- FOURIER SERIES -

A Fourier series breaks down a periodic function (or signal) into a sum of a set of oscillating functions - sines and cosines (or complex exponentials).

Basis of Fourier Series:

The Fourier series of a function is an infinite series of ~~sing~~ sine and cosine functions. So the basis of Fourier Series is based on the principle of periodic functions, trigonometric series and Dirichlet's conditions.

Periodic Function:

A function $f(x)$ is said to be periodic if there exists a positive number T such that $f(x+T) = f(x)$ for all real x. Sine and cosine functions are periodic functions.

We know $\sin(2\pi n + x) = \sin x$ and $\cos(2n\pi + x) = \cos x$, 2π is the period and n is an integer.

Fundamental Period:

The fundamental period of a periodic function $f(x)$, i.e. $f(x+T) = f(x)$, $\forall x$, T is the period of $f(x)$ is its smallest positive period.

Properties: (i) If T is the fundamental period of a periodic function $f(x)$, then nT is also a period of $f(x)$ for an integer $n \neq 0$, i.e. $f(x) = f(x+nT)$

$\sin(2n\pi + x) = \sin x$, here $T = 2\pi$.

(ii) If $f(x)$ and $g(x)$ are periodic functions with common period T then

$af(x) + bg(x)$ is also a periodic function with period T. for some constants a and b.

(iii) If $f(x)$ is a periodic function with period T then $f(ax)$ is periodic with period T/a where a is a constant and $a \neq 0$.

E.g. $f(x) = \sin 3x$. Then $\sin 3x = \sin(2n\pi + 3x) = \sin 3\left(x + \frac{2n\pi}{3}\right)$

$\therefore \; T = \frac{2\pi}{3}$.

(iv) The period of a sum of periodic functions with different periods is the LCM of these periods.

(v) If $f(x) = c$, where c is a constant, then period of $f(x)$ is any positive value of T.

(vi) Any function $f(x)$ defined on a finite interval $[a, b]$ can be extended to a periodic function $f(x)$ of period $T = b - a$, by defining $f(x+T) = f(x)$ for all x.

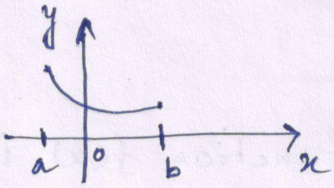


Fig-1

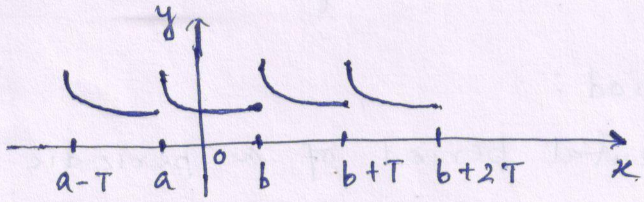


Fig-2.

Fourier - Euler Formula:

A periodic function $f(x)$ with period 2π defined in $(\theta, \theta + 2\pi)$ is the sum of the trigonometric functions given by

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

Fourier Series: Fourier series is a method of representing a complex periodic signal using simplier signals called 'Sinusoids' which are summed to produce an approximation of the original signals. The approximation becomes more accurate as more terms are used.

Fourier series of a function $f(x)$ in $-\pi \le x \le \pi$ under certain conditions is the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, $\{a_n\}$ and $\{b_n\}$ are constants known as Fourier coefficients.

Ex: Suppose a particular function $f(x)$ can be written as an infinite trigonometric series

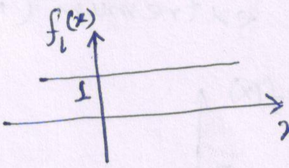
$$f(x) = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx$$

Let $f_1(x) = 1$

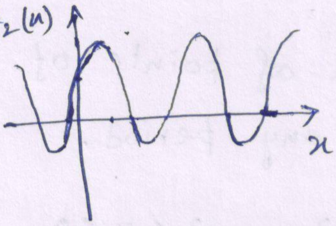
$$f_N(x) = 1 + \sum_{n=1}^{N-1} (-1)^{n+1} \frac{2}{n} \sin nx \quad \text{for } N = 2, 3, 4, \cdots$$

The series $f_N(x)$ converges to a particular shape if more terms are added to it, i.e., if the value of N is increased.

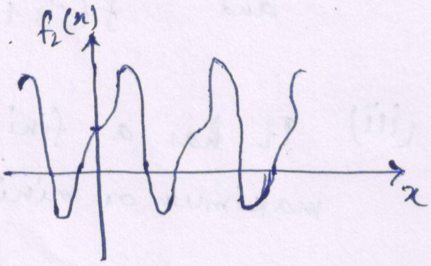
Graphs for $f_N(x)$, $N = 1, 2, 3$ and 6 are given below.



