Spectral Estimation & ASP
Lecture 4: Modern Spectal Estimation

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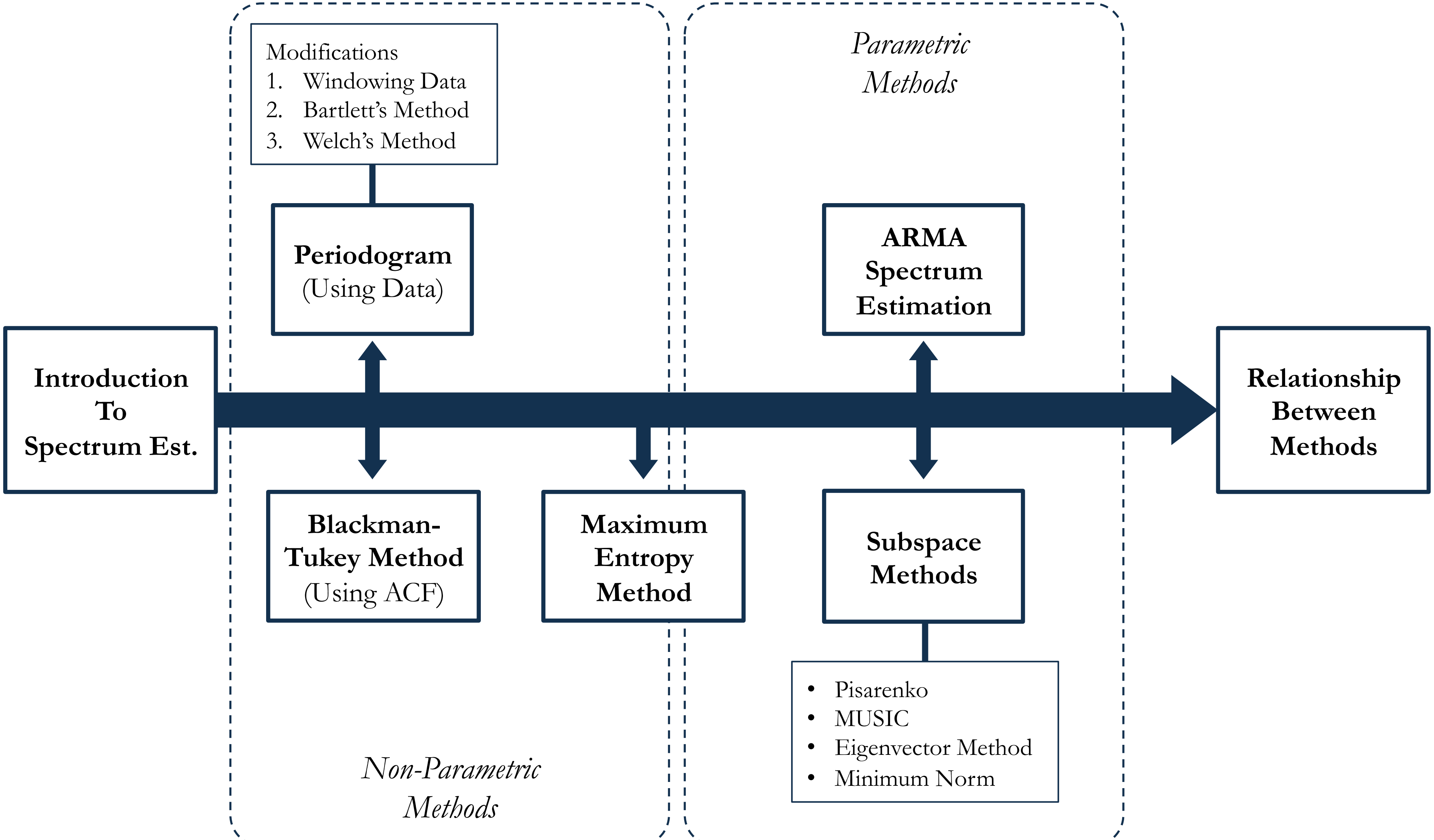
Overview of Spectral Estimation Methods



Periodogram Based Methods

$$\hat{P}_{per}(\omega) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-\jmath n\omega}\right|^2$$

Windowing

Modified Periodogram

$$\hat{P}_{mod}(\omega) = \frac{1}{NU}\left|\sum_{n=0}^{N-1} w(n)x(n)e^{-\jmath n\omega}\right|^2$$

