CHAPTER 1

Infinite Series

1.1 INTRODUCTION

Infinite sequences occur in many situations in scientific and engineering problems. In an infinite series, the number of terms is infinite and if the nth term is known, then we can determine all the terms of the series. Since the number of terms of an infinite series is not finite, the sum of all terms may or may not be finite. If the sum is finite, then the series is called *convergent series* otherwise it is called *divergent series*. The terms of the series form a set called a *sequence*. The terms of a sequence are ordered, i.e. if the nth term is known, then all the terms can be generated.

In this chapter, we first introduce the concept of sequence and then we study the convergence and divergence of an infinite series.

1.2 SEQUENCE

A sequence is a mapping from the set $\mathbb{N}$ (set of natural numbers) to the set $\mathbb{R}$ (the set of real numbers), i.e.

$$x : \mathbb{N} \to \mathbb{R}.$$

The terms of the sequence are denoted by $x_1, x_2, \cdots, x_n, \cdots$. The image of the nth element, x_n, is said to be the nth element of the sequence. A sequence is generally denoted by the symbol $\{x_n\}$ or $\{x_1, x_2, \cdots, x_n, \cdots\}$. The number of terms of a sequence is infinite.

Sometimes, the symbols like $\{u_n\}$, $\{a_n\}$, $\{s_n\}$, etc. may also be used to denote a sequence.

The terms of this sequence are -1, 1, -1, 1, -1, $\cdots$. That is, the sequence contains only two terms -1 and 1.

Again, $\{x_n\}$, where $x_n = \frac{1}{n}, n \in \mathbb{N}$ be the another sequence. The terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$. Here all the terms are distinct. This sequence is known as harmonic sequence.

The sequence $\{2, 2, \cdots\}$ is a constant sequence.

The *range* or *trace* is the set consisting of all distinct elements of a sequence without repetition. Thus, the range may be a finite set or an infinite set.

1.2.1 Bounds of a Sequence

A sequence $\{a_n\}$ is said to be *bounded above* if there exists a number M such that

$$a_n \le M \quad \text{for all} \quad n \in \mathbb{N}$$

M is called the upper bound of the sequence $\{a_n\}$.

A sequence $\{a_n\}$ is said to be *bounded below* if there exists a number m such that

$$a_n \ge m \quad \text{for all} \quad n \in \mathbb{N}$$

m is called the lower bound of $\{a_n\}$.

A sequence is said to be *bounded* if it is bounded both above and below. Thus a sequence $\{a_n\}$ is said to be bounded, if there exist two numbers m and M such that

$$m \le a_n \le M \quad \text{for all} \quad n \in \mathbb{N}.$$

Examples

1. The sequence $\{n^2\}$, i.e. the sequence $\{1, 4, 9, \cdots\}$ is bounded below, but unbounded above, since $n^2 \ge 1$ for all $n \in \mathbb{N}$.
2. The sequence $\{-n\}$ i.e. $\{-1, -2, -3, \cdots\}$ is bounded above and unbounded below, as $-n \le -1$, for all $n \in \mathbb{N}$.
3. The sequence $\left\{\frac{1}{n}\right\}$ is a bounded sequence since, $0 \le a_n \le 1$, where $a_n = \frac{1}{n}$, for all $n \in \mathbb{N}$.
4. Let $a_n = (-1)^n n$, $n \in \mathbb{N}$. The sequence $\{a_n\}$, i.e. $\{-1, 2, -3, 4, \cdots\}$ is unbounded above and unbounded below.

1.2.2 Monotone Sequence

A sequence $\{x_n\}$ is said to be *monotone increasing* (or non-decreasing) iff $x_{n+1} \ge x_n$ for all n.

If $x_{n+1} > x_n$ for all n, then the sequence $\{x_n\}$ is called *strictly monotone increasing*.

A sequence $\{x_n\}$ is said to be *monotone decreasing* (non-increasing) iff $x_{n+1} \le x_n$ for all n.

If $x_{n-1} < x_n$ for all n, then the sequence $\{x_n\}$ is called *strictly monotone decreasing.*

A sequence $\{x_n\}$ is said to be *simply monotone* if it is either monotone increasing or monotone decreasing.

Examples

1. The sequence $\{3, 9, 27, 81, \cdots, 3^n, \cdots\}$ is a strictly monotone increasing sequence.
2. The sequence $\{1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, \cdots\}$ is monotone increasing sequence, but not strictly monotone increasing.
3. The sequence $\left\{\frac{1}{n}\right\}$ i.e. $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ is a strictly monotone decreasing sequence.
4. The sequence $\{-1, 1, -1, 1, -1, 1, \cdots\}$ is neither monotone increasing nor monotone decreasing sequence.

EXAMPLE 1.1 Show that the sequence $\{x_n\}$, where $x_n = \dfrac{n+2}{2n+1}$ is monotone decreasing.

Solution

Since $$x_n = \frac{n+2}{2n+1},\ x_{n+1} = \frac{n+3}{2n+3}.$$

Now, $$x_{n+1} - x_n = \frac{n+3}{2n+3} - \frac{n+2}{2n+1} = \frac{-3}{(2n+3)(2n+1)} < 0,$$

i.e. $$x_{n+1} - x_n < 0$$

or $$x_{n+1} < x_n,$$

Hence the sequence $\{x_n\}$ is monotone decreasing.

EXAMPLE 1.2 Show that the sequence $\left\{\frac{n+1}{n}\right\}$ is monotone increasing.

Solution

Let $$x_n = \frac{n+1}{n}.$$

Therefore $$x_{n+1} = \frac{n+2}{n+1}.$$

Now, $$\frac{x_n+1}{x_n} = \frac{n+2}{n+1} \times \frac{n+1}{n} = \frac{n+2}{n} = 1 + \frac{2}{n} > 1.$$

That is, $x_{n+1} > x_n$. Hence $\{x_n\}$ is monotone increasing sequence.

1.2.3 Convergence and Non-convergence of Sequences

A sequence $\{x_n\}$ is said to *converge* to a number l, if for $\varepsilon > 0$, there exists a positive integer m (depends on ε) such that

$$|x_n - l| < \varepsilon \quad \text{for all} \quad n \geq m.$$

Symbolically,

$$x_n \to l \quad \text{as} \quad n \to \infty$$

or

$$\lim_{n\to\infty} x_n = l.$$

A sequence which is not convergent is called *divergent*.

In the following we define the limit points of a sequence. By finding the limit points, if there be any, we can classify the convergent and divergent sequences.

A number x_i is said to be a *limit point* of a sequence $\{x_n\}$ if every nbd. of x_i contains an infinite number of members of the sequence.

