

A Resolvent Framework for Global and Local Nonlinear Semigroups

Patrick S. Mahon

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Symbols

X	Banach space (X, \cdot)
$w(x, x_t, \theta) : X \times X \times \mathbb{R} \rightarrow \mathbb{R}$	Weighting kernel at x_t , parameterized by $\theta \in \mathbb{R}, x_t \in X$

1 Definitions

Definition 1. *The process- t matrix, read “process until t matrix” or more simply “process until t ”, is*

$$X_t = \begin{bmatrix} x_t \\ x_{t-h} \\ x_{t-2h} \\ \vdots \end{bmatrix}$$

where rows belong to X .

Definition 2. The t -weighting matrix of the process- i is given by

$$W(X_i, x_t, \theta) = \begin{bmatrix} w(x_j, x_t, \theta) & 0 & 0 & \cdots \\ 0 & w(x_{j-h}, x_t, \theta) & 0 & \cdots \\ 0 & 0 & w(x_{j-2h}, x_t, \theta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$W_t^\theta X_i$ is short hand for the the product $W(X_i, x_t, \theta)X_i$

$$W_t^\theta X_i := W(X_i, x_t, \theta) \cdot X_i = \begin{bmatrix} w(x_i, x_t, \theta) \cdot x_i \\ w(x_{i-h}, x_t, \theta) \cdot x_{i-h} \\ w(x_{i-2h}, x_t, \theta) \cdot x_{i-2h} \\ \vdots \end{bmatrix}$$

which is the t -weighting of the process- i .

2 Introduction

2.1 C as a globally linear map

Consider the map

$$C : x_t \mapsto x_{t+h}$$

for $x_i \in X_t \subset X$ and

$$x_{t+h} = Cx_t$$

C may be state dependent, $C(x(t))$, non-autonomous, $C(t)$, or both, $C(t, x(t))$. We denote the possibility of any such case as C_t .

If C_t is globally linear then

$$X_{t+h} = X_t C_t$$

which is solved via,

$$C_t = X_t^{-1} X_{t+h}$$

In an applied time series setting this solution for C_t is the *auto-regressive* or *AR* model for the process X_t .

2.2 C^θ as a locally linear map

If C is not globally linear than the t -weighting can be introduced such that

$$W_t^\theta X_{t+h} = W_t^\theta X_t C_t^\theta$$

where the weighting kernel, w , of W_t^θ is parameterized by θ . The exact local weighting at x_t is achieved as,

$$\lim_{\theta \rightarrow \infty} w(x_i, x_t, \theta) = \delta(x_i - x_t)$$

for all $x_i \in X_t$. Typically the kernel is chosen so $w \sim e^{-\theta}$. The solution for C_t^θ is

$$\begin{aligned} C_t^\theta &= (W_t^\theta X_t)^{-1} W_t^\theta X_{t+h} \\ &= X_t^{-1} (W_t^\theta)^{-1} W_t^\theta X_{t+h} \end{aligned}$$

Taking the limit, W_t^θ reduces to,

$$\lim_{\theta \rightarrow \infty} W_t^\theta = \begin{cases} \delta(x_i - x_t) = 1 & \text{if } i = j \wedge x_i = x_t \\ 0 & \text{if } i \neq j \vee x_i \neq x_t \end{cases}$$

which is a diagonal matrix whose only non-zero entries are 1's corresponding to states arbitrarily close to x_t . If X_t never returns to states arbitrarily close to x_t then W_t^θ reduces to the rank-1 matrix with a single non-zero entry,

$$(W_t^\theta)_{1,1} = 1$$

If X_t is periodic at frequency k the set of all indices i on the diagonal where $(W_t^\theta)_{i,j} = 1$, is

$$\{W_t^\theta\}_1 = \{n \in \mathbb{N} : i = 1 + nk\}$$

If X_t is ergodic than the rank depends on the nature and frequency of close returns to neighbourhoods containing x_t , e.g.

$$U = \{x \in X, \delta = a \in \mathbb{R} : w(x, x_t, \theta) < \delta\}$$

could be used to define ‘‘close’’.

2.2.1 Factorization of C_t^θ

Considering,

$$C_t^\theta = (X_t)^{-1} (W_t^\theta)^{-1} (W_t^\theta X_{t+h}) \tag{1}$$

we take the pseudoinverse of the t -weighting,

$$(W_t^\theta)^{-1} = (W_t^{\theta T} W_t^\theta)^{-1} W_t^{\theta T}$$

and eigen decompose the covariance term and invert for

$$(W_t^\theta)^{-1} = (Q_{w_t} \Lambda_{w_t}^{-1} Q_{w_t}^T) W_t^{\theta T}$$

We can perform the same operations for X_t ,

$$(X_t)^{-1} = (Q_{x_t} \Lambda_{x_t}^{-1} Q_{x_t}^T) X_t^T$$

Substituting into (1) gives

$$\begin{aligned} C_t^\theta &= (Q_{x_t} \Lambda_{x_t}^{-1} Q_{x_t}^T) X_t^T \cdot (Q_{w_t} \Lambda_{w_t}^{-1} Q_{w_t}^T) W_t^{\theta T} \cdot W_t^\theta X_{t+h} \\ &= (Q_{x_t} \Lambda_{x_t}^{-1} Q_{x_t}^T) X_t^T \cdot (Q_{w_t} \Lambda_{w_t}^{-1} Q_{w_t}^T) \cdot (W_t^{\theta T} W_t^\theta) \cdot X_{t+h} \end{aligned} \tag{2}$$

where (\cdot) is simply added for readability and is not the dot product.

