Kinematics Synthesis of Plane Mechanisms or Linkages

Aim:

Design or creation of a mechanism to obtain a desired set of motion characteristics.

Objective

- design of mechanisms to satisfy certain kinematic specification.

- In other words, motion characteristics are given & the mechanism is to be found

Kinematic Synthesis Problems

- Type Synthesis
- Number Synthesis
- Dimensional Synthesis

Kinematic Synthesis of Mechanisms

Synthesis Problems

Kinematic Synthesis

- **Type Synthesis** → Refers to the kind of mechanism selected
- **Number Synthesis** → Deals with the number of links & nos. of joints that are required to obtain a certain mobility
- **Dimensional Synthesis**
  - Motion generation /Body Guidance
  - Path generation
  - Function generation

By Dimensional Synthesis, we mean the determination of kinematic dimensions of the individual links of a mechanism to satisfy specified motion characteristics or specified tasks.

Steps in Kinematics Synthesis of Plane Mechanisms

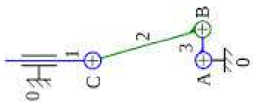
Type Synthesis

↓

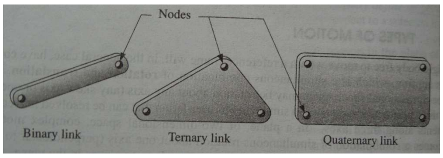
Number Synthesis

↓

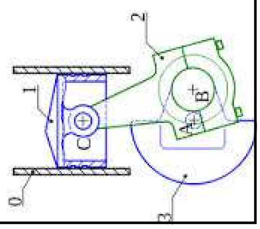
Dimensional Synthesis



What is Kinematic dimensions?



Node-node distance or joint centre to centre distance etc.



Dimensional Synthesis Problems

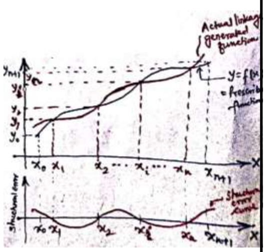
Function generation problem

Synthesis

- Exact Synthesis: By exact synthesis, we mean that the generated function by the physical mechanism fits the desired function at all points in the interval
- Approximate Synthesis: By approximate synthesis, we mean that the generated function by the physical mechanism fits the desired function at a finite number of points in the interval

Accuracy points / Precision points

The points at which the generated and desired functions agree



Structural Error

It is defined as the theoretical difference between the function generated by the synthesized linkage & the function originally prescribed

Structural error is inherent in approximate synthesis

Chebyshev's Spacing of Accuracy Points

Let $y=f(x)$ be the function desired to be generated in an interval $x_0 \le x \le x_{n+1}$:

Let the mechanism generated function be $F(x, R_1, R_2, \ldots, R_k)$ where $R_1, R_2, \ldots R_k$ are design parameters

Structural Error

$$E(x)=f(x)-F(x, R_1, R_2, \ldots, R_k)$$

The best choice for the spacing of accuracy points will be that which gives the min. value of $E(x)$ between any two adjacent points:

However, Chebyshev's spacing of accuracy points can always be taken as a first approximation

A very good trial for the spacing of these precision positions is called Chebyshev Spacing

Chebyshev's Spacing of Accuracy Points

For 'n' precision positions in the range $x_0 \le x \le x_{n+1}$, the Chebyshev's spacing is

$$x_j = \left(\frac{x_{n+1}+x_0}{2}\right) - \frac{(x_{n+1}-x_0)}{2}\cos\left\{\frac{(2j-1)\pi}{2n}\right\} \quad \text{when } j=1,2,\ldots n.$$

Determine the three accuracy points with Chebyshev's spacing for a 4 bar linkage to generate the function $y=x^{0.8}$ in the interval $1 \le x \le 3$,

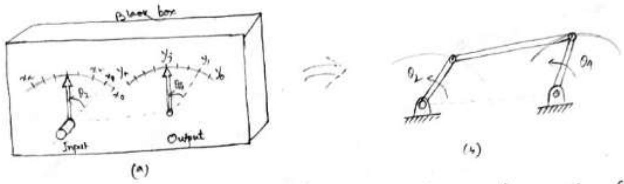


Fig. 1 Function-generator mechanism (a) exterior view, (b) Schematic of the mechanism inside.
(i.e four-bar linkage function generator)

