Convergence and Linear Speed-Up in Stochastic Federated Learning

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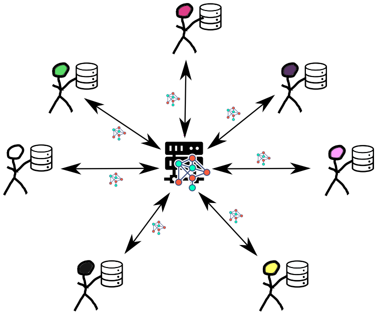
ICSP 2025 – Mini-Symposium
Communication-efficient methods for distributed optimization and federated learning

July 28th, 2025

Federated Learning



Collaborative optimization problem

$$\min_{x \in \mathbb{R}^d} \frac{1}{N} \sum_{c=1}^{N} f_c(x) \quad , \quad f_c(x) = \mathbb{E}_{Z \sim D_c}[F_c(x; Z)]$$

Central Challenges: data and computational heterogeneity

+ slow and difficult-to-establish communication

I. Federated Averaging

Federated Averaging[1]

$$x^\star \in \arg\min_{x \in \mathbb{R}^d} \frac{1}{N} \sum_{c=1}^{N} \mathbb{E}_Z[F_c(x; Z)]$$

At each global iteration

- For $c = 1$ à N in parallel
  - Receive $x^{(t)}$, set $x_c^{(t,0)} = x^{(t)}$
  - For $h = 0$ to $H - 1$

  $$x_c^{(t,h+1)} = x_c^{(t,h)} - \gamma \nabla F_c(x_c^{(t,h)}; Z_c^{(t,h+1)})$$

- Aggregate local models

$$x^{(t+1)} = \frac{1}{N} \sum_{c=1}^{N} x_c^{(t,H)}$$

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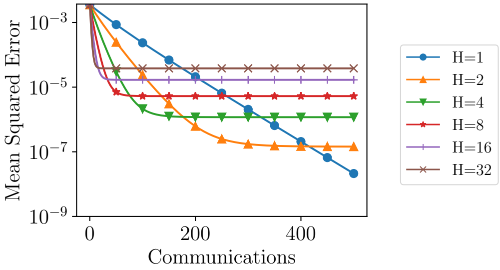
With deterministic gradients:



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Classical analyses of this algorithm

(For L-smooth, μ-strongly convex functions)

Choose your favorite heterogeneity measure

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- average drift[3]: $\zeta = \left\|\frac{1}{NH}\sum_{c=1}^{N}\sum_{h=0}^{H-1} \nabla f(x_c^{(h)}) - \nabla f(x^\star)\right\|^2$

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Show **convergence to a neighborhood** of $x^\star$

$$\|x^{(T)} - x^\star\|^2 \lesssim (1-\gamma\mu)^{HT}\|x^{(0)} - x^\star\|^2 + \boldsymbol{\chi(\gamma, H, \zeta)} \qquad \text{(for some function } \chi)$$

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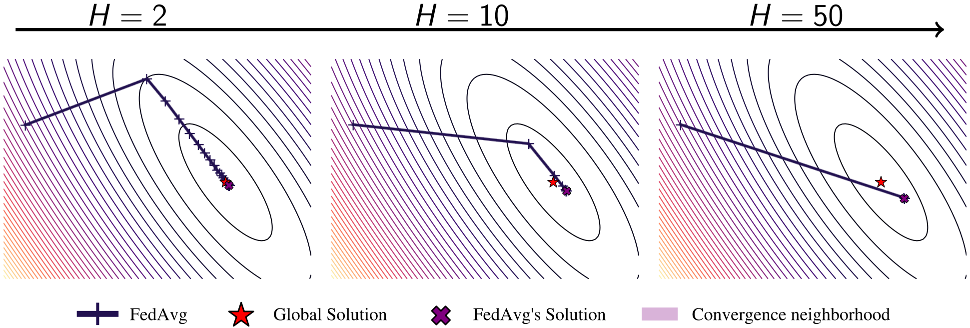

