Homework 18

Mark Petersen

Sections 33,34

33. Sequences

Exercise 33.1. Write the first six terms, and determine the nth term a_n, for each of the following sequences.

a) An arithmetic sequence with first term 5 and common difference -3.
   $5, 2, -1, -4, -7, -10$; $a_n = 5 - 3\,(n-1)$
b) A geometric sequence with first term 4 and common ration 2.
   $4, 8, 16, 32, 64, 128$; $a_n = 4 \cdot 2^{n-1}$
c) An arithmetic sequence with first term $\frac{1}{2}$ and common difference $\frac{3}{4}$
   $\frac{2}{4}, \frac{5}{4}, \frac{8}{4}, \frac{11}{4}, \frac{14}{4}, \frac{17}{4}$; $a_n = \frac{1}{2} + \frac{3}{4}\,(n-1)$
d) A geometric sequence with first term $\frac{3}{5}$ and common ration $\frac{2}{3}$.
   $\frac{3}{5}, \frac{6}{15}, \frac{12}{45}, \frac{24}{135}, \frac{48}{405}, \frac{96}{1215}$; $a_n = \frac{3}{5} \cdot \left(\frac{2}{3}\right)^{n-1}$

Exercise 33.2. Translate the following phrases into symbolic logic.

a) The sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = 3 - 4/n$ converges to $L = 3$.
   $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |3 - 4/n - 3| < \epsilon.$
b) The sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = 6$ does not converge to $L = 3$.
   $\exists \epsilon \in \mathbb{R} > 0, \forall N \in \mathbb{R}, \exists n \in \mathbb{N}, (n > N) \wedge (|6-3| \geq \epsilon)$

Exercise 33.3. Let $a, b, x \in \mathbb{R}$. Prove the following.

a) $\max(a,b) \geq a$ and $\max(a,b) \geq b$.
   Proof: We suppose directly that the function $\max(a,b)$ is defined as
   $$\max(a,b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b > a \end{cases},$$
   then the image of max (a,b) is either a or b. This gives us two cases.
   Case 1. Let the output be a, then $a \geq b$, thus $\max(a,b) \geq a$ and $\max(a,b) \geq b$.
   Case 2. Let the output be b, then $b > a$, thus $\max(a,b) \geq a$ and $\max(a,b) \geq b$.

   ■

   Since both cases hold, the statement is true.
b) $\min(a,b) \leq a$ and $\min(a,b) \leq b$.
   Proof: We suppose directly that the function $\min(a,b)$ is defined as
   $$\min(a,b) = \begin{cases} a & \text{if } a \leq b \\ b & \text{if } b < a \end{cases},$$
   then the image of $\min(a,b)$ is either a or b depending on which one is smaller. This gives us two cases.
   Case 1. Let $\min(a,b) = a$, then $a \leq b$, thus $\min(a,b) \leq a$ and $\min(a,b) \leq b$.
   Case 2. Let $\min(a,b) = b$, then $b < a$, thus $\min(a,b) < a$ and $\min(a,b) \leq b$.

■

Since both cases hold, the statement is true.

c) If $x > \max(a, b)$, then $x > a$ and $x > b$.

We suppose directly that the function $\max(a, b)$ is defined as in part a), and that $x > \max(a, b)$. Since $\max(a, b) \geq a$ and $\max(a, b) \geq b$, we then have that $x > \max(a, b) \geq a$ and $x > \max(a, b) \geq b$. In other words, $x > a$ and $x > b$. Thus the statement is true.

