An Introduction to States and Representations of C*-Algebras

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Abstract

This paper provides an introduction to the theory of states and representations of C*-algebras. Beginning with basic definitions and examples, we explore the structure of state spaces and illustrate the Gelfand-Naimark-Segal (GNS) construction. Connections to quantum mechanics and noncommutative geometry are also discussed.

Contents

1 Introduction

C*-algebras are a generalization of function algebras and play a crucial role in functional analysis, quantum mechanics, and noncommutative geometry. This paper aims to provide an accessible introduction to the concepts of states and their representations. Only a basic level of functional analysis is required to fully grasp all the concepts presented. We are trying to take a tensor product-free approach to keep things as simple and accessible as possible. However, the examples in quantum field theory, namely Fock spaces, require a basic knowledge of tensor products. The quantum field theory section is at the end and can easily be ommitted without consequence.

1.1 Motivation

Discuss the motivation for studying states and their representations. Include their relevance to various mathematical and physical theories.

1.2 Structure of the Paper

Outline the sections of the paper.

2 Preliminaries

2.1 Definition of an Algebra

Definition 2.1. An **algebra** is a vector space A with a multiplication operator $A \times A \to A$ and $(x, y) \mapsto xy$ such that

1. $(xy)z = z(yz)$ for all $x, y, z \in A$
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$ for all $x, y, z \in A$
3. $c(xy) = (cx)y = x(cy)$ for all $c \in \mathbb{C}$ and $x, y \in A$

We say A is **unital** if it has an identity element $e \in A$ such that $ex = xe = x$ for all $x \in A$. We will assume every algebra is unital unless stated otherwise. Further, A is **commutative** or (**abelian**) if $xy = yx$ for all $x, y \in A$.

2.2 Definition of a Banach Algebra

Definition 2.2. A **Banach algebra** A is simultaneously an algebra and a Banach space satisfying

$$\|xy\| \leq \|x\|\|y\| \quad \forall x, y \in A$$

If A is unital then we assume $\|e\| = 1$. Further, we usualy write $1 := e$ for the unital element. We won't work much with Banach algebras exclusively but rather to motivate the definition of a C*-algebra. However, it is important to note an example.

Example 2.3. Let $f, g \in L^1(\mathbb{R})$ and define the algebra's operation as the convolution $f * g \in L^1(\mathbb{R})$ by

$$(f * g)(t) := \int_{\mathbb{R}} f(t - s)g(s)ds \tag{1}$$

Under the convolution, $L^1(\mathbb{R})$ is a commutative Banach algebra but it is not unital.

2.3 Definition of a C*-Algebra

Definition 2.4. A **C*-algebra** is a Banach algebra A equipped with an involution $a \mapsto a^*$ satisfying the C*-identity:

$$\|a^*a\| = \|a\|^2 \quad \text{for all } a \in A.$$

Note that the involution operator is unique. Most C*-algebras are endowed with the opertor norm-induced topology and we will always assume this to be the case unless stated otherwise. For some concrete examples and nonexamples, consider a compact Hausdorff space X. We define $C(X)$ to be all continuous functions $f : X \to \mathbb{C}$. It is crucial that X be compact. The involution on $C(X)$ is defined as complex conjugation denoted $f^* = \overline{f}$. Lets look at some simple examples.

Example 2.5. Let $X = [0, 1]$, then $C(X)$ is a C*-algebra.

Example 2.6. Let $X = S^1$, the unit circle. Then $C(X)$ is a C*-algebra.

Example 2.7. Let $X = \{0, 1, 2, 3\}$. Then $C(X)$ is a C*-algebra.

Example 2.8. Let $X = \mathbb{R}$. Then $C(X)$ is not a C*-algebra because X is not compact.

These are very basic concrete examples of subsets of $\mathbb{C}$, but we also want to consider some much more important examples for our purposes. Arguably the most important example is the C*-algebra $C_0(X)$, the C*-algebra of continuous functions that vanish at infinity over a locally compact Hausdorff space X. This leads us to one of the most important and useful theorems in the theory of operator and C*-algebras. First, a quick definition.

