CHAPTER 4

Mean Value Theorem

4.1 INTRODUCTION

The Lagrange's and Cauchy's Mean Value Theorems (MVTs) are very important results in calculus and they are used in many branches in science and engineering. The general form of MVT due to Taylor and Maclaurin are presented here. These two theorems are used to expand a function into finite and infinite series.

4.2 ROLLE'S THEOREM

Let a function f be defined on a closed interval $[a, b]$. Further suppose that

(i) f is continuous on $[a, b]$,

(ii) f is derivable in the open interval (a, b), and

(iii) $f(a) = f(b)$.

Then there exists at least one value of x say c, where $a < c < b$, such that $f'(c) = 0$.

Note: The three conditions of Rolle's theorem are a set of sufficient conditions. Sometimes all these conditions are not necessary to get the result.

Geometrical interpretation of Rolle's theorem: If the graph, $y = f(x)$ be continuous throughout the interval from a to b; and if the curve has a tangent at every point on it from a to b except possibly at the two extreme points a and b; and has the ordinates at two points a, b equal, then there must exist at least one point on the curve between a and b where the tangent is parallel to the x-axis (Figures 4.1(a) and (b)).

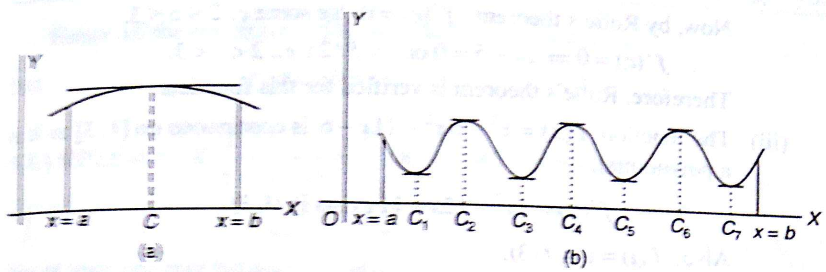

Figure 4.1 Geometrical interpretation of Rolle's theorem.

EXAMPLE 4.1 Verify Rolle's theorem for each of the following cases:

(i) $f(x)=|x|, -1\le x\le 1$; (WBUT 2003)

(ii) $f(x)=x^2-5x+6$ in $2\le x\le 3$ (WBUT 2001)

(iii) $f(x)=x^3-6x^2+11x-6$ in $1\le x\le 3$ (WBUT 2002)

(iv) $f(x)=\sin x$, in $[c, \pi]$

(v) $f(x)=3+(x-1)^{1/3}$ in $0\le x\le 2$.

Solution

(i) The function $f(x)=|x|$ is continuous on [–1, 1]. But, it is not differentiable at $x=0$, shown below.

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-0}{h}$$

$$=\begin{cases}\lim\limits_{h\to 0^-}\dfrac{-h}{h}=\lim\limits_{h\to 0^-}(-1)=-1\\ \lim\limits_{h\to 0^+}\dfrac{h}{h}=\lim\limits_{h\to 0^+}(1)=1.\end{cases}$$

This shows that the left and right derivatives are not equal and hence $f'(x)$ does not exist at $x=0$, i.e. in (–1, 1). Therefore, all the conditions of Rolle's theorem do not be satisfied. Hence Rolle's theorem is not applicable for $f(x)=|x|$.

(ii) The function $f(x)=x^2-5x+6$ is continuous on [2, 3] as it is a polynomial (every polynomial is continuous everywhere).

$$f'(x)=2x-5 \text{ exists for all } x \text{ in } (2, 3).$$

$$f(2)=0=f(3).$$

Hence all the conditions of Rolle's theorem are satisfied for $f(x)=x^2-5x+6$.

Now, by Rolle's theorem, $f'(c)=0$ for some c, $2<c<3$.

$$f'(c)=0 \Rightarrow 2x-5=0 \text{ or } c=5/2 \text{ i.e., } 2<c<3.$$

Therefore, Rolle's theorem is verified for this function.

(iii) The function $f(x)=x^3-6x^2+11x-6$ is continuous on [1, 3] as it is a polynomial.

$$f'(x)=3x^2-12x+11 \text{ exists in } (1, 3).$$

Also, $f(1)=c=f(3)$.

Hence, $f(x)$ satisfied all the conditions of Rolle's theorem.

Now, $f'(c)=0, 1<c<3$ implies $3c^2-12c+11=0$

or $$c=2\pm 1/\sqrt{3}.$$

Hence, $c=2+1/\sqrt{3}, 2-1/\sqrt{3}$ and $1<2+1/\sqrt{3}<3$ and $1<2-1/\sqrt{3}<3$, i.e. $1<c<3$ for both c.

Hence, Rolle's theorem is verified for this function.

(iv) $f(x)=\sin x$ is continuous on $[0, \pi]$.

$f'(x)=\cos x$ exists for all values of x in $(0, \pi)$.

Also, $f(0)=0=f(\pi)$.

Hence $f(x)$ satisfied all the conditions of Rolle's theorem.

Hence $f'(c)=0$ for some c, $0<c<\pi$,

implies $\cos c=0$ or $c=\pi/2$, i.e. $0<\pi/2<\pi$.

Therefore, Rolle's theorem is verified for this function.

(v) Here the function $f(x)=3+(x-1)^{1/3}$ is continuous on [0, 2].

$f'(x)=\dfrac{1}{3}(x-1)^{-2/3}$ does not exist at $x=1$.

That is, $f(x)$ is not differentiable at $x=1$, i.e., in (0, 2).

Thus, the function $f(x)$ does not satisfy all the conditions of Rolle's theorem. Hence, Rolle's theorem is not applicable for this function.

EXAMPLE 4.2 Show that for the function $f(x)=\dfrac{1}{x}+\dfrac{1}{1-x}; 0\le x\le 1$ all the conditions of Rolle's theorem do not satisfied, but $f'(c)=0$ for $0<c<1$.

Solution

The function is continuous in $0<x<1$, not in $0\le x\le 1$, and $f'(x)=\dfrac{1}{(1-x)^2}-\dfrac{1}{x^2}$ exists in $0<x<1$ and $f(0)\neq f(1)$ (both being undefined).

Hence all the conditions of Rolle's theorem do not hold.

But,
$$f'(c) = 0 \text{ when } c = 1/2, 0 < c < 1.$$

EXAMPLE 4.3 If $\frac{a}{5}+\frac{b}{4}+\frac{c}{3}+\frac{d}{2}+e=0$ show that the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

has at least one root between 0 and 1.

Solution

Let $f(x)=\frac{ax^5}{5}+\frac{bx^4}{4}+\frac{cx^3}{3}+\frac{dx^2}{2}+ex, 0 \le x \le 1.$

Therefore,
$$f(0) = 0 \text{ and } f(1)=\frac{a}{5}+\frac{b}{4}+\frac{c}{3}+\frac{d}{2}+e=0$$
(by the given condition).

Thus
$$f(0) = f(1).$$

As $f(x)$ is a polynomial, it is continuous on [0, 1] and its derivative $f'(x) = ax^4 + bx^3 + cx^2 + dx + e$ exists in (0, 1).

$\therefore$ by Rolle's theorem, $f'(c_1) = 0, 0 < c_1 < 1,$

or
$$ac_1^4 + bc_1^3 + c\,c_1^2 + dc_1 + e = 0.$$

Hence $c_1, 0 < c_1 < 1,$ is a root of the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$

EXAMPLE 4.4 Show that the root of the equation $e^x \cos x + 1 = 0$ lies between any pair of roots of $e^x \sin x - 1 = 0$.

Solution

Let $F(x) = e^{-x}(e^x \sin x - 1) = \sin x - e^{-x}$ and a and b be any two roots of $e^x \sin x - 1 = 0$, i.e. $F(a) = F(b) = 0$.