Exercise 33.4. Prove that
$$\lim_{n\to\infty} \frac{2}{n^2} = 0.$$

a) Scratch work:

$$\begin{aligned} |a_n - L| &< \epsilon \\ \left|\frac{2}{n^2} - 0\right| &< \epsilon \\ \frac{2}{n^2} &< \epsilon \\ \frac{2}{\epsilon} &< n^2 \\ \sqrt{\frac{2}{\epsilon}} &< n, \end{aligned}$$

thus we want $N = \sqrt{\frac{2}{\epsilon}}$

Proof: Let $\epsilon \in \mathbb{R} > 0$, and we let $N \in \mathbb{R}$ to be $N = \sqrt{\frac{2}{\epsilon}}$. We suppose directly that $n \in \mathbb{N}$ is greater than N, then

$$\begin{aligned} |a_n - L| &= \left|\frac{2}{n^2} - 0\right| \\ &= \frac{2}{n^2} \\ &< \frac{2}{\sqrt{\frac{2}{\epsilon}}^2} \\ &= \frac{2}{\frac{2}{\epsilon}} \\ &= \epsilon, \end{aligned}$$

thus $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$. Therefore, the limit is 0. ■

Exercise 33.5. Prove that
$$\lim_{n\to\infty} \frac{3n-5}{2n+4} = \frac{3}{2}.$$

a) Scratch work:=

$$\begin{aligned} \left|\frac{3n-5}{2n+4} - \frac{3}{2}\right| &= \left|\frac{2(3n-5) - 3(2n+4)}{2(2n+4)}\right| \\ &= \left|\frac{6n - 10 - 6n - 12}{2(2n+4)}\right| \\ &= \left|\frac{-24}{4n+8}\right| \\ &= \frac{24}{4n+8}, \end{aligned}$$

we want to solve for n such that $\frac{24}{4n+8} < \epsilon$ as follows:

$$\begin{aligned}\frac{24}{4n+8} &< \epsilon\\ \frac{\frac{24}{\epsilon}-8}{4} &< n\\ \frac{6}{\epsilon}-2 &< n\end{aligned}$$

Proof: Let $\epsilon \in \mathbb{R} > 0$, $N = \frac{6}{\epsilon} - 2$ and $n \in \mathbb{N}$. We suppose directly that $n > N$, then

$$\begin{aligned}|a_n - L| &= \left|\frac{3n-5}{2n+4} - \frac{3}{2}\right|\\ &= \left|\frac{24}{4n+8}\right|\\ &< \left|\frac{24}{4\left(\frac{6}{\epsilon}-2\right)+8}\right|\\ &< \left|\frac{24}{\frac{24}{\epsilon}}\right|\\ &< \epsilon,\end{aligned}$$

thus $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$. Therefore, the limit is $\frac{3}{2}$. ■

Exercise 33.6. Prove or disprove: The sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = (n+1)/n$ converges.

a) Scratch Work: (We will prove that it converges to 1)

$$\begin{aligned}\left|\frac{n+1}{n} - 1\right| &= \left|\frac{n+1-n}{n}\right|\\ &= \frac{1}{n},\end{aligned}$$

we want to solve for n such that $\frac{1}{n} < \epsilon$. Thus $\frac{1}{\epsilon} < n$.

Proof: Let $\epsilon \in \mathbb{R} > 0$, $N = \frac{1}{\epsilon}$, and $n \in \mathbb{N}$. We suppose directly that $n > N$, then

$$\begin{aligned}\left|\frac{n+1}{n} - 1\right| &= \frac{1}{n}\\ &< \frac{1}{\frac{1}{\epsilon}}\\ &< \epsilon,\end{aligned}$$

thus $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$. Therefore, the limit is 1. ■

Exercise 33.7. Let $(a_n)_{n\in\mathbb{N}}$ be an arithmetic sequence with first term c and common difference d. Prove that if $d = 0$, the sequence $(a_n)_{n\in\mathbb{N}}$ converges to c.

Proof: Let $\epsilon \in \mathbb{R} > 0$, $N \in \mathbb{R}$, and $n \in \mathbb{N}$. We suppose directly that a_n is an arithmetic sequence with first term c and common difference $d = 0$, and that $n > N$, then

$$\begin{aligned}
|a_n - c| &= |c + d\,(n-1) - c| \\
&= |c - c| \\
&= 0 \\
&< \epsilon,
\end{aligned}$$

thus $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$. Therefore, the limit is c. This is trivially true. ■

Exercise 33.8. Prove that the sequence $(a_n)_{n \in \mathbb{N}}$ defined by $a_n = n$ does not converge to $L = 3$.

