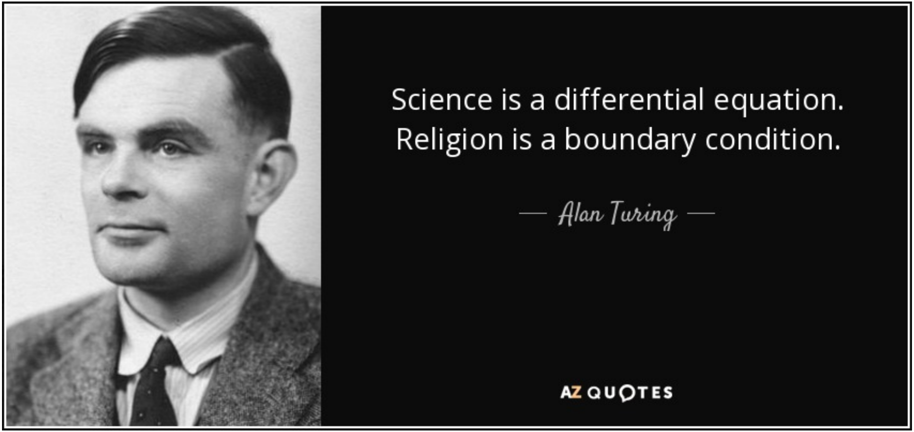
Partial Differential Equations: Lecture 1

Pinaki Pal

Department of Mathematics
National Institute of Technology Durgapur
West Bengal, India

pinaki.pal@maths.nitdgp.ac.in



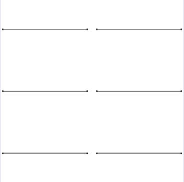
https://www.azquotes.com/quotes/topics/differential-equations.html

Introduction

Definition

An equation involving partial derivative(s) of one or more dependent variables with respect to one or more independent variables is called a partial differential equation (PDE). The dependent variable (s) should be function (s) of at least two independent variables.

Vibrating String

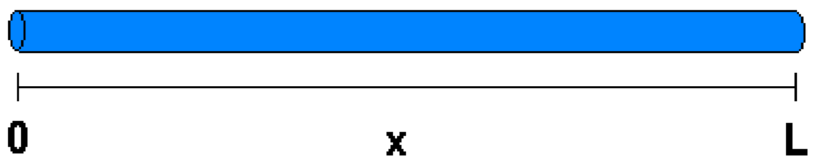


Governing Equations

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \text{ (Wave equation)} \\
\text{BCs}: u(0,t) &= u(L,t) = 0, \\
\text{ICs}: u(x,0) &= f(x), \frac{\partial u}{\partial t}(x,0) = g(x).
\end{aligned}
$$

https://en.wikipedia.org/wiki/String_vibration

Heating of a rod



Governing Equations

$$
\begin{aligned}
\frac{\partial T}{\partial t} &= \alpha \frac{\partial^2 T}{\partial x^2} \text{ (Heat equation)} \\
\text{BCs}: T(0,t) &= 0, \frac{\partial T}{\partial x}(L,t) = H_0, \\
\text{ICs}: T(x,0) &= 0.
\end{aligned}
$$

More examples

$$
\begin{aligned}
\text{Laplace's Equation} : \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= 0 \\
\text{Burger equation} : \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= \mu \frac{\partial^2 u}{\partial x^2} \\
\text{Navier} - \text{Stokes equation} : \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} &= -\nabla p + \nu \nabla^2 \vec{v} + \vec{F} \\
\text{Two dimensional wave equation} : \frac{\partial^2 u}{\partial t^2} &= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \right) \\
\text{Equation of continuity} : \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0
\end{aligned}
$$

Formation of partial differential equations

Elimination of arbitrary constants

Consider the function given by

$$f(x, y, z, c_1, c_2) = 0, \tag{1}$$

where x, y are independent variables, z is dependent variable, c_1 and c_2 are arbitrary constants.
Differential equation (1) with respect to x and y we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0, \tag{2}$$
$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0. \tag{3}$$

Now eliminating c_1 and c_2 from the equations (1), (2) and (3) we get the first order partial equation given by

$$g(x, y, z, p, q) = 0, \tag{4}$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Examples

Form the partial differential equations by eliminating the arbitrary constants c_1 and c_2 from the equations

(i) $z = (x + c_1)(y + c_2)$

(ii) $z = c_1(x + y) + c_2(x - y) + c_1 c_2 t$

Solution (i)

Differentiating with respect to x and y we get

$$\frac{\partial z}{\partial x} = y + c_2 \implies c_2 = p - y, \tag{5}$$

$$\frac{\partial z}{\partial y} = x + c_1 \implies c_1 = q - x. \tag{6}$$

Eliminating c_1 and c_2 from the equations (5), (6) and the given equation we get

$$z = pq,$$

which is the required partial differential equation.

