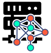
Convergence and Linear Speed-Up in Stochastic Federated Learning

Paul Mangold (CMAP, École polytechnique)

Workshop “Fondements Mathématiques de l’IA”

March 25th, 2025

Federated Learning



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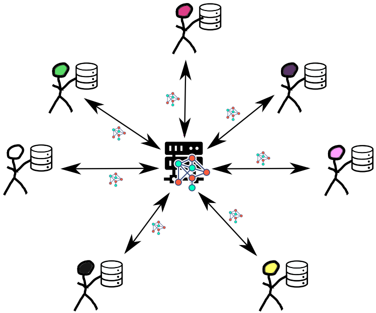

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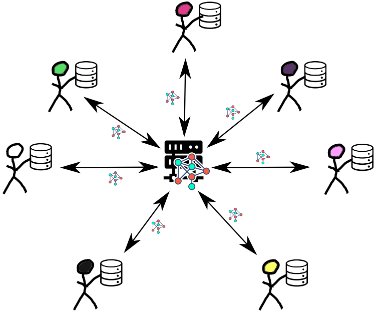

Collaborative optimization problem

$$\min_{x \in \mathbb{R}^d} \frac{1}{N} \sum_{c=1}^{N} f_c(x) \quad , \quad f_c(x) = \mathbb{E}_Z[F_c(x; Z)]$$

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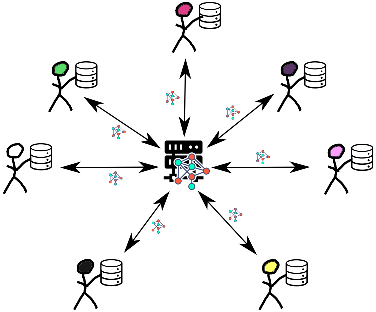

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Problem: data is heterogeneous, communication is expensive

I. Federated Averaging

Federated Averaging[1]

$$x^\star \in \arg\min_{x\in\mathbb{R}^d} \frac{1}{N}\sum_{c=1}^{N} \mathbb{E}_Z[F_c(x;Z)]$$

At each global iteration

- For $c = 1$ à N in parallel
  - Receive $x^{(t)}$, set $x_c^{(t,0)} = x^{(t)}$
  - For $h = 0$ to $H-1$

    $$x_c^{(t,h+1)} = x_c^{(t,h)} - \gamma \nabla F_c(x_c^{(t,h)}; Z_c^{(t,h+1)})$$

- Aggregate local models

$$x^{(t+1)} = \frac{1}{N}\sum_{c=1}^{N} x_c^{(t,H)}$$

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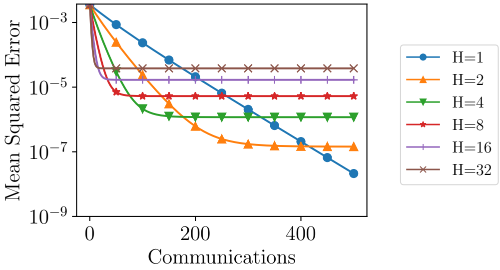
With deterministic gradients:



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Classical analyses of this algorithm

(For L-smooth, μ-strongly convex functions)

Choose your favorite heterogeneity measure

- first-order[1]: $\zeta = \frac{1}{N}\sum_{c=1}^{N} \left\| \nabla f_c(x^\star) - \nabla f(x^\star) \right\|^2$
- second-order[2]: $\zeta = \frac{1}{N}\sum_{c=1}^{N} \left\| \nabla_c^2 f(x^\star) - \nabla^2 f(x^\star) \right\|^2$
- average drift[3]: $\zeta = \left\| \frac{1}{NH}\sum_{c=1}^{N}\sum_{h=0}^{H-1} \nabla f(x_c^{(h)}) - \nabla f(x^\star) \right\|^2$

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[2] A. Khaled and C. Jin. "Faster federated optimization under second-order similarity". In: *arXiv* (2022).

