The Fermi velocity, ie, the velocity of electrons occupying states ("dots") on the surface of the Fermi sphere, and hence are the ones with maximum energy & momentum, is given by

$$V_F = \frac{P_F}{m} = \frac{\hbar k_F}{m} = \frac{4.20}{(r_s/a_0)} \times 10^8 \text{ cm/sec.}$$

This is about 1% of the velocity of light! At temperature T=0! These extremely high velocities are possible because of Pauli's exclusion principle, which disallows electrons from occupying already occupied lower energy levels.

Similarly we can write the Fermi energy as

$$\mathcal{E}_F = \frac{50.1\text{ eV}}{(r_s/a_0)^2},$$

which implies that Fermi energies of metallic elements range from 1.5 to 15 eV.

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What is the total energy of the N-electron system? We must add all e^- energies.

That is, $$E_{tot} = \sum_{k<k_F} \frac{\hbar^2 k^2}{2m}$$

when evaluated,
$$\frac{E}{V} = \frac{1}{\pi^2} \cdot \frac{\hbar^2 k_F^5}{10m}.$$

We already know, $\frac{N}{V} = \frac{k_F^3}{3\pi^2}$.

Fermi energy

$$\therefore \frac{E}{N} = \frac{E}{V}\Big/\frac{N}{V} = \frac{3}{5} \cdot \frac{\hbar^2 k_F^2}{2m} = \frac{3}{5}\mathcal{E}_F$$

Energy per electron

$$\frac{E}{N} = \frac{3}{5} E_F \equiv \frac{3}{5} k_B T_F$$

Fermi temperature.

$\sum_{\vec{k}} F(\vec{k})$, where $F(\vec{k})$ is any function of $\vec{k}$, can be evaluated as follows. Since $\Delta\vec{k}$, the volume occupied by a single "dot" in k-space is $\left(\frac{2\pi}{L}\right)^3$, i.e; $\Delta\vec{k} = \left(\frac{2\pi}{L}\right)^3$, $\frac{V}{8\pi^3} \cdot \Delta\vec{k} = 1$ & so, $\sum_{\vec{k}} F(\vec{k}) = \frac{V}{8\pi^3} \sum F(\vec{k}) \Delta\vec{k}$

Taking the limit $\Delta\vec{k} = 0$, ($V \to \infty$) we get

$$\lim_{V\to\infty} \frac{1}{V} \sum_{\vec{k}} F(\vec{k}) = \int \frac{d\vec{k}}{8\pi^3} F(\vec{k}) = \int \frac{d\vec{k}}{8\pi^3} \cdot \frac{\hbar^2 k^2}{2m}$$

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$$T_F = \frac{\mathcal{E}_F}{k_B}$$

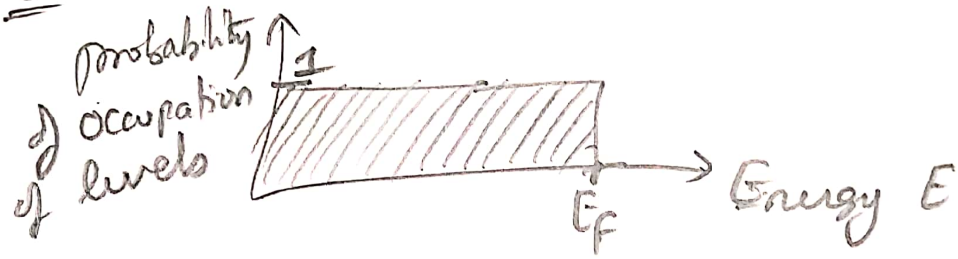
$$= \frac{58.2}{(r_s/a_0)^2} \times 10^4 \text{ K.}$$

That is, even at zero temperature, the electrons in the "Fermi gas" behave as if they are at a temperature of 10,000 K!

A corollary to this observation is that, room temperature (300 K) and zero Kelvin are nearly the same, looking from a temperature of 10,000 K. So we can treat theoretically metals at room temperature as systems at zero Kelvin, simplifying our calculations.

For temperatures $T \neq 0$, if we want to have precise, quantitative results, we must have an idea of the probability of occupation by electrons of states with energy $E > E_F$. Since electrons can gain thermal energy & can move above their Fermi energy states.

At low temperatures, particularly at $T=0$, all the low energy levels are occupied upto the Fermi energy E_F & the levels above are empty. That is the probability of occupation of energy levels with energy $E < E_F$ is $\underline{1}$ & the probability for levels with energy $E > E_F$ is $\underline{0}$.



