Patryk Maciąg 331542 Lista 7

Zad. 2.

1) Teza: $F_{n+1} = \sum_{i=0}^{n} \binom{n-i}{i}$

Indukcja

$n=0 \to F_1 = 1 = \binom{0}{0}$

$n=1 \to F_2 = 1 = \binom{1}{0} + \binom{0}{1}$

Zał. $\forall_{n_0 \leq n}\ F_{n+1} = \sum_{i=0}^{n} \binom{n-i}{i}$.

Pokażemy dla $n+1$.

$$F_{n+2} = F_{n+1} + F_n \overset{zał}{=} \sum_{i=0}^{n} \binom{n-i}{i} + \sum_{i=0}^{n-1} \binom{n-1-i}{i} =$$

$$= \binom{n}{0} + \sum_{i=1}^{n} \binom{n-i}{i} + \sum_{i=1}^{n} \binom{n-i}{i-1} = \binom{n}{0} + \sum_{i=1}^{n} \left[\binom{n-i}{i} + \binom{n-i}{i-1} \right] =$$

$$= \binom{n}{0} + \sum_{i=1}^{n} \binom{n-i+1}{i} = \binom{n+1}{0} + \sum_{i=1}^{n} \binom{n-i+1}{i} = \sum_{i=0}^{n} \binom{n-i+1}{i} =$$

$$= \binom{0}{n+1} + \sum_{i=0}^{n} \binom{n-i+1}{i} = \sum_{i=0}^{n+1} \binom{n-i+1}{i} \quad \blacksquare$$

1) $n=0 \to \sum_{i=0}^{0} \binom{0}{i} F_{i+m} = F_m$

$n=1 \to \sum_{i=0}^{1} \binom{1}{i} F_{i+m} = F_m + F_{m+1} = F_{m+2}$

Teza: $\sum_{i=0}^{n} \binom{n}{i} F_{i+m} = F_{m+2n}$

Zał. $\forall_{n_0 < n}$ — " — . Pokażemy dla n.

$$\sum_{i=0}^{n} \binom{n}{i} F_{i+m} = \sum_{i=0}^{n} \binom{n-1}{i} F_{i+m} + \sum_{i=0}^{n} \binom{n-1}{i-1} F_{i+m} = \sum_{i=0}^{n} \binom{n-1}{i} F_{i+m} + \sum_{i=1}^{n} \binom{n-1}{i-1} F_{m+i} =$$

$$= \binom{n-1}{n} + \sum_{i=0}^{n-1} \binom{n-1}{i} F_{i+m} + \sum_{i=0}^{n-1} \binom{n-1}{i} F_{i+m+1} = 0 + F_{m+2n-2} + F_{m+1+2n-2} =$$

$$= F_{m+2n} \quad \blacksquare$$

Zad. 3.

Zasada włączeń i wyłączeń.

$\frac{(2n)!}{2^n}$ – ~~#~~ wszystkich permutacji (w $(2n)!$ są różne ustawienia tych samych liczb, więc dzielimy przez 2^n

$|A_i| = \frac{(2(n-1))!}{2^{n-1}} \cdot (2n-1) = \frac{(2n-1)!}{2^{n-1}}$ – ustawienie, gdzie

i-te liczby stoją obok siebie (ustawiamy $n-1$ liczb, a parę i możemy ustawić w $2(n-1) - 1 + 2 = 2n-1$ miejscach (pomiędzy $= 2n-3$ oraz początek i koniec)

$|A_i \cap A_j| = \frac{(2(n-2))!}{2^{n-2}} \cdot (2n-3)(2n-2) = \frac{(2n-2)!}{2^{n-2}}$ (par nie rozdzielamy

$$\left|\bigcap_{i=1}^{j} A_i\right| = \frac{(2n-j)!}{2^{n-j}}$$

$$\frac{(2n)!}{2^n} - \sum_{k=1}^{n} (-1)^{k-1} \cdot \binom{n}{k} \cdot \frac{(2n-k)!}{2^{n-k}}$$

Zad. 4.

$a_0 = 1,\ a_1 = 0,\ a_n = \frac{a_{n-1} + a_{n-2}}{2}$

$2a_n - a_{n-1} - a_n = 0$

$2E^2\langle a_n\rangle - E\langle a_n\rangle + 1\langle a_n\rangle = \langle 0\rangle$

$(2E^2 - E - 1) = (E-1)(E+\frac{1}{2})$

$a_n = \alpha \cdot (1)^n + \beta \cdot (-\frac{1}{2})^n,\quad a_0 = \alpha + \beta = 1,\quad a_1 = \alpha - \frac{1}{2}\beta = 0$

$\Rightarrow \alpha = \frac{1}{3},\ \beta = \frac{2}{3}$

$a_n = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$

Zad. 8

$s_n = \sum_{i=1}^{n} i2^i$, $s_n = s_{n-1} + n2^n$

$s_n - s_{n-1} - n2^n = 0$

$(E-1)\langle s_n \rangle - n2^n = 0$

~~$s_n = c2^n$~~ Zauważmy, że jeśli $s_n = c2^n$ dla pewnego $c \in \mathbb{R}$, to mamy warunek na c:

$$c2^n - c2^{n-1} - n2^n = 0$$

$$c = \frac{n2^n}{2^n - 2^{n-1}} = 2n$$

$$s_n = \alpha + \beta 2^n + 2n \cdot 2^n = \alpha + 2^n\beta + n2^{n+1}$$

$s_1 = 2 = \alpha + 2\beta + 4$	$\alpha + 2\beta = -2$
$s_2 = 10 = \alpha + 4\beta + 16$	$\alpha + 4\beta = -6$
	$\alpha = 2$, $\beta = -2$

$$s_n = 2 - 2^{n+1} + n2^{n+1} = 2 + (n-1)2^{n+1}$$

Zad. 5.

