Integral Class 3

Wednesday, January 5, 2022 8:46 AM

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx, \quad n>0$$

Gamma function

Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m>0,\ n>0.$$

Prove that $\int_0^\infty e^{-at}\, t^{n-1}\,dt = \frac{\Gamma(n)}{a^n}$

$at = u \qquad a\,dt = du$

$$= \int_0^\infty e^{-u}\left(\frac{u}{a}\right)^{n-1}\frac{du}{a}$$

$$= \int_0^\infty e^{-u}\,\frac{u^{n-1}}{a^n}\,du$$

$$= \frac{1}{a^n}\boxed{\int_0^\infty e^{-u} u^{n-1}\,du} = \frac{\Gamma(n)}{a^n}.$$

$$\boxed{B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx \quad n>0$$

P.t: $\Gamma(1) = 1$

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$$\Gamma(n) = \int_0$$

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx$$

$$= \lim_{B\to\infty} \int_0^B e^{-x}\,dx \quad ??$$

$$= 1.$$

• $\Gamma(n+1) = n!$

• $\Gamma(n+1) = n\,\Gamma(n)$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1}\,dx$$

$$= \int_0^\infty e^{-x} x^{n}\,dx$$

$$= \int_0^\infty \underline{\underline{x^n}} \cdot \underline{\underline{e^{-x}}}\,dx$$

$$= \lim_{B\to\infty} \left[\int_0^B x^n e^{-x}\,dx \right]$$

$$= \lim_{B\to\infty} \left[x^n \int e^{-x}\,dx - \int \left[n x^{n-1} \int e^{-x}\,dx \right] dx \right]$$

$$= \lim_{B\to\infty} \left[x^n \int e^{-x}\,dx + n \int x^{n-1} e^{-x}\,dx \right]$$

$$= \lim_{B\to\infty} \left[\underbrace{\left[-x^n e^{-x} \right]_0^B}_{0} + n \int_0^B e^{-x} x^{n-1}\,dx \right]$$

$$n\,\Gamma(n)$$

$$= n \int_0^\infty e^{-x} x^{n-1}\, dx = n\,\Gamma(n)$$

- $\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1)$
$= n(n-1)(n-2)\,\Gamma(n-2)$
$= n(n-1)(n-2) \cdots .1 = n!$

(d) $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$

Let $x^2 = t \Rightarrow 2x\,dx = dt$ $\quad x = t^{1/2}$

$dx = \dfrac{dt}{2\sqrt{t}}$

$$= 2\int_0^\infty e^{-t}\, t^{\frac{2n-1}{2}} \cdot \frac{dt}{2 \cdot t^{1/2}}$$

$$= \int_0^\infty e^{-t}\, t^{\frac{2n-1}{2} - \frac{1}{2}}\, dt$$

$\frac{2n-1}{2} - \frac{1}{2} = \frac{2n-2}{2} = n-1$

$$= \int_0^\infty e^{-t}\, t^{n-1}\, dt = \Gamma(n).$$

(e) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$

$\boxed{= B(n,m)}$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

Let $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

Let $x = \sin\theta \rightarrow$

$$B(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\infty} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \quad \boxed{///}$$

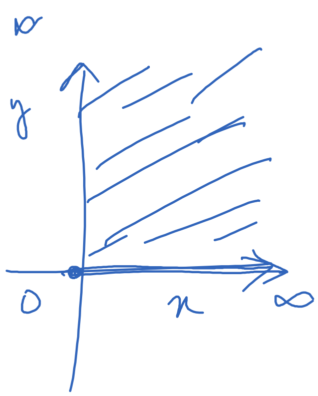
$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2m-1}\, dx$$

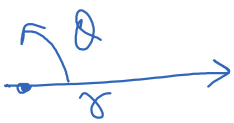
$$\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2n-1}\, dy$$

$$\Gamma(m)\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2m-1}\, dx \cdot 2\int_0^{\infty} e^{-y^2} y^{2n-1}\, dy$$

$$= 4\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)} \cdot x^{2m-1} y^{2n-1}\, \underline{dx\, dy}$$



Let $x = r\cos\theta$, $y = r\sin\theta$ $\Big\}$ $\boxed{dx\, dy = r\, dr\, d\theta}$



$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} \cdot r^{2m-1} \cos^{2m-1}\theta\, r^{2n-1} \sin^{2n-1}\theta \cdot r\, dr\, d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2m-1+2n-1+1} \cdot \cos^{2m-1}\theta \sin^{2n-1}\theta\, dr\, d\theta$$

$$= 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr \cdot 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

$$= 2\int_{r=0} e^{-r}\, r^{\ldots}\, dr \cdot 2\int_{\theta=0} \cos \ldots$$

$$= \Gamma(m+n) \cdot B(m,n)$$

$$* \quad B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} . \quad \blacksquare$$

$$* \quad B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \pi$$

$$B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(1)}$$

$$= \frac{\sqrt{\pi} \cdot \sqrt{\pi}}{1} = \pi .$$

$$\bullet \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \quad \blacksquare$$

Examination date and time 11.01.2022 at 9:00AM to 9:30AM