CHAPTER 7

Functions of Several Variables: Limit, Continuity and Partial Derivatives

7.1 INTRODUCTION

We have already discussed about the limit, continuity and differentiability of a function containing single variable. But, in many practical situations, the functions of several variables may occur. In these situations, computation of limit, testing of continuity and computation of partial derivatives are necessary. In this chapter, we shall mainly consider the problems where two variables will appear. The approach of three and more variables are similar.

Let x, y be two variables and they are connected by a functional relation, say, $z = f(x, y)$, then we say z is a function of x, y. The ordered pair of numbers (x, y) is called a point and the aggregate of the pairs of numbers (x, y) is said to be the *domain* (or region) *of definition* of the function.

The neighbourhood of a point

The set of values x, y other than a, b that satisfy the conditions

$$|x-a|<\delta, \quad |y-b|<\delta$$

where δ is an arbitrary small positive number said to form a neighbourhood of the point (a, b) or sometimes it is called δ-neighbourhood of the point (a, b). That is, δ-neighbourhood of the point (a, b) is the square $(a-\delta, a+\delta; b-\delta, b+\delta)$ where x takes any value from $a-\delta$ to $a+\delta$ except a and y from $b-\delta$ to $b+\delta$ except b.

Also, the δ-neighbourhood of a point (a, b) is the region (circular region)

$$(x-a)^2 + (y-b)^2 < \delta^2.$$

7.2 THE LIMIT OF A FUNCTION

Let $f(x, y)$ be a function defined over a certain domain R. Then the function f is said to tend to a limit l as a point (x, y) tends to (a, b) if for any arbitrary positive number ε, there corresponds to a positive number δ, such that

$$|f(x, y) - l| < \varepsilon$$

for every point (x, y) [other then (a, b)] which satisfies

$$|x-a| < \delta, \quad |y-b| < \delta \quad \text{or} \quad (x-a)^2 + (y-b)^2 < \delta^2.$$

Symbolically, we write

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l \quad \text{or} \quad \lim_{(x, y) \to (a, b)} f(x, y) = l \quad \text{or} \quad f(x, y) \to l \text{ as } (x, y) \to (a, b),$$

l is called the *limit* (the *double limit* or the *simultaneous limit*) of f when x, y tend to a, b simultaneously.

In this definition, we have allowed (x, y) to vary over the region R and approach (a, b). We may often restrict the path along which (x, y) should move. The point (x, y) may approach (a, b) along infinite many paths, but, the important point is that the limit l must be unique, along every possible path.

Thus, if we can find two functions $y = \phi_1(x)$ and $y = \phi_2(x)$ such that the limits of $f(x, \phi_1(x))$ and $f(x, \phi_2(x))$ are different, then the simultaneous limit does not exist.

EXAMPLE 7.1 Show that $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist for

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0. \end{cases}$$

Solution

If we approach the origin along any axis $f(x, y) = 0$.

If we approach the origin along any line $y = mx$ then

$$f(x, y) = f(x, mx) = \frac{mx^3}{x^4 + m^2 x^2} = \frac{mx}{x^2 + m^2} \to 0 \text{ as } x \to 0$$

so any straight line approach gives $\lim_{(x, y) \to (0, 0)} f(x, y) = 0.$

But putting $y = mx^2$

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} f(x, mx^2) = \frac{m}{1 + m^2}$$

which is different for different line (m).

Hence, $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

EXAMPLE 7.2 Show that $\lim\limits_{(x,y)\to(0,0)} xy\dfrac{x^2-y^2}{x^2+y^2}=0.$

Solution

Put $x = r\cos\theta$, $y = r\sin\theta$ to

$$\left|xy\frac{x^2-y^2}{x^2+y^2}-0\right| = \left|r^2\sin\theta\cos\theta\cos 2\theta\right| = \left|\frac{r^2}{4}\sin 4\theta\right|$$

$$\le \frac{r^2}{4} = \frac{x^2+y^2}{4} < \varepsilon$$

if $$\frac{x^2}{4} < \frac{\varepsilon}{2} \text{ and } \frac{y^2}{4} < \frac{\varepsilon}{2}$$

or $$\text{if } |x| < \sqrt{2\varepsilon} = \delta$$

and $$|y| < \sqrt{2\varepsilon} = \delta$$

where $$\delta = \sqrt{2\varepsilon}.$$

$\Rightarrow$ $$\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0.$$

Repeated limits: If a function f is defined in some nbd of (a, b) then the limit $\lim\limits_{y\to b} f(x, y)$ if it exists, is a function of x, say $\phi(x)$. If the limit $\lim\limits_{x\to a}\phi(x)$ exists and is equal to λ, we write

$$\lim_{x\to a}\lim_{y\to b} f(x, y) = \lambda$$

and say that λ is a repeated limit of f as $y \to b$, $x \to a$.

If we change the order of taking the limits, we get the other repeated limits.

$$\lim_{y\to b}\lim_{x\to a} f(x, y) = \lambda \text{ (say)}$$

When first $x \to a$ and then $y \to b$.

These two limits may or may not be equal.

EXAMPLE 7.3 Find the repeated limits of $\dfrac{x^2-y^2}{x^2+y^2}$ at $(0, 0)$.

Solution

We have for all $\lim\limits_{x\to 0}\dfrac{x^2-y^2}{x^2+y^2} = -1.$

$$\therefore \qquad \lim_{y \to 0} \lim_{x \to 0} \frac{x^2 - y^2}{x^2 + y^2} = -1.$$

Again,
$$\lim_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} = 1 \ \forall \ x$$

$$\therefore \qquad \lim_{x \to 0} \lim_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} = 1.$$

Thus, while the two repeated limits exist in this case, they are not equal.

EXAMPLE 7.4 Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$. Show that both the repeated limits exist.

Solution

We have

$$\lim_{x \to 0} f(x, y) = 0 \ \forall \ y$$

and then
$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = 0.$$

Again,
$$\lim_{y \to 0} f(x, y) = 0 \ \forall \ x$$

$$\therefore \qquad \lim_{x \to 0} \lim_{y \to 0} f(x, y) = 0.$$

Hence the repeated limits exist and they are equal.

EXAMPLE 7.5 Show that the limit exists at the origin but the repeated limits do not

$$f(x, y) = \begin{cases} x \sin \dfrac{1}{y} + y \sin \dfrac{1}{x}, & xy \neq 0 \\ 0, & xy = 0. \end{cases}$$

Solution

Since $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$ does not exist

$$\therefore \qquad \lim_{y \to 0} f(x, y) \text{ and } \lim_{x \to 0} f(x, y)$$

do not exist and therefore

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y); \ \lim_{y \to 0} \lim_{x \to 0} f(x, y)$$

do not exist.

Again,

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < |x| + |y| \le 2\sqrt{x^2 + y^2} < \varepsilon$$

if

$$x^2 < \left(\frac{\varepsilon}{4}\right)^2 \text{ and } y^2 < \left(\frac{\varepsilon}{4}\right)^2$$

or

$$|x| < \frac{\varepsilon}{2} = \delta, \; |y| < \frac{\varepsilon}{2} = \delta.$$

Thus for $\varepsilon > 0, \exists\, \delta > 0$ such that

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < \varepsilon$$

when

$$|x| < \delta, \; |y| < \delta$$

Hence

$$\lim_{(x,\, y) \to (0,\, 0)} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x} \right) = 0.$$

7.3 CONTINUITY

A function f is said to be continuous at a point (a, b) of its domain of definition if

$$\lim_{(x,\, y) \to (a,\, b)} f(x, y) = f(a, b).$$

Alternatively: A function f is said to be continuous at a point (a, b) of its domain of definition if for $\varepsilon > 0, \exists$ a *nbd* N of (a, b) such that $|f(x, y) - f(a, b)| < \varepsilon$ for all $(x, y) \in N$.

EXAMPLE 7.6 Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at the origin.

Solution

Let $x = r \cos \theta, y = r \sin \theta$

$$\therefore \quad \left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| = |r \cos \theta \sin \theta| \le r = \sqrt{x^2 + y^2} < \varepsilon$$

if

$$x^2 < \frac{\varepsilon^2}{2} \text{ and } y^2 < \frac{\varepsilon^2}{2}, \text{ or}$$

if $$|x| < \frac{\varepsilon}{\sqrt{2}}, \; |y| < \frac{\varepsilon}{\sqrt{2}}$$

Thus $$\left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| < \varepsilon$$

when $$|x| < \frac{\varepsilon}{\sqrt{2}} = \delta, |y| < \frac{\varepsilon}{\sqrt{2}} = \delta.$$

Thus $$\lim_{(x, y) \to (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0 = f(0, 0).$$

$\therefore$ $$\lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0).$$

Hence, f is continuous at $(0, 0)$.

EXAMPLE 7.7 Show that the function f defined by

$$f(x, y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{when } (x, y) \to (0, 0) \\ 0, & \text{when } (x, y) = (0, 0) \end{cases}$$

is continuous at the origin.

