$$f_i(\boldsymbol{x} + \boldsymbol{c}_i\delta t, t+\delta t) - f_i(\boldsymbol{x},t) = \omega(f_i^{ex} - f_i), \qquad (1)$$

$$g_i(\boldsymbol{x} + \boldsymbol{c}_i\delta t, t+\delta t) - g_i(\boldsymbol{x},t) = \omega_1(g_i^{eq} - g_i) + (\omega-\omega_1)(g_i^{eq} - g_i^*) - w_i q_c \Delta t$$

$$q_c = \alpha \frac{A}{V}(T - T_w)$$

Operator $D = \partial_t + c\,\partial_x$

taylor: (2nd order)

$$\cancel{f_i(x,t)} + \Delta t\, D f_i(x,t) + \frac{\Delta t^2}{2} D^2 f_i(x,t) - \cancel{f_i(x,t)} = \omega(f_i^{ex} - f_i)$$

$$\Delta t\, D g_i + \frac{\Delta t^2}{2} D^2 g_i = \omega_1(g_i^{eq} - g_i) + (\omega-\omega_1)(g_i^{eq} - g_i^*)$$

$$f_i = f_i^0 + \Delta t\, f_i^1 + \Delta t^2 f_i^2 \qquad D^1 = \partial_t^1 + c\,\partial_x$$

$$g_i = g_i^0 + \Delta t\, g_i^1 + \Delta t^2 g_i^2 \qquad \partial_t = \partial_t^1 + \Delta t\,\partial_t^2$$

f: Δt^0: $0 = \omega(f^{ex\,0} - f^0) \Rightarrow \boxed{f^0 = f^{ex\,0}}$ $(= f^{eq\,0})$

Δt^1: $\Delta t\, D f^0 = \Delta t\, \omega(f^{ex} - f^1)$

$$\cancel{\Delta t}\Big(\partial_t^1 f^0 + \underbrace{\Delta t\, \partial_t^2 f^0}_{\mathcal{O}\Delta t^2} + c\,\partial_x f^0\Big) = \cancel{\Delta t}\,\omega(f^{ex\,1} - f^1)$$

$$\partial_t^1 f^0 + c\,\partial_x f^0 = \omega(f^{ex\,1} - f^1)$$

$$\boxed{D^1 f^0 = \omega(f^{ex\,1} - f^1)} \qquad \left(D f^0 = \omega(f^{ex\,1} - f^1) - \Delta t\, \partial_t^2 f^0\right)$$

Δt^2: $\Delta t^2 D f^1 + \frac{\Delta t^2}{2} DD f^0 = \Delta t^2 \omega(f^{ex\,2} - f^2)$

$$\Delta t^2\Big(\partial_t^1 f^1 + \underbrace{\Delta t\, \partial_t^2 f^1}_{\mathcal{O}(\Delta t^3)} + c\,\partial_x f^1\Big) + \frac12 \Delta t^2\Big(\partial_t^1 D f^0 + \underbrace{\Delta t\, \partial_t^2 D f^0}_{\mathcal{O}(\Delta t^3)} + c\,\partial_x D f^0\Big) + \Delta t^2 \partial_t^2 f^0 = \Delta t^2\omega(f^{ex2} - f^2)$$

$$\Delta t^2 \partial_t^2 f^0 + \Delta t^2 D^1 f^1 + \frac12\Delta t^2\Big(\partial_t^1\big(\omega(f^{ex1}-f^1) - \underbrace{\Delta t\, \partial_t^2 f^0}_{\mathcal{O}(\Delta t^3)}\big) + c\,\partial_x\big(\omega(f^{ex1}-f^1) - \underbrace{\Delta t\,\partial_t^2 f^0}_{\mathcal{O}(\Delta t^3)}\big)\Big) = \Delta t^2 \omega(f^{ex1} - f^2)$$

$$\partial_t^2 f^0 + D^1 f^1 + \frac{\omega}{2} D^1(f^{ex\,1} - f^1) = \omega(f^{ex2} - f^2)$$

g: Δt^0: $0 = \omega_1(g^{eq} - g^0) + (\omega-\omega_1)(g^{eq} - g^{*0})$

for arbitrary ω, ω_1: $\boxed{g^0 = g^{*0} = g^{eq}}$

Δt^1: $\Delta t\, D g^0 = -\Delta t\, \omega_1 g^1 - \Delta t(\omega-\omega_1)\, g^{*1}$

$$\boxed{D^1 g^0 = -\omega_1 g^1 - (\omega-\omega_1)\, g^{*1}} - w_i q_c$$

Δt^2: $\Delta t\, D g + \frac{\Delta t^2}{2} DD g = -\Delta t^2 \omega_1 g^2 - \Delta t^2(\omega-\omega_1)\, g^{*2}$

