Adaptive Sig. Proc. & Machine Intel.

Supplement: Circularity of distributions and circularity measures

Danilo Mandic

room 813, ext: 46271

Department of Electrical and Electronic Engineering

Imperial College London, UK

d.mandic@imperial.ac.uk, URL: www.commsp.ee.ic.ac.uk/∼mandic

Circularity

Definition: A complex-valued random is called **circular** if its probability distribution is not dependent on the angle, that is, the distribution is "**rotation invariant**".

For simplicity, we consider univariate complex-valued random variables; the concepts are readily extended to the multivariate case.

Recall that for an iid complex-valued random variable $Z = X + iY$, the pdf

$$\mathcal{P}_Z(z) = \mathcal{P}_X(x)\mathcal{P}_Y(y)$$

On the other hand, in the case of a **rotation invariant** $\mathcal{P}_Z(z)$, its pdf is only be dependent of the **Euclidean distance** from the origin in the complex domain. Therefore, if the random variable Z is circular, we have

$$g(r) = \mathcal{P}_Z(z) = \mathcal{P}_X(x)\mathcal{P}_Y(y)$$

where $r = \sqrt{x^2 + y^2}$ and $g(\cdot)$ is a general function.

Now, lets take the differential of $g(\cdot)$ with respect to x; this gives

$$\frac{\partial g(r)}{\partial x} = \frac{\partial g(r)}{\partial r}\frac{\partial r}{\partial x} \qquad \text{with} \qquad \frac{\partial r}{\partial x} = \frac{x}{r}.$$

Upon replacing $g(r) = \mathcal{P}_X(x)\mathcal{P}_Y(y)$ into the expression above, we have

$$\frac{\partial g(r)}{\partial x} = \frac{\partial g(r)}{\partial r}\frac{x}{r} = \frac{\partial \mathcal{P}_X(x)}{\partial x}\mathcal{P}_Y(y).$$

Dividing both sides of the equation above by $g(r) = \mathcal{P}_X(x)\mathcal{P}_Y(y)$ gives

$$\frac{\partial g(r)}{\partial r}\frac{x}{g(r)r} = \frac{\partial \mathcal{P}_X(x)}{\partial x}\frac{1}{\mathcal{P}_X(x)}.$$

Rearranging the above equation yields

$$\frac{\partial g(r)}{\partial r}\frac{x}{g(r)r} = \frac{\partial \mathcal{P}_X(x)}{\partial x}\frac{1}{\mathcal{P}_X(x)} \rightarrow \frac{\partial g(r)}{\partial r}\frac{1}{g(r)r} = \frac{\partial \mathcal{P}_X(x)}{\partial x}\frac{1}{x\mathcal{P}_X(x)}.$$

Circularity $\looparrowright$ proof of dependence on the radius, contd.

Z is independent and identically distributed (iid)

Now consider the equation below more carefully

$$\underbrace{\frac{\partial g(r)}{\partial r}\frac{1}{g(r)r}}_{\text{This side is dependent on } r} = \underbrace{\frac{\partial \mathcal{P}_X(x)}{\partial x}\frac{1}{x\mathcal{P}_X(x)}}_{\text{This side is only dependent on } x}.$$

Note that the left hand side is dependent on r, that is, it is a function of both x and y. However, the right hand side is only a function of x. Since both sides must be equal to a constant, ν, this results in

$$\frac{\partial g(r)}{\partial r}\frac{1}{g(r)r} = \nu \rightarrow \frac{\partial g(r)}{\partial r}\frac{1}{g(r)} = \nu r$$

Given the differential equation above, we have

$$\frac{\partial g(r)}{\partial r}\frac{1}{g(r)} = \frac{\partial \ln\left(g(r)\right)}{\partial r} = \nu r$$

the solution to which is given by

$$g(r) = Ae^{\frac{1}{2}\nu r}$$

where A is found so that $g(r) = \mathcal{P}_z(z)$ is a pdf, that is, $\int_{-\infty}^{\infty} g(r)\mathrm{d}r = 1$.

Note that this is a form only consistent with the Gaussian distribution.

Circularity
General observations

Consider a general real-valued random variable R and a complex-valued random variable U uniformly distributed on the unit circle, given by

$$u = \cos(\theta) + \jmath\sin(\theta) \qquad \text{where} \qquad \theta \sim [0, 2\pi)$$

that is, θ is independent from R.

Then, the pdf of a complex-valued random variable $Z = RU$ is given by

$$\mathcal{P}_Z(z) = \mathcal{P}_\Theta(\theta)\mathcal{P}_R(r) \rightarrow \mathcal{P}_Z(z) = \frac{1}{2\pi}\mathcal{P}_R(r)$$

where we have used $\forall\theta \in \mathbb{R} : \mathcal{P}_\Theta(\theta) = 1/2\pi$.

In summary:

- The pdf of $Z = RU$ is a function of r only, i.e. the pdf of Z is **rotation invariant**
- A framework for generating samples of circular complex-valued random variables is set
- The the real and imaginary components of Z are uncorrelated, that its

