Integral Calculus 2

Wednesday, December 29, 2021 8:55 AM

MVT for definite integral:

$f(x)$ $\phi(x)$ bounded $m \le f(x) \le M$

$m_1 \le \phi(x) \le M_1$

integrable in $a \le x \le b$, $\phi(x)$ keeps same value $a \le x \le b$.

$$\int_a^b f(x)\phi(x)\,dx = \mu \int_a^b \phi(x)\,dx$$

where $m \le \mu \le M$ $\quad m \le |f(x)| \le M$.

$$\boxed{\mu = f(\xi)} \quad \xi \in [a, b]$$

Simple form $\phi(x) = 1$

$$\int_a^b f(x)\,dx = f(\xi)\int_a^b dx$$

$$\Rightarrow f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example: $k^2 < 1$, show that

$$\frac{\pi}{6} \le \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-k^2\xi^2}}$$

Soln:

$$f \cdot \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad a=0 \quad b=\frac{1}{2}$$

$$\left[0, \frac{1}{2}\right]$$

$$\phi(x) = \frac{1}{\sqrt{1-x^2}} \qquad f(x) = \frac{1}{\sqrt{1-k^2x^2}}$$

$1/2$

$$\varphi(x) = \frac{}{\sqrt{1-x^2}} \qquad f(x) = \frac{}{\sqrt{1-k^2x^2}}$$

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{\sqrt{1-k^2\xi^2}} \underline{\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}} \qquad \xi \in [0, \tfrac{1}{2}]$$

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \left(\sin^{-1} x\right)_0^{1/2} = \pi/6$$

$$= \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-k^2\xi^2}}$$

$$\frac{\pi}{6} \le \underline{\int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}} \le \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-\frac{k^2}{4}}}$$

Proved.

* **μ-test for type I** Suppose $f(x)$ is integrable function $x \geq a$. Then $F = \int_0^\infty f(x)\,dx$ converges absolutely if

$$\lim_{x\to\infty} x^\mu f(x) = \lambda \text{ for some } \underline{\mu > 1}$$

and F diverges if

$$\lim_{x\to\infty} x^\mu f(x) = \lambda\ (\neq 0) \text{ or } \pm\infty \text{ for some } \underline{\mu \le 1}$$

μ-test for type II

Let $f(x)$ is integrable function in an arbitrary interval $(a+\epsilon, b)$

$$F = \int_a^b f(x)\,dx$$

$0 < \epsilon < b-a$ [illegible]

Then F is converges absolutely if

$$\lim_{x \to a+} (x-a)^{\mu} f(x) = \lambda \text{ for some } 0 < \mu < 1$$

and F is diverges if

$$\lim_{x \to a+} (x-a)^{\mu} f(x) = \lambda \ (\neq 0) \text{ or } \pm\infty$$
for some $\mu \geq 1$

Example: $\int_0^{\infty} e^{-x^2} dx$

$$f(x) = e^{-x^2}$$

$$\lim_{x \to \infty} x^2 e^{-x^2} = \lim_{x \to \infty} \frac{x^2}{e^{x^2}} = 0$$

$\int_0^{\infty} e^{-x^2} dx$ is convergent

* $\int_1^{\infty} e^{-x^2} x^n dx$ converges for all values of n

$$f(x) = e^{-x^2} x^n$$

$$\lim_{x \to \infty} x^2 \cdot e^{-x^2} x^n = \lim_{x \to \infty} \frac{x^{n+1}}{e^{x^2}} = 0$$

$\int_1^{\infty} e^{-x^2} x^n dx$ is converges for all values of n

* $\int_0^{\pi} \frac{\sin x}{x^3} dx$ divegers.

$\mu \geq 1$

$$\lim_{x \to 0+} x^{\mu} \frac{\sin x}{x^3}$$

$$\lim_{x \to 0+} x^2 \frac{\sin x}{x^3} = \lim_{x \to 0+} \frac{\sin x}{x} = 1 \checkmark$$

* $\int_0^{\infty} e^{-x} x^{n-1} dx$, $\underline{\underline{n > 0}}$

$f(x) = e^{-x} x^{n-1}$

$= I_1 + I_2$

$I_1 = \int_0^1 e^{-x} x^{n-1} dx$, $I_2 = \int_1^{\infty} e^{-x} x^{n-1} dx$

$$\lim_{x \to 0+} x^{1-n} \cdot e^{-x} x^{n-1} = \lim_{x \to 0+} e^{-x} = 1.$$

$0 < \mu = 1 - n < 1 \Rightarrow 0 < 1 - n < 1$

$\Rightarrow 0 < n < 1.$

$$\lim_{x \to \infty} x^2 \cdot e^{-x} \cdot x^{n-1} = \lim_{x \to \infty} \frac{x^{n+1}}{e^x} = 0$$

$$\boxed{F(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad n > 0}$$

Gamma function

Gamma function

Beta function

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , \qquad m>0,\ n>0$$

$$I_1 = \int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx \quad \propto$$

$$I_2 = \int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\,dx \quad \propto$$

$$\lim_{x\to 0+} x^{1-m} f(x) = \lim_{x\to 0+} x^{1-m}\cdot x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x\to 0+} (1-x)^{n-1}$$

$$= 1$$

$$0 < 1-m < 1 \Rightarrow 0 < m < 1$$

$$\boxed{B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m>0,\ n>0}$$

Beta function