A Resolvent Framework for Global and Local Nonlinear Semigroups

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Contents

Symbols

X	Banach space $(X, \|\| \cdot \|\|)$
$w(x, x_t, \theta) : X \times X \times \mathbb{R} \to \mathbb{R}$	Weighting kernel at x_t, parameterized by $\theta \in \mathbb{R}, x_t \in X$

1 Definitions

Definition 1. *The process-t matrix, read "process until t matrix" or more simply "process until t", is*

$$X_t = \begin{bmatrix} x_t \\ x_{t-h} \\ x_{t-2h} \\ \vdots \end{bmatrix}$$

where rows belong to X.

Definition 2. *The t-weighting matrix of the process-i is given by*

$$W(X_i, x_t, \theta) = \begin{bmatrix} w(x_j, x_t, \theta) & 0 & 0 & \cdots \\ 0 & w(x_{j-h}, x_t, \theta) & 0 & \cdots \\ 0 & 0 & w(x_{j-2h}, x_t, \theta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$W_t^\theta X_i$ is short hand for the the product $W(X_i, x_t, \theta)X_i$

$$W_t^\theta X_i := W(X_i, x_t, \theta) \cdot X_i = \begin{bmatrix} w(x_i, x_t, \theta) \cdot x_i \\ w(x_{i-h}, x_t, \theta) \cdot x_{i-h} \\ w(x_{i-2h}, x_t, \theta) \cdot x_{i-2h} \\ \vdots \end{bmatrix}$$

which is the t-weighting of the process-i.

2 Introduction

2.1 C_t as a globally linear map

Consider the map

$$C : x_t \mapsto x_{t+h}$$

for $x_i \in X_t \subset X$ and

$$x_{t+h} = Cx_t$$

C may be state dependent, $C(x(t))$, non-autonomous, $C(t)$, or both, C(t, x(t)). We denote the possibility of any such case as C_t.

If C_t is globally linear then

$$X_{t+h} = X_t C_t$$

which is solved via,

$$C_t = X_t^{-1} X_{t+h}$$

In an applied time series setting this solution for C_t is the *auto-regressive* or AR model for the process X_t.

2.2 C_t^θ as a locally linear map

If C is not globally linear than the t-weighting can be introduced such that

$$W_t^\theta X_{t+h} = W_t^\theta X_t C_t^\theta$$

where the weighting kernel, w, of W_t^θ is parameterized by θ. The exact local weighting at x_t is achieved as,

$$\lim_{\theta \to \infty} w(x_i, x_t, \theta) = \delta(x_i - x_t)$$

for all $x_i \in X_t$. Typically the kernel is chosen so $w \sim e^{-\theta}$. The solution for C_t^θ is

$$\begin{aligned} C_t^\theta &= \left(W_t^\theta X_t\right)^{-1} W_t^\theta X_{t+h} \\ &= X_t^{-1} (W_t^\theta)^{-1} W_t^\theta X_{t+h} \end{aligned}$$

Taking the limit, W_t^θ reduces to,

$$\lim_{\theta \to \infty} W_t^\theta = \begin{cases} \delta(x_i - x_t) = 1 & \text{if } i = j \wedge x_i = x_t \\ 0 & \text{if } i \neq j \vee x_i \neq x_t \end{cases}$$

which is a diagonal matrix whose only non-zero entries are 1's corresponding to states arbitrarily close to x_t. If X_t never returns to states arbitrarily close to x_t then W_t^θ reduces to the rank-1 matrix with a single non-zero entry,

$$(W_t^\theta)_{1,1} = 1$$

If X_t is periodic at frequency k the set of all indices i on the diagonal where $(W_t^\theta)_{i=j} = 1$, is

$$\{W_t^\theta\}_1 = \{n \in \mathbb{N} : i = 1 + nk\}$$

If X_t is ergodic than the rank depends on the nature and frequency of close returns to neighbourhoods containing x_t, e.g.

$$U = \{x \in X, \delta = a \in \mathbb{R} : w(x, x_t, \theta) < \delta\}$$

could be used to define "close".

3 One Parameter Semigroups

Placing C_t^θ in the theory of semigroups can help us understand the relationship between locality, θ, and the potential higher order terms for some generating process of the elements of X_t. The study of one parameter semigroups provides useful tools that connect the operators which generate the semigroup to the semigroups themselves, by way of *generating theorems*. We start by generalizing C_t as a semigroup.

