# A Resolvent Framework for Global and Local Nonlinear Semigroups

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October 8, 2024

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# Symbols

| | |
|---|---|
| $X$ | Banach space $(X, \lvert\lvert \cdot \rvert\rvert)$ |
| $w(x, x_t, \theta) : X \times X \times \mathbb{R} \to \mathbb{R}$ | Weighting kernel at $x_t$, parameterized by $\theta \in \mathbb{R}, x_t \in X$ |

# 1 Definitions

**Definition 1.** *$X_t$ is a matrix of a sequence of elements, $x \in X$, indexed by $t \geq 0$*

$$X_t = \begin{bmatrix} x_t \\ x_{t-h} \\ x_{t-2h} \\ \vdots \end{bmatrix}$$

**Definition 2.** *The $t$-weighting matrix of $X_i$ is given by*

$$W(X_i, x_t, \theta) = \begin{bmatrix} w(x_j, x_t, \theta) & 0 & 0 & \cdots \\ 0 & w(x_{j-h}, x_t, \theta) & 0 & \cdots \\ 0 & 0 & w(x_{j-2h}, x_t, \theta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

*$W_t^\theta X_i$ is short hand for the the product $W(X_i, x_t, \theta)X_i$*

$$W_t^\theta X_i := W(X_i, x_t, \theta) \cdot X_i = \begin{bmatrix} w(x_i, x_t, \theta) \cdot x_i \\ w(x_{i-h}, x_t, \theta) \cdot x_{i-h} \\ w(x_{i-2h}, x_t, \theta) \cdot x_{i-2h} \\ \vdots \end{bmatrix}$$

# 2 Introduction and Motivation

## 2.1 $C$ as a globally linear map

Consider the map

$$C(\cdot) : X \to X$$

$C$ may describe a time evolving map

$$C(\cdot) : x_t \mapsto x_{t+h} \qquad\qquad h \geq 0,\ x_i \in X$$

and may be state dependent, $C(x)$, non-autonomous, $C(t)$, or both, $C(t, x)$. If, for the autonomous state-independent case

$$x_{t+h} = Cx_t$$

$C$ is globally linear, then,

$$X_{t+h} = X_t C$$
$$C = X_t^{-1} X_{t+h}$$

provides the solution.

In an applied time series setting, this corresponds to an auto-regressive (AR) model for the process $X_t$, where future states depend linearly on past states through the operator $C$. The AR model is a discrete-time example of how a system evolves over time.

## 2.2   $C_t^\theta$ as a locally linear map

If $C$ is not globally linear, it must exhibit either non-autonomous behavior or state dependence. In many empirical settings, implicit non-autonomy arises naturally through state dependence, where $C(x(t))$ depends on the system's evolving state, effectively introducing time dependence via the state trajectory.

To try to account for this dependence we can introduce the $t$-weighting to the AR solution,

$$W_t^\theta X_{t+h} = W_t^\theta X_t C_t^\theta$$
$$C_t^\theta = (W_t^\theta X_t)^{-1} W_t^\theta X_{t+h}$$

To capture local linearity, the weighting kernel $w(x_i, x_t, \theta)$, parameterized by $\theta$, emphasizes states close to $x_t$. As $\theta \to \infty$, the local weighting becomes exact, with:

$$\lim_{\theta \to \infty} w(x_i, x_t, \theta) = \delta(x_i - x_t),$$

where the Dirac delta function selects only states arbitrarily close to $x_t$. This applies to each $x_i \in X_t$, localizing the influence of nearby states.

