1) For rectangular / cartesian coordinate system :-

Let $C(x,y,z)$ is the centre of the CV. at C we assume the properties

$\vec{V} = U\hat{i} + V\hat{j} + W\hat{k}$

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

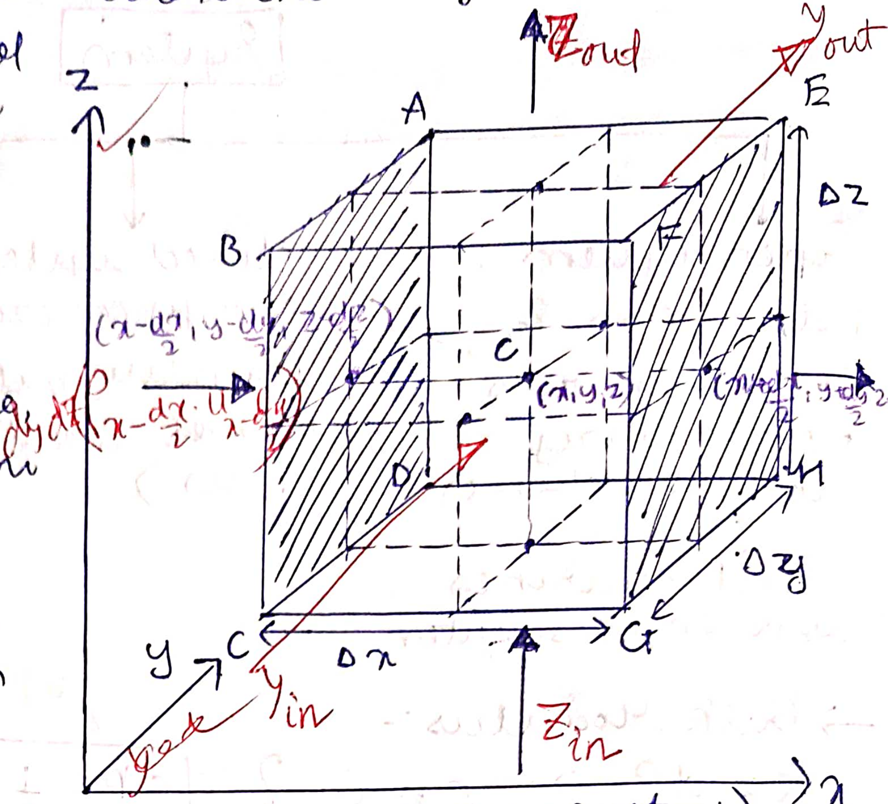
$\rho(x,y,z)$

∴ we know the prop. at centre

∴ we are finding prop. at centre of face FEHG suppose.

$\therefore \rho\Big|_{x+\frac{dx}{2}}$ = density at a dist $x+\frac{dx}{2}$ from origin considering only in x-direction

(keeping y & z dir. variation constant)



$(x,y,z$ are constant$)$

∴ by Taylor expansion :-

$$\rho_{x+\frac{dx}{2}} = \rho + \left(\frac{1}{1!}\right)\left(\frac{\partial \rho}{\partial x}\right)\left(\frac{dx}{2}\right) + \underbrace{\left(\frac{1}{2!}\right)\left(\frac{\partial^2 \rho}{\partial x^2}\right)\left(\frac{dx}{2}\right)^2 + \cdots}_{\text{neglecting higher order terms.}}$$

$$\therefore \rho_{x+\frac{dx}{2}} \simeq \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} \quad \text{--- (1)}$$

∴ for velocity in x-direction :-

$$U_{x+\frac{dx}{2}} = U + \left(\frac{\partial U}{\partial x}\right)\frac{dx}{2} \quad \text{--- (2)}$$

∴ properties at centre of face ABCD suppose.

by Taylor expansion:-

$$\rho_{x-\frac{dx}{2}} \simeq \rho - \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} \quad \text{--- (3)}$$

∴ for velocity in x-direction :-

$$U_{x-\frac{dx}{2}} = U - \left(\frac{\partial U}{\partial x}\right)\frac{dx}{2} \quad \text{--- (4)}$$

∴ net mass flow rate out along x-direction

$= ($~~Volume~~ mass flow rate out $-$ ~~Volume~~ mass flow rate in$)_x$

$$= \left(\rho_{x+\frac{dx}{2}} \cdot u_{x+\frac{dx}{2}} - \rho_{x-\frac{dx}{2}} \cdot u_{x-\frac{dx}{2}}\right) dy\,dz \quad \because \text{from } ①, ②, ③, ④$$

$$= \left[\left(\rho + \frac{\partial \rho}{\partial x} \cdot \frac{dx}{2}\right)\left(u + \frac{\partial u}{\partial x} \cdot \frac{dx}{2}\right) - \left(\rho - \frac{\partial \rho}{\partial x} \cdot \frac{dx}{2}\right)\left(u - \frac{\partial u}{\partial x} \cdot \frac{dx}{2}\right)\right] dy\,dz$$

$$= \left[\cancel{\rho u} + \rho \cdot \frac{\partial u}{\partial x} \cdot \frac{dx}{2} + u \frac{\partial \rho}{\partial x} \frac{dx}{2} - \cancel{\rho u} + \rho \cdot \frac{\partial u}{\partial x} \cdot \frac{dx}{2} + u \cdot \frac{\partial \rho}{\partial x} \cdot \frac{dx}{2}\right] dy\,dz$$

$$= \left[\rho \frac{\partial u}{\partial x} \cdot dx + u \cdot \frac{\partial \rho}{\partial x} \cdot dx\right] dy\,dz \quad \text{(neglecting prod. of derivatives)}$$

$$= \left[\rho \frac{du}{dx} + u \frac{\partial \rho}{\partial x}\right] dx\,dy\,dz.$$

$$= \frac{\partial}{\partial x} (\rho u) \cdot d\forall \quad \text{—— (I)}$$

∴ similarly for face CDHG & ABFE (z-direction)
net mass flow rate out along - z - direction
= (Volume out - volume in) z

$$= \frac{\partial}{\partial z} (\rho w) \cdot d\forall \quad \text{—— (III)}$$

∴ similarly for face AEHD & BFGC (y-direction)
net mass flow rate out along - y - direction.
= (Volume out - volume in) y

$$= \frac{\partial}{\partial y} (\rho v) d\forall \quad \text{—— (II)}$$

∴ from (I), (II), (III)
net mass flow rate out through C.S. along (x, y, z) CV direction = Net flux (mass flow rate out) of N out through C.S.
↓
dx, dy, dz
generate C.S.

∴ Time rate of change of mass within CV.
$$\text{within } C\forall = \frac{\partial}{\partial t} (\rho \cdot d\forall) = \frac{\partial \rho}{\partial t} \cdot d\forall$$
↓
const b/c its C∀.

