A Resolvent Framework for Global and Local Nonlinear Semigroups

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Contents

Symbols

X	Banach space $(X, \|\|\cdot\|\|)$
$w(x, x_t, \theta) : X \times X \times \mathbb{R} \to \mathbb{R}$	Weighting kernel at x_t, parameterized by $\theta \in \mathbb{R}, x_t \in X$

1 Definitions

Definition 1 *The process-t matrix, read “process until t matrix” or more simply “process until t”, is*

$$X_t = \begin{bmatrix} x_t \\ x_{t-h} \\ x_{t-2h} \\ \vdots \end{bmatrix}$$

where rows belong to X.

Definition 2 *The t-weighting matrix of the process-i is given by*

$$W(X_i, x_t, \theta) = \begin{bmatrix} w(x_j, x_t, \theta) & 0 & 0 & \cdots \\ 0 & w(x_{j-h}, x_t, \theta) & 0 & \cdots \\ 0 & 0 & w(x_{j-2h}, x_t, \theta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$W_t^\theta X_i$ is short hand for the the product $W(X_i, x_t, \theta) X_i$

$$W_t^\theta X_i := W(X_i, x_t, \theta) \cdot X_i = \begin{bmatrix} w(x_i, x_t, \theta) \cdot x_i \\ w(x_{i-h}, x_t, \theta) \cdot x_{i-h} \\ w(x_{i-2h}, x_t, \theta) \cdot x_{i-2h} \\ \vdots \end{bmatrix}$$

which is the t-weighting of the process-i.

2 Introduction

2.1 C as a globally linear map

Consider the map

$$C : x_t \mapsto x_{t+h}$$

for $x_i \in X_t \subset X$ and

$$x_{t+h} = C x_t$$

C may be state dependent, $C(x(t))$, non-autonomous, $C(t)$, or both, C(t, x(t)). We denote the possibility of any such case as C_t.

If C_t is globally linear then

$$X_{t+h} = X_t C_t$$

which is solved via,

$$C_t = X_t^{-1} X_{t+h}$$

In an applied time series setting this solution for C_t is the *auto-regressive* or AR model for the process X_t.

2.2 C^θ as a locally linear map

If C is not globally linear than the t-weighting can be introduced such that

$$W_t^\theta X_{t+h} = W_t^\theta X_t C_t^\theta$$

where the weighting kernel, w, of W_t^θ is parameterized by θ. The exact local weighting at x_t is achieved as,

$$\lim_{\theta\to\infty} w(x_i, x_t, \theta) = \delta(x_i - x_t)$$

for all $x_i \in X_t$. Typically the kernel is chosen so $w \sim e^{-\theta}$. The solution for C_t^θ is

$$\begin{aligned} C_t^\theta &= \left(W_t^\theta X_t\right)^{-1} W_t^\theta X_{t+h} \\ &= X_t^{-1}(W_t^\theta)^{-1} W_t^\theta X_{t+h} \end{aligned}$$

Taking the limit, W_t^θ reduces to,

$$\lim_{\theta\to\infty} W_t^\theta = \begin{cases} \delta(x_i - x_t) = 1 & \text{if } i = j \wedge x_i = x_t \\ 0 & \text{if } i \neq j \vee x_i \neq x_t \end{cases}$$

which is a diagonal matrix whose only non-zero entries are 1's corresponding to states arbitrarily close to x_t. If X_t never returns to states arbitrarily close to x_t then W_t^θ reduces to the rank-1 matrix with a single non-zero entry,

$$(W_t^\theta)_{1,1} = 1$$

If X_t is periodic at frequency k the set of all indices i on the diagonal where $(W_t^\theta)_{i=j} = 1$, is

$$\{W_t^\theta\}_1 = \{n \in \mathbb{N} : i = 1 + nk\}$$

If X_t is ergodic than the rank depends on the nature and frequency of close returns to neighbourhoods containing x_t, e.g.

$$U = \{x \in X, \delta = a \in \mathbb{R} : w(x, x_t, \theta) < \delta\}$$

could be used to define "close".