Typically the kernel is chosen so $w \sim e^{-\theta}$. Taking the limit, $W_t^\theta$ reduces to,

$$\lim_{\theta \to \infty} W_t^\theta = \begin{cases} \delta(x_i - x_t) = 1 & \text{if } i = j \wedge x_i = x_t \\ 0 & \text{if } i \neq j \vee x_i \neq x_t \end{cases}$$

which is a diagonal matrix whose only non-zero entries are 1's corresponding to states arbitrarily close to $x_t$. If $X_t$ never returns to states arbitrarily close to $x_t$ then $W_t^\theta$ reduces to the rank-1 matrix with a single non-zero entry,

$$(W_t^\theta)_{1,1} = 1$$

If $X_t$ exhibits periodicity with frequency $k$, then only every $k$-th diagonal element of $W_t^\theta$ remains non-zero, reflecting the periodic return to the same state:

$$(W_t^\theta)_{i,j} = \begin{cases} 1 & \text{if } i = j = t + nk, \, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

For ergodic processes, the rank of $W_t^\theta$ reflects the frequency and nature of returns to neighborhoods around $x_t$, with proximity defined by:

$$U = \{x \in X, \delta \in \mathbb{R} : w(x, x_t, \theta) < \delta\}$$

This provides a natural way to define "closeness" in ergodic settings.

## 3   One Parameter Linear and Nonlinear Semigroups

Semigroup theory provides a powerful framework for understanding the evolution of systems, especially when dealing with local linearity and the influence of higher-order terms. By introducing one-parameter semigroups, we gain tools to connect the operators that generate the semigroup with the semigroup's overall dynamics. This helps us explore the relationship between locality (controlled by $\theta$) and the time evolution of the elements in $X_t$.

## 3.1 $C(h)$ as a one parameter semigroup

We define $C(h)$

$$C(h) : \mathbb{R}_{\geq 0} \times X \to X$$

$\{C(h)\}_{h \geq 0}$ forms a one parameter semigroup if the the following requirements are met:

Associativity:
$$\forall h, s \geq 0 : C(h + s) = C(h)C(s) \tag{1}$$

Identity:
$$C(0) = I \tag{2}$$

In the context of global and local linear maps, the parameter $t$ referred to specific time points in the system's evolution, and $h$ represented the time step or interval over which the system evolved.

In this context, $h$ describes the continuous evolution of the operator family $\{C(h)\}_{h \geq 0}$, functioning similarly to the time step $h$ in the discrete-time models. However, here $h$ represents the time duration over which the system evolves, extending the discrete case into a continuous framework.

### 3.1.1 $C_0$-semigroup

Continuity for one-parameter semigroups is defined by the following properties:

**Uniform Continuity:** A one-parameter semigroup $\{C(h)\}_{h \geq 0}$ is uniformly continuous if:

$$\lim_{h \to 0^+} ||C(h)x - x|| = 0, \quad \forall x \in X \tag{3}$$

Uniform continuity implies that the operator family $C(h)$ behaves "smoothly" as $h \to 0^+$, ensuring that the system's evolution is well-behaved and bounded.

The linear operator $A$ is the infinitesimal generator of a uniformly continuous semigroup if and only if $A$ is a *bounded linear operator*. In this case, the infinitesimal generator $A$ of the semigroup $C(h)$ must satisfy:

$$C(h) = e^{Ah} := \sum_{k=0}^{\infty} \frac{A^k}{k!} h^k \tag{4}$$

This exponential form provides a strong connection to the classical matrix exponential, commonly seen in linear systems.

**Strong Continuity:** If the semigroup is not uniformly continuous but still exhibits strong continuity, it is called a $C_0$-*semigroup*. This requires that for any $x \in X$:

$$\lim_{h \to 0^+} ||C(h)x - x|| = 0, \quad \forall x \in X \tag{5}$$

This property ensures that the semigroup $C(h)$ is *strongly continuous*, even if the system's evolution is not uniformly bounded.

For a strongly continuous semigroup, we define the *infinitesimal generator* $A$, acting on the domain $D(A)$, as:

$$Ax = \lim_{h \to 0^+} \frac{1}{h}(C(h) - I)x \tag{6}$$

The limit must exist for $x \in D(A)$, but unlike the uniformly continuous case, the generator $A$ need not be bounded.

In this more general case, the semigroup $C(h)$ is still governed by the operator $A$, but the series $e^{Ah}$ need not converge if $A$ is unbounded.

The $C_0$-semigroup provides a strong foundation for understanding time evolution in terms of linear operators, but it does not fully capture the role of locality or the higher-order terms in systems like $X_t$. We now extend the $C_0$-semigroup to the more general $C_n$-semigroup, which accounts for higher-order effects via finite differences and their infinitesimal generators.

