq 1$.

b) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots, \quad x \geq 0$. Ans: Convt if $x < 1$, Div if $x \geq 1$.

3. Discuss for convergence of the following series:

a) $\sum\limits_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$. Ans: Convergent.

b) $\sum\limits_{n=1}^{\infty} (1 + \frac{1}{\sqrt{n}})^{-n^{\frac{3}{2}}}$. Ans: Convergent.

4. Test for convergence of the following series:

a) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$. Ans: Convergent.

b) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$. Ans: Not convergent.

c) $\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \cdots \quad a > 0, \ b > 0$. Ans: Convergent.

d) $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - + \cdots$; where $0 < x < 1$. Ans: Convergent.

5. Show that the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$ is absolutely convergent.

6. Define **non-absolutely** convergent series and show that $\sum_{n \geq 1} (-1)^n [\sqrt{n^2+1} - n]$ is non-absolutely convergent series. Can you give some other examples?

7. Show that $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ is convergent.

8. Prove that the series $2 + (2+a) + (2+2a) + (2+3a) + \ldots$ diverges for any real values of a.

9. Consider a series of positive term as $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, can you say that $\sum_{n=1}^{\infty} a_n$ is convergent? Justify your answer.

10. Show that $\sum_{n \geq 2} (-1)^n \frac{1}{n \ log \ n}$ is **non-absolutely** convergent series. Using the Cauchy condensation test, determine the convergence of the following series:

a) $\sum\limits_{n=2}^{\infty} \frac{1}{n \log n}$. Ans: Divergent.

b) $\sum\limits_{n=2}^{\infty} \frac{1}{(n \log n)^2}$. Ans: Convergent.

c) $\sum\limits_{n=2}^{\infty} \frac{1}{(n \log n)^p}$. Ans: Conv if $p > 1$, div if $p \leq 1$.

d) $\sum\limits_{n=2}^{\infty} \frac{1}{(n \log n)(\log(\log n))}$. Ans: Divergent.

4 INTEGRAL CALCULUS

1. First Mean Value Theorem for Definite Integrals: Let $f(x)$ and $\phi(x)$ be two bounded functions integrable on $a \le x \le b$ and let $\phi(x)$ keep the same sign on $[a,b]$, then

$$\int_a^b f(x)\phi(x)dx = \mu \int_a^b \phi(x)dx,$$

where $m \le \mu \le M$, m and M being the greatest lower and least upper bounds of $f(x)$ on $[a,b]$.

Note that here $\mu = f(\xi)$ for some $\xi \in [a,b]$.

2. Mean Value Theorem (Simple form): [Particular case of above choosing $\phi(x) = 1$] If $f(x)$ is continuous on $[a,b]$, then at some point ξ in $[a,b]$,

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx.$$

3. Second Mean Value Theorem for Definite Integrals: [Bonnet's Form] Let $f(x)$ be bounded monotone non-increasing and never negative on $[a,b]$; and let $\phi(x)$ be bounded and integrable on $[a,b]$. Then there exists a value ξ of x on $[a,b]$, such that

$$\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx; \quad a \le \xi \le b.$$

[Weierstrass's Form] Let $f(x)$ be bounded and monotonic on $[a,b]$; and let $\phi(x)$ be bounded and integrable on $[a,b]$. Then there exists at least one value of x, say ξ on $[a,b]$, such that

$$\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx + f(b)\int_\xi^b \phi(x)dx; \quad a \le \xi \le b.$$

Example 1. Show that for $k^2 < 1$,

$$\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6}\cdot\frac{1}{\sqrt{1-\frac{k^2}{4}}}.$$

Solution. Applying first mean-value theorem for integrals which we can do since it satisfies all the conditions. Let $f(x) = \frac{1}{\sqrt{1-k^2x^2}}$ and $\phi(x) = \frac{1}{\sqrt{1-x^2}}$. For $0 \le \xi \le \frac{1}{2}$, we get

