Mark Petersen

Sections 10 and 11

10. Proofs in Set Theory

Exercise 10.1. For each element and set listed below, explain why the element does or does not belong to the set.

a) Is $3 \in \{1, 2, 3, 4, 5, 6, 7\}$?
   Yes, the integer 3 is listed in the set.
b) Is $\pi \in \{1, 2, 3, 4, 5, 6, 7\}$?
   No, π is not listed in the set
c) Is $\pi \in \mathbb{R}$?
   Yes, π is an irrational number, and the set of irrational numbers is a subset of $\mathbb{R}$. Thus $\pi \in \mathbb{R}$.
d) Is $\frac{2}{3} \in \{x \in \mathbb{R} : x < 1\}$?
   Yes, the set $\{x \in \mathbb{R} : x < 1\}$ is the set of all real numbers less than 1. Since $\frac{2}{3} < 1$ and is a real number, it is in the set.
e) Is $\frac{2}{3} \in \{x \in \mathbb{Z} : x < 1\}$?
   No, because $\frac{2}{3}$ is not an integer.

Exercise 10.2. Let

$$A = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 4 \mid (x - y)\}$$

and let

$$B = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \text{ and } y \text{ have the same parity}\}.$$

Prove $A \subseteq B$.

Proof: We suppose directly that $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ and that $4 | (x - y)$. This means that there exists an integer k such that $x - y = 4k = 2(2k)$, or in other words $x - y$ is even. We need to show that the difference of two integers is even if and only if they have the same parity.

$(\implies)$: We suppose contrapositively that two integers a and b do not have the same parity. Without loss in generality let $a = 2l + 1$ and $b = 2m$ for some $l, m \in \mathbb{Z}$. Then

$$\begin{aligned} a - b &= 2l + 1 - 2m \\ &= 2(l - m) + 1, \end{aligned}$$

which is odd. Thus if the difference of two integers is even, then the two integers have the same parity.

$(\impliedby)$: We suppose directly that two integers a and b have the same parity. Then a and b are either both even or odd. Thus we have two cases.

Case 1. Let a and b be even integers such that $a = 2l$ and $b = 2m$ for some $l, m \in \mathbb{Z}$, Then

$$\begin{aligned} a - b &= 2l - 2m \\ &= 2(l - m), \end{aligned}$$

which is even.

Case 2. Let a and b be odd integers such that $a = 2l + 1$ and $b = 2m + 1$ for some $l, m \in \mathbb{Z}$, then

$$\begin{aligned} a - b &= 2l + 1 - 2m - 1 \\ &= 2(l - m), \end{aligned}$$

which is even.

Thus if two integers have the same parity, then their difference is even.

Returning to the main argument of the proof. Since $x - y$ is even, the integers x and y have the same parity and are thus a subset of B. ■

Exercise 10.3. Let X be the set of integers which are congruent to -1 modulo 6 ($X = \{x \in \mathbb{Z} : x \equiv -1 \mod 6\}$) and let Y be the set of integers which are congruent to 2 modulo 3 ($Y = \{y \in \mathbb{Z} : y \equiv 2 \mod 3\}$. Prove $X \subseteq Y$.

Proof: Suppose directly that $x \in \mathbb{Z}$ such that $x \equiv -1 \mod 6$. If follows that $6 \mid x+1$, or equivalently $x = 6k-1$ for some $k \in \mathbb{Z}$. This statement can be modified as follows

$$\begin{aligned} x &= 6k - 1 \\ &= 6k - 1 - 2 + 2 \\ &= 3(2k-1) + 2 \\ &= 3m + 2, \end{aligned}$$

for some $m \in \mathbb{Z}$. Hence $3 \mid x - 2$ or equivalently $x \equiv 2 \mod 3$. Therefore if $x \in X$, then $x \in Y$ and $X \subseteq Y$. ■

Exercise 10.4. Let A and B be sets inside some universal set U.

a) Prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.
Proof: We suppose directly that $x \in \overline{A \cap B}$. Thus $\neg(x \in A \cap B)$. Using De Morgan's law we have $x \in \overline{A}$ or $x \in \overline{B}$. In other words $x \in \overline{A} \cup \overline{B}$. ■
b) Prove that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.
Proof: We suppose directly that $x \in \overline{A} \cup \overline{B}$, which means that $x \in \overline{A}$ or $x \in \overline{B}$. Using De Morgan's law we have $x \not\ni A \cap B$. This is equivalent to $x \in \overline{A \cap B}$. ■
c) Putting the two previous parts together, we have proved that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Exercise 10.5. Let X and Y be sets. Prove $X - (X - Y) \subseteq X \cap Y$.

