Lecture Supplement: Discrete Fourier Transform (DFT)

Introduction

This supplement provides a summary of some basic concepts underpinning the DFT. You are already familiar with these concepts, and therefore this memo provides insight via simulations and block diagrams. Let us start a bit light-hearted, with a list of phobias, from the LA Times.

Phobias: From LA Times

- **Cainophobia**: Fear of new things or ideas.
- **Amaxophobia**: Fear of vehicles.
- **Eosophobia**: Fear of dawn or daylight.
- **Omrophobia**: Fear of rain.
- **Hippophobia**: Fear of horses.
- **Radiophobia**: Fear of radiation, x-rays.
- **Spectraphobia**: Fear of Fourier transforms.
- **Traumatophobia**: Fear of injury.

Our aim is therefore to help mitigate and remove Spectraphobia!

Where do we use the DFT?

The DFT is the most executed DSP algorithm, and is used in an enormous number of applications. Here, we list only some of the applications related to Communications and Signal Processing.

- Spectrum Analysis
- Convolution (Filtering)
- Very Long Word Multiplication (10^9 Digits)
- Modulation, Demodulation (OFDM, OFDMA)
- M-path Channelizers (Filter Banks)
- Source Coding (MP3, MPEG)
- Polynomial Root Finding
- Computerized Axial Tomography (CAT Scan)
- Beamforming (Butler Matrix)
- Solving Differential Equations (Heat Equation)
- Spectroscopy
- Radar

The concept of phasor, time delay and phase delay

Consider a complex exponential $e^{j\omega t} = \cos\omega t + j\sin\omega t$. Then, at some time instant t, the real (in blue) and imaginary (in red) part of the complex exponential are given in Figure 1. Observe the phase shift of $\pi/2$ between the real and imaginary component, independent on the current point in the phasor plane.

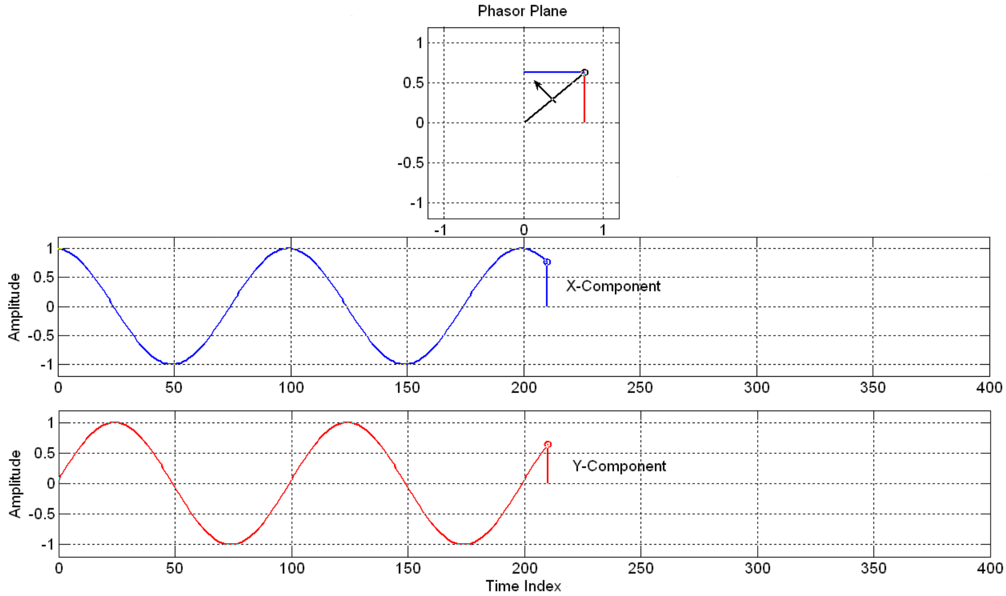

Figure 1: The real (blue) and imaginary (red) part of a complex exponential

Now, consider the same complex exponential, slightly further in time, say at time instant $(t + T_D)$, as shown in Figure 2. The time delay between the complex exponentials in Figure 1 and Figure 2 is clearly reflected in the difference between the angles of the phasor (phase) in the phasor plane, shown in the figures. The relationship between T_D and the phase θ is linear, as we have a complex exponential of a constant frequency.