Thus, x_i is a limit point of a sequence if given any positive number ε, however small, $x_n \in (x_i - \varepsilon, x_i + \varepsilon)$ for an infinite number of values of n, that is $|x_n - x_i| < \varepsilon$ for infinitely many values of n.

1. The constant sequence $\{x_n\}$, where $x_n = 5$ has the only one limit point 5.
2. The limit points of the sequence $\{(-1)^n\}$ are -1 and 1.
3. 0 is the limit point of the sequence $\left\{\frac{1}{n}\right\}$.

A bounded sequence which is not convergent and has at least two limit points, is said to *oscillate finitely.*

A bounded sequence is either convergent or oscillates finitely, but an unbounded sequence is either diverges to ∞ or $-\infty$ or oscillates infinitely.

Examples

1. The sequence $\{1 + (-1)^n\}$, i.e. $\{0, 2, 0, 2, \cdots\}$ oscillates finitely.
2. The sequence $\{n^2\}$ i.e. $\{1, 4, 9, \cdots\}$ diverges to $+\infty$, as
$$\lim_{n\to\infty} n^2 = \infty.$$
3. The sequence $\{-3^n\}$ i.e. $\{-3, -9, -27, \cdots\}$ diverges to $-\infty$, as
$$\lim_{n\to\infty} (-3^n) = -\infty.$$
4. The sequence $\{n\,(-1)^n\}$ oscillates infinitely.
5. The sequence $\left\{\frac{(-1)^{n+1}}{n!}\right\}$ converges to 0, as
$$\lim_{n\to\infty} \frac{(-1)^{n+1}}{n!} = 0.$$

6. The sequence $\left\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \cdots\right\}$ is bounded below but unbounded above.

Here,
$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{1}{n} = 0$$
and
$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} (n+1) = \infty.$$
Thus, the given sequence oscillates infinitely.

7. The sequence $\{(-1)^n\}$ is oscillatory. Since
$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} (-1)^{2n-1} = -1$$
and
$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} (-1)^{2n} = 1$$
that is, the sequence has two limit points, and hence it is oscillatory.

EXAMPLE 1.3 Show that the sequence $\{u_n\}$, where $u_n = 2(-1)^n$ does not converge. (WBUT 2005)

Solution

$$\lim_{n\to\infty} u_{2n-1} = \lim_{n\to\infty} 2(-1)^{2n-1} = \lim_{n\to\infty} (-2) = -2$$
and
$$\lim_{n\to\infty} u_{2n} = \lim_{n\to\infty} 2(-1)^{2n} = 2.$$

Thus, the sequence has two distinct limit points and hence the sequence does not converge. It oscillates finitely.

1.3 INFINITE SERIES

Let $\{a_n\}$ be a real sequence. The series is the sum of the terms of the sequence $\{a_n\}$. Thus, the sum $a_1 + a_2 + a_3 + \cdots$ of all terms is called an *infinite series* and is denoted by $\sum\limits_{n=1}^{\infty} a_n$ or simply Σa_n, a_n is called the nth term of the series.

But, it is difficult to add all the terms of an infinite series in the ordinary way. Thus, we start by associating with the given series, a sequence $\{s_n\}$, where s_n denotes the sum of the first n terms of the series. Therefore,
$$s_n = a_1 + a_2 + \cdots + a_n \text{ for all } n.$$
The sequence $\{s_n\}$ is called the *sequence of partial sums* of the series.

The series $\sum\limits_{n=1}^{\infty} a_n$ is said to converge, diverge to $+\infty$ or $-\infty$ or be oscillatory, accordingly as the sequence $\{s_n\}$ converges, diverges to a limit s, then s is said to be the sum of the series Σa_n and we write $s = \Sigma a_n$.

A necessary condition for convergence of an infinite series is stated as follows.

Theorem 1.1 If Σa_n is convergent, then $\lim_{n\to\infty} a_n = 0$.

But, the converse of this theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$ does not imply the series Σa_n is convergent. But, if $\lim_{n\to\infty} a_n \neq 0$, then we surely conclude that the series Σa_n does not converge. This fact is illustrated in the following example.

EXAMPLE 1.4 Show that the series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$ does not converge.

Solution

Let
$$a_n = \frac{n}{n+1}$$

Therefore,
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0.$$

Thus, the series is not convergent.

In the following, we state a necessary and sufficient condition for the convergence of an infinite series.

Theorem 1.2 (Cauchy's general principle for convergence).

A necessary and sufficient condition for convergence of Σa_n is that, given any $\varepsilon > 0$, we can find a positive integer m, depending on ε such that
$$|s_{n+p} - s_n| < \varepsilon$$
or
$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon$$
for every $n \geq m$ and every positive integer p.

EXAMPLE 1.5 Show that the series $\Sigma \frac{1}{n}$ does not converge.

Solution

If possible, let the series be convergent. Therefore, for any given ε (say, 1/4), there exists a positive integer m, such that
$$\left|\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p}\right| < \varepsilon \text{ for all } n \geq m \text{ and } p \geq 1.$$

Let, in particular, $n = m$ and $p = m$, then
$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p} = \frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{2m} \geq m\cdot\frac{1}{2m} = \frac{1}{2} > \varepsilon.$$

Thus, there is a contradiction. Hence, the given series does not converge.

Note: It may be noted that $\lim_{n\to\infty} \frac{1}{n} = 0$ even though the series $\Sigma \frac{1}{n}$ does not converge.

Geometric series

The positive term series $1 + a + a^2 + \cdots = \Sigma a^n$ is convergent if $a < 1$ and is divergent if $a \geq 1$.

Proof: The nth partial sum

$$s_n = \frac{1-a^{n+1}}{1-a}, \quad a \neq 1 \quad \text{and} \quad s_n = n+1 \text{ if } a = 1.$$

If $a \geq 1$, then $s_n \to \infty$. Therefore Σa^n diverges.

If $a < 1$, then $s_n \to \dfrac{1}{1-a}$ and hence Σa^n converges.

p-series

The series $1+\dfrac{1}{2^p}+\dfrac{1}{3^p}+\cdots = \Sigma\dfrac{1}{n^p}$ is called *p-series*. The condition for convergence of this series is stated below.

Theorem 1.3 The positive term series $\Sigma\dfrac{1}{n^p}$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

From this theorem, we can say that the series

(i) $1+\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\cdots$ diverges

(ii) $1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\cdots$ converges

(iii) $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{4}}+\cdots$ diverges

Some preliminary theorems

Theorem 1.4 If $\Sigma u_n = u$ then $\Sigma cu_n = cu$, where c is any real number independent of n.

Theorem 1.5 If $\Sigma u_n = u$ and $\Sigma v_n = v$, then $\Sigma w_n = u \pm v$ where $w_n = u_n \pm v_n$ for all n.

Theorem 1.6 A positive series converges iff the sequence of its partial sums is bounded above.

