National Institute of Technology Durgapur
Mathematics-II (MAC 02)

Full Marks: 60   Time: 2 Hr (15 minutes extra for uploading)

Important Instructions

- Answer all questions

- Symbols have their usual meanings

- Write your name, roll number, and paper code on the top of the first page of your answer script

- Make a single pdf file of the entire answer script. Give the file name as 'ROLL-NUMBER' and upload.

1. Let $M = \begin{pmatrix} 5 & 2 \\ 0 & 5 \end{pmatrix}$. Find all eigenvalues of M. Does M have two independent eigenvectors? Can M be diagonalized over any field F? [1+2+3]

2. Examine if the set S of all 2×2 real skew-symmetric matrices is a subspace of the vector space $M_{2\times2}(\mathbb{R})$. If so, then find a basis for S. [6]

3. For what values of a and b, the system of equations

$$x + 2y + z = 1, \quad x + y + 3z = b, \quad 2x + ay + 3z = b + 1$$

has (i) only one solution, (ii) no solution, and (iii) infinitely many solutions? [6]

4. Obtain the Fourier series for the function

$$f(x) = \begin{cases} -\pi, & -\pi \leq x < 0 \\ x, & 0 \leq x \leq \pi \end{cases}$$

and its periodic extension outside $[-\pi, \pi]$. Discuss the convergence of the series at $x = 0, \pi$. [6]

5. If $L[f(x)] = F(s)$ and if $\lim\limits_{x \to 0} \dfrac{f(x)}{x}$ exists then show that $L\left[\dfrac{f(x)}{x}\right] = \int_s^\infty F(u)du$. Also find $L\left[\dfrac{1 - \cos 2t}{t}\right]$. [4+2]

6. (a) Solve $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$. [3]

   (b) Solve the ordinary differential equation $y^2(y - xp) = x^4 p^2$ with $p = \frac{dy}{dx}$. [3]

7. (a) Consider the initial value problem $\frac{dy}{dx} = y^k$; $y(0) = 0$ with $k \in (0,1)$. Discuss the existence and uniqueness of the solution in the neighborhood of the origin for all $k \in (0,1)$. [3]

    (b) Find the general solution of the ODE

    $$\left(\frac{dy}{dx}\right)^3 - 4xy\frac{dy}{dx} + 8y^2 = 0.$$

    [3]

8. Solve $(D^2 + 5D + 6)y = \exp(-2x)\sec^2 x[1 + 2\tan(x)]$, where $D = \frac{d}{dx}$. [6]

9. (a) Find the convolution of Fourier Transformation for the functions $f(x) = \sin(bx)$ and $g(x) = \exp(-a|x|)$; $a > 0$. [3]

    (b) Find the Fourier Transformation of $f(x)$, where $f(x) = 1$ if $|x| < a$ and $f(x) = 0$ if $|x| > a$ with $a > 0$. Hence deduce the value of $\int_0^\infty \frac{\sin(ak)}{k}dk$. [3]

10. (a) A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of the state? [3]

    (b) Find the cumulative distribution $F(x)$ of the random variable X with probability mass function $f(x) = \frac{^4C_x}{16}$, $x = 0, 1, 2, 3, 4$. [3]

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