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{\sqrt{1-k^2\xi^2}}\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}.$$

But

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_0^{\frac{1}{2}} = \frac{\pi}{6}.$$

Hence

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{6}.\frac{1}{\sqrt{1-k^2\xi^2}}.$$

Putting $\xi = 1$ and $\xi = \frac{1}{2}$, we get

$$\frac{\pi}{6} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \leq \frac{\pi}{6}.\frac{1}{\sqrt{1-\frac{k^2}{4}}}.$$

■

4. Comparison test 1: If $f(x)$ be a non-negative integrable function when $x \geq a$ and $\int_a^B f(x)\mathrm{dx}$ is bounded above for every $B > a$, then $\int_a^\infty f(x)\mathrm{dx}$ will converge; otherwise it will diverge to ∞.
5. Comparison test 2: If $f(x)$ and $g(x)$ be integrable functions when $x \geq a$ such that $0 \leq f(x) \leq g(x)$, then

$$(i) \quad \int_a^\infty f(x)\mathrm{dx} \text{ converges if } \int_a^\infty g(x)\mathrm{dx} \text{ converges}$$

$$(ii) \quad \int_a^\infty g(x)\mathrm{dx} \text{ diverges if } \int_a^\infty g(x)\mathrm{dx} \text{ diverges.}$$

6. Limit test: Let $f(x)$ and $g(x)$ be integrable functions when $x \geq a$ and $g(x)$ be positive. Then if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lambda \neq 0,$$

the integrals

$F = \int_a^\infty f(x)\mathrm{dx}$ and $G = \int_a^\infty g(x)\mathrm{dx}$ both converge absolutely or both diverge.

7. The μ-test for convergence 1: Let $f(x)$ be an integrable function when $x \geq a$. Then $F = \int_a^\infty f(x)\mathrm{dx}$ converges absolutely if

$$\lim_{x\to\infty} x^\mu f(x) = \lambda, \quad \text{for some} \quad \mu > 1,$$

and F diverges if

$$\lim_{x\to\infty} x^\mu f(x) = \lambda(\neq 0), \quad or \quad \pm\infty; \quad \text{for some} \quad \mu \leq 1.$$

8. The μ-test for convergence 2: Let $f(x)$ be an integrable function in the arbitrary interval $(a+\epsilon, b)$, where $0 < \epsilon < b-a$. Then $F = \int_a^b f(x)\mathrm{dx}$ converges absolutely if

$$\lim_{x\to a+} (x-a)^\mu f(x) = \lambda, \quad \text{for some} \quad 0 < \mu < 1$$

and F diverges if

$$\lim_{x\to a+} (x-a)^\mu f(x) = \lambda(\neq 0), \quad or \quad \pm\infty; \quad \text{for some} \quad \mu \geq 1.$$

Example 2. Show that $\int_0^\infty e^{-x^2}\mathrm{dx}$ converges.

Solution. Applying μ-test,

$$\lim_{x\to\infty} x^2 e^{-x^2} = \lim_{x\to\infty} \frac{x^2}{e^{x^2}} = 0$$

for $\mu = 2 > 1$.

So $\int_0^\infty e^{-x^2} \mathrm{dx}$ is convergent. ∎

Example 3. Show that $\int_1^\infty e^{-x} x^n \mathrm{dx}$ converges for all values of n.

Solution. Applying μ-test,

$$\lim_{x\to\infty} \frac{x^{n+2}}{e^x} = 0$$

for $\mu = 2 > 1$. ∎

Example 4. Show that $\int_1^\infty \frac{\log x}{x^2} \mathrm{dx}$ converges.

Solution. Applying μ-test,

$$\lim_{x\to\infty} x^{\frac{3}{2}} \frac{\log x}{x^{\frac{1}{2}}} = \lim_{x\to\infty} x \log x = 0$$

for $\mu = \frac{3}{2} > 1$. ∎

Example 5. Show that $\int_1^\infty \frac{x^{\frac{3}{2}}}{3x^2+5} \mathrm{dx}$ is divergent.