Now,

(i) $F(x)$ is continuous on $[a, b]$, as $\sin x$ and e^{-x} are both continuous there

(ii) $F'(x) = \cos x + e^{-x}$, exists in (a, b) and

(iii) $F(a) = F(b)$.

Therefore, by Rolle's theorem $F'(c) = 0$ in $a < c < b$.

That is, $\cos c + e^{-c} = 0$ in (a, b)

or
$$e^c \cos c + 1 = 0, a < c < b.$$

Therefore, c, in (a, b), is a root of the equation $e^x \cos x + 1 = 0$.

EXAMPLE 4.5 Show that Rolle's theorem is not applicable to $f(x) = \tan x$ on $[0, \pi]$, although $f(0) = f(\pi)$. (WBUT 2004)

Solution

The function $f(x) = \tan x$ is continuous on $[0, \pi]$ except at $x = \pi/2$ and $f'(x) = \sec^2 x$ does not exist at $x = \pi/2$. But, $f(0) = f(\pi)$. Hence $f(x)$ does not satisfy all the conditions of Rolle's theorem and consequently Rolle's theorem is not applicable for $f(x) = \tan x$ on $[0, \pi]$.

4.3 LAGRANGE'S MEAN VALUE THEOREM

If a function f is

(i) continuous in the closed interval $[a, b]$
(ii) derivable in the open interval (a, b)

Then there exists at least one value of x, say c, such that

$$\frac{f(b) - f(a)}{b - a} = f'(c), \text{ for } a < c < b \tag{4.1}$$

Note: The Lagrange's mean value theorem is known as mean value theorem or in short MVT.

Corollary 4.1 $h - \theta$ form of MVT.

When $b = a + h$ then $c = a + \theta h$, $0 < \theta < 1$ where as $a < c < b$.
Therefore, the Lagrange's MVT in $[a, a + h]$ is

$$f(a + h) - f(a) = hf'(a + \theta h), \; 0 < \theta < 1.$$

Corollary 4.2 Substituting $h = x$ and $a = 0$ to the above result.

$$f(x) - f(0) = xf'(\theta x), \; 0 < \theta < 1 \text{ in the interval } [0, x]$$

or
$$f(x) = f(0) + xf'(\theta x), \; 0 < \theta < 1 \text{ in } [0, x].$$

Geometrical interpretation of MVT

From Figure 4.2, $\dfrac{f(b) - f(a)}{b - a} = \dfrac{BN}{AN} = \tan\underline{|BAN}$. Since $f'(c) = \tan\underline{|CTX}$. From MVT $\tan\underline{|BAN} = \tan\underline{|CTX}$ or $\underline{|BAN} = \underline{|CTX}$, i.e., AB is parallel to CT.

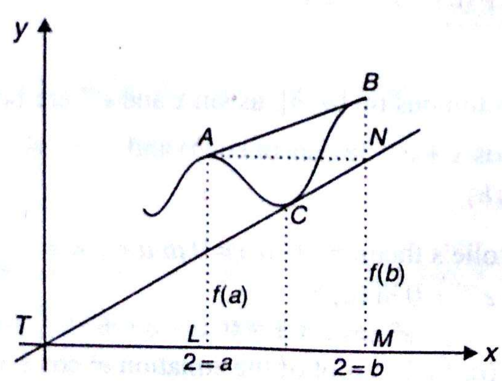

Figure 4.2 Geometrical interpretation of MVT.

Thus, if the graph ACB of $f(x)$ is a continuous curve having everywhere a tangent, then there must be at least one point C between A and B at which the tangent is parallel to the chord AB.

4.3.1 Deduction of Rolle's Theorem from Lagrange's MVT

If $f(x)$ is continuous on $[a, b]$ and derivable in (a, b) then by Lagrange's MVT, we have for at least one c,

$$\frac{f(b)-f(a)}{b-a}=f'(c), \quad a<c<b.$$

In addition, if $f(a)=f(b)$ (which is the third condition of Rolle's Theorem) then $f'(c)=0, a<c<b$. Hence, the Rolle's theorem follows from Lagrange's theorem.

EXAMPLE 4.6 Verify Lagrange's mean value theorem for the following functions:

(i) $f(x)=x^2+3x+2$ in $1 \le x \le 2$

(ii) $f(x)=\dfrac{1}{x}$ in $-1 \le x \le 1$

(iii) $f(x)=1+x^{2/3}$ in $[-8, 1]$

(iv) $f(x)=\begin{cases} x\sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x=0 \end{cases}$ in $[-1, 1]$

Solution

(i) Since $f(x) = x^2 + 3x + 2$ is a polynomial and every polynomial is continuous and differentiable everywhere, $f(x) = x^2 + 3x + 2$ is continuous and differentiable in $1 \le x \le 2$. Also, $f'(x) = 2x + 3$, $1 \le x \le 2$.
Therefore, $f(x)$ satisfies all the conditions of Lagrange's MVT.
Let c, $1 < c < 2$, be any value of x. Then by Lagrange's MVT

$$\frac{f(2)-f(1)}{2-1}=f'(c)$$

or $$\frac{12-6}{1}=2c+3$$

or $$2c+3=6 \text{ or, } c=3/2,$$

i.e. $$1<3/2<2.$$

Hence Lagrange's MVT is verified for this function.

(ii) Here $f(x)=\dfrac{1}{x}$ and $f'(x)=-\dfrac{1}{x^2}$ in $-1 \le x \le 1$. It is easy to observe that both the functions $f(x)$ and $f'(x)$ do not exist at $x = 0$. Thus, $f(x)$ is neither continuous nor derivable on $-1 \le x \le 1$.
Therefore, the Lagrange's MVT is not applicable for this function.

(iii) The function $f(x) = 1 + x^{2/3}$ is continuous on $[-8, 1]$ as the function is finite for all values of x on $[-8, 1]$.

But, $f'(x) = \frac{2}{3}x^{-1/3}$ does not exist at $x = 0$, i.e. $f(x)$ is not derivable in $(-8, 1)$

Hence Lagrange's MVT is not applicable for this function.

(iv) Here $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

To test continuity

Let ε be any pre-assigned positive number and $\delta > 0$ be another number that depends on ε.

Then $$|f(x) - f(0)| = \left|x\sin\frac{1}{x} - 0\right| = |x|\left|\sin\frac{1}{x}\right| \leq |x|$$

as $$\left|\sin\frac{1}{x}\right| \leq 1$$

$\therefore$ $$|f(x) - f(0)| < \varepsilon \quad \text{if} \quad |x| < \varepsilon$$

or $$|f(x) - f(0)| < \varepsilon \quad \text{if} \quad |x - 0| < \delta \qquad \text{where } \delta = \varepsilon.$$

Hence $f(x)$ is continuous at $x = 0$ and obviously it is continuous for all values of x on $[-1, 1]$.

To test differentiability

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x\sin\frac{1}{x} - 0}{x} = \lim_{x\to 0}\sin\frac{1}{x}.$$

But, it is well known that $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist. Therefore, $f(x)$ is not differentiable at $x = 0$ and consequently $f(x)$ is not differentiable in $(-1, 1)$.

Thus Lagrange's MVT is not applicable for this function.

EXAMPLE 4.7 Use Lagrange's MVT to prove the following inequalities:

(i) $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$

(ii) $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$ for all $x > 0$

(iii) $0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1$, for $x > 0$

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(iv) $\dfrac{x}{\sqrt{1-x^2}} \geq \sin^{-1} x \geq x$, if $0 \leq x < 1$

Solution

(i) Let $f(x) = \log(1 + x)$.

Therefore, $f(0) = 0$ and $f'(x) = \dfrac{1}{1+x}$ or $f'(\theta x) = \dfrac{1}{1+\theta x}, 0 < \theta < 1$.

