SUPPLEMENTARY PROBLEMS

28. If $\mathbf{R}(t) = (3t^2-t)\mathbf{i} + (2-6t)\mathbf{j} - 4t\,\mathbf{k}$, find (a) $\int \mathbf{R}(t)\,dt$ and (b) $\int_2^4 \mathbf{R}(t)\,dt$.

Ans. (a) $(t^3-t^2/2)\mathbf{i} + (2t-3t^2)\mathbf{j} - 2t^2\,\mathbf{k} + \mathbf{c}$ (b) $50\mathbf{i} - 32\mathbf{j} - 24\mathbf{k}$

29. Evaluate $\int_0^{\pi/2} (3\sin u\,\mathbf{i} + 2\cos u\,\mathbf{j})\,du$ *Ans.* $3\mathbf{i} + 2\mathbf{j}$

30. If $\mathbf{A}(t) = t\,\mathbf{i} - t^2\,\mathbf{j} + (t-1)\mathbf{k}$ and $\mathbf{B}(t) = 2t^2\,\mathbf{i} + 6t\,\mathbf{k}$, evaluate (a) $\int_0^2 \mathbf{A}\cdot\mathbf{B}\,dt$, (b) $\int_0^2 \mathbf{A}\times\mathbf{B}\,dt$.

Ans. (a) 12 (b) $-24\,\mathbf{i} - \frac{40}{3}\mathbf{j} + \frac{64}{5}\mathbf{k}$

31. Let $\mathbf{A} = t\,\mathbf{i} - 3\mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{C} = 3\mathbf{i} + t\,\mathbf{j} - \mathbf{k}$. Evaluate (a) $\int_1^2 \mathbf{A}\cdot\mathbf{B}\times\mathbf{C}\,dt$, (b) $\int_1^2 \mathbf{A}\times(\mathbf{B}\times\mathbf{C})\,dt$.

Ans. (a) 0 (b) $-\frac{87}{2}\mathbf{i} - \frac{44}{3}\mathbf{j} + \frac{15}{2}\mathbf{k}$

32. The acceleration $\mathbf{a}$ of a particle at any time $t \geqq 0$ is given by $\mathbf{a} = e^{-t}\mathbf{i} - 6(t+1)\mathbf{j} + 3\sin t\,\mathbf{k}$. If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

Ans. $\mathbf{v} = (1-e^{-t})\mathbf{i} - (3t^2+6t)\mathbf{j} + (3-3\cos t)\mathbf{k}$, $\mathbf{r} = (t-1+e^{-t})\mathbf{i} - (t^3+3t^2)\mathbf{j} + (3t-3\sin t)\mathbf{k}$

33. The acceleration $\mathbf{a}$ of an object at any time t is given by $\mathbf{a} = -g\,\mathbf{j}$, where g is a constant. At $t=0$ the velocity is given by $\mathbf{v} = v_0\cos\theta_0\,\mathbf{i} + v_0\sin\theta_0\,\mathbf{j}$ and the displacement $\mathbf{r} = \mathbf{0}$. Find $\mathbf{v}$ and $\mathbf{r}$ at any time $t>0$. This describes the motion of a projectile fired from a cannon inclined at angle θ_0 with the positive x-axis with initial velocity of magnitude v_0.

Ans. $\mathbf{v} = v_0\cos\theta_0\,\mathbf{i} + (v_0\sin\theta_0 - gt)\mathbf{j}$, $\mathbf{r} = (v_0\cos\theta_0)t\,\mathbf{i} + [(v_0\sin\theta_0)t - \frac{1}{2}gt^2]\,\mathbf{j}$.

34. Evaluate $\int_2^3 \mathbf{A}\cdot\frac{d\mathbf{A}}{dt}\,dt$ if $\mathbf{A}(2) = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{A}(3) = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. *Ans.* 10

35. Find the areal velocity of a particle which moves along the path $\mathbf{r} = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$ where a,b,ω are constants and t is time. *Ans.* $\frac{1}{2}ab\omega\,\mathbf{k}$

36. Prove that the squares of the periods of planets in their motion around the sun are proportional to the cubes of the major axes of their elliptical paths (Kepler's third law).

