Advanced Digital Signal Processing

Lecture Supplement: The Method of Lagrange Multipliers

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The method of Lagrange multipliers introduces a new variable (the multiplier) to solve constrained optimisation problems, without first solving the constraint equation for one of the variables.
To illuminate this further, consider a bivariate function $f(x, y)$, given in Figure 1. The curve $f(x, y)$ has a clear global (unconstrained) maximum, however, when we are constrained to find a maximum for which the (x, y) coordinates lie on the constraint curve $g(x, y) = c$, the constrained maximum is an orthogonal projection of the maximum of $f(x, y)$ onto $g(x, y)$. In other words, our task can be formalised as

$$\max f(x, y) \quad \text{subject to} \quad g(x, y) = c$$

To perform this optimisation, we start from a new function

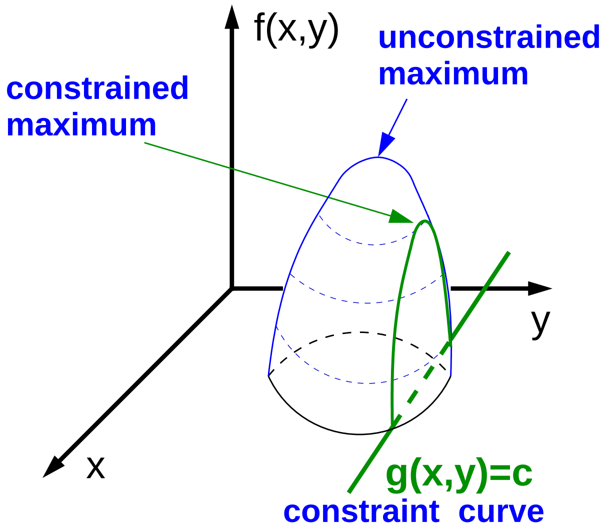

Figure 1: The first insight into Lagrange multipliers for constrained optimisation

$$F(x, y) = f(x, y) - \lambda[g(x, y) - c] \tag{1}$$

where λ is a new variable called the Lagrange multiplier. To find critical points of $F(x, y)$, compute the partial derivatives

$$F'_x = f'_x - \lambda g'_x \quad F'_y = f'_y - \lambda g'_y \quad F'_\lambda = -(g - c) \tag{2}$$

and solve for $F'_x = 0, F'_y = 0, F'_\lambda = 0$, or in other words

$$\begin{aligned} F'_x = f'_x - \lambda g'_x \quad &\rightarrow \quad f'_x = \lambda g'_x \\ F'_y = f'_y - \lambda g'_y \quad &\rightarrow \quad f'_y = \lambda g'_y \\ F'_\lambda = -(g - c) \quad &\rightarrow \quad g = c \end{aligned} \tag{3}$$

Finally, evaluate $f(x, y)$ at each solution of (3) to check if the obtained solution is a minimum or a maximum.

Example 1. The highway agency is planning to build a picnic are along a motorway. It has to be rectangular, with an area of 5000 square meters and it is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing that will be needed to complete the job.

Solution. This constrained optimisation problem is depicted in Figure 2.

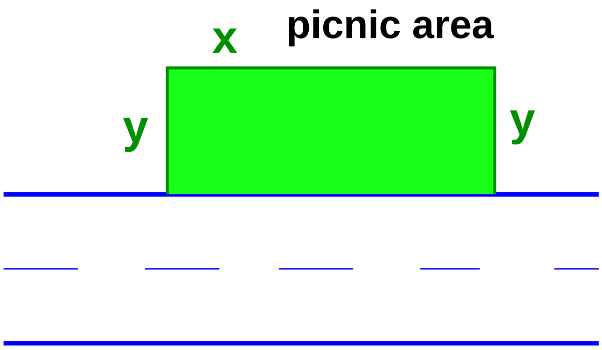

Figure 2: The problem of an area with minimum fencing required

Let $f(x, y)$ denote the amount of fencing required, and x and y the length of the sides of the rectangular picnic area. Then

$$f(x, y) = x + 2y \tag{4}$$

The goal is to minimise $f(x, y)$ subject to the constraint that the area must be 5000 square meters. This gives us the constraint

$$g(x, y) = xy = 5000 \tag{5}$$

Therefore, from (3), we have

$$\begin{aligned} F(x, y) &= f(x, y) - \lambda[g(x, y) - c] \\ &\text{or} \\ F(x, y) &= x + 2y - \lambda[xy - 5000] \\ & f'_x = 1, \quad f'_y = 2, \quad g'_x = y, \quad g'_y = x \end{aligned} \tag{6}$$

