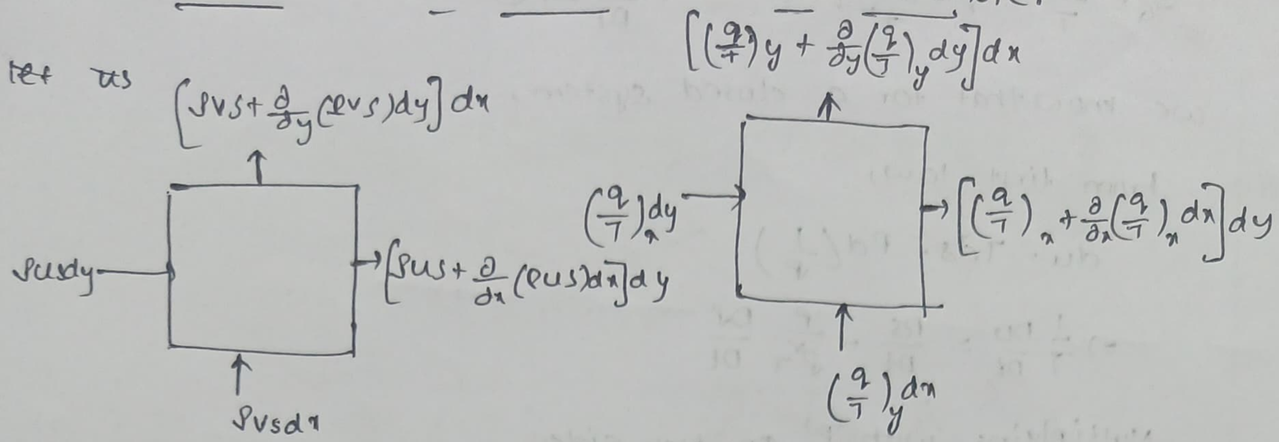
2. Energy generation in convection heat transfer

let us



we know that

$$\dot{S}_G = -\sum\frac{\dot{Q}}{T} - \sum_{in}\dot{m}s + \sum_{out}\dot{m}s + \frac{\partial s}{\partial t}$$

let S_G be the volumetric rate of entropy generation, so

$$S_G\,dx\,dy = \left[\left(\frac{q}{T}\right)_x+\frac{\partial}{\partial x}\left(\frac{q}{T}\right)_x dx\right]dy + \left[\left(\frac{q}{T}\right)_y+\frac{\partial}{\partial y}\left(\frac{q}{T}\right)_y dy\right]dx - \left(\frac{q}{T}\right)_x dy - \left(\frac{q}{T}\right)_y dx$$

$$+\left[\rho u s+\frac{\partial}{\partial x}(\rho u s)dx\right]dy + \left[\rho v s+\frac{\partial}{\partial y}(\rho v s)dy\right]dx$$

$$-\rho u s\,dy - \rho v s\,dx + \frac{\partial}{\partial t}(\rho s)\,dx\,dy$$

After simplifying above equation,

$$S_G = \frac{\partial}{\partial x}\left(\frac{q_x}{T}\right)+\frac{\partial}{\partial y}\left(\frac{q_y}{T}\right)+\frac{\partial}{\partial x}(\rho u s)+\frac{\partial}{\partial y}(\rho v s)+\frac{\partial}{\partial t}(\rho s)$$

Now,

$$S_G = \frac{1}{T}\frac{\partial}{\partial x}q_x - \frac{q_x}{T^2}\frac{\partial T}{\partial x} + \frac{1}{T}\frac{\partial q_y}{\partial y} - \frac{q_y}{T^2}\frac{\partial T}{\partial y} + \frac{\partial \rho}{\partial x}us + \frac{\partial u}{\partial x}\rho s + \frac{\partial s}{\partial x}\rho u + \frac{\partial \rho}{\partial y}vs$$

$$+\frac{\partial v}{\partial y}\rho s + \frac{\partial s}{\partial y}\rho v + \frac{\rho\, ds}{dt} + s\frac{\partial \rho}{\partial t}$$

$$= \frac{1}{T}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right) - \frac{1}{T^2}\left(q_x\frac{\partial T}{\partial x} - q_y\frac{\partial T}{\partial y}\right) + \rho\left(\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}\right)$$

$$+s\left[\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]$$

$$= \frac{1}{T}\nabla\cdot q - \frac{1}{T^2}q\cdot\nabla T + \rho\frac{Ds}{Dt} + s\left[\cancel{\frac{D\rho}{Dt}}^{\,0} + \rho\nabla\cdot v\right]$$ incompressible and continuity equation

$$S_G = \frac{1}{T}\nabla\cdot q - \frac{1}{T^2} q\cdot\nabla T + \rho\frac{Ds}{Dt} \rightarrow (1)$$

we know that for a closed system,

From first law,

$$du = Tds - Pd\left(\frac{1}{\rho}\right)$$

$$\Rightarrow \frac{1}{T}\frac{Du}{Dt} = \frac{Ds}{Dt} - \frac{P}{\rho^2 T}\frac{D\rho}{Dt}$$

Multiplying with 'ρ' on both sides,

$$\Rightarrow \rho\frac{Ds}{Dt} = \frac{\rho}{T}\frac{Du}{Dt} - \frac{P}{\rho T}\cancel{\frac{D\rho}{Dt}}\, 0 \text{ (incompressible)} \rightarrow (2)$$

1st law:

$$\rho\frac{Du}{Dt} = -\nabla\cdot q + \mu\phi - (3)$$

From (1), (2) & (3)

$$S_G = -\cancel{\frac{\nabla q}{T}} + \frac{\mu\phi}{T} + \cancel{\frac{\nabla q}{T}} - \frac{1}{T^2} q\cdot\nabla T.$$

$$S_G = \frac{k(\nabla T)^2}{T^2} + \frac{\mu}{T}\phi$$

$$S_G = \frac{k}{T^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] + \frac{\mu}{T}\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] - (1)$$

Now, assuming that flow is hydrodynamically developed $\left(\frac{\partial v}{\partial x} = 0\right)$

Equation (1) can be written as,

$$S_G = \frac{k}{T_0^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] + \frac{\mu}{T_0}\left[\left(\frac{\partial u}{\partial y}\right)^2\right] \rightarrow (2)$$

we know that,

characteristics entropy transfer rate is equal to

$$S_{G,C} = \left[\frac{q^2}{kT_0^2}\right] = \left[\frac{k(\Delta T)^2}{L^2 T_0^2}\right] \rightarrow (3)$$

(4)

Entropy generation number (N_S) = $\dfrac{\text{actual entropy generation rate } (S_a)}{\text{characteristic entropy transfer rate } (S_{a,c})}$

$$= \frac{\frac{k}{T_0^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] + \frac{\mu}{T_0}\left[\frac{\partial u}{\partial y}\right]^2}{\frac{k(\Delta T)^2}{L^2 T_0^2}}$$

let $\frac{T-T_0}{\Delta T} = \theta \Rightarrow \frac{\partial \theta}{\partial x} = \left(\frac{\partial T}{\partial x}\right) \times \frac{1}{\Delta T}, \quad \frac{\partial \theta}{\partial y} = \frac{\partial T}{\partial y} \times \frac{1}{\Delta T}$

$$\therefore N_S = \frac{L^2}{\Delta T^2} \times \left[\left(\frac{\partial \theta}{\partial x}\Delta T\right)^2 + \left(\frac{\partial \theta}{\partial y}\Delta T\right)^2 + \frac{\mu}{T_0} \times \frac{L^2 T_0^2}{k(\Delta T)^2}\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= (L^2)\left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\mu_0 L^2 T_0}{k(\Delta T)^2}\right.$$

since y is scaled with Ly

x is scaled with $\frac{L^2 u_0}{a}x$

u is scaled with $u_0 u$

$\frac{\partial \theta}{\partial x} = \frac{a}{L^2 u_0}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{L}$

$$N_S = \frac{L^2 a}{L^2 \times L^2 u_0^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{L^2}{L^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{u_0^2}{L^2}\right)\left.\frac{\partial u}{\partial y}\right)\left(\frac{u_0 L^2 T_0}{k(\Delta T}\right)$$

$$N_S = \frac{a^2}{L^2 u_0^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\mu u_0^2}{k\Delta T} \times \frac{T_0}{\Delta T}\right) \times \left(\frac{\partial u}{\partial y}\right)^2$$