Solution

Put $x = r \cos \theta, y = r \sin \theta$

$\therefore$ $$\left| xy \frac{x^2 - y^2}{x^2 + y^2} \right| = \left| r^2 \cos \theta \sin \theta \frac{r^2 (\cos^2 \theta - \sin^2 \theta)}{r^2 (\cos^2 \theta + \sin^2 \theta)} \right|$$

$$= r^2 \left| \cos \theta \sin \theta \cos 2\theta \right| \leq r^2 = x^2 + y^2 < \varepsilon$$

if $$x^2 < \frac{\varepsilon}{2}, \; y^2 < \frac{\varepsilon}{2}$$

or $$\text{if } |x| < \frac{\sqrt{\varepsilon}}{2}, \; |y| < \frac{\sqrt{\varepsilon}}{2} = \delta$$

$\therefore$ $$\left| xy \frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \varepsilon \text{ for } |x| < \frac{\sqrt{\varepsilon}}{2} = \delta, \; |y| < \frac{\sqrt{\varepsilon}}{2} = \delta.$$

Thus $$\lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0).$$

Hence, f is continuous at $(0, 0)$.

7.4 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two independent variables x, y in a region R where the function is defined.

If y is constant, $f(x, y)$ becomes a function of x only. One variable x and its derivative (when exists) is called the partial derivative of $f(x, y)$ with respect to x. We denote it by

$$f_x(x, y) \text{ or } \frac{\partial f}{\partial x} \text{ or } z_x \text{ or } \frac{\partial z}{\partial x}.$$

Thus, $f_x(x,y) = \lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$, if the limit exists.

This formula gives the first order partial derivative of f w.r.t. x at any point (x, y).

In particular, the partial derivative of $f(x, y)$ w.r.t. x at the point (a, b) is given by

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Similarly, if x is constant $f(x, y)$ becomes a function of y alone, whose derivative (when exists) is called the *partial derivative* of $f(x, y)$ w.r.t. y, we denote it by

$$f_y(x, y) \text{ or } \frac{\partial f}{\partial y} \text{ or } z_y \text{ or } \frac{\partial z}{\partial y}.$$

Thus, $f_y(x, y) = \lim\limits_{k \to 0} \dfrac{f(x, y+k) - f(x, y)}{k}$, provided the limit exists.

By the statement "$f(x, y)$ is differentiable", we mean that both the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ exist.

The derivatives of f_x w.r.t. x and w.r.t. y are denoted by $\dfrac{\partial}{\partial x}(f_x)$ or $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$ or $\dfrac{\partial^2 f}{\partial x^2}$ or f_{xx} and $\dfrac{\partial}{\partial y}(f_x)$ or $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ or $\dfrac{\partial^2 f}{\partial y\, \partial x}$ or f_{yx}.

Similarly, the derivatives of f_y are denoted by

$$\frac{\partial}{\partial x}(f_y) \text{ or } \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{ or } \frac{\partial^2 f}{\partial x\, \partial y} \text{ or } f_{xy} \text{ and } \frac{\partial}{\partial y}(f_y) \text{ or } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}.$$

The derivates $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ are called *second order derivatives*. The second order derivatives f_{xy} and f_{yx} are called *mixed derivatives*.

The formulae for second order derivaties at the point (a, b) are given below.

(i) $f_{xx}(a, b) = \lim\limits_{h \to 0} \dfrac{f_x(a+h, b) - f_x(a, b)}{h}$

(ii) $f_{yy}(a, b) = \lim\limits_{k \to 0} \dfrac{f_y(a, b+k) - f_y(a, b)}{k}$

(iii) $f_{xy}(a, b) = \lim\limits_{h \to 0} \dfrac{f_y(a+h, b) - f_y(a, b)}{h}$

(iv) $f_{yx}(a, b) = \lim\limits_{k \to 0} \dfrac{f_x(a, b+k) - f_x(a, b)}{k}$.

EXAMPLE 7.8 If $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

Examine whether $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ exist and are not equal.

Solution

Now $$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$$

$$f_y(h, 0) = \lim_{k \to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k \to 0} \frac{hk(h^2 - k^2)}{k(h^2 + k^2)} = -h$$

and $$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0.$$

$\therefore$ $$f_{xy}(0, 0) = \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

Again, $$f_{yx}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k}.$$

Now, $$f_x(0, k) = \lim_{h \to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h \to 0} \frac{hk(h^2 - k^2)}{h(h^2 + k^2)} = -k$$

and $$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0.$$

$\therefore$ $$f_{yx}(0, 0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1.$$

$\therefore f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ exist but they are not equal.

EXAMPLE 7.9 If $f(x, y) = (x^2 + y^2)\tan^{-1}\dfrac{y}{x}$ when $x \neq 0$ and $f(0, y) = \dfrac{\pi y}{2}$, show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Solution

$$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_y(0+h,0) - f_y(0,0)}{h}$$

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0}\frac{\frac{\pi}{2}k^2 - 0}{k} = 0$$

$$f_y(h,0) = \lim_{k\to 0}\frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0}\frac{(h^2+k^2)\tan^{-1}\frac{k}{h} - 0}{k}$$

$$= \lim_{k\to 0}\frac{(h^2+k^2)\dfrac{1}{1+k^2/h^2}\left(\dfrac{1}{h}\right) + 2k\tan^{-1}\dfrac{k}{h}}{1}$$

$$= \lim_{k\to 0}\left(h + 2k\tan^{-1}\frac{k}{h}\right) = h.$$

$$\therefore \qquad f_{xy}(0,0) = \lim_{h\to 0}\frac{h-0}{h} = 1.$$

Again, $$f_{yx}(0,0) = \lim_{k\to 0}\frac{f_x(0,k) - f_x(0,0)}{k}.$$

Now, $$f_x(0,k) = \lim_{h\to 0}\frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0}\frac{(h^2+k^2)\tan^{-1}\frac{k}{h} - \frac{\pi k^2}{2}}{h}$$

$$= \lim_{h\to 0}\frac{(h^2+k^2)\dfrac{1}{1+\dfrac{k^2}{h^2}}\left(\dfrac{-k}{h^2}\right) + 2h\tan^{-1}\dfrac{k}{h} - 0}{1}$$

$$= -k.$$

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

$$\therefore \qquad f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k \to 0} \frac{-k-0}{k} = -1.$$

Thus, $$f_{xy}(0,0) \neq f_{yx}(0,0).$$

The sufficient conditions for equality of mixed derivatives f_{xy} and f_{yx} are stated in the following two theorems.

Theorem 7.1 *Schwarz's Theorem:* If f_y exists in a certain neighbourhood of a point (a, b) of the domain of definition of a function f and f_{yx} is continuous at (a, b) then $f_{xy}(a, b)$ exists and is equal to $f_{yx}(a, b)$.

Note 1: If f_{xy} and f_{yx} are both continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$ for the assumption of continuity of both these derivatives is a wider assumption.

Note 2: If the conditions of Schwarz's theorem are satisfy then $f_{xy} = f_{yx}$ at a point (a, b). But if the condition are not satisfied, we cannot draw any conclusion regarding the equality of f_{xy} and f_{yx} they may or may not be equal. Thus the conditions are sufficient but not necessary.

Theorem 7.2 *Young's Theorem:* If f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Note: The conditions of this theorem are sufficient but not necessary.

7.4.1 Differentiability of a Function of Two Variables

Let (x, y) and $(x + \delta x, y + \delta y)$ be two neighbourhood points in the domain of definition of a function f. The change δf in the function as the point changes from (x, y) to $(x + \delta x, y + \delta y)$ is given by

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y).$$

The function f is said to be differentiable at (x, y) if the change δf can be expressed in the form

$$\delta f = A\delta x + B\delta y + \delta x\, \phi(\delta x, \delta y) + \delta y\, \psi(\delta x, \delta y) \tag{7.1}$$

where A and B are constants independent of δx, δy and ϕ, ψ are functions of δx, δy tending to zero as δx, δy tend to 0 simultaneously.

If we replace δx, δy by h, k in Equation (7.1), we say that the function is differentiable at a point (a, b) of the domain of definition if it can be expressed as

$$\begin{aligned} df &= f(a + h, b + k) - f(a, b) \\ &= Ah + Bk + h\varphi(h, k) + k\psi(h, k) \end{aligned}$$

where $A = f_x$, $B = f_y$ and φ, ψ are functions of h, k tending to 0 as h, k tend to 0 simultaneously.

EXAMPLE 7.10 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous and possesses partial derivatives at (0, 0) but is not differentiable at (0, 0).

Solution

Differentiability:

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1.$$

Thus the function possesses partial derivatives at (0, 0). If the function is differentiable at (0, 0) then by definition

$$df = f(h, k) - f(0, 0) = Ah + Bk + h\phi + k\psi \qquad (1)$$

where A and B are constants.

i.e. $$A = f_x(0, 0) = 1, \quad B = f_y(0, 0) = -1$$

and ϕ, ψ tend to 0 as $(h, k) \to (0, 0)$.

