Taming Heterogeneity in Federated Linear Stochastic Approximation and Federated Learning

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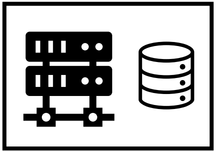
ARGO Seminar

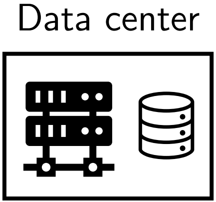
Background on Federated Learning

Data Collection

Data center



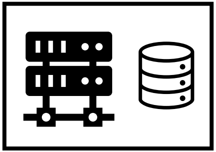
Data Collection



vs.

Data Collection

Data center

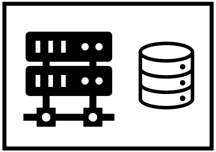


vs.

Data collection *by users*

Data Collection

Data center



vs.

Data collection *by users*

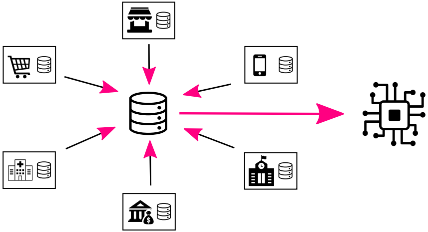
→ **how to use all this data?**

Centralizing in a data center is difficult

Centralizing data is often impossible

- *Privacy*:
  → data may be sensitive (e.g. health records, geolocation)
- *Volume of data*:
  → data may be large (e.g. cameras of self-driving car)
- *Time*:
  → it may be needed to take decisions quickly (e.g. reinforcement learning)

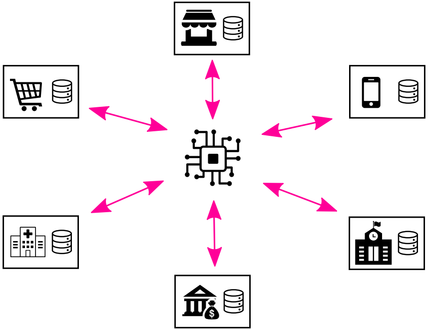
Classical vs Federated Learning



A single optimization problem

$$\min_{\theta \in \mathbb{R}^d} \mathbb{E}_{x,y \sim D} \left[\ell(\theta; x, y) \right]$$

Classical vs Federated Learning



Multiple sub-problems

$$\min_{\theta \in \mathbb{R}^d} \sum_{c=1}^{N} \mathbb{E}_{x^c, y^c \sim \mathcal{D}^c} \left[\ell(\theta; x^c, y^c) \right]$$

$\rightarrow$ but only *one shared solution*

Best Scenario: Homogeneous Data

N local sub-problems

$$\min_{\theta \in \mathbb{R}^d} \mathbb{E}_{x^1, y^1 \sim \mathcal{D}^1} \left[\ell(\theta; x^1, y^1) \right] \to \theta_\star^1$$

$$\min_{\theta \in \mathbb{R}^d} \mathbb{E}_{x^2, y^2 \sim \mathcal{D}^2} \left[\ell(\theta; x^2, y^2) \right] \to \theta_\star^2$$

$$\vdots$$

$$\min_{\theta \in \mathbb{R}^d} \mathbb{E}_{x^N, y^N \sim \mathcal{D}^N} \left[\ell(\theta; x^N, y^N) \right] \to \theta_\star^N$$

Best Scenario: Homogeneous Data

N local sub-problems

$$\min_{\theta \in \mathbb{R}^d} \mathbb{E}_{x^1, y^1 \sim \mathcal{D}^1} \left[\ell(\theta; x^1, y^1) \right] \to \theta_\star^1$$

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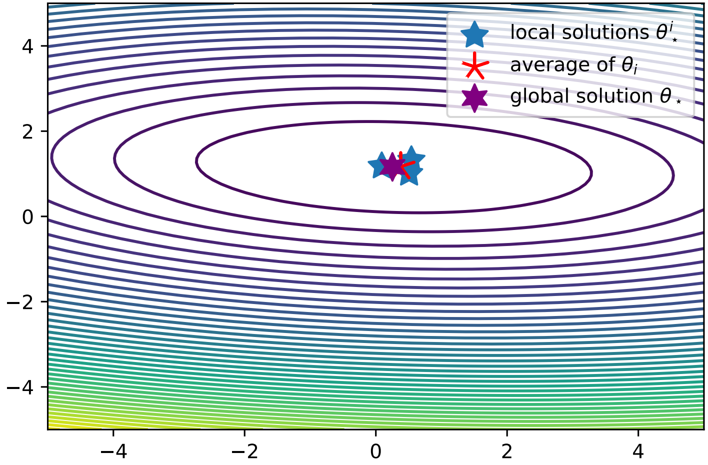
Estimate global solution

