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Linear Function, Limits, and Derivatives

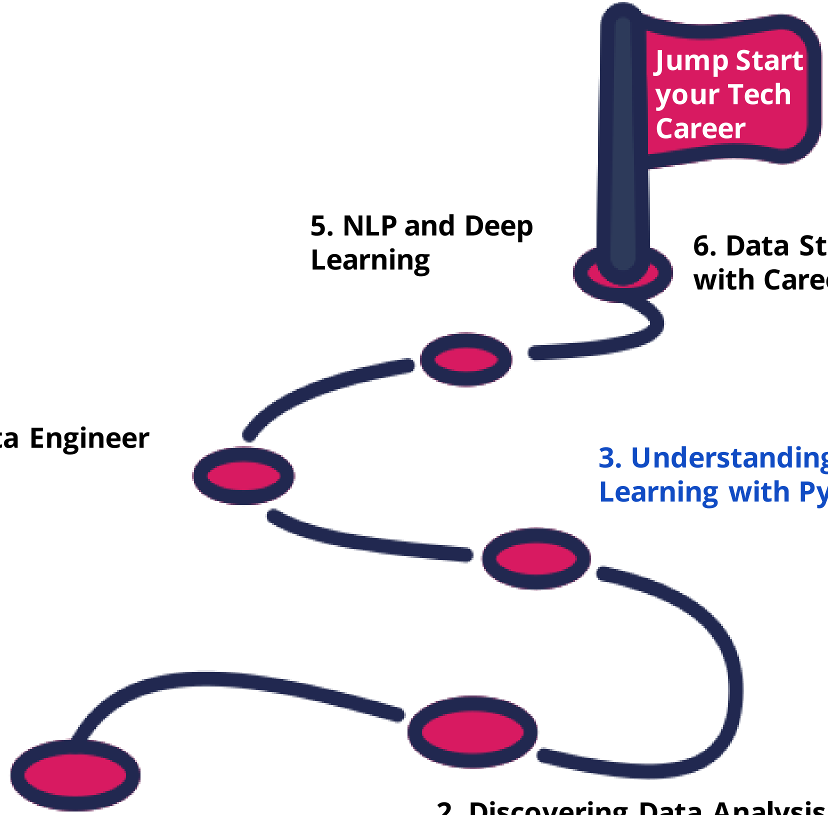
Pre-requisites

Hope you have gone through the self-learning content for this session on the PRISM portal.

By the End of this Session, You Will:

- Understand and explain the fundamental concept in mathematics and data science related to calculus, which is used to analyze the behavior of functions at specific points.
- Apply the knowledge of linear functions and limits to solve problems, demonstrating proficiency in this valuable skill for data science applications.
- Utilize linear functions to model and interpret real-world phenomena, showcasing competence in using this essential skill within the data science context.
- Effectively communicate the results of data analysis, presenting findings in a clear and concise manner to various stakeholders.

What have We Learned So Far?



Poll Time

Q. What is the primary goal of Machine Learning?

a. To automate human tasks and replace human decision-making entirely

b. To understand the underlying principles of human intelligence

c. To develop algorithms that can learn from data and make predictions or decisions without explicit programming

d. To create machines that can think and behave like humans

Poll Time

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c. **To develop algorithms that can learn from data and make predictions or decisions without explicit programming**

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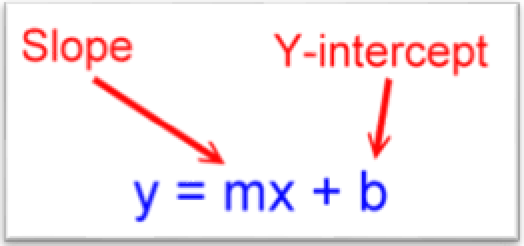
Introduction to Linear Function

What is a Linear Function?

A linear function is a function that has a constant slope. This means that the output of the function changes by a constant amount for every unit change in the input.