Putting, $h = \rho\cos\theta, k = \rho\sin\theta$ in equation (1) and dividing by ρ and taking $\rho \to 0$, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$

or $$\cos\theta \sin\theta = 0$$

which is impossible for arbitrary θ. Thus the function is not differentiable at the origin.

EXAMPLE 7.11 Prove that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at the point (0, 0) but that f_x and f_y both exist at the origin and have the value 0.

Solution

Now

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0.$$

If the function is differentiable at (0, 0) then by definition

$$f(h, k) - f(0, 0) = 0 \cdot h + 0 \cdot k + h\varphi + k\psi$$

where φ and ψ are functions of h and k and tends to 0 as $(h, k) \to (0, 0)$.
Putting $h = \rho \cos \theta$, $k = \rho \sin \theta$ and dividing by ρ we get

$$|\cos \theta \sin \theta|^{1/2} = \phi \cos \theta + \psi \sin \theta.$$

For arbitrary θ, $\rho \to 0$ implies $(h, k) \to (0, 0)$.
Taking limit as $\rho \to 0$

we get
$$|\cos \theta \sin \theta|^{1/2} = 0$$

which is impossible for arbitrary θ.
$\therefore f$ is not differentiable at (0, 0).

Sufficient condition for differentiability

Theorem 7.3 If (a, b) be a point of the domain of definition of a function f such that

(i) f_x is continuous at (a, b)
(ii) f_y exists at (a, b)

then f is differentiable at (a, b).

EXAMPLE 7.12 Show that the function

$$f(x, y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x = 0 = y \end{cases}$$

is differentiable at (0, 0).

Solution

It is obvious $f_x(0, 0) = 0 = f_y(0, 0)$.

Also when $x^2 + y^2 \neq 0$

$$|f_x| = \frac{|x^4 y + 4x^2 y^3 - y^5|}{(x^2 + y^2)} \leq \frac{6(x^2 + y^2)^{5/2}}{(x^2 + y^2)^2} = 6\sqrt{x^2 + y^2}.$$

Similarly,
$$|f_y| \leq 6\sqrt{x^2 + y^2}.$$

Evidently,
$$\lim_{(x, y) \to (0, 0)} f_x(x, y) = 0 = f_x(0, 0).$$

Thus, f_x is continuous at (0, 0) and f_y (0, 0) exists, i.e. f is differentiable at (0, 0).

EXAMPLE 7.13 Given that
$$f(x, y) = xy, \quad \text{if } |y| \leq |x|$$
$$= -xy, \quad \text{if } |x| < |y|$$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Solution

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$$

$$f_x(0,k) = \lim_{h\to 0}\frac{f(h,k)-f(0,k)}{h} = \lim_{h\to 0}\frac{-hk-0}{h} = -k$$

and $$f_y(h,0) = \lim_{k\to 0}\frac{f(h,k)-f(h,0)}{k} = \lim_{k\to 0}\frac{hk-0}{k} = h.$$

Now $$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1$$

and $$f_{yx}(0,0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0}\frac{-k-0}{k} = -1.$$

Hence $f_{xy}(0,0) \neq f_{yx}(0,0)$.

The derivatives calculated in the previous examples, are determined from the definition. But, practically, it is difficult to obtain the derivatives of the first or the higher order from definition. So, the derivatives at any point can be determined by using the rules adopted for single variable, taken other variables as constants. For example, let $z = ax^2 + 2hxy + by^2$.

Then $\frac{\partial z}{\partial x}$ is to be determined by taking y as constant. That is, for this expression a, h, b and y are constants and hence

$$\frac{\partial z}{\partial x} = 2ax + 2hy.$$

Similarly, $\frac{\partial z}{\partial y}$ is to be calculated by taking x as constant.

Thus, $$\frac{\partial z}{\partial y} = 2hx + 2by.$$

The other derivatives are obtained as

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2ax + 2hy) = 2a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(2hx + 2by) = 2b$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(2hx + 2by) = 2h$$

and $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(2ax + 2hy) = 2h.$$

It may be noted that for this problem $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$.

EXAMPLE 7.14 Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ when $u = \tan^{-1} \dfrac{x - y}{x^2 + y^2}$.

Solution

Since $$u = \tan^{-1} \frac{x - y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\dfrac{x - y}{x^2 + y^2}\right)^2} \frac{\partial}{\partial x}\left(\frac{x - y}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2 + (x - y)^2} \cdot \frac{1 \cdot (x^2 + y^2) - (x - y)\, 2x}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2 + 2xy}{(x^2 + y^2)^2 + (x - y)^2}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{1 + \left(\dfrac{x - y}{x^2 + y^2}\right)^2} \frac{\partial}{\partial y}\left(\frac{x - y}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2 + (x - y)^2} \cdot \frac{(-1)\,(x^2 + y^2) - (x - y)\, 2y}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2 + (x - y)^2}.$$

EXAMPLE 7.15 If $U = \sqrt{xy}$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$. (WBUT 2001)

Solution

Here $$U = \sqrt{xy} = x^{1/2}\, y^{1/2}.$$

$$\frac{\partial U}{\partial x} = \frac{1}{2} x^{-1/2} \; y^{1/2}, \quad \frac{\partial U}{\partial y} = \frac{1}{2} x^{1/2} \; y^{-1/2}$$

$\therefore$

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{2}\left(\frac{-1}{2}\right) x^{-3/2} \; y^{1/2} \text{ and } \frac{\partial^2 U}{\partial y^2} = \frac{1}{2}\left(\frac{-1}{2}\right) x^{1/2} \; y^{-3/2}$$

Now,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{-1}{4} x^{-3/2} \; y^{1/2} - \frac{1}{4} x^{1/2} \; y^{-3/2}$$

$$= \frac{-1}{4}\left(\frac{y^{1/2}}{x^{3/2}} + \frac{x^{1/2}}{y^{3/2}}\right)$$

$$= -\frac{1}{4} \frac{x^2 + y^2}{(xy)^{3/2}}.$$

EXAMPLE 7.16 If $x^x \; y^y \; z^z = k$, show that at $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = -\,(x \log(ex))^{-1}.$$

Solution

Taking log on both sides of $x^x \; y^y \; z^z = k$, we get

$$x \log x + y \log y + z \log z = \log k \tag{1}$$

We assume that z is a function of two independent variables x and y. Differentiating equation (1) w.r.t. y treating x as constant, we get

$$\log y + 1 + \frac{\partial z}{\partial y} \log z + z \frac{1}{z} \frac{\partial z}{\partial y} = 0$$

or

$$\frac{\partial z}{\partial y}(1 + \log z) = -\,(1 + \log y)$$

or

$$\frac{\partial z}{\partial y} = -\left(\frac{1 + \log y}{1 + \log z}\right). \tag{2}$$

Similarly,

$$\frac{\partial z}{\partial x} = -\left(\frac{1 + \log x}{1 + \log z}\right). \tag{3}$$

Differentiating equation (1) partially w.r.t. y, we get

$$1 + \log y + \log z \frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = 0.$$

Differentiating w.r.t. x, we get

$$(1+\log z)\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z}\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}.$$

From equations (2) and (3), we see that $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial x} = -1$ at $x = y = z$.

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)} \text{ at } x = y = z$$

$$= -(x \log ex)^{-1}.$$

EXAMPLE 7.17 If $\theta = t^n e^{-r^2/4t}$, find the value of n for which

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}.$$

Solution

Partially differentiating the given relation w.r.t. r, we get

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t} \times \left(-\frac{1}{4t}\right) 2r$$

$$= -\frac{\theta \times 2r}{4t} = -\frac{\theta r}{2t}.$$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\frac{\theta r^3}{2t}.$$

Hence $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2t}\left\{\frac{\partial \theta}{\partial r} r^3 + \theta \cdot 3r^2\right\}$$

$$= -\frac{1}{2t}\left\{-\frac{\theta r}{2t}\cdot r^3 + 3\theta r^2\right\} \qquad \left[\because \frac{\partial \theta}{\partial t} = -\frac{\theta r}{2t}\right]$$

$$= \frac{\theta r^4}{4t^2} - \frac{3}{2}\frac{\theta r^2}{t}$$

or $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\theta r^2}{4t^2} - \frac{3}{2}\frac{\theta}{t}. \tag{1}$$

Again, $$\frac{\partial \theta}{\partial t} = nt^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t}\left(-\frac{r^2}{4}\right)\left(-\frac{1}{t^2}\right)$$

$$= \frac{n\theta}{t} + \frac{\theta r^2}{4t^2}. \tag{2}$$

From equations (1) and (2), we have

$$\frac{\theta r^2}{4t^2} - \frac{3}{2}\frac{\theta}{t} = \frac{n\theta}{t} + \frac{\theta r^2}{4t^2}$$

or

$$n = -\frac{3}{2}.$$

EXAMPLE 7.18 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that

(i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x+y+z}$

(ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right)u = -\dfrac{3}{(x+y+z)^2}$

(iii) $\left(\dfrac{\partial}{\partial x^2} + \dfrac{\partial}{\partial y^2} + \dfrac{\partial}{\partial z^2}\right)^2 u = -\dfrac{9}{(x+y+z)^2}$ (WBUT 2003)

Solution

We know $\quad x^3 + y^3 + z^3 - 3xyz$

$$= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega^2 z),$$

where ω^2 are to the cube roots of unity.