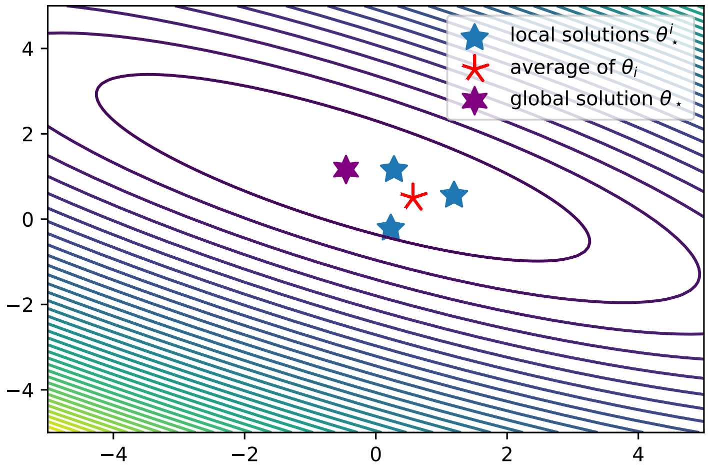
$$\theta_\star = \frac{1}{N} \sum_{c=1}^{N} \theta_\star^c$$

OK if $\mathcal{D}_1 = \mathcal{D}_2 = \cdots = \mathcal{D}_N$

Best Scenario: Homogeneous Data



Failure: Heterogeneous Data



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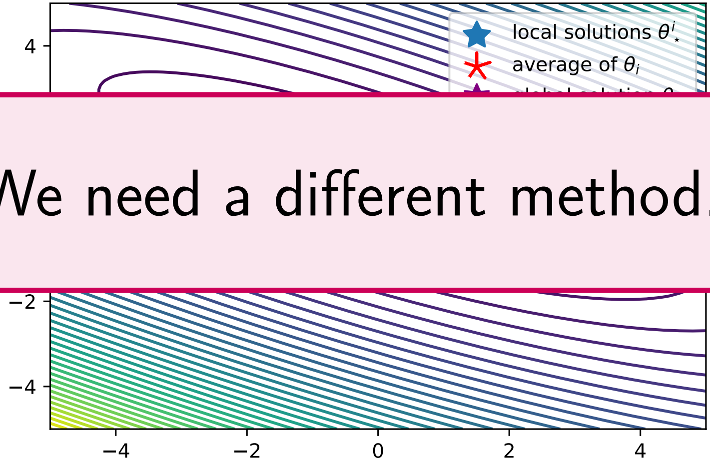

We need a different method...

Federated Optimization

$$\theta_\star \in \arg\min_{\theta \in \mathbb{R}^d} \sum_{c=1}^{N} f^c(\theta) \quad , \qquad \text{where } f^c(\theta) = \mathbb{E}_{x^c, y^c \sim \mathcal{D}^c}\Big[\ell(\theta; x^c, y^c)\Big]$$

[1] Brendan McMahan et al. "Communication-efficient learning of deep networks from decentralized data". In: *AISTATS*. PMLR. 2017, pp. 1273–1282.

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Federated Averaging (or local (S)GD)[1]

