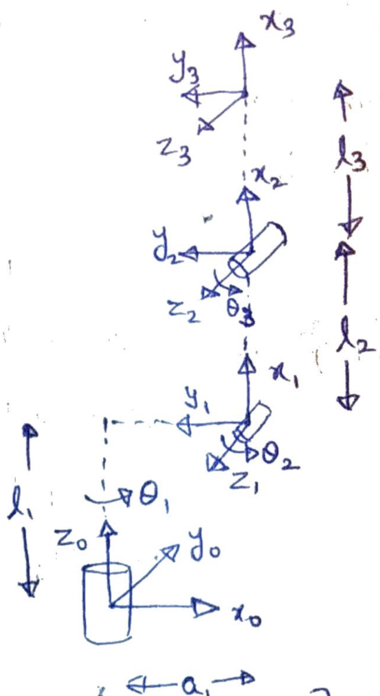
## Part - 1  Forward Kinematics

D-H parameters -

| | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| $J_1$ | $a_1$ | 90 | $l_1$ | $\theta_1$ |
| $J_2$ | $l_2$ | 0 | 0 | $90-\theta_2$ |
| $J_3$ | $l_3$ | 0 | 0 | $\theta_3$ |




$$H_i = \begin{bmatrix} C_{\theta_i} & -S_{\theta_i}C_{\alpha_i} & S_{\theta_i}S_{\alpha_i} & a_i C_{\theta_i} \\ S_{\theta_i} & C_{\theta_i}C_{\alpha_i} & -C_{\theta_i}S_{\alpha_i} & a_i S_{\theta_i} \\ 0 & S_{\alpha_i} & C_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_{\theta_i} \to \cos\theta_i$ $\quad S_{\theta_i} \to \sin\theta_i$

$$H_1 = \begin{bmatrix} C_{\theta_1} & 0 & S_{\theta_1} & a_1 C_{\theta_1} \\ S_{\theta_1} & 0 & -C_{\theta_1} & a_1 S_{\theta_1} \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_2 = \begin{bmatrix} C_{(90-\theta_2)} & -S_{(90-\theta_2)} & 0 & l_2 C_{(90-\theta_2)} \\ S_{(90-\theta_2)} & C_{(90-\theta_2)} & 0 & l_2 S_{(90-\theta_2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} C_{\theta_3} & -S_{\theta_3} & 0 & l_3 C_{\theta_3} \\ S_{\theta_3} & C_{\theta_3} & 0 & l_2 S_{\theta_3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1 \times H_2 = \begin{bmatrix} C_{\theta_1}S_{\theta_2} & -C_{\theta_1}C_{\theta_2} & S_{\theta_1} & a_1 C_{\theta_1} + l_2 C_{\theta_1} S_{\theta_2} \\ S_{\theta_1}S_{\theta_2} & -S_{\theta_1}C_{\theta_2} & -C_{\theta_1} & a_1 S_{\theta_1} + l_2 S_{\theta_1} S_{\theta_2} \\ S_{\theta_2} & C_{\theta_2} & 0 & l_1 + l_2 S_{\theta_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H = H_1 \times H_2 \times H_3 = \begin{bmatrix} C_{\theta_1} S_{(\theta_2-\theta_3)} & -C_{\theta_1} C_{(\theta_2-\theta_3)} & +S_{\theta_1} & C_{\theta_1}(a_1 + l_2 S_{\theta_2} + l_3 S_{(\theta_2-\theta_3)}) \\ S_{\theta_1} S_{(\theta_2-\theta_3)} & -S_{\theta_1} C_{(\theta_2-\theta_3)} & -C_{\theta_1} & S_{\theta_1}(a_1 + l_2 S_{\theta_2} + l_3 S_{(\theta_2-\theta_3)}) \\ S_{(\theta_2-\theta_3)} & +S_{(\theta_2-\theta_3)} & 0 & l_1 + l_2 C_{\theta_2} + l_3 C_{(\theta_2-\theta_3)} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Cartesian:

$x = C_{\theta_1}(a_1 + l_2 S_{\theta_2} + l_3 S_{(\theta_2-\theta_3)})$ $\qquad y = S_{\theta_1}(a_1 + l_2 S_{\theta_2} + l_3 S_{(\theta_2-\theta_3)})$

$z = l_1 + l_2 C_{\theta_2} + l_3 C_{(\theta_2-\theta_3)}$

# Part - 2 Inverse Kinematics

From part-1 $\Rightarrow$ $l_2 \sin\theta_2 + l_3 \sin(\theta_2 - \theta_3) = \frac{x}{\cos\theta_1} - a_1$ —— ①

$\boxed{\theta_1 = \tan^{-1}(y/x)}$

$l_2 \sin\theta_2 + l_3 \sin(\theta_2 - \theta_3) = \frac{y}{\sin\theta_3} - a_1$

$\cancel{l_1} + l_2 \cos\theta_2 + l_3 \cos(\theta_2 - \theta_3) = z - l_1$ —— ②

$①^2 + ②^2 \Rightarrow$ $l_2^2 \sin^2\theta_2 + l_3 \sin^2(\theta_2-\theta_3) + 2 l_2 l_3 \sin\theta_2 \sin(\theta_2-\theta_3) = \left(\frac{x}{\cos\theta_1} - a_1\right)^2$

