(7) Claim:

$\forall n \in \mathbb{N} \left(2+2^2+2^3+\cdots+2^n = 2^{n+1}-2\right)$

let $m = 2+2^2+2^3+\cdots+2^n$

$2m = 2(2+2^2+2^3+\cdots+2^n) = 2^2+2^3+2^4+\cdots+2^n+2^{n+1}$

note that $m = 2m - m =$

$2^2+2^3+2^4+\cdots+2^n+2^{n+1} - (2+2^2+2^3+\cdots+2^n)$

most terms cancel, except for two;

$= 2^{n+1}-2$

Therefore $m = 2^{n+1}-2$ or

$2+2^2+2^3+\cdots+2^n = 2^{n+1}-2$

Alternate proof by induction

rewrite claim as $\displaystyle\sum_{k=1}^{n} 2^k = 2^{n+1}-2$

let $A(n) = \displaystyle\sum_{k=1}^{n} 2^k = 2^{n+1}-2$

show for $n=1$ that claim is True

$A(1) = \displaystyle\sum_{k=1}^{1} 2^k = 2 = 2^{1+1}-2$

$= 4-2 = 2$

so $A(1)$ is True $2=2$ ✓

Assume $A(n) = \displaystyle\sum_{k=1}^{n} 2^k = 2^{n+1}-2$

now show $A(n+1)$ is True $= \displaystyle\sum_{k=1}^{n+1} 2^k = \left(\sum_{k=1}^{n} 2^k\right) + 2^{n+1}$ (a)

⑦ con't

but by Assumtion $A(n)$ is true and $\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$

we subst into eq (a)

$$\left(\sum_{k=1}^{n} 2^k\right) + 2^{n+1} = \left(2^{n+1} - 2\right) + 2^{n+1}$$

$$= 2^{n+2} - 2$$

so $A(n+1) = \sum_{k=1}^{n+1} 2^k = 2^{(n+1)+1} - 2$ is true

hence by mathematical Induction, claim is proved.