Lemma 0.1. *The set of morphisms* $S = \{\emptyset \longrightarrow \{0\},\quad \emptyset \longrightarrow \{0\}^*,\quad \{0,1\}_{disc} \longrightarrow \{0 \to 1\},\quad \{0 \rightrightarrows 1\} \longrightarrow \{0 \to 1\}\}$, *where* $\{0\}^*$ *has 1 object, which is marked, is a set of cofibrations in* Cat^m.

Proof. Case 1. Let's take the first morphism, which adds an unmarked object, then we have a diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \{0\} \arrow[r, "g"'] \arrow[ur, dashed, "f'"] & Y \end{tikzcd}$$

p is essentially surjective and an isofibration, thus for every $a \in X$ there exists $y \in Y$ such that $y = g(0)$ and $p(a) \cong y$. Then there exists $h : y \to p(a)$, wich gives a lift $\phi : a' \to a$, then $p(a') = y$. For $f'(0) = a'$ we have $p(f'(0)) = p(a') = y = g(0)$.

Case 2. In the second case, the morphism adds a marked object and we have a diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \{0\}^* \arrow[r, "g"'] \arrow[ur, dashed, "f'"] & Y \end{tikzcd}$$

Let $y \in \epsilon_Y$ and $y = g(0)$. As a weak equivalence p preserves markings and is surjective on marked objects. For every $a \in \epsilon_X$ there exists $y \in Y$ such that $y = g(0)$ and $p(a) = y$, then for $f'(0) = a$ we have $p \circ f' = g$.

Case 3. In the third case, the arrow is added to the discrete category $\{0,1\}_{disc}$, so we have a diagram:

$$\begin{tikzcd} \{0,1\}_{disc} \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \{0 \to 1\} \arrow[r, "g"'] \arrow[ur, dashed, "f'"] & Y \end{tikzcd}$$

Take $a = f(0), \quad b = f(1)$, on a morphism we will have: $g(0 \to 1) \mapsto p(a) \to p(b)$. p is fully faithful as a weak equivalence, that is why we have the bijection $p\colon \mathrm{Hom}_X(a,b) \ \cong \mathrm{Hom}_Y\big(p(a), p(b)\big)$. Let us take a morphism $k \in \mathrm{Hom}_X(a,b)$, then $p(k) = k'$. Then define $f'(0 \to 1) = k$ and $f'(0) = a$, $f'(1) = b$. Thus f' extends f along $\{0,1\}_{disc} \longrightarrow \{0 \to 1\}$.

Case 4. In the fourth case, we have a morphism, which imposes a relation between two arrows, so we have a diagram:

$$\begin{tikzcd} \{0 \rightrightarrows 1\} \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \{0 \to 1\} \arrow[r, "g"'] \arrow[ur, dashed, "f'"] & Y \end{tikzcd}$$

Take again $a = f(0), \quad b = f(1)$, and let $k, k' : 0 \to 1$ in $\{0 \rightrightarrows 1\}$. Let $\beta = f(k), \beta' = f(k)$ in $\mathrm{Hom}_X(a,b)$.

$$g(k') = p(\beta') = p(\beta) = g(k) = t$$

We again have the bijection $p\colon \mathrm{Hom}_X(a,b) \ \cong \mathrm{Hom}_Y\big(p(a), p(b)\big)$ because p is fully faithful, so $\beta' = \beta$. Define $f'(0) = a$, $f'(1) = b$ and $f'(0 \to 1) = \beta$. Thus f' extends f along $\{0 \rightrightarrows 1\} \longrightarrow \{0 \to 1\}$.