$$\Delta t^2 D^1 g^1 + \Delta t^2 \partial_t^2 g^0 + \frac{\Delta t^2}{2} D^1 D^1 g^0 = -\Delta t^2 \omega_1 g^2 - \Delta t^2(\omega-\omega_1)\, g^{*2}$$

$$D^1 g^1 + \partial_t^2 g^0 - \frac12 D^1\big(\omega_1 g^1 + (\omega-\omega_1)\, g^{*1}\big) = -\omega_1 g^2 - (\omega-\omega_1)\, g^{*2}$$

$$\partial_t \rho = \partial_t \sum f^{eq} = \sum \partial_t f^0$$

$$D^1 f^0 = \omega \left(f^{ex\,1} - f^1 \right) = \partial_t^1 f^0 + c\,\partial_x f^0$$

$$\sum_i : \quad \omega \left(\sum f^{ex\,1} - \sum f^1 \right) = \partial_t^1 \sum f^0 + \partial_x \sum c f^0$$

$$0 \qquad 0 \quad = \partial_t^1 \rho + \partial_x (\rho u) \qquad \boxed{\partial_t \rho + \partial_x(\rho u) = 0}$$

$$D^1 g^0 = -\omega g^1 - (\omega - \omega_1) g^{*\,1} - w_i q_c$$

$$\sum_i : \quad \partial_t \sum g^0 + \partial_x \sum c g^0 = -\omega \sum g^1 - (\omega - \omega_1) \sum g^{*\,1} - \sum w_i q_c$$

$$\partial_t^1(\rho E) + \partial_x q = 0 \qquad 0 - q_c \qquad \boxed{\partial_t^1(\rho E) + \partial_x q = 0} - q_c$$

$$q_c = \alpha \frac{A}{V} (T - T_w) \; : \; T = \frac{1}{c_v}\left(E - \frac{u^2}{2}\right)$$

$$= \frac{1}{c_v}\left(\frac{\sum g_i}{\sum f_i} - \frac{(\sum c_i f_i)^2}{\sum f_i} \right)$$

$$\sum_i c_i : \quad D^1 f^0 = \omega \left(f^{ex\,1} - f^1 \right) = \partial_t^1 f^0 + c\,\partial_x f^0$$

$$\partial_t^1 \sum c_i f_i^0 + \partial_x \sum c_i c_i f_i^0 = \omega \sum c_i f_i^{ex\,1} - \omega \sum_i c_i f_i^1$$

$$\partial_t^1 (\rho u) + \partial_x \rho P^{eq} = 0 - 0$$

$$\partial_t^1 (\rho u) + \partial_x (\rho R T + \rho u^2) = 0 \qquad \boxed{\partial_t^1(\rho u) + \partial_x(p + \rho u^2) = 0}$$

$$\sum_i c_i^2 : \quad \partial_t^1 \sum c_i^2 f_i^0 + \partial_x \sum c_i^3 f_i^0 = \omega \sum c_i^2 f_i^{ex\,1} - \omega \sum c_i^2 f_i^1$$

$$\partial_t \underbrace{P^{eq}}_{\substack{P^{MB} \\ = p + \rho u^2}} + \partial_x \underbrace{Q^{eq}}_{Q^{MB} + \tilde{Q}} = \omega \left(P^{ex,1} - P^1 \right) \qquad Q = c^2 (\rho u) = \rho u$$

$$= p u + \rho u^3 + \rho u (1 - 3RT) - \rho u^3 = p u + \rho u - 3 p u$$

$$\omega \left(P^{ex,1} - P^1 \right) - \partial_x \tilde{Q} = Z = \left(2\,\partial_x u - \tfrac{2}{3}\,\partial_x u \right) p + p \left(\tfrac{2}{3} - \tfrac{R}{c_v} \right) \partial_x u$$

$$= \frac{4}{3} p\,\partial_x u + \left(\frac{2}{3} - \gamma + 1 \right) p\,\partial_x u$$

$$= (3 - \gamma)\, p\,\partial_x u$$

$\sum f_i = \sum f_i^{eq} = \rho \qquad \sum f_i^n = 0 \quad (n \geq 1)$

$\sum c_i f_i = \sum c_i f_i^{eq} = \rho u$

$\sum g_i = \sum g_i^{eq} = \rho E \quad ?$

$\sum g_i = \rho E_n \quad , \quad \sum g_i - \dot{Q}\,\Delta t = \rho E_{n+1}$

$\sum g_{i\,n+1} = \rho E_{n+1}$

$$g_{n+1} - g_n = \omega_1 (g_n^{eq} - g_{in}) + (\omega - \omega_1)(g_n^{eq} - g_n^{*}) - w_i \Delta t\, q_c$$