Averaging

Bartlett's Method

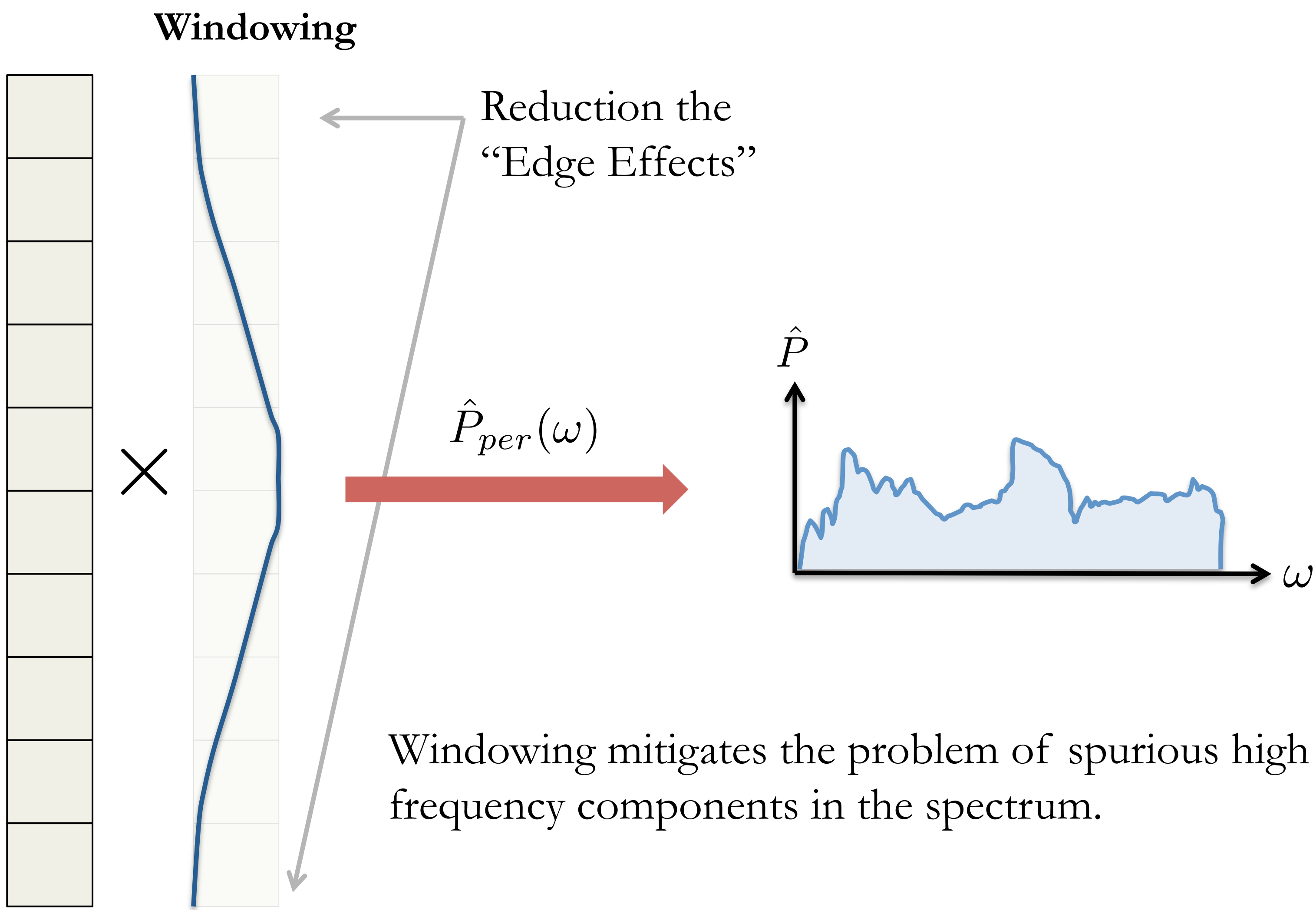
$$\hat{P}_{B}(\omega) = \frac{1}{N}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} x(n+iL)e^{-\jmath n\omega}\right|^2$$