Definition 2.9. For a C*-algebra A, a **character** is a nonzero homomorphism

$$\Phi : A \to \mathbb{C}$$

Theorem 2.10 (Gelfand Representation Theorem)**.** Every commutative unital C*-algebra A is isometrically isomorphic to $C_0(X)$, the algebra of continuous functions on some compact Hausdorff space X, where X is the space of characters.

The utility of the Gelfand Representation Theorem is quite apparent. This allows us to view all commutative C*-algebras as isomorphic. Specifically, for any commutative unital C*-algebra A, there exists an isometric isomorphism

$$\varphi : A \to C_0(X)$$

The importance of this theorem cannot be overstated. This property of unital commutative C*-algebras is often referred to as **Gelfand duality**. A valuable C*-algebra that shows up a lot in physics is the C*-algebra of complex-valued $n \times n$ matrices, denoted $M_n(\mathbb{C})$. Another C*-algebra of particular interest is the bounded linear operators from a Hilbert space onto itself, we denote this $B(H)$ for a Hilbert space H.

In summary, and with some abuse of notation, we can see how much more structure C*-algebras have as compared to typical vector spaces.

$$\text{vector space} \supseteq \text{algebra} \supseteq \text{Banach algebra} \supseteq \text{C*-algebra}$$

2.4 States on C*-Algebras

Definition 2.11. A **state** on a C*-algebra A is a linear functional $\phi : A \to \mathbb{C}$ such that:

$$\phi(a^*a) \geq 0 \text{ for all } a \in A, \quad \text{and } \phi(1) = 1.$$

We require the positive semidefinite condition for ϕ to preserve the positive structure of A. We define the set of all the states on A as the **state space** $S(A)$. It can be shown that $S(A)$ is compact, non-empty, and convex (state spaces are "nice"). There are certain states that are of particular interest. We say a state $\phi \in S(A)$ is **pure** if it is an extreme point of $S(A)$. Note this is well-defined because the Krein-Milman theorem guarantess the existence of extreme points of $S(A)$. States that are not pure are **mixed**.

States of course have the typical properties that we'd expect from a linear functional but they also admit some other properties that may not be so apparent.

Definition 2.12. The **kernel** of a state ϕ is defined as

$$\ker(\phi) = \{a : \phi(a^*a) = 0\}$$

for $a \in A$, a C*-algebra. Furthermore, a state ϕ on a C*-algebra is **faithful** if $\phi(a^*a) = 0$ implies $a = 0$ for all $a \in A$.

It is very important to note that this is different than the kernel of a general linear functional. In a sense, this tells us that faithful states "detect" non-zero elements. As we will see later, a state is faithful if and only if its GNS representation is injective. States have many other properties of interest, particularly

- The state space $S(A)$ is compact in the weak-* topology
- A state is **tracial** if $\phi(ab) = \phi(ba)$ for all $a, b \in A$
- Pure states correspond to rank-1 projections onto a unit vector $\psi \in H$. Specifically, $\phi(A) = \langle \psi, A\psi \rangle$
- A mixed state is a convex combination of pure states

3 Concrete Examples of States

3.1 States on $C(X)$

Recall that for compact space X, we denote the C*-algebra of continuous functions on from $X \to \mathbb{C}$ by $C(X)$. Fix some point $x_0 \in X$. We define the **Dirac state** as $\phi(f) = f(x_0)$. One can easily see that the Dirac state is pure and faithful. Now, consider the C*-algebra $C^*(G)$ which is the algebra generated by the unitary representations of a compact group G (we will talk about representations some more in the next section). Then we define the **Haar state** as

$$\phi(f) = \int_G f(g)d\mu(g) \tag{2}$$

for the Haar measure μ. This state is faithful if G is simple, and tracial if G is abelian. There are of course many other states on $C(X)$ (and $C^*(X)$) but these are of particular interest in quantam mechanics and noncommutative geometry. States on commutative algebras have a very familiar analogue. In fact, for $C(X)$, states are exactly probability measures. To see this, we define the state

$$\phi(f) = \int_X f(x)d\mu(x) \tag{3}$$

with $\mu(X) = 1$. So under the hood, probability measures on commutative algebras are states. If A is commutative, and thus isomporhic to $C(X)$ (by Gelfand duality), then the Riesz Representation Theorem guarantees that ϕ corresponds to a probability measure μ on X. Now what about the noncommutative case? If A is noncommutative, then unfortunatley we can't make a generalization to traditional probability measures and some additional theory is required.