Proof: We want to show that $\exists \epsilon \in \mathbb{R} > 0, \forall N \in \mathbb{R}, \exists n \in \mathbb{N}, n > N \wedge |a_n - L| \geq \epsilon$. Let $\epsilon = 1$ and let $n = \max(N+1, 10)$, then

$$\begin{aligned}
|a_n - L| &= |\max(N+1, 10) - 3| \\
&\geq 7 \\
&> \epsilon,
\end{aligned}$$

thus the sequence doesn't converge to $L = 3$. ■

Exercise 33.9. Prove that $\lim_{n \to \infty} \left(\sqrt{n^2+1} - n\right) = 0$

a) Scratch Work, we want to solve for n in the equation $\left|\sqrt{n^2+1} - n - 0\right| < \epsilon$ as follows:

$$\begin{aligned}
\left|\sqrt{n^2+1} - n - 0\right| &< \epsilon \\
\sqrt{n^2+1} - n &< \epsilon \\
\sqrt{n^2+1} &< \epsilon + n \\
n^2 + 1 &< 2\epsilon n + \epsilon^2 + n^2 \\
1 - \epsilon^2 &< 2\epsilon n \\
\frac{1-\epsilon^2}{2\epsilon} &< n,
\end{aligned}$$

Proof: Let $\epsilon \in \mathbb{R} > 0$, $N = \frac{1-\epsilon^2}{2\epsilon}$, $n \in \mathbb{N}$, and suppose directly that $n > N$, then

$$\begin{aligned}
\left|\sqrt{n^2+1} - n - 0\right| &= \sqrt{n^2+1} - n \\
&< \sqrt{\left(\frac{1-\epsilon^2}{2\epsilon}\right)^2 + 1} - \frac{1-\epsilon^2}{2\epsilon} \\
&= \sqrt{\frac{1 - 2\epsilon^2 + \epsilon^4 + 4\epsilon^2}{4\epsilon^2}} - \frac{1-\epsilon^2}{2\epsilon} \\
&= \sqrt{\frac{\left(1+\epsilon^2\right)^2}{(2\epsilon)^2}} - \frac{1-\epsilon^2}{2\epsilon} \\
&= \frac{1+\epsilon^2}{2\epsilon} - \frac{1-\epsilon^2}{2\epsilon} \\
&= \frac{2\epsilon^2}{2\epsilon} \\
&= \epsilon,
\end{aligned}$$

thus $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$. Therefore, the limit is 0. ■

Exercise 33.10. Let $(a_n)_{n\in\mathbb{N}}$ be a geometric sequence with first term c and common ration r. Prove the following statements.

a) If $|r| < 1$, then a_n converges to 0.

Scratch Work: We want to solve for n in the expression $c \cdot r^{n-1} < \epsilon$. We do this as follows

$$\begin{aligned}
c \cdot r^{n-1} &< \epsilon \\
\ln\left(cr^{n-1}\right) &< \ln(\epsilon) \\
\ln(c) + \ln\left(r^{n-1}\right) &< \ln(\epsilon) \\
\ln(c) - \ln(\epsilon) &< -(n-1)\ln(r) \\
\ln(c) - \ln(\epsilon) - \ln(r) &< -n\ln(r) \\
-\ln(c) + \ln(\epsilon) + \ln(r) &< n\ln(r) \\
\ln\left(\frac{\epsilon}{c}r\right) &< n\ln(r) \\
\frac{\ln\left(\frac{\epsilon}{c}r\right)}{\ln(r)} &< n \\
a &< n,
\end{aligned}$$

with $a = \frac{\ln\left(\frac{\epsilon}{c}r\right)}{\ln(r)}$.

Proof: Let $\epsilon \in \mathbb{R} > 0$, $N = \ln\left(\frac{c}{\epsilon}r\right)/\ln(r)$, $n \in \mathbb{N}$, $a = \frac{\ln\left(\frac{\epsilon}{c}r\right)}{\ln(r)}$, and ln be $\log_r$. We suppose directly that $|r| < 1$ and $n > N$, then

$$\begin{aligned}
|a_n - L| &= \left|c \cdot r^{n-1} - 0\right| \\
&= c \cdot r^{n-1} \\
&< c \cdot r^{a-1} \\
&< c \cdot r^{\frac{\ln\left(\frac{\epsilon}{c}r\right)}{\ln(r)}-1} \\
&< c\frac{\epsilon}{c} \\
&< \epsilon,
\end{aligned}$$

thus $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$. Therefore, the limit is 0. ■

b) If $c \neq 0$ and a_n converges to 0, then $|r| < 1$.