2.2.1 Factorization of C_t^θ

Considering,

$$C_t^\theta = (X_t)^{-1}(W_t^\theta)^{-1}(W_t^\theta X_{t+h}) \tag{1}$$

we take the pseudoinverse of the t-weighting,

$$(W_t^\theta)^{-1} = ({W_t^\theta}^T W_t^\theta)^{-1} {W_t^\theta}^T$$

and eigen decompose the covariance term and invert for

$$(W_t^\theta)^{-1} = (Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t}^{-1} Q_{\mathrm{w}_t}^T) {W_t^\theta}^T$$

We can perform the same operations for X_t,

$$(X_t)^{-1} = (Q_{\mathrm{x}_t} \Lambda_{\mathrm{x}_t}^{-1} Q_{\mathrm{x}_t}^T) X_t^T$$

Substituting into (1) gives

$$\begin{aligned} C_t^\theta &= (Q_{\mathrm{x}_t} \Lambda_{\mathrm{x}_t}^{-1} Q_{\mathrm{x}_t}^T) X_t^T \cdot (Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t} Q_{\mathrm{w}_t}^T) {W_t^\theta}^T \cdot W_t^\theta X_{t+h} \\ &= (Q_{\mathrm{x}_t} \Lambda_{\mathrm{x}_t}^{-1} Q_{\mathrm{x}_t}^T) X_t^T \cdot (Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t}^{-1} Q_{\mathrm{w}_t}^T) \cdot ({W_t^\theta}^T W_t^\theta) \cdot X_{t+h} \end{aligned} \tag{2}$$

where $(\cdot)$ is simply added for readability and is not the dot product.

2.2.2 P_U: The reverberation of C_t^θ

We refer to the covariance of the t-weighting as the ***reverberation*** of X_t at t.

$$\mathrm{P}_U(X_t, t, \theta, \omega) = \mathrm{Cov}(W_t^\theta) = {W_t^\theta}^T W_t^\theta$$

where P is the Greek capital rho.

$\mathrm{P}_U(X_t, t, \theta, \omega)$ can be thought of as defining a fuzzy elliptic neighbourhood about x_t that describes close returns to x_t. It is "fuzzy" not in the set theoretic sense, although that could be an interesting extension, but for $\theta < \infty$ all $x \in X$ are included with some x having a greater weight than others.

For example, given the kernel

$$w \sim e^{-\theta||x - x_t||} \quad \theta \in \mathbb{R}$$

when $\theta = 0$ all members of X are weighted equally where

$$\begin{aligned} W_t^0 &= \begin{bmatrix} w(x_t, x_t, 0) & 0 & \dots \\ 0 & w(x_t, x_{t-h}, 0) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= I \end{aligned}$$

which gives us the global linear map

$$\begin{aligned} C_t^0 &= (X_t)^{-1}(W_t^0)^{-1} W_t^0 X_{t_t+h} \\ &= (X_t)^{-1} X_{t_t+h} \end{aligned}$$

Conversely, in the $\lim \theta \to \infty$ the only elements that are prominent members of the neighbourhood are those that can be made arbitrarily close; $\epsilon - \delta$ reasoning clarifies.

We may define some small and bounded reverberative neighbourhood centered on x_t,

$$\{x_i \in X : x_i \in U(x_t, \epsilon) \leftrightarrow ||x_j - x_t|| < \epsilon\}$$

whose members are prominent in x_t's neighbourhood. For any $\epsilon > 0$, it can be shown there exists a θ_0 such that $\forall \theta > \theta_0$ the reverberation of states x_j outside the bounded neighborhood $U(x_t, \epsilon)$ are less participatory for some level $\delta > 0$

$$\sup_{x_j \in U(x_t, \epsilon)} ||({W_t^\theta}^T W_t^\theta)\, x_j|| < \delta$$

In all cases, in the limit we have $w(x_i) = \delta(x_i - x_t)$ and the measure of the members comprising the reverberation collapses to 0.

$$\inf \lim_{\theta \to \infty} \mu(\{x \in X : \mathrm{P} x_i = 0\}) = 0$$

We assume the parties aren't very fun.

2.2.3 P_λ: The reverberative map of C_t^θ

In 2, the modulating term of the reverberation is the inverse of the the reverberation,

$$\mathrm{P}_\lambda(X_t, t, \theta, \omega) = Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t}^{-1} Q_{\mathrm{w}_t}^T$$

which we refer to as the ***reverberative map*** of X_t at t. Here, Q_{w_t} determines the directions in X along which X_t reverberates more or less strongly with Λ_{w_t} corresponding to the magnitude of reverberation along such directions.

Taking the inverse of the reverberation re-scales the eigenvalues by the reciprocal. The result is an amplification of the weakest effects of the reverberation and a reduction in the strongest.

The effect of the product of the reverberative map and the reverberation,

$$I = (Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t}^{-1} {Q_{\mathrm{w}_t}}^T)({W_t^0}^{-1} W_t^0)$$

ensures no one is left behind when placing greater emphasis on the more prominent members of the reverberative neighbourhood.