□

Theorem 0.2. *Theorem 1.12 (Small Object Argument). Let κ be an infinite cardinal number and I a set of morphisms of a cocomplete category C . If the domains of all morphisms of I are κ-small with respect to I-attachments, then there is a weak factorization system (L, R) where L is the class of retracts of relative I-cell complexes and $R = I^{\square_r}$.*

Lemma 0.3. *Let $f\colon (A, \varepsilon_A) \to (B, \varepsilon_B)$ be a cofibration. Then f is injective on objects and reflects markings.*

Proof. From the small-object-argument we get the factorisation:

$$(A, \varepsilon_A) \xrightarrow{j} (C, \varepsilon_C) \xrightarrow{k} (B, \varepsilon_B)$$

where $j \in L_S,\ k \in R_S$. Then we have the lift (because f has the LLP against k):

$$\begin{tikzcd}
(A, \varepsilon_A) \arrow[r, "j"] \arrow[d, "f"'] & (C, \varepsilon_C) \arrow[d, "k"] \\
(B, \varepsilon_B) \arrow[r, "id_B"'] \arrow[ur, dashed, "d"] & (B, \varepsilon_B)
\end{tikzcd}$$

where $k \circ d = id_B$ and $d \circ f = j$. Let $a_1, a_2 \in Ob(A)$ and $f\,(a_1) = f\,(a_2)$, then: $j\,(a_1) = \delta f\,(a_1) = \delta f\,(a_2) = j\,(a_2)$. As a cofibration, j can be rewritten as the colimit of a transfinite sequence (composite) of the generators (maps from the set S), which are injective on objects and reflect markings, so j is injective on objects. For $a_1 = a_2$, we get the injectivity of f.
Let $a \in ObA$ and assume that $f(a) \in \epsilon_B$. We have $\delta f\,(a) = j\,(a)$, where j reflects markings, so $a \in \epsilon_A$, while δ preserves markings, thus f reflects markings.

□

Definition 0.4. (https://ncatlab.org/nlab/show/cocomplete+category)
A category C is cocomplete if it has all small colimits: that is, if every small diagram $F : D \to C$ where D is a small category has a colimit in C. Equivalently, a category C is cocomplete if it has all small wide pushouts and an initial object.

Definition 0.5. (https://ncatlab.org/nlab/show/small+object)
An object X of a category is small if it is κ-compact for some regular cardinal κ (and therefore also for all greater regular cardinals).

Definition 0.6. (https://ncatlab.org/nlab/show/compact+object) Let C be a locally small category that admits filtered colimits. Then an object $X \in C$ is compact, or finitely presented or finitely presentable, or of finite presentation, if the corepresentable functor

$$\mathrm{Hom}_C(X, -) : C \to \mathrm{Set}$$

preserves these filtered colimits. This means that for every filtered category D and every functor $F : D \to C$, the canonical morphism

$$\xrightarrow[d]{\lim_d C} (X, F(d)) \overset{\approx}{\rightrightarrows} C(X, \lim_d F(d))$$

is an isomorphism. More generally, if K is a regular cardinal, then an object X such that $C(X, -)$ commutes with K-filtered colimits is called K-compact, or K-presented, or K-presentable. An object which is K-compact for some regular K is called a small object.

Lemma 0.7. *Let $f : (A, \varepsilon_A) \longrightarrow (B, \varepsilon_B)$ be injective on objects and reflect marking, then f is a cofibration in Cat^m*

Proof. Cat^m inherits cocompleteness from Cat, because the structure of markings does not impact the existence of colimits. Let L_S be a saturation of $S = \{\emptyset \longrightarrow \{0\}, \emptyset \longrightarrow \{0\}^*, \{0,1\}_{disc} \longrightarrow \{0 \to 1\}, \{0 \rightrightarrows 1\} \longrightarrow \{0 \to 1\}\}$.

The number of objects and morphisms is finite in every domain of every morphism in S, so they are $\aleph_0$-small in Cat^m. $\aleph_0$ is a regular cardinal.

Then, from the small-object argument, we can construct a sequence

$$(A_0, \varepsilon_0) \xrightarrow{f_0} (A_1, \varepsilon_1) \xrightarrow{f_1} \cdots \xrightarrow{f_\alpha} (A_{\alpha+1}, \varepsilon_{\alpha+1}) \cdots$$

for $(\alpha < \lambda)$.

Let $f'_\alpha : (A_\alpha, \varepsilon_\alpha) \to (B, \epsilon_B)$ for every $(\alpha < \lambda)$, where $\lambda \leq \left|Mor(B)\right|^+$.