$$\sum g_{n+1} - \sum g_n = \omega_1 \left(\sum g_n^{eq} - \sum g_{in}\right) + (\omega - \omega_1)\left(\sum g_n^{eq} - \sum g_n^{*}\right) - \Delta t\, q_c \sum w_i$$

$$\cancel{\rho E_n} - q_c \Delta t - \cancel{\rho E_n} = \omega_1 \cancel{(\rho E_n - \rho E_n)} + (\omega - \omega_1)\cancel{(\rho E_n - \rho E_n)} - \Delta t\, q_c$$

$$\Delta t D g_i + \frac{\Delta t^2}{2} D^2 g_i = \omega_1 (g_i^{eq} - g_i) + (\omega - \omega_1)(g_i^{eq} - g_i^{*}) - \Delta t\, S$$

$S = S_0 + \Delta t\, S_1 + \dots$

$\Delta t^0: \quad \omega_1 (g^{eq} - g_0) + (\omega - \omega_1)(g^{eq} - g_0^{*}) = 0$ for arbitrary ω, ω_1: $g_0 = g_0^{*} = g^{eq}$

$\Delta t^1: \quad \partial_t^1 g_0 + c\partial_x g_0 = -\omega g_1 - (\omega - \omega_1) g_1^{*} - S_0$

(I) $\sum: \quad \partial_t^1 (\rho E) + \partial_x q^{eq} = 0 \qquad - \sum S_0$

$\Delta t^2: \quad \partial_t^2 g_0 + \partial_t^1 g_1 + c\partial_x g_1 - \frac{1}{2}\omega D_1 g_1 - \frac{1}{2}(\omega - \omega_1) D_1 g_1^{*} - D_1 S_0 = -\omega_1 g_2 - (\omega - \omega_1) g_2^{*} - S_1$

$\sum: \quad \partial_t^2 (\rho E) + 0 + \partial_x q^1 - \frac{\omega}{2}\partial_x q^1 - \frac{1}{2}(\omega - \omega_1)\partial_x q_1^{*} - \partial_t^1 \sum S_0 - \partial_x \sum c S_0 = 0 - 0 - \sum S_1$

(II) $\quad \partial_t^2 (\rho E) + \partial_x \left((1 - \frac{\omega}{2}) q^1 - \frac{\omega - \omega_1}{2} q^{1*} \right) - \partial_t^1 \sum S_0 - \partial_x \sum c S_0 = -\sum S_1$

(I) + Δt (II) $\quad \partial_t (\rho E) + \partial_x g^{eq} + \Delta t\, \partial_x \left((1 - \frac{\omega}{2}) q^1 - \frac{\omega - \omega_1}{2} q^{1*} \right) \underbrace{- \Delta t\, \partial_t^1 \sum S_0 - \Delta t\, \partial_x \sum c S_0 = -\Delta t \sum S_1 - \sum S_0}_{= \dot{q}_c}$

$$\boxed{\sum S_0 + \Delta t \sum S_1 - \Delta t\, \partial_t^1 \sum S_0 - \Delta t\, \partial_x \sum c S_0 = \dot{q}_c}$$

e.g. $\quad S_{0i} = W_i \dot{q}_c \qquad \sum S_1 = \partial_t^1 \sum S_0 + \partial_x \sum c S_0$

$W_0 = \frac{2}{3}$

$W_1 = W_{-1} = \frac{1}{6}$

$\qquad = \partial_t^1 \dot{q}_c + \partial_x \underbrace{(S_{0,1} - S_{0,-1})}_{0}$

$S_i = W_i \left(\dot{q}_c + \Delta t \frac{\partial \dot{q}_c}{\partial t} \right)$

$$D D g_0 = D \left(\partial_t^1 g_0 + \Delta t\, \partial_t^2 g_0 + c\partial_x g_0 \right)$$

$$= \partial_t^1 \partial_t^1 g_0 + \Delta t\, \cancel{\partial_t^1 \partial_t^2} g_0 + \partial_t^1 c \partial_x g_0 + \Delta t\, \cancel{\partial_t^2 \partial_t^1} g_0 + \Delta t^2\, \cancel{\partial_t^2 \partial_t^2} g_0 + \Delta t\, \cancel{\partial_t^2 c \partial_x} g_0$$

$$+ c\partial_x \partial_t^1 g_0 + \Delta t\, \cancel{c \partial_x \partial_t^2} g_0 + c^2 \partial_x \partial_x g_0$$

$$= D_1 D_1 g_0$$