- For each $t = 0 \ldots$:
  - Set $\theta_{t,0}^c = \theta_t$
  - For each agent c, do H gradient updates:

$$\theta_{t,h+1}^c = \theta_{t,h}^c - \eta \nabla f^c(\theta_{t,h}^c)$$

- Aggregate models: $\theta_{t+1} = \frac{1}{N} \sum_{c=1}^{N} \theta_{t,H}^c$

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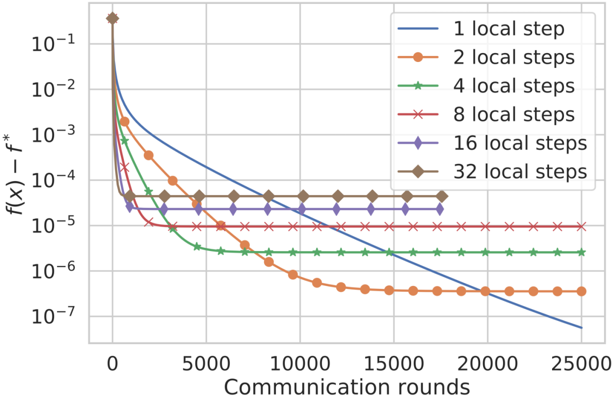
Communication and Sample Complexity

Local Training vs. Precision



(Figure from Ahmed Khaled, Konstantin Mishchenko, and Peter Richtarik. "Tighter Theory for Local SGD on Identical and Heterogeneous Data". In: *AISTATS*. 2020, pp. 4519–4529)

Beyond Federated Optimization: Federated TD and LSA

Some problems do not fit this framework...

Example: TD Learning with linear approximation (I)

In Federated TD learning, N agent use a shared policy π in N different environments:

$$S_0^c = s, A_k^c \sim \pi(\cdot|S_k^c), \text{ and } S_{k+1}^c \sim P_{\mathrm{MDP}}^c(\cdot|S_k^c, A_k^c)$$

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Goal: estimate its value in each environment, for $s \in \mathcal{S}$,

$$V^{c,\pi}(s) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k r^c(S_k^c, A_k^c)\right]$$

where r^c is a reward obtained by agent c

Some problems do not fit this framework...

Example: TD Learning with linear approximation (II)

Idea: build a *shared estimate* of all values

$$V^{c,\pi}(s) \approx \theta^\top \varphi(s)$$

using $\theta \in \mathbb{R}^d$ and embedding $\varphi : \mathcal{S} \to \mathbb{R}^d$

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Is this meaningful to use a shared estimate? Yes, because:

- If agents are homogeneous, it reduces sample complexity
- If agents are heterogeneous, it may reduce bias of local data

Linear Stochastic Approximation

Special case: only one agent

TD (with linear approx.) can be seen as solving a linear system

$$\bar{A}\theta_\star = \bar{b}$$

where $\bar{A}$ and $\bar{b}$ are known through stochastic estimates $A(Z)$, $b(Z)$ for a sequence of random variables Z

... variance of $A(Z)$ and $b(Z)$ are typically very large

... and $\bar{A}$ is not symmetric

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Note: It is inefficient to cast it as a minimization problem with loss $\|\bar{A}\theta_\star - \bar{b}\|^2$
$\rightarrow$ This requires a different method, with a different analysis

Algorithm for LSA

Initialize $\theta_0 \in \mathbb{R}^d$

for $t = 0$ to $T - 1$ **do**

Observe Z_t and update:

$$\theta_t = \theta_{t-1} - \eta(A(Z_t)\theta_{t-1} - b(Z_t))$$

end for

Context, analysis of TD (I)[2]

Initialize $\theta_0 \in \mathbb{R}^d$
for $t = 0$ to $T - 1$ **do**
    Observe $Z_{t,h}^c$ and update: $\theta_t = \theta_{t-1} - \eta(A(Z_t)\theta_{t-1} - b(Z_t))$
end for

[2] Sergey Samsonov et al. "Improved High-Probability Bounds for the Temporal Difference Learning Algorithm via Exponential Stability". In: *COLT*. PMLR. 2024, pp. 4511–4547.

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Stochastic Expansion

We may write: $\theta_t - \theta_\star = (\mathrm{Id} - \eta A(Z_t))(\theta_{t-1} - \theta_\star) - \eta\varepsilon(Z_t)$

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Assumptions

- Oracle: i.i.d sequence Z_t's such that $\mathbb{E}[A(Z_t)] = \bar{A}$, and $\mathbb{E}[b(Z_t)] = \bar{b}$
- Exponential stability: $\mathbb{E}[\| \prod_{t=\ell}^{k}(\mathrm{Id} - \eta A(Z_t))\|^2] \leq (1-\eta a)^{k-\ell}$ for some $a > 0$
- Noise $\varepsilon(Z) = (A(Z) - \bar{A})\theta_\star + (b(Z) - \bar{b})$ has finite variance $\sigma_\star^2$

[2] Sergey Samsonov et al. "Improved High-Probability Bounds for the Temporal Difference Learning Algorithm via Exponential Stability". In: *COLT*. PMLR. 2024, pp. 4511–4547.