$\therefore \quad u = \log(x+y+z) + \log(x+\omega y+\omega^2 z) + \log(x+\omega^2 y+\omega^2 z)$

(i) $\therefore$

$$\frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{1}{x+\omega y+\omega^2 z} + \frac{1}{x+\omega^2 y+\omega z} \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y+z} + \frac{\omega}{x+\omega y+\omega^2 z} + \frac{\omega^2}{x+\omega^2 y+\omega z} \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x+y+z} + \frac{\omega^2}{x+\omega y+\omega^2 z} + \frac{\omega}{x+\omega^2 y+\omega z} \quad (3)$$

Adding equations (1), (2) and (3) we get

$$\therefore \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z} \qquad [\because 1+\omega+\omega^2 = 0]$$

(ii) Now $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left[\dfrac{1}{x+y+z} + \dfrac{1}{x+\omega y+\omega^2 z} + \dfrac{1}{x+\omega^2 y+\omega z}\right]$

$$= -\frac{1}{(x+y+z)^2} - \frac{1}{(x+\omega y+\omega^2 z)^2} - \frac{1}{(x+\omega^2 y+\omega z)^2}.$$

Similarly, we have

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y+z)^2} - \frac{\omega^2}{(x+\omega y+\omega^2 z)^2} - \frac{\omega^4}{(x+\omega^2 y+\omega z)^2}$$

and $$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x+y+z)^2} - \frac{\omega^4}{(x+\omega y+\omega^2 z)^2} - \frac{\omega^2}{(x+\omega^2 y+\omega z)^2}.$$

Adding above three equations, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}.$$

(iii) Now, $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= -\frac{9}{(x+y+z)^2}.$$

EXAMPLE 7.19 If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$

prove that $$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta} \text{ and } \frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$$

hence deduce that $$\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = 0.$$

Solution

Given $$x = e^{r\cos\theta}\cos(r\sin\theta) \quad (1)$$

$$y = e^{r\cos\theta}\sin(r\sin\theta). \quad (2)$$

Differentiating equation (1) partially w.r.t. r, we get

$$\frac{\partial x}{\partial r} = e^{r\cos\theta}\cos(r\sin\theta)\cos\theta - e^{r\cos\theta}\sin(r\sin\theta)\sin\theta \quad (3)$$

Differentiating equation (1) partially w.r.t. θ, we get

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-\sin\theta\cdot r)\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta)\times r\cos\theta$$

$$= -r(e^{r\cos\theta})\{\sin\theta\cos(r\sin\theta) + \sin(r\sin\theta)\cos\theta\} \quad (4)$$

$$\frac{\partial y}{\partial r}=e^{r\cos\theta}(\cos\theta)\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)\sin\theta. \tag{5}$$

Differentiating equation (2) partially w.r.t. θ, we get

$$\frac{\partial y}{\partial\theta}=e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)(r\cos\theta)$$

$$=re^{r\cos\theta}\{-\sin\theta\sin(r\sin\theta)+\cos(r\sin\theta)\cos\theta\} \tag{6}$$

∴ From equations (3) and (6), we have

$$\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial\theta} \tag{7}$$

and from equations (4) and (5), we have

$$\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial\theta}. \tag{8}$$

Differentiating equation (7) partially w.r.t. r, we get

$$\frac{\partial^2 x}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{\partial\theta}\right)$$

$$=-\frac{1}{r^2}\frac{\partial y}{\partial\theta}+\frac{1}{r}\frac{\partial^2 y}{\partial r\partial\theta}$$

$$=-\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r}\frac{\partial^2 y}{\partial r\partial\theta}. \tag{9}$$

[using equation (7)]

Equation (8) can be written as,

$$\frac{\partial x}{\partial\theta}=-r\frac{\partial y}{\partial r}.$$

Differentiating w.r.t. θ, we get

$$\frac{\partial^2 x}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(-r\frac{\partial y}{\partial r}\right)=-r\frac{\partial^2 y}{\partial\theta\partial r}$$

or
$$\frac{1}{r^2}\frac{\partial^2 x}{\partial\theta^2}=-\frac{1}{r}\frac{\partial^2 y}{\partial\theta\partial r}. \tag{10}$$

Adding equations (9) and (10), we get

$$\frac{\partial^2 x}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 x}{\partial\theta^2}=-\frac{1}{r}\frac{\partial x}{\partial r}$$

or
$$\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial\theta^2}=0.$$

EXAMPLE 7.20 If $z = f(x+ay) + \phi(x-ay)$ show that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Solution

We have $$z = f(x+ay) + \phi(x-ay). \tag{1}$$

Differentiating w.r.t. x, we get

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay).$$

Again differentiating w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay). \tag{2}$$

Differentiating equation (1) twice w.r.t. y, we get

$$\frac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay)$$

and $$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay) \tag{3}$$

From equations (2) and (3), we have

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

EXAMPLE 7.21 Suppose $v = f(u)$, where $u = (x^2 + y^2)\tan^{-1}(y/x)$.

Prove that $$x\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial y} = (x^2+y^2)\, f'(u).$$

Solution

Differentiating w.r.t. x, we get $$v = f(u) \tag{1}$$

$$\frac{\partial v}{\partial x} = f'(u)\frac{\partial u}{\partial x}$$

$$= f'(u)\left\{2x\tan^{-1}\left(\frac{y}{x}\right) + (x^2+y^2)\frac{\left(-\frac{y}{x^2}\right)}{1+\frac{y^2}{x^2}}\right\}$$

$$= f'(u)\left\{2x\tan^{-1}\left(\frac{y}{x}\right) - y\right\}$$

$$y\frac{\partial v}{\partial x} = f'(u)\left\{2xy\tan^{-1}\left(\frac{y}{x}\right) - y^2\right\} \qquad (2)$$

or

Differentiating equation (1) w.r.t. y, we get

$$\frac{\partial v}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

$$= f'(u)\left\{2y\tan^{-1}\frac{y}{x} + (x^2+y^2)\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right\}$$

$$= f'(u)\left\{2y\tan^{-1}\frac{y}{x} + x\right\}$$

or

$$x\frac{\partial v}{\partial y} = f'(u)\left\{2yx\tan^{-1}\frac{y}{x} + x^2\right\}. \qquad (3)$$

From equations (2) and (3)

$$-x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = f'(u)\{x^2+y^2\}$$

or

$$y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} = (x^2+y^2)f'(u)$$

or

$$y\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial x} = (x^2+y^2)f'(u). \qquad [\text{as } v = f(u)]$$

EXAMPLE 7.22 Let $z = f(u)$ where $u = x^m y^n$ show that

$$nx\frac{\partial z}{\partial x} - my\frac{\partial z}{\partial y} = 0$$

and hence deduce that

$$n^2x\frac{\partial z}{\partial x} - m^2y\frac{\partial z}{\partial y} + n^2x^2\frac{\partial^2 z}{\partial x^2} - m^2y^2\frac{\partial^2 z}{\partial y^2} = 0.$$

Solution

$$z = f(u) \qquad (1)$$

Differentiating w.r.t. x we get

$$\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} = f'(u)\,mx^{m-1}y^n$$

or

$$nx\frac{\partial z}{\partial x} = f'(u)\,nm\,x^m y^n. \qquad (2)$$

Differentiating equation (1) w.r.t. y we get

$$\frac{\partial z}{\partial y} = f'(u)\, x^m\, n\, y^{n-1}$$

or
$$my\frac{\partial z}{\partial y} = f'(u)\, mn\, x^m y^n. \tag{3}$$

From equations (2) and (3), we get

$$nx\frac{\partial z}{\partial x} - my\frac{\partial z}{\partial y} = 0. \tag{4}$$

Differentiating equation (4) w.r.t. x, we get

$$nx\frac{\partial^2 z}{\partial x^2} + n\frac{\partial z}{\partial x} - my\frac{\partial^2 z}{\partial x \partial y} = 0.$$

Multiplying both sides by nx, we get

$$n^2x^2\frac{\partial^2 z}{\partial x^2} + n^2x\frac{\partial z}{\partial x} - mn\, xy\frac{\partial^2 z}{\partial x \partial y} = 0. \tag{5}$$

Differentiating equation (4) w.r.t. y, we get

$$nx\frac{\partial^2 z}{\partial y \partial x} - m\frac{\partial z}{\partial y} - my\frac{\partial^2 z}{\partial y^2} = 0.$$