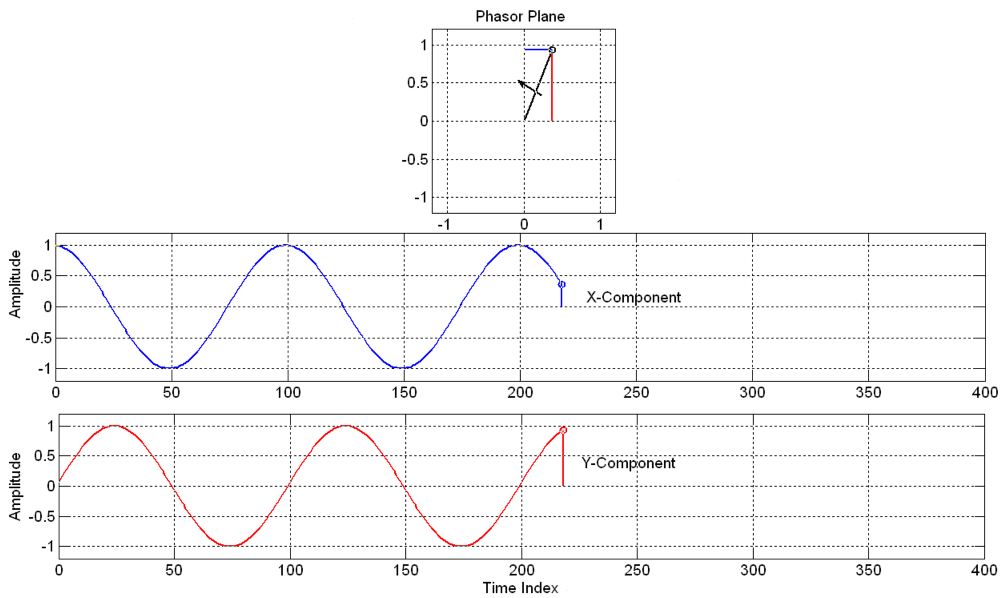

Figure 2: The real (blue) and imaginary (red) part of a complex exponential

Phase delay is frequency-dependent

As shown in Figure 3 the phase θ of a phasor can be calculated as $\theta = 2\pi T_D f$, where T_D denotes the time delay and f the frequency of the phasor. However, it is important to note that if we consider the same time delay T_D for two sinewaves of different frequencies, f_1 and f_2, then their phases are frequency-dependent. Indeed, $\theta_1 = 2\pi T_D f_1$ and $\theta_2 = 2\pi T_D f_2$.

This fact may be quite obvious but is often forgotten when considering e.g. *time x bandwidth* products and other more advanced concepts in spectrum estimation.

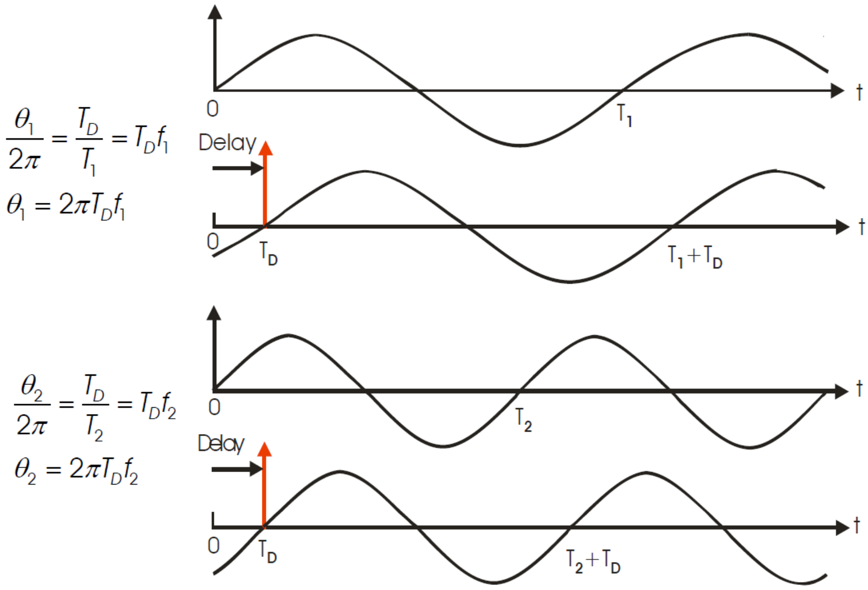

Figure 3: For the same time delay, the phase is frequency-dependent!