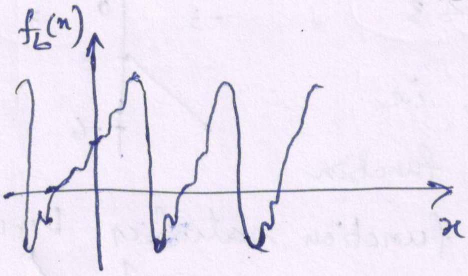
$f_1(x) = 1$



$f_2(x) = 1 + 2\sin x$

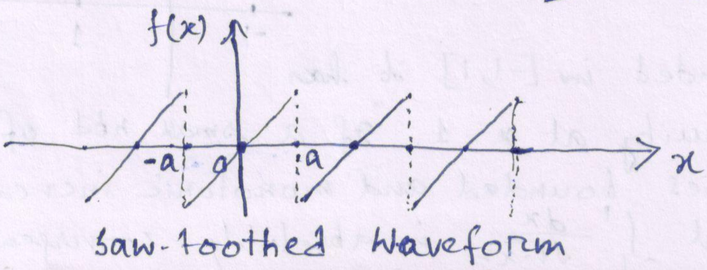


$f_3(x) = 1 + 2\sin x - \sin 2x$



$f_6(x) = 1 + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \frac{2}{5}\sin 5x$

Note: If more and more terms of $f_N(x)$ or possibly an infinite number of terms in the series of $f(x)$ are considered then the graph tends to be a sawtooth like curve.



Saw-toothed Waveform

$f(x) = x, \quad -a < x \le a$

and $f(x + 2a) = f(x), \ \forall x$

Dirichlet's Conditions:

A series corresponding to a function exists does not ensure that the series converges and even if it converges, it is not necessary that its sum will be the function for which it has been generated. The possibility of expansion of a function $f(x)$ defined in $[-T, T]$ in Fourier series depends on the existence of some integrals. These integrals, again, exists if $f(x)$ satisfies Dirichlet's Conditions.

A periodic function $f(x)$ defined on $[-T, T]$ is said to satisfy Dirichlet's conditions if:

(i) It is single-valued and uniformly bounded, i.e. $|f(x)| \le M$, $-T \le x \le T$ and M is a constant.

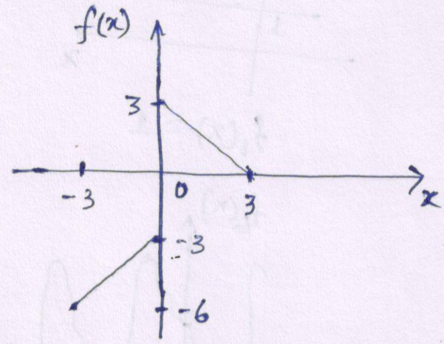
(ii) It has atmost a finite number of discontinuities and all of them are first kind within a period, i.e. at each discontinuity ξ, the function $f(x)$ has a finite limit on the left and on the right, i.e.

$$f(\xi - 0) = \lim_{\varepsilon \to 0} f(\xi - \varepsilon)$$

and $$f(\xi + 0) = \lim_{\varepsilon \to 0} f(\xi + \varepsilon), \quad \varepsilon > 0$$

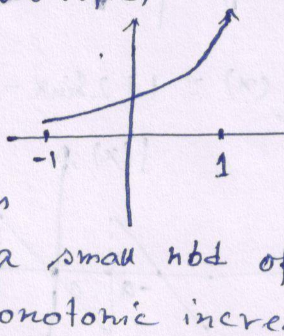
(iii) It has a finite number of points of strict extremum (ie. maximum or minimum) in any period.

Ex. 1. Let $f(x) = \begin{cases} x-3, & -3 \le x < 0 \\ 3-x, & 0 \le x \le 3 \end{cases}$



From the graph we see $-6 \le f(x) \le 3$, i.e. $f(x)$ is bounded in $[-3, 3]$. Also the function has the discontinuity at $x = 0$. So this function satisfies Dirichlet's Condition.

Ex. 2. Let $f(x) = \dfrac{1}{\sqrt{1-x}}$ $\quad -1 \le x \le 1$.



Though this function is not bounded in $[-1, 1]$ it has one point of infinite discontinuity at $x = 1$. If a small nbd of $x = 1$ is excluded the function becomes bounded and monotonic increasing. Moreover, the improper integral $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$ is absolutely convergent.

Convergence Theorem:

If a function $f(x)$ satisfies Dirichlet's conditions in $[-T, T]$ then its corresponding Fourier series converges to $f(x)$ at any point $x \in (-T, T)$, where $f(x)$ is continuous and converges to $\frac{1}{2}[f(x-0)+f(x+0)]$ at a point $x \in (-T, T)$, where $f(x)$ has an ordinary discontinuity. In particular, if $f(-T+0)$ and $f(T-0)$ exist, the series converges to $\frac{1}{2}[f(-T+0)+f(T-0)]$ at $x = T$ and also at $x = -T$.

Evaluation of Fourier coefficients:

Consider a periodic function $f(x)$ with a period 2π satisfying all the three Dirichlet's conditions. Let it be expressed by the trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \text{——} \ (1)$$

where a_0, $\{a_n\}$ and $\{b_n\}$ are Fourier coefficients, converges, and the series ~~are~~ converges to a continuous function $f(x)$ on $[-\pi, \pi]$.

