Convergence and Linear Speed-Up in Stochastic Federated Learning

Paul Mangold (CMAP, École Polytechnique)

Séminaire — Université Paris–Dauphine

May 27th, 2025

Me and my research

... about me

My research journey

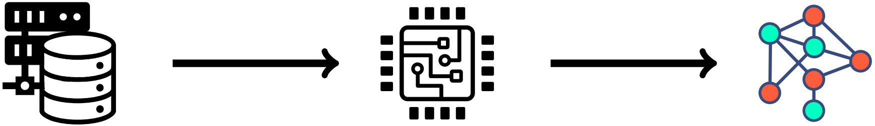
- Research themes: stochastic optimization, privacy, fairness, federated learning, reinforcement learning...
- 2023–present Post-doctoral Researcher (CMAP, École Polytechnique, Paris):
  - Federated (reinforcement) learning
- 2020–2023 PhD (MAGNET team, Inria Lille):
  - Differentially private optimization and fairness
- before: studied at ENS de Lyon

Me and my research

... about my (past) research

Optimization for Machine Learning

Data Optimization Model



$$\min_{x \in \mathbb{R}^d} f(x)$$

Overview of My Research

I. Differentially Private Optimization and Fairness } PhD

II. Federated Stochastic Optimization

III. Federated Reinforcement Learning

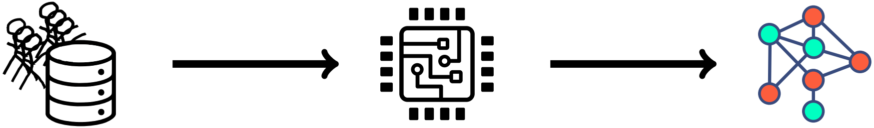
} Post-doc

I. Differentially Private Optimization

Sensitive Data

Optimization

Sensitive Model



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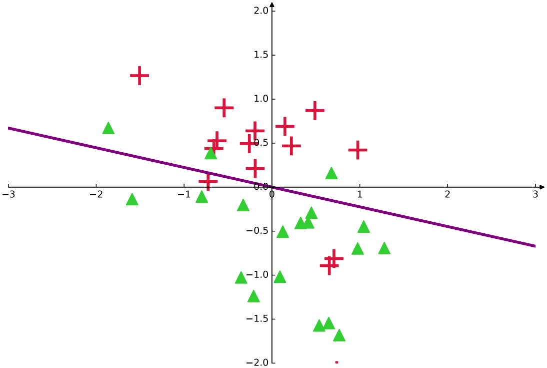
Why is the model sensitive?

Membership Inference:

“ guess if an individual was in the training data ”



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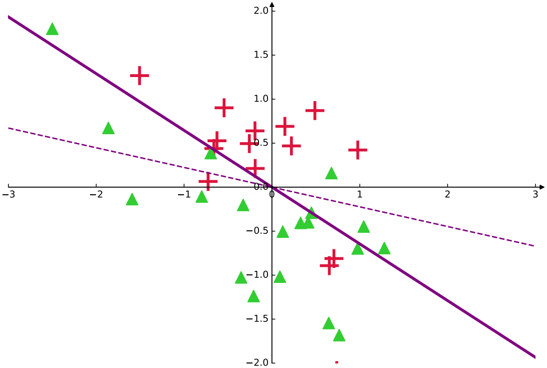
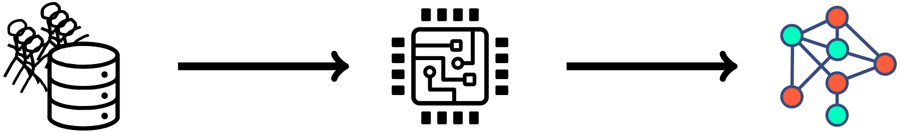


I. Differentially Private Optimization

Sensitive Data — Optimization — **Sensitive** Model



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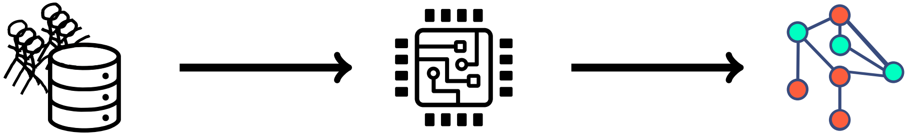
I. Differentially Private Optimization