1.4 COMPARISON TESTS FOR POSITIVE TERM SERIES

1.4.1 (First Type)

If Σu_n and Σv_n are two positive term series and $k(\neq 0)$, a fixed positive real number (independent of n) and there exists a positive integer m such that $u_n \leq k\, v_n$ for all $n \geq m$, then

(i) Σu_n converges if Σv_n converges, and

(ii) Σv_n diverges if Σu_n diverges.

1.4.2 Limit Form

If Σu_n and Σv_n are two positive term series such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, where l is a non-zero finite number, then the two series converge or diverge together.

EXAMPLE 1.6 Test the convergence of the series

$$\frac{1}{1.2^2}+\frac{1}{2.3^2}+\frac{1}{3.4^2}+\cdots$$

Solution

Let the series be Σu_n, where

$$u_n = \frac{1}{n(n+1)^2}.$$

Let

$$v_n = \frac{1}{n^3}.$$

Then

$$\lim_{x\to\infty}\frac{u_n}{v_n} = \lim_{x\to\infty}\frac{1}{n(n+1)^2}\times n^3 = \lim_{x\to\infty}\frac{1}{(1+1/n)^2} = 1 \neq 0.$$

The series Σv_n is convergent, therefore Σu_n is also convergent by comparison test.

EXAMPLE 1.7 Test the convergence of the series

$$\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\cdots$$

Solution

Let the series be Σu_n where

$$u_n = \frac{1+2+3+\cdots+(n+1)}{(n+1)^3}.$$

Since

$$1+2+3+\cdots+(n+1) = \frac{(n+1)(n+2)}{2}$$

so

$$u_n = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{n+2}{2(n+1)^2}.$$

Let

$$v_n = \frac{1}{n}.$$

Then

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n+2}{2(n+1)^2}\cdot n = \lim_{n\to\infty}\frac{1+2/n^2}{2(1+1/n)^2} = \frac{1}{2} \neq 0.$$

But Σv_n is divergent, hence Σu_n is convergent by comparison test.

EXAMPLE 1.8 Test the convergence of the series $\Sigma(\sqrt[3]{n^3+1}-n)$.

(WBUT 2003, 2007)

Solution

Let $$u_n = \sqrt[3]{n^3+1}-n.$$

Then $$u_n = (n^3+1)^{1/3} - n = \{n^3\,(1+1/n^3)\}^{1/3} - n$$

$$= n\left[1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{1/3\,(1/3-1)}{2!}\left(\frac{1}{n^3}\right)^2+\cdots\right]-n$$

$$= \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}+\cdots\right]$$

Let $$\Sigma\, v_n = \Sigma\frac{1}{n^2}.$$

Then $$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\dfrac{1}{n^2}\left[\dfrac{1}{3}-\dfrac{1}{9}\cdot\dfrac{1}{n^3}+\cdots\right]}{\dfrac{1}{n^2}} = \frac{1}{3}\neq 0.$$

The series $\Sigma\, v_n = \Sigma\dfrac{1}{n^2}$ is convergent, hence by comparison test, the series $\Sigma\, u_n$ is convergent.

EXAMPLE 1.9 Test the convergence of the series $\Sigma\, u_n$ where

$$u_n = \sqrt{n^4+1}-\sqrt{n^4-1}.$$ (WBUT 2008)

Solution

Let $$v_n = \frac{1}{n^2}.$$

$$u_n = \sqrt{n^4+1}-\sqrt{n^4-1} = \frac{(\sqrt{n^4+1})^2-(\sqrt{n^4-1})^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}.$$

Then

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \lim_{n\to\infty}\frac{2}{\sqrt{1+1/n^4}+\sqrt{1-1/n^4}} = 1\neq 0.$$

But $\Sigma v_n = \Sigma \dfrac{1}{n^2}$ is convergent. Therefore, Σu_n is convergent by comparison test.

EXAMPLE 1.10 Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin \frac{1}{n}.$$ (WBUT 2001)

Solution

Let $$u_n = \frac{1}{\sqrt{n}} \sin \frac{1}{n} \text{ and } v_n = \frac{1}{n^{3/2}}.$$

Then $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}} \sin \frac{1}{n}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 \neq 0.$$

But $\Sigma v_n = \Sigma \dfrac{1}{n^{3/2}}$, here $p = 3/2 > 1$, is convergent. Hence by comparison test, the series Σu_n is convergent.

EXAMPLE 1.11 Test the convergence of the following series.

$$\sin\left(\frac{1}{1^{3/2}}\right) + \sin\left(\frac{1}{2^{3/2}}\right) + \sin\left(\frac{1}{3^{3/2}}\right) + \sin\left(\frac{1}{4^{3/2}}\right) + \cdots$$

(WBUT 2005)

Solution

Let $$u_n = \sin\left(\frac{1}{n^{3/2}}\right) \text{ and } v_n = \frac{1}{n^{3/2}}.$$

Then $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^{3/2}}\right)}{\frac{1}{n^{3/2}}} = \lim_{x\to 0} \frac{\sin x}{x}, \left(\text{where } x = \frac{1}{n^{3/2}}\right) = 1 \neq 0.$$

But $\Sigma v_n = \Sigma \dfrac{1}{n^{3/2}}$ is convergent. Therefore, the series Σu_n is convergent by comparison test.

EXAMPLE 1.12 Test the convergence of the series

$$\frac{\sqrt{1}}{a\cdot 1^{3/2} + b} + \frac{\sqrt{2}}{a\cdot 2^{3/2} + b} + \frac{\sqrt{3}}{a\cdot 3^{3/2} + b} + \cdots, \; a > 0.$$

(WBUT 2004)

Solution

Let $$u_n = \frac{\sqrt{n}}{a \cdot n^{3/2} + b} \text{ and } v_n = \frac{1}{n}.$$

But $$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{\sqrt{n}}{a \cdot n^{3/2} + b} \cdot n = \lim_{n \to \infty} \frac{n^{3/2}}{a \cdot n^{3/2} + b}$$

$$= \lim_{n \to \infty} \frac{1}{a + b\,n^{-3/2}} = \frac{1}{a} \neq 0.$$

Here $\Sigma v_n = \Sigma \frac{1}{n}$ is a divergent series. Hence by comparison test, the series Σu_n, i.e. given series is divergent.

EXAMPLE 1.13 Test the convergence of the series

$$\Sigma \frac{1}{n \log n}, n > 2.$$

Solution

Let $$u_n = \frac{1}{n \log n}.$$

We know for $n > 2$, $n \log n > n$

or $$\frac{1}{n \log n} < \frac{1}{n}$$

Thus, $$\Sigma \frac{1}{n \log n} < \Sigma \frac{1}{n}.$$

Also, we know $\Sigma \frac{1}{n}$ is a divergent series. Hence by comparison test (first form) the series $\Sigma \frac{1}{n \log n}$ is divergent.