Different forms of Fourier and inverse Fourier transforms

Fourier transform was originally introduced for continuous-time signals and was subsequently extended for sampled-data (discrete time Fourier transform – DTFT) and for periodic sampled data. The kind of Fourier transform for periodic sampled data is called the Discrete Fourier Transform (DFT). Figure 4 illustrates the forms of **Fourier analysis** for all the cases. Once a time series is Fourier-transformed, this allows us to analyse the spectral contents in the data. Figure

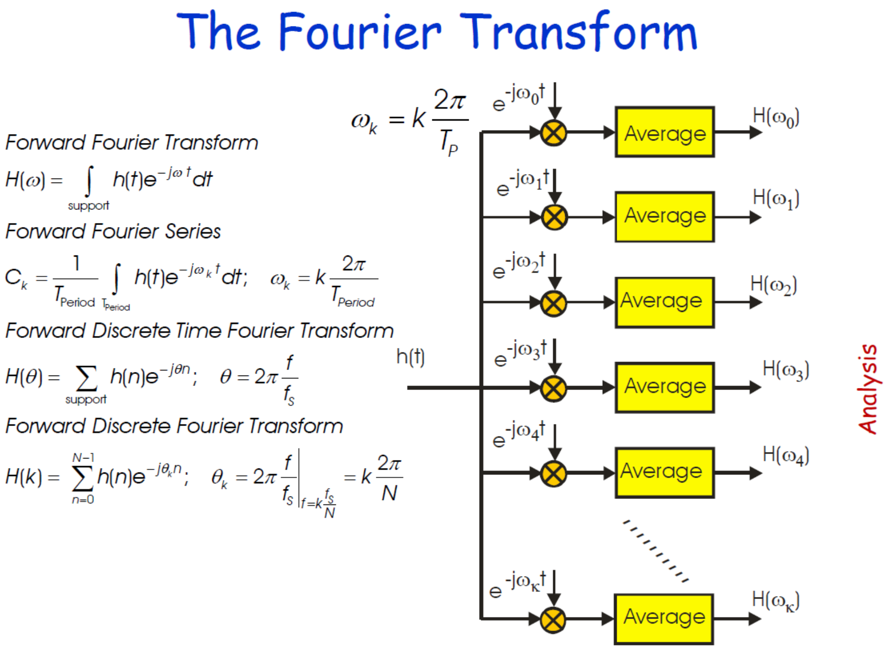

Figure 4: Different forms of the Fourier transform

5 shows the various forms of inverse Fourier transforms. The process of building up the original signal from its Fourier representation is called the **Fourier synthesis**. Both the **analysis** and **synthesis** have many practical applications, as shown in Figure 6.

