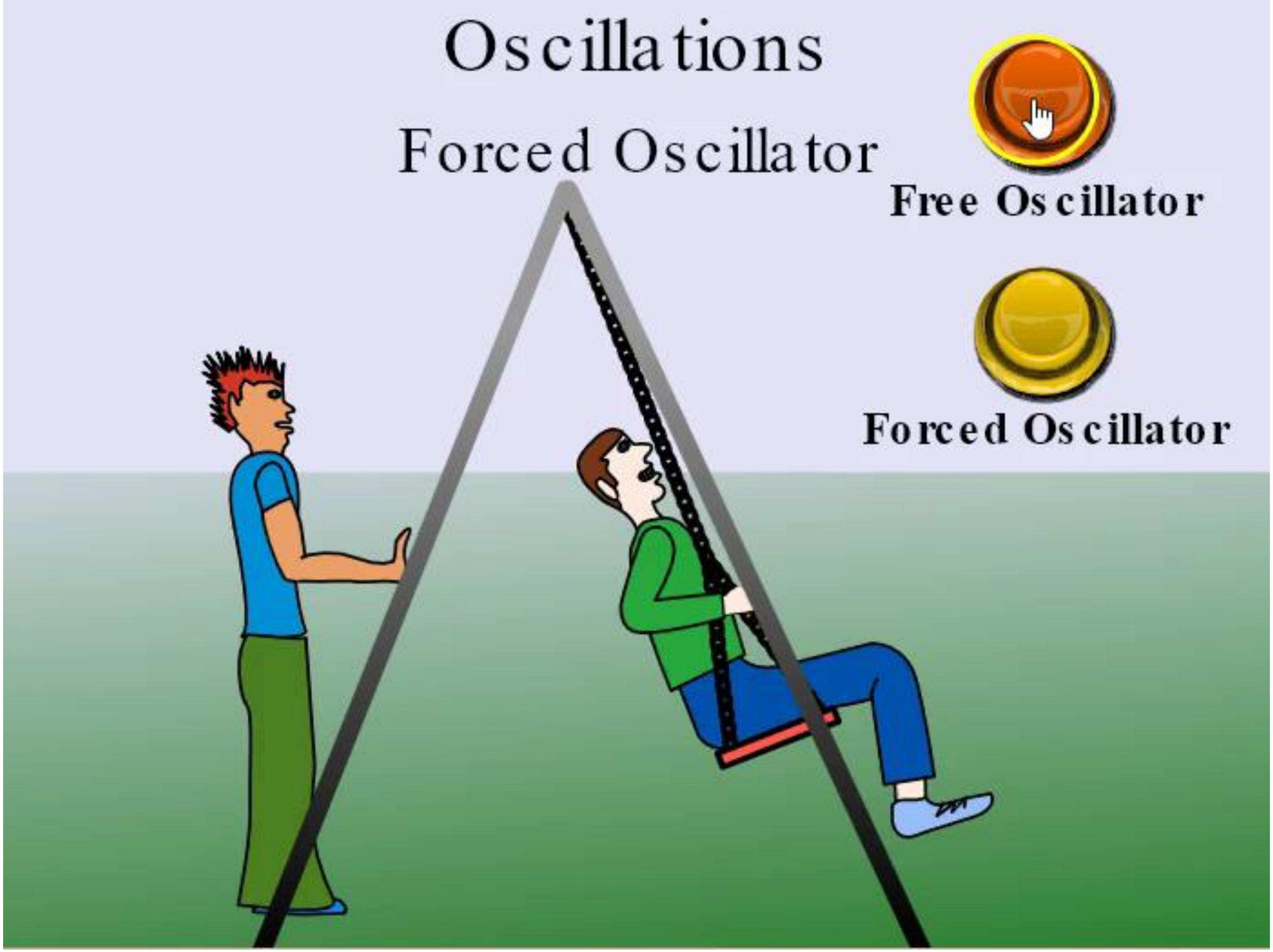
Forced Oscillator



Damped Vibration

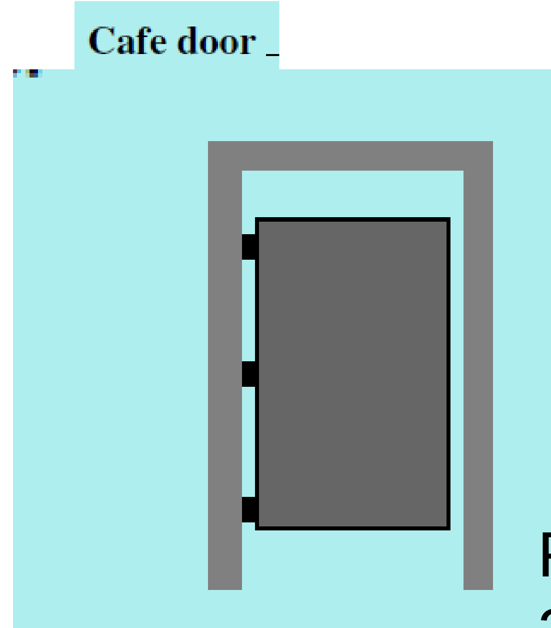


Figure. 2

A cafe door on three hinges with damper in the lower hinge