2.3 $\{C^\theta(t)\}_{t\geq 0}$ as a nonlinear semigroup

We can place C_t in the more general context of operator theory. In this case we are no longer reasoning about strictly empirical observations, but considering members of the C_0 semigroup $\{X_t\} = \{C(t)\}_{t\geq 0}$.

The infinitesimal generator A for the semigroup $C(t)\}_{t\geq 0}$ is

$$A(t)x(t) = \lim_{h\to 0} \lim_{\theta\to\infty} \frac{C^\theta(t)x(t) - x(t+h)}{h}$$

where $D(A) \in X$ is the subspace where A is a well defined infinitesimal generator. A is a well defined infinitesimal generator if its resolvent operator,

$$R(\lambda, A) = (\lambda I - A)^{-1} \qquad \lambda \in \mathbb{C} \tag{3}$$

provides a non empty set of λ where the inverse is defined. This is the resolvent set of A, denoted $\rho(A)$.

We take three cases: (i) $\theta = 0$, (ii) $\theta \in (0, \infty)$, (iii) $\theta \to \infty$.

For $\theta = 0$, as noted above, we have

$$\begin{aligned} A(t)x(t) &= \lim_{h\to 0^+} \frac{C^0(t)x(t) - x(t+h)}{h} \\ &= \lim_{h\to 0^+} (X_t^{-1} X_{t+h} x(t) - x(t+h)) \frac{1}{h} \end{aligned}$$

C may require higher forms,

$$C_h^0(t) = X_t^{-1} X_{t+h} = I + hC_h^0(t) + O(h^2)$$

where we can account for the quadratic

$$C_{2h}^0(t) = {X_t}^{-1} X_{t+2h} = I + h^2 C_{2h}^0(t) + 0(h^3)$$

and generally

$$
\begin{aligned}
C_h^0(t) &= I + hC_h^0(t) + 0(h^2) + I + h^2 C_{2h}^0(t) + 0(h^3) + I + h^3 C_{3h}^0(t) + 0(h^4) + \dots \\
&= \sum_{n=0}^{\infty} nI + h^n C_n h^0(t) + O(h^{n+1}) \\
&= \sum_{n=0}^{\infty} nI + h^n C_{nh}^0(t) + \sum_{n=0}^{\infty} O(h^{n+1}) \\
&= \sum_{n=0}^{\infty} X_t^{-1} X_{t+nh} + R_h(t,n)
\end{aligned}
$$

If

$$
\lim_{h\to 0^+} R_h(t,n) = 0
$$

then,

$$
\begin{aligned}
\lim_{h\to 0^+} C_h^0(t) &= \lim_{h\to 0^+} \sum_{n=0}^{\infty} X_t^{-1} X_{t+nh} + \lim_{h\to 0^+} R_h(t,n) \\
C^0(t) &= \sum_{n=0}^{\infty} X_t^{-1} X_{t+nh}
\end{aligned}
$$

and $\{X_t\} = \{C^0(t)\}_{t\geq 0}$ is a ***globally nonlinear*** semigroup.

On the other hand, if

$$
\lim_{h\to 0^+} \lim_{\theta\to\infty} R_h(t,n,\theta) = 0
$$

for

$$
\begin{aligned}
\lim_{h\to 0^+} \lim_{\theta\to\infty} C_h^\theta(t) &= \lim_{h\to 0^+} \lim_{\theta\to\infty} \sum_{n=0}^{\infty} X_t^{-1} (W_t^\theta)^{-1} W_t^\theta X_{t+nh} + \lim_{h\to 0^+} \lim_{\theta\to\infty} R_h(t,n,\theta) \\
C^0(t) &= \sum_{n=0}^{\infty} X_t^{-1} (W_t^\theta)^{-1} W_t^\theta X_{t+nh}
\end{aligned}
$$

then $\{X_t\} = \{C^\theta(t)\}_{t\geq 0}$ is a ***locally nonlinear semigroup***.

Here the reverberation,

$$
P_U(X_t, t, \theta, w) = {W_t^\theta}^T W_t^\theta
$$

and it's reveberative map,

$$
P_\lambda(X_t, t, \theta, w) = ({W_t^\theta}^T W_t^\theta)^{-1} = Q_{\mathrm{w}_t^\theta} \Lambda_{\mathrm{w}_t^\theta}^{-1} Q_{\mathrm{w}_t^\theta}^T
$$

describe cumulative strength and direction of all higher order terms at $x(t)$.