Differential and Complex Algebraic Geometry

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Introduction

First and foremost I want to clarify that by no means am I an expert on the presented topics or any topics in mathematics for that matter. I have wanted to study complex geometry for quite some time but was unable take a course or do research in the area. So this is me exploring complex geometry and some of the topics that are parallel. As it happens, this requires a full course on differential geometry and Lie groups. I'm taking these notes from 3-4 books as well as supplemental notes found online. Of course these will all be cited at the end. The end goal is to prepare myself for spectral/noncommutative geometry. However, the functional analysis side of things would be a novel on it's own. Some functional analysis will pop up here and there in these notes but only in the context of geometry. I am not going to go too deep into operator algebras or anything like that. Most of this information is new to me at the time of writing. Also worth noting, I am writing this with a VERY casual approach so my language and verbiage may not be as formal or precise, I'm writing what my brain is thinking. Lots of proofs will be ommitted but more the simpler ones I will probably include. Some of the proofs just require results that are beyond the scope of the current section, but I will of course cite the books that contain the proofs. I am documenting my journey of learning geometry and whatever may branch off of it that interests me. Join me as I ascend to a geometry god.

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1 The Exponential Map and Matrix Groups

1.1 The Matrix Exponential

Given an $n \times n$ matrix A, we want to find a way to find e^A. We can actually just do this with the usual power series:

$$e^A = I_n + \sum_{p \geq 1} \frac{A^p}{p!} = \sum_{p \geq 0} \frac{A^p}{p!} \tag{1}$$

Using an inductive proof, we can show that this is well defined. But we won't write it out here.

Example 1.1

Consider the matrix

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

We want to find a way to express the powers A^n. We can factor out a θ to see

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^2 = -\theta^2 I_2$$

Now, let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we have

$$\begin{aligned} A^{4n} &= \theta^{4n} I_2 \\ A^{4n+1} &= \theta^{4n+1} J \\ A^{4n+2} &= -\theta^{4n+2} I_2 \end{aligned}$$

and so on. This means we can now express e^A as a power series

$$e^A = I_2 + \frac{\theta}{1!} J - \frac{\theta^2}{2!} I_2 \dots$$

Writing this out we will see that we actually get the power series for cosine and sine, thus

$$e^A = \cos\theta I_2 + \sin\theta J$$

or equivalently

$$e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So we see e^A is in fact a rotation matrix.

This is actually a general fact. If we have a skew-symmetric matrix A, then e^A is an orthogonal matrix with determinant 1. In fact, EVERY rotation matrix is of this form. To be explicit, the exponential map from the set of skew-symmetric matrices to the set of rotation matrices is surjective. But note that the exponential map is NOT surjective in general.

Proposition 1.1

Let A and U be (real or complex) matrices and assume U is invertible. Then

$$e^{UAU^{-1}} = Ue^{A}U^{-1}$$

This is pretty obvious and its easily proven using an inductive proof. But I hate induction so of course I will not include the proof here, although its only a few lines. Now we will look at another important result that will be important when we start to look at some spectral properties of the exponential map.

Proposition 1.2

Given any complex $n \times n$ matrix A, there is an invertible matrix P and an upper-triangular matrix T such that

$$A = PTP^{-1}$$

Proof. (Sketch) Induct on n if $f : \mathbb{C} \to \mathbb{C}$ is a linear map then there exists some basis with respect to f which can be represented as an upper-triangular matrix. $\square$

But note this proof is very technical.

The exponential operator has some nice properties. Some are expected but some properties that we might expect to hold are actually not true.

Proposition 1.3: Properties of Exponential Operator

- If $\{\lambda_i\}_n$ are the eigenvalues of A, then $\{e^{\lambda_i}\}_n$ are the eigenvalues of e^A
- $\det(e^A) = e^{\text{tr}(A)}$
- If A and B commute under multplication, then $e^{A+B} = e^A e^B$

1.2 Matrix Lie Groups

First, recall the "usual" matrix groups.