When drawn on a common (x, y) graph it is usually expressed as:

$$y = mx + b$$



Or, in a formal function definition:

$$f(x) = mx + b$$

The variable ***m*** holds the slope of this line. The variable ***b*** holds the y-coordinate for the spot where the line crosses the y-axis. This point is called the ***'y-intercept'***.

Example of Linear Function

Question:

A linear function models the relationship between the price of a product and the number of units sold. The equation of the function is $y = mx + b$, where $m = 1$ and $b = 5$. If 10 units are sold, what is the price of the product?

Steps in solution:

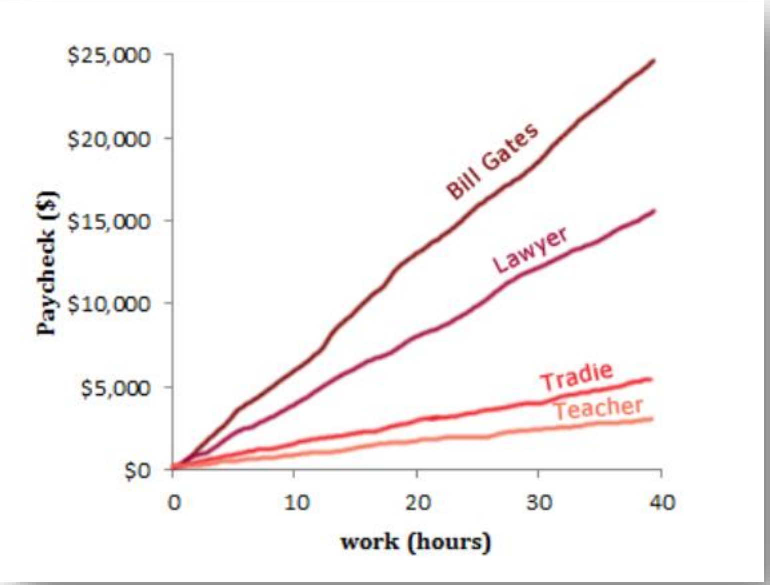
1. Since the slope of the function is 1, we know that the price increases by \$1 for every unit sold.
2. Since the y-intercept is \$5, we know that the price is \$5 when no units are sold.
3. Therefore, the price of the product when 10 units are sold is \$5 + \$10 = \$15.

Answer:

The price of the product, when 10 units are sold, is \$15.

Linear Function in Real World

- The relationship between the amount of money you earn and the number of hours you work.



The amount of money you earn is a linear function of the number of hours you work. This means that for every additional hour you work, you will earn a constant amount of money.

Demo – Linear Function

Pop Quiz

Q. A linear function is represented by the equation $y = mx + b$, where m and b are constants. Which of the following statements is true about the function?

a. The slope of the function is m

b. The y-intercept of the function is b

c. The function passes through the point $(0, b)$

d. All of the listed

Pop Quiz

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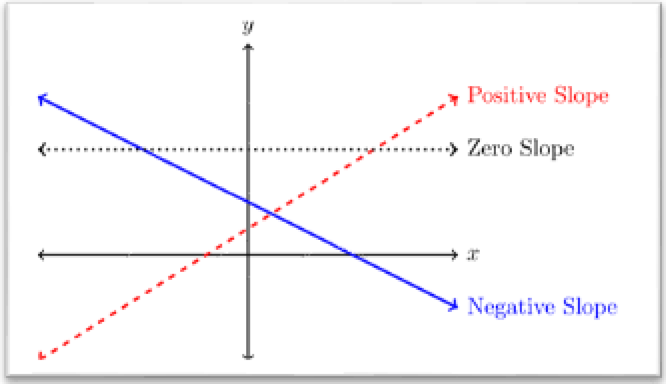
Slopes in Linear Function

The slope, m of a (non-vertical) linear function f(x) which passes through any two points (x1,y1), (x2,y2), can be found using the formula:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}}$$

Types of Slope:

Slope Type	Description
Positive	The line goes up as we move to the right.
Negative	The line goes down as we move to the right.
Zero	The line is horizontal.