3.2 States on $M_2(\mathbb{C})$

In quantum mechanics, a density matrix is a postivie semidefinite operator ρ with operator norm 1. The state corresponding to a given density matrix is

$$\phi(A) = \mathrm{Tr}(\rho A) \tag{4}$$

Recall that is ρ is a rank-1 projection, then we can recover the pure state exactly by taking the inner-product $\phi(A) = \langle \psi, A\psi \rangle$ for a unit vector $\psi \in H$. If ρ is not rank-1, then it is a mixed state. These mixed states have a special relationship to the probability of quantum states which we will discuss later.

Another simple example of a state on $M_2(\mathbb{C})$ would be a rank-one projection. Consider a state

$$\varphi(A) = \langle Av, v \rangle \tag{5}$$

for a unit vector $v \in \mathbb{C}^2$ and the usual inner-product. To give a quick example, consider the vector $v = [0, 1]^T$ and the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, we can compute the state $\varphi(A)$ by taking the usual inner product

$$\varphi(A) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \tag{6}$$

Thus our state just returns the A_{11} entry and is a valid state. One can also convince them-self that this state is in fact pure. In physics, this state corresponds to the expectation value of A in the pure quantum state represented by v.

3.3 States on $B(H)$

For noncommutative C*-algebras, the theory changes but there is still a relationship to probability. Consider the C*-algebra of bounded linear operators on a Hilbert space $B(H)$. Recall that the involution on $B(H)$ is the adjoint operator. For a quantum state ϕ on $B(H)$, we can define a probability distribution for each observable. In this context, observables are self-adjoint elements of $B(H)$. Suppose we have an observable A with the given spectral decomposition

$$A = \sum_i \lambda_i P_i \tag{7}$$

where the P_i are projections onto the eigenspaces of A. Then we can find the probability of a given λ_i by $\mathbb{P}(A = \lambda_i) = \phi(P_i)$. So the quantum state doesn't assign a probability distribution to the whole space, but rather assigns a probability distribution to each observable in $B(H)$. If we have two observables A and B that do not commute, then there does not exist a single probability distribution that describes both simultaneously.

In summary, states on commutative C*-algebras correspond to probability measures either directly or by the Riesz Representation Theorem. States on noncommutative C*-algebras cannot assign a global probability measure, but rather each observable is assigned a probability measure. So the commutative case lets us make a global generalization while the noncommutative case only allows local consistency. We have only just scratched the surface of states here but hopefully the reader can start to understand the importance they have in quantum mechanics and noncommutative geometry.

4 The Gelfand-Naimark-Segal (GNS) Construction

4.1 Overview of the Construction

The Gelfand-Naimark-Segal (GNS) Construction is a very important method in the theory of operator algebras. On a high level, the idea is to take a state ϕ on a C*-algebra and transform it into a representation of the whole algebra over a Hilbert space. First, we define representations (technically these are *-representations).

Definition 4.1. A **representation** of a C*-algebra A over a Hilbert space H is a map $\pi : A \to B(H)$ such that

- π is a ring homomorphism and carries the involution from A into the involution on operators in $B(H)$
- π is nondegenerate and thus unit-preserving ($\pi(1) = 1$)

Definition 4.2. Let π be a representation of a C*-algebra A on a Hilbert space H. An element $\xi \in H$ is called a **cyclic vector** if the set of vectors

$$\{\pi(a) \cdot \xi : a \in A\} \tag{8}$$

is norm dense in H. Specifically, the range of the representation $\pi(x)\xi$ is dense in H with respect to the induced norm. When this is the case, π is a **cyclic representation**.

The ultimate goal of the GNS construction is to take an abstract state and realize it as a representation on a Hilbert space. Lets do this step by step.

1. We take the algebra A and define a pre-Hilbert space H_A on it where the vectors in H_A are equivalence classes of elements of A with the semi inner-product $\langle a, b\rangle_\phi = \phi(b^* a)$ for a state ϕ on A.

