Homework 7

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Sections 12 and 13

12. Set Proofs in Logic

Exercise 12.1. For each natural number n, define the set $S_n = \{x \in \mathbb{Z} : x^2 = n\}$. Prove the following statement: If $n = 4$, then $S_n = \{2, -2\}$.

Proof: We assume directly that $n = 4$, then $S_4 = \{2, -2\}$. Hence $S_4 \subseteq \{2, -2\}$ and $S_4 \supseteq \{2, -2\}$. ■

Exercise 12.2. Give a complete proof for Proposition 12.4, using the sketched outline.

Prop: Given sets S and T, we have $S \subseteq T$ if and only if $S = S \cap T$.

Proof: This is a biconditional compound statement so we must show both ways.

$(\implies)$: Assume that $S \subseteq T$. Therefore if $x \in S$, then $x \in T$.

$(\subseteq)$: Assume that $x \in S$, then $x \in T$ since $S \subseteq T$. Therefore $x \in S$ and $x \in T$. In other words $x \in S \cap T$. Hence $S \subseteq S \cap T$.

$(\supseteq)$: Assume that $x \in S \cap T$, then $x \in S$ and $x \in T$. Since $x \in S$, $S \subseteq S \cap T$.

Thus we have shown that if $S \subseteq T$, then $S = S \cap T$.

$(\impliedby)$: Assume that $S = S \cap T$. We show that $S \subseteq T$.

$(\subseteq)$: Assume $z \in S$, then $z \in S \cap T$. Therefore $z \in S$ and $z \in T$. Since every element of S is an element of T, $S \subseteq T$. ■

Exercise 12.3. Consider the statement:

Let S and T be sets. Then $S \subseteq T$ if and only if $T = S \cup T$.

Outline a proof of the statement.

Proof Outline:

$(\implies)$: Assume $S \subseteq T$.

$(\subseteq)$: We first show $T \subseteq S \cup T$.

Assume $x \in T$.

$\vdots$

Conclude that $x \in S \cup T$.

Conclude that $T \subseteq S \cup T$.

$(\supseteq)$: We next show $T \supseteq S \cup T$

Assume $z \in S \cup T$.

$\vdots$

Conclude that $x \in T$.

Conclude that $T \supseteq S \cup T$.

$(\impliedby)$: Assume $T = S \cup T$.

$(\subseteq)$: We finally show $S \subseteq T$.

Assume $x \in S$.

$\vdots$

Conclude that $x \in T$.

Conclude that $S \subseteq T$. ■

Exercise 12.4. Let S and T be sets. Prove the following.

a) If $S \cap T = T \cup S$, then $S = T$
Proof: We suppose directly that $S \cap T = T \cup S$. This means that if $x \in S$ or $x \in T$, then $x \in S$ and $x \in T$.
$(\subseteq)$:We first show that $S \subseteq T$. We assume that $x \in S$; therefore, $x \in S \cap T$ since $S \cap T = T \cup S$. Because $x \in S \cap T$, $x \in S$ and $x \in T$. Hence $S \subseteq T$.
$(\supseteq)$:We next show that $T \subseteq S$. We assume that $x \in T$; therefore, $x \in S \cap T$ since $S \cap T = T \cup S$. Because $x \in S \cap T$, $x \in S$ and $x \in T$. Hence $T \subseteq S$.
Thus if $S \cap T = T \cup S$, then $S = T$. ■

b) If $S \times T = T \times S$ and both S and T are nonempty, then $S = T$.
Proof: We suppose directly that $S \times T = T \times S$ and that both $S \neq \emptyset$ and $T \neq \emptyset$.
$(\subseteq)$: We first show that $S \subseteq T$. We assume that $s \in S$, then for some $x = (s, t) \in S \times T$ with $t \in T$, $x \in T \times S$. In other words $(s, t) \in T \times S$. Hence $s \in T$. Therefore $S \subseteq T$.
$(\supseteq)$: We next show that $T \subseteq S$. We assume that $t \in T$, then for some $x = (t, s) \in T \times S$ with $s \in S$, $x \in S \times T$. In other words $(t, s) \in S \times T$. Hence $t \in S$. Therefore $T \subseteq S$.
Thus if $S \times T = T \times S$ and both S and T are nonempty, then $S = T$. ■

Exercise 12.5. Let S and T be sets. Prove that $S = T$ if and only if $S - T = T - S$.