Context, analysis of TD (II)[3]

Stochastic Expansion

$$\theta_T - \theta_\star = \Gamma_{1:T}(\theta_0 - \theta_\star) + \eta \sum_{t=1}^{T} \Gamma_{t+1:T}\varepsilon(Z_t)$$

Where $\Gamma_{t:t'}$ "accumulates the updates" from t to t':

$$\Gamma_{t:t'} = (\mathrm{Id} - \eta A(Z_{t'}))(\mathrm{Id} - \eta A(Z_{t'-1}))\cdots(\mathrm{Id} - \eta A(Z_t))$$

[3] Sergey Samsonov et al. "Improved High-Probability Bounds for the Temporal Difference Learning Algorithm via Exponential Stability". In: *COLT*. PMLR. 2024, pp. 4511–4547.

Context, analysis of TD (III)[4]

Stochastic Expansion

$$\theta_T - \theta_\star = \Gamma_{1:T}(\theta_0 - \theta_\star) + \eta \sum_{t=1}^{T} \Gamma_{t+1:T}\varepsilon(Z_t)$$

Using $\mathbb{E}[\|\Gamma_{t:t'} u\|^2] \leq (1-\eta a)^{t'-t+1}\|u\|^2$ to bound each term:

$$\mathbb{E}[\|\theta_T - \theta_\star\|^2] \leq (1-\eta a)^T \|\theta_0 - \theta_\star\|^2 + \frac{\eta \sigma_\star^2}{a}$$

[4] Sergey Samsonov et al. "Improved High-Probability Bounds for the Temporal Difference Learning Algorithm via Exponential Stability". In: *COLT*. PMLR. 2024, pp. 4511–4547.

Federated LSA

Take $\bar{A}^c, \bar{b}^c$ such that $\bar{A}^c \theta_\star^c = \bar{b}^c$ for $c = 1..N$

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Goal: solve collaboratively

$$\left(\frac{1}{N} \sum_{c=1}^{N} \bar{A}^c \right) \theta_\star = \frac{1}{N} \sum_{c=1}^{N} \bar{b}^c$$

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$$\left(\frac{1}{N} \sum_{c=1}^{N} \bar{A}^c \right) \theta_\star = \frac{1}{N} \sum_{c=1}^{N} \bar{b}^c$$

Assumptions

- $\theta_\star$ and $\theta_\star^c$ are unique, and $\bar{A}^c$ and $\bar{b}^c$ are split among N agents
- Oracle: i.i.d sequence Z_t^c's such that $\mathbb{E}[A(Z_t^c)] = \bar{A}^c$, and $\mathbb{E}[b(Z_t^c)] = \bar{b}^c$
- Exponential stability: $\mathbb{E}[\| \prod_{t=\ell}^{k} (\mathrm{Id} - \eta A^c(Z_t^c)) \|^2] \leq (1 - \eta a)^{k-\ell}$ for $a > 0$
- Noise $\varepsilon^c(Z) = (A^c(Z) - \bar{A}^c)\theta_\star^c + (b^c(Z) - \bar{b}^c)$ has variance bounded by $\sigma_\star^2$

Solving Federated LSA

Paul Mangold et al. "SCAFFLSA: Taming Heterogeneity in Federated Linear Stochastic Approximation and TD Learning". In: *NeurIPS* (2024)

FedLSA Algorithm

for $t = 0$ to $T - 1$ **do**

Initialize $\theta_{t,0} = \theta_t$

for each agent $c = 1..N$ **do**

for $h = 1$ to H **do**

Observe $Z^c_{t,h}$ and perform local update:

$$\theta_{t,h} = \theta^c_{t,h-1} - \eta(A^c(Z^c_{t,h})\theta^c_{t,h-1} - b^c(Z^c_{t,h}))$$

end for

end for

Aggregate local updates $\theta_{t+1} = \frac{1}{N}\sum_{c=1}^{N} \theta^c_{t,H}$

end for

Analysis of FedLSA

Stochastic Expansion (over one communication round)

$$\theta_t - \theta_\star = \frac{1}{N}\sum_{c=1}^{N} \Gamma^c_{t,1:H}(\theta_{t-1} - \theta_\star) + \frac{1}{N}\sum_{c=1}^{N}(\mathrm{Id} - \Gamma^c_{t,1:H})(\theta^c_\star - \theta_\star)$$
$$+ \frac{\eta}{N}\sum_{c=1}^{N}\sum_{h=1}^{H} \Gamma^c_{t,h+1:H}\varepsilon^c(Z^c_t)$$