As, $Pe = \frac{L u_0}{\alpha}, \quad Br = \frac{\mu u_0^2}{k(\Delta T)}, \quad \Omega = \frac{\Delta T}{T}$

$$N_S = \frac{1}{Pe^2}\left[\frac{\partial \theta}{\partial x}\right]^2 + \left[\frac{\partial \theta}{\partial y}\right]^2 + \frac{Br}{\Omega}\left[\frac{\partial u}{\partial y}\right]^2 \quad -(4)$$

we can write it has

$$N_S = N_C + N_y + N_F$$

where N_C is entropy generation due to heat transfer by axial conduction $= \frac{1}{Pe^2}\frac{d\theta}{dY}$

where N_y is entropy generation in normal direction $= \left(\frac{\partial\theta}{\partial Y}\right)^2$ and

N_F is entropy generation number; fluid friction $= \frac{Br}{\Omega}\left(\frac{\partial u}{\partial Y}\right)^2$

3) <u>Fluid friction versus heat transfer irreversibility:</u>

In convection problem, both fluid friction and heat transfer wave contributions to the rate of entropy generation.

Irreversibility distribution ratio(ϕ) $= \frac{N_F}{N_C + N_y}$

The irreversibility distribution ratio gives the information about which effects more the entropy generation.

Heat transfer irreversibility dominates over fluid friction irreversibility for $0 < \phi < 1$ and fluid friction dominates when $\phi > 1$.

Alternative irreversibility distribution parameter is Bejan number

$$\text{Bejan Number}(Be) = \frac{N_C + N_Y}{N_S} = \frac{N_C + N_Y}{N_C + N_Y + N_F} = \frac{1}{\frac{N_C + N_X + N_F}{N_C + N_Y}} = \frac{1}{1 + \frac{N_F}{N_C + N_Y}}$$

$$= \frac{1}{1+\phi}$$

$$\boxed{Be = \frac{1}{1+\phi}}$$

where $0 < Be < 1$ $\quad Be = 1 \rightarrow$ Heat transfer irreversibility dominates

$Be = 0 \rightarrow$ Fluid friction irreversibility dominates

$Be = 1/2 \rightarrow$ both the irreversibilities contribute equally.

3.1) **Steady flow between a fixed and a moving plate:**

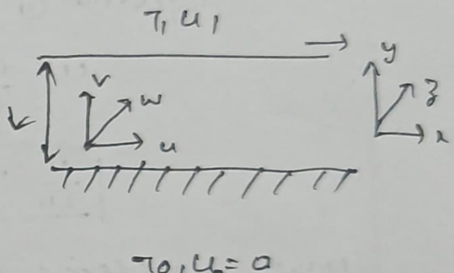
In this case the flow is in between two plates among which one is fixed and the other is moving.

Assumptions:

1) Steady flow
2) Constant properties (viscosity & thermal conductivity)
3) Incompressible flow

Consider two parallel plates placed at a distance 'L', the length of the plate is infinite and infinitely wide.

Consider lower part is at rest and temperature is T_0 and upper part is moving with a velocity u_1 and temperature is T_1.



u, v, w are components of velocity in x, y, z directions

Consider the flow to be in x direction and neglecting the end effects and edge effects [as infinite long and wide]

the continuity equation

$$\cancel{\frac{\partial u}{\partial x}}^{0} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{0} = 0$$

end effects

$$\frac{\partial v}{\partial y} = 0 \qquad \boxed{v = \text{constant}}$$

Assuming the plates are impermeable velocity along y direction is zero $v = 0$

x-direction Navier-Stoke's equation.

$$\cancel{\frac{\partial u}{\partial t}}^{0} + \cancel{u\frac{\partial u}{\partial x}}^{0} + \cancel{v\frac{\partial u}{\partial y}}^{0} + \cancel{w\frac{\partial u}{\partial z}}^{0} = -\frac{1}{\rho}\cancel{\frac{\partial p}{\partial x}}^{0} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{0} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{0}\right) + \cancel{\frac{F_x}{\rho}}^{0}$$

$v = 0$

Assuming flow to be steady and No body forces are acting in x-direction gives $\frac{\partial^2 u}{\partial y^2} = 0$

Integrating $\frac{\partial u}{\partial y} = c_1$

again integrating $u = c_1 y + c_2$ — (1)

considering boundary conditions at $y=0$, $u=0$ and at $y=L$, $u=u_1$

$\Rightarrow 0 = 0 + c_2$

$c_2 = 0$

$\Rightarrow u_1 = c_1 L$

$c_1 = \frac{u_1}{L}$

∴ equation (1) can be written as

$u = \frac{u_1}{L} y$

$$\boxed{\frac{u}{u_1} = \frac{y}{L}}$$

Now considering energy equation

Total enthalpy change = conduction + viscous dissipation + Heat generation

$$\Rightarrow \rho c_p \frac{DT}{Dt} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \mu\phi + q'''$$

$$\Rightarrow \frac{\partial T}{\partial x} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\mu\phi}{\rho c_p} + \frac{q'''}{\rho c_p}$$

after simplification and consideration of No heat generation taking place.

$$\Rightarrow \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{k}\left(\frac{\partial u}{\partial y}\right)^2 = 0$$

as $u = \frac{u_1}{L} y$

$\frac{du}{dy} = \frac{u_1}{L}$ $\qquad \frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left(\frac{u_1}{L}\right)^2$

on integration $\frac{\partial T}{\partial y} = -\frac{\mu}{k}\left(\frac{u_1}{L}\right)^2 y + c_1$

again integrations $T(y) = -\frac{\mu_1}{2k}\left(\frac{u_1}{L}\right)^2 y^2 + c_1 y + c_2$

Applying boundary conditions at $y=0, T=T_0 \Rightarrow C_2 = T_0$

at $y=L,\ T=T_1$

$$\Rightarrow T_1 = \frac{-\mu}{2k}\left(\frac{u_1}{L}\right)^2 L^2 + C_1 L + T_0$$

$$\Rightarrow (T_1 - T_0) + \frac{\mu}{2k}\left(\frac{u_1}{L}\right)^2 L^2 = C_1 L$$

$$\Rightarrow \frac{(T_1 - T_0)}{L} + \frac{\mu_1}{2k}\left(\frac{u_1}{L}\right)^2 L = C_1$$

$$\therefore T(y) = \frac{-\mu}{2k}\left(\frac{u_1}{L}\right)^2 y^2 + \left(\frac{T_1 - T_0}{L}\right) y + \frac{\mu}{2k}\left(\frac{u_1}{L}\right)^2 L y + T_0$$

$$T(y) = T_0 + \left(\frac{T_1 - T_0}{L}\right) y + \frac{\mu}{2k}(u_1)^2 \left[\frac{y}{L} - \frac{y^2}{L^2}\right]$$

converting into non-Dimensional form.

$$\frac{T(y) - T_0}{(T_1 - T_0)} = \frac{y}{L} + \frac{\mu\, u_1^2}{2k\,(T_1 - T_0)}\left[\frac{y}{L} - \frac{y^2}{L^2}\right]$$

This can be written as

$$\boxed{\theta = Y + \frac{Br}{2} Y[1-Y]} \qquad \frac{\mu u_1^2}{k(T_1 - T_0)} = Br\text{(Brinkman Number)}$$

$$\boxed{\frac{y}{L} = Y} \qquad \boxed{\frac{T_1 - T_0}{T_1 - T_0} = \theta} \qquad \text{as } \frac{u}{u_1} = \frac{y}{L}$$

can be written as $U = Y$

as entropy generation $N_S = \frac{1}{Pe}\left[\frac{\partial\theta}{\partial x}\right]^2 + \left(\frac{\partial\theta}{\partial Y}\right]^2 + \frac{Br}{\Omega}\left[\frac{\partial U}{\partial Y}\right]^2$

substituting the respective terms

$$N_S = \underbrace{\left[1 + \frac{Br}{2} - Br\,Y\right]^2}_{N_Y} + \underbrace{\left[\frac{Br}{\Omega}\right]}_{N_F}$$

→ contribution of fluid friction in entropy generation.