Then by MVT, for the interval $[0, x]$, we have

$$f(x) = f(0) + x f'(\theta x), \quad 0 < \theta < 1$$

or $$\log(1+x) = \frac{x}{1+\theta x}. \tag{1}$$

Since $0 < \theta < 1, 0 < \theta x < x$ $\qquad (\because x > 0)$

or $1 < 1 + \theta x < 1 + x$ or $1 > \dfrac{x}{1+\theta x} > \dfrac{1}{1+x}$

or $\dfrac{x}{1+x} < \dfrac{1}{1+\theta x} < x$ or $\dfrac{x}{1+x} < \log(1+x) < x$ $\qquad$ [Using (1)]

(ii) From previous examples, we have

$$\frac{x}{1+x} < \log(1+x) < x$$

or $\dfrac{1+x}{x} > \dfrac{1}{\log(1+x)} > \dfrac{1}{x}$ or $0 < \dfrac{1}{\log(1+x)} - \dfrac{1}{x} < 1.$

(iii) Let $f(x) = e^x \quad \therefore f'(x) = e^x, f(0) = 1$

From MVT, on $[0, x]$, we have

$$f(x) = f(0) + x f'(\theta x), \quad 0 < \theta < 1$$

or $$e^x = 1 + x\theta x \text{ or } \frac{e^x - 1}{x} = e^{\theta x}$$

or $$\theta = \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right).$$

Since, $\quad 0 < \theta < 1, \quad 0 < \dfrac{1}{x}\log\left(\dfrac{e^x - 1}{x}\right) < 1.$

(iv) Let $f(x) = \sin^{-1}x$.

$\therefore \qquad f(0) = 0, \; f'(x) = \dfrac{1}{\sqrt{1-x^2}}.$

From MVT, on $[0, x]$, we have

$$f(x) = f(0) + x f'(\theta x), \quad 0 < \theta < 1$$

or
$$\sin^{-1} x = \frac{x}{\sqrt{1-(\theta x)^2}}, \quad 0 < \theta < 1.$$

Since, $0 < \theta < 1, 0 \le \theta x \le x$ as $0 \le x < 1$

or $\quad -x^2 \le -(\theta x)^2 \le 0 \quad$ or $\quad 1 - x^2 \le 1 - (\theta x)^2 \le 1$

$$1 \le \frac{1}{\sqrt{1-(\theta x)^2}} \le \frac{1}{\sqrt{1-x^2}} \quad \text{or} \quad x \le \sin^{-1} x \le \frac{x}{\sqrt{1-x^2}}.$$

EXAMPLE 4.8 Prove that if $0 < a < b$ then

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}.$$

Solution

Let $f(x) = \tan^{-1}x$. $f(x)$ satisfies all the conditions of Lagrange's theorem on $[a, b]$.

Therefore, by Lagrange's MVT we have

$$\frac{f(b) - f(a)}{b - a} = f'(c), a < c < b$$

or
$$\frac{\tan^{-1} b - \tan^{-1} a}{b - a} = \frac{1}{1+c^2}, a < c < b.$$

Now,
$$a < c < b \quad \text{or} \quad a^2 + 1 < c^2 + 1 < b^2 + 1$$

or
$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

or
$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b - a} < \frac{1}{1+a^2}$$

or
$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

EXAMPLE 4.9 In the MVT applied to $f(x)$ in $[0, h]$, i.e. in

$$f(h) = f(0) + hf'(\theta h), 0 < \theta < 1$$

prove that $\lim\limits_{h \to 0^+} \theta = \frac{1}{2}$ when $f(x) = \cos x$.

Solution

Here
$$f(x) = \cos x.$$

$$f(0) = 1 \text{ and } f'(x) = -\sin x$$

Now, from $f(h) = f(0) + hf'(\theta h)$, we have

$$\cos h = 1 + h(-\sin \theta h) \text{ or } \frac{\cos h - 1}{h} = -\sin \theta h$$

or
$$\frac{2\sin^2(h/2)}{h} = \sin \theta h \text{ or } \left(\frac{\sin(h/2)}{h/2}\right)^2 = 2\frac{\sin \theta h}{h}$$

or
$$2\lim_{h \to 0^+}\left(\frac{\sin \theta h}{\theta h} \cdot \theta\right) = \lim_{h \to 0^+}\left(\frac{\sin(h/2)}{h/2}\right)^2$$

or
$$2\lim_{h \to 0^+} \theta = 1 \quad \left[\because \lim_{h \to 0^+}\left(\frac{\sin \theta h}{\theta h}\right) = 1\right].$$

Hence $\lim_{h \to 0^+} \theta = \frac{1}{2}$.

EXAMPLE 4.10 Apply Lagrange's mean value theorem, find the derivative of a function assuming that the derivatives which occur are continuous.

Solution

The Lagrange's MVT on $[a, a + h]$ is

$$f(a + h) - f(a) = hf'(a + \theta h), 0 < \theta < 1$$

or
$$f'(a + \theta h) = \frac{f(a + h) - f(a)}{h}.$$

Taking limit $h \to 0$, we get

$$\lim_{h \to 0} f'(a + \theta h) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

or
$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$\left[\because f'(x) \text{ is continuous (given), so } \lim_{h \to 0} f'(a + h) = f'(a)\right]$$

This is the well known formula for derivative.

EXAMPLE 4.11 Use Lagrange's MVT to prove that, if $f(x)$ is continuous and $f'(x) > 0$, then $f(x)$ is an increasing function.

Solution

Let $f(x)$ be continuous on $[a, b]$ and differentiable in (a, b) where $a < b$. Then by Lagrange's MVT,

$$f(b) - f(a) = (b - a)\, f'(c), \qquad a < c < b.$$

Since $f'(x) > 0$ on $[a, b]$ and $b > a$, the right hand side of the above equation is positive, and hence

$$f(b) - f(a) > 0 \text{ or } f(b) > f(a)$$

Thus f is an increasing function on $[a, b]$.

EXAMPLE 4.12 Use MVT to prove that

$$\sin 46^\circ \simeq \frac{1}{2}\sqrt{2}\left(1 + \frac{\pi}{180}\right)$$

Is this estimate high or how? (WBUT 2003)

Solution

Let $f(x) = \sin x$. The Lagrange's MVT on $[a, a + h]$ is

$$f(a+h) = f(a) + hf'(a + \theta h), 0 < \theta < 1.$$

Let $a = 45^\circ$, $h = 1^\circ$.

Then $$f(a) = f(45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}.$$

$$f'(a) = \cos x,\ f'(a + \theta h) = f'(45^\circ + \theta.1^\circ) = \cos(45^\circ + \theta^\circ).$$

Therefore, $$\sin(45^\circ + 1^\circ) = \frac{1}{\sqrt{2}} + 1^\circ \times \cos(45^\circ + \theta^\circ)$$

or
$$\sin 46^\circ = \frac{1}{\sqrt{2}} + \frac{\pi}{180}\cos(45^\circ + \theta^\circ)$$

$$\simeq \frac{1}{\sqrt{2}} + \frac{\pi}{180}\cos 45^\circ \left[\because \theta^\circ \text{ is too small compared to } 45^\circ\right]$$

$$\simeq \frac{1}{\sqrt{2}} + \frac{\pi}{180}\cdot\frac{1}{\sqrt{2}}$$

$$\simeq \frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{180}\right)$$

Hence $\sin 46^\circ \simeq \frac{1}{2}\sqrt{2}\left(1 + \frac{\pi}{180}\right)$.

For $\cos(45^\circ + \theta^\circ) < \cos 45^\circ$.

$\therefore$ Exact value of $\sin 46^\circ <$ approximate value of $\sin 46^\circ$.

Thus the estimate is high.