+ Overlapping windows

Welch's Method

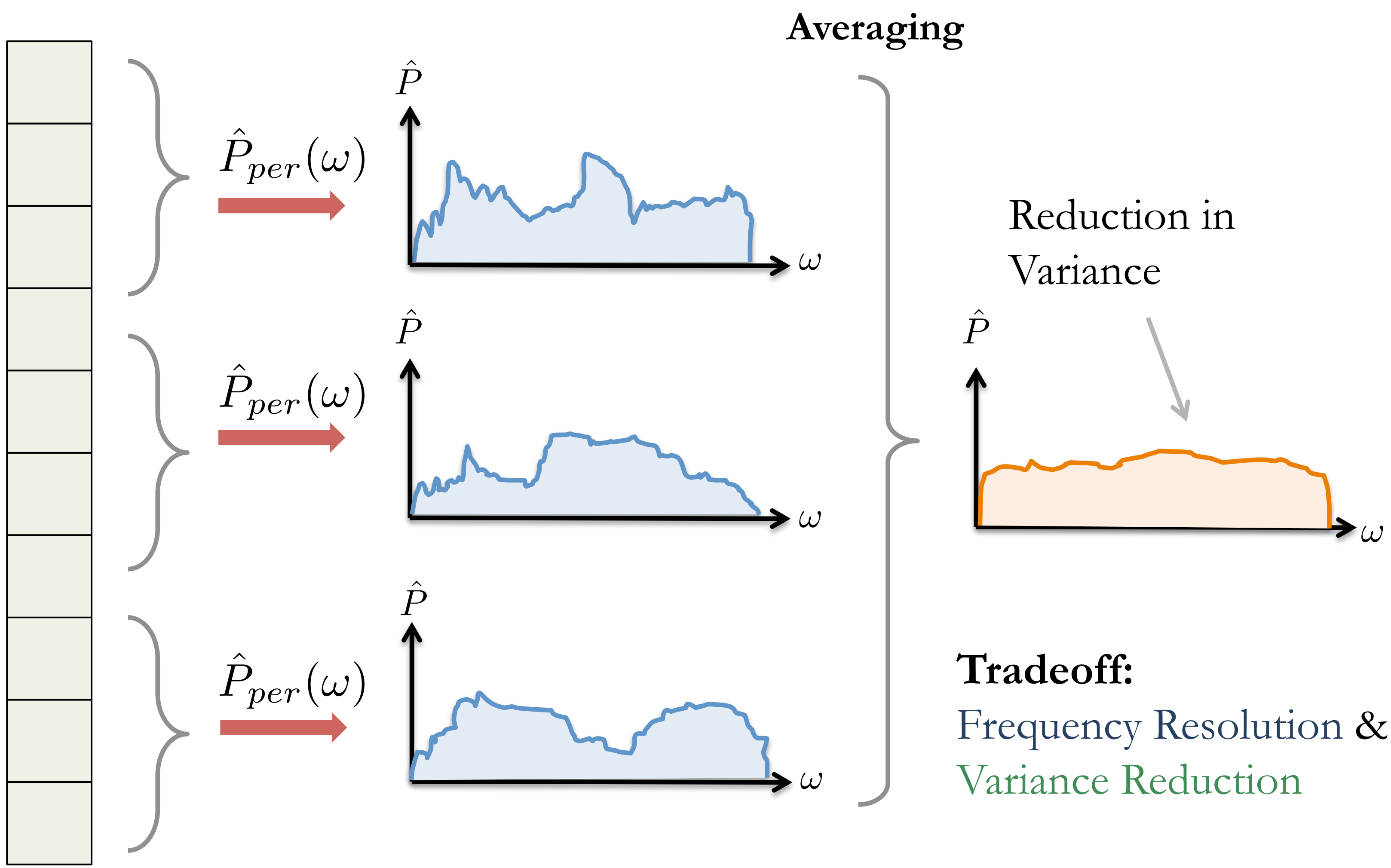
$$\hat{P}_{W}(\omega) = \frac{1}{KLU}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} w(n)x(n+iD)e^{-\jmath n\omega}\right|^2$$

Modified Periodogram



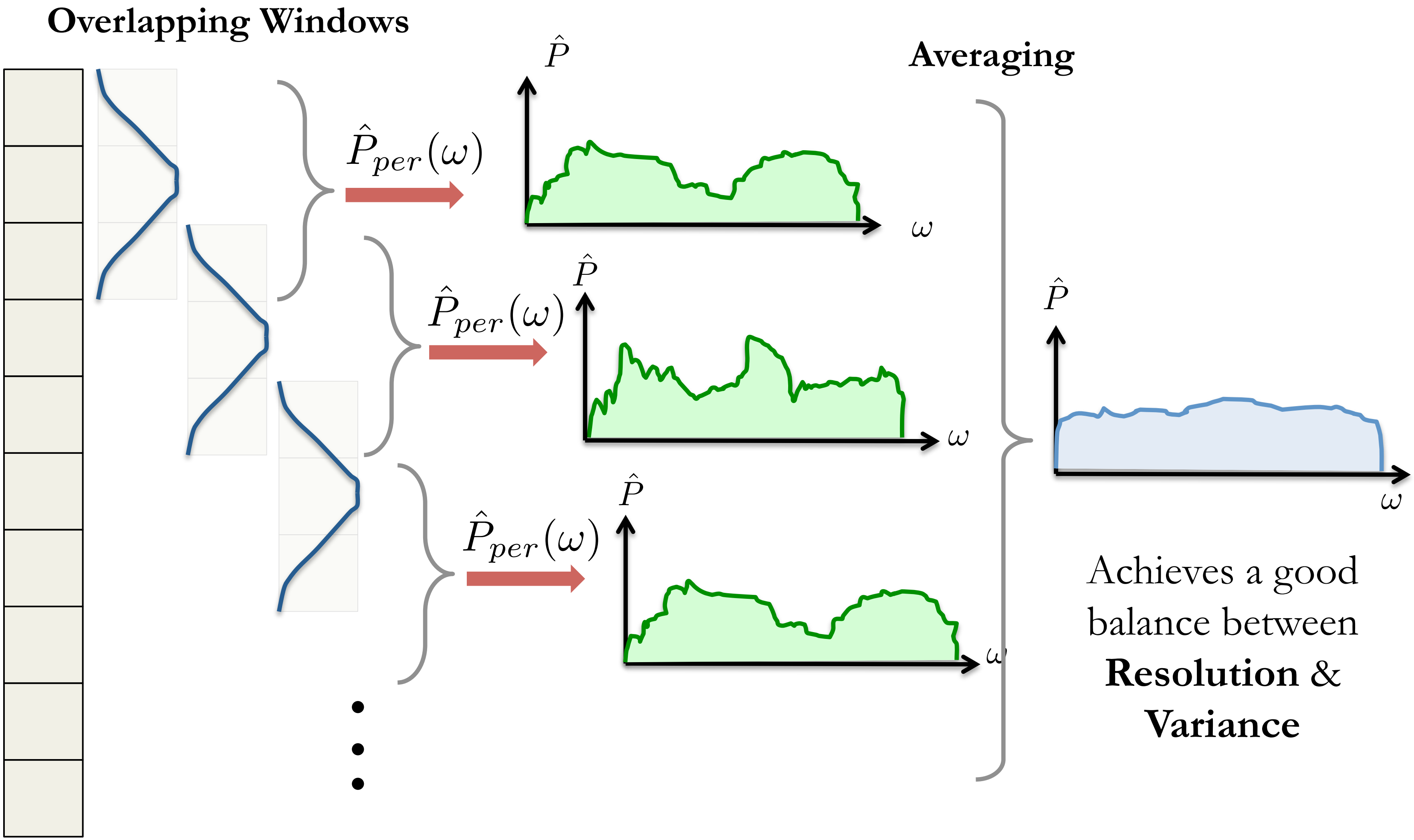
Windowing mitigates the problem of spurious high frequency components in the spectrum.