Integrating both sides of the equation w.r. to x we have

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right]$$

Since $\displaystyle\int_{-\pi}^{\pi} \cos nx\,dx = \left.\frac{\sin nx}{n}\right|_{-\pi}^{\pi} = \frac{2}{n}\sin n\pi = 0$ and

$$\int_{-\pi}^{\pi} \sin nx\,dx = -\left.\frac{\cos nx}{n}\right|_{-\pi}^{\pi} = \frac{1}{n}\left[-\cos n\pi + \cos(-n\pi)\right] = 0.$$

$$\therefore \int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0 \Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx. \quad -(2)$$

Multiplying both sides of (1) by $\cos mx$ and $\sin mx$ separately and integrating from $-\pi$ to π we have

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\right]$$

and

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\sin mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\sin mx\,dx\right]$$

Now the integral $\int_{-\pi}^{\pi}\cos px\cos qx\,dx = \frac{1}{2}\left[\int_{-\pi}^{\pi}\cos(p+q)x\,dx + \int_{-\pi}^{\pi}\cos(p-q)x\,dx\right]$

$$= \frac{1}{p+q}\sin(p+q)\pi + \frac{1}{p-q}\sin(p-q)\pi.$$

Clearly first term becomes zero for all values of p and q and the second term also be zero for $p \neq q$. For $p = q$ the integral

$$\frac{1}{2}\int_{-\pi}^{\pi}\cos(p-q)\,dx = \frac{2}{2}\int_{-\pi}^{\pi}dx = \pi.$$

$$\therefore \int_{-\pi}^{\pi}\cos px\cos qx\,dx = \begin{cases} 0 & \text{for } p \neq q \\ \pi & \text{for } p = q \end{cases}$$

Similarly $\int_{-\pi}^{\pi}\sin px\cos qx\,dx = 0$ for all $p \neq q$

and

$$\int_{-\pi}^{\pi}\sin px\sin qx\,dx = \begin{cases} 0 & \text{for } p \neq q \\ \pi & \text{for } p = q. \end{cases}$$

Using the above result we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \text{and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Example 1. Expand the Fourier series $f(x) = x + x^2, \ -\pi < x < \pi$

Hence prove that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Solution: First compute the Fourier coefficients.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1+2x)\frac{\sin nx}{n}\,dx\right]$$

$$= -\frac{1}{\pi}\left[\frac{1+2x}{n}\left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi}\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= -\frac{1}{\pi n^2}\left[\left\{(1+2\pi)\cos n\pi - (1-2\pi)\cos n\pi\right\} + 0\right]$$

$$= \frac{4}{n^2}\cos n\pi.$$

i.e. $$a_n = \begin{cases} \dfrac{4}{n^2}, & n \text{ is even} \\ -\dfrac{4}{n^2}, & n \text{ is odd.}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= -\frac{1}{\pi}(x+x^2)\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}(1+2x)\frac{\cos nx}{n}\,dx$$

$$= -\frac{2\cos n\pi}{n} + \frac{1+2x}{n\pi}\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} 2\frac{\sin nx}{n}\,dx$$

$$= -\frac{2\cos n\pi}{n}$$ [$\sin nx$ is an odd function in $-\pi \le x \le$

(4)

i.e. $b_n = \begin{cases} \frac{2}{n}, & n \text{ is odd} \\ -\frac{2}{n}, & n \text{ is even.} \end{cases}$

Therefore, the Fourier series of $f(x) = x + x^2$ can be written as

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2} \cdot \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos n\pi \cos nx + \frac{2}{n} \cos n\pi \sin nx \right)$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots \right)$$

$$+ 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \right)$$

Again $f(\pi - 0) = \underset{h \to 0}{Lt}\, f(\pi - h) = \underset{h \to 0}{Lt} \left[\pi - h + (\pi - h)^2 \right]$

$$= \underset{h \to 0}{Lt} \left[\pi - h^2 + \pi^2 - 2\pi h + h^2 \right] = \pi + \pi^2.$$

$$f(-\pi + 0) = \underset{h \to 0}{Lt}\, f(-\pi + h) = \underset{h \to 0}{Lt} \left[(-\pi + h) + (-\pi + h)^2 \right]$$

$$= \underset{h \to 0}{Lt} \left[-\pi + h + \pi^2 - 2\pi h + h^2 \right] = -\pi + \pi^2.$$

Now $\frac{1}{2} \left[f(-\pi + 0) + f(\pi - 0) \right] = \pi^2$

Therefore at $x = \pi$ we have

$$\pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots \right) + 2 \times 0$$

$$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Note: Here $f(x) = x + x^2$ is a continuous function and satisfies Dirichlet's conditions in $-\pi < x < \pi$. Hence the series is equal to $f(x)$ at any point in $-\pi < x < \pi$. i.e.

$$x + x^2 = \frac{\pi^2}{\ } - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots \right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots \right)$$

Example 2. Obtain the Fourier series of $x \sin x$, $-\pi \le x \le \pi$ and deduce that $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