Proof: We assume by contradiction that $c \neq 0$, a_n converges to 0 and that $|r| \geq 1$. There are two cases to consider, when $|r| = 1$ and $|r| > 1$. If the statement is false under any case, then the entire statement is false. So we only consider the first case.

Case 1. We suppose that $|r| = 1$, then $a_n = c \cdot (\pm 1)^{n-1}$. If $r = 1$, then a_n converges trivially to c as shown similarly in exercise 33.7. If $r = -1$, then $a_n = (-c)^{n-1}$ which does not converge as shown similarly in proposition 33.16.

Since a_n does not converge to 0 when $|r| = 1$, this is a contradiction. Therefore, the original statement is true. Thus if $c \neq 0$ and a_n converges to 0, then $|r| < 1$.

■

c) If $c > 0$ and $r > 1$, then a_n diverges.

Scratch Work: Let's solve for n in the equation $cr^{n-1} = L$ as follows

$$\begin{aligned} cr^{n-1} &= L \\ \ln\left(r^{n-1}\right) &= \ln\left(\frac{L}{c}\right) \\ (n-1)\ln(r) &= \ln\left(\frac{L}{c}\right) \\ n\ln(r) - \ln(r) &= \ln\left(\frac{L}{c}\right) \\ n &= \frac{\ln\left(\frac{Lr}{c}\right)}{\ln(r)}, \end{aligned}$$

note that when $n > \frac{\ln\left(\frac{Lr}{c}\right)}{\ln(r)}$, then $cr^{n-1} > L$ since $r > 0$.

Proof: Let $\epsilon = cr$ and $n \in \mathbb{N}$ be chosen such that $n > \max\left(N, \frac{\ln\left(\frac{Lr}{c}\right)}{\ln(r)} + 1\right)$, then

$$\begin{aligned} \left|cr^{n-1} - L\right| &\geq cr^{\frac{\ln\left(\frac{Lr}{c}\right)}{\ln(r)}+1} - L \\ &= cr^{\frac{\ln\left(\frac{Lr}{c}\right)}{\ln(r)}} + cr - L \\ &= L - L + cr \\ &= cr, \end{aligned}$$

thus, $\forall L \in \mathbb{R}, \exists \epsilon \in \mathbb{R} > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, n > N \wedge \left|cr^{n-1} - L\right| \geq \epsilon$ when $r > 0$ and $c > 0$. Therefore, the statement is true. ■

34. SERIES

Exercise 34.1. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by the rule $a_n = n$. Find the first 6 terms of the sequence of partial sums s_n. Conjecture a simple formula for s_n and prove it.

a_n	1	2	3	4	5	6
s_n	1	3	6	10	15	21

a)

Conjecture: Let the sequence $(a_n)_{n\in\mathbb{N}}$ be given by the rule $a_n = n$, then the partial sum $s_n = \frac{n(n+1)}{2}$.

Proof: Let the sequence $(a_n)_{n\in\mathbb{N}}$ be given by the rule $a_n = n$, we want to show that the open sentence

$$P(n) : \text{the partial sum } s_n = \frac{n(n+1)}{2}$$

is true. We work this by induction.

Base Case: We verify $P(1)$ which is

$$s_1 = a_1 = 1 = \frac{1(1+1)}{2},$$

thus it is true.

Induction: Let $k \in \mathbb{N}$. We assume by induction that $P(k)$ is true, which is the statement $s_k = \frac{k(k+1)}{2}$, and we want to show that $P(k+1)$ is true. Well,

$$\begin{aligned} s_{k+1} &= s_k + a_{k+1} \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}, \end{aligned}$$

thus $P(k+1)$ is true. Therefore $s_n = \frac{n(n+1)}{2}$. ■

Exercise 34.2. Let $c, d \in \mathbb{R}$ and let $(a_n)_{n\in\mathbb{N}}$ be the arithmetic sequence defined by $a_n = c + (n-1)d$. Find a formula for the nth partial sum $s_n = \sum_{k=1}^{n} a_k$ and prove it.

