Homework 15

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Sections 28 and 29

28. Definitions regarding cardinality

Exercise 28.1. Declare whether the following statements are true or false, with proof/reason or counterexample:

a) All finite sets have the same cardinality.
Disproof: We will disprove this statement with an example. Let $A = \{1\}$, then $|A| \neq |\emptyset|$. Therefore, finite sets can have different cardinality. ■
b) If $f : A \to B$ is a function between two sets, then $|f| = |A|$.
Proof: We suppose directly that $f : A \to B$, then f is the relation $f = \{(a, b) : a \in A, b = f(a)\}$. Since every element of A is a left coordinate of an element of f once, then for every $a \in A$ there is exactly one element in f such that $(a, b) \in f$. And for every $(a, b) \in f$ there is exactly one element in A such that $a \in A$. Thus $|f| = |A|$. ■
c) Every subset of $\mathbb{N}$ is countably infinite.
Disproof: We will disprove this statement with a counter example. Consider the set $\emptyset$ which has no elements and is a subset of $\mathbb{N}$. Since the emptyset is a finite set, there exists a subset of $\mathbb{N}$ that is not infinite. Therefore, not every subset of $\mathbb{N}$ is countably infinite. ■
d) Every subset of an infinite set has cardinality $\aleph_0$.
Disproof: This is similar to the previous problem. Consider the emptyset. ■
e) If $f : A \to B$ is a surjective function then $|f| = |B|$.
Disproof: We will show that there exists sets A, B and a surjective function f such that $|f| \neq |B|$. Let $A = \{1, 2\}$, $B = \{1\}$, and

$$f = \{(1, 1), (1, 2)\},$$

then $|f| \neq |B|$. Therefore the statement is false. ■

Exercise 28.2. Define $h : (0, \infty) \to (0, 1)$ by the rule

$$h(x) = \frac{x}{x+1}.$$

Verify that h is a bijection. What does this say about the cardinality of these open intervals?

Proof: To show that h is a bijection, we will show that it is both injective and surjective.

Injective: We assume contrapositively that $h(m) = h(n)$ for some $m, n \in (0, \infty)$, then

$$\begin{aligned} h(m) &= h(n) \\ \frac{m}{m+1} &= \frac{n}{n+1} \\ mn + m &= mn + n \\ m &= n, \end{aligned}$$

hence g is injective.

Surjective: We assume directly that $k \in (0,1)$. We solve for $\ell \in (0,\infty)$ as follows

$$\begin{aligned}\frac{\ell}{\ell+1} &= k \\ \ell &= k\ell + k \\ \ell(1-k) &= k \\ \ell &= \frac{k}{1-k}.\end{aligned}$$

Plugging ℓ into h yields

$$\begin{aligned}h(\ell) &= h\left(\frac{k}{1-k}\right) \\ &= k.\end{aligned}$$

Therefore, h is surjective. Since h is both injective and surjective, it is bijective. Since h is bijective we know that the domain and codomain have the same cardinality. ■

Exercise 28.3. Prove that the set of those natural numbers with exactly one digit equal to 7 is countably infinite. For instance, the number 103792 has exactly one of its digits equal to 7, while 8772 has two digits equal to seven.

Proof: Let $S \subseteq \mathbb{N}$ be the subset of $\mathbb{N}$ with exactly one digit equal to 7. Then S is a countable set. Let $A = \{7, 70, 700, 7000, \cdots\}$which is a non-repeating infinite series. In other words, it is countably infinite. Since every element of A has only one digit equal to 7 and is a natural number, we know that $A \subseteq S$. Therefore, S must be infinite, and since it is a subset of $\mathbb{N}$, S must be countably infinite. ■

Exercise 28.4. Consider the set

$$S = \left\{x \in \mathbb{Z} : x = a^2 + b^2 \text{ for some } a, b \in \mathbb{Z}\right\}.$$

Prove that $|S| = |\mathbb{N}|$.

Proof: We show this directly. It has been shown that $|\mathbb{Z}| = |\mathbb{N}|$, thus $\mathbb{Z}$ is a countably infinite set. Consider the set

$$\begin{aligned}A &= \left\{x \in \mathbb{Z}\, x = a^2 \text{ for some } a \in \mathbb{Z}\right\} \\ &= \{0, 2, 4, 9, 16, \cdots\}\end{aligned}$$

which is a countably infinite set. Since $A \subseteq S$, we know that S must be an infinite set. And since $S \subseteq \mathbb{Z}$, it must be a countably infinite set. Thus $|S| = |\mathbb{N}|$. ■

Exercise 28.5. Prove that the function in Theorem 28.4 is a bijection. The function being

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ -(n-1)/2 & \text{if } n \text{ is odd.}\end{cases}$$

Proof: By using the pasting together theorem, we will show directly that $f(n)$ is a bijection. Let Q_1 be the set of even natural numbers, Q_2 be the set of odd natural numbers, P_1 be the set of positive integers, P_2 be the set of non-positive integers, $f_1 : Q_1 \to P_1$ defined as $f_1(n) = n/2$, and $f_2 : Q_2 \to P_2$ defined as $f_2(n) = -(n-1)/2$. Since Q_1 and Q_2 form a partition of $\mathbb{N}$ and P_1 and P_2 form a partition of $\mathbb{Z}$, we just need to show that f_1 and f_2 are a bijection by showing that they are both injective and surjective.

