HW 8

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Sections 14 and 15

14. More Examples of Induction

Exercise 14.1. Prove that $n! > 3^n$ for each natural number $n > 6$.

Proof: We wish to prove that the open sentence

$$P(n):\ n! > 3^n$$

is true for each $n > 6$ with $n \in \mathbb{N}$. We work by induction.

Base Case: We verify that $P(7)$ is true, as follows:

$$7! = 5040 > 2187 = 3^7.$$

Inductive Step: Let $k \in \mathbb{N} > 6$. Assume that $P(k)$ is true. So we now know that $k! > 3^k$. We wish to prove $P(k+1)$, which states that

$$(k+1)! > 3^{k+1}.$$

In order to do this we examine $(k+1)!$ closely.

$$\begin{aligned}
(k+1)! &= (k+1)(k!) \quad \text{(by definition of factorial)}\\
&> (k+1)\,3^k \quad \text{(Since we assume } P(k)\text{)}\\
&= (3+(k-2))\,3^k \quad \text{(Since } k > 6\text{)}\\
&= 3 \cdot 3^k + (k-2)\,3^k\\
&> 3 \cdot 3^k \quad \text{(Since } (k-2)\,3^k > 0\text{)}\\
&= 3^{k+1}.
\end{aligned}$$

Hence $P(k+1)$ in true. Therefore, by mathematical induction, $P(n)$ is true for each $n > 6$. ∎

Exercise 14.2. Prove that if n is any natural number greater than 5, then $n! > n^3$.

Proof: We wish to prove that the open sentence

$$P(n):\ n! > n^3$$

is true for each $n \in \mathbb{Z} > 5$ (I've seen the instructor use this notation. Is it common?). We work by induction.

Base Case: We wish to verify $P(6)$ is true as follows:

$$6! = 720 > 216 = 6^3.$$

Inductive Step: Let $k \in \mathbb{Z} > 5$. Assume that $P(k)$ is true. Se we know that $k! > k^3$. We wish to prove $P(k+1)$, which states that

$$(k+1)! > (k+1)^3.$$

To show this, we begin by manipulating the left hand side.

$$\begin{aligned}
(k+1)! &= (k+1)(k!) \quad \text{(By definition of factorail)} \\
&> (k+1)k^3 \quad \text{(Since we assume } P(k)\text{)} \\
&= k^3 + k^2 \cdot k^2 \\
&> k^3 + (3+1)k^2 \quad \text{(Since } k^2 > 4\text{)} \\
&= k^3 + 3k^2 + k \cdot k \\
&> k^3 + 3k^2 + (3+1)k \quad \text{(Since } k > 4 \text{ by our assumpiton)} \\
&= k^3 + 3k^2 + 3k + k \\
&> k^3 + 3k^2 + 2k + 1 \quad \text{(Since } k > 1\text{)} \\
&= (k+1)^3,
\end{aligned}$$

hence $P(k+1)$ is true. Therefore, by induction $P(n)$ is true for each $n > 5$. ■

Exercise 14.3. Prove that for each $n \in \mathbb{N}$, we have $3^n \geq n^3$.

Proof: We wish to prove that the open sentence

$$P(n): \ 3^n \geq n^3$$

is true for each $n \in \mathbb{N}$. We work by induction.

Base Cases: For this proof we will establish several base cases. We will verify $P(1)$, $P(2)$ and $P(3)$. We verify them as follows: for $P(1)$ we have

$$3^1 = 3 \geq 1^3,$$

which is true; for $P(2)$ we have

$$3^2 = 9 \geq 8 = 2^3,$$

which is true; and for $P(3)$ we have

$$3^3 = 27 \geq 27 = 3^3,$$

which is true.