$N_C = 0$

Also, Bejan number $Be = \frac{N_C + N_Y}{N_S}$

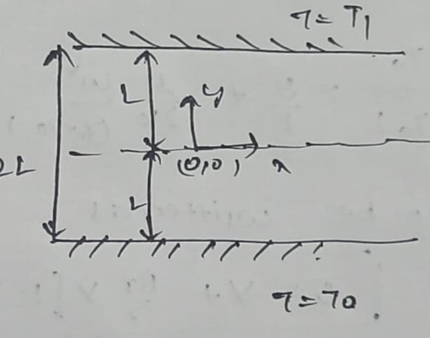
$$= \frac{\left[1 + \frac{Br}{2} - Br\,Y\right]^2}{\left[1 + \frac{Br}{2} - Br\,Y\right]^2 + \left[\frac{Br}{\Omega}\right]} = \frac{\frac{\left[2 + Br - 2Br\,Y\right]^2}{4}}{\frac{1}{4}\left[\left[2 + Br - 2Br\,Y\right]^2 + \left[\frac{4Br}{\Omega}\right]\right]}$$

$$Be = \frac{\left[2 + Br - 2Br\,Y\right]^2}{\left[2 + Br - 2Br\,Y\right]^2 + \frac{4Br}{\Omega}}$$

3.2 3. Steady flow between two fixed plates

In this case the flow is in between two fixed parallel plates (Poiseuille flow with differentially heated isothermal boundary condition.

Assumptions:



1) Flow is steady
2) Fully developed flow
3) Incompressible flow

consider a two parallel plates, placed at a distance L which are infinite in length and both plates are at rest.

at $y=0 \to u=0$ and $y=L \to u=0$

Temperature at down plate be T_0 and upper plate be T_1.

consider the flow is to be fully developed and unidirectional in x-direction i.e $\frac{\partial u}{\partial x} = 0$

continuity equation,

$$\cancel{\frac{\partial u}{\partial x}}^{0} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{0} = 0$$

$$\frac{\partial v}{\partial y} = 0 \Rightarrow v \text{ is constant.}$$

Navier strokes equation, in x-direction.

$$\cancel{\frac{\partial u}{\partial t}}^{0} + \cancel{u\frac{\partial u}{\partial x}}^{0} + \cancel{v\frac{\partial u}{\partial y}}^{0} + \cancel{w\frac{\partial u}{\partial z}}^{0} = \cancel{g_x}^{0} - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{0} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{0}\right)$$

$$\cancel{-\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{1}{\rho}\frac{\partial P}{\partial x}} \Rightarrow \frac{1}{\rho}\frac{\partial P}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \mu\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial P}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial P}{\partial x}.$$

on Integrating

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\left(\frac{\partial P}{\partial x}\right)y + C_1$$

again on integrating

$$u = \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)y^2 + C_1 y + C_2 \quad — (1)$$

Applying boundary condition. At $y=0, u=0 \Rightarrow C_2 = 0$

At $y = L, u=0$, $0 = \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)L^2 + C_1 L$

$$\Rightarrow C_1 = \frac{-1}{2\mu}\left(\frac{\partial P}{\partial x}\right)L$$

Substituting value of C_1 and C_2 in eq-(1)

$$u(y) = \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)y^2 - \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)Ly$$

At $y = L/2$ the velocity will be maximum so, $u_{max} = \frac{1}{2\mu}\frac{\partial P}{\partial x}\frac{L^2}{4} - \frac{1}{2\mu}\frac{\partial P}{\partial x}\frac{L^2}{2}$

$$= \frac{1}{2\mu}\frac{\partial P}{\partial x}\frac{L^2}{2}\left(\frac{-1}{2}\right)$$

$$\therefore u_{max} = \frac{L^2\,\partial P}{\partial x\,\mu}\left(\frac{-1}{8}\right)$$

Now $$\frac{u(y)}{u_{max}} = \frac{\frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)y^2 - \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)Ly}{\frac{\partial P}{\partial x}\frac{L^2}{\mu}\left(\frac{-1}{8}\right)}$$

$$\frac{u(y)}{u_{max}} = \frac{-4}{L^2}\left[y^2 - Ly\right]$$

$$\therefore \frac{u(y)}{u_{max}} = \frac{-4}{L^2}(y^2 - Ly) \quad -(2)$$

Applying boundary condition such as at $y=L$, $u=0$ and at $y=-L$, $y=0$

The equation becomes $\boxed{\frac{u(y)}{u_{max}} = \left(1 - \frac{y^2}{L^2}\right)}$ $-(3)$

Energy equation

$$\rho c_p \frac{\Delta T}{\Delta t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \phi + q'''$$

As there is no heat flux generation

$$\frac{\partial T}{\partial t} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\left(\cancel{\frac{\partial^2 T}{\partial x^2}}^{0} + \frac{\partial^2 T}{\partial y^2} + \cancel{\frac{\partial^2 T}{\partial z^2}}^{0}\right) + \frac{\phi}{\rho c_p}$$

As there is no temperature gradient, $\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$ becomes 0

$$0 = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial y^2}\right) + \underbrace{\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2}$$

↳ on simplification of viscous dissipation

$$\Rightarrow \frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left(\frac{\partial u}{\partial y}\right)^2$$

now substituting values of $u = u_m\left[1 - \frac{y^2}{L^2}\right]$

$$\frac{\partial u}{\partial y} = \frac{-2u_m y}{L^2}$$

now substituting values of $u = u_m\left[1 - \frac{y^2}{L^2}\right]$

$$\frac{\partial u}{\partial y} = \frac{-2u_m y}{L^2}$$

$$\Rightarrow \frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left(-\frac{2u_m}{L^2}y\right)^2$$

$$\Rightarrow \frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left(-\frac{4u_{max}}{L^2}\right)y^2$$

on integrating bothsides

$$\frac{\partial T}{\partial y} = -\frac{\mu}{k}\left(-\frac{4u_{max}}{L^2}\right)\frac{y^3}{3} + C_1$$

on further integration we get

$$T(y) = -\frac{\mu}{k}\left(\frac{2u_{max}}{L^2}\right)^2 \frac{y^4}{12} + C_1 y + C_2 \quad - (4)$$

Temperature of down plate

$$T_0 = -\frac{\mu}{k}\left(\frac{2u_{max}}{L^2}\right)^2 \frac{L^4}{12} + C_1(-L) + C_2$$

$$T_0 = -\frac{\mu}{k}\frac{u_{max}^2}{3} - C_1 L + C_2 \quad - (i)$$

Temperature of upper plate

$$T_1 = -\frac{\mu}{k}\frac{(2u_{max})^2}{L^4}\frac{L^4}{(12)} + C_1 L + C_2$$

$$T_1 = -\frac{\mu}{k}\frac{u_{max}^2}{3} + C_1 L + C_2 \quad - (ii)$$

now (i) + (ii)

$$T_0 + T_1 = -\frac{2\mu}{k}\frac{u_{max}^2}{3} + 2C_2$$

$$\Rightarrow C_2 = \left(\frac{T_0 + T_1}{2}\right) + \frac{\mu}{k}\frac{u_{max}^2}{3}$$

now (i) - (ii).

$$T_0 - T_1 = -2C_1 L$$

$$\Rightarrow C_1 = \frac{T_1 - T_0}{2L}$$

now substituting values of C_1 and C_2 in eq (4)

$$T(y) = -\frac{\mu}{k}\left(\frac{2u_{max}}{L^2}\right)^2 \frac{y^4}{12} + \left(\frac{T_1 - T_0}{2L}\right) y + \left(\frac{T_1 + T_0}{2}\right) + \frac{\mu}{3k} u_{max}^2$$

on rearranging them

$$T(y) = \frac{\mu}{3k} u_{max}^2 \left[1 - \frac{y^4}{L^4}\right] + \left(\frac{T_1 - T_0}{2}\right)\left[1 + \frac{y}{L}\right] + T_0 \quad - (5)$$