(f_1) : **Injective**: We assume contrapositively that $f_1(a) = f_1(b)$ with $a, b \in Q_1$, then

$$\begin{aligned} f_1(a) &= f_1(b) \\ a/2 &= b/2 \\ a &= b, \end{aligned}$$

thus it is injective.

Surjective: We assume directly that $k \in P_1$, and let $a = 2k$, (which is an even natural number), then

$$f_1(2k) = k,$$

thus f_1 is surjective. Since f_1 is both injective and surjective, it is bijective.

(f_2) :**Injective**: We assume contrapositively that $f_2(c) = f_2(d)$ with $c, d \in Q_2$, then

$$\begin{aligned} f_2(c) &= f_2(d) \\ -(c-1)/2 &= -(d-1)/2 \\ c &= d, \end{aligned}$$

thus it is injective.

Surjective: We assume directly that $m \in P_2$, and let $c = -2m + 1$(which is an element of Q_2), then

$$\begin{aligned} f_2(c) &= -(-2m + 1 - 1)/2 \\ &= 2m/2 \\ &= m, \end{aligned}$$

thus f_2 is surjective. Since it is both injective and surjective, it is bijective.

Since f_1 and f_2 are bijections, their domains form a partition of the domain of f and their codomains form a partition of the codomain of f, we can glue f_1 and f_2 to form the function f, which would then be a bijective function. ∎

Exercise 28.6. Prove that $|\mathbb{R}| = |(0,1)|$.

Proof: Let $f : \mathbb{R} \to (0,1)$ be defined as

$$f(x) = \frac{\arctan(x) + \frac{\pi}{2}}{\pi},$$

and consider the function $f^{-1} : (0,1) \to \mathbb{R}$ be defined as

$$f^{-1}(x) = \tan\left(\pi x - \frac{\pi}{2}\right),$$

then

$$\begin{aligned} f^{-1} \circ f(x) &= f^{-1}(f(x)) \\ &= \tan\left(\pi\left(\frac{\arctan(x) + \frac{\pi}{2}}{\pi}\right) - \frac{\pi}{2}\right) \\ &= \tan\left(\arctan(x) + \frac{\pi}{2} - \frac{\pi}{2}\right) \\ &= x, \end{aligned}$$

thus f^{-1} is the inverse function of f. This means that f is bijective. Since f is bijective, $|\mathbb{R}| = |(0,1)|$. ∎

Exercise 28.7. Prove Corollary 28.14

Proof: We suppose directly that A is a countable set. Then A is either finite or countably infinite. This gives us two cases to consider.

Case 1. Suppose A countably infinite. Then any $B \subseteq A$ is either infinite or finite. If B is infinite, then (according to theorem 28.13) it is countably infinite and thus countable. If B is finite then it is still a countable set. Thus in either case B is a countable set.

Case 2. Suppose A is finite. Then any $B \subseteq A$ must also be finite since B cannot have more elements than A. Thus B is countable.

Since in any possible situation of A and B, we get that B is countable, the Corollary is true.

$\blacksquare$

29. More examples of countable sets

Exercise 29.1. Finish the proof of Theorem 29.1

Proof: We have two cases left to consider: (a) both S and T' are finite, or (b) one of them is finite and the other infinite.

Case 1. Assume directly that S and T' are finite sets. Then $|S| = s$ and $|T'| = t$ for some $s, t \in \mathbb{N}$. Which means $|S \cup T'| = s + t \in \mathbb{N}$ and their union is a finite set, since it has a finite cardinality. Therefore, $S \cup T'$ are countable.

Case 2. With no loss in generality, we assume that S is finite and T' is infinite. Thus we can list S in a finite list $S = (s_1, s_2, \cdots, s_n)$ with $n < \infty$, and T' in a non repeating infinite list as $T' = \{t_1, t_2, t_2 \cdots\}$. Since $S \cap T' = \emptyset$, We can write $S \cup T'$ as a non repeating, infinite list as $S \cup T' = \{s_1, s_2, \cdots, s_n, t_1, t_2, t_2, \cdots\}$. Thus $S \cup T'$ is countably infinite.

Since both cases hold, we the union of two countable sets is again countable. ■

Exercise 29.2. Prove that $\{0, 1\} \times \mathbb{N}$ is countably infinite.