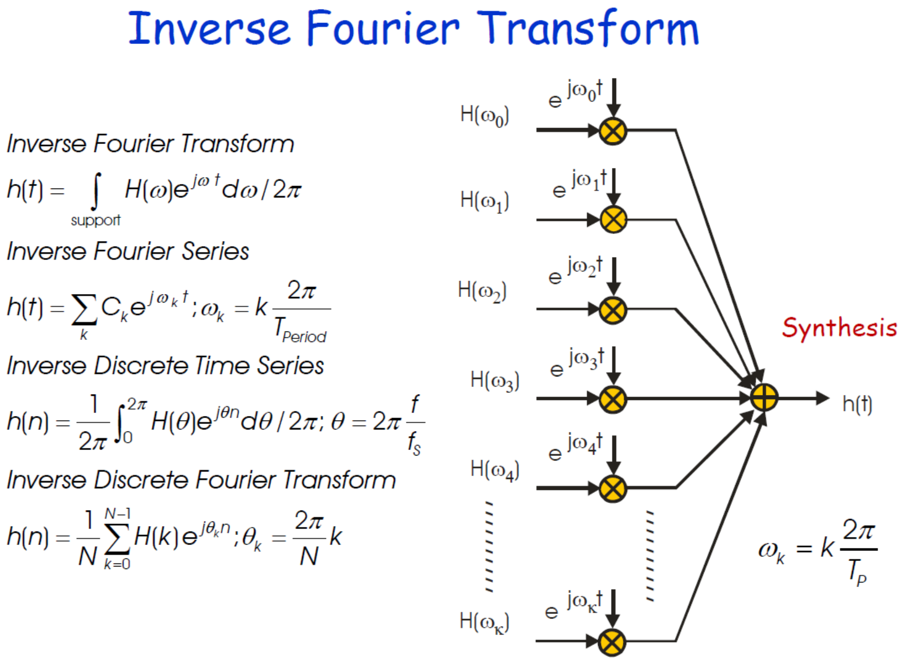

Figure 5: Different forms of the inverse Fourier transform

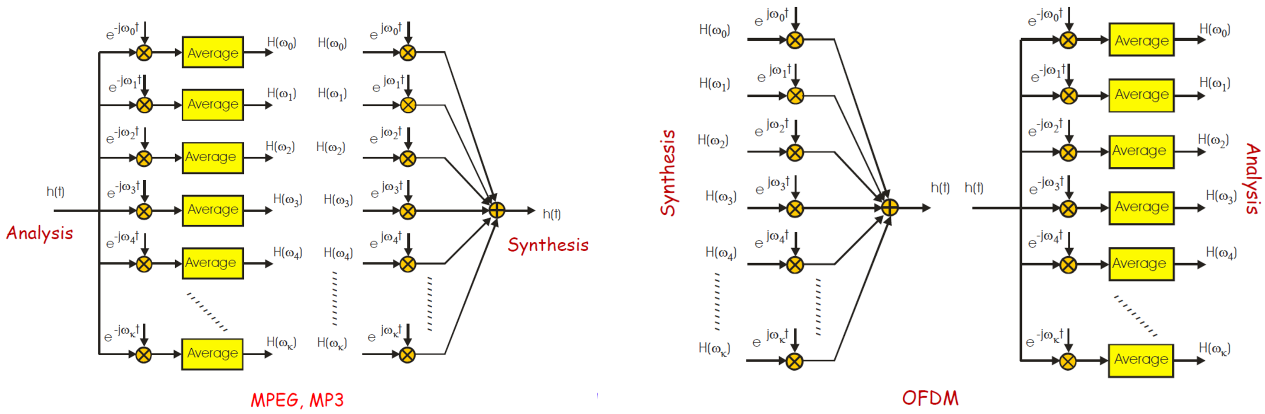

Figure 6: Applications of the Fourier analysis and synthesis

The DFT is in general complex-valued

Most of the time we only consider the **magnitude spectrum**, and it is easy to forget that the DFT is inherently complex-valued. It is real valued only for some special kinds of signals (e.g. real and symmetric). It is therefore useful to remind ourselves that for real valued data, the DFT in general produces a complex valued time series, consisting of pairs of sines of cosines, which are the basis functions of the DFT.
To illustrate this point, in Figure 7 we consider a time-domain Dirac pulse, $\delta(n)$ and its discrete Fourier transform (top panel), then a time delayed Dirac pulse, $\delta(n-1)$ and its DFT (middle panel), and the same Dirac pulse delayed by 2 samples, $\delta(n-2)$ and its DFT (bottom panel).
We have used the DFT time-shifting property, given by

$$DFT\big(x(n-T_D)\big) = e^{-\jmath \frac{2\pi}{N} T_D k} DFT(x) \tag{1}$$

For the case of a Dirac at $n = 0$, the DFT is equal to unity, so that, from (1), the DFT of delayed Dirac becomes

$$DFT\Big(\delta(n-T_D)\Big) = \cos\left(\frac{2\pi}{N} T_D k\right) - \jmath \sin\left(\frac{2\pi}{N} T_D k\right) \tag{2}$$

as shown in Figure 7. Notice that since $\sin^2 + \cos^2 = 1$ for any frequency, the magnitude spectra of all the DFTs in Figure 7 will be unity, the same as that for the Dirac at zero. Also compare with Figure 3.