Inductively add 1 to every $\alpha < \lambda$, so that for $\alpha + 1$ we get the next "options":
For each unmarked object such that $b \in Ob(B)$, $b \notin Ob(A)$ we construct a pushout:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A_\alpha \arrow[d, "j_b"] \\ \{0\} \arrow[r] & A_\alpha \cup \{0\} \end{tikzcd}$$

so in this case we have $A_{\alpha+1} = A_\alpha \cup \{0\}$.
The same way we add a marked object such that $b' \in \epsilon_B$, $b' \notin Ob(A)$:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A_\alpha \arrow[d, "j_{b'}"] \\ \{0\}^* \arrow[r] & A_\alpha \cup \{0\}^* \end{tikzcd}$$

so $A_{\alpha+1} = A_\alpha \cup \{0\}^*$ and $0 \in \epsilon_{A_{\alpha+1}}$.

To add a missing morphism g, such that for each pair $a, a' \in Ob(A_\alpha)$, $g : f'_\alpha(a) \to f'_\alpha(a')$, where $g \notin \mathrm{Hom}_B(f'_\alpha(a), f'_\alpha(a'))$, we need to use the third morpism:

$$\begin{tikzcd} \{0,1\}_{disc} \arrow[r, "t"] \arrow[d] & A_\alpha \arrow[d, "j_h"] \\ \{0 \to 1\} \arrow[r, "t'"] & A_\alpha \cup \{0 \to 1\} \end{tikzcd}$$

then $t(0) = a$, $t(1) = a'$, $t'(0) = a$, $t'(1) = a'$, $t'(0 \to 1) = g$, so for $g' : a \to a'$ we get $f'_{\alpha+1}(g') = g$, thus $g \in \mathrm{Hom}_B(f'_{\alpha+1}(a), f'_{\alpha+1}(a'))$.

For each pair of arrows such that, $\beta, \beta' : c \to c'$, $\beta, \beta' \in A_\alpha$ and $f'_\alpha(\beta) = f'_\alpha(\beta')$ we construct a pushout:

$$\begin{tikzcd} \{0 \rightrightarrows 1\} \arrow[r] \arrow[d] & A_\alpha \arrow[d, "j_{\beta,\beta'}"] \\ \{0 \to 1\} \arrow[r] & A_\alpha \cup (\beta \sim \beta') \end{tikzcd}$$

β and β' become equal in $A_{\alpha+1}$.

At stage $\alpha + 1$ we attach every morphism whose domain fits inside A_α. The colimit defining that attachment exists because Cat^m is cocomplete. The resulting map $f_{\alpha+1} : A_{\alpha+1} \to B$ is injective on objects and reflects markings, because only a set of pushouts is taken at step α, and because the sets $Ob(B)$ and $Mor(B)$ are themselves sets. In a cocomplete category, taking pushouts one after another is equivalent to taking them all at once. Hence $(A_{\alpha+1}, \varepsilon_{\alpha+1})$ exists.

At the limit stage, for $\beta \in LIM$ we have a colimit, which is defined by the unions: $\mathrm{Ob}\, A_\beta = \bigcup_{\alpha<\beta} \mathrm{Ob}\, A_\alpha$, $\mathrm{Mor}\, A_\beta = \bigcup_{\alpha<\beta} \mathrm{Mor}\, A_\alpha$, $\varepsilon_\beta = \bigcup_{\alpha<\beta} \varepsilon_\alpha$, $(A_\beta, \varepsilon_\beta) = \mathrm{colim}_{\alpha<\beta} (A_\alpha, \varepsilon_\alpha)$.

Lemma 0.8. *Let* $H = \{j_0 : (\emptyset, \emptyset) \longrightarrow (1, \{0\}), j_1 : (\{0,1\}_{disc}, \{0\}) \longrightarrow ([1], \{0\}),$ $j_2 : (\{0 \rightrightarrows 1\}, \{0\}) \longrightarrow ([1], \{0\}), j_3 : (\{0 \xrightarrow{\cong} 1\}, \{0\}) \longrightarrow (\{0 \xrightarrow{\cong} 1\}, \{0,1\})\}$

