SUPPLEMENTARY PROBLEMS

42. If $\phi = 2xz^4 - x^2y$, find $\nabla\phi$ and $|\nabla\phi|$ at the point $(2,-2,-1)$. *Ans.* $10\mathbf{i} - 4\mathbf{j} - 16\mathbf{k}$, $2\sqrt{93}$

43. If $\mathbf{A} = 2x^2\mathbf{i} - 3yz\mathbf{j} + xz^2\mathbf{k}$ and $\phi = 2z - x^3y$, find $\mathbf{A}\cdot\nabla\phi$ and $\mathbf{A}\times\nabla\phi$ at the point $(1,-1,1)$.
Ans. 5, $7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$

44. If $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find (a) $\nabla(F+G)$ and (b) $\nabla(FG)$ at the point $(1,0,-2)$.
Ans. (a) $-4\mathbf{i} + 9\mathbf{j} + \mathbf{k}$, (b) $-8\mathbf{j}$

45. Find $\nabla|\mathbf{r}|^3$. *Ans.* $3r\mathbf{r}$

46. Prove $\nabla f(r) = \dfrac{f'(r)\,\mathbf{r}}{r}$.

47. Evaluate $\nabla(3r^2 - 4\sqrt{r} + \dfrac{6}{\sqrt[3]{r}})$. *Ans.* $(6 - 2r^{-3/2} - 2r^{-7/3})\,\mathbf{r}$

48. If $\nabla U = 2r^4\mathbf{r}$, find U. *Ans.* $r^6/3$ + constant

49. Find $\phi(r)$ such that $\nabla\phi = \dfrac{\mathbf{r}}{r^5}$ and $\phi(1) = 0$. *Ans.* $\phi(r) = \dfrac{1}{3}(1 - \dfrac{1}{r^3})$

50. Find $\nabla\psi$ where $\psi = (x^2 + y^2 + z^2)\,e^{-\sqrt{x^2+y^2+z^2}}$. *Ans.* $(2 - r)\,e^{-r}\,\mathbf{r}$

51. If $\nabla\phi = 2xyz^3\,\mathbf{i} + x^2z^3\,\mathbf{j} + 3x^2yz^2\,\mathbf{k}$, find $\phi(x,y,z)$ if $\phi(1,-2,2) = 4$. *Ans.* $\phi = x^2yz^3 + 20$

52. If $\nabla\psi = (y^2 - 2xyz^3)\mathbf{i} + (3 + 2xy - x^2z^3)\mathbf{j} + (6z^3 - 3x^2yz^2)\mathbf{k}$, find ψ.
Ans. $\psi = xy^2 - x^2yz^3 + 3y + (3/2)\,z^4$ + constant

53. If U is a differentiable function of x,y,z, prove $\nabla U\cdot d\mathbf{r} = dU$.

54. If F is a differentiable function of x,y,z,t where x,y,z are differentiable functions of t, prove that
$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla F\cdot\frac{d\mathbf{r}}{dt}$$

55. If $\mathbf{A}$ is a constant vector, prove $\nabla(\mathbf{r}\cdot\mathbf{A}) = \mathbf{A}$.

56. If $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, show that $d\mathbf{A} = (\nabla A_1\cdot d\mathbf{r})\mathbf{i} + (\nabla A_2\cdot d\mathbf{r})\mathbf{j} + (\nabla A_3\cdot d\mathbf{r})\mathbf{k}$.

57. Prove $\nabla(\dfrac{F}{G}) = \dfrac{G\nabla F - F\nabla G}{G^2}$ if $G \neq 0$.

58. Find a unit vector which is perpendicular to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(1,2,5)$. *Ans.* $\dfrac{2\mathbf{i} + 4\mathbf{j} - \mathbf{k}}{\pm\sqrt{21}}$

59. Find the unit outward drawn normal to the surface $(x-1)^2 + y^2 + (z+2)^2 = 9$ at the point $(3,1,-4)$.
Ans. $(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$

60. Find an equation for the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point $(1,-3,2)$.
Ans. $2x - y - 3z + 1 = 0$