37. If $\mathbf{A} = (2y+3)\mathbf{i} + xz\,\mathbf{j} + (yz-x)\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ along the following paths C:

(a) $x=2t^2$, $y=t$, $z=t^3$ from $t=0$ to $t=1$,

(b) the straight lines from (0,0,0) to (0,0,1), then to (0,1,1), and then to (2,1,1),

(c) the straight line joining (0,0,0) and (2,1,1).

Ans. (a) 288/35 (b) 10 (c) 8

38. If $\mathbf{F} = (5xy-6x^2)\mathbf{i} + (2y-4x)\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C in the xy plane, $y=x^3$ from the point (1,1) to (2,8). *Ans.* 35

39. If $\mathbf{F} = (2x+y)\mathbf{i} + (3y-x)\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane consisting of the straight lines from (0,0) to (2,0) and then to (3,2). *Ans.* 11

40. Find the work done in moving a particle in the force field $\mathbf{F} = 3x^2\,\mathbf{i} + (2xz-y)\mathbf{j} + z\,\mathbf{k}$ along

(a) the straight line from (0,0,0) to (2,1,3).

(b) the space curve $x=2t^2$, $y=t$, $z=4t^2-t$ from $t=0$ to $t=1$.

(c) the curve defined by $x^2=4y$, $3x^3=8z$ from $x=0$ to $x=2$.

Ans. (a) 16 (b) 14.2 (c) 16

41. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (x-3y)\mathbf{i} + (y-2x)\mathbf{j}$ and C is the closed curve in the xy plane, $x = 2\cos t$, $y = 3\sin t$ from $t=0$ to $t = 2\pi$. *Ans.* 6π, if C is traversed in the positive (counterclockwise) direction.

42. If $\mathbf{T}$ is a unit tangent vector to the curve C, $\mathbf{r} = \mathbf{r}(u)$, show that the work done in moving a particle in a force field $\mathbf{F}$ along C is given by $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where s is the arc length.

43. If $\mathbf{F} = (2x+y^2)\mathbf{i} + (3y-4x)\mathbf{j}$, evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around the triangle C of Figure 1, (a) in the indicated direction, (b) opposite to the indicated direction. *Ans.* $(a)\ -14/3$ $(b)\ 14/3$

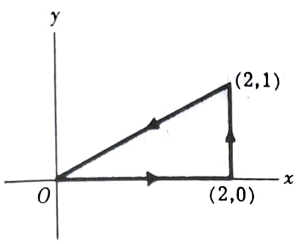

Fig. 1

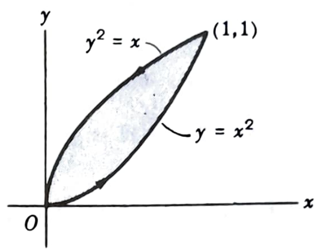

Fig. 2

44. Evaluate $\oint_C \mathbf{A}\cdot d\mathbf{r}$ around the closed curve C of Fig. 2 above if $\mathbf{A} = (x-y)\mathbf{i} + (x+y)\mathbf{j}$. *Ans.* $2/3$

45. If $\mathbf{A} = (y-2x)\mathbf{i} + (3x+2y)\mathbf{j}$, compute the circulation of $\mathbf{A}$ about a circle C in the xy plane with centre at the origin and radius 2, if C is traversed in the positive direction. *Ans.* 8π

46. (a) If $\mathbf{A} = (4xy - 3x^2z^2)\mathbf{i} + 2x^2\mathbf{j} - 2x^3z\,\mathbf{k}$, prove that $\int_C \mathbf{A}\cdot d\mathbf{r}$ is independent of the curve C joining two given points. (b) Show that there is a differentiable function ϕ such that $\mathbf{A} = \nabla\phi$ and find it.
Ans. $(b)\ \phi = 2x^2y - x^3z^2 + \text{constant}$