EXAMPLE 1.14 Test the convergence of the series $\Sigma \frac{1}{n^{1+1/n}}$.

Solution

Let $$u_n = \frac{1}{n^{1+1/n}} \quad \text{and} \quad v_n = \frac{1}{n}.$$

Now, $$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{n^{1+1/n}} \cdot n = \lim_{n \to \infty} \frac{1}{n^{1/n}}.$$

Now, let us consider $y = n^{1/n}$. Taking logarithm of both sides, we get

$$\log y = \frac{1}{n}\log n.$$

Therefore,
$$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \frac{\log n}{n} = 0.$$

or
$$\log\left(\lim_{n\to\infty} y\right) = 0$$

or
$$\lim_{n\to\infty} y = e^0 = 1$$

or
$$\lim_{n\to\infty} n^{1/n} = 1$$

Thus,
$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0.$$

But, $\Sigma v_n = \Sigma \frac{1}{n}$ is a divergent series. Hence by comparison test Σu_n, i.e. the given series is divergent.

1.4.3 Comparison Test (Second Type)

If Σu_n and Σv_n are two positive term series and there exists a positive integer m such that $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$ for all $n \geq m$, then

(i) Σu_n is convergent if Σv_n is convergent,
(ii) Σv_n is divergent if Σu_n is divergent.

EXAMPLE 1.15 Test the convergence of the series

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \cdots \qquad \text{(WBUT 2005)}$$

Solution

Let the series be $1 + \sum_{n=1}^{\infty} u_n$, where $u_n = \frac{n^n}{(n+1)^{n+1}}$.

Let $v_n = \frac{1}{n}$ since the difference between the degrees of denominator and numerator is 1.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \cdot n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{1+n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{1+n}} \cdot \frac{1}{1+\frac{1}{n}}$$

$$= \frac{1}{e} \cdot \frac{1}{1+0} = \frac{1}{e} \neq 0 \left[\because \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e\right]$$

Since the series $\Sigma\, v_n = \Sigma\, \frac{1}{n}$ is divergent, by comparison test, the given series $1 + \Sigma\, u_n$ is divergent.

1.5 D'ALEMBERT'S RATIO TEST

Statement. If $\Sigma\, u_n$ is a positive term series, such that $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then the series

(i) converges, if $l < 1$,
(ii) diverges, if $l > 1$, and
(iii) the test fails, if $l = 1$.

EXAMPLE 1.16 Test the convergence of the series

$$1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots$$

Solution

Let $\sum\limits_{n=1}^{\infty} u_n$ be the given series.

Then $$u_n = \frac{2n-1}{n!}.$$

$\therefore$ $$u_{n+1} = \frac{2n+1}{(n+1)!}$$

and $$\frac{u_{n+1}}{u_n} = \frac{2n+1}{(n+1)!} \times \frac{n!}{2n-1} = \frac{2n+1}{(n+1)\,(2n-1)}$$

Now, $$\lim_{n\to\infty} \frac{u_{n+1}}{v_n} = \lim_{n\to\infty} \frac{2n+1}{(n+1)\,(2n-1)} = \lim_{n\to\infty} \frac{\frac{1}{n}\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(2-\frac{1}{n}\right)} = 0 < 1.$$

By D'Alembert's ratio test, $\Sigma\, u_n$ is convergent.

EXAMPLE 1.17 Test the convergence of the series $x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots, x>0.$

Solution

Let Σu_n be the given series. Then $u_n=\frac{x^n}{n}$. Since $x>0$, the series is positive term.

Now,
$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}=\lim_{n\to\infty}\frac{x^n}{n+1}=\lim_{n\to\infty}\frac{1}{1+1/n}=x.$$

Hence, by D'Alembert's ratio test, Σu_n is convergent if $x<1$, Σu_n is divergent if $x>1$.

When $x=1$, the series becomes $1+\frac{1}{2}+\frac{1}{3}+\cdots$, which is divergent. Hence, the given series is convergent if $x<1$ and divergent if $x\geq 1$.

EXAMPLE 1.18 Test the convergence of the series

$$\Sigma\frac{n^2-1}{n^2+1}x^n, x>0. \qquad \text{(WBUT 2006)}$$

Solution

Let
$$u_n=\frac{n^2-1}{n^2+1}x^n.$$

Then
$$u_{n+1}=\frac{(n+1)^2-1}{(n+1)^2+1}x^{n+1}$$

$\therefore$
$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n^2+1}{n^2-1}\times\frac{(n+1)^2-1}{(n+1)^2+1}\frac{x^{n+1}}{x^n}$$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n^2}\right)}{\left(1-\frac{1}{n^2}\right)}\cdot\frac{\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}x=x.$$

Hence by D'Alembert's ratio test, the series converges if $x<1$ and diverges if $x>1$.

The test fails when $x=1$. But, when $x=1$, then

$$u_n=\frac{n^2-1}{n^2+1}$$

and
$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}}=1\neq 0.$$

Hence by necessary conditions, the series is divergent. Thus, the series converges if $x < 1$ and diverges if $x \geq 1$.

EXAMPLE 1.19 Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n!2^n}{n^n}. \qquad \text{(WBUT 2003)}$$

Solution

Let $\Sigma\, u_n$ be the given series, where

$$u_n = \frac{n!2^n}{n^n}.$$

Then $$u_{n+1} = \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}}$$

and $$\frac{u_{n+1}}{u_n} = \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!2^n} = \frac{2n^n}{(n+1)^n}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1$$

$$\left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

Hence by D'Alembert's ratio tests, the given series is convergent.

EXAMPLE 1.20 Test the convergence of the series

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \cdots \qquad \text{(WBUT 2002, 2007)}$$

Solution

Let $\Sigma\, u_n$ be the given series, where

$$u_n = \left(\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots (n+1)}\right)^2.$$

Then $$u_{n+1} = \left(\frac{1\cdot 2\cdot 3\cdots n\,(n+1)}{3\cdot 5\cdot 7\cdots (2n+1)\,(2n+3)}\right)^2.$$

and $$\frac{u_{n+1}}{u_n} = \left(\frac{1\cdot 2\cdot 3\cdots n\,(n+1)}{3\cdot 5\cdot 7\cdots (2n+1)\,(2n+3)}\right)^2 \times \left(\frac{3\cdot 5\cdot 7\cdots (2n+1)}{1\cdot 2\cdot 3\cdots n}\right)^2$$

$$= \left(\frac{n+1}{2n+3}\right)^2$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{2n+3}\right)^2 = \lim_{n\to\infty} \left(\frac{1+\dfrac{1}{n}}{2+\dfrac{3}{n}}\right)^2 = \left(\frac{1}{2}\right)^2 < 1.$$

Hence by D'Alembert's ratio tests, the series is convergent.