Sensitive Data → **Noisy** Optimization → **Private** Model



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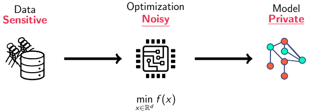
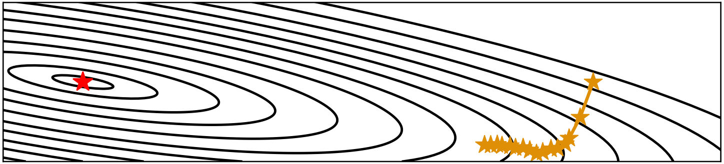
I. Differentially Private Optimization



Private gradient descent

$$x^{(t+1)} = x^{(t)} - \gamma\left(\nabla f(x^{(t)}) + \mathcal{N}(0; \sigma^2 I)\right)$$



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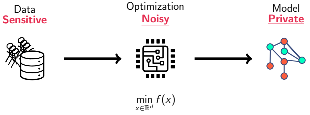
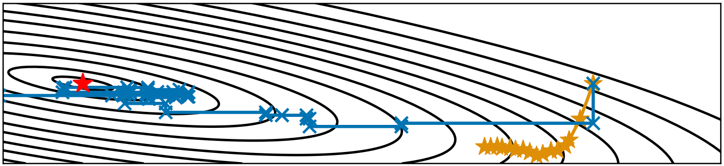


Private **coordinate descent**[1,2]

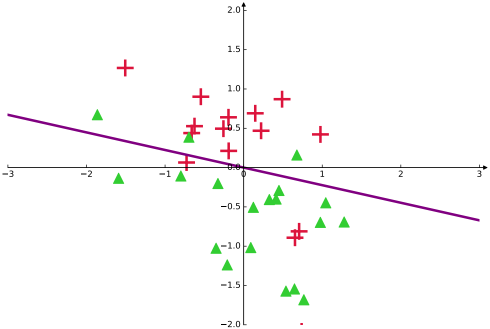
$$x_j^{(t+1)} = x_j^{(t)} - \gamma_j \left(\nabla_j f(x^{(t)}) + \mathcal{N}(0; \sigma_j^2) \right)$$



[1] P. Mangold et al. "Differentially private coordinate descent for composite empirical risk minimization". In: **ICML**. 2022.

[2] P. Mangold et al. "High-dimensional private empirical risk minimization by greedy coordinate descent". In: **AISTATS**. 2023.

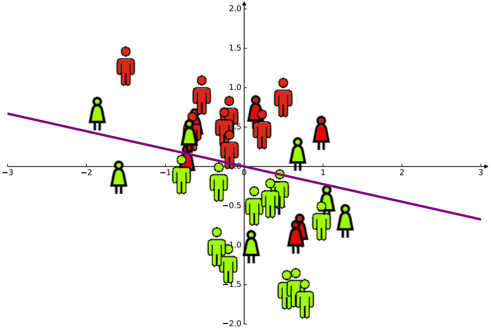
I. What about fairness?



Group Fairness:

All groups must be treated similarly

I. What about fairness?



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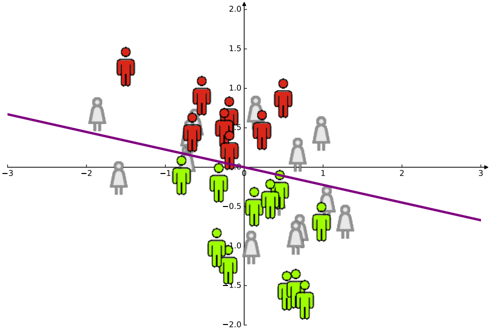

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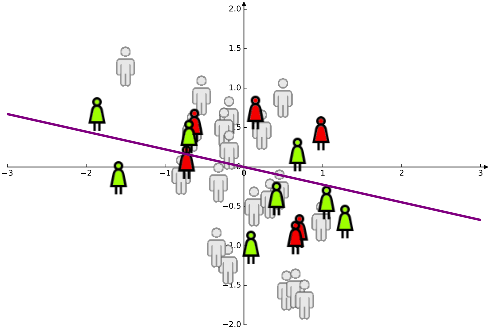

Group Fairness:

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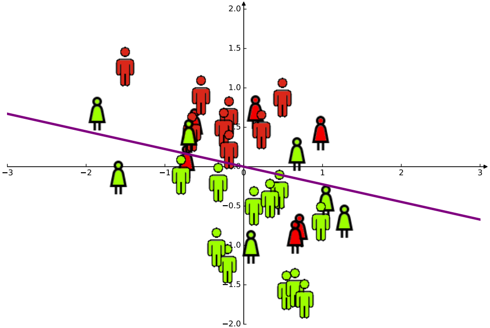
I. Fairness... and Privacy?