Example of Slope

<u>Positive slope:</u>

The linear equation $y = 2x + 1$ has a positive slope.

This means that as the value of x increases, the value of y also increases. For example, when $x = 0$, $y = 1$, and when $x = 1$, $y = 3$. This shows that the y-value increases by 2 for every 1 unit increase in the x-value.

<u>Negative slope:</u>

The linear equation $y = -3x + 4$ has a negative slope.

This means that as the value of x increases, the value of y decreases. For example, when $x = 0$, $y = 4$, and when $x = 1$, $y = 1$. This shows that the y-value decreases by 3 for every 1 unit increase in the x-value.

Example of Slope

<u>**Zero slope:**</u>

A horizontal line has a zero slope.

This means that the y-value does not change as the x-value changes. For example, the line y = 5 is a horizontal line with a zero slope. Regardless of the value of x, y will always be equal to 5.

<u>**Undefined slope:**</u>

A vertical line has an undefined slope.

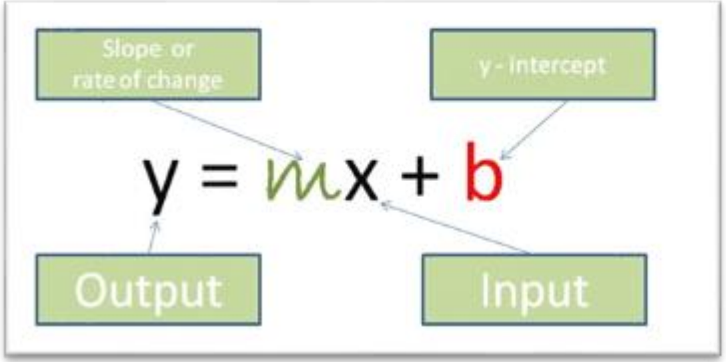
This means that the slope cannot be calculated because the line is perpendicular to the x-axis. For example, the line x = 3 is a vertical line with an undefined slope. As x changes, y can take any value, but the slope cannot be determined.

Case Study: Effect of Weight on Medication Needs

- A doctor wants to determine the relationship between the patient's weight and the amount of medication they need to take.
- They collect data on the patient's weight and the amount of medication they need to take for each visit for the past year.
- They plot the data on a graph and calculate the slope of the line.
- The slope is negative, which means that as the patient's weight increases, the amount of medication they need to take decreases.
- The doctor can use this information to prescribe the correct amount of medication for their patients in the future.

Slope Intercept Form

A slope-intercept form is a form of a linear equation that is written in the following way:



Where:
y is the dependent variable
x is the independent variable
m is the slope of the line
b is the y-intercept

Pop Quiz

Q. A car rental agency charges a flat rate of $40 per day and an additional $0.25 per mile driven. Write the equation in slope-intercept form (y = mx + b) to represent the total cost (y) as a function of the number of miles driven (x).

a. y = 0.25x

b. y = 60x + 0.25

c. y = 40x + 0.25

d. y = 0.25x + 40

Pop Quiz

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a. $y = 0.25x$

b. $y = 60x + 0.25$

c. $y = 40x + 0.25$

d. $y = 0.25x + 40$


Q&A

Introduction to Limits in Calculus

A limit is the value that a function approaches as the input approaches a certain value.

Example:

Let's say we have a function that represents the price of a product as a function of the number of products sold.

Suppose we have a function that represents the price of a product as a function of the number of products sold. The function is:

price = 100 / (number_of_products + 1)

This function tells us that the price of the product decreases as the number of products sold increases.

As the number of products sold approaches infinity, the price of the product approaches zero.

This means that the maximum price that the product could sell for is zero. We can use limits to calculate this value.

The limit of the function as the number of products sold approaches infinity is zero. This means that the maximum price that the product could sell for is zero.

Benefits of Limit in Data Analysis

They can be used to understand the behavior of functions.