EXAMPLE 1.21 Examine the convergence of the series

$$1+\frac{x^2}{2^p}+\frac{x^4}{4^p}+\frac{x^6}{6^p}+\cdots$$

Solution

Let the given series be $1+\sum_{n=1}^{\infty} u_n$,

where $$u_n = \frac{x^{2n}}{(2n)^p}.$$

Then $$u_{n+1} = \frac{x^{2n+2}}{(2n+2)^p}$$

and $$\frac{u_{n+1}}{u_n} = \frac{x^{2n+2}}{(2n+2)^p} \times \frac{(2n)^p}{x^{2n}} = x^2 \frac{n^p}{(n+1)^p}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x^2 \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} x^2 \left(\frac{1}{1+1/n}\right)^p = x^2.$$

Thus, by D'Alembert's ratio test, the given series is convergent if $x^2 < 1$ or $|x| < 1$ and divergent if $|x| > 1$.

When $x = 1$,

then $$u_n = \frac{1}{(2n)^p} < \frac{1}{n^p}$$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n)^p} = 0, \; p > 0$$

since $$\lim_{n\to\infty} u_n = 0.$$

The series may or may not be converge when $x = 1$. But, $\Sigma u_n < \Sigma \frac{1}{n^p}$ and $\Sigma \frac{1}{n^p}$ is convergent when $p > 1$. Thus, the given series is convergent when $x = 1$ and $p > 1$.

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EXAMPLE 1.22 Test the convergence of the series

$$\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \frac{4^2 \cdot 5^2}{4!} + \cdots$$

Solution

Let Σu_n be the given series.

Then $$u_n = \frac{n^2 (n+1)^2}{n!}$$

$\therefore$ $$u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$$

and $$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2 (n+2)^2}{(n+1)!} \times \frac{n!}{n^2 (n+1)^2} = \frac{(n+2)^2}{(n+1)n^2}$$

Now, $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+2)^2}{n^2 (n+1)} = \lim_{n\to\infty} \frac{1}{n+1}\left(1+\frac{2}{n}\right)^2 = 0 < 1.$$

Hence the given is convergent by D'Alembert's ratio test.

EXAMPLE 1.23 Examine the convergence and divergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots \infty \qquad \text{(WBUT 2003)}$$

Solution

Let $\sum_{n=0}^{\infty} u_n$ be the given series, where $u_n = \frac{x^n}{n^2+1}$. Then $u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$.

Now, $$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{x^n} = x \cdot \frac{n^2+1}{(n+1)^2+1}$$

$\therefore$ $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x \cdot \frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}} = x.$$

Hence, by D'Alembert's ratio test, the series is convergent if $x < 1$ and divergent if $x > 1$.

When $x = 1$, then

$$u_n = \frac{1}{n^2+1} < \frac{1}{n^2}$$

$\therefore$ $$\Sigma u_n < \Sigma \frac{1}{n^2}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{2n+3}\right)^2 = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right)^2 = \left(\frac{1}{2}\right)^2 < 1.$$

Hence by D'Alembert's ratio tests, the series is convergent.

EXAMPLE 1.21 Examine the convergence of the series

$$1 + \frac{x^2}{2^p} + \frac{x^4}{4^p} + \frac{x^6}{6^p} + \cdots$$

Solution

Let the given series be $1 + \sum_{n=1}^{\infty} u_n$,

where $$u_n = \frac{x^{2n}}{(2n)^p}.$$

Then $$u_{n+1} = \frac{x^{2n+2}}{(2n+2)^p}$$

and $$\frac{u_{n+1}}{u_n} = \frac{x^{2n+2}}{(2n+2)^p} \times \frac{(2n)^p}{x^{2n}} = x^2 \frac{n^p}{(n+1)^p}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x^2 \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} x^2 \left(\frac{1}{1+1/n}\right)^p = x^2.$$

Thus, by D'Alembert's ratio test, the given series is convergent if $x^2 < 1$ or $|x| < 1$ and divergent if $|x| > 1$.

When $x = 1$,

then $$u_n = \frac{1}{(2n)^p} < \frac{1}{n^p}$$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n)^p} = 0, \; p > 0$$

since $$\lim_{n\to\infty} u_n = 0.$$

The series may or may not be converge when $x = 1$. But, $\Sigma u_n < \Sigma \frac{1}{n^p}$ and

$\frac{1}{}$ ... gent when $p > 1$. Thus, the given series is convergent when $x = 1$

EXAMPLE 1.22 Test the convergence of the series

$$\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \frac{4^2 \cdot 5^2}{4!} + \cdots$$

Solution

Let Σu_n be the given series.

Then $$u_n = \frac{n^2 (n+1)^2}{n!}$$

$\therefore$ $$u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$$

and $$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2 (n+2)^2}{(n+1)!} \times \frac{n!}{n^2 (n+1)^2} = \frac{(n+2)^2}{(n+1)n^2}$$

Now, $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+2)^2}{n^2 (n+1)} = \lim_{n\to\infty} \frac{1}{n+1}\left(1+\frac{2}{n}\right)^2 = 0 < 1.$$

Hence the given is convergent by D'Alembert's ratio test.

EXAMPLE 1.23 Examine the convergence and divergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots \infty. \qquad \text{(WBUT 2003)}$$

Solution

Let $\sum_{n=0}^{\infty} u_n$ be the given series, where $u_n = \frac{x^n}{n^2+1}$. Then $u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$.

u . x^{n+1} n^2+1 n^2+1

But, $\Sigma \frac{1}{n^2}$ is a convergent series, hence by comparison test (first form) the series $\Sigma \frac{1}{n^2+1}$ is convergent. Hence the given series is convergent if $x \le 1$ and divergent if $x > 1$.

EXAMPLE 1.24 Examine the convergence of the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$$

Solution

Let the given series be $\sum_{n=1}^{\infty} u_n$, where $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$.

Then
$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

and
$$\frac{u_{n+1}}{u_n} = \frac{x^{2n}}{(n+2)\sqrt{n+1}} \times \frac{(n+1)\sqrt{n}}{x^{2n-2}} = x^2 \frac{n+1}{n+2} \frac{\sqrt{n}}{\sqrt{n+1}}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x^2 \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \times \frac{1}{\sqrt{1+\frac{1}{n}}} = x^2.$$

Hence, by D'Alembert's ratio test the given series is convergent when $x^2 < 1$, i.e., $|x| < 1$ and divergent when $|x| > 1$.