Induction Step: Let $k \in \mathbb{Z}$. We assume that $P(k)$ is true for $k \geq 3$. So we know that $3^k \geq k^3$. We wish to show that $P(k+1)$, which states

$$3^{k+1} \geq (k+1)^3,$$

is true. To do this we begin by manipulating the left hand side.

$$\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&\geq 3 \cdot k^3 \quad \text{(Since we assume } P(k)\text{)} \\
&= k^3 + 2k \cdot k^2 \\
&\geq k^3 + (3+2)k^2, \quad \text{(Sice we assume } k \geq 3,\ 2k \geq 5\text{)} \\
&= k^3 + 3k^2 + 2k \cdot k \\
&\geq k^3 + 3k^2 + (3+1)k \quad \text{(Sice we assume } k \geq 3,\ 2k \geq 4\text{)} \\
&= k^3 + 3k^2 + 3k + k \\
&\geq k^3 + 3k^3 + 3k + 1 \\
&= (k+1)^3,
\end{aligned}$$

hence $P(k+1)$ is true. Therefore, by induction $P(n)$ is true for each $n \in \mathbb{Z}$. ■

Exercise 14.4. Prove that for any $n \in \mathbb{N}$ with $n \geq 2$, if $P_1, \ldots, P_n$ are statements, then

$$\neg(P_1 \wedge \cdots \wedge P_n) \equiv (\neg P_1) \vee \cdots \vee (\neg P_n).$$

Proof: We wish to prove the open sentence

$$H(n): \neg(P_1 \wedge \cdots \wedge P_n) \equiv (\neg P_1) \vee \cdots \vee (\neg P_n),$$

is true for any $n \in \mathbb{N} \geq 2$. We work this by induction.

Base Case: We verify that $P(2)$ is true. We know from De Morgans law that

$$\neg(P_1 \wedge P_2) = (\neg P_1) \vee (\neg P_2),$$

so this is true.

Induction Step: Let $k \in \mathbb{Z} \geq 2$. We assume that $P(k)$ is true. So we know that

$$\neg(P_1 \wedge \cdots \wedge P_k) \equiv (\neg P_1) \vee \cdots \vee (\neg P_k),$$

is true. We wish to show that $P(k+1)$ is true. To show this, let $Q(k)$ be the open statement

$$Q(k): P_1 \wedge \cdots \wedge P_k.$$

The open statement $P(k+1)$ is

$$\neg(P_1 \wedge \cdots \wedge P_k \wedge P_{k+1}) \equiv (\neg P_1) \vee \cdots \vee (\neg P_k) \vee (\neg P_{k+1}),$$

to show that it is true we work with the left hand side

$$\begin{aligned}
\neg(P_1 \wedge \cdots \wedge P_k \wedge P_{k+1}) &= \neg(Q(n) \wedge P_{k+1}) \\
&\equiv (\neg Q(n)) \vee (\neg P_{k+1}), \quad \text{(Using De Morgans Law)} \\
&= ((\neg P_1) \vee \cdots \vee (\neg P_k)) \vee (\neg P_{k+1}) \\
&= (\neg P_1) \vee \cdots \vee (\neg P_k) \vee (\neg P_{k+1}),
\end{aligned}$$

hence $P(k+1)$. Therefore we have shown by induction that $P(n)$ is true for any $n \in \mathbb{N} \geq 2$. ■

Exercise 14.5. Prove that for any $n \in \mathbb{N}$, if $x_1, \ldots, x_n \in \mathbb{R}$, then

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|.$$

Proof: We wish to prove that the open sentence

$$P(n): \left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|$$

for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$ is true. We work this by induction.

Base Cases: We wish to verify that $P(1)$ and $P(2)$ are true. For $P(1)$ it follows that

$$|x_1| \leq |x_1|,$$

is true. For $P(2)$ it follows that

$$|x_1 + x_2| \leq |x_1| + |x_2|,$$

is true by the triangle inequality theorem.

Induction Step: For $k \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$, we assume that $P(k)$ is true, which is the statement

$$\left|\sum_{i=1}^{k} x_i\right| \leq \sum_{i=1}^{k} |x_i|.$$

We wish to show that $P(k+1)$ is true, which is stated as

$$\left|\sum_{i=1}^{k+1} x_i\right| \leq \sum_{i=1}^{k+1} |x_i|.$$

Do do this be begin with the left hand side and manipulate it

$$\begin{aligned}
\left|\sum_{i=1}^{k+1} x_i\right| &= \left|\sum_{i=1}^{k} x_i + x_{k+1}\right| \\
&\leq \left|\sum_{i=1}^{k} x_i\right| + |x_{k+1}| \quad \text{(By the triangle inequality theorem)} \\
&\leq \sum_{i=1}^{k} |x_i| + |x_{k+1}| \quad \text{(By our assumpiton of } P(k)\text{)} \\
&\leq \sum_{i=1}^{k+1} |x_i|,
\end{aligned}$$

hence $P(k+1)$ is true. Therefore the open sentence $P(n)$ is true for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$. ■