3.1 C_t as a one parameter semigroup

We define C_t

$$C(t) : t \to \mathbb{G} \qquad\qquad t \in \mathbb{R}_{\geq 0}$$

where we take the group $\mathbb{G}$ as $\{X_t\}_{t\geq 0}$

$$C(t) : t \to X_{t+h} \qquad\qquad X_{t+h} \in X$$

with $(X, ||\cdot||)$ being Banach.

$\{C(t)\}_{t\geq 0}$ forms a one parameter semigroup if the the following requirements are met:

Associativity:

$$\forall t, s \geq 0 : C(t+s) = C(t)C(s) \tag{1}$$

Identity:

$$C(0) = I \tag{2}$$

3.2 C_0-semigroup

Continuity for one parameter semigroups are often defined by the following properties:

Strongly Continuous:

For the *infinitesimal generator*, A, defined on the domain $D(A)$

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(C(t) - I)x \tag{3}$$

the limit must exist.

The strongly continuous one parameter semigroups is denoted as the C_0-semigroup. This semigroup provides a generalization of the exponential in the case where it is also uniformly continuous;

Uniform continuity:

$$\lim_{t\downarrow 0} ||C(t)x(0) - x(0)|| \to 0 \qquad\qquad \forall x(0) \in X \tag{4}$$

The linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. In this case the infinitesimal generator A of the semigroup $C(t)$ must satisfy

$$C(t) = e^{At} := \sum_{0}^{\infty} \frac{A^k}{k!} t^k$$

However if $C(t)$ is strongly but not uniformly continuous then A is not bounded and e^{At} need not converge.

The C_0-semigroup provides a good, however insufficient, basis for understanding the relationships of locality and the higher order terms of the generating process for X_t. Specifically, it fails to

3.3 C_0^n-semigroup

C_0-semigroup to be a special case of a more general strongly continuous one parameter semigroup, the nonlinear C_0^n-semigroup, whose infinitesimal generator expresses higher order effects in the limit of ever smaller finite differences.

We both guide and justify the definition such a semigroup on the basis that the exponential of the infinitesimal generator yields a well known generalization of the Taylor series that uses the infinitesimal limit of finite differences. This is the Hille series.

The upshot is that the C_0^n-semigroup and it's infinitesimal generator can carry intuition regarding continuity and nonlinearity over to pragmatic operator theory approaches, wherein we are better suited to investigate locality.

3.3.1 The Hille generator for nonlinear semigroups

Consider the finite difference operator,

$$\Delta_h^k x(t) = x(t) - x(t - kh)$$

We relate the map,

$$x(t + kh) = C_h^k(t)x(t)$$

to the difference operator:

$$\begin{aligned} \Delta_h^k x(t) &= x(t) - x(t - hk) \\ &= \left(C_h^k(t)x(t) - x(t)\right) \\ \Delta_h^{-k} x(t + kh) &= \left(C_h^k(t) - I\right) x(t) \end{aligned} \tag{5}$$

We generalize the definition of strong continuity, (3); the infinitesimal generator A_h for a strongly continuous nonlinear semigroup satisfies

$$\begin{aligned} A_h x(t) &= \lim_{t \to 0^+} \frac{1}{t} \sum_{k=0}^{\infty} (C_h^k(t) - I)x(t) \\ x(t + h) &= \lim_{t \to 0^+} \frac{1}{t} \sum_{k=0}^{\infty} \Delta_h^k x(t + kh) \end{aligned}$$

wherever this limit exists. The domain of A, $D(A)$, is the set of $x \in X$ for which this limit exists. Furthermore we define uniform continuity, (4), where if

$$\lim_{t \to 0^+} \lim_{h \to 0^+} || \sum_{k=0}^{\infty} C_h^k(t)x_0 - x_0|| \to 0$$

then the semigroup is uniformly continuous.

We define the higher powers of the infinitesimal generator,

$$A_h^k = \frac{\Delta_h^k}{h^k}$$

and restate uniformly continuous semigroup as the exponential according to a new definition

$$T(t) = e^{At} := \sum_{0}^{\infty} \frac{A^k}{k!} t^k$$

4 Applications

4.0.1 Factorization of C_t^θ

Considering,

$$C_t^\theta = (X_t)^{-1} (W_t^\theta)^{-1} (W_t^\theta X_{t+h}) \tag{6}$$

we take the pseudoinverse of the t-weighting,

$$(W_t^\theta)^{-1} = ({W_t^\theta}^T W_t^\theta)^{-1} {W_t^\theta}^T$$

and eigen decompose the covariance term and invert for

$$(W_t^\theta)^{-1} = (Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t}^{-1} Q_{\mathrm{w}_t}^T) {W_t^\theta}^T$$