2. To construct the Hilbert space, first, consider N, the left kernel of A

$$N = \{a \in A : \varphi(a^*a) = 0\} \tag{9}$$

Then the complete Hilbert space H can be made by taking the completion of A/N.

3. Now the algebra A can act on H space by operators. For each $a \in A$ we define the operator (representation) $\pi_\varphi(a)$ as acting on the Hilbert space H (the completion of the pre-Hilbert space). This representation is explicitly realized as $\pi_\varphi(a) \cdot b = a \cdot b$ for $a, b \in H$. Recall that the representation maps from $\pi_\phi : A \to B(H)$.

4. The construction of H gives us a cyclic vector $\xi \in H$ generated by the identity of A. This vector is crucial because all of H is generated by the action of A onto this vector. Particularly, all vectors in H can be written $\pi_\phi(a) \cdot \xi$ for some $a \in A$.

5. The state ϕ corresponds to the expectation of the representations $\pi_\phi(a)$ acting on ξ. Specifically, $\phi(a) = \langle \pi_\phi(a) \cdot \xi, \xi \rangle$.

Thus our state ϕ, can now be realized as a real-valued expectation on H. Now we have all the ingredients we need to explicitly define the GNS construction.

Theorem 4.3 (Gelfand-Naimark-Segal (GNS) Construction)**.** Given a state ϕ on a C*-algebra A, there is a *-representation π of A acting on a Hilbert space H with a cyclic vector ξ such that

$$\phi(a) = \langle \pi_\phi(a)\xi, \xi \rangle \tag{10}$$

for every $a \in A$.

Note that the GNS construction is unique up to unitary equivalence. We don't necessarily need the details on how this works but in essence, it means that for any two GNS representations with the same state on the same C*-algebra, there exists a unitary operator that transforms one into the other.

Theorem 4.4. A state ϕ on a C*-algebra A is faithful if and only if its GNS representation $\pi_\phi : A \to B(H)$ is injective.

This is a nice result that is important to keep in mind but won't see much application in the remainder of this paper.

4.2 Examples

Example 4.5. Lets step through a GNS construction for $C([0, 1])$ with a simple state. One can easily verify that $C([0, 1])$ is a C*-algebra under pointwise addition, pointwise multiplication, and the usual suprememum norm. The involution on $C([0, 1])$ complex conjugation for continuous functions but because we are only considering real-valued functions, here, we have the identity involution $f^* = f$. First, lets define a simple state on $C([0, 1])$. Lets consider the state φ which is just a point evaluation, say $x_0 \in [0, 1]$, specifically, $\varphi(f) = f(x_0)$. So we see every $f \in C([0, 1])$ is a linear functional, we just need to ensure positive definiteness. So we check $\varphi(f^*f) = f(x_0)^2 \geq 0$. Thus φ as we defined it, is a valid state on $C([0, 1])$. Now we need to constuct the appropriate pre-Hilbert space H_A. The elements of H_A will be equivalence classes of functions on $C([0, 1])$ with the inner-product given by pointwise multiplication of point evaluations

$$\langle f, g \rangle = \varphi(g^*f) = g(x_0)f(x_0) \tag{11}$$

We take the completion of H_A to turn it into a proper Hilbert space. Next, we define a the representation π_φ of the algebra $C([0,1])$ on the Hilbert space H. So we define the operator $\pi_\varphi(f)$ as

$$\pi_\varphi : C([0,1]) \to B(\mathbb{C}) \tag{12}$$

$$\pi_\varphi(f)\xi \mapsto f\xi \tag{13}$$

for the cyclic vector ξ. In this example, ξ is any function that is nonzero when evaluated at x_0. For simplicity we can even choose $\xi = 1$. So when we take $f \in C([0,1])$ and apply the cyclic vector ξ, we get

$$\pi_\varphi(f)\xi = f(x_0)\xi \tag{14}$$

So the algebra $C([0,1])$ acts on the Hilbert space by multiplication, specifically, by multiplying the cyclic vector by the function evaluated at x_0. Here, our cyclic vector ξ generates the entire Hilbert space under the action of the algebra. In other words, every vector (function) in $C([0,1])$ can be generated by multiplying an element $f \in C([0,1])$ by the cyclic vector $\xi = 1$. Moreover, for any function $f \in C([0,1])$, the action on ξ is