Definition 1.1: Common Lie Groups and Lie Algebras

- $GL(n, \mathbb{R})$: The group of all real invertible $n \times n$ matrices. This is the *general linear group.*
- $SL(n, \mathbb{R})$: The group of all real invertible $n \times n$ matrices with determinant $+1$. This is the *special linear group.* Note that this is a subgroup of the general linear group.
- $O(n)$: The group of all real orthogonal $n \times n$ matrices. This is the *orthogonal group.*
- $SO(n)$: The group of all real orthogonal $n \times n$ matrices with determinant $+1$. This is

the *special orthogonal group*. Note this is a subgroup of the orthogonal group.

- $\mathfrak{sl}(n, \mathbb{R})$: The vector space of real $n \times n$ matrices with null trace.
- $\mathfrak{so}(n)$: The vector space of real skew-symmetric $n \times n$ matrices.

The groups from above are more than just groups, they have additional topological structure. They are topological spaces (viewed as subspaces of $\mathbb{R}^{n^2}$) with smooth operations, specifically, the inverse and multplication operations are continuous. Further, they are in fact smooth manifolds (we will define this later). These are examples of *Lie groups*; groups that are simultaneously topological spaces, and smooth manifolds. The above vector spaces are *Lie algebras*; tangent spaces at the identity of the respective group. The algebraic structure on Lie algebras is well defined, we will see later how far this defition extends.

Definition 1.2: Lie Bracket

The *Lie bracket* of a Lie algebra is the commutator

$$[A, B] = AB - BA$$

Note that if A and B commute, their Lie bracket is trivial. Later, we will see how this corresponds to conservative vector fields on manifolds. There is a really cool connection between Lie Groups and their Lie algebras, in fact, the exponential map is a map from the Lie algebra to the Lie group.

$$\exp : \mathfrak{so}(n) \to SO(n) \tag{2}$$

$$\exp : \mathfrak{sl}(n, \mathbb{R}) \to SL(n, \mathbb{R}) \tag{3}$$

This is really neat because it lets us parameterize the Lie group elements by the Lie algebra elements, which sounds weird but it will be really convenient later.

Well what happened to the Lie algebras $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{o}(n)$? As it turns out, these Lie algebras are actually equivalent to some already familiar vector spaces. It happens that $\mathfrak{gl}(n, \mathbb{R})$ is just the vector space of all real $n \times n$ matrices, while $\mathfrak{o}(n) = \mathfrak{so}(n)$.

Sometimes these maps can be explicity computed, giving rise to lots of application in kinematics, robotics, etc.

Proposition 1.4: Rodrigues Formula

The exponential map $\exp : \mathfrak{so}(3) \to SO(3)$ is given by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1 - \cos\theta)}{\theta^2} A^2$$

if $\theta \neq 0$, with $e^{0_3} = I_3$.

Proof. Notice that

$$A^2 = -\theta^2 I_3 + B$$

and $AB = BA = 0$. From there we can easily deduce the fact that

$$A^3 = \theta^2 A$$

Now we just write the power series for e^A

$$e^A = I_3 + \sum_{p\geq 1} \frac{A^p}{p!}$$

Some painful calculation will give us the following

$$\begin{aligned}
e^A &= I_3 + \sum_{p\geq 0} \frac{A^{2p+1}}{(2p+1)!} + \sum_{p\geq 1} \frac{A^{2p}}{(2p)!} \\
&= I_3 + \sum_{p\geq 0} \frac{(-1)^p \theta^{2p}}{(2p+1)!} A + \sum_{p\geq 1} \frac{(-1)^{p-1}\theta^{2(p-1)}}{(2p)!} A \\
&= I_3 + \frac{A}{\theta} \sum_{p\geq 0} \frac{(-1)^p \theta^{2p+1}}{(2p+1)!} - \frac{A^2}{\theta^2} \sum_{p\geq 1} \frac{(-1)^p \theta^{2p}}{(2p)!} \\
&= I_3 + \frac{\sin\theta}{\theta} A - \frac{A^2}{\theta^2} \sum_{p\geq 0} \frac{(-1)^p \theta^{2p}}{(2p)!} + \frac{A^2}{\theta^2} \\
&= I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2
\end{aligned}$$

□

It looks worse than it is, its really just moving things around to get the power series for sine and cosine. Although I do not want to do it again.