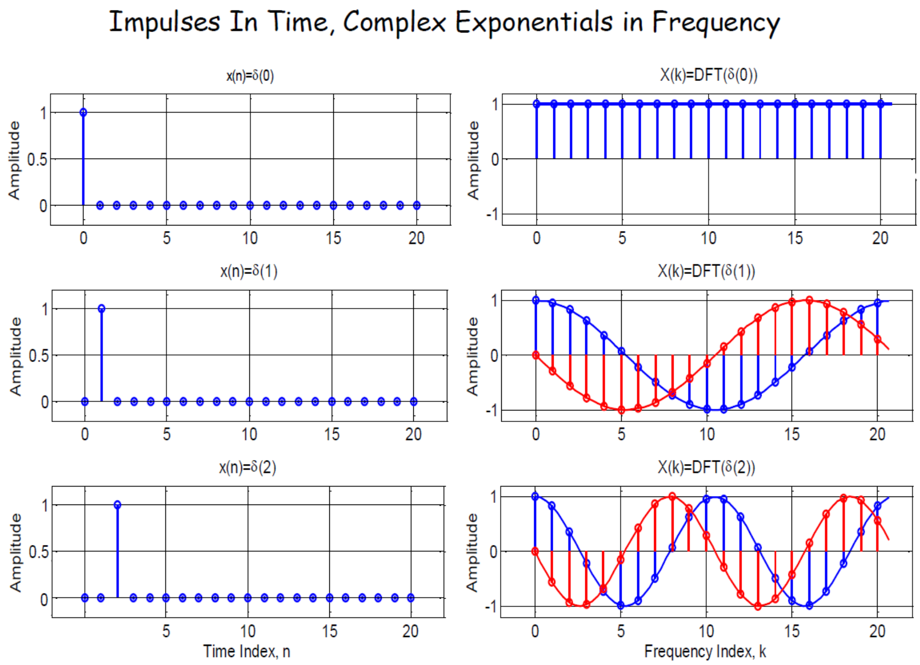

Figure 7: Time domain delayed Dirac pulses and their DFTs. The real components of the DFT are in blue (cosines) and the imaginary ones in red (sines)

The complementary situation is depicted in Figure 8, where the inputs are complex exponentials at frequencies $\omega = 0$, $\omega = 1$, and $\omega = 2$. From Figure 8, comment on the spectra of a pure sine or cosine, and how to obtain them from complex exponentials.

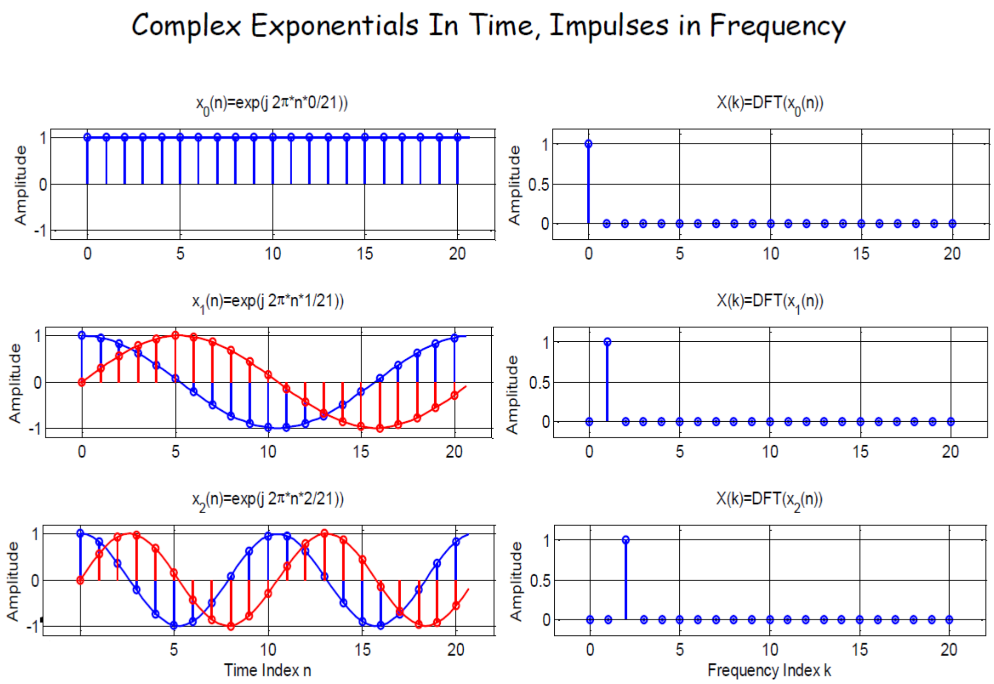

Figure 8: Time domain complex exponentials and their DFTs. The real components of the input are in blue (cosines) and the imaginary ones in red (sines)

