Principles of Program Analysis:

A Sampler of Approaches

Transparencies based on Chapter 1 of the book: Flemming Nielson, Hanne Riis Nielson and Chris Hankin: Principles of Program Analysis. Springer Verlag 2005.

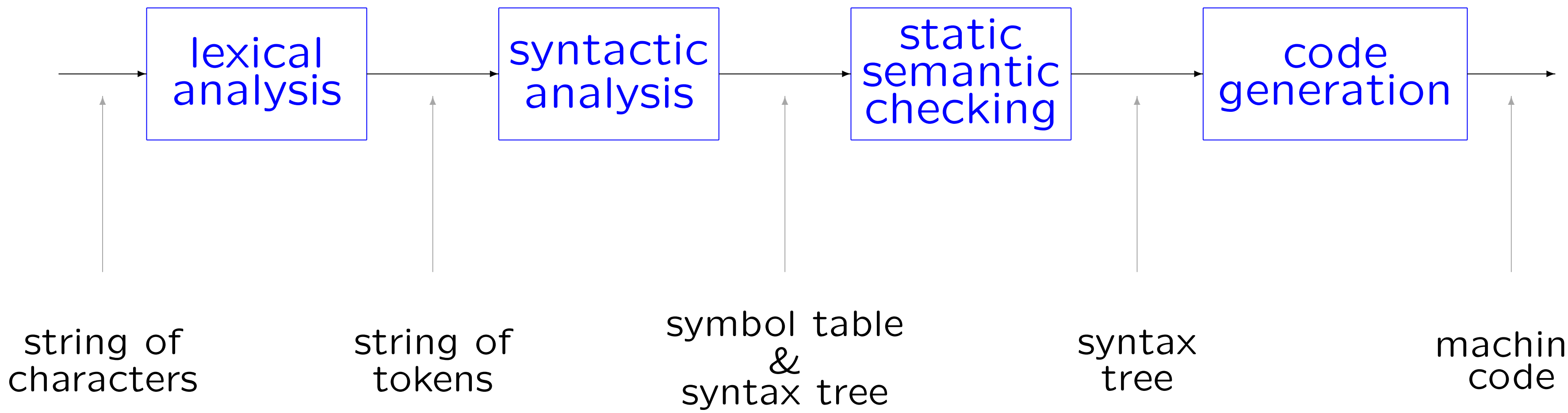
Compiler Optimisation

The classical use of program analysis is to facilitate the construction of compilers generating "optimal" code.

We begin by outlining the structure of optimising compilers.

We then prepare the setting for a worked example where we "optimise" a naive implementation of Algol-like arrays in a C-like language by performing a series of analyses and transformations.

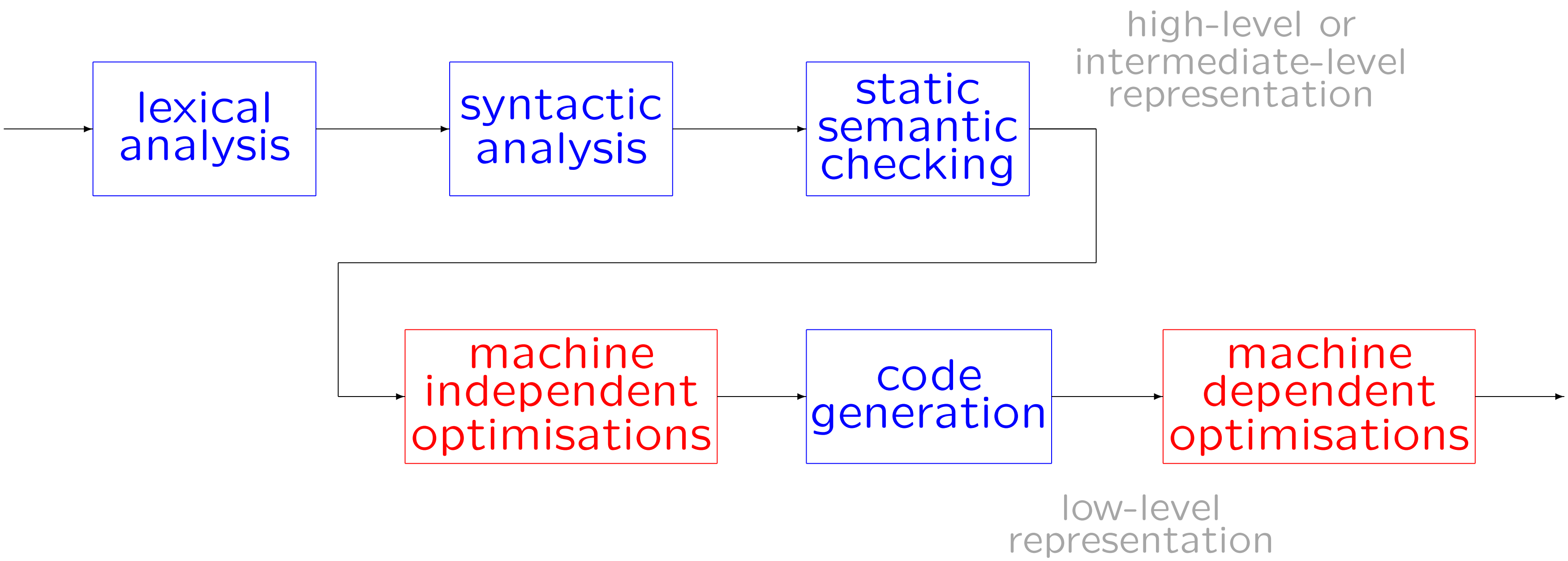
The structure of a simple compiler



Characteristics of a simple compiler:

- many phases – one or more passes
- the compiler is fast – but the code is not very efficient

The structure of an optimising compiler

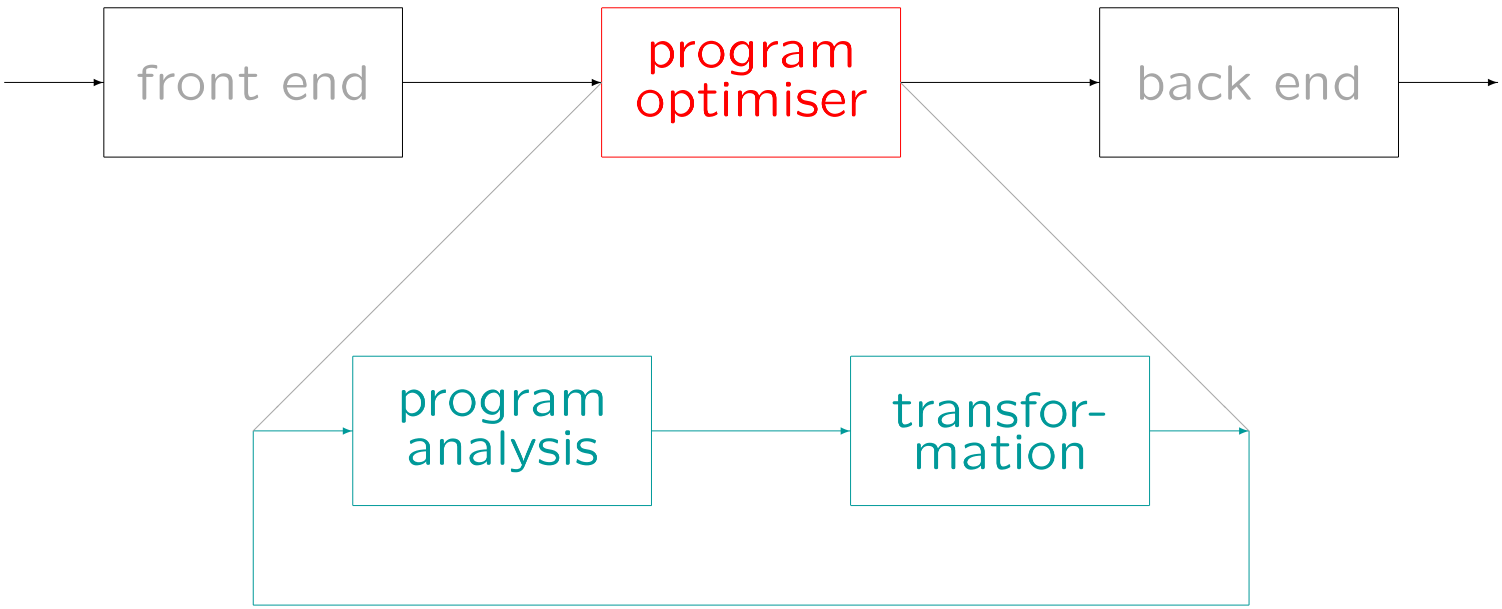


Characteristics of the optimising compiler:

- high-level optimisations: easy to adapt to new architectures

- low-level optimisations: less likely to port to new architectures

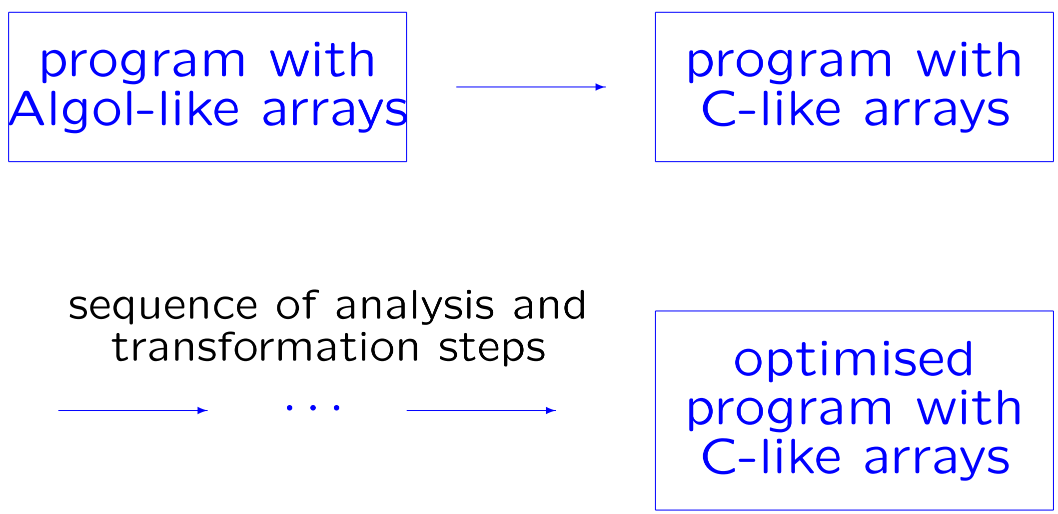
The structure of the optimisation phase



Avoid redundant computations: reuse available results, move loop invariant computations out of loops, ...