∴ As per law of conservation of mass :- From RTT, $\frac{dm}{dt} = \frac{\partial}{\partial t}\iiint_{CV} \rho\, d\forall + \iint_{CS} \rho \vec{V} \cdot d\vec{A}$

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] d\forall + \frac{\partial \rho}{\partial t} \cdot d\forall = \frac{dm}{dt}$$

$$\therefore \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t}\right] d\forall = 0 \quad \left[\because \frac{dm}{dt} = 0 \text{ in C.M}\right]$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial}{\partial z}(\rho w) = 0/\not{\forall}^{0} \text{ as } \partial\forall \text{ = constant.}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{V} = 0} \text{ where } \vec{V} = u\hat{i} + v\hat{j} + w\hat{k}, \ \& \ \vec{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \quad \text{--- (A)}$$

$$\therefore \frac{\partial \rho}{\partial t} + \rho\vec{\nabla}\cdot\vec{V} + (\vec{V}\cdot\vec{\nabla})\rho = 0$$

$$\therefore \left[\frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}\right]\rho + \rho\vec{\nabla}\cdot\vec{V} = 0$$

$$\therefore \boxed{\frac{D\rho}{Dt} + \rho\cdot\vec{\nabla}\cdot\vec{V} = 0} \quad \text{--- (B)}$$

$$\text{here } \underbrace{\frac{D}{Dt}}_{\text{material derivative}} = \underbrace{\frac{\partial}{\partial t}}_{\substack{\text{local derivative} \\ \downarrow \\ \text{unsteady part}}} + \underbrace{(\vec{V}\cdot\vec{\nabla})}_{\substack{\text{advective} \\ \downarrow \\ \text{convective part}}}$$

$\frac{\partial \rho}{\partial t}$ = ~~lock~~ fix your eyes ~~position~~ at a certain pt. & observe the ~~characteristics~~ change in density (ρ) with time.

$\frac{D\rho}{Dt}$ = ~~observes a property at a particular pt~~

= Fix your eyes at a fluid particle and observe the change in density.

Hence.

Rate of change of mass within a CV	+ Net mass flow rate out through a C.S.	= 0

{ $\rho \times$ spd = intensity }

{ volume at each face is constant as the x, y, z coordinates are not changing their values with that direction }

① for incompressible flow :- (not changing with space) w.r.t time

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + 0 \qquad \frac{D\rho}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\rho = 0$$

→ for steady flow :- (not changing with time) :-

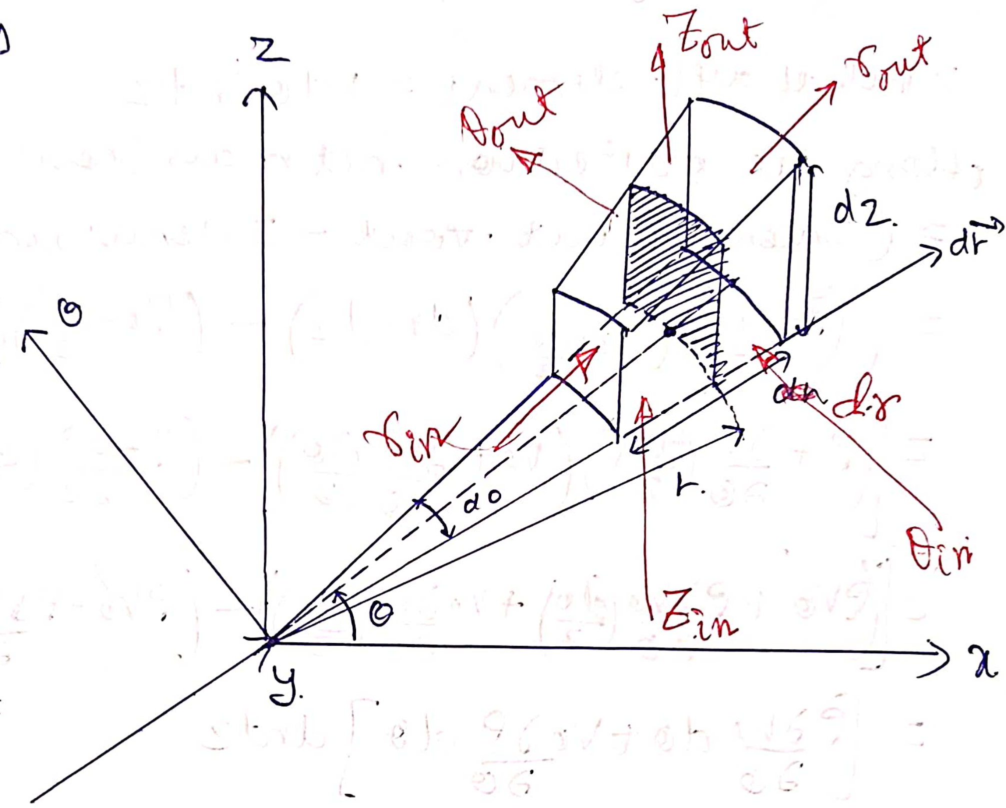
$$\cancel{\frac{D\rho}{Dt}}^{0} + \rho\cdot\vec{\nabla}\cdot V = 0$$

$$\vec{\nabla}\cdot\vec{V} = 0 \Rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

② For steady flow :- From eqn. A $\Rightarrow$ $\cancel{\frac{\partial \rho}{\partial t}}^{0} + \vec{\nabla}\cdot\rho\vec{V} = 0 \ \therefore \ \vec{\nabla}\cdot\rho\vec{V} = 0$

2) Equation of continuity in cylindrical coordinate syst. (r, θ, z):-

Here r, θ are varying and z = constant.
Hence volume in & vol. out vary in case of direction in r.
We assume the following properties at the surface element considered:-



$\vec{V} = v_r e_r + v_\theta e_\theta + v_z e_z$

$\rho(x, y, z)$

$\vec{A} = A_r e_r + A_\theta e_\theta + A_z e_z$

∴ considering each surface:-

∴ Along the r direction; net mass flow rate:-

= (Intensity out − Intensity in) r × vol. (in general).

