CHAPTER 8

Maxima and Minima

8.1 INTRODUCTION

Finding out extrema (maxima and/or minima) of a function (with or without some constraints) is very important subject in every branch of engineering, science, social science, medicine, etc. Students have already solved problems on maxima and minima of functions of a single variable. Here, we shall discuss maxima and minima of functions containing two independent variables.

8.2 TOTAL DIFFERENTIAL

Let $u = f(x, y)$ be a function of two independent variables. Also, let Δx and Δy be their increment, then $\Delta u = f(x + \Delta x,\ y + \Delta y) - f(x, y)$ is the increment of u or f.

Now, by chain rule (Chapter 7), we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{8.1}$$

The quantity du or df is called the total differential or simply differential (first order) of u or f.

The above equation (8.1) can also be expressed as

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y. \tag{8.2}$$

This result can be generalised for n variables as stated below:

If
$$u = f(x_1, x_2, \ldots, x_n)$$

then
$$du = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n.$$

The equation (8.1) can be written as

$$du = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} \right) u \text{ or } d = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} \right)$$

Now, the second order differential d^2u is given by

$$d^2u = d(du) = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} \right)\left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} \right) u$$

$$= \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} \right)^2 u$$

$$= \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x \partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}(dy)^2$$

$\left(\text{Assuming } \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x} \right)$.

EXAMPLE 8.1 Let $u = x^3 + xy^2$. Find du and d^2u.

Solution

Here

$$\frac{\partial u}{\partial x} = 3x^2 + y^2, \quad \frac{\partial u}{\partial y} = 2xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial x \partial y} = 2y, \quad \frac{\partial^2 u}{\partial y^2} = 2x.$$

$\therefore$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = (3x^2 + y^2)dx + (2xy)dy$$

and

$$d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x \partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}(dy)^2$$

$$= 6x(dx)^2 + 4y\,dx\,dy + 2x(dy)^2.$$

EXAMPLE 8.2 Let $f(x, y) = xy + x^2 + y^2$. Find the change in f when the increments of x and y are respectively 0.01 and 0.05 at $x = 2, y = 1$.

Solution

The change in f is given by

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

Here

$$f(x, y) = xy + x^2 + y^2.$$

Thus, $$\frac{\partial f}{\partial x} = y + 2x \text{ and } \frac{\partial f}{\partial y} = x + 2y.$$

$\therefore$ $$\Delta x = 0.01,\ \Delta y = 0.05 \text{ and at } (2, 1), \frac{\partial f}{\partial x} = 5,\ \frac{\partial f}{\partial y} = 4.$$

Hence change in f is

$$\Delta f = 5 \times 0.01 + 4 \times 0.05 = 0.25.$$

8.3 DEFINITION OF MAXIMA AND MINIMA

Let $f(x, y)$ be a function of two independent variables x and y. Let us further assume that f is *continuous* and *finite* for all values of x and y in the neighbourhood of their values a, b. Then the value of $f(a, b)$ is said to be a maximum or a minimum if

$$f(a+h, b+k) - f(a, b) \tag{8.3}$$

keeps some sign (positive or negative) for all positive or negative small values of h and k. The point (a, b) is called *extreme point*.

The values $f(a, b)$ is maximum if the difference shown in equation (8.3) is negative for all positive and negative small values of h and k; and minimum if the difference shown in equation (8.3) is positive for all positive and negative small values of h and k.

8.4 THE NECESSARY CONDITION FOR EXTREMA

Let $f(x, y)$ be *continuous* and *differentiable* function of two variables x and y. Then by Taylor's theorem for two variables,

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \text{ terms of second and higher order.}$$

$$f(x+h, y+k) - f(x, y) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \text{ terms of second and higher order.} \tag{8.4}$$

But, for extrema

$$f(x+h, y+k) - f(x, y) \tag{8.5}$$

must preserve same sign.

For sufficiently small values of h and k, the sign of equation (8.5) depends on the sign of $h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$, and the sign of this first degree terms depends on the signs of h and k.

Hence the necessary condition for the existence of extrema is $h\dfrac{\partial f}{\partial x}+k\dfrac{\partial f}{\partial y}=0$.

But, h and k are independent increments of x and y respectively, so the above equation must hold if

$$\frac{\partial f}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}=0. \tag{8.6}$$

Solve these equations for x and y, we get the values of variable for which $f(x, y)$ is either a maximum or a minimum.

It may be noted that the condition of equation (8.6) is necessary and not sufficient for the continuous and differentiable function.

Note: 1. The function $f(x, y) = |x| + |y|$ has a minimum value at $(0, 0)$ even though the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ do not exist at $(0, 0)$.

2. If $f(x, y) = 0$, if $x = 0, y = 0$
$= 1$, elsewhere

Then both the partial derivatives exist at the origin but $f(0, 0)$ is not an extreme value. Thus the conditions are only necessary, but, not sufficient.

8.5 THE SUFFICIENT CONDITION FOR EXTREMA (THE LAGRANGE'S CONDITION)

Let $f(x, y)$ be a function of two independent variables x and y. We assume that f is continuous and differentiable at the neighbourhood of (a, b).

Let $$r=\frac{\partial^2 f}{\partial x^2},\quad s=\frac{\partial^2 f}{\partial x\partial y} \text{ and } t=\frac{\partial^2 f}{\partial y^2} \text{ at } x=a,\ y=b.$$

As a set of necessary conditions for a maximum or minimum at (a, b) we have $\dfrac{\partial f}{\partial x}=0$ and $\dfrac{\partial f}{\partial y}=0$ at (a, b).

