Notes

Preston Malen

January 2024

Introduction

First and foremost I want to clarify that by no means am I an expert on the presented topics or any topics in mathematics for that matter. I have wanted to study complex geometry for quite some time but was unable take a course or do research in the area. So this is me exploring complex geometry and some of the topics that are parallel. Most of this information is new to me at the time of writing. Also worth noting, I am writing this with a VERY casual approach so my language and verbiage may not be as formal or precise, I'm writing what my brain is thinking. I am documenting my journey of learning complex geometry and whatever may branch off of it that interests me.

Contents

1 Foundations of Differential Geometry

There are a lot of important and foundational concepts in differential geometry but we will only explore a few. We really just want to establish the differential calculus on manifolds and perhaps some Riemannian structure. I'm assuming there is some previous knowledge of some basic defintions like manifolds, charts, transition maps etc.

1.1 Algebraic Preliminaries

To go into anymore depth, we have to take a quick detour and establish some algebraic structure.

Definition 1.1. *Let V be a vector space (over an arbitrary field $\mathbb{K}$). The k-fold product $V \times V \times \ldots \times V$ is denoted as V^k. A function $T : V^k \to \mathbb{K}$ is* **multilinear** *if for each i for $1 \leq 1 \leq k$ we have:*

- $T(v_1, \ldots, v_i + v_i', \ldots, v_k) = T(v_1, \ldots, v_i, \ldots, v_k) + T(v_1, \ldots, v_i', \ldots, v_k)$
- $T(v_1, \ldots, av_i, \ldots, v_k) = aT(v_1, \ldots, v_i, \ldots v_k)$

A multilinear function $T : V^k \to \mathbb{K}$ is a k-**tensor** *on V.*

The set of all k-tensors, denoted $\mathfrak{J}^k(V)$, becomes a vector space over $\mathbb{R}$ if for $S, T \in \mathfrak{J}^k(V)$ and some $a \in \mathbb{R}$, we have:

- $(S + T)(v_i) = S(v_i) + T(v_i)$
- $(aS)(v_i) = a \cdot S(v_i)$

You may have noticed that the dual of V, denoted V^* is exactly $\mathfrak{J}^k(V)$. This is more than a convenient coincidence, we will see later that this actually lets us define other vector spaces in terms of $\mathfrak{J}^k(V)$. To connect the spaces in $\mathfrak{J}^k(V)$ we have to introduce a general notion of a product.

Definition 1.2. *Let $S \in \mathfrak{J}^m(V)$ and $T \in \mathfrak{J}^n(V)$. We define the* **tensor product** $S \otimes T \in \mathfrak{J}^{m+n}(V)$ *by:*

$$S \otimes T(v_1, \ldots, v_k, v_{m+1}, \ldots, v_{m+n}) = S(v_1, \ldots, v_k) \cdot T(v_{m+1}, \ldots, v_{m+n})$$

It is extremely important to note that the tensor product does NOT commute in general. Changing the order of the factors will yield two entirely different spaces. However, it does benefit from associativity and some other convenient properties:

$$\begin{aligned}
(S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T \\
S \otimes (T_1 + T_2) &= S \otimes T_1 + S \otimes T_2 \\
(aS) \otimes T &= S \otimes (aT) = a(S \otimes T) \\
(S \otimes T) \otimes U &= S \otimes (T \otimes U)
\end{aligned}$$

To make more sense of tensors and their structure, we introduce some more exterior algebra. We need to introduce a special type of tensor first.

Definition 1.3. *An* **alternating tensor** *of degree k on a vector space V is a map $T : V \times \ldots \times V \to \mathbb{K}$ such that:*

- $T(u_1, \ldots, u_i, \ldots, u_k) = -T(u_1, \ldots, u_j, \ldots, u_i, \ldots, u_k)$
- $T(\lambda_1 v_1 + \lambda_2 v_2, u_2, \ldots, u_k) = \lambda_1 T(v_1, u_2, \ldots, u_k) + \lambda_2 T(v_2, u_2, \ldots, u_k)$

There is nothing crazy going on here but it is worth noting that the alternating property switches the u_i and u_j. It hints at some sort of ani-symmetric property but not quite. Now we can look at another important "collection" of tensors.