Solution: First compute the Fourier coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx = 2 \quad \left[\text{Since } x\sin x \text{ is even}\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx \quad \left[\text{Since } x\sin x\cos nx \text{ is even function}\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{\pi}\left\{ x\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi} - \int_0^{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(1-n)x}{1-n}\right]dx\right\}$$

$$= -\left\{\frac{\cos(1-n)\pi}{1-n} + \frac{\cos(n+1)x}{n+1}\right\} + \frac{1}{\pi}\left[\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}$$

$$= -\left\{\frac{(-1)^{1-n}}{1-n} + \frac{(-1)^{n+1}}{n+1}\right\} = -(-1)^{n+1}\left\{\frac{1}{1-n} + \frac{1}{n+1}\right\} = \frac{2(-1)^n}{1-n^2}, \quad n \ne 1$$

For $n = 1$, $\quad a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{x\sin 2x}{2}\,dx$

$$= \frac{1}{\pi}\left[-\frac{x\cos 2x}{2}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos 2x}{2}\,dx\right] = \frac{1}{\pi}\left[-\frac{\pi}{2} + \frac{1}{2}\,\frac{\sin 2x}{2}\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2} + 0\right] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\sin nx\,dx = 0$$

Since $x \sin x \sin nx$ is odd f^n

(5)

$f(x) = x \sin x$ satisfies Dirichlet's conditions in $(-\pi, \pi)$ and it is a continuous function; The Fourier series expansion is

$$x \sin x = \frac{1}{2}\cdot 2 + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty}\left\{\frac{2(-1)^n}{1-n^2}\cos nx + 0 \sin nx\right\}$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2}\cos nx$$

$$= 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{2\cdot 4} + \frac{\cos 4x}{3\cdot 5} \cdots\right\}$$

Putting $x = \pi/2$ we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos \pi/2 - 2\left\{\frac{\cos \pi}{1\cdot 3} - \frac{\cos \frac{3\pi}{2}}{2\cdot 4} + \frac{\cos 2\pi}{3\cdot 5} - \cdots\right\}$$

or $$\frac{\pi}{2} = 1 - 2\left\{-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots\right\}$$

$$\Rightarrow \frac{1}{2} + \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \pi/4$$

Fourier Series in an Arbitrary Interval:

* change of interval $[-T, T]$

suppose $f(x)$ has period $2T$, i.e. $f(x+2T) = f(x)$, $\forall x$

Assume that x varies from $-T$ to T and t varies from $-\pi$ to π. Then $\frac{x}{2T} = \frac{t}{2\pi} \Rightarrow t = \frac{\pi x}{T}$

Let $g(t) = f(x) = f\left(\frac{T}{\pi}t\right)$

Suppose $g(t)$ is a periodic function with period 2π. Then by definition of Fourier series

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\,dt$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin nt\,dt \qquad n \geq 1.$$

Substituting $t = \frac{\pi}{T}x$, $dt = \frac{\pi}{T}dx$ we have

$$g(t) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right)\right]$$

where $a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\,dx$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(\frac{n\pi x}{T}\right)dx$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\left(\frac{n\pi x}{T}\right)dx \qquad \text{for all integers } n \geq 1.$$

Change of Interval: $c \leq x \leq c+2T$

Suppose $f(x)$ has a period $2T$ in $c \leq x \leq c+2T$

then the fourier series expansion of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right)\right]$$

where $a_0 = \frac{1}{T}\int_{c}^{c+2T} f(x)\,dx$

$$a_n = \frac{1}{T}\int_{c}^{c+2T} f(x)\cos\frac{n\pi x}{T}\,dx$$

$$b_n = \frac{1}{T}\int_{c}^{c+2T} f(x)\sin\frac{n\pi x}{T}\,dx$$

Example 1: Find the Fourier series for $f(x) = 1 - x^2$ in $-1 \le x \le 1$.

Sol^n: First we find the Fourier coefficients.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= 2 - \frac{2}{3} = \frac{4}{3}.$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx = \int_{-1}^{1}(1-x^2)\cos n\pi x\,dx$$

$$= 2\int_{0}^{1}(1-x^2)\cos n\pi x\,dx \qquad [(1-x^2)\cos n\pi x \text{ is even}]$$

$$= 2\left[(1-x^2)\frac{\sin n\pi x}{n\pi}\Big|_0^1 - \int_0^1(-2x)\frac{\sin n\pi x}{n\pi}\,dx\right]$$

$$= 2\left[0 + \frac{2}{n\pi}\int_0^1 x\sin(n\pi x)\,dx\right]$$

$$= \frac{4}{n\pi}\left[-x\frac{\cos n\pi x}{n\pi}\Big|_0^1 - \int_0^1 1\cdot\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right]$$

$$= -\frac{4}{n^2\pi^2}(-1)^n + \frac{4}{n^2\pi^2}\frac{\sin n\pi x}{n\pi}\Big|_0^1$$

$$= \frac{4}{n^2\pi^2}(-1)^{n+1}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{1}(1-x^2)\sin n\pi x\,dx = 0$$

since $(1-x^2)\sin n\pi x$ is an odd function.