Conjecture: Let $c, d \in \mathbb{R}$ and let the sequence $(a_n)_{n\in\mathbb{N}}$ be defined by $a_n = c + (n-1)d$, then the partial sum $s_n = cn + d\left(\frac{n(n-1)}{2}\right)$.

Proof: We wish to show that $s_n = cn + d$ for all $n \in \mathbb{N}$. We work this by induction.

Base Case: We verify s_1 as follows:

$$\begin{aligned} s_1 &= a_1 \\ &= c + (1-1)d \\ &= c \cdot 1 + d\left(\frac{1(1-1)}{2}\right), \end{aligned}$$

thus s_1 is true.

Induction Step: Let $k \in \mathbb{N}$, we suppose that $s_k = ck + d\left(\frac{k(k-1)}{2}\right)$, and we want to show that $s_{k+1} = c(k+1) + d\left(\frac{(k+1)(k)}{2}\right)$. We do this by looking at s_{k+1}.

$$\begin{aligned}
s_{k+1} &= s_k + a_{k+1} \\
&= ck + d\left(\frac{k(k-1)}{2}\right) + c + (k+1-1)d \\
&= c(k+1) + d\left(\frac{k(k-1)}{2}\right) + kd \\
&= c(k+1) + d\left(\frac{k(k-1)}{2} + \frac{2k}{2}\right) \\
&= c(k+1) + d\left(\frac{k(k-1)+2k}{2}\right) \\
&= c(k+1) + d\left(\frac{k^2-k+2k}{2}\right) \\
&= c(k+1) + d\left(\frac{(k+1)(k)}{2}\right),
\end{aligned}$$

thus $s_{k+1} = c(k+1) + d\left(\frac{(k+1)(k)}{2}\right)$. Therefore, $s_n = cn + d\left(\frac{n(n-1)}{2}\right)$. ■

Exercise 34.3. Give a complete proof that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, by filling in the missing details from Example 34.5.

a) Prove by induction that $s_n = 1 - 1/2^n$.
Proof: We wish to show that $s_n = 1 - 1/2^n$ for all $n \in \mathbb{N}$. We work this by induction.
Base Case: We verify that $s_1 = 1 - 1/2^1$. We start with s_1

$$\begin{aligned}
s_1 &= a_1 \\
&= \frac{1}{2^1} \\
&= 1 - 1/2,
\end{aligned}$$

thus $s_1 = 1 - 1/2^1$.
Induction Step: Let $k \in \mathbb{N}$, we suppose directly that $s_k = 1 - \frac{1}{2^k}$ and we wish to show that $s_{k+1} = 1 - \frac{1}{2^{k+1}}$. We do this by looking at s_{k+1}.

$$\begin{aligned}
s_{k+1} &= s_k + a_{k+1} \\
&= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\
&= 1 + \frac{1-2}{2^{k+1}} \\
&= 1 - \frac{1}{2^{k+1}},
\end{aligned}$$

thus $s_{k+1} = 1 - \frac{1}{2^{k+1}}$. Therefore, $s_n = 1 - \frac{1}{2^n}$. ■

b) Prove that $\lim_{n\to\infty} s_n = 1$.
Scratch Work: We want to find an n such that $|s_n - 1| = \epsilon$. We do this

as follows.

$$\begin{aligned}|s_n - 1| &= \epsilon \\ \left|1 - \frac{1}{2^n} - 1\right| &= \epsilon \\ \left|-\frac{1}{2^n}\right| &= \epsilon \\ \frac{1}{2^n} &= \epsilon \\ \frac{1}{\epsilon} &= 2^n \\ \ln\left(\frac{1}{\epsilon}\right) &= n\ln(2) \\ \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)} &= n.\end{aligned}$$

