CHAPTER 6

SHEARING FORCE AND BENDING MOMENT

Shearing Force

Consider a horizontal beam AB (Fig. 6.1) which is simply supported at each end and which carries a single load W at the centre. Such a load which is assumed to act at a point, is called a concentrated or point load. Since the beam is in equilibrium, the reactions R_A and R_B of the supports at A and B, respectively, will each be $W/2$ units.

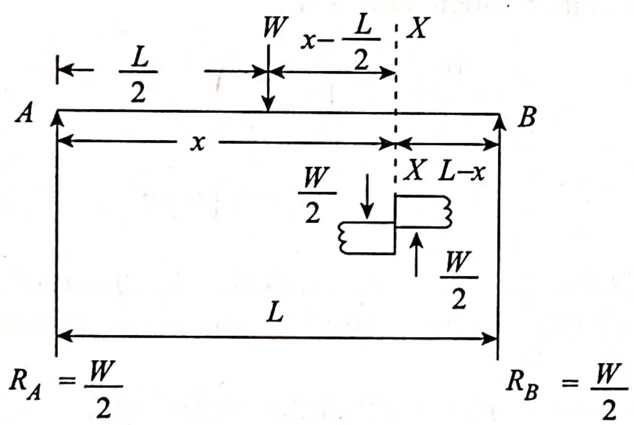


Figure 6.1

Now let us consider any section XX of the beam, distance x from the left-hand end A. It will be noticed that two forces are acting on the part of the beam to the left of the section XX. These forces are:

i) The reaction $R_A = W/2$ acting vertically upwards
ii) The concentrated load W acting vertically downwards

There is, therefore a resultant force of

$$W - \frac{W}{2} = \frac{W}{2}$$

tending to shear the left hand part of the beam downwards across $X - X$. This resultant force is called the shearing force of the beam at the section $X - X$.

For equilibrium, the resultant of the forces acting on the part of the beam to the right of $X-X$ must be equal in magnitude, but part of the beam to the left of $X-X$. As a matter of fact, it will be noticed that the only force acting on the right-hand side of section $X-X$ of the beam is the reaction $R_B = W/2$ tending to shear this part of the beam upwards across. We may, therefore conclude that:

'The shearing force at a particular section of a beam is the resultant force, of all the forces acting on either side of that section.'

Bending Moment

Now let us think again of the part of beam to the left of the section $X-X$ (Fig. 6.1). Taking moments about $X-X$, we get

$$\text{Moment due to reaction } R_A = \frac{W}{2}x \quad \text{(clockwise)}$$

$$\text{Moment due to load } W = W\left(x-\frac{L}{2}\right) \quad \text{(anticlockwise)}$$

There is therefore, a resultant moment at $X-X$,

$$= \frac{W}{2}x - W\left(x-\frac{L}{2}\right)$$

$$= \frac{W}{2}(L-x) \quad \text{(clockwise)}$$

Acting upon the beam in a *clockwise* direction, and the left-hand part of the beam will tend to rotate about XX in the direction of this resultant moment. The resultant moment is called the *bending moment* at the section $X-X$.

For equilibrium, the resultant moment of all the forces acting on the part of the beam to the right of $X-X$ must be equal in magnitude, but opposite in direction to the resultant moment of all the forces acting on the part of the beam to the right of $X-X$. It will be noticed that the only force acting on this part of the beam in the reaction $R_B = W/2$ acting upwards at the end B, its distance being $(L-x)$ from $X-X$.

Now taking moments about $X-X$, we get:

$$\text{Moment due to reaction } R_B = \frac{W}{2}(L-x) \quad \text{(anticlockwise).}$$

Hence, the right-hand part of the beam will tend to rotate about $X-X$ in the direction of this moment.

It should therefore be emphasised that there can only be one value for the bending moment (and also for the shearing force) at a particular section of a beam regardless of which side of the section is considered.

From the foregoing discussion we may then conclude that:

'The bending moment at a particular section of a beam is the resultant moment about a section of all the forces acting on *either side* of the section.'

Sign convention for the shearing force and bending moment:

i) *Shear Force*

There are several conventions in use for mathematical sign with the magnitude of a shearing force and with the magnitude of a bending moment, but the system given below appears to be most popular.

The shearing force (SF) which tends to slide the left-hand side of the section of the beam *upwards* relative to the right-hand side will be considered *positive*, and that which tends to slide the left-hand side of the section of the beam *downwards* relative to the right-hand side will be considered *negative* (see Fig. 6.2).

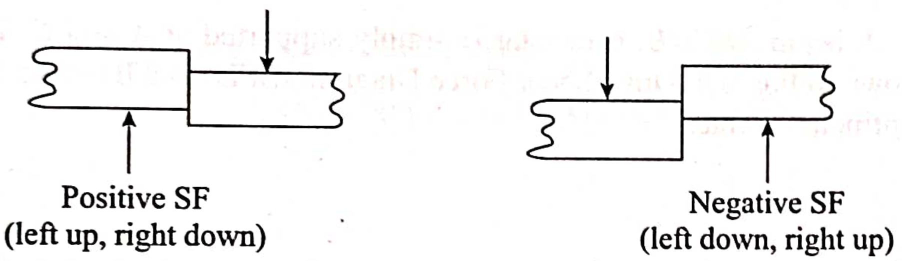


Figure 6.2

Hence, the rule is:

Left up, right down is positive
Left down, right up is negative

ii) *Bending Moment:*

The bending moment (BM) which tends to cause the beam concave upwards, sometimes called *sagging*, will be considered *positive* and that which tends to cause the beam concave *downwards*, sometimes called *hogging*, will be considers *negative* (see Fig. 6.3). Notice that this diagram also gives the sign convention for cantilever beams.

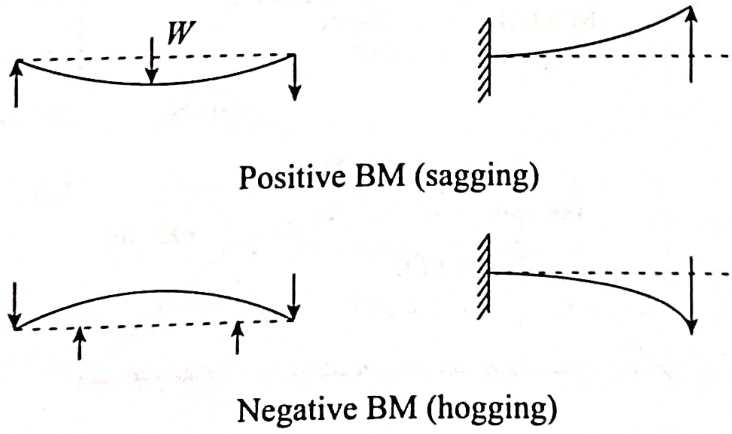


Figure 6.3

Shearing Force and Bending Moment Diagrams

Diagrams indicating the value of the shearing force and the bending moment at any section along a loaded beam or cantilever are called Shear Force Diagram and Bending Moment Diagram. These diagrams are drawn immediately underneath the loading diagram to same horizontal scale. In each case positive values are plotted upwards and negative values downwards and, in addition, the principal values should be shown on both diagrams. Also scale for drawing (for example, 5 kN = 10 mm to suit the size of paper) and for Bending Moment Diagram (for example, 15 kNm = 10 mm to suit the size of paper) should be shown beneath each diagram.

Method: In case of beams always start from the left hand end and for cantilevers from the free end.

EXAMPLE 6.1: A beam *ABCDE*, 6 m long is simply supported at *A* and *E*. It carries concentrated loads as shown in Fig. 6.4. Draw Shear Force Diagram (SFD) and Bending Moment Diagram (BMD) showing principal values.

SOLUTION:

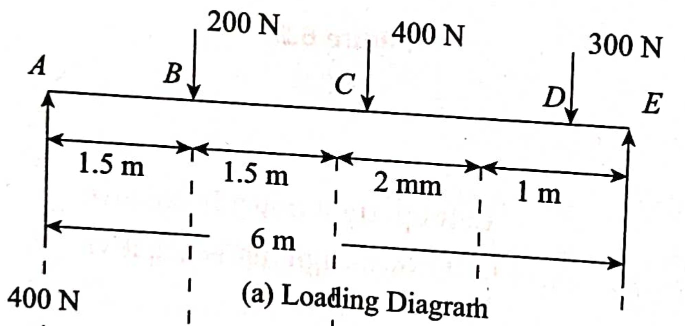


Figure 6.4

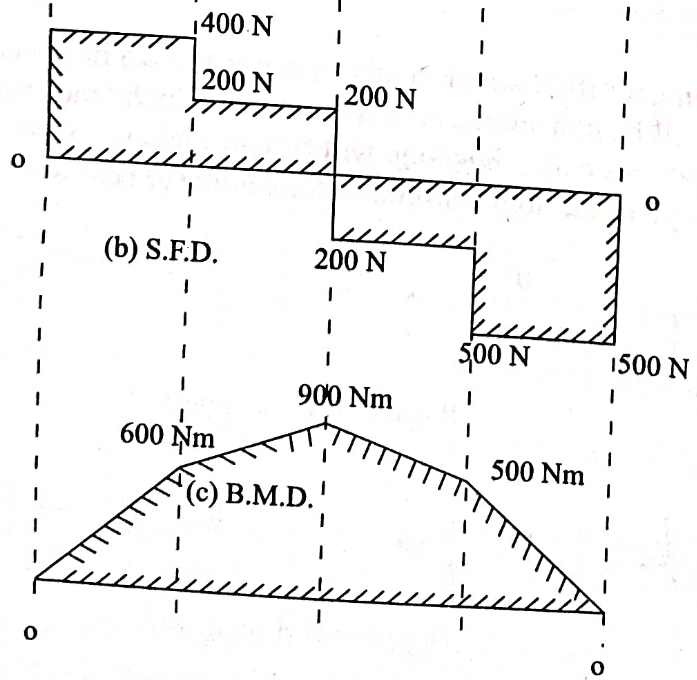


Figure 6.5

Calculations:

Beam Reactions

Taking moments about A, and working in N and m, we get

$$R_E \times 6 = (200 \times 1.5) + (400 \times 3) + (300 \times 5)$$

$$= 300 + 1200 + 1500 = 3000$$

$$\therefore \quad R_E = \frac{3000}{6} = 500 \text{ N}$$

Now, $\quad R_A + R_E = \text{Sum of loads}$

$$R_A = 200 + 400 + 300 - 500 = 400 \text{ N}$$

SFD Calculations:

Starting from left point A,

SF at $A = 0$ rising to 400 N

SF at $B = 400$ falls to $400 - 200 = 200$ N

SF at $C = 200$ N falls to $+200 - 400 = -200$ N

SF at $D = -200$ N falls to $-200 - 300 = -500$ N

SF at $E = -500$ N rises to $-500 + 500 = 0$

BMD Calculations:

BM at $A = 0$

BM at $B = +(400 \times 1.5) = +600$ Nm

BM at $C = +(400 \times 3) - (200 \times 1.5) = 1200 - 300$

$= +900$ Nm

BM at $D = +(400 \times 5) - (200 \times 3.5) - (400 \times 2)$

$= +2000 + 700 - 800 = +500$ Nm

BM at $E = +(400 \times 6) - (200 \times 4.5) - (400 \times 3)$

$= -(300 \times 1) = +2400 - 900 - 1200 - 300 = 0$

Note: Scales for drawing SFD and BMD should be chosen such that it suits the size of the paper and place allotted for diagrams.

EXAMPLE 6.2: An overhang beam (refer Fig. 6.6) is loaded as shown. Draw to seek the shearing force and bending moment diagram and mark all the important points.

SOLUTION:

Reactions: Taking moments about A and working in kN and m units

$$R_B \times 3 + \times 0.3 = (10 \times 0.9) + (15 \times 2.1) + (10 \times 3.6)$$

$$R_B \times 3 = 9 + 31.5 + 36 - 1.5 = 75$$

$$R_B = \frac{75}{3} = 25 \text{ kN}$$

Now $\quad R_A + R_B = \text{Sum of loads}$

$$R_A + 25 = 5 + 10 + 15 + 10 = 40 \text{ kN}$$

$$\therefore \quad R_A = 40 - 25 = 15 \text{ kN}$$

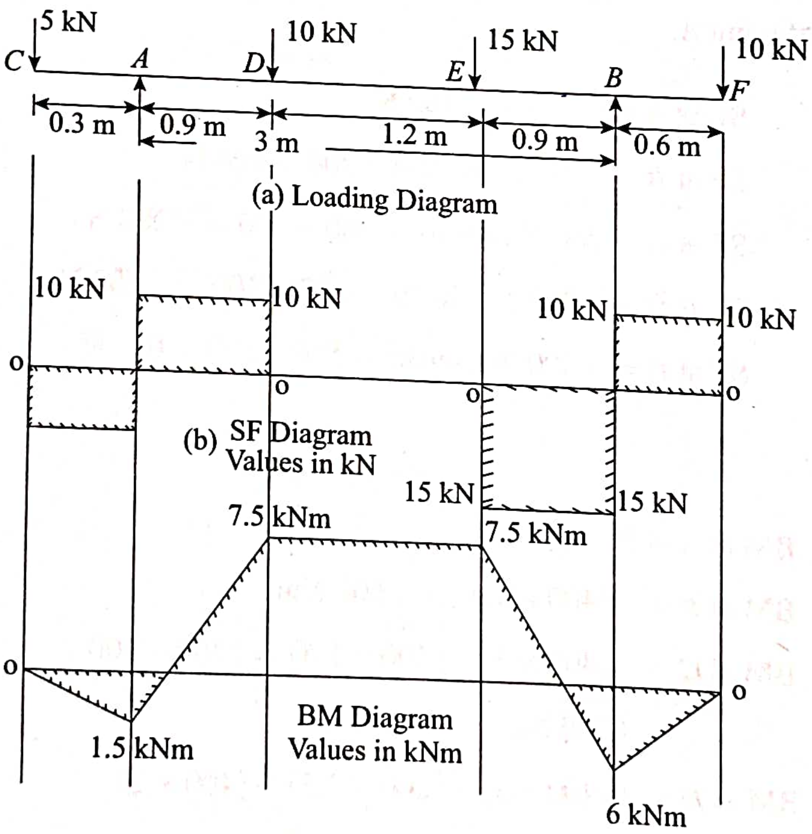


Figure 6.6

Figure 6.7

SFD Calculations:

$$\text{At } C = 0 \text{ falls to } -5 \text{ kN}$$

$$\text{At } A = -5 \text{ rising to } -5 + 15 = +10 \text{ kN}$$

$$\text{At } D = +10 \text{ kN falls to } +10 - 10 = 0$$

EXAMPLE 6.2: An overhang beam (refer Fig. 6.6) is loaded as shown. Draw to seek the shearing force and bending moment diagram and mark all the important points.

SOLUTION:

Reactions: Taking moments about A and working in kN and m units

$$R_B \times 3 + \times 0.3 = (10 \times 0.9) + (15 \times 2.1) + (10 \times 3.6)$$

$$R_B \times 3 = 9 + 31.5 + 36 - 1.5 = 75$$

$$R_B = \frac{75}{3} = 25 \text{ kN}$$

Now $\quad R_A + R_B = \text{Sum of loads}$

$$R_A + 25 = 5 + 10 + 15 + 10 = 40 \text{ kN}$$

$$\therefore \quad R_A = 40 - 25 = 15 \text{ kN}$$

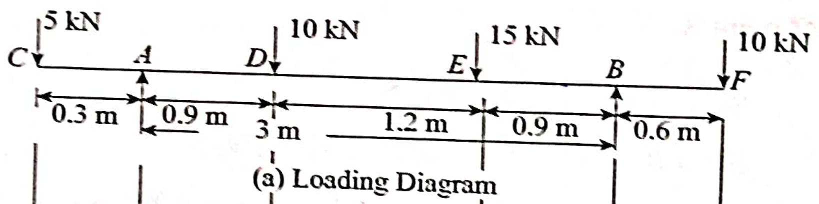


Figure 6.6

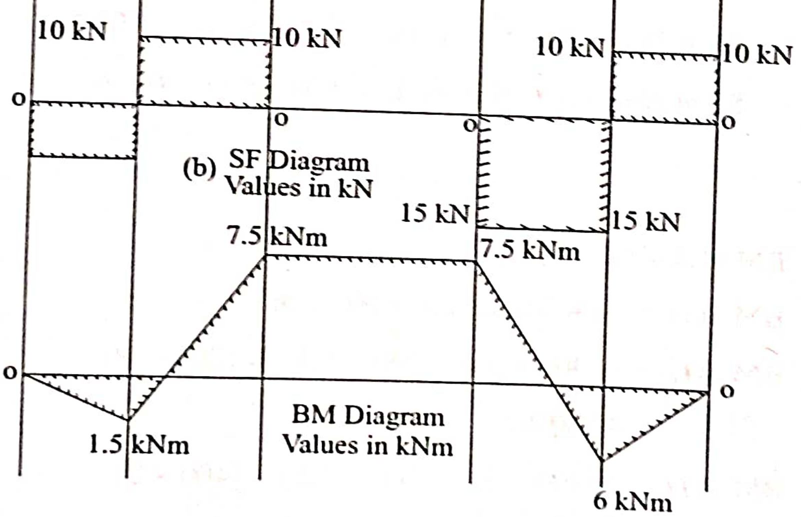


Figure 6.7

SFD Calculations:

At $C = 0$ falls to -5 kN

At $A = -5$ rising to $-5 + 15 = +10$ kN

At $D = +10$ kN falls to $+10 - 10 = 0$

At $E = 0$ falls, to $0 - 15 = -15$ kN

At $B = -15$ kN rises to $-15 + 25 = +10$ kN

At $F = +10$ falls to $+10 - 10 = 0$

BMD Calculations:

$$\text{At } C = 0$$

$$\text{At } A = -5 \times 0.3 = -1.5 \text{ kNm}$$

$$\text{At } D = (-5 \times 1.2) + (15 \times 0.9) = +7.5 \text{ kNm}$$

$$\text{At } E = (-5 \times 2.4) + (15 \times 2.1) - (10 \times 1.2) = +7.5 \text{ kNm}$$

$$\text{At } B = (-5 \times 3.3) + (15 \times 3) - (10 \times 2.1) - (15 \times 0.9) = -6 \text{ kNm}$$

$$\text{At } F = (-5 \times 3.9) + (15 \times 3.6) - (10 \times 2.7) - (15 \times 1.5) + (25 \times 0.6) = 0$$

The above values are then plotted to scale (for both SFD & BMD) and indicating beneath diagram beams carrying uniformly distributed loads.

EXAMPLE 6.3: A 20 m long beam is simply supported between B and F. It carries uniformly distributed loads of 200 N/m along overhang AB and 100 N/m run over the length EF as shown in Fig. (6.8(a)). It also carries concentrated loads (refer Fig. 6.8(b)). Draw Shear Force and Bending Moment diagrams for the loaded beam and mark all important values.

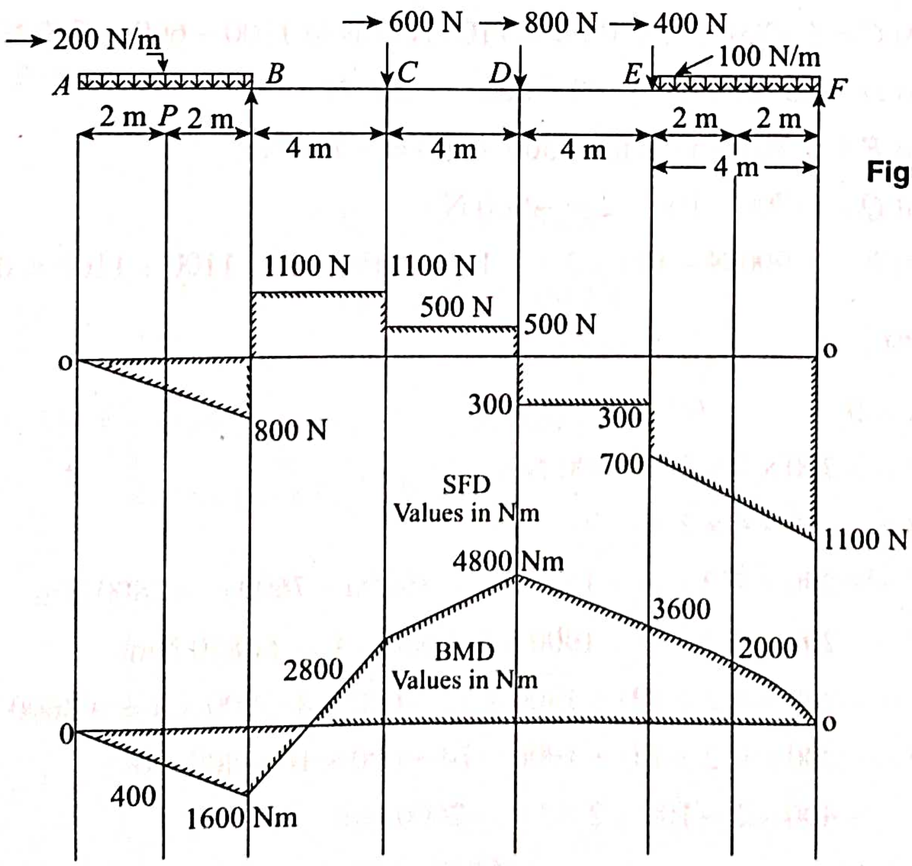


Figure 6.8(a)

Figure 6.8(b)

Beam Reactions:

Taking moments about F and working in N and m units, we get:

$$R_B \times 16 = (200 \times 4 \times 18) + (600 \times 12) + (800 \times 8)$$
$$+ (400 \times 4) + (100 \times 4 \times 2)$$
$$= 14400 + 7200 + 6400 + 1600 + 800 = 30400$$

$$\therefore \quad R_B = \frac{30400}{16} = 1900 \text{ N}$$

Now, $\quad R_B + R_F = \text{Sum of loads}$

$$1900 + R_F = (200 \times 4) + 600 + 800 + 400 + (100 \times 4)$$

$$\therefore \quad R_F = 3000 - 1900 = 1100 \text{ N}$$

SFD Calculations:

At $A = 0$

For uniformly distributed load, let us take mid-point P and Q so that we know exactly whether it will be a straight line, concave or convex.

At $P = -200 \times 2 = -400$ N

At $B = -200 \times 4 = -800$ N

At $C = (-200 \times 4) + 1900 = 1100$ N falls to $1100 - 600 = 500$ N

At $D = 500$ N falls to $500 - 800 = -300$ N

At $E = -300$ N falls to $-300 - 400 = -700$ N

At $Q = -700 - 100 \times 2 = -900$ N

At $F = -900$ N $- 100 \times 2 = -1100$ N rises to $-1100 + 1100 = 0$

BMD Calculations:

At $A = 0$

At $P = -200 \times 2 \times 2 = -400$ Nm

At $B = -200 \times 4 \times 2 = -1600$ Nm

At $C = -200 \times 4(2+4) + 1900 \times 4 = 19200 + 7600 = +2800$ Nm

At $D = -200 \times 4(2+8) + 1900 \times 8 - 600 \times 4 = +4800$ Nm

At $E = -200 \times 4(2+12) + 1900 \times 12 - 600 \times 8 - 800 \times 4 = +3600$ Nm

At $Q = -200 \times 4(2+14) + 1900 \times 14 - 600 \times 10 - 800 \times 6$
$\quad - 400 \times 2 - 100 \times 2 \times 1 = +2000$ Nm

At $F = 0$

EXAMPLE 6.4: For the loaded beam in Fig. 6.9, draw BM and SF diagrams and show important points.