Where $\Gamma^c_{t,h:h'}$ "accumulates local updates", round t, from h to h',

$$\Gamma^c_{t,h:h'} = (\mathrm{Id} - \eta A^c(Z^c_{t,h'}))(\mathrm{Id} - \eta A^c(Z^c_{t,h'-1}))\cdots(\mathrm{Id} - \eta A^c(Z^c_{t,h}))$$

Analysis of FedLSA

We can characterize the bias of FedLSA:

$$\theta_{\infty}^{\mathrm{bias}} = \frac{1}{N}\sum_{c=1}^{N}(\mathrm{Id} - \bar{\Gamma}_{t,1:H})^{-1}(\mathrm{Id} - (\mathrm{Id} - \eta \bar{A}^{c})^{H})\{\theta_{\star}^{c} - \theta_{\star}\}$$

where $\bar{\Gamma}_{t,1:H} = \frac{1}{N}\sum_{c=1}^{N}(\mathrm{Id} - \eta \bar{A}^{c})^{H}$

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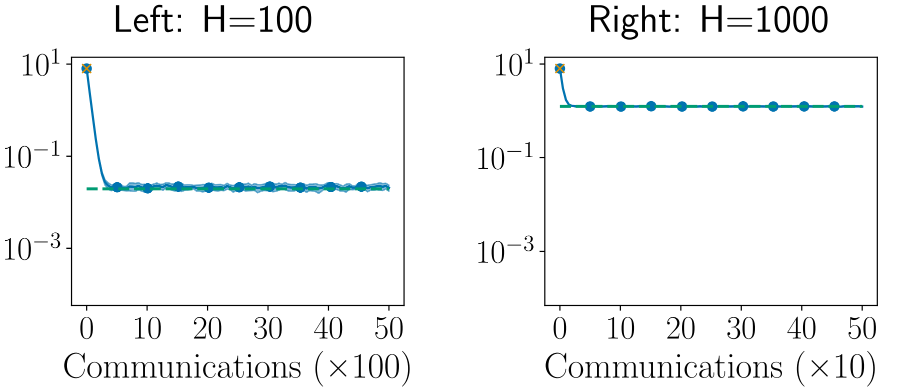
where $\bar{\Gamma}_{t,1:H} = \frac{1}{N}\sum_{c=1}^{N}(\mathrm{Id} - \eta \bar{A}^c)^H$

And give a convergence rate

$$\mathbb{E}\left[\|\theta_t - \theta_\infty^{bias} - \theta_\star\|^2\right] = O\left((1-\eta a)^{Ht}\|\theta_0 - \theta_\star\|^2 + \frac{\eta\sigma_\star^2}{Na}\right)$$

Numerical Illustration ($N = 100$ agents)



Blue line: FedLSA's mean squared error
Green line: FedLSA's bias as predicted by our theory

Problem: heterogeneity requires lots of communications

To achieve $\mathbb{E}\left[\|\theta_T - \theta_\star\|^2\right] \leq \epsilon^2$, we need

- $\frac{\eta\sigma_\star^2}{Na} \leq \epsilon^2$ $\quad\rightarrow \eta = \frac{Na\epsilon^2}{\sigma_\star^2}$
- $\|\theta_T^{\text{bias}}\|^2 \leq \epsilon^2$ $\quad\rightarrow H = \frac{\sigma_\star^2}{N\epsilon\Delta_{\text{het}}}$
- $(1-\eta a)^{HT}\|\theta_0 - \theta_\star\|^2 \leq \epsilon^2$ $\quad\rightarrow T = \frac{\Delta_{\text{het}}}{a^2\epsilon}\log\frac{\|\theta_0-\theta_\star\|}{\epsilon}$

where $\Delta_{\text{het}} = \frac{1}{N}\sum_{c=1}^{N}\|\theta_\star - \theta_\star^c\|$