a) $a_{n+2} = 2a_{n+1} - a_n + 3^n - 1$, gdy $a_0 = a_1 = 0$

$a_{n+2} - 2a_{n+1} + a_n - 3^n + 1 = 0$

$(E^2 - 2E + 1)\langle a_n \rangle = \langle 0 \rangle$

$(E-3)\langle -3^n \rangle = \cancel{\quad} \langle -3^{n+1} + 3^{n+1} \rangle = \langle 0 \rangle$

$(E-1)\langle 1 \rangle = \langle 1 - 1 \rangle = \langle 0 \rangle$

$(E-1)^3 (E-3)$ anihiluje całość

$a_n = \alpha 1^n + \beta \cdot n \cdot 1^n + \gamma \cdot n^2 \cdot 1^n + \delta \cdot 3^n$

$$\left.\begin{aligned} a_0 &= 0 = \alpha + \delta \\ a_1 &= 0 = \alpha + \beta + \gamma + 3\delta \\ a_2 &= 0 = \alpha + 2\beta + 4\gamma + 9\delta \\ a_3 &= 2 = \alpha + 3\beta + 9\gamma + 27\delta \end{aligned}\right\} \left\{\begin{aligned} \alpha &= -\tfrac{1}{4} \\ \beta &= 0 \\ \gamma &= -\tfrac{1}{2} \\ \delta &= \tfrac{1}{4} \end{aligned}\right. \Rightarrow a_n = -\frac{1}{4} - \frac{1}{2}n^2 + \frac{1}{4}3^n$$

b) $a_{n+2} = 4a_{n+1} - 4a_n + n2^{n+1}$, gdy $a_0 = a_1 = 1$

$a_{n+2} - 4a_{n+1} + 4a_n - n2^{n+1}$

$(E^2 - 4E + 4)\langle a_n \rangle = \langle 0 \rangle$

$(E-2)^2 \langle n2^{n+1} \rangle = (E-2)\langle (n+1)2^{n+2} - n2^{n+2} \rangle = (E-2)\langle 2^{n+2} \rangle =$
$= \langle 0 \rangle$

$(E-2)^4$ anihiluje całość

$a_n = \alpha \cdot 2^n + \beta n 2^n + \gamma n^2 2^n + \delta n^3 2^n$

c) $a_{n+2} = 2^{n+1} - a_{n+1} - a_n$, gdy $a_0 = a_1 = 1$

$a_{n+2} + a_{n+1} + a_n - 2^{n+1} = 0$

$\Delta = 1 - 4 = -3$

$(E^2 + E + 1)(E - 2)$ anihiluje całość

$E_0 = \frac{-1 \pm \sqrt{-3}}{2}$, $\sqrt{-3} = \sqrt{3}i$

$(E-2)\left\{\left(E - \left(\frac{-1-\sqrt{3}i}{2}\right)\right)\left(E + \left(\frac{-1+\sqrt{3}i}{2}\right)\right)\right\}$

$a_n = \alpha 2^n + \beta\left(\frac{-1-\sqrt{3}i}{2}\right)^n + \gamma\left(\frac{-1+\sqrt{3}i}{2}\right)^n$

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Zad. 13.

a) $b_n = n a_n$

$$B(x) = \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} k a_k x^k = x \sum_{k=0}^{\infty} k a_k x^{k-1} = x \sum_{k=0}^{\infty} (a_k \cdot x^k)' = xA'(x)$$

b) $c_n = \frac{a_n}{n}, \; c_0 = 0$

$$C(x) = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{a_k}{k} x^k = \int_0^x \sum_{k=1}^{\infty} a_k t^{k-1} dt =$$

$$= \int_0^x \frac{1}{t} \sum_{k=0}^{\infty} a_k \cdot t^k dt = \int_0^x \frac{A(t) - a_0}{t} \rightsquigarrow \left(\int A(x) = \underbrace{\int a_0}_{x a_0} + \sum_{k=1}^{\infty} \int a_k t^k \right)$$

c) $s_n = a_0 + a_1 + a_2 + \ldots + a_n$

$$S(x) = \sum_{k=0}^{\infty} s_k x^k = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} a_l \right) x^k = \left(\sum_{k=0}^{\infty} a_k x^k \right) \left(\sum_{l=0}^{\infty} x^l \right) =$$

$$= A(x) \cdot \frac{1}{1-x} = \frac{A(x)}{1-x}$$

d) $d_n = \begin{cases} a_n & \text{gdy } n = 2k \\ 0 & \text{gdy } n = 2k+1 \end{cases}$

$$D(x) = \sum_{k=0}^{\infty} d_k x^k = \frac{A(x) + A(-x)}{2}$$

$$A(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$$

$$A(-x) = a_0 x^0 - a_1 x^1 + a_2 x^2 + \ldots$$