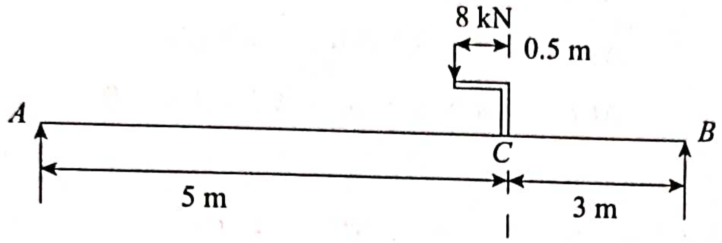


Figure 6.9(a)

This beam (6.9b) is equivalent to shown in (6.9a)

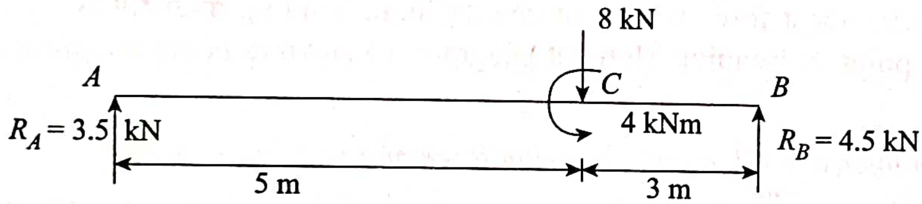


Figure 6.9(b)

Reactions:

Taking moments about *B*,

$$R_A \times 8 = 8 \times 3 + 4 = 28$$

$$\therefore \quad R_A = \frac{28}{8} = 3.5 \text{ kN}$$

$$R_B = 8 - 3.5 = 4.5 \text{ kN}$$

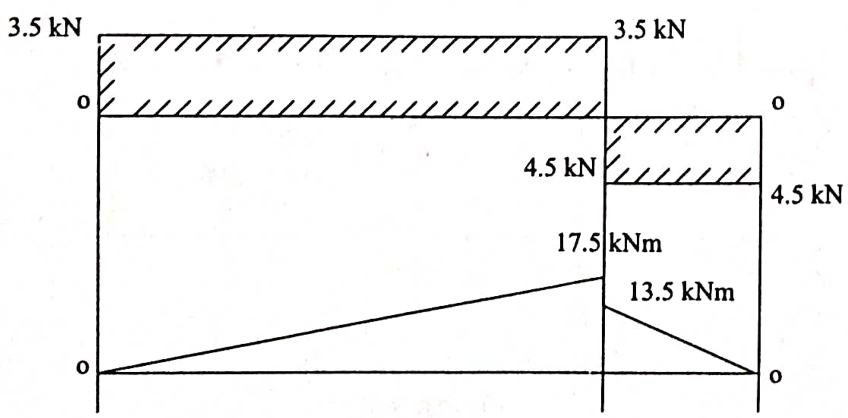


Figure 6.9(c)

SFD Calculations:

$$\text{At } A = 0 \text{ rises to } +3.5 \text{ kN}$$
$$\text{At } C = 3.5 \text{ to } 3.5 - 8 = -4.5 \text{ kN}$$
$$\text{At } B = -4.5 \text{ rises to } -4.5 + 4.5 = 0$$

BMD Calculations:

$$\text{At } A = 0$$
$$\text{At } C = +3.5 \times 5 = 17.5 \text{ kNm}$$
$$\text{Also at } C = 17.5 - 4 = 13.5 \text{ kNm}$$

Remember: i) where shear force is zero or change sign, bending moment is maximum; ii) where B.M. is zero that point on Bending Moment Diagram is known as *point of contraflexure* or *point of inflexion*.

Relation between intensity of loading, shearing force and bending moment:

Consider a short length, dx of a beam Fig. 6.10, carrying a uniformly distributed load w per unit length. Over this length, let the shear force change from F to $F + dF$ and the bending moment change from M to $M + dM$.

Equating vertical force on the element,

$$F + wdx = F + dF$$

$$\therefore w = \frac{dF}{dx} \qquad \text{(i)}$$

Taking moments about the right-hand end of the element,

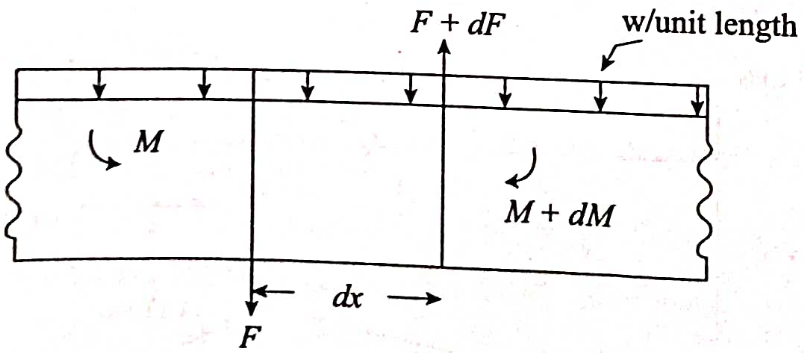


Figure 6.10

$$M + F.dx + w.dx\frac{dx}{2} = M + dM$$

$$F.dx = dM \qquad \text{ignoring the second order of small quantities}$$

$$\therefore \quad F = \frac{dM}{dw} \tag{ii}$$

Therefore, intensity of loading is the rate of change of shearing force and shearing force is the rate of change of bending moment. This latter relation shows that the maximum bending moment occurs where the shearing force is zero. Combining Eqns. (i) and (ii)

$$w = \frac{dF}{dx} = \frac{d^2M}{dx^2}.$$

Cantilevers

EXAMPLE 6.5: Draw SF and BM diagrams for the cantilever loaded as shown in Fig. (6.11) and show important values on both diagrams.

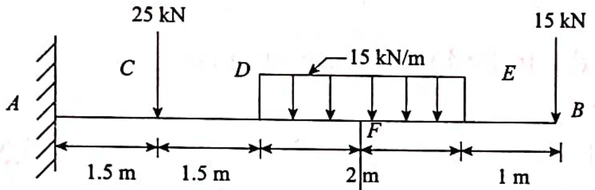


Figure 6.11

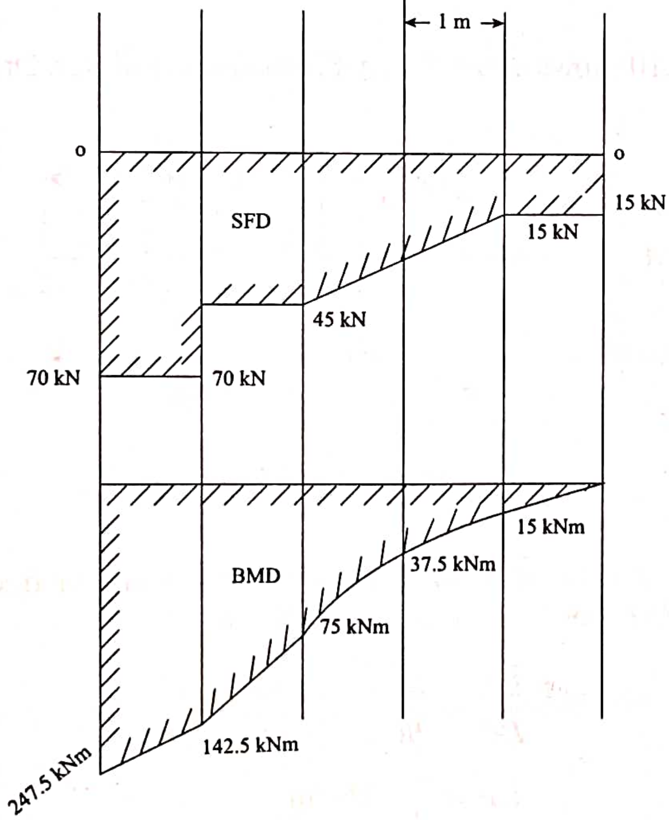


Figure 6.12

SF Calculations:

$$\text{SF at } B = 0 \text{ falling to } -15 \text{ kN}$$

$$\text{SF at } E = -15 \text{ kN}$$

$$\text{SF at } D = -15 \text{ kN} - 15 \times 2 = -45 \text{ kN}$$

$$\text{SF at } C = -45 \text{ kN falls to } -45 - 25 = -70 \text{ kN}$$

$$\text{SF at } A = -70 \text{ kN}$$

BM Calculations:

$$\text{BM at } = 0$$

$$\text{BM at } E = -15 \times 1 = -15 \text{ kNm}$$

$$\text{BM at } D = -15 \times 3 - 15 \times 2 \times 1 = -75 \text{ kNm}$$

$$\text{BM at } C = -15 \times 4.5 - 15 \times 2 \times 2.5 = -142.5 \text{ kNm}$$

$$\text{BM at } A = -15 \times 6 - 15 \times 2 \times 4 - 25 \times 1.5 = -247.5 \text{ kNm}$$

Take extra mid-point F on u.d.l. to find the nature of curve

$$\text{BM at } F = -15 \times 2 - 15 \times 1 \times \frac{1}{2} = -37.5 \text{ kNm}$$

EXAMPLE 6.6: Figure 6.10 shows a partial variable loaded cantilever. Draw SF and BM diagrams.

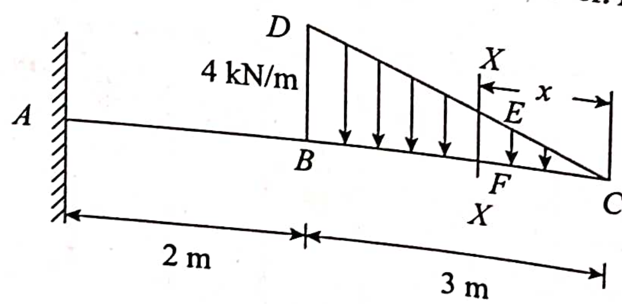


Figure 6.13(a)

Consider a section $X - X$ between B and C at a distance x from the free end C. ΔBCD is similar to ΔCEF For Rate of loading at $X - X$

$$\frac{EF}{DB} = \frac{FC}{BC} \quad ; \quad \frac{EF}{4} = \frac{x}{3}$$

$$EF = \frac{4}{3}x \text{ kN/m}$$

Now,

$$\text{Shear force at } X-X = \text{Total load from } C \text{ to } F$$

$$= \frac{1}{2} \times \frac{4x}{3} \times x = \frac{2x^2}{3} \text{ kN} \qquad \text{(i)}$$

For SF at C, put $x = 0$ in Eqn. (i) $\therefore$ SF at $C = 0$
SF at B, put $x = -3$ in Eqn. (i)

$$\therefore \quad \text{SF at } B = \frac{2 \times 3^2}{3} = -6 \text{ kN}$$

$$\text{SF at } A = -6 \text{ kN}$$

BM Calculations:

$$\text{BM at } X-X = -\frac{2x^2}{3} \times \frac{x}{3} = \frac{-2x^3}{9} \text{ kN/m} \qquad \text{(ii)}$$

Equation (ii) is that of a cubic parabola

$$\text{BM at } C = 0$$

BM at B, put $x = 3$ in Eq. (ii) $= \dfrac{-2(3)^3}{9} = -6$ kNm

$$\text{BM at } A = \text{Total load on } BC \times \left(2 + \frac{3}{3}\right)$$

$$= -\left(\frac{1}{2} \times 4 \times 3\right) \times (2+1)$$

$$= -18 \text{ kNm}$$

Between A and B, BMD will be a straight line.

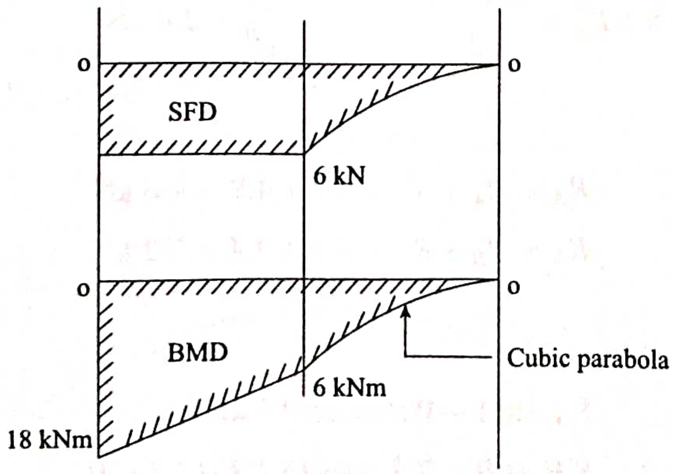


Figure 6.13(b)

EXAMPLE 6.7: In a gradually varying loaded beam as shown in Fig. 6.14, the span of simply supported beam is 6 m. Load at B is 1.6 kN/m run and gradually increases to 4 kN/m run at the other end.

Determine the position and amount of maximum bending moment. Also draw the shear force and bending moment diagram.

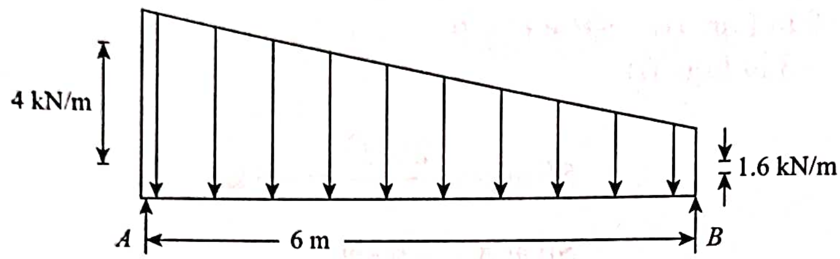


Figure 6.14

Let us split the beam in two parts:

i) a rectangular of length 6 m and height 1.6 kN/m run.
ii) a triangular zero at B and $4 - 1.6 = 2.4$ kN/m at A.

Let R'_A and R'_B be reactions due to part (i)

$$R'_A = R'_B = \frac{1.6 \times 6}{2} = 4.8 \text{ kN}$$

Let R''_A and R''_B be reactions due to part (ii)

Taking moments about B,

$$R''_A \times 6 = \frac{2.4 \times 6}{2} \times \frac{2}{3} \times 6 \qquad \therefore \quad R''_A = 4.8 \text{ kN}$$

$$R''_A + R''_B = \text{Total triangular beam (part ii)}$$

$$4.8 + R''_B = \frac{2.4 \times 6}{2} \qquad \therefore \quad R''_B = 2.4 \text{ kN}$$

Net reactions:

$$R_A = R'_A + R'' = 4.8 + 4.8 = 9.6 \text{ kN}$$

$$R_B = R'_B + R''_B = 4.8 + 2.4 = 7.2 \text{ kN}$$

SFD Calculations:

S.F. at $A = 0$ rises to 9.6 kN

S.F. at $B = 7.2$ rises to $7.2 - 7.2 = 0$

BMD Calculations:

$$\text{Obviously, BM at } A = 0$$

$$\text{BM at } B = 0$$

Maximum bending moment occurs where SF changes sign.

Let P be the point where maximum bending moment occurs and S.F. changes sign.

Let x be the distance between P and B.

As, SF is zero, therefore,

$$0 = -7.2 + 1.6x + \frac{2.4x}{6} \times x \times \frac{1}{2}$$

$$0 = -7.2 + 1.6x + 0.2x^2$$

$$0.2x^2 + 1.6x - 7.2 = 0$$

$$x^2 + 8x - 36 = 0$$

$$x = \frac{-8 \pm \sqrt{64 + 4 \times 36}}{2} = \frac{-8 \pm 14.42}{2}$$

$$= 3.21 \text{ m.}$$

$$\text{Maximum BM} = +7.2 \times 3.21 - 1.6 \times 3.21 \times \frac{3.21}{2} - \frac{2.4 \times 3.21}{6} \times \frac{3.21}{3}$$

$$- 23.112 - 8.24 - 1.37 = 13.5 \text{ kNm} \quad \text{Ans.}$$

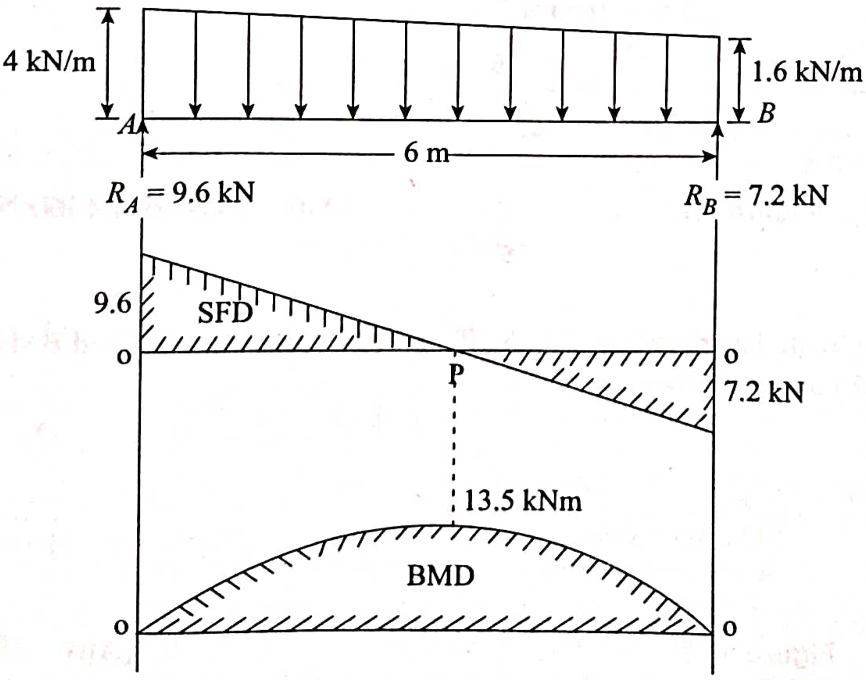


Figure 6.15

Exercise

6.1 Draw and SF and BM diagrams for the loaded cantilever as shown is in Fig. (6.16) and state BM at *D*.

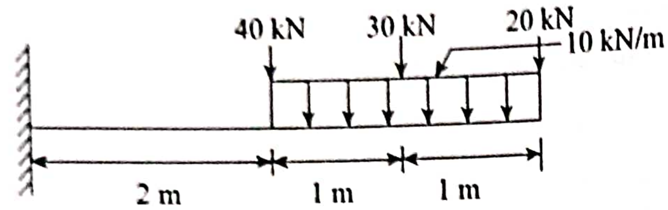


Figure 6.16

[Ans (310 kNm)]

6.2 A beam *AB* of length 8 m is simply supported at *A* and *B*. It carries a load which varies from 2 kN/m at *A* to 5 kN at *B*. Draw SF and BM diagrams. State what is maximum bending moment and its position. Show all principal values on SF and BM on diagrams

[Ans (28.1 kNm) at 4.3m from A]

6.3 A beam *AB* is hinged at *A* and simply supported at *B*. A rigid bracket is attached to the middle point *C* of the beam is loaded as shown in Fig. 6.17. Draw SF and BM diagrams showing important values. What is the maximum BM and its position.

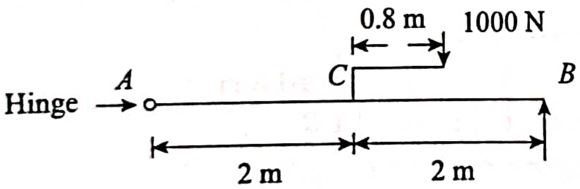


Figure 6.17

[Ans Max. BM 1400 Nm at 2 m from A]

6.4 For the beam loaded as shown in Fig. 6.18, at 2 m from *A* draw SF and BM diagrams. Indicate maximum BM and its position.

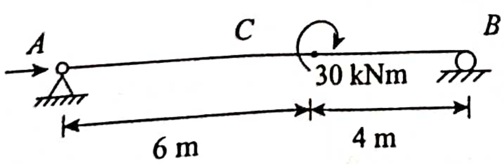


Figure 6.18

[Ans 18 Nm, 6 m from A]

6.5 Draw the SF and BM diagram for beam shown in Fig. 6.19. Locate the point of contraflexure.

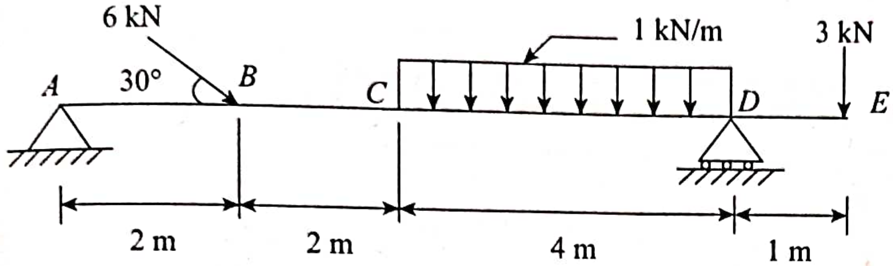


Figure 6.19

[Ans 0.8 m from D]

6.6 For the loaded beam shown in Fig. 6.20. Draw SF and BM diagrams and indicate all principal values on both diagrams. Also find maximum bending moment and its position.

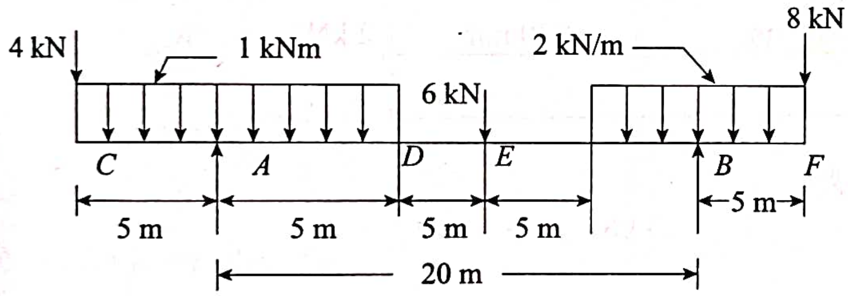


Figure 6.20

[Ans 65 kNm 5 m from *F*]

6.7 Draw SF and BM diagrams for a cantilever loaded as shown in Fig. 6.21. Show all the principal values on both diagrams.

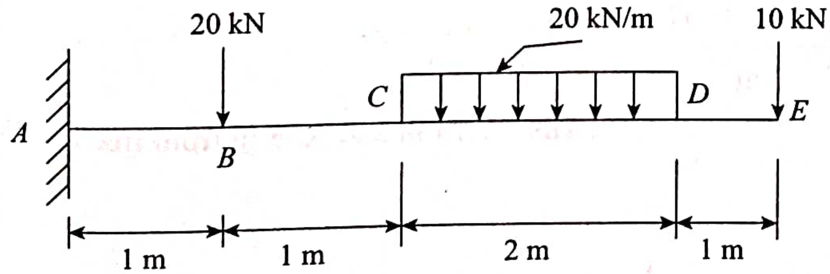


Figure 6.21

[Ans Max BM = 190 kNm]

6.8 Draw SF and BM diagrams for loaded cantilever as shown in Fig. 6.22 and indicate position and value of maximum BM.

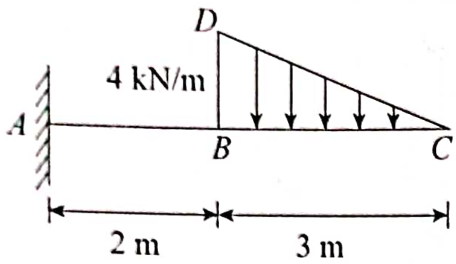


Figure 6.22

[**Ans** 18 kNm at *A*]

6.9 A simply supported beam of 5 m span carries a triangular load of total 30 kN. Draw SF and BM diagrams and indicate the maximum value of BM and its position.

[**Ans** 25 kNm at centre]

6.10 Draw the SF and BM diagrams for the cantilever beam shown in Fig. 6.23. The vertical load of 2 kN at *C* is partly supported by a force of 3 kN at the prop *B*. State: a) the reaction at the built in end; b) the maximum BM and its position and c) where the BM is zero.