4.4 CAUCHY'S MEAN VALUE THEOREM

If two functions f and g

(i) be both continuous on $a \le x \le b$

(ii) are both differentiable in $a < x < b$

(iii) $g'(x)$ does not vanish at any value of x in $a < x < b$, then there exists at least one value of x, say c, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad \text{for } a < c < b. \tag{4.2}$$

Corollary 4.3 $\theta - h$ form of Cauchy's MVT

Let $b = a + h$. Then $c = a + \theta h$, $0 < \theta < 1$.
Then from Equation (4.2)

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \quad 0<\theta<1 \tag{4.3}$$

on $[a, a+h]$.

Corollary 4.4 Cauchy's MVT on $[0, x]$.

Substituting $h = x$ and $a = 0$ to the above expression.

Then
$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(\theta x)}{g'(\theta x)}, \quad 0<\theta<1.$$

4.4.1 Reduction of Lagrange's MVT from Cauchy's MVT

The Cauchy's MVT for the functions f and g on $[a, b]$ is

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad a<c<b.$$

We consider $g(x) = x$ which is continuous and differentiable everywhere, and $g'(x) = 1$.

Then the above expression reduces to

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1}, \quad a<c<b$$

or
$$f(b)-f(a) = (b-a)\, f'(c), \quad a<c<b$$

This is the well known form of Lagrange's MVT.

EXAMPLE 4.13 Verify Cauchy's mean value theorem for the following functions:

(i) $f(x) = 2x^2$, $g(x) = 4x+1$, for $1 \le x \le 2$

(ii) $f(x) = \sin x$, $g(x) = x^3$, for $-1 \le x \le 1$.

Solution

(i) The functions $f(x) = 2x^2$ and $g(x) = 4x+1$ both are continuous on $[1, 2]$.

$$f'(x) = 4x \text{ and } g'(x) = 4 \text{ both exist in } (1, 2)$$

Also, $$g'(x) \neq 0 \text{ for all } x.$$

Thus Cauchy's MVT is applicable for these functions.

Now, by Cauchy's MVT,

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}, 1 < c < 2$$

or $$\frac{8-2}{9-5} = \frac{4c}{4} \text{ or } \frac{6}{4} = c \text{ or } c = \frac{3}{2}.$$

That is c lies between 1 and 2. Hence Cauchy's MVT is verified for these functions.

(ii) Here both $f(x) = \sin x$ and $g(x) = x^3$ are continuous on $[-1, 1]$ and both are differentiable in $(-1, 1)$, where $f'(x) = \cos x$ and $g'(x) = 3x^2$. But, $g'(x) = 0$ at $x = 0$. Thus, all the conditions of Cauchy's MVT do not satisfy. Hence Cauchy's MVT is not applicable for these functions.

EXAMPLE 4.14 Use Cauchy's MVT to prove that $0 < \frac{1}{x} \log\left(\frac{e^x - 1}{x}\right) < 1.$

Solution

Let $f(x) = e^x$ and $g(x) = x$. $f'(x) = e^x$ and $g'(x) = 1$.

The Cauchy's MVT in $[0, x]$ is

$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(\theta x)}{g'(\theta x)}, 0 < \theta < 1$$

or $$\frac{e^x - 1}{x - 0} = \frac{e^{\theta x}}{1} \text{ or } e^{\theta x} = \frac{e^x - 1}{x}$$

or $$\theta x = \log\left(\frac{e^x - 1}{x}\right), 0 < \theta < 1 \text{ or } \theta = \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right).$$

Since, $$0 < \theta < 1, 0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1.$$

4.5 GENERALISED MEAN VALUE THEOREM: TAYLOR'S THEOREM

If a function f be such that

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[a, a+h]$

(ii) the nth derivative f^n exists in the open interval $(a, a+h)$, then there exists at least one number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n,$$

(i) If $R_n = \dfrac{h^n}{n!} f^n\,(a+\theta h), 0<\theta<1,$ then the theorem is called Taylor's theorem with Lagrange's form of remainder.

(ii) If $R_n = \dfrac{h^n\,(1-\theta)^{n-1}}{(n-1)!} f^n\,(a+\theta h), 0<\theta<1,$ then the theorem is known as Taylor's theorem with Cauchy's form of remainder.

4.5.1 Alternative form of Taylor's Theorem

If a function f be such that

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[a, x]$.
(ii) the nth derivative f^n exists in the open interval (a, x), then there exists at least one member θ, $0<\theta<1$, such that

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}\,(a) + R_n,$$

(i) $R_n = \dfrac{(x-a)^n}{n!} f^n\,(a+\theta\,(x-a)), 0<\theta<1,$ due to Lagrange.

(ii) $R_n = \dfrac{(x-a)^n\,(1-\theta)^{n-1}}{(n-1)!} f^n\,(a+\theta\,(x-a)), 0<\theta<1,$ due to Cauchy.

This form of Taylor's theorem is used to expand the functions $f(x)$ in powers of $(x-a)$, i.e. about the point a. The Taylor's theorem is also called the Mean Value Theorem of nth order. The above form of the Taylor's series is the finite form.

EXAMPLE 4.15 Apply Cauchy's MVT to the functions $f(x) = e^x$ and $g(x) = e^{-x}$ in $[x, x+h]$ and obtain the value of θ. Interpret your result.

(WBUT 2003)

Solution

Here $f(x) = e^x$ and $g(x) = e^{-x}$

Therefore, $f'(x) = e^x$ and $g'(x) = -e^{-x}$

The Cauchy's MVT on $[x, x+h]$ is

$$\frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f'(x+\theta h)}{g'(x+\theta h)}, 0<\theta<1,$$

or
$$\frac{e^{x+h}-e^x}{e^{-(x+h)}-e^{-x}} = \frac{e^{x+\theta h}}{-e^{-(x+\theta h)}}$$

or
$$\frac{e^x\,(e^h-1)}{e^{-x}\,(e^{-h}-1)} = \frac{e^x\cdot e^{\theta h}}{-e^{-x}\cdot e^{-\theta h}}$$

or
$$\frac{e^h - 1}{e^{-h}(1 - e^h)} = -e^{2\theta h}$$

or
$$e^{2\theta h} = e^h \quad \text{or} \quad 2\theta h = h \quad \text{or} \quad 2\theta = 1, \quad \text{i.e.} \quad \theta = \frac{1}{2}.$$

It may be noted that the condition $0 < \theta < 1$ is satisfied by this θ and it is independent of both x and h.

Note: The condition (i) of Taylor's theorem implies that $f, f', f'', \ldots, f^{n-1}$ all are continuous on the interval $[a, a+h]$.

A particular case of Taylor's theorem known as Maclaurin's theorem is stated below.

4.5.2 Maclaurin's Theorem

If a function f be such that

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[0, x]$,

(ii) the nth derivative f^n exists in $(0, x)$, then there exists at least one number θ, where $0 < \theta < 1$, such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$$

where $R_n = \dfrac{x^n}{n!}f^n(\theta x), 0 < \theta < 1$ the theorem is known as Maclaurin's theorem with Lagrange's form of remainder, and when $R_n = \dfrac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x), 0 < \theta < 1$, the theorem is known as Maclaurin's theorem with Cauchy's form of remainder.

Note: The Maclaurin's theorem is used to expand a function in powers of x, i.e. to expand in the neighbourhood of the origin.

EXAMPLE 4.16 Expand $7x^3 + 4x + 8$ in powers of $x - 1$.

Solution

Let $f(x) = 7x^3 + 4x + 8$ and $a = 1$, $f(1) = 19$.

$f'(x) = 21x^2 + 4$ $\qquad f'(1) = 25$

$f''(x) = 42x$ $\qquad f''(1) = 42$

$f'''(x) = 42$ $\qquad f'''(1) = 42$

$f^{iv}(x) = 0$ $\qquad f^{iv}(1) = 0$.