When $x^2 = 1$, then

$$u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n\sqrt{n}\left(1+\frac{1}{n}\right)}.$$

Let
$$v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n\sqrt{n}\left(1+\frac{1}{n}\right)} \times n^{3/2} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \ne 0.$$

Thus, Σu_n and Σv_n behave alike.

Again, $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ is a convergent series, hence by comparison test the given series is convergent.

Thus, the given series is convergent when $|x| \le 1$ and divergent when $|x| > 1$.

EXAMPLE 1.25 Test the series

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\cdots$$

Solution

Let Σu_n be the given series, where $u_n=\dfrac{n^{n-1}x^{n-1}}{n!}$.

Then
$$u_{n+1}=\frac{(n+1)^n x^n}{(n+1)!}$$

and
$$\frac{u_{n+1}}{u_n}=\frac{(n+1)^n\ x^n}{(n+1)!}\times\frac{n!}{n^{n-1}\ n^{n-1}}=\frac{(n+1)^{n-1}}{n^{n-1}}\,x.$$

$\therefore$
$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}x\left(1+\frac{1}{n}\right)^{n-1}=xe.$$

$$\left[\because\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n-1}=e.\right]$$

Hence, by D'Alembert's test, the series is convergent if $xe<1$, i.e. $x<\dfrac{1}{e}$ and divergent if $x>\dfrac{1}{e}$.

When $xe=1$, i.e. $x=\dfrac{1}{e}$ then

$$u_n=\frac{n^{n-1}}{n!\,e^{n-1}}.$$

Again
$$\frac{n^{n-1}}{n!}=\frac{n}{n}\,\frac{n}{n-1}\,\frac{n}{n-2}\cdots\frac{n}{2}<1\ \text{(as each term is <1)}.$$

$\therefore$
$$u_n<\frac{1}{e^{n-1}},\ \text{i.e.,}\ \Sigma u_n<\Sigma\frac{1}{e^{n-1}}.$$

The series $\Sigma\dfrac{1}{e^{n-1}}=\Sigma\left(\dfrac{1}{e}\right)^{n-1}$ is a geometric series with common ratio $\dfrac{1}{e}\ (<1)$, therefore it is convergent. Hence by comparison test Σu_n is convergent when $x=\dfrac{1}{e}$. Thus the given series is convergent when $x\le\dfrac{1}{e}$ and divergent when $x>\dfrac{1}{e}$.

EXAMPLE 1.26 Discuss the convergence of the series

$$x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{6}+\cdots$$

Solution

Let the series be $x+\sum_{n=1}^{\infty} u_n$, where

$$u_n=\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{x^{2n+1}}{2n+1}.$$

Therefore, $$u_{n+1}=\frac{1\cdot 3\cdots(2n-1)(2n+1)}{2\cdot 4\cdots(2n)(2n+2)}\frac{x^{2n+3}}{2n+3}.$$

Now,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1\cdot 3\cdots(2n-1)(2n+1)}{2\cdot 4\cdots(2n)(2n+2)}\frac{x^{2n+3}}{2n+3}\times\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\frac{2n+1}{x^{2n+1}}$$

$$=\lim_{n\to\infty}\frac{2n+1}{2n+2}\frac{2n+1}{2n+3}x^2$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{2n}}{1+\frac{1}{n}}\frac{1+\frac{1}{2n}}{1+\frac{3}{2n}}x^2=x^2.$$

Hence, by D'Alembert's ratio test, the series is convergent if $x^2<1$, i.e. $|x|<1$ and divergent if $|x|>1$.

EXAMPLE 1.27 Discuss the convergence of the series

$$\frac{1}{3}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}}+\cdots$$

Solution

We know, $$\left(\frac{1}{3}\right)^{1+\frac{1}{2}}<\left(\frac{1}{3}\right)^2$$

$$\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}}<\left(\frac{1}{3}\right)^3$$

$$\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}<\left(\frac{1}{3}\right)^4 \text{ and so on.}$$

Thus, $$\frac{1}{3}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}+\cdots$$

$$<\frac{1}{3}+\left(\frac{1}{3}\right)^{2}+\left(\frac{1}{3}\right)^{3}+\left(\frac{1}{3}\right)^{4}+\cdots$$

The series $\frac{1}{3}+\left(\frac{1}{3}\right)^{2}+\left(\frac{1}{3}\right)^{3}+\left(\frac{1}{3}\right)^{4}+\cdots$ is a geometric series with common ratio $\frac{1}{3}<1$, hence it is convergent. Therefore, by comparison test the given series is convergent.

EXAMPLE 1.28 Test the convergence of the series

$$\frac{1+x}{1!}+\frac{(1+2x)^2}{2!}+\frac{(1+3x)^3}{3!}+\cdots,\quad x>0.$$

Solution

Let the given series be Σu_n, where $u_n=\dfrac{(1+nx)^n}{n!}$.

Now, $$u_{n+1}=\frac{\{1+(n+1)\,x\}^{n+1}}{(n+1)!}$$

$$\therefore\quad \frac{u_{n+1}}{u_n}=\frac{\{1+(n+1)\,x\}^{n+1}}{(n+1)!}\times\frac{n!}{(1+nx)^n}=\frac{\{(n+1)x\}^{n+1}\left\{1+\dfrac{1}{(n+1)\,x}\right\}^{n+1}}{(n+1)\,(nx)^n\left\{1+\dfrac{1}{nx}\right\}^n}$$

$$=x\left(\frac{n+1}{n}\right)^n\frac{\left\{1+\dfrac{1}{(n+1)\,x}\right\}^{n+1}}{\left\{1+\dfrac{1}{nx}\right\}^n}$$

Now, $$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}x\left(1+\frac{1}{n}\right)^n\frac{\left\{1+\dfrac{1}{(n+1)\,x}\right\}^{n+1}}{\left\{1+\dfrac{1}{nx}\right\}^n}=xe\cdot\frac{e^{1/x}}{e^{1/x}}=xe.$$

Hence by D'Alembert's ratio test the given series is convergent if $xe<1$, i.e. $x<\frac{1}{e}$ and divergent if $x>\frac{1}{e}$.

EXAMPLE 1.29 Discuss the convergence of the series

$$\sum_{n=1}^{\infty} n^4 e^{-n^2}. \qquad \text{(WBUT 2005)}$$

Solution

Let $u_n = n^4 e^{-n^2}$. Then the given series is Σu_n.

$\therefore$
$$u_{n+1} = (n+1)^4 e^{-(n+1)^2}.$$

Now,
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^4 e^{-(n+1)^2}}{n^4 \cdot e^{-n^2}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 e^{-(n^2+2n+1)+n^2}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^4 e^{-2n} \cdot e^{-1} = 1.0 \cdot e^{-1} = 0 < 1.$$

$$[\because e^{-\infty} = 0]$$

Hence by D'Alembert's ratio test the given series is convergent.

