Expansion

Q1>

$f(x) = e^x \cos x$

$f'(x) = e^x \cos x + e^x \sin x = f(x) - e^x \sin x$

$f''(x) = f'(x) - e^x \sin x - e^x \cos x = f'(x) - f(x) - e^x \sin x$

$f'''(x) = f''(x) - f'(x) - e^x \sin x - e^x \cos x = f''(x) - f'(x) - f(x) - e^x \sin x$

Similarly $f''''(x) = f'''(x) - f''(x) - f'(x) - f(x) - e^x \sin x$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0) + \frac{x^4}{24} f'''(0) + \ldots$

$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \ldots$

Q2)

$$f(x) = \sqrt{1+x+2x^2}$$

$$f'(x) = \frac{1+4x}{2\sqrt{1+x+2x^2}} = \frac{1+4x}{2f(x)}$$

$$f''(x) = \frac{1}{2}\times\frac{4f(x) - 2(1+4x)\,f'(x)}{(f(x))^2} = \frac{2}{f(x)} - \frac{(f'(x))^2}{f(x)}$$

$$f'''(x) = \frac{-2f'(x)}{(f(x))^2} - \frac{2f'(x)f(x) - f'(x)^3}{f(x)^2}$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\sqrt{1+x+2x^2} = 1 + \frac{x}{2} + \frac{7x^2}{8} - \frac{x^3}{3!} \times \frac{15}{8}$$

$$= 1 + \frac{x}{2} + \frac{7}{8}x^2 - \frac{5x^3}{16} + \dots$$

Q3)

$$f(x) = \sin x \quad ; \left(x - \frac{\pi}{2}\right)$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f''''(x) = \sin x$$

$$f(x) = f\left(\frac{\pi}{2}\right) + \left(x-\frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{(x-\pi/2)^2}{2} f''\left(\frac{\pi}{2}\right) + \frac{(x-\pi/2)^3}{3!} f'''\left(\frac{\pi}{2}\right)$$

$$+ \frac{(x-\pi/4)^4}{4!} f''''(\pi/2) + \dots$$

$$\sin x = 1 - \frac{1}{2}\left(x - \pi/2\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 + \dots$$

$$\sin 91^\circ = \sin(0.50555\,\pi) = 1 - \frac{1}{2}\left(0.50555\,\pi - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(0.5055 - \frac{\pi}{2}\right)^4$$

$$=$$

Q4) $f(x) = e^{a\sin^{-1}x}$

$$f'(x) = \frac{a e^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{\frac{a^2 e^{a\sin^{-1}x}}{\sqrt{1-x^2}} \times \sqrt{1-x^2} + \frac{\cancel{2}x}{2\sqrt{1-x^2}} \cdot a e^{a\sin^{-1}x}}{1-x^2}$$

$$= \frac{a^2 e^{a\sin^{-1}x}}{1-x^2} + \frac{x}{\sqrt{1-x^2}} a e^{a\sin^{-1}x}$$

$$f'''(x) = \frac{\frac{a^2 \cdot a e^{a\sin^{-1}x}}{\sqrt{1-x^2}}(1-x^2) - (-2x) a^2 e^{a\sin^{-1}x}}{1-x^2}$$

$$+ \frac{a e^{a\sin^{-1}x}}{\sqrt{1-x^2}} \times \frac{x}{\sqrt{1-x^2}} + a e^{a\sin^{-1}x}\left(\frac{\sqrt{1-x^2} - x(-2x)}{1-x^2}\right)$$

$$f'''(x) = \frac{a^3 e^{a\sin^{-1}x}}{\sqrt{1-x^2}} + \frac{2x \cdot a^2 e^{a\sin^{-1}x}}{1-x^2} + \frac{x a^2 e^{a\sin^{-1}x}}{1-x^2} + a e^{a\sin^{-1}x}\left(\frac{\sqrt{1-x^2}+2x^2}{1-x^2}\right)$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$= 1 + ax + \frac{x^2}{2} \times a^2 + \frac{x^3}{3!} \times (a^3 + a) + \cdots$$

$$e^{a\sin^{-1}x} = 1 + ax + \frac{(ax)^2}{2} + \frac{a(1+a^2)x^3}{3!} + \cdots$$

$$\theta = \cancel{a}\sin^{-1}x \Rightarrow x = \sin\theta$$

Taking $a=1$

$$e^{\theta} = 1 + \sin\theta + \frac{\sin^2\theta}{2} + \frac{2 \times \sin^3\theta}{3!} + \cdots$$

Q5) a) $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(1+\theta x) \quad 0<\theta<1$$

