Patryk Maciąg 331542 Lista 1

Zad. 1

$\lim_{n\to\infty} \frac{a_n}{b_n} < \infty \Rightarrow a_n = O(b_n)$

a) $a_n = \frac{2n^{81,2} + 3n^{45,1}}{4n^{23,3} + 5n^{11,3}}$

$\lim_{n\to\infty} \frac{a_n}{n^k} = \lim_{n\to\infty} \frac{2n^{81,2} + 3n^{45,1}}{4n^{23,3+k} + 5n^{11,3+k}} < \infty$

$81,2 - 23,3 - k \leq 0$

$k \geq 57,9$

$k_{min} = 57,9$

b) $a_n = 5^{\log_2 n}$

$\lim_{n\to\infty} \frac{5^{\log_2 n}}{n^k} = \lim_{n\to\infty} \frac{2^{\log_2 5 \cdot \log_2 n}}{n^k} = \lim_{n\to\infty} \frac{n^{\log_2 5}}{n^k} < \infty$

$k \geq \log_2 5$

$k_{min} = \log_2 5$

c) $a_n = (1{,}001)^n$

$$\lim_{n\to\infty} \frac{1{,}001^n}{n^k} = \lim_{n\to\infty} \log \frac{1{,}001^n}{n^k} = \lim_{n\to\infty} \left(\log 1{,}001^n - \log n^k\right) =$$

$$= \lim_{n\to\infty} n\left(\log 1{,}001 - k\frac{\log n}{n}\right) = \infty$$

$$k \in \emptyset$$

d) $a_n = n \log^3 n$

$$\lim_{n\to\infty} \frac{n\log^3 n}{n^k} = \cancel{\lim_{n\to\infty} \frac{\log^3 n + 3n\log^2 n \cdot \frac{\log e}{n}}{k n^{k-1}}} =$$

$$= \lim_{n\to\infty} \frac{\log^3 n}{n^{k-1}}$$

Wiemy, że $\forall_{x>0}\ \log^3 n = O(n^x)$.

Zatem:

$k-1>0$

$k>1 \Rightarrow k_{min} \in \emptyset$

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Zad. 2.

$(\log n = \log_2 n)$

$\log n$

$(\log n)^n = 2^{\log \log n \cdot n}$

$n^{\log n} = 2^{\log^2 n}$

$\log(n^n) = n \log n$

$3^{\log n} = 2^{\log 3 \cdot \log n} = n^{\log 3}$

n

$n^2 = 2^{2 \log n}$

$2^{\sqrt{n}}$

$1.01^n = 2^{\log 1.01 \cdot n}$

0.99^n

$(1 + \frac{1}{n})^n \longrightarrow e$

$0.99^n, (1+\frac{1}{n})^n, \log n, n, (\log n)^n, 3^{\log n}, n^2, n^{\log n}, 2^{\sqrt{n}},$

$1{,}01^n, \log(n^n)$

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Zad. 5.

a) $\underbrace{f = o(g)}_{\text{zał.}} \Rightarrow f = O(g)$

z def. $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$

def. Cauchy'ego
$\Rightarrow \quad \forall_{\varepsilon>0} \exists_{n_0>0} \forall_{n>n_0} \left| \frac{f(n)}{g(n)} - 0 \right| \leqslant \varepsilon$

$\Leftrightarrow \quad \forall_{\varepsilon>0} \exists_{n_0>0} \forall_{n>n_0} |f(n)| \leqslant \varepsilon \cdot |g(n)|$

z def.
$\Leftrightarrow \quad f(n) = O(g(n))$

b) $f \sim g \Rightarrow f = \Theta(g)$

z def.

