# Hamiltonian for the Ising Model with Fixed Sum of Spins

## Contents

# 1 Introduction

This document derives the Hamiltonian for an Ising model where the spins $s_i$ are constrained to sum to 1, and each spin takes values in the interval $[0, 1]$. Two Hamiltonian's are considered, $H_+$ and $H_-$. $H_+$ is minimized where the product of spins is maximized and maximized when said product is minimized. $H_-$ is the converse, minimized where the product is minimized and maximized where the product is maximized

# 2 Definitions

Given a set of spins $\{s_1, s_2, \ldots, s_N\}$, the arithmetic mean (AM) and geometric mean (GM) are given by:

$$\text{AM} = \frac{1}{N}\sum_{i=1}^{N} s_i$$

$$\text{GM} = \left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}$$

# 3 Hamiltonian as the Ratios of AM and GM

## 3.1 GM:AM Ratio is $H_+$

We define the Hamiltonian $H_+$ as the ratio of the geometric mean to the arithmetic mean. This ratio is:

$$\frac{\text{GM}}{\text{AM}} = \frac{\left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}}{\frac{1}{N}\sum_{i=1}^{N} s_i} = \left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}} \cdot \frac{N}{\sum_{i=1}^{N} s_i}$$

Taking the natural logarithm of this ratio, we get:

$$\ln\left(\frac{\text{GM}}{\text{AM}}\right) = \ln\left(\left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}} \cdot \frac{N}{\sum_{i=1}^{N} s_i}\right)$$

This simplifies to:

$$\ln\left(\frac{\mathrm{GM}}{\mathrm{AM}}\right) = \frac{1}{N}\sum_{i=1}^{N}\ln(s_i) + \ln(N) - \ln\left(\sum_{i=1}^{N} s_i\right)$$

Multiplying both sides by $N$ to form the Hamiltonian $H_+$:

$$H_+ = N\ln\left(\frac{\mathrm{GM}}{\mathrm{AM}}\right) = \sum_{i=1}^{N}\ln(s_i) + N\ln(N) - N\ln\left(\sum_{i=1}^{N} s_i\right)$$

Applying the constraint that the sum of the spins equals 1:

$$\sum_{i=1}^{N} s_i = 1$$

we get:

$$H_+ = \sum_{i=1}^{N}\ln(s_i) + N\ln(N) - N\ln(1) = \sum_{i=1}^{N}\ln(s_i) + N\ln(N)$$

### 3.2 AM:GM Ratio is $H_-$

We define the Hamiltonian $H_-$ as the ratio of the arithmetic mean to the geometric mean. This ratio is:

$$\frac{\mathrm{AM}}{\mathrm{GM}} = \frac{\frac{1}{N}\sum_{i=1}^{N} s_i}{\left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}} = \frac{1}{N}\sum_{i=1}^{N} s_i \cdot \left(\prod_{i=1}^{N} s_i\right)^{-\frac{1}{N}}$$

Taking the natural logarithm of this ratio, we get:

$$\ln\left(\frac{\mathrm{AM}}{\mathrm{GM}}\right) = \ln\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) - \ln\left(\left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}\right)$$

This simplifies to:

$$\ln\left(\frac{\mathrm{AM}}{\mathrm{GM}}\right) = \ln\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) - \frac{1}{N}\sum_{i=1}^{N}\ln(s_i)$$

Multiplying both sides by $N$ to form the Hamiltonian $H_-$:

$$H_- = N\ln\left(\frac{\mathrm{AM}}{\mathrm{GM}}\right) = N\ln\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) - \sum_{i=1}^{N}\ln(s_i)$$

Applying the constraint that the sum of the spins equals 1:

$$\sum_{i=1}^{N} s_i = 1$$

Substituting this into the Hamiltonian, we get:

$$H_- = N \ln\left(\frac{1}{N}\right) - \sum_{i=1}^{N} \ln(s_i) = -N\ln(N) - \sum_{i=1}^{N} \ln(s_i)$$

and have,

$$-H_+ = H_-$$

### 3.3 Minimization of $H_+$ and $H_-$

To see when $H_-$ is minimized, consider the case where one spin is 1 and the rest are 0:

$$s_1 = 1 \quad \text{and} \quad s_i = 0 \quad \text{for} \quad i \neq 1$$

Substituting these values into $H_-$:

$$H_- = -N\ln(N) - \left(\ln(1) + \sum_{i=2}^{N} \ln(0)\right)$$

Since $\ln(1) = 0$ and $\ln(0) \to -\infty$:

$$H_- \to -N\ln(N) + (N-1)(-\infty) \to -\infty$$

This indicates that $H_-$ tends to $-\infty$ as the product of the spins is minimized (i.e., when one spin is 1 and the rest are 0).