$= (\rho_{out} \cdot V_{out} \cdot \forall_{out} - \rho_{in} \cdot V_{in} \cdot \forall_{in})$

$= \rho_{r+\frac{dr}{2}} \cdot V_{r+\frac{dr}{2}} \cdot \left(r+\frac{dr}{2}\right) d\theta dz - \rho_{r-\frac{dr}{2}} \cdot V_{r-\frac{dr}{2}} \left(r-\frac{dr}{2}\right) d\theta dz$

$$= \left[\left(\rho + \frac{\partial \rho}{\partial r}\left(\frac{dr}{2}\right)\right)\left(v_r + \frac{\partial v_r}{\partial r}\left(\frac{dr}{2}\right)\right)\left(r+\frac{dr}{2}\right) - \left(\rho - \frac{\partial \rho}{\partial r}\left(\frac{dr}{2}\right)\right)\left(v_r - \frac{\partial v_r}{\partial r}\left(\frac{dr}{2}\right)\right)\left(r-\frac{dr}{2}\right)\right] d\theta\, dz$$

$$= \left[\left(\rho v_r + \rho\frac{\partial v_r}{\partial r}\left(\frac{dr}{2}\right) + v_r\frac{\partial \rho}{\partial r}\left(\frac{dr}{2}\right)\right)\left(r+\frac{dr}{2}\right) - \left(\rho v_r - \rho\frac{\partial v_r}{\partial r}\left(\frac{dr}{2}\right) - v_r\frac{\partial \rho}{\partial r}\left(\frac{dr}{2}\right)\right)\left(r-\frac{dr}{2}\right)\right] d\theta dz$$

$$= \left[\rho v_r dr + \rho\frac{\partial v_r}{\partial r}\left(\frac{dr}{\cancel{2}}\right)(\cancel{2}r) + v_r\frac{\partial \rho}{\partial r}\left(\frac{dr}{\cancel{2}}\right)(\cancel{2}r)\right] d\theta dz$$

$$= \left[\rho v_r + \rho\frac{\partial v_r}{\partial r} \cdot r + v_r\frac{\partial \rho}{\partial r} \cdot r\right] dr\, d\theta\, dz.$$

$$= \left[\rho v_r + r\left(\rho\frac{\partial v_r}{\partial r} + v_r\frac{\partial \rho}{\partial r}\right)\right] dr\, d\theta\, dz$$

$$= \left(\rho v_r + r\frac{\partial}{\partial r}(\rho v_r)\right) dr\,d\theta\,dz \text{ ——— (I)}$$

$\Rightarrow$ vol. of diff. element $= r\,d\theta\,dr\,dz = d\forall$

Along the θ direction, net mass flow rate :-

$$= (\text{Intensity out} \cdot \forall \text{out} - \text{Intensity in} \cdot \forall \text{in})_\theta$$

$$= \left(\left(\rho_{\theta+\frac{d\theta}{2}}\right)\left(v_{\theta\,\theta+\frac{d\theta}{2}}\right)(dr \cdot dz) - \left(\rho_{\theta-\frac{d\theta}{2}}\right)\left(v_{\theta\,\theta-\frac{d\theta}{2}}\right)\right) dr \cdot dz$$

$$= \left[\left(\rho + \frac{\partial \rho}{\partial \theta}\left(\frac{d\theta}{2}\right)\right)\left(v_\theta + \frac{\partial v_\theta}{\partial \theta}\left(\frac{d\theta}{2}\right)\right) - \left(\rho - \frac{\partial \rho}{\partial \theta}\left(\frac{d\theta}{2}\right)\right)\left(v_\theta - \frac{\partial v_\theta}{\partial \theta}\left(\frac{d\theta}{2}\right)\right)\right] dr\,dz$$

$$= \left[\left(\cancel{\rho v_\theta} + \rho\frac{\partial v_\theta}{\partial \theta}\left(\frac{d\theta}{2}\right) + v_\theta\frac{\partial \rho}{\partial \theta}\left(\frac{d\theta}{2}\right)\right) - \left(\cancel{\rho v_\theta} - \rho\frac{\partial v_\theta}{\partial \theta}\left(\frac{d\theta}{2}\right) - v_\theta\frac{\partial \rho}{\partial \theta}\left(\frac{d\theta}{2}\right)\right)\right] dr\,dz$$

$$= \left[\rho\frac{\partial v_\theta}{\partial \theta}d\theta + v_\theta\frac{\partial \rho}{\partial \theta}d\theta\right] dr\,dz$$

$$= \frac{\partial}{\partial \theta}(\rho v_\theta)\, dr\,d\theta\,dz \text{ ——— (II)}$$

Along the z direction, net mass flow rate :-

$$= (\text{Intensity out} \cdot \forall \text{out} - \text{Intensity in} \cdot \forall \text{in})_z$$

$$= \left[\left(\rho_{z+\frac{dz}{2}}\right)\left(v_{z\,z+\frac{dz}{2}}\right) r\,dr\,d\theta - \left(\rho_{z-\frac{dz}{2}}\right)\left(v_{z\,z-\frac{dz}{2}}\right) r\,dr\,d\theta\right]$$

$$= \left[\left(\rho + \frac{\partial \rho}{\partial z}\left(\frac{dz}{2}\right)\right)\left(v_z + \frac{\partial v_z}{\partial z}\left(\frac{dz}{2}\right)\right) - \left(\rho - \frac{\partial \rho}{\partial z}\left(\frac{dz}{2}\right)\right)\left(v_z - \frac{\partial v_z}{\partial z}\left(\frac{dz}{2}\right)\right)\right] r\,dr\,d\theta$$

$$= \left[\left(\cancel{\rho v_z} + \rho\frac{\partial v_z}{\partial z}\left(\frac{dz}{2}\right) + v_z\frac{\partial \rho}{\partial z}\left(\frac{dz}{2}\right)\right) - \left(\cancel{\rho v_z} - \rho\frac{\partial v_z}{\partial z}\left(\frac{dz}{2}\right) - v_z\frac{\partial \rho}{\partial z}\left(\frac{dz}{2}\right)\right)\right] r\,dr\,d\theta$$

$$= \left[\rho\frac{\partial v_z}{\partial z} + v_z\frac{\partial \rho}{\partial z}\right] r\,dr\,d\theta\,dz$$

$$= r\frac{\partial}{\partial z}(\rho v_z)\, dr\,d\theta\,dz \text{ ——— (III)}$$

from (I) (II) (III), net mass flow through C.S:

$$= \left[\left(\rho v_r + r\frac{\partial}{\partial r}(\rho v_r)\right) dr\, d\theta\, dz\right] + \left[\frac{\partial}{\partial \theta}(\rho v_\theta)\, dr\, d\theta\, dz\right] + \left[r\frac{\partial}{\partial z}(\rho v_z)\, dr\, d\theta\, dz\right]$$

$$= \left[\rho v_r + r\frac{\partial}{\partial r}(\rho v_r) + \frac{\partial}{\partial \theta}(\rho v_\theta) + r\frac{\partial}{\partial z}(\rho v_z)\right] dr\, d\theta\, dz$$

$$= \left[\frac{\partial}{\partial r}(r \cdot \rho v_r) + \frac{\partial}{\partial \theta}(\rho v_\theta) + r\frac{\partial}{\partial z}(\rho v_z)\right] dr\, d\theta\, dz \text{ —— (IV)}$$

∵ Time rate of change of mass within CV $= \frac{\partial}{\partial t}(\rho \cdot r\, d\theta\, dr\, dz) = \frac{\partial \rho}{\partial t} \cdot r\, d\theta\, dr\, dz$ —— (V)