Multiplying both sides by my, we get

$$mn\, xy\frac{\partial^2 z}{\partial x \partial y} - m^2y\frac{\partial z}{\partial y} - m^2y^2\frac{\partial^2 z}{\partial y^2} = 0. \tag{6}$$

Adding equations (5) and (6), we get the required result.

$$n^2x^2\frac{\partial^2 z}{\partial x^2} + n^2x\frac{\partial z}{\partial y} - m^2y\frac{\partial z}{\partial y} - m^2y^2\frac{\partial^2 z}{\partial y^2} = 0.$$

EXAMPLE 7.23 If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

Solution

Since
$$u = e^{xyz}$$

$$\frac{\partial u}{\partial z} = xy\, e^{xyz},$$

$$\frac{\partial^2 u}{\partial y \partial z} = xe^{xyz} + xy\,(xze^{xyz}) = (x + x^2yz)\, e^{xyz}$$

and
$$\frac{\partial^2 u}{\partial x \partial y \partial z} = (1 + 2xyz)\, e^{xyz} + (x + x^2 yz)\,(yz)\, e^{xyz}$$

$$= (1 + 2xyz + xyz + x^2 y^2 z^2)\, e^{xyz}$$

$$= (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}.$$

7.5 HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

If every term of a function is of same degree, say, n, then the function is said to be a *homogeneous function* of degree n. If the function has two independent variables x and y then the degree of a term is the algebraic sum of the indices of x and y.

That is the degree of $x^3y^{1/2}$ is 5/2 and that of $\frac{x^3}{y^2}$ is one.

Alternatively, a function $f(x, y, z)$ is said to be homogeneous of degree n, if

$$f(tx, ty, tz) = t^n\, f(x, y, z),$$

for every positive value of t. In case of two variables, this expression is

$$f(tx, ty) = t^n\, f(x, y).$$

A homogeneous function of three variables can also be written in the following forms:

$$f(x, y, z) = x^n\, f_1\left(\frac{y}{x}, \frac{z}{x}\right)$$

or
$$f(x, y, z) = y^n\, f_2\left(\frac{x}{y}, \frac{z}{y}\right)$$

or
$$f(x, y, z) = z^n\, f_3\left(\frac{x}{z}, \frac{y}{z}\right)$$

where f_1, f_2, f_3 are some functions depending on f.

Similarly a homogeneous function of two variables can be written as

$$f(x, y) = x^n\, \phi\left(\frac{y}{x}\right)$$

or
$$f(x, y) = y^n\, \psi\left(\frac{x}{y}\right).$$

For example, the function

$$f(x, y) = x^2 + 2y^2 = x^2\left(1 + \frac{2y^2}{x^2}\right)$$

is a homogeneous function of degree two, while the function

$$f(x, y, z) = x^3 + \frac{y^4}{z} + z^3 = x^3 \left\{1 + \frac{y^4}{x^3 z} + \frac{z^3}{x^3}\right\}$$

is a homogeneous function of degree 3.

Theorem 7.4 *Euler's theorem (case of two variables)*

If $u = f(x, y)$ be a homogeneous function in x and y of degree n and if u has continuous partial derivatives, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof: Since $u = f(x, y)$ is a homogeneous function in x and y of degree n, we may write it as

$$u = x^n \phi\left(\frac{y}{x}\right) \tag{1}$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

or

$$x\frac{\partial u}{\partial x} = nx^n \phi\left(\frac{y}{x}\right) - x^{n-1} y\phi'\left(\frac{y}{x}\right). \tag{2}$$

Again from equation (1),

$$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right) \times \frac{1}{x}$$

or

$$y\frac{\partial u}{\partial y} = x^{n-1} y\phi'\left(\frac{y}{x}\right) \tag{3}$$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) = nf(x, y) = nu.$$

Hence proved.

Theorem 7.5 *Euler's theorem (case of three variables)*

If $f(x, y, z)$ be a homogeneous function in x, y, z of degree n, having continuous partial derivatives then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf.$$

The converse of this theorem is also valid. That is, $f(x, y, z)$ admits of continuous partial derivatives and satisfies the relation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf$$

where n is any number. Therefore, f is a homogeneous function of degree n.

EXAMPLE 7.24 Verify Euler's theorem for the function

$$u=f(x,y)=ax^2+2hxy+by^2.$$

Solution

Here
$$u=f(x,y)=x^2\left\{a+2h\left(\frac{y}{x}\right)+b\left(\frac{y}{x}\right)^2\right\}$$

$$=x^2\,\phi\left(\frac{y}{x}\right)\text{(say)}$$

$\therefore$ u is a homogeneous function of degree two. So according to Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u. \tag{1}$$

Now
$$\frac{\partial u}{\partial x}=2ax+2hy \text{ and } \frac{\partial u}{\partial y}=2hx+2by$$

$\therefore$
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(2ax^2+2hxy)+(2hxy+2by^2)$$

$$=2\,(ax^2+2hxy+2by^2)$$

$$=2u.$$

Hence verified.

EXAMPLE 7.25 Let u be a homogeneous function in x and y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n\,(n-1)\,u,$$

where all partial derivatives of the first and the second order are continuous. The above result can also be written as

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u=n\,(n-1)\,u.$$

Solution

Since u is a homogeneous function of x and y of degree n, we have by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu. \tag{1}$$

Differentiating equation (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

or $$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}. \tag{2}$$

Differentiating equation (1) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}. \tag{3}$$

Multiplying equation (2) by x and equation (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = (n-1)\left\{x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right\}$$

$$\left[\text{using } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right]$$

$$= n(n-1)\,u \qquad \text{[using equation (1)]}$$

Hence proved.

EXAMPLE 7.26 Let $u = \sin^{-1}\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$ show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u). \quad \text{(WBUT 2001)}$$

Solution

Let $$v(x, y) = \sin u = \sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}.$$

Putting $x = tx$ and $y = ty$
we get

$$v(tx, ty) = t^{\left(\frac{1}{3}-\frac{1}{2}\right)\frac{1}{2}}\, v(x, y) = t^{-\frac{1}{12}}\, v(x, y).$$

$\therefore$ v is a homogeneous function of degree $\left(-\dfrac{1}{12}\right)$.

∴ By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=-\frac{1}{12}v$$

or

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=-\frac{1}{12}\sin u$$

or

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\frac{\sin u}{\cos u}=-\frac{1}{12}\tan u. \quad (1)$$

Differentiating equation (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial x} \quad (2)$$

Again differentiating equation (1) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial y}. \quad (3)$$

Using equation (2) × x + equation (3) × y, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)-\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=-\frac{1}{12}\sec^2 u\left(-\frac{1}{12}\tan u\right)+\frac{1}{12}\tan u$$

$$=\frac{1}{144}(1+\tan^2 u)\ \tan u+\frac{1}{12}\tan u$$

$$=\left(\frac{1}{144}+\frac{1}{12}+\frac{1}{144}\tan^2 u\right)\tan u$$

$$=\left(\frac{13}{144}+\frac{1}{144}\tan^2 u\right)\tan u$$

$$=\frac{\tan u}{144}(13+\tan^2 u).$$

Hence proved.

EXAMPLE 7.27 If $u=\dfrac{(x^2+y^2)^n}{2n(2n-1)}+xf\left(\dfrac{y}{x}\right)+g\left(\dfrac{y}{x}\right)$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(x^2+y^2)^n.$$

Solution

Let
$$u = P + Q + R$$
where

$$P = \frac{(x^2+y^2)^n}{2n\,(2n-1)} = \text{ a homogeneous function of degree } 2n$$

$$Q = xf\left(\frac{y}{x}\right) = \text{ a homogeneous function of degree } 1$$

$$R = g\left(\frac{y}{x}\right) = \text{ a homogeneous function of degree } 0.$$

∴ By Euler's theorem on P, Q, R we have

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = 2nP \tag{1}$$

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} = 1 \cdot Q \tag{2}$$

and
$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} = 0 \cdot R \tag{3}$$

Adding, we get

$$x\frac{\partial}{\partial x}(P+Q+R) + y\frac{\partial}{\partial y}(P+Q+R)$$

$$= 2nP + Q$$

or
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2n\,(x^2+y^2)^n}{2n\,(2n-1)} + xf\left(\frac{y}{x}\right)$$

$$= 2nP + Q. \tag{4}$$

Differentiating equation (4) partially w.r.t. x and multiplying by x, we get

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = 2nx\frac{\partial P}{\partial x} + x\frac{\partial Q}{\partial x}. \tag{5}$$

Again, differentiating equation (4) partially w.r.t. y and multiplying by y, we get

$$xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2ny\frac{\partial P}{\partial y} + y\frac{\partial Q}{\partial y}. \tag{6}$$

Adding equations (5) and (6), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2n\left(x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}\right) + \left(x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y}\right) - \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= 2n\,(2nP) + Q - (2nP + Q)$$

using equations (1), (2) and (3)

$$= (2n - 1)\,2nP$$

$$= 2n\,(2n-1)\,\frac{(x^2 + y^2)^n}{2n\,(2n-1)} = (x^2 + y^2)^n.$$

Hence proved.