Solution. Applying μ-test,

$$\lim_{x\to\infty} x^{\frac{1}{2}} f(x) = \lim_{x\to\infty} x^{\frac{1}{2}} \frac{x^{\frac{3}{2}}}{3x^2+5} = \lim_{x\to\infty} \frac{x^2}{3x^2+5} = \frac{1}{3}$$

for $\mu = \frac{1}{2} < 1$. ∎

Example 6. Show that $\int_0^\pi \frac{\sin x}{x^3} \mathrm{dx}$ is diverges.

Solution. By μ-test, since

$$\lim_{x\to 0+} x^2 \frac{\sin x}{x^3} = \lim_{x\to 0+} \frac{\sin x}{x} = 1$$

∎

9. Gamma Function: Let us discuss the convergence of

$$\int_0^\infty e^{-x}x^{n-1}\mathrm{dx}, \quad n > 0. \tag{4.1}$$

We write, $f(x) = e^{-x}x^{n-1}$, $I_1 = \int_0^1 e^{-x}x^{n-1}\mathrm{dx}$ and $I_2 = \int_1^\infty e^{-x}x^{n-1}\mathrm{dx}$.

The part I_1 is proper when $n \geq 1$ and improper but absolutely convergent when $0 < n < 1$ by the following test.

By second μ-test,

$$\lim_{x\to 0+} x^{1-n}f(x) = \lim_{x\to 0+} x^{1-n}e^{-x}x^{n-1} = \lim_{x\to 0+} e^{-x} = 1,$$

for $0 < \mu = 1-n < 1$, i.e., for $0 < n < 1$.

The part I_2 also converges absolutely for all values of n by first μ-test,

$$\lim_{x\to\infty} x^2 f(x) = \lim_{x\to\infty} x^2 e^{-x}x^{n-1} = \lim_{x\to\infty} \frac{x^{n+1}}{e^x} = 0.$$

Thus equation (4.1) converges for $n > 0$. This is called gamma function denoted by $\Gamma(n)$.

Hence

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\mathrm{dx}, \quad n > 0.$$

10. Beta Function: Next, let us discuss the convergence of

$$\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{dx}, \quad m > 0, \quad n > 0. \tag{4.2}$$

This is a proper integral when $m, n \geq 1$ but is improper at the lower limit when $m < 1$, at the upper limit when $n < 1$. We, therefore, split it into two parts $I_1 + I_2$ where

$$I_1 = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\mathrm{dx} \quad \text{and} \quad I_2 = \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\mathrm{dx}.$$

We write, $f(x) = x^{m-1}(1-x)^{n-1}$. Now I_1 converges for $0 < m < 1$, diverges when $m \leq 0$, by second μ-test

$$\lim_{x\to 0+} x^{1-m}f(x) = \lim_{x\to 0+} x^{1-m}x^{m-1}(1-x)^{n-1} = \lim_{x\to 0+}(1-x)^{n-1} = 1,$$

for $\mu = 1-m$ and for convergence $0 < \mu = 1-m < 1$ that is $0 < m < 1$.

Also

$$\lim_{x\to 0+} xf(x) = \lim_{x\to 0+} xx^{m-1}(1-x)^{n-1} = \lim_{x\to 0+} x^m(1-x)^{n-1} = \begin{cases} 1 & \text{when} \quad m = 0, \\ \infty & \text{when} \quad m < 0. \end{cases}$$

Next if we make the change of variable $x = 1-y$, the second integral reduces to the first with m and n interchanged. Hence we may draw the same conclusion as before with n in place of m. Thus equation (4.2) converges for $m, n > 0$. This is called Beta function denoted by $\beta(m,n)$, or,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{dx} \quad \text{for} \;\; m, n > 0.$$