Therefore, $f(a+h, b+k)-f(a, b)=\dfrac{1}{2!}(rh^2+2shk+tk^2)+$ terms of third and higher order. (8.7)

By taking h and k sufficiently small, the sign of left hand side of equation (8.7) is depends on $rh^2+2shk+tk^2$. Thus, f has a maximum if $I = rh^2+2shk+tk^2$ is negative and a minimum if I is positive.

Now,

$$I = rh^2+2shk+tk^2$$

$$= \frac{1}{r}\left[r^2h^2 + 2srhk + rtk^2\right]$$

$$= \frac{1}{r}[(rh + sk)^2 + (rt - s^2)k^2].$$

Thus, we observed that, if $rt - s^2 > 0$ then sign of I is same as r. Hence, the expression I is positive, i.e. f is minimum if

$$rt - s^2 > 0 \text{ and } r > 0 \tag{8.8}$$

and the expression I is negative, i.e. f is maximum if

$$rt - s^2 > 0 \text{ and } r < 0. \tag{8.9}$$

Stationary point or critical point

A point (a, b) is said to be *stationary point* or *critical point* of a function $f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Saddle point

A point (a, b) is saddle point of a function $f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, but, $f(x, y)$ has neither a maximum nor a minimum at (a, b).

Thus, every saddle point is a stationary point, but converse is not true (see Figure 8.1)

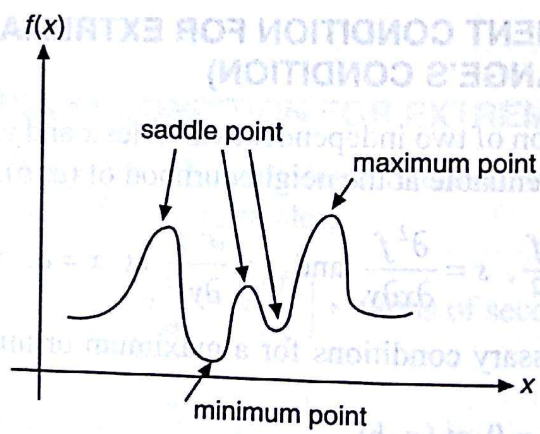

Figure 8.1 Stationary points for $y = f(x)$.

8.6 WORKED-OUT EXAMPLES

EXAMPLE 8.3 Find the maximum and minimum values of the function $f(x, y) = x^3 + y^3 - 3axy$.

(WBUT 2002, 2008)

Solution

Here

$$f(x, y) = x^3 + y^3 - 3axy.$$

Therefore, at stationary points

$$f_x = 3x^2 - 3ay = 0,\ f_y = 3y^2 - 3ax = 0.$$

That is,

$$x^2 = ay \text{ and } y^2 = ax,$$

or

$$x^4 = a^2y^2 = a^2\,ax = a^3x \text{ or } x(x^3 - a^3) = 0$$

or

$$x = 0, a.$$

Similarly,

$$y = 0, a.$$

Thus the stationary points are $(0, a)$ and $(0, 0)$.

Now,

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3a.$$

At (a,a)

$$f_{xx} = 6a, f_{yy} = 6a \text{ and } f_{xy} = -3a.$$

Also

$$f_{xx} \cdot f_{yy} - (f_{xy})^2 = 36a^2 - 9a^2 > 0.$$

Again, $f_{xx} = 6a$ is positive or negative depending on whether a is positive or negative. Hence f has a maximum or a minimum value at $x = y = a$ depending on whether a is negative or positive.
The optimum value is $f(a, a) = -a^3$.

At $(0, 0)$

$$f_{xx} = 0,\ f_{yy} = 0,\ f_{xy} = -3a \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = -9a^2 < 0.$$

Hence, $(0, 0)$ is not an extreme point.

EXAMPLE 8.4 Find the maximum and minimum values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$, and also find saddle points if any.

Solution

$$f_x(x, y) = 3x^2 - 3 = 0 \text{ when } x = \pm 1$$

$$f_y(x, y) = 3y^2 - 12 = 0 \text{ when } y = \pm 2.$$

Thus the function has four stationary points $(1, 2)$; $(-1, 2)$; $(1, -2)$; $(-1, -2)$.

Now,

$$f_{xx}(x, y) = 6x,\ f_{xy}(x, y) = 0,\ f_{yy}(x, y) = 6y.$$

At $(1, 2)$

$$f_{xx} = 6 > 0 \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = 72 > 0.$$

Hence, $(1, 2)$ is a point of minimum of the function.

At $(-1, 2)$

$$f_{xx} = -6 \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = -72 < 0.$$

Hence, the function has neither maximum nor minimum at $(-1, 2)$.

At $(1, -2)$

$$f_{xx} = 6 \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = -72 < 0.$$

Hence, the function has neither maximum nor minimum at $(1, -2)$.

At $(-1,-2)$

$$f_{xx} = -6 \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = 72 > 0.$$

Hence the function has a maximum value at $(-1,-2)$. Therefore, the maximum value is 38 and minimum value is 2 and the saddle points are $(-1, 2)$ and $(1,-2)$.

EXAMPLE 8.5 Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^4 + y^4$ is a minimum at the origin.

