Homework 16

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Sections 30 and 31

30. Uncountable Sets

Exercise 30.1. Let $a, b \in \mathbb{R}$ with $a < b$. Construct a bijection $f : (0,1) \to (a,b)$, and prove it is a bijection.

Proof: Let f be a function defined as

$$f(x) = a + x \cdot (b-a),$$

and we wish to show that it is a bijection. This is done by proving that it is both injective and surjective.

Injective: We suppose contrapositively that $f(m) = f(k)$ for some $m, k \in (0,1)$, then

$$\begin{aligned} f(m) &= f(k) \\ a + m(b-a) &= a + k(b-a) \\ mb - ma &= kb - ka \\ b(m-k) &= a(m-k), \end{aligned}$$

since $a < b$, this equality holds only when $m - k = 0$. Thus, $m = k$. Therefore, f is injective.

Surjective: We suppose directly that $y \in (a,b)$, and we want to find a $z \in x(0,1)$ such that $f(z) = y$. We do this by solving for z as follows

$$\begin{aligned} y &= a + z(b-a) \\ y - a &= z(b-a) \\ \frac{y-a}{b-a} &= z, \end{aligned}$$

since $b - a > y - a > 0$, we know that $1 > \frac{y-a}{b-a} > 0$. Plugging in the expression for z into f yields

$$\begin{aligned} f(z) &= a + \frac{y-a}{b-a}(b-a) \\ &= a + y - a \\ &= y, \end{aligned}$$

thus, f is surjective. Since it is both surjective and injective, we know that f is a bijection. ■

Exercise 30.2. Prove that the interval $[0,1)$ has continuum cardinality, by creating a bijection $[0,1) \to (0,1)$.

Proof: We define a piecewise bijection $f : [0,1) \to (0,1)$. Let $S = \{1/(n+1) : n \in \mathbb{Z} \geq 0\} \subsetneq [0,1)$. Now define f by the rule

$$f(x) = \begin{cases} x & \text{if } x \notin S \\ 1/(n+2) & \text{if } x = 1/(n+1) \in S \end{cases}.$$

It is easy to see that f is a bijection from $[0,1) - S \to (0,1) - S$ (as it is essentially the identity function on this set). It is also a bijection from $S \to$

$(0,1) \cap S$. By pasting together we have a bijection. Thus, $[0,1)$ has continuum cardinality since $(0,1)$ has continuum cardinality. ■

Exercise 30.3. Prove that the interval $[0,1]$ has continuum cardinality.

Proof: Let $S = (0-\delta, 1+\delta)$ and $T = (0+\delta, 1-\delta)$ with $\delta \in \mathbb{R} - \{0\}$. Then $T \subseteq [0,1] \subseteq S$. Since S and T are both open intervals in $\mathbb{R}$, it has been proven that they have continuum cardinality. Since a set cannot have less elements than it's subset, nor can a subset have more elements than the set of which it is a subset, the set $[0,1]$ must have continuum cardinality. ■

Exercise 30.4. Prove that the irrational numbers are uncountable.

Proof: According to theorem 29.1. If S and T are countable sets, then $S \cup T$ is countable. The contrapositive of this is if $S \cup T$ is uncountable, then S or T is uncountable. The rational numbers $\mathbb{R}$ is the union of the irrational and rational numbers, i.e. $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ where $\mathbb{I}$ denotes the set of irrational numbers. Since $\mathbb{R}$ is an uncountable set, either $\mathbb{Q}$ and/or $\mathbb{I}$ is an uncountable set. It was already shown that $\mathbb{Q}$ is a countable set, thus $\mathbb{I}$ must be an uncountable set. ■

Exercise 30.5. Prove or disprove: The set $\mathbb{C}$ of complex numbers is uncountable.

Proof: Since $\mathbb{R} \subsetneq \mathbb{C}$, and $\mathbb{R}$ is an uncountable set, then according to theorem 30.6, the set $\mathbb{C}$ is uncountable. ■

Exercise 30.6. We defined a product of two sets A and B to be the collection of the ordered pairs from A and B.

Let $A_1, A_2, A_3, \ldots$ be sets. Define the product $\prod_{i=1}^{-\infty} A_i = A_1 \times A_2 \times A_3 \times \cdots$ to be the set of ordered sequences

$$\{(a_1, a_2, a_3, \ldots) : a_i \in A_i \text{ for each integer } i \geq 1\}.$$

We showed previously that a finite product of countable sets is countable. Show that the countable product $\prod_{i=1}^{\infty} \{0,1\} = \{0,1\} \times \{0,1\} \times \{0,1\} \times \cdots$ is not countable.