Group Fairness and Privacy:

Perturbing the model can have a disparate impact[1]



→ but, under some assumptions, this impact remains bounded[2]

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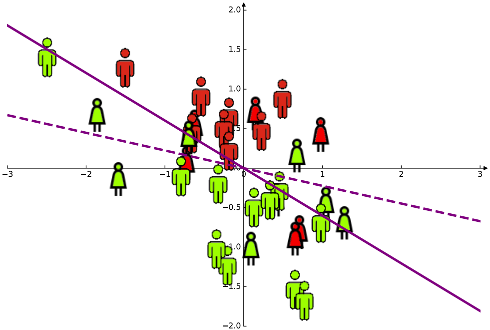

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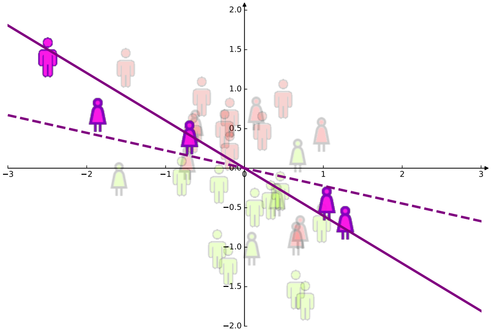

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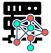
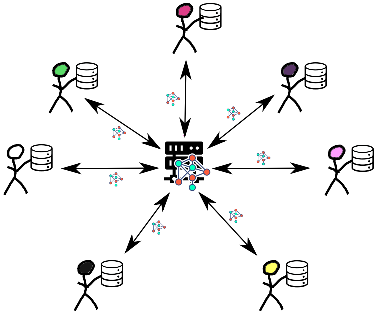


II. Federated Optimization



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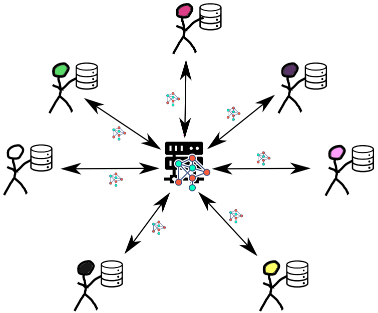

Collaborative Optimization

$$\min_{x\in\mathbb{R}^d} \frac{1}{N}\sum_{c=1}^{N} f_c(x) \quad , \quad f_c(x) = \mathbb{E}_Z[F_c(x;Z)]$$

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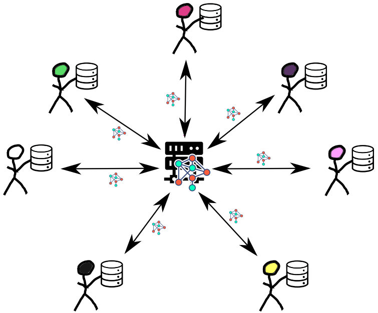

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Central Challenges: data and computational heterogeneity

+ slow and difficult-to-establish communication

II. Federated Optimization

- Theoretical analysis of Federated Averaging[1] and Scaffold[2]
  - First proof showing linear acceleration with the number of clients!
  - Federated methods that correct bias... are still biased!

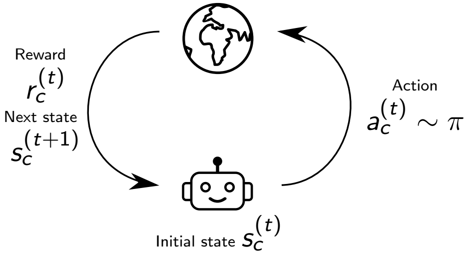
$\Rightarrow$ more details in the second part of this presentation

[1] P. Mangold et al. "Refined Analysis of Federated Averaging's Bias and Federated Richardson-Romberg Extrapolation". In: **AISTATS**. 2025.

[2] P. Mangold et al. "Scaffold with Stochastic Gradients: New Analysis with Linear Speed-Up". In: **ICML**. 2025.

III. Federated Reinforcement Learning

Each agent c operates in its environment independently of others:



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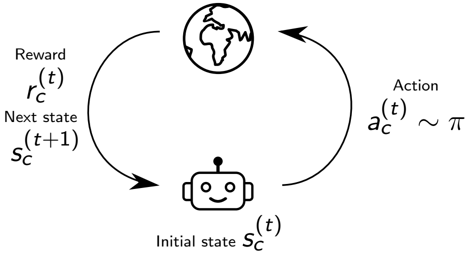

Some of my work in this area:

- federated TD Learning[1]
- federated value iteration[2]
- federated deep RL for vehicular communications[3]

