# A Resolvent Framework for Global and Local Nonlinear Semigroups

Patrick S. Mahon

September 29, 2024

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## Symbols

| | |
|---|---|
| $X$ | Banach space $(X, \|\| \cdot \|\|)$ |
| $w(x, x_t, \theta) : X \times X \times \mathbb{R} \to \mathbb{R}$ | Weighting kernel at $x_t$, parameterized by $\theta \in \mathbb{R}, x_t \in X$ |

## 1 Definitions

**Definition 1.** *The process-t matrix, read "process until $t$ matrix" or more simply "process until $t$", is*

$$X_t = \begin{bmatrix} x_t \\ x_{t-h} \\ x_{t-2h} \\ \vdots \end{bmatrix}$$

*where rows belong to $X$.*

**Definition 2.** *The t-weighting matrix of the process-i is given by*

$$W(X_i, x_t, \theta) = \begin{bmatrix} w(x_j, x_t, \theta) & 0 & 0 & \cdots \\ 0 & w(x_{j-h}, x_t, \theta) & 0 & \cdots \\ 0 & 0 & w(x_{j-2h}, x_t, \theta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$W_t^\theta X_i$ *is short hand for the the product* $W(X_i, x_t, \theta) X_i$

$$W_t^\theta X_i := W(X_i, x_t, \theta) \cdot X_i = \begin{bmatrix} w(x_i, x_t, \theta) \cdot x_i \\ w(x_{i-h}, x_t, \theta) \cdot x_{i-h} \\ w(x_{i-2h}, x_t, \theta) \cdot x_{i-2h} \\ \vdots \end{bmatrix}$$

*which is the t-weighting of the process-i.*

# 2 Introduction

## 2.1 $C_t$ as a globally linear map

Consider the map

$$C : x_t \mapsto x_{t+h}$$

for $x_i \in X_t \subset X$ and

$$x_{t+h} = C x_t$$

C may be state dependent, $C(x(t))$, non-autonomous, $C(t)$, or both, C(t, x(t)). We denote the possibility of any such case as $C_t$.

If $C_t$ is globally linear then

$$X_{t+h} = X_t C_t$$

which is solved via,

$$C_t = X_t^{-1} X_{t+h}$$

In an applied time series setting this solution for $C_t$ is the *auto-regressive* or $AR$ model for the process $X_t$.

## 2.2 $C_t^\theta$ as a locally linear map

If $C$ is not globally linear than the $t$-weighting can be introduced such that

$$W_t^\theta X_{t+h} = W_t^\theta X_t C_t^\theta$$

where the weighting kernel, $w$, of $W_t^\theta$ is parameterized by $\theta$. The exact local weighting at $x_t$ is achieved as,

$$\lim_{\theta\to\infty} w(x_i, x_t, \theta) = \delta(x_i - x_t)$$

for all $x_i \in X_t$. Typically the kernel is chosen so $w \sim e^{-\theta}$. The solution for $C_t^\theta$ is

$$\begin{aligned} C_t^\theta &= \left(W_t^\theta X_t\right)^{-1} W_t^\theta X_{t+h} \\ &= X_t^{-1}(W_t^\theta)^{-1} W_t^\theta X_{t+h} \end{aligned}$$

Taking the limit, $W_t^\theta$ reduces to,

$$\lim_{\theta\to\infty} W_t^\theta = \begin{cases} \delta(x_i - x_t) = 1 & \text{if } i = j \wedge x_i = x_t \\ 0 & \text{if } i \neq j \vee x_i \neq x_t \end{cases}$$

which is a diagonal matrix whose only non-zero entries are 1's corresponding to states arbitrarily close to $x_t$. If $X_t$ never returns to states arbitrarily close to $x_t$ then $W_t^\theta$ reduces to the rank-1 matrix with a single non-zero entry,

$$(W_t^\theta)_{1,1} = 1$$

If $X_t$ is periodic at frequency $k$ the set of all indices $i$ on the diagonal where $(W_t^\theta)_{i=j} = 1$, is

$$\{W_t^\theta\}_1 = \{n \in \mathbb{N} : i = 1 + nk\}$$

If $X_t$ is ergodic than the rank depends on the nature and frequency of close returns to neighbourhoods containing $x_t$, e.g.

$$U = \{x \in X, \delta = a \in \mathbb{R} : w(x, x_t, \theta) < \delta\}$$

could be used to define "close".

# 3 One Parameter Semigroups

Placing $C_t^\theta$ in the theory of semigroups can help us understand the relationship between locality, $\theta$, and the potential higher order terms for some generating process of the elements of $X_t$. The study of one parameter semigroups provides useful tools that connect the operators which generate the semigroup to the semigroups themselves, by way of *generating theorems*. We start by generalizing $C_t$ as a semigroup.

## 3.1 $C_t$ as a one parameter semigroup

We define $C_t$

$$C(t) : t \to \mathbb{G} \qquad\qquad t \in \mathbb{R}_{\geq 0}$$

where we take the group $\mathbb{G}$ as $\{X_t\}_{t\geq 0}$

$$C(t) : t \to X_{t+h} \qquad\qquad X_{t+h} \in X$$

with $(X, ||\cdot||)$ being Banach.

