1. Using matrix operations, describe the solutions for the following family of equations:

x + 2y - 3z = 5
2x + y - 3z = 13
-x + y = -8

First I rewrite the equations to make sure that variables are in the same order in each and to make any 0, 1, or negative coefficients more clear:

1x + 2y + -3z = 5
2x + 1y + -3z = 13
-1x + 1y + 0z = -8

Then I restate it in matrix form, as Ax = b. I want to solve for x.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ -8 \end{bmatrix}$$ (or Ax = b), so x = A^{-1}b.

Now I need to find the inverse of my 3X3 matrix (A^{-1}). I recall that the inverse of a matrix is the 1/determinant of the matrix * the matrix adjoint.

Let me find the determinant of my matrix by rewriting the first two columns, then taking the sum of "blue" products and subtracting the "red" products

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix}\begin{matrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{matrix}$$

(1 * 1 * 0) + (2 * -3 * -1) + (-3 * 2 * 1) – (-1 * 1 * -3) – (1 * -3 * 1) – (0 * 2 * 2)
0 + 6 + -6 - 3 - -3 - 0
0

To find the matrix adjoint, I would have to calculate a matrix of minors and cofactor matrix. However, since my determinant is 0, I can't multiply by 1/0, which is undefined, so I stop here.

Instead, I'll try to get my matrix into reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{array}\right]$$

I'll add row three to row one and make that sum the new row one:

$$\left[\begin{array}{ccc|c} 0 & 3 & -3 & -3 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{array}\right]$$

Now I'll double row three and add that to row two, to replace row two:

$$\left[\begin{array}{ccc|c} 0 & 3 & -3 & -3 \\ 0 & 3 & -3 & -3 \\ -1 & 1 & 0 & -8 \end{array}\right]$$

I'll multiply row three by -1 for a new row three:

$$\left[\begin{array}{ccc|c} 0 & 3 & -3 & -3 \\ 0 & 3 & -3 & -3 \\ 1 & -1 & 0 & 8 \end{array}\right]$$

Now I'll subtract row two from row one to replace row one:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & -3 \\ 1 & -1 & 0 & 8 \end{array}\right]$$

And divide row two by 3:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 8 \end{array}\right]$$

I change around the order to make it fit the convention:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 8 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Almost there! I add row two to row one for a new row 1.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x and y are pivot variables, and z is free

I can rewrite a new system of equations that's much easier:

$x - z = 7$, or $x = z + 7$

$y - z = -1$, or $y = z - 1$

The number of solutions is infinite: pick your z, subtract one and you have your y. Or add 7, and you have your x. In a three-dimensional world where the z axis comes toward you positively, perpendicular to your body plane with you facing the origin in positive z space, the solution would be a line that would start far in front of you, in negative z space, far to your right and a bit above you, in the XY quadrant I, and as it approached you, it would angle to the left and down.

2. Provide a solution for #1, using R functions of your choice.
First, I enter my A (3 X 3 matrix) and b (constant vector):

```
A<-rbind(c(1,2,-3),c(2,1,-3),c(-1,1,0))
b<-c(5,13,-8)
```

I then ask R to solve the set of equations:

```
solve(A,b)
```

This produces an error:

```
Error in solve.default(A, b) :
  Lapack routine dgesv: system is exactly singular: U[3,3] = 0
```

An unsurprising error, given that the determinant is 0. I instead use the pracma package's rref tool to give me the reduced row echelon form.

```
install.packages("pracma")
library(pracma)
rref(cbind(A,b))
```

This gives me my reduced row echelon form, which allows me to come up with my solution set of equations:

```
             b
[1,] 1 0 -1  7
[2,] 0 1 -1 -1
[3,] 0 0  0  0
```

This evaluates to:
$x - z = 7$ (or $x = z + 7$)
$y - z = -1$ (or $y = z - 1$)

In short, a much faster way to solve the problem!

3. Solve for AB by hand:

$$A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$$

AB = $\begin{bmatrix} (4 * 1) + (-3 * 3) & (4 * 4) + (-3 * -2) \\ (-3 * 1) + (5 * 3) & (-3 * 4) + (5 * -2) \\ (0 * 1) + (1 * 3) & (0 * 4) + (1 * -2) \end{bmatrix}$

AB = $\begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

4. Solve AB from #3 using R functions of your choice.

```
A<-rbind(c(4,-3),c(-3,5),c(0,1))
B<-rbind(c(1,4),c(3,-2))
> A %*% B
     [,1] [,2]
[1,]   -5   22
[2,]   12  -22
[3,]    3   -2
```