Hence continuity in r-θ-z is:-

$= \text{(IV)} + \text{(V)} = 0$

$$\boxed{= \left[\left(\frac{\partial}{\partial r}(r \cdot \rho v_r) + \frac{\partial}{\partial \theta}(\rho v_\theta) + r\frac{\partial}{\partial z}(\rho v_z)\right) + \left(r\frac{\partial \rho}{\partial t}\right)\right] dr\, d\theta\, dz = 0}$$

also:

$$\therefore \boxed{\left[\frac{\partial \rho}{\partial t}\right] + \left[\frac{1}{r}\cdot\frac{\partial}{\partial r}(r \cdot \rho v_r) + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z)\right] = 0}$$

$$\therefore \rightarrow \boxed{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0} \rightarrow \text{in general.}$$

∴ for r, θ, z system:-

$$\vec{\nabla} = e_r\frac{\partial}{\partial r} + \frac{1}{r}e_\theta\frac{\partial}{\partial \theta} + e_z\frac{\partial}{\partial z}$$

$$\vec{V} = v_r e_r + v_\theta e_\theta + v_z e_z$$

for x-y-z system:-

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

→ mass flow through the differential area :-

$dm = \rho v\, ds$

ρ → density, v → velocity, ds → cross section area.

1) eq. of continuity in x-y-z coordinates :-

$$\boxed{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial}{\partial z}(\rho w)\right)}$$

∴ for $(x, y, z) \longrightarrow (r\cos\theta, r\sin\theta, z)$

$\downarrow$ $r = (x^2+y^2)^{1/2}$; $\theta = \tan^{-1}(y/x)$

2)
$$\boxed{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \left(\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}\right)}$$

i.e. eq. of continuity in r-θ-z coordinates

∴ for $(x, y, z) \longrightarrow (r\sin\theta\sin\phi, r\sin\theta\cos\phi, r\cos\theta)$

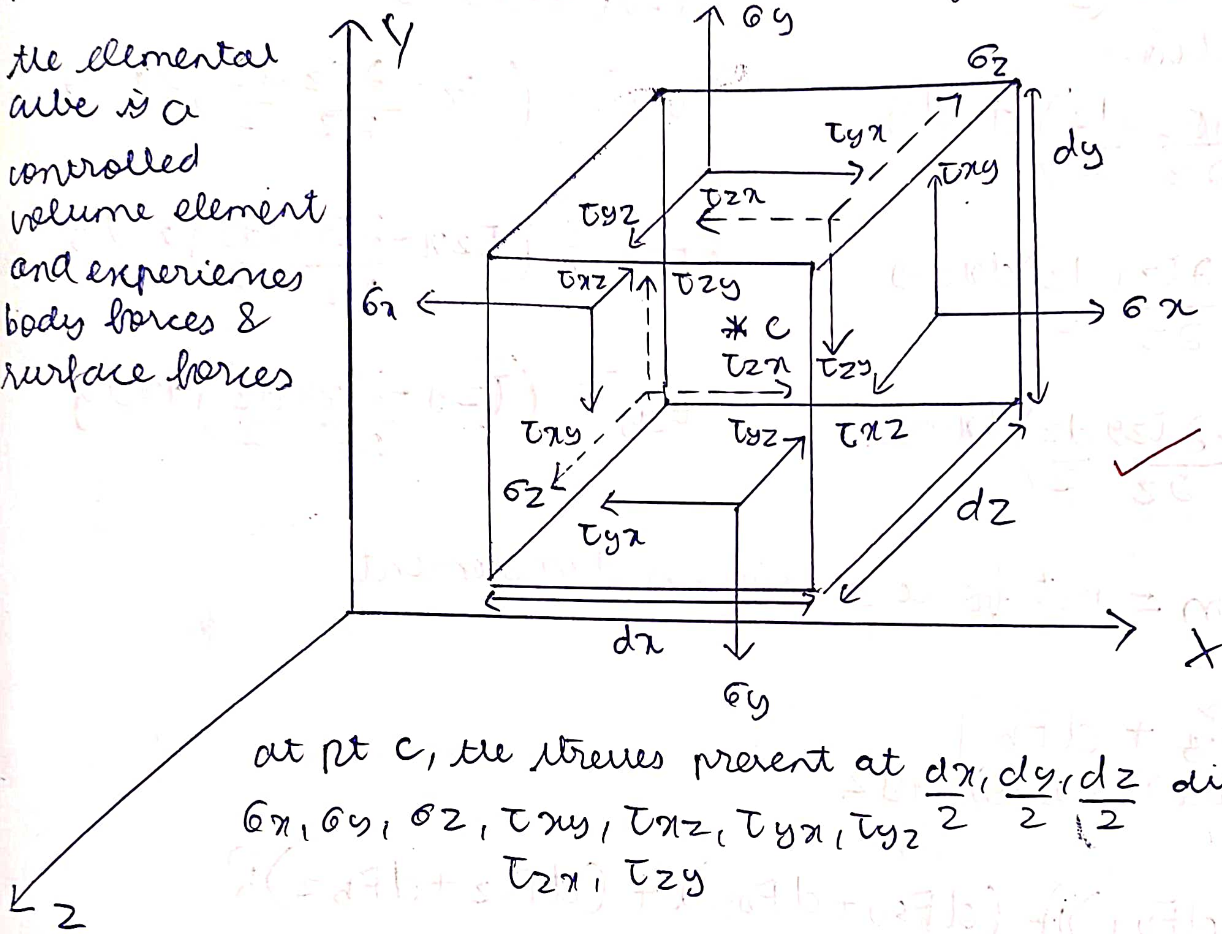
$\downarrow$ $r = (x^2+y^2+z^2)^{1/2}$; $\theta = \tan^{-1}\dfrac{(x^2+y^2)^{1/2}}{z}$; $\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$

3)
$$\boxed{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \left(\frac{1}{r^2}\cdot\frac{\partial(r^2 \rho v_r)}{\partial r} + \frac{1}{r\sin\theta}\cdot\frac{\partial(\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial \phi}(\rho v_\phi)\right)}$$

i.e. eq of continuity in r-θ-φ coordinates

→ Navier-Stokes Equation:-
Its a Partial differential equation that describes the flow of incompressible fluids & frictionless fluids.

the elemental cube is a controlled volume element and experiences body forces & surface forces



at pt C, the stresses present at $\frac{dx}{2}, \frac{dy}{2}, \frac{dz}{2}$ will be
$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}$
τ_{zx}, τ_{zy}