Solution: Control variates (SCAFFLSA)[5]

for $t = 0$ to $T - 1$ **do**
  Initialize $\theta_{t,0} = \theta_t$
  for each agent $c = 1..N$ **do**
    for $h = 1$ to H **do**
      Observe $Z^c_{t,h}$ and perform local update:
$$\theta_{t,h} = \theta^c_{t,h-1} - \eta(A^c(Z^c_{t,h})\theta^c_{t,h-1} - b^c(Z^c_{t,h}) - \xi_t)$$
    end for
  end for
  Aggregate local updates $\theta_{t+1} = \frac{1}{N}\sum_{c=1}^{N} \theta^c_{t,H}$
  Update control variate $\xi_{t+1} = \xi_t - \frac{1}{\eta H}(\theta_{t+1} - \theta^c_{t,H})$
end for

[5] Based on Sai Praneeth Karimireddy et al. "Scaffold: Stochastic controlled averaging for federated learning". In: *ICML*. PMLR. 2020, pp. 5132–5143

Theoretical analysis

We prove, assuming $H \leq \frac{a}{\eta \max_c \|\bar{A}^c\|^2}$

$$\mathbb{E}[\|\theta_T - \theta_\star\|^2] \lesssim \left(1 - \frac{\eta a H}{2}\right)^T \psi_0 + \frac{\eta \sigma_\star^2}{N a}$$

with $\psi_0 = \|\theta_0 - \theta_\star\|^2 + \frac{\eta^2 H^2}{N} \sum_{c=1}^N \|\bar{A}^c(\theta_\star^c - \theta_\star)\|^2$

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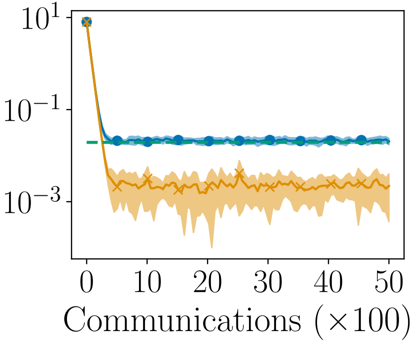
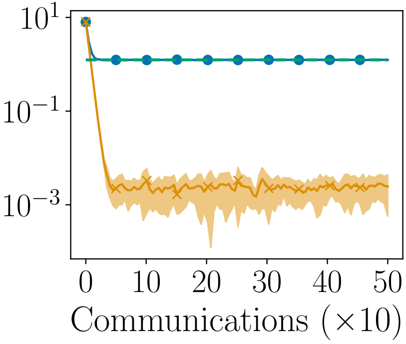
Note on analysis
Direct analysis “à la LSA” does not work. We need a “Lyapunov” analysis, and to carefully study covariances of control variates to obtain linear speed-up.

Numerical Illustration ($N = 100$ agents)

Left: H=100



Right: H=1000



Blue line: FedLSA's mean squared error

Orange line: SCAFFLSA's mean squared error

Communication Complexity

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- $\frac{\eta\sigma_\star^2}{Na} \leq \epsilon^2$ $\rightarrow \eta = \frac{Na\epsilon^2}{\sigma_\star^2}$
- $H \leq \frac{a}{\eta \max_c \|\bar{A}^c\|^2}$ $\rightarrow H = \frac{\sigma_\star^2}{N\epsilon^2 \max_c \|\bar{A}^c\|^2}$
- $(1 - \frac{\eta a H}{2})^T \psi_0 \leq \epsilon^2$ $\rightarrow T = \frac{2 \max_c \|\bar{A}^c\|^2}{a^2} \log \frac{\psi_0}{\epsilon}$

$\rightarrow H \propto 1/N\epsilon^2$ rather than $1/N\epsilon$, and T independent on ϵ

What about the Non-Linear Case?