61. Find equations for the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(2,-1,5)$.
Ans. $4x - 2y - z = 5$, $\dfrac{x-2}{4} = \dfrac{y+1}{-2} = \dfrac{z-5}{-1}$ or $x = 4t + 2$, $y = -2t - 1$, $z = -t + 5$

62. Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
Ans. $376/7$

63. Find the directional derivative of $P = 4e^{2x-y+z}$ at the point $(1,1,-1)$ in a direction toward the point $(-3,5,6)$. *Ans.* $-20/9$

64. In what direction from the point (1,3,2) is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum? *Ans.* In the direction of the vector $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, $2\sqrt{14}$

65. Find the values of the constants a,b,c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1,2,−1) has a maximum of magnitude 64 in a direction parallel to the z axis. *Ans.* $a = 6$, $b = 24$, $c = -8$

66. Find the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1,−2,1).
Ans. $\text{arc cos } \dfrac{3}{\sqrt{14}\sqrt{21}} = \text{arc cos } \dfrac{\sqrt{6}}{14} = 79°55'$

67. Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1,−1,2). *Ans.* $a = 5/2$, $b = 1$

68. (a) Let u and v be differentiable functions of x,y and z. Show that a necessary and sufficient condition that u and v are functionally related by the equation $F(u,v) = 0$ is that $\nabla u \times \nabla v = \mathbf{0}$.
(b) Determine whether $u = \text{arc tan}\, x + \text{arc tan}\, y$ and $v = \dfrac{x+y}{1-xy}$ are functionally related.
Ans. (b) Yes ($v = \tan u$)

69. (a) Show that a necessary and sufficient condition that $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ be functionally related through the equation $F(u,v,w) = 0$ is $\nabla u \cdot \nabla v \times \nabla w = 0$.
(b) Express $\nabla u \cdot \nabla v \times \nabla w$ in determinant form. This determinant is called the Jacobian of u,v,w with respect to x,y,z and is written $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ or $J\left(\dfrac{u,v,w}{x,y,z}\right)$.
(c) Determine whether $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $w = xy + yz + zx$ are functionally related.

Ans. (b) $\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$ (c) Yes ($u^2 - v - 2w = 0$)

70. If $\mathbf{A} = 3xyz^2\,\mathbf{i} + 2xy^3\,\mathbf{j} - x^2yz\,\mathbf{k}$ and $\phi = 3x^2 - yz$, find (a) $\nabla\cdot\mathbf{A}$, (b) $\mathbf{A}\cdot\nabla\phi$, (c) $\nabla\cdot(\phi\mathbf{A})$, (d) $\nabla\cdot(\nabla\phi)$, at the point (1,−1,1). *Ans.* (a) 4, (b) −15, (c) 1, (d) 6

71. Evaluate div $(2x^2z\,\mathbf{i} - xy^2z\,\mathbf{j} + 3yz^2\,\mathbf{k})$. *Ans.* $4xz - 2xyz + 6yz$

72. If $\phi = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$, find $\nabla^2\phi$. *Ans.* $6z + 24xy - 2z^3 - 6y^2z$

73. Evaluate $\nabla^2(\ln r)$. *Ans.* $1/r^2$

74. Prove $\nabla^2 r^n = n(n+1)r^{n-2}$ where n is a constant.

75. If $\mathbf{F} = (3x^2y - z)\mathbf{i} + (xz^3 + y^4)\mathbf{j} - 2x^3z^2\mathbf{k}$, find $\nabla(\nabla\cdot\mathbf{F})$ at the point (2,−1,0). *Ans.* $-6\mathbf{i} + 24\mathbf{j} - 32\mathbf{k}$

76. If $\boldsymbol{\omega}$ is a constant vector and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, prove that div $\mathbf{v} = 0$.

