An Introduction to States and Representations of C*-Algebras

Preston Malen

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Abstract

This paper provides an introduction to the theory of states and representations of C*-algebras. Beginning with basic definitions and examples, we explore the structure of state spaces and illustrate the Gelfand-Naimark-Segal (GNS) construction. Connections to quantum mechanics and noncommutative geometry are also discussed.

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1 Introduction

C*-algebras are a generalization of function algebras and play a crucial role in functional analysis, quantum mechanics, and noncommutative geometry. This paper aims to provide an accessible introduction to the concepts of states and their representations.

1.1 Motivation

Discuss the motivation for studying states and their representations. Include their relevance to various mathematical and physical theories.

1.2 Structure of the Paper

Outline the sections of the paper.

2 Preliminaries

2.1 Definition of an Algebra

Definition 2.1. An **algebra** is a vector space A with a multiplication operator $A \times A \to A$ and $(x, y) \mapsto xy$ such that

1. $(xy)z = z(yz)$ for all $x, y, z \in A$
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$ for all $x, y, z \in A$
3. $c(xy) = (cx)y = x(cy)$ for all $c \in \mathbb{C}$ and $x, y \in A$

We say A is **unital** if it has an identity element $e \in A$ such that $ex = xe = x$ for all $x \in A$. Further, A is **abelian** or (**commutative**) if $xy = yx$ for all $x, y \in A$.

2.2 Definition of a Banach Algebra

Definition 2.2. A **Banach algebra** is an algebra A which is norm complete, thus a Banach space, such that

$$\|xy\| \leq \|x\|\|y\| \quad \forall x, y \in A$$

If A is unital then we assume $\|e\| = 1$. Further, we usualy write $1 := e$ for the unital element. We won't work much with Banach algebras exclusively but rather to motivate the definition of a C*-algebra. However, it is important to note an example.

Example 2.3. Let $f, g \in L^1(\mathbb{R})$ and define the multiplication operation as the convolution $f * g \in L^1(\mathbb{R})$ by

$$(f * g)(t) := \int_{\mathbb{R}} f(t - s)g(s)ds$$

Under the convolution, $L^1(\mathbb{R})$ is a commutative Banach algebra but it is not unital.

2.3 Definition of a C*-Algebra

Definition 2.4. A **C*-algebra** is a Banach algebra A equipped with an involution $a \mapsto a^*$ satisfying the C*-identity:

$$\|a^*a\| = \|a\|^2 \quad \text{for all } a \in A.$$

Note that the involution operator is unique. For some concrete examples and nonexamples, consider a compact Hausdorff space X. We define $C(X)$ to be all continuous functions $f : X \to \mathbb{C}$. It is crucial that X be compact for our purposes. The involution on $C(X)$ is defined as complex conjugation denoted $f^* = \overline{f}$. Lets look at some simple examples.

Example 2.5. Let $X = [0, 1]$, then $C(X)$ is a C*-algebra.

Example 2.6. Let $X = S^1$, the unit circle. Then $C(X)$ is a C*-algebra.

Example 2.7. Let $X = \{0, 1, 2, 3\}$. Then $C(X)$ is a C*-algebra.

Example 2.8. Let $X = \mathbb{R}$. Then $C(X)$ is not a C*-algebra because X is not compact.

These are very basic concrete examples of subsets of $\mathbb{C}$, but we also want to consider some much more important examples for our purposes. One that shows up a lot in physics is the C*-algebra of complex-valued $n \times n$ matrices, denoted $M_n(\mathbb{C})$. Another C*-algebra of particular interest is the bounded linear operators from a Hilbert space onto itself, we denote this $B(H)$ for a Hilbert space H. One may also explore the C*-algebras of continuous compactly supported functions and continuous functions vanishing at infinity. Should these arise in any examples we will give a rigorous definition.

In summary, and with some abuse of notation, we can see how much more structure C*-algebras have as compared to typical vector spaces

$$\text{vector space} \supseteq \text{algebra} \supseteq \text{Banach algebra} \supseteq \text{C*-algebra}$$

2.4 States on C*-Algebras

Definition 2.9. A **state** on a C*-algebra A is a linear functional $\phi : A \to \mathbb{C}$ such that:

$$\phi(a^*a) \geq 0 \quad \text{for all } a \in A, \quad \text{and } \phi(1) = 1.$$

3 Concrete Examples of States

3.1 States on $C(X)$

Explain how states correspond to probability measures.

3.2 States on $M_2(\mathbb{C})$

Describe the geometric structure of the state space, highlighting pure and mixed states.

4 The Gelfand-Naimark-Segal (GNS) Construction

4.1 Overview of the Construction

Explain the process of constructing a Hilbert space representation of A using a state ϕ.

4.2 Examples

Example 4.1. GNS construction for $C([0,1])$ with a specific state.

Example 4.2. GNS construction for $M_2(\mathbb{C})$.

5 Applications and Connections

5.1 Quantum Mechanics

Discuss how states correspond to physical states and observables.

5.2 Noncommutative Geometry

Briefly describe the role of states in measuring noncommutative spaces.

6 Conclusion and Future Directions

Summarize the main points and suggest further topics for exploration.

A Proofs of Key Results

Provide detailed proofs for any key theorems mentioned in the paper.

References

[1] William Arveson, *An Invitation to C*-Algebras.*

[2] Gerald J. Murphy, *C*-Algebras and Operator Theory.*