For a marked category (C, I) *the existance of the lifts:*

$$\begin{tikzcd}
(\emptyset,\emptyset) \arrow[rr] \arrow[dr, "j_0"'] & & (\mathcal{C},\mathcal{I}) \\
& (\{0\},\{0\}) \arrow[ur, dashed] &
\end{tikzcd}$$

$$\begin{tikzcd}
(\{0,1\},\{0\}) \arrow[rr] \arrow[dr, "j_1"'] & & (\mathcal{C},\mathcal{I}) \\
& ([1],\{0\}) \arrow[ur, dashed] &
\end{tikzcd}$$

$$\begin{tikzcd}
(0 \rightrightarrows 1,\{0\}) \arrow[rr] \arrow[dr, "j_2"'] & & (\mathcal{C},\mathcal{I}) \\
& ([1],\{0\}) \arrow[ur, dashed] &
\end{tikzcd}$$

$$\begin{tikzcd}
(0 \xrightarrow{\cong} 1,\{0\}) \arrow[rr] \arrow[dr, "j_3"'] & & (\mathcal{C},\mathcal{I}) \\
& (0 \xrightarrow{\cong} 1,\{0,1\}) \arrow[ur, dashed] &
\end{tikzcd}$$

is equivalent to these two conditions:

1. *C has at least one marked object, and*
2. *an object of* C *is marked iff it is initial.*

Proof. $1 \Rightarrow 2$
Case 1. We have a diagram (with a lift):

$$\begin{tikzcd}
(\emptyset,\emptyset) \arrow[r, "f"] \arrow[d, "j_0"'] & (\mathcal{C},\mathcal{I}) \\
(1,\{0\}) \arrow[ur, dashed, "f'"] &
\end{tikzcd}$$

The lifting property guarantees the existence of f', which preserves the markings, thus the image of 0 in C is marked.

Case 2. Choose some $i \in I$. We have a lift:

$$\begin{tikzcd}
(\{0,1\}_{\text{disc}},\{0\}) \arrow[r, "f"] \arrow[d, "j_1"'] & (\mathcal{C},\mathcal{I}) \\
([1],\{0\}) \arrow[ur, dashed, "f'"] &
\end{tikzcd}$$

where $f(0) = i$, $f(1) = c \in C$, $f'(0) = i$, $f'(1) = c$ and we also have $f'(0 \to 1) = \beta : i \to c$ for some β. Then the unique morphism from the marked object exists.

Case 3. Let $\beta, \beta' : 0 \to 1$.

$$\begin{tikzcd}
(0 \rightrightarrows 1,\{0\}) \arrow[r, "f"] \arrow[d, "j_2"'] & (\mathcal{C},\mathcal{I}) \\
([1],\{0\}) \arrow[ur, dashed, "f'"] &
\end{tikzcd}$$

Then let $f(0) := i, f(1) := c$ and $f(\beta) = \beta, f(\beta') = \beta'$. The lift imposes a relation on these two morphisms, hence $\beta = \beta'$, and the arrow from the image of the marked object is unique.

Case 4. Let $c \in C$ be an arbitrary initial object.

$$\begin{tikzcd}
((0 \overset{\cong}{\rightrightarrows} 1, \{0\}) \arrow[r, "f"] \arrow[d, "j_3"'] & (\mathcal{C}, \mathcal{I}) \\
(0 \overset{\cong}{\rightrightarrows} 1, \{0,1\}) \arrow[ur, dashed, "f'"] &
\end{tikzcd}$$

Let $f(0) = i$, $f(1) = c$, $f'(0) = i$, $f'(0 \overset{\cong}{\to} 1) = \beta$. By the lifting property, f' sends the marked object 1 to a marked object of C, by the lifting property, it must be c. Therefore, every initial object c is marked.

$2 \Rightarrow 1$ Case 1. Let i be the initial and marked in C, then

$$\begin{tikzcd}
(\emptyset, \emptyset) \arrow[r, "f"] \arrow[d, "j_0"'] & (\mathcal{C}, \mathcal{I}) \\
(1, \{0\}) \arrow[ur, dashed, "f'"] &
\end{tikzcd}$$

define $f'(0) = i$. Then f' sends marked objects to merked objects, thus it is a morphism in Cat^m.

