Partial Differential Equations: Lecture 2

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Classification of First Order PDE

- **Linear PDE:** Any first order PDE of the form

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$$

(i.e., linear in p and q with coefficients $P(x, y)$, $Q(x, y)$, $R(x, y)$ and $S(x, y)$ are function of x and y only)
Example: $py - qx = z(x + y) + xy$

- **Semi-Linear PDE:** Any first order PDE of the form

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

(i.e., linear in p and q with coefficients $P(x, y)$, $Q(x, y)$ are functions of x and y only, but the function $R(x, y, z)$ is function of x, y and z.)
Example: $px + qy = x^2y^3z^4$

- **Quasi-Linear PDE:** Any first order PDE of the form

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

(i.e., linear in p and q with coefficients $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are functions of x, y and z.)
Example: $(x^2 + y^2 + z^2)p - xyzq = z^3x + y^2$

- **Non-Linear PDE:** Any functional relation of x, y, z, p, q, given by $f(x, y, z, p, q) = 0$, which does not comes under the above cases.
Examples: $p^2q^3 = z^2 \exp^{x^2+y^2}$, $pq = 1$, $p^2 + q^2 = z^2(x^2 + y^2)$.

Lagrange's method of solution

An equation of the form

$$Pp + Qq = R, \tag{1}$$

where P, Q and R are functions of x, y and z, is called Lagrange's equation (Quasi-linear PDE).

We have already seen that the relation $f(u, v) = 0$ containing the arbitrary function f satisfies Lagrange's equation when $P = \frac{\partial(u,v)}{\partial(y,z)}$, $Q = \frac{\partial(u,v)}{\partial(z,x)}$, $R = \frac{\partial(u,v)}{\partial(x,y)}$. Then $f(u, v) = 0$ is a solution of (1).

Now for finding the solution of Lagrange's equation (1) for given P, Q and R, we require to determine the functions u and v such that $P = \frac{\partial(u,v)}{\partial(y,z)}$, $Q = \frac{\partial(u,v)}{\partial(z,x)}$, $R = \frac{\partial(u,v)}{\partial(x,y)}$.

For that let us consider the surfaces $u = c_1$ and $v = c_2$. From these two equations we can easily derive

$$\begin{aligned} \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz &= 0 \\ \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz &= 0. \end{aligned}$$

From the above two equations we can find the simultaneous differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2}$$

We then observe that $u = c_1$ and $v = c_2$ are the solutions of the differential equations (2). Therefore to find the solution of the Lagrange's equation, we need to solve the simultaneous differential equations (2). Note that the equations (2) are called the Lagrange's subsidiary equations.

Example 1

Find the solution of the PDE $(mz - ny)p + (nx - lz)q = ly - mx$.

Solution

Lagrange's subsidiary equation

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow \quad l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow \quad lx + my + nz = c_1$

and $\quad x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow \quad x^2 + y^2 + z^2 = c_2$

$\therefore \quad u = lx + my + nz$

and $v = x^2 + y^2 + z^2$

$\Rightarrow$ Solution of the Lagrange's eqn. is

$\phi(lx + my + nz,\ x^2 + y^2 + z^2) = 0$, where ϕ is an arbitrary function.

Example 2

Solve the PDE $(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$.

Solution

Lagrange's subsidiary equations are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2zx} \quad \text{—— (1)}$$

From the last two terms of (1) we get

$$\frac{dy}{-2xy} = \frac{dz}{-2zx}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

Integrating we get

$$\Rightarrow \frac{y}{z} = c_1 \quad \text{—— (2)}, \quad c_1 \text{ is an arbitrary const}$$

Again from (1),

$$\frac{dy}{-2xy} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{2x(y^2+z^2-x^2) + 2y(-2xy) + 2z(-2xz)}$$

$$\Rightarrow \frac{dy}{-2xy} = \frac{d(x^2+y^2+z^2)}{2x(-x^2+y^2+z^2-2y^2-2z^2)}$$

$$\Rightarrow \frac{dy}{-2xy} = \frac{d(x^2+y^2+z^2)}{-2x(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{dy}{y} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

Integrating we get, $\frac{x^2+y^2+z^2}{y} = c_2$. —— (3)

$\therefore$ Solution of the PDE is

$\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{y}\right) = 0$, where ϕ is an arbitrary function.

Example 3

Solve the PDE $zp - zq = z^2 + (x+y)^2$.