For $H_+$, when all spins are equal, $s_i = s$ for all $i$, and $\sum_{i=1}^{N} s_i = 1$ implies $s = \frac{1}{N}$:

$$H_+ = \sum_{i=1}^{N} \ln\left(\frac{1}{N} \cdot N\right) - N\ln(N) = -N\ln(N)$$

and so $H_+$ is minimized where $\sigma_i = \sigma_j$ for all spins $i, j$.

## 4 Combined Hamiltonian $H_\delta$

We can construct a combined Hamiltonian $H_\delta$ as an interpolation of $H_+$ and $H_-$:

$$H_\delta = (1-\delta)H_+ + \delta H_-$$

Substituting the expressions for $H_+$ and $H_-$:

$$H_\delta = (1-\delta)\left(\sum_{i=1}^{N} \ln(s_i) + N\ln(N)\right) + \delta\left(-N\ln(N) - \sum_{i=1}^{N} \ln(s_i)\right)$$

Simplifying, we get:

$$H_\delta = (1-2\delta)\left(\sum_{i=1}^{N}\ln(s_i) + N\ln(N)\right)$$

Here $H_\delta = H_+$ for $\delta = 0$ and $H_\delta = H_-$ for $\delta = 1$.

# 5 Lorenz Curve and Gini Coefficient

The Lorenz curve $L(x,\delta)$ for the combined Hamiltonian is given by:

$$L(x,\delta) = (1-\delta)x + \delta \cdot \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

The Gini coefficient $G(\delta)$ is calculated as:

$$G(\delta) = 1 - 2\int_0^1 L(x,\delta)\,dx$$

Integrating the Lorenz curve:

$$\int_0^1 L(x,\delta)\,dx = \int_0^1 \left((1-\delta)x + \delta \cdot \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}\right) dx$$

Since $\delta \cdot 0$ is zero for $0 \leq x < 1$:

$$\int_0^1 L(x,\delta)\,dx = \int_0^1 (1-\delta)x\,dx + \delta \int_0^1 \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases} dx$$

For the first part:

$$\int_0^1 (1-\delta)x\,dx = (1-\delta)\left[\frac{x^2}{2}\right]_0^1 = (1-\delta)\cdot\frac{1}{2}$$

For the second part:

$$\delta\int_0^1 \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases} dx = \delta\cdot 0 + \delta\cdot\int_0^0 dx = 0$$

So:

$$\int_0^1 L(x,\delta)\,dx = (1-\delta)\cdot\frac{1}{2}$$

The Gini coefficient is:

$$G(\delta) = 1 - 2\left((1-\delta)\cdot\frac{1}{2}\right) = 1-(1-\delta) = \delta$$

providing the rather intuitive result that,

$$G(\delta) = \delta$$

# 6 Alternative Formulation of the Combined Hamiltonian

In this section, we introduce a new formulation of the Hamiltonian $H_\delta$ which interpolates between the ratio of the arithmetic mean (AM) to the geometric mean (GM) and its inverse, depending on a parameter $\delta$. The Hamiltonian is defined as:

$$H_\delta = \left(\frac{\text{AM}}{\text{GM}}\right)^\delta$$

where $\delta \in [-1, 1]$.

## 6.1 Hamiltonian Formulation

The Hamiltonian $H$ is formulated as:

$$H_\delta = \left(\frac{\frac{1}{N}\sum_{i=1}^N s_i}{\left(\prod_{i=1}^N s_i\right)^{\frac{1}{N}}}\right)^\delta$$

Simplifying, we get:

$$H_\delta = \left(\frac{1}{N}\sum_{i=1}^N s_i\right)^\delta \cdot \left(\left(\prod_{i=1}^N s_i\right)^{-\frac{1}{N}}\right)^\delta$$

Further simplifying:

$$H_\delta = \left(\frac{1}{N}\sum_{i=1}^N s_i\right)^\delta \cdot \left(\prod_{i=1}^N s_i\right)^{-\frac{\delta}{N}}$$

## 6.2 Behavior at Extremes of $\delta$

For $\delta = 1$:

$$H_\delta = \frac{\text{AM}}{\text{GM}} = H_-$$

For $\delta = -1$:

$$H_\delta = \frac{\text{GM}}{\text{AM}} = H_+$$

For $\delta = 0$:

$$H_\delta = 1$$

For $0 < \delta < 1$, the Hamiltonian favors a more egalitarian distribution, but not perfectly uniform. For $-1 < \delta < 0$, the Hamiltonian favors a more totalitarian distribution, but not perfectly concentrated in one spin.