Bartlett's Method



Tradeoff:
Frequency Resolution &
Variance Reduction

Welch's Method



Achieves a good balance between **Resolution** & **Variance**

Blackman-Tukey Method

The Periodogram can also be expressed as:

$$\hat{P}_{per}(\omega) = \sum_{k=-N+1}^{N-1} \hat{r}(k) e^{-\jmath k\omega}$$

Autocorrelation Estimates at large lags are **unreliable**

$$\hat{P}_{BT}(\omega) = \sum_{k=-M}^{M} w(k)\hat{r}(k) e^{-\jmath k\omega}$$

Lags: $M < N - 1$

Windowing

Next: Can we **extrapolate the autocorrelation** estimates for lags $k > M$?

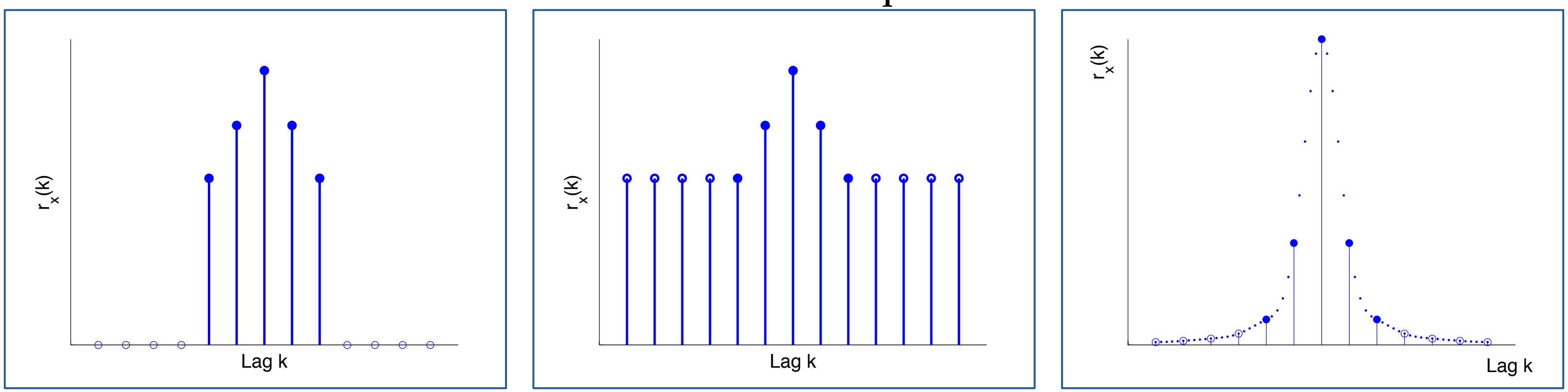
Maximum Entropy Method

How can we **extrapolate the autocorrelation** estimates with imposing the **least amount of structure on the data**?

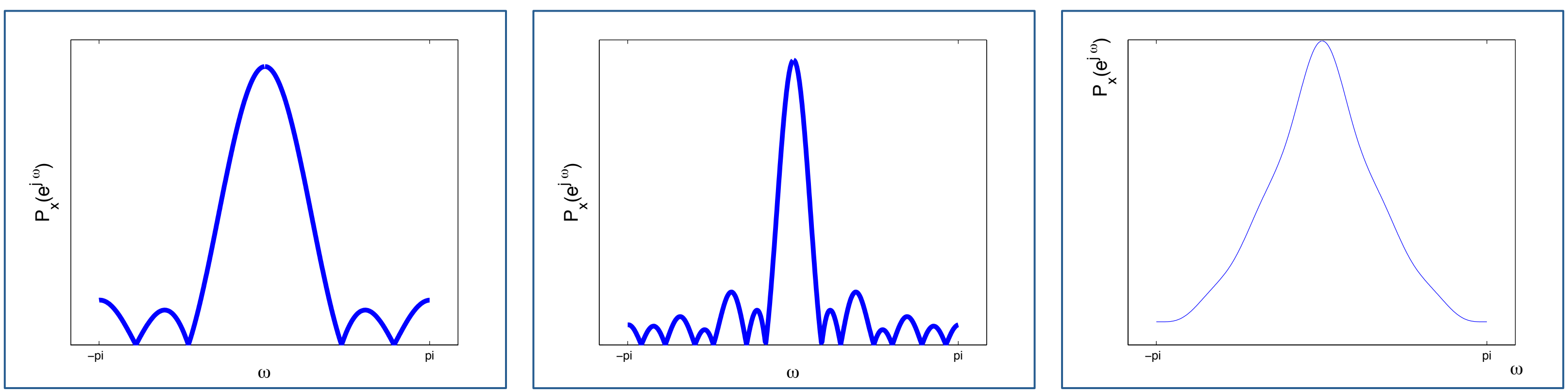
$\Longrightarrow$ Maximize the randomness $\Longrightarrow$ **Maximize Entropy**