Figure 1. In order to counteract dampening forces, this dad needs to keep pushing the swing. (credit:

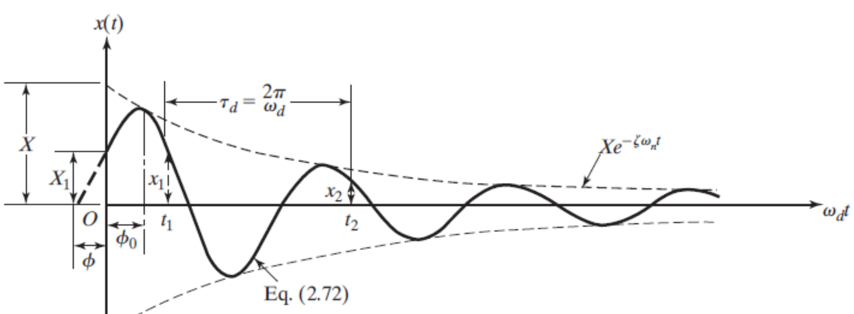
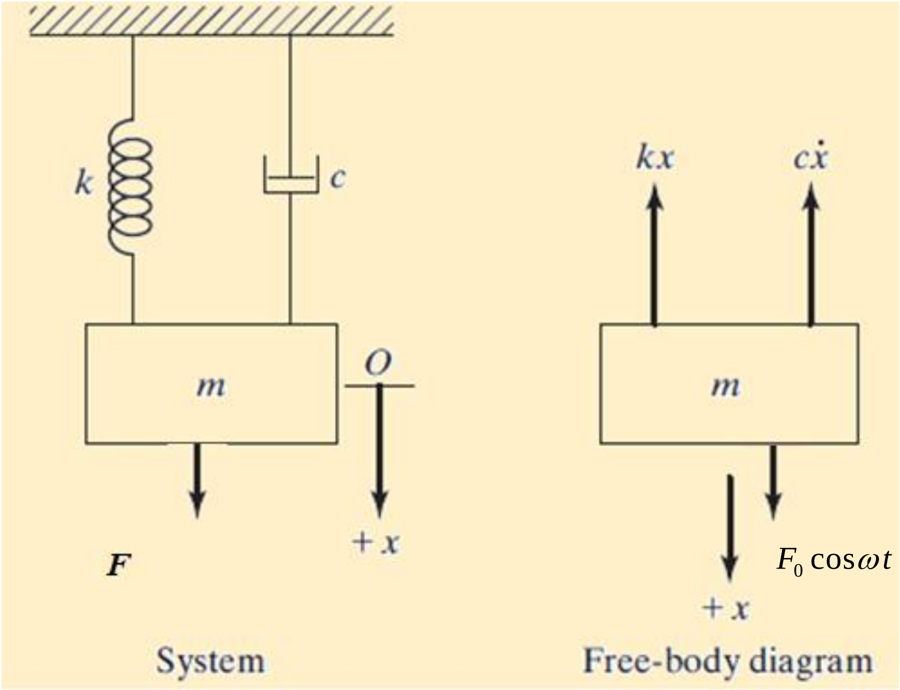