47. (a) Prove that $\mathbf{F} = (y^2\cos x + z^3)\mathbf{i} + (2y\sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k}$ is a conservative force field.
(b) Find the scalar potential for $\mathbf{F}$.
(c) Find the work done in moving an object in this field from $(0,1,-1)$ to $(\pi/2, -1, 2)$.
Ans. $(b)\ \phi = y^2\sin x + xz^3 - 4y + 2z + \text{constant}$ $(c)\ 15 + 4\pi$

48. Prove that $\mathbf{F} = r^2\mathbf{r}$ is conservative and find the scalar potential. *Ans.* $\phi = \dfrac{r^4}{4} + \text{constant}$

49. Determine whether the force field $\mathbf{F} = 2xz\,\mathbf{i} + (x^2-y)\mathbf{j} + (2z - x^2)\mathbf{k}$ is conservative or non-conservative.
Ans. non-conservative

50. Show that the work done on a particle in moving it from A to B equals its change in kinetic energies at these points whether the force field is conservative or not.

51. Evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ along the curve $x^2+y^2 = 1$, $z = 1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$ if $\mathbf{A} = (yz+2x)\mathbf{i} + xz\,\mathbf{j} + (xy+2z)\mathbf{k}$. *Ans.* 1

52. (a) If $\mathbf{E} = r\,\mathbf{r}$, is there a function ϕ such that $\mathbf{E} = -\nabla\phi$? If so, find it. (b) Evaluate $\oint_C \mathbf{E}\cdot d\mathbf{r}$ if C is any simple closed curve. *Ans.* $(a)\ \phi = -\dfrac{r^3}{3} + \text{constant}$ $(b)\ 0$

53. Show that $(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2\sin y)\,dy + x\sin y\,dz$ is an exact differential. Hence

solve the differential equation $(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2\sin y)\,dy + x\sin y\,dz = 0$.
Ans. $x^2\cos y + xz\sin y = \text{constant}$

54. Solve (a) $(e^{-y} + 3x^2y^2)\,dx + (2x^3y - xe^{-y})\,dy = 0$,
(b) $(z - e^{-x}\sin y)\,dx + (1 + e^{-x}\cos y)\,dy + (x - 8z)\,dz = 0$.
Ans. (a) $xe^{-y} + x^3y^2 = \text{constant}$ (b) $xz + e^{-x}\sin y + y - 4z^2 = \text{constant}$

55. If $\phi = 2xy^2z + x^2y$, evaluate $\int_C \phi\, d\mathbf{r}$ where C
(a) is the curve $x = t,\ y = t^2,\ z = t^3$ from $t = 0$ to $t = 1$
(b) consists of the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).
Ans. (a) $\frac{19}{45}\mathbf{i} + \frac{11}{15}\mathbf{j} + \frac{75}{77}\mathbf{k}$ (b) $\frac{1}{2}\mathbf{j} + 2\mathbf{k}$

56. If $\mathbf{F} = 2y\,\mathbf{i} - z\,\mathbf{j} + x\,\mathbf{k}$, evaluate $\int_C \mathbf{F}\times d\mathbf{r}$ along the curve $x = \cos t,\ y = \sin t,\ z = 2\cos t$ from $t = 0$ to $t = \pi/2$. *Ans.* $(2 - \frac{\pi}{4})\mathbf{i} + (\pi - \frac{1}{2})\mathbf{j}$

57. If $\mathbf{A} = (3x + y)\mathbf{i} - x\mathbf{j} + (y - 2)\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, evaluate $\oint_C (\mathbf{A}\times\mathbf{B})\times d\mathbf{r}$ around the circle in the xy plane having centre at the origin and radius 2 traversed in the positive direction. *Ans.* $4\pi(7\mathbf{i} + 3\mathbf{j})$

58. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$ for each of the following cases.
(a) $\mathbf{A} = y\,\mathbf{i} + 2x\,\mathbf{j} - z\,\mathbf{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.
(b) $\mathbf{A} = (x + y^2)\mathbf{i} - 2x\,\mathbf{j} + 2yz\,\mathbf{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.
Ans. (a) 108 (b) 81

59. If $\mathbf{F} = 2y\,\mathbf{i} - z\,\mathbf{j} + x^2\,\mathbf{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$. *Ans.* 132

60. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$, if $\mathbf{A} = 6z\,\mathbf{i} + (2x + y)\mathbf{j} - x\,\mathbf{k}$. *Ans.* 18π

61. Evaluate $\iint_S \mathbf{r}\cdot\mathbf{n}\,dS$ over: (a) the surface S of the unit cube bounded by the coordinate planes and the planes $x = 1,\ y = 1,\ z = 1$; (b) the surface of a sphere of radius a with centre at (0,0,0).
Ans. (a) 3 (b) $4\pi a^3$

62. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$ over the entire surface of the region above the xy plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\mathbf{A} = 4xz\,\mathbf{i} + xyz^2\,\mathbf{j} + 3z\,\mathbf{k}$. *Ans.* 320π

63. (a) Let R be the projection of a surface S on the xy plane. Prove that the surface area of S is given by $\iint_R \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\,dx\,dy$ if the equation for S is $z = f(x,y)$.

(*b*) What is the surface area if S has the equation $F(x,y,z)=0$? *Ans.* $\displaystyle\iint_R \frac{\sqrt{(\frac{\partial F}{\partial x})^2+(\frac{\partial F}{\partial y})^2+(\frac{\partial F}{\partial z})^2}}{|\frac{\partial F}{\partial z}|}\,dx\,dy$

64. Find the surface area of the plane $x+2y+2z=12$ cut off by: (*a*) $x=0, y=0, x=1, y=1$; (*b*) $x=0, y=0$, and $x^2+y^2=16$. *Ans.* (*a*) $3/2$ (*b*) 6π

65. Find the surface area of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. *Ans.* $16a^2$

66. Evaluate (*a*) $\displaystyle\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ and (*b*) $\displaystyle\iint_S \phi\,\mathbf{n}\,dS$ if $\mathbf{F}=(x+2y)\mathbf{i}-3z\,\mathbf{j}+x\,\mathbf{k}$, $\phi=4x+3y-2z$, and S is the surface of $2x+y+2z=6$ bounded by $x=0, x=1, y=0$ and $y=2$. *Ans.* (*a*) 1 (*b*) $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$

67. Solve the preceding problem if S is the surface of $2x+y+2z=6$ bounded by $x=0, y=0$, and $z=0$. *Ans.* (*a*) $9/2$ (*b*) $72\mathbf{i}+36\mathbf{j}+72\mathbf{k}$

68. Evaluate $\displaystyle\iint_R \sqrt{x^2+y^2}\,dx\,dy$ over the region R in the xy plane bounded by $x^2+y^2=36$. *Ans.* 144π

69. Evaluate $\displaystyle\iiint_V (2x+y)\,dV$, where V is the closed region bounded by the cylinder $z=4-x^2$ and the planes $x=0, y=0, y=2$ and $z=0$. *Ans.* $80/3$

70. If $\mathbf{F}=(2x^2-3z)\mathbf{i}-2xy\,\mathbf{j}-4x\,\mathbf{k}$, evaluate (*a*) $\displaystyle\iiint_V \nabla\cdot\mathbf{F}\,dV$ and (*b*) $\displaystyle\iiint_V \nabla\times\mathbf{F}\,dV$, where V is the closed region bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$. *Ans.* (*a*) $\frac{8}{3}$ (*b*) $\frac{8}{3}(\mathbf{j}-\mathbf{k})$