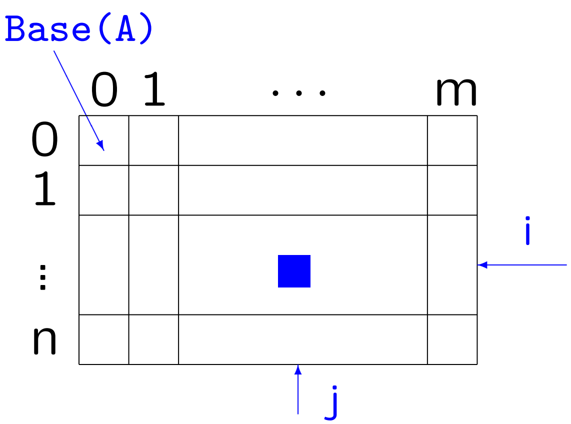
Avoid superfluous computations: results known not to be needed, results known already at compile time, ...

Example: Array Optimisation



Array representation: Algol vs. C

`A: array [0:n, 0:m] of integer`



Accessing the (i,j)'th element of `A`:

in Algol:

`A[i,j]`

in C:

`Cont(Base(A) + i * (m+1) + j)`

An example program and its naive realisation

Algol-like arrays:

```
i := 0;
while i <= n do
   j := 0;
   while j <= m do
      A[i,j] := B[i,j] + C[i,j];
      j := j+1
   od;
   i := i+1
od
```

C-like arrays:

```
i := 0;
while i <= n do
   j := 0;
   while j <= m do
      temp := Base(A) + i * (m+1) + j;
      Cont(temp) := Cont(Base(B) + i * (m+1) + j)
                   + Cont(Base(C) + i * (m+1) + j);
      j := j+1
   od;
   i := i+1
od
```

Available Expressions analysis and Common Subexpression Elimination

```
i := 0;
while i <= n do
   j := 0;
   while j <= m do
      temp := Base(A) + i*(m+1) + j;
      Cont(temp) := Cont(Base(B) + i*(m+1) + j)
                    + Cont(Base(C) + i*(m+1) + j);
      j := j+1
   od;
   i := i+1
od
```

first computation

re-computations

```
t1 := i * (m+1) + j;
temp := Base(A) + t1;
Cont(temp) := Cont(Base(B)+t1)
               + Cont(Base(C)+t1);
```

Detection of Loop Invariants and Invariant Code Motion

loop invariant

```
i := 0;
while i <= n do
   j := 0;
   while j <= m do
      t1 := i * (m+1) + j;
      temp := Base(A) + t1;
      Cont(temp) := Cont(Base(B) + t1)
                    + Cont(Base(C) + t1);
      j := j+1
   od;
   i := i+1
od
```

```
t2 := i * (m+1);
while j <= m do
   t1 := t2 + j;
   temp := ...
   Cont(temp) := ...
   j := ...
od
```

Detection of Induction Variables and Reduction of Strength

```
i := 0;
while i <= n do
   j := 0;
   t2 := i * (m+1);
   while j <= m do
      t1 := t2 + j;
      temp := Base(A) + t1;
      Cont(temp) := Cont(Base(B) + t1)
                  + Cont(Base(C) + t1);
      j := j+1
   od;
   i := i+1
od
```

induction variable

```
i := 0;
t3 := 0;
while i <= n do
   j := 0;
   t2 := t3;
   while j <= m do ...  od
   i := i + 1;
   t3 := t3 + (m+1)
od
```

Equivalent Expressions analysis and Copy Propagation

```
i := 0;
t3 := 0;
while i <= n do
   j := 0;
   t2 := t3;
   while j <= m do
      t1 := t2 + j;
      temp := Base(A) + t1;
      Cont(temp) := Cont(Base(B) + t1)
                  + Cont(Base(C) + t1);
      j := j+1
   od;
   i := i+1;
   t3 := t3 + (m+1)
od
```

t2 = t3

```
while j <= m do
   t1 := t3 + j;
   temp := ...;
   Cont(temp) := ...;
   j := ...
od
```

Live Variables analysis and Dead Code Elimination

```
i := 0;
t3 := 0;
while i <= n do
   j := 0;
   t2 := t3;
   while j <= m do
      t1 := t3 + j;
      temp := Base(A) + t1;
      Cont(temp) := Cont(Base(B) + t1)
                    + Cont(Base(C) + t1);
      j := j+1
   od;
   i := i+1;
   t3 := t3 + (m+1)
od
```

dead variable

```
i := 0;
t3 := 0;
while i <= n do
   j := 0;
   while j <= m do
      t1 := t3 + j;
      temp := Base(A) + t1;
      Cont(temp) := Cont(Base(B) + t1)
                    + Cont(Base(C) + t1);
      j := j+1
   od;
   i := i+1;
   t3 := t3 + (m+1)
od
```

Summary of analyses and transformations

Analysis	Transformation
Available expressions analysis	Common subexpression elimination
Detection of loop invariants	Invariant code motion
Detection of induction variables	Strength reduction
Equivalent expression analysis	Copy propagation
Live variables analysis	Dead code elimination