$l_2^2 \cos^2\theta_2 + l_3 \cos^2(\theta_2-\theta_3) + 2 l_2 l_3 \cos\theta_2 \cos(\theta_2-\theta_3) = (z - l_1)^2$

$l_2^2 + l_3^2 + 2 l_2 l_3 \cos\theta_3 = \left(\frac{x}{\cos\theta_1} - a_1\right)^2 + (z - l_1)^2$

$\cos\theta_3 = \frac{\left(\frac{x}{\cos\theta_1} - a_1\right)^2 + (z-l_1)^2 - l_2^2 - l_3^2}{2 l_2 l_3}$

$\rightarrow$ $\boxed{\theta_3 = \cos^{-1}\left(\frac{\left(\frac{x}{\cos\theta_1} - a_1\right)^2 + (z-l_1)^2 - l_2^2 - l_3^2}{2 l_2 l_3}\right)}$

② $\Rightarrow$ $z - l_1 = l_2 \cos\theta_2 + l_3(\cos\theta_2 \cos\theta_3 + \sin\theta_2 \sin\theta_3)$

$z - l_1 - \cos\theta_2 (l_2 + l_3 \cos\theta_3) = l_3 \sin\theta_3 \sin\theta_2$

$\frac{z - l_1}{l_3 \sin\theta_3} - \cos\theta_2 \left(\frac{l_2 + l_3 \cos\theta_3}{l_3 \sin\theta_3}\right) = \sqrt{1 - \cos^2\theta_2}$

• sq on both sides

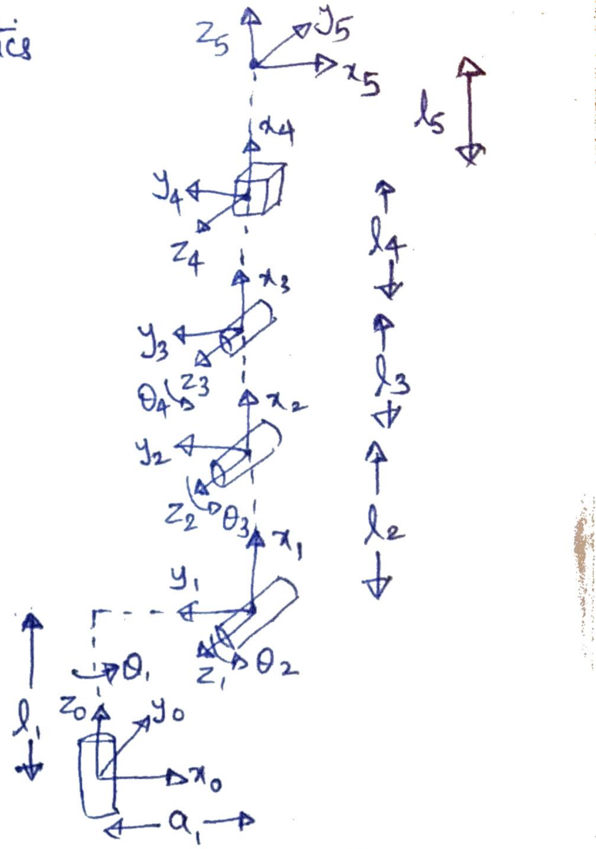
$\underbrace{\left(\frac{z - l_1}{l_3 \sin\theta_3}\right)^2}_{a} + \underbrace{\left(\frac{l_2 + l_3 \cos\theta_3}{l_3 \sin\theta_3}\right)^2}_{b} \cos^2\theta_2 - 2\cos\theta_2 \left(\frac{l_2 + l_3 \cos\theta_3}{l_3 \sin\theta_3}\right)\left(\frac{z - l_1}{l_3 \sin\theta_3}\right) = 1 - \cos^2\theta_2$

$\cos^2\theta_2 (b)^2 - 2\cos\theta_2 (a)(b) + a^2 = 1 - \cos^2\theta_2$

$\cos^2\theta_2 (b^2+1) - 2\cos\theta_2 (ab) + a^2 - 1 = 0$ $\rightarrow$ quadratic equation with 2 solutions

$\theta_2 = \cos^{-1}\left(\frac{2ab \pm \sqrt{4a^2b^2 - 4(b^2+1)(a^2-1)}}{2(b^2+1)}\right)$ $\Rightarrow$ $\boxed{\cos^{-1}\left(\frac{2ab \pm \sqrt{b^2 - a^2 + 1}}{2(b^2+1)}\right) = \theta_2}$

Part-3 Numerical Inverse Kinematics




D-H parameters –

| | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| $J_1$ | $a_1$ | 90 | $l_1$ | $\theta_1$ |
| $J_2$ | $l_2$ | 0 | 0 | $90-\theta_2$ |
| $J_3$ | $l_3$ | 0 | 0 | $\theta_3$ |
| $J_4$ | $l_4$ | 0 | 0 | $\theta_4$ |
| $J_5$ | 0 | -90 | $l_5$ | 0 |