2.2.2 P_U : The reverberation of C_t^θ

We refer to the covariance of the t -weighting as the **reverberation** of X_t at t .

$$P_U(X_t, t, \theta, \omega) = \text{Cov}(W_t^\theta) = W_t^{\theta T} W_t^\theta$$

where P is the Greek capital rho.

$P_U(X_t, t, \theta, \omega)$ can be thought of as defining a fuzzy elliptic neighbourhood about x_t that describes close returns to x_t . It is “fuzzy” not in the set theoretic sense, although that could be an interesting extension, but for $\theta < \infty$ all $x \in X$ are included with some x having a greater weight than others.

For example, given the kernel

$$w \sim e^{-\theta \|x - x_t\|} \quad \theta \in \mathbb{R}$$

when $\theta = 0$ all members of X are weighted equally where

$$\begin{aligned} W_t^0 &= \begin{bmatrix} w(x_t, x_t, 0) & 0 & \dots \\ 0 & w(x_t, x_{t-h}, 0) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= I \end{aligned}$$

which gives us the global linear map

$$\begin{aligned} C_t^0 &= (X_t)^{-1} (W_t^0)^{-1} W_t^0 X_{t+h} \\ &= (X_t)^{-1} X_{t+h} \end{aligned}$$

Conversely, in the $\lim \theta \rightarrow \infty$ the only elements that are prominent members of the neighbourhood are those that can be made arbitrarily close; $\epsilon - \delta$ reasoning clarifies.

We may define some small and bounded reverberative neighbourhood centered on x_t ,

$$\{x_i \in X : x_i \in U(x_t, \epsilon) \leftrightarrow \|x_j - x_t\| < \epsilon\}$$

whose members are prominent in x_t 's neighbourhood. For any $\epsilon > 0$, it can be shown there exists a θ_0 such that $\forall \theta > \theta_0$ the reverberation of states x_j outside the bounded neighborhood $U(x_t, \epsilon)$ are less participatory for some level $\delta > 0$

$$\sup_{x_j \in U(x_t, \epsilon)} \|(W_t^{\theta T} W_t^\theta) x_j\| < \delta$$

In all cases, in the limit we have $w(x_i) = \delta(x_i - x_t)$ and the measure of the members comprising the reverberation collapses to 0.

$$\inf \lim_{\theta \rightarrow \infty} \mu(\{x \in X : P x_i = 0\}) = 0$$

We assume the parties aren't very fun.

2.2.3 P_λ : The reverberative map of C_t^θ

In 2, the modulating term of the reverberation is the inverse of the the reverberation,

$$P_\lambda(X_t, t, \theta, \omega) = Q_{w_t} \Lambda_{w_t}^{-1} Q_{w_t}^T$$

which we refer to as the **reverberative map** of X_t at t . Here, Q_{w_t} determines the directions in X along which X_t reverberates more or less strongly with Λ_{w_t} corresponding to the magnitude of reverberation along such directions.

Taking the inverse of the reverberation re-scales the eigenvalues by the reciprocal. The result is an amplification of the weakest effects of the reverberation and a reduction in the strongest.

The effect of the product of the reverberative map and the reverberation,

$$I = (Q_{w_t} \Lambda_{w_t}^{-1} Q_{w_t}^T)(W_t^{0^{-1}} W_t^0)$$

ensures no one is left behind when placing greater emphasis on the more prominent members of the reverberative neighbourhood.

2.3 $\{C(t, \theta(t))\}_{t \geq 0}$ as a single parameter nonlinear semigroup

We can place C_t^θ in the more general context of group theory to yield a notion of of an infinitesimal generator for X_t . In this case we are no longer reasoning about strictly empirical observations, but considering members of some two parameter semigroup $\{C(t, \theta)\}_{t \geq 0, \theta \geq 0}$. We consider the case where θ is dependent on t and reduce to a one parameter semigroup, $\{C(t, \theta(t))\}_{t \geq 0}$.