1.6 CAUCHY'S ROOT TEST

Statement: If Σu_n is a positive term series, such that $\lim_{n\to\infty} (u_n)^{1/n} = l$, then the series

(i) converges, if $l < 1$
(ii) diverges, if $l > 1$, and
(iii) test fails, if $l = 1$

EXAMPLE 1.30 Test for convergence of the series

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}. \qquad \text{(WBUT 2004)}$$

Solution

Let Σu_n be the given series, where

$$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$$

$\therefore$
$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

$$\left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

Hence by Cauchy's root test, the series is convergent.

EXAMPLE 1.31 Examine the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots \quad \text{(WBUT 2001, 2005)}$$

Solution

Let the series be Σu_n, where $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$.

$$\therefore \qquad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left\{\left(\frac{n+1}{n}\right)^n - 1\right\}\right]^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}^{-1}$$

$$= (1+0)^{-1}(e-1)^{-1} = \frac{1}{e-1} < 1$$

$$\left[\text{as } 2 < e < 3 \text{ or } 1 < e-1 < 2 \text{ or } \frac{1}{e-1} < 1\right]$$

Hence, the given series is convergent by Cauchy's root test.

EXAMPLE 1.32 Test for convergence of the series

$$r + \frac{r^2}{2^2} + \frac{r^3}{3^3} + \frac{r^4}{4^4} + \cdots, \quad r > 0.$$

Solution

Let the given series be Σu_n, where $u_n = \frac{r^n}{n^n}$.

$$\therefore \qquad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\frac{r}{n} = 0 < 1$$

Hence, by Cauchy's root test, the series is convergent.

EXAMPLE 1.33 Test for convergence of the series

$$\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots + \left(\frac{n}{2n+1}\right)^n + \cdots$$

Solution

Let the series be Σu_n, where $u_n = \left(\dfrac{n}{2n+1}\right)^n$.

Now, $$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+1/n} \frac{1}{2} < 1.$$

Hence, by Cauchy's root test, the series is convergent.

EXAMPLE 1.34 Examine the convergence of the series Σu_n, where $u_n = 2^{-n-(-1)^n}$.

Solution

Here $$u_n = 2^{-n-(-1)^n}.$$

$$\therefore \quad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left\{2^{-n-(1)^n}\right\}^{1/n} = \lim_{n\to\infty} 2^{-1-\frac{(-1)^n}{n}} = 2^{-1} < 1.$$

Hence, by Cauchy's root test, the given series is convergent.

1.7 ALTERNATING SERIES

A series whose terms are alternatively positive and negative is called *alternating series*.

For example, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is an alternating series.

There are many formulae available to test the convergence of alternating series among them Leibnitz's test is simple and widely used, which is stated below.

Leibnitz's test: Let the alternating series be $u_1 - u_2 + u_3 - u_4 + \cdots$ ($u_n > 0$ for all n) and

(i) $u_{n+1} \le u_n$ for all n ($\{u_n\}$ is monotonic decreasing)

(ii) $\lim\limits_{n\to\infty} u_n = 0$,

then the series is convergent.

It may be noted that Leibnitz's test can test the convergence of an alternating series.

EXAMPLE 1.35 Show that the series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \cdots$ converges for $p > 0$.

Solution

Let $u_n = \dfrac{1}{n^p}$. Then $u_{n+1} = \dfrac{1}{(n+1)^p}$.

Now, $$\frac{u_{n+1}}{u_n} = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1}\right)^p < 1 \text{ if } p > 0.$$

$\therefore$ $$u_{n+1} < u_n \text{ for } p > 0.$$

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^p} = 0 \text{ if } p > 0.$$

Hence by Leibnitz's test the given series is convergent when $p > 0$.

Note: From the above example we note the following:

(i) the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent

(ii) the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is convergent

(iii) the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is convergent.

EXAMPLE 1.36 Show that the following series is convergent

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \cdots \text{ } (x \neq \text{any negative integer}).$$

Solution

Let $u_n = \frac{1}{x+n}$. Then $u_{n+1} = \frac{1}{x+(n+1)}$.

Hence, $$\frac{u_{n+1}}{u_n} = \frac{x+n}{x+(n+1)} < 1$$

or $$u_{n+1} < u_n$$

i.e. $\{u_n\}$ is monotonic decreasing.

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{x+n} = 0$$

Hence by Leibnitz's test, the given series is convergent.

EXAMPLE 1.37 Show that the following series is convergent

$$1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$$

Solution

Let the series be $u_1 - u_2 + u_3 - u_4 + \cdots$, where $u_n = \frac{1}{n\sqrt{n}}$.

Then $$u_{n+1} = \frac{1}{(n+1)\sqrt{n+1}}.$$

Now, $$\frac{u_n}{u_{n+1}} = \frac{(n+1)\sqrt{n+1}}{n\sqrt{n}} = \left(1+\frac{1}{n}\right)\sqrt{\left(1+\frac{1}{n}\right)} > 1$$

or $u_n > u_{n+1}$. Therefore, $\{u_n\}$ is monotone decreasing.

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0.$$

Hence, by Leibnitz's test, the given series is convergent.

EXAMPLE 1.38 Show that the following series is convergent

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \cdots$$

Solution

Let the given series be $u_1 - u_2 + u_3 - \cdots$, where $u_n = \dfrac{\log(n+1)}{(n+1)^2}$.

Then $$u_{n+1} = \frac{\log(n+2)}{(n+2)^2}.$$

Now,

$$u_{n+1} - u_n = \frac{\log(n+2)}{(n+2)^2} - \frac{\log(n+1)}{(n+1)^2} = \frac{(n+1)^2\log(n+2) - (n+2)^2\log(n+1)}{\{(n+2)(n+1)\}^2} < 0$$

i.e. $u_{n+1} < u_n$. Therefore, $\{u_n\}$ is monotone decreasing.

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{\log(n+1)}{(n+1)^2} = \lim_{n\to\infty}\frac{\log(n+1)}{(n+1)} \times \frac{1}{n+1} = 0 \times 0 = 0.$$

Hence, by Leibnitz's test the series is convergent.

EXAMPLE 1.39 Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}.$$ (WBUT 2002)

Solution

The given series is $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$, i.e. the series is alternating.

$[\because \cos n\pi = (-)^n]$

Let $u_n = \dfrac{1}{n^2+1}$. Then obviously $\lim\limits_{n\to\infty} u_n = 0$.

Now, $$\frac{u_{n+1}}{u_n} = \frac{n^2+1}{(n+1)^2+1} < 1, \text{ i.e. } u_{n+1} < u_n$$

Therefore, the sequence $\{u_n\}$ is divergent. Hence, by Leibnitz's test, the given series is convergent.