Solution

Now,

$$f_x(x, y) = 2x - 2y + 4x^3 = 0$$

if

$$x = 0, y = 0$$

$$f_y(x, y) = -2x + 2y + 4y^3 = 0$$

if

$$x = 0, y = 0$$

$$f_{xx}(x, y) = 2 + 12x^2, f_{xx}(0, 0) = 2$$

$$f_{yy}(x, y) = 2 + 12y^2, f_{yy}(0, 0) = 2$$

$$f_{xy}(x, y) = -2$$

Now,

$$f_{xx} \cdot f_{yy} - (f_{xy})^2 = 4 - 4 = 0.$$

Therefore, we cannot conclude about extremum at $(0, 0)$. Thus we arrive at a doubtful case and it requires further investigation.

$f(x, y)$ can be written as

$$f(x, y) = (x - y)^2 + x^4 + y^4$$

and

$$f(x, y) - f(0, 0) = (x - y)^2 + x^4 + y^4$$

which is greater than zero for all values of (x, y). Thus f has a minimum value at the origin.

EXAMPLE 8.6 Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$ has neither a maximum nor a minimum at the origin.

Solution

Now,

$$f_x(x, y) = 2x - 2y + 3x^2 + 5x^4$$

and

$$f_y(x, y) = -2x + 2y - 3y^2$$

at extrema,

$$f_x = 0 \text{ and } f_y = 0.$$

$$\therefore \quad 2x - 2y + 3x^2 + 5x^4 = 0 \text{ and } -2x + 2y - 3y^2 = 0.$$

Solving these two equations, we get

$$x = 0, y = 0.$$

$$\therefore \quad f_{xx}(x, y) = 2 + 6x + 20x^3, \; f_{xx}(0, 0) = 2$$

$$f_{yy}(x, y) = 2 - 6y, \; f_{yy}(0, 0) = 2$$

$$f_{xy}(x, y) = -2, \; f_{xy}(0, 0) = -2$$

$$\therefore \quad f_{xx}(0, 0) f_{yy}(0, 0) - f_{yy}^2(0, 0) = 0.$$

So, there is a doubtful case. Thus we apply direct method. The function can be written as

$$f(x, y) = (x-y)^2 + x^5 + (x-y)(x^2+xy+y^2)$$

or

$$f(x, y) - f(0, 0) = (x-y)^2 + x^5 + (x-y)(x^2+xy+y^2).$$

Now, we put $x = 2$ and $y = 1$, then we have

$$f(2, 1) - f(0, 0) = 40 > 0$$

and

$$f(1, 2) - f(0, 0) = -5 < 0.$$

Therefore near (0, 0), the sign of $f(x, y) - f(0, 0)$ is not fixed, i.e. (0, 0) is not a minimum or maximum point.

EXAMPLE 8.7 Find the maximum value of the function

$f(x, y) = \sin\frac{x}{2} \sin\frac{y}{2} \sin\frac{x+y}{2}$ defined on the triangular area bounded by the coordinate axes and the line $x + y = 2\pi$.

Solution

We have for maximum and minimum

$$f_x(x, y) = \sin\frac{y}{2}\left[\frac{1}{2}\cos\frac{x}{2}\sin\frac{x+y}{2} + \frac{1}{2}\sin\frac{x}{2}\cos\frac{x+y}{2}\right]$$

which gives

$$\tan\frac{x+y}{2} = -\tan\frac{x}{2} \qquad (1)$$

and

$$f_y(x, y) = \sin\frac{x}{2}\left[\frac{1}{2}\cos\frac{y}{2}\sin\frac{x+y}{2} + \frac{1}{2}\sin\frac{y}{2}\cos\frac{x+y}{2}\right] = 0$$

or

$$\tan\frac{x+y}{2} = -\tan\frac{y}{2}. \qquad (2)$$

From equations (1) and (2), we get

$$\tan\frac{x}{2} = \tan\frac{y}{2} \text{ or } \frac{x}{2} = \frac{y}{2} \text{ or } x = y$$

Again from equation (1)

$$\tan\left(\frac{2x}{2}\right) = -\tan\frac{x}{2}$$

or

$$\tan x = \tan\left(\pi - \frac{x}{2}\right)$$

or

$$x = \pi - \frac{x}{2} \text{ or } \frac{3x}{2} = \pi \text{ or } x = \frac{2\pi}{3}$$

and

$$y = \frac{2\pi}{3}.$$

The point $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ lies inside the triangle, defined in the problem.

Now, $$f_x(x, y) = \frac{1}{2}\sin\frac{y}{2}\sin\left(x + \frac{y}{2}\right)$$

and $$f_y(x, y) = \frac{1}{2}\sin\frac{x}{2}\sin\left(y + \frac{x}{2}\right).$$

$\therefore$ $$f_{xx}(x, y) = \frac{1}{2}\sin\frac{y}{2}\cos\left(x + \frac{y}{2}\right) = \frac{1}{2}\sin\frac{\pi}{3}\cos\pi$$

$$= \frac{-\sqrt{3}}{4} \text{ at } x = y = \frac{2\pi}{3}.$$

$$f_{yy}(x, y) = \frac{1}{2}\sin\frac{x}{2}\cos\left(y + \frac{x}{2}\right) = \frac{-\sqrt{3}}{4} \text{ at } x = y = \frac{2\pi}{3}$$

$$f_{yx}(x, y) = \frac{1}{4}\cos\frac{y}{2}\sin\left(x + \frac{y}{2}\right) + \frac{1}{4}\sin\frac{y}{2}\cos\left(x + \frac{y}{2}\right)$$

$\therefore$ $$f_{xy}\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \frac{1}{4}\cos\frac{\pi}{3}\sin\pi + \frac{1}{4}\sin\frac{\pi}{3}\cos\pi$$

$$= -\frac{1}{4}\frac{\sqrt{3}}{2}.$$

$\therefore$ $$f_{xx}f_{yy} - (f_{xy})^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{8}\right)^2$$

$$= 3 - \frac{3}{64} = \frac{189}{64} > 0$$

and $$f_{xx} = -\sqrt{3} < 0.$$

Hence there is a maximum at $x = y = \frac{2\pi}{3}$ and the maximum value is

$$f_{\max} = \sin\frac{\pi}{3}\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{8}.$$

8.7 LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Let $u = f(x, y)$ be a function of 2 variables x and y which are connected by the equation $\phi(x, y) = 0$.

