Homework 14

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Sections 26 and 27

26. Composition of Functions

Exercise 26.1. Let $f : A \to B$ and $g : B \to C$ be functions.

a) Prove that if f and g are injective, then $g \circ f$ is injective.
Proof: We assume that f and g are injective, then contrapositively if $g(x) = g(y)$ then $x = y$. We have two cases

Case 1. Let $x, y \in \text{im}(f)$, then there exists some $j, k \in A$ such that $x = f(j)$ and $y = f(k)$. Since $x = y$, then $f(j) = f(k)$, and because f is injective we know that $j = k$. Thus if $g(f(j)) = g(f(k))$ then $j = k$. Which shows that $g \circ f$ is injective.

Case 2. Let $x, y \notin \text{im}(f)$, then $g(x)$ and $g(y)$ are not in the image of $g \circ f$. Thus we do not need to consider them.

■

b) Prove that if $g \circ f$ is surjective, then g is surjective.
Proof: We suppose directly that $g \circ f : A \to C$ is surjective. Then for every $c \in C$, there exists an $a \in A$ such that $g \circ f(a) = c$. Since the function f maps elements from $A \to B$, we know that $f(a) \in B$. Let $f(a) = b$. Then there exists a $b \in B$ such that $g(b) = c$. This element b is simply $f(a)$. ■

Exercise 26.2. Let $f : A \to B$ be a function. Prove that $f \circ id_A = f$.

Proof: To show that $f \circ id_A = f$, we need to ensure that their domains and codomains are equal and that the sets of their relations are equal.

Domains and Codomains:

We suppose directly that $f : A \to B$ and that $id_A : A \to A$, then by the definition of the composition of functions $f \circ id_A : A \to A$. Thus their domains and Codomains are equal.

Equal Sets:

We suppose directly that the identity function maps as follows: $id_A(a) = a$ where $a \in A$, and we assume that $f(a) = b$ with $b \in B$. Then $f \circ id_A = f(id_A(a)) = f(a) = b$. This shows that you get the same output for every input; therefore, the functions are equal . ■

Exercise 26.3. Prove or disprove: If $f : A \to B$ and $g : B \to C$ are functions, and g is surjective, then $g \circ f$ is surjective.

Disproof: We wish to disprove this statement. We assume the negation, that there exists a $c \in C$ such that for all $a \in A$, $g \circ f(a) \neq c$. We show this by example. Let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 2\}$, f be the relation

$$f = \{(1, 1)\},$$

and g be the relation

$$g = \{(1, 1), (2, 2)\},$$

then there exists no $a \in A$ such that $g \circ f(a) = 2$. Therefore the statement is false. ■

Exercise 26.4. Prove or disprove: If $f : A \to B$ and $g : B \to C$ are functions, and $g \circ f$ is injective, then g is injective.

Disproof: We wish to disprove this statement by showing that there exists an $b_1, b_2 \in B$ such that $b_1 \neq b_2$ but $g(b_1) = g(b_2)$. We assume directly that $g \circ f$ is injective, and give an example to disprove the statement. Let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1\}$ and the functions be defined as

$$f = \{(1, 1)\},$$

$$g = \{(1, 1), (2, 1)\}$$

and

$$g \circ f = \{(1, 1)\}.$$

Since $g(1) = g(2)$, but $1 \neq 2$, g is not injective. Therefore, the statement is false. ∎

Exercise 26.5. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if f and g are both bijective, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: We assume directly that f and g are bijective functions, then we know that their inverses $f^{-1} : B \to A$ and $g^{-1} : C \to B$ exists and are bijective functions according to Theorem 26.20. According to Theorem 26.12, the composition of bijective functions is a bijective function. This means that $f^{-1} \circ g^{-1} : C \to A$ is a bijective function and has an inverse. It's relation is the set

$$f^{-1} \circ g^{-1} = \left\{ \left(f^{-1} \circ g^{-1}(c), c \right) : c \in C \right\}.$$