Which one has the "flattest" PSD?

Autocorrelation Sequences



Power Spectral Density (PSD)



Maximum Entropy Method

Entropy of Gaussian random process $x(n)$ with PSD $P_{xx}(\omega)$:

$$H(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_{xx}(\omega)\, d\omega$$

Goal: Find extrapolated autocorrelation values $r_e(k)$ to maximize the entropy:

$$\frac{\partial H(x)}{\partial r_e^*(k)} = 0, \quad \text{for: } |k| > p$$

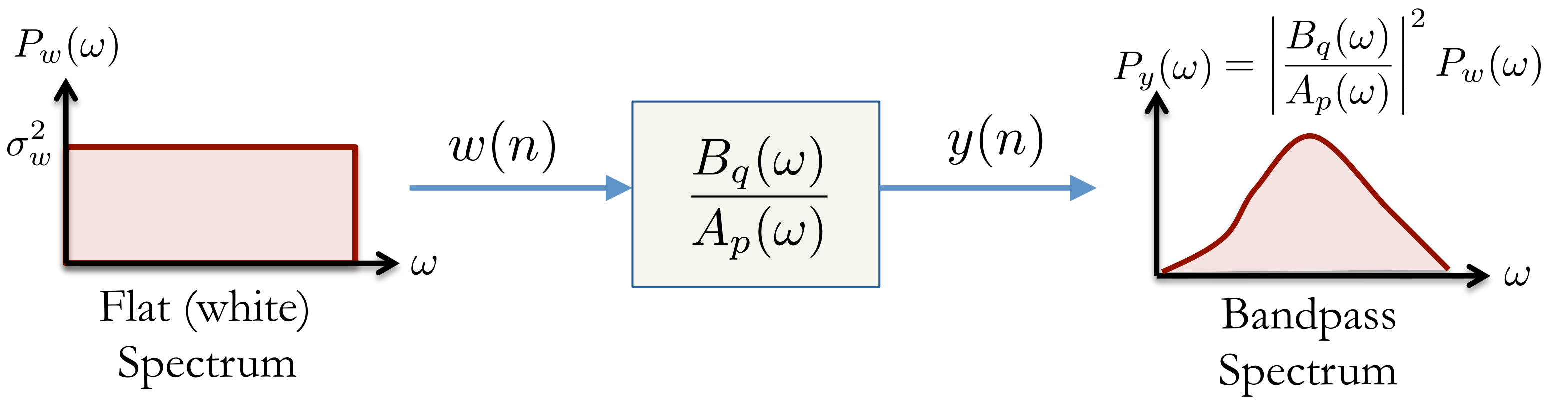
**Refer to handout for the full derivation*

$$\boxed{\hat{P}_{mem}(\omega) = \frac{\sigma_w^2}{\left|1 + \sum_{k=1}^{p} \hat{a}_k e^{-\jmath k\omega}\right|^2}}$$

Estimated using the Yule-Walker Method

The MEM method is **identical to the all-pole AR(p) spectrum** although **no assumptions were made** about the model of the data (except Gaussianity).

ARMA Method



$$y(n) = \underbrace{-\sum_{k=1}^{p} a_k y(n-k)}_{\substack{\textbf{Autoregressive}\\ \text{AR(p)}}} + \underbrace{\sum_{k=0}^{q} b_k w(n-k)}_{\substack{\textbf{Moving Average}\\ \text{MA(q)}}}$$

$$\hat{P}_{ARMA}(\omega) = \frac{\left|\sum_{k=0}^{q} \hat{b}_k e^{-\jmath k\omega}\right|^2}{\left|1 + \sum_{k=1}^{p} \hat{a}_k e^{-\jmath k\omega}\right|^2}$$

Subspace Methods: Introduction

$$x(n) = A_1 e^{\jmath n\omega_1} + w(n)$$

$$A_1 = |A_1|e^{\jmath\Phi}$$
$$w(n) \sim \mathcal{N}(0, \sigma_w^2)$$

$$\boxed{\mathbf{x} = A_1\mathbf{e}_1 + \mathbf{w}}$$

$$\mathbf{x} = [x(0),\, x(1),\, \ldots,\, x(M-1)]^T$$
$$\mathbf{e}_1 = [1,\, e^{\jmath\omega_1},\, \ldots,\, e^{\jmath\omega_1(M-1)}]^T$$