Case 2. Let $f(0) = i$, where i is initial. Define $f'(0) = f(0)$, $f'(1) = f(1)$, $f'(0 \to 1) = f(0) \to f(1)$. Then there is a commutative diagram:

$$\begin{tikzcd}
(\{0,1\}_{\text{disc}}, \{0\}) \arrow[r, "f"] \arrow[d, "j_1"'] & (\mathcal{C}, \mathcal{I}) \\
([1], \{0\}) \arrow[ur, dashed, "f'"] &
\end{tikzcd}$$

which preserves markings. Case 3. Let $f(0) = i$, where i is initial. Let $\beta, \beta' : f(0) \to f(1)$. Define $f'(0) = f(0)$, $f'(1) = f(1)$, because i is initial we get $\beta = \beta'$, thus $f'(\beta) = f'(\beta')$. Then the diagram is commutative: Let $\beta, \beta' : 0 \to 1$.

$$\begin{tikzcd}
(0 \rightrightarrows 1, \{0\}) \arrow[r, "f"] \arrow[d, "j_2"'] & (\mathcal{C}, \mathcal{I}) \\
([1], \{0\}) \arrow[ur, dashed, "f'"] &
\end{tikzcd}$$

Case 4. Let $c \in C$ be isomorphic to an arbitrary initial object, then c is marked and initial. Let $f(0) = i$, $f(1) = c$, $f(0 \overset{\cong}{\to} 1) = \beta = i \overset{\cong}{\to} c$.

$$\begin{tikzcd}
((0 \overset{\cong}{\rightrightarrows} 1, \{0\}) \arrow[r, "f"] \arrow[d, "j_3"'] & (\mathcal{C}, \mathcal{I}) \\
(0 \overset{\cong}{\rightrightarrows} 1, \{0,1\}) \arrow[ur, dashed, "f'"] &
\end{tikzcd}$$

Define $f'(0) = i$, $f'(1) = c$, $f'(0) = i$, $f'(0 \overset{\cong}{\to} 1) = \beta$. Because both objects in the codomain are marked, f' preserves markings, thus is a morphism in Cat^m, the diagram commutes.

$\square$

Lemma 0.9. *Every morphism $h \in L_S$ is injective on objects and reflects markings.*

Proof. L_S is the "smallest" set containing S, which is also closed under pushouts, transfinite compositions (sequences), and retracts.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "t"] \\ B \arrow[r, "u"] & Y \end{tikzcd}$$

Assume that $g \in L_S$ is injective on objects and reflects markings.

1. In Cat, pushouts on objects are produced as a quotient of disjoint unions:

$$Ob(Y) = Ob(B) \sqcup Ob(X)/\sim$$

where the equivalence is defined as $g(a) \sim f(a) \quad$ for all $a \in Ob(A)$. For every $b, b' \in Ob(B)$, for which $u(b) = u(b')$, there is an object $g(a) \in Ob(X)$, where g is injective on objects. Then $b = b'$ and u is injective on objects. Assume $y \in \epsilon_Y$, then, there is some $b'' \in \epsilon_B$ such that $u(b'') = y$ or there is some $a' \in \epsilon_A$ such that $g(a') = x \in \epsilon_X$, such that $y = t(x)$, because t preserves markings and g reflects markings. Thus $b'' = f(a') \in \epsilon_B$ and u reflects markings.

2. Retracts of injective functions are injective: Let $h : X \to A$ and $h \circ g$. Let $x_1, x_2 \in Ob(X)$.
$g(h(x_1)) = g(h(x_2))$ then $x_1 = (g \circ h)(x_1) = (g \circ h)(x_2) = x_2$
Therefore, h is injective on objects. For $x\epsilon_X$, then $h(x)$ is marked, because h preserves markings and $g(h(x))$ is marked, because g reflects markings. Then h also reflects markings.

3. Transfinite compositions

$$(A_0, \varepsilon_0) \xrightarrow{f_0} (A_1, \varepsilon_1) \xrightarrow{f_1} \cdots \xrightarrow{f_\alpha} (A_{\alpha+1}, \varepsilon_{\alpha+1}) \cdots$$

for $\qquad (\alpha < \lambda)$.