They can be used to make predictions.

They can be used to calculate derivatives.

They are a powerful tool for understanding and analyzing data.

Scenario for the Upcoming Poll

Scenario:

A company collects data on the number of units sold and the price of its product for each month for the past year.

The data shows a positive correlation between the two variables, meaning that as the number of units sold increases, the price of the product also increases.

The company wants to know if this trend will continue in the future, or if the price of the product will eventually plateau.

The company can use limits to determine the limit of the price of the product as the number of units sold approaches infinity.

This will give them an idea of what the price of the product will be in the future.

Pop Quiz

Q. Which of the following methods can be used to determine the limit of the price of the product as the number of units sold approaches infinity? (Select the answer as per the given scenario)

a. Calculate the slope of the line

b. Plot the data on a graph and look at the trend

c. Use limits to calculate the value of the function as the input approaches infinity

d. All of the listed

Pop Quiz

Q. Which of the following methods can be used to determine the limit of the price of the product as the number of units sold approaches infinity? (Select the answer as per the given scenario)

a. Calculate the slope of the line

b. Plot the data on a graph and look at the trend

c. Use limits to calculate the value of the function as the input approaches infinity

d. All of the listed

Derivatives of Linear Function

The derivative of a linear function is the slope of the function.

This can be shown by differentiating the linear function with respect to x.

The derivative of a linear function of the form ***y = mx + c*** is given by ***dy/dx = m***.

This can be shown by differentiating the function as follows:

$$dy/dx = d/dx\,(mx + c)$$

$$= m * d/dx\,(x) + d/dx\,(c)$$

$$= m * 1 + 0$$

$$= m$$

Example

The function $y = 9x + 10$ is a linear function. The slope of the function is 9.

The derivative of the function can be found by differentiating the function as follows:

$$dy/dx = d/dx\,(9x + 10)$$

$$= 9 * d/dx\,(x) + d/dx\,(10)$$

$$= 9 * 1 + 0 = 9$$

Derivatives of Various Linear Functions

Function	Derivative
y = mx + b	m
y = x	1
y = -x	-1
y = x^2	2x
y = -x^2	-2x
y = x^3	3x^2
y = -x^3	-3x^2
y = x^4	4x^3
y = -x^4	-4x^3

Pop Quiz

Q. What is the derivative of the linear function $y = mx + b$?

a. $mx + b$

b. m

c. b

d. None of the listed

Pop Quiz

Q. What is the derivative of the linear function $y = mx + b$?

a. $mx + b$

b. m

c. b

d. None of the listed

Laws of Derivatives

First Law of Derivatives:

The first law of derivatives states that the derivative of a function $f(x)$ is the limit of the difference quotient as h approaches 0:

$$\mathbf{f'(x) = \lim_{h \to 0} (f(x+h) - f(x)) / h}$$

Two major rules which come under the first law of derivatives are:

1. **Power Rule:** The power rule states that if you have a function of form $f(x) = x^n$, where "n" is a constant exponent, the derivative of the function $f'(x)$ is given by:

$$f'(x) = n * x^{(n-1)}$$

For example:

If $f(x) = x^3$, then $f'(x) = 3 * x^{(3-1)} = 3x^2$.

2. Constant Multiple Rule: The constant multiple rules states that if you have a function $f(x)$ multiplied by a constant "c," then the derivative of the function $f'(x)$ is given by:

$$f'(x) = c * f(x)$$

For example:

If $f(x) = 2x^2$, then $f'(x) = 2 * 2x^2 = 4x^2$.

Laws of Derivatives

Second Law of Derivatives:

The second law of derivatives states that the derivative of a function f(x), also known as the second derivative, is the limit of the difference quotient of the first derivative as h approaches 0:

$$f''(x) = \lim_{h \to 0} (f'(x+h) - f'(x)) / h$$

For example:

If

$$f(x) = 3x^2,$$

we first find the first derivative:

$$f'(x) = d/dx\,(3x^2) = 2 * 3x^{(2-1)} = 6x$$

Now, we find the second derivative by taking the derivative of

$f'(x)$: $f''(x) = d/dx\,(6x) = 6$

So, the second derivative of

$f(x) = 3x^2$ is $f''(x) = 6$.