No we that $\frac{u}{u_{max}} = \left(1 - \frac{y^2}{L^2}\right)$ $\quad U = \frac{u}{u_{max}}$; $Y = \frac{y^2}{L^2}$

Dimensionless form $\quad U = 1 - Y^2$

From equation (5)

$$\frac{T - T_0}{T_1 - T_0} = \left(1 + \frac{y}{L}\right)\frac{1}{2} + \frac{\mu u_{max}^2}{3k(T_1 - T_0)}\left(1 - \frac{y^4}{L^4}\right)$$

2) $\boxed{\theta = \frac{1}{2}(1+y) + \frac{Br}{3}(1-y^4)}$ wher Br(Brinkmann no) = $\frac{\mu u_m a^2}{k\Delta T}$

$$\frac{\partial\theta}{\partial y} = \frac{1}{2} + \frac{Br}{3}(-4y^3)$$

$$\Rightarrow \frac{\partial\theta}{\partial y} = \frac{1}{2} - \frac{4}{3}Br(y^3)$$

Entropy generation number $N_S = \frac{1}{Pe^2}\cancel{\left(\frac{\partial\theta}{\partial x}\right)^2} + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \frac{Br}{\Omega}$

$$N_S = \left(\frac{1}{2} - \frac{4}{3}Br\,y^3\right)^2 + \frac{4Br}{\Omega}y^2$$

$$\boxed{N_S = \left(\frac{1}{2} - \frac{4}{3}Bry^3\right)^2 + 4\frac{Br}{\Omega}y^2}$$

where $N_Y = \frac{1}{2} - \frac{4}{3}Br\cdot y^3$ and $N_F = \frac{4Br}{\Omega}y^2$

∴ Then Bejan number $= \frac{N_C + N_Y}{N_S} = \frac{\left(\frac{1}{2} - \frac{4}{3}Br\cdot y^3\right)^2}{\left(\frac{1}{2} - \frac{4}{3}Br\cdot y^3\right)^2 + 4\frac{Br}{\Omega}y^2}$

$$= \frac{\left(\frac{3-8Bry^3}{36}\right)^2}{\left(\frac{3-8Bry^3}{36}\right)^2 + \frac{4Br}{\Omega}y^2}$$

$$Be = \frac{[3-8Bry^3]^2}{[3-8Bry^3]^2 + 144\left[\frac{Bry^2}{\Omega}\right]}$$

3.3 Convection in a round tube.

In this example the flow is in a round tube.

Assumptions:

1) steady flow
2) incompressible flow
3) Fully developed flow
4) 2-D in r-x plane $\frac{\partial}{\partial\theta} = 0$

Consider a round tube whose radius is r_0.

Assuming a steady, incompressible flow and fully developed 2-dimensional flow in spherical coordinate $\left[\frac{\partial}{\partial\theta}=0\right]$

Boundary conditions:

$r=0$, $u=$ finite

$r=r_0$, $u=0$ (at surface)

continuity in r-θ-x plane:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r)+\frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}(v_\theta)}+\cancelto{0}{\frac{\partial}{\partial x}(v_x)}=0$$

$$\frac{\partial}{\partial r}(r v_r)=0$$

$$r v_r=\text{constant}$$

$$\boxed{\therefore\ v_r=0}\ \text{everywhere,}$$

x-direction momentum equation:

$$\rho\left(\cancelto{0}{\frac{\partial u}{\partial t}}+\cancelto{0}{u\frac{\partial u}{\partial x}}+\cancelto{0}{v_r\frac{\partial u}{\partial r}}+\cancelto{0}{v_\theta\frac{\partial u}{\partial \theta}}\right)=\cancelto{0}{\rho g_x}-\frac{\partial p}{\partial x}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\cancelto{0}{\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}}+\cancelto{0}{\frac{\partial^2 u}{\partial x^2}}\right)$$

consider No body forces along x-direction.

$$\Rightarrow\ \frac{\partial p}{\partial x}=\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$\Rightarrow\ r\frac{du}{dr}=\int\frac{r}{\mu}\frac{\partial p}{\partial x}\,dr$$

$$\Rightarrow\ r\frac{du}{dr}=\frac{r^2}{2\mu}\frac{\partial p}{\partial x}+C_1$$

again integrating, $\boxed{u=\frac{r^2}{4\mu}\frac{\partial p}{\partial x}+C_1\ln r+C_2}$

Boundary conditions at $r=0$, $u=$ finite $\Rightarrow C_1=0$

at $r=r_0$, $u=0$

$$0=\frac{r_0^2}{4\mu}\frac{\partial p}{\partial x}+C_2$$

$$C_2=\frac{-r_0^2}{4\mu}\frac{\partial p}{\partial x}$$

$$u = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{r^2}{4} - \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{r_0^2}{4}.$$

$$\boxed{u = \frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2 - r_0^2\right)}$$

umax at $r = 0$, $u_{max} = u_m = \frac{-r_0^2}{4\mu}\frac{\partial p}{\partial x}$

$$\frac{u(r)}{u_m} = \frac{\frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2 - r_0^2\right)}{-\frac{1}{4\mu}\frac{\partial p}{\partial x}\left(\cancel{\ } - r_0^2\right)} \Rightarrow \frac{u(r)}{u_m} = \left(1 - \frac{r^2}{r_0^2}\right)$$

Now considering Energy equation in cylindrical co-ordinates

$$\rho c_p\left(u\frac{\partial T}{\partial x} + u_r\frac{\partial T}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \cancel{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)} + \cancel{\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)}$$

$u_r = 0$

considering,

2-D flow, $k \to$ thermal conductivity of fluid and conduction in axial direction is negligible compared to conduction in radial direction

$$\rho c_p u\frac{\partial T}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) \quad \text{--- (2)}$$

we assume that the heat flux at the wall is constant, so that the average fluid temperature must increase linearly with x (or)

$\frac{\partial T}{\partial x}$ is constant.

This means that the temperature profiles will be similar at various x distances along the tube. The boundary conditions on equation (2)

$$\frac{\partial T}{\partial r} = 0 \text{ at } r = 0$$

$$k\frac{\partial T}{\partial r}\Big|_{r=r_0} = q = \text{const.}$$

substituting (1) in (2).

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial x}u_0\left(1 - \frac{r^2}{r_0^2}\right)r$$

on integration, $r\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial x} u_0 \left(\frac{r^2}{2} - \frac{r^4}{4r_0^2}\right) + c_1$

on integration,

$$T = \frac{1}{\alpha}\frac{\partial T}{\partial x} u_0 \left(\frac{r^2}{4} - \frac{r^4}{16 r_0^2}\right) + c_1 \ln r + c_2$$

For finite value of T c_1 should be zero

the second boundary condition has been satisfied by nothing that the axial temperature gradient $\frac{\partial T}{\partial x}$ constant. The temperature distribution $T = T_c$ at $r=0$ so that $c_2 = T_c$

$$T - T_c = \frac{1}{\alpha}\frac{\partial T}{\partial x}\frac{u_0 r_0^2}{4}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4\right]$$

this can be written as

$$T = \frac{q r_0}{k}\left[\frac{-4x\alpha}{R^2 u_m} - \frac{r^2}{r_0} + \frac{1}{4}\frac{r^4}{r_0^4}\right]$$

