Lemma 0.1. The set of morphisms $S = \{\emptyset \longrightarrow \{0\}, \emptyset \longrightarrow \{0\}^*, \{0,1\}_{disc} \longrightarrow \{0 \rightarrow 1\}, \{0 \rightarrow 1\}\} \}$, where $\{0\}^*$ has 1 object, which is marked, is a set of cofibrations in Cat^m .

Proof. Case 1. Let's take the first morphism, which adds an unmarked object, then we have a diagram:

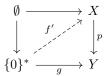
$$\emptyset \xrightarrow{f'} X$$

$$\downarrow f' \qquad \downarrow p$$

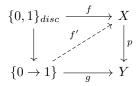
$$\{0\} \xrightarrow{g} Y$$

p is essentially surjective and an isofibration, thus for every $a \in X$ there exists $y \in Y$ such that y = g(0) and $p(a) \cong y$. Then there exists $h: y \to p(a)$, wich gives a lift $\phi: a' \to a$, then p(a') = y. For f'(0) = a' we have p(f'(0)) = p(a') = y = g(0).

Case 2. In the second case, the morphism adds a marked object and we have a diagram:



Let $y \in \epsilon_Y$ and y = g(0). As a weak equivalence p preserves markings and is surjective on marked objects. For every $a \in \epsilon_X$ there exists $y \in Y$ such that y = g(0) and p(a) = y, then for f'(0) = a we have $p \circ f' = g$. Case 3. In the third case, the arrow is added to the discrete category $\{0, 1\}_{disc}$, so we have a diagram:



Take a = f(0), b = f(1), on a morphism we will have: $g(0 \to 1) \mapsto p(a) \to p(b)$. p is fully faithful as a weak equivalence, that is why we have the bijection p: $\operatorname{Hom}_X(a,b) \cong \operatorname{Hom}_Y(p(a),p(b))$. Let us take a morphism $k \in \operatorname{Hom}_X(a,b)$, then p(k) = k'. Then define $f'(0 \to 1) = k$ and f'(0) = a, f'(1) = b. Thus f' extends f along $\{0,1\}_{disc} \longrightarrow \{0 \to 1\}$.

Case 4. In the fourth case, we have a morphism, which imposes a relation between two arrows, so we have a diagram:

$$\begin{cases}
0 \Rightarrow 1 \end{cases} \xrightarrow{f} X \\
\downarrow \qquad \qquad \downarrow^{p'} \downarrow^{p} \\
\{0 \Rightarrow 1 \} \xrightarrow{g} Y$$

Take again a = f(0), b = f(1), and let $k, k' : 0 \to 1$ in $\{0 \Rightarrow 1\}$. Let $\beta = f(k), \beta' = f(k)$ in $Hom_X(a, b)$.

$$g(k') = p(\beta') = p(\beta) = g(k) = t$$

We again have the bijection p: $\operatorname{Hom}_X(a,b) \cong \operatorname{Hom}_Y(p(a),p(b))$ because p is fully faithful, so $\beta' = \beta$. Define f'(0) = a, f'(1) = b and $f'(0 \to 1) = \beta$. Thus f' extends f along $\{0 \to 1\} \longrightarrow \{0 \to 1\}$.

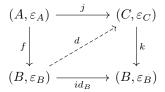
Theorem 0.2. Theorem 1.12 (Small Object Argument). Let κ be an infinite cardinal number and I a set of morphisms of a cocomplete category C. If the domains of all morphisms of I are κ -small with respect to I-attachments, then there is a weak factorization system (L,R) where L is the class of retracts of relative I-cell complexes and $R = I^{\square_r}$.

Lemma 0.3. Let $f:(A,\varepsilon_A)\to (B,\varepsilon_B)$ be a cofibration. Then f is injective on objects and reflects markings.

Proof. From the small-object-argument we get the factorisation:

$$(A, \varepsilon_A) \xrightarrow{j} (C, \varepsilon_C) \xrightarrow{k} (B, \varepsilon_B)$$

where $j \in L_S$, $k \in R_S$. Then we have the lift (because f has the LLP against k):



where $k \circ d = id_B$ and $d \circ f = j$. Let $a_1, a_2 \in Ob(A)$ and $f(a_1) = f(a_2)$, then: $j(a_1) = \delta f(a_1) = \delta f(a_2) = j(a_2)$. As a cofibration, j can be rewritten as the colimit of a transfinite sequence (composite) of the generators (maps from the set S), which are injective on objects and reflect markings, so j is injective on objects. For $a_1 = a_2$, we get the injectivity of f.