Proof: Assume directly that $m \in X - (X - Y)$, then $m \in X$ and $m \notin X - Y$. From $m \notin X - Y$, we can deduce that $m \notin X$ or $m \in Y$. However, since $m \in X$, m must be an element of Y in order for $m \notin X - Y$. Hence m is in X and Y and $X - (X - Y) \subseteq X \cap Y$. ■

Exercise 10.6. Given a set X, show that $X \cup \emptyset = X$.

Proof: Assume directly that $m \in X \cup \emptyset$, which means that $m \in X$ or $m \in \emptyset$. Since the empty set doesn't have any elements, m must be an element of X. Thus $X \cup \emptyset = X$. ■

Exercise 10.7. Let $n \in \mathbb{Z}$. Prove that

$$\{x \in \mathbb{Z} : n \mid x\} = \{x \in \mathbb{Z} : x \equiv 0 \mod n\}.$$

Proof: This is a biconditional statement so we prove both ways.
($\subseteq$) : We directly suppose that $x \in \mathbb{Z}$ and that $n \mid x$. This is equivalent to $x = nk$ for some $k \in \mathbb{Z}$, which is also equivalent to $x - 0 = nk$ for some $k \in \mathbb{Z}$. In other words $n \mid x - 0$. Hence $x \equiv 0 \mod n$. Hence $\{x \in \mathbb{Z} : n \mid x\} \subseteq \{x \in \mathbb{Z} : x \equiv 0 \mod n\}$.
($\supseteq$) : We directly suppose that $x \in \mathbb{Z}$ and that $x \equiv 0 \mod n$, which can be stated as $n \mid x - 0$ or in other words $x - 0 = nk$ for some $k \in \mathbb{Z}$. This statement can be reduced to $x = nk$ and equivalently stated as $n \mid x$. Hence $\{x \in \mathbb{Z} : x \equiv 0 \mod n\} \subseteq \{x \in \mathbb{Z} : n \mid x\}$.
By proving both ways, we have shown $\{x \in \mathbb{Z} : n \mid x\} = \{x \in \mathbb{Z} : x \equiv 0 \mod n\}$. ■

Exercise 10.8. Let A, B, and C be sets. Prove

$$A - (B \cap C) \subseteq (A - B) \cup (A - C).$$

Is the other inclusion true?

Proof: We suppose directly that $m \in A - (B \cap C)$ which indicates that $m \in A$ and $m \notin B \cap C$. In other words $m \in A$ and $m \in \overline{B \cap C}$ ($m \in A \cap \overline{B \cap C}$). Using De Morgans law proved in exercise 10.4 we can say that $m \in A \cap \left(\overline{B} \cup \overline{C}\right)$. Using the distributive law we get $m \in \left(A \cap \overline{B}\right) \cup \left(A \cap \overline{C}\right)$. Which is equivalent to $(A - B) \cup (A - C)$. Since we have only used equivalent expressions in deriving this proof, $A - (B \cap C) = (A - B) \cup (A - C)$. ■

Exercise 10.9. Prove that $\cup_{n \in \mathbb{Z}} \{m \in \mathbb{Z} : m \leq n\} = \mathbb{Z}$.

Proof: Since this is a set equality we must prove that $\cup_{n \in \mathbb{Z}} \{m \in \mathbb{Z} : m \leq n\} \subseteq \mathbb{Z}$ and $\cup_{n \in \mathbb{Z}} \{m \in \mathbb{Z} : m \leq n\} \supseteq \mathbb{Z}$. To show this, set S_n denote the set $\{m \in \mathbb{Z} : m \leq n\}$ for some $n \in \mathbb{Z}$.

$(\subseteq)$: Suppose directly that $x \in \cup_{n\in\mathbb{Z}} S_n$, by the construction of the set $\{m \in \mathbb{Z} : m \leq n\}$, x must be an element of $\mathbb{Z}$.

$(\supseteq)$: Suppose directly that $x \in \mathbb{Z}$, then for all $x \in \mathbb{Z}$ there is some set S_n for which $x \in S_n$. If we let $n = x$ then $x \in S_n$ and must also be an element of $\cup_{n\in\mathbb{Z}} S_n$.