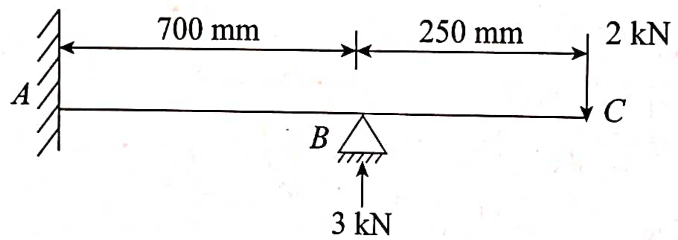


Figure 6.23

[**Ans** a) 1 kN; b) 0.5 kNm; c) At *B* downward]

6.11 Draw SF and BM diagrams for the beam shown in Fig. 6.24. Determine the points of contraflexure. What is the maximum BM.

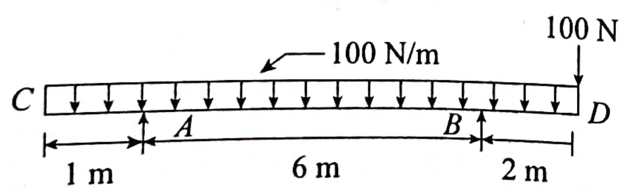


Figure 6.24

[**Ans** 7.79 m and 3.38 m from free end *D*]

CHAPTER 9

BENDING OF BEAMS

The shearing forces and bending moments at the various sections of a loaded beam give rise to shearing and bending stresses in the beam and, by making a number of assumptions, the relationship existing between the stresses, the strains, the bending moment, the curvature as well as the elasticity of the beam will be established in the following sections of this chapter.

If a beam $ABCD$ of uniform section is subjected to bending moment M the beam will bend with radius R, subtending an angle θ at sector $P'OQ'$. After bending the beam, has taken shape as $A'B'C'D'$ and centroidal axis PQ as $P'Q'$. Now upper longitudinal layer is subjected to compression and has become $A'B'$. The lowermost layer DC has become $D'C'$ and is in tension. Since at top it is compression and at bottom it is tension, so obviously at centroidal axis, there will be no stress and is called neutral axis. Here $P'Q'$ is neutral axis and layers of fibres here suffer no stress or strain due to bending. Consequently, its original length PQ remain, unchanged, i.e., $PQ = P'Q'$.

Before we derive any relationship between stresses, we have to make a number of *assumptions* which are given below:

i) The material obeys Hooke's law and is within elastic limit.
ii) The beam is initially straight and unstressed.
iii) The material is homogeneous and isotropic.
iv) The beam bends in the form of a circular arc.
v) The value of modulus of elasticity E for the material of the beam has the same value in compression as in tension.
vi) Plane transverse sections of the beam before bending remain plane after bending.

Relationship Between Curvature and Strain

After application of bending moment M, the radius of curvature R of the bent beam is measured from the centre O to the neutral surface. If θ is the angle is radius subtended by the arc $A'B'$ at centre O, then since the neutral surface remains unchanged in length then,

$$\text{Line } PQ = \text{Arc } P'Q'$$
$$P'Q' = R\theta$$

Now let us consider the bottom layer of the beam distance y from the neutral surface.

$$\text{Length of } DC \text{ before bending} = PQ = P'Q' = R\theta$$

$$\text{Length of } DC \text{ after bending} = D'C' = (R+y)\theta$$

$$\therefore \quad \text{Extension of } DC = (R+y)\theta - R\theta$$

$$= y\theta$$

$$\text{Strain in } DC,\ \varepsilon = \frac{\text{Extension}}{\text{Original length}} = \frac{y\theta}{R\theta}$$

$$\therefore \quad \varepsilon = \frac{y}{R}$$

Also, strain in DC, $\varepsilon = \dfrac{\sigma}{E}$

$$\therefore \quad \frac{\sigma}{E} = \frac{y}{R};\ \frac{\sigma}{y} = \frac{E}{R} \qquad \text{(i)}$$

Since, Young's modulus E is constant for the material of the beam, and R is also constant for the particular curvature considered, the stress σ varies across the depth of the beam from zero at the neutral axis XX to a maximum value at the outer layers of fibres. Since, we are only considering sections that are symmetrical about the neutral axis XX, the distance y from the neutral axis XX to the outer layers of fibres is equal to half the overall depth of the beam. The stress distribution is drawn in Fig. 9.1 where compressive stresses are plotted to the left of axis $Y-Y$, and tensile stresses to the right.

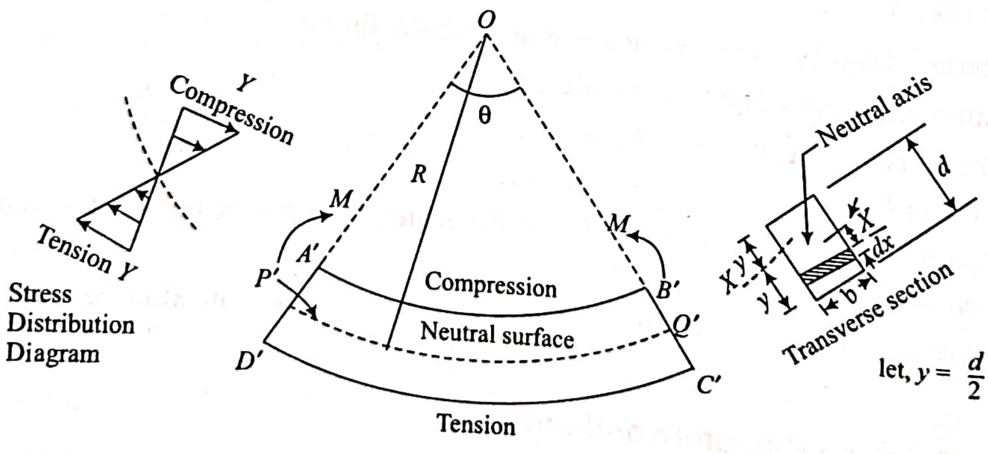


Figure 9.1

Moment of Resistance

The moment of resistance of a beam is the resultant moment about the neutral axis of the internal tensile and compressive forces resisting the applied bending moment M.

Referring again to Fig. 9.1, let us consider an elemental strip of the cross section of the beam at a distance x from the neutral axis XX and of thickness dx.

If σ_1 is the stress at the strip then, since the stresses at the various layers of the beam section are proportional to their distances from the neutral axis XX.

$$\frac{\sigma_1}{\sigma} = \frac{x}{y}; \qquad \sigma \text{ is the stress at } y.$$

$$\therefore \quad \sigma_1 = \sigma\frac{x}{y}$$

$$\text{Force (F) on strip} = \text{Stress} \times \text{Area} = \sigma\frac{x}{y} \times b\,dx$$

Moment of this force about the neutral axis XX

$$= \text{Force} \times \text{Distance from the neutral axis } XX$$

$$= \left(\sigma\frac{x}{y} \times b\,dx\right)x = \frac{\sigma}{y}bx^2\,dx$$

From Eqn. (i), $= \frac{E}{R}bx^2\,dx$

The total moment of resistance is the sum of the moments of all the elemental stripes above and below the neutral axis (NA), XX, i.e., from $x = y = d/2$ to $x = y = -d/2$ and is given by:

$$M = \int_{-d/2}^{+d/2} \frac{E}{R}bx^2\,dx \quad \therefore \quad M = \frac{E}{R}\int_{-d/2}^{+d/2} bx^2\,dx$$

$$M = \frac{E}{R}\left[\frac{bx^3}{3}\right]_{-d/2}^{+d/2}$$

$$\text{i.e.,} \quad M = \frac{E}{R} \times \frac{bd^3}{12}$$

The quantity $\frac{bd^3}{12}$ is called the second moment of area of the section about NA and is denoted by I.

$$\therefore \quad M = \frac{E}{R} \times I$$

$$\frac{M}{I} = \frac{E}{R} \qquad \text{(ii)}$$

Combining Eqn. (i) and (ii)

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{Y}$$

This is known as the *simple bending equation.*

Note: If we had taken any other section, for example, circular, the result would have been same. Only I has to be determined for a particular section.

Modulus of Section

$$\frac{M}{I}=\frac{\sigma}{y} \quad \text{or} \quad M=\frac{I}{y}\sigma$$

Since y represents the distance of the outer layer of fibres from NA, the ratio of the second moment of area (moment of inertia) I to y is called modulus of section and is denoted by τ.

$$\text{Thus,} \quad Z=\frac{I}{y} \qquad \therefore \quad M=Z\sigma \tag{iii}$$

Equation (iii) is very important since it enables us to determine readily the permissible or maximum bending moment on a beam.

Since the beam of Fig. 9.1 is of rectangular section.

$$I=\frac{bd^3}{12} \quad \text{and} \quad y=\frac{d}{2}$$

$$\therefore \quad \text{Modulus of section, } Z=\frac{I}{Y}=\frac{bd^3}{12}\times\frac{2}{d}=\frac{bd^2}{6}$$

Note: EI is known as flexural rigidity.

EXAMPLE 9.1: A mild steel bar of 35 mm × 15 mm section and 1.2 m length is simply supported at its ends with the 35 mm edge horizontal. If a load of 100 N is applied at the centre of the bar. determine the maximum stress, produced in the material due to bending.

SOLUTION:

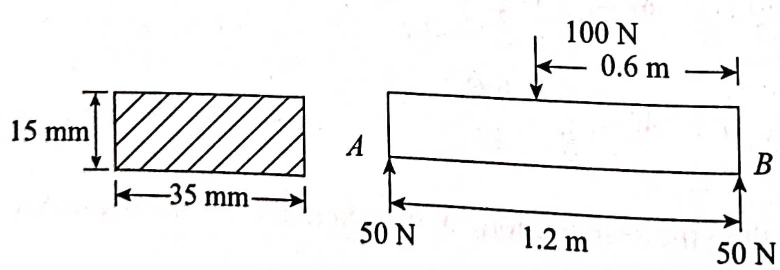


Figure 9.2

$$I=\frac{bd^3}{12}=\frac{35\times 15^3}{12}=9843.75 \text{ mm}^4$$

$$\text{Reactions, } R_A=R_B=\frac{100}{2}=50 \text{ N}$$

$$M=50\times 0.6=30 \text{ Nm}=30000 \text{ Nmm}$$

Maximum stress will occur at 7.5 mm from NA.

$$y = 7.5 \text{ mm}$$

$$\text{Now,} \quad \frac{M}{I} = \frac{\sigma}{y} \qquad \therefore \quad \sigma = \frac{M}{I}y$$

$$\sigma = \frac{30000}{9843.75} \times 7.5 = 22.86 \text{ N/mm}^2 = 22.86 \text{ MN/m}^2$$

EXAMPLE 9.2: A tubular frame member of 85 mm external diameter is subjected to a bending moment of 600 Nm. If the stress set up in the material due to bending is 50 MN/m^2, find the internal diameter of the member.

SOLUTION:

Let D and d be the external and internal diameters of the tubular frame.

$$\text{Then,} \quad I = \frac{\pi\,(D^4 - d^4)}{64}, \qquad \sigma = 50 \text{ MN/m}^2 = 50 \text{ N/mm}^2$$

$$\frac{M}{I} = \frac{\sigma}{y}; \qquad \text{Here} \quad y = \frac{85}{2} = 42.5$$

$$M = 600 \text{ Nm} = 600000 \text{ Nmm}, \quad I = \frac{\pi^2(85^4 - d^4)}{64} = \frac{\pi(52200625 - d^4)}{64}$$

Substituting

$$\frac{600000 \times 64}{\pi(52200625 - d^4)} = \frac{50}{42.5}$$

$$\pi(52200625 - d^4) \times 50 = 600000 \times 64 \times 42.5$$

$$52200625 - d^4 = \frac{600000 \times 64 \times 42.5}{\pi \times 50}$$

$$= 10385454.5$$

$$\text{or} \quad d^4 = 52200625 - 10385454.5$$

$$\therefore \quad d^4 = 41815170.5$$

$$\therefore \quad d = 80.41 \text{ mm.} \quad \textbf{Ans}$$

EXAMPLE 9.3: A beam, the cross-section of which is shown in Fig. 9.3, acts as a cantilever, which projects 1.8 m from the wall. The cantilever carries a load of 5 kN at the free end. Calculate the maximum bending stress.

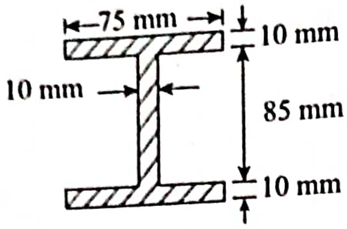


Figure 9.3

SOLUTION:

Since the beam cross section is symmetrical about X and Y axis, therefore I can be obtained as follows:

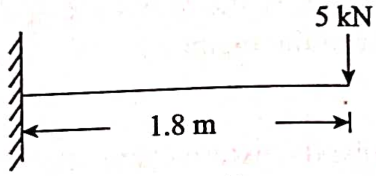


Figure 9.4

$$I = \frac{75\times(85+20)^3}{12} - \frac{(75-10)(85)^3}{12}$$

$$= \frac{75\times 1157625}{12} - \frac{65\times 614125}{12}$$

$$= 7235156.25 - 3326510.42$$

$$I = 3908645.83 \text{ mm}^4$$

$$M = 5\times 1000\times 1.8\times 1000 = 9000000 \text{ mm}^4$$

$$y = 10 + \frac{85}{2} = 52.5$$

Now, $\dfrac{M}{I} = \dfrac{\sigma}{y}$ $\quad\therefore\quad \sigma = \dfrac{M}{I}y$

$$\sigma = \frac{9000000}{3908645.83}\times 52.5 = 120.89 \text{ N/mm}^2$$

or $\quad \sigma = 120.89 \text{ MN/m}^2$. **Ans**

EXAMPLE 9.4: A spring steel strip, 25 mm wide and 1.5 mm thick, is bent to an arc of a circle of 2 m radius. Calculate the bending moment necessary and the maximum stress set up. E for steel $= 200 \text{ GN/m}^2$.

SOLUTION:

$$b = 25 \text{ mm};\quad d = 1.5 \text{ mm}$$

$$\therefore\quad I = \frac{bd^3}{12} = \frac{25\times 1.5^3}{12} = 7.03 \text{ mm}^4 = \frac{7.03}{10^{12}} \text{ m}^4$$

Distance from N.A. to extreme layer of fibres } $= y = \frac{d}{2} = \frac{1.5}{2}$ mm

$$y = 0.75 \text{ mm} = \frac{0.75}{10^3} \text{ m}$$

$$\text{Now} \quad \frac{M}{I} = \frac{E}{R} \quad \text{OR} \quad M = \frac{EI}{R}$$

$$M = \frac{200 \times 10^9 \times 7.03}{2 \times 10^{12}} \left[\frac{N}{m^2} \times \frac{m^4}{m}\right]$$

∴ Bending moment $= 0.703$ Nm.

$$\text{For maximum stress} \quad \sigma = \frac{Ey}{R} = \frac{200 \times 10^9 \times 0.75}{2 \times 10^3} \left[\frac{N}{m^2} \times \frac{m}{m}\right]$$

$$\sigma = 75 \times 10^6 \text{ N/m}^2 = 75 \text{ MN/m}^2.$$

Example 9.5: The beam cross section shown in Fig. 9.5 is subjected to a bending moment of 12 kNm. Calculate the maximum stress and its nature indicating its place.

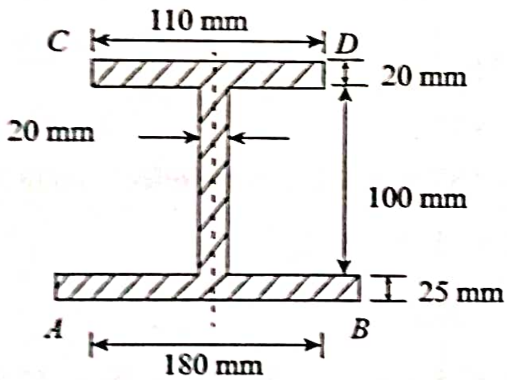


Figure 9.5

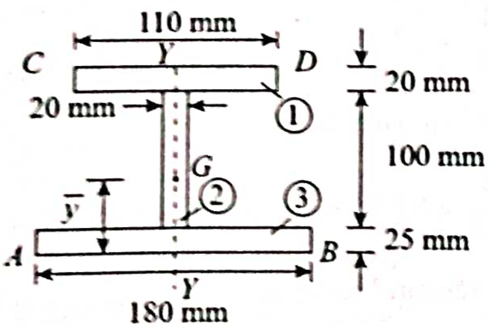


Figure 9.6

We see that the section is symmetrical about $Y-Y$ axis. Let the centre of gravity G be at $\bar{Y}$ from AB. In order to find G, let us divide the section into three simple rectangles, i.e., 1, 2 & 3.

$$a_1 = 110 \times 20 = 2200\ \text{mm}^2; y_1 = 135,\ a_2 = 100 \times 20 = 2000\ \text{mm}^2;$$

$$y_2 = 75;\ a_3 = 180 \times 25 = 4500\ \text{mm}^2;\ y_3 = 12.5\ \text{mm}$$

$$\text{Total area, } A = 2200 + 2000 + 4500 = 8700\ \text{mm}^2$$

$$\bar{y} = \frac{2200 \times 135 + 2000 \times 75 + 4500 \times 12.5}{8700} = \frac{503250}{8700}$$

$$\therefore\quad \bar{y} = 57.84\ \text{mm}$$

Now to find I of the whole section, using parallel axis theorem:

$$I = \left[\frac{110 \times 20^3}{12} + 2200\ (135 - 57.84)^2\right]$$

$$+ \left[\frac{20 \times 100^3}{12} + 2000\ (75 - 57.84)^2\right]$$

$$+ \left[\frac{180 \times 25^3}{12} + 4500\ (12.5 - 57.84)^2\right]$$

$$= [73333.3 + 13098064.3] + [1666666.7 + 588931.2]$$

$$+ [234375 + 15054595.2]$$

$$\therefore\quad I = 30715965.7\ \text{mm}^4$$

$$\frac{M}{I} = \frac{\sigma}{y} \qquad y_t = 57.84,\ y_c = (100 + 25 + 20) - 57.84 = 87.16\ \text{mm}$$

t for tension, c for compression

$$\sigma = \frac{M}{I} y$$

$$\sigma_t = \frac{12000000}{30715965.7} \times 57.84 = 22.596\ \text{N/mm}^2$$

$$= 22.6\ \text{MN/m}^2\ \text{(tensile at } AB)$$

$$\sigma_c = \frac{12000000}{30715965.7} \times 87.16 = 34.05\ \text{N/mm}^2$$

$$= 34\ \text{MN/m}^2\ \text{(compressive at } CD)$$

Hence, maximum stress is 34 MN/m^2 at CD of compressive nature. **Ans**

EXAMPLE 9.6: The horizontal beam of channel cross section as shown in Fig. 9.7 is 2.8 m long and is simply supported at the ends. Determine the maximum uniformly distributed load it can carry if the tensile and compressive stresses must not exceed 35 and 52 N/mm^2, respectively. Neglect the weight of channel.

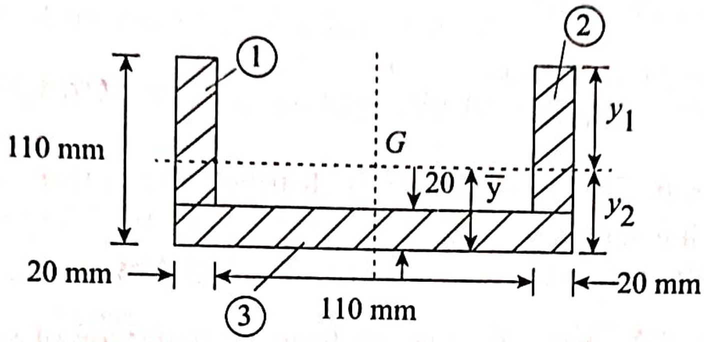


Figure 9.7

In order to find position of G, let us divide the channel in three simple rectangles. Being symmetrical about Y axis, we have to only find $\bar{y}$.

$$a_1 = 90 \times 20 = 1800;\ y_1 = 65 \text{ mm};\ a_2 = 1800 \text{ mm}^2,$$

$$y_2 = 65 \text{ mm},\ a_3 = 150 \times 20 = 3000 \text{ mm}^2;\ y_3 = 10 \text{ mm}.$$

$$\text{Total area, } A = 1800 + 1800 + 3000 = 6600 \text{ mm}^2$$

$$\bar{y} = \frac{1800 \times 65 + 1800 \times 65 + 3000 \times 10}{6600} = 40 \text{ mm}$$

$$\bar{y} = Y_2 = 40 \text{ mm},\ y_1 = 110 - 40 = 70 \text{ mm}$$

$$I = \left[\frac{20 \times 90^3}{12} + 1800\,(65 - 40)^2\right]$$

$$\times 2 + \left[\frac{150 \times 20^3}{12} + 3000\,(10 - 40)^2\right]$$

$$= [1215000 + 1125000] \times 2 + [100000 + 2700000]$$

$$= 4680000 + 2800000 = 7480000 \text{ mm}^4$$

$$\sigma = \frac{M}{I}y \qquad M = 1.4w \times \frac{1.4}{2}$$

$$= 0.98w \text{ Nm}$$

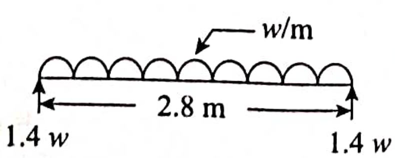


Figure 9.8

Since $y_1 > y_2$, so stress will be greater at y_1 of

$$52 = \frac{0.98w \times 1000}{7480000} \times 70 \ldots \text{(i)} \qquad \therefore \quad w = 5669.97\ \text{N/m}$$

Now taking lower side, i.e., $y_2 = 40$

$$35 = \frac{0.98w \times 1000}{7480000} \times 40 \ldots \text{(ii)} \qquad \therefore \quad w = 6678.6\ \text{N/m}$$

Obviously, if we choose 6.678 kN/m uniformly distributed load then stress in equation (i) will exceed 52 N/mm^2 which is not permissible.

Hence, safe uniformly distributed load $(w) = 5.67$ kN/m **Ans.**

EXAMPLE 9.7: A rectangular beam is to be cut from a circular log of wood of diameter D. Find the dimensions of the strongest section in bending.

SOLUTION:

$$\frac{M}{I} = \frac{\sigma}{y} \quad \text{or} \quad M = \sigma \frac{I}{y} = \sigma Z$$

Let the strongest section cut out of the circular log of diameter D be of width b and depth d.

The section modulus for it is:

$$Z = \frac{bd^2}{6}$$

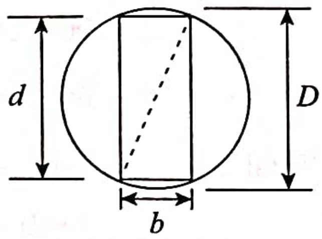


Figure 9.9

Also $b^2 + d^2 = D^2$ or $d^2 = D^2 - b^2$

$$Z = \frac{b\,(D^2 - b^2)}{6} = \frac{b\,D^2 - b^3}{6}$$

For strongest section Z should be maximum. Therefore, $\dfrac{dZ}{db} = 0$.

$$\frac{1}{6}(D^2 - 3b^2) = 0$$

$$D^2 - 3b^2 = 0 \quad \text{or} \quad 3b^2 = D^2$$

$$b^2 = \frac{D^2}{3} \quad \text{or} \quad b = \frac{D}{\sqrt{3}}$$

$$\text{But} \quad d^2 = D^2 - b^2$$

$$= D^2 - \frac{D^2}{3} = \frac{2D^2}{3}$$

For strongest section $d = D\sqrt{\frac{2}{3}}$ **Ans**

EXAMPLE 9.8: A horizontal cantilever 3.2 m long is having rectangular cross section 65 mm wide throughout its length. Its depth varies from 55 mm at free end to 200 mm at fixed end. A load of 6 kN at free end is applied, determine the position of maximum stress induced in the section. Also find the value of maximum bending stress.