Let $a \in ObA$ and assume that $f(a) \in \epsilon_B$. We have $\delta f(a) = j(a)$, where j reflects markings, so $a \in \epsilon_A$, while δ preserves markings, thus f reflects markings.

Definition 0.4. (https://ncatlab.org/nlab/show/cocomplete+category)

A category C is cocomplete if it has all small colimits: that is, if every small diagram $F: D \to C$ where D is a small category has a colimit in C. Equivalently, a category C is cocomplete if it has all small wide pushouts and an initial object.

Definition 0.5. (https://ncatlab.org/nlab/show/small+object)

An object X of a category is small if it is κ -compact for some regular cardinal κ (and therefore also for all greater regular cardinals).

Definition 0.6. (https://ncatlab.org/nlab/show/compact+object) Let C be a locally small category that admits filtered colimits. Then an object $X \in C$ is compact, or finitely presented or finitely presentable, or of finite presentation, if the corepresentable functor

$$\operatorname{Hom}_C(X,-):C\to\operatorname{Set}$$

preserves these filtered colimits. This means that for every filtered category D and every functor $F: D \to C$, the canonical morphism

$$\xrightarrow{\lim_{d} C} (X, F(d)) \stackrel{\approx}{\rightrightarrows} C(X, \lim_{d} F(d))$$

is an isomorphism. More generally, if K is a regular cardinal, then an object X such that C(X, -) commutes with K-filtered colimits is called K-compact, or K-presented, or K-presentable. An object which is K-compact for some regular K is called a small object.

Lemma 0.7. Let $f:(A, \varepsilon_A) \longrightarrow (B, \varepsilon_B)$ be injective on objects and reflect marking, then f is a cofibration in Cat^m

Proof. Cat^m inherits cocompleteness from Cat, because the structure of markings does not impact the existence of colimits. Let L_S be a saturation of $S = \{\emptyset \longrightarrow \{0\}, \emptyset \longrightarrow \{0\}^*, \{0, 1\}_{disc} \longrightarrow \{0 \rightarrow 1\}, \{0 \Rightarrow 1\} \longrightarrow \{0 \rightarrow 1\}\}.$

The number of objects and morphisms is finite in every domain of every morphism in S, so they are \aleph_0 -small in Cat^m . \aleph_0 is a regular cardinal.

Then, from the small-object argument, we can construct a sequence

$$(A_0, \varepsilon_0) \xrightarrow{f_0} (A_1, \varepsilon_1) \xrightarrow{f_1} \cdots \xrightarrow{f_{\alpha}} (A_{\alpha+1}, \varepsilon_{\alpha+1}) \cdots$$

for $(\alpha < \lambda)$.

Let
$$f'_{\alpha}: (A_{\alpha}, \varepsilon_{\alpha}) \to (B, \epsilon_B)$$
 for every $(\alpha < \lambda)$, where $\lambda \leq |Mor(B)|^+$.

Inductively add 1 to every $\alpha < \lambda$, so that for $\alpha + 1$ we get the next "options": For each unmarked object such that $b \in Ob(B)$, $b \notin Ob(A)$ we construct a pushout:

$$\emptyset \longrightarrow A_{\alpha}$$

$$\downarrow j_{b}$$

$$\{0\} \longrightarrow A_{\alpha} \cup \{0\}$$

so in this case we have $A_{\alpha+1} = A_{\alpha} \cup \{0\}$.

The same way we add a marked object such that $b' \in \epsilon_B$, $b' \notin Ob(A)$:

$$\emptyset \longrightarrow A_{\alpha}$$

$$\downarrow j_{b'}$$

$$\{0\}^* \longrightarrow A_{\alpha} \cup \{0\}^*$$

so $A_{\alpha+1} = A_{\alpha} \cup \{0\}^*$ and $0 \in \epsilon_{A_{\alpha+1}}$.