The Essence of Program Analysis

Program analysis offers techniques for predicting
statically at compile-time
safe & efficient approximations
to the set of configurations or behaviours arising
dynamically at run-time

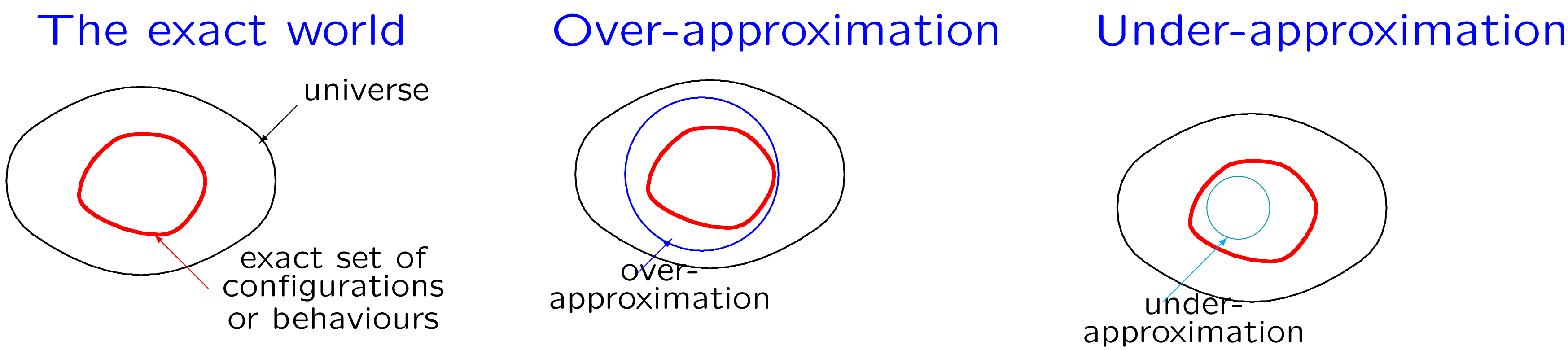
we cannot expect exact answers!

Safe: faithful to the semantics

Efficient: implementation with

– good time performance and
– low space consumption

The Nature of Approximation



Slogans: Err on the safe side!
Trade precision for efficiency!

Approaches to Program Analysis

A family of techniques . . .

- data flow analysis
- constraint based analysis
- abstract interpretation
- type and effect systems
- . . .
- flow logic:
  a unifying framework

. . . that differ in their focus:

- algorithmic methods
- semantic foundations
- language paradigms
  - imperative/procedural
  - object oriented
  - logical
  - functional
  - concurrent/distributive
  - mobile
  - . . .

Data Flow Analysis

- **Technique:** Data Flow Analysis
- **Example:** Reaching Definitions analysis
  - idea
  - constructing an equation system
  - solving the equations
  - theoretical underpinnings

Example program

Program with labels for elementary blocks:

$[y := x]^1$;

$[z := 1]^2$;

while $[y > 0]^3$ **do**

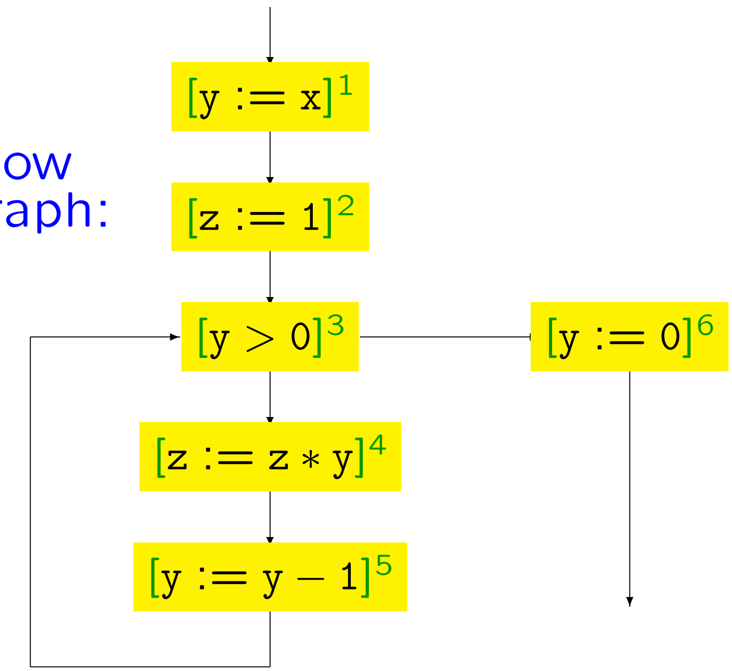
  $[z := z * y]^4$;

  $[y := y - 1]^5$

od;