Proof: Let $f : \mathbb{N} \to \prod_{i=1}^{\infty} \{0,1\}$ and let

$$\text{dig}\,(i) = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{if } i = 0. \end{cases}$$

We will show that f is not surjective. Write $f(n)$ as the ordered tuple $(a_1, a_2, a_3, \ldots)$ where $a_i \in \{0,1\}$, and let $x \in \prod_{i=1}^{\infty} \{0,1\}$ be written as the ordered tuple $(b_1, b_2, b_2, \ldots)$ where $b_n = \text{dig}\,(a_n)$. In other words, the nth digit of x is the digit change of the nth digit of $f(n)$. Hence $x \neq f(n)$ for each $n \in \mathbb{N}$. Therefore f is not surjective, as x is not in the image.

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31. INJECTIONS AND CARDINALITIES

Exercise 31.1. Answer each of the following true or false problems, proving your answer.

a) Every uncountable set has the same cardinality as $(0, 1)$
Disproof: We know that $|\mathbb{R}| = |(0, 1)|$. From theorem 31.5 we know that $|\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$. This shows that the uncountable set $\mathcal{P}(\mathbb{R})$ has a larger cardinality than $(0, 1)$. ■
b) Let A and B be sets. If $A \subseteq B$, then $|A| \leq |B|$.
Proof: Let $f : A \to B$ be the identity function. Since $A \subseteq B$, we know that f is either surjective (if $A = B$) or nor surjective if $A \neq B$. If it is surjective, then $|A| = |B|$, and if it is injective, then $|A| \leq |B|$. In either case, we have that $|A| \leq |B|$. ■
c) For sets A and B, if $A \subsetneq B$, then $|A| < |B|$.
Disproof: Consider the sets $\mathbb{N}$ and $\mathbb{Z}$ that have the relation $\mathbb{N} \subsetneq \mathbb{Z}$; however, it has been shown that $|\mathbb{N}| = |\mathbb{Z}|$. This example proves that the statement is false. ■
d) Given sets A, B, and C, if $A \subseteq B \subseteq C$ and both A and C are countably infinite, then B is countably infinite.
Proof: Since B is a subset of C, there exists an injective function $f : B \to C$ that is the identity function, thus $|B| \leq |C|$. Also, since $A \subseteq B$, then there exists an injective function $g : A \to C$ that can be the identity function, thus $|A| \leq |C|$. Hence $|A| \leq |B| \leq |C|$. Which shows that B is countably infinite. ■
e) No subsets of $\mathbb{R}$ has smaller cardinality than $\mathbb{R}$.
Disproof: We disprove this statement with a simple example. Consider the empty set $\emptyset \subsetneq \mathbb{R}$. Surely $|\emptyset| < |\mathbb{R}|$. Thus the statement is false. ■
f) For sets S and T, if $|S| < |T|$ and S is finite, then T is infinite.
Disproof: We disprove this statement with a counterexample. Let $S = \emptyset$ and $T = \{0\}$, then $|S| < |T|$, but T is not infinite. This is a contradiction to the statement, thus the statement is false. ■
g) For sets S and T, if $|S| < |T|$ and S is countable, then T is uncountable.
Disproof: We disprove this statement with a counterexample. Let $S = \emptyset$ and $T = \{0\}$, then $|S| < |T|$, S and T are both finite sets and are thus both countable. This is a contradiction to the statement, thus it is false. ■
h) For sets S and T, if $|S| < |T|$ and S is countably infinite, then T is uncountable.
Proof: A countably infinite set has cardinality $\aleph_0$, the next largest cardinality is $\aleph_1$ which pertains to a set that is uncountable. Since $|S| < |T|$, then T must be uncountable since $|T| \geq \aleph_1$. ■
i) For any set S, there exists another set T such that $|S| < |T|$.
Proof: We assume directly that S is a set, then according to theorem 31.5, $|S| < |\mathcal{P}(S)|$. By letting $T = \mathcal{P}(S)$, we have shown that there exists another set T such that $|S| < |T|$. ■

Exercise 31.2. Let $S = \{a, b, c, d, e\}$ and let $g : S \to \mathcal{P}(S)$ be defined by the rule $g(a) = \{b, d\}$, $g(b) = \{a, c, e\}$, $g(c) = \{a, c, d, e\}$, $g(d) = \emptyset$, $g(e) = \{e\}$. List the elements of the barber set $B = \{s \in S : s \notin g(s)\}$. Why is it not in the image of g?

$B = \{a, b, d\}$.