~~Contro~~

$H_1, H_2, H_3$ same as part 1

$$H_4 = \begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & l_4\cos\theta_4 \\ \sin\theta_4 & \cos\theta_4 & 0 & -l_4\sin\theta_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$a_1 = 50 \quad l_1 = 400 \quad l_2 = 400$

$l_3 = 350 \quad l_4 = 50$

$$H_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & l_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After finding $H = H_1 \times H_2 \times H_3 \times H_4 \times H_5$

$$\Rightarrow x = \cos\theta_1\left(a_1 + l_2\sin\theta_2 + l_3\sin(\theta_2+\theta_3) - l_4\sin(\theta_2+\theta_3+\theta_4)\right) + l_5\sin\theta_1$$

$$y = \sin\theta_1\left(a_1 + l_2\sin\theta_2 + l_3\sin(\theta_2+\theta_3) - l_4\sin(\theta_2+\theta_3+\theta_4)\right) - l_5\cos\theta_1$$

$$z = l_1 + l_2\cos\theta_2 + l_3\cos(\theta_2+\theta_3) + l_4\cos(\theta_2+\theta_3+\theta_4)$$

$$\frac{y + l_5\cos\theta_1}{\sin\theta_1} = A \quad — ②$$

$$\frac{x - l_5\sin\theta_1}{\cos\theta_1} = A \quad ①$$

solving ① & ② $\Rightarrow$ $\boxed{\theta_1 = \sin^{-1}\left(\frac{x l_5 \pm y\sqrt{l_5^2 + x^2 - y^2}}{x^2 - y^2}\right)}$

We have 4 unknown variables and just 3 equations, hence, it is solved by newton–Raphson method by calculating psuedo-inverse of Jacobian $J^+ = (J^T \times J)^{-1} J^T$

We finally have $\Delta\theta = J^+ \times \Delta p$

$p = (x, y, z)^T$

$\theta = (\theta_1, \theta_2, \theta_3, \theta_4, l_5)$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} & \frac{\partial x}{\partial \theta_4} & \frac{\partial x}{\partial l_5} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} & \frac{\partial y}{\partial \theta_4} & \frac{\partial y}{\partial l_5} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \frac{\partial z}{\partial \theta_3} & \frac{\partial z}{\partial \theta_4} & \frac{\partial z}{\partial l_5} \end{bmatrix}$$

$\frac{\partial x}{\partial \theta_1} = -\sin\theta_1(----) + l_5\cos\theta_1$ ; $\frac{\partial x}{\partial \theta_2} = \cos\theta_1(l_2\cos\theta_2 + l_3\cos(\theta_2+\theta_3) - l_4\cos(\theta_2+\theta_3+\theta_4))$

$\frac{\partial x}{\partial \theta_3} = \cos\theta_1(l_3\cos(\theta_2+\theta_3) - l_4\cos(\theta_2+\theta_3+\theta_4))$ ; $\frac{\partial x}{\partial \theta_4} = -l_4\cos\theta_1\cos(\theta_2+\theta_3+\theta_4)$

$\frac{\partial x}{\partial l_5} = \sin\theta_1$ | $\frac{\partial y}{\partial \theta_1} = \cos\theta_1(----) + l_5\sin\theta_1$ ; $\frac{\partial y}{\partial \theta_2} = \sin\theta_1(l_2\cos\theta_2 + l_3\cos(\theta_2+\theta_3) - l_4\cos(\theta_2+\theta_3+\theta_4))$

$\frac{\partial y}{\partial \theta_3} = \sin\theta_1(l_3\cos(\theta_2+\theta_3) - l_4\cos(\theta_2+\theta_3+\theta_4))$ ; $\frac{\partial y}{\partial \theta_4} = -l_4\sin\theta_1\cos(\theta_2+\theta_3+\theta_4)$

$\frac{\partial y}{\partial l_5} = -\cos\theta_1$ | $\frac{\partial z}{\partial \theta_1} = 0$ ; $\frac{\partial z}{\partial \theta_2} = -l_2\sin\theta_2 - l_3\sin(\theta_2+\theta_3) - l_4\sin(\theta_2+\theta_3+\theta_4)$

$\frac{\partial z}{\partial \theta_3} = -l_3\sin(\theta_2+\theta_3) - l_4\sin(\theta_2+\theta_3+\theta_4)$ ; $\frac{\partial z}{\partial \theta_4} = -l_4\sin(\theta_2+\theta_3+\theta_4$

$\frac{\partial z}{\partial l_5} = 0$

Steps to find the solution :—

1) Give an initial guess for $\theta(\theta_1, \theta_2, \theta_3, \theta_4, l_5)$.
2) Calculate $p(x, y, z)$ for $\theta$.
3) Get final $p'$ from the user.
4) Calculate error $e = p' - p$
5) Calculate Jacobian $J$
6) Get the pseudo-inverse of $J \to J^+$
7) Find new $\theta'$ which is $\theta + J^+ \times e$
8) Repeat 1-7 until $e \leq 0.01$