Figure 3. Door Closure: over dampe

Forced Vibrations

- Occurs when object is vibrated at frequency other than natural frequency of object
- We continue the discussion of the last section, and now consider the presence of a periodic external force:



$$m\ddot{x}(t) + C\dot{x}(t) + kx(t) = F_0\, cos\, \omega\, t$$

Forced Vibrations with Damping

- Consider the equation below for damped motion and external forcing function $F_0 \cos \omega t$.
- $m\ddot{x} + C\dot{x} + kx = F_0 \cos \omega t$

  normalize the equation of motion

  $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 \cos \omega t, \;\; f_0 = F_0 / m$

- The general solution of this equation has the form

$$x(t) = x_h + x_p$$

where the general solution of the homogeneous equation is x_h

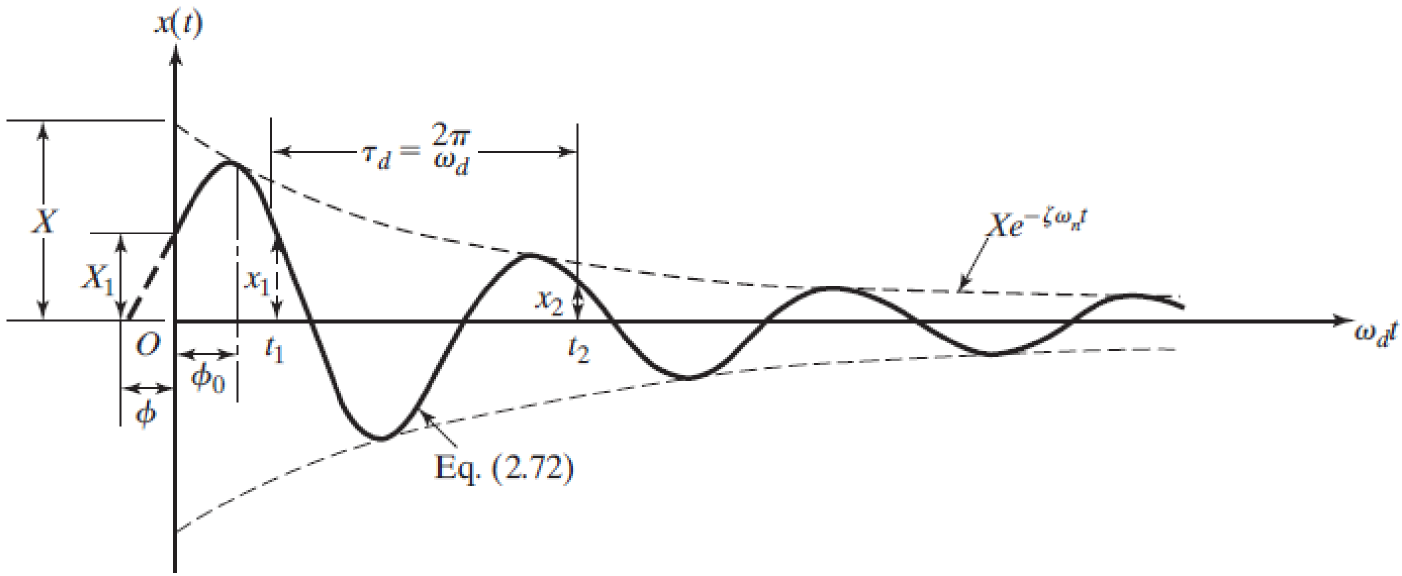
$$\ddot{x}_h + 2\zeta\omega_n \dot{x}_h + \omega_n^2 x_h = 0$$

and the particular solution of the nonhomogeneous equation is x_p

$$\ddot{x}_p + 2\zeta\omega_n \dot{x}_p + \omega_n^2 x_p = f_0 \cos \omega t$$

Recall the homogeneous solution of the underdamped system

$$x_h = Ce^{-\zeta\omega_n t}\cos\left(\omega_d t - \phi\right) \text{ or } x_h = e^{-\zeta\omega_n t}\left(A_1 \sin\omega_d t + A_2 \cos\omega_d t\right)$$