Therefore we get $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x$

$$= \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

Exemple 2: Find the Fourier series of

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$$

$f(x)$ is a periodic function of period 10.

Sol^n: First find the Fourier coefficients.

$$a_0 = \frac{1}{5}\int_{-5}^{5} f(x)\,dx = \frac{1}{5}\left[\int_{-5}^{0} 0 \cdot dx + \int_{0}^{5} 3\,dx\right] = \frac{1}{5} \times 3 \times 5 = 3.$$

$$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx = \frac{1}{5}\int_{0}^{5} 3\cos\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left[\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right]_0^5 = 0$$

$$b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx = \frac{3}{5}\int_{0}^{5}\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left[-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right]_0^5 = -\frac{3}{n\pi}\left[\cos n\pi - 1\right]$$

So the Fourier series is

$$\frac{3}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin\frac{n\pi x}{5}$$

$$= \frac{3}{2} + \frac{6}{\pi}\left\{\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} \cdots\right\}$$

Here we see that $f(x)$ is bounded in $[-5,5]$ and monotonic. So $f(x)$ satisfies Dirichlet's conditions in $(-5,5)$. It is discontinuous at $x=0$. Therefore at $x=0$ the series converges to

$$\frac{1}{2}\left\{\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-}\right\} = \frac{1}{2}\left\{\lim_{x\to -5+} 0 + \lim_{x\to 5^-} 3\right\} = \frac{3}{2}.$$

At the end points $x=-5$ and $x=5$ the Fourier series also converges to

$$\frac{1}{2}\left[\lim_{x\to -5^+} f(x) + \lim_{x\to 5^-} f(x)\right] = \frac{1}{2}\left[\lim_{x\to -5+} 0 + \lim_{x\to 5^*} 3\right] = \frac{3}{2}$$

(7)

Fourier series for Even and Odd functions:

Suppose $f(x)$ is an even periodic function with period $2T$, defined in $[-T, T]$.

Then the Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right]$$

where $$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\,dx = \frac{2}{T}\int_{0}^{T} f(x)\,dx$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\,dx$$

$$= \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{n\pi x}{T}\,dx \quad \left[\text{Since } f(x)\cos\frac{n\pi x}{T} \text{ is an even function}\right]$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx \quad \left[\text{Since } f(x)\sin\frac{n\pi x}{T} \text{ is an odd function.}\right.$$

$$= 0$$

In summary $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T}$$

which is known as Fourier cosine series where

$$a_0 = \frac{2}{T}\int_{0}^{T} f(x)\,dx$$

$$a_n = \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{n\pi x}{T}\,dx \qquad n \geq 1$$

Fourier Series for odd functions:

Suppose $f(x)$ is an odd periodic function with period $2T$ defined in $[-T, T]$.

Here $$a_0 = \frac{1}{2T}\int_{-T}^{T} f(x)\,dx = 0 \quad [\text{Since } f(x) \text{ is odd}]$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\,dx$$

$$= 0 \quad \left[\text{Since } f(x)\cos\frac{n\pi x}{T} \text{ is odd } f^n\right]$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx$$

$$= \frac{2}{T}\int_{0}^{T} f(x)\sin\frac{n\pi x}{T}\,dx \quad \left[\text{Since } f(x)\cos\frac{n\pi x}{T} \text{ is even } f^n\right].$$

Therefore

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{T}\,dx$$

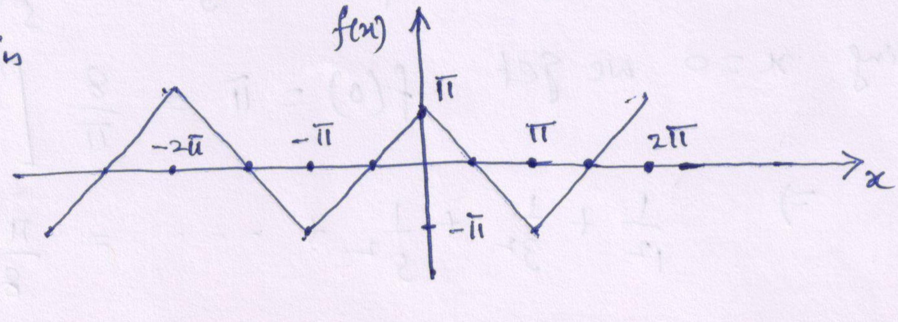
where $$b_n = \frac{2}{T}\int_{0}^{T} f(x)\sin\frac{n\pi x}{T}\,dx, \quad n \geq 1.$$

Example 1. Find the Fourier series of the function

$$f(x) = \begin{cases} \pi + 2x & -\pi < x < 0 \\ \pi - 2x & 0 \leq x \leq \pi \end{cases}$$

the deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Sol^n: Graph of $f(x)$ is



From the graph we see $f(x)$ is an even function.