Let $c \in C$, $a \in A$ and $b \in B$ then

$$\begin{aligned} f^{-1} \circ g^{-1}(c) &= a \\ f^{-}(b) &= a, \end{aligned}$$

Thus $f(a) = b$, $g(b) = c$ and $g \circ f(a) = c$. In other words

$$\begin{aligned} \left(f^{-1} \circ g^{-1} \right) \circ (g \circ f) &= f^{-1} \circ g^{-1} \circ g \circ f, \\ &= f^{-1} \circ id_B \circ f \\ &= f^{-1} \circ f \\ &= id_A \end{aligned}$$

which shows that $g \circ f$ is the inverse of $f^{-1} \circ g^{-1}$. Therefore

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

∎

Exercise 26.6. Prove that the function $f : \mathbb{R} - \{5\} \to \mathbb{R} - \{3\}$ given by

$$f(x) = \frac{3x + 1}{x - 5}$$

is bijective. Find $f^{-1}(y)$ for $y \in \mathbb{R} - \{3\}$.

Proof: To show that the function is bijective, we need to show that it is both injective and surjective.

Injective: We suppose contrapositively that $f(a) = f(b)$ with $a, b \in \mathbb{R} - \{5\}$. Then

$$\begin{aligned}
\frac{3a+1}{a-5} &= \frac{3b+1}{b-5} \\
(3a+1)(b-5) &= (3b+1)(a-5) \\
3ab - 15a + b - 5 &= 3ba - 15b + a - 5 \\
-16a &= -16b \\
a &= b,
\end{aligned}$$

hence the function is injective.

Surjective: We suppose directly that $y \in \mathbb{R} - \{3\}$. Setting y to be equal to the output of the function and solving for x yields

$$\begin{aligned}
y &= \frac{3x+1}{x-5} \\
yx - y5 &= 3x + 1 \\
x(y-3) &= 5y + 1 \\
x &= \frac{5y+1}{y-3},
\end{aligned}$$

which is a valid element since $y \neq 3$. Plugging in x to $f(x)$ yields

$$f(x) = f\left(\frac{5y+1}{y-3}\right) = y,$$

therefore, f is surjective.

Since the function is both injective and surjective, the function is bijective. ■

Now that we have shown that the function is bijective, we know that it's inverse exists. Using our calculations in the surjective step of the proof we get

$$f^{-1} : \mathbb{R} - \{3\} \to \mathbb{R} - \{5\}$$

is define as

$$f^{-1} = \frac{5y+1}{y-3},$$

with $y \in \mathbb{R} - \{3\}$.

Exercise 26.7. Let $A = \{1, 2, 3\}$ and let $f : A \to A$ be given as

$$f = \{(1,2), (2,3), (3,1)\}.$$

a) Determine f^{-1}.

$$f^{-1} = \{(1,3), (2,1), (3,2)\}$$

b) Determine $f \circ f$.

$$\begin{aligned}
f \circ f &= \{(1,3), (2,1), (3,2)\} \\
&= f^{-1}.
\end{aligned}$$

c) Determine $f \circ f \circ f$.

$$\begin{aligned}
f \circ f \circ f &= f \circ f^{-1} \\
&= id_A
\end{aligned}$$

d) Define

$$f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}.$$

Determine f^n, as a collection of ordered pairs, for each natural number n.

$$f^n = \begin{cases} f & \text{if } n \equiv 1 \bmod 3 \\ f^{-1} & \text{if } n \equiv 2 \bmod 3 \\ id_A & \text{if } n \equiv 3 \bmod 3 \end{cases}$$

27. Additional Facts About Functions

Exercise 27.1. Prove theorem 27.2

Proof: There are two parts to the theorem. We first prove that if B is finite, then $|\text{im}(f)| \leq |B|$. We suppose directly that A and B are sets, $f : A \to B$ and that B is finite. Since f is a function, $\text{im}(f) = \{f(a) : a \in A\}$ and must be a subset of B. Since a subset of a set cannot have more elements than the set, we have that $|\text{im}(f)| \leq |B|$.

We now show that f is surjective if and only if $|\text{im}(f)| = |B|$ under the assumption that B is finite.

$(\Longrightarrow)$: We suppose directly that B is finite and that f is surjective. We already know that $|\text{im}(f)| \leq |B|$. However, since f is surjective, we know that for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Since $\text{im}(f) = \{f(a) : a \in A\}$, $\text{im}(f) = B$. Therefore $|\text{im}(f)| = |B|$.