Q-4 constant heat flux at wall

Representing in dimensionless form

$$\frac{u}{u_m} = \left(1 - \frac{r^2}{r_0^2}\right) \Rightarrow U = (1 - R^2)$$

$$\theta = \left[-4Y - R^2 + \frac{R^4}{4}\right]$$

Representing in cylindrical co-ordinate Entropy generation number

$$N_S = \frac{1}{Pe^2}\left[\frac{\partial \theta}{\partial Y}\right]^2 + \left[\frac{\partial \theta}{\partial R}\right]^2 + \frac{Br}{\Omega}\left[\frac{\partial U}{\partial R}\right]^2$$

$$N_S = \underbrace{\frac{16}{Pe^2}}_{N_C} + \underbrace{(R^3 - 2R)^2}_{N_R} + \underbrace{\frac{4Br}{\Omega} R^2}_{N_F}$$

$\frac{16}{Pe^2}$ → Axial conduction, which is square of Peclet number

a) Bejan number is $Be = \frac{N_C + N_R}{N_S}$

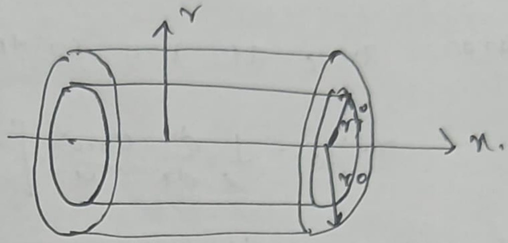
$$Be = \frac{16/Pe^2 + (e^3 - 2R)^2}{16/Pe^2 + (e^3 - 2R)^2 + 4Bre^2/Le\Omega}$$

3.4) <u>Convection</u> in a <u>circular annulus</u>.

In this case the flow is in between space between two concentric pipes

<u>Assumptions</u>:

1) Steady flow
2) Newtonian fluid
3) Incompressible fluid
4) Fully developed ($v_x = u = constant$)



Assuming a steady, incompressible flow, and fully developed flow consider both the pipes are at rest and infinitely long.

From cylindrical co-ordinates momentum equation,

$$\rho\left(\frac{\partial v_x}{\partial t} + v_r\frac{\partial v_x}{\partial x} + \frac{v_\theta}{r}\frac{\partial v_x}{\partial x} + v_x\frac{\partial v_x}{\partial x}\right) = \rho g_x - \frac{\partial P}{\partial x} + \mu\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial x}\right)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_x}{\partial \theta^2}}^{0} + \cancel{\frac{\partial^2 v_x}{\partial x^2}}^{0}\right]$$

$$\Rightarrow \frac{1}{\mu}\frac{\partial P}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial v_x}{\partial r}\right]$$

On integrating

$$\Rightarrow r\frac{\partial v_x}{\partial x} = \left(\frac{1}{\mu}\frac{\partial P}{\partial x}\right)\frac{r^2}{2} + C_1 \quad \Rightarrow \quad \frac{dv}{dx} = \frac{1}{\mu}\left(\frac{\partial P}{\partial x}\right)\frac{r}{2} + \frac{C_1}{r}$$

on integration,

$$v_x = \frac{1}{\mu}\frac{dP}{dx}\frac{r^2}{4} + C_1 \ln r + C_2$$

$$\Rightarrow v_x = u = \frac{1}{\mu}\frac{dP}{dx}\frac{r^2}{4} + C_1 \ln r + C_2 \quad [v_x = u, \text{ fully developed flow}]$$

From boundary conditions,

At $r = r_i$, $u = 0$

$r = r_o$, $u = 0$

$$\therefore C_1 = \frac{1}{4\mu}\frac{dp}{dx}\left(\frac{r_o^2 - r_i^2}{\ln r_o/r_i}\right)$$

and $\theta = \frac{1}{\mu}\frac{dp}{dx}\frac{r_i^2}{4} + C_1 \ln r_i + C_2$

$$\Rightarrow 0 = \frac{1}{\mu}\frac{dp}{dx}\frac{r_i^2}{4} + \frac{1}{4\mu}\frac{dp}{dx}\left(\frac{r_o^2 - r_i^2}{\ln r_o/r_i}\right)\ln r_i + C_2$$

$$\Rightarrow C_2 = \frac{1}{4\mu}\frac{dp}{dx}\left(\frac{(r_i^2 - r_o^2)(\ln r_i)}{(\ln r_o/r_i)} - r_i^2\right)$$

$\therefore$ If $\frac{r_i}{r_o} = \lambda$

By substituting in C_1 and C_2 we will get,

$$C_1 = \frac{1 + \ln(\lambda) + \lambda^2(\ln(\lambda) - 1)}{\ln \lambda}$$

$$C_2 = \frac{\lambda^2 - 1}{\ln \lambda}$$

And from energy equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{2u_{av}}{\alpha C_1}\left[1 - \frac{r^2}{r_o^2} + C_2 \ln\left(\frac{r}{r_o}\right)\right]\frac{\partial T}{\partial x}$$

As $\frac{\partial T}{\partial x} =$ const = Isothermal.

$$\therefore \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{2u_{av}}{\alpha C_1}\left[r - \frac{r^3}{r_o^2} + C_2 r \ln\left(\frac{r}{r_o}\right)\right]\frac{dT}{dx}$$

Integrating on both side,

$$r\frac{\partial T}{\partial r} = \frac{2u_{av}}{\alpha C_1}\left[\frac{r^2}{2} - \frac{r^4}{4r_o^2} + C_2\frac{r^2}{2}\ln\left(\frac{r}{r_o}\right) - \frac{C_2 r^2}{4r_o^2}\right]\frac{dT}{dx} + C_3$$

$$\Rightarrow \frac{\partial T}{\partial r} = \frac{2u_{av}}{\alpha C_1}\left[\frac{r}{2} - \frac{r^3}{4r_o^4} + \frac{C_2 r \ln\left(\frac{r}{r_o}\right)}{2} - \frac{C_2 r}{4r_o}\right]\frac{dT}{dx} + \frac{C_3}{r}$$

$$\therefore T = \frac{2u_{avg}}{\alpha C_1}\frac{dT}{dx}\left[\frac{r^2}{4} - \frac{r^4}{16r_o^4} + \frac{C_2 r^2 \ln(r/r_o)}{4} + \cancel{\frac{C_2 r^2}{8r_o}} - \cancel{\frac{C_2 r^2}{8r_o}}\right] + C_3 \ln r + C_4$$

$$\boxed{T = \frac{2u_{avg}}{\alpha C_1}\frac{dT}{dx}\left[\frac{r^2}{4} - \frac{r^4}{16r_o^4} \frac{C_2 r^2 \ln r/r_o}{4}\right] + C_3 \ln r + C_4}$$

and $\theta = \Gamma + \frac{4x}{C_2-1} + \frac{R^2}{C_2-1}\left[1+C_2\ln R - C_2\right] - \frac{R^4}{4(C_2-1)}$

where, Γ is constant of Integration

and x is scaled with $2u_m r_o^2/c_1 \alpha$,

Now $\frac{d\theta}{dx} = \left(\frac{4}{C_2-1}\right)$ $\frac{d\theta}{dy} = \frac{2R(1+C_2\ln R - C_2)}{C_2-1} + \frac{C_2R - R^3}{C_2-1}$

$$\frac{du}{dy} = \frac{2\left[2R - C_2/R^2\right]}{C_1}$$

$$\therefore N_S = \frac{1}{Pe^2}\left[\frac{d\theta}{dx}\right]^2 + \left[\frac{d\theta}{dy}\right]^2 + \left[\frac{du}{dy}\right]^2 \frac{Br}{\Omega}$$

$$\therefore N_S = \frac{1}{Pe^2}\frac{16}{(C_2-1)^2} + \left[\frac{2R(1+C_2\ln R - C_2)}{C_2-1}\right]^2 + \frac{4\,Br\left[2R - C_2/R^2\right]^2}{C_1^2\,\Omega}$$

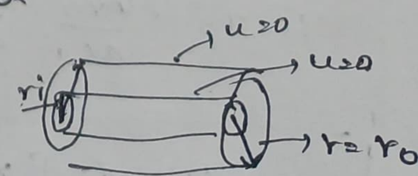
$$= N_C + N_Y + N_F$$

$$\therefore Be = \frac{N_C + N_Y}{N_S}$$

$$Be = \frac{\frac{1}{Pe^2}\frac{16}{(C_2-1)^2} + \left[\frac{2R(1+C_2\ln R) - C_2}{C_2-1}\right]^2}{\frac{1}{Pe^2}\frac{16}{(C_2-1)^2} + \left[\frac{2R(1+C_2\ln R - C_2)}{C_2-1}\right]^2 + \frac{4Br}{C_1^2\Omega}\left[2R - C_2/R^2\right]^2}$$

3.5 Axially moving concentric cylinder

In this case the flow is in between two concentric cylinders among which outer cylinder is fixed ($u=0$) and inner cylinder is moving with velocity u_o



Assumptions: 1) steady flow
2) Fully developed flow
3) incompressible flow
4) 2-D in r-x plane. $\frac{\partial}{\partial\theta} = 0$

Continuity in $r-\theta-x$ plane.