Proof: This is a biconditional compound statement, we will show both ways.
$(\Longrightarrow)$: We assume directly that $S = T$.
$(\subseteq)$: We first show that $S - T \subseteq T - S$. Let $x \in S - T$, then $x \in S$ and $x \notin T$. This is a contradiction to the assumption that $S = T$, since x must be an element of S and T. Therefore $S - T$ has no elements and is the empty set. Hence $S - T \subseteq T - S$ since the empty set is a subset of every set.
$(\supseteq)$: We can similarly show that $T - S \subseteq S - T$.
$(\Longleftarrow)$: We assume directly that $S - T = T - S$.
$(\subseteq)$: We first show that $S \subseteq T$. We assume that $x \in S$. This means that either $x \in S - T$ or $x \notin S - T$.
We show that if $x \in S$, then $x \notin S - T$. Assume by contradiction that $x \in S$ and $x \in S - T$, then $x \notin T$. If $x \in S - T$ then $x \in T - S$ since $S - T = T - S$. This means that $x \in T$ and $x \notin S$. Which is a contradiction. Hence if $x \in S$ then $x \notin S - T$. Therefore $x \notin T - S$. For this to be true, x must be an element of T since $x \in S$. Hence $S \subseteq T$.
$(\supseteq)$: We can similarly show that $T \subseteq S$. ■

Exercise 12.6. Let S and T be sets. Prove or disprove: $S = T$ if and only if $S - T \subseteq T$.

Proof: This is a biconditional compound statement, we will show both ways.
$(\Longrightarrow)$: We assume directly that $S = T$; therefore $x \in S$, $x \in T$ and $S - T = \emptyset$. Since the empty set is a subset of every set, $S - T \subseteq T$. Hence, if $S = T$, then $S - T \subseteq T$.
$(\Longleftarrow)$: We assume directly that $S - T \subseteq T$.
$(\subseteq)$: We first show that $S \subseteq T$. Let $x \in S - T$, then $x \in S$ and $x \notin T$, but since $S - T \subseteq T$, $x \in T$. This is a contradiction. Therefore, there are no elements in $S - T$. This means that $\forall x \in S, x \in T$. Hence $S \subseteq T$.
$(\supseteq)$: We next show that $T \subseteq S$. Let $x \in S - T$, then $x \in S$ and $x \notin T$, but since $S - T \subseteq T$, $x \in T$. This is a contradiction. Therefore, there are no elements in $S - T$. This means that $\forall x \in T, x \in S$. Hence $T \subseteq S$. ■

Exercise 12.7. Let S be a set. Prove that $\emptyset \times S = \emptyset$.

Proof: We assume by contradiction that $\emptyset \times S \neq \emptyset$ so that $x = (a, b) \in \emptyset \times S$. Which means that $a \in \emptyset$ and $b \in S$. This is a contradiction since there can't be an element a in the empty set. Hence $\emptyset \times S = \emptyset$. ■

Exercise 12.8. For sets S and T, show that $S \times T = \emptyset$ if and only if $S = \emptyset$ or $T = \emptyset$.

Proof: This is a biconditional compound statement, we will show both ways.
($\implies$): We assume by contradiction that $S \times T = \emptyset$, $S \neq \emptyset$, and $T \neq \emptyset$. Therefore there is an $s \in S$ and $t \in T$ such that $x = (s, t) \in S \times T$ which is a contradiction since $S \times T = \emptyset$. Hence if $S \times T = \emptyset$, then $S = \emptyset$ or $T = \emptyset$.
($\impliedby$): Without loss in generality we assume that $S = \emptyset$ or $T = \emptyset$, then as shown in exercise 12.7, $S \times T = \emptyset$.
Thus $S \times T = \emptyset$ if and only if $S = \emptyset$ or $T = \emptyset$. ■

Exercise 12.9. Consider the statement: Given sets A, B, C if $A \times B \subseteq B \times C$ and $B \neq \emptyset$, then $A \subseteq C$.

Proof: We assume directly that $A \times B \subseteq B \times C$ and $B \neq \emptyset$. Let $x = (a, b) \in A \times B$ where $a \in A$ and $b \in B$. Since $A \times B \subseteq B \times C$, $x \in B \times C$. Thus $a \in B$ and $b \in C$. This shows that $A \subseteq B$ and $B \subseteq C$. Hence $A \subseteq C$. Thus given sets A, B, C if $A \times B \subseteq B \times C$ and $B \neq \emptyset$, then $A \subseteq C$. ■

Exercise 12.10. Prove or disprove the converse of Theorem 12.7.
Thm: Given four sets A, B, C, D, if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.
Converse: Given four sets A, B, C, D, if $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$.