## 3.2 $C_n$-semigroup

The $C_0$-semigroup can be viewed as a special case of a more general continuous one-parameter semigroup, the nonlinear $C_n$-semigroup. This semigroup's infinitesimal generator expresses higher-order effects through the limit of ever smaller finite differences.

We guide and justify the definition of such a semigroup based on the fact that the exponential of the infinitesimal generator yields a well-known generalization of the Taylor series—using the infinitesimal limit of finite differences—known as the Hille series.

The key advantage of the $C_n$-semigroup and its infinitesimal generator is that they carry intuition regarding continuity and nonlinearity into a more pragmatic operator-theoretic framework. This makes the $C_n$-semigroup particularly well-suited for investigating local dynamics and the role of locality in nonlinear systems.

### 3.2.1 The Hille Operator for Nonlinear Semigroups

We can relate the forward differences to the semigroup where:

$$\Delta_h^1 x(t) = x(t + h) - x(t) = C_1(h)x(t) - x(t).$$

For higher-order differences, the difference operator series is:

$$\begin{aligned}
\Delta_h^n x(t) &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x(t + kh) \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_k(h)x(t)
\end{aligned} \tag{7}$$

If we define the $n$-order generator of a strongly continuous nonlinear semigroup:

$$A^n x = \lim_{h \to 0^+} \frac{1}{h^n}(C_n(h) - I)x,$$

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then the higher-order generators, $\{A^n\}_{n\geq 2}$, describe higher-order terms of the semigroup, capturing corrections to the first-order dynamics $A^1$.

We can express local continuity of the nonlinear $C_n$-semigroup as the matrix exponential of the $n$-order generators,

$$C_n(h) = e^{Ah} := \sum_{k=0}^{n} \frac{h^k}{k!} A^k$$

$$:= \lim_{h\to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} \frac{C^n(h) - I}{h^n}$$

In the limit

$$\lim_{h\to 0^+} C_n(h) = \lim_{h\to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} A^k$$

$$H = \lim_{h\to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} \frac{C^n(h) - I}{h^n}$$

We call $H$ the *Hille operator*. Upon application, $Hx(t)$, this operator yields a special case of the Hille series:

$$Hx(t) = \lim_{h\to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} \frac{C^k(h) - I}{h^k} x(t)$$

$$= \lim_{h\to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} \frac{C^k(h)x(t) - x(t)}{h^k}$$

$$= \lim_{h\to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} \frac{x(t + kh) - x(t)}{h^k}$$

$$x(t + h) = \lim_{h\to 0^+} \sum_{k=0}^{\infty} \frac{\Delta_h^k x(t)}{k!}$$

The notation can be a little confusing here. In our formulation we identify $h$ as both the span over which we apply the operator and the higher-order infinitesimals. This offers the simplification of the time dependent and infinitesimal terms to yield the local expansion at $t$. Whereas in the more general Hille series,

$$x(a + t) = \lim_{h\to 0^+} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\Delta_h^k x(a)}{h^k}$$

$t$ is the independent time parameter and $h$ is the infinitesimal step size. The simplification merely arises from the fact that, for a one parameter semigroup, $a = t \in \mathbb{R}_{\geq 0}$

$H$ provides a way to generalize the semigroup dynamics to include higher-order effects via the Hille like expansion, defining the locally continuous action of a one-parameter nonlinear semigroup.

This generalization, where each term captures the evolution of the semigroup through increasingly higher-order dynamics, allows us to track the local nonlinear behavior of the system, much like a Taylor expansion tracks higher-order terms in a polynomial.

If the $C_n$-semigroup possesses uniform continuity for $t \in \mathbb{R}_{\geq 0}$, then,

$$C_n(t) = e^{(e^{Ah})t} := \sum_{m=0}^{\infty} \frac{t^m}{m!} \left( \lim_{h \to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} A^k \right)^m$$

where $t$ is the global parameter governing the semigroup's evolution in time and the inner sum $\sum_{k=0}^{n} \frac{h^k}{k!} A^k$ represents the local action up to $n$-orders over an infinitesimal $h$.