Determine the three accuracy points with Chebyshev's spacing for a 4 bar linkage to generate the function $y=x^{0.8}$ in the interval $1 \le x \le 3$,

Here $n=3$; $x_0 = 1$; $x_{3+1} = x_4 = 3$

$$x_j = \left(\frac{x_{n+1}+x_0}{2}\right) - \left(\frac{x_{n+1}-x_0}{2}\right)\cos\left\{\frac{(2j-1)\pi}{2n}\right\} \quad \text{where } j=1,2,3$$

$$x_j = \left(\frac{x_4+x_0}{2}\right) - \left(\frac{x_4-x_0}{2}\right)\cos\left\{\frac{(2j-1)\pi}{6}\right\}$$

$$x_1 = \left(\frac{3+1}{2}\right) - \left(\frac{3-1}{2}\right)\cos\left\{\frac{(2-1)\pi}{2\times 3}\right\} = 2-\cos\pi/6 = 1.134$$

$$x_2 = \left(\frac{3+1}{2}\right) - \left(\frac{3-1}{2}\right)\cos\left\{\frac{(4-1)\pi}{2\times 3}\right\} = 2-\cos\pi/2 = 2$$

$$x_3 = \left(\frac{3+1}{2}\right) - \left(\frac{3-1}{2}\right)\cos\left\{\frac{(6-1)\pi}{6}\right\} = 2-\cos 5\pi/6 = 2.866$$

Accuracy pts.

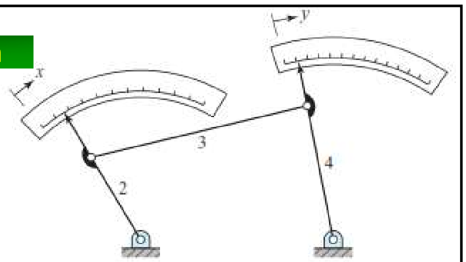
The corresponding values of 'y' to be

$$y_1 = x^{0.8} = (1.134)^{0.8} = 1.106$$

$$y_2 = (2)^{0.8} = 1.741$$

$$y_3 = (2.666)^{0.8} = 2.322.$$

Scale Factor for Input & Output motion



Mechanized variables: θ_2 & θ_4

Functional variables: 'x' & 'y'

The orientation of the driver link (θ_2) represents the independent variable 'x'

The orientation of the driven link (θ_4) represents the dependent variable 'y'.

The mechanized variables θ_2 & θ_4 are proportional to the functional variables 'x' & 'y'.

The relation betn Δx and $\Delta\theta_2$ & that betn Δy and $\Delta\theta_4$ is usually assumed to be linear.

With the mappings betn function variable space (x, y) and mechanism joint space (θ_2, θ_4) known, we can map the three function precision points to corresponding precision joint angles.

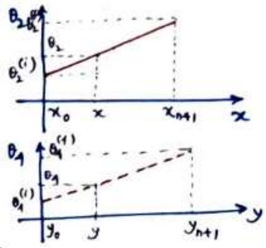
Let $\theta_2^{(i)}$ be the initial value of θ_2 representing x_0

$\theta_4^{(i)}$ be " " " of θ_4 " $y_0 = f(x_0)$

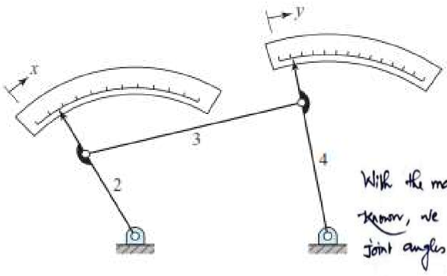
The input & output scale factors m_x & m_y resp. are defined as:

$\therefore$ Scale factor $m_x = \frac{\Delta\theta_2}{\Delta x} = \frac{\theta_2^{(f)} - \theta_2^{(i)}}{x_{n+1} - x_0} = \frac{\theta_2 - \theta_2^{(i)}}{x - x_0}$

$m_y = \frac{\Delta\theta_4}{\Delta y} = \frac{\theta_4^{(f)} - \theta_4^{(i)}}{y_{n+1} - y_0} = \frac{\theta_4 - \theta_4^{(i)}}{y - y_0}$



The superscripts 'i' & 'f' denote the initial & final values of θ_2 & θ_4.