Proof: According to theorem 29.3, the Cartesian product of two countably infinite sets is a countably infinite set. Then $\mathbb{N} \times \mathbb{N}$ is a countably infinite set. Since $\{0, 1\} \times \mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$ and $\{0, 1\} \times \mathbb{N}$ is infinite, then according to theorem 28.13, the set $\{0, 1\} \times \mathbb{N}$ is countably infinite. ■

Exercise 29.3. Let A and B be countable sets. Prove that $A \times B$ is countable.

Proof: We have three cases to consider: A and B are countably infinite, A and B are finite, and one set is finite and the other is countably infinite.

Case 1. We assume directly that A and B are countably infinite, then according to theorem 29.3, $A \times B$ is countably infinite, and thus countable.

Case 2. We assume directly that A and B are finite. Let $|A| = a$ and $|B| = b$ for some $b, a \in \mathbb{N}$. Then $|A \times B| = ab$. Since $ab \in \mathbb{N}$, then $A \times B$ is a finite set and thus countable.

Case 3. Without loss in generality, we assume directly that A is finite and B is countably infinite. Since the Cartesian product of two countably infinite sets is countably infinite, we know that $k < |A \times B| \leq |\mathbb{N}|$ for some $k \in \mathbb{Z} > 0$. If A is the empty set, then $|A \times B| = 0$ and is finite. If $A \neq \emptyset$, then $A \times B$ is infinite and thus countably infinite since it is the subset of some countably infinite set.

■

Since all cases hold, the statement is true.

Exercise 29.4. Let $n \geq 2$ be an integer, and let $A_1, A_2, \ldots, A_n$ be countable sets. Prove that $A_1 \times A_2 \times \cdots A_n$

Proof: We want to show that the open sentence

$$P(n): \ A_1 \times A_2 \times \cdots A_n \text{ is a countable set if } A_1, A_2, \ldots, A_n \text{ are countable sets}$$

with $n \geq 2$. We show this by induction.

Base Case: $P(2)$ was proven in the previous exercise.

Induction Step: We assume that $P(k)$ is true for some $k \in \mathbb{N} \geq 2$ and that $A_1, A_2, \ldots, A_n$ are countable sets , and we want to show that $P(k+1)$ is true. Let $A_1 \times A_2 \times \cdots A_{k+1} = B$ which is a countable set, then $B \times A_{k+1}$ is a countable set since the Cartesian product of two countable sets are countable. Thus the open sentence is true. ■

Exercise 29.5. Prove $|\mathbb{Z}\times\mathbb{N}| = |\mathbb{Q}|$.

Proof: According to Theorem 29.3, the Cartesian product of two countably infinite sets is again countably infinite, and according to Corollary 29.7. The set $\mathbb{Q}$ is countably infinite. Therefore, $|\mathbb{Z}\times\mathbb{N}| = |\mathbb{Q}|$. ■

Exercise 29.6. Prove that if $A_1, A_2, \ldots$ are pairwise disjoint, countably infinite sets, then $\cup_{i=1}^{\infty} A_i$ is countably infinite.

Proof: We suppose directly that $A_1, A_2, \ldots$ are pairwise disjoint, countably infinite sets. Then there exists a bijection $f_i : \{i\}\times\mathbb{N} \to A_i$ where the index is over $\mathbb{N}$. Since the sets A_i are pairwise disjoint, the co-domain of f_i is disjoint. Also, the domain of f_i is disjoint from any other function. We can glue the functions together to form the function $f : \mathbb{N}\times\mathbb{N} \to \cup_{i=1}^{\infty} A_i$ defined as

$$f\left(a, b\right) = f_a\left(b\right).$$

Since we have a bijective map $f : \mathbb{N}\times\mathbb{N} \to \cup_{i=1}^{\infty} A_i$, we know that $|\mathbb{N}\times\mathbb{N}| = |\cup_{i=1}^{\infty} A_i|$. Since $\mathbb{N}\times\mathbb{N}$ is countably infinite, $\cup_{i=1}^{\infty} A_i$ must be countably infinite. ■

Exercise 29.7. Prove that the set of all finite subsets of $\mathbb{N}$ is countably infinite.

Proof: Let $S_i = \{T \subseteq \mathbb{N} : \forall x \in T, x \leq i\}$, then all the elements of S_i are finite subsets of $\mathbb{N}$. Let $s_{i,j} \in S_i$ be the j^{th} element of the set S_i. We can place these elements in a list

$$\left\{s_{0,1}, s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}, \cdots\right\}.$$

Let M be the set after removing the duplicate entries in the list, then the elements of M form a non repeating infinite list. Thus M is countably infinite. Therefore, the set of all finite subsets of $\mathbb{N}$ is countably infinite. ■