Definition 1.4. *The* k**-th** *exterior power* $\Lambda^k V$ *of a finite dimensional vector space* V *is the dual space of the vector space of alternating tensors of degree* k *on* V*. Elements of* $\Lambda^k V$ *are* k**-vectors***.*

Now we have this dual space of tensors but we need additional structure on it. We introduce another product

Definition 1.5. *Given* $v_1, \ldots, u_k \in V$*, the* **exterior product** *or* **wedge product** $v_1 \wedge \ldots \wedge v_k \in \Lambda^k V$ *is the linear map to* $\mathbb{K}$ *which, on analternating tensor* T *takes the value:*

$$(v_1 \wedge \ldots \wedge v_k)(T) = T(v_1, \ldots, v_k)$$

The exterior product has a few important properties:

- it is linear in each variable independently
- internchanging two variables changes the sign of the product
- two variables are the same, the product vanishes
- taking the product of tensors results in a tensor of a larger rank than either factor

One may be tempted to ask why not just use the tensor product? This wouldn't work as the tensor product is for taking products of "spaces" while the exterior product allows us to take products of k-vectors.

1.2 Integration on Chains

2 Complex Manifolds

2.1 Complex Analysis Review

Before we can go too deep in Kähler manifolds (and some Riemannian geometry in general), we need to have a quick review of complex analysis.

2.1.1 Holomorphic Functions

Definition 2.1. *Given an open set $\Omega \in \mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is* **analytic** *if for all $z_0 \in \Omega$ there exists a ball of radius $\varepsilon > 0$ about z_0 such that f has a well-defined power series:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall z \in B_\varepsilon(z_0)$$

Some refer to this definition as a **holomorphic** function but this is technically incorrect. Analyticity is a local property defined about some ε-neighbordhood whereas being holomorphic is a little more general. The differentiable property of complex functions is defined only pointwise. A classical example of this difference would be a function that is only differentiable about a line and thus not analytic. However, in complex analysis these two definitions (analytic and holomorphic) happen to be equivalent so they are often used interchangeabley. The proof is not too difficult but we omit it here. We will however establish the exact definition of holomorphic.

Definition 2.2. *A function $f : \Omega \to \mathbb{C}$, expressed as $f(x, y) = u(x, y) + iv(x, y)$, is* **holomorphic** *if it satisfies the Cauchy-Riemannan equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$$

In fact, we can generalize this to $\mathbb{C}^n$. Now let $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^n$. This function is holomorphic on $\mathbb{C}^n$ if it satisfies the generalized Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}, \quad \text{for } i = 1, 2, \ldots, n$$

In laymen's terms, f is holomorphic in every "direction" or coordinate. This gives us a perfect segue into how the complex Jacobian matrix for a holomorphic function f is related to and defined by the Cauchy-Riemann equations:

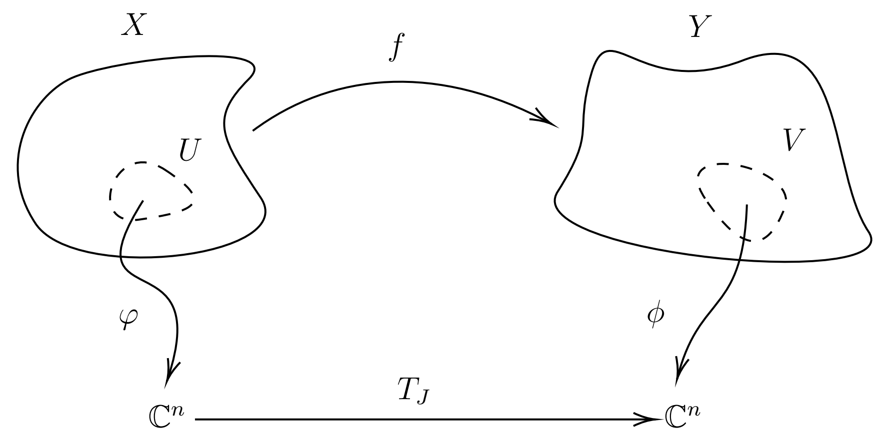
$$J(f)(z) := \left(\frac{\partial f_i}{\partial z_j}(z) \right), \ 1 \leq i \leq n, \ 1 \leq j \leq m$$

Recall from differential geometry that Jacobian matrices define transition maps. The complex Jacobian matrices also play a part in complex manifolds as we will see later.