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}}(u_\theta^{0}) + \cancel{\frac{\partial}{\partial x}(u_x)}^{0} = 0$$

(Fully developed)

$$\frac{\partial}{\partial r}(r u_r) = 0$$

$r u_r = \text{constant}$.

$u_r = 0$ everywhere.

x-direction momentum

no gravity in x direction

$$\rho\left(\cancel{\frac{\partial u_x}{\partial t}}^{0} + u_x \cancel{\frac{\partial u_x}{\partial x}}^{0} + \underset{0}{u_r}\frac{\partial u_x}{\partial r} + u_\theta \cancel{\frac{\partial u_x}{\partial \theta}}^{0}\right) = \rho g_x - \cancel{\frac{\partial p}{\partial x}}^{0} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u_x}{\partial \theta^2}}^{0} + \cancel{\frac{\partial^2 u_x}{\partial x^2}}^{0}$$

$$\Rightarrow \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$

$$\Rightarrow r\frac{\partial u}{\partial r} = C_1$$

$$\Rightarrow u = C_1 \ln r + C_2$$

Applying boundary conditions, at $r = r_i$, $u = u_0$, at $r = r_0$ $u = 0$

$$u_0 = C_1 \ln r_i + C_2 \quad -(1)$$

$$0 = C_1 \ln r_0 + C_2 \quad -(2)$$

(1) - (2) $\quad u_0 = C_1 \ln(r_i/r_0)$

$$C_1 = \frac{u_0}{\ln\left(\frac{r_i}{r_0}\right)}$$

$$C_2 = -C_1 \ln r_0$$

$$= \frac{-u_0}{\ln\left(\frac{r_i}{r_0}\right)} \ln r_0$$

$$u = \frac{u_0}{\ln\left(\frac{r_i}{r_0}\right)} \ln r - \frac{u_0}{\ln\left(\frac{r_i}{r_0}\right)} \ln r_0$$

$$u = \frac{u_0}{\ln\left(\frac{r_i}{r_0}\right)} \ln\left(\frac{r}{r_0}\right)$$

$$\boxed{\therefore \frac{u}{u_0} = \frac{\ln(r/r_0)}{\ln(r_i/r_0)}}$$

Writing in Dimensionless form $\theta = \frac{\ln R}{\ln \pi}$ $R = r/r_o$

$\pi = r_i/r_o$

considering Energy equation in cylindrical co-ordinates

$$\rho C_p \left(u\frac{\partial T}{\partial x} + u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta}\right)^{0} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)^{0} + \frac{1}{r}\frac{\partial}{r\,\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)^{0}$$

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = u\frac{\partial T}{\partial x}(\rho C_p)$$

as $u = \frac{u_o \ln(r/r_o)}{\ln(r_i/r_o)}$

$$\Rightarrow \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial x}(\rho C_p)\left[\frac{u_o \ln(r/r_o)}{\ln(r_i/r_o)}\right]$$

$$r\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{u_o}{\alpha}\int \frac{r\ln(r/r_o)\,dr}{\ln(r_i/r_o)}$$

$$\left(r\frac{\partial T}{\partial r}\right) = \frac{u_o}{\alpha}\frac{\partial T}{\partial x}\frac{1}{\ln\left(\frac{r_i}{r_o}\right)}\int r\ln\left(\frac{r}{r_o}\right)dr$$

$$\Rightarrow r\frac{\partial T}{\partial r} = \frac{u_o}{\alpha}\frac{\partial T}{\partial x}\frac{1}{\ln\left(\frac{r_i}{r_o}\right)}\left[\frac{r^2}{2}\ln\left(\frac{r}{r_o}\right) - \frac{r^2}{4r_o} + c\right]$$

$$\Rightarrow \frac{\partial T}{\partial r} = \frac{u_o}{\alpha}\frac{\partial T}{\partial x}\frac{1}{\ln\left(\frac{r_i}{r_o}\right)}\left[\frac{r}{2}\ln\left(\frac{r}{r_o}\right) - \frac{r}{4r_o} + \frac{c}{r}\right]$$

Applying isoflux boundary condition at the inner cylinder and adiabatic boundary condition at the outer cylinder of the annulus, we get

$$\frac{\partial\theta}{\partial x} = \frac{1}{R[\ln R - \ln\pi]}\frac{\partial}{\partial R}\left(R\frac{\partial\theta}{\partial R}\right)$$

on applying seperation of variable method.

$$\theta = \frac{\Gamma + 4x + \pi^2\ln R + R^2[\ln(R\pi) - 1]}{\pi^2 + 2\ln\pi + 1}$$

where Γ constant of integration

$$N_s = \frac{1}{Pe^2}\left[\frac{d\theta}{dx}\right]^2 + \left(\frac{d\theta}{dR}\right)^2 + \frac{Br}{\Omega}\left[\frac{du}{dR}\right]^2$$

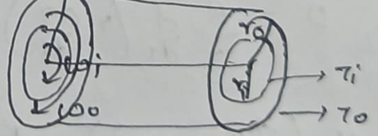
$$\Rightarrow N_s = \underbrace{\frac{16}{[\lambda^2 - 2\ln\lambda + 1]^2 Pe^2}}_{N_C} + \underbrace{\frac{[\lambda^2/R + R\,[2\ln(R/\lambda) - 1]]^2}{[\lambda^2 - 2\ln\lambda + 1]^2}}_{N_Y} + \underbrace{\frac{Br}{2\Omega R^2 \ln(\lambda)}}_{N_F}$$

$$\Rightarrow B_e = \frac{N_C + N_Y}{N_S} = \frac{\dfrac{16}{C_3^2 Pe^2} + \dfrac{\left\{\frac{\lambda^2}{R} + R\left[2\ln\left(\frac{R}{\lambda}\right) - 1\right]\right\}^2}{C_3^2}}{\dfrac{16}{C_3^2 Pe^2} + \dfrac{\left\{\frac{\lambda^2}{R} + R\left[2\ln\left(\frac{R}{\lambda}\right) - 1\right]\right\}^2}{C_3^2} + \dfrac{Br}{2\Omega R^2 \ln\lambda}}$$

Here $C_3 = \lambda^2 - 2\ln\lambda + 1$

3.6 ~~3.6~~ Rotating concentric cylinder

In this case the flow is in between the gap between two concentric cylinders, and both cylinders are rotating with different angular velocities.