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2}\sin(1+\theta x)$$

$$\sin(1+\theta x)\frac{h^2}{2} = \sin x + h\cos x - \sin(x+h)$$

$$\therefore \frac{h^2}{2} \geq \sin x + h\cos x - \sin(x+h)$$

b) $f(x) = \sin x$ $\quad h = \theta$, $a = \frac{\pi}{4}$ $\quad b = \theta + \frac{\pi}{4}$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f''''(x) = \sin x$

$$f(b) = f(a) + h f'(a) + \frac{h^2}{2} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

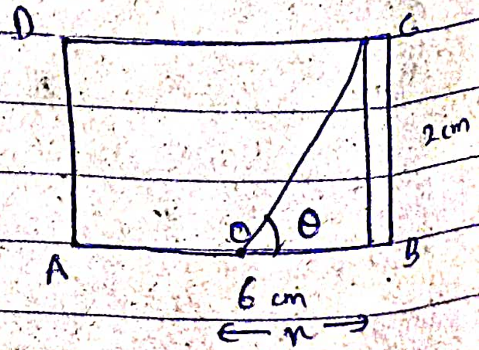
$$\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{\sqrt{2}} + \frac{\theta}{\sqrt{2}} - \frac{\theta^2}{2\sqrt{2}} - \frac{\theta^3}{3!\sqrt{2}} + \frac{\theta^4}{4!\sqrt{2}} + \cdots$$

$$= \frac{1}{\sqrt{2}}\left[1+\theta - \frac{\theta^2}{2} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots\right]$$

Q7)
$$pV^{1.4} = c$$
$$\log p + 1.4 \log V = \log c$$
On ~~deter~~ diff.
$$\frac{1}{p} dp + \frac{1.4}{V} dV = 0$$
Since,
$$-\frac{1}{2} = \frac{dV}{V} \times 100$$
$$\frac{dp}{p} = -1.4 \times \left(\frac{-1}{200}\right) = \frac{\cancel{14}^{7}}{\cancel{2000}} = 0.007$$
$$\boxed{\frac{dp}{p} \times 100 = 0.7\%}$$

Q8)
$$f(x) = V = \frac{4}{3} \pi x^3$$
$$dV = 4\pi x^2 \cdot dx$$
$$dV = 4 \times \pi \times 1.9 \times 1.9 \times (-0.1)$$
$$dV = -4.53416$$

Q9)
$$\tan\theta = \frac{2}{x}$$
$$\sec^2\theta \, d\theta = \frac{-2}{x^2} dx$$



$$d\theta = \frac{-2}{x^2} \cos^2\theta \cdot dx \cdot 1$$

$$= \frac{-\cancel{2}}{\cancel{4}2} \times \tan^2\theta \cos^2\theta \, dx \qquad x = \frac{2}{\tan\theta}$$

$$d\theta = \frac{-1}{2} \sin^2\theta \cdot dx \cdot 1$$

converting to radian.

$$d\theta = \frac{-1}{\cancel{2}} \times \sin^2\theta \times \frac{1}{10\cancel{0}} \times \frac{\overset{9}{\cancel{180}}}{\pi} \qquad \left(dx = \frac{1}{100}\right)$$

$$d\theta = \frac{-9}{10\pi} \sin^2\theta$$

Q 10) Considering

$$f(mx) = f(mx + x - x)$$

$$f(mx) = f(x + (mx - x))$$

comparing

$$f(b) = f(a+h)$$

$$f(b) = f(a) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

$$f(mx) = f(x) + (mx - x) f'(x) + \frac{(mx-x)^2}{2!} f''(x) + \ldots$$

Q11) Given,

$$x = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \dots \infty$$

considering

$$x = f(y) = \log(1+y)$$

$$f'(y) = \frac{1}{1+y}, \quad f^2(y) = \frac{-1}{(1+y)^2}$$

$$f^3(y) = \frac{2}{(1+y)^3}, \quad f^4(y) = \frac{-6}{(1+y)^4}$$

$$f(y) = f(0) + y f'(0) + \frac{y^2}{2} f''(0) + \frac{y^3}{3!} f'''(0) + \frac{y^4}{4!} f''''(0) + \dots$$

$$= 0 + y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots$$

Hence,

$$x = \log(1+y)$$

$$1+y = e^x$$

$$f(x) = y = e^x - 1$$

$$f'(x) = e^x = f''(x) = f'''(x) + \dots$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \dots$$

$$y = 0 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Hence proved

Q12) $f(x) = \log \sec x$

$f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x$

$f''(x) = \sec^2 x$

$f'''(x) = 2\sec^2 x \tan x$

$f''''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\log \sec x = 0 + x \times 0 + \frac{x^2}{2} \times 1 + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!} \times 2 + \dots$$

$$\log \sec x = \frac{x^2}{2} + \frac{x^4}{4 \times 3} + \dots$$

Q13) $f(x) = \cos x$

$b = 32^\circ$ $a = 30^\circ$ $h = 2^\circ = 2 \times \frac{\pi}{180} = \frac{\pi}{90}$