We define a function

$$F = f + \lambda\phi$$

and consider all the variables x, y and λ are independent, λ is called *Lagrange's multiplier*.

At a stationary point of F, $dF = 0$.

Therefore,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial \lambda}d\lambda = 0.$$

Since dx, dy, $d\lambda$ are arbitrary (independent), the above equation will satisfy only if

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial \lambda} = 0.$$

The solutions of these equations give the stationary points of F. It can be shown that these stationary points are the stationary points of f.

A stationary point will be an extreme point of f if d^2F or d^2f keeps the same sign and will be a maximum or minimum depending on whether d^2F or d^2f is negative or positive.

The above results can also be extended for the function of three or more variables. Suppose $u = f(x, y, z)$ and the independent variables x, y and z are connected by $\phi(x, y, z) = 0$ and $\psi(x, y, z) = 0$. Then we consider

$$F = f + \lambda_1\phi + \lambda_2\psi$$

where λ_1, λ_2 are the Lagrange's multipliers.

As in the previous case, the stationary points of F as well as f, are obtained by solving the following equations

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0,\ \frac{\partial F}{\partial \lambda_1} = 0,\ \frac{\partial F}{\partial \lambda_2} = 0.$$

The function f will be maximum or minimum according as d^2F or d^2f is negative or positive.

EXAMPLE 8.8 Show that the greatest value of x^3y^2 and $x + y = k$ (k is a constant) is $108\,k^5/3125$.

Solution

Let

$$u = x^3y^2$$

and

$$F = x^3\,y^2 + \lambda(x + y - k)$$

$$F = u + \lambda(x + y - k)$$

where λ is Lagrange's multiplier.

At stationary point $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial \lambda} = 0.$

That is,
$$F_x = 3x^2y^2 + \lambda = 0$$
$$F_y = 2x^3y + \lambda = 0$$
$$F_z = x + y - k = 0.$$

First two equations give $3y = 2x$ or $y = \dfrac{2x}{3}$.

Substitute this value in $x + y = k$. Then $x + \dfrac{2x}{3} = k$

or
$$\frac{5x}{3} = k \text{ or } x = \frac{5k}{3}.$$

Therefore,
$$x = \frac{3k}{5} \text{ and } y = \frac{2x}{3} = \frac{2k}{5}.$$

Now, $\quad F_x = 3x^2y^2,\ F_y = 2x^3y,\ F_{xx} = 6xy^2,\ F_{yy} = 2x^3,\ F_{xy} = 6x^2y.$

At $\left(\dfrac{3k}{5}, \dfrac{2k}{5}\right)$,

$$F_{xx} = 6\cdot\frac{3k}{5}\cdot\frac{4k^2}{25} = \frac{72k^3}{125},\ F_{yy} = 2\frac{27k^3}{125} = \frac{54k^3}{125}$$

and
$$F_{xy} = 6\cdot\frac{9k}{5}\cdot\frac{2k^2}{25} = \frac{108k^3}{125}.$$

Also,
$$dx + dy = 0 \text{ or } dy = -dx \text{ since } x + y = k.$$

Now,
$$d^2F = F_{xx}(dx)^2 + 2F_{xy}\,dxdy + F_{yy}(dy)^2$$
$$= \frac{72k^3}{125}dx + 2\frac{108k^3}{125}dx(-dx) + \frac{54k^3}{125}(-dx)^2$$
$$= -\frac{90}{125}k^3(dx)^2 < 0.$$

Hence F i.e. u is maximum at $x = \dfrac{3k}{5}$, $y = \dfrac{2k}{5}$ and the maximum value is

$$\left(\frac{3k}{5}\right)^3\left(\frac{2k}{5}\right)^2 = \frac{108k^5}{3125}.$$

EXAMPLE 8.9 **Find the maximum value of $2x + y$ $(x, y > 0)$ where x and y satisfy the equation $x^2 + xy + y^2 = 3$.**

Solution

Let $$u = 2x + y$$

and $$F = u + \lambda(x^2 + xy + y^2 - 3) = 2x + y + \lambda(x^2 + xy + y^2 - 3),$$

where λ is Lagrange's multiplier.

$$F_x = 2 + 2x\lambda + \lambda y$$
$$F_y = 1 + \lambda x + 2\lambda y$$
$$F_\lambda = x^2 + xy + y^2 - 3.$$

At stationary points,

$$F_x = 2 + 2x\lambda + \lambda y = 0$$
$$F_y = 1 + \lambda x + 2\lambda y = 0$$
$$F_\lambda = x^2 + xy + y^2 - 3 = 0.$$

Solving the first two equations for x and y, we get $x = -\frac{1}{\lambda}$, $y = 0$.