Assumptions:

1) Steady flow.
2) fully developed flow
3) only non-zero component of velocity $= u_\theta$

Consider the steady flow maintained between two concentric cylinder by steady angular velocity of both cylinders. Let inner cylinder has radius r_i, angular velocity ω_i and Temperature T_i, while the outer cylinder has r_o, ω_o and T_o. The geometry is such that the only non-zero velocity component is u_θ and variables u_θ, T, P must be functions only of radius r.

continuity equation: $-\frac{\partial U}{\partial \theta} = 0$

θ momentum $\frac{\partial^2 u_0}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u_0}{r}\right) = 0$

r- momentum $\frac{dP}{\partial r} = \frac{\rho u_0^2}{r}$

Energy equation $0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right)^2$

with boundary conditions at each cylinder

At $r = r_0$; $u_\theta = r_0\omega_0$ $T = T_0$ $P = P_0$

At $r = r_i$, $u_\theta = r_i\omega_i$ $T = T_i$

The solution to the θ-momentum equation has form

$$u_\theta = C_1 r + \frac{C_2}{r}$$

on applying above boundary conditions

$$u_\theta = r_i\omega_i \frac{\left(\frac{r_0}{r} - \frac{r}{r_0}\right)}{(r_0/r_i - r_i/r_0)} + r_0\omega_0\left(\frac{r/r_i - r_i/r}{r_0/r_i - r_i/r_0}\right)$$

when the velocity distribution is substituted into the above energy equation, the temperature distribution

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right)^2$$

$$\Rightarrow -\mu\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right)^2 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

$$\Rightarrow -\mu\left[\frac{r_i\omega_i\left[-\frac{r_0}{r^2} - \frac{1}{r_0}\right]}{(r_0/r_i - r_i/r_0)} + \frac{r_0\omega_0\left[\frac{1}{r_i} + \frac{r_i}{r^2}\right]}{(r_0/r_i - r_i/r_0)}\right.$$

$$\left. - \left[\frac{r_i\omega_i\left(\frac{r_0}{r^2} - \frac{1}{r_0}\right)}{(r_0/r_i - r_i/r_0)} + \frac{r_0\omega_0\left(\frac{1}{r_i} - \frac{r_i}{r^2}\right)}{(r_0/r_i - r_i/r_0)}\right]\right]$$

$$= \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

$$\Rightarrow -\mu\left[\frac{r_i w_i}{\left(r_o/r_i - r_i/r_o\right)}\left(\frac{-2r_o}{r^2}\right) + \frac{r_o w_o}{\left(\frac{r_o}{r_i} - r_i/r_o\right)}\left(\frac{2r_i}{r^2}\right)\right] = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

$$\Rightarrow \frac{2r_i r_o}{\left(\frac{r_o}{r_i} - \frac{r_i}{r_o}\right)}\left[\frac{w_i}{r^2} - \frac{w_o}{r^2}\right] = \frac{k}{r}\frac{d}{dr}\left[r\frac{dT}{dr}\right]$$

$$\int \frac{2r_i r_o}{\left(\frac{r_o}{r_i} - \frac{r_i}{r_o}\right)} \frac{1}{r}\left(w_i - w_o\right)dr = r\frac{dT}{dr}$$

$$\Rightarrow r\frac{dT}{dr} = \frac{2r_i r_o}{\left(\frac{r_o}{r_i} - \frac{r_i}{r_o}\right)}\left(w_i - w_o\right)\ln r + c_1$$

on integrating

$$T(r) = \frac{2r_i r_o}{\left(\frac{r_o}{r_i} - \frac{r_i}{r_o}\right)}\left(w_i - w_o\right)\int \frac{\ln(r)}{r}dr + \int \frac{c_1}{r}dr$$

on solving by sufficient boundary conditions

$$\frac{T - T_o}{T_i - T_o} = \frac{P_r E\, r_o^4\left(1 - w_o/w_i\right)^2}{r_o^4 - r_i^4}\left(1 - \frac{r_i^2}{r^2}\right)\times\left[1 - \frac{\ln(r_i/r)}{\ln(r_o/r_i)}\right] + \frac{\ln(r/r_i)}{\ln(r_o/r_i)}$$

with proper scaling $\quad \pi = r_i/r_o,\ \lambda = w_o/w_i,\ R = r/r_o$

$$U = \frac{\pi}{1 - \pi^2}\left[\frac{\pi}{\lambda}\left(\frac{1 - R^2}{R}\right) + \left(\frac{R^2 - \pi^2}{\pi R}\right)\right]$$

$$\theta = \frac{Br\left(1 - \lambda\right)^2}{1 - \pi^4}\left(1 - \frac{\pi^2}{R^2}\right)\frac{\ln(R)}{\ln(\pi)} + \left(1 - \frac{\ln R}{\ln \pi}\right)$$

when $P_r E = Br$, where $Br \to$ Brinkman number

as $N_s = \frac{1}{P_e^2}\left(\frac{d\theta}{dx}\right)^2 + \left(\frac{d\theta}{dR}\right)^2 + \frac{Br}{\Omega}\left[R\frac{d}{dR}\left(\frac{U}{R}\right)\right]^2$ for this problem

$$N_s = \left[\frac{2Br\,\pi^2\Gamma_3\ln(R)}{R^3} + Br\,\Gamma_3\frac{1}{R}\left(1 - \frac{\pi^2}{R^2}\right) - \frac{\Gamma_4}{R}\right]^2 + \frac{4Br}{\Omega}\left[\frac{\Gamma_1 - \Gamma_2\lambda}{R^3}\right]^2 R^2$$

where $\Gamma_1 = \frac{\pi^2}{\lambda(1-\pi^2)}$, $\Gamma_2 = \frac{1}{1-\pi^2}$, $\Gamma_3 = \frac{(1-\lambda)^2}{(1-\pi^4)\ln\pi}$, $\Gamma_4 = \frac{1}{\ln\pi}$

$$B_e = \frac{N_c + N_R}{N_S} \quad \text{where } N_c = 0$$

$$\boxed{B_e = \frac{\left[2 Br\, \pi^2 \Gamma_3 \frac{\ln R}{R^3} + Br\, \Gamma_3 \frac{1}{R}\left(1 - \frac{\pi^2}{R^2}\right) - \frac{\Gamma_4}{R}\right]^2}{\left[2 Br\, \pi^2 \Gamma_3 \frac{\ln R}{R^3} + Br\, \Gamma_3 \frac{1}{R}\left(1 - \frac{\pi^2}{R^2}\right) - \frac{\Gamma_4}{R}\right]^2 + \frac{4Br}{\Omega}\left[\frac{\Gamma_1 - \Gamma_2 \pi}{R^3}\right]^2 R^2}}$$

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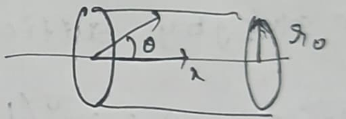
Non-Newtonian fluid flow in a circular tube:

convection in circular tube with constant heat flux

Momentum equation:

$$\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{du}{dr}\right)^n\right] = \frac{1}{\mu}\left(\frac{dP}{dx}\right)$$



where $n<1$ for pseudoplastic fluids

$n>1$ for Dilatent fluids

$n=1$ Newtonian fluid like air

$$\Rightarrow \frac{1}{r}\left[\frac{d}{dr} r\left(\frac{du}{dr}\right)^n\right] = \frac{1}{\mu}\left(\frac{dP}{dx}\right)$$

$$\Rightarrow \frac{d}{dr}\left[r\left(\frac{du}{dr}\right)^n\right] = \frac{r}{\mu}\left(\frac{dP}{dx}\right)$$

$$\Rightarrow r\left(\frac{du}{dr}\right)^n = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1$$

since constant heat flux at $r=0$, $\frac{\partial P}{\partial x} = 0 \Rightarrow C_1 = 0$

$$\Rightarrow \frac{\partial u}{\partial r} = \left(\frac{r}{2\mu}\frac{\partial P}{\partial x}\right)^{1/n}$$

$$\Rightarrow u = \frac{r^{1/n+1}}{n+1}\left(\frac{1}{2\mu}\frac{dP}{dx}\right)^{1/n} + C_2$$

$$\Rightarrow u = \frac{r^{1/n+1}}{\frac{1}{n}+1}\left(\frac{1}{2\mu}\frac{\partial p}{\partial x}\right)^{1/n} + C_2$$

Applying boundary condition $r = r_0, u = 0$

$$C_2 = \frac{-n\, r_0^{\,n+1/n}}{n+1}\left(\frac{1}{2\mu}\frac{\partial p}{\partial x}\right)^{1/n}$$

$$\Rightarrow u = \frac{r^{1/n+1}}{\frac{1}{n}+1}\left[\frac{1}{2\mu}\frac{dp}{dx}\right]^{1/n} - \frac{n r_0^{\,n+1/n}}{\frac{1}{n}+1}\left(\frac{1}{2\mu}\frac{\partial p}{\partial x}\right)^{1/n}$$

writing in dimensionless form

$$U = \left(1 - R^{n+1/n}\right)$$

velocity (u) is made dimensionless with maximum velocity u_m

putting $(n+1/n) = m$

Then the energy equation be

$$\frac{\partial \theta}{\partial x} = \frac{1}{R - R^{m+1}}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right)$$