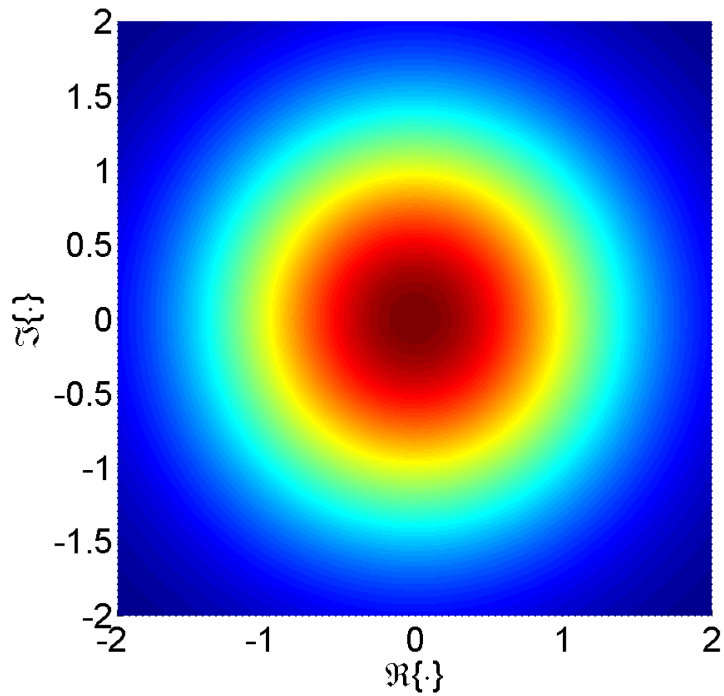
$$E[XY] = E[r^2\cos(\theta)\sin(\theta)] = 0$$

- Remember that orthogonality and independence are two different concepts, apart from the case of Gaussian random variables

Circularity
Some circular distributions

Circular complex-valued random variables

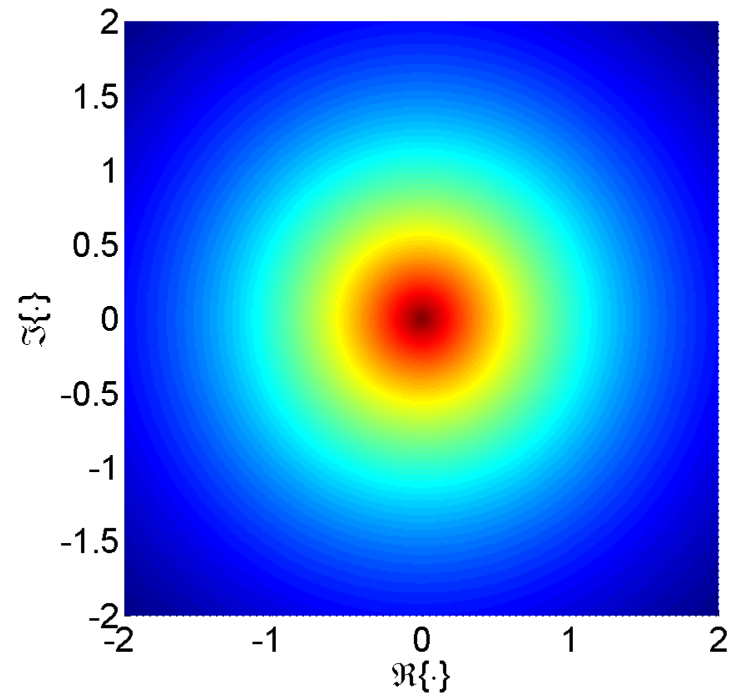
The distribution of R is Rayleigh. Thus, the distributions of the real and imaginary parts are Gaussian.



circular Rayleigh distribution

The distribution of R is exponential

$$\mathcal{P}_R(r) = \lambda e^{-\lambda r}, \ \lambda = 1$$



circular exponential distribution

Circularity
A noncircular distribution

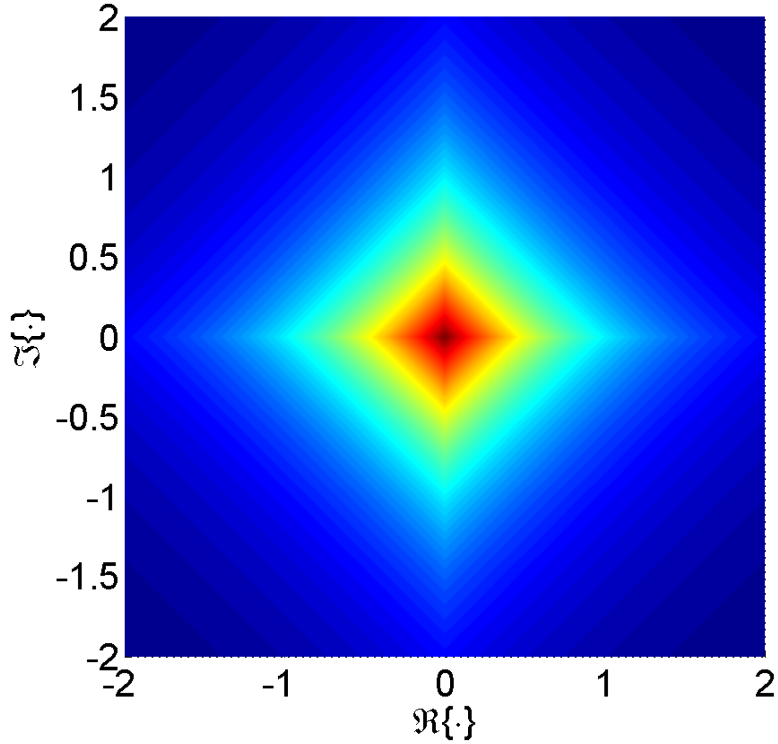
Independent real & imaginary distributions but not circular!

Distributions of the real and imaginary part are **independent Laplace distributions**

$$\mathcal{P}_X(x) = \frac{1}{2}e^{-|x|} \text{ and } \frac{1}{2}\mathcal{P}_Y(y) = \frac{1}{2}e^{-|y|}$$

Thus,

$$\mathcal{P}_Z(z = x + \jmath y) = \frac{1}{4}e^{-(|x|+|y|)}$$



Although the distributions on the real and imaginary axes are independent and hence uncorrelated, the resulting distribution is not rotation invariant, that is, it is non-circular.