What is the probability of occupation of energy levels when $T \neq 0$? The answer to this question is given by the Fermi-Dirac distribution function:

$$f(E) = \frac{1}{\exp\left((E-\mu)/k_B T\right) + 1}$$

probability of occupation — $f(E)$

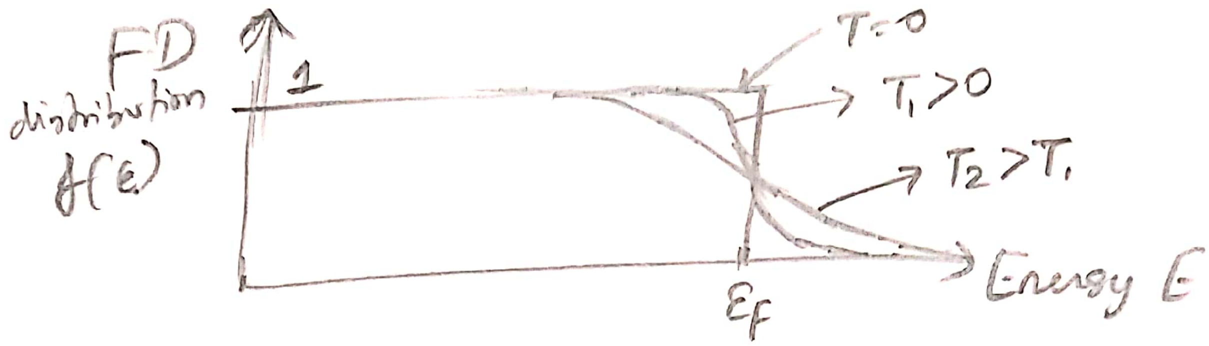
energy of the energy level whose occupation probability we want — E

Boltzmann const — k_B

temperature — T

Chemical potential $\approx E_F$, Fermi energy near $T=0$ — μ

At temperature $T \neq 0$, the F-D distribution looks like below:



Since $f(E)$ gives the probability of occupation of a given level of energy E, the total number of electrons in a given system at a temp T & chemical potential μ is

$$N = \sum_{i} f_i = \sum \frac{1}{\exp\left((E-\mu)/k_B T\right)+1}$$

$i \to$ levels

This connects T, N & μ.

The total energy of a system is

$$U = 2\sum_{\vec{k}} E(\vec{k})\, f(E(\vec{k}))$$

$f \to$ FD distribution

$\vec{k} \to$ wave vectors of electrons

$E \to$ energy

This can be written as an integral if we allow $V \to \infty$:

$$u = \frac{U}{V} = \int \frac{d\vec{k}}{4\pi^3} \mathcal{E}(\vec{k}) f(\mathcal{E}(\vec{k}))$$

energy density.

and

$$n = \frac{N}{V} = \int \frac{d\vec{k}}{4\pi^3} f(\mathcal{E}(\vec{k}))$$

It is important to note that these $\int$'s are over k-space or the momentum space, whereas the integrands enter as functions of energy. It is useful to do the integration over energy, instead of momentum. That is, we go from $d\vec{k}$ to $d\mathcal{E}$. Since $d\vec{k} = dk_x dk_y dk_z$ & in spherical coordinates, it becomes $d\vec{k} = 4\pi k^2 dk$, we get

$$\int \frac{d\vec{k}}{4\pi^3} \underbrace{F(\mathcal{E}(\vec{k}))}_{\text{any } F} = \int_0^\infty \frac{4\pi k^2}{4\pi^3} F(\mathcal{E}) |dk|$$

$$= \int \frac{k^2 dk}{\pi^2} F(\mathcal{E})$$

using $\mathcal{E} = \frac{\hbar^2 k^2}{2m}$,

$d\mathcal{E} = \frac{2\hbar^2 k dk}{2m}$

or $d\mathcal{E} = \frac{2\hbar^2}{2m} \cdot \sqrt{\frac{2m\mathcal{E}}{\hbar^2}} \cdot dk$

Substituting,

$$\int_0^\infty \frac{k^2\,dk}{\pi^2}\,F(\varepsilon) = \int_{-\infty}^{\infty} \frac{2m\varepsilon}{\hbar^2}\cdot\frac{m}{\hbar^2}\sqrt{\frac{\hbar^2}{2m\varepsilon}}\,\frac{d\varepsilon}{\pi^2}\,F(\varepsilon)$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[\frac{m}{\hbar^2\pi^2}\sqrt{\frac{2m\varepsilon}{\hbar^2}}\right]}_{\Downarrow} F(\varepsilon).d\varepsilon$$

$g(\varepsilon)$ = Density of states [calculated by going from dk to $d\varepsilon$, momentum to energy].

$$= \int_{-\infty}^{\infty} g(\varepsilon)\,F(\varepsilon)\,d\varepsilon$$

DOS per unit volume

$g(\varepsilon)d\varepsilon = \frac{1}{\text{Volume}} \times$ Number of one electron energy levels in the energy range ε to $\varepsilon + d\varepsilon$.