# 7 Transformation and Equivalence of Hamiltonians

In this section, we show the equivalence between the combined Hamiltonian $H_{\delta_0} = (1-\delta_0)H_+ + \delta_0 H_-$ and the new formulation $H_{\delta_1} = \left(\frac{\text{AM}}{\text{GM}}\right)_1^\delta$ under an appropriate transformation of $\delta$.

## 7.1 Combined Hamiltonian

The combined Hamiltonian is defined as:

$$H_{\delta_0} = (1-\delta_0)H_+ + \delta_0 H_-$$

where:

$$H_+ = \sum_{i=1}^{N} \ln(s_i) + N\ln(N)$$

$$H_- = -N\ln(N) - \sum_{i=1}^{N} \ln(s_i)$$

Substituting $H_+$ and $H_-$:

$$H_{\delta_0} = (1-\delta_0)\left(\sum_{i=1}^{N} \ln(s_i) + N\ln(N)\right) + \delta_0\left(-N\ln(N) - \sum_{i=1}^{N} \ln(s_i)\right)$$

Simplifying:

$$H_{\delta_0} = (1-2\delta_0)\left(\sum_{i=1}^{N} \ln(s_i) + N\ln(N)\right)$$

## 7.2 Transformation and Equivalence

To show the equivalence, we apply the transformation $\delta' = 1-2\delta$. Setting $\delta' = 1-2\delta$:

$$1-2\delta = \delta'$$

Solving for $\delta$:

$$\delta = \frac{1-\delta'}{2}$$

Substituting $\delta = \frac{1-\delta'}{2}$ into the reformulated Hamiltonian:

$$H = \left(\frac{1}{N}\sum_{i=1}^{N} s_i\right)^{\frac{1-\delta'}{2}} \cdot \left(\left(\prod_{i=1}^{N} s_i\right)^{-\frac{1-\delta'}{2N}}\right)$$

Simplifying using exponent and logarithm relationship:

$$H = e^{\frac{1-\delta'}{2}\left(\ln\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) - \frac{1}{N}\sum_{i=1}^{N}\ln(s_i)\right)}$$

$$H = e^{\frac{1-\delta'}{2}\left(\ln\left(\frac{\text{AM}}{\text{GM}}\right)\right)}$$

Final equivalence:

$$H = \left(\frac{\text{AM}}{\text{GM}}\right)^{\frac{1-\delta'}{2}}$$

Thus, under the transformation $\delta' = 1 - 2\delta$, the combined Hamiltonian $H = (1-\delta)H_+ + \delta H_-$ is equivalent to the new formulation $H = \left(\frac{\text{AM}}{\text{GM}}\right)^{\delta}$.

### 7.3 Hamiltonian and Parameterization

The Hamiltonian is defined as:

$$H = \left(\frac{\text{AM}}{\text{GM}}\right)^{\delta}$$

where:

$$\text{AM} = \frac{1}{N}\sum_{i=1}^{N} s_i$$

$$\text{GM} = \left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}$$

Taking the natural logarithm of $H$:

$$\ln H = \delta \ln\left(\frac{\text{AM}}{\text{GM}}\right)$$

Expressing $\ln\left(\frac{\text{AM}}{\text{GM}}\right)$:

$$\ln\left(\frac{\text{AM}}{\text{GM}}\right) = \ln\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) - \frac{1}{N}\sum_{i=1}^{N} \ln s_i$$

Thus:

$$\ln H = \delta\left(\ln\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) - \frac{1}{N}\sum_{i=1}^{N} \ln s_i\right)$$

## 8 Entropy in Terms of $\delta$ in the Microcanonical Ensemble

In this section, we derive an expression for the entropy $S$ in terms of the parameter $\delta$ under the assumptions of the microcanonical ensemble, where the total energy $E$ is fixed.

## 8.1 Hamiltonian and Spin Distribution

The Hamiltonian is given by:

$$H = \left(\frac{\text{AM}}{\text{GM}}\right)^{\delta}$$

where the arithmetic mean (AM) and geometric mean (GM) are defined as:

$$\text{AM} = \frac{1}{N}\sum_{i=1}^{N} s_i$$

$$\text{GM} = \left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}$$

Given the fixed total energy $E$:

$$\sum_{i=1}^{N} s_i = E$$

## 8.2 Entropy and Probability Distribution

The entropy $S$ for a set of spins $\{s_i\}$ is:

$$S = -\sum_{i=1}^{N} p_i \ln p_i$$

where $p_i = \frac{s_i}{\sum_{j=1}^{N} s_j} = \frac{s_i}{E}$. Thus, the entropy becomes:

$$S = -\sum_{i=1}^{N} \frac{s_i}{E} \ln\left(\frac{s_i}{E}\right)$$

## 8.3 Entropy in Terms of $\delta$

First, express $H$ in terms of $s_i$:

$$H = \left(\frac{\frac{E}{N}}{\left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}}\right)^{\delta} = \left(\frac{E}{N \cdot \text{GM}}\right)^{\delta}$$