Chain Rule

The chain rule states that the derivative of a composite function is equal to the product of the derivative of the outer function evaluated at the inner function's output and the derivative of the inner function evaluated at the input.

Let f and g be two functions such that g(x) is differentiable at x and f is differentiable at g(x).

Then the chain rule states that the derivative of the composite function h’(x) is given by:

$$\boxed{h'(x) = g'\left(f(x)\right) \cdot f'(x)}$$

Example of Chain Rule

- Write the function as a composite function of $f(u) = u^4$ and $g(x) = 3x^2 + 2$.
- Find the derivative of $f(u)$ and $g(x)$.
- Apply the chain rule:

$$f'(g(x)) = 4(3x^2 + 2)^3$$

$$g'(x) = 6x$$

$$dy/dx = f'(g(x)) * g'(x) = 24x(3x^2 + 2)^3$$

Pop Quiz

Q. Find the derivative of the function $y = 3x^2 + 2x$ using the chain rule.

a. $y' = 6x + 2$

b. $y' = 3x^2 + 2x$

c. $y' = 6x$

d. $y' = 3$

Pop Quiz

Q. Find the derivative of the function y = 3x^2 + 2x using the chain rule.

a. y' = 6x + 2

b. y' = 3x^2 + 2x

c. y' = 6x

d. y' = 3


Q&A

Summary

- Defined the fundamental concept in mathematics and data science related to calculus, which allows you to analyze the behavior of functions at specific points.

- Solved problems, demonstrating proficiency in this valuable skill of calculus for data science applications.

- Utilized linear functions to model and interpret real-world phenomena, showcasing your competence in using this essential skill within the data science context.

- Presented the results of data analysis, so you can demonstrate your findings in a clear and concise manner to various stakeholders.

THANK YOU!

Please complete your assessments and review the self-learning content for this session on the **PRISM** portal.

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Integrals and Optimization

Pre-requisites

Hope you have gone through the self-learning content for this session on the PRISM portal.

By the End of this Session, You Will:

Core Concepts

- Understand the power of mathematical modeling in solving problems.
- Gain the ability to think critically and solve problems creatively.

Analytical Skills Development

- Utilize linear optimization to improve the efficiency of your business.
- Model real-world phenomena using optimization.
- Communicate the results of data analysis.
- Think critically about data and its implications.

What’s in It for Me?

Pop Quiz

Q. Which of the following is the correct way to write the constraint that the number of workers must be at least 10 in a linear optimization problem?

a. number_of_workers≥10

b. number_of_workers≤10

c. 10≤number_of_workers

d. number_of_workers=10

Pop Quiz

Q. Which of the following is the correct way to write the constraint that the number of workers must be at least 10 in a linear optimization problem?

a. **number_of_workers≥10**

b. number_of_workers≤10

c. 10≤number_of_workers

d. number_of_workers=10

Introduction to Integrals

Introduction to Integrals

Integrals are mathematical tools that can be used to calculate the area under a curve.

Example:

$$I = \int_1^2 (4x + 1)\, dx$$

The first step is to use the basic rule of definite integral:

$$\int_a^b f(x)\, dx = |F(x)|_a^b = [F(b) - F(a)]$$

This rule states that the definite integral of a function f(x) between the points a and b is equal to the absolute value of the function evaluated at b minus the absolute value of the function evaluated at a.

In the example, the function f(x) is 4x + 1, the points a and b are 1 and 2, and the absolute value of a function is its distance from zero.

Introduction to Integrals

The next step is to evaluate the integral:

$$I = 4 \int_1^2 x \, dx + \int_1^2 1 \, dx$$

The integral of $4x + 1$ is $2x^2 + x$, and the integral of 1 is x.