$f'(x) = -\sin x$; $f''(x) = -\cos x$ $f'''(x) = \sin x$

$$f(32^\circ) = f(30^\circ) + \frac{\pi}{90} f'(30^\circ) + \left(\frac{\pi}{90}\right)^2 \frac{1}{2} f''(30^\circ)$$

$$\cos 32^\circ = \frac{\sqrt{3}}{2} - \frac{\pi}{90} \times \frac{1}{2} - \frac{\pi^2}{90^2} \times \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$= 0.8660 - 0.0174 - 0.0005$

$= 0.8487$

Q 14) $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$b = 31^\circ$, $a = 30^\circ$ $h = 1^\circ = \frac{\pi}{180}$

$$f(31^\circ) = f(30^\circ) + \frac{\pi}{180} f'(30^\circ) + \left(\frac{\pi}{180}\right)^2 f''(30^\circ)$$

$$= \frac{1}{2} + \frac{\pi}{180} \times \frac{\sqrt{3}}{2} - \frac{\pi^2}{180^2} \times \frac{1}{2}$$

$$= 0.5 + 0.0151 - 0.0001$$

$$\sin 31^\circ = 0.515$$

Q15) $f(x) = \frac{e^x}{e^x+1}$

$$f'(x) = \frac{e^x(e^x+1) - e^{2x}}{(e^x+1)^2}$$

$$f'(x) = \frac{e^x}{(e^x+1)^2}$$

$$f''(x) = \frac{e^x(e^x+1)^2 - e^x \cdot 2(e^x+1)e^x}{(e^x+1)^4} = \frac{e^x - e^{2x}}{(e^x+1)^3}$$

$$f'''(x) = \frac{(e^x - 2e^{2x})(e^x+1)^3 - (e^x - e^{2x})3(e^x+1)^2 e^x}{(e^x+1)^6}$$

$$f'''(x) = \frac{e^{2x} + e^x - 2e^{3x} - 2e^{2x} - 3e^{2x} + 3e^{3x}}{(e^x+1)^4} = \frac{e^{3x} - 4e^{2x} + e^x}{(e^x+1)^4}$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\frac{e^x}{e^x+1} = 1 + \frac{x}{4} + \frac{x^2}{2} \times 0 + \frac{x^3}{3!}\left(\frac{-2}{2^4}\right) + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^3}{3! \times 8} + \cdots$$

Q16> $f(x) = \log_{10} x$, $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$

$b = 404$ $h = 4$ $a = 400$ | $f'''(x) = \frac{2}{x^3}$

$$f(404) = f(4) + 400 f'(4) + \frac{(400)^2}{2} f''(4) + \frac{400^3}{3!} f'''(x)$$

$$\log_{10} 404 = 0.6021 + \frac{\overset{100}{\cancel{400}}}{\cancel{4}} + \frac{400^2}{2} \times \frac{-1}{4^2} + \frac{400^3}{3!} \times \frac{2}{4^3}$$

$$= 0.6021 + 100 \{ - \frac{\overset{50}{\cancel{100}} \times 100}{\cancel{2}} + \frac{400^3}{3 \times \cancel{2}} \times \cancel{2}$$

$$\cancel{= 0.6021 + 100 \left[1 - 50 \right.}$$

$$= 0.6021 + 100 - 5000 + \frac{1000000}{3}$$

$$f(404) = f(400) + 4 f'(400) + \frac{4^2}{2} f''(400)$$

$$\log_{10} 404 = \log_{10} 400 + 4 \times \frac{1}{400} - \frac{4^2}{2 \times} \frac{1}{400^2}$$

$$= \log_{10} 4 + 2 + \frac{1}{100} - \frac{1}{2 \times 100^2}$$

$$= 0.6021 + 2 + 0.01 - 0.00005$$

$$\log_{10}(404) = 2.61205$$

Q 17)

$$f(x) = \tan^{-1} x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$f'''(x) = \frac{-2(1+x^2)^2 + 2x \cdot 2(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{-2 - 2x^2 + 8x^2 + 8x^4}{(1+x^2)^3}$$

$$f^3(x) = \frac{8x^4 + 6x^2 - 2}{(1+x^2)^3} = \frac{2[4x^4 + 3x^2 - 1]}{(1+x^2)^3}$$