∴ for x plane:-

i) +ve x direction:- — (I)

$$F_{\sigma x}^{+} = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \cdot \frac{dx}{2}\right) dy\,dz$$

$$F_{\tau xy}^{+} = \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \frac{dx}{2}\right) dy\,dz$$

$$F_{\tau xz}^{+} = \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} \frac{dx}{2}\right) dy\,dz.$$

ii) -ve x direction:-

$$F_{\sigma x}^{-} = \left(\sigma_x - \frac{\partial \sigma_x}{\partial x} \cdot \frac{dx}{2}\right) dy\,dz$$

$$F_{\tau xy}^{-} = \left(\tau_{xy} - \frac{\partial \tau_{xy}}{\partial x} \cdot \frac{dx}{2}\right) dy\,dz$$

$$F_{\tau xz}^{-} = \left(\tau_{xz} - \frac{\partial \tau_{xz}}{\partial x} \cdot \frac{dx}{2}\right) dy\,dz$$

for y plane:- — (II)

i) +ve y-direction

$$F_{\tau yx}^{+} = \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \frac{dy}{2}\right) dx\,dz$$

$$F_{\tau yz}^{+} = \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} \cdot \frac{dy}{2}\right) dx\,dz$$

ii) -ve y-direction

$$F_{\tau yx}^{-} = \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y} \frac{dy}{2}\right) dx\,dz$$

$$F_{\tau yz}^{-} = \left(\tau_{yz} - \frac{\partial \tau_{yz}}{\partial y} \cdot \frac{dy}{2}\right) dx\,dz$$

$$F_{\sigma y}^{+} = \left(\sigma_y + \frac{\partial \sigma_y}{\partial y}\cdot\frac{dy}{2}\right) dx\,dz \qquad F_{\sigma y}^{-} = \left(\sigma_y - \frac{\partial \sigma_y}{\partial y}\cdot\frac{dy}{2}\right) dx\,dz$$

for z-plane :- —— (III)

i) +ve z-direction

$$F_{\sigma z}^{+} = \left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\cdot\frac{dz}{2}\right) dx\,dy$$

$$F_{\tau zx}^{+} = \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{dz}{2}\right) dx\,dy$$

$$F_{\tau zy}^{+} = \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}\frac{dz}{2}\right) dx\,dy$$

ii) -ve z-direction

$$F_{\sigma z}^{-} = \left(\sigma_z - \frac{\partial \sigma_z}{\partial z}\frac{dz}{2}\right) dx\,dy$$

$$F_{\tau zx}^{-} = \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\cdot\frac{dz}{2}\right) dx\,dy$$

$$F_{\tau zy}^{-} = \left(\tau_{zy} - \frac{\partial \tau_{zy}}{\partial z}\cdot\frac{dz}{2}\right) dx\,dy$$

$\because d\vec{F} = \frac{D\vec{V}}{Dt}\cdot dm$ = net force acting on dm element

$$d\vec{F} = \Big[\underset{(\text{surface})}{d\vec{F}_s} + \underset{(\text{body})}{d\vec{F}_b}\Big]_{xyz}$$

$$\therefore d\vec{F} = (dF_{sx} + dF_{bx})\hat{i} + (dF_{sy} + dF_{by})\hat{j} + (dF_{sz} + dF_{bz})\hat{k}$$

$\therefore$ along x-direction

$$dF_{sx} = F_{\sigma x}^{+} + F_{\sigma x}^{-} + F_{\tau yx}^{+} + F_{\tau yx}^{-} + F_{\tau zx}^{+} + F_{\tau zx}^{-} \quad (\text{from I, II, III})$$

$$= \left[\left(\cancel{\sigma_x} + \frac{\partial \sigma_x}{\partial x}\cdot\frac{dx}{2}\right) - \left(\cancel{\sigma_x} - \frac{\partial \sigma_x}{\partial x}\frac{dx}{2}\right)\right] dy\,dz \;+$$

$$\left[\left(\cancel{\tau_{yx}} + \frac{\partial \tau_{yx}}{\partial y}\frac{dy}{2}\right) - \left(\cancel{\tau_{yx}} - \frac{\partial \tau_{yx}}{\partial y}\cdot\frac{dy}{2}\right)\right] dx\,dz \;+$$

$$\left[\left(\cancel{\tau_{yz}} + \frac{\partial \tau_{yz}}{\partial z}\cdot\frac{dz}{2}\right) - \left(\cancel{\tau_{yz}} - \frac{\partial \tau_{yz}}{\partial z}\cdot\frac{dz}{2}\right)\right] dx\,dy$$

$$dF_{sx} = \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] dx\,dy\,dz \quad \text{—— (1)}$$

also: $dF_{bx} = (\rho\, d\forall)\, g_x$

∴ along y-direction:-

$$dF_{sy} = F_{\sigma y}^{+} + F_{\sigma y}^{-} + F_{\tau xy}^{+} + F_{\tau xy}^{-} + F_{\tau zy}^{+} + F_{\tau zy}^{-} \text{ (from (I) (II) (III))}$$

$$= \left[\left(\cancel{\sigma_y} + \frac{\partial \sigma_y}{\partial y}\frac{dy}{2}\right) - \left(\cancel{\sigma_y} - \frac{\partial \sigma_y}{\partial y}\frac{dy}{2}\right)\right] dx\,dz +$$

$$\left[\left(\cancel{\tau_{xy}} + \frac{\partial \tau_{xy}}{\partial x}\cdot\frac{dx}{2}\right) - \left(\cancel{\tau_{xy}} - \frac{\partial \tau_{xy}}{\partial x}\cdot\frac{dx}{2}\right)\right] dy\,z +$$

$$\left[\left(\cancel{\tau_{zy}} + \frac{\partial \tau_{zy}}{\partial z}\frac{dz}{2}\right) - \left(\cancel{\tau_{zy}} - \frac{\partial \tau_{zy}}{\partial z}\cdot\frac{dz}{2}\right)\right] dx\,dy$$

$$dF_{sy} = \left[\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial zy}{\partial z}\right] dx\,dy\,dz \quad \text{—— (2)}$$

also $dF_{bx} = (\rho\, d\forall) g_y$

∴ along z-direction:-

$$dF_{sz} = F_{\sigma z}^{+} + F_{\sigma z}^{-} + F_{\tau xz}^{+} + F_{\tau xz}^{-} + F_{\tau yz}^{+} + F_{\tau yz}^{-} \text{ ((I) (II) (III))}$$

$$= \left[\left(\cancel{\sigma_z} + \frac{\partial \sigma_z}{\partial z}\frac{dz}{2}\right) - \left(\cancel{\sigma_z} + \frac{\partial \sigma_z}{\partial z}\frac{dz}{2}\right)\right] dx\,dy +$$

$$\left[\left(\cancel{\tau_{xz}} + \frac{\partial \tau_{xz}}{\partial x}\frac{dx}{2}\right) - \left(\cancel{\tau_{xz}} - \frac{\partial \tau_{xz}}{\partial x}\cdot\frac{dx}{2}\right)\right] dy\,dz +$$

$$\left[\left(\cancel{\tau_{yz}} + \frac{\partial \tau_{yz}}{\partial y}\frac{dy}{2}\right) - \left(\cancel{\tau_{yz}} - \frac{\partial \tau_{yz}}{\partial y}\cdot\frac{dy}{2}\right)\right] dx\,dz$$

$$dF_{sz} = \left[\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right] dx\,dy\,dz \quad \text{—— (3)}$$

also. $dF_{bz} = (\rho\, d\forall) g_z$

$$\therefore d\vec{F} = \left(\rho d\forall \frac{DU}{Dt}\right)\hat{i} + \left(\rho d\forall \frac{DV}{Dt}\right)\hat{j} + \left(\rho d\forall \frac{DW}{Dt}\right)\hat{k} \quad \text{—— (4)}$$