Definition 7 (Gamma function). *The Gamma function denoted by* $\Gamma(n)$ *is defined by*

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt,$$

$x > 0.$

Definition 8 (Beta function). *The Beta function denoted by* $\beta(m, n)$ *is defined by*

$$\beta(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1} dt,$$

$x > 0,\ y > 0.$

Example 7. Show that

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1} \mathrm{dx} = (b-a)^{m+n-1}\ B(m, n), \quad m, \quad n > 0.$$

Solution. Put $x = a\cos^2\theta + b\sin^2\theta$, then

$$\begin{aligned}\int_a^b (x-a)^{m-1}(b-x)^{n-1} \mathrm{dx} &= \int_0^{\frac{\pi}{2}} (b-a)^{m+n-1} \sin^{2m-1}\theta \cos^{2n-1}\theta \mathrm{d}\theta \\ &= (b-a)^{m+n-1} B(m, n).\end{aligned}$$

■

Example 8. Show that

$$\int_0^\infty x^{\frac{1}{2}} e^{-x^3} \mathrm{dx} = \frac{\sqrt{\pi}}{3}.$$

Solution. Put $x^3 = z$, then

$$\begin{aligned}\int_0^\infty x^{\frac{1}{2}} e^{-x^3} \mathrm{dx} &= \frac{1}{3}\int_0^\infty z^{-\frac{1}{2}} e^{-z} \mathrm{dz} \\ &= \frac{1}{3}\Gamma(\frac{1}{2}) \\ &= \frac{\sqrt{\pi}}{3}.\end{aligned}$$

■

4.1 Problem Set

1. Do the following integrals exist? If exist, find the value:
   a) $\int_0^\infty \frac{1}{1+x^2} dx$ Ans: $\frac{\pi}{2}$
   b) $\int_0^\infty \frac{1}{x^2} dx$ Ans: $\times$
   c) $\int_0^\infty \sin x dx$ Ans: $\times$
   d) $\int_0^\infty e^{-x^2} dx$ Ans: Using β and Γ

e) $\int_2^\infty \frac{1}{x\log x}dx$ Ans: $\times$

f) $\int_{-\infty}^\infty xe^{-x^2}dx$ Ans: 0

g) $\int_0^\infty e^{-ax}\sin bx\ dx$ Ans:

h) $\int_0^1 \frac{dx}{x}$ Ans: $\times$

i) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ Ans: $\frac{\pi}{2}$

j) $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1+\cot^2 x}}$ Ans:

k) $\int_0^4 \frac{dx}{2x-8}$ Ans: $\times$

2. Examine the convergence of the improper integral

$$\int_0^\infty \frac{x^{p-1}}{1+x}\text{dx},$$

where $p \in \mathbb{R}$.

3. Prove that $\int_{-1}^1 \frac{dx}{x^3}$ exists in Cauchy principal value sense but not in general sense.

4. Prove the following relations ($a>0$, $x>0$, $y>0$, n being positive integer):

a) $\int_0^\infty e^{-at}t^{x-1}dt = \frac{\Gamma(x)}{a^x}$. Hint: Let $at = u$

b) $\beta(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt$. Hint: Put $\frac{1}{1+t} = u$

c) $\Gamma(1) = 1$, $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n+1) = n!$.

d) $\Gamma(x) = 2\int_0^\infty e^{-y^2}y^{2x-1}dy$. Hint: Let $t = x^2$

e) $\beta(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\cos^{2y-1}\theta d\theta$. Hint: Let $t = \sin^2\theta$

f) $\beta(m,n) = \beta(n,m)$.

g) $\beta(x,y) = \beta(x+1,y) + \beta(x,y+1)$.

h) $\beta(\frac{1}{2},\frac{1}{2}) = \pi$.