Substituting these values in the third equation, we get

$$\frac{1}{\lambda^2} - 3 = 0 \text{ or } \frac{1}{\lambda} = \pm\sqrt{3}.$$

Thus, $x = \pm\sqrt{3}$ and $y = 0$ are the stationary points.

Now, we calculate second order derivative of F.

$$F_{xx} = 2\lambda,\ F_{yy} = 2\lambda,\ F_{xy} = \lambda.$$

From the equation $x^2 + xy + y^2 = 3$, we have

$$2xdx + xdy + ydx + 2ydy = 0$$

or $$(2x + y)dx + (x + 2y)dy = 0$$

or $$\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = -2 \text{ at } (\pm\sqrt{3}, 0).$$

Thus, $$dy = -2dx \text{ at } (\pm\sqrt{3}, 0).$$

Now, $$d^2F = F_{xx}(dx)^2 + 2F_{xy}dxdy + F_{yy}(dy)^2$$

$$= 2\lambda(dx)^2 + 2\lambda dx(-2dx) + 2\lambda(-2dx)^2$$

$$= 6\lambda(dx)^2 = -\frac{6}{x}(dx)^2.$$

Thus, $$d^2F = -\frac{6}{\sqrt{3}}(dx)^2 < 0 \text{ when } x = \sqrt{3}$$

and $$d^2F = \frac{6}{\sqrt{3}}(dx)^2 > 0 \text{ when } x = -\sqrt{3}.$$

$\therefore$ F, i.e. $u = 2x + y$ is maximum at $x = \sqrt{3}$ and the maximum value is $2\sqrt{3}$.

EXAMPLE 8.10 Find the maximum and minimum distances of the point (4, 5) from the circle $x^2 + y^2 = 4$.

Solution

Let (x, y) be any point on the given circle. Then the distance between (x, y) and $(4, 5)$ is $d = \sqrt{(x-4)^2 + (y-5)^2}$ or $d^2 = (x-4)^2 + (y-5)^2$, which is to be maximized or minimized subject to the condition $x^2 + y^2 = 4$.

Let $\quad F = d^2 + \lambda(x^2 + y^2 - 4) = (x-4)^2 + (y-5)^2 + \lambda(x^2 + y^2 - 4)$,

where λ is Lagrange's multiplier.

Now,
$$F_x = 2(x-4) + 2\lambda x$$
$$F_y = 2(y-5) + 2\lambda y$$
$$F_\lambda = x^2 + y^2 - 4.$$

At stationary points,
$$F_x = 2(x-4) + 2\lambda x = 0$$
$$F_y = 2(y-5) + 2\lambda y = 0$$
$$F_\lambda = x^2 + y^2 - 4 = 0.$$

Solving first two equations, we get
$$x = \frac{4}{1+\lambda} \text{ and } y = \frac{5}{1+\lambda}.$$

Substituting these values in the third equation, we get
$$\frac{16}{(1+\lambda)^2} + \frac{25}{(1+\lambda)^2} = 4 \text{ or } (1+\lambda)^2 = \frac{41}{4} \text{ or } 1+\lambda = \pm\frac{\sqrt{41}}{2}.$$

Therefore,
$$x = \pm\frac{8}{\sqrt{41}} \text{ and } y = \pm\frac{10}{\sqrt{41}}.$$

Thus the stationary points are $\left(\frac{8}{\sqrt{41}}, \frac{10}{\sqrt{41}}\right)$ and $\left(-\frac{8}{\sqrt{41}}, -\frac{10}{\sqrt{41}}\right)$.

The second order derivatives are
$$F_{xx} = 2 + 2\lambda,\ F_{yy} = 2 + 2\lambda,\ F_{xy} = 0.$$

Again,
$$2x\,dx + 2y\,dy = 0 \text{ or } dy = -\frac{x}{y}dx.$$

Now,
$$d^2F = F_{xx}(dx)^2 + 2F_{xy}\,dxdy + F_{yy}(dy)^2$$
$$= 2(1+\lambda)(dx)^2 + 2(1+\lambda)\frac{x^2}{y^2}(dx)^2$$

$$= 2(1 + \lambda)\left\{1 + \frac{x^2}{y^2}\right\}(dx)^2.$$

The term $2\left\{1 + \frac{x^2}{y^2}\right\}(dx)^2$ is positive.

Therefore, $d^2F < 0$ if $1 + \lambda < 0$, i.e. if $1 + \lambda = -\frac{\sqrt{41}}{2}$ and $d^2F > 0$ if $1 + \lambda = \frac{\sqrt{41}}{2}$.

Hence, F or d^2 or d is maximum when $1 + \lambda = -\frac{\sqrt{41}}{2}$ or $x = -\frac{8}{\sqrt{41}}$ and $y = -\frac{10}{\sqrt{41}}$ and is minimum when $1 + \lambda = \frac{\sqrt{41}}{2}$ or $x = +\frac{8}{\sqrt{41}}$ and $y = +\frac{10}{\sqrt{41}}$.

