HW 17

Section 32

31. The Schroder-Bernstein Theorem

Exercise 31.1. Let X, Y, and Z be sets. Prove that if $X \subseteq Y \subseteq Z$, and $|X| = |Z|$, then $|X| = |Y|$ as well.

Proof: We suppose directly that $X \subseteq Y \subseteq Z$ and that $|X| = |Z|$. Then there is a bijective function $f : X \to Z$. Since $S \subseteq X$, then $f \mid_{f^{-1}(S)}: X \to S$ restricted to the preimage of S is a bijective function. Since there is a bijective function from $X \to S$, then $|X| = |Y|$. ■

Exercise 31.2. Prove that $[5, 16)$ and $(0, \infty)$ have the same cardinalities.

Proof: To show that the sets $A = [5, 16)$ and $B = (0, \infty)$ have the same cardinalities, we will show that there are two injective functions $f : A \to B$ and $g : B \to A$, and use the Schroder-Bernstein Theorem to show that the sets A and B have the same cardinalities.

$(f : A \to B)$ **Injective**: Since $A \subseteq B$, let $f = id_A$ such that $f(x) = x$. This is injective since the identity function is injective.

$(g : B \to A)$ **Injective**: Let g be defined as $\arctan(x) + 5$ which has a codomain of $C = \left(0, 5 + \frac{\pi}{2}\right)$ which is a subset of A. To show that it is injective, we assume contrapositively that $g(b_1) = g(b_2)$ for some $b_1, b_2 \in B$. Then

$$\begin{aligned} g(b_1) &= g(b_2) \\ \arctan(b_1) + 5 &= \arctan(b_2) + 5 \\ \arctan(b_1) &= \arctan(b_2). \end{aligned}$$

Since the function $\arctan$ is injective, we have that $b_1 = b_2$. Therefore, the function g is injective. Since the both g and f are injective, then A and B have the same cardinalities. ■

Exercise 31.3. Given sets A and B, prove that if there is an injection $f : A \to B$ and a surjection $g : A \to B$, then $|A| = |B|$.

Proof: We assume directly that f is an injection and g is a surjection from A to B. Since g is surjective, we know that for all $b \in B$, there exists and $a \in A$, such that $g(a) = b$. Let $C \subseteq A$ be the subset of A such that $g|_C$ is a bijection, then $|C| = |B|$. In addition, since $C \subseteq A$, $|C| \leq |A|$. Also, because the injection f exists, we know that $|A| \leq |B|$. Due to the facts that $|C| \leq |A|$, $|C| = |B|$, and $|A| \leq |B|$ we get that $|A| = |B|$. ■

Exercise 31.4. Complete the proof in case 2 of the Schroder-Bernstein theorem, by showing that $f|_{A_2}$ is a function from A_2 to B_2, and also that it is bijective.

Proof: We suppose directly that the sets A_2 and B_2 correspond to the chain that never loops and has an ultimate ancestor in A. Let $a_i \in A_2$ and $b_i \in B_2$ with a_0 being the ultimate ancestor in A_2. Using $f|_{A_2}$ and $g|_{B_2}$, we get the chain

$$a_0 \mapsto b_0 \mapsto a_1 \mapsto b_1 \mapsto \cdots \mapsto a_n \mapsto b_n,$$

with . Since f is injective, then $f|_{A_2}$ is injective. From the chain we can see that any $b_i \in B_2$ is simply $f(a_i)$, thus $f|_{A_2}$ is surjective. Since the function $f|_{A_2}$ is both injective and surjective, it is bijective. ■

Exercise 31.5. In exercise 31.6 we showed that $|R| \leq |\mathcal{P}(\mathbb{N})|$. Here is another way to do that.

Define a function $f : (0,1) \to \mathcal{P}(\mathbb{N})$, by sending (the decimal expansion of) a real number $0.a_1a_2a_3\ldots$ (not ending in repeating 9's) to the set

$$\{a_1, 10a_2, 100a_3, \ldots\} - \{0\} \subseteq \mathbb{N}.$$

Prove that this is an injective function.

Proof: We suppose contrapositively that $f(x) = f(y)$ for some $x, y \in (0,1)$, then

$$f(x) = f(y)$$
$$\{x_1, 10x_2, 100x_3, \ldots\} = \{y_1, 10y_2, 100y_3, \ldots\}.$$

The two sets can only be equivalent if they have the same elements. Since the elements of the sets are constructed uniquely from the decimal digits of x and y, this means that x and y must have the same decimal digits. And, since $0 < x, y < 1$, we get that $x = y$ since their decimal values are equivalent. Thus the function is injective. ■