Differential and Complex Algebraic Geometry

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Introduction

First and foremost I want to clarify that by no means am I an expert on the presented topics or any topics in mathematics for that matter. I have wanted to study complex geometry for quite some time but was unable take a course or do research in the area. So this is me exploring complex geometry and some of the topics that are parallel. As it happens, this requires a full course on differential geometry and Lie groups. I'm taking these notes from 3-4 books as well as supplemental notes found online. Of course these will all be cited at the end. The end goal is to prepare myself for spectral/noncommutative geometry. However, the functional analysis side of things would be a novel on it's own. Some functional analysis will pop up here and there in these notes but only in the context of geometry. I am not going to go too deep into operator algebras or anything like that. Most of this information is new to me at the time of writing. Also worth noting, I am writing this with a VERY casual approach so my language and verbiage may not be as formal or precise, I'm writing what my brain is thinking. Lots of proofs will be ommitted but more the simpler ones I will probably include. Some of the proofs just require results that are beyond the scope of the current section, but I will of course cite the books that contain the proofs. I am documenting my journey of learning geometry and whatever may branch off of it that interests me. Join me as I ascend to a geometry god.

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1 The Exponential Map and Matrix Groups

1.1 The Matrix Exponential

Given an $n \times n$ matrix A, we want to find a way to find e^A. We can actually just do this with the usual power series:

$$e^A = I_n + \sum_{p \geq 1} \frac{A^p}{p!} = \sum_{p \geq 0} \frac{A^p}{p!} \tag{1}$$

Using an inductive proof, we can show that this is well defined. But we won't write it out here.

Example 1.1

Consider the matrix

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

We want to find a way to express the powers A^n. We can factor out a θ to see

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^2 = -\theta^2 I_2$$

Now, let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we have

$$\begin{aligned} A^{4n} &= \theta^{4n} I_2 \\ A^{4n+1} &= \theta^{4n+1} J \\ A^{4n+2} &= -\theta^{4n+2} I_2 \end{aligned}$$

and so on. This means we can now express e^A as a power series

$$e^A = I_2 + \frac{\theta}{1!} J - \frac{\theta^2}{2!} I_2 \ldots$$

Writing this out we will see that we actually get the power series for cosine and sine, thus

$$e^A = \cos\theta I_2 + \sin\theta J$$

or equivalently

$$e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So we see e^A is in fact a rotation matrix.

This is actually a general fact. If we have a skew-symmetric matrix A, then e^A is an orthogonal matrix with determinant 1. In fact, EVERY rotation matrix is of this form. To be explicit, the exponential map from the set of skew-symmetric matrices to the set of rotation matrices is surjective. But note that the exponential map is NOT surjective in general.

Proposition 1.1

Let A and U be (real or complex) matrices and assume U is invertible. Then

$$e^{UAU^{-1}} = Ue^{A}U^{-1}$$

This is pretty obvious and its easily proven using an inductive proof. But I hate induction so of course I will not include the proof here, although its only a few lines. Now we will look at another important result that will be important when we start to look at some spectral properties of the exponential map.

Proposition 1.2

Given any complex $n \times n$ matrix A, there is an invertible matrix P and an upper-triangular matrix T such that

$$A = PTP^{-1}$$

Proof. (Sketch) Induct on n if $f : \mathbb{C} \to \mathbb{C}$ is a linear map then there exists some basis with respect to f which can be represented as an upper-triangular matrix. $\square$

But note this proof is very technical.

The exponential operator has some nice properties. Some are expected but some properties that we might expect to hold are actually not true.

Proposition 1.3: Properties of Exponential Operator

- If $\{\lambda_i\}_n$ are the eigenvalues of A, then $\{e^{\lambda_i}\}_n$ are the eigenvalues of e^A
- $\det(e^A) = e^{\mathrm{tr}(A)}$
- If A and B commute under multplication, then $e^{A+B} = e^A e^B$

1.2 Matrix Lie Groups

First, recall the "usual" matrix groups.

Definition 1.1: Common Lie Groups and Lie Algebras

- $GL(n, \mathbb{R})$: The group of all real invertible $n \times n$ matrices. This is the *general linear group.*
- $SL(n, \mathbb{R})$: The group of all real invertible $n \times n$ matrices with determinant $+1$. This is the *special linear group.* Note that this is a subgroup of the general linear group.
- $O(n)$: The group of all real orthogonal $n \times n$ matrices. This is the *orthogonal group.*
- $SO(n)$: The group of all real orthogonal $n \times n$ matrices with determinant $+1$. This is

the *special orthogonal group*. Note this is a subgroup of the orthogonal group.