It is also worth mentioning *Hermitian matrices*. They are a complex equivalent of orthogonal matrices, namely, a matrix A is Hermitian if $A = \overline{A^T}$. There are similar matrix groups for such matrices. Usually we are concernd about The groups $H(n)$, $HP(n)$, and $HPD(n)$. These are the groups of Hermitian matrices, Hermitian positive semidefinite matrices, and the group of Hermitian positive definite matrices, respectively. We will talk about this in the next section but Hermitian matrices are very important and worth mentioning, they will show up a lot more when we look into Kähler stuff. We also have the unitary and special unitary matrix group but we will define these once they show up.

1.2.1 Symmetric and Positive Definite Matrices

Now we will backtrack a little bit and look at another important class of matrices, and consequently, groups. A matrix is *positive semidefinite* if all of its eigenvalues are nonnegative. The spectrum of positive definite matrices is used extensively in optimization and convex analysis, it also has some neat geometric implications we will see later. I assume familiarity of symmetric matries by now, my first grade students could identity one. Now for some counterintuitive notation. We denote the vector space (not necessarily group) of symmetric $n \times n$ matrices by $S(n)$ and the vector space of symmetric positive define matrices by $SPD(n)$. Yes I know the notation is the same we used for matrix Lie groups, and we used the fraktur symbols for the vector spaces and algebras. But just remember these are vector spaces, I will try my best to remember myself and remind the reader. It is also not uncommon for this notation to use boldface characters, it is however a pain to type sometimes.

Proposition 1.5

For every symmetric matrix B, the matrix e^B is symmetric positive definite. For every symmetric positive definite matrix A, there is a unique symmetric matrix B such that $A = e^B$.

Proof. Two pages and some technical linear algebra, I still don't understand it completely. $\square$

2 Adjoint Representations and Derivatives

This section is mainly for motivating the Lie bracket and its implications. We will use lots of matrix groups from the last section so its important to understand these groups and their properties. There is going to be a lot going on in this section and may be the most difficult section thus far so take your time, it took me a hot minute to get throught this.

2.1 Adjoint maps

Let the group of all real $n \times n$ matrices be denoted $M_n(\mathbb{R})$. For some matrix $A \in M_n(\mathbb{R})$, define two maps L_A and R_A (left and right lol) such that

$$L_A(B) = AB$$
$$R_A(B) = BA$$

for all $B \in M_n(\mathbb{R})$. Now we want to define the adjoint. So for some $A \in GL(n, \mathbb{R})$, we have

$$\mathbf{Ad}_A : M_n(\mathbb{R}) \to M_n(\mathbb{R})$$

Which is really just a conjugation in disguise

$$\mathbf{Ad}_A(B) = ABA^{-1} \tag{4}$$

This is convenient because we are somewhat familiar with group conjugation and normal subgroups and what not. It is also worth noting that $\mathbf{Ad}_A = L_A \circ R_{A^{-1}}$. This is obviously very easy to check. If we restrict $\mathbf{Ad}_A$ to $GL(n, \mathbb{R})$ then $\mathbf{Ad}_A$ is a group isomorphism. Note this map is not linear as $GL(n, R)$ is not a vector space. However, we still have sufficient structure to define the derivative of the adjoint map.