Then by Taylor's theorem

$$f(x) = f(1) + (x-1)\,f'(1) + \frac{(x-1)^2}{2!}\,f''(1) + \frac{(x-1)^3}{3!}\,f'''(1) + \frac{(x-1)^4}{4!}\,f^{\text{iv}}(1)$$

$$= 19 + 25\,(x-1) + 42 \times \frac{(x-1)^2}{2!} + 42 \times \frac{(x-1)^3}{3!}$$

$$= 19 + 25\,(x-1) + 21\,(x-1)^2 + 7\,(x-1)^3.$$

EXAMPLE 4.17 Apply Maclaurin's theorem to the function $f(x) = (1+x)^4$ to deduce $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.

Solution

Here $f(x) = (1+x)^4$. Then $f(0) = 1$.

$f'(x) = 4(1+x)^3$	$f'(0) = 4$
$f''(x) = 12(1+x)^2$	$f''(0) = 12$
$f'''(x) = 24(1+x)$	$f'''(0) = 24$
$f^{\text{iv}}(x) = 24$	$f^{\text{iv}}(0) = 24$
	$f^{\text{v}}(x) = 0.$

Then by Maclaurin's theorem

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}\,f''(0) + \frac{x^3}{3!}\,f'''(0) + \frac{x^4}{4!}\,f^{\text{iv}}(0)$$

or $$(1+x)^4 = 1 + 4x + \frac{x^2}{2!} \times 12 + \frac{x^3}{3!} \times 24 + \frac{x^4}{4!} \times 24$$

or $$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

EXAMPLE 4.18 Obtain the expansions of the following functions with the remainder in Lagrange's form

(i) $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4\,e^{\theta x}$,

(ii) $\cos x + \sin x = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4\,(\sin\theta x + \cos\theta x)$,

(iii) $(x+h)^{3/2} = x^{3/2} + \frac{3}{2}x^{1/2}\,h + \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{h^2}{2!}\,\frac{1}{\sqrt{(x+\theta h)}}$

(iv) $a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \cdots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!}\,a^{\theta x}\,(\log a)^n$.

Solution

(i) Let $f(x)=e^x, f(0)=1.$

$$f'(x)=e^x, f'(0)=1, f''(x)=e^x, f''(0)=1.$$

Similarly, $f'''(0)=1, f^{iv}(x)=e^x$ and $f^{iv}(\theta x)=e^{\theta x}, 0<\theta<1.$

Then by Maclaurin's theorem up to fifth term, with Lagrange's form of remainder

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(\theta x), 0<\theta<1,$$

or
$$e^x=1+x\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}e^{\theta x}, 0<\theta<1.$$

(ii) Let $f(x)=\cos x+\sin x, f(0)=1$

$f'(x)=-\sin x+\cos x,\quad f'(0)=+1$

$f''(x)=-\cos x-\sin x,\quad f''(0)=-1$

$f'''(x)=\sin x-\cos x,\quad f'''(0)=-1$

$f^{iv}(x)=\cos x+\sin x,\quad f^{iv}(\theta x)=\cos\theta x+\sin\theta x$

Therefore, by Maclaurin's theorem with Lagrange's form of remainder,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(\theta x), 0<\theta<1$$

or
$$\cos x+\sin x=1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}(\cos\theta x+\sin\theta x), 0<\theta<1.$$

(iii) Let
$$f(h)=(x+h)^{3/2}, f(0)=x^{3/2}$$

$$f'(h)=\frac{3}{2}(x+h)^{1/2}, f'(0)=\frac{3}{2}x^{1/2}$$

$$f''(h)=\frac{3}{2}\cdot\frac{1}{2}(x+h)^{1/2}, f''(\theta h)=\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{(x+\theta h)}}$$

The Maclaurin's theorem in powers of h with Lagrange's form of remainder

$$f(h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(\theta h)$$

$$=x^{3/2}+\frac{3}{2}h\,x^{1/2}+\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{h^2}{2!}\cdot\frac{1}{\sqrt{(x+\theta h)}}, 0<\theta<1.$$

(iv) Let $$f(x) = a^x,\ f(0) = 1$$

$$f'(x) = a^x \cdot \log a,\ f'(0) = \log a$$

$$f''(x) = a^x (\log a)^2,\ f''(0) = (\log a)^2$$

Similarly, $$f^{n-1}(x) = a^x (\log a)^{n-1},\ f^{n-1}(0) = (\log a)^{n-1}$$

$$f^n(x) = a^x (\log a)^n,\ f^n(\theta x) = a^{\theta x} (\log a)^n.$$

Therefore, by Maclaurin's theorem with Lagrange's form of remainder,

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!} f''(0)+\cdots+\frac{x^{n-1}}{(n-1)!} f^{n-1}(0)+\frac{x^n}{n!} f^n(\theta x), 0<\theta<1$$

or

$$a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \cdots + \frac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \frac{x^n}{n!} e^{\theta x} (\log a)^n, 0<\theta<1.$$

EXAMPLE 4.19 Use MVT, to prove that, if $0 \le x \le 1$,

$$\left|\log(1+x) - x + \frac{1}{2}x^2\right| \le \frac{1}{3}x^3.$$

Solution

Let $f(x) = \log(1+x),\ f(0) = 0$

$$f'(x) = \frac{1}{1+x},\ f'(0) = 1$$

$$f''(x) = \frac{1}{(1+x)^2},\ f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3},\ f'''(\theta x) = \frac{2}{(1+\theta x)^3}.$$

Now, by Maclaurin's finite series up to fourth term,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x), 0<\theta<1$$

or $$\log(1+x) = 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!} \cdot \frac{2}{(1+\theta x)^3}, 0<\theta<1$$

or $$\log(1+x) - x + \frac{x^2}{2} = \frac{x^3}{3} \cdot \frac{1}{(1+\theta x)^3}, 0<\theta<1$$

Since $0<\theta<1$ and $0 \le x \le 1, 0 \le \theta x \le 1$.

Therefore, $$(1+\theta x)^3 \geq 1 \quad \text{or} \quad \frac{1}{(1+\theta x)^3} \leq 1.$$

Hence $$\left| \log(1+x) - x + \frac{x^2}{2} \right| \leq \frac{1}{3} x^3.$$

EXAMPLE 4.20 If f'' is continuous on some neighbourhood of c, prove that

$$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Solution

From Taylor's series in $[c, c+h]$ we have

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c + \theta_1 h), 0 < \theta_1 < 1 \tag{1}$$

This result is true for all h, negative or positive or 0. Replacing h by $-h$ in equation (1), we get

$$f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!} f''(c - \theta_2 h), 0 < \theta_2 < 1 \tag{2}$$

Adding (1) and (2) we get

$$f(c+h) + f(c-h) = 2f(c) + \frac{h^2}{2!}\{f''(c+\theta_1 h) + f''(c-\theta_2 h)\}$$

or $$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = \frac{1}{2!} \lim_{h \to 0} \{f''(c+\theta_1 h) + f''(c-\theta_2 h)\}$$

$$= \frac{1}{2} \times \{2 f''(c)\}$$

Because f'' is continuous in the neighbourhood of c,

$$\lim_{h \to 0} f''(c+\theta_1 h) = f''(c)$$

and also $$\lim_{h \to 0} f''(c-\theta_2 h) = f''(c)]$$

Hence $$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

4.6 MACLAURIN'S INFINITE SERIES

If (i) f be defined on $[-h, h]$

(ii) for each positive integer n, $f^n(x)$ exists for $-h \leq x \leq h$

(iii) $\lim_{n \to \infty} R_n = 0$ for each x in $[-h, h]$, where R_n is the remainder after n terms, then for each x in $[-h, h]$

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n\ (0) + \cdots$$

This series is called Maclaurin's infinite series expansion of $f(x)$ about $x = 0$.

EXAMPLE 4.21 Obtain the Maclaurin's infinite series for sin x and show that the series is valid for all real x.

