Linear Programming Problems: Lecture 1

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Linear Programming Problem (LPP): Definition

Optimize (Maximize or Minimize): $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$,

subject to

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &\quad (\leq, =, \geq) \quad b_1, && (1)\\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &\quad (\leq, =, \geq) \quad b_2, && (2)\\
\vdots \qquad \vdots \qquad \ldots \qquad \vdots &\qquad\qquad\qquad\quad \vdots && \\
A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &\quad (\leq, =, \geq) \quad b_m, && (3)\\
x_1, x_2, \ldots, x_n &\qquad \geq 0, && (4)
\end{aligned}
$$

where $c_1, c_2, \ldots, c_n$, A_{ij}'s are constants and $x_1, x_2, \ldots, x_n$ are variables (**decision variables**). The function Z is called **Objective Function**, the **equations** or **inequations** given in (1) - (3) are called the **constraints** and (4) provide the **non-negativity restrictions** of the LPP.

Mathematical Formulation: Examples

Diet Problem

A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units per kg of vitamin A and 1 unit per kg of vitamin C while food II contains 1 unit per kg of vitamin A and 2 units per kg of vitamin C. If costs of per kg of the foods I and II are Rs. 50 and 70 respectively. Formulate the problem as LPP.

Production Problem

A firm can produce three types of cloths, say C_1, C_2 and C_3. Three kinds of wool are required for it, say, red wool, blue wool and green wool. One unit of C_1 needs 2 meters of red wool and 3 meters of blue wool; one unit of C_2 needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool; and one unit of C_3 needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 16 meters of red wool, 20 meters of green wool and 30 meters of blue wool. The income obtained from each unit of the cloths C_1, C_2 and C_3 are Rs. 6, 10 and 8 respectively. Formulate the problem as LLP to maximize the profit.

Basic Concepts

Consider the LPP defined in slide number 2. The following important concepts will be used frequently in the subsequent discussion.

Solution

A set of values of the variables $x_1, x_2, \ldots, x_n$ is called a **solution** of the LPP, if it satisfies the **constraints** of the LPP.

Feasible Solution

A set of values of the variables $x_1, x_2, \ldots, x_n$ is called a **feasible solution** of the LPP, if it satisfies the **constraints** as well as **non-negativity restrictions** of the LPP.

Infeasible Solution

A set of values of the variables $x_1, x_2, \ldots, x_n$ is called a **infeasible solution** of the LPP, if it satisfies the **constraints** but does not satisfy the **non-negativity restrictions** of the LPP.

Basic Concepts

Feasible Region

The common region of $\mathbb{R}^n$ determined by the **constraints** and **non-negativity restrictions** of the LPP is called the **feasible region**. Each point in this region is a feasible solution of the problem.

Optimal Feasible Solution

A feasible solution of a LPP is said to be **optimal feasible solution**, if it also optimizes (maximizes or minimizes) the objective function.

Convex Set

A set is said to be a **convex set**, if every point on the line segment joining any two points of the set lies in it.

Theorem

The set of all feasible solutions of a LPP forms a convex set.

Fundamental Extreme Point Theorem

An optimal solution of a LPP, if it exists, occurs at one of the extreme (corner) points of the convex region of the set of all feasible solutions of the LPP.

Graphical Method

Solve the following LPP graphically.

1. Maximize $Z = 5x + 3y$,
   subject to

$$\begin{aligned} 3x + 5y &\leq 15, \\ 5x + 2y &\leq 10, \\ x, y &\geq 0. \end{aligned}$$

2. Minimize $Z = 18x + 10y$,
   subject to

$$\begin{aligned} 4x + y &\geq 20, \\ 2x + 3y &\leq 30, \\ x, y &\geq 0. \end{aligned}$$