Patryk Maciąg 331542 Lista2

Zad. 1

$a \in \mathbb{R} \setminus \mathbb{Q}$

$n \in \mathbb{Z}_+$

1) Teza: $\lfloor an \rfloor + \lfloor (1-a)n \rfloor = n-1$

D-d:

$$n \in \mathbb{Z}_+ \wedge a \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow a \cdot n \notin \mathbb{Z}$$

$$\lfloor a \cdot n \rfloor = \lceil an \rceil - 1 \quad (\text{dla } x \notin \mathbb{Z} \ \lfloor x \rfloor - \lceil x \rceil = -1)$$

$$\lfloor an \rfloor + \lfloor (1-a)n \rfloor = \lfloor an \rfloor + \lfloor n - an \rfloor \overset{n \in \mathbb{Z}_+}{=} \lfloor an \rfloor + n + \lfloor -an \rfloor =$$

$$\overset{\lfloor x \rfloor = -\lceil x \rceil}{=} \lfloor an \rfloor + n - \lceil an \rceil = n-1 \quad \blacksquare$$

2) Teza: $\lceil an \rceil + \lceil (1-a)n \rceil = n+1$

D-d:

$$\lceil an \rceil + \lceil (1-a)n \rceil = \lceil an \rceil + n + \lceil -an \rceil =$$

$$= \lceil an \rceil + n - \lfloor an \rfloor = n+1 \quad \blacksquare$$

Zad. 2.

$x \in \mathbb{R} \quad m \in \mathbb{N}$

$$S = \left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x+1}{m} \right\rfloor + \dots + \left\lfloor \frac{x+m-1}{m} \right\rfloor = ?$$

$r = [0, m)$

$k \in \mathbb{Z}, \ k = \left\lfloor \frac{x}{m} \right\rfloor, \ r = x - km$

$x = k \cdot m + r$

$n \in \{0, \dots, m-1\}$

$$a_n = \left\lfloor \frac{x+n}{m} \right\rfloor = \left\lfloor \frac{km + r + n}{m} \right\rfloor = k + \left\lfloor \frac{r+n}{m} \right\rfloor$$

$||$

k dla $r+n < m$ $\quad \vee \quad$ $k+1$ dla $r+n \geq m$

$$S = m \cdot k + (m-1) - (m - \lfloor r \rfloor) + 1 = m \cdot k + \lfloor r \rfloor = \lfloor x \rfloor$$

Zad. 4.

a) $f_n = f_{n-1} + 3^n$ dla $n > 1$; $f_1 = 3$

$$f_n = 3^n + 3^{n-1} + \dots + 3 = \sum_{k=1}^{n} 3^k = \frac{1-3^n}{1-3} \cdot 3 =$$

$$= \frac{3}{2}\left(1 - 3^n\right) = \frac{1}{2}\left(3^{n+1} - 3\right)$$

b) $h_n = h_{n-1} + (-1)^{n+1} n$ dla $n > 1$; $h_1 = 1$

$h_2 = -1$, $h_3 = 2$, $h_4 = -2 \dots$

$$h_n = (-1)^{n+1}\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)$$

D-d

$h_1 = 1$

$h_2 = -1$

Zał. $\forall_{n_0 < n}\ h_{n_0} = (-1)^{n_0+1}\left(\left\lfloor \frac{n_0+1}{2} \right\rfloor\right)$

1) $n = 2k$

$$h_n = h_{n-1} + (-1)^{n+1} n \overset{\text{zał}}{=} (-1)^n\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + (-1)^{n+1} n =$$

$$= (-1)^n\left(\left\lfloor \frac{n}{2} \right\rfloor \cancel{+n}\right) \overset{n=2k}{=} (-1)^n\left(\frac{n}{2} - n\right) =$$

$$= (-1)^n\left(-\frac{n}{2}\right) = (-1)^{n+1}\left(\frac{n}{2}\right) \overset{n=2k}{=} (-1)^{n+1}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \overset{n=2k}{=} (-1)^{n+1}\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)$$

2) $n = 2k+1$

$$h_n = h_{n-1} + (-1)^{n+1} \overset{\text{zał}}{=} h_{n-2} + (-1)^n (n-1) + (-1)^{n+1} n =$$

$$\overset{\text{zał}}{=} (-1)^{n-1}\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + (-1)^n (n-1) + (-1)^{n+1} n = (-1)^{n-1}\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right.$$

$$\left. - (n-1) + n\right) \overset{n=2k+1}{=} (-1)^{n-1}\left(\frac{n-1}{2} + 1\right) = (-1)^{n-1} \cdot (-1)^2 \left(\frac{n+1}{2}\right) \overset{n=2k+1}{=} (-1)^{n+1}\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)$$

c) $l_n = l_{n-1} l_{n-2}$ dla $n \geq 2$ i $l_1 = l_2 = 2$

$$l_n = l_{n-1} l_{n-2} = l_{n-2} l_{n-3} l_{n-2} = (l_{n-2})^2 l_{n-3} =$$

$$= (l_{n-3} l_{n-4})^2 l_{n-3} = (l_{n-3})^3 (l_{n-4})^2 = \ldots$$

$$= (l_{n-4} l_{n-5})^3 (l_{n-4})^2 = (l_{n-4})^5 (l_{n-5})^3 = \ldots =$$

$$= l_{n-(n-2)}^{F_{n-1}} l_{n-(n-1)}^{F_{n-2}} = l_2^{F_{n-1}} l_1^{F_{n-2}} = 2^{F_n}$$