Solution

Let $f(x) = \sin x,$ $\qquad f(0) = 0$

$f'(x) = \cos x = \sin(\pi/2 + x),$ $\qquad f'(0) = 1$

$f''(x) = \cos(\pi/2 + x) = \sin(2.\pi/2 + x),$ $\qquad f''(0) = 0$

$f'''(x) = \cos(2.\pi/2 + x) = \sin(3.\pi/2 + x),$ $\qquad f'''(0) = -1$

and so on.

Therefore, $\qquad f^n\ (x) = \sin(n\pi/2 + x),\ f^n(0) = \sin(n\pi/2)$

and $\qquad f^n\ (\theta x) = \sin(n\pi/2 + \theta x).$

Also, R_n (the Lagrange's form of remainder)

$$= \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \sin(n\pi/2 + \theta x),\ 0 < \theta < 1.$$

Maclaurin's infinite series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

or
$$\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

This series converges to sin x for all x, iff $R_n \to 0$ as $n \to \infty$ for all x.

Now,
$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \left| \frac{x^n}{n!} \sin(n\pi/2 + \theta x) \right|$$

$$\le \lim_{n\to\infty} \frac{|x|^n}{n!} = 0 \text{ for all } x$$

$$[\because |\sin(n\pi/2 + \theta x)| \le 1]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ for all } x.$$

EXAMPLE 4.22 Find the expansion of $f(x) = (1 + x)^m$, where m is a negative fraction.

Solution

Here $f(x) = (1 + x)^m$, where $x \neq -1$

$$f'(x) = m(1 + x)^{m-1}$$

$$f''(x) = m(m-1)(1 + x)^{m-2}$$

....

$$f^k(x) = m(m-1)(m-2) \cdots (m-k+1)(1 + x)^{m-k}$$

$$f^k(0) = m(m-1)(m-2) \cdots (m-k+1)$$

$$f^n(x) = m(m-1) \cdots (m-n+1)\, x^{m-n}, x \neq -1.$$

We consider Cauchy's remainder after n terms

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f(1 + \theta x), \quad 0 < \theta < 1$$

$$= \frac{x^n (1-\theta)^{n-1}}{(n-1)!} m(m-1)(m-1) \cdots (m-n+1)(1 + \theta x)^{m-n}$$

$$= \frac{m(m-1)(m-1) \cdots (m-n+1)}{(n-1)!} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1 + \theta x)^{m-1}.$$

We see that

$$\lim_{n \to \infty} \frac{m(m-1): \cdots (m-n+1)}{(n-1)!} x^n = 0 \quad \text{for} \quad |x|^n < 1$$

and $$\frac{1-\theta}{1+\theta x} < 1, \text{ so that } \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0 \text{ as } n \to \infty.$$

Also, $$(1 + \theta x)^{m-1} < (1 + |x|)^{m-1}, \quad m > 1, 0 < \theta < 1$$

and $$(1 + \theta x)^{m-1} = \frac{1}{(1 + \theta x)^{1-m}} < \frac{1}{(1 - |x|)^{1-m}}, \quad \text{when } m < 1.$$

Thus $R_n \to 0$ when $n \to \infty$ for $|x| < 1$.

Hence $$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \cdots, \quad \text{for } |x| < 1.$$

EXAMPLE 4.23 Find the expansion of $\log_e (1 + x)$ in a power series of x and indicate the region of validity of the expansion.

Solution

Let

$$f(x) = \log_e (1 + x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -1$$

$$f'''(x) = \frac{2\cdot 1}{(1+x)^3} \qquad f'''(0) = 2!$$

$$f^{\text{iv}}(x) = \frac{-3\cdot 2\cdot 1}{(1+x)^4} \qquad f^{\text{iv}}(0) = -3!$$

and in this way

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Taking Lagrange's form of remainder,

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{(-1)^{n-1} x^n}{n(1+\theta x)^n} = (-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$$

Therefore, $f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!} f''(0) + \cdots + \dfrac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$

or,
$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + R_n$$

Case I: When $0 \le x \le 1$, then $0 < \theta x < x \le 1$ and

$$|R_n| = \left|\frac{x^n}{n}\right| \left|\frac{1}{(1+\theta x)^n}\right| \le \frac{x^n}{n} \le \frac{1}{n} \text{ since } 0 \le x \le 1.$$

Also, $\dfrac{1}{n} \to 0$ as $n \to \infty$.

Therefore, $R_n \to 0$ as $n \to \infty$ for $0 \le x \le 1$.

Thus the conditions of Maclaurin's infinite series expansion are satisfied for $0 \le x \le 1$.

Case II: When $-1 < x < 0$.

In this case x may or may not be numerically less than $1+\theta x$, so that nothing can be said about the limit of $\left(\frac{x}{1+\theta x}\right)^n$ when $n \to \infty$. Thus from Lagrange's form of remainder, we cannot draw any definite conclusion. Now, we consider Cauchy's form of remainder,

$$R_n = \frac{x^n (1-\theta)^{n-1} f^n(\theta x)}{(n-1)!} = \frac{(-1)^{n-1} x^n (1-\theta)^{n-1}}{(1+\theta x)^n}$$

$$= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \frac{1}{1+\theta x}.$$

Now, $1-\theta < 1+\theta x$ so that $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$.

Also, $x^n \to 0$ as $n \to \infty$ and $\frac{1}{1+\theta x} < \frac{1}{1-|x|}$ and moreover it is independent of n.

Thus $R_n \to 0$ as $n \to \infty$.

Hence, the conditions of Maclaurin's series expansion are satisfied also when $-1 < x < 0$.

Thus $$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ for } -1 < x \le 1.$$

EXAMPLE 4.24 Expand e^x in powers of x in infinite series. (WBUT 2004)

Solution

Let
$$f(x) = e^x,\ f(0) = 1.$$

and in this way
$$f'(x) = e^x,\ f'(0) = 1,\ f''(x) = e^x,\ f''(0) = 1,$$

$$f^n(x) = e^x,\ f^n(\theta x) = e^{\theta x}.$$

The Maclaurin's series for $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

where $$R_n = \frac{x^n}{n!} f^n(\theta x),\quad 0 < \theta < 1 \quad \text{(Due to Lagrange's)}$$

or,
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and
$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} e^{\theta x}.$$

Now,
$$\lim_{n \to \infty} |R_n| = \lim_{n \to \infty} \frac{|x|^n}{n!} e^{\theta x}$$

$$= e^{\theta x} \lim_{n \to \infty} \frac{|x|^n}{n!} = 0. \qquad \left[\because \lim_{n \to \infty} \frac{|x|^n}{n!} = 0\right]$$

4.7 INDETERMINATE FORM: L'HOSPITAL RULE

Suppose we have to calculate the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$. If $\lim_{x \to a} g(x)$ is 0 but $\lim_{x \to a} f(x)$ is non-zero then this limit does not exist. But, when both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are 0 then this limit is of the form $\frac{0}{0}$ and it may have a define value. The form $\frac{0}{0}$ is called an indeterminate form. Other types of indeterminate forms are

$$\frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^{\pm\infty}, \infty^0, \text{etc.}$$

The limit of the form $\frac{0}{0}$ can be obtained by L'Hospital rule.

(a) $\frac{0}{0}$ *form*

L'Hospital rule: If $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, a+h]$,
(ii) derivable in the open interval $(a, a+h)$, and
(iii) $f(a) = g(a) = 0$,

then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ provided the limit exists.}$$

Note: If
$$f(a) = f'(a) = \cdots = f^{n-1}(a) = 0$$

and
$$g(a) = g'(a) = \cdots = g^{n-1}(a) = 0,$$

but,
$$\lim_{x \to a} g^n(a) \neq 0$$

then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^n(x)}{g^n(x)}.$$

It may be noted that the evaluation of the limit of the forms $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, $1^{\pm\infty}$, ∞^0, etc., depends on the evaluation of the limit of the form $\frac{0}{0}$.