The final step is to simplify the integral:

$$I = 4 [(2)^2 - (1)^2] + [(2) - (1)] = 2(4 - 1) + (2 - 1) = 7$$

Therefore, the integral of the function $4x + 1$ between points 1 and 2 is equal to 7.

Need for Integrals in Different Scenarios

Need	Description
Calculating the area under a curve	Integrals can be used to calculate the area under a curve, which can be useful for tasks such as finding the average value of a function or determining the probability of a certain event occurring.
Finding the volume of a solid	Integrals can also be used to find the volume of a solid, which can be useful for tasks such as calculating the amount of water in a tank or determining the amount of material needed to build a structure.
Solving differential equations	Integrals are often used to solve differential equations, which are equations that involve the derivative of a function. Differential equations are used in a variety of fields, such as physics, engineering, and economics.
Machine learning	Integrals are also used in machine learning, which is a field of computer science that deals with the development of algorithms that can learn from data. Integrals can be used to calculate the error of a machine learning model, which can be used to improve the performance of the model.

Integrals of Various Linear Functions

Function	Integral
y = mx + b	mx^2/2 + bx + C
y = x	x^2/2 + C
y = -x	-x^2/2 + C
y = x^2	x^3/3 + C
y = -x^2	-x^3/3 + C
y = x^3	x^4/4 + C
y = -x^3	-x^4/4 + C
y = x^4	x^5/5 + C
y = -x^4	-x^5/5 + C

Pop Quiz

Q. What is the integral of the function $y = x^2$?

a. $x^3/3$

b. $x^3 + 1$

c. $2x^3/3$

d. $3x^3/2$

Pop Quiz

Q. What is the integral of the function y = x^2?

a. x^3/3 + C

b. x^3 + 1+ C

c. 2x^3/3+ C

d. 3x^3/2+ C

Demo – Integrals


Q&A

Optimization

- A mathematical **optimization** problem is one in which some function is either maximized or minimized relative to a given set of alternatives.

- The function to be minimized or maximized is called the objective function, and the set of alternatives is called the feasible region (or **constraint region**).

- **Linear programming** is an extremely powerful tool for addressing a wide range of applied optimization problems.

<u>**Types of Optimization:**</u>

1. Graphical optimization involves plotting the function and looking for the minimum or maximum value. This method is only feasible for simple functions, as it can be difficult to plot complex functions.

2. Numerical optimization involves using an algorithm to find the minimum or maximum value of a function.

Example of Linear Optimization

PLASTIC CUP FACTORY: Scenario

A local family-owned plastic cup manufacturer wants to optimize their production mix in order to maximize their profit.

They produce personalized beer mugs and champagne glasses.

The profit on a case of beer mugs is $25, while the profit on a case of champagne glasses is $20.

The cups are manufactured with a machine called a plastic extruder which feeds on plastic resins.

Each case of beer mugs requires 20 lbs. of plastic resins to produce, while champagne glasses require 12 lbs. per case.

The daily supply of plastic resins is limited to at most 1800 pounds. About 15 cases of either product can be produced per hour.

At the moment, the family wants to limit their workday to 8 hours.

Example of Linear Optimization

PLASTIC CUP FACTORY : Solution

Step 1: Identifying Decision Variables - These variables represent the quantifiable decisions that must be made to determine the daily production schedule.

The best way to identify the decision variables is to put oneself in the shoes of the decision maker and then ask the question, “What do I need to know in order to make this thing work?”

Decision Variables:

B = Number of cases of beer mugs to be produced daily.
C = Number of cases of champagne glasses to be produced daily.

Step 2: Identify the objective Function.

Maximize profit where profit = 25B + 20C

Example of Linear Optimization

Step 3: Determine Explicit and Implicit Constraints.

<u>Explicit Constraints:</u>

The explicit constraints are those that are explicitly given in the problem statement. In the problem under consideration, there are explicit constraints on the amount of resin and the number of work hours that are available on a daily basis.