Let us focus on the particular solution of

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

normalize the equation of motion

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos \omega t, \quad f_0 = F_0 / m$$

$$f(t) = f_0 \operatorname{Re}\left[e^{i\omega t}\right]$$

$\therefore$ solve for $z(t)$ from $\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = f_0 e^{i\omega t}$

and the solution is the real part of $z(t)$; $x(t) = \operatorname{Re}\left[z(t)\right]$

<u>Assume</u> the solution to have the same form as the forcing function

$z(t) = Z(i\omega)e^{i\omega t}$ (same frequency as the input w/ different mag. and phase)

$$\left(-\omega^2 + i2\zeta\omega\omega_n + \omega_n^2\right)Z(i\omega)e^{i\omega t} = f_0 e^{i\omega t}$$

$$Z(i\omega) = \frac{f_0}{\omega_n^2 - \omega^2 + i2\zeta\omega\omega_n} = \frac{f_0 / \omega_n^2}{1 - (\omega / \omega_n)^2 + i2\zeta\omega / \omega_n}$$

$$= \frac{F_0}{k\left[1 - (\omega / \omega_n)^2 + i2\zeta\omega / \omega_n\right]}$$

$$z(t)=\frac{F_0}{k\left[1-r^2+i2\zeta r\right]}e^{i\omega t}=H(i\omega)F_0e^{i\omega t}$$

$$\therefore x(t)=\mathrm{Re}\left[\frac{F_0}{k\left[1-r^2+i2\zeta r\right]}e^{i\omega t}\right],\ r=\omega/\omega_n$$

If $H(i\omega)=\dfrac{1}{k\left[1-r^2+i2\zeta r\right]}=\left|H(i\omega)\right|e^{i\theta}$ is the frequency response

$$\therefore x(t)=F_0\left|H(i\omega)\right|\cos(\omega t+\theta)$$

where $\left|H(i\omega)\right|=\dfrac{1}{k\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}=$ magnitude

$\theta=\tan^{-1}\dfrac{-2\zeta r}{1-r^2}=$ phase

The system modulates the harmonic input by

the magnitude $\left|H(i\omega)\right|$ and phase $\measuredangle H(i\omega)$

$$\therefore x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi) + F_0\left|H(i\omega)\right|\cos(\omega t + \theta)$$

$$\text{or } x(t) = e^{-\zeta\omega_n t}\left(A_1 \sin\omega_d t + A_2 \cos\omega_d t\right) + F_0\left|H(i\omega)\right|\cos(\omega t + \theta)$$

The initial conditions will be used to determine C, ϕ or A_1, A_2

They will be <u>different</u> from those of free response

because the <u>transient term now is partly due to the excitation force</u>

and partly due to the initial conditions

Now

$$\lim_{t \to \infty} x_h(t) = 0$$

Thus $x_h(t)$ is called the **transient solution**.

Note however that after sufficient time the system oscillate with same frequency as forcing function. For this reason, $x_p(t)$ is called the **steady-state solution**, or **forced response**.

Amplitude Analysis of Forced Response

- The amplitude R of the steady state solution

$$R = \frac{F_0}{k\sqrt{(1-r^2)^2 + (2\varsigma r)^2}}$$

depends on the driving frequency ω. For low-frequency excitation we have

$$\lim_{\omega \to 0} R = \lim_{\omega \to 0} \frac{F_0}{k\sqrt{(1-r^2)^2 + (2\varsigma r)^2}} = \frac{F_0}{k} \qquad \because r = \frac{\omega}{\omega_n}$$

Note that F_0/k is the static displacement of the spring produced by force F_0.

- For high frequency excitation,

$$\lim_{\omega \to \infty} R = \lim_{\omega \to \infty} \frac{F_0}{k\sqrt{(1-r^2)^2 + (2\varsigma r)^2}} = 0$$

Maximum Amplitude of Forced Response

- Thus $$\lim_{\omega\to 0} R = F_0/k, \quad \lim_{\omega\to\infty} R = 0$$

- At an intermediate value of ω, the amplitude R may have a maximum value. To find this frequency ω, differentiate the denominator of R and set the result equal to zero.