EXAMPLE 4.25 Evaluate the following limits:

(i) $\lim_{x \to 0} \frac{\sin x}{x}$ (ii) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\log(1+x)}$

(iii) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ (iv) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

Solution

(i) $\lim_{x \to 0} \frac{\sin x}{x} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

(ii) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\log(1+x)} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{e^x + \cos x}{1/(1+x)} = \frac{1+1}{1} = 2.$$

(iii) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{e^x - \cos x\, e^{\sin x}}{1 - \cos x} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x + \sin x\, e^{\sin x} - \cos^2 x\, e^{\sin x}}{\sin x} \left(\frac{0}{0} \text{ form}\right)$$

(Again applying L'Hospital rule)

$$= \lim_{x \to 0} \frac{e^x + \cos x\, e^{\sin x} + \sin x \cos x\, e^{\sin x} - \cos^3 x\, e^{\sin x} + 2 \sin x \cos x\, e^{\sin x}}{\cos x}$$

$$= \frac{1 + 1 + 0 - 1 + 0}{1} = 1.$$

(iv) $$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x}\left(\frac{0}{0}\text{ form}\right) = \lim_{x\to 0}\frac{\sin x - x\cos x}{x\cos x - \sin x\cos x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{\cos x - \cos x + x\sin x}{\cos x - x\sin x - \cos 2x}\left(\frac{0}{0}\text{ form}\right) \quad \left[\because \sin x\cos x = \frac{1}{2}\sin 2x\right]$$

$$= \lim_{x\to 0}\frac{\sin x + x\cos x}{-\sin x - \sin x - x\cos x + 2\sin 2x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{\cos x + \cos x - x\sin x}{-2\cos x - \cos x + x\sin x + 4\cos 2x}$$

$$= \frac{1+1}{-2+4} = 2.$$

(b) $\frac{\infty}{\infty}$ *form*: This form is similar to the form $\frac{0}{0}\cdot$ The same technique is applicable for this form, i.e. if

$$\lim_{x\to a} f(x) = \infty, \text{ and } \lim_{x\to a} g(x) = \infty$$

then $$\lim_{x\to a}\frac{f(x)}{g(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}, \text{ provided } \lim_{x\to a} g'(x) \text{ exists.}$$

EXAMPLE 4.26 Find the value of $\lim_{x\to\infty}\frac{\log(1+x)}{x}$.

Solution

$$\lim_{x\to\infty}\frac{\log(1+x)}{x}\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}[\log(1+x)]}{\frac{d}{dx}[x]} = \lim_{x\to\infty}\frac{\frac{1}{1+x}}{1} = \lim_{x\to\infty}\frac{1}{1+x} = 0.$$

(c) $0\cdot\infty$ *form*: This form can be converted to either in $\frac{0}{0}$ form or in $\frac{\infty}{\infty}$ form.

That is, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} f(x)\,g(x)$ can be written in the following two forms:

(i) $$\lim_{x\to a} f(x)\,g(x) = \lim_{x\to a}\frac{f(x)}{1/g(x)} \quad \left(\frac{0}{0}\text{ form}\right)$$

(ii) $$\lim_{x\to a} f(x)\,g(x) = \lim_{x\to a}\frac{1/f(x)}{g(x)} \quad \left(\frac{\infty}{\infty}\text{ form}\right).$$

EXAMPLE 4.27 Find $\lim\limits_{x\to 0} \cot x \cdot \log \dfrac{1+x}{1-x}$.

Solution

$$\lim_{x\to 0} \cot x \cdot \log \frac{1+x}{1-x} \quad (\infty \cdot 0 \text{ form})$$

$$= \lim_{x\to 0} \frac{\log(1+x) - \log(1-x)}{\tan x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\dfrac{1}{1+x} + \dfrac{1}{1-x}}{\sec^2 x} = 2.$$

(d) $\infty - \infty$ *form*: If $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$ then $\lim\limits_{x\to a} [f(x) - g(x)]$ is of the form $\infty - \infty$. This form can be written as

$$\lim_{x\to a} \frac{1/g(x) - 1/f(x)}{1/\{g(x)\, f(x)\}} \quad \left(\frac{0}{0}\text{ form}\right).$$

EXAMPLE 4.28 Find the following limits:

(i) $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$ (ii) $\lim\limits_{x\to 0} \left\{\dfrac{1}{x} - \dfrac{\log(1+x)}{x^2}\right\}$

(iii) $\lim\limits_{x\to 1} \left\{\dfrac{x}{x-1} - \dfrac{1}{\log x}\right\}$.

Solution

(i) $$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \quad (\infty - \infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\sin x - x}{x \sin x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{1+1} = 0.$$

(ii) $\lim_{x\to 0}\left\{\frac{1}{x}-\frac{\log(1+x)}{x^2}\right\}$ $\quad(\infty-\infty \text{ form})$

$$= \lim_{x\to 0}\frac{x-\log(1+x)}{x^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x} = \lim_{x\to 0}\frac{1}{2(1+x)} = \frac{1}{2}.$$

(iii) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)$ $\quad(\infty-\infty \text{ form})$

$$= \lim_{x\to 1}\frac{x\log x-(x-1)}{(x-1)\log x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 1}\frac{\log x + x\cdot\frac{1}{x}-1}{\log x+(x-1)\cdot\frac{1}{x}} = \lim_{x\to 1}\frac{x\log x}{x\log x + x - 1} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 1}\frac{\log x + x\cdot\frac{1}{x}}{\log x + x\cdot\frac{1}{x}+1} = \frac{1}{2}.$$

(e) *Other forms*: $0^0, 1^{\pm\infty}, \infty^0$: These forms occur when the limit is of the form $\lim_{x\to a}[f(x)]^{g(x)}$. This limit can be evaluated by taking logarithm of the function $[f(x)]^{g(x)}$. That is, if we take $y=[f(x)]^{g(x)}$ then $\log y = g(x)\log f(x)$ and $\lim_{x\to a}\log y = \lim_{x\to a} g(x)\log f(x)$ or $\log(\lim_{x\to a} y) = \lim_{x\to a} g(x)\log f(x)$. The right hand side becomes one of the forms discussed earlier.

EXAMPLE 4.29 Evaluate the following limits:

(i) $\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^{1/x}$, $\quad$ (ii) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$,

(iii) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$, $\quad$ (iv) $\lim_{x\to\infty} x^{1/x}$,

(v) $\lim_{x\to \pi/2}(\sin x)^{\tan x}$.

Solution

(i) The given limit is of the form 1^∞.