Exercise 14.6. The *Fibonacci numbers* are a collection of natural numbers labeled F_1, F_2, $F_3 \ldots$ and defined by the rule

$$F_1 = F_2 = 1,$$

and for $n > 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

a) Write down the first fifteen Fibonacci numbers.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

b) Prove by induction that for each $n \geq 1$,

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

Proof: We wish to show that the open sentence

$$P(n): \sum_{i=1}^{n} F_i = F_{n+2} - 1$$

is true for any $n \in \mathbb{N} \geq 1$. We work this by induction.
Base Case: We verify that $P(1)$ is true

$$1 = 2 - 1 = F_{1+2} - 1.$$

Induction Step: For any $k \in \mathbb{N}$, we assume that $P(k)$ is true, which is the statement

$$\sum_{i=1}^{k} F_i = F_{k+2} - 1.$$

We want to show that $P(k+1)$ is true with is the statement

$$\sum_{i=1}^{k+1} F_i = F_{k+2+1} - 1.$$

We do this by examining more closely the left hand side

$$\begin{aligned}\sum_{i=1}^{k+1} F_i &= \sum_{i=1}^{k} F_i + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1}, \quad \text{(Since we assume that } P(k) \text{ is true)}, \\ &= F_{k+2+1} - 1, \quad \text{(According to the definition of Fibonacci numbers, and since } k \geq 1),\end{aligned}$$

hence $P(k+1)$ is true.
Therefore the open sentence $P(n)$ is true for any $n \in \mathbb{N}$. ■

c) Prove by induction that for each $n \in \mathbb{N}$,

$$\sum_{i}^{n} F_i^2 = F_n F_{n+1}.$$

Proof: We wish to prove that the open sentence

$$P(n): \sum_{i}^{n} F_i^2 = F_n F_{n+1},$$

is true for any $n \in \mathbb{N}$. We work this by induction.
Base Case: We verify that $P(1)$ is true as follows

$$F_1^2 = 1^2 = 1 \cdot 1 = F_1 F_2.$$

Induction Step: Let $k \in \mathbb{N}$. We assume that $P(k)$ is true, which is the statement

$$\sum_{i}^{k} F_i^2 = F_k F_{k+1}.$$

We want to show that $P(k+1)$ is true, which is the statement

$$\sum_{i}^{k+1} F_i^2 = F_{k+1} F_{k+2}.$$

We show that it is true by manipulating the left hand side as follows:

$$\begin{aligned}\sum_{i}^{k+1} F_i^2 &= \sum_{i}^{k} F_i^2 + F_{k+1}^2 \\ &= F_k F_{k+1} + F_{k+1}^2 \quad \text{(Since we assume } P(k) \text{ to be true)} \\ &= F_{k+1}(F_k + F_{k+1}) \\ &= F_{k+1} F_{k+2} \quad \text{(According to the definition of Fibonacci numbers, and since } k \geq 1),\end{aligned}$$

hence $P(k+1)$ is true.
Therefore the open sentence $P(n)$ is true for each $n \in \mathbb{N}$. ■

Exercise 14.7. Using the definition of the Fibonacci numbers form the previous problem, prove by induction that for any integer $n > 12$ that $F_n > n^2$.

Proof: We wish to show that the open sentence

$$P(n): F_n > n^2 \text{ and } F_{n-1} > (n-1)^2,$$

for any $n \in \mathbb{N} > 13$. We work this by induction.
Base Case: We verify that $P(14)$ is true

$$\begin{aligned}F_{13} &= 233 > 169 = 13^2. \\ F_{14} &= 377 > 196 = 14^2\end{aligned}$$

Induction Step: Let $k \in \mathbb{N} > 13$. We assume that $P(k)$ is true, which is the statement

$$F_k > k^2 \text{ and } F_{h-1} > (k-1)^2 .$$

We wish to show that $P(k+1)$ is true, which is the statement

$$F_{k+1} > (k+1)^2 \text{ and } F_k > (k)^2 .$$

From our assumption that $P(k)$ is true, we know that $F_k > k^2$, thus we only need to show that $F_{k+1} > (k+1)^2$. To do this we manipulate the term F_{k+1} as follows:

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} \\ &> k^2 + (k-1)^2 \quad \text{(By assumption that } P(k) \text{ is true)} \\ &= k^2 + k^2 - 2k + 1 \\ &= k^2 + k(k-2) + 1 \\ &> k^2 + 2k + 1 \quad \text{(Since } k > 13,\ k-2{>}2\text{)} \\ &= (k+1)^2 , \end{aligned}$$

hence $P(k+1)$ is true. Therefore $P(n)$ is true for all $n \in \mathbb{N} > 13$, and thus for any integer $n > 12$, $F_n > n^2$. ■

15. Strong Induction

Exercise 15.1. Prove by induction that for each integer $n > 5$, it is possible to subdivide an equilateral triangle into n equilateral triangles.

Proof: We wish to show by strong induction that for each integer $n > 5$ the open sentence

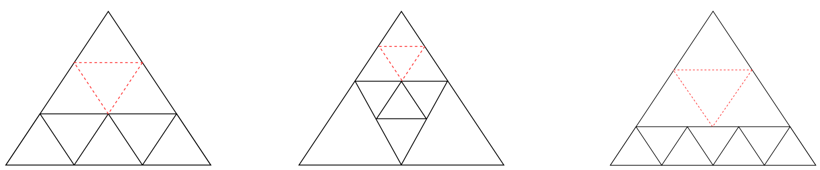
$P(n)$: it is possible to subdivide an equilateral triangles into n equilateral triangles.

Let $Q(n)$ be the open sentence

$$Q(n) : P(6) \wedge \cdots \wedge P(n).$$

We work by strong induction to show $P(n)$ is true for each $n \in \mathbb{N}$.

Base Cases: We verify that this is true for $P(6)$, $P(7)$, $P(8)$ by considering the images.



(a) (b) (c)

Figure 1: The red triangle in each image is to show how the triangle can be potentially subdivided further to have 3 more equilateral triangles. From left to right we show how an equilateral triangle can be subdivided into 6, 7, and 8 smaller equilateral triangles.

Strong Induction Step: Let $k \in \mathbb{N} > 8$ and assume $P(6) \wedge \cdots \wedge P(k)$ is true. We want to show that $P(k+1)$ is true. From the base cases and considering fig 1, we know that we can sub divide the equilateral triangle with 6, 7, and 8 sub equilateral triangles to have 9, 10, respectively 11 sub divided equilateral triangles, as indicated by the red dashed triangle. This means that I can recursively construct an equilateral triangle with $6+3k$, $7+3k$, or $8+3k$ sub divided equilateral triangles for any $k \in \mathbb{N}$. Since I know that $P(k+1-3)$, is true, I can subdivide the equilateral triangle corresponding to $P(k-2)$ even more by the same process described so that $P(k+1)$ has $k+1$ equilateral triangle. Hence $P(k+1)$ is true. Therefore, the open sentence $P(n)$ is true for all $n \in \mathbb{N} > 5$. ■

Exercise 15.2. For the network of nine cities with one-way roads below, find a route that visits all nine cities. Do this using the method in the proof of Proposition 15.1, letting X be the city denoted by A.

Let $U = \{A, B, C, D, E, F, G, H, I\}$, $S = \{Y \in U$: There is a path from Y to $A\} = \{D, F, H\}$, and $T = \{Y \in U$: There is a path from A to $Y\} = \{B, C, E, G, I\}$. According to Proposition 15.1 we know that there is a valid path is subset S and T. Therefore we can start in either S or T, follow the valid path and move to A. From A we can more into the other subset T or S and finish the valid path in the other subset. Let us start in S then move to A, lastly to T. This path is

$$\underbrace{D \to H \to F}_{\in S} \to A \to \underbrace{C \to B \to I \to E \to G}_{\in T}.$$

Exercise 15.3. Do the following

a) Prove that every integer $n > 13$ can be written as $n = 3x_n + 8y_n$ for some integers $x_n, y_n \geq 0$ where x_n and y_n depend on n.