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Show **convergence to a neighborhood** of $x^\star$

$$\|x^{(T)} - x^\star\|^2 \lesssim (1-\gamma\mu)^{HT}\|x^{(0)} - x^\star\|^2 + \boldsymbol{\chi(\gamma, H, \zeta)} \qquad \text{(for some function } \chi\text{)}$$

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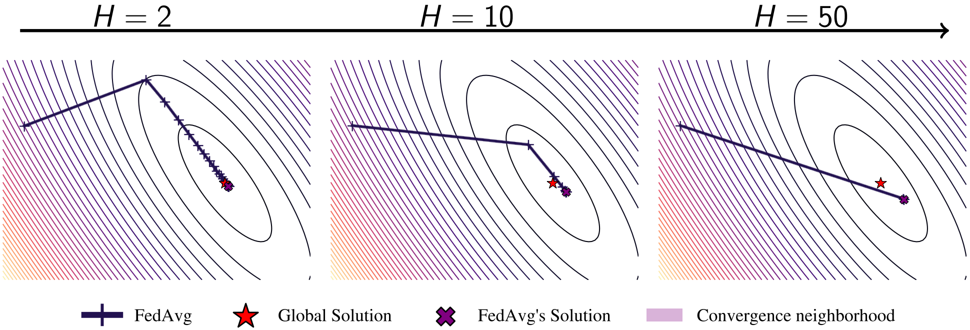
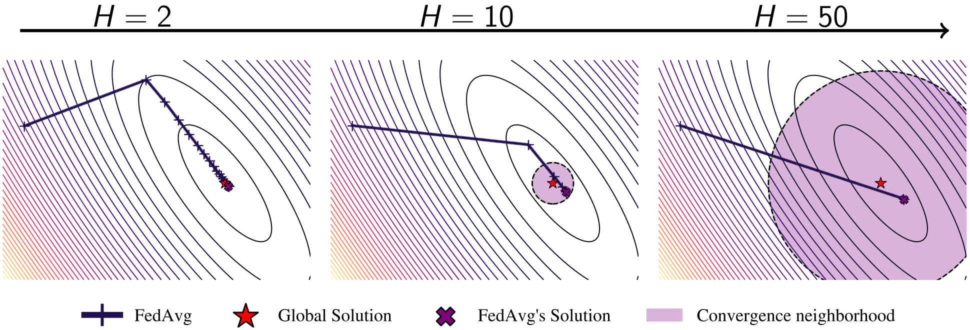


When the number of local iterations increases, bias incrases



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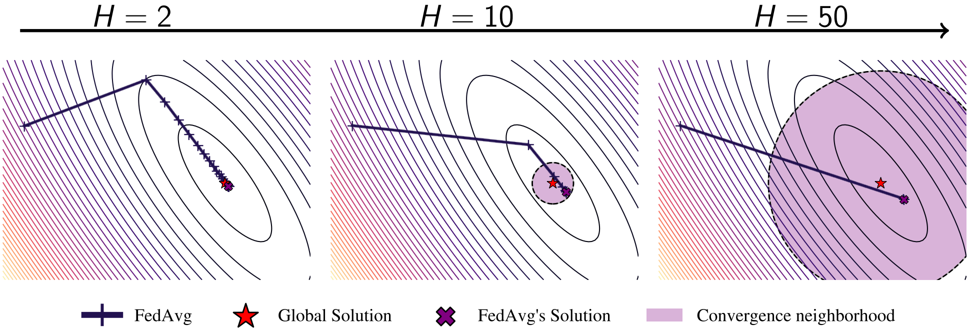
... but the bound is oblivious to problem's geometry



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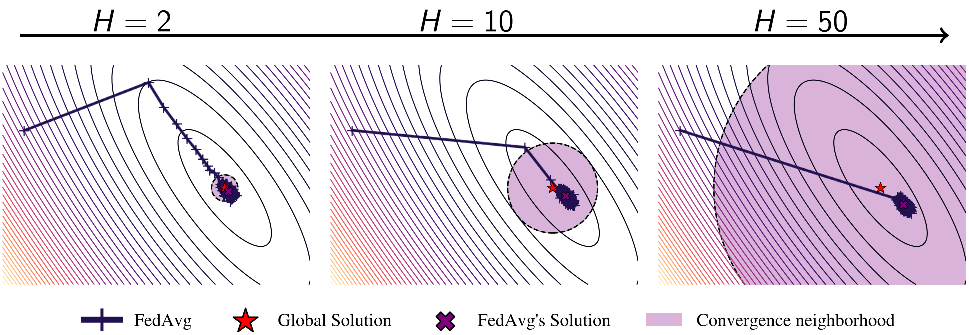