<u>Explicit Constraints:</u>

Resin constraint: 20B + 12C 1800
Work hours constraint: 1 15B + 1 15C 8

<u>Implicit Constraints:</u>

These are constraints that are not explicitly given in the problem statement but are present nonetheless. In the cup factory problem, it is clear that one cannot have negative cases of beer mugs and champagne glasses. That is, both B and C must be non-negative quantities.

Implicit Constraints:

0 <= B, 0 <= C.

Example of Linear Optimization

The entire model for the cup factory problem can now be succinctly stated as:

$$
\begin{aligned}
P : \max\ & 25B + 20C \\
\text{subject to }\ & 20B + 12C \leq 1800 \\
& (1/15)*B + (1/15)*C \leq 8 \\
& 0 \leq B, C
\end{aligned}
$$

Pop Quiz

Q. Which of the following is the correct way to write the objective function for a linear optimization problem that minimizes the cost of transporting goods from one location to another?

a. Minimize $cost$

b. Minimize $cost = distance \times price$

c. Minimize $distance \times price$

d. Minimize $cost + distance \times price$

Pop Quiz

Q. Which of the following is the correct way to write the objective function for a linear optimization problem that minimizes the cost of transporting goods from one location to another?

a. Minimize $cost$

b. Minimize $cost = distance \times price$

c. **Minimize $distance \times price$**

d. Minimize $cost + distance \times price$

Demo – Linear Optimization

Linear Optimization of Quadratic Equation

Linear optimization is a process of finding the best solution to a problem where the objective function is linear, and the constraints are linear.

Assume quadratic equations are equations of the form:

$$f(x) = ax^2 + bx + c$$

Linear Optimization of Quadratic Equation

Global Minima/Maxima of a given equation:

- The global minima of a quadratic equation is the point on the parabola where the function f(x) is at its lowest value.

- The global maxima of a quadratic equation is the point on the parabola where the function f(x) is at its highest value.

Local Minima/Maxima of a given equation:

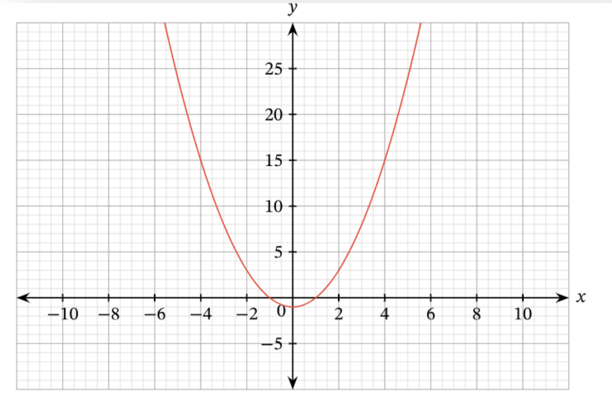
- The local minima of a quadratic equation are the points on the parabola where the function f(x) is decreasing but not necessarily at its lowest value.

- The local maxima of a quadratic equation are the points on the parabola where the function f(x) is increasing but not necessarily at its highest value.

Linear Optimization of Quadratic Equation

For example, let's consider the quadratic equation:

$$f(x) = x^2 - 1$$



Inferences generated from the co-ordinate plot:

1. The critical point of this function is x = 0. The value of the function at x = 0 is f(0) = -1.

2. The value of the function at x = -1 is f(-1) = 2. The value of the function at x = 1 is f(1) = 2.

3. Therefore, the global minima of the function is f(0) = -1. The global maxima of the function does not exist.

Linear Optimization of Quadratic Equation

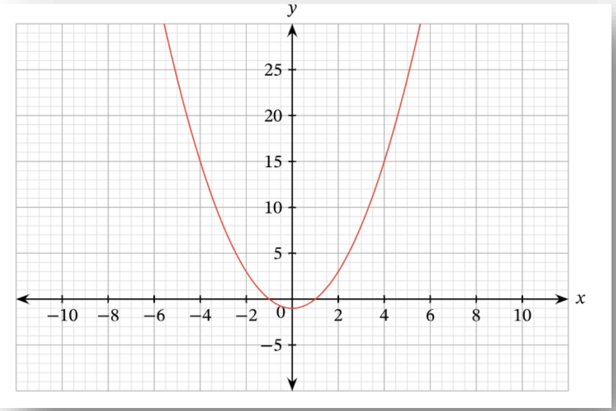
4. The local minima of the function is f(-1) = 2. The local maxima of the function is f(1) = 2.

5. The function has a minimum value of -1 at x = 0.

6. The function does not have a maximum value.

7. The function has a local minimum value of 2 at x = -1.

8. The function has a local maximum value of 2 at x = 1.



Pop Quiz

Q. A quadratic function has the following formula:

$$f(x) = x^2 - 2x + 1$$

Which of the following points is a local minimum of the function?

a. (-1, 2)

b. (0, 1)

c. (1, 0)

d. (2, 1)

Pop Quiz

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$$\mathbf{f(x) = x^2 - 2x + 1}$$

Which of the following points is a local minimum of the function?

a. (-1, 2)

b. (0, 1)

c. (1, 0)

d. (2, 1)

Benefits of Linear Optimization

Improved decision-making

Linear optimization can help businesses to make better decisions by providing them with insights into how their resources are being used and how they can be used more efficiently.

For example, a company might use linear optimization to determine the optimal price to charge for its products or to select the best suppliers.

Improved planning

Linear optimization can help businesses to improve their planning by providing them with a way to model and simulate different scenarios.

For example, a company might use linear optimization to determine the impact of a change in demand on its production schedule or to assess the risk of a disruption to its supply chain.

Benefits of Linear Optimization

Increased efficiency

Linear optimization can help businesses to increase their efficiency by ensuring that they are using their resources in the most optimal way possible.

For example, a company might use linear optimization to determine the optimal way to allocate its workforce or schedule its production.

Reduced costs

Linear optimization can help businesses to reduce their costs by minimizing the amount of resources that they use.

For example, a company might use linear optimization to determine the optimal way to transport its goods or purchase raw materials.

Real World Use Cases

- **Airlines:** Airlines use linear optimization to determine the optimal way to allocate their aircraft and crew. This can help them to reduce costs and improve the efficiency of their operations.

- **Retailers:** Retailers use linear optimization to determine the optimal way to allocate their inventory. This can help them to reduce stockouts and improve the profitability of their sales.

- **Logistics companies:** Logistics companies use linear optimization to determine the optimal way to transport goods. This can help them to reduce costs and improve the speed of delivery.

- **Manufacturing companies:** Manufacturing companies use linear optimization to determine the optimal way to schedule their production. This can help them to reduce costs and improve the efficiency of their operations.

Pop Quiz

Q. Which of the following is not an example of a linear optimization problem?

a. Minimize the cost of transporting goods from one location to another

b. Maximize the profit from selling a product

c. Minimize the number of workers needed to complete a project

d. Maximize the number of customers served by a call center

Pop Quiz

Q. Which of the following is not an example of a linear optimization problem?

a. Minimize the cost of transporting goods from one location to another

b. Maximize the profit from selling a product

c. Minimize the number of workers needed to complete a project

d. Maximize the number of customers served by a call center


Q&A

Summary

- Linear functions are a fundamental concept in mathematics and are used in many different areas, such as computer science, economics, and engineering.
- The derivative of a function can be used to find the slope of the function's graph, and the integral of a function can be used to find the area under the function's graph.
- Optimization is a powerful tool that can be used to solve a wide variety of problems, such as finding the shortest path between two points or the most profitable way to allocate resources.

Session Feedback

THANK YOU

Please complete your assessments and review the self-learning content for this session on the **PRISM** portal.

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