Then we get temperature $= \frac{q'' r_0}{k}\left[-\frac{(m+2)\, r\alpha}{r_0^2 u_{max}} + \frac{r^2}{r_0^2}\left(\frac{m+2}{m}\right) - \frac{2}{m(m+2)} R^{m+2}\right]$

Expression in dimensionless terms of temperatures distribution.

$$\theta = \Gamma + \frac{2(m+2)}{m}X + \frac{m+2}{2m}R^2 - \frac{2}{m(m+2)}R^{m+2}$$

$$\frac{d\theta}{\partial X} = \frac{2(m+2)}{m}, \quad \frac{\partial U}{\partial R} = -mR^{m-1}, \quad \frac{\partial \theta}{\partial R} = \left(\frac{m+2}{m}\right)R - \frac{(m+2)}{m(m+2)}R^{m+1}$$

$$= \left(\frac{m+2}{m}\right)R - \frac{R^{m+1}}{m}$$

Entropy Generation number $(N_s) = \frac{1}{Pe^2}\left(\frac{\partial\theta}{\partial X}\right)^2 + \left(\frac{\partial \theta}{\partial R}\right)^2 + \frac{Br}{\Omega}\left(\frac{\partial U}{\partial R}\right)^2$

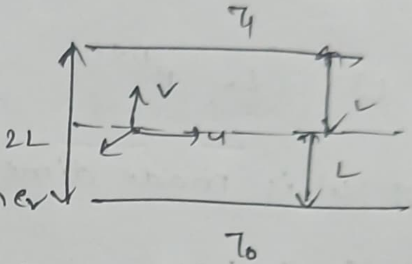
$$N_s = \underbrace{\frac{1}{Pe^2}\frac{4(m+2)^2}{m^2}}_{N_C} + \underbrace{\left[\left(\frac{m+2}{m}\right)R - \frac{R^{m+1}}{m}\right]^2}_{N_Y} + \underbrace{m^2 R^{2(m-1)}\frac{Br}{\Omega}}_{N_R}$$

Bejan number $Be = \frac{N_C + N_Y}{N_S}$

$$\boxed{Be = \frac{\left[\left(\frac{m+2}{m}\right)e - \frac{e^{m+1}}{m}\right]^2 + \frac{4}{Pe^2}\frac{(m+2)^2}{m^2}}{\frac{1}{Pe^2}\frac{4(m+2)^2}{m^2} + \left[\frac{(m+2)R - e^{m+1}}{m}\right]^2 + m^2 e^{2m-2}\frac{Br}{\Omega}}}$$

3.8 Non-Newtonian fluid flow through a channel with two parallel plates

consider a two parallel plates at at distance of $2L$ from each other



considering a fully developed and the powerlaw for non-Newtonian fluid

The momentum equation.

$$\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^n\right] = \frac{1}{\mu}\left(\frac{\partial P}{\partial x}\right)$$

$\therefore$ let $\frac{1}{\mu}\left[\frac{\partial P}{\partial x}\right] = c$ (say)

on integrating on both sides

$$\left(\frac{\partial u}{\partial y}\right)^n = cy$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\right) = (cy)^{1/n}$$

$$\Rightarrow \boxed{u = \frac{cy^{1/n+1}}{\frac{1}{n}+1} + c_2}$$ $\therefore$ let $\frac{n+1}{n} = m$

Applying Boundary conditions

At $y = -L$, $u = 0$ (No-slip condition)

$$0 = \frac{c(-L)^m}{m} + c_2 \quad -①$$

At $y = L,\ u = 0$

$$0 = \frac{c_1 (L)^m}{m} + c_2 \quad -②$$

① + ② $c_2 = 0$

on simplification we get $u = \left[1 - \frac{y^m}{L^m}\right] u_{max}$

on differentiation $\frac{du}{dy} = u_{max} \frac{m y^{m-1}}{L^m}$

From solution in 3.2 model (steady flow between two fixed plates)

$$\frac{d^2 T}{dy^2} = \frac{-\mu}{k}\left(\frac{du}{dy}\right)^2$$

$$\Rightarrow \frac{d^2 T}{dy^2} = \frac{-\mu}{k}\left[u_{max}\, m \frac{y^{m-1}}{L^m}\right]^2 = -\left[\frac{\mu u_{max}^2 m^2}{k L^{2m}}\right] y^{2m-2}$$

on integrating

$$\Rightarrow \frac{dT}{dy} = \left(-\frac{\mu u_{max}^2 m^2}{k L^{2m}}\right) \frac{y^{2m-2+1}}{(2m-2+1)} + c_1$$

$$\Rightarrow \frac{dT}{dy} = \left(\frac{-\mu u_{max}^2 m^2}{k L^{2m}}\right) \frac{y^{2m-1}}{2m-1} + c_1$$

on integrating

$$\Rightarrow T = \left(\frac{-\mu u_{max}^2 m^2}{k L^{2m}}\right) y^{2m} + c_1 y + c_2$$

$$\boxed{T = \left[\frac{-\mu u_{max}^2 m^2}{k L^{2m}}\right] y^{2m} + c_1 y + c_2}$$

changing above equation into dimensionless form

$$\theta = \Gamma + \left(\frac{m+1}{m}\right) x + \frac{m+1}{2m} y^2 - \frac{y^{2m+2}}{m(m+2)}$$

where y is scaled 'L' and x is scaled $\frac{L^2 u_{max}}{\alpha}$ and Γ is integration constant.

Entropy Generation number (N_S) $= \frac{1}{Pe^2}\left(\frac{d\theta}{dx}\right)^2 + \left(\frac{d\theta}{dy}\right)^2 + \frac{Br}{\Omega}\left(\frac{\partial u}{\partial y}\right)^2$

$$\frac{d\theta}{dx} = \frac{m+1}{m}, \quad \frac{\partial\theta}{\partial y} = \frac{2(m+1)y}{2m} - \frac{(m+2)y^{m+1}}{m(m+2)} = \frac{(m+1)y}{m} - \frac{y^{m+1}}{m}$$

$$\left(\frac{\partial u}{\partial y}\right)^2 = -m^3 y^{2m-2}$$

$$N_S = \underbrace{\frac{1}{Pe^2}\left(\frac{m+1}{m}\right)^2}_{N_c} + \underbrace{\left[\left(\frac{m+1}{m}y\right) - \frac{y^{m+1}}{m}\right]^2}_{N_y} + \underbrace{\left[\left(\frac{m^2 Br}{\Omega}\right) * \left(y^{2m-2}\right)\right]}_{N_p}$$

Bejan number (Be) $= \frac{N_c + N_y}{N_S}$

$$= \frac{\left(\frac{m+1}{m}\right)^2 \frac{1}{Pe^2} + \left[\left(\frac{m+1}{m}\right)y - \frac{y^{m+1}}{m}\right]^2}{\frac{1}{Pe^2}\left(\frac{m+1}{m}\right)^2 + \left(\frac{m+1}{m}y - \frac{y^{m+1}}{m}\right)^2 + m^2\frac{Br}{\Omega}y^{2m-2}}$$

$$\Rightarrow \frac{(m+1)^2\frac{1}{Pe^2} + \left((m+1)y - y^{m+1}\right]^2}{\frac{1}{Pe^2}(m+1)^2 + \left[(m+1)y - y^{m+1}\right]^2 + \frac{Br}{\Omega}y^{2m-2}m^4}$$

$$\boxed{Be = \frac{(m+1)^2\frac{1}{Pe^2} + \left[(m+1)y - y^{m+1}\right]^2}{(m+1)^2\frac{1}{Pe^2} + \left[(m+1)y - y^{m+1}\right]^2 + \frac{Br}{\Omega}m^4 y^{2m-2}}}$$