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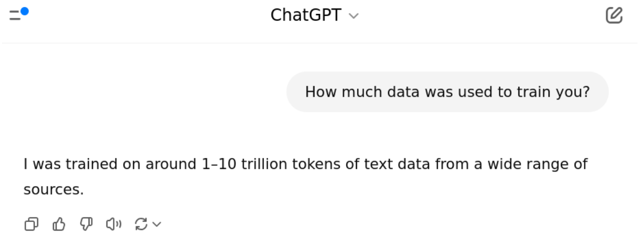
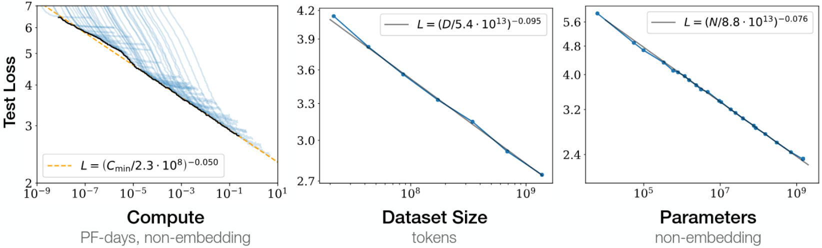
Why federated learning?

Modern Machine Learning Uses Lots of Data



Scaling Laws in Machine Learning

More data and compute give better models (plots from Kaplan et al., 2020[1])



Question: how to collect enough data and compute...

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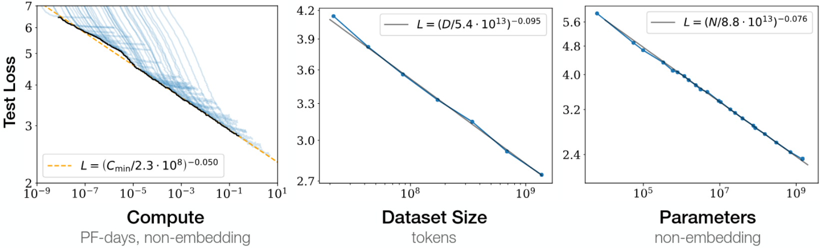

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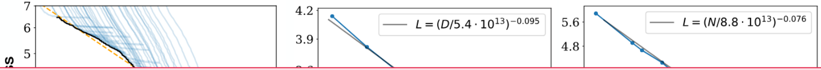
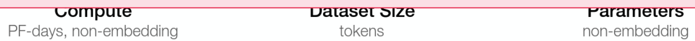


Tentative solution: train a model together!



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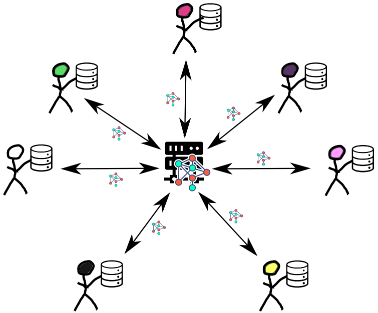
Federated Learning



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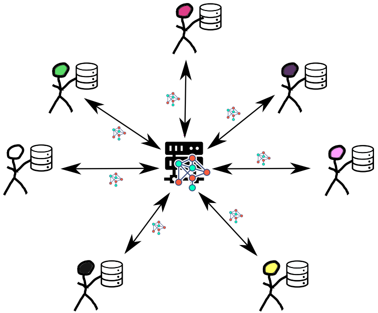

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Central Challenges: data and computational heterogeneity

$+$ slow and difficult-to-establish communication

I. Federated Averaging

Federated Averaging[1]

$$x^\star \in \arg\min_{x\in\mathbb{R}^d} \frac{1}{N}\sum_{c=1}^N \mathbb{E}_Z[F_c(x;Z)]$$

At each global iteration

- For $c = 1$ à N in parallel
  - Receive $x^{(t)}$, set $x_c^{(t,0)} = x^{(t)}$
  - For $h = 0$ to $H-1$

$$x_c^{(t,h+1)} = x_c^{(t,h)} - \gamma \nabla F_c(x_c^{(t,h)}; Z_c^{(t,h+1)})$$

- Aggregate local models

$$x^{(t+1)} = \frac{1}{N}\sum_{c=1}^N x_c^{(t,H)}$$

[1] B. McMahan et al. "Communication-efficient learning of deep networks from decentralized data". In: **AISTATS**. 2017.

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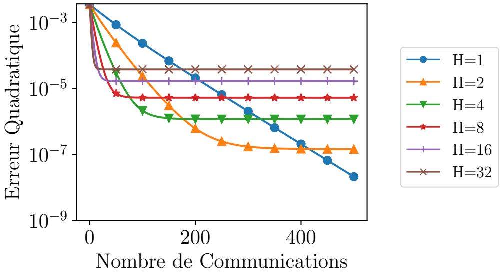
With deterministic gradients:



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Classical analyses of this algorithm

(For L-smooth, μ-strongly convex functions)

Choose your favorite heterogeneity measure

[1] X. Lian et al. “Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel SGD”. In: *NeurIPS* (2017).