2.2 Complex Manifolds

Now we have all the tools to study complex manifolds via their charts, transition maps, and morphisms. First we will look at the complex equivalents of some foundational definitions in differential geometry. We start by defining an analogue for an atlas.

Definition 2.3. *A* **holomorphic atlas** *on a differentiable manifold M is an atlas $\{(\Omega_i, \varphi_i)\}$ such that $\Omega_i \cong \varphi_i(\Omega_i) \subseteq \mathbb{C}^n$ and the transition functions are holomorphic.*



A map $f : X \to Y$ between differentiable manifolds, is holomorphic if the map:

$$\phi \cdot f \cdot \varphi^{-1} : \varphi(f^{-1}(U) \cap U)) \to \phi(V)$$

is holomorphic. Here the map T_J is the transition map given by the Jacobian matrix. Note that φ and ϕ map to a respective open subset of $\mathbb{C}^n$ (perhaps all of $\mathbb{C}^n$).

Definition 2.4. *Two differentiable manifolds X and Y are* **biholomorphic** *(or isomorphic) if there exists a holomorphic homeomorphism between them.*

So in the picture above, if f is a holomorphic homeomorphism, that would imply X and Y are biholomorphic.

Another really important topic in the study of manifolds is the idea of a submanifold. In a very loose sense, we can think of these as subsets of manifolds.

Definition 2.5. *Let X be a complex manifold with dimension n and $Y \subseteq X$ a differentiable manifold of dimension $2k$. Y is a* **complex submanifold** *of X of dimension k if there exists a holomorphic atlas $\{(U_i, \varphi_i)\}$ of X such that $\varphi : U_i \cap Y \cong \varphi(U_i) \cap \mathbb{C}^k$ is a biholomorphism.*

We identify $\mathbb{C}^k \subseteq \mathbb{C}^n$ as $(z_1, \ldots, z_k, 0, \ldots, 0)$ with zeroes to the n-th coordinate.

2.2.1 Algebra of Complex Manifolds

Now we will shift gears just a little and look at some very foundational objects in complex geometry and algebraic geometry in general. These are some pretty heavy duty ideas and they are not easy to grasp by any means. We will try to give visual representations as much as possible, but note that it may take a lot of time to understand some of the following theory and that's totally normal.

Definition 2.6. *Let X be a topological space. A* **presheaf** *F of sets on X conatins the following:*

- *For each open set U of X, there exists a set $F(U)$ denoted $\Gamma(U, F)$. Elements of this set are* **sections** *of F over U. The sections of F over X are called the* **global sections** *of F.*

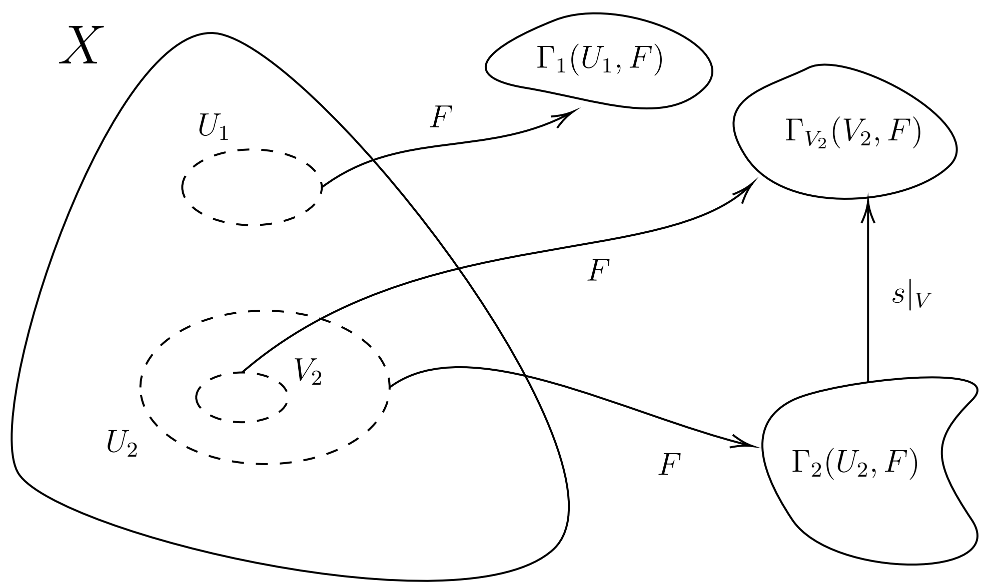
- *For each inclusion of open sets $V \subseteq U$, there is a function $\mathrm{res}_{V,U} : F(U) \to F(V)$. These are* **restriction morphisms***. If $s \in F(U)$, this its restriction $\mathrm{res}_{V,U}$ is denoted $s\mid_V$.*