1.7.1 Absolute and Conditional Convergence

A series $u_1 - u_2 + u_3 - u_4 + \cdots$ ($u_n > 0$ for all n) is said to be *absolutely convergent* if the series $u_1 + u_2 + u_3 + u_4 + \cdots$ of positive term is convergent.

A series $u_1 - u_2 + u_3 - u_4 + \cdots$ ($u_n > 0$ for all n) is called *conditionally convergent* if $u_1 + u_2 + u_3 + u_4 + \cdots$ of positive term is not convergent.

A conditionally convergent series is also called a *semi-convergent* or a *non-absolutely convergent* series.

Theorem 1.7 Every absolutely convergent series is convergent.

EXAMPLE 1.40 Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Solution

Let the given series be $u_1 - u_2 + u_3 - u_4 + \cdots$. Then $u_n = \frac{1}{n}$.

Now,
$$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1.$$

i.e.
$$u_{n+1} < u_n.$$

Therefore, $\{u_n\}$ is monotone decreasing.

Also,
$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

Hence, by Leibnitz's test, the given series is convergent. But, the series $u_1 + u_2 + u_3 + \cdots$ i.e., $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (the p-series with $p = 1$) is divergent.

Hence, the given series is conditionally convergent.

EXAMPLE 1.41 Show that the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is absolutely convergent.

Solution

The positive term series corresponding to the given series is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

The above series is a p-series with $p = 2 > 1$. Therefore, it is convergent.

Hence, the given series is absolutely convergent.

EXAMPLE 1.42 Show that for any fixed value of x, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent.

Solution

Let $$u_n = \frac{\sin nx}{n^2}.$$

Therefore, $$|u_n| = \frac{|\sin nx|}{n^2} \le \frac{1}{n^2} \text{ for all } n.$$

That is, $\Sigma |u_n| \le \Sigma \frac{1}{n^2}$. But, the series $\Sigma \frac{1}{n^2}$ is convergent.

Hence, by comparison test the series $\Sigma |u_n|$ is convergent. Therefore, the given series is absolutely convergent.

EXAMPLE 1.43 Show that the series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converges absolutely for all values of x.

Solution

Let $$u_n = \frac{x^n}{n!}.$$

$$\therefore \quad \lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} \times \frac{n!}{|x|^n} = \lim_{n \to \infty} \frac{|x|}{n+1} = 0 < 1 \text{ for all } x.$$

Hence, by D'Alembert's ratio test the series $\Sigma |u_n|$ is convergent. Therefore, the given series is absolutely convergent for all x.

Note: We know that if the series Σu_n is convergent, then $\lim_{n \to \infty} u_n = 0$. Thus from the above example, we have $\lim_{n \to \infty} \frac{x^n}{n!} = 0$.

EXAMPLE 1.44 Show that the series $\Sigma \frac{(-1)^{n+1}}{n^p}$ is absolutely convergent for $p > 1$, but, conditionally convergent for $0 < p \le 1$.

Solution

Let the given series be Σu_n, where $u_n = \frac{(-1)^{n+1}}{n^p}$.

Now, $|u_n| = \frac{1}{n^p}$. Then $\Sigma |u_n| = \Sigma \frac{1}{n^p}$, the p-series, which is convergent for $p > 1$ and divergent if $p \le 1$.

Therefore, the given series is absolutely convergent if $p > 1$ and $\Sigma |u_n|$ is divergent if $p \le 1$.

Let $$p \le 1.$$

Then $$\frac{|u_n|}{|u_{n+1}|} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p > 1, \text{i.e.} |u_n| > |u_{n+1}|.$$

Also, $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n^p} = 0, \quad \text{if } p > 0.$$

Therefore, $$\Sigma u_n = \Sigma \frac{(-1)^{n+1}}{n^p} \text{ is convergent if } p > 0.$$

Hence the given series is absolutely convergent if $p > 1$ and conditionally convergent if $0 < p \le 1$.

EXAMPLE 1.45 Test the convergence of the series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$$

Solution

Let the series be $\Sigma\, u_n$, where $u_n = \dfrac{n+1}{n}(-1)^{n+1}$.

Now, $$\frac{|u_{n+1}|}{|u_n|} = \frac{n+2}{n+1} \times \frac{n}{n+1} = \frac{n^2+2n}{n^2+2n+1} < 1, \text{ i.e. } |u_{n+1}| < |u_n|.$$

Therefore, $\{|u_n|\}$ is monotone decreasing.

But, $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) 1 \ne 0.$$

Therefore, the conditions of Leibnitz's test are not satisfied, and hence the given series does not converge.

EXAMPLE 1.46 Prove that the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

is absolutely convergent when $|x| < 1$ and conditionally convergent when $x = 1$.

(WBUT 2001)

Solution

Let the series be $\Sigma\, u_n$, when $u_n = (-1)^{n+1}\dfrac{x^n}{n}$.

Now, $$\Sigma |u_n| = \frac{|x|^n}{n}$$

$$\therefore \quad \lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n \to \infty} \frac{|x|^{n+1}}{n+1} \times \frac{n}{|x|^n}$$

$$= \lim_{n \to \infty} \frac{n|x|}{n+1} = \lim_{n \to \infty} \frac{|x|}{1+\frac{1}{n}} = |x|.$$

If $|x|<1$ then by D'Alembert's ratio test $\Sigma|u_n|$ is convergent, i.e. Σu_n is absolutely convergent. But, when $x=1$ then the series $1-\frac{1}{2}+\frac{1}{3}-\cdots$ is convergent by Leibnitz's test while the series $1+\frac{1}{2}+\frac{1}{3}+\cdots$, the p-series with $p = 1$, is divergent. Hence the given series is conditionally convergent.

EXERCISES

Short Answer Questions

(Section A)

1. The series $2 + 2^2 + 2^3 + 2^4 + \cdots$ is .... (convergent/divergent)
2. The series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$ is .... (convergent/divergent)
3. The series $1 + r + r^2 + r^3 + \cdots$ is convergent if $r <$ ....
4. If the series $\Sigma\, u_n$ is convergent, then $\lim\limits_{n \to \infty} u_n$ ....
5. The limit point of the sequence $\left\{\frac{1}{n}\right\}$ is ....
6. The sequence $\left\{\frac{n+1}{n}\right\}$ is monotone ....
7. If $\Sigma\, u_n$ is a convergent series, then $\lim\limits_{n \to \infty} nu_n =$ ....
8. The series $\sum\limits_{n=1}^{\infty} \frac{\log n}{\sqrt{n+1}}$ is .... (convergent/divergent)
9. The series $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n\,(n-1)}}$ is .... (convergent/divergent)
10. The series $\sum\limits_{n=1}^{\infty} \frac{n}{3n^3-2}$ is .... (convergent/divergent)