$$\pi_\varphi(f)\xi = f(x_0) \tag{15}$$

This is somewhat trivial in this example but of course the choice of cyclic vector can be very difficult in general. Finally, we note that the state φ corresponds to the expectation value of the representation acting on the cyclic vector. Specifically, we have

$$\varphi(f)\xi = \langle \pi_\varphi(f)\xi, \xi\rangle \tag{16}$$

In our case, this boils down to

$$\varphi(f) = \langle f(x_0), 1\rangle = f(x_0) \tag{17}$$

This is just our function evaluated at x_0 which is exactly what our state does. Despite this being a "simple" example, we still had to do a lot of work. When the choice of pre-Hilbert space and cyclic vector are apparent, the GNS construction of any C*-algebra may look similar to this example.

Example 4.6. The GNS construction for $H = M_2(\mathbb{C})$ is simple but because we are dealing with matrices, it may give a better intuition of what an arbitray GNS construction would look like for linear operators. It is easy to see that $M_2(\mathbb{C})$ is a C*-algebra under the usual matrix addition and multiplication and with involution given by the conjugate transpose. As for states on $M_2(\mathbb{C})$, there are a few "natural" choices. We will use the trace state

$$\varphi(A) = \frac{1}{2}|\operatorname{Tr}(A)| \tag{18}$$

Note that the trace state is faithful. Using the trace state φ, we can build the pre-Hilbert space by taking the inner product

$$\langle A, B\rangle = \varphi(A^*B) = \frac{1}{2}|\operatorname{Tr}(A^*B)| \tag{19}$$

and finding the quotient of $M_2(\mathbb{C})$ by the kernel of the inner-product. Because H is finite dimensional, the inner-product is strictly positive definite which introduces a lot of simplicity. Note that because our action is faithful, φ has a trivial null space, which simplifies the calculations a lot. It is a well known fact that $M_2(\mathbb{C})/\{0\} \cong M_2(\mathbb{C})$. Thus the pre-Hilbert space is just $M_2(\mathbb{C})$ with the inner product given by the trace state, which is in fact a Hilbert space. This makes things simple because we don't need to compute any difficult closures (this can get very cumbersome in general), essentially saving us a step. Next, we need to define our representation. In this case it is convenient to compute the "trivial" representation $\pi_\varphi : M_2(\mathbb{C}) \to B(M_2(\mathbb{C}))$ defined by $\pi_\varphi(A)(B) = AB$ for all $A, B \in M_2(\mathbb{C})$ (left multiplication by matrices). This is trivial in the fact that it isn't providing any additional structure. This representation simply states the fact that matrices act on the Hilbert space $M_2(\mathbb{C})$ by left multiplication, which is exactly the multiplication operation we expect. If the reader picked up on the pattern, it will be no surprise that for the cyclic vector, we just take the identity matrix $\xi = I_2$. So in a concrete sense, the state representation is

$$\pi_\varphi(A)\xi = A\xi \tag{20}$$

which gives us a natural extension to

$$\varphi(A) = \langle \pi_\varphi(A)\xi, \xi \rangle = \langle A\xi, \xi \rangle = \langle A, I_2 \rangle = \frac{1}{2}|\operatorname{Tr}(A^*)| \tag{21}$$

which is exactly what we would expect. This example was potentially easier for some and more difficult for others. But between the two basic examples, hopefully the importance of the GNS construction is illustrated. In a nutshell, we can think of the GNS construction as a way to express a state on a C*-algebra as an inner product in a Hilbert space using a representation of the algebra, and a cyclic vector from the Hilbert space.

5 Applications and Connections

5.1 Quantum Mechanics

States are a key concept in quantum mechanics. We saw earlier how states on $B(H)$ assign probability distributions to observables. We also looked at the trace state on the algebra $M_2(\mathbb{C})$ and saw how it relates to density matrices. Now, we will look at some non-trivial states in quantum mechanics that are of equal or greater importance. We will start with some background on some important operators, then see the C*-algebras they generate and ultimately work through the GNS construction of the Fock spaces.