Taking the logarithm of $H$:

$$\ln H = \delta\left(\ln\frac{E}{N} - \ln \text{GM}\right)$$

Recall that:

$$\ln \mathrm{GM} = \frac{1}{N}\sum_{i=1}^{N} \ln s_i$$

So:

$$\ln H = \delta \left( \ln \frac{E}{N} - \frac{1}{N}\sum_{i=1}^{N} \ln s_i \right)$$

The entropy $S$ is:

$$S = -\sum_{i=1}^{N} \frac{s_i}{E} \ln s_i + \ln E$$

Given:

$$\ln H = \delta \left( \ln \frac{E}{N} - \frac{1}{N}\sum_{i=1}^{N} \ln s_i \right)$$

We rearrange terms to express $\ln s_i$ in terms of $\delta$:

$$-\frac{1}{N}\sum_{i=1}^{N} \ln s_i = \frac{\ln E}{N} - \frac{\ln N}{N} - \frac{\ln H}{\delta}$$

Thus:

$$S = -\sum_{i=1}^{N} \frac{s_i}{E} \ln s_i + \ln E = \delta \ln \frac{E}{N} + \ln E = \delta \ln N + \ln E$$

In summary:

$$S = (1+\delta)\ln E - \delta \ln N$$

### 8.4 Summary

Under the assumptions of the microcanonical ensemble with fixed total energy $E$, the entropy $S$ in terms of $\delta$ is given by:

$$S(\delta) = (1+\delta)\ln E - \delta \ln N$$

This expression reflects how the parameter $\delta$ influences the entropy, interpolating between minimal entropy (concentrated distribution) and maximal entropy (uniform distribution).

## 9 Entropy Maximization in Microcanonical and Canonical Ensembles

In this section, we derive the relationship between entropy $S$, the Hamiltonian $H$, and the parameter $\delta$ in both the microcanonical and canonical ensembles. We show that entropy is maximized when $\delta = 1$ in the microcanonical ensemble and when $\delta = -1$ in the canonical ensemble.

## 9.1 Microcanonical Ensemble (Fixed Total Energy)

In the microcanonical ensemble, the total energy $E$ is fixed.

### 9.1.1 Hamiltonian

The Hamiltonian is given by:

$$H = \left(\frac{\text{AM}}{\text{GM}}\right)^{\delta}$$

where:

$$\text{AM} = \frac{1}{N}\sum_{i=1}^{N} s_i = \frac{E}{N}$$

$$\text{GM} = \left(\prod_{i=1}^{N} s_i\right)^{\frac{1}{N}}$$

### 9.1.2 Entropy in Terms of Spin Distribution

The entropy $S$ for a set of spins $\{s_i\}$ is given by:

$$S = -\sum_{i=1}^{N} p_i \ln p_i$$

Since $p_i = \frac{s_i}{E}$, we have:

$$S = -\sum_{i=1}^{N} \frac{s_i}{E} \ln\left(\frac{s_i}{E}\right) = -\sum_{i=1}^{N} \frac{s_i}{E}(\ln s_i - \ln E)$$

$$S = -\sum_{i=1}^{N} \frac{s_i}{E} \ln s_i + \ln E$$

### 9.1.3 Relate Entropy to $\delta$

Given:

$$\ln H = \delta\left(\ln\frac{E}{N} - \ln \text{GM}\right)$$

where:

$$\ln \text{GM} = \frac{1}{N}\sum_{i=1}^{N} \ln s_i$$

Thus:

$$\ln H = \delta\left(\ln\frac{E}{N} - \frac{1}{N}\sum_{i=1}^{N} \ln s_i\right)$$

Using the expression for entropy:

$$S = -\sum_{i=1}^{N} \frac{s_i}{E} \ln s_i + \ln E$$

Isolate the $\ln s_i$ term:

$$-\frac{1}{N} \sum_{i=1}^{N} \ln s_i = \frac{\ln E}{N} - \frac{\ln N}{N} - \frac{\ln H}{\delta}$$

Rewriting entropy $S$ in terms of $\delta$:

$$S = -\sum_{i=1}^{N} \frac{s_i}{E} \ln s_i + \ln E = \delta \ln \frac{E}{N} + \ln E = (1+\delta) \ln E - \delta \ln N$$

Thus, in the microcanonical ensemble:

$$S(\delta) = (1+\delta) \ln E - \delta \ln N$$

### 9.1.4 Maximizing Entropy

1. **For $\delta = 1$**:

$$S(1) = 2 \ln E - \ln N$$

Entropy is maximized.