The restriction morphisms must satisfy two additional properties:

- *For every open $U \subseteq X$, the restriction morphism $\mathrm{res}_{U,U} : F(U) \to F(U)$ is the identity morphism on $F(U)$.*

- *For three open sets $W \subseteq V \subseteq U$, then we have $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$.*

Now for a concrete example, consider a set U, we can then assign the set $C^0(U)$ of all continuous real-valued functions on U. The restriction maps here are given by restricting a continuous function on U to a an open subset $V \subseteq U$. In this case, $C^0(U)$ is a presheaf on the set U.



Pictured above we have a visual of a presheaf defined on open subsets of a topological space X. Here, the family sets $F(U_i)$ are a presheaf over X. We also see the restriction morphism $s\mid_V : \Gamma_2(U_2, F) \to \Gamma_{V_2}(V_2, F)$. Elements of any given Γ are sections of F on the given U.

Presheaves are a great tool but they lack a little bit of power. We really want to extend this idea to an entire space or set to retrieve global information.

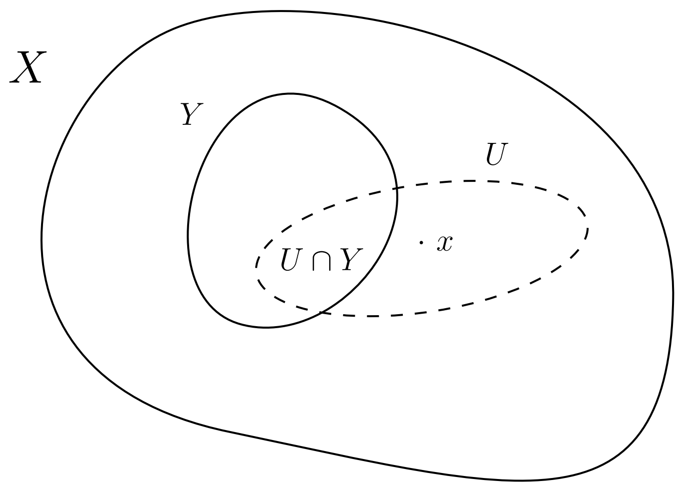
Definition 2.7. *A presheaf is a* **sheaf** *if it satisfies the following:*

- *(Locality) Let U be an open set, $\{U_i\}$ is an open cover of U with $U_i \subseteq U$ for all i, and $s, t \in F(U)$ are sections. If $s\mid_{U_i} = t\mid_{U_i}$ for all i then $s = t$.*

- *(Gluing) Let U be an open set, $\{U_i\}$ is an open cover of U with $U_i \subseteq U$ for all i, and $\{s_i \in F(U_i)\}$ is a family of sections. If all pairs of sections agree on the overlap of their domains, $s_i\mid_{U_i \cap U_j} = s_j\mid_{U_i \cap U_j}$ for all indices i, j, then there exists a section $s \in F(U)$ such that $s_i\mid_{U_i} = s_i$ for all i.*

In a nutshell, presheaves assign data to sets locally while sheaves let us extend this globally. We will mostly be concerned with a sheaf of holomorphic functions $\mathcal{H}(-)$. However, there will certainly be more applications. Yes that is a lot to take in.

Definition 2.8. *Let X be a complex manifold. An* **analytic subvariety** *of X is a closed subset $Y \subseteq X$ such that for each $x \in X$, there exists an open neighbordhood $U \in X$ such that $U \cap Y$ is of the form $U \cap Y = \{x \in U \mid f_1(x) = \ldots = f_k(x) = 0\}$ for finitely many holomorphic functions f_i in the sheaf $\mathcal{H}(U)$.*



Typically, we have the subset Y ahead of time. We can then choose the x and U accordingly to make Y into an analytic subvariety. So we want to choose points that the elements of $U \cap Y$ are solutions to a finite family of holomorphic functions.