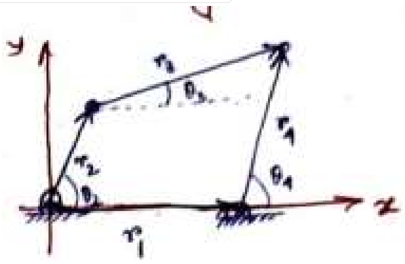
$\theta_2 - \theta_2^{(i)} = m_x(x - x_0) \Rightarrow \theta_2 = \theta_2^{(i)} + m_x(x - x_0)$ where $m_x = \frac{\theta_2^{(f)} - \theta_2^{(i)}}{x_{n+1} - x_0}$

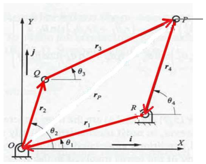
$\theta_4 - \theta_4^{(i)} = m_y(y - y_0) \Rightarrow \theta_4 = \theta_4^{(i)} + m_y(y - y_0)$ where $m_y = \frac{\theta_4^{(f)} - \theta_4^{(i)}}{y_{n+1} - y_0}$

With the mappings betn function variable space (x, y) and mechanism joint space (θ_2, θ_4) known, we can map the three function precision points to corresponding precision joint angles.

Displacement Analysis of 4R linkage

Freudenstein's Method





Loop Closure Equation in Scalar Form

Two scalar eqns.

$r_2\cos\theta_2 + r_3\cos\theta_3 = r_1\cos\theta_1 + r_4\cos\theta_4 \Rightarrow r_2\cos\theta_2 + r_3\cos\theta_3 = r_1 + r_4\cos\theta_4$

$r_2\sin\theta_2 + r_3\sin\theta_3 = r_1\sin\theta_1 + r_4\sin\theta_4 \Rightarrow r_2\sin\theta_2 + r_3\sin\theta_3 = r_4\sin\theta_4$

as $\theta_1 = 0°$

$r_3\cos\theta_3 = r_1 + r_4\cos\theta_4 - r_2\cos\theta_2$

$r_3\sin\theta_3 = r_4\sin\theta_4 - r_2\sin\theta_2$

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$r_3 \cos\theta_3 = r_1 + r_4 \cos\theta_4 - r_2 \cos\theta_2$

$r_3 \sin\theta_3 = r_4 \sin\theta_4 - r_2 \sin\theta_2$

$$r_3^2 = (r_1 + r_4\cos\theta_4 - r_2\cos\theta_2)^2 + (r_4\sin\theta_4 - r_2\sin\theta_2)^2$$

$$r_3^2 = r_1^2 + r_2^2 + r_4^2 - 2r_1r_2\cos\theta_2 + 2r_1r_4\cos\theta_4 - 2r_2r_4\cos\theta_2\cos\theta_4 - 2r_2r_4\sin\theta_2\sin\theta_4$$

$$r_3^2 = r_1^2 + r_2^2 + r_4^2 - 2r_1r_2\cos\theta_2 + 2r_1r_4\cos\theta_4 - 2r_2r_4\cos(\theta_2 - \theta_4)$$

$$2r_1r_4\cos\theta_4 - 2r_1r_2\cos\theta_2 + (r_1^2 + r_2^2 + r_4^2 - r_3^2) = 2r_2r_4\cos(\theta_2 - \theta_4)$$

$$\frac{r_1}{r_2}\cos\theta_4 - \frac{r_1}{r_4}\cos\theta_2 + \frac{r_1^2 + r_2^2 + r_4^2 - r_3^2}{2r_2r_4} = \cos(\theta_2 - \theta_4)$$

$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos(\theta_2 - \theta_4) \quad \text{------ (A)} \quad \text{F}$$

FREUDENSTEIN's eq^n

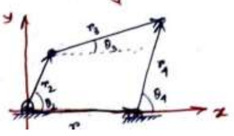
FREUDENSTEIN's METHOD: Function Generation with Three Accuracy Points.

With three accuracy points, the number of design parameters that can be determined is three.