5.1.1 Weyl and Majorna Operators

First we introduce the bosonic creation $a^\dagger(f)$ and annihilation $a(f)$ operators. If these operators satisfy the **canonical commutation relations**:

$$[a(f), a^\dagger(g)] = \langle f, g \rangle \tag{22}$$
$$[a(f), a(g)] = 0 \tag{23}$$
$$[a^\dagger(f), a^\dagger(g)] = 0 \tag{24}$$

Where $[-,-]$ is the Lie bracket. Then we can define the **Weyl operator**:

$$W(f) = e^{i(a(f)+a^\dagger(f))} \tag{25}$$

Where f and g are functions from a hilbert space H. Weyl operators satisfy the **Weyl commutation relations** given by:

$$W(f)W(g) = e^{-\frac{i}{2}\operatorname{Im}\langle f,g\rangle} W(f+g) \tag{26}$$

Weyl operators satisfy a lot of convenient properties but it is sufficient to note they are unitary operators, specificially:

$$W(f)^* = W(-f)$$
$$W(0) = 1$$

From a physics perspective, Weyl generate phase space transformations. They also describe coherent states and quantum optical transformations. These are beyond the scope of this survey but it is important to note that Weyl operators play a key role in physics.

We saw that Weyl operators satisfy the canonical commutation relations and thus the question arises; are there anticommutation relations and further, corresponding operators? First, for the creation and annihilation operators, we let $c(f)$ and $c^\dagger(f)$ be the fermionic creation and annihilation operators respectively. We define the **canonical anticommutation relations** under the anticommutor operator $\{A, B\} = AB + BA$ (sometimes called the Jordan bracket, the positive analogue of the Lie bracket):

$$\{c(f), c^\dagger(g)\} = \langle f, g\rangle \tag{27}$$
$$\{c(f), c(g)\} = 0 \tag{28}$$
$$\{c^\dagger(f), c^\dagger(g)\} = 0 \tag{29}$$

Given these relations, we can define the anticommutative analogue to Weyl operators. The **unitary fermionic coherent operator** is defined as:

$$U(f) = e^{c^\dagger(f) - c(f)} \tag{30}$$

However, the exponential is usually dropped and the unitary fermionic coherent operator becomes the **Majorna operator**:

$$\gamma(f) = c(f) + c^\dagger(f) \tag{31}$$

Majorna operators satisfy the **Clifford algebra relations**:

$$\{\gamma(f), \gamma(g)\} = 2\operatorname{Re}\langle f, g\rangle \tag{32}$$

Majorna operators are used in fermionic systems, particularly in topological phases of matter and quantum computing.

5.1.2 CCR and CAR Algebras

It might not be obvious how Weyl and Majorna operators show up in the theory of C*-algebras but now we demonstrate how they generate two of the most important algebras in physics. Given a Hilbert space H, the Weyl operators on H generate the **Canonical Commutation Relations algebra (CCR)**. Under the Weyl commutation relations, the CCR algebra is a C*-algebra. Similarly, the Majorna operators under the

Clifford algebra relations generate the **Canonical Anticommutation Relations algebra (CAR)**. These algebras alone are of great interest in C* theory and could be studied extensively, however, we introduce them in anticipation of the construction of the Fock spaces. In the CAR algebra, Majorana operators are used to model systems with fermionic particles. They generate the algebra in a similar way that Weyl operators generate the CCR algebra, but they obey different anticommutation relations. These operators are critical in the study of fermions in quantum mechanics. The CAR algebra is associated with fermionic systems, where particles obey Fermi-Dirac statistics. The key difference between the CAR and CCR algebras is that in fermionic systems, the creation and annihilation operators anticommute rather than commute.

5.1.3 Fock Spaces as Representations

Fermionic and Bosonic

5.1.4 GNS Contstruction of the Fock Spaces

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5.2 Noncommutative Geometry

Briefly describe the role of states in measuring noncommutative spaces.

6 Conclusion and Future Directions

Summarize the main points and suggest further topics for exploration.

A Proofs of Key Results

Provide detailed proofs for any key theorems mentioned in the paper.

References

[1] William Arveson, *An Invitation to C*-Algebras.*

[2] Gerald J. Murphy, *C*-Algebras and Operator Theory.*