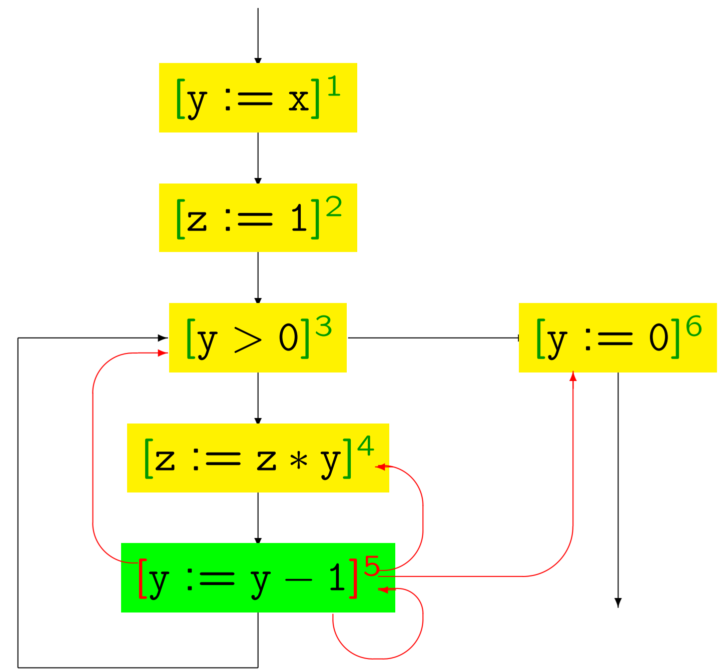
$[y := 0]^6$

Flow graph:



Example: Reaching Definitions

The assignment $[x := a]^{\ell}$ reaches ℓ' if there is an execution where x was last assigned at ℓ



Reaching Definitions analysis (1)

$$\{(x,?),(y,?),(z,?)\}$$

$[y := x]^1$;

$$\{(x,?),(y,1),(z,?)\}$$

$[z := 1]^2$;

$$\{(x,?),(y,1),(z,2)\}$$

while $[y > 0]^3$ **do**

$$\{(x,?),(y,1),(z,2)\}$$

  $[z := z * y]^4$;

  $[y := y - 1]^5$

od;

$$\{(x,?),(y,1),(z,2)\}$$

$[y := 0]^6$

Reaching Definitions analysis (2)

	$\leftarrow$ $\{(x,?),(y,?),(z,?)\}$
$[y := x]^1;$	
	$\leftarrow$ $\{(x,?),(y,1),(z,?)\}$
$[z := 1]^2;$	
	$\leftarrow$ $\{(x,?),(y,1),(z,2)\} \cup \{(y,5),(z,4)\}$
while $[y > 0]^3$ **do**	
	$\leftarrow$ $\{(x,?),(y,1),(z,2)\}$
$\quad[z := z * y]^4;$	
	$\leftarrow$ $\{(x,?),(y,1),(z,4)\}$
$\quad[y := y - 1]^5$	
	$\leftarrow$ $\{(x,?),(y,5),(z,4)\}$
od;	
	$\leftarrow$ $\{(x,?),(y,1),(z,2)\}$
$[y := 0]^6$	
	$\leftarrow$

Reaching Definitions analysis (3)

$$\{(x,?),(y,?),(z,?)\}$$

$[y := x]^1;$

$$\{(x,?),(y,1),(z,?)\}$$

$[z := 1]^2;$

$$\{(x,?),(y,1),(y,5),(z,2),(z,4)\} \cup \{(y,5),(z,4)\}$$

$\texttt{while}\ [y > 0]^3\ \texttt{do}$

$$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$$

$\quad [z := z * y]^4;$

$$\{(x,?),(y,1),(y,5),(z,4)\}$$

$\quad [y := y - 1]^5$

$$\{(x,?),(y,5),(z,4)\}$$

$\texttt{od};$

$$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$$

$[y := 0]^6$

The best solution

	$\{(x,?),(y,?),(z,?)\}$
$[y := x]^1$;	
	$\{(x,?),(y,1),(z,?)\}$
$[z := 1]^2$;	
	$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$
while $[y > 0]^3$ **do**	
	$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$
$[z := z * y]^4$;	
	$\{(x,?),(y,1),(y,5),(z,4)\}$
$[y := y - 1]^5$	
	$\{(x,?),(y,5),(z,4)\}$
od;	
	$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$
$[y := 0]^6$	
	$\{(x,?),(y,6),(z,2),(z,4)\}$