First find the Fourier coefficients.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - 2x)\,dx = \frac{2}{\pi}\Big[\pi x - x^2\Big]_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - 2x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - 2x)\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} -2\,\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{2}{n}\cdot -\frac{\cos nx}{n}\Big|_0^{\pi}\right] = -\frac{4}{\pi n^2}(\cos n\pi - 1)$$

$$= \frac{4}{\pi n^2}\left\{1 - (-1)^n\right\}, \quad n \geq 1$$

$\therefore$ The Fourier series of $f(x)$ is

$$\sum_{n=1}^{\infty} \frac{4}{\pi n^2}\left\{1 - (-1)^n\right\}\cos nx$$

$$= \frac{4}{\pi}\left(\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \cdots\right)$$

$$= \frac{8}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos x}{3^2} + \frac{\cos x}{5^2} + \cdots\right]$$

From the graph we see that $f(x)$ is bounded and monotonic in $[-\pi, \pi]$. So $f(x)$ satisfies Dirichlet's conditions.

So $$f(x) = \frac{8}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Putting $x = 0$ we get $f(0) = \pi = \frac{8}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$

$$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Half-range Series:

Suppose a function is defined in the range $(0, T)$ instead of the full range $(-T, T)$. But in some applications, it is required to expand this function $f(x)$ in a series of sine or cosine terms only. This series is termed as a half-range Fourier Series.

If $f(x)$ is taken to be an odd function, its Fourier series expansion will consist of only sine terms, whereas if $f(x)$ is taken to be an even function, then its Fourier series expansion will consist of a constant and cosine terms. Therefore $f(x)$ can be rewritten as

$$f(x) = \begin{cases} \text{sum of only sine terms}, & \text{if } f(x) \text{ is odd } f^n. \\ \text{constant + sum of only cosine terms}, & \text{if } f(x) \text{ is even } f^n. \end{cases}$$

Thus, for for a given function defined in $(0, T)$, we can have two different Fourier series with period $2T$ such that the two series have the same value in $(0, T)$ but values with opposite signs in the interval $(-T, 0)$.

Half range sine series:

Suppose a function $f(x)$ is defined in $(0, T)$ and it is required to express $f(x)$ as a sine series with period $2T$. By defining $f(x)$ as

$$f(x) = -f(-x) \quad \forall\, x \text{ in } (-T, 0)$$

Then the extended function is an odd function in $(-T, T)$.

Thus the Fourier series of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{T}, \text{ where}$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin \frac{n\pi x}{T}\, dx, \quad n \geq 1 \text{ and integer}$$

Half range cosine series:

Suppose a function $f(x)$ is defined in $(0,T)$ and it is required to express $f(x)$ as a cosine series with period $2T$. Let us define $f(x)$ as

$$f(x) = f(-x) \quad \forall x \text{ in } (-T,0)$$

Then the extended function is an even function in $(-T,T)$. Thus, the Fourier series of the function is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T}$$

where,
$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{n\pi x}{T}\,dx, \quad n \geq 1 \text{ and integer}$$

Example 1. Determine the half-range Fourier sine series for

$$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x \leq \frac{1}{2} \\ 3 - 3/4, & \frac{1}{2} < x < 1. \end{cases}$$

Sol^n: Here $f(x)$ is defined in $(0,1)$ Then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

where
$$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx$$

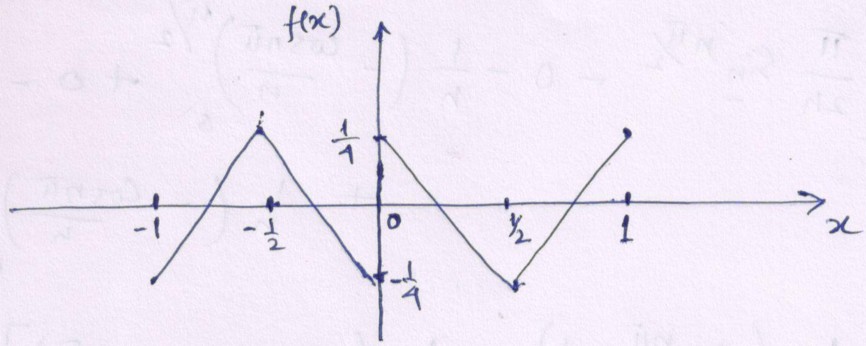
$$= 2\left[\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{1/2}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$= 2\left[\left(x - \frac{1}{4}\right)\frac{\cos n\pi x}{n\pi}\Big|_0^{1/2} - \int_0^{1/2}\frac{\cos n\pi x}{n\pi}dx - \left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi}\Big|_{1/2}^{1} + \int_{1/2}^{1}\frac{\cos n\pi x}{n\pi}d\right.$$

$$= 2\left[\frac{\cos\frac{n\pi}{2}}{4n\pi} + \frac{1}{4n\pi} - \frac{\sin\frac{n\pi}{2}}{(n\pi)^2} - \frac{\cos\frac{n\pi}{2}}{4n\pi} - \frac{\cos n\pi}{4n\pi} - \frac{\sin\frac{n\pi}{2}}{(n\pi)^2}\right]$$

$$b_n = \frac{1}{2n\pi}\left\{1-(-1)^n\right\} - \frac{4\sin\frac{n\pi}{2}}{n^2\pi^2}.$$

The graph of $f(x)$ on $(-1, 1)$ is shown below.