EXAMPLE 7.28 If $u = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xf\left(\frac{y}{x}\right)$$

and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0. \qquad \text{(WBUT 2004)}$$

Solution

Let $u = v + w$, where $v = xf\left(\frac{y}{x}\right)$ and $w = g\left(\frac{y}{x}\right)$. Both v and w are homogeneous functions of degree 1 and degree 0 respectively.

Then by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v \text{ and } x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0.w.$$

Adding these equations, we get

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = v$$

or

$$x\frac{\partial}{\partial x}(v + w) + y\frac{\partial}{\partial y}(v + w) = v$$

or

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xf\left(\frac{y}{x}\right). \qquad (1)$$

Differentiating equation (1) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

or

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} = xf\left(\frac{y}{x}\right) - yf'\left(\frac{y}{x}\right). \qquad (2)$$

Again differentiating equation (2) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\,\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

or
$$xy\frac{\partial^2 u}{\partial y\,\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=yf'\left(\frac{y}{x}\right).$$

Adding equations (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=xf\left(\frac{y}{x}\right)$$

or
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+xf\left(\frac{y}{x}\right)=xf\left(\frac{y}{x}\right)$$

[using equation (1)

or
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

EXAMPLE 7.29 If $r^2=x^2+y^2+z^2$ and $v=r^m$, then show that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=m\,(m+1)\,r^{m-2}$$

More generally, if $v=f(r)$, then prove that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=f''(r)+\frac{2}{r}f'(r).$$

Solution

First Part:

$$\frac{\partial v}{\partial x}=mr^{m-1}\frac{\partial r}{\partial x}=\frac{m\,x\,r^{m-1}}{r}=m\,x\,r^{m-2}$$

$$\left[\text{as } r^2=x^2+y^2+z^2 \text{ or } 2r\frac{\partial r}{\partial x}=2x\right]$$

Again
$$\frac{\partial^2 v}{\partial x^2}=m\,r^{m-2}+mx\,(m-2)\,r^{m-3}\frac{\partial r}{\partial x}$$

$$=m\,r^{m-2}+m\,(m-2)\,x\,r^{m-3}\,\frac{x}{r}$$

$$=m\,r^{m-2}+m\,(m-2)\,r^{m-4}.$$

Similarly,
$$\frac{\partial^2 v}{\partial y^2}=m\,r^{m-2}+m\,(m-2)\,y^2\,r^{m-4}$$

and $$\frac{\partial^2 v}{\partial z^2} = m r^{m-2} + m(m-2) z^2 r^{m-4}.$$

Adding the above results, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 3m r^{m-2} + m(m-2) r^{m-4}(x^2 + y^2 + z^2)$$

$$= 3m r^{m-2} + m(m-2) r^{m-2}$$

$$= m(m+1) r^{m-2}.$$

If $v = f(r)$

then $$\frac{\partial v}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$$

and $$\frac{\partial^2 v}{\partial x^2} = \frac{1}{r} f'(r) - \frac{x}{r^2}\frac{\partial r}{\partial x} f'(r) + \frac{x}{r} f''(r)\frac{\partial r}{\partial x}$$

$$= \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r).$$

Similarly, $$\frac{\partial^2 v}{\partial y^2} = \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r)$$

and $$\frac{\partial^2 v}{\partial z^2} = \frac{1}{r} f'(r) - \frac{z^2}{r^3} f'(r) + \frac{z^2}{r^2} f''(r).$$

Adding the above results, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{3}{r} f'(r) - \frac{1}{r^3} f'(r)(x^2+y^2+z) + \frac{1}{r^2} f''(r)(x^2+y^2+z^2)$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + \frac{1}{r^2} r^2 f''(r)$$

$$= \frac{2}{r} f'(r) + f''(r).$$

EXAMPLE 7.30 If $v = 3(ax+by+cz)^2 - (x^2+y^2+z^2)$ and $a^2+b^2+c^2 = 1$, then prove that $v_{xx} + v_{yy} + v_{zz} = 0$.

Solution

We have

$$v = 3(ax+by+cz)^2 - (x^2+y^2+z^2). \tag{1}$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial v}{\partial x} = 3\cdot 2\,(ax+by+cz)\,a - 2x$$

and $$\frac{\partial^2 v}{\partial x^2} = 6a^2 - 2.$$

Similarly, $$\frac{\partial^2 v}{\partial y^2} = 6b^2 - 2 \text{ and } \frac{\partial^2 v}{\partial z^2} = 6c^2 - 2.$$

$\therefore$ $$v_{xx} + v_{yy} + v_{zz} = 6(a^2 + b^2 + c^2) - 6 = 6 - 6 = 0.$$

Hence proved.

EXAMPLE 7.31 If $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Solution

Let u be a function of x, y and z.

Differentiating partially,

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad (1)$$

w.r.t. x, we get

$$\frac{2x}{a^2+u} - \frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x} - \frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x} - \frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x} = 0$$

or $$\frac{\partial u}{\partial x} = \frac{2x}{a^2+u}\Big/ A$$

where $$A = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}.$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{2y}{(a^2+u)}\Big/ A \text{ and } \frac{\partial u}{\partial z} = \frac{2z}{(a^2+u)}\Big/ A.$$

$\therefore$ $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

$$= \frac{4}{A^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] = \frac{4}{A^2}\cdot A = \frac{4}{A}. \qquad (2)$$

Again, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{2}{A}\left(\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right)$$

$$= \frac{2}{A}. \tag{3}$$

[using equation (1)]

∴ From equations (2) and (3)

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right).$$

EXAMPLE 7.32 If $x^2 + y^2 + z^2 - 2xyz = 1$, show that

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

Solution

Differentiating $x^2 + y^2 + z^2 - 2xyz = 1$, we get

$$2x\,dx + 2y\,dy + 2z\,dz - 2\,(yz\,dx + xz\,dy + xy\,dx) = 0$$

or
$$(x - yz)\,dx + (y - zx)\,dy + (z - xy)\,dz = 0. \tag{1}$$

Now
$$x^2 + y^2 + z^2 - 2xyz = 1$$

or
$$x^2 - 2xyz + y^2z^2 + y^2 + z^2 - y^2z^2 = 1$$

or
$$(x - yz)^2 = 1 - y^2 - z^2 + y^2z^2$$
$$= (1 - y^2) - z^2\,(1 - y^2) = (1 - y^2)\,(1 - z^2)$$

or
$$x - yz = \pm\sqrt{(1 - y^2)\,(1 - z^2)}.$$

Similarly,
$$y - zx = \pm\sqrt{(1 - x^2)\,(1 - z^2)}$$

and
$$z - xy = \pm\sqrt{(1 - x^2)\,(1 - y^2)}.$$

∴ Form equation (1)

$$\pm\sqrt{(1-z^2)(1-y^2)}\,dx \pm \sqrt{(1-x^2)(1-z^2)}\,dy \pm \sqrt{(1-x^2)(1-y^2)}\,dz = 0.$$

Dividing both sides by $-\sqrt{(1-z^2)(1-y^2)(1-z^2)}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

EXAMPLE 7.33 If v be a homogeneous function in x, y, z of degree n, prove that $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}$ are each a homogeneous function in x, y, z of degree $(n-1)$.

Solution

Since v is a homogeneouos function of degree n.

$\therefore$ By Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=nv.$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial v}{\partial x}+x\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+y\frac{\partial^2 v}{\partial x\partial y}+z\frac{\partial^2 v}{\partial x\partial z}=n\frac{\partial v}{\partial x}$$

or
$$x\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)+z\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}\right)=(n-1)\frac{\partial v}{\partial x}.$$

This shows that $\frac{\partial v}{\partial x}$ is a homogeneous function in x, y, z of degree $(n-1)$.

Similarly $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial z}$ are homogeneous functions of degree $(n-1)$.