Example: Four-bar Function Generators with Three Accuracy Points

Loop-closure eq^n or vector loop eq^n

$$\vec{r_2} + \vec{r_3} = \vec{r_1} + \vec{r_4}$$



$$K_1\cos\theta_4 - K_2\cos\theta_2 + K_3 = \cos(\theta_2 - \theta_4) \quad \text{------ (A) Freudenstein's } eq^n$$

Substituting the three related pairs $(\theta_2^{(1)} \,\&\, \theta_4^{(1)})$, $(\theta_2^{(2)} \,\&\, \theta_4^{(2)})$ and $(\theta_2^{(3)} \,\&\, \theta_4^{(3)})$ successively in eq^n (A), we obtain three linear simultaneous eq^ns in K_1, K_2 & K_3.

$$K_1\cos\theta_4^{(1)} - K_2\cos\theta_2^{(1)} + K_3 = \cos(\theta_2^{(1)} - \theta_4^{(1)}) \quad \text{------ (I)}$$

$$K_1\cos\theta_4^{(2)} - K_2\cos\theta_2^{(2)} + K_3 = \cos(\theta_2^{(2)} - \theta_4^{(2)}) \quad \text{------ (II)}$$

$$K_1\cos\theta_4^{(3)} - K_2\cos\theta_2^{(3)} + K_3 = \cos(\theta_2^{(3)} - \theta_4^{(3)}) \quad \text{------ (III)}$$

Solving above linear eq^ns, we get the K_1, K_2, & K_3 i.e. link length ratios (design parameters)

Manual Method:

(II) − (I): $K_1[\cos\theta_4^{(2)} - \cos\theta_4^{(1)}] - K_2[\cos\theta_2^{(2)} - \cos\theta_2^{(1)}] = \cos(\theta_2^{(2)} - \theta_4^{(2)}) - \cos(\theta_2^{(1)} - \theta_4^{(1)})$

(III) − (I): $K_1[\cos\theta_4^{(3)} - \cos\theta_4^{(1)}] - K_2[\cos\theta_2^{(3)} - \cos\theta_2^{(1)}] = \cos(\theta_2^{(3)} - \theta_4^{(3)}) - \cos(\theta_2^{(1)} - \theta_4^{(1)})$.

Solve two eq^n for two unknowns

Example # 2

Determine the lengths of the links of a 4R planar mechanism to generate $y=\log_{10}x$ in the interval $1 \le x \le 10$. The length of the smallest link is 5 cm. Use three accuracy points with Chebyshev's spacing. Assume initial and final value of input angle are 45 deg. and 105 deg. respectively whereas initial and final value of output angle are 135 deg. and 225 deg. respectively.

$45^\circ \le \theta_2 \le 105^\circ$; $135^\circ \le \theta_4 \le 225^\circ$

Given data: $n = 3$, $x_0 = 1$, $x_4 = 10$

$$x_j = \left(\frac{x_{n+1}+x_0}{2}\right) - \left(\frac{x_{n+1}-x_0}{2}\right)\cos\left\{\frac{(2j-1)\pi}{2n}\right\}$$

i.e. $$x_j = \left(\frac{x_4+x_0}{2}\right) - \left(\frac{x_4-x_0}{2}\right)\cos\left\{\frac{(2j-1)\pi}{6}\right\}$$

$$x_j = \frac{11}{2} - 4.5\cos\left((2j-1)\pi/6\right)$$

$x_1 = 5.5 - 4.5\cos\pi/6 = 1.6 \Rightarrow y_1 = \log_{10}(1.6) = 0.204$

$x_2 = 5.5 - 4.5\cos\pi/2 = 5.5 \Rightarrow y_2 = \log_{10}(5.5) = 0.741$

$x_3 = 5.5 - 4.5\cos 5\pi/6 = 9.4 \Rightarrow y_3 = \log_{10}(9.4) = 0.974$

$x_0 = 1 \Rightarrow y_0 = \log_{10}(1) = 0$

$x_4 = 10 \Rightarrow y_4 = \log_{10}(10) = 1.$

Scale Factor for Input & Output motion

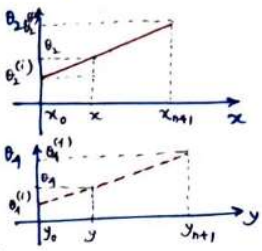
Let $\theta_2^{(i)}$ be the initial value of θ_2 representing x_0

$\theta_4^{(i)}$ be " " " of θ_4 " $y_0 = f(x_0)$

The input & output scale factors m_x & m_y resp. are defined as:

∴ Scale factor $m_x = \frac{\Delta\theta_2}{\Delta x} = \frac{\theta_2^{(f)} - \theta_2^{(i)}}{x_{n+1} - x_0} = \frac{\theta_2 - \theta_2^{(i)}}{x - x_0}$

$m_y = \frac{\Delta\theta_4}{\Delta y} = \frac{\theta_4^{(f)} - \theta_4^{(i)}}{y_{n+1} - y_0} = \frac{\theta_4 - \theta_4^{(i)}}{y - y_0}$



The superscripts 'i' & 'f' denote the initial & final values of θ_2 & θ_4.

$$m_x = \frac{105 - 45^\circ}{10 - 1} = \frac{\theta_2 - 45^\circ}{x - 1} \quad ; \quad m_y = \frac{225 - 135}{1 - 0} = \frac{\theta_4 - 135}{y - 0}$$

$$\theta_2 = \left(\frac{60}{9}\right)(x-1) + 45 \quad ; \quad \theta_4 = 90(y-0) + 135$$

$$\theta_2 = \left(\frac{60}{9}\right)(x-1) + 45 \quad ; \quad \theta_4 = 90(y-0) + 135$$

Accuracy Points or Precision Point

Position	x_j	$\theta_2^{(j)}$	y_j	θ_4	$\theta_4^{(j)}$
1	1.6	49°	0.204	~~63.36~~	153.36
2	5.5	75°	0.741	~~111.69~~	201.69
3	9.4	101°	0.974	~~132.66~~	222.66

Freudenstein's eqn

$$K_1 \cos\theta_4 - K_2 \cos\theta_2 + K_3 = \cos(\theta_2 - \theta_4)$$

$$K_1 \cos 153.36 - K_2 \cos 49° + K_3 = \cos(49 - 153.36)$$

$$K_1 \cos 201.69 - K_2 \cos 75° + K_3 = \cos(75 - 201.69)$$

$$K_1 \cos 222.66 - K_2 \cos 101° + K_3 = \cos(101 - 222.66)$$

Solving

$K_1 = 2.0$; $K_2 = -0.7015$; $K_3 = 1.081$

$$K_1 = 2.0 \; ; \; K_2 = -0.7015 \; ; \; K_3 = 1.081$$

$$K_1 = \frac{r_1}{r_2} = 2 \; ; \; K_2 = \frac{r_1}{r_4} = -0.7015$$

$$r_2 = r_1/2 \; ; \; r_4 = -\frac{r_1}{0.7015}$$

$$K_3 = \frac{r_1^2 + r_2^2 + r_4^2 - r_3^2}{2 r_2 r_4} = 1.081$$

$$\frac{r_1^2 + \frac{r_1^2}{4} + \left(\frac{r_1}{0.7015}\right)^2 - r_3^2}{\cancel{2}\, r_1/\cancel{2} \cdot \left(-\frac{r_1}{0.7015}\right)} = 1.081$$

$$+\left[\frac{r_1^2 + r_1^2/4 + \left(\frac{r_1}{0.7015}\right)^2 - r_3^2}{r_1^2}\right] = -\frac{1.081}{0.7015}$$

$$1 + \frac{1}{4} + \left(\frac{1}{0.7015}\right)^2 - \left(\frac{r_3}{r_1}\right)^2 = -\frac{1.081}{0.7015}$$

$$\left(\frac{r_3}{r_1}\right)^2 = (2.1962)^2 \quad \therefore \quad \frac{r_3}{r_1} = 2.1962$$

$$\frac{r_1}{r_3} = 0.462$$

$$\left|\frac{r_1}{r_2}\right| = 2 \; ; \; \left|\frac{r_1}{r_4}\right| = +0.7015, \quad \left|\frac{r_1}{r_3}\right| = 0.462$$

From above it is clear that $\left[r_2 < r_1 < r_4 < r_3 \right]$

$$r_2 < r_1$$
$$< r_4 \quad \text{as } r_1 < r_4$$
$$< r_3 \quad \text{as } r_1 < r_3$$

$\therefore$ r_2 is the smallest link

$r_2 = 5$ cm

$r_1 = 10$ cm ; $r_4 = 14.2$ cm, $r_3 = 21.85$ cm

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Dimensional Synthesis

Function Generation Problem

Computer Aided Synthesis of Planar Mechanisms

Write MATLAB code for solving following problems.