77. Prove $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi$.

78. If $U = 3x^2y$, $V = xz^2 - 2y$ evaluate grad [(grad U)·(grad V)]. *Ans.* $(6yz^2 - 12x)\mathbf{i} + 6xz^2\mathbf{j} + 12xyz\,\mathbf{k}$

79. Evaluate $\nabla\cdot(r^3\mathbf{r})$. *Ans.* $6r^3$

80. Evaluate $\nabla\cdot[r\nabla(1/r^3)]$. *Ans.* $3r^{-4}$

81. Evaluate $\nabla^2[\nabla\cdot(\mathbf{r}/r^2)]$. *Ans.* $2r^{-4}$

82. If $\mathbf{A} = \mathbf{r}/r$, find grad div $\mathbf{A}$. *Ans.* $-2r^{-3}\,\mathbf{r}$

83. (a) Prove $\nabla^2 f(r) = \dfrac{d^2f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr}$. (b) Find $f(r)$ such that $\nabla^2 f(r) = 0$.
Ans. $f(r) = A + B/r$ where A and B are arbitrary constants.

84. Prove that the vector $\mathbf{A} = 3y^4z^2\,\mathbf{i} + 4x^3z^2\,\mathbf{j} - 3x^2y^2\,\mathbf{k}$ is solenoidal.

85. Show that $\mathbf{A} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$ is not solenoidal but $\mathbf{B} = xyz^2\mathbf{A}$ is solenoidal.

86. Find the most general differentiable function $f(r)$ so that $f(r)\mathbf{r}$ is solenoidal.
Ans. $f(r) = C/r^3$ where C is an arbitrary constant.

87. Show that the vector field $\mathbf{V} = \dfrac{-x\,\mathbf{i} - y\,\mathbf{j}}{\sqrt{x^2+y^2}}$ is a "sink field". Plot and give a physical interpretation.

88. If U and V are differentiable scalar fields, prove that $\nabla U \times \nabla V$ is solenoidal.

89. If $\mathbf{A} = 2xz^2\,\mathbf{i} - yz\,\mathbf{j} + 3xz^3\,\mathbf{k}$ and $\phi = x^2yz$, find
(a) $\nabla\times\mathbf{A}$, (b) curl $(\phi\mathbf{A})$, (c) $\nabla\times(\nabla\times\mathbf{A})$, (d) $\nabla[\mathbf{A}\cdot\text{curl }\mathbf{A}]$, (e) curl grad $(\phi\mathbf{A})$ at the point (1,1,1).
Ans. (a) $\mathbf{i}+\mathbf{j}$, (b) $5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$, (c) $5\mathbf{i} + 3\mathbf{k}$, (d) $-2\mathbf{i} + \mathbf{j} + 8\mathbf{k}$, (e) $\mathbf{0}$

90. If $F = x^2yz$, $G = xy - 3z^2$, find (a) $\nabla[(\nabla F)\cdot(\nabla G)]$, (b) $\nabla\cdot[(\nabla F)\times(\nabla G)]$, (c) $\nabla\times[(\nabla F)\times(\nabla G)]$.
Ans. (a) $(2y^2z + 3x^2z - 12xyz)\mathbf{i} + (4xyz - 6x^2z)\mathbf{j} + (2xy^2 + x^3 - 6x^2y)\mathbf{k}$
(b) 0
(c) $(x^2z - 24xyz)\mathbf{i} - (12x^2z + 2xyz)\mathbf{j} + (2xy^2 + 12yz^2 + x^3)\mathbf{k}$

91. Evaluate $\nabla\times(\mathbf{r}/r^2)$. *Ans.* $\mathbf{0}$

92. For what value of the constant a will the vector $\mathbf{A} = (axy - z^3)\,\mathbf{i} + (a-2)x^2\,\mathbf{j} + (1-a)xz^2\,\mathbf{k}$ have its curl identically equal to zero? *Ans.* $a = 4$

93. Prove curl $(\phi \text{ grad } \phi) = \mathbf{0}$.

94. Graph the vector fields $\mathbf{A} = x\,\mathbf{i} + y\,\mathbf{j}$ and $\mathbf{B} = y\,\mathbf{i} - x\,\mathbf{j}$. Compute the divergence and curl of each vector field and explain the physical significance of the results obtained.