∴ comparing in each direction:-

from ⑤ & ①:-

$$\rho \cancel{d\forall} \frac{DU}{Dt} = dF_{sx} + dF_{bx} \quad (\because dx\,dy\,dz = d\forall)$$

$$= \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \cancel{dx\,dy\,dz} + \rho \cancel{d\forall}\, g_x$$

$$\therefore \frac{DU}{Dt} = \frac{1}{\rho}\cdot\frac{\partial \sigma_x}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} + g_x \quad \text{—— (A)}$$

from ⑤ & ②:-

$$\rho \cancel{d\forall} \frac{DV}{Dt} = dF_{sy} + dF_{by} \quad (\because dx\,dy\,dz = d\forall)$$

$$= \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y}\right) \cancel{dx\,dy\,dz} + \rho \cancel{d\forall}\, g_y$$

$$\therefore \frac{DV}{Dt} = \frac{1}{\rho}\frac{\partial \sigma_y}{\partial y} + \frac{1}{\rho}\cdot\frac{\partial \tau_{xy}}{\partial x} + \frac{1}{\rho}\cdot\frac{\partial \tau_{zy}}{\partial z} + g_y \quad \text{—— (B)}$$

from ⑤ & ③:-

$$\rho \cancel{d\forall} \frac{DW}{Dt} = dF_{sz} + dF_{bz} \quad (\because dx\,dy\,dz = d\forall)$$

$$= \left(\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right) \cancel{dx\,dy\,dz} + \rho \cancel{d\forall}\, g_z$$

$$\frac{DW}{Dt} = \frac{1}{\rho}\cdot\frac{\partial \sigma_z}{\partial z} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial y} + g_z \quad \text{—— (C)}$$

Also (no need to know proof):-

$$\left.\begin{aligned} \tau_{yx} &= \tau_{xy} = \mu\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) \\ \tau_{zx} &= \tau_{xz} = \mu\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right) \\ \tau_{zy} &= \tau_{yz} = \mu\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) \end{aligned}\right\} \text{—— } (\alpha)$$

$$\left.\begin{aligned} \sigma_x &= -P - \frac{2}{3}\mu \vec{\nabla}\cdot\vec{V} + 2\mu \frac{\partial U}{\partial x} \\ \sigma_y &= -P - \frac{2}{3}\mu \vec{\nabla}\cdot\vec{V} + 2\mu \frac{\partial V}{\partial y} \\ \sigma_z &= -P - \frac{2}{3}\mu \vec{\nabla}\cdot\vec{V} + 2\mu \frac{\partial W}{\partial z} \end{aligned}\right\} \leftarrow (\beta)$$

Hydrostatic   volume dilat^n linear dilatation here in β

p = thermodynamic pressure

$(2/3)\mu\vec{\nabla}\cdot\vec{V}$ = volume dilation

$2\mu \dfrac{\partial \text{ velocity in direction}}{\partial \text{ direction}}$ = linear dilation

$\Big\}$ (μ = absolute or dynamic viscosity)

from ⓐ & ⓑ in Ⓐ, Ⓑ, Ⓒ :- $\{ \nu = \mu/\rho$ = kinematic viscosity $\}$

$$\frac{DU}{Dt} = \frac{1}{\rho}\frac{\partial}{\partial x}\left[-P-\frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}+2\mu\frac{du}{dx}\right]+\frac{1}{\rho}\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\mu\right]+\frac{1}{\rho}\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+g_x \quad \text{— ⓐ}$$

$$\frac{DV}{Dt} = \frac{1}{\rho}\frac{\partial}{\partial y}\left[-P-\frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}+2\mu\frac{dv}{dy}\right]+\frac{1}{\rho}\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{1}{\rho}\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+g_y \quad \text{— ⓑ}$$

$$\frac{DW}{Dt} = \frac{1}{\rho}\frac{\partial}{\partial z}\left[-P-\frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}+2\mu\frac{dw}{dz}\right]+\frac{1}{\rho}\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\frac{1}{\rho}\frac{1}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+g_z \quad \text{— ⓒ}$$

considering incompressible fluid :- $\vec{\nabla}\cdot\vec{V} = 0$ (refer RTT)

$$\therefore \frac{DU}{Dt} = g_x + \frac{1}{\rho}\frac{\partial(-P)}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] = g_x - \frac{\partial P}{\partial x}\cdot\frac{1}{\rho} + \nu\nabla^2 u \quad \text{— (a.1)}$$

$$\frac{DV}{Dt} = g_y + \frac{1}{\rho}\frac{\partial(-P)}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right] = g_y - \frac{\partial P}{\partial y}\cdot\frac{1}{\rho} + \nu\nabla^2 v \quad \text{— (b.1)}$$

$$\frac{DW}{Dt} = g_z + \frac{1}{\rho}\frac{\partial(-P)}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right] = g_z - \frac{\partial P}{\partial z}\cdot\frac{1}{\rho} + \nu\nabla^2 w \quad \text{— (c.1)}$$

from (a·1) (b·1) (c·1) :-

$$\boxed{\frac{D\vec{V}}{Dt} = \vec{g} - \frac{1}{\rho}\vec{\nabla}.P + \nu\nabla^2\vec{V}}$$

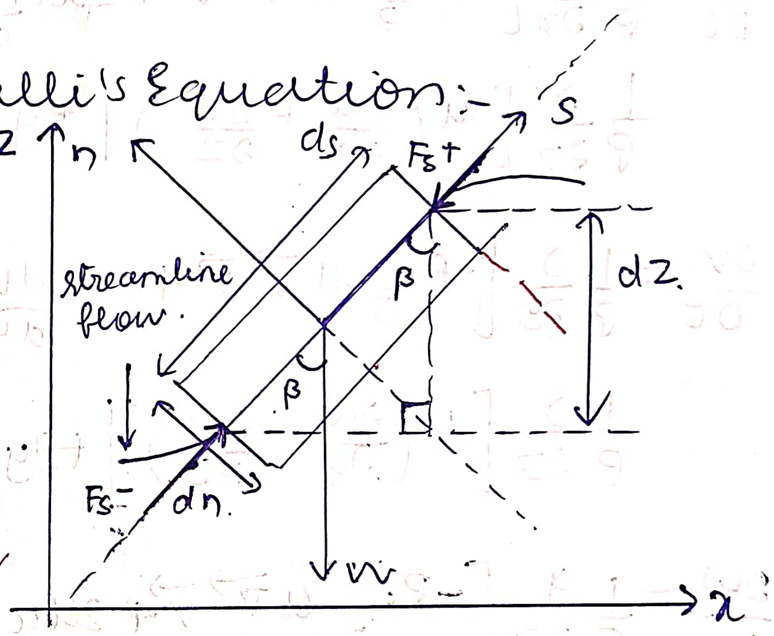
$\frac{D\vec{V}}{Dt}$ → inertial force per unit volume

$\vec{g}$ → gravitational force.

