A Resolvent Framework for Global and Local Nonlinear Semigroups

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Symbols

X	Banach space $(X, \|\|\cdot\|\|)$
$w(x, x_t, \theta) : X \times X \times \mathbb{R} \to \mathbb{R}$	Weighting kernel at x_t, parameterized by $\theta \in \mathbb{R}, x_t \in X$

1 Definitions

Definition 1. *The process-t matrix, read "process until t matrix" or more simply "process until t", is*

$$X_t = \begin{bmatrix} x_t \\ x_{t-h} \\ x_{t-2h} \\ \vdots \end{bmatrix}$$

where rows belong to X.

Definition 2. *The t-weighting matrix of the process-i is given by*

$$W(X_i, x_t, \theta) = \begin{bmatrix} w(x_j, x_t, \theta) & 0 & 0 & \cdots \\ 0 & w(x_{j-h}, x_t, \theta) & 0 & \cdots \\ 0 & 0 & w(x_{j-2h}, x_t, \theta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$W_t^\theta X_i$ is short hand for the the product $W(X_i, x_t, \theta)X_i$

$$W_t^\theta X_i := W(X_i, x_t, \theta) \cdot X_i = \begin{bmatrix} w(x_i, x_t, \theta) \cdot x_i \\ w(x_{i-h}, x_t, \theta) \cdot x_{i-h} \\ w(x_{i-2h}, x_t, \theta) \cdot x_{i-2h} \\ \vdots \end{bmatrix}$$

which is the t-weighting of the process-i.

2 Introduction

2.1 C_t as a globally linear map

Consider the map

$$C : x_t \mapsto x_{t+h}$$

for $x_i \in X_t \subset X$ and

$$x_{t+h} = Cx_t$$

C may be state dependent, $C(x(t))$, non-autonomous, $C(t)$, or both, C(t, x(t)). We denote the possibility of any such case as C_t.

If C_t is globally linear then

$$X_{t+h} = X_t C_t$$

which is solved via,

$$C_t = X_t^{-1} X_{t+h}$$

In an applied time series setting this solution for C_t is the *auto-regressive* or AR model for the process X_t.

2.2 C_t^θ as a locally linear map

If C is not globally linear than the t-weighting can be introduced such that

$$W_t^\theta X_{t+h} = W_t^\theta X_t C_t^\theta$$

where the weighting kernel, w, of W_t^θ is parameterized by θ. The exact local weighting at x_t is achieved as,

$$\lim_{\theta\to\infty} w(x_i, x_t, \theta) = \delta(x_i - x_t)$$

for all $x_i \in X_t$. Typically the kernel is chosen so $w \sim e^{-\theta}$. The solution for C_t^θ is

$$\begin{aligned} C_t^\theta &= \left(W_t^\theta X_t\right)^{-1} W_t^\theta X_{t+h} \\ &= X_t^{-1} (W_t^\theta)^{-1} W_t^\theta X_{t+h} \end{aligned}$$

Taking the limit, W_t^θ reduces to,

$$\lim_{\theta\to\infty} W_t^\theta = \begin{cases} \delta(x_i - x_t) = 1 & \text{if } i = j \wedge x_i = x_t \\ 0 & \text{if } i \neq j \vee x_i \neq x_t \end{cases}$$

which is a diagonal matrix whose only non-zero entries are 1's corresponding to states arbitrarily close to x_t. If X_t never returns to states arbitrarily close to x_t then W_t^θ reduces to the rank-1 matrix with a single non-zero entry,

$$(W_t^\theta)_{1,1} = 1$$

If X_t is periodic at frequency k the set of all indices i on the diagonal where $(W_t^\theta)_{i=j} = 1$, is

$$\{W_t^\theta\}_1 = \{n \in \mathbb{N} : i = 1 + nk\}$$

If X_t is ergodic than the rank depends on the nature and frequency of close returns to neighbourhoods containing x_t, e.g.

$$U = \{x \in X, \delta = a \in \mathbb{R} : w(x, x_t, \theta) < \delta\}$$

could be used to define “close”.