Autocorrelation: Matrix

$$E(\mathbf{x}\mathbf{x}^H) = \mathbf{R}_{xx} = \underbrace{|A_1|^2\mathbf{e}_1\mathbf{e}_1^H}_{\mathbf{R}_s} + \underbrace{\sigma_w^2\mathbf{I}}_{\mathbf{R}_n}$$

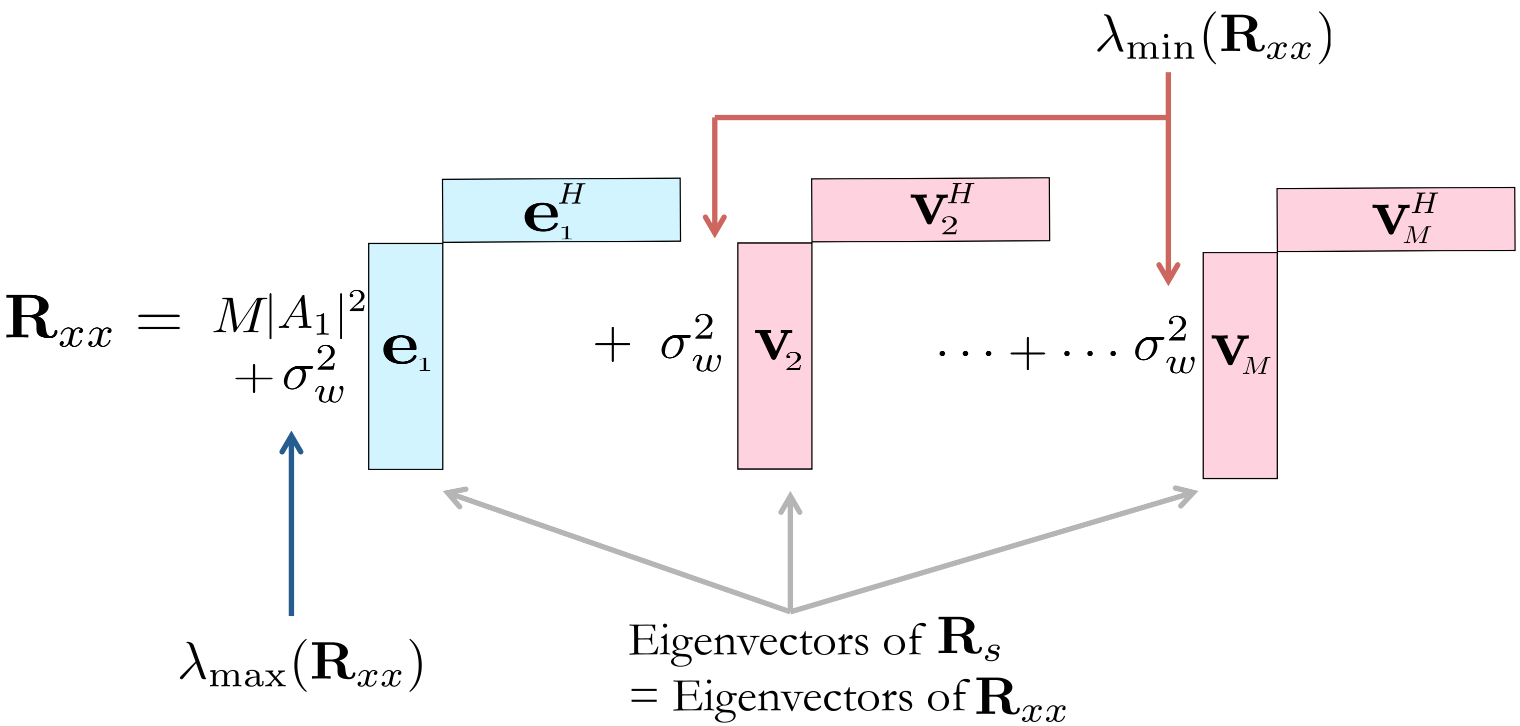
Signal Autocorrelation
Rank 1
Single non-zero Eigenvalue $= M|A_1|^2$

Noise Autocorrelation
Rank M
All Eigenvalues $= \sigma_w^2$

Decomposing the Autocorrelation Matrix

$\mathbf{R}_s = |A_1|^2 \mathbf{e}_1 \mathbf{e}_1^H$ is Hermitian. Remaining M-1 eigenvectors are orthogonal to $\mathbf{e}_1$

$$\mathbf{e}_1^H \mathbf{v}_i = 0, \quad i = 2, \ldots M$$



Can we use the idea that $\mathbf{e}_1^H \mathbf{v}_i = 0$, to somehow estimate the power spectrum?

