Differential and Complex Algebraic Geometry

Preston Malen

January 2024

Introduction

First and foremost I want to clarify that by no means am I an expert on the presented topics or any topics in mathematics for that matter. I have wanted to study complex geometry for quite some time but was unable take a course or do research in the area. So this is me exploring complex geometry and some of the topics that are parallel. As it happens, this requires a full course on differential geometry and Lie groups. I'm taking these notes from 3-4 books as well as supplemental notes found online. Of course these will all be cited at the end. The end goal is to prepare myself for spectral/noncommutative geometry. However, the functional analysis side of things would be a novel on it's own. Some functional analysis will pop up here and there in these notes but only in the context of geometry. I am not going to go too deep into operator algebras or anything like that. Most of this information is new to me at the time of writing. Also worth noting, I am writing this with a VERY casual approach so my language and verbiage may not be as formal or precise, I'm writing what my brain is thinking. Lots of proofs will be ommitted but more the simpler ones I will probably include. Some of the proofs just require results that are beyond the scope of the current section, but I will of course cite the books that contain the proofs. I am documenting my journey of learning geometry and whatever may branch off of it that interests me.

Contents

1 The Matrix Exponential

1.1 The Exponential Map and Matrix Groups

Given an $n \times n$ matrix A, we want to find a way to find e^A. We can actually just do this with the usual power series:

$$e^A = I_n + \sum_{p \geq 1} \frac{A^p}{p!} = \sum_{p \geq 0} \frac{A^p}{p!} \tag{1}$$

Using an inductive proof, we can show that this is well defined. But we won't write it out here.

Example 1.1

Consider the matrix

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

We want to find a way to express the powers A^n. We can factor out a θ to see

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^2 = -\theta^2 I_2$$

Now, let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we have

$$\begin{aligned} A^{4n} &= \theta^{4n} I_2 \\ A^{4n+1} &= \theta^{4n+1} J \\ A^{4n+2} &= -\theta^{4n+2} I_2 \end{aligned}$$

and so on. This means we can now express e^A as a power series

$$e^A = I_2 + \frac{\theta}{1!} J - \frac{\theta^2}{2!} I_2 \dots$$

Writing this out we will see that we actually get the power series for cosine and sine, thus

$$e^A = \cos\theta I_2 + \sin\theta J$$

or equivalently

$$e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So we see e^A is in fact a rotation matrix.

This is actually a general fact. If we have a skew-symmetric matrix A, then e^A is an orthogonal matrix with determinant 1, or $e^A \in SO_n(\mathbb{F})$. In fact, EVERY rotation matrix is of this form. To be explicit, the exponential map from the set of skew-symmetric matrices to the set of rotation matrices is surjective. But note that the exponential map is NOT surjective in general.

Proposition 1.1

Let A and U be (real or complex) matrices and assume U is invertible. Then

$$e^{UAU^{-1}} = Ue^{A}U^{-1}$$

This is pretty obvious and its easily proven using an inductive proof. But I hate induction so of course I will not include the proof here, although its only a few lines. Now we will look at another important result that will be important when we start to look at some spectral properties of the exponential map.

Proposition 1.2

Given any complex $n \times n$ matrix A,