The set $S(n) = \{m \in \mathbb{Z} : m \leq n\}$, is the set of all integers that are less than or equal to the integer n such that $S(n) \subseteq \mathbb{N}$. Since we are taking the union of $S(n)$ for all $n \in \mathbb{Z}$ one of the sets $S(n)$ must be equal to $\mathbb{Z}$. Hence the union of these sets must include the union with $\mathbb{Z}$ which is the largest set. ■

11. Existence Proofs and Counterexamples

Exercise 11.1. Prove the following

a) There exists $a, b \in \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.
Proof: Let $a = b = 1$, then $a^b = 1^1 = 1$ which is an element of $\mathbb{Q}$. ■
b) There exists $a, b \in \mathbb{Q}$ such that $a^b \in \mathbb{R} - \mathbb{Q}$.
Proof: Let $a = 2$ and $b = \frac{1}{2}$, then $a^b = 2^{(1/2)} = \sqrt{2}$ which is an irrational number according to a proof in a previous assignment. ■
c) There exists $a, b \in \mathbb{R} - \mathbb{Q}$ such that $a^b \in \mathbb{R} - \mathbb{Q}$.
Proof: We note that $\sqrt{2}^{\sqrt{2}} \cdot \sqrt{2}^{1-\sqrt{2}} = \sqrt{2}$. This implies that either $\sqrt{2}^{\sqrt{2}}$ or $\sqrt{2}^{1-\sqrt{2}}$ must be irrational since $\sqrt{2}$ is irrational and the product of two rational numbers is rational. Hence $a = \sqrt{2}$ and b is either $\sqrt{2}$ or $1 - \sqrt{2}$. ■
d) There exists $a \in \mathbb{Q}$ and $b \in \mathbb{R} - \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.
Proof: We show this with a non constructive proof. ■
Case 1. Let $a = 2$ and $b = \sqrt{2}$, if $a^b \in \mathbb{Q}$ then the statement is true.
Case 2. Let $a = 2^{\sqrt{2}}$ and $b = \sqrt{2}$, then $a^b = 2^{\sqrt{2} \cdot \sqrt{2}} = 2^2 = 4$ which is a rational number.
e) There exists $a \in \mathbb{Q}$ and $b \in \mathbb{R} - \mathbb{Q}$ such that $a^b \in \mathbb{R} - \mathbb{Q}$.
Proof: We show this non-constructively. ■
Case 1. Let $a = 2$ and $b = \frac{1}{\sqrt{2}}$, if $a^b \in \mathbb{R} - \mathbb{Q}$, then the statement is true. Otherwise, $2^{\frac{1}{\sqrt{2}}}$ is rational
Case 2. If $2^{\frac{1}{\sqrt{2}}}$ is rational, let $a = 2^{\frac{1}{\sqrt{2}}}$ and $b = \frac{1}{\sqrt{2}}$, then $a^b = 2^{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}} = 2^{\frac{1}{2}} = \sqrt{2}$ which is an irrational number.
f) There exists $a \in \mathbb{R} - \mathbb{Q}$ and $b \in \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.
Proof: Let $a = \sqrt{2}$ and $b = 2$, then $a^b = \sqrt{2}^2 = 2$ which is a rational number. ■
g) There exists $a \in \mathbb{R} - \mathbb{Q}$ and $b \in \mathbb{Q}$ such that $a^b \in \mathbb{R} - \mathbb{Q}$.
Proof: Let $a = \sqrt{2}$ and $b = 1$, then $a^b = \sqrt{2}^1 = \sqrt{2}$ which is an irrational number. ■

Exercise 11.2. Prove or disprove: given $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$, then $xy \in \mathbb{R} - \mathbb{Q}$.

Disproof: Let $x = 0$ then $xy = 0$ which is a rational number and hence not in $xy \in \mathbb{R} - \mathbb{Q}$. ■

Exercise 11.3. Prove or disprove: Let $s \in \mathbb{Z}$. If $6s - 3$ is odd, then s is odd.

Disproof: We suppose that s is even and that $6s - 3$ is odd. Since s is even, it can be written as $s = 2k$ for some $k \in \mathbb{Z}$. Substituting this into $6s - 3$ yields

$$\begin{aligned} 6s - 3 &= 6(2k) - 3 \\ &= 2(6k) - 4 + 1 \\ &= 2(6k - 2) + 1, \end{aligned}$$

which is odd. Hence the statement if $6s - 3$ is odd, then s is odd, is false. ■

Exercise 11.4. Prove or disprove: There exists an integer x such that $x^2 + x$ is odd.

