Hamiltonian for the Ising Model with Fixed Sum of Spins

Introduction

This document derives the Hamiltonian for an Ising model where the spins s_i are constrained to sum to 1, and each spin takes values in the interval $[0, 1]$. The Hamiltonian is defined to be minimized when the volume of the hypercube formed by the spins is maximized, and maximized when the volume is minimized.

Definitions

Given a set of spins $\{s_1, s_2, \ldots, s_N\}$ such that:

$$\sum_{i=1}^{N} s_i = 1$$

the arithmetic mean (AM) and geometric mean (GM) are given by:

$$\mathrm{AM} = \frac{1}{N}$$

$$\mathrm{GM} = \left(\prod_{i=1}^{N} s_i \right)^{\frac{1}{N}}$$

Hamiltonian as the Ratio of GM to AM

We define the Hamiltonian H as the ratio of the geometric mean to the arithmetic mean. This ratio is:

$$\frac{\mathrm{GM}}{\mathrm{AM}} = \left(\prod_{i=1}^{N} s_i \right)^{\frac{1}{N}} \cdot N$$

Taking the natural logarithm of this ratio, we get:

$$\ln \left(\frac{\mathrm{GM}}{\mathrm{AM}} \right) = \ln \left(\left(\prod_{i=1}^{N} s_i \right)^{\frac{1}{N}} \cdot N \right)$$

This simplifies to:

$$\ln\left(\frac{\text{GM}}{\text{AM}}\right) = \ln\left(\left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}\right) + \ln(N)$$

Further simplification gives:

$$\ln\left(\frac{\text{GM}}{\text{AM}}\right) = \frac{1}{N}\sum_{i=1}^{N} \ln(s_i) + \ln(N)$$

Multiplying both sides by N to form the Hamiltonian H:

$$H = N\ln\left(\frac{\text{GM}}{\text{AM}}\right) = \sum_{i=1}^{N} \ln(s_i) + N\ln(N)$$

Summary

The Hamiltonian for the Ising model with the given constraints is:

$$H = \sum_{i=1}^{N} \ln(s_i) + N\ln(N)$$

This Hamiltonian is minimized when the geometric mean is maximized (i.e., when all spins are equal) and maximized when the geometric mean is minimized (i.e., when one spin is 1 and the rest are 0).