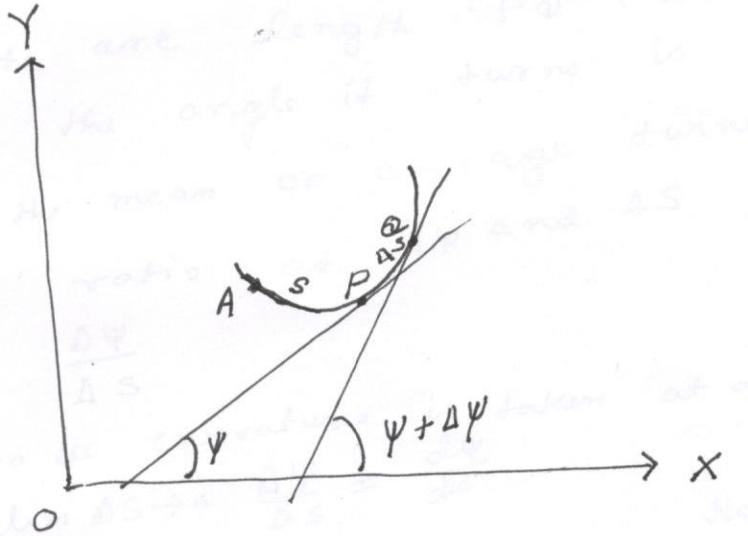
Curvature :- curvature of a curve at a particular point gives definite numerical measure of bending of a curve at that point.

Procedure :- By drawing tangents at adjacent points.



Description of the diagram:-

Suppose 'A' be a point on the curve then 'P' and 'Q' are two other points on the curve.

The arc length AP is 'S'

the arc length AQ is more the 'AP' so it was assumed 'AQ' has the length '$S+\Delta S$'

So the arc length 'PQ' is ΔS

Tangent at point 'P' makes angle ψ with X axis.

Tangent at point 'Q' makes angle $\psi + \Delta\psi$ with X axis.

So the arc length 'PQ' is ΔS

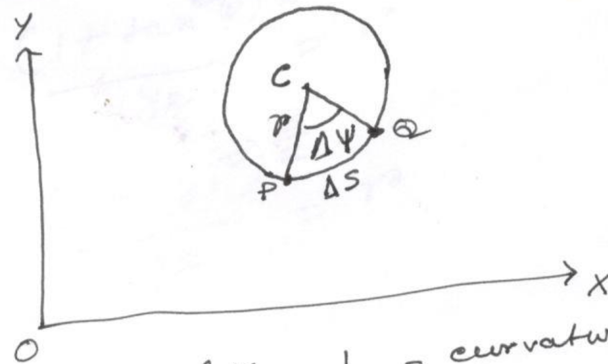
and the angle it turns is $\Delta\psi$

so the mean or average turning is the ratio of $\Delta\psi$ and ΔS

is $\dfrac{\Delta\psi}{\Delta S}$

so the curvature is taken at a point

$\lim_{\Delta S \to 0} \dfrac{\Delta\psi}{\Delta S} = \dfrac{d\psi}{ds}$



Now we extend 'PQ' & make a circle with centre 'C'

$PQ = \Delta S = r \cdot \Delta\psi$

so $\dfrac{d\psi}{ds} = \dfrac{1}{r}$ = curvature

so radius of curvature $\dfrac{ds}{d\psi} = r$

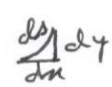
So radius of curvature is $r = \frac{ds}{d\psi}$