$$\cancel{f(x)} = \cancel{f(}\ f^4(x) = 2\left[\frac{(16x^3 + 6x)(1+x^2)^3 - (4x^4 + 3x^2 - 1)\, 3(1+x^2) \cdot 2x}{(1+x^2)^6}\right]$$

$$f^4(x) = 2\left[\frac{16x^3 + 6x + 16x^5 + 6x^3 - 24x^5 - 24x^7 - 18x^3 - 18x^5 + 6x + 6x^3}{(1+x^2)^4}\right]$$

$$= \cancel{2 \quad 24x^4}$$

$$f(x) = f(0) + x f'(x) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^4(0) + \frac{x^5}{5!} f^5(0)$$

$$= 0 + x + \frac{x^2}{2} \times 0 - \frac{x^3}{3} + \frac{x^4}{4!} \times 0 - \frac{x^5}{5}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5}$$

MVT

(Q1) i) $f(x) = x(x+3)e^{-x/2}$ in $-3 \le x \le 0$

$f(x) = (x^2+3x)e^{-x/2}$

$f(b) = f(a)$

$f(0) = f(-3)$

$0 = 0$

$f'(x) = (2x+3)e^{-x/2} - \frac{1}{2}e^{-x/2}(x^2+3x)$

$0 = (2x+3)e^{-x/2} - \frac{1}{2}e^{-x/2}(x^2+3x)$

$e^{-x/2}\left[2x+3 - \frac{1}{2}(x^2+3x)\right] = 0$

$2x+3 = \frac{1}{2}(x^2+3x)$

$4x+6 = x^2+3x$

$x^2 - x + 6 = 0$

$x^2 - 3x + 2x + 6 = 0$

$(x-3)(x+2) = 0$

$x = -2, 3$

$\therefore -2 \in [-3, 0]$

Hence verified.

ii) $f(x) = e^x(\sin x - \cos x)$, in $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$

$f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x)$

$= e^x(+2\cos x)$

$f'(x) = +2e^x \cos x = 0$

$\cancel{e^x}\ \cos x = 0$

$x = \frac{\pi}{2} \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

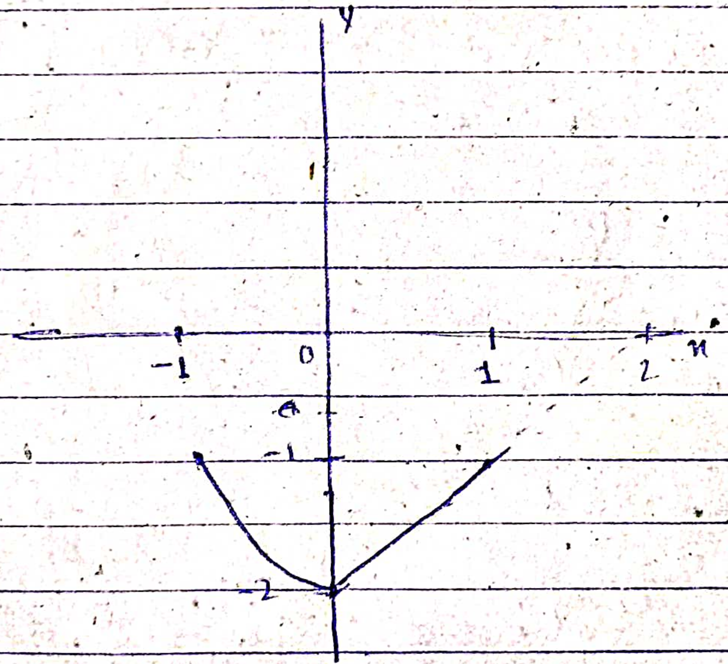
and.

$$f\left(\frac{\pi}{4}\right) = e^{\pi/4} \times 0 = 0$$

$$f\left(\frac{5\pi}{4}\right) = 0$$

Both conditions ~~verfy~~ verifies rolle's therorem

iii) $f(x) = \begin{cases} x^2 - 2, & -1 \le x \le 0 \\ x - 2, & 0 \le x \le 1 \end{cases}$



from graph it is clear at ~~x~~ y = -2, it is not differentiable.

Hence Rolle's theorem is not verified

Q 27

~~Ax~~ $x^3 - 6x^2 + c = 0$ $[0, 4]$

$f(x) = x^3 - 6x^2 + c$

$f'(x) = 3x^2 - 12x$

$3x^2 - 12x = 0$

$x(3x - 12) = 0$

$\boxed{x = 0}$ $x = 4$

When ~~$x = 4$~~

Since at $x = 4$ slope is 0, then putting $x = 4$ in function $c = 32$, ~~too~~.