Patryk Maciąg 331542 Lista 2

Zad. 5.

a) $a_0 = 1$ $a_n = \frac{2}{a_{n-1}}$

$a_1 = \frac{2}{1}$ $a_2 = \frac{2}{2}$ $a_3 = \frac{2}{1}$ $a_4 = \frac{2}{2}$

$a_n = (n\%2)+1$

D-d:

$a_0 = 1 = 1\%2+1$

Zał. $\forall_{n_0<n}$ $a_{n_0} = (n_0\%2)+1$

Dla n:

$$a_n = \frac{2}{a_{n-1}} = \frac{2}{((n-1)\%2)+1}$$

1) $n=2k \Rightarrow (n-1)\%2+1 = 2 \Rightarrow a_n = 1 = 0+1 = (n\%2)+1$

2) $n = 2k+1 \Rightarrow (n-1)\%2+1 = 1 \Rightarrow a_n = 2 = 1+1 =$
$= (n\%2)+1$

b) $b_0 = 0$ $b_n = \frac{1}{1+b_{n-1}}$

$b_1 = \frac{1}{1+0} = 1$ $b_2 = \frac{1}{1+1} = \frac{1}{2}$ $b_3 = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$ $b_4 = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$

$$b_n = \frac{F_n}{F_{n+1}}$$

$b_0 = 0 = \frac{0}{1} = \frac{F_0}{F_1}$

Zał. $\forall_{n_0<n}$ $b_{n_0} = \frac{F_{n_0}}{F_{n_0+1}}$

Dla n:

$$b_n = \frac{1}{1+b_{n-1}} = \frac{1}{1+\frac{F_{n-1}}{F_n}} = \frac{1}{\frac{F_{n-1}+F_n}{F_n}} = \frac{F_n}{F_{n+1}}$$ $\blacksquare$

c) $c_0 = 1 \qquad c_n = \sum_{i=0}^{n-1} c_i$

$c_1 = 1 \qquad c_2 = 2 \qquad c_3 = 4 \qquad c_4 = 8$

$c_n = 2^{n-1}$ dla $n > 0$

D-d)

$c_1 = 1 = 2^0$

Zał. $\forall_{n_0 < n} \ c_{n_0} = 2^{n_0 - 1}$

Pokażmy dla n

$\cancel{c_n = \sum_{i=0}^{n-1} c_i = \sum_{i=0}^{n-1} 2^{i-1} =}$

$c_n = \sum_{i=0}^{n-1} c_i = c_{n-1} + \sum_{i=0}^{n-2} c_i \overset{\text{def}}{=} c_{n-1} + c_{n-1} =$

$= 2c_{n-1} = 2 \cdot 2^{n-2} = 2^{n-1}$ $\square$

d) $d_0 = 1 \qquad d_1 = 2 \qquad d_n = \frac{(d_{n-1})^2}{d_{n-2}}$

$d_2 = \frac{4}{1} \qquad d_3 = \frac{16}{2} = 8 \qquad d_4 = \frac{64}{4} = 16 \qquad d_5 = \frac{16^2}{8} = 32$

$d_n = 2^n$

D-d)

Zał. $\forall_{n_0 < n} \ d_{n_0} = 2^{n_0}$

Pokażmy dla n:

$d_n = \frac{(d_{n-1})^2}{d_{n-2}} \overset{\text{zał}}{=} \frac{(2^{n-1})^2}{2^{n-2}} = 2^n$ $\square$

Zad. 6.

a) $y_0 = y_1 = 1 \qquad y_n = \dfrac{(y_{n-1})^2 + y_{n-2}}{y_{n-1} + y_{n-2}}$

$y_n = 1$

$\cancel{y_0 = y_1 = 1}$ D-d)

$y_0 = y_1 = 1$

Zał. $\forall_{n_0 < n} \quad y_{n_0} = 1$

Dla n:

$$y_n = \frac{(y_{n-1})^2 + y_{n-2}}{y_{n-1} + y_{n-2}} \overset{zał}{=} \frac{1^2 + 1}{1+1} = \frac{2}{2} = 1 \qquad \blacksquare$$

b) $z_0 = 1 \qquad z_1 = 2 \qquad z_n = \dfrac{(z_{n-1})^2 - 1}{z_{n-2}}$

$z_n = n+1$

D-d)

