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Math 290
Winter 2017
Sample Exam 1

Note that the first 10 questions are true-false. Mark A for true, B for false. Questions 11 through 20 are multiple choice–mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly. Questions 1 to 20 are worth 2.5 points each, and questions 21-25 are worth 10 points each.

True-false questions

1. For any set A, the empty set is an element of the power set of A.
2. For any sets A and B, we have $A - B \subseteq A$.
3. Let I be the set of natural numbers, and for each $i \in I$ let A_i be the closed interval in the real numbers $[1/i, i^2 + 1]$. Then
$$\bigcap_{i \in I} A_i = [1, 2].$$
4. Let $A = \{1, 2, 3\}$. Then A is a subset of the power set of A.
5. If $a \equiv 3 \pmod 5$, then $a^2 \equiv 4 \pmod 5$.
6. Let A, B, and C be sets. Then $A - (B \cup C) = (A - B) \cap (A - C)$.
7. The converse of the statement "If x is even, then $x + 1$ is odd," is the statement "If $x + 1$ is even, then x is odd."
8. The negation of the statement "There exists $x \in \mathbb{R}$, $x^2 - 1 < 0$," is the statement "For all $x \in \mathbb{R}$, $x^2 - 1 < 0$."
9. The statement $P \wedge (\neg P)$ is a tautology.
10. Let A and B be sets. If A has seven elements, $A \cup B$ has ten elements, and $A - B$ has two elements, then B must contain eight elements.

Multiple choice section

11. For the following proof, determine which of the statements given below is being proved.

Proof. Assume a and b are odd integers. Then $a = 2k + 1$ and $b = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. Then $ab^2 = (2k+1)(2\ell+1)^2 = 8kl^2 + 8kl + 2k + 4l^2 + 4l + 1 = 2(4kl^2 + 4kl + k + 2l^2 + 2l) + 1$. Since $4kl^2 + 4kl + k + 2l^2 + 2l \in \mathbb{Z}$, we see that ab^2 is odd. □

a) If a or b is even, then ab^2 is even.

b) If a and b are even, then ab^2 is even.

c) If ab^2 is even, then a and b are even.

d) If ab^2 is even, then a is even or b is even.

e) None of the above.

12. Let A be a set with 5 elements. Which of the following cannot exist:

a) A subset of the power set of A with six elements.

b) An element of the power set of A with six elements.

c) An element of A containing six elements.

d) Any of the above can exist, for suitable sets A.

e) None of (a) through (c) can exist, no matter what A is.

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13. Which of the following has a vacuous proof?

a) Let $n \in \mathbb{Z}$. If $|n| < 1$ then $5n + 3$ is odd.

b) Let $n \in \mathbb{Z}$. If $2n + 1$ is odd, then $n^2 + 1 > 0$.

c) Let $x \in \mathbb{R}$. If $x^2 - 2x + 3 < 0$, then $2x + 3 < 5$.

d) Let $x \in \mathbb{R}$. If $-x > 0$, then $x^2 + 3 > 3$.

e) None of the above.

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14. Which of the following statements has a trivial proof.

a) Let $x \in \mathbb{N}$. If $x > 0$ then $x^2 > x$. b) Let $x \in \mathbb{N}$. If $x > 3$ then $2x$ is even.

c) Let $x \in \mathbb{N}$. If $x < 2$ then $x^2 + 1$ is even. d) Let $x \in \mathbb{N}$. If $2x$ is even, then x is odd.

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15. Evaluate the following proof:

Theorem: Let $n \in \mathbb{Z}$. If $3n - 8$ is odd, then n is odd.

Proof. Let $n \in \mathbb{Z}$. Assume that n is odd. Then $n = 2k + 1$ for some integer k. Then

$$3n - 8 = 3(2k + 1) - 8 = 6k + 3 - 8 = 6k - 5 = 2(3k - 3) + 1.$$

Since $3k - 3 \in \mathbb{Z}$, we know that $3n - 8$ is odd. $\square$

a) The proof and the theorem are correct.

b) The proof proves the converse of the given statement.

c) The proof proves the contrapositive of the given statement.

d) The proof contains arithmetic mistakes, which make it incorrect.

e) None of the above.

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16. Let $A = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. The number of elements in the power set of A is

a) 3 b) 4 c) 6 d) 8 e) 16 f) 64

17. Let $x \in \mathbb{Z}$. The contrapositive of the open sentence "If x is even then $3x+7$ is odd." is

a) If x is odd then $3x+7$ is even. b) If $3x+7$ is odd then x is even.
c) If $3x+7$ is even then x is odd. d) If $3x+7$ is even, then x is even.
e) x is odd or $3x+7$ is odd. f) x is odd or $3x+7$ is even.

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18. Let x and y be integers. The negation of the open sentence "If xy is even then x is even or y is even" is

a) If x is odd and y is odd, then xy is odd. b) If x is even or y is even, then xy is even.
c) If xy is odd, then x is even and y is even. d) xy is even and x is odd and y is odd.
e) xy is even and (x is odd or y is odd). f) xy is odd and (x is even or y is even).
g) xy is odd and (x is odd and y is odd).

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19. If you wish to prove a statement of the form "If P, then (Q or R),", which of the following would **not** be a good way to begin.

a) Assume P.

b) Assume $(\neg P) \wedge (Q \vee R)$.

c) Assume $(\neg Q) \wedge (\neg R)$.

d) Assume $P \wedge (\neg Q) \wedge (\neg R)$.

e) None of the above: all of these would be acceptable ways to begin.

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20. The following is a theorem proved in "Cohomology of number fields" (pg. 75) by J. Neukirch.

Theorem: Let G be a finite group, and let A, B be G-modules. If A is cohomologically trivial or B is divisible, then $\hom(A, B)$ is cohomologically trivial.

Suppose that we know that G is a finite group, A and B are G-modules, and that $\hom(A, B)$ is not cohomologically trivial. Which of the following must be true? (Think about the contrapositive.)

a) A is cohomologically trivial and B is divisible.

b) A is cohomologically trivial or B is divisible.

c) A is not cohomologically trivial or B is divisible.

d) A is not cohomologically trivial or B is not divisible.

e) A is not cohomologically trivial and B is not divisible.

Written Answer Section

21. Construct a truth table for $(P \Rightarrow Q) \Rightarrow (\neg R)$.

P	Q	R						

22. Prove that if n is an even integer, then $3n + 2$ is even in each of the following three ways: (i) a direct proof, (ii) a contrapositive proof, and (iii) a proof by contradiction.

23. Prove the following statement. If x and y are rational, $x \neq 0$, and z is irrational, then $\frac{y+z}{x}$ is irrational.

24. Let A, B, C be sets. Prove (with justification for every step) that
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

25. Give examples of three sets A, B, and C such that $A \in B$, $B \subseteq C$, and $A \nsubseteq C$.