The recursive application of the matrix exponential provides a mechanism for capturing higher-order corrections to local continuity. Uniform continuity is obtained if the generators $A^n$ remain bounded, extending the notion of smooth evolution across higher-order dynamics.

**Aside:** In a way the recursive operation of the exponential extends the semigroup across different measures. The first application, $e^{Ah}$, yields the local semigroup action at a set of measure-0. The second, $e^{(e^{Ah})t}$ yields the action across a trajectory set of measure-1, We could apply once more for a manifold of measure-2,

$$\exp(A, h, 3) = e^{(e^{(e^{Ah})t_1})t_2}$$

This could be explored elsewhere though as a formalization of multiparameter semigroups and investigate connections to concepts in other fields like the exponential map in Riemannian geometry, which takes a tangent vector (local direction) and maps it to a geodesic (global trajectory). This could provide a nice supplemental perspective when investigating locality and curvature across dimensions.

Taking the limit as $n \to \infty$ allows us to transition from discrete higher-order semigroups to continuous generators, capturing nonlinear interactions across all scales.

### 3.2.2 $C_\infty$-semigroup

If taking $n$ at the limit $\lim \to \infty$ we argue for a continuous integration over the set of local operators,

$$\lim_{n \to \infty} \lim_{h \to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} A^k = \lim_{h \to 0^+} \lim_{n \to \infty} \sum_{k=0}^{n} \frac{h^k}{k!} A^k$$

$$\lim_{n \to \infty} \lim_{h \to 0^+} \sum_{k=0}^{n} \frac{h^k}{k!} A^k = \lim_{h \to 0^+} \int_0^{\infty} \frac{h^r}{\Gamma(r+1)} A^r dr$$

Consider the continuous forward difference operator,

$$\Delta_h^r x(t) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} x(t+kh)$$

We redefine $A^n$ in terms of the continuous order parameter $r \in \mathbb{R}_{\geq 0}$,

$$A^r x(t) = \lim_{h \to 0^+} \frac{1}{h^r} (C^r(h) - I)x(t)$$

If $C^r(h)x(t) = x(t + rh)$, then

$$\begin{aligned}
A^r x(t) &= \lim_{h \to 0^+} \frac{1}{h^r} (C^r(h) - I)x(t) \\
&= \lim_{h \to 0^+} \frac{1}{h^r} (C^r(h)x(t) - x(t)) \\
&= \lim_{h \to 0^+} \frac{1}{h^r} (x(t + rh) - x(t)) \\
&= \lim_{h \to 0^+} \frac{\Delta_h^r}{h^r} x(t)
\end{aligned}$$

The matrix exponential

$$\begin{aligned}
A(h) &= e^{\int Ah} \\
&:= \int_0^\infty \frac{h^r}{\Gamma(r+1)} A^r dr \\
&:= \int_0^\infty \frac{h^r}{\Gamma(r+1)} \frac{\Delta_h^r}{h^r} dr \\
&:= \int_0^\infty \frac{\Delta_h^r}{\Gamma(r+1)} dr
\end{aligned}$$

yields the strongly nonlinear Hille operator in the limit,

$$\begin{aligned}
H &= \lim_{h \to 0^+} e^{\int Ah} \\
&= \lim_{h \to 0^+} \int_0^\infty \frac{\Delta_h^r}{\Gamma(r+1)} dr
\end{aligned}$$

which we take as the infinitesimal generator for the $C_\infty$-semigroup

$$\begin{aligned}
C_\infty(t) &= e^{Ht} \\
&:= \sum_{n=0}^\infty \frac{t^n}{n!} \int_0^\infty \frac{\lim_{h \to 0^+} \Delta_h^r}{\Gamma(r+1)} dr
\end{aligned}$$

## 3.3 The resolvents of the $C_n$ and $C_\infty$ semigroups

The resolvent

$$R(\lambda, H) = \lim_{h \to 0^+} (\lambda I - H)^{-1}$$

defines the spectrum and growth behavior of the semigroup.