$(\Longleftarrow)$: We suppose directly that $|\text{im}(f)| = |B|$. Since $f : A \to B$ we know that $\text{im}(f) \subseteq B$. A subset of a set cannot contain any elements that are not in the original set. So, since $|\text{im}(f)| = |B|$, $\text{im}(f)$ has as many elements as B, which means that $\text{im}(f) = B$. Thus, for every $b \in B$, there exists an element $a \in A$ such that $f(a) = b$. Therefore, f is surjective. ■

Exercise 27.2. Prove that the functions f_1 and f_2 defined in Example 27.12 are both bijections. We partition $\mathbb{Z} = \{P_1, P_2\}$, where P_1 is the set of positive integers, and P_2 is the set of nonpositive integers. We partition $\mathbb{N} = \{Q_1, Q_2\}$, where Q_1 is the set of even natural numbers, and Q_2 is the set of odd natural numbers.

a) Prove that the function $f_1 : P_1 \to Q_1$ defined as $f_1(n) = 2n$ is a bijective map.
   Proof: In order to show this, we must show that the function is injective and surjective.
   Injective:
   We suppose contrapositively that $f_1(x) = f_1(y)$ with $x, y \in P_1$. Then

$$\begin{aligned} f_1(x) &= f_1(y) \\ 2x &= 2y \\ x &= y, \end{aligned}$$

   hence it is injective.
   Surjective:
   We suppose directly that $a \in Q_1$, then we can write $a = 2n$. Solving for n gives us $n = \frac{a}{2}$. Since $a \in Q_1$, it is an even natural number and divisible by 2. Thus n is a natural number and an element of P_1. Therefore

$$f_1\left(\frac{a}{2}\right) = a.$$

   This shows that for every element of Q_1, there is an element $m \in P_1$ such that $f_1(m) = a$. Hence, f_1 is surjective.
   Since f_1 is surjective and injective, it is bijective. ■

b) Prove that the function $f_2 : P_2 \to Q_2$ defined as $f_2(n) = 1 - 2n$ is a bijective map.

Proof: In order to show this, we must show that the function is injective and surjective.

Injective:

We suppose contrapositively that $f_2(x) = f_2(y)$ with $x, y \in P_1$. Then

$$\begin{aligned} f_1(x) &= f_1(y) \\ 1 - 2x &= 1 - 2y \\ x &= y, \end{aligned}$$

hence it is injective.

Surjective:

We suppose directly that $a \in Q_2$, then we can write $a = 1 - 2m$ for some $m \in \mathbb{Z}$. Solving for m gives us $m = -\frac{a-1}{2}$. Since $a \in Q_2$, it is an odd natural number and can be written as $a = 2k + 1$ with $k \in \mathbb{Z} \geq 0$. Substituting this into $m = -\frac{a-1}{2}$ yields

$$\begin{aligned} m &= -\frac{2k+1-1}{2} \\ &= -\frac{2k}{2} \\ &- k \end{aligned}$$

which shows that m is really an element of P_2. Therefore

$$f_2\left(-\frac{a-1}{2}\right) = a.$$

This shows that for every element of Q_2, there is an element $j \in P_2$ such that $f_2(j) = a$. Hence, f_2 is surjective.

Since f_2 is surjective and injective, it is bijective. ■

Exercise 27.3. Give an example of a bijective function $f : \mathbb{Z} \to \{0, 1\} \times \mathbb{N}$ and include a proof that it is bijective.

We partition $\mathbb{Z}$ into P_1 and P_2 where P_1 are the positive integers and P_2 are the negative integers. We also partition $\{0, 1\} \times \mathbb{N}$ into the sets

$$Q_1 = \{(0, n) \, : \, n \in \mathbb{N}\}$$

and

$$Q_2 = \{(1, n) \, : \, n \in \mathbb{N}\}.$$

We then define the functions $f_1 : P_1 \to Q_1$ and $f_2 : P_2 \to Q_2$ as $f_1(a) = (0, a)$ and $f_2(b) = (1, -b + 1)$. Lastly we define the function

$$f = \begin{cases} f_1(x) & \text{if } x \in P_1 \\ f_2(x) & \text{if } x \in P_2 \end{cases}.$$