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FedAvg (with stochastic gradients) converges![1]

(For thrice derivable, L-smooth, μ-strongly convex functions)

- FedAvg converges to a stationary distribution $\pi^{(\gamma,H)}$
  - denoting $x^{(t)} \sim \psi_{x^{(t)}}$, we have

$$\mathcal{W}_2(\psi_{x^{(t)}}; \pi^{(\gamma,H)}) \leq (1-\gamma\mu)^{Ht} \mathcal{W}_2(\psi_{x^{(0)}}; \pi^{(\gamma,H)})$$

  - where $\mathcal{W}_2$ is the second order Wasserstein distance

[1] P. Mangold et al. "Refined Analysis of Federated Averaging's Bias and Federated Richardson-Romberg Extrapolation". In: **AISTATS**. 2025.

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- FedAvg's iterates covariance is

$$\int (x - x^\star)(x - x^\star)^\top \pi^{(\gamma,H)}(\mathrm{d}x) = \boxed{\frac{\gamma}{N} C(x^\star)} + O(\gamma^{3/2} H)$$

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$$- \boxed{\frac{\gamma}{2N}\nabla^2 f(x^\star)^{-1}\nabla^3 f(x^\star)A^{-1}C(x^\star)} + O(\gamma^{3/2}H)$$

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Heterogeneity bias

vanishes when $\nabla^2 f_c(x^\star) = \nabla^2 f(x^\star)$
or when $\nabla f_c(x^\star) = \nabla f(x^\star)$

th, μ-s

Stochasticity bias

$A = I \otimes \nabla^2 f(x^\star) + \nabla^2 f(x^\star) \otimes I$
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II. Correcting heterogeneity: Scaffold

Scaffold[1]

$x^\star \in \arg\min_{x \in \mathbb{R}^d} \frac{1}{N} \sum_{c=1}^{N} \mathbb{E}_Z[F_c(x; Z)]$

At each global iteration

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  - For $h = 0$ to $H-1$

$$x_c^{(t,h+1)} = x_c^{(t,h)} - \gamma\big(\nabla F_c(x_c^{(t,h)}; Z_c^{(t,h+1)}) + \boldsymbol{\xi_c^{(t)}}\big)$$

- Aggregate models, update control variates

$$x^{(t+1)} = \frac{1}{N} \sum_{c=1}^{N} x_c^{(t,H)}$$

$$\boldsymbol{\xi_c^{(t+1)} = \xi_c^{(t)} + \frac{1}{\gamma H}(\theta_c^{t,H} - \theta^{(t+1)})}$$

[1] S. P. Karimireddy et al. "Scaffold: Stochastic controlled averaging for federated learning". In: **International conference on machine learning**. PMLR. 2020, pp. 5132–5143.

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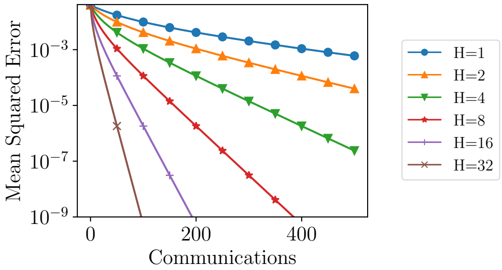

$\rightarrow$ No more heterogeneity bias!

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Scaffold also converges ![1]

(For L-smooth, μ-strongly convex functions with $\nabla^3 f(x)$ bounded by Q)

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- Scaffold's variance is close to FedAvg's variance

$$\int (x - x^\star)(x - x^\star)^\top \pi^{(\gamma,H)}(\mathrm{d}x) = \boxed{\frac{\gamma}{N} C(x^\star)} + O(\gamma^{3/2})$$