When $B = I$, the map $d(\mathbf{Ad}_A)_I$ is linear and we denote it as Ad_A and it is similarly defined as $\mathrm{Ad}_A(X) = AXA^{-1}$ for all $X \in \mathfrak{gl}(n, \mathbb{R})$. Remember that this just means X is a real $n \times n$ (not necessarily invertible) matrix. Also note that $\mathrm{Ad}_A \in GL(\mathfrak{gl}(n, \mathbb{R}))$, so the multplication operation is just given by composition

$$\mathrm{Ad}_{AB} = \mathrm{Ad}_A \circ \mathrm{Ad}_B$$

This means that $A \mapsto \mathrm{Ad}_A$ is a group homomorphism of $GL(\mathfrak{gl}(n, \mathbb{R}))$. This map is so special that it is called the *adjoint representation* of $GL(n, \mathbb{R})$. We make a small notation change for the adjoint

$$\mathrm{Ad} : GL(n, \mathbb{R}) \to GL(\mathfrak{gl}(n, \mathbb{R}))$$

If the derivative of Ad exists at I, it is a linear map denoted

$$d(\mathrm{Ad})_I : \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{Hom}(\mathfrak{gl}(n, \mathbb{R}), \mathfrak{gl}(n, \mathbb{R})) \tag{5}$$

And this is good because its exactly what we expect. The derivative is a linear approximation so it makes sense that the derivative operator here is linear, and when evaluated, lives in the homset of the Lie algebra.

Now we will look at a beautiful formlua that relates the adjoint and exponential mapas as we'd expect.

Proposition 2.1

For any $X \in M_n(\mathbb{R}) = \mathfrak{gl}(n, \mathbb{R})$, we have

$$\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X} = \sum_{k=0}^{\infty} \frac{(\mathrm{ad}_X)^k}{k!}$$

The proof for this really isnt too bad but I don't want to write it out.

2.2 Derivative of the Exponential Map

Ok now we are getting a little closer to doing some actual geometry, but not quite.

There does exist an explicit formula for the exponential map but the computation isn't exactly nice so I just present it here without proof.

Proposition 2.2: Derivative of exp

$$d(\exp)_A = e^A \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (\mathrm{ad}_A)^k$$

Or similarly, in terms of the "right" and "left" operators, we get

$$d(\exp) = e^{L_A} \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} (L_A - R_A)^j$$

We now have enough background on the exponential and adjoint operator to move forward and start to study manifolds and geometry in general.

3 Manifolds and Lie Groups

Now we will give explicit defitions for manifolds, Lie groups, and Lie algebras. There is a lot to talk about here and there is easily enough out there to fill an entire book so we will only talk about what is necessary in the context of geometry.

3.1 Embedded Manifolds

Before we explicitly define manifolds, we need to look at a simplified version which we call *embedded manifolds*. In a huge nutshell, a manifold is some kind of space that looks like $\mathbb{R}^n$ locallly. Now we will explicitly define a manifold and unpack the defition.

Definition 3.1: Manifold

Given two integers N, m with $N \geq m \geq 1$, an *m-dimensional smooth manifold in* $\mathbb{R}^N$ (we will now just say "manifold"), is a nonempty subset $M \subseteq \mathbb{R}^N$ such that for every point $p \in M$ there are two open subsets $\Omega \subseteq \mathbb{R}^m$ and $U \subseteq M$ with $p \in U$, and a smooth function $\varphi : \Omega \to \mathbb{R}^N$ such that φ is a homeomorphism, $\varphi'(t_0)$ is injective, where $t_0 = \varphi^{-1}(p)$.

The function $\varphi : \Omega \to U$ is a *local parameterization* of M at p. If $0_m \in \Omega$ and $\varphi(0_m) = p$, we say that φ is *centered* at p.

Although the defition is lengthy, it is very precise and actually a little more intuitive than it may initially seem. Also note that because φ is a homeomorphism, it admits a smooth inverse $\varphi^{-1} : U \to \Omega$ called a *chart*. These charts are what allow us to travel between $\mathbb{R}^n$ and a given manifold. The charts tell us how to translate the language of the manifold onto the base field. Lets look at a pretty picture of a manifold in $\mathbb{R}^n$.

