Time Series and Forecasting
ARMA Models

Maria Eduarda Silva

FEP.UP

ARMA models

- Data is generally not well represented by AR or MA models
- Combine both $\rightarrow$ ARMA models
- Predictors include both **lagged values of Y_t** and **lagged errors**
- ARMA(1,1)

$$Y_t = aY_{t-1} + be_{t-1} + e_t$$

e_t iid, $|a| < 1$, $|b| < 1$

ARMA models

- ARMA(p, q)

$$Y_t = a_1 Y_{t-1} + \ldots + a_p Y_{t-p} + b_1 e_{t-1} + \ldots + b_q e_{t-q} + e_t$$

- $a_p \neq 0$, $b_q \neq 0$, p and q are called the AR and MA orders, respectively.
- Conditions on coefficients to ensure **stationarity and invertibility**
  - the roots of the (autoregressive) polynomial $\phi(z) = 1 - a_1 z - \ldots - a_p z^p$ are outside the unit circle $\longrightarrow$ stationarity
  - the roots of the polynomial (ma) $\theta(z) = 1 + b_1 z + \ldots + b_q z^q$ are outside the unit circle $\longrightarrow$ invertibility

Invertibility and Causality-example

$$Y_t = 1.5Y_{t-1} - 0.75Y_{t-2} - 0.5e_{t-1} + 0.4e_{t-2} + e_t$$
$$Y_t - 1.5Y_{t-1} + 0.75Y_{t-2} = 0.5e_{t-1} + e_t$$
$$(1 - 0.75B + 0.75B^2)Y_t = (1 + 0.5B)e_t$$
$$\phi(z) = 1 - 0.75z + 0.75z^2 \longrightarrow \text{ARpolynomial}$$
$$\theta(z) = 1 + 0.5z \longrightarrow \text{MApolynomial}$$

```
z=c(1,-1.5,0.75)
abs(polyroot(z))^2
1.333333 1.333333

z=c(1,0.5)
abs(polyroot(z))^2
4
```

Parameter redundancy or over -parametrization

$Y_t = 0.5Y_{t-1} - 0.5e_{t-1} + e_t$

Parameter redundancy or over -parametrization

$Y_t = 0.5Y_{t-1} - 0.5e_{t-1} + e_t$ looks like an ARMA(1,1) but it is just white noise

However if you fit an ARMA(1,1) to 150 iid r.v. the parameters are significant!!

It is easy to fit an overly complex ARMA model to data!!

```
x=rnorm(150)
arima(x,order=c(1,0,1))

Call:
arima(x = x, order = c(1, 0, 1))

Coefficients:
          ar1     ma1  intercept
      -0.7185  0.8148     0.0965
s.e.   0.1645  0.1327     0.0796
```

Checking for parameter redundancy

$$\phi(B)Y_t = \theta(B)e_t$$

The polynomials $\phi(z)$ and $\theta(z)$ must not have common roots.

$$Y_t = 0.3Y_{t-1} + 0.4Y_{t-2} + e_t + 0.5e_{t-1}$$

ARMA$(2, 1)$?

```
> AR=c(1,-0.3,-0.4)
> polyroot(AR)
[1]  1.25-0i -2.00+0i
> MA=c(1,0.5)
> polyroot(MA)
[1] -2+0i
```

$\phi(z) = (1 - 0.8z)(1 + 0.5z)$ $\theta(z) = (1 + 0.5z)$ One common root -2, ie, one commom factor which can be cancelled out.

So in fact ARMA(1,0) $Y_t = 0.8Y_{t-1} + e_t$

ACF of ARMA(1,1)- example

ARMA models

- Yule-Walker equations

$$\gamma_k = a_1\gamma_{k-1} + a_2\gamma_{k-2} + \ldots + a_p\gamma_{k-p}$$

  $k \geq \max p, q$

- ACF decays to zero
- PACF decays to zero
- Difficult to determine p and q from SACF and SPACF $\longrightarrow$ use automatic criteria such as AIC, AICc, BIC

Causal and Invertible representations of ARMA(p,q)

Causal form or infinite MA

$$Y_t = \phi(B)^{-1}\ \theta(B)\ e_t = \psi(B)\ e_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}$$

Example: $Y_t = .9Y_{t-1} + .5e_{t-1} + e_t$

```
ARMAtoMA(ar=0.9, m=0.5,10) #first 10 psi-weights

1.40 1.26 1.13 1.02 0.92 0.83 0.74 0.67 0.60
```

Causal and Invertible representations of ARMA(p,q)

Invertible form or infinite AR

$$e_t = \theta(B)^{-1}\ \phi(B)^{-1}\ Y_t = \pi(B)Y_t = \sum_{j=0}^{\infty} \pi_j Y_{t-j}$$

Example: $Y_t = .9Y_{t-1} + .5e_{t-1} + e_t$
reverse the role of Y_t and e_t:
$e_t = -0.5e_{t-1} + Y_t - 0.9Y_{t-1}$

```
ARMAtoMA(ar=0.5, m=-0.9,10)
 [1] -1.400000000  0.700000000 -0.350000000
 [4]  0.175000000 -0.087500000  0.043750000
 [7] -0.021875000  0.010937500 -0.005468750
[10]  0.002734375
```