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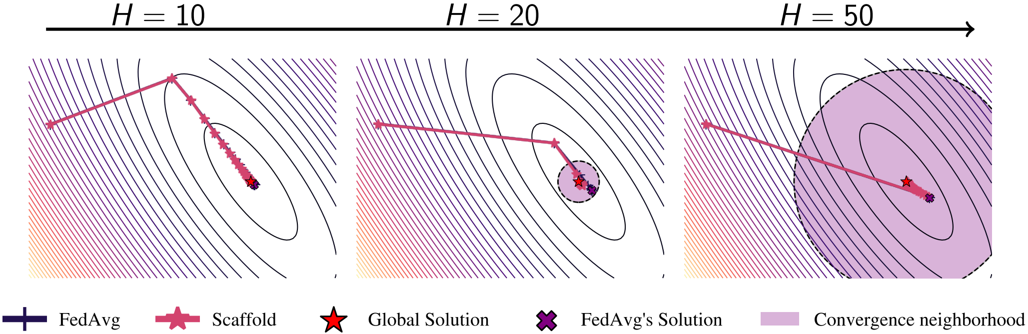

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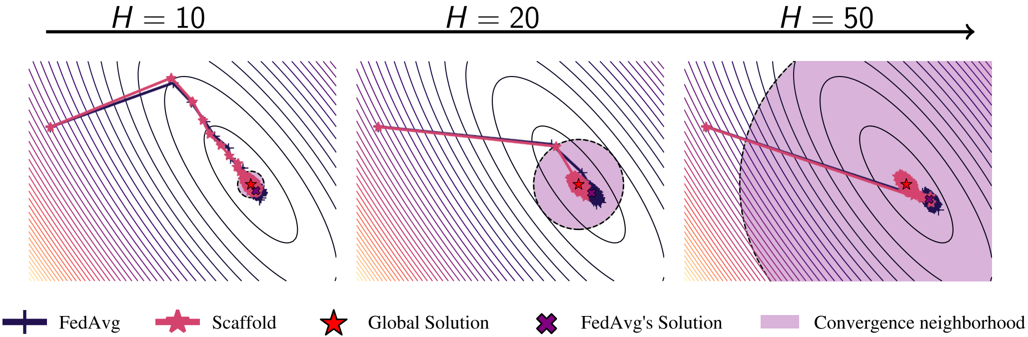
... and its variance is similar to FedAvg!



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New Convergence Rate for Scaffold

(For L-smooth, μ-strongly convex functions with $\nabla^3 f(x)$ bounded by Q)

$$\mathbb{E}\left[\|x^{(T)} - x^\star\|^2\right] \lesssim \left(1 - \frac{\gamma\mu}{4}\right)^{HT} \left\{\|x^{(0)} - x^\star\|^2 + 2\gamma^2 H^2 \zeta^2 + \frac{\sigma_\star^2}{L\mu}\right\} + \frac{\gamma}{\boldsymbol{N}\mu}\sigma_\star^2 + \frac{\gamma^{3/2} Q}{\mu^{5/2}}\sigma_\star^3 + \frac{\gamma^3 H Q^2}{\mu^3}\sigma_\star^4$$

where

- $\sigma_\star^2 = \mathbb{E}[\frac{1}{N}\sum_{c=1}^N \|\nabla F_c^Z(x^\star) - \nabla f_c(x^\star)\|^2$ is the variance at $x^\star$
- $\zeta^2 = \frac{1}{N}\sum_{c=1}^N \|\nabla f_c^Z(x^\star)\|^2$ measures gradient heterogeneity

Linear Speed-Up!

As long as N is not too large, one can obtain $\mathbb{E}\left[\|x^{(T)} - x^{\star}\|^2\right] \leq \epsilon^2$ with

$$\#\text{grad per client} = \widetilde{O}\Big(\frac{\sigma_{\star}^2}{\boldsymbol{N}\mu^2\epsilon^2}\log\left(\frac{1}{\epsilon}\right)\Big)$$

Conclusion

- FedAvg and Scaffold converge (even with stochastic gradients)
- This allows to derive new analyses for these problems, with exact first-order expression for bias
- And we proved that Scaffold has:
  - variance similar to FedAvg's variance
  - *linear speed-up* in the number of clients!!

Thank you!

Check the papers:

- P. Mangold et al. "Refined Analysis of Federated Averaging's Bias and Federated Richardson-Romberg Extrapolation". In: **AISTATS**. 2025
- P. Mangold et al. "Scaffold with Stochastic Gradients: New Analysis with Linear Speed-Up". In: **arxiv preprint**. 2025

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- `https://pmangold.fr/research.php?page=talks`