$$D_n = (1-r^2)^2 + (2\zeta r)^2$$

$$\frac{dD_n}{dr} = -4r(1-r^2) + 8\zeta^2 r = -(1-r^2) + 2\zeta^2 = 0$$

$$r_{\max}{}^2 = 1 - 2\zeta^2$$

$$\therefore \omega_{\max} = \omega_n \sqrt{1-2\zeta^2}$$

Note $\omega_{max} < \omega_n$, and ω_{max} is close to ω_n for small *C*. The maximum value of *R* is

$$R_{\max} = \frac{F_0}{k 2\zeta \sqrt{2(1-\zeta^2)}}$$

Maximum Amplitude for Imaginary ω_{max}

- We have

and
$$\omega_{\max}^2 = \omega_n^2\left(1-2\zeta^2\right)$$

$$R_{\max} = \frac{F_0}{k2\zeta\sqrt{2(1-\zeta^2)}}$$

Peak occurs when $\zeta \le \frac{1}{\sqrt{2}}$,this is called Amplitude Resonance

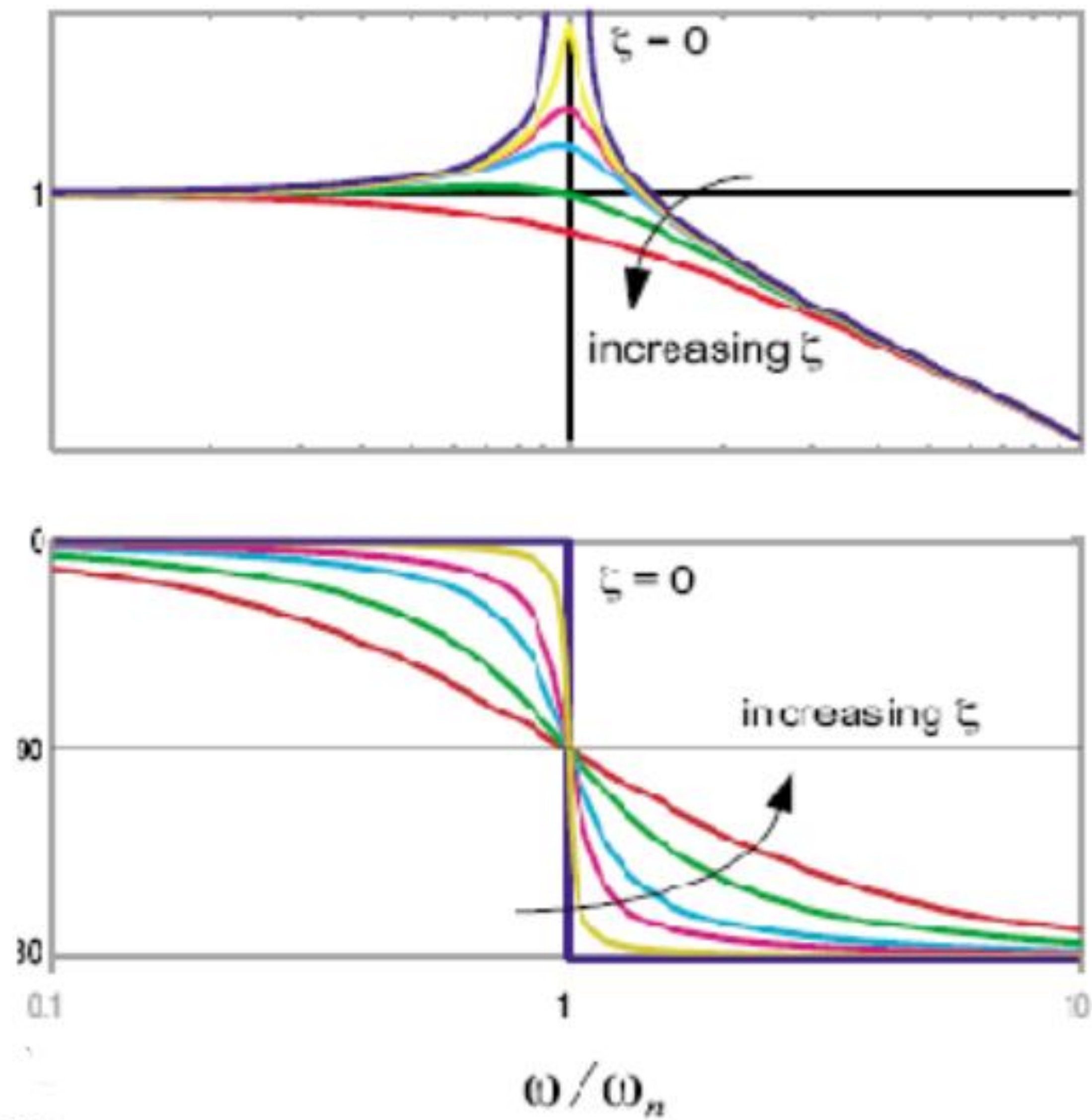
If , $\zeta > \frac{1}{\sqrt{2}}$ then $\omega_{\max}$ is imaginary. In this case, $R_{\max} = F_0/k$,