Proof: We wish to show that f is a bijective function. According tot he Pasting Together Theorem, if f_1 and f_2 are bijective maps whose domains form a partition of the domain of f, and whose codomains form a parition of the codomain of f, then f is a bijective function. Since $\mathbb{Z} = P_1 \cup P_2$ and $P_1 \cap P_2 = \emptyset$, the set $\{P_1, P_2\}$ is a partition of the domain$\mathbb{Z}$ and since $\{0, 1\} \times \mathbb{N} = Q_1 \cup Q_2$ and $Q_1 \cap Q_2 = \emptyset$, the set $\{Q_1, Q_2\}$ is a partition of the codomain $\{0, 1\} \times \mathbb{N} = Q_1 \cup Q_2$. All that is left to show is that f_1 and f_2 are bijective maps.

(f_1) : To show that f_1 is bijective, we need to show that it is injective and surjective.

Injective: We assume contrapositively that $f_1(a_1) = f_2(a_2)$ for some $a_1, a_2 \in P_1$. Then

$$\begin{aligned} f_1(a_1) &= f_2(a_2) \\ (0, a_1) &= (0, a_2), \end{aligned}$$

which shows that $a_1 = a_2$. Hence f_1 is injective.

Surjective: Let $(0,b) \in Q_1$. We wish to find an $a \in P_1$ such that $f_1(a) = (0,b)$. To do this we solve for a.

$$\begin{aligned}(0,b) &= f_1(a)\\ &= (0,a),\end{aligned}$$

thus by letting $a = b$ we get that $f_1(b) = (0,b)$. Hence f_1 is surjective.

(f_2) : To show that f_2 is bijective, we need to show that it is injective and surjective.

Injective: We assume contrapositively that $f_2(x_1) = f(x_2)$ for some $x_1, x_2 \in P_2$. Then

$$\begin{aligned}f_2(x_1) &= f(x_2)\\ (1, -x_1 + 1) &= (1, -x_2 + 1),\end{aligned}$$

which is only possible if $x_1 = x_2$. Hence f_2 is injective.

Surjective: Let $(1,y) \in Q_2$. We wish to fine a $x \in P_2$ such that $f_2(x) = (1,y)$. To do this we solve for x,

$$(1,y) = (1, -x+1)$$

which means that $y = -x + 1$. Solving for x yields $x = -y + 1$ which is an element of P_2. So, by letting $x = -y + 1$ we get

$$f_2(-y+1) = (1,y),$$

therefore, it is surjective.

Since f_2 is injective and surjective, it is bijective.

Since f_1 and f_2 are bijective whose domains and codomains form a partition of the domain and codomain of f, the function f is bijective.

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Exercise 27.4. Let $A = \{n \in \mathbb{Z} : -3 \leq n \leq 3\}$, and let $f : A \to \mathbb{Z}$ be defined by $f(x) = x^2 + 2x + 2$.

a) Write f as a set of ordered pairs.

$$f = \{(-3,5), (-2,2), (-1,1), (0,2), (1,5), (2,10), (3,17)\}$$

b) Find the image of f
   $\operatorname{im}(f) = \{1, 2, 5, 10, 17\}$
c) Find a subset C of A so that $f|_C$ is injective and $\operatorname{im}(f|_C) = \operatorname{im}(f)$
   $C = \{-1, 0, 1, 2, 3\}$

Exercise 27.5. Let $f : A \to B$ be an injective function, as let S be an arbitrary subset of A

a) Prove that $f|_S : S \to B$ is injective.
   Proof: We suppose directly that $S \subseteq A$. Since A is injective, then we know that for all $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$. Let $s_1, s_2 \in S$, then $s_1, s_2 \in A$. Therefore, if $s_1 \neq s_2$, then $f(a_2) \neq f(s_2)$. ■
b) Prove that $\hat{f}$ is a bijection.
   Proof: Let $T = \operatorname{im}(f)$ and $R = \{a \in A : f(a) \in T\}$. We suppose directly that $\hat{f} = f_{|R}$. Therefore, for every $t \in T$, there exists an $a \in A$ such that $f(a) = t$. This shows that $\hat{f}$ is surjective. Using the results from the previous proof, since f is injective and $R \subseteq A$, then $f|_R$ is also injective. Therefore, $\hat{f}$ is bijective. ■