Solution (ii)

Given equation is $z = c_1(x+y) + c_2(x-y) + c_1c_2t$. Differentiating with respect to x, y and t respectively we get

$$\frac{\partial z}{\partial x} = c_1 + c_2, \ \frac{\partial z}{\partial y} = c_1 - c_2 \text{ and } \frac{\partial z}{\partial t} = c_1c_2.$$

Eliminating the arbitrary constants we get

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = 4\frac{\partial z}{\partial t},$$

which is the required PDE.

Exercise

Form the partial differential equations by eliminating the arbitrary constants from the following equations

(i) $z = c_1 x + c_1^2 y^2 + c^2$

(ii) $z = c_1 x e^y + \frac{1}{2} c_1^2 e^{2y} + c_2$

(iii) $\frac{x^2}{c_1^2} + \frac{y^2}{c_2^2} + \frac{z^2}{c_3^2} = 1$ ✗

(iv) $z = ax + by + a^2 + b^2$

(v) $z = (x - a)^2 + (y - b)^2$

(vi) $z = (x^2 + a^2)(y^2 + b^2)$

Elimination of arbitrary functions

Consider the relation

$$f(u, v) = 0, \tag{7}$$

where u and v are unknown functions of x, y and z and f is an arbitrary function.

Differentiating the equation (7) with respect to x, y respectively we get

$$\begin{aligned} \frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right] &= 0 \qquad (8) \\ \text{and } \frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right] &= 0, \qquad (9) \end{aligned}$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from the equations (8) and (9) we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0.$$

Simplifying we can write it in the form

$$Pp + Qq = R, \tag{10}$$

where

$$\begin{aligned} P &= \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = \frac{\partial(u, v)}{\partial(y, z)}, \\ Q &= \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = \frac{\partial(u, v)}{\partial(z, x)}, \\ R &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial(u, v)}{\partial(x, y)}. \end{aligned}$$

The equation (10) is known as Legrange's equation.

Examples

Obtain the partial differential equation by eliminating arbitrary function f and g

(i) $z = (x+y)f(x^2-y^2)$

(ii) $ax + by + cz = f(x^2+y^2+z^2), (iii) y = f(x-at) + g(x+at)$

Solution (i)

$$z = (x+y)\, f(x^2-y^2)$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = (x+y)\, f'(x^2-y^2) \times 2x + f(x^2-y^2) \quad \text{— (1)}$$

$$\frac{\partial z}{\partial y} = (x+y)\, f'(x^2-y^2) \times (-2y) + f(x^2-y^2) \quad \text{— (2)}$$

$(1) \times y + (2) \times x \Rightarrow$

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = y\, f(x^2-y^2) + x\, f(x^2-y^2)$$

$$= (x+y)\, f(x^2-y^2)$$

$$= z$$

Solution (ii)

$$ax + by + cz = f(x^2+y^2+z^2) \quad \text{--- (1)}$$

Differentiating w.r.t. x, y respectively we get

$$a + c\frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\cdot\left(2x + 2z\frac{\partial z}{\partial x}\right) \quad \text{--- (2)}$$

$$b + c\frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\cdot\left(2y + 2z\frac{\partial z}{\partial y}\right) \quad \text{--- (3)}$$

(2) $\div$ (3) $\Rightarrow$

$$\frac{a+cp}{b+cq} = \frac{x+zp}{y+zq}, \quad p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}$$

Solution (iii)

$$y = f(x-at) + g(x+at) \longleftarrow (1)$$

Differentiating w.r.t. x and t

$$\frac{\partial y}{\partial x} = f'(x-at) + g'(x+at)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x-at) + g''(x+at) \longleftarrow (2)$$

$$\frac{\partial y}{\partial t} = -af'(x-at) + ag'(x+at)$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 f''(x-at) + a^2 g''(x+at) \longleftarrow (3)$$

Eliminating f & g from (2) & (3) we get

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}}$$

Exercise

Form the PDEs by eliminating the arbitrary functions from the following relations:

1. $z = y^2 + 2f(\frac{1}{x} + logy)$.
2. $\phi(x + y + z, x^2 + y^2 - z^2) = 0$.
3. $y = f(x + at) + xg(x + at)$
4. $z = e^{ax+by} f(ax - by)$
5. $z = \phi(x^2 - y^2)$.
6. $f(xy + z^2, x + y + z) = 0$.