$\frac{1}{\rho}\vec{\nabla}.P$ → pressure force.

$\nu\nabla^2\vec{V}$ → viscous force per unit volume.

→ Euler's Equation & Bernoulli's Equation :-



Consider a streamlined flow as shown in figure with the coordinate axis n-s.

Considering a streamline element, along the s-axis :-

$$F_s^+ = \left(P + \frac{\partial P}{\partial s}\cdot\frac{ds}{2}\right) dn\,dx \quad \text{—— ①}$$

$$F_s^- = \left(P - \frac{\partial P}{\partial s}\cdot\frac{ds}{2}\right) dn\,dx \quad \text{—— ②}$$

Using Newton's 2nd law of motion :-

$\vec{F}_{net} = M\cdot\vec{a}$ ∴ considering along s-axis :-

$$F_s^- - F_s^+ - W\cos\beta = M a_s$$

$$\left[\left(P - \frac{\partial P}{\partial s}\cdot\frac{ds}{2}\right) - \left(P + \frac{\partial P}{\partial s}\cdot\frac{ds}{2}\right)\right] dn\,dx - (\rho d\forall g)\cos\beta = (\rho d\forall)\, a_s$$

$$-\frac{\partial P}{\partial s}\cdot ds\, dn\, dx - (\rho d\forall g)\left(\frac{dz}{ds}\right) = (\rho d\forall)\, a_s \quad \left(\because \cos\beta = dz/ds \ \& \ dn\,dx\,ds = d\forall\right)$$

$$-\frac{1}{\rho}\cdot\frac{\partial P}{\partial s} - g\cdot\frac{dz}{ds} = a_s = \left.\frac{DV}{Dt}\right|_s \quad \text{—— ①}$$

$$\because \frac{DV}{Dt} = \frac{\partial V}{\partial t} + V_s\frac{dV}{ds} + \cancel{V_n}^{0}\frac{dV}{dn} \quad \left(\because \text{streamline element is flowing in s-direction only} \ \therefore V_n = 0\right)$$

$$\left.\frac{DV}{Dt}\right|_s = \frac{\partial V}{\partial t} + V_s\frac{dV}{ds} = \frac{\partial V}{\partial t} + \frac{V dV}{ds} \quad \left(\because \vec{V} = V_s\hat{s} + \cancel{V_n}^{0}\hat{n} \ \therefore V = V_s\right) \text{—— ②}$$

from ① & ② :-

$$\left(-\frac{1}{\rho}\frac{\partial P}{\partial s} - g\frac{dz}{ds} = \frac{\partial V}{\partial t} + \frac{VdV}{ds}\right) \times ds.$$

$$\therefore \frac{\partial V}{\partial t}.ds + VdV + \frac{1}{\rho}\frac{\partial P}{\partial s}.ds + g\,dz = 0 \quad \text{— ③}$$

$$\because VdV = d(V^2/2) \quad \text{— ④}$$

$\because$ let $s = s(x, y, z)$

$$\therefore \text{absolute derivative} = ds = \frac{\partial s}{\partial x}.dx + \frac{\partial s}{\partial y}.dy + \frac{\partial s}{\partial z}.dz$$

Exact differential

$\therefore$ for $P = P(n, s)$

$$\therefore \text{absolute derivative} = dP = \frac{\partial P}{\partial s}.ds + \frac{\partial P}{\partial n}.dn$$

$\because$ ~~along N-axis, $\partial P/\partial n = 0$~~

$$\therefore dP = \frac{\partial P}{\partial s}.ds \quad \text{— ⑤}$$

from ③ ④ ⑤ :-

$$\frac{\partial V}{\partial t}.ds + d\left(\frac{V^2}{2}\right) + \frac{dP}{\rho} + d(gz) = 0 \quad (\because g \text{ is a constant})$$

considering steady flow. $(\partial V/\partial t) = 0$

$$\therefore \boxed{\frac{dP}{\rho} + d\left(\frac{V^2}{2}\right) + d(gz) = 0} \longrightarrow \text{Euler's equation.}$$

This eq. holds true for the assumptions :-

1) only pressure & gravity forces are dominant & all other forces are negligible. i.e. inviscid fluid.
2) Flow is steady.
3) The fluid is ideal & non-viscous.

→ deriving Bernoulli's equation from euler's equation :-
for an incompressible fluid, ρ is constant :-

$\therefore$ integrating Euler's equation throughout :-

$$\boxed{\frac{P}{\rho} + gz + \frac{V^2}{2} = \text{constant}} \quad \text{— ①}$$

$$\boxed{\frac{P}{\gamma} + z + \frac{V^2}{2g} = \text{constant}} \longrightarrow ②$$

① & ② are the different form of Bernoulli's equation or energy equation where each terms in ② represents the energy per unit weight of the fluid.

Hence assumptions made Bernoulli's equation:-

i) The fluid is incompressible
ii) The flow is steady & continuous
iii) The fluid is non-viscous
iv) The flow is irrotational
v) The gravity & pressure forces are only considered
vi) The equation is applicable along a stream line only
vii) The velocity is uniform over the cross-section.

→ Limitations of Bernoulli's Equation :-

i) The fluid is not ideal. Hence, due to no slip condition, the velocity at the fixed boundary is zero & it does not increase.
ii) It means, the velocity across the section is not uniform
iii) Practically, in addition to gravity & pressure forces, some forces like the viscous force is also involved in the motion of the fluid.
iv) When the fluid passes from 1 section to another, some energy addition or subtraction

→ [illegible]

→ Energy Equation / First Law of Thermodynamics

When we apply energy conservation of the 1st law of thermodynamics to a moving fluid, the statement changes to:-

$$\left\{\begin{array}{l}\text{Rate of change}\\ \text{of energy inside}\\ \text{the fluid element}\end{array}\right\} = \left\{\begin{array}{l}\text{Net flux}\\ \text{of heat}\\ \text{into the}\\ \text{element}\end{array}\right\} + \left\{\begin{array}{l}\text{Rate of work done}\\ \text{on the element due}\\ \text{to body \&}\\ \text{surface forces}\end{array}\right\}$$

i.e.

$$\frac{dE}{dt} = \frac{\delta Q}{dt} - \frac{\delta W_{net}}{dt} \quad \text{——— (I)}$$

∵ Q & W are path f^n, hence δ is used. (Inexact differential)