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Show **convergence to a neighborhood** of $x^\star$

$$\|x^{(T)} - x^\star\|^2 \lesssim (1-\gamma\mu)^{HT}\|x^{(0)} - x^\star\|^2 + \boldsymbol{\chi(\gamma, H, \zeta)} \qquad \text{(for some function } \chi\text{)}$$

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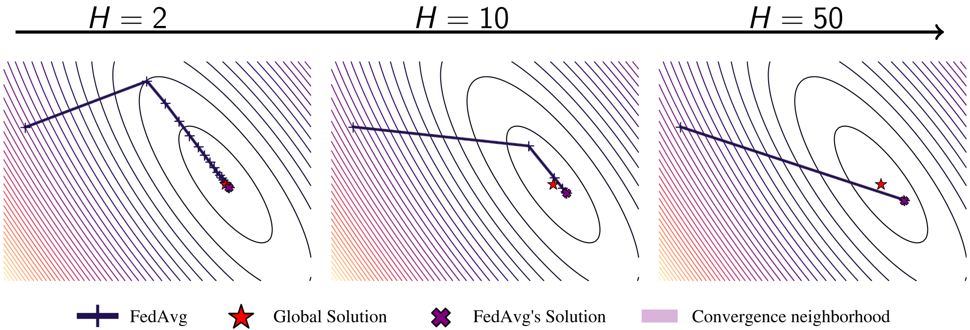

When the number of local iterations increases, bias incrases

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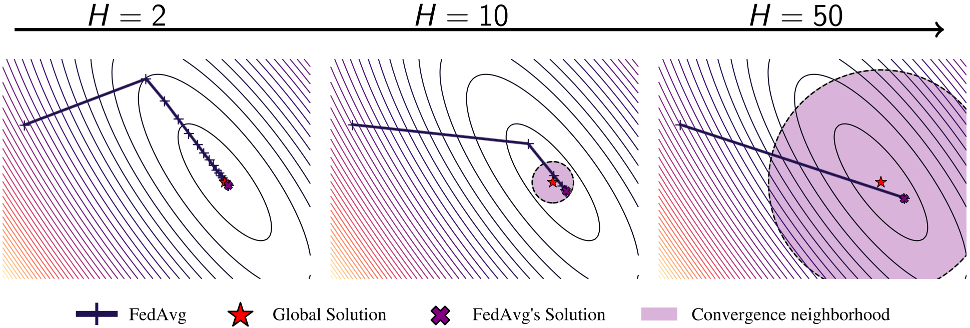

When the number of local iterations increases, bias incrases

Remark: It seems that iterates converge in some way?

Federated Averaging as Fixed Point Iteration

Remark that

$$x_c^{(t,h+1)} - y_c^{(t,h+1)} = x_c^{(t,h)} - y_c^{(t,h)} - \gamma(\nabla f_c(x_c^{(t,h)}) - \nabla f_c(y_c^{(t,h)}))$$

Thus

$$\|x_c^{(t+1)} - y_c^{(t+1)}\| \leq (1-\gamma\mu)^H \|x_c^{(t)} - y_c^{(t)}\|$$

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$\Rightarrow$ deterministic FedAvg converges to a unique point[1]

[1] G. Malinovskiy et al. "From local SGD to local fixed-point methods for federated learning". In: **ICML**. 2020.

Open Question: What about the Stochastic Case?