EXAMPLE 7.34 If V be a homogeneous function in x, y, z of degree n and if $V=f(X, Y, Z)$ where X, Y, Z are respectively $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$, show that

$$X\frac{\partial V}{\partial X}+Y\frac{\partial V}{\partial Y}+Z\frac{\partial V}{\partial Z}=\frac{n}{n-1}V.$$

Solution

Given $V=f(X, Y, Z)$

$$\therefore \quad \frac{\partial V}{\partial x}=\frac{\partial V}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial V}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial V}{\partial Z}\frac{\partial Z}{\partial x}$$

$$\frac{\partial V}{\partial y}=\frac{\partial V}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial V}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial V}{\partial Z}\frac{\partial Z}{\partial y}$$

$$\frac{\partial V}{\partial z}=\frac{\partial V}{\partial X}\frac{\partial X}{\partial z}+\frac{\partial V}{\partial Y}\frac{\partial Y}{\partial z}+\frac{\partial V}{\partial Z}\frac{\partial Z}{\partial z}$$

$$\therefore \quad x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\frac{\partial V}{\partial x}\left(x\frac{\partial X}{\partial x}+y\frac{\partial X}{\partial y}+z\frac{\partial X}{\partial z}\right)$$
$$+\frac{\partial V}{\partial y}\left(x\frac{\partial Y}{\partial x}+y\frac{\partial Y}{\partial y}+z\frac{\partial Y}{\partial z}\right)+\frac{\partial V}{\partial z}\left(x\frac{\partial Z}{\partial x}+y\frac{\partial Z}{\partial y}+z\frac{\partial Z}{\partial z}\right)$$

or

$$nV = \frac{\partial V}{\partial X}(n-1)X + \frac{\partial V}{\partial Y}(n-1)Y + \frac{\partial V}{\partial Z}(n-1)Z.$$

Since V is a homogeneous function of degree n and $\frac{\partial V}{\partial x}$ i.e. X, Y and Z are homogeneous functions of degree $(n-1)$.

or

$$X\frac{\partial V}{\partial X} + Y\frac{\partial V}{\partial Y} + Z\frac{\partial V}{\partial Z} = \frac{n}{n-1}V.$$

7.6 CHAIN RULES

If $z = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$; f, ϕ, ψ are all differentiable functions, then the composite function $z = f(\phi(t), \psi(t)) = F(t)$, say, is a differentiable function of t and the total derivative is given by

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{7.2}$$

also the total differential

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{7.3}$$

Again, if $z = f(x, y)$ be a differentiable function in x and y where $x = \phi(u, v)$, $y = \psi(u, v)$ are also differentiable functions of u and v, then the composite function $z = f(\phi, \psi) = F(u, v)$, say, is a differentiable function of u, v whose partial derivatives $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ are given by

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \tag{7.4}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \tag{7.5}$$

and the total differential dz of z is given by

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{7.6}$$

Formula of equations (7.2), (7.4) and (7.5) are known as chain rules.
From Equations (7.3) and (7.6) it may be observed that if $z = f(x, y) = 0$ then $dz = 0$ and hence

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

or
$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{f_x}{f_y}.$$

This formula may be used to find $\frac{dy}{dx}$ of an implicit function.

EXAMPLE 7.35 Find $\frac{dy}{dx}$ when $x^3 - 3xy + 2y^3 = 0$.

Solution

Let
$$f(x, y) = x^3 - 3xy + 2y^3.$$

Now,
$$f_x = 3x^2 - 3y \text{ and } f_y = -3x + 6y^2$$

Therefore,
$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 - 3y}{-3x + 6y^2} = \frac{x^2 - y}{x - 2y^2}.$$

EXAMPLE 7.36 If $x^y y^x = c$, find the value of $\frac{dy}{dx}$.

Solution

Taking log of $x^y y^x = c$ both sides.
$$y \log x + x \log y = \log c.$$

Let
$$f(x, y) = y \log x + x \log y - \log c.$$

$\therefore$
$$f_x = y\frac{1}{x} + \log y \text{ and } f_y = \log x + x\frac{1}{y}.$$

Hence,
$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\dfrac{y}{x} + \log y}{\log x + \dfrac{x}{y}} = -\frac{y(y + x \log y)}{x(x + y \log x)}.$$

EXAMPLE 7.37 If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that
$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - yx)\frac{\partial u}{\partial z} = 0. \quad \text{(WBUT 20}$$

Solution

Let
$$r = x^2 + 2yz, \ s = y^2 + 2zx.$$

$\therefore$
$$u = f(r, s)$$

Now,
$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}=\frac{\partial u}{\partial r}(2x)+\frac{\partial u}{\partial s}(2z)$$
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}=\frac{\partial u}{\partial r}(2z)+\frac{\partial u}{\partial s}(2y)$$
and
$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}=\frac{\partial u}{\partial r}(2y)+\frac{\partial u}{\partial s}(2x).$$
Now,
$$(y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}$$
$$=\frac{\partial u}{\partial r}\{2x(y^2-zx)+2z(x^2-yz)+2y(z^2-xy)\}$$
$$+\frac{\partial u}{\partial s}\{2z(y^2-zx)+2y(x^2-yz)+2x(z^2-xy)\}$$
$$=0.$$

EXAMPLE 7.38 If $z=\sin uv$ where $u=3x^2$ and $v=\log x$, find $\dfrac{dz}{dx}$.

(WBUT 2004)

Solution

Since z is a function of u and v,
$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\frac{du}{dx}+\frac{\partial z}{\partial v}\frac{dv}{dx}$$
$$=v\cos uv\cdot 6x+u\cos uv\,\frac{1}{x}$$
$$=6xv\cos uv+\frac{u}{x}\cos uv$$
$$=(6x\log x+3x)\cos(3x^2\log x).$$

EXAMPLE 7.39 If $z=u^3+v^3$, where $u=\sin xy$ and $v=y^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution

Since z is a function of u, v and u, v are functions of x and y.

Therefore,
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$=3u^2\cdot y\cos xy+3v^2\cdot 0$$
$$=3yu^2\cos xy$$
$$=3y\sin^2(xy)\cos xy$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$
$$= 3u^2 \cdot x\cos xy + 3v^2 \cdot 2y$$
$$= 3u^2 x\cos xy + 6v^2 y$$
$$= 3x\sin^2 xy\cos xy + 6y^5.$$

EXAMPLE 7.40 If $z = f(u, v)$ where $u = x^2 - 2xy - y^2$ and $v = y$, show that $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$ can be transformed into $\frac{\partial z}{\partial v} = 0$.

Solution

Since $z = f(u, v)$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$
$$= \frac{\partial f}{\partial u}(2x - 2y) + 0$$
$$= -2(x-y)\frac{\partial f}{\partial u}$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$
$$= \frac{\partial f}{\partial u}(-2x - 2y) + \frac{\partial f}{\partial v}\cdot 1$$
$$= -2(x+y)\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}.$$

Now,
$$(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0 \text{ becomes}$$
$$2(x^2 - y^2)\frac{\partial f}{\partial u} - 2(x^2 - y^2)\frac{\partial f}{\partial u} + (x-y)\frac{\partial f}{\partial u} = 0$$

or
$$(x-y)\frac{\partial f}{\partial v} = 0$$

or
$$\frac{\partial f}{\partial v} = 0 \text{ if } x \neq y$$

or
$$\frac{\partial z}{\partial v} = 0 \text{ since } z = f(u, v).$$

Hence proved.

EXAMPLE 7.41 If $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ where u and v are functions of x and y, prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

where $x = r\cos\theta, y = r\sin\theta$.

Solution

Given
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

and
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad (2)$$

Now,
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$$

$$= \cos\theta\frac{\partial u}{\partial x} - \sin\theta\frac{\partial v}{\partial x}. \qquad \text{[using equation (2)]}$$

Again
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$= r\left(-\sin\theta\frac{\partial v}{\partial x} + \cos\theta\frac{\partial u}{\partial x}\right) \qquad \text{[by equation (1)]}$$

$\therefore$ From the above two relations, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Now,
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

$$= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$

and
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$= -r\left(\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial v}{\partial x}\right)$$

and hence
$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

or
$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

EXAMPLE 7.42 If $u = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$, then prove that

(i) $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(ii) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$ (WUBT 2002)

Solution

(i) $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\,\frac{\partial u}{\partial x} + \sin\theta\,\frac{\partial u}{\partial y}$$

and $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\sin\theta\,\frac{\partial u}{\partial x} + r\cos\theta\,\frac{\partial u}{\partial y}.$$

$$\therefore \quad \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right)^2 + \left(-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

(ii) We use the operators $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ as

$$\frac{\partial}{\partial r} = \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y}$$

and $$\frac{\partial}{\partial \theta} = -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}$$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(\cos\theta\,\frac{\partial u}{\partial x} + \sin\theta\,\frac{\partial u}{\partial y}\right)$$

$$= \cos\theta\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + \sin\theta\,\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) \quad [r, \theta \text{ are independent}]$$

$$= \cos\theta \left[\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right]\left(\frac{\partial u}{\partial x}\right) + \sin\theta \left[\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right]\left(\frac{\partial u}{\partial y}\right)$$

$$= \cos^2\theta \frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta \frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 u}{\partial y^2}.$$