Back to FedAvg

$$\theta_\star \in \arg\min_{\theta \in \mathbb{R}^d} \sum_{c=1}^{N} f^c(\theta) \ , \qquad \text{where } f^c(\theta) = \mathbb{E}_{x^c, y^c \sim \mathcal{D}^c}\left[\ell(\theta; x^c, y^c)\right]$$

Federated Averaging (or local (S)GD)[6]

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- Aggregate models: $\theta_{t+1} = \frac{1}{N} \sum_{c=1}^{N} \theta^c_{t,H}$

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F

What can we say about the bias?

$$\theta^c_{t,h+1} = \theta^c_{t,h} - \eta \nabla f^c(\theta^c_{t,h})$$

- Aggregate models: $\theta_{t+1} = \frac{1}{N} \sum_{c=1}^{N} \theta^c_{t,H}$

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For Quadratics ($f^c(\theta) = (1/2)\theta^\top \bar{A}^c \theta + \bar{b}^c \theta$)

The bias is the same as FedLSA

$$\theta_\infty^{\text{bias}} = \frac{1}{N}\sum_{c=1}^{N}(\mathrm{Id} - \bar{\Gamma}_{t,1:H})^{-1}(\mathrm{Id} - (\mathrm{Id} - \eta \bar{A}^c)^H)\{\theta_\star^c - \theta_\star\}$$

where $\bar{\Gamma}_{t,1:H} = \frac{1}{N}\sum_{c=1}^{N}(\mathrm{Id} - \eta \bar{A}^c)^H$

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And we can give first order expansion:

$$\theta_\infty^{\text{bias}} = \frac{\eta(H-1)}{2N}\sum_{c=1}^{N}\nabla^2 f^c(\theta_\star)^{-1}(\nabla^2 f^c(\theta_\star) - \nabla^2 f(\theta_\star))\nabla f^c(\theta_\star) + O(\eta^2 H^2)$$

In the General Case

(Strongly convex and smooth functions f^c)

Bias is in *two* parts!

$$\theta_\infty^{\text{bias}} = \frac{\eta(H-1)}{2N}\sum_{c=1}^{N} \nabla^2 f^c(\theta_\star)^{-1}(\nabla^2 f^c(\theta_\star) - \nabla^2 f^c(\theta_\star))\nabla f^c(\theta_\star)$$
$$+ \frac{\eta}{2N}\nabla^2 f^c(\theta_\star)^{-1}\nabla^3 f(\theta_\star)\mathbf{A}\mathcal{C}(\theta_\star) + O(\eta^{3/2}H + \eta^2 H^2)$$

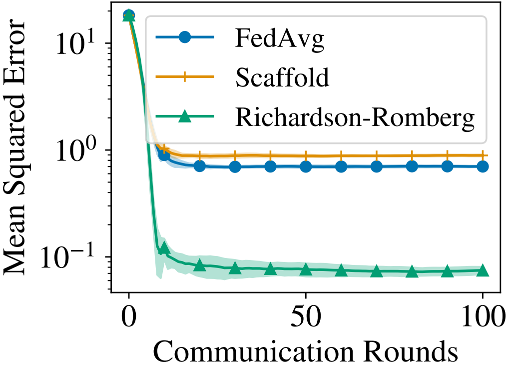
where:

- $\mathbf{A} = (\mathrm{Id} \otimes \nabla^2 f(\theta_\star) + \nabla^2 f(\theta_\star) \otimes \mathrm{Id})^{-1}$
- $\mathcal{C}(\theta_\star)$ is the gradient's covariance at $\theta_\star$

A new Federated Method?

(A bit of teasing on Richardson-Romberg)

Running FedAvg with step sizes η and 2η, we can correct the bias:



$\rightarrow$ it seems Scaffold cannot correct bias due to stochasticity!

Conclusion and Perspectives

Summary:

- We studied FedLSA's communication complexity
- We extended control variates methods to FedLSA
- We showed that both methods have linear speed-up (up to bias)
- We proved first-order expansion of FedAvg's bias

Perspectives:

- SCAFFLSA's analysis is good for small step-size: what about larger steps?
- Direct analysis of SCAFFLSA "à la FedLSA"?
- Removing hyperparameters?
- Asynchronous federated learning?

Thank you!

Questions?

See the papers:

P. Mangold, S. Samsonov, S. Labbi, I. Levin, R. Alami, A. Naumov, and E. Moulines. "SCAFFLSA: Taming Heterogeneity in Federated Linear Stochastic Approximation and TD Learning". In: *NeurIPS* (2024)

On FedAvg and Richardson-Romberg (with E. Moulines, A. Durmus, A. Dieuleveut and S. Samsonov): soon on arXiv!