- $\mathfrak{sl}(n, \mathbb{R})$: The vector space of real $n \times n$ matrices with null trace.
- $\mathfrak{so}(n)$: The vector space of real skew-symmetric $n \times n$ matrices.

The groups from above are more than just groups, they have additional topological structure. They are topological spaces (viewed as subspaces of $\mathbb{R}^{n^2}$) with smooth operations, specifically, the inverse and multplication operations are continuous. Further, they are in fact smooth manifolds (we will define this later). These are examples of *Lie groups*; groups that are simultaneously topological spaces, and smooth manifolds. The above vector spaces are *Lie algebras*; tangent spaces at the identity of the respective group. The algebraic structure on Lie algebras is well defined, we will see later how far this defition extends.

Definition 1.2: Lie Bracket

The *Lie bracket* of a Lie algebra is the commutator

$$[A, B] = AB - BA$$

Note that if A and B commute, their Lie bracket is trivial. Later, we will see how this corresponds to conservative vector fields on manifolds. There is a really cool connection between Lie Groups and their Lie algebras, in fact, the exponential map is a map from the Lie algebra to the Lie group.

$$\exp : \mathfrak{so}(n) \to SO(n) \tag{2}$$

$$\exp : \mathfrak{sl}(n, \mathbb{R}) \to SL(n, \mathbb{R}) \tag{3}$$

This is really neat because it lets us parameterize the Lie group elements by the Lie algebra elements, which sounds weird but it will be really convenient later.

Well what happened to the Lie algebras $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{o}(n)$? As it turns out, these Lie algebras are actually equivalent to some already familiar vector spaces. It happens that $\mathfrak{gl}(n, \mathbb{R})$ is just the vector space of all real $n \times n$ matrices, while $\mathfrak{o}(n) = \mathfrak{so}(n)$.

Sometimes these maps can be explicity computed, giving rise to lots of application in kinematics, robotics, etc.

Proposition 1.4: Rodrigues Formula

The exponential map $\exp : \mathfrak{so}(3) \to SO(3)$ is given by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1 - \cos\theta)}{\theta^2} A^2$$

if $\theta \neq 0$, with $e^{0_3} = I_3$.

Proof. Notice that

$$A^2 = -\theta^2 I_3 + B$$

and $AB = BA = 0$. From there we can easily deduce the fact that

$$A^3 = \theta^2 A$$

Now we just write the power series for e^A

$$e^A = I_3 + \sum_{p\geq 1} \frac{A^p}{p!}$$

Some painful calculation will give us the following

$$\begin{aligned}
e^A &= I_3 + \sum_{p\geq 0} \frac{A^{2p+1}}{(2p+1)!} + \sum_{p\geq 1} \frac{A^{2p}}{(2p)!} \\
&= I_3 + \sum_{p\geq 0} \frac{(-1)^p \theta^{2p}}{(2p+1)!} A + \sum_{p\geq 1} \frac{(-1)^{p-1}\theta^{2(p-1)}}{(2p)!} A \\
&= I_3 + \frac{A}{\theta} \sum_{p\geq 0} \frac{(-1)^p \theta^{2p+1}}{(2p+1)!} - \frac{A^2}{\theta^2} \sum_{p\geq 1} \frac{(-1)^p \theta^{2p}}{(2p)!} \\
&= I_3 + \frac{\sin\theta}{\theta} A - \frac{A^2}{\theta^2} \sum_{p\geq 0} \frac{(-1)^p \theta^{2p}}{(2p)!} + \frac{A^2}{\theta^2} \\
&= I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2
\end{aligned}$$

□

It looks worse than it is, its really just moving things around to get the power series for sine and cosine. Although I do not want to do it again.

1.2.1 Symmetric and Positive Definite Matrices

Now we will backtrack a little bit and look at another important class of matrices, and consequently, groups. A matrix is *positive semidefinite* if all of its eigenvalues are nonnegative. The spectrum of positive definite matrices is used extensively in optimization and convex analysis, it also has some neat geometric implications we will see later. I assume familiarity of symmetric matries by now, my first grade students could identity one. Now for some counterintuitive notation. We denote the vector space (not necessarily group) of symmetric $n \times n$ matrices by $S(n)$ and the vector space of symmetric positive define matrices by $SPD(n)$. Yes I know the notation is the same we used for matrix Lie groups, and we used the fraktur symbols for the vector spaces and algebras. But just remember these are vector spaces, I will try my best to remember myself and remind the reader. It is also not uncommon for this notation to use boldface characters, it is however a pain to type sometimes.

Proposition 1.5

For every symmetric matrix B, the matrix e^B is symmetric positive definite. For every symmetric positive definite matrix A, there is a unique symmetric matrix B such that $A = e^B$.

Proof. Two pages and some technical linear algebra, I still don't understand it completely.

□