Proof: We assume directly that $A \times B \subseteq C \times D$. Let $x = (a, b) \in A \times B$ where $a \in A$ and $b \in B$. Since $A \times B \subseteq C \times D$, then $x \in C \times D$. Thus $a \in C$ and $b \in D$. This shows that $A \subseteq C$ and $B \subseteq D$. Thus if $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$. ■

13. Mathematical Induction

Exercise 13.1. Prove that for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} (2i-1) = n^2.$$

Proof: We prove this by induction on $n \in \mathbb{N}$.
Let $P(n)$ be the open sentence. $P(n) : \sum_{i=1}^{n} (2i-1) = n^2$.
Base Case: We verify that $P(1)$ is true. $2 \cdot 1 - 1 = 1$.
Inductive Step: Let $k \in \mathbb{N}$. We assume $P(k)$.

$$\begin{aligned}\sum_{i=1}^{k+1} (2i-1) &= \sum_{i}^{k} (2i-1) + 2(k+1) - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2,\end{aligned}$$

hence $P(k) \implies P(k+1)$. Thus $\sum_{i=1}^{n} (2i-1) = n^2$. ■

Exercise 13.2. Prove that for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}.$$

Proof: We prove this by induction on $n \in \mathbb{N}$.
Let $P(n)$ be the open sentence. $P(n) : \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$.
Base Case: We verify that $P(1)$ is true. $\frac{1}{(2\cdot1-1)(2\cdot1+1)} = \frac{1}{3} = \frac{1}{2\cdot1+1}$.
Induction Step: Let $k \in \mathbb{N}$. We assume $P(k)$. It follows that

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} &= \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\ &= \frac{(k+1)}{2(k+1)+1},\end{aligned}$$

hence if $P(k)$, then $P(k+1)$. Thus $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$. ■

Exercise 13.3. Prove that for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof: We prove this by induction on $n \in \mathbb{N}$.
Let $P(n)$ be the open sentence; $P(n) : \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
Base Case: We verify that $P(1)$ is true.

$$\begin{aligned}1^2 &= \frac{1(1+1)(2\cdot1+1)}{6} \\ 1 &= 1.\end{aligned}$$

Induction Step: Let $k \in \mathbb{N}$. We assume $P(k)$. It follows that

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6}{6}(k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)\left(2k^2 + k + 6k + 6\right)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6},
\end{aligned}$$

hence if $P(k)$ is true, then $P(k+1)$ is true. Thus $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. ■

Exercise 13.4. This has two parts.

a) Prove that for each $n, k \in \mathbb{N}$,

$$n < 3^n.$$

Proof: We prove this by induction.
Let $P(n)$ be the open sentence; $P(n) : n < 3^n$.
Base Case: We will show that $P(1)$ is true. $P(1)$: $1 < 3^1$ which is true.
Induction Step: We assume $P(k)$. It follows that $3^{k+1} = 3^k + 3^k + 3^k$. We assume $3^k > k$, thus $3^k + 3^k + k < 3^{k+1}$. Since $1 < 3^k + 3^k$, then $k + 1 < 3^{k+1}$. Hence $P(n)$ is true. ■

b) Prove that for each $n \in \mathbb{Z}$, $n < 3^n$.
Proof: We prove by splitting up the domain of n, and ensure that $n < 3^n$ for each subset of $\mathbb{Z}$. ■

Case 1. We assume that $n \in \mathbb{N}$, then $n < 3^n$. We proved this in the first part of this exercise.
Case 2. We assume that $n = 0$, then $0 < 3^0$.
Case 3. We assume that $-n \in \mathbb{N}$. Under this condition, $3^n > 0$ for all n. Since $n < 0 < 3^n$, $n < 3^n$. Therefore if $-n \in \mathbb{N}$, then $n < 3^n$.

Thus we have shown that for each $n \in \mathbb{Z}$, $n < 3^n$.

Exercise 13.5. Let $x \in \mathbb{R} - \{1\}$. Prove that for each $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}.$$

Proof: We prove this by induction.
Let $P(n)$ be the open sentence; $P(n) : \sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$.
Base Case: We will show that $P(1)$ is true.

$$\begin{aligned}
x^1 + 1 &= \frac{1 - x^2}{1 - x} \\
&= \frac{(1-x)(1+x)}{1-x} \\
&= 1 + x,
\end{aligned}$$

which is true.