SOLUTION:

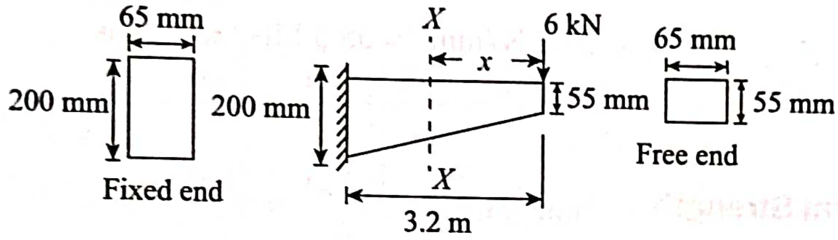


Figure 9.10

Let us consider $X-X$ section at a distance of x from the free end.

$$dx\text{, depth at section } X-X = 55 + \frac{145x}{3200} = \frac{176000+145x}{3200}$$

$$\text{Sectional modulus } Z = \frac{bdx^2}{6} = \frac{65}{6}(176000+145x)^2 \times \frac{1}{3200^2}$$

$$Z = (176000+145x) \times 1.058 \times 10^{-6}$$

$$\sigma = \frac{M}{Z} = \frac{6000x}{(176000+145x)^2 \times 1.058 \times 10^{-6}}$$

$$= \frac{6000 \times 0.945 \times 10^6 x}{(176000+145x)^2} = \frac{5670 \times 10^6 x}{(176000+145x)^2}$$

For maximum stress

$$\frac{d\sigma}{dx} = \frac{d}{dx}\left\{\frac{5670 \times 10^6 x}{(176000+145x)^2}\right\}$$

$$= \frac{5670 \times 10^6 (176000+145x)^2 + 2(176000+145x)145 \times 5670 \times 10^6 x}{(176000+145x)^4}$$

$$176000 + 145x = 290x$$

$$x = 1213.8 \text{ mm}$$

The maximum stress will be at 1213.8 mm from the free end. **Ans**

For maximum stress:

$$\sigma = \frac{6000x}{(176000 + 145x)^2 \times 1.058 \times 10^{-6}}$$

$$\text{(Substituting for } x) = \frac{6000 \times 1213.8 \times 0.945 \times 10^6}{(176000 + 145 \times 1213.8)^2} = \frac{6882246 \times 10^6}{1.24 \times 10^{11}}$$

$$= 55.5 \text{ N/mm}^2 = 55.5 \text{ MN/m}^2 \quad \textbf{Ans}$$

Beams of Uniform Strength

In simply supported beams, carrying a uniformly distributed load, the maximum bending moment will occur at its centre. As we go near the supports, the bending moment reduces untill it become zero. Therefore, it is in interest to save the material near supports. Lot of material can be saved by designing beams of uniform strength. Naturally, we will have to reduce the section gradually untill supports. The section of a beam of uniform strength may be varied in the following ways:

a) By keeping the width uniform and varying depth
b) By keeping the depth uniform and varying width
c) By varying both width and depth.

Generally, uniform strength is maintained by keeping the width uniform and varying the depth.

EXAMPLE 9.9: A simply supported beam of 2.6 m span has a constant width of 120 mm throughout its length with varying depth of 160 mm at the centre to minimum at the ends as shown in Figure 9.11. The beam is carrying a concentrated load P at its centre.

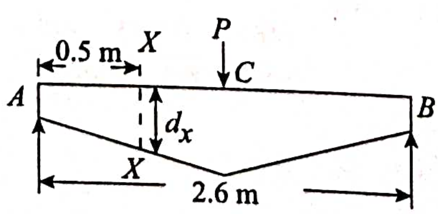


Figure 9.11

Find the minimum depth of the beam at a section 0.5 m from the left hand support, such that the maximum bending stress at this section is equal to that at the mid-span of the beam.

SOLUTION:

$$\text{Width} = 120 \text{ mm}, \text{ span } = 2.6 \text{ m} = 2600 \text{ mm}$$
$$\text{depth at centre } d_c = 160 \text{ mm.}$$

Let depth at 0.5 m from A be d_x at X and σ_x be the bending stress at X.

Reaction $R_A = R_B = \dfrac{P}{2}$

Bending moment at C, $M_c = \dfrac{P}{2} \times 1300 = 650 \text{ P}$

and bending moment at X, $M_x = \dfrac{P}{2} \times 500 = 250 \text{ P}$

Section modulus at centre of beam,

$$Z_c = \frac{b.d_c^2}{6} = \frac{120\,(160)^2}{6} = 512000 \text{ mm}^3$$

$$Z_x = \frac{b.d_x^2}{6} = \frac{120 d_x^2}{6} = 20\, dx^2 \text{mm}^3$$

Bending moment at $C\,(M_c)$,

$$650P = \sigma_c \times Z_c = \sigma_c \times 512000$$

$$\sigma_c = \frac{650P}{512000} \qquad \text{(i)}$$

Similarly, for stress at X,

$$\text{Bending moment at } X, \quad 250P = \sigma_x \times Z_x = \sigma_x \times 20 d_x^2$$

$$\sigma_y = \frac{250P}{20 d_x^2} = \frac{12.5P}{d_x^2} \qquad \text{(ii)}$$

for uniform strength,

$$\sigma_c = \sigma_x$$

Equating (i) and (ii)

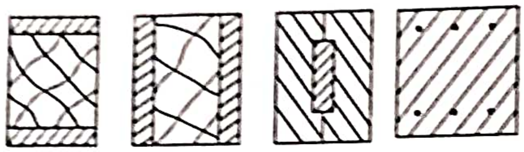
$$\frac{650P}{512000} = \frac{12.5P}{d_x^2}$$

$$d_x^2 = \frac{12.5 \times 512000}{650} = 9846$$

$$\therefore \quad d_x = 99.23 \text{ mm} \quad \textbf{Ans}$$

Composite Beams or Flitched Beams

A composite beam is one which consists of two or more materials rigidly fixed together throughout their length. Generally, wooden beams are reinforced with steel plates to make them stronger. The reinforcing material should have higher modulus of elasticity then the material of the beam. Some examples are given in Figure 9.12.



Composite beam

Figure 9.12

If M_1 and M_2 are the parts of the applied bending moment M carried by the two materials, then

$$M_1 + M_2 = M \tag{i}$$

Here the radius of curvature of the two materials will be same as they are fixed together,

$$R_1 = R_2$$

$$\frac{E_1 I_1}{M_1} = \frac{E_2 I_2}{M_2} \quad \text{or} \quad \frac{M_1}{M_2} = \frac{E_1 I_1}{E_1 I_2} \tag{ii}$$

M_1 and M_2 can be determined from Eqns. (i) and (ii) and the stresses in the two materials are then given by

$$\sigma_1 = \frac{M_1}{Z_1} \quad \text{and} \quad \sigma_2 = \frac{M_2}{Z_2}$$

Alternately, the composite section may be replaced by an equivalent homogeneous section. Thus, the section shown in Fig. 9.13(a) is equivalent to the section in Fig. 9.13(b) composed entirely of material (1) or to the section in Fig. 9.13(c) composed entirely of material (2). Where $\frac{E_2}{E_1} = n$.

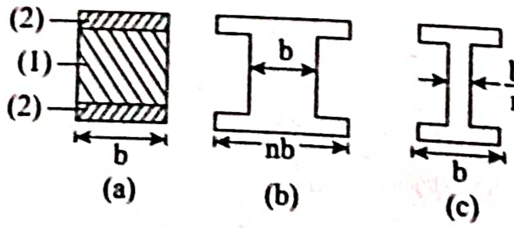


Figure 9.13

In this method first I of the equivalent section is obtained. Then stresses may be obtained from $\sigma = \frac{M}{I} y$, these being the stresses which would exist in the homogeneous section. In section (b), the actual stresses in material (2) would be n times those in the equivalent section and in section (c), the actual stresses in material (1) would be $1/n$ times those in the equivalent section.

In this analysis, it has been assumed that the compound sections are all symmetrical about the plane of bending, otherwise twisting of the section would occur.

EXAMPLE 9.10: A flitched beam consists of a wooden joist 110 mm wide and 220 mm deep reinforced by two steel plates 12 mm thick and 220 mm deep as shown in Fig. 9.14. If the maximum stress in wooden joist is 10 N/mm^2, determine the corresponding stress induced in steel. Also find the moment of resistance of composite section. E for steel = 200 GN/m^2 and E for wood = 12 GN/m^2.

SOLUTION:

$$\text{Modular ratio } m = \frac{E_s}{E_w} = \frac{200}{12} = 16.7$$

Maximum stress in wood = 10 N/mm^2

Corresponding maximum stress in steel,

$$\sigma_s = m\sigma_w = 16.7 \times 10 = 167 \text{ N/mm}^2 \quad \textbf{Ans}$$

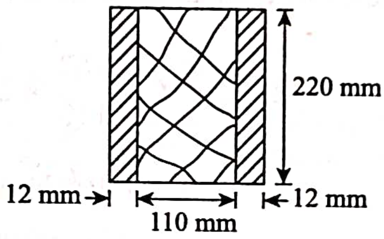


Figure 9.14

$$\text{Total moment of resistance} = \text{Moment of resistance of wood} + \text{Moment of resistance by steel}$$

$$= \frac{\sigma_w I_w}{y} + \frac{\sigma_s I_s}{y}$$

$$\text{Moment of inertia of wood} = I_w = \frac{110 \times 220^3}{12} = 97606666.7 \text{ mm}^4$$

$$\text{Moment of inertia of steel} = I_s = 2 \times \frac{12 \times 220^3}{12} = 21296000 \text{ mm}^4$$

$$\text{Moment of resistance by wood} = M_w = \frac{\sigma_w}{y} \times I_w = \frac{10}{110} \times 97606666.7$$

$$= 8873333.3 \text{ Nmm}$$

$$\text{Moment of resistance by steel, } M_s = \frac{\sigma_s}{y} \times I_s$$

$$= \frac{167}{110} \times 21296000 = 32331200 \text{ Nmm}$$

$$\text{Total moment of resistance} = M_w + M_s = 8873333.3 + 32331200$$

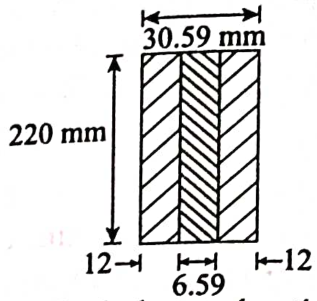
$$= 41204533.3 \text{ Nmm}$$

$$= 41.2 \times 10^6 \text{ Nmm} \quad \textbf{Ans}$$

Alternative method

Equivalent width in terms of steel

$$b_s = \frac{b_w}{m} = \frac{110}{16.7} = 6.59 \text{ mm}$$



Equivalent steel section

Figure 9.15

$$\text{Moment of inertia, } I = \frac{bd^3}{12} = \frac{30.59 \times 220^3}{12}$$

$$= 27143526.7 \text{ mm}^4$$

Moment of resistance of equivalent steel section

$$M = \frac{\sigma_s}{y} I = \frac{167}{110} \times 27143526.7 = 41208809 \text{ Nmm}$$

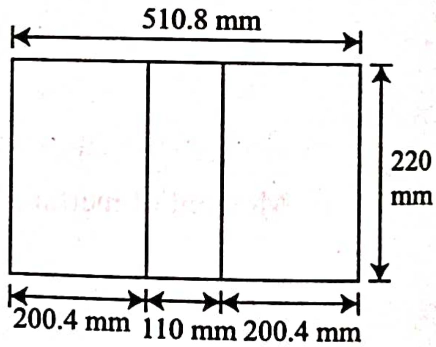
$$= 41.2 \times 10^6 \text{ Nmm} \quad \textbf{Ans}$$

Note: Same as before.

Another alternative method

Equivalent width of steel in terms of wood,

$$b_w = mb_s = 16.7 \times 12 = 200.4 \text{ mm}$$



Equivalent wooden section

Figure 9.16

$$\text{Moment of inertia } I = \frac{bd^3}{12} = \frac{510.8 \times 220^3}{12}$$

$$= 453249867 \text{ mm}^4$$

Moment of resistance of equivalent wooden section,

$$M = \frac{\sigma_w}{y} \times I = \frac{10 \times 453249867}{110} = 41.2 \times 10^6 \text{ Nmm} \quad \textbf{Ans}$$

Note: Same as before.

EXAMPLE 9.11: A wooden beam 120 mm wide and 180 mm deep is to reinforced by two plates 120 mm × 10 mm and 100 × 5 mm. The thicker plate is secured to the top and thinner one to the bottom surface as shown in Fig. 9.17. The permissible stress in steel is 140 MPa and the value of modular ratio is 18. Calculate moment of resistance of strengthened section. Also determine maximum stress in wood (timber).

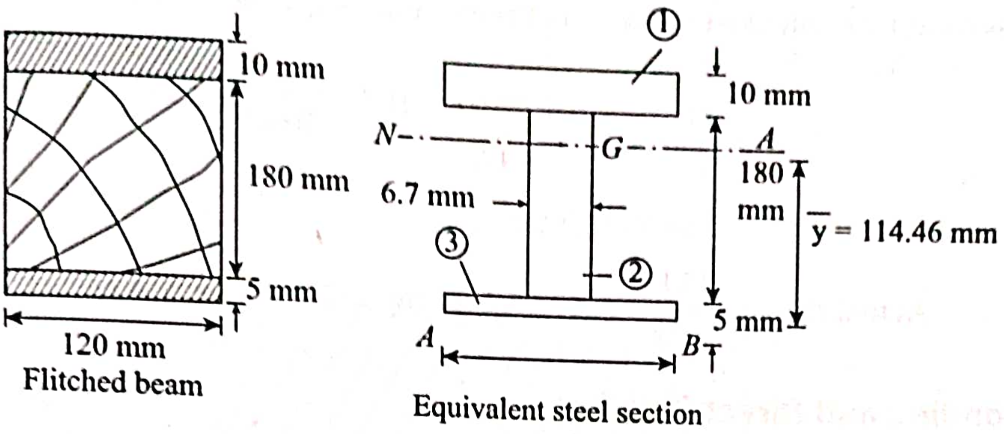


Figure 9.17

Let us find the equivalent steel section of flitched beam.

$$\text{Equivalent width of timber in steel} = \frac{bt}{m} = \frac{120}{18} = 6.7 \text{ mm}$$

To find position of neutral axis, first find the centre of gravity of equivalent steel section from AB. For that we split equivalent steel section in three simple rectangles 1, 2 & 3. $a_1 = 120 \times 10 = 1200 \text{ mm}^2$; $y_1 = 190$ mm; $a_2 = 180 \times 6.7 = 1206 \text{ mm}^2$; $y_2 = 95$ mm; $a_3 = 120 \times 5 = 600 \text{ mm}^2$; $y_3 = 2.5$ mm.

$$\text{Total area} = 1200 + 1206 + 600 = 3006 \text{ mm}^2$$

$$\bar{Y} \text{ (from } AB) = \frac{1200 \times 190 + 1206 \times 95 + 600 \times 2.5}{3006} = 114.46 \text{ mm}$$

$$I = \left[\frac{120 \times 10^3}{12} + 1200(190 - 114.46)^2\right]$$

$$+ \left[\frac{6.7 \times 180^3}{12} + 1206(95 - 114.46)^2\right]$$

$$+ \left[\frac{120 \times 5^3}{12} + 600(2.5 - 114.46)^2\right]$$

$$= [10000 + 3423775] + [3256200 + 456702]$$

$$+ [1250 + 7521025] = 14668952 \text{ mm}^4$$

$$= 14.67 \times 10^6 \text{ mm}^4$$

$$M = \sigma_s \times \frac{I}{y_{\max}}; \qquad \bar{y} \text{ from } AB = 114.46 \text{ is max}$$

$$= 140 \times \frac{14668952}{114.46} = 17942104 \text{ Nmm}$$

$$= 17.94 \text{ kNm} \quad \textbf{Ans}$$

Maximum stress in timber will exist at the top of bottom flange, so $y_{t_{max}} = 114.6 - 5 = 109.6$ mm

$$\sigma_{t_{max}} = \frac{M}{I} \times Y_{t_{max}} = \frac{17.94 \times 10^6}{14.67 \times 10^6} \times 109.6$$

$$= 134.03 \text{ N/mm}^2$$

$$\text{Actual } \sigma_{t_{max}} = \frac{134.03}{18} = 7.45 \text{ N/mm}^2 = 7.45 \text{ MPa.} \quad \textbf{Ans}$$

Combined Bending and Direct Stresses

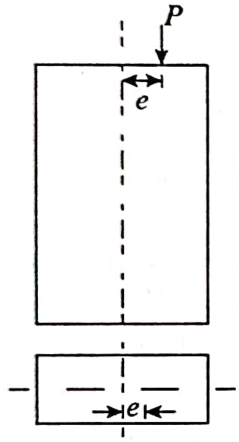


Figure 9.18

In Fig. 9.18, a short column is shown with eccentric loading. In this case there will be two stresses (i) direct stress $\frac{P}{A}$ and (ii) bending stress i.e., $\frac{P.e}{I} \times$ distance of load from the neutral axis.

A combination of bending and direct stresses may occur in a variety of circumstances but in every case, the stresses due to bending moment and direct load may be calculated separately and the results combined to give the resultant stresses. Thus $\sigma = \sigma_d \pm \sigma_b$ where σ_d and σ_b are the direct and bending stresses. The shape of the resultant stress distribution diagram will depend on whether σ_b is greater or less than σ_d. Figure 9.18 shows a bar which is subjected to an axial load P and a bending moment due to load being eccentric. Figures 9.19(a), (b) and (c) show the possible forms of the resultant stress distribution.

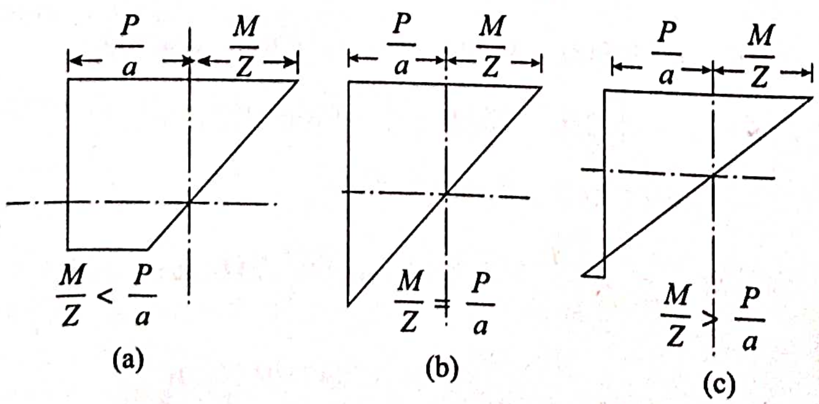


Figure 9.19

EXAMPLE 9.12: A steel flat 180 mm wide and 30 mm thick is subjected to a pull of 180 kN, which is off the geometrical axis by 4 mm in the plane which bisects the thickness. Determine the maximum and minimum stresses induced in the section.

SOLUTION:

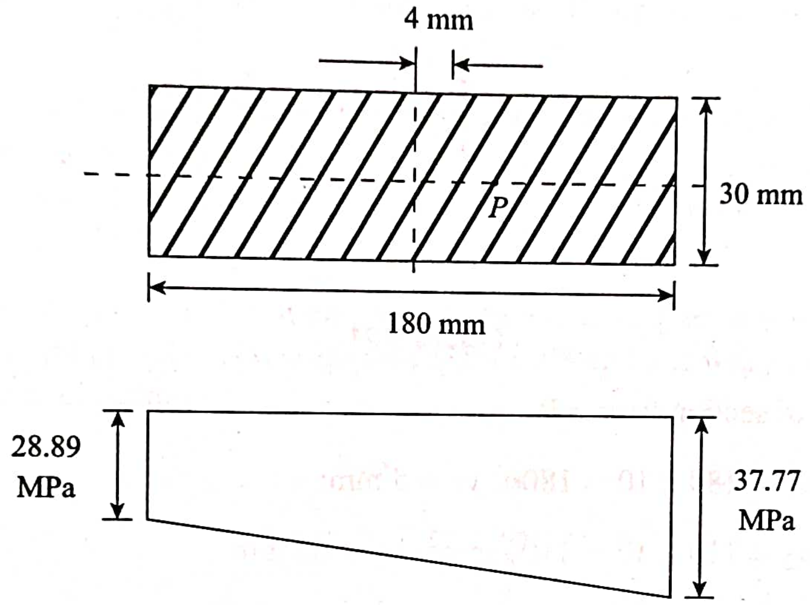


Figure 9.20

$$P = 180000 \text{ N}$$

$$A = 180 \times 30 = 5400 \text{ mm}^2, \quad P = 180000 \text{ N}$$

$$M = P.e = 180000 \times 4 = 720000 \text{ Nmm}$$

$$Z = \frac{bd^2}{6} = \frac{30 \times 180^2}{6} = 162000 \text{ mm}^3$$

$$\sigma_d = \frac{P}{A} = \frac{180000}{5400} = 33.33 \text{ N/mm}^2$$

$$\sigma_b = \frac{M}{Z} = \frac{720000}{162000} = 4.44 \text{ N/mm}^2$$

$$\sigma_{\max} = \sigma_d + \sigma_b = 33.33 + 4.44 = 37.77 \text{ N/mm}^2 = 37.77 \text{ MPa}$$

$$\sigma_{\min} = \sigma_d - \sigma_b = 33.33 - 4.44 = 28.89 \text{ N/mm}^2 = 28.89 \text{ MPa}$$

EXAMPLE 9.13: A mild steel *T* section as shown in Fig. 9.21 carries a load of 100 kN in the central plane bisecting the web at 45 mm from the base. Determine the maximum and minimum stresses induced in the section.