Disproof: We show that if $x \in \mathbb{Z}$, then $x^2 + x$ is even. In fact this statement is trivially true. We show this by factoring $x^2 + x$

$$\begin{aligned} x^2 + x &= x(x + 1) \\ &= xy, \end{aligned}$$

where $y = x + 1$ and has a different parity than x. According to lemma A.1, the product of two integers with different parity is an even integer. This means that since x and y have different parity, then xy is even. Hence $x^2 + x$ is always even. ■

Exercise 11.5. Prove or disprove: Given any positive rational number a, there is an irrational number $x \in (0, a)$.

Proof: We suppose directly that $a \in \mathbb{Q}$ and that $a > 0$. Let y be a positive irrational number and let x be a positive rational number such that $x > y$. According to lemma A.3, $xy \in \mathbb{R} - \mathbb{Q}$, and is positive. Since $x > y$,

$0 < \frac{y}{x} < 1$, and $0 < \frac{y}{x}a < a$. This means that $\frac{y}{x}a$ is an irrational number in the interval $(0, a)$. This is only possible since there is no smallest positive rational number according to lemma A.2. ■

Exercise 11.6. Prove that for any two real numbers $x < y$, there exists a rational number in the interval (x, y).

Proof: We suppose directly that $y > x$. According to lemma A.5, there exists a number $\ell \in \mathbb{Z}$ such that $10^{-\ell+1}y$ is more than 1 away from $10^{-\ell+1}x$. This means that there is an integer $z \in \mathbb{Z}$ such that $10^{-\ell+1}y > z > 10^{-\ell+1}x$. Multiplying everything by $10^{\ell-1}$ yields

$$y > z \cdot 10^{\ell-1} > x,$$

where $z \cdot 10^{\ell-1}$ is a rational number.

■

APPENDIX

Lemma A.1. *The product of two integers with different parity is an even integer.*

Proof: We suppose directly that $x = 2k$ and $y = 2m + 1$ for some $k, m \in \mathbb{Z}$, then

$$\begin{aligned} xy &= 2k\left(2m+1\right) \\ &= 2\left(2km + k\right), \end{aligned}$$

which is even. ■

Lemma A.2. *There is no smallest positive rational number.*

Proof: We suppose by contradiction that $z \in \mathbb{Q}$ is the smallest positive rational number. Let $a = \frac{1}{b}$ for some $b \in \mathbb{Z} - \{0\}$, then $az = \frac{z}{b}$ which is smaller than z which contradicts our assumption that z is the smallest positive rational number. Hence there is no smallest positive rational number. ■

Lemma A.3. *The product of a nonzero rational number and an irrational number is irrational.*

Proof: We assume by contradiction that $x \in \mathbb{Q} - \{0\}$, $y \in \mathbb{R} - \mathbb{Q}$ and $xy \in \mathbb{Q}$. The rational number x can be written as $\frac{a}{b}$ for some $a, b \in \mathbb{Z} - \{0\}$, and the rational number xy can be written as $\frac{c}{d}$ for some $c \in \mathbb{Z}$ and $d \in \mathbb{Z} - \{0\}$. Solving for y yields

$$\begin{aligned} y &= xy/x \\ &= \frac{c}{d}\frac{b}{a} \\ &= \frac{cb}{da}, \end{aligned}$$

which is rational. This is contradictory since y was stated to be irrational. Therefore if $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$, then $xy \in \mathbb{R} - \mathbb{Q}$. ■

Lemma A.4. *For any two real numbers x and y where $y > x$, their difference $y - x$ has at least one nonzero decimal digit.*

Proof: Suppose directly that $y > x$, then $y - x > 0$. The decimal expansion of $y - x$ can be written as $d_k d_{k-1} \dots d_1 d_0 . d_{-1} d_{-2} \dots$ where $d_k = 10^k m$ for some $m \in \mathbb{Z}$. ■

Lemma A.5. *For any two real numbers x and y where $y > x$, there is a number ℓ such that $10^{-\ell+1} y$ is more than 1 away form $10^{-\ell+1}$.*

Proof: Suppose directly that $y > x$. From lemma A.4, we know that $y - x$ can be written as $d_k d_{k-1} \dots d_1 d_0 . d_{-1} d_{-2} \dots$. Let d_ℓ be the non-zero decimal digit with the greatest index ℓ, e.g. $d_\ell = d_k$ if $d_k \neq 0$. Since $d_\ell = m 10^\ell$ for some $m \in \mathbb{Z}$, $d_\ell > 10^{\ell-1}$. Therefore $y - x > 10^{\ell-1}$. Multiplying both sides by $10^{-\ell+1}$ yields

$$10^{-\ell+1} y - 10^{-\ell+1} x > 1.$$

This shows that $10^{-\ell+1} y$ is more than 1 away from $10^{-\ell+1} x$. ■