which occurs at $\omega = 0$, and R is a monotone decreasing function of ω.

$$\left|H(i\omega)\right| = \frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}$$

Frequency response plot
(Bode diagram)

$$\theta = \tan^{-1}\left(\frac{-2\zeta r}{1-r^2}\right)$$



Amplitude Resonance

Resonance is defined to be the vibration response at $\omega = \omega_n$, regardless whether the damping ratio is zero. At this point, the phase shift of the response is $-\pi/2$.

The resonant frequency will give the peak amplitude for the response only when ζ =0. For $0 < \zeta < 1/\sqrt{2}$,the peak amplitude will be at $\omega = \omega_n \sqrt{1-2\zeta^2}$, slightly before ω_n. For $\zeta \geq 1/\sqrt{2}$, there is no peak but the max. value of the output is equal to the input for the dc signal (of course, for this normalized transfer function).

Velocity Resonance

$$Z(t) = \frac{F_0 e^{i\omega t}}{k[(1-r^2)+(i2\varsigma r)]},$$

$$v = \frac{dz}{dt} = \frac{F_0 i\omega e^{i\omega t}}{k[(1-r^2)+(i2\varsigma r)} = \frac{F_0 e^{i\omega t}}{k[-i(\frac{1}{\omega}-\frac{\omega}{\omega_n^2})+(2\varsigma/\omega_n)]}$$

$$= \frac{F_0 e^{i(\omega t-\alpha)}}{k[(\frac{1}{\omega}-\frac{\omega}{\omega_n^2})^2+(2\varsigma/\omega_n)^2]^{1/2}}$$

So

$$v = \frac{F_0}{k[(\frac{1}{\omega}-\frac{\omega}{\omega_n^2})^2+(2\varsigma/\omega_n)^2]^{1/2}}\cos(\omega t-\alpha) = \frac{F_0}{|z_m|}\cos(\omega t-\alpha)$$

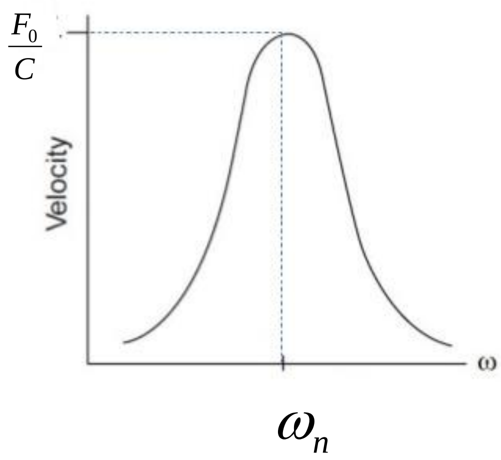
Where

$$z_m = [(2\varsigma k/\omega_n) - ik(\frac{1}{\omega}-\frac{\omega}{\omega_n^2})]$$

Velocity Resonance

$$v = \frac{F_0}{k[(\frac{1}{\omega} - \frac{\omega}{\omega_n^2})^2 + (2\varsigma / \omega_n)^2]} \cos(\omega t - \alpha)$$

Velocity v of Forced Oscillator versus Driving Frequency ω



Then velocity Amplitude

$$V = \frac{F_0}{k[(\frac{1}{\omega} - \frac{\omega}{\omega_n^2})^2 + (2\varsigma / \omega_n)^2]^{\frac{1}{2}}}$$