To add a missing morphism g, such that for each pair $a, a' \in Ob(A_{\alpha}), g : f'_{\alpha}(a) \to f'_{\alpha}(a')$, where $g \notin \operatorname{Hom}_B(f'_{\alpha}(a), f'_{\alpha}(a'))$, we need to use the third morpism:

$$\begin{cases} \{0,1\}_{disc} & \xrightarrow{\quad t \quad} A_{\alpha} \\ \downarrow & \downarrow_{j_h} \\ \{0 \! \to \! 1\} & \xrightarrow{\quad t' \quad} A_{\alpha} \! \cup \! \{0 \! \to \! 1\}$$

then t(0) = a, t(1) = a', t'(0) = a, t'(1) = a', $t'(0 \to 1) = g$, so for $g' : a \to a'$ we get $f'_{\alpha+1}(g') = g$, thus $g \in \text{Hom}_B(f'_{\alpha+1}(a), f'_{\alpha+1}(a'))$.

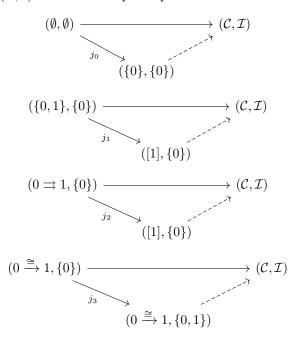
For each pair of arrows such that, $\beta, \beta': c \to c', \beta, \beta' \in A_{\alpha}$ and $f'_{\alpha}(\beta) = f'_{\alpha}(\beta')$ we construct a pushout:

 β and β' become equal in $A_{\alpha+1}$.

At stage $\alpha+1$ we attach every morphism whose domain fits inside A_{α} . The colimit defining that attachment exists because Cat^m is cocomplete. The resulting map $f_{\alpha+1}:A_{\alpha+1}\to B$ is injective on objects and reflects markings, because only a set of pushouts is taken at step α , and because the sets Ob(B) and Mor(B) are themselves sets. In a cocomplete category, taking pushouts one after another is equivalent to taking them all at once. Hence $(A_{\alpha+1}, \varepsilon_{\alpha+1})$ exists.

At the limit stage, for $\beta \in LIM$ we have a colimit, which is defined by the unions: Ob $A_{\beta} = \bigcup_{\alpha < \beta} \operatorname{Ob} A_{\alpha}$, Mor $A_{\beta} = \bigcup_{\alpha < \beta} \operatorname{Mor} A_{\alpha}$, $\varepsilon_{\beta} = \bigcup_{\alpha < \beta} \varepsilon_{\alpha}$, $(A_{\beta}, \varepsilon_{\beta}) = \operatorname{colim}_{\alpha < \beta} (A_{\alpha}, \varepsilon_{\alpha})$.

Lemma 0.8. Let $H = \{j_0 : (\emptyset, \emptyset) \longrightarrow (1, \{0\}), j_1 : (\{0, 1\}_{disc}, \{0\}) \longrightarrow ([1], \{0\}), j_2 : (\{0 \Rightarrow 1\}, \{0\}) \longrightarrow ([1], \{0\}), j_3 : (\{0 \xrightarrow{\cong} 1\}, \{0\}) \longrightarrow (\{0 \xrightarrow{\cong} 1\}, \{0, 1\})\}$ For a marked category (C, I) the existance of the lifts:

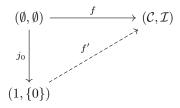


is equivalent to these two conditions:

- 1. C has at least one marked object, and
- 2. an object of C is marked iff it is initial.

Proof. $1 \Rightarrow 2$

Case 1. We have a diagram (with a lift):



The lifting property guarantees the existence of f', which preserves the markings, thus the image of 0 in C is marked.

Case 2. Choose some $i \in I$. We have a lift:

$$(\{0,1\}_{\mathrm{disc}},\{0\}) \xrightarrow{f} (\mathcal{C},\mathcal{I})$$

$$\downarrow^{j_1} \qquad \qquad f'$$

$$([1],\{0\})$$

where f(0) = i, $f(1) = c \in C$, f'(0) = i, f'(1) = c and we also have $f'(0 \to 1) = \beta : i \to c$ for some β . Then the unique morphism from the marked object exists.

Case 3. Let $\beta, \beta' : 0 \to 1$.