Since we know that the electronic density

$$n = \frac{k_F^3}{3\pi^2}$$ & Fermi energy $$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m},$$

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we can use n & $\mathcal{E}_F$ to simplify the "appearance" of $g(\mathcal{E})d\mathcal{E}$:

$$g(\mathcal{E}) = \frac{m}{\hbar^2\pi^2}\sqrt{\frac{2m\mathcal{E}}{\hbar^2}} = \left[\frac{m}{\hbar^2\pi^2}\sqrt{\frac{2m}{\hbar^2}}\right]\sqrt{\mathcal{E}}.$$

$$\mathcal{E}_F^{3/2} = \frac{\hbar^3 k_f^3}{(2m)^{3/2}} \quad \& \quad \frac{n}{\mathcal{E}_F^{3/2}} = \frac{\cancel{k_f^3}/3\pi^2}{\hbar^3\cancel{k_f^3}/(2m)^{3/2}}$$

$$= \frac{(2m)^{3/2}}{3\hbar^3 \cdot \pi^2} = \frac{1}{3}\cdot\frac{2m}{\hbar^2\cdot\pi^2}\sqrt{\frac{2m}{\hbar^2}} = \frac{2}{3}\left[\frac{m}{\hbar^2\pi^2}\sqrt{\frac{2m}{\hbar^2}}\right]$$

$$\therefore \boxed{g(\mathcal{E})} = \frac{2}{3}\frac{n}{\mathcal{E}_F^{3/2}}\sqrt{\mathcal{E}} = \boxed{\frac{2}{3}\left(\frac{n}{\mathcal{E}_F}\right)\sqrt{\frac{\mathcal{E}}{\mathcal{E}_F}}}.$$

dimension $\equiv \dfrac{e^- \text{ density}}{\text{energy}}$

We can now get back to evaluating the energy density & number density of electrons when $T \neq 0$:

$$u = \int \frac{d\vec{k}}{4\pi^3}\,\mathcal{E}(\vec{k})\,f(\mathcal{E}(\vec{k}))$$

$$\& \quad n = \int \frac{d\vec{k}}{4\pi^3}\,f(\mathcal{E}(\vec{k}))$$

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where the $\int k$ is over k-space.
we can write this $\int k$ as one over energy by introducing the density of states:

$$u = \int \boxed{\frac{d\vec{k}}{4\pi^3}} \, \underbrace{\mathcal{E}(\vec{k})}_{\text{energy}} \, \underbrace{f(\mathcal{E}(\vec{k}))}_{\text{FD distr. fn}} = \int \boxed{d\varepsilon \, g(\varepsilon)} \, \varepsilon \cdot f(\varepsilon)$$

for energy density

and

$$n = \int \boxed{\frac{d\vec{k}}{4\pi^3}} \, f(\mathcal{E}(\vec{k})) = \int \boxed{d\varepsilon \, g(\varepsilon)} \, f(\varepsilon)$$

This form of equations for evaluating u & n is important because only $g(\varepsilon)$ depends on the geometry of the substance studied (1-D, 2-D or 3-D) and can be modified accordingly.

—

Our goal, as in Drude's model, is to evaluate the specific heat of electron gas. This involves evaluating $C_V = \left(\frac{\partial u}{\partial T}\right)_V$.

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Since the details of calculation are quite involved, we can just write down the answer for u, the energy density:

$$\boxed{u = u_0 + \frac{\pi^2}{6}(k_B T)^2 g(\varepsilon_F)}$$

u_0: ground state density at $T=0$

$g(\varepsilon_F)$: DOS at Energy ε_F.

Therefore, C_V is

$$C_V = \left(\frac{\partial u}{\partial T}\right)_{n,V} = \frac{\pi^2}{3} k_B^2 T \cdot g(\varepsilon_F)$$

for free electrons, $g(\varepsilon_F)$ is $\frac{3}{2} \cdot \frac{n}{\varepsilon_F}$ (see above)

$\therefore$ C_V for free electrons is

$$\boxed{C_V = \frac{\pi^2}{2}\left(\frac{k_B T}{\varepsilon_F}\right) \cdot n k_B}$$

$\frac{k_B T}{\varepsilon_F} \approx 0.01$

Compare this to $\frac{3}{2} n k_B$ from classical kinetic theory of gases. This explains why C_V of e^-s is $1/100^{th}$ of $\frac{3}{2} n k_B$ from classical Physics.