We are interested in the resolvent set that provides a well defined inverse,

$$\rho(H) = \left\{ \lambda \in \mathbb{C} : \lim_{h \to 0^+} (\lambda I - H)^{-1} \right\}$$

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For the discrete case we have,

$$R(\lambda, H_n) = \lim_{h \to 0^+} (\lambda I - H)^{-1}$$

$$= \left( \lambda I - \lim_{h \to 0^+} \sum_n \frac{\Delta_h^n}{n!} \right)^{-1} \qquad n \in \mathbb{N}$$

and for the continuum,

$$R(\lambda, H_\infty) = \left( \lambda I - \lim_{h \to 0^+} \int_0^\infty \frac{\Delta_h^r}{\Gamma(r+1)} dr \right)^{-1} \qquad r \in \mathbb{R}_{\geq 0}$$

We need $\lambda \in \rho(H)$ such that

$$\rho(H) = \mathbb{C}/\sigma(H)$$

where $\sigma(h)$ represents the spectrum of the Hille operator generating the semigroup

The resolvent set $\rho(H)$ is crucial for understanding the stability and long-term behavior of the semigroup. In particular, the spectral bound,

$$s(H) = \sup\{\Re(\lambda) : \lambda \in \sigma(H)\}$$

governs the growth rate of the semigroup. If

$$s(H) \leq 0$$

then the semigroup is stable and contracts over time, while positive values indicate exponential growth.

# 4  $C_n$-semigroup Operating On Matrices

A relatively trivial right translation of $C_n$-semigroup allows meaningful operation on state space matrices of trajectories in $X$. We reframe

$$x(t + kh) = C_k(h)x(t)$$

as a right translation

$$X(t + kh) = X(t)C_k(h)$$

of the semigroup.

We have,

$$\Delta_h^n X(t) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} X(t + kh)$$

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and define the right $n$-order generator of a strongly continuous nonlinear semigroup:

$$XA^n = \lim_{h \to 0^+} \frac{1}{h^n} X(C_n(h) - I)$$

The matrix exponential requiring right multiplication is

$$C(h) = e^{hA} := \sum_{k=0}^{n} A^k \frac{h^k}{k!}$$

$$:= \lim_{h \to 0^+} \sum_{k=0}^{n} \frac{C^n(h) - I}{h^n} \frac{h^k}{k!}$$

where the corresponding Hille operator

$$\lim_{h \to 0^+} C(h) = \lim_{h \to 0^+} \sum_{k=0}^{n} A^k \frac{h^k}{k!}$$

$$H = \lim_{h \to 0^+} \sum_{k=0}^{n} \frac{C^n(h) - I}{h^n} \frac{h^k}{k!}$$

which upon right application gives,

$$X(t)H = \lim_{h \to 0^+} \sum_{k=0}^{n} X(t) \frac{C^k(h) - I}{h^k} \frac{h^k}{k!}$$

$$= \lim_{h \to 0^+} \sum_{k=0}^{n} \frac{X(t)C^k(h) - X(t)}{h^k} \frac{h^k}{k!}$$

$$= \lim_{h \to 0^+} \sum_{k=0}^{n} \frac{X(t + kh) - X(t)}{h^k} \frac{h^k}{k!}$$

$$X(t + h) = \lim_{h \to 0^+} \sum_{k=0}^{\infty} \frac{\Delta_h^k X(t)}{k!}$$

Uniform continuity is given by

$$C_n(t) = e^{t(e^{hA})} := \sum_{m=0}^{\infty} \left( \sum_{k=0}^{n} A^k \frac{h^k}{k!} \right)^m \frac{t^m}{m!}$$

As before we consider the achievement of the limit of $n$ as a continuous extension,

$$H = \lim_{h \to 0^+} e^{\int hA}$$

$$= \int_0^{\infty} A^r \frac{h^r}{\Gamma(r+1)} dr$$

$$= \lim_{h \to 0^+} \int_0^{\infty} \frac{\Delta_h^r}{\Gamma(r+1)} dr$$

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