3 One Parameter Semigroups

Placing C_t^θ in the theory of semigroups can help us understand the relationship between locality, θ, and the potential higher order terms for some generating process of the elements of X_t. The study of one parameter semigroups provides useful tools that connect the operators which generate the semigroup to the semigroups themselves, by way of *generating theorems*. We start by generalizing C_t as a semigroup.

3.1 C_t as a one parameter semigroup

We define C_t

$$C(t) : t \to \mathbb{G} \qquad\qquad t \in \mathbb{R}_{\geq 0}$$

where we take the group $\mathbb{G}$ as $\{X_t\}_{t\geq 0}$

$$C(t) : t \to X_{t+h} \qquad\qquad X_{t+h} \in X$$

with $(X, ||\cdot||)$ being Banach.

$\{C(t)\}_{t\geq 0}$ forms a one parameter semigroup if the the following requirements are met:

Associativity:

$$\forall t, s \geq 0 : C(t+s) = C(t)C(s) \tag{1}$$

Identity:

$$C(0) = I \tag{2}$$

3.2 C_0-semigroup

Continuity for one parameter semigroups are often defined by the following properties:

Strongly Continuous:

For the *infinitesimal generator*, A, defined on the domain $D(A)$

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(C(t) - I)x \tag{3}$$

the limit must exist.

The strongly continuous one parameter semigroups is denoted as the C_0-semigroup. This semigroup provides a generalization of the exponential in the case where it is also uniformly continuous;

Uniform continuity:

$$\lim_{t\downarrow 0} ||C(t)x(0) - x(0)|| \to 0 \qquad \forall x(0) \in X \tag{4}$$

The linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. In this case the infinitesimal generator A of the semigroup $C(t)$ must satisfy

$$C(t) = e^{At} := \sum_{0}^{\infty} \frac{A^k}{k!} t^k$$

However if $C(t)$ is strongly but not uniformly continuous then A is not bounded and e^{At} need not converge.

The C_0-semigroup provides a good, however insufficient, basis for understanding the relationships of locality and the higher order terms of the generator for X_t.

3.3 C_0^n-semigroup

C_0-semigroup to be a special case of a more general strongly continuous one parameter semigroup, the nonlinear C_0^n-semigroup, whose infinitesimal generator expresses higher order effects in the limit of ever smaller finite differences.

We both guide and justify the definition such a semigroup on the basis that the exponential of the infinitesimal generator yields a well known generalization of the Taylor series that uses the infinitesimal limit of finite differences. This is the Hille series.

The upshot is that the C_0^n-semigroup and it's infinitesimal generator can carry intuition regarding continuity and nonlinearity over to pragmatic operator theory approaches, wherein we are better suited to investigate locality.

3.3.1 The Hille Operator for nonlinear semigroups

We can relate the forward differences to the map of C where,

$$\Delta_h^1 x(t) = x(t+h) - x(t) = C_h^1 x(t) - x(t)$$

.

For higher order differences the difference operator series is

$$\begin{aligned} \Delta_h^n x(t) &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x(t+kh) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_h^k x(t) \end{aligned} \tag{5}$$

If we define the n-order generator,

$$A_h^n = \frac{\Delta_h^n}{h^n}$$

The the n-order generators of the semigroup, A_h^n, describe higher-order terms of the expansion when viewed as finite-difference corrections to the first-order dynamics, A_h^1, capturing more detailed behavior of the system.

A semigroup can be defined over the sum of the n-order generators,

$$\begin{aligned} C_h(t) = e^{A_h t} &:= \sum_{n=0}^{\infty} \frac{t^n}{n!} A_h^n \\ &:= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n}{h^n} \end{aligned} \tag{6}$$

In the limit of difference size h we get

$$\begin{aligned} \lim_{h \to 0^+} C_h(t) &= \lim_{h \to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n}{h^n} \\ H(t) &= \lim_{h \to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n}{h^n} \end{aligned}$$

where we call $H(t)$ the *Hille operator*. On application to $x(t)$, this operator yields the Hille series expansion

$$H(t)x(t) = \lim_{h \to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n x(t)}{h^n}$$

Thus, the Hille Operator provides a way to generalize the semigroup dynamics to include higher-order effects via the Hille Series, defining a strongly and uniformly continuous one parameter nonlinear semigroup. We denote the order of the semigroup as C_0^n-semigroup. Where n is non finite we have the strongly nonlinear C_0^∞-semigroup.