A safe solution — but not the best

$\{(x,?),(y,?),(z,?)\}$

$[y := x]^1;$

$\{(x,?),(y,1),(z,?)\}$

$[z := 1]^2;$

$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$

$\texttt{while}\ [y > 0]^3\ \texttt{do}$

$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$

$\quad [z := z * y]^4;$

$\{(x,?),(y,1),(y,5),\boxed{(z,2)},(z,4)\}$

$\quad [y := y - 1]^5$

$\{(x,?),\boxed{(y,1)},(y,5),\boxed{(z,2)},(z,4)\}$

$\texttt{od};$

$\{(x,?),(y,1),(y,5),(z,2),(z,4)\}$

$[y := 0]^6$

$\{(x,?),(y,6),(z,2),(z,4)\}$

An unsafe solution

$\{(x,?),(y,?),(z,?)\}$

$[y := x]^1;$

$\{(x,?),(y,1),(z,?)\}$

$[z := 1]^2;$

$\{(x,?),(y,1),(z,2),(y,5),(z,4)\}$

$\mathtt{while}\ [y > 0]^3\ \mathtt{do}$

$\{(x,?),\boxed{(y,1)},(y,5),\boxed{(z,2)},(z,4)\}$

$\quad [z := z * y]^4;$

$\{(x,?),\boxed{(y,1)},(y,5),(z,4)\}$

$\quad [y := y - 1]^5$

$\{(x,?),(y,5),(z,4)\}$

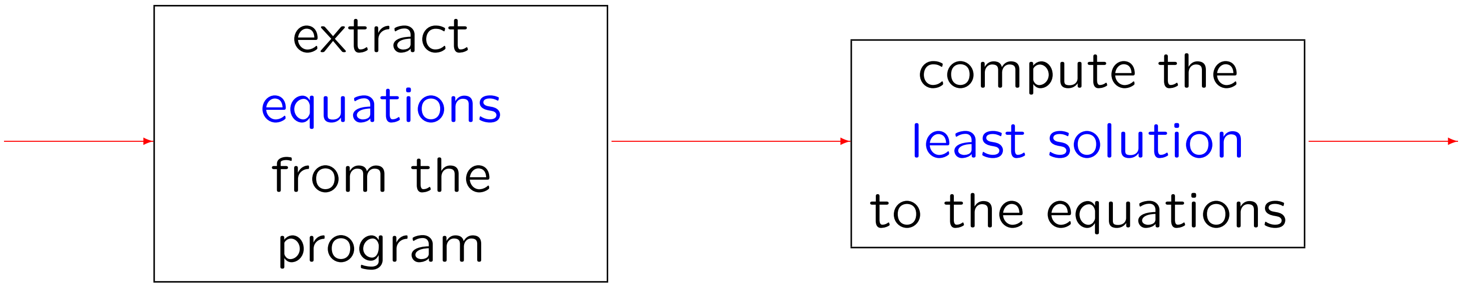
$\mathtt{od};$

$\{(x,?),\boxed{(y,1)},(y,5),\boxed{(z,2)},(z,4)\}$

$[y := 0]^6$

$\{(x,?),(y,6),(z,2),(z,4)\}$

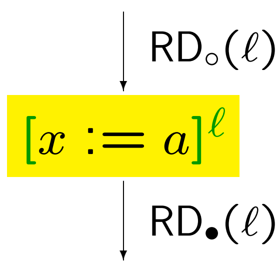
How to automate the analysis



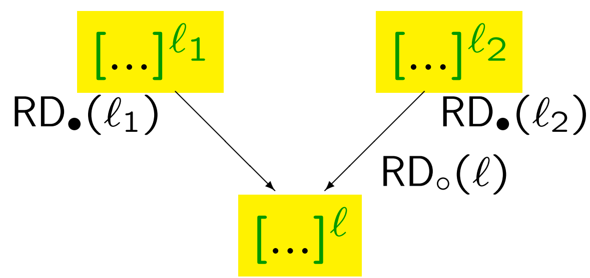
Analysis information:

- $\mathsf{RD}_{\circ}(\ell)$: information available at the entry of block ℓ
- $\mathsf{RD}_{\bullet}(\ell)$: information available at the exit of block ℓ

Two kinds of equations



$$\mathsf{RD}_\circ(\ell) \setminus \{(x,\ell') \mid \ell' \in \mathbf{Lab}\} \cup \{(x,\ell)\} = \mathsf{RD}_\bullet(\ell)$$



$$\mathsf{RD}_\bullet(\ell_1) \cup \mathsf{RD}_\bullet(\ell_2) = \mathsf{RD}_\circ(\ell)$$

Flow through assignments and tests

$[y := x]^1;$ $\longleftarrow$ $RD_\bullet(1) = RD_\circ(1) \setminus \{(y,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(y,1)\}$

$[z := 1]^2;$ $\longleftarrow$ $RD_\bullet(2) = RD_\circ(2) \setminus \{(z,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(z,2)\}$

while $[y > 0]^3$ **do** $\longleftarrow$ $RD_\bullet(3) = RD_\circ(3)$

$[z := z * y]^4;$ $\longleftarrow$ $RD_\bullet(4) = RD_\circ(4) \setminus \{(z,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(z,4)\}$

$[y := y - 1]^5$ $\longleftarrow$ $RD_\bullet(5) = RD_\circ(5) \setminus \{(y,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(y,5)\}$

od; $\longleftarrow$

$[y := 0]^6$ $\longleftarrow$ $RD_\bullet(6) = RD_\circ(6) \setminus \{(y,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(y,6)\}$

Lab $= \{1,2,3,4,5,6\}$ $\longleftarrow$ 6 equations in $RD_\circ(1), \cdots, RD_\bullet(6)$

Flow along the control

$$RD_\circ(1) = \{(x,?),(y,?),(z,?)\}$$

$[y := x]^1$;

$$RD_\circ(2) = RD_\bullet(1)$$

$[z := 1]^2$;

$$RD_\circ(3) = RD_\bullet(2) \cup RD_\bullet(5)$$

while $[y > 0]^3$ **do**

$$RD_\circ(4) = RD_\bullet(3)$$

$[z := z * y]^4$;

$$RD_\circ(5) = RD_\bullet(4)$$

$[y := y - 1]^5$

od;