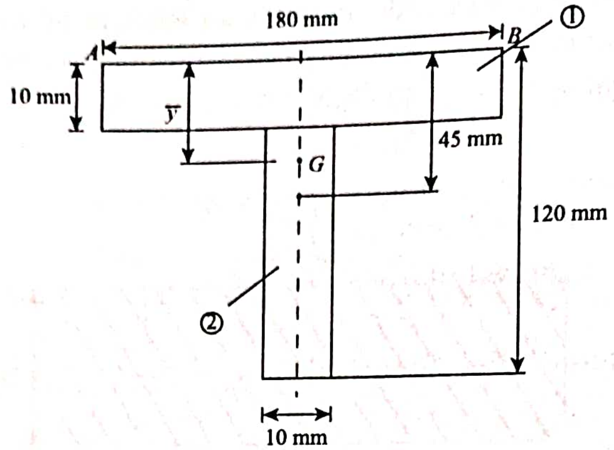


Figure 9.21

Let us first find the G of section from AB,

$$a_1 = 180 \times 10 = 1800;\ y_1 = 5 \text{ mm};$$

$$a_2 = 110 \times 10 = 1100 \text{ mm}^2;\ y_2 = 65 \text{ mm}$$

$$\text{Total area } = 1800 + 1100 = 2900 \text{ mm}^2$$

$$\bar{y} = \frac{1800 \times 5 + 1100 \times 65}{2900} = 27.76 \text{ mm} = y_t; \quad y_c = 92.24$$

$$I = \left[\frac{180 \times 10^3}{12} + 1800(27.76 - 5)^2\right]$$

$$+ \left[\frac{10 \times 110^3}{12} + 1100(27.76 - 65)^2\right]$$

$$= [15000 + 932432] + [1109167 + 1525499] = 3582098 \text{ mm}^4$$

Eccentricity of loading, $e = 45 - 27.76 = 17.24$ mm

$$M = P.e = 100000 \times 17.24 = 1724000 \text{ Nmm}$$

$$\sigma_c = \frac{M.y_c}{I} = \frac{1724000 \times 92.24}{3582098} = 44.39 \text{ MPa}$$

$$\sigma_t = \frac{M.y_t}{I} = \frac{1724000 \times 27.76}{3582098} = 13.36 \text{ MPa}$$

$$\sigma_d = \frac{P}{A} = \frac{100000}{2900} = 34.48 \text{ MPa}$$

$\sigma_{max} = 34.48 + 44.39 = 78.87$ MPa **Ans**

$\sigma_{min} = 34.48 - 13.36 = 21.12$ MPa **Ans**

$$\begin{bmatrix} \sigma_{max} = \sigma_d + \sigma_c \\ \sigma_{min} = \sigma_d - \sigma_t \end{bmatrix}$$

Modulus of rupture:

If a metallic beam, simply supported at its ends, is loaded by a gradually increasing transverse central load till rupture or breaking takes place, the actual stresses at the outer layers at rupture or breaking, are not the ones calculated by the bending equation: $\frac{M}{I} = \frac{\sigma}{Y}$, since the condition of elasticity assumed in this formula has ceased to exist. However, the stress calculated from the equation:

$\frac{M}{I} \cdot y = \frac{M}{Z} = \sigma$ or say σ_r, is known as *transverse rupture stress* or *modulus* or *rupture*. Since, this test is generally applied to different qualities of cast iron or timber beams, the modulus of rupture σ_r so calculated is called a guide to compare the strength of various qualities of these materials. This test is conducted on brittle materials.

EXAMPLE 9.14: A 150 mm × 150 mm pine wood beam was supported at its ends on a 4.5 m span and loaded as shown in Fig. 9.22. The beam failed when a 8 kN load was placed at 1.5 m from each end. Find the modulus of rupture.

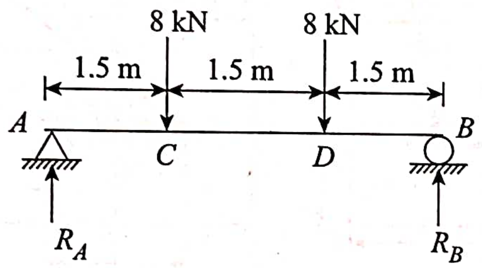


Figure 9.22

Beam is of square section, side = 150 mm = b

$$\text{Section modulus} = \frac{b^3}{6}$$

$$= \frac{150^3}{6} = 56.25 \times 10 \text{ mm}^4$$

Reactions $R_A = R_B = 8$ kN

Portion CD of the beam is under pure bending,

$$M_C = M_D = 8 \times 1.5 = 12 \text{ kNm}$$

$$\text{So maximum, } M = 12 \times 10^6 \text{ Nmm}$$

$$\text{Modulus of rupture, } \sigma_m = \frac{M}{Z}$$

$$= \frac{12 \times 10^6}{56.25 \times 10^4}$$

$$= 21.33 \text{ N/mm}^2 \quad \textbf{Ans}$$

CHAPTER 10

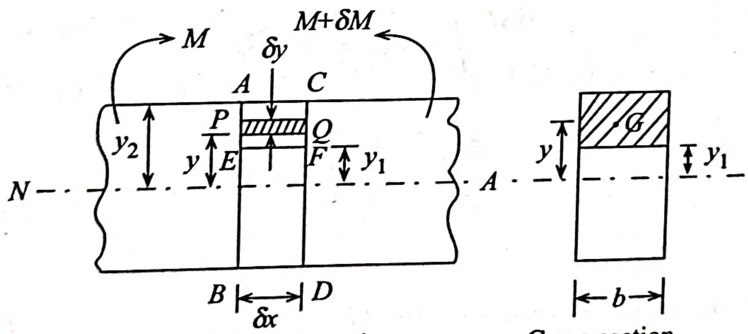
SHEAR STRESSES IN BEAMS

Normally, beams are designed for bending stresses and then checked for shear stresses, as in several situations arise in design in which mode of failure is likely to be shear rather than bending. For example, wooden beams which are weak in shear along the planes parallel to the grain of the wood, and thin webbed beams, where if the web is excessively thin, it would not have sufficient stiffness and stability to hold its shape and it would fail due to shearing stress.

Therefore, it is necessary to study shear stress distribution as detailed under.

On application of shear force, the shear stress on the cross section tends to slide the transverse elements of the beam and the complimentary shear stress of elements act on it (for example, wooden beams when tested to destruction). Though the mean shear stress is equal to shear force divided by cross-sectional area but the shear stress in fact is not uniform in the cross section. It is zero on top and bottom of section.

In order to derive an expression for shear stress at any point in the cross section of beam, let us consider any normal section *AB* of a beam where bending moment and shear force be *M* and *F* act respectively. Now consider another section *CD* at a distance δx from *AB*, where bending moment and shear force are $M + \delta M$ and $F + \delta F$, respectively



Longitudinal section

Cross section

Figure 10.1

Now consider any elementary stripe *PQ* of thickness δy at a distance *Y* from neutral axis.

$$\text{Bending stress at } P = \frac{M}{I}y$$

$$\text{Bending stress at } Q = \frac{M + \delta M}{I}y$$

Therefore, in the web, the shear stress τ varies with respect to y follows a parabolic curve. Also, τ increases as y decreases. At $y = 0$, τ is maximum.

Substituting in Eqn. (i),

$$\tau_{max} = \frac{F}{8Ib}\left[B(D^2 - d^2) + bd^2\right]$$

On the junction of the web and flange, $y = \frac{d}{2}$,

Substituting in Eqn. (i)

$$\tau = \frac{F}{8Ib}\left[B(D^2 - d^2) - b(d^2 - \frac{d^2}{4})\right]$$

$$= \frac{FB}{8Ib}(D^2 - d^2)$$

Of course in web also shear stress distribution is parabolic. It may be noted that abrupt change in width of flange at bottom surface of the flange, the numerical value of shearing stress suddenly changes from $\frac{F}{8I}(D^2 - d^2)$ to $\frac{B}{b} \times \frac{F}{8I}(D^2 - d^2)$ in the proportion $\frac{B}{b}$ (i.e., increases).

EXAMPLE 10.1: An R.S.J. is of *I* section of overall height 200 mm and flange width 125 mm. The web thickness is 7 mm and the flange thickness 11 mm. The standard taper on the flange may be neglected and all corners may be assumed sharp. The beam is subjected to transverse loads acting parallel to the web, and at one section the shear force is 100 kN. Determine the maximum vertical shearing stress in the web at this section.

Also determine the shear stress at top and bottom of the flange.

SOLUTION:

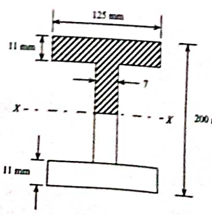


Figure 10.5

Shear force, $F = 100$ kN, Working in metre & N,

$$I_{xx} = \left[\frac{0.125 \times 0.2^3}{12} - \frac{(0.125 - 0.007) \times (0.2 - 0.022)^3}{12}\right]$$

$$= \left[8.83 \times 10^{-5} - 5.546 \times 10^{-5}\right]$$

$$= 2.784 \times 10^{-5}\ \text{m}^4$$

Maximum stress will be in the web at neutral axis

Now, $\tau = \frac{F.A.\bar{y}}{Ib}$

$$\tau_{max} = \frac{100000(0.125 \times .011 \times 0.945 + 0.007 \times 0.089 \times 0.0445)}{2.784 \times 10^{-5}}$$

$= 80.78\ \text{MN/m}^2$ **Ans**

Shear stress at the top and bottom of the web,

$$\tau = \frac{100000 \times 0.125 \times .011 \times 0.0945}{2.784 \times 10^{-5}} = 66.6\ \text{MN/m}^2$$ **Ans**

EXAMPLE 10.2: A cast iron beam with flange and web section is 250 mm deep overall. The top flange is 125 mm × 50 mm deep, the bottom flange 200 mm × 50 mm deep and the web is 40 mm thick. If the transverse shearing force is 140 kN. Calculate the consequent shear stress is the web at the top and bottom junctions with flanges and maximum shear stress and sketch a diagram showing the variation of shear stresses over the depth of the beam.

SOLUTION:

Working in *m* and *N* throughout;

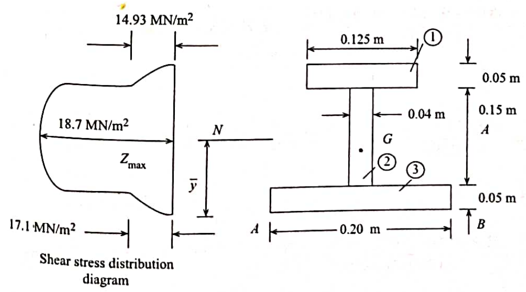


Figure 10.6

For $\bar{y}$ from AB,

$$a_1 = 0.125 \times 0.05 = 6.25 \times 10^{-3};\ y_1 = 0.225\ \text{m}$$

$$a_2 = 0.04 \times 0.15 = 6 \times 10^{-3};\ y_2 = 0.125\ \text{m}$$

$$a_3 = 0.20 \times 0.05 = 0.01;\ y_3 = 0.025\ \text{m}$$

$$A = a_1 + a_2 + a_3 = 6.25 \times 10^{-3} + 6 \times 10^{-3} + 0.01$$

$$= 0.02225\ \text{m}^2$$

$$\bar{y} = \frac{6.25 \times 10^{-3} \times 0.225 + 6 \times 10^{-3} \times 0.125 + 0.01 \times 0.025}{0.02225}$$

$$= 0.1083\ \text{m}$$

$$I = \left[\frac{0.125 \times 0.05^3}{12} + 0.125 \times 0.05 \times (0.1083 - 0.225)^2\right] + \left[\frac{0.04 \times 0.15^3}{12} + 0.15 \times 0.04 \times (0.1083 - 0.125)^2\right] + \left[\frac{0.2 \times 0.05^3}{12} + 0.2 \times 0.05 \times (0.1083 - 0.025)^2\right] = 0.000171\ \text{m}^4$$

Shear stress at top of web, $\tau = \dfrac{FA\bar{y}}{Ib}$

$$\tau = \frac{140000 \times 0.125 \times 0.05 \times 0.1167}{0.000171 \times 0.04} = 14.93\ \text{MN/m}^2 \quad \textbf{Ans}$$

Shear stress at bottom of web,

$$\tau = \frac{140000 \times 0.2 \times 0.05 \times 0.0833}{0.000171 \times 0.04} = 17.1\ \text{MN/m}^2 \quad \textbf{Ans}$$

Max stress shear stress,

$$\tau_{max} = \frac{140000 \times [0.125 \times 0.05(0.1167) + 0.04 \times 0.0917 \times 0.04585}{0.000171 \times 0.04} = 18.5\ \text{MN/m}^2 \quad \textbf{Ans}$$

Hint:

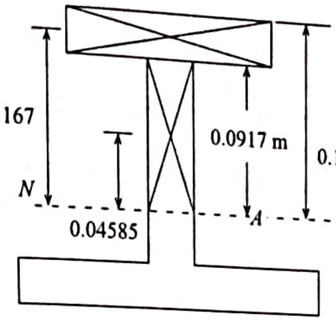


Figure 10.7

EXAMPLE 10.3: Figure 10.8 shows the section of a Tee-beam made of a uniform material, which is subjected to a shear force of 200 kN and a bending moment of 25 kNm. Calculate (a) the maximum bending stress and (b) the maximum shear giving sketches to show the form of stress distribution in each case.

SOLUTION:

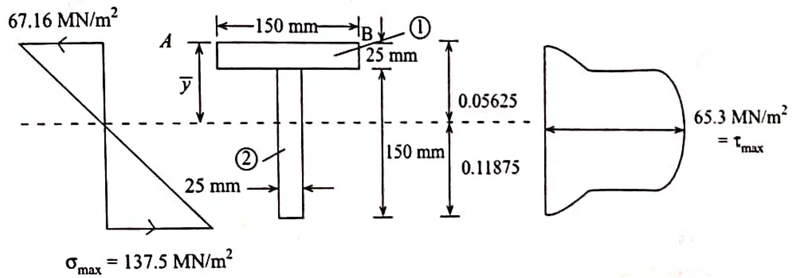


Figure 10.8

Let us first find position of C.G. from AB,

$$a_1 = 0.150 \times 0.025 = 3.75 \times 10^{-3}\ \text{m}^2;\ y_1 = 0.0125\ \text{m};\ a_2 = 0.150 \times 0.025 = 3.75 \times 10^{-3}\ \text{m}^2$$

$$y_2 = 0.1\ \text{m};\ A = a_1 + a_2 = 0.0075$$

$$\bar{y} = \frac{3.75 \times 10^{-3} \times 0.0125 + 3.75 \times 10^{-3} \times 0.1}{0.0075} = 0.05625\ \text{m}$$

$$I_{xx} = \left[\frac{0.15 \times 0.025^3}{12} + 3.75 \times 10^{-3}(0.05625 - 0.0125)^2\right] + \left[\frac{0.025 \times 0.15^3}{12} + 3.75 \times 10^{-3}(0.1 - 0.05625)^2\right]$$

$= 2158 \times 10^{-8}\ \text{m}^4$

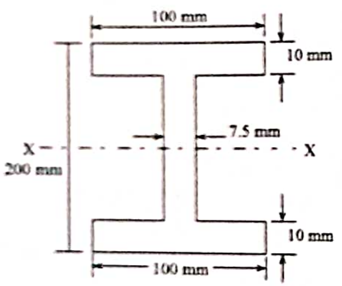
$$\sigma_{max} = \frac{M}{I} y_{max} = \frac{25000}{2158 \times 10^{-8}} \times 0.11875 = 137.5\ \text{MN/m}^2 \quad \textbf{Ans}$$

$$\sigma_{min} = \frac{M}{I} y_{min} = \frac{25000}{2158 \times 10^{-8}} \times 0.05625 = 65.16\ \text{MN/m}^2 \quad \textbf{Ans}$$

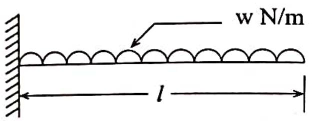
$$\tau = \frac{FA\bar{y}}{Ib};\ \tau_{max} = \frac{200000 \times [0.15 \times .025 \times 0.04375 + 0.025 \times 0.03125 \times 0.0156]}{2158 \times 10^{-8} \times 0.025}$$

$= 65.3\ \text{MN/m}^2$ **Ans**

EXAMPLE 10.4: A cantilever of *I* Section 200 mm × 100 mm has rectangular flanges 10 mm thick web and 7.5 mm thick. It carries a uniformly distributed load. Determine the length of the cantilever if the maximum bending stress is three times the maximum shearing stress. What is the ratio of the stresses halfway along the length of cantilever?



(a)



(b)

Figure 10.9

SOLUTION:

$$I_{xx} = \left[\frac{0.1 \times 0.2^3}{12} - \frac{0.0925 \times 0.18^3}{12} = 2.17 \times 10^{-5}\ \text{m}^4\right]$$

$$\text{Maximum BM} = \frac{wl^2}{2},\ \sigma_{max} = \frac{M}{I} y = \frac{wl^2 \times 0.1}{2 \times 2.17 \times 10^{-5}} = 2304.15\ wl^2$$

τ_{max} is at neutral axis & $\tau = \frac{FA\bar{y}}{Ib}$, Shear force(max) $= wl$

$$\tau_{max} = \frac{wl \times (0.1 \times 0.01 \times 0.095 + 0.090 \times .0075 \times 0.045)}{2.17 \times 10^{-5} \times 0.0075}$$

$$= \frac{wl(9.5 \times 10^{-5} + 3.0375 \times 10^{-5})}{0.016275 \times 10^{-5}} = \frac{wl \times 10^{-5}(12.54)}{0.016275 \times 10^{-5}} \quad \text{(i)}$$

$= 770.5\ wl$

Since σ_{max} is three times τ_{max};

$$\therefore \quad 3 \times 770.5 \cancel{w} l = 2304.15 \cancel{w} l^2$$

$$l = 1\ \text{m} \quad \textbf{Ans}$$

$$\therefore \quad \sigma_{max} = 2304.15\ w(1)^2 = 2304.15\ w$$

$$\tau_{max} = 770.5\ w \times 1 = 770.5\ w$$

Ratio of stresses half-way:

$$M = \frac{w(0.5)^2}{2} = 0.125\ w \therefore \sigma = \frac{0.125\ w}{I} \times 0.1 = \frac{0.0125w}{I}$$

$$F = w \times 0.5 = 0.5w \quad \text{From Eq(i)} \quad \tau = \frac{0.5w \times 10^{-5}(12.54)}{I \times 0.0675} = 10^{-5} \times 836 \frac{w}{I}$$

$$\text{Required ratios} = \frac{\sigma}{\tau} = \frac{0.0125w/I}{836 \times 10^{-5}\ w/I} = 1.5 \quad \textbf{Ans}$$

EXAMPLE 10.5: Find the maximum shear stress in a hollow circular section of 100 mm external diameter and 75 mm internal when subjected to a total shearing force of 160 kN.

SOLUTION:

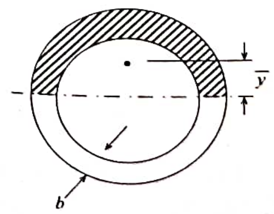


Figure 10.10

$$\tau = \frac{FA\bar{y}}{Ib} \quad \text{(i)}$$

$$b = 2(R - r)$$

$$\text{Shaded area} = \frac{\pi}{2}(R^2 - r^2)$$

$$\bar{y} = \frac{\frac{\pi R^2}{2} \cdot \frac{4R}{3\pi} - \frac{\pi r^2}{2} \cdot \frac{4r}{3\pi}}{\frac{\pi}{2}(R^2 - r^2)}$$

$$\bar{y} = \frac{(R^3 - r^3)}{3\pi(R^2 - r^2)}$$

Substituting in Eqn. (i)

$$\tau = \frac{F \times \frac{\pi}{2}(R^2 - r^2)}{I \times 2(R - r)} \times \frac{4(R^3 - r^3)}{3\pi(R^2 - r^2)}$$

$$= \frac{F}{3I} \cdot \frac{R^3 - r^3}{R - r} \qquad \text{(ii)}$$

$$I = \frac{\pi}{64}(0.1^4 - 0.075^4) = 0.00000336 \text{ m}^4$$

$$R = \frac{0.1}{2} = 0.05; \; r = \frac{0.075}{2} = 0.375$$

Substituting in Eqn. (ii): $\therefore$ $$\tau = \frac{160000}{3 \times 0.00000336} \times \frac{0.05^3 - 0.0375^3}{0.05 - 0.0375}$$

$$= 91.7 \text{ MN/m}^2$$ **Ans**

Exercise

10.1 Plot the distribution of shear stress over the section shown in Fig. 10.11, which is subjected to a shearing force of 300 kN, giving essential values.

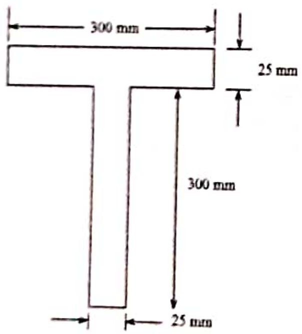


Figure 10.11

[Ans $\tau_{max} = 51.5 \text{ MN/m}^2$**]**

10.2 A transverse shear force F and a bending moment FL are applied to a uniform beam having the symmetrical cross section shown in Fig. 10.12. If the ratio of the transverse shear stress at the natural axis to the maximum direct stress due to bending is not to be less then 0.5 determine the maximum permissible value of l in terms of a.

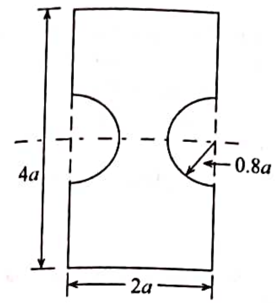


Figure 10.12

[Ans $l = 9.15a$**]**

10.3 A rolled steel section is shown in Fig. 10.13, it is subjected to a vertical force of 20 kN. Determine shear stress at points A, B and C of the section.

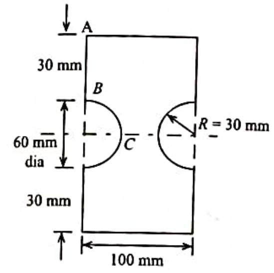


Figure 10.13

[Ans $\tau_A = 0,\ \tau_B = 1.96 \text{ N/mm}^2,\ \tau_c = 5.88 \text{ N/mm}^2$**]**

10.4 An extruded aluminium alloy section is of shape and dimensions as per Fig. 10.14. If a vertical force on the section is 4 kN. Find the dimension d, if the average shear stress in the section is 8 MN/m^2. What is the shear stress at neutral axis?

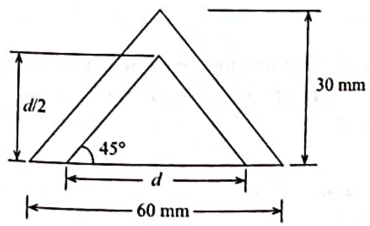


Figure 10.14

[Ans 20 mm; τ at neutral axis $= 11.4 \text{ MN/m}^2$**]**

10.5 A T-section beam symmetrical about a vertical axis, is made with a top flange 100 mm wide and 14 mm thick to which a vertical web plate 150 mm deep and 10 mm wide is welded. At a certain point, the total shearing force is 40 kN. Calculate the percentage shear carried by the vertical web and the shearing force per metre run in the welded section.

[Ans 94.2%; 308 kN]

10.6 An I Section shown in Fig. 10.15 is used as a beam. Determine the percentage of shear force resisted by web if the beam is subjected to a shear force F.

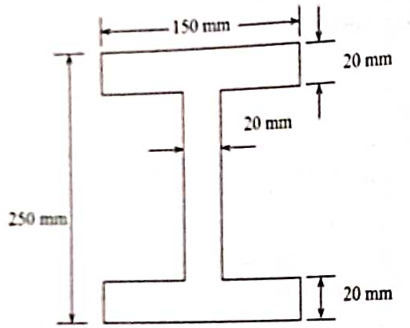


Figure 10.15

[Ans 92.53%]

10.7 A beam of triangular section having base width 200 mm and height of 300 mm is subjected to a shear force 3 kN. Determine the value of maximum shear stress and sketch the shear stress distribution diagram along the depth of beam.

[Ans Hint: $\tau_{max} = \tau_{mean} \times \frac{3}{2}$; 150 kN/m^2]

10.8 A beam of channel section as shown in Fig. 10.16, is subjected at a vertical section, a shear force of 50 kN. Draw shear stress distribution diagram. Find the ratio of maximum and mean shear stresses.

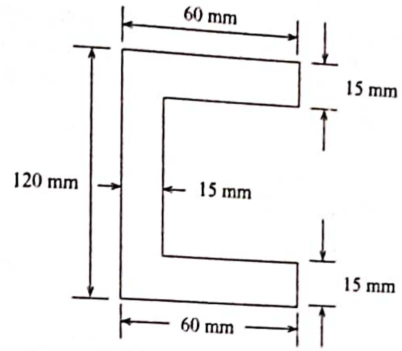


Figure 10.16

[Ans 2.2]

10.9 Find the maximum shear stress induced by a force of 4 kN is the vertical section of a hollow beam of a square section, if the outside width is 100 mm and the thickness is 20 mm.