We can perform the same operations for X_t,

$$(X_t)^{-1} = (Q_{\mathrm{x}_t} \Lambda_{\mathrm{x}_t}^{-1} Q_{\mathrm{x}_t}^T) X_t^T$$

Substituting into (6) gives

$$\begin{aligned} C_t^\theta &= (Q_{\mathrm{x}_t} \Lambda_{\mathrm{x}_t}^{-1} Q_{\mathrm{x}_t}^T) X_t^T \cdot (Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t} Q_{\mathrm{w}_t}^T) {W_t^\theta}^T \cdot W_t^\theta X_{t+h} \\ &= (Q_{\mathrm{x}_t} \Lambda_{\mathrm{x}_t}^{-1} Q_{\mathrm{x}_t}^T) X_t^T \cdot (Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t}^{-1} Q_{\mathrm{w}_t}^T) \cdot ({W_t^\theta}^T W_t^\theta) \cdot X_{t+h} \end{aligned} \tag{7}$$

where $(\cdot)$ is simply added for readability and is not the dot product.

4.0.2 P_U: The reverberation of C_t^θ

We refer to the covariance of the t-weighting as the ***reverberation*** of X_t at t.

$$\mathrm{P}_U(X_t, t, \theta, \omega) = \mathrm{Cov}(W_t^\theta) = {W_t^\theta}^T W_t^\theta$$

where P is the Greek capital rho.

$\mathrm{P}_U(X_t, t, \theta, \omega)$ can be thought of as defining a fuzzy elliptic neighbourhood about x_t that describes close returns to x_t. It is "fuzzy" not in the set theoretic sense, although that could be an interesting extension, but for $\theta < \infty$ all $x \in X$ are included with some x having a greater weight than others.

For example, given the kernel

$$w \sim e^{-\theta ||x - x_t||} \quad \theta \in \mathbb{R}$$

when $\theta = 0$ all members of X are weighted equally where

$$\begin{aligned} W_t^0 &= \begin{bmatrix} w(x_t, x_t, 0) & 0 & \dots \\ 0 & w(x_t, x_{t-h}, 0) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= I \end{aligned}$$

which gives us the global linear map

$$\begin{aligned} C_t^0 &= (X_t)^{-1}(W_t^0)^{-1} W_t^0 X_{t_t+h} \\ &= (X_t)^{-1} X_{t_t+h} \end{aligned}$$

Conversely, in the $\lim \theta \to \infty$ the only elements that are prominent members of the neighbourhood are those that can be made arbitrarily close; $\epsilon - \delta$ reasoning clarifies.

We may define some small and bounded reverberative neighbourhood centered on x_t,

$$\{x_i \in X : x_i \in U(x_t, \epsilon) \leftrightarrow ||x_j - x_t|| < \epsilon\}$$

whose members are prominent in x_t's neighbourhood. For any $\epsilon > 0$, it can be shown there exists a θ_0 such that $\forall \theta > \theta_0$ the reverberation of states x_j outside the bounded neighborhood $U(x_t, \epsilon)$ are less participatory for some level $\delta > 0$

$$\sup_{x_j \in U(x_t, \epsilon)} ||({W_t^\theta}^T W_t^\theta)\, x_j|| < \delta$$

In all cases, in the limit we have $w(x_i) = \delta(x_i - x_t)$ and the measure of the members comprising the reverberation collapses to 0.

$$\inf \lim_{\theta \to \infty} \mu(\{x \in X : \mathrm{P} x_i = 0\}) = 0$$

We assume the parties aren't very fun.

4.0.3 P_λ: The reverberative map of C_t^θ

In 7, the modulating term of the reverberation is the inverse of the the reverberation,

$$\mathrm{P}_\lambda(X_t, t, \theta, \omega) = Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t}^{-1} Q_{\mathrm{w}_t}^T$$

which we refer to as the ***reverberative map*** of X_t at t. Here, Q_{w_t} determines the directions in X along which X_t reverberates more or less strongly with Λ_{w_t} corresponding to the magnitude of reverberation along such directions.

Taking the inverse of the reverberation re-scales the eigenvalues by the reciprocal. The result is an amplification of the weakest effects of the reverberation and a reduction in the strongest.

The effect of the product of the reverberative map and the reverberation,

$$I = (Q_{\mathrm{w}_t} \Lambda_{\mathrm{w}_t}^{-1} {Q_{\mathrm{w}_t}}^T)({W_t^0}^{-1} W_t^0)$$

ensures no one is left behind when placing greater emphasis on the more prominent members of the reverberative neighbourhood.