Solution

Given PDE : $zp - zq = z^2 + (x+y)^2$

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2} \quad \text{——} (1)$$

From the first two terms of (1) we get

$$\frac{dx}{z} = \frac{dy}{-z} \Rightarrow dx = -dy$$

$\Rightarrow$ $x + y = c_1$ —(2) (on integration), c_1 is an arbitrary constant.

Now from first and third terms of (1) we get

$$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$$

$$\Rightarrow \frac{dx}{z} = \frac{dz}{z^2+c_1^2} \quad \text{(Using (2))}$$

$$\Rightarrow dx = \frac{z}{z^2+c_1^2}\,dz$$

Integrating, $x = \frac{1}{2}\log(z^2+c_1^2) + c_2$.

$$\Rightarrow x - \frac{1}{2}\log\{z^2+(x+y)^2\} = c_2. \quad (\because c_1 = x+y)$$

$\therefore$ Solution of the PDE is

$$\phi\left(x+y,\ x - \frac{1}{2}\log\{z^2+(x+y)^2\}\right) = 0,$$

where ϕ is an arbitrary ~~constant~~ function.

Example 4

Solve the PDE $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

Solution

Given PDE

$$(x^2-yz)\,p + (y^2-zx)\,q = z^2-xy \quad \text{--- (1)}$$

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}. \quad \text{--- (2)}$$

$$\Rightarrow \frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-z^2-zx+xy} = \frac{dz-dx}{z^2-xy-x^2+yz}.$$

$$\Rightarrow \frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} = \frac{d(z-x)}{(z-x)(x+y+z)}$$

$$\Rightarrow \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x} \quad \text{--- (3)}$$

Considering first two terms of (3) we get

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating we get, $\dfrac{x-y}{y-z} = c_1$, c_1 an arbitrary constant. --- (4)

Again considering last two terms of (3) we get.

$$\frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x}$$

$\Rightarrow$ Integrating we get, $\dfrac{y-z}{z-x} = c_2$, c_2 are arbitrary cnt. --- (5)

$\therefore$ Solution of the PDE is

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0, \text{ where } \phi \text{ is an arbitrary } f^n.$$

Example 5

Solve the PDE $\frac{y-z}{yz}p + \frac{z-x}{xz}q = \frac{x-y}{xy}$.

Solution

Given PDE

$$\frac{y-z}{yz}\,p + \frac{z-x}{zx}\,q = \frac{x-y}{xy} \quad \text{—— (1)}$$

$$\Rightarrow \left(\frac{1}{z}-\frac{1}{y}\right)p + \left(\frac{1}{x}-\frac{1}{z}\right)q = \left(\frac{1}{y}-\frac{1}{x}\right)$$

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{\frac{1}{z}-\frac{1}{y}} = \frac{dy}{\frac{1}{x}-\frac{1}{z}} = \frac{dz}{\frac{1}{y}-\frac{1}{x}}$$

$$= \frac{dx+dy+dz}{\left(\cancel{\frac{1}{z}}-\cancel{\frac{1}{y}}+\cancel{\frac{1}{x}}-\cancel{\frac{1}{z}}+\cancel{\frac{1}{y}}-\cancel{\frac{1}{x}}\right)} = \frac{dx+dy+dz}{0}$$

$\Rightarrow dx+dy+dz = 0$, $\Rightarrow$ Integrating we get, $x+y+z = c_1$, c_1 is an arbitrary constant.

Again each of the terms of (1) are equal to

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}\left(\frac{1}{z}-\frac{1}{y}\right)+\frac{1}{y}\left(\frac{1}{x}-\frac{1}{z}\right)+\frac{1}{z}\left(\frac{1}{y}-\frac{1}{x}\right)}$$

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\Rightarrow \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$$

Integrating we get; $xyz = c_2$, c_2 is an arbitrary constant

$\therefore$ $\phi(x+y+z,\, xyz) = 0$ is the sol^n of the PDE, where ϕ is an arbitrary ~~constant~~ function.

Exercise

Solve the following Lagrange's equations:

(i) $x^2p + y^2q = z^2$.

(ii) $y^2zp + zx^2q = xy^2$.

(iii) $(y + z)p + (z + x)q = x + y$.

(iv) $p\cos(x + y) + q\sin(x + y) = z$.

(v) $p - 2q = 3x^2\sin(2x + y)$.

(vi) $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)$.

(vii) $z(x + y)p + z(x - y)q = x^2 + y^2$.

(viii) $(y^2 + yz + z^2)p + (z^2 + zx + x^2)q = x^2 + xy + y^2$.

(ix) $p + 3q = 5z + tan(y - 3x)$.

(x) $yzp + zxq = xy$.