$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 1 \overset{\text{def Cauchy'ego}}{\Longrightarrow} \forall_{\varepsilon>0} \exists_{n_0>0} \forall_{n>n_0} \left| \frac{f(n)}{g(n)} - 1 \right| \leqslant \varepsilon$

$-\varepsilon \leqslant \frac{f(n)}{g(n)} - 1 \leqslant \varepsilon$

$1 - \varepsilon \leqslant \frac{f(n)}{g(n)} \leqslant 1 + \varepsilon$

$(1-\varepsilon)\, g(n) \leq f(n) \leqslant (1+\varepsilon) \cdot g(n) \quad (*)$

(*) Zachodzi dla dowolnego $\varepsilon > 0$.
Czyli w szczególności dla $1-\varepsilon > 0$ i $1+\varepsilon > 0$

Niech $c = 1-\varepsilon$ i $d = 1+\varepsilon$.

$$\exists_{c>0\ d>0\ n_0>0} \forall_{n \geq n_0}\ c \cdot g(n) \leq f(n) \leq d \cdot g(n)$$

$$\overset{def}{\iff} f(n) = \Theta(g)$$

c) $f = O(g) \iff g = \Omega(f)$

$$f = O(g) \overset{def}{\iff} \exists_{c>0}^{n_0>0 \forall n > n_0} |f(n)| \leq c\,|g(n)|$$

$$\iff \exists_{c>0}\ g(n) \geq \frac{1}{c}|f(n)| \iff g = \Omega(f)$$

d) $f = O(g) \wedge g = O(f) \iff g = \Omega(f)$

$$f = O(g) \overset{def}{\iff} \exists_{c>0} |f(n)| \leq c\,|g(n)|$$

$$g = O(f) \overset{def}{\iff} \exists_{d>0} \underbrace{|g(n)| \leq d\,|f(n)|}_{n_0 \forall_{n \geq n_0}}$$

Zatem:

$$\exists_{c>0\ d>0\ n_0} \forall_{n \geq n_0} \frac{1}{c}|f(n)| \leq g(n) \leq d\,|f(n)|$$

$$\overset{def}{\iff} g = \Theta(f)$$

[illegible]

symbol	czy przechodnie?	czy symetryczne?
$\odot$	TAK	NIE
$\bigcirc$	TAK	NIE
$\sim$	TAK	TAK
$\circledsim$	TAK	TAK
Ω	TAK	NIE

Patryk Maciąży 331542 Lista 1.

Zad. 6.

$$e^{\frac{1}{n}} = 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

Dowód:

$$\left(\text{Szereg Maclaurina: } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

$$e^{\frac{1}{n}} = \sum_{i=0}^{\infty} \frac{\left(\frac{1}{n}\right)^i}{i!} = 1 + \frac{1}{n} + \sum_{i=2}^{\infty} \frac{1}{n^i i!} \leqslant 1 + \frac{1}{n} + \sum_{i=2}^{\infty} \frac{1}{n^2 2^i} =$$

$$= 1 + \frac{1}{n} + \frac{1}{n^2} \sum_{i=2}^{\infty} \frac{1}{2^i} = 1 + \frac{1}{n} + \frac{\frac{1}{4}}{1-\frac{1}{2}} \cdot \frac{1}{n^2} =$$

$$= 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right) \quad \blacksquare$$

Zad. 7.

Najgorszy możliwy przypadek:

najpierw wykonujemy $(n-1)$ porównań, potem $(n-2)$, $(n-3)$, itd.

Liczba porównań $= (n-1) + (n-2) + (n-3) + \ldots + 0 =$

$$= \sum_{i=1}^{n} (n-i) = n^2 - \sum_{i=1}^{n} i = n^2 - \frac{(n+1)n}{2} =$$

$$= \frac{n^2 - n}{2} = O(n^2)$$

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Zad. 14

$x \geqslant 0$

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor \leqslant \sqrt{\lfloor x \rfloor} < \lfloor \sqrt{\lfloor x \rfloor} \rfloor + 1$$

$$(\lfloor \sqrt{\lfloor x \rfloor} \rfloor)^2 \leqslant \overset{\leq x}{\lfloor x \rfloor} < (\lfloor \sqrt{\lfloor x \rfloor} \rfloor + 1)^2$$

$$(\lfloor \sqrt{\lfloor x \rfloor} \rfloor)^2 \leqslant x < (\lfloor \sqrt{\lfloor x \rfloor} \rfloor + 1)^2$$

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor \leqslant \sqrt{x} < \lfloor \sqrt{\lfloor x \rfloor} \rfloor + 1$$

$$\sqrt{\lfloor x \rfloor} = \lfloor \sqrt{x} \rfloor$$

$\boxed{\forall_{n \in N}\ \forall_{y \in R}\ n \leqslant \lfloor y \rfloor \Rightarrow n \leqslant y}$