Proof: We wish to show that the open sentence

$P(n):$ every integer n>13 can be written as n=3x_n+8y_n for some integers x_n,y_n ≥0 where x_n and y_n depend on n

is true. We work this by strong induction. For this let $Q(n)$ be the statement

$$P(14) \wedge \ldots \wedge P(n).$$

Base Case: We verify that $P(14)$ is true.

$$\begin{aligned} 14 &= 6+8 \\ &= 3(2)+8(1) \\ &= 3x_{14}+8y_{14}, \end{aligned}$$

with $x_{14}=2$ and $y_{14}=1$.
Induction Step: Let $k \in \mathbb{N} > 13$. We assume that $Q(k)$ is true. We want to show that $P(k+1)$ is true, which is the statement

$$k+1 = 3x_{k+1}+8y_{k+1}.$$

Manipulating the left hand side gives us

$$\begin{aligned} k+1 &= k-2+2+1 \\ &= 3x_{k-2}+8y_{k-2}+3, \quad \text{(Since we know that } P(k-2)\text{) is true)} \\ &= 3(x_{k-2}+1)+8y_{k-2} \\ &= 3x_{k+1}+8y_{k+1}, \end{aligned}$$

where $x_{k+1} = x_{k-2}+1$ and $y_{k+1} = y_{k-2}$. Hence $P(k+1)$ is true. Therefore the open sentence $P(n)$ is true for every integer $n > 13$. ■

b) Prove that 13 cannot be written as $3x+8y$ for any integers $x, y \geq 0$.
Proof: We suppose by contradiction that 13 can be written as $3x+8y$ for any integers $x, y \geq 0$. We know that $8 \cdot 2 = 16 > 13$. This gives us two cases for y. ■

Case 1. Assume $y=0$ so that $13=3x$. However since $3\cdot4 < 13 < 3\cdot5$, there is no integer x greater than or equal to 0 that satisfies this equation.

Case 2. Assume $y=1$ so that $13=3x+8$. Which means $5=3x$. However since $3\cdot1 < 5 < 3\cdot2$, there is not integer x greater than or equal to 0 that satisfies this equation.

Since neither case holds, the statement that 13 can be written as $3x+8y$ is false for any integers $x, y \geq 0$. Therefore, 13 cannot be written as $3x+8y$ for any integers $x, y \geq 0$.

Exercise 15.4. Let $n \in \mathbb{N}$. Prove by induction that $n = 2^{k_n} m_n$ for some nonnegative $k_n \in \mathbb{Z}$ and some odd $m_n \in \mathbb{N}$.

Proof: We want to show that the open sentence

$$P(n): \ n = 2^{k_n} m_n$$

where $n \in \mathbb{N}$, nonnegative $k_n \in \mathbb{Z} \geq 0$ and some odd $m_n \in \mathbb{N}$. We do this in two cases. The first case is if n is odd, and the second is if n is even.

Case 1. Let $k \in \mathbb{N}$, and assume that k is odd. Then $k = k = 2^0 k = 2^{k_n} m_n$ where $k_n = 0$ and $m_n = k$.

Case 2. Let ℓ be the even natural number and $Q(\ell)$ be the open sentence

$$P(2) \wedge P(4) \wedge \ldots \wedge P(\ell-2) \wedge P(\ell).$$

We assume that $Q(\ell)$ is true, and want to show that $P(\ell+2)$ is true, which is the statement

$$\ell+2=2^{k_{\ell+2}}m_{\ell+2}.$$

We do this by manipulating the left hand side

$$\begin{aligned}\ell+2&=2\alpha \quad \text{(For some } \alpha\in\mathbb{N} \text{ such that } \alpha<\ell \text{ and since } \ell+2 \text{ is an even natural number)}\\ &=2\left(2^{k_\alpha}m_\alpha\right) \quad \text{(Since we assum } P(\alpha))\\ &=2^{k_\alpha+1}m_\alpha,\end{aligned}$$

hence $P(\ell+2)$ is true. Therefore if ℓ is an even natural number, then $P(\ell)$ is true.

Since both cases are true. The open sentence $P(n)$ is true for all $n\in\mathbb{N}$.

■

Exercise 15.5. Prove that for each natural number $n>43$, we can write

$$n=6x_n+9y_n+20z_n$$

for some nonnegative integers x_n, y_n, z_n. Then prove that 43 cannot be written in this from.

Proof: We want to show that for each natural number $n>43$ the open sentence

$$P(n):\ n=6x_n+9y_n+20z_n$$

is true for some nonnegative integers x_n, y_n, z_n. We work this by induction.