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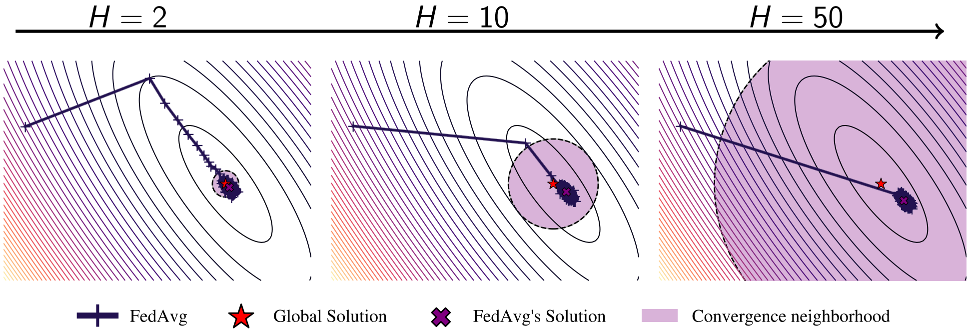
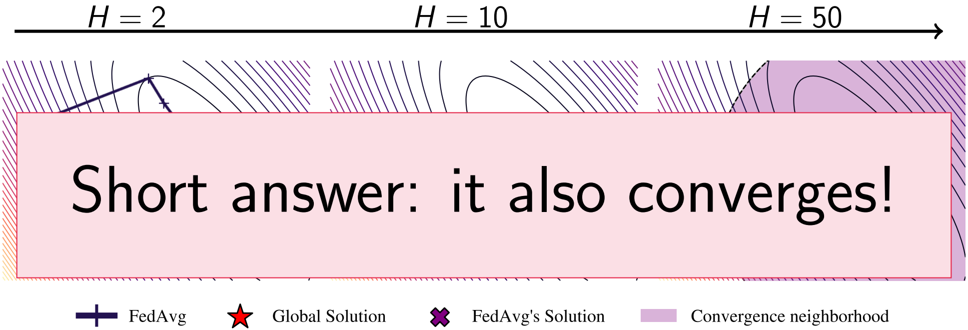


Open Question: What about the Stochastic Case?



Short answer: it also converges!

FedAvg (with stochastic gradients) converges![1]

(For thrice derivable, L-smooth, μ-strongly convex functions)

- FedAvg converges to a stationary distribution $\pi^{(\gamma,H)}$
  - denoting $x^{(t)} \sim \psi_{x^{(t)}}$, we have

$$\mathcal{W}_2(\psi_{x^{(t)}}; \pi^{(\gamma,H)}) \leq (1 - \gamma\mu)^{Ht} \mathcal{W}_2(\psi_{x^{(0)}}; \pi^{(\gamma,H)})$$

  - where $\mathcal{W}_2$ is the second order Wasserstein distance

[1] P. Mangold et al. "Refined Analysis of Federated Averaging's Bias and Federated Richardson-Romberg Extrapolation". In: **AISTATS**. 2025.

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- FedAvg's iterates covariance is

$$\int (x - x^\star)(x - x^\star)^\top \pi^{(\gamma,H)}(\mathrm{d}x) = \boxed{\frac{\gamma}{N} C(x^\star)} + O(\gamma^{3/2} H)$$

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(F … ngly convex functions)

Linear speed-up !

variance decreases in $1/N$

$C(x^\star)$ is ∇F^Z's covariance at $x^\star$

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- We can now give an **exact expansion of the bias**

$$\int x\pi^{(\gamma,H)}(\mathrm{d}x) = x^\star + \boxed{\frac{\gamma(H-1)}{2N}\sum_{c=1}^{N}\nabla^2 f(x^\star)^{-1}\big(\nabla^2 f_c(x^\star) - \nabla^2 f(x^\star)\big)\nabla f_c(x^\star)}$$
$$- \boxed{\frac{\gamma}{2N}\nabla^2 f(x^\star)^{-1}\nabla^3 f(x^\star)A^{-1}C(x^\star)} + O(\gamma^{3/2}H)$$

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FedAvg (with stochastic gradients)[1]

oth, μ-st

Heterogeneity bias

vanishes when $\nabla^2 f_c(x^\star) = \nabla^2 f(x^\star)$

or when $\nabla f_c(x^\star) = \nabla f(x^\star)$

Stochasticity bias

$A = I \otimes \nabla^2 f(x^\star) + \nabla^2 f(x^\star) \otimes I$

$C(x^\star)$ is ∇F^Z's covariance at $x^\star$

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Correcting the Bias

Novel Algorithm: Federated Richardson-Romberg Extrapolation

Run FedAvg twice:

- with step size γ: global iterates $x_\gamma^{(t)}$
- with step size 2γ: global iterates $x_{2\gamma}^{(t)}$

We can combine the iterates

$$\chi_{\mathrm{RR}}^{(t)} = 2x_\gamma^{(t)} - x_{2\gamma}^{(t)}$$

Correcting the Bias

Novel Algorithm: Federated Richardson-Romberg Extrapolation

Run FedAvg twice:

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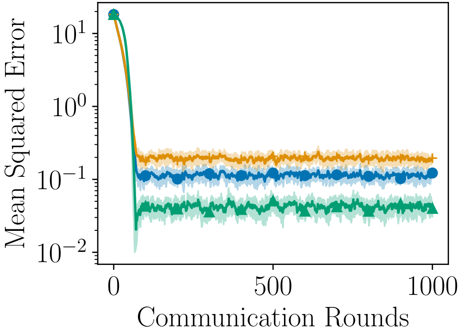
Theorem: $\mathbb{E}[\chi_{\mathrm{RR}}^{(t)}] = x_\star + O(\gamma^2 H^2 + \gamma^{3/2} H)$