When the number of local iterations increases, bias incrases

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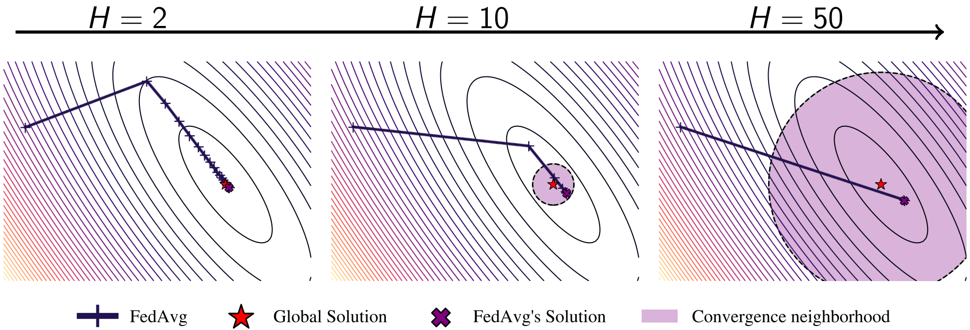

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Remark: It seems that iterates converge in some way?

Federated Averaging as Fixed Point Iteration

Remark that, starting with $x_c^{(t)}, y_c^{(t)} \in \mathbb{R}^d$,

$$x_c^{(t,h+1)} - y_c^{(t,h+1)} = x_c^{(t,h)} - y_c^{(t,h)} - \gamma(\nabla f_c(x_c^{(t,h)}) - \nabla f_c(y_c^{(t,h)}))$$

Thus

$$\|x_c^{(t+1)} - y_c^{(t+1)}\| \leq (1 - \gamma\mu)^H \|x_c^{(t)} - y_c^{(t)}\|$$

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$\Rightarrow$ deterministic FedAvg converges to a unique point[1]

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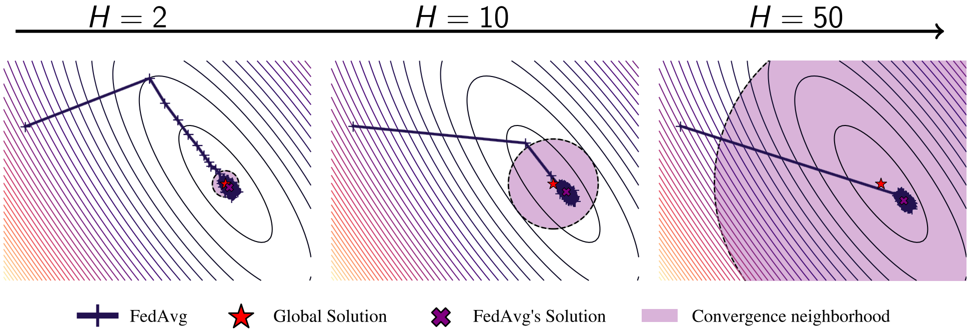

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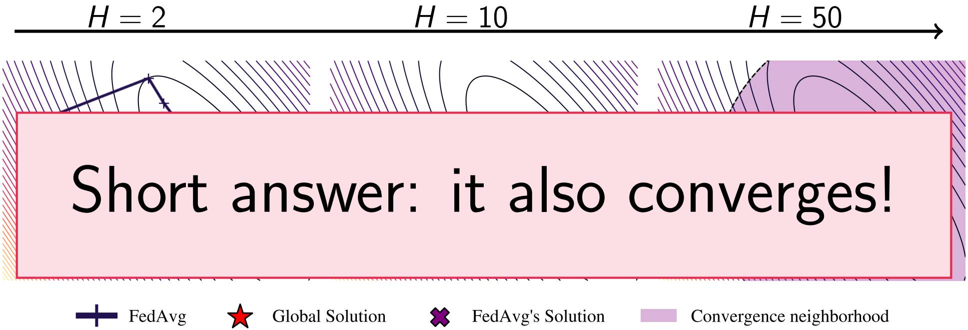

Short answer: it also converges!

