Hamiltonian for the Ising Model

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1 Introduction

This document provides a detailed derivation and explanation of the Hamiltonian for the Ising model where the Hamiltonian is minimized at the maximal possible volume of a hypercube with the perimeter fixed to the total sum of the spins, and maximized where the volume of the hypercube is minimized.

2 Maximizing Volume of a Hypercube

2.1 Arithmetic Mean and Geometric Mean

Given a set of n non-negative numbers $x_1, x_2, \ldots, x_n$ with a fixed sum S, the arithmetic mean (AM) and geometric mean (GM) are defined as:

$$\text{AM} = \frac{x_1 + x_2 + \ldots + x_n}{n} \tag{1}$$

$$\text{GM} = \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} \tag{2}$$

According to the AM-GM inequality:

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} \tag{3}$$

Equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

2.2 Maximizing Volume

The volume (or hypervolume) of a set of non-negative numbers is maximized when all the numbers are equal. Given:

$$nx = S \tag{4}$$

The common value x is:

$$x = \frac{S}{n} \tag{5}$$

Thus, the maximal volume is:

$$V = \left(\frac{S}{n}\right)^n \tag{6}$$

3 Hamiltonian for the Ising Model

3.1 Hamiltonian Minimization and Maximization

To define a Hamiltonian H that is minimized at the maximal possible volume of a hypercube with the perimeter fixed to the total sum of the spins, and maximized where the volume is minimized, we use the negative logarithm of the volume:

$$H = -\ln\left(\prod_{i=1}^{N} s_i\right) = -\sum_{i=1}^{N} \ln(s_i) \tag{7}$$

3.2 Ratio of Value to Arithmetic Mean

Starting from the perspective that the Hamiltonian at a specific site i is the ratio of the value of site i to the arithmetic mean:

$$\frac{s_i}{\text{AM}} = s_i \cdot N \tag{8}$$

The Hamiltonian at site i is:

$$H_i = -\ln(s_i \cdot N) \tag{9}$$

The total Hamiltonian is:

$$H = -\sum_{i=1}^{N} \ln(s_i \cdot N) \tag{10}$$

This expands to:

$$H = -\sum_{i=1}^{N} (\ln(s_i) + \ln(N)) = -\sum_{i=1}^{N} \ln(s_i) - N\ln(N) \tag{11}$$

4 Final Hamiltonian

By recognizing the Hamiltonian as an externally coupled field, we can write the final Hamiltonian as:

$$H = -\sum_{i=1}^{N} \ln(s_i) - N\ln(N) \tag{12}$$

Or equivalently:

$$H = \sum_{i=1}^{N} \ln(s_i \cdot N) \tag{13}$$