[Ans 1.35 MN/m^2]

10.10 The beams of cross section (shown in Fig. 10.17) has shear force of 5.3 kN. Determine shear stress at points A and B.

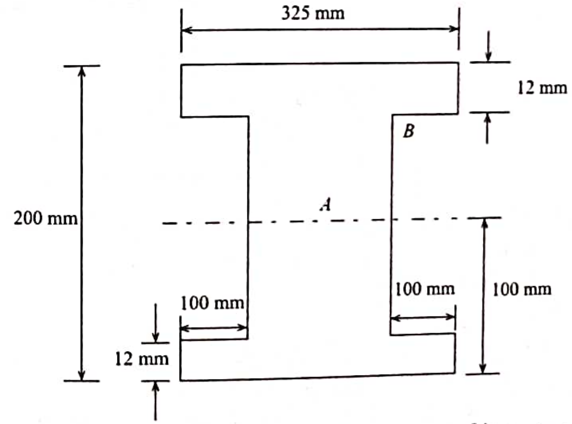


Figure 10.17

[Ans $\tau_A = 287$ kPa; $\tau_B = 396$ kPa]

CHAPTER 11

TORSION

When a cylindrical shaft is subjected to equal and opposite couples at the end, shear stresses develop in the shaft whether it is rotating or in equilibrium. Our objective is to derive an equation connecting torque, modulus of rigidity, length, polar moment of inertia, length, shear strain, shear stress, radius and length. For that we have to make certain important assumptions before going ahead.

Assumptions

i) The material is homogeneous and isotropic.
ii) Twist along the shaft is uniform.
iii) The shear stress is directly proportional to the radial distance from the axis of shaft.
iv) Transverse planes of the shaft remain plane.
v) The modulus of rigidity is same throughout the material.
vi) All radii remain straight.
vii) The distortion along the shaft is uniform throughout.

Now consider a circular shaft of outer radius R, fixed at one end. A torque T is applied at the force end as shown in Fig. 11.1.

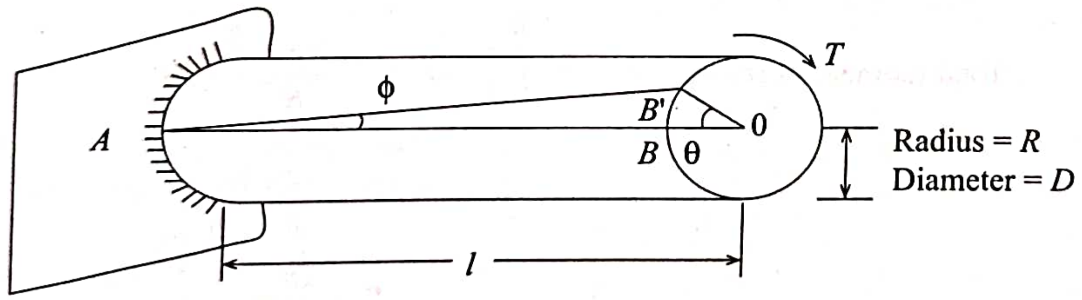


Figure 11.1

Let us take a straight line AB, which after application of torque T in clockwise direction takes the new position as AB' so that shear strain ϕ as shown in Fig. 11.1 is set up. At the same time angle θ (in radians) is made at the free end as $\angle BOB'$.

$$\text{Now shear strain} = \phi$$

$$\therefore BB' = l\phi, \quad \text{Also } BB' = R\theta$$

$$\text{Therefore, } l\phi = R\theta, \quad \text{or} \quad \phi = \frac{R\theta}{l} \tag{i}$$

$$\text{Also we know shear strain } \phi = \frac{\text{Shear stress at outer radius}}{\text{Modulus of rigidity}} = \frac{\tau}{C} \tag{ii}$$

Equating Eqns. (i) and (ii)

$$\frac{R\theta}{l} = \frac{\tau}{C}, \quad \text{We get } \frac{C\theta}{l} = \frac{\tau}{R} \tag{iii}$$

Torsional Moment of Resistance

Take an elementary ring at radius r of thickness dr. Now as per assumption shear stress τ' at r is proportional to shear stress τ at radius R.

$$\therefore \frac{\tau}{R} = \frac{\tau'}{r} \qquad \therefore \tau' = \frac{\tau}{R}.r$$

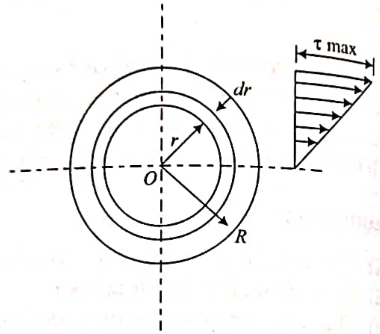


Figure 11.2

$$\text{Shear force acting on elementary ring} = \tau' 2\pi r dr$$

$$\text{Substituting for } \tau', \text{ shear force acting on elementary ring} = \frac{\tau}{R} 2\pi r^2 dr$$

$$\text{Moment of this shear force at centre } O = \frac{\tau}{R} 2\pi r^3 dr$$

$$\text{Total moment of resistance, } T = \int_0^R \frac{\tau}{R} 2\pi r^3 dr = \frac{2\pi\tau}{R}\int_0^R r^3 dr$$

$$= \frac{2\pi\tau}{R}\left[\frac{r^4}{4}\right]_0^R$$

$$\therefore \quad T = \frac{2\pi\tau}{R} \times \frac{R^4}{4} = \frac{\pi\tau R^4}{2R}$$

$$\text{Now} \quad \frac{\pi R^4}{2} = \frac{\pi\left(\frac{D}{2}\right)^4}{2} = \frac{\pi D^4}{32}$$

$$\frac{\pi D^4}{32} = J, \quad \text{Polar moment of inertia}$$

[As we know from perpendicular axis theorem, $I_{XX} + I_{YY} = I_{ZZ}$ $\therefore \frac{\pi D^4}{64} + \frac{\pi D^4}{64} = \frac{\pi D^4}{32}$
I_{ZZ} is known as polar moment of inertia]

$$\text{Hence,} \quad T = \frac{\tau J}{R}; \quad \text{or} \quad \frac{T}{J} = \frac{\tau}{R} \tag{iv}$$

Combining Eqns. (iii) and (iv), we get torsion equation:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{l}$$

Power Transmitted: If T is torque and N is rotational speed in revolutions per minute, then,

$$\text{Power,} \quad P = \frac{T \times 2\pi N}{60 \times 1000} \text{ kW, if } T \text{ is in Nm.}$$

Torsional Rigidity: The stiffness or torsional rigidity $= \frac{T}{\theta r} = CJ$, CJ is known as torsional rigidity, which can also be defined as the ratio of the torsion to angle of twist (in radians) per unit length of the shaft.

EXAMPLE 11.1: A circular shaft of 60 mm diameter transmits torque from one shaft to another. Find the safe torque, which the shaft can transmit, if the shear stress is not to exceed 50 MPa.

SOLUTION:

$$\frac{T}{J} = \frac{\tau}{R} \quad \text{or,} \quad T = \frac{J}{R}\tau = \frac{\pi d^4}{32.d/2}\tau = \frac{\pi d^3}{16}\tau$$

$$\therefore \quad \text{Safe torque} = \frac{\pi(60)^3}{16} \times 50 = 2119500 \text{ Nmm}$$

$$= 2.1195 \text{ kNm} \quad \textbf{Ans}$$

EXAMPLE 11.2: A solid circular shaft is to transmit a torque of 15 kNm. If the permissible shear stress is 55 MPa, find the diameter of the shaft.

SOLUTION:

$$\frac{T}{J} = \frac{\tau}{R} \quad \text{or,} \quad T = \frac{J}{R}\tau = \frac{\pi d^3}{16}\tau$$

$$T = 15000\text{Nm}, \qquad \tau = 55 \text{ MPa}, d = ?$$

$$\text{Substituting,} \quad 15000 = \frac{\pi d^3}{16} \times 55 \times 10^6$$

$$\therefore \quad d^3 = \frac{15000 \times 16}{\pi \times 55 \times 10^6} = 1.39 \times 10^{-3}$$

$$\text{or,} \quad d = 0.112 \text{ m} = 111.6 \text{ mm} \quad \textbf{Ans}$$

EXAMPLE 11.3: Torque is required to transmit through a hollow shaft of external diameter 100 mm and internal diameter 55 mm. If the permissible shear stress in 55 MPa, what is the safe torque that can be transmitted?

SOLUTION:

$$J = \frac{\pi(D^4 - d^4)}{32}$$

$$T = J \times \frac{\tau}{D} \times 2 = \frac{\pi(D^4 - d^4)}{16 \times D} \times 55$$

$$= \frac{\pi(100^4 - 55^4)}{16 \times 100} \times 55 = \frac{\pi(100000000 - 9150625) \times 55}{16 \times 100}$$

$$= 9806054 \text{ Nmm} = 9.8 \text{ kNm} \quad \textbf{Ans}$$

EXAMPLE 11.4: A solid steel shaft 60 mm diameter and 800 mm long transmits 35 kW at 200 r.p.m. Calculate: i) the maximum shear stress produced; ii) the angle of twist in degrees and iii) the shear stress at a radius of 25 mm. $C = 80$ GN/m^2

SOLUTION:

$$P = \frac{2\pi NT}{60} \quad \text{or,} \quad T = \frac{P \times 60}{2\pi N}$$

i) $$T = \frac{35000 \times 60}{2\pi \times 200} = 1671.97 \text{ Nm}$$

$$\frac{T}{J} = \frac{\tau}{r} \quad \therefore \quad \tau = \frac{T}{J}r = \frac{1671.97 \times 1000 \times 30 \times 32}{\pi(60)^4}$$

$$= 39.44 \text{ N/mm}^2 = 39.44 \text{ MPa} \quad \textbf{Ans}$$

Because maximum stress will be at radius $\frac{60}{2} = 30$ mm

ii) $$\frac{T}{J} = \frac{C\theta}{l} \quad \therefore \quad \theta = \frac{T.l}{C.J} = \frac{1671.97 \times 1000 \times 800 \times 32}{80000 \times \pi(60)^4} = 0.01315 \text{ radians}$$

$$= \frac{180}{\pi} \times 0.01315 = 0.754° \quad \text{Ans}$$

iii) $$\frac{\tau}{r} = \frac{\tau_1}{r_1}; \frac{39.44}{30} = \frac{\tau_1}{25} \quad \therefore \tau_1 = \frac{39.44 \times 25}{30} = 32.87 \text{ N/mm}^2$$

$$= 32.87 \text{ MPa} \quad \textbf{Ans}$$

EXAMPLE 11.5: Compare the weights of hollow and solid shaft with equal lengths to transmit a given torque for the same maximum shear stress if the inside diameter is $\frac{2}{3}$ of the outside.

SOLUTION:

$$\frac{W_{\text{hollow}}}{W_{\text{solid}}} = \frac{\pi/4\left\{D_o^2 - \left(\frac{2}{3}D_o\right)^2\right\} \times l \times \text{density}}{\pi/4\, D^4 \times l \times \text{density}}$$

D_o = Outside diameter of hollow shaft

D = Diameter of solid shaft

$$= \frac{D_o^2 - \frac{4}{9}D_o^2}{D^2} = \frac{5D_o^2}{9D^2} \qquad \text{(i)}$$

Now, $T_{\text{hollow}} = T_{\text{solid}}$;

$$T_{\text{hollow}} = \frac{\tau \times \pi\left\{D_o^4 - \left(\frac{2}{3}D_o\right)^4\right\} \times 2}{32\, D_o} = \frac{\tau\pi\left(\frac{65}{81}D_o^4\right)}{16 D_o}$$

$$= 0.1575 D_o^3 \tau$$

$$T_{\text{solid}} = \frac{\tau}{D} \times \frac{\pi}{32}(D)^4 \times 2 = 0.19625 D^3 \tau$$

Equating, $T_{\text{solid}} = T_{\text{hollow}}$

$$0.19625 D^3 \tau = 0.1575 D_o^3 \tau$$

$$\frac{D_o^3}{D^3} = 1.246 \quad \therefore \quad D_o = 1.076D$$

Substituting in Equation (i) $= \frac{W_{\text{hollow}}}{W_{\text{solid}}} = \frac{5(1.0760D)^2}{9D^2} = 0.643$ **Ans**

EXAMPLE 11.6: A hollow marine propeller shaft turning at 110 rev/min is required to propel a vessel at 47 km/h for the expenditure of 6.4 MW, the efficiency of the propeller being 68 per cent. The diameter ratio of the shaft is to be $\frac{2}{3}$ and the direct stress due to thrust is not to exceed 8 MN/m^2. Calculate: (a) the shaft diameter and (b) the maximum shearing stress due to the torque.

SOLUTION:

$$\text{Output power} = \frac{47 \times 10^3}{3600} \times P \text{ Watt}$$

where P is the propulsive force in N.

$$\therefore \quad \frac{47 \times 10^3}{3600}P = 0.68 \times 6.4 \times 10^6$$

$$\therefore \quad P = 334 \text{ kN}$$

$$\therefore \quad 334 \times 10^3 = \frac{\pi}{4}(D^2 - d^2) \times 8 \times 10^6$$

where D and d are outside and inside diameters, respectively

$$334 \times 10^3 = \frac{\pi}{4} \times \frac{5}{9} D^2 \times 8 \times 10^6; \text{ because } d = \frac{2}{3} D$$

$$\therefore \quad D = 0.3093 \text{ m and } d = 0.2062 \text{ m} \quad \textbf{Ans}$$

$$T = \frac{6.4 \times 10^6 \times 10}{2\pi \times 110} = 556000 \text{ Nm}$$

$$\therefore \quad 556000 = \tau \frac{J}{R} = \tau \times \frac{\pi}{16}(D^4 - d^4)$$

$$= \tau \times \frac{\pi}{16}\left[\frac{0.3093^4 - 0.2062^4}{0.3093}\right]$$

Solving, we get

$$\therefore \quad \tau = 128 \text{ MN/m}^2 \quad \textbf{Ans}$$

EXAMPLE 11.7: A solid shaft 220 mm diameter has the same cross-sectional area of that of a hollow shaft of the same material with inside diameter of 140 mm. Find the ratio of the power transmitted by the two shafts at the same speed.

SOLUTION:

D_h = External dia of the hollow shaft

d_n = 140 mm, the internal dia of the hollow shaft

Because the two shafts have the same area,

$$\frac{\pi}{4}(D_h^2 - 140^2) = \frac{\pi}{4} \times 220^2$$

$$D_h^2 - 19600 = 48400$$

$$D_h^2 = 48400 + 19600 = 68000$$

$$\therefore \quad D_h = 260.77 \text{ mm}$$

Ratio of power transmitted by the two shafts,

$$\frac{P_{\text{hollow}}}{P_{\text{solid}}} = \frac{T_{\text{hollow}}}{T_{\text{solid}}} = \frac{Z_{\text{hollow}}}{Z_{\text{solid}}}$$

$$Z_{\text{hollow}} = \frac{\pi(D_h^4 - d_n^4)}{16 D_h} = \frac{\pi(260.77^4 - 140^4)}{16 \times 260.77} = 3190915.75 \text{ mm}^3$$

$$Z_{\text{solid}} = \frac{\pi(220)^3}{16} = 2089670 \text{ mm}^3$$

$$\frac{P_{\text{hollow}}}{P_{\text{solid}}} = \frac{3190915.75}{2089670} = 1.527 \quad \textbf{Ans}$$

EXAMPLE 11.8: A hollow shaft is to transmit 320 kW at 100 r.p.m. If the shear stress is not to exceed 65 N/mm^2 and the internal diameter is 0.5 of the external diameter, determine the external and internal diameters, assuming the maximum torque is 1.5 times the mean torque.

SOLUTION:

D_h = External diameter of the hollow shaft

$D_i = 0.5\, D_h, \quad P = 320$ kW

$$P = \frac{2\pi N T_{\text{mean}}}{60} = \frac{2 \times 100 \times T_{\text{mean}}}{60}$$

$$T_{\text{mean}} = \frac{320000 \times 60 \times 1000}{2\pi \times 100} = 30573.2 \text{ Nm}$$

$$T_{\max} = 1.5 \times 30573.2 = 45859.8 \text{ Nm}$$

$$= 45859.8 \times 1000 = 45859800 \text{ Nmm}$$

$$Z_n = \frac{\pi(D_h^4 - D_i^4)}{16 D_h} = \frac{\pi(D_h^4 - (0.5 D_h)^4)}{16 D_h}$$

$$= \frac{\pi \times 0.9375 D_h^4}{16 D_h} = 0.184 D_h^3$$

$$T_{\max} = \tau \cdot Z_h = 65 \times 0.184 D_h^3$$

$$45859800 = 65 \times 0.184 D_h^3$$

$$\therefore \quad D_h = (3834431.4)^{1/3} = 156.52 \text{ mm} \quad \textbf{Ans}$$

$\therefore$ Internal diameter, $D_i = 156.52 \times 0.5 = 78.26$ mm **Ans**

EXAMPLE 11.9: A shaft shown in Fig. 11.3 rotates at 220 r.p.m. with 35 kW and 20 kW taken off at A and B, respectively and 50 kW applied at C. Find the maximum shear stress developed in the shaft and the angle of twist (degrees) of the gear A relative to gear C. Take $C = 85$ GN/m^2.

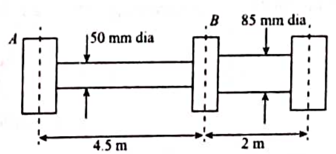


Figure 11.3

SOLUTION:

Shaft between B and C: Power of the shaft = 50 kW, let the torque in this part be T_{bc}

$$\text{Power} = \frac{2\pi NT}{60}; \quad 50000 = \frac{2\pi \times 220 T_{bc}}{60}$$

$$\therefore \quad T_{bc} = 2171.4 \text{ Nm} = 2171400 \text{ Nmm}$$

Let σ_s be the maximum shear stress in this part (between B and C) of the shaft.

$$\tau_s \frac{\pi d^3}{16} = T_{bc} \quad \text{or} \quad \tau_s = \frac{16 T_{bc}}{\pi d^3}$$

$$\tau_s = \frac{16 \times 2171400}{\pi 85^3} = \frac{34742400}{1928352.5} = 18 \text{ N/mm}^2$$

Shaft between B and A.

$$\text{Power of the shaft} = 50 - 20 = 30 \text{ kW}$$

$$\text{For } T_{ab},\ 30000 = \frac{2\pi \times 220 \times Tab}{60}$$

$$\therefore \quad T_{ab} = 1302.84 \text{ Nm}$$

Let τ_s' be the maximum shear stress in this part of the shaft,

$$\tau_s' \times \frac{\pi d^3}{16} = T_{ab} \quad \therefore \quad \tau' = \frac{T_{ab} \times 16}{\pi d^3}$$

$$\therefore \quad \tau' = \frac{16 \times 1302.84 \times 1000}{\pi (50)^3} = 53.11 \text{ N/mm}^2 \quad \textbf{Ans}$$

Therefore, maximum shear stress occurs at the 50 mm diameter shaft.

Twist of the Shaft

Let θ_{bc} be the twist of the shaft BC,

$$\theta_{bc} = \frac{l}{C} \cdot \frac{T_{bc}}{J} = \frac{2000}{85000} \times \frac{2171400 \times 32}{\pi (50)^4}$$

$$= 0.0833 \text{ radians}$$

Now let θ_{ab} be the twist of the shaft AB,

$$\theta_{ab} = \frac{l'}{C} \times \frac{T_{ab}}{J} = \frac{4500}{85000} \times \frac{1302.84 \times 1000 \times 32}{\pi (85)^4}$$

$$= 0.013466 \text{ radians}$$

Hence, angle of twist of A with respect to C

$$= \theta_{bc} + \theta_{ab} = 0.0833 + 0.013466 = 0.096766 \text{ radians}$$

$$= \frac{0.096766 \times 180}{\pi} = 5.55° = 5°33' \quad \textbf{Ans}$$

EXAMPLE 11.10: A maximum shear stress of 160 MN/m^2 is induced in a hollow shaft of 120 mm and 70 mm external and internal diameters, respectively. What maximum shear stress will be developed in a solid shaft of the same weight, material and length, subjected to the same torque?

SOLUTION:

$$R_H = \frac{120}{2} = 60 \text{ mm}, \quad r_H = \frac{70}{2} = 35 \text{ mm}$$

Let τ be the maximum shear stress in the hollow shaft and τ_1 be the maximum shear stress in the solid shaft

$$T_H = \frac{\tau}{R_H} \times \frac{\pi}{2}(R_H^4 - r_H^4);\ T_s = \frac{\tau_1 \times \pi R_s}{2};\ R_s \text{ is the radius of solid shaft.}$$

Since the torque is same,

$$\frac{\tau}{R_H} \times \frac{\pi}{2}\left(R_H^4 - r_H^4\right) = \frac{\tau_1 \times \pi R_s}{2} \tag{i}$$

putting the values of τ, R_H and r_H in Eqn. (i)

$$\frac{160}{60} \times \frac{\cancel{\pi}}{\cancel{2}}(60^4 - 35^4) = \tau_1 \times \frac{\cancel{\pi}}{\cancel{2}} R_s^3$$

$$2.67(12960000 - 1500625) = \tau_1 R_s^3$$

$$30596531 = \tau_1 R_s^3 \tag{ii}$$

Because the weight, length and material is same,

$$\text{Cross-sectional area of hollow shaft} = \text{Cross-sectional area of solid shaft}$$

$$\cancel{\pi}(R_H^2 - r_H^2) = \cancel{\pi} R_s^2$$

$$60^2 - 35^2 = R_s^2 \quad \therefore \quad R_s = \sqrt{3600 - 1225}$$

$$\therefore \quad R_s = 48.73 \text{ mm}$$

Substituting in Eqn. (ii),

$$30596531 = \tau_1 (48.73)^3$$

$$\therefore \quad \tau_1 = 264.41 \text{ N/mm}^2$$

$$\frac{\text{Maximum shear stress in solid shaft}}{\text{Maximum shear stress in hollow shaft}} = \frac{264.41}{160} = 1.65 \quad \textbf{Ans}$$

Hence, hollow shaft is 1.65 stronger than solid shaft.

EXAMPLE 11.11: The stepped steel shaft shown in Fig. 11.4 is subjected to a torque T at free end and a Torque $(1.7T)$ in the opposite direction at the junction of the two sizes.

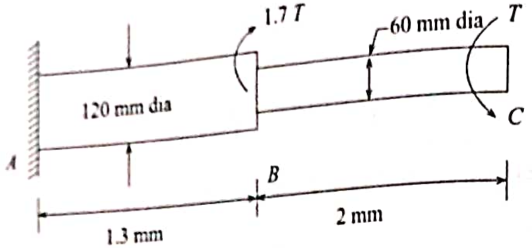


Figure 11.4

What is the total angle of twist at the free end, if maximum shear stress in the shaft is limited to 80 MPa? Take $C = 85\ GN/m^2$.

Solution:

Now the torques at B and C are in opposite directions, therefore the effect of these two torques will be studied independently (sum of the two twists, one in clockwise direction and the other in anticlockwise direction).