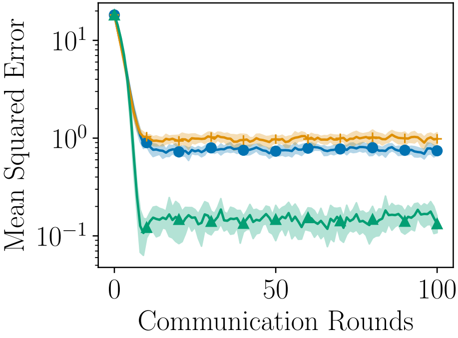
$\rightarrow$ bias is effectively reduced!!

$$\chi_{\mathrm{RR}}^{(t)} = 2x_\gamma^{(t)} - x_{2\gamma}^{(t)}$$

Numerical Illustration: FedAvg



(a) $H = 10$



(b) $H = 100$

Blue: FedAvg, Orange: Scaffold, Green: Federated Richardson-Romberg

II. Correcting heterogeneity with Scaffold

Scaffold[1]

$$x^\star \in \arg\min_{x\in\mathbb{R}^d} \frac{1}{N}\sum_{c=1}^N \mathbb{E}_Z[F_c(x;Z)]$$

At each global iteration

- For $c = 1$ to N in parallel
  - Receive $x^{(t)}$, set $x_c^{(t,0)} = x^{(t)}$
  - For $h = 0$ to $H-1$

$$x_c^{(t,h+1)} = x_c^{(t,h)} - \gamma\big(\nabla F_c(x_c^{(t,h)}; Z_c^{(t,h+1)}) + \boldsymbol{\xi_c^{(t)}}\big)$$

- Aggregate models, update control variates

$$x^{(t+1)} = \frac{1}{N}\sum_{c=1}^N x_c^{(t,H)}$$

$$\boldsymbol{\xi_c^{(t+1)} = \xi_c^{(t)} + \frac{1}{\gamma H}(\theta_c^{t,H} - \theta^{(t+1)})}$$

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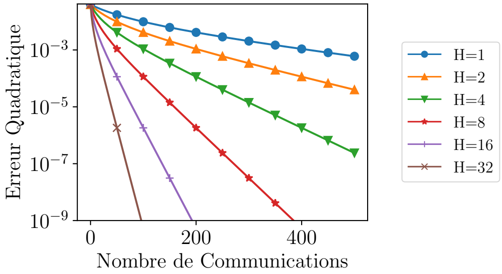

$\rightarrow$ No more heterogeneity bias!

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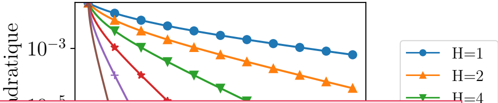

Open Question:

Does linear speed-up remain with control variates?

$\xi_c^{(t+1)} = \xi_c^{(t)} + \frac{1}{\gamma H}(\theta_c^{t,H} - \theta^{(t+1)})$

$\rightarrow$ No more heterogeneity bias!

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Scaffold also converges ![1]

(For L-smooth, μ-strongly convex functions with $\nabla^3 f(x)$ bounded by Q)

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$$\int x\pi^{(\gamma,H)}(\mathrm{d}x) = x^\star - \boxed{\frac{\gamma}{2N}\nabla^2 f(x^\star)^{-1}\nabla^3 f(x^\star)A^{-1}C(x^\star)} + O(\gamma^{3/2})$$