Multiple Sinusoids

Consider: $x(n) = A_1 e^{\jmath n \omega_1} + A_2 e^{\jmath n \omega_2} + w(n)$

$$\mathbf{R}_{xx} = \mathbf{E}\mathbf{P}\mathbf{E}^H + \sigma_w^2 \mathbf{I} \qquad \mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2]\,,\ \mathbf{P} = \mathrm{diag}(|A_1|^2, |A_2|^2)$$

The first 2 eigenvalues of $\mathbf{R}_{xx}$ are $\lambda_i^s + \sigma_w^2$

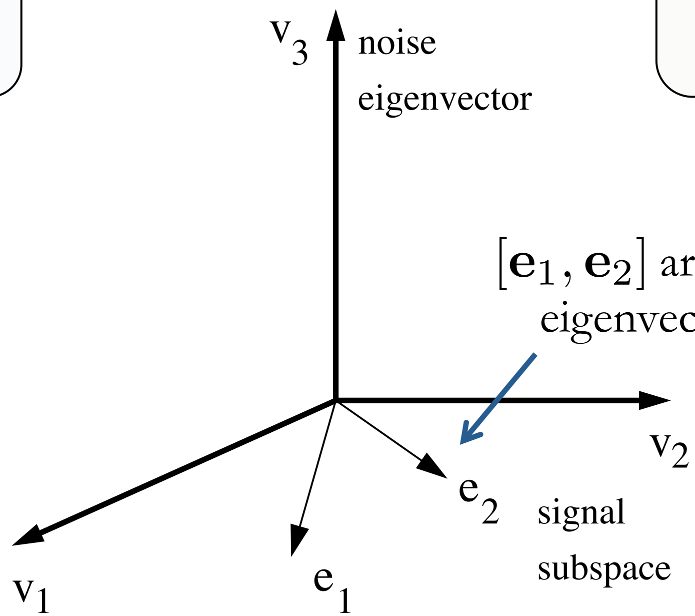
The remaining are σ_w^2

Rank 2

- Signal eigevectors span a 2D subspace
- Noise eigevectos span a (M-2) dim. subspace

The signal and noise subspaces are orthogonal

$[\mathbf{e}_1, \mathbf{e}_2]$ are not the signal eigenvectors in this case.



Subspace Methods

Extending to p sinusoids. $\mathbf{R}_{xx} = \mathbf{E}\mathbf{P}\mathbf{E}^H + \sigma_w^2 \mathbf{I}$

$$\mathbf{E} = [\mathbf{e}_1, \ldots, \mathbf{e}_p]\,,\ \mathbf{P} = \mathsf{diag}(|A_1|^2, \ldots, |A_p|^2)$$

Using $\mathbf{e}_i^H \mathbf{v}_k = 0 \quad \begin{cases} i = 1, \ldots, p \\ k = p+1, \ldots, M \end{cases}$

$\Longrightarrow$ PSD estimation can be performed as: $\hat{P}_{sub}(\omega) = \dfrac{1}{\sum_{i=p+1}^{M} \alpha_i \left|\mathbf{e}^H \mathbf{v}_i\right|^2}$

Pisarenko Harmonic Decomposition

$$\boxed{\hat{P}_{PHD}(\omega) = \frac{1}{\left|\mathbf{e}^H \mathbf{v}_{\min}\right|^2}}$$

MUltiple Signal Classification (MUSIC)

$$\boxed{\hat{P}_{MU}(\omega) = \frac{1}{\sum_{i=p+1}^{M} \left|\mathbf{e}^H \mathbf{v}_i\right|^2}}$$

EigenVector Method

$$\boxed{\hat{P}_{EV}(\omega) = \frac{1}{\sum_{i=p+1}^{M} \frac{1}{\lambda_i} \left|\mathbf{e}^H \mathbf{v}_i\right|^2}}$$

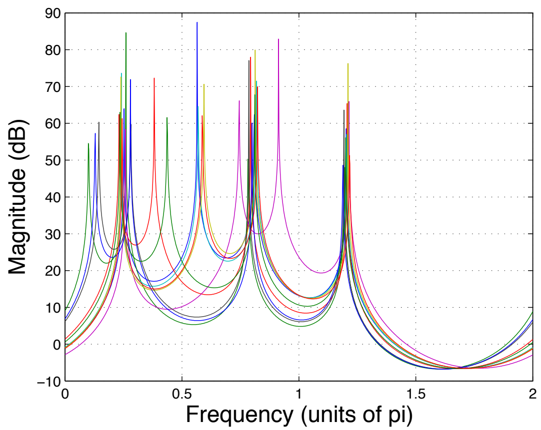
Minimum Norm Method

$$\boxed{\hat{P}_{MN}(\omega) = \frac{1}{\left|\mathbf{e}^H \mathbf{a}\right|^2}}$$

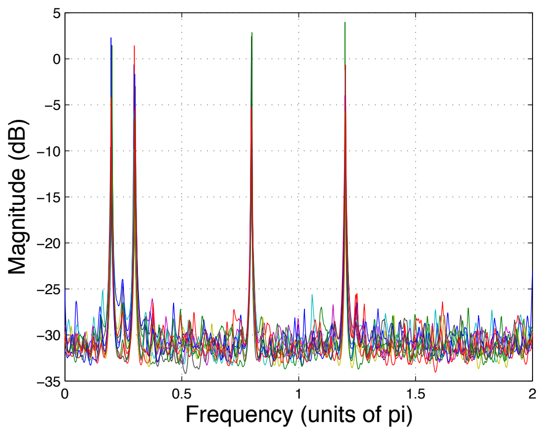
$\mathbf{a} \in$ Noise Subspace & has min. norm