Therefore, the maximum distance is

$$d = \sqrt{\left(\frac{8}{\sqrt{41}} + 4\right)^2 + \left(\frac{10}{\sqrt{41}} + 5\right)^2} = \sqrt{45 + 4\sqrt{41}}$$

and the minimum distance is

$$d = \sqrt{\left(\frac{8}{\sqrt{41}} - 4\right)^2 + \left(\frac{10}{\sqrt{41}} - 5\right)^2} = \sqrt{45 - 4\sqrt{41}}.$$

EXAMPLE 8.11 Find the area of the greatest rectangle that can be inscribed in an ellipse.

or

Find the maximum value of $4xy$ subject to the condition $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

Let $2x$ and $2y$ be the length and breadth of the rectangle (see Figure 8.2)

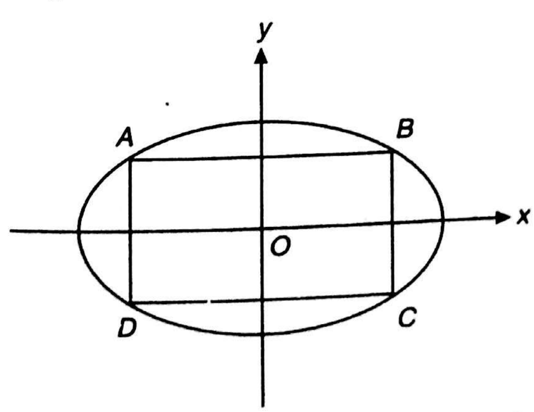

Figure 8.2 A rectangle inscribed inside the ellipse.

Then its area is $A = 4xy$, where x and y satisfy the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Now, the problem is to find the maximum value of A subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $$F = A+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=4xy+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

where λ is Lagrange's multiplier.

$$F_x = 4y+2\lambda x/a^2,\quad F_y = 4x+2\lambda y/b^2,\quad F_\lambda = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1.$$

At stationary points,

$$F_x = 4y+2\lambda x/a^2=0$$
$$F_y = 4x+2\lambda y/b^2=0$$
$$F_\lambda = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

Multiplying the first equation by x and the second equation by y and adding, we get

$$4xy+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0 \quad\text{or}\quad 4xy+\lambda=0\ \left[\text{as } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right].$$

Therefore,

$$xy=-\lambda/4 \quad\text{or}\quad y=-\frac{\lambda}{4x}.$$

From first equation

$$-\frac{\lambda}{2x}+\frac{\lambda x}{a^2}=0$$

or $$2x^2=a^2 \quad\text{or}\quad x=\pm\frac{a}{\sqrt{2}}.$$

Similarly, $$y=\pm\frac{b}{\sqrt{2}}.$$

Thus, the stationary points are $\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$, $\left(\frac{a}{\sqrt{2}},\frac{-b}{\sqrt{2}}\right)$, $\left(\frac{-a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$, $\left(\frac{-a}{\sqrt{2}},\frac{-b}{\sqrt{2}}\right)$.

Now, $$F_{xx} = 2\lambda / a^2,\ F_{yy} = 2\lambda / b^2,\ F_{xy} = 4.$$

Also, $$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy = 0 \qquad \left[\text{since } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right]$$

or $$dy = -\frac{b^2}{a^2}\frac{x}{y}dx.$$

$$\therefore \quad d^2F = F_{xx}(dx)^2 + 2F_{xy}\,dxdy + F_{yy}(dy)^2$$

$$= \frac{2\lambda}{a^2}(dx)^2 + 2.4\,dx\left(-\frac{b^2}{a^2}\frac{x}{y}\right)dx + \frac{2\lambda}{b^2}\left(-\frac{b^2}{a^2}\frac{x}{y}\right)^2(dx)^2$$

$$= \left[\frac{2\lambda}{a^2} - \frac{8b^2}{a^2}\frac{x}{y} + 2\lambda\frac{b^2}{a^4}\left(\frac{x^2}{y^2}\right)\right](dx)^2.$$

If $x = a/\sqrt{2},\ y = b/\sqrt{2}$ then $\lambda = -4xy = -2ab$ and hence

$$d^2F = \left[-\frac{4ab}{a^2} - \frac{8b^2}{a^2}\frac{a}{b} - 4ab\frac{b^2}{a^4}\frac{a^2}{b^2}\right](dx)^2$$

$$= -\frac{16b}{a}(dx)^2 < 0.$$

Thus F, i.e. A is maximum when $x = \dfrac{a}{\sqrt{2}},\ y = \dfrac{b}{\sqrt{2}}$.

Therefore, the dimensions of the rectangle is $a\sqrt{2}$ and $b\sqrt{2}$ and its maximum area is $2ab$.

EXAMPLE 8.12 Divide 10 into two parts such that the sum of whose squares is as small as possible. Method of Lagrange's multiplier may be used.

Solution

Let x and y be two parts. That is, $x + y = 10$.

The sum of squares of them is $x^2 + y^2$.

Now, the problem is to minimized $x^2 + y^2$ subject to the condition $x + y = 10$.

Let $$F = x^2 + y^2 + \lambda(x + y - 10).$$

$$\therefore \quad F_x = 2x + \lambda,\quad F_y = 2y + \lambda,\quad F_\lambda = x + y - 10.$$

At stationary points,

$$F_x = 2x + \lambda = 0$$
$$F_y = 2y + \lambda = 0$$
$$F_\lambda = x + y - 10 = 0.$$

The first two equations give $x = -\lambda/2,\ y = -\lambda/2$.

Putting these values in the third equation we get $\lambda = -10$. Therefore, $x = 5, y = 5$.
Again,

$$F_{xx} = 2,\ F_{yy} = 2,\ F_{xy} = 0 \text{ and } dx + dy = 0 \qquad [\because x + y = 10]$$

$$\therefore \quad d^2F = F_{xx}(dx)^2 + 2F_{xy}\,dx\,dy + F_{yy}(dy)^2$$
$$= 2(dx)^2 + 2(dx)^2 = 4(dx)^2 > 0.$$

Thus, F i.e. $x^2 + y^2$ is minimum when $x = 5$, $y = 5$. Hence the required parts are 5, 5.