V attains its maximum value

$$V_{\max} = \frac{F_0}{(k2\varsigma / \omega_n)} = \frac{F_0}{C} \qquad \because \mathrm{C} = 2\zeta m \omega_n$$

when $$(\frac{1}{\omega} - \frac{\omega}{\omega_n^2}) = 0 \Rightarrow \omega = \omega_n$$

Power relation in forced vibration and resonance

The power of the driver is the rate at which it does work. In steady –state, the instantaneous power of the driver in forced vibration is

$$P = (F_0 \cos \omega t).\frac{dx}{dt} = \frac{F_0^2}{|z_m|} \cos \omega t . \cos(\omega t - \alpha),$$

Hence

$$P = \frac{F_0^2}{|z_m|}\left(\cos^2 \omega t \cos \alpha + \sin \omega t \cos \omega t \sin \alpha\right)$$

$$= \frac{F_0^2}{|z_m|}\left(\cos^2 \omega t \cos \alpha + \frac{1}{2} \sin 2\omega t \sin \alpha\right),$$

The average power over a complete cycle is $P_{av} = \frac{1}{T}\int_0^T P dt$

Since $\langle \cos^2 \omega t \rangle = \dfrac{1}{2}$ and $\langle \sin 2\omega t \rangle = 0$

Then

$$P_{av} = \frac{F_0^{\,2}}{2|z_m|}\cos\alpha = \frac{F_0}{\sqrt{2}} . \frac{F_0}{\sqrt{2}.|z_m|} .\cos\alpha = F_{rms} . v_{rms} . \cos\alpha$$

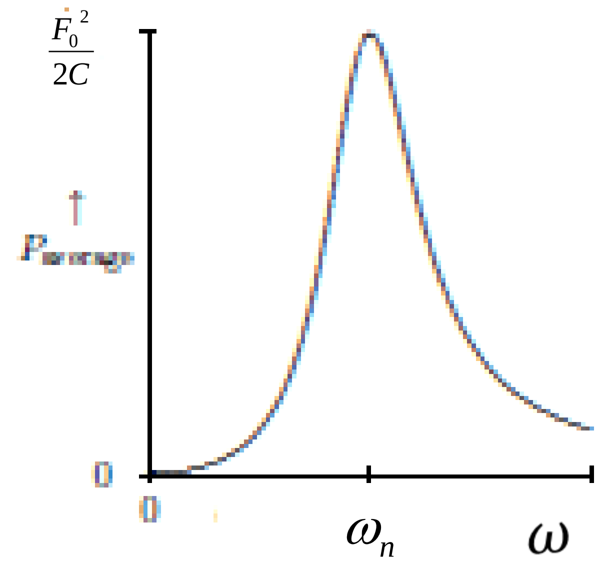
Power factor

Now at resonance

$\omega = \omega_n$, so $|z_m| = C$

Then average power at resonaces is

$$(P_{av})_r = \frac{F_0^{\,2}}{2C}$$



$$P_{av} = \frac{F_0^{\,2}}{2\left|z_m\right|}\cos\alpha = \frac{F_0^{\,2}}{2\left|z_m\right|}\cdot\frac{C}{\left|z_m\right|} = \frac{F_0^{\,2}C}{2\left|z_m\right|^2},$$

$$\frac{P_{av}}{(P_{av})_r} = \frac{C^2}{k^2[(\frac{1}{\omega}-\frac{\omega}{\omega_n{}^2})^2+(2\zeta/\omega_n)^2]}$$

$$= \frac{C^2}{[m^2k^2/m^2(\frac{1}{\omega}-\frac{\omega}{\omega_n{}^2})^2+(2\zeta k/\omega_n)^2]}$$

$$= \frac{C^2}{[m^2\omega_n{}^2(\frac{\omega_n}{\omega}-\frac{\omega}{\omega_n})^2+(C)^2]}$$

$$= \frac{C^2}{[m^2\omega_n{}^2\Delta^2+(C)^2]}$$

Where $\Delta = \left(\frac{\omega_n}{\omega}-\frac{\omega}{\omega_n}\right)$

$$\frac{P_{av}}{(P_{av})_r} = \frac{C^2}{[m^2 \omega_n{}^2 \Delta^2 + (C)^2]}$$

now we define the bandwidth/ half power frequency of the oscilation by the relation

$$\frac{P_{av}}{(P_{av})_r} = \frac{C^2}{[m^2 \omega_n{}^2 \Delta^2 + (C)^2]} = \frac{1}{2}$$