Exercise 27.6. Let $f : A \to B$ be a function.

a) Prove that f is surjective if and only if $f^{-1}(\{b\}) \neq \emptyset$.
Proof: This is a biconditional statement, we must show both ways.
$(\implies)$: We assume directly that f is surjective. Then for all $b \in B$ there exists an $a \in A$ such that $f(a) = b$. Since there is at least one $a \in A$ such that $f(a) = b$, then the preimage of $\{b\}$ cannot cannot be empty since it contains a. This holds for any $b \in B$.
$(\impliedby)$: We assume directly that $f^{-1}(\{b\}) \neq \emptyset$ for all $b \in B$. Since the preimage is not empty, then there exists an $a \in A$ such that $f(b) = a$. Since this holds for all b, the function is surjective.
By proving both ways, we have shown that the biconditional statement is true. ■

Exercise 27.7. Let $f : A \to B$ be a function, and let $X, Y \subseteq A$ and $C, D \subseteq B$.

a) Prove or disprove: $f(X \cup Y) = f(X) \cup f(Y)$
Proof: We suppose directly that $f : A \to B$ and that $X, Y \subseteq A$.
$(\subseteq)$: Let $a \in f(X \cup Y)$, then $a = f(x)$ or $a = f(y)$ for some $x \in X$ and $y \in Y$. This means that $a \in f(X)$ or $a \in f(Y)$. In other words, $a \in f(X) \cup f(Y)$.
$(\supseteq)$: Let $b \in f(X) \cup f(Y)$, then $b \in f(X)$ or $b \in f(Y)$. This means that $b = f(x)$ or $b = f(y)$ for some $x \in X$ and $y \in X$. Then $b \in f(X \cup Y)$.
Since $f(X \cup Y) \subseteq f(X) \cup f(Y)$ and $f(X \cup Y) \supseteq f(x) \cup f(Y)$, then $f(X \cup Y) = f(X) \cup f(Y)$. ■
b) Prove or disprove: $f(X \cap Y) = f(X) \cap f(Y)$.
Disproof: We will show that there exists an element in $f(X) \cap f(Y)$ that is not in $f(X \cap Y)$. Let $X = \{1\}$, $Y = \{2\}$, $f(1) = 2$ and $f(2) = 2$, then

$$\begin{aligned} f(X \cap Y) &= f(\emptyset) \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} f(X) \cap f(Y) &= \{2\} \cap \{2\} \\ &\{2\}. \end{aligned}$$

Since $\{2\} \neq \emptyset$, this shows that there is an element in $f(X) \cap f(Y)$ that is not in $f(X \cap Y)$. ■
c) Prove or disprove: $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
Proof: Since this is an equality statement between sets, we will show that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$ and $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.
$(\subseteq)$: We suppose directly that $x \in f^{-1}(C \cup D)$, then $f(x) \in C \cup D$. This means that $f(x) \in C$ and/or $f(x) \in D$. Which is equivalent to $f(x) \in C \cup D$. In other words, $x \in f^{-1}(C \cup D)$. Thus $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.
$(\supseteq)$: We suppose directly that $x \in f^{-1}(C) \cup f^{-1}(D)$. Which means that $f(x) \in C$ and/or $f(x) \in D$. In other words, $f(x) \in C \cup D$. This is equivalent to $x \in f^{-1}(C \cup D)$. Thus $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.
Since both $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$ and $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$ are true, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$. ■
d) Prove or disprove: $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
Proof: Since this is an equality statement between sets, we will show that $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$ and $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.
$(\subseteq)$: We suppose directly that $x \in f^{-1}(C \cap D)$, then $f(x) \in C \cap D$. In other words, $f(x) \in C$ and $f(x) \in D$. This means that x is an element

of $f^{-1}(C)$ and $f^{-1}(D)$. Which is equivalent to $x \in f^{-1}(C) \cap f^{-1}(D)$. Thus $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.
$(\supseteq)$: We suppose directly that $x \in f^{-1}(C) \cap f^{-1}(D)$. Then $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. In other words, $f(x) \in C$ and $f(x) \in D$. Which means that $f(x) \in C \cap D$, or that $x \in f^{-1}(C \cap D)$. Thus $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.
Since $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$ and $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$, we know that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. $\blacksquare$