Now we know $\tan\psi = \frac{dy}{dx} = y_1$

$$\tan\psi = \frac{dy}{dx}$$

Now differentiate both sides with respect to s we get

$$\sec^2\psi \, \frac{d\psi}{ds} = \frac{d}{dx}(y_1)\,\frac{dx}{ds}$$



$$= y_2 \cos\psi$$

$$\text{or } \sec^3\psi \, \frac{d\psi}{ds} = y_2$$

$$\text{or } \frac{\sec^3\psi}{y_2} = \frac{ds}{d\psi} = r$$

$$\text{or } \frac{(1+\tan^2\psi)^{\frac{3}{2}}}{y_2} = r$$

$$\text{or } \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = r$$

Find the radius of curvature at the point (x, y) for the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

we know r = radius of curvature $= \dfrac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

differentiating both sides

$$\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = 0$$

$$\text{or} \quad \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = -\frac{2}{3} x^{-\frac{1}{3}}$$

$$\text{or} \quad \frac{dy}{dx} = -\frac{\cancel{2}}{\cancel{3}} x^{-\frac{1}{3}} \cdot \frac{\cancel{3}}{\cancel{2}} y^{\frac{1}{3}}$$

$$= -x^{-\frac{1}{3}} y^{\frac{1}{3}}$$

$$\frac{d^2y}{dx^2} = +\frac{1}{3} x^{-\frac{4}{3}} y^{\frac{1}{3}} + x^{-\frac{1}{3}} \cdot \frac{1}{3} y^{-\frac{2}{3}} \cdot \frac{dy}{dx}$$

$$= +\frac{1}{3} x^{-\frac{4}{3}} y^{\frac{1}{3}} + \frac{1}{3} x^{-\frac{1}{3}} \cdot y^{-\frac{2}{3}} \cdot x^{-\frac{1}{3}} y^{\frac{1}{3}}$$

$$= +\frac{1}{3} \left[x^{-\frac{4}{3}} y^{\frac{1}{3}} + x^{-\frac{2}{3}} y^{-\frac{1}{3}} \right]$$

$$= \frac{1}{3} x^{-\frac{2}{3}} y^{-\frac{1}{3}} \left[1 + x^{-\frac{2}{3}} y^{\frac{2}{3}} \right]$$

$$r = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\left(1 + x^{-\frac{2}{3}} y^{\frac{2}{3}}\right)^{\frac{3}{2}}}{\frac{1}{3} x^{-\frac{2}{3}} y^{-\frac{1}{3}} \left[x^{-\frac{2}{3}} y^{\frac{2}{3}} + 1 \right]}$$

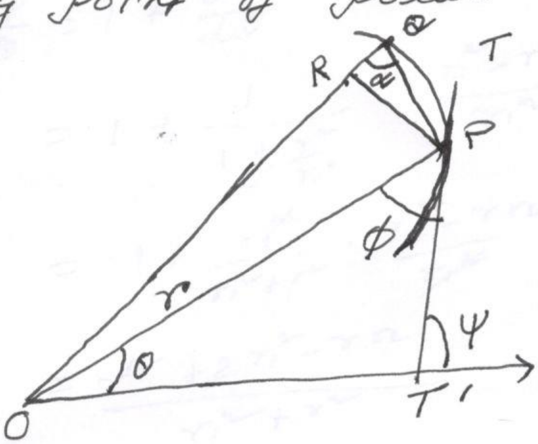
$$= 3 \cdot x^{\frac{2}{3}} y^{\frac{1}{3}} \cdot \left(1 + x^{-\frac{2}{3}} y^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

$$= 3 x^{\frac{1}{3}} \cdot y^{\frac{1}{3}} \left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

$$= 3 x^{\frac{1}{3}} y^{\frac{1}{3}} a^{\frac{1}{3}} = 3 \sqrt[3]{x y a}$$

Radius of curvature (Polar form)

Angle between Radius vector & tangent.
at any point of polar curve.



suppose P be a point (r, θ)

" Q also a neighbouring point $(r+\Delta r, \theta+\Delta\theta)$

The angle between the radius vector & tangent is ϕ

Now if point Q approaches to P and the angle α becomes

mean $\Delta\theta \to 0$

ϕ.