$$RD_\circ(6) = RD_\bullet(3)$$

$[y := 0]^6$

Lab $= \{1,2,3,4,5,6\}$

6 equations in $RD_\circ(1), \cdots, RD_\bullet(6)$

Summary of equation system

$$\begin{aligned}
\mathsf{RD}_\bullet(1) &= \mathsf{RD}_\circ(1) \setminus \{(y,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(y,1)\}\\
\mathsf{RD}_\bullet(2) &= \mathsf{RD}_\circ(2) \setminus \{(z,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(z,2)\}\\
\mathsf{RD}_\bullet(3) &= \mathsf{RD}_\circ(3)\\
\mathsf{RD}_\bullet(4) &= \mathsf{RD}_\circ(4) \setminus \{(z,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(z,4)\}\\
\mathsf{RD}_\bullet(5) &= \mathsf{RD}_\circ(5) \setminus \{(y,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(y,5)\}\\
\mathsf{RD}_\bullet(6) &= \mathsf{RD}_\circ(6) \setminus \{(y,\ell) \mid \ell \in \mathbf{Lab}\} \cup \{(y,6)\}
\end{aligned}$$

$$\begin{aligned}
\mathsf{RD}_\circ(1) &= \{(x,?),(y,?),(z,?)\}\\
\mathsf{RD}_\circ(2) &= \mathsf{RD}_\bullet(1)\\
\mathsf{RD}_\circ(3) &= \mathsf{RD}_\bullet(2) \cup \mathsf{RD}_\bullet(5)\\
\mathsf{RD}_\circ(4) &= \mathsf{RD}_\bullet(3)\\
\mathsf{RD}_\circ(5) &= \mathsf{RD}_\bullet(4)\\
\mathsf{RD}_\circ(6) &= \mathsf{RD}_\bullet(3)
\end{aligned}$$

- 12 sets: $\mathsf{RD}_\circ(1), \cdots, \mathsf{RD}_\bullet(6)$
  all being subsets of $\mathbf{Var} \times \mathbf{Lab}$
- 12 equations:
  $\vec{\mathsf{RD}}_j = \mathsf{F}_j(\mathsf{RD}_\circ(1), \cdots, \mathsf{RD}_\bullet(6))$
- one function:
  $\mathsf{F} : \mathcal{P}(\mathbf{Var} \times \mathbf{Lab})^{12} \to \mathcal{P}(\mathbf{Var} \times \mathbf{Lab})^{12}$
- we want the least fixed point of F — this is the best solution to the equation system

How to solve the equations

A simple iterative algorithm

- Initialisation
  $\mathsf{RD}_1 := \emptyset; \cdots ; \mathsf{RD}_{12} := \emptyset;$

- Iteration
  $\texttt{while}\ \mathsf{RD}_j \neq \mathsf{F}_j(\mathsf{RD}_1, \cdots, \mathsf{RD}_{12})$ for some j
  $\texttt{do}$
  $\quad \mathsf{RD}_j := \mathsf{F}_j(\mathsf{RD}_1, \cdots, \mathsf{RD}_{12})$

$\boxed{\text{The algorithm terminates and computes the least fixed point of F.}}$

The example equations

$RD_\circ$	1	2	3	4	5	6
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$x_?, y_?, z_?$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	$x_?, y_?, z_?$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	$x_?, y_?, z_?$	$x_?, y_1, z_?$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	$x_?, y_?, z_?$	$x_?, y_1, z_?$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	$x_?, y_?, z_?$	$x_?, y_1, z_?$	$x_?, y_1, z_2$	$\emptyset$	$\emptyset$	$\emptyset$
6	$x_?, y_?, z_?$	$x_?, y_1, z_?$	$x_?, y_1, z_2$	$\emptyset$	$\emptyset$	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$RD_\bullet$	1	2	3	4	5	6
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	$x_?, y_1, z_?$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	$x_?, y_1, z_?$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4	$x_?, y_1, z_?$	$x_?, y_1, z_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	$x_?, y_1, z_?$	$x_?, y_1, z_2$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	$x_?, y_1, z_?$	$x_?, y_1, z_2$	$x_?, y_1, z_2$	$\emptyset$	$\emptyset$	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The equations:

$$\begin{aligned}
RD_\bullet(1) &= RD_\circ(1) \setminus \{(y,\ell) \mid \cdots\} \cup \{(y,1)\}\\
RD_\bullet(2) &= RD_\circ(2) \setminus \{(z,\ell) \mid \cdots\} \cup \{(z,2)\}\\
RD_\bullet(3) &= RD_\circ(3)\\
RD_\bullet(4) &= RD_\circ(4) \setminus \{(z,\ell) \mid \cdots\} \cup \{(z,4)\}\\
RD_\bullet(5) &= RD_\circ(5) \setminus \{(y,\ell) \mid \cdots\} \cup \{(y,5)\}\\
RD_\bullet(6) &= RD_\circ(6) \setminus \{(y,\ell) \mid \cdots\} \cup \{(y,6)\}
\end{aligned}$$

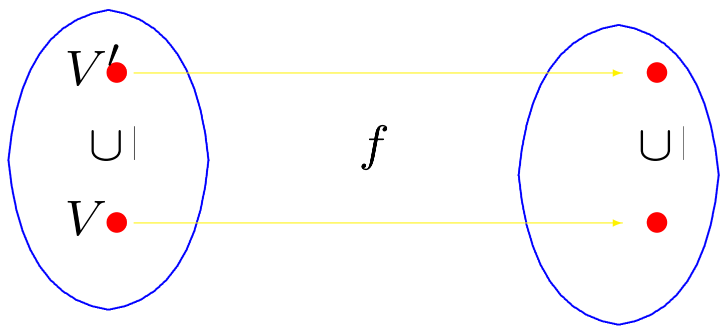
$$\begin{aligned}
RD_\circ(1) &= \{(x,?),(y,?),(z,?)\}\\
RD_\circ(2) &= RD_\bullet(1)\\
RD_\circ(3) &= RD_\bullet(2) \cup RD_\bullet(5)\\
RD_\circ(4) &= RD_\bullet(3)\\
RD_\circ(5) &= RD_\bullet(4)\\
RD_\circ(6) &= RD_\bullet(3)
\end{aligned}$$