95. If $\mathbf{A} = x^2z\,\mathbf{i} + yz^3\,\mathbf{j} - 3xy\,\mathbf{k}$, $\mathbf{B} = y^2\,\mathbf{i} - yz\,\mathbf{j} + 2x\,\mathbf{k}$ and $\phi = 2x^2 + yz$, find
(a) $\mathbf{A}\cdot(\nabla\phi)$, (b) $(\mathbf{A}\cdot\nabla)\phi$, (c) $(\mathbf{A}\cdot\nabla)\mathbf{B}$, (d) $\mathbf{B}(\mathbf{A}\cdot\nabla)$, (e) $(\nabla\cdot\mathbf{A})\mathbf{B}$.
Ans. (a) $4x^3z + yz^4 - 3xy^2$, (b) $4x^3z + yz^4 - 3xy^2$ (same as (a)),
(c) $2y^2z^3\,\mathbf{i} + (3xy^2 - yz^4)\mathbf{j} + 2x^2z\,\mathbf{k}$,
(d) the operator $(x^2y^2z\,\mathbf{i} - x^2yz^2\,\mathbf{j} + 2x^3z\,\mathbf{k})\dfrac{\partial}{\partial x} + (y^3z^3\,\mathbf{i} - y^2z^4\,\mathbf{j} + 2xyz^3\,\mathbf{k})\dfrac{\partial}{\partial y}$

$$+ (-3xy^3\,\mathbf{i} + 3xy^2z\,\mathbf{j} - 6x^2y\,\mathbf{k})\frac{\partial}{\partial z}$$

(e) $(2xy^2z + y^2z^3)\mathbf{i} - (2xyz^2 + yz^4)\mathbf{j} + (4x^2z + 2xz^3)\mathbf{k}$

96. If $\mathbf{A} = yz^2\,\mathbf{i} - 3xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$, $\mathbf{B} = 3x\,\mathbf{i} + 4z\,\mathbf{j} - xy\,\mathbf{k}$ and $\phi = xyz$, find
(a) $\mathbf{A}\times(\nabla\phi)$, (b) $(\mathbf{A}\times\nabla)\phi$, (c) $(\nabla\times\mathbf{A})\times\mathbf{B}$, (d) $\mathbf{B}\cdot\nabla\times\mathbf{A}$.
Ans. (a) $-5x^2yz^2\,\mathbf{i} + xy^2z^2\,\mathbf{j} + 4xyz^3\,\mathbf{k}$
(b) $-5x^2yz^2\,\mathbf{i} + xy^2z^2\,\mathbf{j} + 4xyz^3\,\mathbf{k}$ (same as (a))
(c) $16z^3\,\mathbf{i} + (8x^2yz - 12xz^2)\mathbf{j} + 32xz^2\,\mathbf{k}$ (d) $24x^2z + 4xyz^2$

97. Find $\mathbf{A}\times(\nabla\times\mathbf{B})$ and $(\mathbf{A}\times\nabla)\times\mathbf{B}$ at the point (1,−1,2), if $\mathbf{A} = xz^2\,\mathbf{i} + 2y\,\mathbf{j} - 3xz\,\mathbf{k}$ and $\mathbf{B} = 3xz\,\mathbf{i} + 2yz\,\mathbf{j} -$
Ans. $\mathbf{A}\times(\nabla\times\mathbf{B}) = 18\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}$, $(\mathbf{A}\times\nabla)\times\mathbf{B} = 4\mathbf{j} + 76\mathbf{k}$

98. Prove $(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$.

99. Prove $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$.

100. Prove $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$.

101. Prove $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$.

102. Show that $\mathbf{A} = (6xy + z^3)\mathbf{i} + (3x^2 - z)\mathbf{j} + (3xz^2 - y)\mathbf{k}$ is irrotational. Find ϕ such that $\mathbf{A} = \nabla\phi$.
Ans. $\phi = 3x^2y + xz^3 - yz + \text{constant}$

103. Show that $\mathbf{E} = \mathbf{r}/r^2$ is irrotational. Find ϕ such that $\mathbf{E} = -\nabla\phi$ and such that $\phi(a) = 0$ where $a > 0$.
Ans. $\phi = \ln(a/r)$

104. If $\mathbf{A}$ and $\mathbf{B}$ are irrotational, prove that $\mathbf{A}\times\mathbf{B}$ is solenoidal.

105. If $f(r)$ is differentiable, prove that $f(r)\mathbf{r}$ is irrotational.

106. Is there a differentiable vector function $\mathbf{V}$ such that (a) curl $\mathbf{V} = \mathbf{r}$, (b) curl $\mathbf{V} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$? If so, find $\mathbf{V}$.
Ans. (a) No, (b) $\mathbf{V} = 3x\,\mathbf{j} + (2y - x)\mathbf{k} + \nabla\phi$, where ϕ is an arbitrary twice differentiable function.