FedAvg (with stochastic gradients) converges![1]

(For thrice derivable, L-smooth, μ-strongly convex functions)

- FedAvg converges to a stationary distribution $\pi^{(\gamma,H)}$
  - denoting $x^{(t)} \sim \psi_{x^{(t)}}$, we have

$$\mathcal{W}_2(\psi_{x^{(t)}}; \pi^{(\gamma,H)}) \leq (1-\gamma\mu)^{Ht} \mathcal{W}_2(\psi_{x^{(0)}}; \pi^{(\gamma,H)})$$

  - where $\mathcal{W}_2$ is the second order Wasserstein distance

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- FedAvg's iterates covariance is

$$\int (x - x^\star)(x - x^\star)^\top \pi^{(\gamma,H)}(\mathrm{d}x) = \boxed{\frac{\gamma}{N} \boldsymbol{A} C(x^\star)} + O(\gamma^{3/2} H)$$

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(For t… trongly convex functions)

Linear speed-up !

variance decreases in $1/N$
variance scales in γ

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- We can now give an **exact expansion of the bias**

$$\int x\pi^{(\gamma,H)}(\mathrm{d}x) = x^\star + \boxed{\frac{\gamma(H-1)}{2N}\sum_{c=1}^{N}\nabla^2 f(x^\star)^{-1}\big(\nabla^2 f_c(x^\star) - \nabla^2 f(x^\star)\big)\nabla f_c(x^\star)}$$

$$- \boxed{\frac{\gamma}{2N}\nabla^2 f(x^\star)^{-1}\nabla^3 f(x^\star)\boldsymbol{A}C(x^\star)} + O(\gamma^{3/2}H)$$

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Heterogeneity bias

vanishes when $\nabla^2 f_c(x^\star) = \nabla^2 f(x^\star)$
or when $\nabla f_c(x^\star) = \nabla f(x^\star)$

Stochasticity bias

$\boldsymbol{A} = (I \otimes \nabla^2 f(x^\star) + \nabla^2 f(x^\star) \otimes I)^{-1}$
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Correcting the Bias

Novel Algorithm: Federated Richardson-Romberg Extrapolation

Run FedAvg twice:

- with step size γ: global iterates $x_\gamma^{(t)}$
- with step size 2γ: global iterates $x_{2\gamma}^{(t)}$

We can combine the iterates

$$\chi_{\mathrm{RR}}^{(t)} = 2x_\gamma^{(t)} - x_{2\gamma}^{(t)}$$

Correcting the Bias

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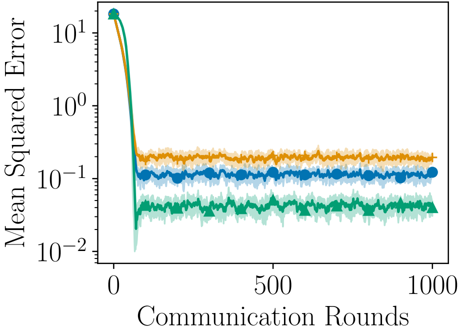
Theorem: $\mathbb{E}[\chi_{\mathrm{RR}}^{(t)}] = x_\star + O(\gamma^2 H^2 + \gamma^{3/2} H)$

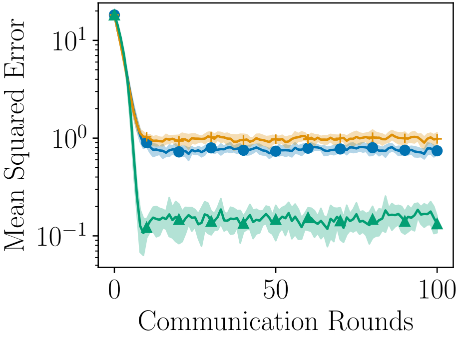
$\rightarrow$ bias is effectively reduced!!

$$\chi_{\mathrm{RR}}^{(t)} = 2x_\gamma^{(t)} - x_{2\gamma}^{(t)}$$

Numerical Illustration: FedAvg



(a) $H = 10$



(b) $H = 100$

Blue: FedAvg, Orange: Scaffold, Green: Federated Richardson-Romberg

II. Correcting heterogeneity: Scaffold

Scaffold[1]

(*without global step size)

$$x^\star \in \arg\min_{x\in\mathbb{R}^d} \frac{1}{N}\sum_{c=1}^N \mathbb{E}_Z[F_c(x;Z)]$$

At each global iteration

- For $c = 1$ to N in parallel
  - Receive $x^{(t)}$, set $x_c^{(t,0)} = x^{(t)}$
  - For $h = 0$ to $H-1$

$$x_c^{(t,h+1)} = x_c^{(t,h)} - \gamma\big(\nabla F_c(x_c^{(t,h)}; Z_c^{(t,h+1)}) + \boldsymbol{\xi_c^{(t)}}\big)$$