E is a point f^n, hence d is used

from Reynold's Transportation Theorem, we know that:-

$$\left.\frac{dN}{dt}\right|_{CM} = \frac{\partial}{\partial t}\iiint_{CV} \rho\eta \, d\forall + \iint_{C.S.} \rho\eta\vec{V}\cdot d\vec{A} \quad \text{——— (1)}$$

N = extensive property = E = m e. ——— (2)

where e = specific energy

= energy per unit mass

$= u + \frac{V^2}{2} + gz$ (Bernoulli's eq (1))

where u = internal energy per unit mass

$V^2/2$ = kinetic energy per unit mass

gz = Potential energy per unit mass

$\eta = \frac{E}{m} = e$ = extensive property per unit mass ——— (3)

∴ from (1) (2) (3)

$$\left.\frac{dE}{dt}\right|_{CM} = \frac{\partial}{\partial t}\iiint_{CV} \rho e \, d\forall + \iint_{C.S.} \rho e\vec{V}\cdot d\vec{A} \quad \text{——— (II)}$$

also.

$$\frac{\delta Q}{dt} = \frac{\delta Q}{dm} \cdot \frac{dm}{dt} = \dot{m} \cdot \frac{\delta Q}{dm} = \dot{m} q \quad \text{—— (III)}$$

also.

$W_{net} = W_{sh} + W_{press} + W_{shear} + \cancel{W_{other}}^{\,0}$ = net work done on element

$\delta W_s = W_{shaft}$ —— (i)

$\delta W_{press} = W_{pressure\ diff} = \iint_{cs} P\, d\vec{A} \cdot d\vec{s}$ —— (ii)

$\delta W_{shear} = (\vec{\vec{\tau}} \times d\vec{A}) \cdot \vec{V} = W_{shear\ force}$.

where τ = tensor (meaning has more than 1 direction hence 2 arrows on its head)

also $\vec{\vec{\tau}} \times d\vec{A}$ = vector as a Tensor × vector = vector.

$\therefore W_{shear} = 0$ if a) $\tau = 0$ for inviscid flow, b) $d\vec{A} \perp^r \vec{V}$ —— (iii)

$\therefore \delta W_{net} = \delta W_s + \delta W_{press} + \delta W_{shear}$

$$= \delta W_s + \iint_{cs} P\, dA \cdot d\vec{s} + 0$$

$$\therefore \frac{\delta W_{net}}{dt} = \frac{\delta w_s}{dt} + \iint_{cs} P\, d\vec{A} \cdot \frac{d\vec{s}}{dt} = \frac{\delta w_s}{dt} + \iint_{cs.} P\, d\vec{A} \cdot \vec{V}$$

$$= \frac{\delta w_s}{dm} \cdot \frac{dm}{dt} + \iint_{cs} P\, d\vec{A} \cdot \frac{d\vec{s}}{dt}$$

$$\frac{\delta W_{net}}{dt} = \dot{m} w_s + \iint_{c.s.} P\, d\vec{A} \cdot \vec{V} \quad \text{—— (IV)}$$

from (I) (II) (III) (IV) :-

$$\frac{\partial}{\partial t} \iiint_{cv} \rho e\, d\forall + \iint_{cs} \rho e \vec{V} \cdot d\vec{A} = \dot{m} q - \dot{m} w_s - \iint_{c.s} P\, d\vec{A} \cdot \vec{V}$$

for steady flow $\frac{\partial}{\partial t} \iiint_{c.v} \rho e\, d\forall = 0$

$$\therefore \dot{m}q = \dot{m}w_s + \iint_{c.s.} \rho e \vec{V}\cdot d\vec{A} + \iint_{c.s.} p\, d\vec{A}\cdot\vec{V}$$

~~assuming properties at a section are uniform~~

$\therefore v = 1/\rho$ = specific volume.

$\therefore \rho v = 1$ & $\because \vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

$$\therefore \dot{m}q = \dot{m}w_s + \iint_{c.s.} \rho e \vec{V}\cdot d\vec{A} + \iint_{c.s.} p\rho v(\vec{V}\cdot d\vec{A})$$

$$\therefore q = w_s + \frac{1}{\dot{m}}\left[\iint_{c.s.} e(\rho\vec{V}\cdot d\vec{A}) + \iint_{c.s.} pv(\rho\vec{V}\cdot d\vec{A})\right]$$

$$q = w_s + \frac{1}{\dot{m}}\iint_{c.s.} e\,d\dot{m} + \frac{1}{\dot{m}}\iint_{c.s.} pv\,d\dot{m}$$

$$q = w_s + \frac{1}{\dot{m}}\iint_{A1\to \text{state }1}^{A2\to \text{state }2} (e + pv)\,d\dot{m}$$ } assuming properties at a section are uniform.

$$\therefore q = w_s + \left[e + \frac{p}{\rho}\right]_{A1}^{A2}$$

$$q = w_s + \left(e_2 + \frac{p_2}{\rho_2}\right) - \left(e_1 + \frac{p_1}{\rho_1}\right)$$

$$\therefore q + \left(e_1 + \frac{p_1}{\rho_1}\right) = w_s + \left(e_2 + \frac{p_2}{\rho_2}\right)$$

$$q + \left[u_1 + \frac{V_1^2}{2} + gz_1 + \frac{p_1}{\rho_1}\right] = w_s + \left[u_2 + \frac{V_2^2}{2} + gz_2 + \frac{p_2}{\rho_2}\right] \quad \text{—— (V)}$$

$\because$ enthalpy $= h = u + pv \Rightarrow$ & $v = 1/\rho$

$$\left.\begin{aligned} h_1 &= u_1 + \frac{p_1}{\rho_1} \\ h_2 &= u_2 + \frac{p_2}{\rho_2} \end{aligned}\right\} \longrightarrow \text{putting in (V)}$$

$$\therefore \boxed{h_1 + \frac{v_1^2}{2} + gz_1 + q = w_s + h_2 + \frac{v_2^2}{2} + gz_2}$$

$\therefore$ → deriving Bernoulli's eq. using energy conservation (or) 1st law of thermodynamics :-

$$h_1 + \frac{v_1^2}{2} + gz_1 + q = w_s + h_2 + \frac{v_2^2}{2} + gz_2$$

$$\therefore U_1 + \frac{P_1}{\rho_1} + \frac{v_1^2}{2} + gz_1 + q = w_s + U_2 + \frac{P_1}{\rho_2} + \frac{v_2^2}{2} + gz_2$$

$$\therefore \frac{P_1}{\rho_1} + \frac{v_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{v_2^2}{2} + gz_2 + \left[w_s - q + U_2 - U_1\right]$$

$$\frac{P_1}{\rho_1} + \frac{v_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{v_2^2}{2} + gz_2 + \left[\cancel{\Delta U}^{-w_s} - q + w_s\right]$$

$$\therefore \frac{P_1}{\rho_1} + \frac{v_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{v_2^2}{2} + gz_2$$

$$\therefore \boxed{\frac{P}{\rho} + \frac{V^2}{2} + gZ = \text{constent}}$$