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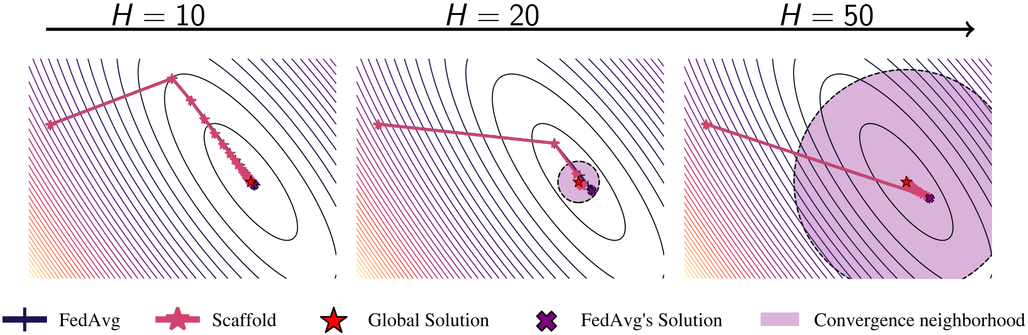
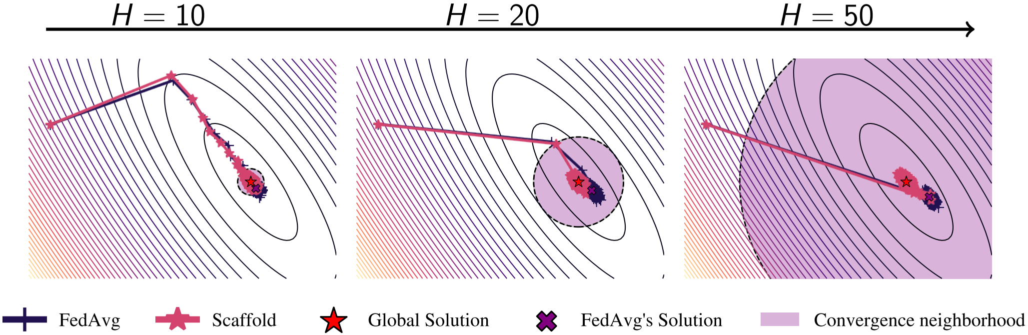


Scaffold converges to the right point

... and its variance is similar to FedAvg!



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New Convergence Rate for Scaffold

(For L-smooth, μ-strongly convex functions with $\nabla^3 f(x)$ bounded by Q)

$$\mathbb{E}\left[\|x^{(T)} - x^\star\|^2\right] \lesssim \left(1 - \frac{\gamma\mu}{4}\right)^{HT} \left\{\|x^{(0)} - x^\star\|^2 + 2\gamma^2 H^2 \zeta^2 + \frac{\sigma_\star^2}{L\mu}\right\} + \frac{\gamma}{\boldsymbol{N}\mu}\sigma_\star^2 + \frac{\gamma^{3/2} Q}{\mu^{5/2}}\sigma_\star^3 + \frac{\gamma^3 H Q^2}{\mu^3}\sigma_\star^4$$

where

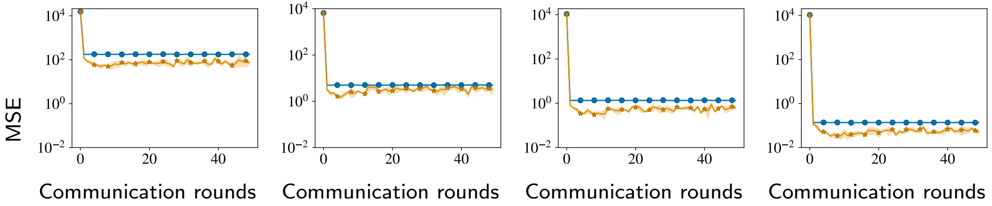
- $\sigma_\star^2 = \mathbb{E}[\frac{1}{N}\sum_{c=1}^N \|\nabla F_c^Z(x^\star) - \nabla f_c(x^\star)\|^2$ is the variance at $x^\star$
- $\zeta^2 = \frac{1}{N}\sum_{c=1}^N \|\nabla f_c^Z(x^\star)\|^2$ measures gradient heterogeneity

Linear Speed-Up!

As long as N is not too large, one can obtain $\mathbb{E}\left[\|x^{(T)} - x^\star\|^2\right] \leq \epsilon^2$ with

$$\#\text{grad per client} = \widetilde{O}\Big(\frac{\sigma_\star^2}{\boldsymbol{N}\mu^2\epsilon^2}\log\left(\frac{1}{\epsilon}\right)\Big)$$

Numerical Illustration: Speed-Up of Scaffold



(a) $N = 10$ (b) $N = 100$ (c) $N = 1000$ (d) $N = 10000$

Blue: FedAvg, Orange: Scaffold

Conclusion

- FedAvg and Scaffold converge (even with stochastic gradients)
- This allows to derive new analyses for these problems, with exact first-order expression for bias
- And we proved that Scaffold has:
  - variance similar to FedAvg's variance
  - *linear speed-up* in the number of clients!!

  But... Scaffold is still biased: some good directions for future. :)

Thank you!

Papers related to this presentation:

- P. Mangold et al. "Refined Analysis of Federated Averaging's Bias and Federated Richardson-Romberg Extrapolation". In: **AISTATS**. 2025
- P. Mangold et al. "Scaffold with Stochastic Gradients: New Analysis with Linear Speed-Up". In: **ICML**. 2025