- Aggregate models, update control variates

$$x^{(t+1)} = \frac{1}{N}\sum_{c=1}^N x_c^{(t,H)}$$

$$\boldsymbol{\xi_c^{(t+1)} = \xi_c^{(t)} + \frac{1}{\gamma H}\left(x_c^{t,H} - x^{(t+1)}\right)}$$

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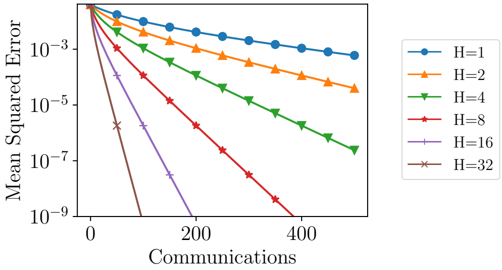

$\rightarrow$ No more heterogeneity bias!

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Scaffold also converges ![1]

(For L-smooth, μ-strongly convex functions with $\nabla^3 f(x)$ bounded by Q)

- Scaffold converges if $\gamma HL \leq 1$, towards a distribution $\pi^{(\gamma,H)}$
  - denoting $(x^{(t)}, \xi^{(t)}_{1:N}) \sim \psi_{(x^{(t)},\xi^{(t)}_{1:N})}$, we have

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$\Rightarrow$ **but it is still biased**

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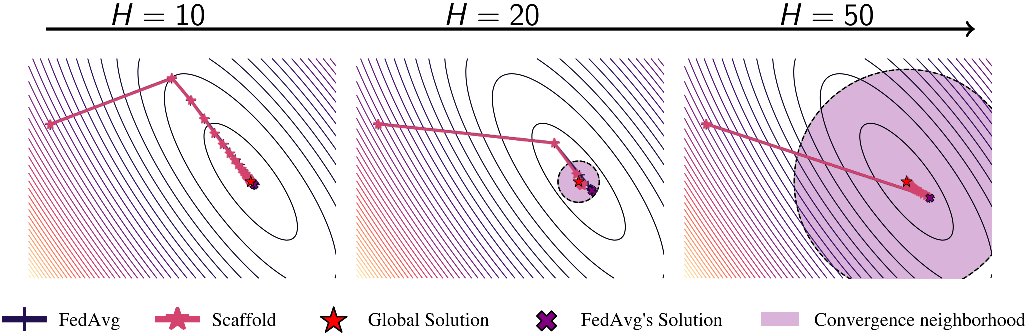

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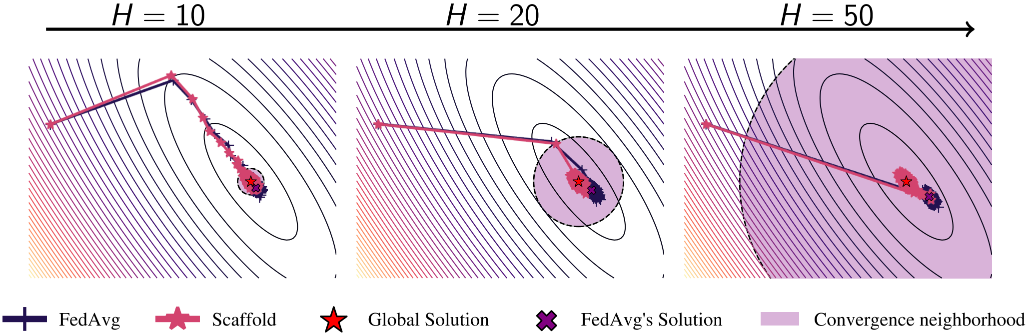
... and its variance is similar to FedAvg!



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Bounding the Covariance

Define covariance matrices

$$\bar{\Sigma}^{x} \triangleq \int \left(x - x_{\star}\right)^{\otimes 2} \pi^{(\gamma,H)}(\mathrm{d}x, \mathrm{d}\Xi)$$

$$\bar{\Sigma}^{\xi}_{(c,c')} \triangleq \int \left(\xi_c - \xi_c^{\star}\right)\left(\xi_{c'} - \xi_c^{\star}\right)^{\top} \pi^{(\gamma,H)}(\mathrm{d}x, \mathrm{d}\Xi)$$

$$\bar{\Sigma}^{x,\xi}_{(c)} \triangleq \int \left(x - x_{\star}\right)\left(\xi_c - \xi_c^{\star}\right)^{\top} \pi^{(\gamma,H)}(\mathrm{d}x, \mathrm{d}\Xi)$$