Obviously the function satisfies Dirichlet's conditions but discontinuous at $x=0$. At $x=0$ the series converges to

$$\frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = \frac{1}{2}\left[-\frac{1}{4}+\frac{1}{4}\right] = 0.$$

The series is

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\left\{1-(-1)^n\right\} - \frac{4\sin\frac{n\pi}{2}}{n^2\pi^2}\right]\sin n\pi x$$

$$= \left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x + \left(\frac{1}{3\pi}+\frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi}-\frac{4}{5^2\pi^2}\right)\sin 5\pi x + \cdots$$

Example 2. Determine the half-range Fourier cosine series of

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$$

Sol^n. Here $f(x)$ is defined in $(0, \pi)$ i.e. $T = \pi$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right]$$

$$= \frac{4}{\pi}\int^{\pi/2} x\,dx = \frac{4}{\pi}\,\frac{x^2}{2}\Big|^{\pi/2} = \pi/2$$

(01)

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\frac{\sin nx}{n}\,dx + (\pi - x)\frac{\sin nx}{n}\Big|_{\frac{\pi}{2}}^{\pi} + \int_{\pi/2}^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} - 0 - \frac{1}{n}\left(-\frac{\cos nx}{n}\right)_0^{\pi/2} + 0 - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n}\left(-\frac{\cos nx}{n}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\left(\cos\frac{n\pi}{2} - 1\right) - \frac{1}{n^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

Therefore the half-range cosine series is given by

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos nx$$

Example 3. Expand the following function $f(x)$ into a Fourier cosine series

$$f(x) = \begin{cases} 1, & 0 \le x < \pi/3 \\ 0 & \frac{1}{3}\pi \le x \le \frac{2\pi}{3} \\ -1 & \frac{2}{3}\pi < x \le \pi \end{cases}$$

Sol^n: To get the Fourier cosine series we extend the f^n in the interval $[-\pi, 0]$ by $f(-x) = f(x)$. The extended function become an even function in $[-\pi, \pi]$.

Now $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/3} dx + \int_{\pi/3}^{\frac{2\pi}{3}} 0\,dx + \int_{\frac{2\pi}{3}}^{\pi}(-1)\,dx\right] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/3}\cos nx\, dx + \int_{\pi/3}^{2\pi/3} 0.\cos nx\, dx + \int_{2\pi/3}^{\pi}(-1)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin nx}{n}\Big|_0^{\pi/3} - \frac{\sin nx}{n}\Big|_{2\pi/3}^{\pi}\right]$$

$$= \frac{2}{n\pi}\left[\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right] \qquad n \geq 1 \text{ and integer}$$

So the Fourier cosine series is

$$0 + \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\cos nx$$

The function satisfies Dirichlet's conditions on $[-\pi, \pi]$, so the Fourier series is convergent.

Complex Form of Fourier Series:

The general form of Fourier series of a periodic function $f(x)$ of period $2T$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{T} + b_n\sin\frac{n\pi x}{T}\right]$$

Now, we know $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}\left[\frac{a_n}{2}\left(e^{i\frac{n\pi x}{T}} + e^{-i\frac{n\pi x}{T}}\right) + \frac{b_n}{2i}\left(e^{i\frac{n\pi x}{T}} - e^{-\frac{in\pi x}{T}}\right)\right]$$

$$= C_0 + \sum_{n=1}^{\infty}\left[C_n e^{i\frac{n\pi x}{T}} + C_{-n}e^{-i\frac{n\pi x}{T}}\right]$$

(11)

$$c_n = \frac{1}{2}(a_n - i b_n)$$

$$= \frac{1}{2T}\left[\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\,dx - i\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx\right]$$

$$= \frac{1}{2T}\int_{-T}^{T} f(x)\, e^{-i\frac{n\pi x}{T}}\,dx$$

$$c_{-n} = \frac{1}{2}(a_n + i b_n)$$

$$= \frac{1}{2T}\left[\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\,dx + i\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx\right]$$

$$= \frac{1}{2T}\int_{-T}^{T} f(x)\, e^{i\frac{n\pi x}{T}}\,dx$$

and $c_0 = \frac{a_0}{2} = \frac{1}{2T}\int_{-T}^{T} f(x)\,dx = \frac{1}{2T}\int_{-T}^{T} f(x)\,e^{i0}\,dx$

Thus the Fourier series can be expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{T}}$$

where $\left[c_n = \frac{1}{2T}\int_{-T}^{T} f(x)\, e^{-i\frac{n\pi x}{T}}\,dx\right.$

for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

This is known as complex form of Fourier series of a function $f(x)$.

~~Example 1~~. Note: The complex form of Fourier series is true when $f(x)$ is continuous at x and satisfies the Dirichlet's conditions. If $f(x)$ is discontinuous at x then the series converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ i.e.

$$f(x) \to \frac{1}{2}[f(x+0) + f(x-0)].$$

Example 1. Find the complex Fourier series of $f(x) = x$ for $-\pi < x < \pi$.