Hence it cannot have distinct roots in $[0, 4]$

Q 37 i)

$f(x) = lx^2 + mx + n$

$f'(x) = 2lx + m$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2lc + m = \frac{lb^2 + mb + n - (la^2 + ma + n)}{b - a}$$

$$= \frac{l(b^2 - a^2) + m(b - a)}{b - a}$$

$2lc + m = l(b + a) + m$

$$c = \frac{b + a}{2}$$

$\therefore c \in [a, b]$ Hence proved

ii) $f(x) = \log x$ in $[1, e] = [1, 2.77]$

$f'(x) = \frac{1}{x}$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log_e e - \log 1}{e-1}$$

$$\frac{1}{c} = \frac{1}{e-1}$$

$$c = e-1$$

$$c = 1.77$$

$$c \in [1, e]$$

Q4) $y = x^n$

let, $m = \frac{k^n}{k} = k^{n-1}$ = slope

$$y' = n x^{n-1}$$

$$k^{n-1} = n x^{n-1}$$

$$\frac{k^{n-1}}{n} = x^{n-1}$$

$$\frac{k}{n^{n-1}} = x$$

$$x = k n^{1-n}$$

$$y = x^n = k^n n^{n(1-n)}$$

So, required point is $(k n^{1-n}, k^n n^{n(1-n)})$

Q5>

$$f(x) = x + \frac{1}{x}$$

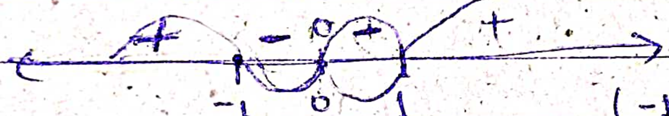
$$f'(x) = 1 - \frac{1}{x^2}$$

$$0 = 1 - \frac{1}{x^2}$$

$$\frac{1}{x^2} = 1$$

$$x^2 = 1$$

$$x = \pm 1$$



$(-1,0) \cup (0,1)$

Hence $f(x)$ is decreasing in $(-1,0) \cup (0,1)$

and increasing in $(-\infty,-1] \cup (1,\infty)$.

Q7)

$f(x) = \sin x$

$g(x) = \cos x$

$$\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{1}{\cancel{-}1} = \frac{\cos c}{\cancel{-}\sin c}$$

$$1 = \cot c$$

$$c = \frac{\pi}{4}$$

$$\therefore c \in \left[0, \frac{\pi}{2}\right]$$

Hence verified

Q8)

$g(x) = \frac{1}{\sqrt{x}}$ $\qquad$ $f(x) = \sqrt{x}$

$g'(x) = \frac{-1}{2x^{3/2}}$ $\qquad$ $f'(x) = \frac{1}{2\sqrt{x}}$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c^{3/2}}}$$

$$\cancel{-}\frac{\cancel{(\sqrt{b} - \sqrt{a})}\sqrt{ab}}{\cancel{(\sqrt{b} - \sqrt{a})}} = \cancel{-}\frac{\cancel{2}c^{3/2}}{\cancel{2}c^{1/2}}$$

$$\sqrt{ab} = c$$ Verified

Q9)

$$f(x) = \log$$

$$f(b) = f(a) + (b-a) \quad f'(a+\theta h)$$

$$\log b = \log a + \frac{(b-a)}{a+\theta h}$$

$$\log \frac{b}{a} = \frac{b-a}{a+\theta h} \quad -(1)$$

$$\log \frac{b}{a} = \frac{a\left(\frac{b}{a}-1\right)}{a+\theta h}$$

Since $0<\theta<1$

$$\log \frac{b}{a} < \frac{\cancel{a}}{\cancel{a}}\left(\frac{b}{a}-1\right)$$

and from (1)

$$\log \frac{b}{a} = \frac{b\left(1-\frac{a}{b}\right)}{a+\theta h}$$

again $0 < \theta < 1$

$\log \frac{b}{a} \approx \frac{b}{a}\left(1 - \frac{a}{b}\right)$

$\therefore \left(1 - \frac{a}{b}\right) < \log \frac{b}{a} < \left(\frac{b}{a} - 1\right)$

Proved

Q 10)

$f(x) = \log x \qquad f'(x) = \frac{1}{x}$

$g(x) = x^2 \qquad , g'(x) = 2x$

Using C. M. V. T

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\log b - \log a}{b^2 - a^2} = \frac{1}{2c^2}$$

$$\log \frac{b}{a} = \frac{b^2 - a^2}{2c^2}$$