Expansion of Covariance

$$\bar{\Sigma}^x = \tfrac{\gamma}{N}\boldsymbol{A}\mathcal{C}(x_\star) + O(\gamma^2 H + \gamma^{3/2})$$

$$\bar{\Sigma}^{x,\xi}_{(c)} = \tfrac{\gamma}{N}\boldsymbol{A}\mathcal{C}(x_\star)(\nabla^2 f_c(c)x_\star - \nabla^2 f(x_\star)) + \tfrac{\gamma}{N}(\mathcal{C}_c(x_\star) - \mathcal{C}(x_\star)) + O(\gamma^2 H + \gamma^{3/2})$$

$$\bar{\Sigma}^{\xi}_{(c,c)} = (1 - \tfrac{2}{N})\tfrac{1}{H}\mathcal{C}_c(x_\star) + \tfrac{1}{NH}\mathcal{C}(x_\star) + O(\gamma)$$

$$\bar{\Sigma}^{\xi}_{(c,c')} = \tfrac{1}{NH}(\mathcal{C}(x_\star) - \mathcal{C}_c(x_\star) - \mathcal{C}_{c'}(x_\star)) + O(\gamma)$$

where

$$\boldsymbol{A} \triangleq (Id \otimes \nabla^2 f(x_\star) + \nabla^2 f(x_\star) \otimes Id)^{-1}$$

$$\mathcal{C}_c(x_\star) \triangleq \mathbb{E}\Big[\big(\nabla F_c^{Z_c}(x_\star) - \nabla f_c(x_\star)\big)^{\otimes 2}\Big] \quad \mathcal{C}(x_\star) \triangleq \frac{1}{N}\sum_{c=1}^{N}\mathcal{C}_c(x_\star)$$

New Convergence Rate for Scaffold

(For L-smooth, μ-strongly convex functions with $\nabla^3 f(x)$ bounded by Q)

$$\mathbb{E}\left[\|x^{(T)} - x^\star\|^2\right] \lesssim \left(1 - \frac{\gamma\mu}{4}\right)^{HT} \left\{\|x^{(0)} - x^\star\|^2 + 2\gamma^2 H^2 \zeta^2 + \frac{\sigma_\star^2}{L\mu}\right\} + \frac{\gamma}{\boldsymbol{N}\mu}\sigma_\star^2 + \frac{\gamma^{3/2} Q}{\mu^{5/2}}\sigma_\star^3 + \frac{\gamma^3 H Q^2}{\mu^3}\sigma_\star^4$$

where

- $\sigma_\star^2 = \mathbb{E}[\frac{1}{N}\sum_{c=1}^N \|\nabla F_c^Z(x^\star) - \nabla f_c(x^\star)\|^2$ is the variance at $x^\star$
- $\zeta^2 = \frac{1}{N}\sum_{c=1}^N \|\nabla f_c^Z(x^\star)\|^2$ measures gradient heterogeneity

Linear Speed-Up!

As long as N is not too large, one can obtain $\mathbb{E}\left[\|x^{(T)} - x^\star\|^2\right] \leq \epsilon^2$ with

$$\#\text{grad per client} = \widetilde{O}\Big(\frac{\sigma_\star^2}{\boldsymbol{N}\mu^2\epsilon^2}\log\left(\frac{1}{\epsilon}\right)\Big)$$

Conclusion

- FedAvg and Scaffold converge (even with stochastic gradients)
- This allows to derive new analyses for these problems, with exact first-order expression for bias
- And we proved that Scaffold has:
  - variance similar to FedAvg's variance
  - *linear speed-up* in the number of clients!!
- But: Scaffold is still biased
  $\Rightarrow$ **Need for algorithms tailored for FL <u>and</u> stochasticity!**

Thank you!

Check the papers:

- P. Mangold et al. "Refined Analysis of Constant Step Size Federated Averaging and Federated Richardson-Romberg Extrapolation". In: **AISTATS**. 2025
- P. Mangold et al. "Scaffold with Stochastic Gradients: New Analysis with Linear Speed-Up". In: **ICML**. 2025

Find this presentation on my website:

- `https://pmangold.fr/research.php?page=talks`