Base Cases: We verify that $P(44)$, $P(45)$, $P(46)$, $P(47)$, $P(48)$, and $P(49)$ are true

$$\begin{aligned}P(44)&:44=6\cdot4+9\cdot0+20\cdot1\\ P(45)&:45=6\cdot0+9\cdot5+20\cdot0\\ P(46)&:46=6\cdot1+9\cdot0+20\cdot2\\ P(47)&:47=6\cdot1+9\cdot3+20\cdot1\\ P(48)&:48=6\cdot5+9\cdot2+20\cdot0\\ P(49)&:49=6\cdot0+9\cdot1+20\cdot2\end{aligned}$$

Induction Step: We assume that for each natural number $k\geq49$ the open sentence

$$Q:\ P(49)\wedge P(50)\wedge\ldots\wedge P(k),$$

is true. We wish to show that $P(k+1)$, which is the statement

$$k+1=6x_n+9y_n+20z_n,$$

is true. We begin by manipulating the left hand side

$$\begin{aligned}k+1&=(k-5)+6\\ &=6x_{k-5}+9y_{k-5}+20z_{k-5}+6\\ &=6\left(x_{k-5}+1\right)+9y_{k-5}+20z_{k-5},\end{aligned}$$

hence $P(k+1)$ is true. Therefore, $P(n)$ is true for each natural number $n>43$.

■

Next we prove that 43 cannot be written in the form described.

Proof: We assume by contradiction that $43=6x+9y+20z$ for some x,y,z nonnegative integers. We can easily see that z must be less than 3, since $20\cdot3=60>43$. This leaves us with three cases.

Case 1. Assume that $z = 2$ so that $43 = 6x + 9y + 40$ which can be written as $3 = 6x + 9y$ which isn't possible since $3 < 6 < 9$. So there is no nonnegative integer such that $3 = 6x + 9y$.

Case 2. Assume that $z = 1$ so that $43 = 6x+9y+20$ which can be written as $23 = 6x+9y$. From this we can see that $y < 3$ since $9 \cdot 3 = 27 > 23$. This gives us three cases.

- *Case* i. Assume that $y = 2$, then $23 = 6x + 18$ which can be written as $5 = 6x$. But 6 does not divide 5. So this is false.
- *Case* ii. Assume that $y = 1$, then $23 = 6x + 9$ which can be written as $14 = 6x$. But 6 does not divide 14. So this is false.
- *Case* iii. Assume that $y = 0$, then $23 = 6x$. But 6 does not divide 14 so this is false.

Since all sub cases are false, case 2 must be false.

Case 3. Assume that $z = 0$ so that $43 = 6x + 9y$. Note that $9 \cdot 2m = 6 \cdot 3m$ for some $m \in \mathbb{N}$. This means that for every even positive natural number value of y, the term $9y = 6x$ holds for some x. We also note since $9 \cdot 5 = 45 > 43$, y must be less than 5. These two facts give us to only have to look at the y for the values $\{0, 1, 3\}$.

- *Case* i. Assume $y = 3$ so that $43 = 6x+27$ which can be written as $16 = 6x$. Since 6 doesn't divide 16. This is false.
- *Case* ii. Assume $y = 1$ so that $43 = 6x + 9$ which can be written as $34 = 6x$. Since 6 doesn't divide 16. This is false.
- *Case* iii. Assume $y = 0$ so that $43 = 6x$. Since 6 doesn't divide 16. This is false.

Since all sub cases are false, case 3 must be false.

This all cases are false, the statement $43 = 6x + 9y + 20z$ for some x, y, z nonnegative integers is false. Therefore, $43 \neq 6x + 9y + 20z$ for some x, y, z nonnegative integers is false.

■

Exercise 15.6. Find the largest postage that cannot be paid that cannot be paid with 4, 10, and 15 stamps. Prove that your answer is correct.

This can be done by proving the statement two statements. The first is that every integer $n > 29$ can be written in the form $n = 4x_n + 10y_n + 15z_n$ for some x_n, y_n, z_n nonnegative integers where the subscript denotes dependency on n, and the second is that $21 \neq 4x_{30} + 10y_{30} + 15z_{30}$ for some x_{30}, y_{30}, z_{30} nonnegative integers. We begin by proving the first statement.

Proof: We want to show that the for all integers $n > 30$, the open sentence

$$P(n): \; n = 4x_n + 10y_n + 15z_n,$$

is true for some x_n, y_n, z_n nonnegative integers. We work this by induction.