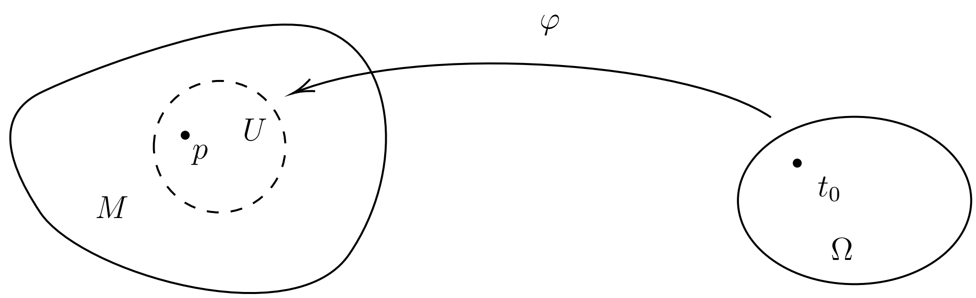

Figure 1: A manifold M and a local parameterization φ.

This visualization is key and will be paramount to understanding how we do analysis on manifolds. It is worth noting too that M is a topological space and U is a subspace endowed with the subspace topology. Perhaps the most basic example of a manifold would be the unit sphere S^2 embedded in $\mathbb{R}^3$. It is pretty easy to check the conditions for a manifold but the arithmetic is a little tedious (stereographic projections ew).

In fact, every open subset of $\mathbb{R}^n$ is a manifold in a somewhat trivial way. For the parameterization, we just choose the inclusion map and the rest is trivial. A great example would be the Lie group $GL(n, \mathbb{R})$. Of course it is a manifold by definition but assuming no Lie group structure, we can still view $GL(n,\mathbb{R})$ as an open subset of $\mathbb{R}^{n^2}$ as its complement is closed; the set of invertible matrices is the inverse image of the determinant funtion, which is continuous of course (this is worth proving if you haven't already).

Example 3.1: $GL(n,\mathbb{C})$ as a manifold

For every $A \in M_n(\mathbb{C})$, construct an analogous $2n \times 2n$ real matrix such that every entry $a+ib$ in A is replaced by the matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

We can represent this map by

$$\Phi : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$$

$$A \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

This map is obviously a group isomorphism so we can view $GL(n,\mathbb{C})$ as a subgroup of $GL(2n,\mathbb{R})$ and as a manifold in $\mathbb{R}^{(2n)^2}$.

Now we have a decent understanding of some properties of manifolds and their charts. Now, we will look at how to jump between manifolds and the inverse images of their respective charts.

Definition 3.2: Transition Maps

Given an m-dimensional manifold M in $\mathbb{R}^n$, for every $p \in M$ and any two parameterizations $\varphi_1 : \Omega_1 \to U_1$ and $\varphi_2 : \Omega_2 \to U_2$ of M at p, if $U_1 \cap U_2 \neq \varnothing$, the map $\varphi_2^{-1} \circ \varphi_1 : \varphi_1^{-1}(U_1 \cap U_2) \to \varphi_2^{-1}(U_1 \cap U_2)$ is a smooth diffeomorphism. These maps are called *transition maps*.

In a nutshell, if there is some overlap of the images of parameterizations, given sufficient conditions, we can construct a transition map to take elements from the first open subset Ω_1, to another Ω_2. Note that these are maps between open subsets of $\mathbb{R}^n$ and NOT maps between manifolds (we will talk about that later).

In general, it can be difficult to prove a given space is a manifold in an explicit manner, so we want to have alternate ways of characterizing and identifying manifolds. This leads us to a great proposition that will be extremely useful.

Proposition 3.1

butthole