Now that we have reminded ourselves that the DFT is complex-valued, we can revisit the Fourier transforms, from the continuous time one through to the DFT. The top panel in Figure 9 shows a continuous-time real-valued function $h(t)$ (in yellow) and its Fourier transform (in green). Notice that we have now plotted both the real and imaginary parts of FT and not the magnitude spectrum. The second panel shows a sampled version of the signal $h(t)$ and its Fourier transform, the so called Discrete Time Fourier Transform (DTFT). Notice that since the signal is now real and discrete in time the DTFT is continuous and periodic. The third panel illustrates the idea behind the Fourier series expansion. The continuous time signal $h(t)$ is periodic, and therefore its FT is discrete. The bottom panel combines the panels 1 – 3, the signal is now sampled (discrete time) and periodic. Because the signal is sampled its spectrum if periodic and because the signal is periodic its spectrum is discrete, giving the Discrete Fourier transform (DFT).

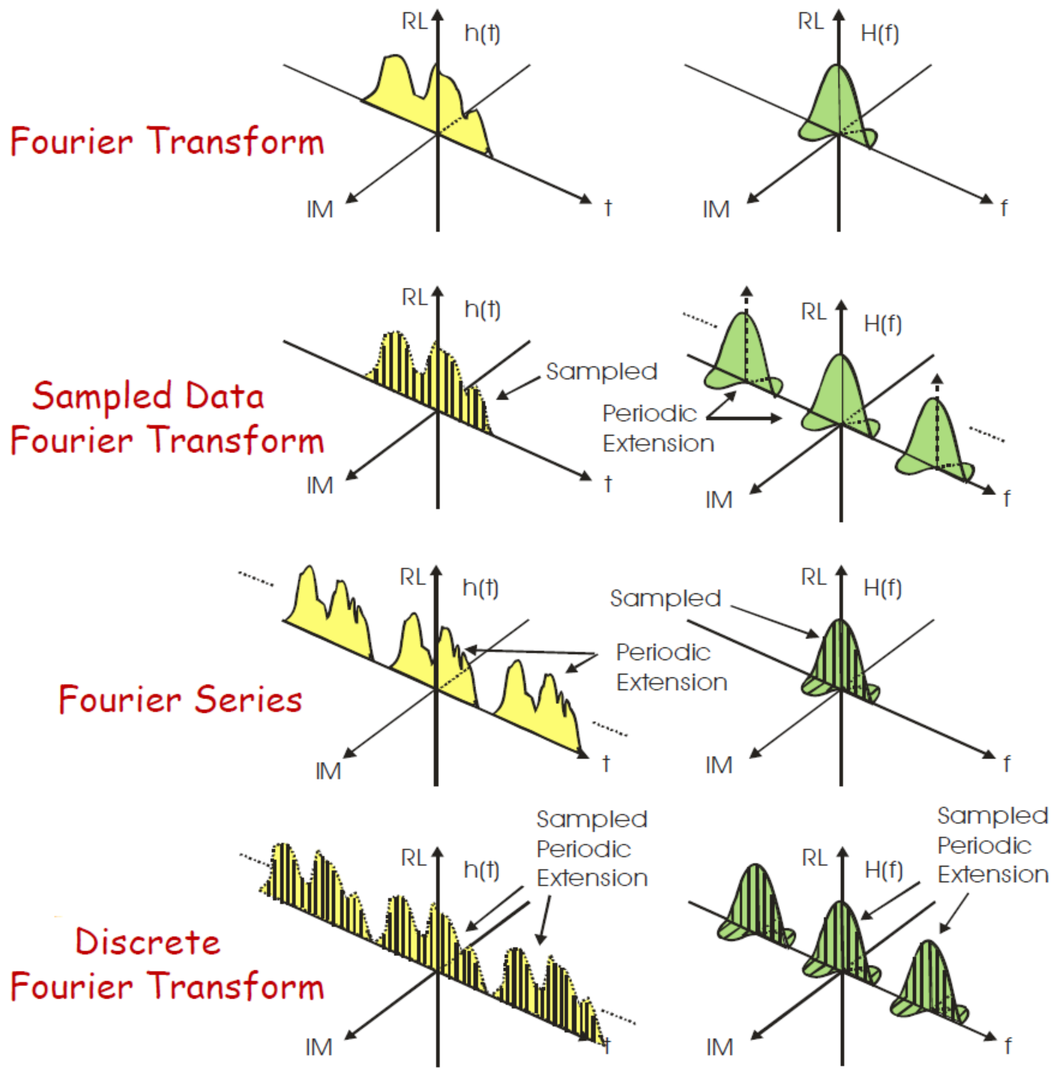

Figure 9: Fourier transforms of a real-valued signal which is sampled and undergoes a periodic extension. Notice that the Fourier transforms are complex valued. Notation: RL denotes the real part and IM the imaginary part