To find value of T at C: It must be noted that torque in AB will induce more stress in BC because of smaller diameter between B and C. Hence, let us first calculate the torque in BC because it is less stressed than permissible in AB.

$$T = \frac{\pi}{16}\tau(D_{BC})^3 = \frac{\pi}{16} \times 80 \times 60^3 = 3391200 \text{ Nmm}$$

Polar moment of inertia,

$$J_{AB} = \frac{\pi}{32}(D_{AB})^4 = \frac{\pi}{32}(120)^4 = 20347200 \text{ mm}^4$$

$$J_{BC} = \frac{\pi}{32}(D_{BC})^4 = \frac{\pi}{32}(60)^4 = 1271700 \text{ mm}^4$$

For angle of twist due to T at C

$$\theta = \frac{T.l}{J.C} = \frac{T}{C}\left(\frac{l_{AB}}{J_{AB}} + \frac{l_{BC}}{J_{BC}}\right)$$

$$= \frac{3391200}{85000}\left(\frac{1300}{20347200} + \frac{2000}{1271700}\right)$$

$$= 39.9\left(6.39 \times 10^{-5} + 157.27 \times 10^{-5}\right)$$

$$= 39.9 \times 163.66 \times 10^{-5} = 0.0653 \text{ radians}$$

Now angle of twist at C due to torque $1.7T$ at B,

$$\theta = \frac{T}{C} \times \frac{l_{AB}}{J_{AB}} = \frac{1.7T \times 1300}{85000 \times 20347200}$$

Substituting for

$$\theta = \frac{1.7 \times 3391200 \times 1300}{85000 \times 20347200} = 0.00433 \text{ radians}$$

Hence, angle of twist $= 0.0653 - 0.00433 = 0.06097$ radians

$$= \frac{0.06097 \times 180}{\pi} = 3.495° \quad \textbf{Ans}$$

Composite Shaft

In a composite shaft, there are two or more shafts of different materials fixed together. The applied torque is shared by each material.

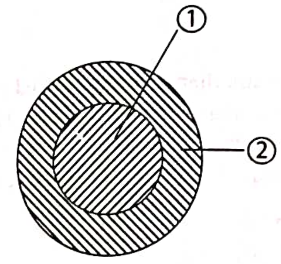


Figure 11.5

Total torque T is shared as torque T_1 by material number 1 and T_2 by material number 2.

$$\therefore \quad T_1 + T_2 = T \qquad \text{(i)}$$

Since the angle of twist of the two shafts is same,

$$\theta_1 = \theta_2$$

$$\frac{T_1}{C_1 J_1} = \frac{T_2}{CJ_2} \quad \text{or} \quad \frac{T_1}{T_1} = \frac{C_1 J_1}{C_2 J_2} \qquad \text{(ii)}$$

Solving Eqns. (i) and (ii) T_1 and T_2 can be found out and the stresses in the two materials are given by:

$$\tau_1 = \frac{T_1}{Z_1}; \quad \tau_2 = \frac{T_2}{Z_2}$$

Twisting Beyond the Limit of Proportionality

If the torque applied to a shaft is sufficient to cause yielding in the material, the relation between the shear stress and the angle of twist is assumed to be similar to that between the direct stress and angle of bending for an overstrained beam. Thus, the stress is proportional to the radians up to the limit of proportionality, after which it remains constant over the remainder of the section of the shaft.

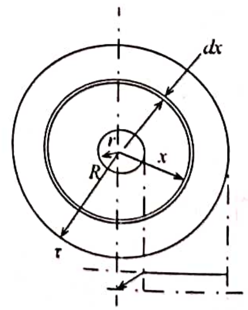


Figure 11.6

Let us consider a shaft section of radius R Fig. (11.6), which is subjected to a torque sufficient to cause yielding to a radius r; let the shear stress at the limit of proportionality be τ.

For the plastic part, then,

$$T = \tau Z = \tau \times \frac{\pi}{2} r^3$$

For the plastic part, the torque on an elementary ring of radius x and thickness dx is $z \times 2\pi x \times dx \times x$.

$$\text{Total torque on plastic part} = \int_r^R \tau \times 2\pi x^2 dx$$
$$= \tau \times \frac{2}{3}\pi(R^3 - r^3)$$

The total torque carried by the shaft is then the sum of torques carried by the elastic and plastic parts.

Note: T. J is called torsional rigidity.

Polar Modulus:

$$Z_P = \frac{J}{R} = \text{Polar modulus}$$

EXAMPLE 11.12: A composite shaft consists of a steel rod of 70 mm diameter surrounded by a closely fitting tube of brass. Find the outside diameter of the tube so that when a torque of 1200 Nm is applied to the composite shaft, it will be shared equally by the two materials. C for steel = 85 GN/m^2, C for brass = 45 GN/m^2. Find also the maximum shear stress in each material and common angle of twist in a length of 3.8 m.

SOLUTION:

Total torque, $T = T_s + T_b = T_s + T_s$ ($\because T_s = T_b$ given)

$$T_s = \frac{T}{2} = \frac{1200000}{2} = 600000 \text{ Nmm}$$
$$T_b = T_s = 600000 \text{ Nmm}$$
$$\frac{T}{J} = \frac{C\theta}{l} \quad \text{OR,} \quad T = \frac{C.\theta.J}{l}$$
$$\text{For steel,} \quad T_s = \frac{C_s\theta_s J_s}{l_s}$$
$$\text{For brass,} \quad T_b = \frac{C_b\theta_b J_b}{l_b}$$
$$\text{since} \quad T_s = T_b; \quad \therefore \quad \frac{C_s\theta_s J_s}{l_s} = \frac{C_b\theta_b J_b}{l_b}$$

As we know $l_s = l_b$

$$\therefore \quad C_s\theta_s J_s = C_b\theta_b J_b$$

But in composite shaft, the angle of twist in each shaft is same.

$$\therefore \quad \theta_s = \theta_b$$
$$\text{OR,} \quad C_s J_s = C_b J_b$$
$$85000 \times \frac{\pi}{32}(70)^4 = 45000 \times \frac{\pi}{32}[D^4 - 70^4]$$

where D is the outside diameter of brass tube

$$2.04 \times 10^{12} = 45000\left[D^4 - 24010000\right]$$

$$D^4 - 24010000 = 4.533 \times 10^9$$
$$D^4 = 45333333 + 24010000$$
$$D^4 = 69343333, \quad \text{OR,} \quad D = 91.25 \text{ mm} \quad \textbf{Ans}$$

For maximum shear stresses:

$$\tau = \frac{T \times R}{J}; \quad \text{For steel,} \ \tau_s \frac{T_s \times d/2}{J_s}$$
$$\tau_s = \frac{600000 \times \frac{70}{2}}{\frac{\pi}{32}(70)^4} = \frac{600000 \times 16}{\pi(70)^3} = 8.913 \text{ N/mm}^2 \quad \textbf{Ans}$$
$$\tau_b = \frac{T_b \times \left(\frac{D}{2}\right)}{J_b} = \frac{600000 \times \left(\frac{D}{2}\right)}{\frac{\pi}{32}(D^4 - 70^4)} = \frac{300000D}{0.1067(D^4 - 24010000)}$$
$$= \frac{300000 \times 91.25}{0.1067 \times (91.25^4 - 24010000)} = \frac{27375000}{4835819.3} = 5.66 \text{ N/mm}^2 \quad \textbf{Ans}$$

Common angle of twist,

$$T_s = \frac{C_s \times \theta_s \times J_s}{l_s} = \frac{85000 \times \theta_s \times \frac{\pi}{32}(70)^4}{3800} = 52699580.6\,\theta_s$$

substituting for T_s,

$$600000 = 52699580.6\theta_s$$
$$\therefore \quad \theta_s = 0.0114 \text{ radians}$$
$$= \frac{0.0114 \times 180}{\pi} = 0.6535° \quad \textbf{Ans}$$

Torsion of a Tapering Shaft

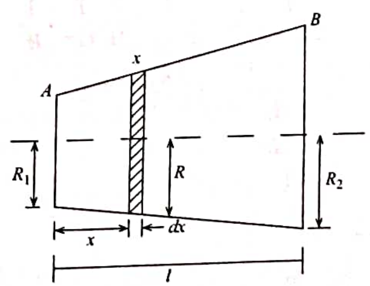


Figure 11.7

Let the tapered shaft in Fig. 11.7 be subjected to a torque T.

τ_1 = Maximum shear stress at A

τ_2 = Maximum shear stress at B

The shear stress at a distance x and section $X = \tau$, R is the radius at section x.

Now, $T = \tau_1 \times \frac{\pi}{2} R_1^3 = \tau_2 \times \frac{\pi}{2} R_2^3 = \tau \times \frac{\pi}{2} \times R^3$

or, $\tau_1 R_1^3 = \tau_2 R_2^3 = \tau R^3$

Angle of twist of small length dx,

$$d\theta = \frac{Tdx}{CJ} = \frac{Tdx}{C \times \frac{\pi}{2} R^4} = \frac{2Tdx}{C\pi R^4}$$

Also, $$R = R_1 + \left[\frac{R_2 - R_1}{l}\right] \times x = R_1 + kx \qquad \text{(i)}$$

where $k = \frac{R_2 - R_1}{l}$, k is constant for this shaft.

$$d\theta = \frac{2T}{C\pi} \cdot \frac{dx}{(R_1 + kx)^4} \qquad \text{(ii)}$$

$\therefore$ Total angle of twist for length l of the shaft $= \theta \int d\theta$

$$= \int_0^l \frac{2T}{C\pi} \cdot \frac{dx}{(R_1 + kx)^4} = -\frac{2T}{C\pi} \cdot \frac{1}{3k} \left[\frac{1}{(R_1 + kx)^3}\right]_0^l$$

$$= -\frac{2T}{C\pi} \cdot \frac{1}{3k} \left[\frac{1}{(R_1 + kl)^3} - \frac{1}{R_1^3}\right]$$

Now, $k = \frac{R_2 - R_1}{l}$ or, $kl = R_2 - R_1$; OR $R_2 = R_1 + kl$

$$\therefore \quad \theta = -\frac{2}{3k} \cdot \frac{T}{C\pi} \left[\frac{1}{R_2^3} - \frac{1}{R_1^3}\right] = \frac{2}{3k} \cdot \frac{T}{C\pi} \left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right]$$

$$\theta = \frac{2T}{3C\pi} \left(\frac{l}{R_2 - R_1}\right) \left(\frac{R_2^3 - R_1^3}{R_1^3 R_2^3}\right)$$

$$\theta = \frac{2}{3} \cdot \frac{Tl}{C\pi} \left[\frac{R_1^2 + R_1 R_2 + R_2^2}{R_1^3 R_2^3}\right] \qquad \text{(iii)}$$

In case of a shaft of uniform radius R,

$$R_1 = R_2 = R$$

$$\theta = \frac{2}{3} \cdot \frac{Tl}{C\pi} \left[\frac{R^2 + R^2 + R^2}{R^6}\right] = \frac{2}{3} \cdot \frac{Tl}{C\pi} \left[\frac{3R^2}{R^6}\right]$$

$$\theta = \frac{2}{3} \cdot \frac{Tl}{C\pi} \times \frac{3}{R^4} = \frac{Tl}{CJ},$$

which is same as before.

EXAMPLE 11.13: A 1.2 m long shaft tapers uniformly from a diameter of 100 mm to a diameter of 140 mm. If the shaft transmits a torque of 18 kNm, find:

i) Angle of twist

ii) Maximum shear stress developed. Take $C = 85\ \text{GN/m}^2$

SOLUTION:

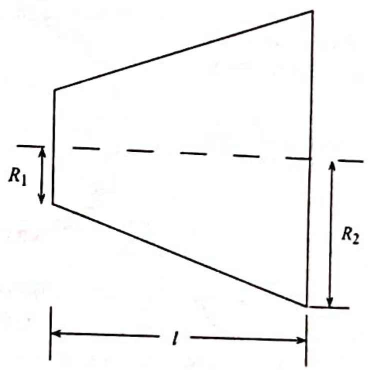


Figure 11.8

$$R_2 = \frac{140}{2} = 70\ \text{mm}, \quad R_1 = \frac{100}{2} = 50\ \text{mm}$$

$T = 18000$ Nm, $l = 1.2$ m, Now working in N and m;

Angle of twist, θ:

$$\theta = \frac{2}{3} \cdot \frac{Tl}{C\pi} \left[\frac{R_1^2 + R_1 R_2 + R_2^2}{R_1^3 R_2^3}\right]; \quad \text{working in } N \text{ and } m$$

$$= \frac{2}{3} \times \frac{18000 \times 1.2}{85 \times 10^9 \pi} \left[\frac{0.05^2 + 0.05 \times 0.07 + 0.07^2}{0.05^3 \times 0.07^3}\right]$$

$$= 5.3953 \times 10^{-8} \left[\frac{2.5 \times 10^{-3} + 3.5 \times 10^{-3} + 4.9 \times 10^{-3}}{1.25 \times 10^{-4} \times 3.43 \times 10^{-4}}\right]$$

$$= 5.3953 \times 10^{-8} \left[\frac{10.9 \times 10^{-3}}{4.2875 \times 10^{-8}}\right] = 0.01372 \text{ radian}$$

$$= \frac{0.01372 \times 180}{\pi} = 0.786° \quad \textbf{Ans}$$

Maximum shear stress developed:

$$T_{max} = \tau_{max} \times \frac{\pi}{16} \times D_1^3$$

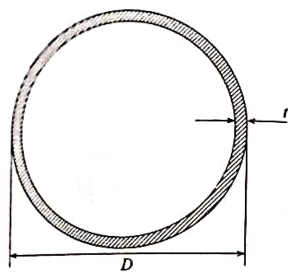
Because maximum shear stress occurs at the smallest diameter.

$$18 \times 10^3 = \tau_{max} \times \frac{\pi}{16}(0.1)^3$$

$$\therefore \quad \tau_{max} = 91.72\ \text{MN/m}^2 \quad \textbf{Ans}$$

Thin Circular Tube Subjected to Torsion

Consider a thin circular tube of external diameter D and thickness t. Whereas t is very small compared to diameter D.



Thin circular tube

Figure 11.9

J = Area of the section × Square of radius

$$= \pi Dt \left(\frac{D}{2}\right)^2 = \frac{\pi D^3 t}{4}$$

$$T = \tau \times \frac{J}{R} = \tau \times \frac{\pi D^3 t \times 2}{4 \times D}$$

$$\therefore \quad T = \frac{\tau \pi D^2 t}{2} \qquad \text{(i)}$$

and

$$\theta = \frac{T.l}{C.J} = \frac{T.l.4}{C.\pi D^3 t} = \frac{4Tl}{\pi D^3 tC}$$

$$\therefore \quad \theta = \frac{4Tl}{\pi D^3 tC} \qquad \text{(ii)}$$

$$\text{Strength weight ratio} = \frac{T}{W}$$

$W = \pi D t l w$ where w is the weight density of the material.

$$T = \frac{\tau \pi D^2 t}{2}$$

$$\therefore \quad \frac{T}{W} = \frac{\tau \pi D^2 t}{2\pi D t l w} = \frac{\tau D}{2lw}$$

$$\text{Hence,} \quad \frac{T}{W} = \frac{\tau D}{2lw} \qquad \text{(iii)}$$

EXAMPLE 11.14: A thin steel tube of 90 mm diameter is 3 mm thick. If the allowable shear stress is 75 MN/m^2 and C = 82 GN/m^2, find:

i) Safe twisting moment that can be applied to the tube;
ii) The twist in a length of 550 mm.

SOLUTION:

$$D = 90\text{ mm} = 0.09\text{ m}$$
$$t = 3\text{ mm} = 0.003\text{ m}$$

i) Safe twisting moment T:

$$T = \frac{\tau \pi D^2 t}{2} = \frac{75 \times 10^6 \times (0.09)^2 (0.003)}{2}$$
$$= 911.25\text{ Nm} \quad \textbf{Ans}$$

ii) Angular twist θ:

$$\theta = \frac{4Tl}{\pi D^3 tC} = \frac{4 \times 911.25 \times 0.55}{\pi (0.09)^3 (0.003) \times 82 \times 10^9}$$
$$= 0.00356 \text{ radians}$$
$$= \frac{0.00356 \times 180}{\pi} = 0.204° \quad \textbf{Ans}$$

EXAMPLE 11.15: A shaft *LMN* of 550 mm length and 45 mm external diameter is having a hole for a part of its length *LM*, of a 22 mm diameter and for the remaining length *MN* having a hole of 32 mm diameter. If the shear stress is not to exceed 80 N/mm^2, find the maximum power, the shaft can transmit at a speed of 250 r.p.m.

If the angle of twist in the length of 22 mm diameter hole is equal to that in the 32 mm diameter hole, find the length of the shaft that has been bored to 22 mm and the length of the shaft that has been bored to 32 mm diameter.

$$T = \frac{\pi}{16} \tau \left[\frac{d_o^4 - d_i^4}{d_o}\right]$$

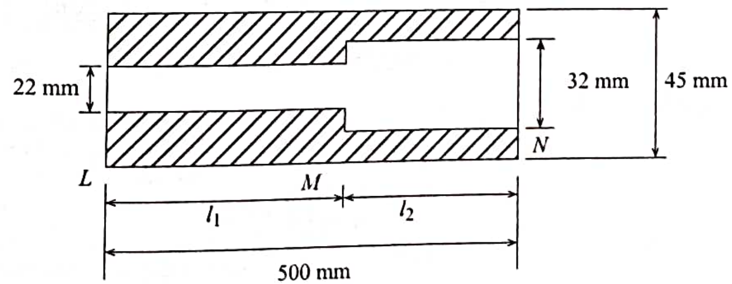


Figure 11.10

CHAPTER 16

THEORIES OF ELASTIC FAILURE

A material is regarded as failed if it is loaded beyond the elastic limit and permanent deformation occurs when the particles of a material separate from each other (as in use of brittle material) accompanied by considerable plastic deformation.

It is easy to guess failure of material when the material is subjected to simple stress followed by an axial loading. But when the material is subjected to complex stresses followed by biaxial or triaxial loading then it is difficult to predict the failure of material.

For complex systems, we shall discuss five important theories of failure. In these five theories, the complex state has been related to the elastic limit in simple tension or compression.

As we have studied that in any complex loading system, three principal stresses exist: σ_1, σ_2 and σ_3 such that $\sigma_1 > \sigma_2 > \sigma_3$. σ_1 is the maximum principal stress, σ_2 is the intermediate principal stress end σ_3 is the minimum principal stress.

In 2-D stress system we will discuss the following five theories of failure:

1. Maximum Principal Stress Theory (Rankine's Theory)
2. Maximum Shear Stress Theory (Guest's or Tresca's Theory)
3. Maximum Principal Strain Theory (St. Venant's Theory)
4. Maximum Strain Energy Theory (Haigh's Theory)
5. Maximum Shear Strain Energy Theory (Von Mises's Theory):

Now we shall discuss these theories in details.

1. Maximum Principal Stress Theory (Rankine's Theory)

According to this theory, failure will occur when the maximum principal tensile stress (σ_1) in the complex system reaches the value of the maximum stress at the elastic limit (σ_{et}) in simple tension or the minimum principal stress (that is, the maximum principal compressive stress) reaches the elastic limit stress (σ_{ec}) in simple compression, i.e.

$\sigma_1 = \sigma_{et}$ (in simple tension) $= \sigma_o = \sigma_u$ (yield stress is simple tension or compression)

$\sigma_3 = \sigma_{ec}$ (in simple compression)

σ_3 means numerical value of σ_3.

Graphical representation is shown in Fig. 16.1.

This theory is found to have good results when applied to brittle materials such as cast iron.

Note: Working stress, $\sigma_1 = \pm\frac{\sigma_u}{\text{Factor of Safety}}$ and $\sigma_2 = \pm\frac{\sigma_u}{F.O.S}$

where σ_y, σ_x, τ direct and shear stresses on given planes is the complex system, σ_1 = maximum principal stress

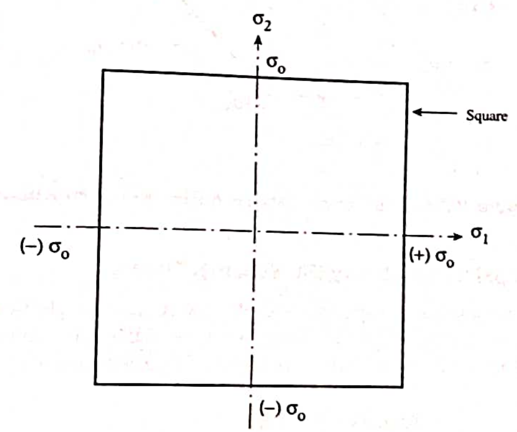


Figure 16.1

2. Maximum Shear Stress Theory (Guest or Teresa's Theory)

According to this theory the failure occurs when the maximum shear stress τ_{max} reaches the value of the maximum shear stress in simple tension at the elastic limit, i.e.,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{et}}{2} = \frac{\sigma_u}{2} \quad \text{in simple tension}$$

or

$$\sigma_1 - \sigma_3 = \sigma_{et} = \sigma_u$$

While designing σ_{et} is replaced by the safe stress.

This theory does not give accurate results for the state of stress of pure shear in which the maximum amount of shear in developed.

This theory is preferred in case of ductile materials such as mild steel. In 3-D stress system,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

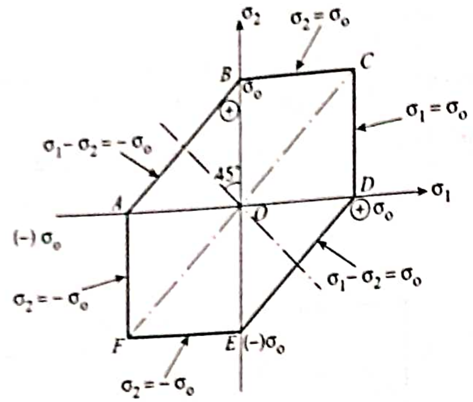


Figure 16.2 Graphical representation of maximum shear stress theory

3. Maximum Principal Strain Theory (St. Venant's Theory)

Failure occurs when the greatest principal strain reaches the strain at the elastic limit in a simple test. According to this theory, failure of material occurs when the maximum strain in the complex stress system equals the value of maximum strain at yield point in simple tension or compression test.

$$\varepsilon_{\text{complex}} = \varepsilon_{\text{simple}}$$

$$\frac{1}{E}(\sigma_1 - \mu\sigma_2) = \pm\frac{\sigma_0}{E}$$

$$\sigma_1 - \mu\sigma_2 = \pm\sigma_0 \quad \text{(for tensile test)}$$

$$\text{Also} \quad \sigma_2 - \mu\sigma_1 = \pm\sigma_0 \quad \text{(for compressive test)}$$

Except for brittle material, this theory does not match with experimental results, and so finds little general support.

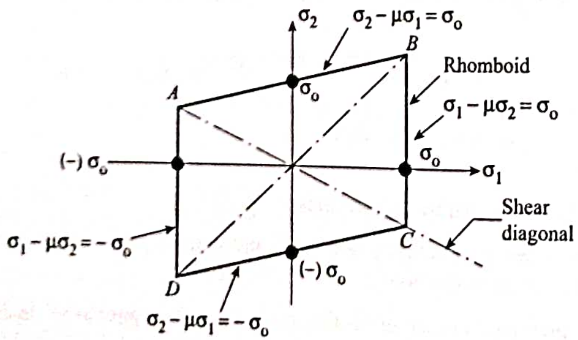


Figure 16.3 Graphical representation of maximum principal strain theory

By plotting the above equation a rhomboid is obtained.