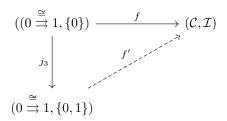
$$(0 \rightrightarrows 1, \{0\}) \xrightarrow{f} (\mathcal{C}, \mathcal{I})$$

$$\downarrow^{j_2} \qquad \qquad f'$$

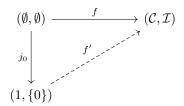
$$([1], \{0\})$$

Then let f(0) := i, f(1) := c and $f(\beta) = \beta$, $f(\beta') = \beta'$. The lift imposes a relation on these two morphisms, hence $\beta = \beta'$, and the arrow from the image of the marked object is unique.

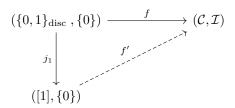
Case 4. Let $c \in C$ be an arbitrary initial object.



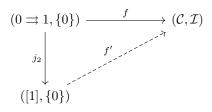
Let f(0) = i, f(1) = c, f'(0) = i, $f'(0 \stackrel{\cong}{\to} 1) = \beta$. By the lifting property, f' sends the marked object 1 to a marked object of C, by the lifting property, it must be c. Therefore, every initial object c is marked. $2 \Rightarrow 1$ Case 1. Let i be the initial and marked in C, then



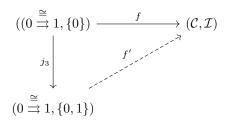
define f'(0) = i. Then f' sends marked objects to merked objects, thus it is a morphism in Cat^m . Case 2. Let f(0) = i, where i is initial. Define f'(0) = f(0), f'(1) = f(1), $f'(0 \to 1) = f(0) \to f(1)$. Then there is a commutative diagram:



which preserves markings. Case 3. Let f(0) = i, where i is initial. Let $\beta, \beta' : f(0) \to f(1)$. Define f'(0) = f(0), f'(1) = f(1), because i is initial we get $\beta = \beta'$, thus $f'(\beta) = f'(\beta')$. Then the diagram is commutative: Let $\beta, \beta' : 0 \to 1$.



Case 4. Let $c \in C$ be isomorphic to an arbitrary initial object, then c is marked and initial. Let f(0) = i, f(1) = c, $f(0 \stackrel{\cong}{\to} 1) = \beta = i \stackrel{\cong}{\to} c$.



Define f'(0) = i, f'(1) = c, f'(0) = i, $f'(0 \stackrel{\cong}{\to} 1) = \beta$. Because both objects in the codomain are marked, f' preserves markings, thus is a morphism in Cat^m , the diagram commutes.

Lemma 0.9. Every morphism $h \in L_S$ is injective on objects and reflects markings.

Proof. L_S is the "smallest" set containing S, which is also closed under pushouts, transfinite compositions (sequences), and retracts.

$$\begin{array}{ccc}
A & \xrightarrow{g} & X \\
\downarrow^f & & \downarrow^t \\
B & \xrightarrow{u} & Y
\end{array}$$

Assume that $g \in L_S$ is injective on objects and reflects markings.

1. In Cat, pushouts on objects are produced as a quotient of disjoint unions:

$$Ob(Y) = Ob(B) \sqcup Ob(X) / \sim$$

where the equivalence is defined as $g(a) \sim f(a)$ for all $a \in Ob(A)$. For every $b, b' \in Ob(B)$, for which u(b) = u(b'), there is an object $g(a) \in Ob(X)$, where g is injective on objects. Then b = b' and u is injective on objects. Assume $y \in \epsilon_Y$, then, there is some $b'' \in \epsilon_B$ such that u(b'') = y or there is some $a' \in \epsilon_A$ such that $g(a') = x \in \epsilon_X$, such that y = t(x), because t preserves markings and t reflects markings. Thus $t'' = f(a') \in \epsilon_B$ and t reflects markings.

2. Retracts of injective functions are injective: Let $h: X \to A$ and $h \circ g$. Let $x_1, x_2 \in Ob(X)$. $g(h(x_1)) = g(h(x_2))$ then $x_1 = (g \circ h)(x_1) = (g \circ h)(x_2) = x_2$

Therefore, h is injective on objects. For $x \in X$, then h(x) is marked, because h preserves markings and g(h(x)) is marked, because g reflects markings. Then h also reflects markings.

3. Transfinite compositions

$$(A_0, \varepsilon_0) \xrightarrow{f_0} (A_1, \varepsilon_1) \xrightarrow{f_1} \cdots \xrightarrow{f_{\alpha}} (A_{\alpha+1}, \varepsilon_{\alpha+1}) \cdots$$

for $(\alpha < \lambda)$.