So at half power frequencies

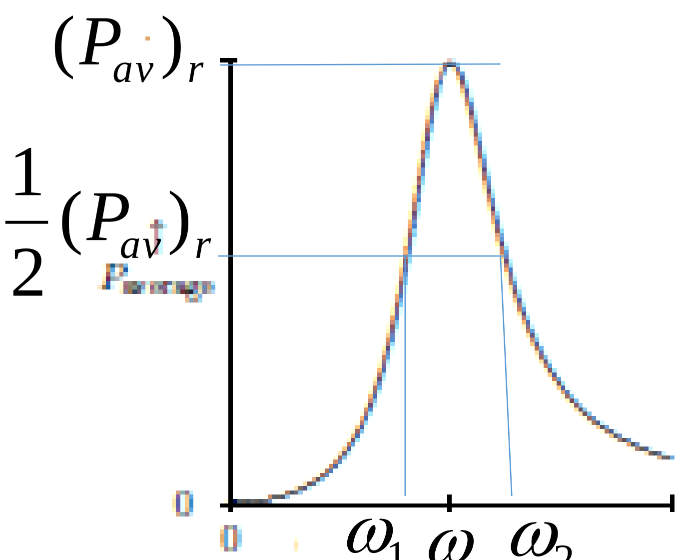
$$\frac{C^2}{[m^2 \omega_n{}^2 \Delta^2 + (C)^2]} = \frac{1}{2}$$

$$or, \quad m^2 \omega_n{}^2 \Delta^2 + (C)^2 = 2C^2$$

$$or, \quad m^2 \omega_n{}^2 \Delta^2 = C^2$$

$$or, \quad \omega_n \Delta = \pm C / m$$

$$or, \quad \omega^2 \mp C / m\omega - \omega_n^2 = 0$$



Thus accepting only positive roots, we have

$$\omega_1 = \sqrt{\frac{C^2}{4m^2} + \omega_n^2} - \frac{C}{2m}$$

$$\omega_2 = \sqrt{\frac{C^2}{4m^2} + \omega_n^2} + \frac{C}{2m}$$

Sharpness of resonance $S_r = \frac{1}{|\Delta|} = \frac{\omega_n m}{C} = \frac{\omega_n}{\omega_2 - \omega_1}$

The frequency range $(\omega_2 - \omega_1)$ is called the *half-power bandwidth/bandwidth.*

Q factor is defined as

$$Q = 2\pi . \frac{\text{maximum kinetic energy at resonance}}{\text{energy dissipated per cycle at resonance}}$$

$$\text{maximum kinetic energy at resonance} = \frac{1}{2} m V_{\max}^2 = \frac{m F_0^2}{2C^2}$$

$$\text{energy dissipated } \frac{\mathrm{dw}}{\mathrm{dt}} = \mathrm{C}\left(\frac{dx}{dt}\right).\frac{dx}{dt} = C\left(\frac{dx}{dt}\right)^2$$

$$Now \;\; v = \frac{F_0}{|z_m|}\cos(\omega t - \alpha), Therefore$$

$$\text{energy dissipated} = C\left(\frac{dx}{dt}\right)^2 = C\,\frac{F_0^{\;2}}{|z_m|^2}\cos^2(\omega t - \alpha)$$

$$\textit{avergae} \text{ of enrgy dissipated in a complete cycle at resonance is} = C.\frac{F_0^{\;2}}{|C|^2}.\frac{1}{2}$$

$$\textit{total energy} \text{ dissipated at resonance in a complete cycle is} = \frac{F_0^{\;2}}{2C}.T = \frac{F_0^{\;2}}{2C}.\frac{2\pi}{\omega_n}$$

$$Q = 2\pi.\frac{\dfrac{m F_0^2}{2C^2}}{\dfrac{F_0^{\;2}}{C}.\dfrac{\pi}{\omega_n}} = \frac{\omega_n m}{C} = S_r$$