Why does it work? (1)

A function $f : \mathcal{P}(S) \to \mathcal{P}(S)$ is a monotone function if

$$V \subseteq V' \quad \Rightarrow \quad f(V) \subseteq f(V')$$

(the larger the argument – the larger the result)



Why does it work? (2)

A set L equipped with an ordering $\subseteq$ satisfies the Ascending Chain Condition if all chains

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$$

stabilise, that is, if there exists some n such that $V_n = V_{n+1} = V_{n+2} = \cdots$

If S is a finite set then $\mathcal{P}(S)$ equipped with the subset ordering $\subseteq$ satisfies the Ascending Chain Condition — the chains cannot grow forever since each element is a subset of a finite set.

Fact

For a given program $\mathbf{Var} \times \mathbf{Lab}$ will be a finite set so $\mathcal{P}(\mathbf{Var} \times \mathbf{Lab})$ with the subset ordering satisfies the Ascending Chain Condition.

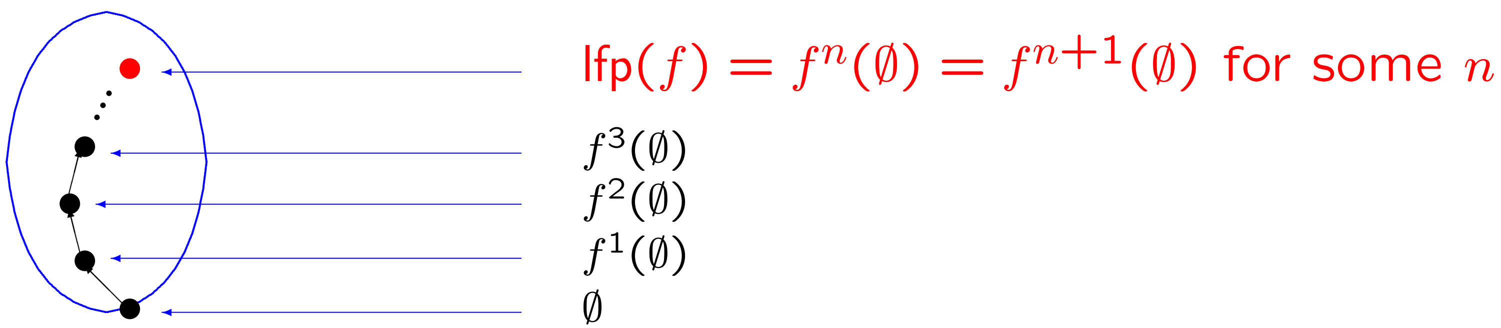
Why does it work? (3)

Let $f : \mathcal{P}(S) \to \mathcal{P}(S)$ be a monotone function. Then

$$\emptyset \subseteq f(\emptyset) \subseteq f^2(\emptyset) \subseteq f^3(\emptyset) \subseteq \cdots$$

Assume that S is a finite set; then the Ascending Chain Condition is satisfied. This means that the chain cannot be growing infinitely so there exists n such that $f^n(\emptyset) = f^{n+1}(\emptyset) = \cdots$

$f^n(\emptyset)$ is the least fixed point of f



Correctness of the algorithm

- Initialisation
  $\mathsf{RD}_1 := \emptyset; \cdots ; \mathsf{RD}_{12} := \emptyset;$
  Invariant: $\vec{\mathsf{RD}} \subseteq \mathsf{F}^n(\vec{\emptyset})$ since $\vec{\mathsf{RD}} = \vec{\emptyset}$ is the least element

- Iteration
  while $\mathsf{RD}_j \neq \mathsf{F}_j(\mathsf{RD}_1, \cdots, \mathsf{RD}_{12})$ for some j
  do assume $\vec{\mathsf{RD}}$ is $\vec{\mathsf{RD}}'$ and $\vec{\mathsf{RD}}' \subseteq \mathsf{F}^n(\vec{\emptyset})$
  $\mathsf{RD}_j := \mathsf{F}_j(\mathsf{RD}_1, \cdots, \mathsf{RD}_{12})$
  then $\vec{\mathsf{RD}} \subseteq \mathsf{F}(\vec{\mathsf{RD}}') \subseteq \mathsf{F}^{n+1}(\vec{\emptyset}) = \mathsf{F}^n(\vec{\emptyset})$ when $\mathsf{lfp}(\mathsf{F}) = \mathsf{F}^n(\vec{\emptyset})$

If the algorithm terminates then it computes the least fixed point of F.

The algorithm terminates because $\mathsf{RD}_j \subset \mathsf{F}_j(\mathsf{RD}_1, \cdots, \mathsf{RD}_{12})$ is only possible finitely many times since $\mathcal{P}(\mathbf{Var} \times \mathbf{Lab})^{12}$ satisfies the Ascending Chain Condition.