Determine the lengths of the links of a 4R planar mechanism to generate y=log10x in the interval 1≤x≤10. The length of the smallest link is 50 mm. Use three accuracy points with Chebyshev's spacing. Assume initial and final value of input angle are 45 deg. and 105 deg. respectively whereas initial and final value of output angle are 135 deg. and 225 deg. respectively.

Determine the lengths of the links of a four bar mechanism to generate $y=x^{1.6}$ over the range 1≤x≤4 using three accuracy points with Chebyshev's spacing. The length of the smallest link is 30 mm. Assume initial and final value of input angle are 30 deg. and 120 deg. respectively whereas initial and final value of output angle are 60 deg. and 150 deg. respectively.

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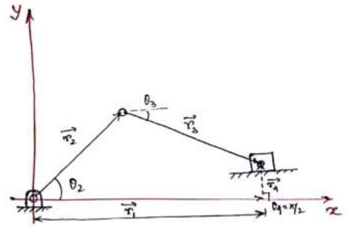
Synthesis of the Slider-Crank Mechanism with three accuracy points



Loop-closure eqn

$$\vec{r_2} + \vec{r_3} = \vec{r_1} + \vec{r_4}$$

Scalar component of the eqn.

$$r_2\cos\theta_2 + r_3\cos\theta_3 = r_1\cos\theta_1 + r_4\cos\theta_4$$

$$r_2\sin\theta_2 + r_3\sin\theta_3 = r_1\sin\theta_1 + r_4\sin\theta_4$$

where $\theta_1 = 0°$, $\theta_4 = \pi/2$

r_1 is variable

$$r_3\cos\theta_3 = r_1 + 0 - r_2\cos\theta_2$$

$$r_3\sin\theta_3 = 0 + r_4 - r_2\sin\theta_2$$

Squaring & adding

$$r_3^2 = (r_1 - r_2\cos\theta_2)^2 + (r_4 - r_2\sin\theta_2)^2$$

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$$r_3\cos\theta_3 = r_1 + 0 - r_2\cos\theta_2$$

$$r_3\sin\theta_3 = 0 + r_4 - r_2\sin\theta_2$$

Squaring & adding

$$r_3^2 = (r_1 - r_2\cos\theta_2)^2 + (r_4 - r_2\sin\theta_2)^2$$

$$r_3^2 = r_1^2 + r_2^2 + r_4^2 - 2r_1r_2\cos\theta_2 - 2r_2r_4\sin\theta_2$$

$$2r_1r_2\cos\theta_2 + 2r_2r_4\sin\theta_2 - (r_2^2 - r_3^2 + r_4^2) = r_1^2$$

$K_1 s\cos\theta_2 + K_2\sin\theta_2 - K_3 = s^2$, where $K_1 = 2r_2$

$K_2 = 2r_2r_4$

$K_3 = r_2^2 - r_3^2 + r_4^2$

Substituting the three related pairs

$[\theta_2^{(1)}, s^{(1)}]$, $[\theta_2^{(2)}, s^{(2)}]$ & $[\theta_2^{(3)}, s^{(3)}]$

Variable $r_1 = s$ (Sliding)

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$$K_1 s \cos\theta_2 + K_2 \sin\theta_2 - K_3 = s^2$$

Successively in above eqn, we obtain three linear simultaneous eq. in K_1, K_2 & K_3.

$$\left.\begin{aligned} K_1 s^{(1)} \cos\theta_2^{(1)} + K_2 \sin\theta_2^{(1)} - K_3 &= \{s^{(1)}\}^2 \\ K_1 s^{(2)} \cos\theta_2^{(2)} + K_2 \sin\theta_2^{(2)} - K_3 &= \{s^{(2)}\}^2 \\ K_1 s^{(3)} \cos\theta_2^{(3)} + K_2 \sin\theta_2^{(3)} - K_3 &= \{s^{(3)}\}^2 \end{aligned}\right\} \text{-------} (I)$$

Dimensional Synthesis

Function Generation Problem

Computer Aided Synthesis of Planar Mechanisms

Write MATLAB code for solving following problems.

Synthesize a slider crank mechanism in which the slider displacement is proportional to the square of the crank angular displacement in the interval $40° \leq \theta_2 \leq 130°$. The initial and final value of slider position are 10 cm and 3 cm respectively from the reference frame fixed at crank. Use the three point Chebyshev spacing.

- Hand calculation
- MATLAB code