2. **For $\delta = 0$**:

$$S(0) = \ln E$$

3. **For $\delta = -1$**:

$$S(-1) = 0$$

**Conclusion:** In the microcanonical ensemble with fixed total energy, entropy is maximized when $\delta = 1$, corresponding to a uniform distribution of resources.

## 9.2 Canonical Ensemble (Unbounded Energy)

In the canonical ensemble, the energy is not fixed but can fluctuate, characterized by a temperature parameter $T$.

### 9.2.1 Partition Function

The partition function $Z$ is given by:

$$Z = \sum_{i} e^{-\beta E_i}$$

where $\beta = \frac{1}{k_B T}$.

### 9.2.2 Average Energy

The average energy $\langle E \rangle$ is the expected value of the energy, given by:

$$\langle E \rangle = \sum_i E_i \frac{e^{-\beta E_i}}{Z} = -\frac{\partial \ln Z}{\partial \beta}$$

### 9.2.3 Entropy

The entropy $S$ in the canonical ensemble is given by:

$$S = k_B \left( \ln Z + \beta \langle E \rangle \right)$$

This expression for entropy is derived from the thermodynamic potential for the canonical ensemble, known as the Helmholtz free energy $F$:

$$F = \langle E \rangle - TS$$

Rearranging for $S$:

$$S = \frac{\langle E \rangle - F}{T}$$

Using $F = -k_B T \ln Z$, we get:

$$S = \frac{\langle E \rangle + k_B T \ln Z}{T} = k_B \left( \ln Z + \beta \langle E \rangle \right)$$

### 9.2.4 Hamiltonian in Terms of $\delta$

For the system with the Hamiltonian given by:

$$H = \left( \frac{\mathrm{AM}}{\mathrm{GM}} \right)^{\delta}$$

where:

$$\mathrm{AM} = \frac{1}{N} \sum_{i=1}^{N} E_i = \frac{\langle E \rangle}{N}$$

$$\mathrm{GM} = \left( \prod_{i=1}^{N} E_i \right)^{\frac{1}{N}}$$

The Hamiltonian can be rewritten as:

$$H = \left( \frac{\frac{\langle E \rangle}{N}}{\left( \prod_{i=1}^{N} E_i \right)^{\frac{1}{N}}} \right)^{\delta} = \left( \frac{\langle E \rangle}{\left( \prod_{i=1}^{N} E_i \right)^{\frac{1}{N}}} \right)^{\delta}$$

Taking the natural logarithm:

$$\ln H = \delta \left( \ln \langle E \rangle - \frac{1}{N} \sum_{i=1}^{N} \ln E_i \right)$$

### 9.2.5 Connecting Entropy to $\delta$

To understand why $\delta = -1$ maximizes the entropy in the canonical ensemble, consider the following:

- The Boltzmann distribution maximizes entropy subject to the constraint of fixed average energy $\langle E \rangle$. This distribution is characterized by an exponential decay of probabilities with energy, reflecting a concentrated distribution.

- When $\delta = -1$, the Hamiltonian becomes:

$$H = \left(\frac{\mathrm{GM}}{\mathrm{AM}}\right) = \left(\frac{\left(\prod_{i=1}^{N} E_i\right)^{\frac{1}{N}}}{\frac{1}{N}\sum_{i=1}^{N} E_i}\right)$$

  This form emphasizes the geometric mean over the arithmetic mean, favoring a distribution where energies are more sharply peaked, aligning with the Boltzmann distribution.

### 9.2.6 Maximizing Entropy

The entropy $S$ in the canonical ensemble can be expressed in terms of the distribution of $E_i$:

$$S = -k_B \sum_i p_i \ln p_i$$

where $p_i = \frac{e^{-\beta E_i}}{Z}$. For $\delta = -1$, the distribution of $E_i$ reflects the natural exponential distribution, which maximizes entropy in the canonical ensemble.

### 9.2.7 Conclusion

For the canonical ensemble, the entropy $S$ is maximized when $\delta = -1$, as this configuration leads to the Boltzmann distribution. The Boltzmann distribution is the one that maximizes the entropy for a given average energy $\langle E \rangle$. This reflects a concentrated distribution of energies, emphasizing the geometric mean over the arithmetic mean, which is characteristic of the canonical ensemble with unbounded energy.