i) $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

j) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

5. Express $\int_0^1 t^m(1-t^n)^p dt$ in terms of Beta function and hence evaluate $\int_0^1 t^5(1-t^3)^9 dt$.

6. Evaluate:

a) $\int_0^1 x^3(1-x)^{\frac{1}{2}}dx$ Ans:

b) $\int_0^1 x(1-x)^7 dx$ Ans:

c) $\int_0^\infty x^2 e^{-x^2}dx$ Ans:

d) $\int_0^1 t^3(1-t^2)^{\frac{5}{2}}dt$ Ans:

e) $\int_0^\infty x^4 e^{-x}dx$ Ans:

f) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}d\theta$ Ans:

7. Prove that:

a) $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

b) $\int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \Gamma(n)\Gamma(1-n)$

c) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}d\theta = \frac{\pi}{\sqrt{2}}$ using $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \sqrt{2}\pi$

d) $\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^4\theta d\theta = \frac{1}{120}$

e) $\int_0^{\infty} x^n e^{-a^2x^2} \mathrm{dx} = \frac{1}{2a^{n+1}}\Gamma(\frac{n+1}{2})$ Hint: Put $ax = \sqrt{t}$

f) $\beta(n,n) = \frac{\sqrt{\pi}\Gamma(n)}{2^{2n-1}\Gamma(n+\frac{1}{2})}$.

g) $2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2}) = \Gamma(2n)\sqrt{\pi}$. (This is known as **duplication** formula.)

5 Useful Formulas

1. $\int \sqrt{a^2+x^2}\mathrm{dx} = \dfrac{x\sqrt{a^2+x^2}}{2} + \dfrac{a^2}{2}\sinh^{-1}\frac{x}{a} + C.$
2. $\int \sqrt{x^2-a^2}\mathrm{dx} = \dfrac{x\sqrt{x^2-a^2}}{2} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{x}{a} + C.$
3. $\int \sqrt{a^2-x^2}\mathrm{dx}d = \dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + C.$
4. $\int \dfrac{\mathrm{dx}}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C.$
5. $\int \sin^{-1}x\mathrm{dx} = x\sin^{-1}x + \sqrt{1-x^2} + C$
6. $\cosh x = \dfrac{e^x - e^{-x}}{2}$, $\sinh x = \dfrac{e^x + e^{-x}}{2}$, $\dfrac{d}{dx}(\cosh x) = \sinh x$, $\dfrac{d}{dx}(\sinh x) = \cosh x$, $\cosh^2 x - \sinh^2 x = 1$.
7. $$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \qquad \lim_{n\to\infty}(1+\frac{1}{n})^n = e \text{ and } \lim_{n\to\infty}(1+\frac{x}{n})^n = e^x.$$
8. The equation of a **cardiod** is $r = a(1+\cos\theta)$ and shape of the equation is in Figure 5.1. Similarly the equation of another **cardiod** is $r = a(1-\cos\theta)$ and shape of the equation is in Figure 5.2. The pole of the cardiod is the origin.

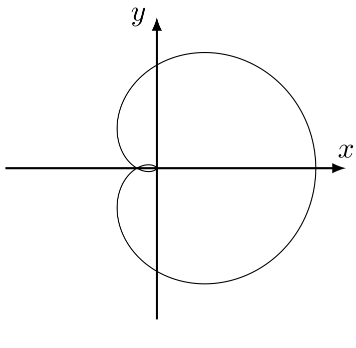

Figure 5.1: cardiod: $r = a(1+\cos\theta)$

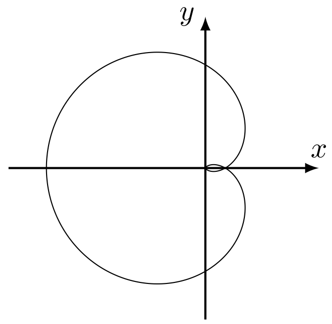

Figure 5.2: cardiod: $r = a(1 - \cos\theta)$

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