$\{C(t)\}_{t\geq 0}$ forms a one parameter semigroup if the the following requirements are met:

Associativity:

$$\forall t, s \geq 0 : C(t+s) = C(t)C(s) \tag{1}$$

Identity:

$$C(0) = I \tag{2}$$

### 3.2 $C_0$-semigroup

Continuity for one parameter semigroups are often defined by the following properties:

Strongly Continuous:

For the *infinitesimal generator*, $A$, defined on the domain $D(A)$

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(C(t) - I)x \tag{3}$$

the limit must exist.

The strongly continuous one parameter semigroups is denoted as the $C_0$-semigroup. This semigroup provides a generalization of the exponential in the case where it is also uniformly continuous;

Uniform continuity:

$$\lim_{t\downarrow 0} ||C(t)x(0) - x(0)|| \to 0 \qquad \forall x(0) \in X \tag{4}$$

The linear operator $A$ is the infinitesimal generator of a uniformly continuous semigroup if and only if $A$ is a bounded linear operator. In this case the infinitesimal generator $A$ of the semigroup $C(t)$ must satisfy

$$C(t) = e^{At} := \sum_{0}^{\infty} \frac{A^k}{k!} t^k$$

However if $C(t)$ is strongly but not uniformly continuous then $A$ is not bounded and $e^{At}$ need not converge.

The $C_0$-semigroup provides a good, however insufficient, basis for understanding the relationships of locality and the higher order terms of the generator for $X_t$.

### 3.3 $C_0^n$-semigroup

$C_0$-semigroup to be a special case of a more general strongly continuous one parameter semigroup, the nonlinear $C_0^n$-semigroup, whose infinitesimal generator expresses higher order effects in the limit of ever smaller finite differences.

We both guide and justify the definition such a semigroup on the basis that the exponential of the infinitesimal generator yields a well known generalization of the Taylor series that uses the infinitesimal limit of finite differences. This is the Hille series.

The upshot is that the $C_0^n$-semigroup and it's infinitesimal generator can carry intuition regarding continuity and nonlinearity over to pragmatic operator theory approaches, wherein we are better suited to investigate locality.

#### 3.3.1 The Hille Operator for nonlinear semigroups

We can relate the forward differences to the map of $C$ where,

$$\Delta_h^1 x(t) = x(t+h) - x(t) = C_h^1 x(t) - x(t)$$

.

For higher order differences the difference operator series is

$$\begin{aligned} \Delta_h^n x(t) &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x(t+kh) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_h^k x(t) \end{aligned} \tag{5}$$

If we define the $n$-order generator,

$$A_h^n = \frac{\Delta_h^n}{h^n}$$

The the $n$-order generators of the semigroup, $A_h^n$, describe higher-order terms of the expansion when viewed as finite-difference corrections to the first-order dynamics, $A_h^1$, capturing more detailed behavior of the system.

A semigroup can be defined over the sum of the $n$-order generators,

$$\begin{aligned} C_h(t) = e^{A_h t} &:= \sum_{n=0}^{\infty} \frac{t^n}{n!} A_h^n \\ &:= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n}{h^n} \end{aligned} \tag{6}$$

In the limit of difference size $h$ we get

$$\begin{aligned} \lim_{h \to 0^+} C_h(t) &= \lim_{h \to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n}{h^n} \\ H(t) &= \lim_{h \to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n}{h^n} \end{aligned}$$

where we call $H(t)$ the *Hille operator*. On application to $x(t)$, this operator yields the Hille series expansion

$$H(t)x(t) = \lim_{h \to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n x(t)}{h^n}$$

Thus, the Hille Operator provides a way to generalize the semigroup dynamics to include higher-order effects via the Hille Series, defining a strongly and uniformly continuous one parameter nonlinear semigroup. We denote the order of the semigroup as $C_0^n$-semigroup. Where $n$ is non finite we have the strongly nonlinear $C_0^\infty$-semigroup.

We can reintroduce (5) so,

$$\begin{aligned} H(t) &= \lim_{h\to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n}{h^n} \\ &= \lim_{h\to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_h^k}{h^n} \end{aligned}$$

We can now reintroduce a parameterization of local dependencies to $H(t)$. We have

$$x(t+kh) = C_h^k(t)x(t)$$

where

$$\begin{aligned} W_t^\theta X_{t+kh} &= W_t^\theta X_t C_h^k(t) \\ X_t^{-1} W_t^{\theta^{-1}} W_t^\theta X_{t+kh} &= C_h^k(t) \end{aligned}$$

Therefor

$$\begin{aligned} H(t) &= \lim_{h\to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_h^k}{h^n} \\ &= \lim_{h\to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} X_t^{-1} W_t^{\theta^{-1}} W_t^\theta X_{t+kh}}{h^n} \\ &= \lim_{h\to 0^+} X_t^{-1} W_t^{\theta^{-1}} W_t^\theta \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}}{h^n} X_{t+kh} \end{aligned}$$