Comparison of the 4 Subspace Methods

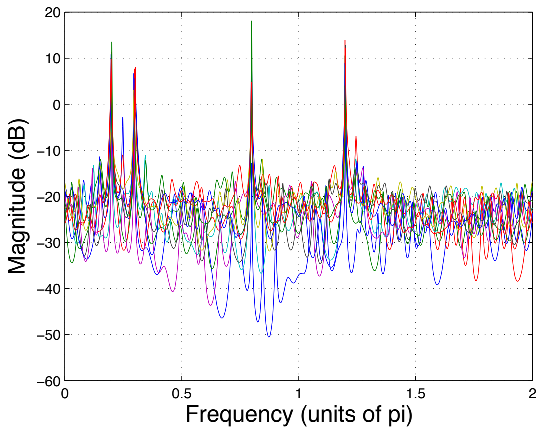
Pisarenko



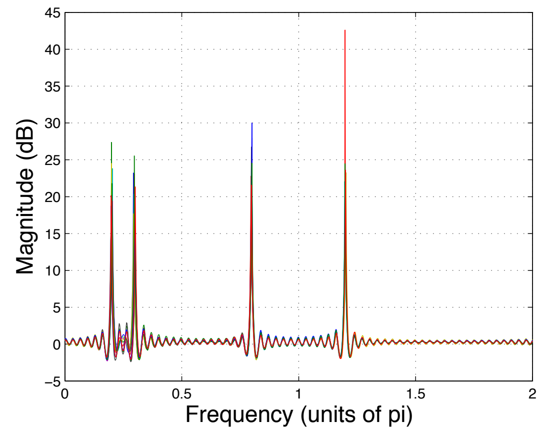
MUSIC



EigenVector



Minimum Norm



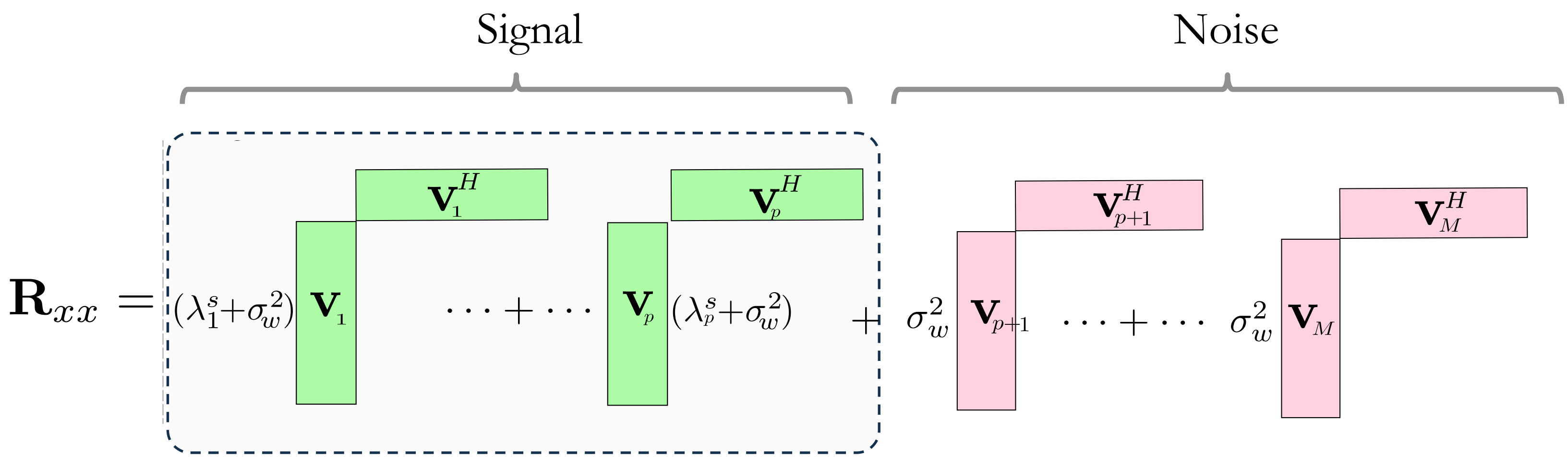
Overlay of 10 different realizations of 4 complex sinusoids in white noise.

Pisarenko only needs a 5 x 5 correlation matrix

A 64 x 64 correlation matrix was used for other methods

Except for Pisarenko's method, all other estimates are correct!

Principle Components Spectral Estimation



$$\hat{\mathbf{R}}_{xx} \approx \hat{\mathbf{R}}_s = \sum_{i=1}^{p} \lambda_i \mathbf{v}_i \mathbf{v}_i^H$$

Linear Algebra terms: **We impose a rank p constraint** on $\mathbf{R}_{xx}$

Can we de-noise the signal by discarding the noise eigenvectors $[\mathbf{v}_{p+1}, \ldots, \mathbf{v}_M]$?

Principal component analysis (PCA) **can be used with Blackman–Tukey, maximum entropy method** and **AR spectrum estimation**.

Summary of the Different Methods