DFT and Data Compression

One important application of DFT is in data compression. To illustrate the usefulness of DFT in this area, consider the time-domain signal in Figure 10 (top). This was an attempt to draw the engine of a steam train. The Fourier coefficients of this time-domain signal are shown in panel 2, in red. As expected, due to many sharp edges the Fourier spectrum contains high-frequency components, so Fourier coefficients populate the whole frequency plane. The bottom two plots show the representation using only the first 42 Fourier coefficients (bottom plot) and the reconstructed Choo-Choo from this truncated representation on the third panel. Observe a good match between the original drawing (in blue) and the representation based on truncated DFT (in red). This is the very basis for the use of DFT in data compression, see also Figure 6 (left).

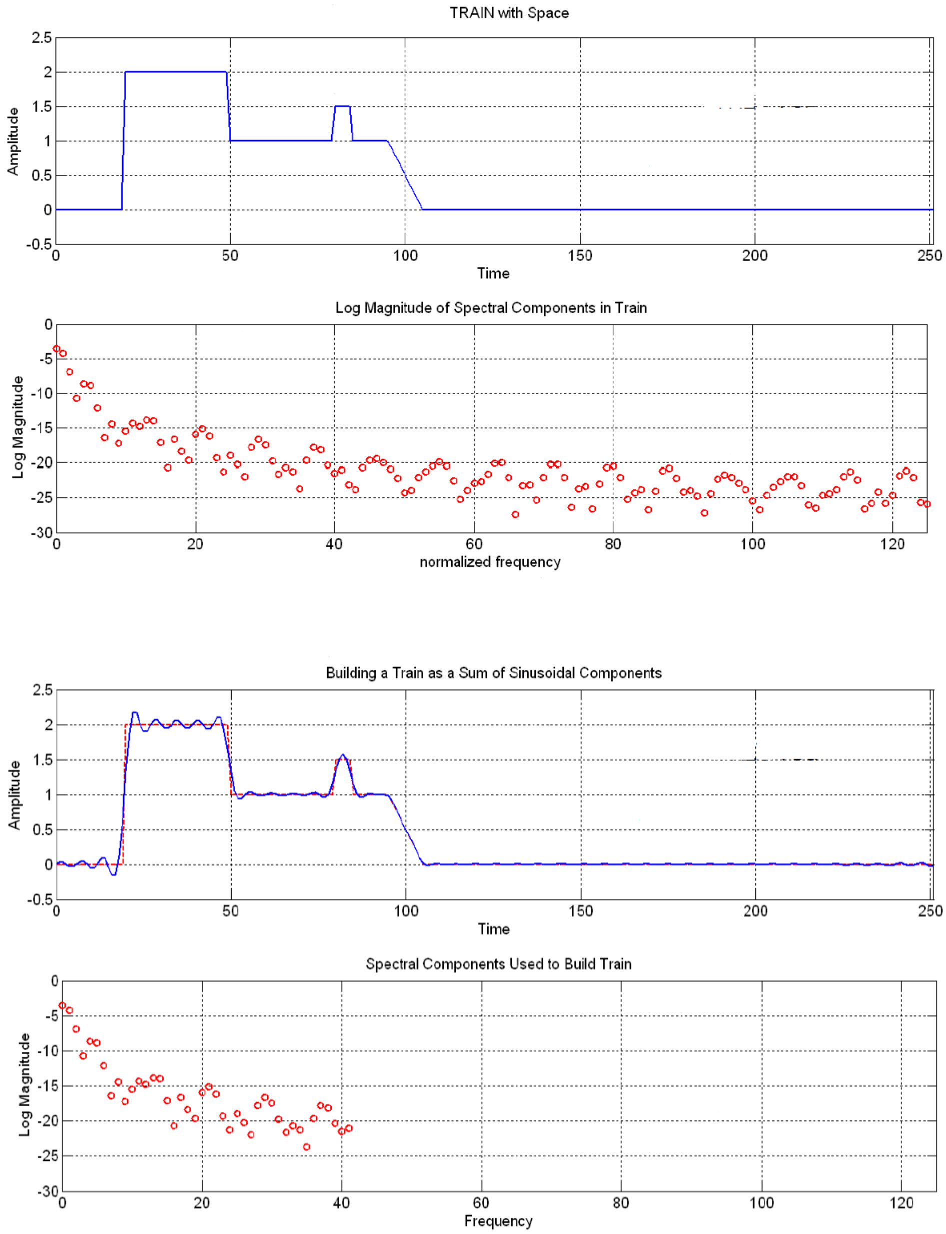