So we can write $\tan\phi = \lim_{\Delta\theta\to 0} \tan\alpha$

$OP = r$, $OQ = r+\Delta r$, $QR = r+\Delta r - r\cos\Delta\theta$

$PR = r\sin\Delta\theta$ $OR = r\cos\Delta\theta$

so $\tan\alpha = \dfrac{r\sin\Delta\theta}{r+\Delta r - r\cos\Delta\theta} = \dfrac{r\sin\Delta\theta}{r(1-\cos\Delta\theta)+\Delta r}$

$\tan\alpha = \dfrac{r\frac{\sin\Delta\theta}{\Delta\theta}}{r\left(\frac{1-\cos\Delta\theta}{\Delta\theta}\right)+\frac{\Delta r}{\Delta\theta}}$

$= \lim_{\Delta\theta\to 0} \tan\phi = \dfrac{r}{\frac{dr}{d\theta}}$

$= \dfrac{r}{f'(\theta)}$

$\tan\phi = \frac{r}{r_1}$ $\qquad\qquad \phi = \tan^{-1}\frac{r}{r_1}$

and $\psi = \theta + \phi$

$$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$= 1 + \frac{1}{1+\frac{r^2}{r_1^2}} \cdot \frac{r_1^2 - r r_2}{r_1^2}$$

$$= 1 + \frac{\cancel{r_1^2}}{r_1^2+r^2} \cdot \frac{r_1^2 - r r_2}{\cancel{r_1^2}}$$

$$= \frac{r^2 + 2r_1^2 - r r_2}{r_1^2 + r^2}$$

Now

$x = r\cos\theta \quad = f(\theta)\cos\theta \qquad ds^2 = dx^2 + dy^2$

$y = r\sin\theta \quad = f(\theta)\cdot\sin\theta$

so $f'(\theta)\cdot\cos\theta - \sin\theta\cdot f(\theta) = \frac{dx}{d\theta}$

$f'(\theta)\sin\theta + \cos\theta\cdot f(\theta) = \frac{dy}{d\theta}$

$f'(\theta)^2\cos^2\theta - 2f'(\theta)f(\theta)\cos\theta\sin\theta + f'(\theta)^2\sin^2\theta = \left(\frac{dx}{d\theta}\right)^2$

$f'(\theta)^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + f'(\theta)^2\cos^2\theta = \left(\frac{dy}{d\theta}\right)^2$

Now $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + r_1^2 = \left(\frac{ds}{d\theta}\right)^2$

so $\frac{ds}{d\theta} = (r^2 + r_1^2)^{\frac{1}{2}}$

$$\frac{ds}{d\psi} = \frac{\frac{ds}{d\theta}}{\frac{d\psi}{d\theta}} = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - r r_2}$$

Find the radius of curvature at point θ of the curve

$$r = a \cdot e^{\theta \cdot \cot\alpha}$$

$$r_1 = a \cdot \cot\alpha \, e^{\theta \cdot \cot\alpha} = \frac{dr}{d\theta}$$

$$r_2 = a \cot\alpha \cdot \cot\alpha \cdot e^{\theta \cdot \cot\alpha}$$

$$= a \cot^2\alpha \, e^{\theta \cot\alpha}$$

so radius of curvature

$$= \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - r r_2} = \frac{\left(a^2 e^{2\theta \cot\alpha} + a^2 \cot^2\alpha \, e^{2\theta \cot\alpha}\right)^{\frac{3}{2}}}{a^2 e^{2\theta \cot\alpha} + 2a^2 \cot^2\alpha \, e^{2\theta \cot\alpha} - a^2 \cot^2\alpha \, e^{2\theta \cot\alpha}}$$

$$= \frac{\left(a^2 e^{2\theta \cot\alpha}\right)^{\frac{3}{2}} \left(1 + \cot^2\alpha\right)^{\frac{3}{2}}}{a^2 e^{2\theta \cot\alpha} \left[1 + \cot^2\alpha\right]}$$

$$= a e^{\theta \cot\alpha} \cdot \frac{\operatorname{cosec}^3\alpha}{\operatorname{cosec}\alpha} = r \operatorname{cosec}\alpha$$

$$= \frac{r}{\sin\alpha}$$