Proof: Let $\epsilon \in \mathbb{R} > 0$, $N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}$, and $n \in \mathbb{N}$. We suppose directly that $n > N$. Then

$$\begin{aligned}|s_n - 1| &= \left|1 - \frac{1}{2^n} - 1\right| \\ &= \left|-\frac{1}{2^n}\right| \\ &= \frac{1}{2^n} \\ &< \frac{1}{2^{\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}}} \\ &< \frac{1}{\frac{1}{\epsilon}} \\ &< \epsilon,\end{aligned}$$

thus $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$, therefore, the limit is 1. ■

Exercise 34.4. Prove or disprove: The series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges.

a) The first step is to find a partial sum $s_n = \sum_{k=1}^{n} \frac{1}{3^n}$.

Conjecture: Let $a_n = \frac{1}{3^n}$ for $n \in \mathbb{N}$, then $s_n = \sum_{k=1}^{n} a_n = b\frac{(1-b^n)}{(1-b)}$, with $b = \frac{1}{3}$.

Proof: Let $b = \frac{1}{3}$. We wish to show that $s_n = b\frac{(1-b^n)}{(1-b)}$ for all $n \in \mathbb{N}$. We work this by induction.

Base Case: We verify that $s_1 = b\frac{(1-b^1)}{(1-b)}$ as follows:

$$\begin{aligned}s_1 &= a_1 \\ &= \frac{1}{3} \\ &= \frac{1}{3}\left(\frac{1-\left(\frac{1}{3}\right)^1}{1-\frac{1}{3}}\right),\end{aligned}$$

thus $s_1 = b\frac{(1-b^1)}{(1-b)}$.

Induction Step: Let $k \in \mathbb{N}$. We suppose directly that $s_k = b\frac{\left(1-b^k\right)}{(1-b)}$, and we want to show that $s_{k+1} = b\frac{\left(1-b^{k+1}\right)}{(1-b)}$. We begin with the definition of s_{k+1}.

$$\begin{aligned}
s_{k+1} &= s_k + a_{k+1} \\
&= b\frac{\left(1-b^k\right)}{(1-b)} + b^{k+1} \\
&= b\frac{\left(1-b^k\right)}{(1-b)} + b^{k+1}\left(\frac{1-b}{1-b}\right) \\
&= \frac{b\left(1-b^k\right) + b^{k+1}(1-b)}{1-b} \\
&= \frac{b - b\cdot b^k + b^{k+1} - b^{k+1}\cdot b}{1-b} \\
&= \frac{b - b\cdot b^{k+1}}{1-b} \\
&= b\left(\frac{1-b^{k+1}}{1-b}\right),
\end{aligned}$$

thus $s_{k+1} = b\frac{\left(1-b^{k+1}\right)}{(1-b)}$. Therefore, $s_n = b\frac{(1-b^n)}{(1-b)}$. ■

b) Let $b = \frac{1}{3}$, $s_n = b\frac{(1-b^n)}{(1-b)}$ and $n \in \mathbb{N}$. We want to show that $\lim_{n\to\infty} s_n = \frac{1}{2}$.

Scratch Work: We want to solve for n in the equation $\left|s_n - \frac{1}{2}\right| = \epsilon$ with $\epsilon \in \mathbb{R} > 0$. We do this as follows:

$$\begin{aligned}
\left|s_n - \frac{1}{2}\right| &= \epsilon \\
\left|b\frac{(1-b^n)}{(1-b)} - \frac{1}{2}\right| &= \epsilon \\
\frac{1}{2} - b\frac{(1-b^n)}{(1-b)} &= \epsilon \\
\frac{1}{2} - \epsilon &= b\frac{(1-b^n)}{(1-b)} \\
\frac{(1-b)}{b}\left(\frac{1}{2}-\epsilon\right) &= 1 - b^n \\
1 - \frac{(1-b)}{b}\left(\frac{1}{2}-\epsilon\right) &= b^n \\
\ln\left(1 - \frac{(1-b)}{b}\left(\frac{1}{2}-\epsilon\right)\right)/\ln(b) &= n.
\end{aligned}$$