Sol^n:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{T}} = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

For $n = 0$, $\quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, e^{i.0}\, dx = 0$

For $n \neq 0$, $\quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, e^{-inx}\, dx$

$$= \frac{1}{2\pi}\left[x\frac{e^{-inx}}{-in}\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 1.\frac{e^{-inx}}{-in}\, dx\right]$$

$$= \frac{1}{2\pi}\left[\pi\frac{(e^{-in\pi}+e^{in\pi})}{-in} - \frac{1}{(in)^2}\left[e^{-inx}\right]_{-\pi}^{\pi}\right]$$

$$= -\frac{1}{2in}\left(e^{-in\pi}+e^{in\pi}\right) - \frac{1}{2\pi(in)^2}\left[e^{-in\pi} - e^{in\pi}\right]$$

$$= -\frac{1}{2in}\, 2\cos n\pi - \frac{1}{2\pi(in)^2}(-2i\sin n\pi)$$

$$= -\frac{1}{in}(-1)^n - 0 = \frac{i}{n}(-1)^n$$

Therefore $\quad f(x) = \sum_{n=-\infty}^{\infty} \frac{i}{n}(-1)^n e^{inx}$

i.e. $\quad x = i\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n} e^{inx}$

Parseval's Identity

If $\quad \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right)$

is the Fourier series of a function $f(x)$ and the Fourier series converges uniformly in $(-T, T)$, then

$$\frac{1}{T}\int_{-T}^{T}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof: By Fourier series expansion, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right)$$

Multiplying both sides by $f(x)$ and integrating over the interval $(-T, T)$ we get

$$\int_{-T}^{T}\{f(x)\}^2\,dx = \frac{a_0}{2}\int_{-T}^{T} f(x)\,dx$$

$$+ \int_{-T}^{T}\left[\sum_{n=1}^{\infty}\left\{a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right\} f(x)\right]dx$$

Since the Fourier series is uniformly convergent in $(-T, T)$, the right-hand side can be integrated term by term, and hence, we have

$$\int_{-T}^{T}\{f(x)\}^2\,dx = \frac{a_0}{2}\int_{-T}^{T} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\,dx\right.$$

$$\left. + b_n\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx\right]$$

$$= \frac{a_0^2}{2}T + \sum_{n=1}^{\infty}\left[a_n(Ta_n) + b_n(Tb_n)\right]$$

Since $a_0 = \dfrac{1}{T}\displaystyle\int_{-T}^{T} f(x)\,dx$, $a_n = \dfrac{1}{T}\displaystyle\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\,dx$, $b_n = \dfrac{1}{T}\displaystyle\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx$

Therefore, $\quad \dfrac{1}{T}\displaystyle\int_{-T}^{T} f(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$

Note:

1. The sum of the squares of the moduli of the complex Fourier coefficients is equal to the average value of $|f(x)|^2$ within the period. That is

$$\frac{1}{T}\int_{a}^{a+T} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |C_n|^2$$

2. For fourier half range sine series $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{T}$ the ~~Pa~~ Parseval's identity is represented by

$$\frac{2}{T}\int_0^T \{f(x)\}^2\,dx = \sum_{n=1}^{\infty} b_n^2 ,$$ since here $f(x)$ is extended to an even function.

3. For Fourier half range cosine series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T}$ the Parseval's identity is represented as

$$\frac{2}{T}\int_0^T \{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 ,$$ since here $f(x)$ is extended to an odd function.

Example 1: Prove that the Fourier cosine series for the function $f(x) = x$ in $0 \le x \le 2$ is

$$f(x) = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi}{2}x + \frac{1}{5^2}\cos\frac{5\pi}{2}x + \cdots\right)$$

Hence determine the sum of the series by Parseval's identity.

Sol^n: The fourier cosine series of $f(x)$ in $0 \le x \le 2$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

where $$a_0 = \frac{2}{2}\int_0^2 x\,dx = 2$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx = \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \sin\frac{n\pi x}{2}\,dx\right]$$

$$= \left[0 + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_0^2\right]$$

$$= \frac{4}{n^2\pi^2}\left(\cos n\pi x - 1\right)$$

$$\therefore \quad f(x) = x = \frac{2}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$

$$\text{or} \quad x = 1 - \frac{8}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

Again, from Parseval's identity

$$\frac{1}{2}\int_{-2}^{2} x^2\,dx = \frac{2^2}{2} + \sum_{n=1}^{\infty}\frac{4^2}{\pi^4 n^4}(\cos n\pi - 1)^2$$

$$\text{or} \quad \frac{1}{2}\times 2\,\frac{x^3}{3}\Big|_0^2 = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4}\cdots\right)$$

$$\frac{8}{3} - 2 = \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\cdots\right)$$

$$\Rightarrow \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

Now, let $S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right)$$

$$= \frac{\pi^2}{96} + \frac{1}{2^4}S$$

$$\Rightarrow \left(1 - \frac{1}{2^4}\right)S = \frac{\pi^2}{96} \Rightarrow S = \frac{\pi^2}{\cancel{96}_{\,6}} \times \frac{\cancel{16}}{15} = \frac{\pi^2}{90}$$

$$\therefore \quad \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^2}{90}$$