$z_0 = 0+1 = 1$

Zał. $\forall\, n_0 < n \quad z_{n_0} = n_0 + 1$

Dla n:

$$z_n = \frac{(z_{n-1})^2 - 1}{z_{n-2}} \overset{zał}{=} \frac{((n-1)+1)^2 - 1}{(n-2)+1} =$$

$$= \frac{n^2 - 1}{n-1} = \frac{(n-1)(n+1)}{n-1} = n+1 \qquad \blacksquare$$

c) $t_0 = 0 \quad t_1 = 1 \quad t_n = \dfrac{(t_{n-1} - t_{n-2} + 3)^2}{4}$

$t_n = n^2$

D-d

$t_0 = 0 = 0^2$

$t_1 = 1 = 1^2$

Zał. $\forall_{n_0 < n} \; t_{n_0} = n_0^2$

Dow:

$$t_n = \frac{(t_{n-1} - t_{n-2} + 3)^2}{4} = \frac{((n-1)^2 - (n-2)^2 + 3)^2}{4} = \frac{(n^2 - 2n + 1 - n^2 + 4n - 4 + 3)^2}{4} =$$

$$= \frac{(2n)^2}{4} = \frac{4n^2}{4} = n^2 \quad \blacksquare$$

Patryk Maciony 331542 Lista 2

Zad. 13.

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

Teza:

$$H_n = \frac{1}{n}\left(H_{n-1} + H_{n-2} + \dots + H_1\right) + 1 \quad \text{dla } n > 1$$

D-d)

$$H_n = 1 + \frac{1}{n}\sum_{i=1}^{n-1} H_i \overset{\cdot n}{\Longrightarrow} nH_n - n = \sum_{i=1}^{n-1} H_i$$

Indukcja:

$$n = 2 \longrightarrow 2H_2 - 2 = 2\left(1 + \frac{1}{2}\right) - 2 = 1 = H_1 = \sum_{i=1}^{1} H_i$$

Zał. $\forall_{n_0 < n} \quad n_0 H_{n_0} - n_0 = \sum_{i=1}^{n_0 - 1} H_i$

Dla n:

$$\sum_{i=1}^{n-1} H_i = H_{n-1} + \sum_{i=1}^{n-2} H_i \overset{\text{zał}}{=} H_{n-1} + (n-1)H_{n-1} - (n-1) =$$

$$= nH_{n-1} + n + 1 = n\Big(\underbrace{H_{n-1} + \frac{1}{n}}_{H_n} - \frac{1}{n}\Big) - (n-1) =$$

$$= nH_n - 1 - (n-1) = nH_n - n$$

∎

Zad. 14.

a) $F_0 + F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$

D-d)

$F_0 = 0 = 1 - 1 = F_{0+2} - 1$

Zał. $\forall_{n_0 < n} \sum_{i=0}^{n_0} F_i = F_{n_0+2} - 1$

Dla n:

$\sum_{i=0}^{n} F_i = \sum_{i=0}^{n-1} F_i + F_n = F_{n+1} - 1 + F_n = F_{n+2} - 1$

b) $F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}$ dla $n \geq 1$

D-d)

$n = 1 \rightarrow F_1 = F_{2\cdot 1} = 1$

Zał. $\forall_{n_0 < n} F_{2n_0} = F_1 + F_3 + \ldots + F_{2n_0-1}$

Dla n:

$F_{2n} = F_1 + F_3 + \ldots + F_{2n-1}$

$F_{2n} = F_{2n-1} + F_{2n-2} = F_{2n-1} + (F_1 + F_3 + \ldots + F_{2n-3})$

$\blacksquare$

c) $F_0^2 + F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$

D-d

1° $n=0 \rightarrow F_0^2 = 0 = F_0 F_{0+1}$

Zał. $\forall_{n_0 < n} \; F_0^2 + F_1^2 + \ldots + F_{n_0}^2 = F_{n_0} F_{n_0+1}$

Dla n:

$$F_n F_{n+1} = F_n (F_{n-1} + F_n) = F_n^2 + F_n F_{n-1} =$$

$$= \sum_{i=0}^{n-1} F_i^2 + F_n^2 = \sum_{i=0}^{n} F_i^2 \quad \blacksquare$$

d) $F_n F_{n+2} = F_{n+1}^2 + (-1)^{n+1}$

D-d

$n=0 \longrightarrow F_0 F_{0+2} = 0 = F_{0+1}^2 + (-1)^{0+1}$

Zał. $\forall_{n_0 < n} \; F_{n_0} F_{n_0+2} = F_{n_0+1}^2 + (-1)^{n_0+1}$

Dla n:

$$F_n F_{n+2} = F_n (F_n + F_{n+1}) = F_n^2 + F_n F_{n+1} =$$

$$= F_n^2 + F_n (F_{n-1} + F_n) = F_n^2 + F_{n-1} F_n + F_n^2 =$$

$$\overset{\text{zał}}{=} F_n^2 + F_{n-1} F_n + F_{n-1} F_{n+1} - (-1)^n =$$

$$= F_n^2 + F_{n-1} F_n + F_{n-1}(F_{n-1} + F_n) + (-1)^{n+1} =$$

$$= F_n^2 + F_{n-1} F_n + F_{n-1}^2 + F_{n-1} F_n + (-1)^{n+1} =$$

$$= \underbrace{F_n^2 + 2 F_{n-1} F_n + F_{n-1}^2}_{F_n^2} + (-1)^{n+1} = F_n^2 + (-1)^{n+1}$$

$\blacksquare$