Base Case: We verify that $P(22), P(23)$, $P(24)$, $P(25)$ are true.

$$\begin{aligned}
P(22) &: 22 = 4 \cdot 3 + 10 \cdot 1 + 15 \cdot 0 \\
P(23) &: 23 = 4 \cdot 2 + 10 \cdot 0 + 15 \cdot 1 \\
P(24) &: 24 = 4 \cdot 1 + 10 \cdot 2 + 15 \cdot 0 \\
P(25) &: 25 = 4 \cdot 0 + 10 \cdot 1 + 15 \cdot 1
\end{aligned}$$

Induction Step: Let $k \in \mathbb{N} \geq 25$, and the open sentence $Q(k)$ be

$$Q(k): \; P(25) \wedge \ldots \wedge P(k).$$

We assume that $Q(k)$ is true and want to show that $P(k+1)$ is true. We show this as follows:

$$\begin{aligned} k+1 &= (k-3)+4 \\ &= 4x_{k-3} + 10y_{k-3} + 15z_{k-3} + 4 \\ &= 4(x_{k-3}+1) + 10y_{k-3} + 15z_{k-3}, \end{aligned}$$

hence $P(k+1)$ is true. Therefore the open sentence $P(n)$ is true for all integers $n > 30$. ■

We now want to show that 21 cannot be written as $4x + 10y + 15z$ for some x, y, z nonnegative integers.

Proof: We assume by contradiction that $21 = 4x + 10y + 15z$ for some x, y, z nonnegative integers. Since $15 \cdot 2 > 21$, z must be smaller than 2. This gives us the following cases.

Case 1. Assume that $z = 1$ so that $21 = 4x + 10y + 15$ which can be written as $6 = 4x + 10y$. Since $10 > 6$, y must equal 0. This leaves us with $6 = 4x$. However, $4 \nmid 6$, thus this is false.

Case 2. Assume that $z = 0$ so that $21 = 4x + 10y$. We only have to consider the cases $y = \{0, 1, 2\}$ since $10 \cdot 3 > 29$. Also, since $4 \cdot 5 = 10 \cdot 2$ we really only need to consider the cases $y = \{0, 1\}$.

Case i. Assume $y = 1$ so that $21 = 4x + 10$ which can be written as $11 = 4x$. This is false since $4 \nmid 11$.

Case ii. Assume $y = 0$ so that $21 = 4x$. This is false since $4 \nmid 21$.

Since both sub cases are false, the second case is false.

Since both case are false statements, $21 \neq 4x + 10y + 15z$ or some x, y, z nonnegative integers.

■

Exercise 15.7. Recall the definition of the Fibonacci numbers from Exercise 14.6. Prove that every positive integer is a sum of one or more distinct Fibonacci numbers.

Proof: We wish to show that the open sentence

$P(n)$: every positive integer is a sum of one or more distinct Fibonacci numbers.

Let $Q(n)$ be the statement

$$Q(n): \ P(1) \wedge \ldots \wedge P(n).$$

We work this by induction.

Base Case: We verify $P(1)$.

$$1 = 1 = F_1,$$

where F_n is the n^{th} Fibonacci number in the sequence.

Induction Step: We assume that for every positive integer k that $Q(k)$ is true. We want to show that $P(k+1)$ is true, which is the statement that $k+1$ can be written as a sum of one or more distinct Fibonacci numbers. We know that for each $k > 0$, we can find some m so that $F_m \leq k+1 < F_{m+1}$. This allows us to write

$$k+1 = F_m + (k+1-F_m).$$

We have two cases to consider. When $k+1-F_m = 0$ and $k+1-F_m > 0$.

Case 1. Assume that $k+1-F_m = 0$, then $k+1 = F_m$.

Case 2. Assume that $k+1-F_m > 0$ so that $k+1 = F_m + y$ where $y = k+1-F_m$. Since $y < k+1$, we know that $y < F_m$, and since we assume that $Q(k)$ is true, $P(y)$ must be true. Since $F_m > y$, F_m cannot be a term in the sum of Fibonacci numbers that add up to y. Thus $F_m + y$ can be written as the sum of distinct Fibonacci numbers. Hence this is true.

Since both cases are true, the statement $P(k+1)$ is true. Therefore $P(n)$ is true for all positive integers n.

■