Proof: Let $\epsilon \in \mathbb{R} > 0$, $n \in \mathbb{N}$, $N = \ln\left(1 - \frac{(1-b)}{b}\left(\frac{1}{2}-\epsilon\right)\right)/\ln(b)$, and $t = \ln\left(1 - \frac{(1-b)}{b}\left(\frac{1}{2}-\epsilon\right)\right)/\ln(b)$. We quickly note that $b^t = 1 -$

$\frac{(1-b)}{b}\left(\frac{1}{2}-\epsilon\right)$. We assume directly that $n > N$, then

$$\begin{aligned}\left|s_n - \frac{1}{2}\right| &= \left|b\frac{(1-b^n)}{(1-b)} - \frac{1}{2}\right| \\ &= \frac{1}{2} - b\frac{(1-b^n)}{(1-b)} \\ &< \frac{1}{2} - b\frac{(1-b^t)}{(1-b)} \\ &= \frac{1}{2} - b\frac{\left(1 - 1 - \frac{(1-b)}{b}\left(\frac{1}{2}-\epsilon\right)\right)}{(1-b)} \\ &= \frac{1}{2} - \frac{1}{2} + \epsilon \\ &= \epsilon,\end{aligned}$$

thus $\left|s_n - \frac{1}{2}\right| < \epsilon$. Therefore, $\lim_{n\to\infty} s_n = \frac{1}{2}$. ■

Exercise 34.5. In the exercise we will show that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. Throughout the exercise, let $s_n = \sum_{k=1}^{n} \frac{1}{k}$ be the nth partial sum, for each integer $n \geq 1$.

a) For each $n \geq 1$, define

$$t_n = \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k}.$$

Prove that $t_n \geq \frac{1}{2}$, for each $n \geq 1$.

Proof: We suppose directly that $n \geq 1$ and that $t_n = \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k}$. Then for each n, we sum over $\left(2^n - 2^{n-1} - 1 + 1\right)$ terms. Which is simply $\left(2^n - 2^{n-1}\right)$ terms. In each summation, the smallest term is always the last term to be added. Thus

$$\begin{aligned}t_n &= \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k} \\ &\geq \left(2^n - 2^{n-1}\right)\frac{1}{2^n} \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Therefore $t_n \geq \frac{1}{2}$ for all $n \geq 1$. ■

b) Show that $s_{2^n} \geq n/2$, for each $n \geq 0$, by induction.

Proof: Let $n \geq 1$ and $t_n = \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k}$. We note that

$$\begin{aligned}s_{2^{n+1}} &= \sum_{k=1}^{2^n} \frac{1}{k} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \\ &= s_{2^n} + t_{n+1}.\end{aligned}$$

and we want to show that $s_{2^n} \geq n/2$ for all $n \geq 0$. We work this by induction.

Base Case: We verify that $s_{2^0} \geq 0/2$.

$$\begin{aligned}s_{2^0} &= s_1 \\ &= \frac{1}{1} \\ &\geq 0.\end{aligned}$$

Induction Step: Let $k \geq 0$. We suppose by induction that $s_{2^k} \geq k/2$, and we want to show that $s_{2^{k+1}} \geq (k+1)/2$. We do this as follows:

$$\begin{aligned} s_{2^{k+1}} &= s_{2^k} + t_{n+1} \\ &\geq k/2 + t_{n+1} \\ &\geq k/2 + 1/2 \\ &= (k+1)/2. \end{aligned}$$

Thus $s_{2^{k+1}} \geq (k+1)/2$. Therefore, $s_{2^n} \geq n/2$. ■

c) Now show that the harmonic series does not converge.
Proof: We suppose directly that $s_n = \sum_{k=1}^{n} \frac{1}{k}$. We can write this sum as $s_{2^k} = \sum_{k=0}^{2^k} t_k$ where t_k was defined in part 1. By definition $t_m \geq \frac{1}{2}$ for all $m \in \mathbb{Z} > 0$, we know that

$$\lim_{n\to\infty} t_n \neq 0.$$

Thus, using corollary 34.7, $\sum_{k=0}^{2^k} t_k$ does not converge. Therefore the series s_n does not converge. ■