4. Maximum Strain Energy Theory (Haigh's Theory)

Failure occurs when the energy stored per unit volume in a strained material reaches the strain energy per unit volume at the elastic limit in a simple tension test, i.e., the maximum energy a body can store without permanent deformation in a fixed quantity, irrespective of the manner in which it is strained.

$$\frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2\right) = \frac{\sigma_0^2}{2E}$$

or
$$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \sigma_0^2$$

or
$$\left(\frac{\sigma_1}{\sigma_0}\right)^2 + \left(\frac{\sigma_2}{\sigma_0}\right)^2 - 2\mu\left(\frac{\sigma_1}{\sigma_0}\right)\cdot\left(\frac{\sigma_2}{\sigma_0}\right) = 1$$

This is the equation of an ellipse with semi-major and semi-minor axes $\frac{\sigma_0}{\sqrt{1-\mu}}$; $\frac{\sigma_0}{\sqrt{1+\mu}}$ respectively each at 45°. This theory gives good results for ductile materials to the coordinate axes as shown graphically in Fig. 16.4.

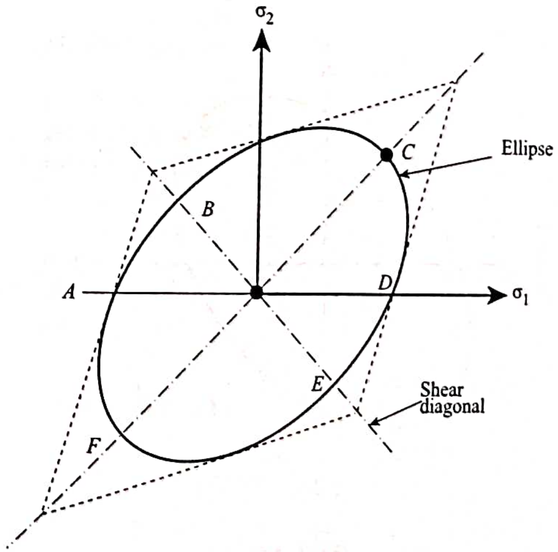


Figure 16.4

5. Maximum Shear Strain Energy Theory (Von Mises and Henkey's Theory)

This is also known as *distortion energy theory*. Failure occurs when the shear strain energy per unit volume in a strained material reaches the shear strain energy per unit volume at the elastic limit in a simple tension test. this is similar to the preceding theory but it is assumed that the volumetric strain energy plays no part in producing elastic failures.

(a) For 3D stress system:

$$\frac{1}{12C}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=\frac{\sigma_0^2}{6C}$$

$$\text{or}\quad (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_0^2$$

(b) For 2D stress system, putting $\sigma_3 = 0$, we get

$$\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2=\sigma_0^2$$

$$\left(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2\right)^{\frac{1}{2}}=\sigma_0$$

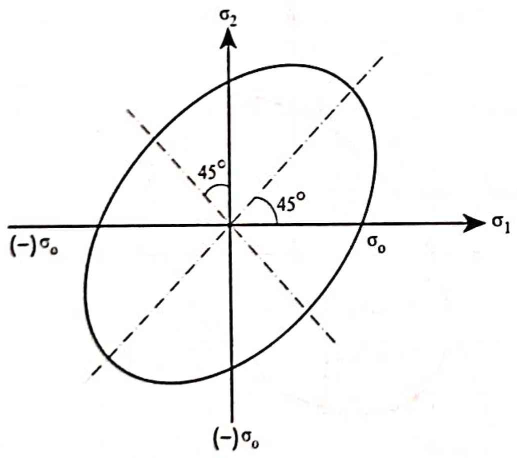


Figure 16.5

This theory is good for ductile materials.

This is the best theory of the above five theories.

When $\sigma_t = \sigma_c$ and $\varepsilon_t = \varepsilon_c$.

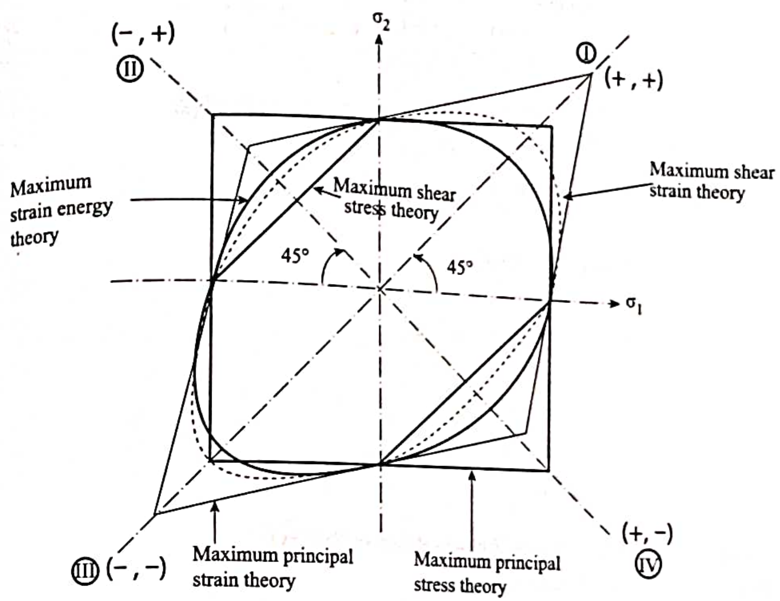


Figure 16.6 Graphical representation of various theories of failure on the same diagrams

EXAMPLE 16.1: The load on a bolt is an axial pull of 15 kN together with transverse shear force of 7 kN. Estimate the diameter of the bolt. Using all theories of failure. $\sigma_{ut} = 280\ \text{N/mm}^2$, factor of safety = 4. Poisson's ratio $(\mu) = 0.3$

SOLUTION:

Allowable simple tensile stress $= \dfrac{280}{4} = 70\ \text{N/mm}^2 = \sigma_0$

Let d = diameter of bolt (core dia).

$$\text{Then, normal stress is} = \frac{15000}{\frac{\pi}{4}d^2} = \frac{60000}{\pi d^2} = \sigma$$

$$\text{and shear stress}\quad (\tau) = \frac{7000}{\frac{\pi}{4}d^2} = \frac{28000}{\pi d^2}$$

Therefore, principle stresses are

$$\sigma_1,\sigma_2=\frac{\sigma_x}{2}\pm\sqrt{\left(\frac{\sigma_x}{2}\right)^2+\tau^2}\quad\text{and}\quad \sigma_3=0$$

$$\sigma_1\sigma_2 = \frac{60000}{2\pi d^2} \pm \sqrt{\left(\frac{60000}{2\pi d^2}\right)^2 + \left(\frac{28000}{\pi d^2}\right)^2}$$

$$\sigma_1 = \frac{30000}{\pi d^2} + \frac{1}{\pi d^2}\sqrt{900000000+784000000}$$

$$= \frac{30000}{\pi d^2} + \frac{1}{\pi d^2} \times 41036.6$$

$$= \frac{71036.6}{\pi d^2}$$

$$\sigma_2 = \frac{30000}{\pi d^2} - \frac{41036.6}{\pi d^2}$$

$$= -\frac{110.36}{\pi d^2}$$

$$\sigma_3 = 0$$

1. Applying Rankine Theory (Principal Stress Theory)

Maximum principal stress in bolt

$$\sigma_1 = \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2+4\tau^2}$$

$$= \frac{60000}{2\pi d^2} + \frac{1}{2}\sqrt{\left(\frac{60000}{\pi d^2}\right)^2 + 4\left(\frac{28000}{\pi d^2}\right)^2}$$

$$= \frac{30000}{\pi d^2} + \frac{1}{2\pi d^2}\sqrt{3600000000+3136000000}$$

$$= \frac{30000}{\pi d^2} + \frac{1}{2\pi d^2} \times 82073.14$$

$$\sigma_1 = \frac{71036.57}{\pi d^2}$$

Maximum stream in simple tension = 70 N/mm^2 = σ_0

Now equating $\sigma_1 = \sigma_0$

$$\frac{71036.57}{\pi d^2} = 70$$

$$\therefore \quad d^2 = 323.2$$

$$\text{or} \quad d = 17.977 = 18 \text{ mm}$$

2. Maximum Shear Stress Theory (Guest on Tresca's Theory)

$$\tau_{max} = \frac{1}{2}\sqrt{\sigma^2+4\tau^2}$$

$$= \frac{1}{2}\sqrt{\left(\frac{60000}{\pi d^2}\right)^2 + 4\left(\frac{28000}{\pi d^2}\right)^2}$$

$$= \frac{1}{2\pi d^2}\sqrt{3600000000+3136000000}$$

$$= \frac{1}{2\pi d^2} \times 82073.14$$

$$= \frac{41036.6}{\pi d^2}$$

Now $\tau = \frac{\sigma_2 - \sigma_1}{2} = \frac{\sigma}{2} = \frac{10}{2} = 35 \text{ N/mm}^2$

$$\therefore \quad \frac{41036.6}{\pi d^2} = 35$$

$$d^2 = \frac{41036.6}{35 \times \pi} = 373.4$$

$$\therefore \quad d = 19.32 \text{ mm}$$

3. Maximum Strain Energy Theory (Von Mises & Hencky Theory) or Distortion Theory

$$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \sigma_0^2$$

$$\left(\frac{71036.6}{\pi d^2}\right)^2 + \left(-\frac{11036}{\pi d^2}\right)^2 + 2\times 0.3\left(\frac{71036.6}{\pi d^2}\right)\left(\frac{11036}{\pi d^2}\right) = 70^2$$

$$\frac{5046198540}{\pi^2 d^4} + \frac{121793296}{\pi^2 d^4} + \frac{4703759506}{\pi^2 d^4} = 4900$$

$$\frac{9871751392}{\pi^2 d^4} = 4900$$

$$\frac{1001232443}{4900} = d^4$$

$$d^4 = 204333$$

$$d = 21.26 \text{ mm}$$

4. Maximum Strain Energy Theory (Haigh Theory)

$$(\sigma_1 - \sigma_2)^2 + \sigma_2^2 = \sigma_0^2$$

$$\left(\frac{71036.6}{\pi d^2} + \frac{11036}{\pi d^2}\right)^2 + \left(-\frac{11036}{\pi d^2}\right) = 70^2$$

$$\left(\frac{82072}{\pi d^2}\right)^2 + \frac{121793296}{\pi^2 d^4} = 4900$$

$$\frac{1}{\pi^2 d^4}(6735813184 + 121793296) = 4900$$

$$\frac{6747992480}{\pi^2 \times 4900} = d^4$$

$$d^4 = 139675$$

$$d = 19.33 \text{ mm}$$

Maximum diameter d is 21.26 mm

So the answer is **21.26** mm

EXAMPLE 16.2: A subject is subjected to a maximum torque of 12 kNm and a maximum bending moment of 8.5 kNm at a particular section. If the allowable equivalent stress in simple tension is 170 MN/m^2, find the diameter of the shaft according to the maximum shear strain theory.

Maximum torque, $T = 12$ kNm

Maximum bending moment, $M = 8.5$ kNm

Allowable equivalent stress in simple tension,

$$\sigma_t = 170 \text{ MN/m}^2$$

$$M = \frac{\pi}{32} d^3 \sigma_b$$

$$\sigma_b = \frac{M \times 32}{\pi d^3} \quad \text{and} \quad T = \tau \times \frac{\pi}{16} d^3$$

$$\therefore \quad \tau = \frac{16T}{\pi d^3}$$

Principal stresses are given by:

$$\sigma_1, \sigma_2 = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2}$$

$$= \frac{1}{2}\left[\sigma_b \pm \sqrt{\sigma_b^2 + 4\tau^2}\right]$$

$$= \frac{1}{2}\left[\frac{32M}{\pi d^3} \pm \sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + \left(\frac{32T}{\pi d^3}\right)^2}\right]$$

$$= \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

$$\sigma_1 = \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

$$\sigma_2 = 0$$

$$\text{and} \quad \sigma_3 = \frac{16}{\pi d^3}\left[M - \sqrt{M^2 + T^2}\right]$$

According to the maximum shear stress theory,

$$\sigma_t = \sigma_1 - \sigma_3 = \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right] - \frac{16}{\pi d^3}\left[M - \sqrt{M^2 + T^2}\right]$$

$$= \frac{32}{\pi d^3}[M^2 + T^2]$$

$$d^3 = \frac{32 \times 10^3}{\pi \times 160 \times 10^6} \times \sqrt{8.5^2 + 12^2}$$

$$d^3 = 6.37 \times 10^{-5} \times 14.705$$

$$d^3 = 0.000937 \quad \therefore d = 0.0978 \text{ m}$$

$$d = 97.8 \text{ mm} \quad \textbf{Ans}$$

EXAMPLE 16.3: A steel shaft is subjected to an end thrust of 90 MPa and the maximum shearing stress on the surface arising from torsion is 68 MPa. The yield point of the material in simple tension was found to be 300 MPa. Calculate the factor of safety of the shaft according to the following theories:

i) Maximum shear stress theory
ii) Maximum distortion energy theory

(AMIE Summer 2000)

SOLUTION:

Given $\sigma_1 = ?$ $\sigma_2 = 0$, $\sigma_3 = -90$ MN/m^2

$$\tau_{max} = 60 \text{ MN/m}^2;\ \sigma_{t}t = 300 \text{ MN/m}^2$$

i) Maximum shear stress theory

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 60$$

$$\text{or}\quad \sigma - (-90) = 120 \quad \text{or} \quad \sigma_1 = 30 \text{ MN/m}^2$$

$$\text{Also}\quad \sigma_1 - \sigma_3 = \sigma_t$$

$$30 - (-90) = \sigma_t \quad \therefore \quad \sigma_t = 120 \text{ MN/m}^2$$

$$\text{F. O. S.} = \frac{\sigma_{et}}{\sigma_t} = \frac{300}{120} = 2.5 \quad \textbf{Ans}$$

ii) Maximum distortion energy theory

$$\sigma_t^2 = \sigma_l^2 + \sigma_3^2 - \sigma_3\sigma_1$$

$$= 30^2 + (-90)^2 - (-90)(30)$$

$$= 11700$$

$$\sigma_t = 108.17 \text{ MN/m}^2$$

$$\therefore \quad \text{F.O.S.} = \frac{\sigma_{et}}{\sigma_t} = \frac{300}{108.17} = 2.77 \quad \textbf{Ans}$$

EXAMPLE 16.4: In a steel member, at a point the major principal stress is 200 MN/m^2 and the minor principal stress is compressive. If the tensile yield point of the steel is 250 MN/m^2, find the value of the minor principal stress at which yielding will commence, according to each of the following criteria of failure:

(i) Maximum shearing stress,

(ii) Maximum total strain energy, and

(iii) Maximum shear strain energy

Take, Poisson's ratio = 0.3

SOLUTION:

$\sigma_1 = 200$ MN/m^2

Yield point stress = $\sigma_e = 250$ MN/m^2

Minor principal stress, σ_2:

i) Maximum shearing stress criterion:

$$\sigma_1 - \sigma_2 = \sigma_e$$

$$\sigma_2 = \sigma_1 - \sigma_e = 200 - 250 = -50 \text{ MN/m}^2$$

$$\sigma_2 = 50 \text{ MN/m}^2 \quad \text{(Compressive)} \quad \textbf{Ans}$$

ii) Maximum total strain energy criterion:

$$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \sigma_e^2 \quad (\text{or } \sigma_0^2)$$

$$(200)^2 + \sigma_2^2 - 2 \times 0.3 \times 200 \times \sigma_2 = 250^2$$

$$40000 + \sigma_2^2 - 120\,\sigma_2 = 62500$$

$$\sigma_2^2 - 120\sigma - 62500 + 40000 = 0$$

$$\sigma_2^2 - 1206 - 22500 = 0$$

$$\sigma_2 = \frac{+120 - \sqrt{14400 + 90000}}{2} \quad \text{(only −ve sign is taken as } \sigma_2 \text{ is to be compressive)}$$

$$\sigma_2 = \frac{+120 - \sqrt{104400}}{2}$$

$$= \frac{+120 - 323}{2}$$

$$= 101.5 \text{ MN/m}^2 \quad \text{(Compressive)} \quad \textbf{Ans}$$

iii) Maximum shear strain energy criterion:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_e^2$$

putting $\sigma_3 = 0$, we get

$$(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2 = 2\sigma_e^2$$

$$\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_1^2 = 2\sigma_e^2$$

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_2^2$$

$$(200)^2 + \sigma_2^2 - 200\,\sigma_2 = (250)^2$$

$$\sigma_2^2 + 40000 - 62500 - 200\sigma_2 = 0$$

$$\sigma_2^2 - 200\sigma_2 - 22500 = 0$$

$$\sigma_2 = \frac{+200 - \sqrt{40000 + 90000}}{2}$$

$$= -80.27\ \text{MN/m}^2 \quad \text{(Compressive)} \quad \textbf{Ans}$$

EXAMPLE 16.5: At a point two of the principal stresses are 150 N/mm² and 100 N/mm². Determine the safe range of the third principal stress at the point by five different theories. Take E = 200 GPa and μ = 0.25, failure stress in tension test to be 220 N/mm². Failure stress in tension and compression is the same.

SOLUTION:

a) Maximum principal stress theory

$$\sigma_1 = \sigma_y = 220\ \text{N/mm}^2 \quad (\sigma_y = \text{yield stress})$$

$$\sigma_3 = \sigma_y' = -220\ \text{N/mm}^2$$

Hence, the range is $-220 \leq \sigma \leq 220$

b) Maximum strain theory

$$\sigma_1 - 0.3\,(150 + 100) = \sigma_y = 220$$

$$\sigma_1 = 220 + 75 = 295\ \text{N/mm}^2$$

Also, $\sigma_3 - 0.3\,(150 + 100) = \sigma' y = -220$

$$\sigma_3 = -220 + 75 = -145\ \text{N/mm}^2$$

Hence, the range is $-145 \leq \sigma \leq 220\ \text{N/mm}^2$

c) Maximum strain energy theory

$$\sigma^2 + (100)^2 + (150)^2 - 2 \times 0.3\,(150\sigma + 100\sigma + 150 \times 100) = 220^2$$

$$\sigma^2 + 10000 + 22500 - 150\sigma - 9000 = 48400$$

$$\sigma^2 - 150\sigma - 24800 = 0$$

$$\sigma = \frac{150 \pm \sqrt{22500 + 99600}}{2}$$

$$\sigma = \frac{150 \pm 349.4}{2}$$

$$\therefore \quad \sigma_1 = 249.7,\ \sigma_2 = -99.7$$

Hence, the range is $-99.7 \leq \sigma \leq 249.7$.

d) Maximum shear stress theory

$$\sigma_1 - 100 = \sigma_y = 210$$

$$\sigma_1 = 220 + 100 = 320\ \text{N/mm}^2$$

Also, $150 - \sigma_3 = 220$

$$\sigma_3 = 150 - 220 = -70\ \text{N/mm}^2$$

Hence, the range is $-70 \leq \sigma \leq 320\ \text{N/mm}^2$

e) Maximum distortion theory or shear strain theory

$$(\sigma - 100)^2 + (\sigma - 150)^2 + (150 - 100)^2 = 2\,(220)^2$$

$$\sigma^2 + 10000 - 200\sigma + \sigma^2 + 22500 - 300\sigma + 2500 = 96800$$

$$2\sigma^2 - 500\sigma - 61800 = 0$$

$$\sigma^2 - 250\sigma - 30900 = 0$$

$$\sigma = \frac{250 \pm \sqrt{62500 + 123600}}{2}$$

$$\sigma = \frac{+250 \pm 431.4}{2}$$

$$\sigma_1 = 340.7,\ \sigma_3 - 181.4$$

Hence, the range is $-181.4 \leq \sigma \leq 340.7\ \text{N/mm}^2$.

EXAMPLE 16.6: Principal stresses at a point in an elastic material are 120 MPa tensile, 60 MPa tensile and 20 MPa compressive. Determine the factor of safety against the failure based on various theories. The elastic limit in simple tension is 240 MPa and Poisson's ratio 0.3.

SOLUTION:

i) Maximum principal stress theory

Failure takes place when the maximum principal stress reaches the value of maximum stress at that limit.

Thus, maximum principal stress, σ = 120 MPa

So factor of safety $= \dfrac{240}{120} = 2$

ii) Maximum shear stress theory

$$\sigma = 120 - (-20) = 140\ \text{MPa}$$

$$\text{Factor of safety} = \frac{240}{140} = 1.71$$

iii) Maximum principal strain theory

$$\sigma_1 - \mu\sigma_2 - \mu\sigma_3 = 120 - 0.3 \times 60 - 0.3(-20) = 108 \text{ MPa}$$

$$\text{Factor of safety} = \frac{240}{108} = 2.03$$

iv) Maximum strain energy theory

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$= 120^2 + 60^2 + (-20)^2 - 2 \times 0.3[120 \times 60 + 60(-20) + (-20 \times 120)]$$

$$= 14400 + 3600 + 400 - 3600$$

$$= 14800$$

$$\therefore \quad \sigma = 121.6$$

$$\text{Factor of safety} = \frac{240}{121.6} = 1.97$$

v) Maximum shear strain energy theory

$$2\sigma^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

$$= (120 - 60)^2 + [60 - (-20)]^2 + [-20 - 120]^2$$

$$= 3600 + 6400 + 19600$$

$$= 29600$$

$$\sigma^2 = \frac{29600}{2} = 14800$$

$$\therefore \quad \sigma = 121.65 \text{ MPa}$$

$$\text{Factor of safety} = \frac{240}{121.65} = 1.97$$

EXAMPLE 16.7: A solid shaft transmits 1000 kW at 300 r.p.m. Maximum torque is 2 times the mean. The shaft is subjects to a bending moment, which is 1.5 times the mean torque. The shaft is of ductile material for which the permissible tensile and shear stresses are 120 MPa and 60 MPa respectively. Determine the shaft diameter using suitable theory of failure. Give justification of the theory used.

[AMI.E. Sec B Winter 94]

SOLUTION:
Principal stresses:

$$\sigma_b = \frac{32M}{\pi d^3}; \quad \tau = \frac{16T}{\pi d^3}$$

$$\text{Principal stresses} = \frac{1}{2}\left[\sigma_b \pm \sqrt{\sigma_b^2 + 4\tau^2}\right]$$

$$= \frac{1}{2}\left[\frac{32M}{\pi d^3} \pm \frac{32}{\pi d^3}\sqrt{M^2 + T^2}\right]$$

$$= \frac{16}{\pi d^3}\left[M \pm \sqrt{M^2 + T^2}\right]$$

$$\sigma_1 = \frac{16}{\pi d^3}\left(M + \sqrt{M^2 + T^2}\right), \sigma_2 = 0 \quad \text{and} \quad \sigma_3 = \frac{16}{\pi d^3}\left(M - \sqrt{M^2 + T^2}\right)$$

Best theory is the distortion energy theory, which gives

$$\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 = \sigma_t^2$$

$$\text{or} \quad \left(\frac{16}{\pi d^3}\right)^2 \left(M + \sqrt{M^2 + T^2}\right)^2 + \left(M - \sqrt{M^2 + T^2}\right)^2 \times$$

$$\left(\frac{16}{\pi d^3}\right)^2 - \left(M^2 - M^2 + T^2\right)\left(\frac{16}{\pi d^3}\right)^2 = \sigma_t^2$$

$$\text{or} \quad \left(\frac{16}{\pi d^3}\right)^2 \left(2M^2 + 2M^2 + 2T^2 + T^2\right) = \sigma_t^2$$

$$\text{or} \quad \left(\frac{16}{\pi d^3}\right)^2 \left(4M^2 + 3T^2\right) = \sigma_t^2$$

$$\sigma_t = \frac{32}{\pi d^3}\sqrt{M^2 + \frac{3}{4}T^2} \qquad \text{(i)}$$

$$\frac{2\pi N T_{mean}}{60} = 1000 \times 10^3$$

$$\frac{2\pi \times 300 \times T_{mean}}{60} = 1000 \times 10^3$$

$$\therefore \quad T_{mean} = \frac{100 \times 10^3}{\pi} = \frac{100}{\pi} \text{ kNm}$$