Therefore the three Lagrange equations are given by

$$1 = \lambda y \qquad 2 = \lambda x \qquad xy = 5000 \tag{7}$$

We therefore have $\lambda = 1/y$ and $\lambda = 2/x$. Since $x \neq 0$ and $y \neq 0$, we have

$$\frac{1}{y} = \frac{2}{x} \quad \rightarrow \quad x = 2y \tag{8}$$

We can now substitute $x = 2y$ into the third Lagrange equation ($xy = 5000$) to obtain

$$2y^2 = 5000 \quad \text{or} \quad y = \pm 50 \tag{9}$$

Now, use $y = 50$ and put into the equation $x = 2y$ to obtain $x = 100$. Therefore, the lengths $x = 100$, $y = 50$ minimise the function $f(x, y) = x + 2y$ subject to the constraint that the picnic area $g(x, y) = xy = 5000$. The optimal picnic area will therefore be 100 meters wide (along the motorway), will extend 50 meters back from the road, and will require $100 + 50 + 50 = 200$ meters of fencing.

Example 2. (interior design) An interior designer wishes to design a desk for an oval-shaped space of area $x^2 + 4y^2 = 4$ which will seat the maximum number of people. This can be formalised as: *Find the rectangle of maximal perimeter, inscribed in the ellipse* $x^2 + 4y^2 = 4$.

Solution. Our constraint is given by $g(x, y) = x^2 + 4y^2 = 4$, as depicted in Figure .

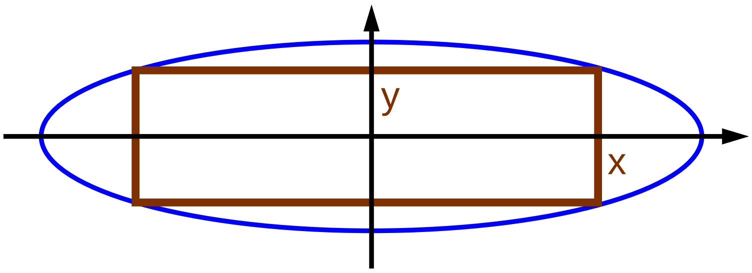

Figure 3: Interior design problem. The useful area of a room is in the shape of an ellipse and the interior designer wishes to choose a table which will seat a maximum number of people. Hence the perimeter of such a table has is be maximised.

The perimeter of the table $P(x, y) = 4x + 4y$, so that

$$F(x, y, \lambda) = 4x + 4y - \lambda(x^2 + 4y^2 - 4) \tag{10}$$

Therefore, $P'_x = \lambda g'_x, P'_y = \lambda g'_y$. In our case, this gives us $x = 4y$ (try yourselves).

Solve to give: $x = 4/\sqrt{5}, P = 4\sqrt{5}$. Therefore, the table which fits the oval area of 4 units and seats maximum number of people will have the sides $x = 4/\sqrt{5}$ and $y = 1/\sqrt{5}$.

Example 3. (management) A company allocates $£600,000$ to spend on advertising and research. Their estimate is that by spending x thousand pounds on advertising and y thousand pounds on research, they will sell a total of $f(x, y) = 30x^{4/5}y^{1/3}$ of their products. How much should the company spend on research and advertising to achieve this?

Solution. We will work in units of $£1000$. Then, the objective function to maximise the revenue is given by

$$f(x, y) = 30x^{4/5}y^{1/3} \tag{11}$$

and the constraint equation is

$$g(x, y) = x + y = 600 \qquad \text{that is their spending budget is} \quad £600,000 \tag{12}$$

The Lagrangian function $F(x, y, \lambda)$ therefore becomes

$$F(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - c] = 30x^{4/5}y^{1/3} - \lambda[x + y - 600) \tag{13}$$

The stationary points now become

$$\begin{array}{rclcl} F'_x & = & 24x^{-1/5}y^{1/3} - \lambda & = & 0 \\ F'_y & = & 10x^{4/5}y^{-2/3} - \lambda & = & 0 \\ F'_\lambda & = & 600 - x - y & = & 0 \end{array} \tag{14}$$

Solve for λ to obtain $\lambda = 24x^{-1/5}y^{1/3} = 10x^{4/5}y^{-2/3}$. Therefore

$$x = 2.4y \tag{15}$$

Substitute into the constraint (third equation in (14) to give $600 - 2.4y - y = 0$ or $y = 600/3.4 = 176.47$. Then $x = 423.53$ and $\lambda = 40.15$.

The maximum sales total is therefore $f(423.53, 176.47) = 21,257.83$ units of the product.

It is important to notice that if we change the advertising and research budget, we can use the Lagrange multiplier λ to estimate the resulting maximum sales total. For example, suppose that the budget is increased by 1% to $£606,000$. This is an increase by 6 in the constraint, since we are working in units of $£1000$. Therefore the maximum sales total will increase by about

$$6\lambda = 6 \times 40.15 = 204.90 \tag{16}$$

which is an increase of about 1.13% on the previous sales maximum.