Proof: We suppose by contradiction that $B = \{s \in S\, s \notin g(s)\}$ and that $B = g(x)$ for some $x \in S$. Then there are two cases:

Case 1. Let $x \in B$, this can't be the case by the definition of B.
Case 2. Let $x \notin B$, well then $x \in B$ according to the definition of B.

Since both cases lead to a contradiction, the statement must be false. Thus B can never be in the image of g.

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Exercise 31.3. Find a set with cardinality bigger than that of $\mathbb{R}$. Then find a set with cardinality bigger than that.

The power sets $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}(\mathcal{P}(\mathbb{R}))$.

Exercise 31.4. Theorem 27.5 says that for finite sets A and B, if $|A| = |B|$ and $f : A \to B$ is a function, then f is injective if and only if f is surjective. Prove that this fails for infinite sets, by proving the following.

a) Find and infinite set S and a function $f : S \to S$ that is injective but not surjective.

   Let $S = \mathbb{Z}$ and consider the piecewise function

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 2(-x) + 1 & \text{if } x < 0 \end{cases}$$

   which is injective but not surjective. (The problem doesn't say that I have to prove it)

b) Find an infinite set S and a function $g : S \to S$ that is surjective but not injective.

   Let $S = \mathbb{Z}$ and consider the piecewise function

$$g(x) = \begin{cases} x - 1 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$$

   which is surjective and not injective.

Exercise 31.5. Let A and B be sets with $f : A \to B$ a bijection. Define a new map $g : \mathcal{P}(A) \to \mathcal{P}(B)$ by the rule $g(S) = \{f(s) : s \in S\}$, where $S \subseteq A$ is an arbitrary element of $\mathcal{P}(A)$. Prove that g is a bijection. Conclude that if $|A| = |B|$ then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

Proof: To show that the function g is a bijection, we must show that it is injective and surjective.

Injective: We suppose contrapositively that $g(M) = g(N)$ for some $M, N \in \mathcal{P}(A)$, then

$$\begin{aligned} g(M) &= g(N) \\ \{f(m) : m \in M\} &= \{f(n) : n \in N\}, \end{aligned}$$

since the function f is a bijection, $f(m) = f(n)$ only if $m = n$. Thus, for $g(M) = g(N)$, M must equal N. Therefore, g is injective.

Surjective: Let $X \in \mathcal{P}(B)$, then $X \subseteq B$ and there exists a corresponding $Y \subseteq A$ such that $f(A) = B$ since f is bijective. Thus $g(Y) = \{f(y) : y \in Y\} = B$. Therefore, g is surjective.

Since g is both surjective and injective, then it is bijective. And because there is a bijection from $\mathcal{P}(A) \to \mathcal{P}(B)$ we see that if $|A| = |B|$ then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$. ■

Example 31.6. Define a function $f : \mathbb{R} \to \mathcal{P}(\mathbb{Q})$ by the rule

$$f(x) = \{q \in \mathbb{Q} : q \leq x\}.$$

Prove that f in injective.

Proof: We suppose contrapositively that $f(m) = f(n)$ for some $m, n \in \mathbb{R}$, then

$$f(m) = f(n)$$
$$\{q \in \mathbb{Q} : q \le m\} = \{q \in \mathbb{Q} : q \le n\},$$

Because there is a rational number between any two real numbers $x < y$, and the two sets contain all of the same elements, $m = n$. Therefore f is injective. ■

Since f is injective, $|\mathbb{R}| \le \mathcal{P}(\mathbb{Q})$. We know that $|\mathbb{Q}| = |\mathbb{N}|$. Using the result from the previous exercise, we get that $|\mathcal{P}(\mathbb{Q})| = |\mathcal{P}(\mathbb{N})|$. Therefore $|\mathbb{R}| \le |\mathcal{P}(\mathbb{N})|$.

Example 31.7. Let A and B be nonempty sets. Prove that there exists an injection $f : A \to B$ if and only if there exists a surjection $g : B \to A$.

Proof: Since the statement is biconditional, we must show both ways.

$(\implies)$: We suppose directly that $f : A \to B$ is an injection. Then $|A| \le |B|$. Since the cardinality of B is greater or equal to the cardinality of A, then there exists a function $g : B \to A$ that is surjective. The function g can be a piecewise function that includes $\hat{f}^{-1}$.

$(\impliedby)$: We suppose directly that $g : B \to A$ is surjective. Then for all $a \in A$, there exists a $b \in B$ such that $g(b) = a$. We can then define a function $f : A \to B$ by the rule that $f(a) =$one of the elements which mapped to a. And f would be an injective map.

Since both implications are true, the biconditional statement is true. ■