To satisfy this condition the following must hold:

$$(i) \text{ Associativity: } C(t+s, \theta(t+s)) = C(t, \theta(t))C(s, \theta(s)) \quad \forall t, s \geq 0 \quad (3)$$

$$(ii) \text{ Identity at 0: } C(0, \theta(0)) = I \quad (4)$$

$$(iii) \text{ Weak Continuity: } \lim_{t \rightarrow 0^+} C(t, \theta(t))x = x \quad \forall x \in X \quad (5)$$

An additional requirement may be imposed on growth,

$$(iv) \text{ Boundedness: } \|C(t, \theta(t))\| \leq M e^{\omega t} \quad \forall t \geq 0, \exists \omega \in \mathbb{R} \quad (6)$$

If the semigroup is to be strongly continuous, i.e. a C_0 -semigroup, then there must be an infinitesimal generator A satisfying

$$Ax = \lim_{t \rightarrow 0^+} \frac{C(t, \theta(t))x - x}{t} \quad \forall t \geq 0$$

In an applied settings it may be useful to relax (iv) to capture a more general class. We deal with conditions (i)-(iii) and subsequently apply (iv)

2.3.1 Basic conditions

We define our one parameter group

$$C(t, \theta(t)) : t \rightarrow \mathbb{G} \quad t \in \mathbb{R}$$

We take \mathbb{G} to be $\{X_t\}_{t \geq 0}$ to get the one parameter semigroup

$$C(t, \theta(t)) : t \rightarrow X_{t+h} \quad X_{t+h} \in X$$

where $(X, \|\cdot\|)$ is Banach. We further specify

$$\{C(t, \theta(t))\}_{t \geq 0} = \{(X_t)^{-1}(W_t^{\theta_t})^{-1}W_t^{\theta_t}X_{t+h}\}_{t \geq 0} \quad \theta_t \in \mathbb{R}_{\geq 0} \quad (7)$$

for the full definition. We address the previously stated requirements in turn.

Proof.

(i) *Associativity:* $C(t+s, \theta(t+s)) = C(t, \theta(t))C(s, \theta(s))$

Going from left to right,

$$C(t+s, \theta(t+s)) = (X_{t+s})^{-1}(W_{t+s}^{\theta_{t+s}})^{-1}W_{t+s}^{\theta_{t+s}}X_{t+s+h}$$

[TODO]

In the reverse direction,

$$\begin{aligned} C(t, \theta(t))C(s, \theta(s)) &= \\ &X_t^{-1}(W_t^{\theta_t})^{-1}W_t^{\theta_t}X_{t+h}X_s^{-1}W_s^{\theta_s-1}W_s^{\theta_s}X_{s+h} \end{aligned}$$

[TODO]

Therefore

$$C(t+s, \theta(t+s)) = C(t, \theta(t))C(s, \theta(s))$$

Identity at 0 [TODO] *Weak Continuity* [TODO]

□

3 Random Snippets

For $\theta = 0, \forall t \geq 0$ we have a the linear

$$\begin{aligned} A(t,)x(t) &= \lim_{h \rightarrow 0^+} \frac{C^0(t)x(t) - x(t+h)}{h} \\ &= \lim_{h \rightarrow 0^+} (X_t^{-1}X_{t+h}x(t) - x(t+h))\frac{1}{h} \end{aligned}$$

C may require higher forms,

$$C_h^0(t) = X_t^{-1} X_{t+h} = I + hC_h^0(t) + O(h^2)$$

where we can account for the quadratic

$$C_{2h}^0(t) = X_t^{-1} X_{t+2h} = I + h^2 C_{2h}^0(t) + O(h^3)$$

and generally

$$\begin{aligned} C_h^0(t) &= I + hC_h^0(t) + O(h^2) + I + h^2 C_{2h}^0(t) + O(h^3) + I + h^3 C_{3h}^0(t) + O(h^4) + \dots \\ &= \sum_{n=0}^{\infty} nI + h^n C_n h^0(t) + O(h^{n+1}) \\ &= \sum_{n=0}^{\infty} nI + h^n C_{nh}^0(t) + \sum_{n=0}^{\infty} O(h^{n+1}) \\ &= \sum_{n=0}^{\infty} X_t^{-1} X_{t+nh} + R_h(t, n) \end{aligned}$$

If

$$\lim_{h \rightarrow 0^+} R_h(t, n) = 0$$

then,

$$\begin{aligned} \lim_{h \rightarrow 0^+} C_h^0(t) &= \lim_{h \rightarrow 0^+} \sum_{n=0}^{\infty} X_t^{-1} X_{t+nh} + \lim_{h \rightarrow 0^+} R_h(t, n) \\ C^0(t) &= \sum_{n=0}^{\infty} X_t^{-1} X_{t+nh} \end{aligned}$$

and $\{X_t\} = \{C^0(t)\}_{t \geq 0}$ is a **globally nonlinear** semigroup.

On the other hand, if

$$\lim_{h \rightarrow 0^+} \lim_{\theta \rightarrow \infty} R_h(t, n, \theta) = 0$$

for

$$\begin{aligned} \lim_{h \rightarrow 0^+} \lim_{\theta \rightarrow \infty} C_h^\theta(t) &= \lim_{h \rightarrow 0^+} \lim_{\theta \rightarrow \infty} \sum_{n=0}^{\infty} X_t^{-1} (W_t^\theta)^{-1} W_t^\theta X_{t+nh} + \lim_{h \rightarrow 0^+} \lim_{\theta \rightarrow \infty} R_h(t, n, \theta) \\ C^0(t) &= \sum_{n=0}^{\infty} X_t^{-1} (W_t^\theta)^{-1} W_t^\theta X_{t+nh} \end{aligned}$$

then $\{X_t\} = \{C^\theta(t)\}_{t \geq 0}$ is a **locally nonlinear semigroup**.