Similarly,

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left\{-r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}\right\}$$

$$= -r \frac{\partial}{\partial \theta}(\sin\theta) \frac{\partial u}{\partial x} - r\sin\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$

$$+ r \frac{\partial}{\partial \theta}(\cos\theta) \frac{\partial u}{\partial y} + r\cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)$$

$$= -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta \frac{\partial u}{\partial y} - r\sin\theta\left[-r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}\right]\left(\frac{\partial u}{\partial x}\right)$$

$$+ r\cos\theta\left[-r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}\right]\left(\frac{\partial u}{\partial y}\right)$$

$$= -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta \frac{\partial u}{\partial y} + r^2\sin^2\theta \frac{\partial^2 u}{\partial x^2} - r^2 \sin\theta\cos\theta \frac{\partial^2 u}{\partial y \partial x}$$

$$- r^2\sin\theta\cos\theta \frac{\partial^2 u}{\partial x \partial y} + r^2\cos^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$= -r\frac{\partial u}{\partial r} + r^2\sin^2\theta \frac{\partial^2 u}{\partial x^2} + r^2\cos^2\theta \frac{\partial^2 u}{\partial y^2} - 2r^2\sin\theta\cos\theta \frac{\partial^2 u}{\partial y \partial x}.$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \sin^2\theta \frac{\partial^2 u}{\partial x^2} + \cos^2\theta \frac{\partial^2 u}{\partial y^2} - 2\sin\theta\cos\theta \frac{\partial^2 u}{\partial x \partial y}.$$

Hence by adding, we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

EXAMPLE 7.43 By the transfromation $\xi = a + \alpha x + \beta y, \eta = b - \beta x + \alpha y$ where a, b, α, β are constants and $\alpha^2 + \beta^2 = 1$ the function $u(x, y)$ is transformed to $U(\xi, \eta)$. Prove that

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 = U_{xx}U_{yy} - U_{xy}^2.$$

Solution

Since the function $u(x, y)$ transformed to $U(\xi, \eta)$, so we have

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \alpha \frac{\partial U}{\partial \xi} - \beta \frac{\partial U}{\partial \eta}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \left(\alpha \frac{\partial}{\partial \xi} - \beta \frac{\partial}{\partial \eta}\right)\left(\alpha \frac{\partial U}{\partial \xi} - \beta \frac{\partial U}{\partial \eta}\right)$$

$$= \alpha^2 U_{\xi\xi} - 2\alpha\beta U_{\xi\eta} + \beta^2 U_{\eta\eta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial y} = \beta U_\xi + \alpha U_\eta,$$

$$\frac{\partial^2 u}{\partial y^2} = \beta^2 U_{\xi\xi} + 2\alpha\beta U_{\xi\eta} + \alpha^2 U_{\eta\eta},$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\alpha \frac{\partial}{\partial \xi} - \beta \frac{\partial}{\partial \eta}\right)\left(\beta \frac{\partial u}{\partial \xi} + \alpha \frac{\partial u}{\partial \eta}\right)$$

$$= \alpha\beta (U_{\xi\xi} - U_{\eta\eta}) + (\alpha^2 - \beta^2) U_{\xi\eta}.$$

Now,

$$U_{xx} U_{yy} - U_{xy}^2 = \alpha^2 \beta^2 U_{\xi\xi}^2 - 2\alpha\beta^3 U_{\xi\eta} U_{\xi\xi} + \beta^4 U_{\xi\xi} U_{\eta\eta} + \alpha^4 U_{\xi\xi} U_{\eta\eta}$$

$$- 2\alpha^3 \beta U_{\xi\eta} U_{\eta\eta} + \alpha^2 \beta^2 U_{\eta\eta}^2 + 2\alpha^3 \beta U_{\xi\xi} U_{\eta\eta}$$

$$- 4\alpha^2 \beta^2 U_{\xi\eta}^2 + 2\alpha\beta^3 U_{\eta\eta} U_{\xi\eta} - \alpha^2 \beta^2 (U_{\xi\xi}^2 + U_{\eta\eta}^2 - 2U_{\xi\xi} U_{\eta\eta})$$

$$- (\alpha^2 - \beta^2) U_{\xi\eta}^2 - 2\alpha\beta (\alpha^2 - \beta^2) U_{\xi\eta} (U_{\xi\xi} - U_{\eta\eta})$$

$$= \alpha^2 \beta^2 (U_{\xi\xi}^2 + U_{\eta\eta}^2) + (\alpha^4 + \beta^4)(U_{\xi\xi} U_{\eta\eta}) + 2\alpha\beta(\alpha^2 - \beta^2)$$

$$\times U_{\xi\eta}(U_{\xi\xi} - U_{\eta\eta}) - 4\alpha^2 \beta^2 U_{\xi\eta}^2 - \alpha^2 \beta^2 (U_{\xi\xi}^2 + U_{\eta\eta}^2)$$

$$+ 2\alpha^2 \beta^2 U_{\xi\xi} U_{\eta\eta} - (1 - 4\alpha^2 \beta^2) U_{\xi\eta}^2$$

$$- 2\alpha\beta (\alpha^2 - \beta^2) U_{\xi\eta} (U_{\xi\xi} - U_{\eta\eta})$$

$$= (1 - 2\alpha^2 \beta^2)(U_{\xi\xi} U_{\eta\eta}) - 4\alpha^2 \beta^2 U_{\xi\eta}^2 + 2\alpha^2 \beta^2 U_{\xi\xi} U_{\eta\eta}$$

$$- U_{\xi\eta}^2 + 4\alpha^2 \beta^2 U_{\xi\eta}^2$$

$$= U_{\xi\xi} U_{\eta\eta} - U_{\xi\eta}^2.$$

EXAMPLE 7.44 Let u be a function of x and y satisfying $x = \theta\cos\alpha - \phi\sin\alpha$, $y = \theta\sin\alpha + \phi\cos\alpha$ (α = constant). Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2}.$$

Solution

Since u is a function of x and y therefore

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \cos\alpha\frac{\partial u}{\partial x} + \sin\alpha\frac{\partial u}{\partial y}$$

and

$$\frac{\partial}{\partial \theta} \equiv \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}.$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial u}{\partial x} + \sin\alpha\frac{\partial u}{\partial y}\right)$$

$$= \cos^2\alpha\frac{\partial^2 u}{\partial x^2} + 2\cos\alpha\sin\alpha\frac{\partial^2 u}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 u}{\partial y^2}. \qquad (1)$$

Again,

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial x}(-\sin\alpha) + \frac{\partial u}{\partial y}(\cos\alpha)$$

or

$$\frac{\partial}{\partial \phi} \equiv -\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}.$$

Now,

$$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial u}{\partial x} + \cos\alpha\frac{\partial u}{\partial y}\right)$$

$$= \sin^2\alpha\frac{\partial^2 u}{\partial x^2} + \cos^2\alpha\frac{\partial^2 u}{\partial y^2} - 2\sin\alpha\cos\alpha\frac{\partial^2 u}{\partial x\partial y}. \qquad (2)$$

Adding equations (1) and (2), we get

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = \frac{\partial^2 u}{\partial x^2}(\sin^2\alpha + \cos^2\alpha) + \frac{\partial^2 u}{\partial y^2}(\cos^2\alpha + \sin^2\alpha)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Since dx, dy, dz are independent, so by equating we get

$$\frac{\partial f}{\partial x} = \mu P, \frac{\partial f}{\partial y} = \mu Q, \frac{\partial f}{\partial z} = \mu R.$$

Now,
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial \mu}{\partial y} P + \mu \frac{\partial P}{\partial y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial \mu}{\partial x} Q + \mu \frac{\partial Q}{\partial x}$$

$\therefore$
$$\frac{\partial \mu}{\partial y} P + \mu \frac{\partial P}{\partial y} = \frac{\partial \mu}{\partial x} Q + \mu \frac{\partial Q}{\partial x}$$

or
$$\frac{\partial \mu}{\partial y} P - \frac{\partial \mu}{\partial x} Q = \mu \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

Similarly,
$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial \mu}{\partial x} R + \mu \frac{\partial R}{\partial x}$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial \mu}{\partial z} P + \mu \frac{\partial P}{\partial z}$$

$\therefore$
$$\frac{\partial \mu}{\partial x} R + \mu \frac{\partial R}{\partial x} = \frac{\partial \mu}{\partial z} P + \mu \frac{\partial P}{\partial z}$$

or
$$\frac{\partial \mu}{\partial x} R - \frac{\partial \mu}{\partial z} P = \mu \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)$$

and finally
$$\frac{\partial \mu}{\partial y} R - \frac{\partial \mu}{\partial z} Q = \mu \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right).$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial R}{\partial z} \right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

$$= \frac{P}{\mu}\left(\frac{\partial \mu}{\partial y} R - \frac{\partial \mu}{\partial z} Q \right) + \frac{Q}{\mu}\left(\frac{\partial \mu}{\partial z} P - \frac{\partial \mu}{\partial x} R \right) + \frac{R}{\mu}\left(\frac{\partial \mu}{\partial x} Q - \frac{\partial \mu}{\partial y} P \right)$$

$$= \frac{1}{\mu}\left[\frac{\partial \mu}{\partial y}(PR - RP) + \frac{\partial \mu}{\partial z}(-PQ + QP) + \frac{\partial \mu}{\partial x}(-RQ + QR) \right]$$

$$= 0.$$