EXAMPLE 8.13 Find a point in the plane $x + 2y + 3z = 13$ nearest to the point $(1, 1, 1)$ using the method of Lagrange's multiplier. (WBUT 2002)

Solution

Let (x, y, z) be any point on the plane. Then the distance between (x, y, z) and $(1, 1, 1)$ is $d = \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$.

Let $\quad F = d^2 + \lambda(x + 2y + 3z - 13)$
$$= (x-1)^2 + (y-1)^2 + (z-1)^2 + \lambda(x + 2y + 3z - 13).$$

Therefore, at stationary points,

$$F_x = 2(x-1) + \lambda = 0 \quad \text{or} \quad x = \frac{2-\lambda}{2}$$

$$F_y = 2(y-1) + 2\lambda = 0 \quad \text{or} \quad y = \frac{2-2\lambda}{2}$$

$$F_z = 2(z-1) + 3\lambda = 0 \quad \text{or} \quad z = \frac{2-3\lambda}{2}$$

$$F_\lambda = x + 2y + 3z - 13 = 0.$$

Putting the values of x, y, z in $x + 2y + 3z - 13 = 0$.

Then $\dfrac{2-\lambda}{2} + 2\dfrac{2-2\lambda}{2} + 3\dfrac{2-3\lambda}{2} = 13$ or $-14\lambda = 14$ or $\lambda = -1$.

$$\therefore \quad x = \frac{3}{2},\ y = 2,\ z = \frac{5}{2}.$$

Now, $\quad d^2 = (x-1)^2 + (y-1)^2 + (z-1)^2$

$$= (x-1)^2 + (y-1)^2 + \left(\frac{13 - x - 2y}{3} - 1\right)^2$$

$$= (x-1)^2 + (y-1)^2 + \left(\frac{10 - x - 2y}{3}\right)^2 = f(x, y) \qquad \text{(say)}$$

$$f_x = 2(x-1) - \frac{2}{9}(10-x-2y), \quad f_{xx} = 2 + \frac{2}{9} = \frac{20}{9}$$

$$f_y = 2(y-1) - \frac{4}{9}(10-x-2y), \quad f_{yy} = 2 + \frac{8}{9} = \frac{26}{9}$$

$$f_{xy} = -\frac{2}{9}(-2) = \frac{4}{9}.$$

Therefore, $$f_{xx} \cdot f_{yy} - (f_{xy})^2 = \frac{20}{9} \cdot \frac{26}{9} - \left(\frac{4}{9}\right)^2 > 0$$

and $$f_{xx} = \frac{20}{9} > 0.$$

Thus, f, i.e. d is minimum when $x = 3/2$, $y = 2$ and $z = 5/2$, which is the required point on the plane.

EXAMPLE 8.14 Find the volume of the largest rectangular parallelepiped that can be inscribed in the ethipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, using Lagrange's method of multipliers.

(WBUT 2002)

Solution

Let $2x$, $2y$, $2z$ be the length, breath and height of the rectangular parallelepiped. Therefore, its volume is $V = 8xyz$.

Now, the problem is to find the greatest value of $V = 8xyz$ subject to the conditions

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \; x > 0, y > 0, z > 0. \tag{1}$$

Let $$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

Now, at stationary points

$$\left.\begin{aligned} F_x &= 8yz + \frac{2x\lambda}{a^2} = 0 \\ F_y &= 8zx + \frac{2y\lambda}{b^2} = 0 \\ F_z &= 8xy + \frac{2z\lambda}{c^2} = 0. \end{aligned}\right\} \tag{2}$$

Multiplying the above equations by x, y, z and adding, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$$

or $$24xyz + 2\lambda = 0 \qquad \text{[Using equation (1)]}$$

or $$\lambda = -12\,xyz. \qquad (3)$$

Hence from (2), $x=\dfrac{a}{\sqrt{3}},\ y=\dfrac{b}{\sqrt{3}},\ z=\dfrac{c}{\sqrt{3}}$ and $\lambda=-\dfrac{4abc}{\sqrt{3}}$.

Differentiating (1) w.r.t. x, taking x and y are independent variables and z is dependent variable, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \text{ or } \frac{\partial z}{\partial x}=-\frac{x}{z}\frac{c^2}{a^2}.$$

Similarly, $$\frac{\partial z}{\partial y}=-\frac{y}{z}\frac{c^2}{a^2}.$$

Now, $$V_x = 8yz + 8xy\frac{\partial z}{\partial x} = 8yz + 8xy\left(-\frac{c^2}{a^2}\frac{x}{z}\right)$$

$$= 8yz - \frac{8x^2yc^2}{a^2z}.$$

$$V_{xx} = 8y\frac{\partial z}{\partial x} - \frac{16c^2xy}{a^2z} + \frac{8x^2yc^2}{a^2z^2}\frac{\partial z}{\partial x}$$

$$= -8y\frac{x}{z}\frac{c^2}{a^2} - \frac{16c^2xy}{a^2z} + \frac{8x^2yc^2}{a^2z^2}\left(-\frac{x}{z}\frac{c^2}{a^2}\right)$$

$$= -8\frac{b}{\sqrt{3}}\cdot\frac{c}{a} - 16\frac{c}{a}\cdot\frac{b}{\sqrt{3}} - 8\frac{b}{\sqrt{3}}\frac{c}{a} \qquad \left[\because \frac{x}{a}=\frac{1}{\sqrt{3}}=\frac{z}{c}\right]$$

$$= -32\frac{bc}{a\sqrt{3}} < 0.$$

Similarly, $$V_{yy} = -32\frac{ac}{a\sqrt{3}}.$$

$$V_{xy} = 8z + 8y\frac{\partial z}{\partial y} - \frac{8x^2c^2}{a^2z} - \frac{8x^2c^2y}{a^2}\left(-\frac{1}{z^2}\frac{\partial z}{\partial y}\right)$$