Induction Step: Let $k \in \mathbb{N}$. We assume $P(k)$. It follows that

$$\begin{aligned}\sum_{i=0}^{k+1} x^i &= \sum_{i=0}^{k} x^i + x^{k+1} \\ &= \frac{1-x^{k+1}}{1-x} + x^{k+1} \\ &= \frac{1-x^{k+1}}{1-x} + \frac{(1-x)\left(x^{k+1}\right)}{1-x} \\ &= \frac{1-x^{k+1}+x^{k+1}-x^{k+2}}{1-x} \\ &= \frac{1-x^{(k+1)+1}}{1-x},\end{aligned}$$

thus if $P(k)$ is true, then $P(k+1)$ is true. Hence $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$. ■

Exercise 13.6. Let $x \in \mathbb{R}$ and assume $x > -1$. Prove that for each $n \in \mathbb{N}$,

$$(1+x)^n \geq 1 + nx.$$

Proof: We prove this by induction and assume that $x \in \mathbb{R}$, $x > -1$, and $n \in \mathbb{N}$.

Let $P(n)$ be the open sentence $P(n) : (1+x)^n \geq 1 + nx$.

Base Case: We will show that $P(1)$ is true.

$$\begin{aligned}(1+x)^1 &\geq 1 + 1 \cdot x \\ 1 + x &\geq 1 + x,\end{aligned}$$

which is true.

Induction Step: Let $k \in \mathbb{N}$. We assume $P(k)$. It follows

$$(1+x)^{k+1} = (1+x)^k (1+x).$$

Using the assumption that $(1+x)^k \geq 1 + kx$, we get

$$\begin{aligned}(1+x)^{k+1} &\geq (1+x)(1+kx) \\ &\geq 1 + kx + x + kx^2 \\ &\geq 1 + x(k+1) + kx^2 \\ &\geq 1 + x(k+1),\end{aligned}$$

using the fact that $kx^2 \geq 0$. Thus if $P(k)$ is true then $P(k+1)$ is true. Therefore, $P(n)$ is true. ■

Exercise 13.7. Let S be any nonempty set of natural numbers. Prove that S has a least element.

Proof: We suppose directly that S is any nonempty subset of $\mathbb{N}$. Since $\mathbb{N}$ is a finite set, any of it's subsets other than the empty set is finite. Therefore S is a finite set. In addition, since $\mathbb{N} \subseteq \mathbb{R}$, S is a nonempty finite subset of $\mathbb{R}$. According to proposition A.1, S must have a least element. ■

Exercise 13.8. Prove the following variation of the pigeonhole principle.

Let $m \in \mathbb{N} \cup \{0\}$, let $n \in \mathbb{N}$, and assume $m < n$. If we suppose m objects are placed in n bins, conclude that some bin does not contain any object.

Proof: We prove this by induction and assume that $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, and $m < n$.

Let $P(n)$ be the open sentence; $P(n)$: If m objects are placed in n bins, than some bin does not contain any object.

Base Case: We will show that $P(1)$ is true. We assume directly that $n = 1$, then $m = 0$ since $m < n$ and $m \in \mathbb{N} \cup \{0\}$. This means that we have no objects to place in one bin. Therefore the bin is empty.

Induction Step: Let $k \in \mathbb{N}$. We assume $P(k)$, and suppose we have $k + 1$ bins and $m < k + 1$ objects to place in the bins. Let S denote the set of object such that $|S| = m$, and let x be one of those objects, i.e. $x \in S$. Let $T = S - \{x\}$ such that $|T| = m - 1 = \ell$, then $\ell < k$. We know that if the ℓ objects are placed in k bins, at least one of the bins is empty. Since we have $k + 1$ bins, if the ℓ objects are placed in $k + 1$ bins, then at least two of the bins are empty. By placing the object x in the $k + 1$ bins, two things can happen:

Case 1. We place x into a bin that already has an object, and then at least two of the bins are empty.

Case 2. We place x into a bin that doesn't have an object, and then at least one bin is empty.

Regardless of the case, a bin remains empty. Hence if $P(k)$ is true, then $P(k + 1)$ is true. Thus the open sentence $P(n)$ is true.

■

Exercise 13.9.

Appendix

Proposition A.1. *Let A be a finite nonempty set of real numbers. Then A has a least element.*

Proof: The proof is shown in the book. See proposition 13.11. ■