We can reintroduce (5) so,

$$\begin{aligned} H(t) &= \lim_{h\to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\Delta_h^n}{h^n} \\ &= \lim_{h\to 0^+} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_h^k}{h^n} \end{aligned}$$

3.4 The resolvent of C_0^n

The resolvent

$$R(\lambda, H(t)) = (\lambda I - H(t))^{-1}$$

is important because it tells us whether the Hille operator $H(t)$ has an inverse for certain values of λ. This can help define the spectrum and growth behavior of the C_0^n semigroup.

We are interested in the resolvent set that provides a well defined inverse,

$$\rho(H) = \left\{ n \in \mathbb{Z}, \lambda \in \mathbb{C} : \lim_{h\to 0^+} (\lambda I - H(t))^{-1} \right\}$$

We have

$$\begin{aligned} R(\lambda, H(t)) &= \lim_{h\to 0^+} (\lambda I - H(t))^{-1} \\ &= \lim_{h\to 0^+} \left(\lambda I - \sum_n \frac{t^n}{n!} \frac{A_h^n}{h^n} \right)^{-1} \end{aligned}$$

so,

$$0 \neq \lim_{h\to 0^+} |\lambda I - H(t)|$$

We want $\lambda \in \rho(H(t))$ where

$$\begin{aligned} \rho(H(t)) &= \mathbb{C}/\sigma\left(H(t)\right) \\ &= \mathbb{C}/\sigma \left(\lim_{h\to 0^+} \sum_n \frac{t^n}{n!} \frac{A_h^n}{h^n} \right) \end{aligned}$$

There are a number of options to resolve $\sigma(H(t))$ in non-empirical settings. These include, among others, methods from functional calculus, operator, theory, and perturbation theory.

4 An Empirical Hille Operator

In an empirical setting the coefficients, a_n, of the operator,

$$H(t) = \lim_{h\to 0^+} \sum_n a^n \frac{\Delta_h^n}{h^n}$$

may not ensure convergence. In such cases we have to consider how to treat the distribution of coefficients,

$$P(a_n) \sim f(n)$$

assuming it is some function of the order n. In this case the only required constraint is normalization

$$\sum_n P(a_n) = 1$$

Beyond this many natural systems may or may not exhibit further constraints including bounded higher order moments. How should we, *a priori*, evaluate methods to estimate $P(a_n)$

If we seek the least biased estimate of the dynamics described by $H(t)$ then we should adopt a maximum entropy approach. Following such a solution we may assess the spectra of the ensemble, $\sigma(< H(t)x(t) >)$, where

$$< H(t)x(t) >= \sum_n P(a_n)\, a_n \frac{\Delta_h^n x(t)}{h^n}$$

4.1 Maximum Entropy Estimates of $H(t)$

Given a set of trajectory data, $\{X_t\}_t \geq 0$, we propose the following application of the empirical Hille operator,

$$\begin{aligned}
H(t)x(t) &= \sum_n P(a_n)\, a_n \frac{\Delta_h^n x(t)}{h^n} \\
&= \sum_n P(a_n)\, a_n \frac{\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} C_h^k(t)}{h^n} \\
&= \sum_n P(a_n)\, a_n \frac{\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} X_t^{-1} X_{t+kh}}{h^n} \\
&= \sum_n X_t^{-1} P(a_n)\, a_n \frac{\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} X_{t+kh}}{h^n}
\end{aligned}$$

The entropy of $P(a_n)$ is given by

$$S(P) = -\sum_n P(a_n) \log P(a_n)$$

where,

$$\sum_n P(a_n) = 1$$

We seek the solution to the Lagrangian,

$$\mathcal{L} = -\sum_n P(a_n) \log P(a_n) + \lambda \left(\sum_n P(a_n) - 1 \right)$$