Let $y=\left(\frac{\sin x}{x}\right)^{1/x}$. Both sides taking logarithm we get,

$$\log y=\frac{1}{x}\log\left(\frac{\sin x}{x}\right).$$

Now, $\lim_{x\to 0^+}\log y=\lim_{x\to 0^+}\frac{\log\left(\frac{\sin x}{x}\right)}{x}\quad\left(\frac{0}{0}\text{ form}\right)\quad\left[\because \lim_{x\to 0^+}\frac{\sin x}{x}=1\right]$

$$=\lim_{x\to 0^+}\frac{\frac{x}{\sin x}\,\frac{x\cos x-\sin x}{x^2}}{1}=\lim_{x\to 0^+}\frac{x\cos x-\sin x}{x\sin x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0^+}\frac{\cos x-x\sin x-\cos x}{\sin x+x\cos x}=\lim_{x\to 0^+}\frac{-x\sin x}{\sin x+x\cos x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0^+}\frac{-\sin x-x\cos x}{\cos x+\cos x-x\sin x}=\frac{0}{2}=0$$

or $\log\left(\lim_{x\to 0^+}y\right)=0$ or $\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^{1/x}=e^{\circ}=1.$

(ii) Let $y=\left(1+\frac{2}{x}\right)^x$. Taking logarithm both sides.

$$\log y=x\log\left(1+\frac{2}{x}\right).$$

$\therefore\quad \lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{\log\left(1+\frac{2}{x}\right)}{1/x}\quad\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+2/x}\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{2}{1+\frac{2}{x}}=2$$

i.e. $\lim_{x\to\infty}\log y=2$ or $\log\left(\lim_{x\to\infty}y\right)=2$

or $$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2.$$

(iii) Let $y=\left(\dfrac{\tan x}{x}\right)^{1/x^2}$. Then $\log y=\dfrac{1}{x^2}\log\left(\dfrac{\tan x}{x}\right)$

$$\therefore \lim_{x\to 0}\log y=\lim_{x\to 0}\frac{\log\left(\dfrac{\tan x}{x}\right)}{x^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\left[\because \lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x\cos x}\quad\left(\frac{0}{0}\text{ form}\right)=\lim_{x\to 0}\frac{\cos x}{\cos x - x\sin x}=1\right]$$

$$=\lim_{x\to 0}\frac{\dfrac{x}{\tan x}\cdot\dfrac{x\sec^2 x-\tan x}{x^2}}{2x}=\lim_{x\to 0}\frac{x-\dfrac{1}{2}\sin 2x}{x^2\sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{1-\cos 2x}{2x\sin 2x+2x^2\cos 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{2\sin 2x}{2\sin 2x+4x\cos 2x+4x\cos 2x-4x^2\sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{4\cos 2x}{4\cos 2x+8\cos 2x-12\,x\sin 2x-8x\sin 2x-8x^2\cos 2x}$$

$$=\frac{4}{12}=\frac{1}{3}.$$

$$\therefore \qquad \lim_{x\to 0}\log y=\frac{1}{3} \text{ or } \log\left\{\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}\right\}=\frac{1}{3}$$

or $$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}=e^{1/3}.$$

(iv) $\lim_{x\to\infty} x^{1/x}$ $\quad(\infty^0$ form).

Let $y=x^{1/x}$. Taking logarithm both sides, we get

$$\log y=\frac{1}{x}\log x.$$

$$\therefore \lim_{x\to\infty} \log y = \lim_{x\to\infty} \frac{\log x}{x}\left(\frac{\infty}{\infty} \text{ form}\right) = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

$$\therefore \quad \log\left(\lim_{x\to\infty} y\right) = 0 \quad \text{or} \quad \lim_{x\to\infty} x^{1/x} = e^{\circ} = 1.$$

(v) $\lim_{x\to\pi/2} (\sin)^{\tan x}\ (1^{\infty})$.

Let $y = (\sin x)^{\tan x}$. Taking logarithm both sides, we get

$$\log y = \tan x \log \sin x.$$

$$\therefore \quad \lim_{x\to\pi/2} \log y = \lim_{x\to\pi/2} \tan x \log \sin x \qquad (\infty \cdot 0 \text{ form})$$

$$= \lim_{x\to\pi/2} \frac{\log \sin x}{\cot x}\left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to\pi/2} \frac{\frac{1}{\sin x}\cdot \cos x}{-\operatorname{cosec}^2 x} = \lim_{x\to\pi/2} (-\cos x \sin x) = 0.$$

$$\therefore \quad \lim_{x\to\pi/2} \log y = 0 \quad \text{or} \quad \log\left(\lim_{x\to\pi/2} y\right) = 0$$

$$\text{or} \quad \lim_{x\to\pi/2} (\sin x)^{\tan x} = e^{\circ} = 1.$$

EXAMPLE 4.30

(i) Find the value of $\lim_{x\to\infty}\left[x - \sqrt{(x-a)(x-b)}\right]$

(ii) Determine a such that

$$\lim_{x\to 0} \frac{a \sin x - \sin 2x}{\tan^3 x} \text{ exists and it is equal to 1.}$$

(iii) Find the values of a, b, c such that

$$\lim_{x\to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x \sin x} = 2.$$

Solution

(i) Substituting $x = \frac{1}{y}$. When $x \to \infty$ then $y \to 0$.

Now, $\lim_{x\to\infty}\left[x - \sqrt{(x-a)(x-b)}\right] \qquad (\infty - \infty \text{ form})$

$$= \lim_{y \to 0}\left[\frac{1}{y} - \sqrt{\left(\frac{1}{y} - a\right)\left(\frac{1}{y} - b\right)}\right]$$

$$= \lim_{y \to 0}\left[\frac{1 - \sqrt{(1-ay)(1-by)}}{y}\right] \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{y \to 0} \frac{-\frac{d}{dy}\left[\sqrt{(1-ay)(1-by)}\right]}{1}$$

$$= \lim_{y \to 0}\left\{-\frac{1}{2}\frac{1}{\sqrt{(1-ay)(1-by)}}[-(a+b)+2aby]\right\}$$

$$= \frac{a+b}{2}.$$

(ii) Let $\quad l = \lim\limits_{x \to 0} \dfrac{a \sin x - \sin 2x}{\tan^3 x} \quad \left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{a \cos x - 2\cos 2x}{3\tan^2 x \cdot \sec^2 x}.$$

Here the numerator is $a - 2$ (when $x \to 0$), but, denominator is 0. So, if $a - 2$ is not equal to 0 then the limit does not exist. But, the limit is finite.

Therefore, $a - 2 = 0 \quad \text{or} \quad a = 2.$

(iii) Let $l = \lim\limits_{x \to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x \sin x}$

Here the numerator is $a - b + c$ and the denominator is 0. For a finite limit

$$a - b + c = 0 \tag{1}$$

$\therefore \quad l = \lim\limits_{x \to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x \sin x} \quad \left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}.$$

Again, denominator is zero, but, numerator is $a - c$ and it should be zero for finite limit.

Thus, $$a - c = 0 \tag{2}$$

Now, $\quad l = \lim\limits_{x \to 0} \dfrac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x} \quad \left(\dfrac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{ae^x + b\cos x - ce^{-x}}{\cos x + \cos x - x\sin x}$$

$$= \frac{a+b+c}{2}.$$

Given that $l = 2$. Therefore,

$$\frac{a+b+c}{2} = 2 \quad \text{or,} \quad a+b+c = 4. \tag{3}$$

Solving equations (1), (2) and (3), we get

$$a = 1, b = 2, c = 1.$$

These are the required values of a, b and c.

EXERCISES

Short Answer Questions

(Section A)

1. In the mean value theorem $f(x+h) = f(x) + hf'(x+\theta h)$, if $f(x) = ax^2 + bx + c$, $a \neq 0$ then $\theta =$ ....
2. In the mean value theorem $f(a+h) = f(a) + hf'(a+\theta h)$, if $a = 1$, $h = 3$ and $f(x) = \sqrt{x}$, then $\theta =$ ....
3. If $f(x) = x^2$, $\phi(x) = x$, $a \le x \le b$ then the value of c in terms of a, b in Cauchy's mean value theorem is ....
4. Applying Cauchy's mean value theorem to the functions $f(x) = e^x$ and $g(x) = e^{-x}$ in the interval $[a, b]$ then c is ....
5. The value of c in the MVT

   $f(b) - f(a) = (b-a)\, f'(c)$, if $f(x) = x(x-1)\;(x-2)$, $a = 0$, $b = \frac{1}{2}$ is ....
6. The value of $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)^{1/x}$ is ....
7. $\lim\limits_{x \to \infty} x^{1/x}$ is equal to ....
8. $\lim\limits_{x \to 0} (\cos 2x)^{1/x^2}$ is equal to ....
9. The value of $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$ is equal to ....
10. The value of $\lim\limits_{x \to \infty} \left[x - \sqrt{(x-a)(x-b)}\right] =$ ....