Figure 10: Fourier transform of a train drawing which contains sharp edges (high frequency components). Top two diagrams: the original drawing and its DFT coefficients. Bottom two diagrams: the drawing reconstructed from a subset of DFT coefficients.

The Need for the Averaging of Fourier Spectra

In many applications we measure signals immersed in noise, which makes it difficult to identify the signal of interest. One way to enhance the utility of DFT in such cases is through the averaging of individual DFTs (for consecutive segments of data), as shown in Figure 11. Observe how the effects of noise on the Fourier spectrum decreases with the degree of averaging. Compare with the example from your coursework (where you are investigating power spectra).

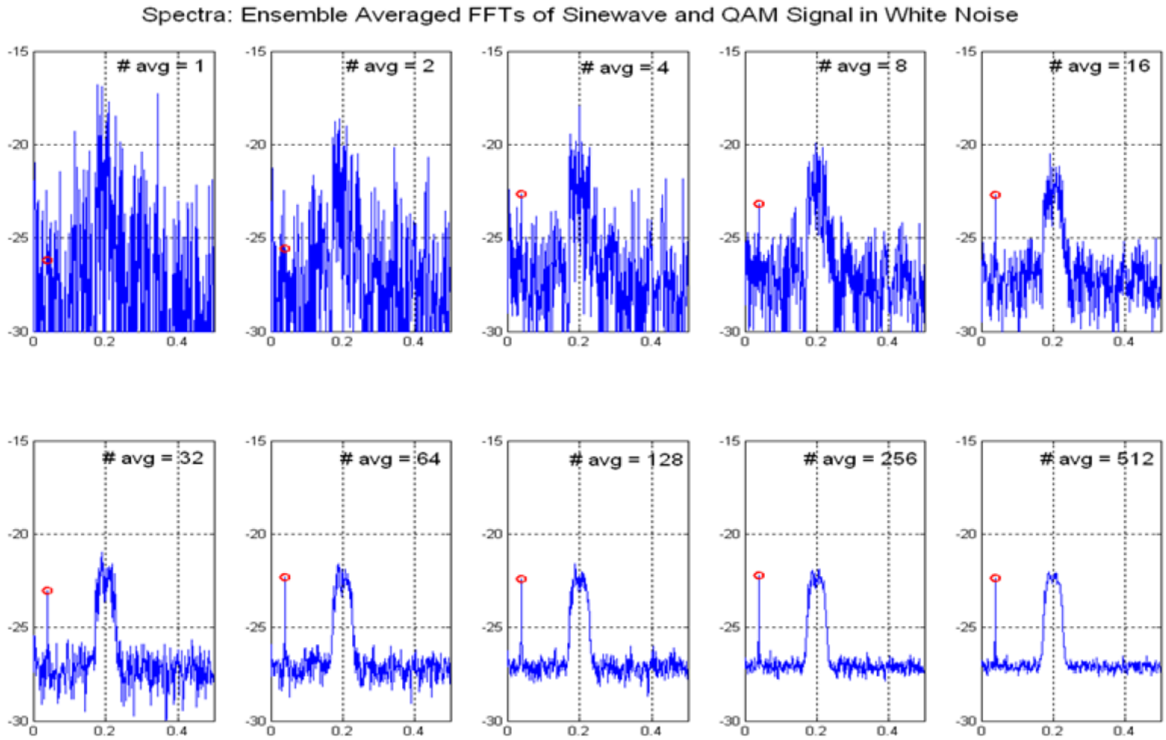

Figure 11: Averaging of DFT spectra makes it possible to identify both a sinewave (red circle) and a bandpass QAM signal