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$$= 8z - \frac{8x^2c^2}{a^2z} - 8y\frac{y\,c^2}{z\,b^2} - \frac{8x^2c^2y}{a^2z^2}\frac{y\,c^2}{z\,b^2}$$

$$= -\frac{16c}{\sqrt{3}}.$$

$$\therefore\ V_{xx}.V_{yy} - (V_{xy})^2 = 32\frac{bc}{a\sqrt{3}} \cdot 32\frac{ac}{b\sqrt{3}} - \frac{(16)^2c^2}{3}$$

$$= 256c^2 > 0.$$

Therefore, V is maximum when $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$ and the greatest volume of the rectangular parallelepiped is $8xyz = \frac{8abc}{3\sqrt{3}}$.

EXAMPLE 8.15 Finding the maximum of the function $u = xyz$ under the condition $x + y + z = S$, S is a given constant, prove that the inequality.

$$(x + y + z)^3 \geq 27xyz \ \text{ for } x \geq 0, y \geq 0, z \geq 0.$$

Solution

Let $F = u + \lambda(x + y + z - S) = xyz + \lambda(x + y + z - S)$.

Therefore, at stationary points

$$F_x = yz + \lambda = 0$$
$$F_y = xz + \lambda = 0$$
$$F_z = xy + \lambda = 0$$
$$F_\lambda = x + y + z - S = 0.$$

Multiplying the first, the second, and the third equations by x, y and z respectively and adding, we get

$$3xyz + \lambda(x + y + z) = 0$$

or $$3u + \lambda S = 0 \quad \text{or} \quad \lambda = -3u/S.$$

Therefore, $$xyz + \lambda x = 0 \quad \text{or} \quad u + \lambda x = 0$$

or $$x = -\frac{u}{\lambda} = \frac{u}{3u/S} = S/3.$$

Similarly, $$y = S/3,\ z = S/3.$$

Here, $x + y + z = S$. Let x, y be the independent and z be the dependent variables. Then $1 + \frac{\partial z}{\partial x} = 0$ or $\frac{\partial z}{\partial x} = -1$.

Similarly, $$\frac{\partial z}{\partial y} = -1.$$

Now, $$u_x = yz + xy\frac{\partial z}{\partial x} = yz - xy$$

$$u_{xx} = y\frac{\partial z}{\partial x} - y = -2y = 2S/3 < 0, \text{ at } (S/3, S/3, S/3).$$

Similarly, $$u_{yy} = -2S/3.$$

$$u_{xy} = z + y\frac{\partial z}{\partial y} - x = z - y - x = -S/3 .$$

$\therefore$ $$u_{xx} \cdot u_{yy} - (u_{xy})^2 = \frac{4S^2}{9} - \frac{S^2}{9} = \frac{S^2}{3} > 0.$$

Hence u is maximum at $(S/3, S/3, S/3)$ and maximum value is $\frac{S^3}{27}$.

Thus, $$xyz \le \frac{S^3}{27} \quad \text{or} \quad S^3 \ge 27xyz$$

$$(x+y+z)^3 \ge 27xyz.$$

Hence proved.

EXAMPLE 8.16 Show that the maximum and minimum values of $r^2 = x^2 + y^2$ where $ax^2 + 2hxy^2 + by^2 = 1$ are given by the roots of the quadratic

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2.$$

Solution

Let $$F = r^2 + \lambda(ax^2 + 2hxy + by^2 - 1)$$
$$= x^2 + y^2 + \lambda(ax^2 + 2hxy + by^2 - 1),$$
where λ is Lagrange's multiplier.
At stationary points

$$F_x = 2x + 2\lambda ax + 2\lambda hy = 0 \tag{1}$$
$$F_y = 2y + 2\lambda hx + 2\lambda by = 0 \tag{2}$$
$$F_\lambda = ax^2 + 2hxy + by^2 - 1 = 0. \tag{3}$$

Multiplying equations (1) and (2) by x and y and adding, we get

$$(x^2 + y^2) + \lambda(ax^2 + 2hxy + by^2) = 0$$

or $$r^2 + \lambda.1 = 0 \quad \text{or} \quad \lambda = -r^2. \qquad \text{[Using equation (3)]}$$

From equation (1),

$$x(1 + a\lambda) + hy\lambda = 0$$

$$x(1 - ar^2) = hyr^2 \tag{4}$$

or

and from equation (2),

$$y(1 + b\lambda) + hx\lambda = 0$$

or

$$y(1 - br^2) = hxr^2. \tag{5}$$

Multiplying equations (4) and (5), we get

$$xy(1 - ar^2)(1 - br^2) = xyh^2r^4$$

or

$$(1 - ar^2)(1 - br^2) = h^2r^4$$

or

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2.$$

The roots of this equation give the maximum and the minimum values.