107. Show that solutions to Maxwell's equations

$$\nabla\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \qquad \nabla\cdot\mathbf{H} = 0, \qquad \nabla\cdot\mathbf{E} = 4\pi\rho$$

where ρ is a function of x,y,z and c is the velocity of light, assumed constant, are given by

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{H} = \nabla\times\mathbf{A}$$

where $\mathbf{A}$ and ϕ, called the *vector and scalar potentials* respectively, satisfy the equations

$$(1)\ \nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0, \qquad (2)\ \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \qquad (3)\ \nabla^2\mathbf{A} = \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}$$

108. (a) Given the dyadic $\mathbf{\Phi} = \mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k}$, evaluate $\mathbf{r}\cdot(\mathbf{\Phi}\cdot\mathbf{r})$ and $(\mathbf{r}\cdot\mathbf{\Phi})\cdot\mathbf{r}$. (b) Is there any ambiguity in writing $\mathbf{r}\cdot\mathbf{\Phi}\cdot\mathbf{r}$? (c) What does $\mathbf{r}\cdot\mathbf{\Phi}\cdot\mathbf{r} = 1$ represent geometrically?
Ans. (a) $\mathbf{r}\cdot(\mathbf{\Phi}\cdot\mathbf{r}) = (\mathbf{r}\cdot\mathbf{\Phi})\cdot\mathbf{r} = x^2 + y^2 + z^2$, (b) No, (c) Sphere of radius one with centre at the origin.

109. (a) If $\mathbf{A} = xz\,\mathbf{i} - y^2\,\mathbf{j} + yz^2\,\mathbf{k}$ and $\mathbf{B} = 2z^2\,\mathbf{i} - xy\,\mathbf{j} + y^3\,\mathbf{k}$, give a possible significance to $(\mathbf{A}\times\nabla)\mathbf{B}$ at the point $(1,-1,1)$.
(b) Is it possible to write the result as $\mathbf{A}\times(\nabla\mathbf{B})$ by use of dyadics?
Ans. (a) $-4\,\mathbf{i}\,\mathbf{i} - \mathbf{i}\,\mathbf{j} + 3\,\mathbf{i}\,\mathbf{k} - \mathbf{j}\,\mathbf{j} - 4\,\mathbf{j}\,\mathbf{i} + 3\,\mathbf{k}\,\mathbf{k}$
(b) Yes, if the operations are suitably performed.

110. Prove that $\phi(x,y,z) = x^2 + y^2 + z^2$ is a scalar invariant under a rotation of axes.

111. If $\mathbf{A}(x,y,z)$ is an invariant differentiable vector field with respect to a rotation of axes, prove that (a) div $\mathbf{A}$ and (b) curl $\mathbf{A}$ are invariant scalar and vector fields respectively under the transformation.

112. Solve equations (3) of Solved Problem 38 for x,y,z in terms of x',y',z'.
Ans. $x = l_{11}x' + l_{21}y' + l_{31}z'$, $y = l_{12}x' + l_{22}y' + l_{32}z'$, $z = l_{13}x' + l_{23}y' + l_{33}z'$

113. If $\mathbf{A}$ and $\mathbf{B}$ are invariant under rotation show that $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ are also invariant.

114. Show that under a rotation

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \mathbf{i}'\frac{\partial}{\partial x'} + \mathbf{j}'\frac{\partial}{\partial y'} + \mathbf{k}'\frac{\partial}{\partial z'} = \nabla'$$

115. Show that the Laplacian operator is invariant under a rotation.