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**Exercise 0.1.** Answer the following

a) Let $f : \mathbb{Z}_7 \to \{-1, 1\}$ be given by $f(\bar{a}) = (-1)^a$. Is $f$ well defined?

For $f$ to be well defined, every element in the domain can only be mapped to one element in the codomain. The integers 1 and 8 are elements of $\bar{1}$, but they are mapped to different elements in the codomain.

$$(-1)^1 = -1$$
$$(-1)^8 = 1$$

thus the function is not well defined.

b) Let $i$ be the complex number satisfying $i^2 = -1$. Is the function $g : \mathbb{Z}_4 \to \mathbb{C}$ given by $g(\bar{a}) = i^a$ well defined?

Yes, every element $m \in \bar{a}$ can be written as $a + k4$ for some $k \in \mathbb{Z}$. Thus

$$\begin{aligned} i^m &= i^{(a+k4)} \\ &= i^a \cdot i^{k4} \\ &= i^a \cdot \left(1^k\right) \\ &= i^a \end{aligned}$$

This shows that every representation of the domain is mapped to the same element in the codomain, and therefore, the function is well defined.

c) Consider the new definition of addition of fractions as $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. Is $h(x, y) = x \oplus y$ a well defined function from $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$?

No, we will show this with a simple counterexample. Let $a = \frac{1}{2}$, $b = \frac{3}{8}$, and $a' = \frac{2}{4}$ such that $a = a'$. In other words $(a, b) = (a', b)$. We note that

$$\begin{aligned} a + b &= \frac{1}{2} + \frac{3}{8} = \frac{4}{10} = \frac{2}{5} \\ a' + b &= \frac{2}{4} + \frac{3}{8} = \frac{5}{12}. \end{aligned}$$

Since $\frac{2}{5} \neq \frac{5}{12}$ , this shows that the same element in the domain has more than one image. Therefore, the function is not well defined.

**Exercise 0.2.** Prove that the set $\mathbb{R}$ is uncountable. Make no assumptions about the uncountability of any other sets. You may use the fact that $|(0, 1)| = |\mathbb{R}|$.

*Proof:* Since $|(0, 1)| = |\mathbb{R}|$, it is sufficient to show that the set $(0, 1)$ is uncountable. We can show that it is uncountable by showing that there does not exist a surjection from $\mathbb{N}$ to $(0, 1)$ since $\mathbb{N}$ is a countable set. To do this, let $f : \mathbb{N} \to (0, 1)$ be the arbitrary function defined as

$$f(x) = 0.x_1x_2x_3 \cdots$$

where $x_i$ represents the decimal digits that are not all 9. The codomain of the function is $U \subseteq (0, 1)$ by construction. The decimal expansion of any number in the codomain $(0, 1)$ can be written as $0.a_1a_2a_3 \cdots$ with $a_i \in \mathbb{N}$. To show that $f$ is not surjective, we must show that there exists at least one element in the codomain $k$, such that $f(x) \neq k$ for all $x \in \mathbb{N}$. Let $k \in (0,1)$ be written as $0.k_1 k_2 k_2 \cdots$ where the digit $k_i$ is defined as

$$k_i = \begin{cases} 7 & \text{if } g(i) \neq 7 \\ 0 & \text{otherwise} \end{cases}$$

with $g(i)$ being the function that returns the $i^{th}$ digit of $f(i)$. For example, if $f(3) = 0.123000\cdots$ then $g(i) = 3$ and $k_i = 7$. Then $k \notin f(\mathbb{R})$, thus there is no surjective function $f : \mathbb{N} \to (0,1)$. Thus $|\mathbb{N}| < |(0,1)|$, and since $\mathbb{N}$ is countably infinite, $(0,1)$ must be uncountable. Therefore $|\mathbb{R}|$ is uncountable, and since $|(0,1)| = |\mathbb{R}|$. ■

**Exercise 0.3.** Let $f : \mathbb{R} - \{7\} \to \mathbb{R} - \{5\}$ be the function $f(x) = \frac{5x-2}{x-7}$.

a) Prove that $f : \mathbb{R} - \{7\} \to \mathbb{R} - \{5\}$ is a function.
*Proof:* To show that $f$ is a function, we must show that every element in the domain is mapped to an element in the codomain. To do this, we assume directly that $f(x) = \frac{5x-2}{x-7}$. Since $7 \notin \mathbb{R} - \{7\}$, the denominator of $\frac{5x-2}{x-7}$ is never zero and thus $f(x)$ does not go to infinity. We also need to make sure that $f(x) \neq 5$ for all $x$ in the domain. To do this, we solve for $x$ as follows

$$\begin{aligned} \frac{5x-2}{x-7} &= 5 \\ 5x-2 &= 5(x-7) \\ 5x-2 &= 5x-35 \\ -2 &= -35, \end{aligned}$$

which is a contradiction, thus $f(x)$ can never equal 5. Lastly, since any real number divided by a another real number not equal to 0 is another real number, we know that $\frac{5x-2}{x-7} \in \mathbb{R} - \{5\}$ for any $x \in \mathbb{R} - \{7\}$. Therefore, $f$ is a function. ■

b) Prove that $f$ is bijective.
*Proof:* To show that $f$ is bijective, we must show that it is a function, injective, and surjective. We have already shown that $f$ is a function, so we only need to show that it is injective and surjective.
**Injective**: We suppose contrapostively that $f(m) = f(k)$ for some $m, k \in \mathbb{R} - \{7\}$, then

$$\begin{aligned} f(m) &= f(k) \\ \frac{5m-2}{m-7} &= \frac{5k-2}{k-7} \\ (5m-2)(k-7) &= (5k-2)(m-7) \\ 5mk - 35m - 2k + 14 &= 5mk - 35k - 2m + 14 \\ 33k &= 33m \\ k &= m, \end{aligned}$$

since $k = m$, the function is injective.
**Surjective**: We suppose directly that $f(x) = \frac{5x-2}{x-7}$. To show that it is surjective, we need to show that for every $b \in \mathbb{R} - \{5\}$, there exists an $a \in \mathbb{R} - \{7\}$ such that $f(a) = b$. To do this, we begin by solving for $a$ as

follows

$$\begin{aligned}
\frac{5a-2}{a-7} &= b \\
5a-2 &= ba-7b \\
5a-ba &= 2-7b \\
a\left(5-b\right) &= 2-7b \\
a &= \frac{2-7b}{5-b},
\end{aligned}$$

since $b$ can never equal 5, $a$ is never infinity. We must also show that $a \neq 7$ for any $b \in \mathbb{R} - \{5\}$. This is done by assuming that $7 = \frac{2-7b}{5-b}$ for some $b$, and showing that this can never happen by solving for $b$ as follows:

$$\begin{aligned}
\frac{2-7b}{5-b} &= 7 \\
2-7b &= 35-7b \\
2 &= 35,
\end{aligned}$$

which is a contradiction, thus $7 \neq \frac{2-7b}{5-b}$ for all $b \in \mathbb{R} - \{5\}$.
Plugging in $a$ into the function yields

$$\begin{aligned}
f\left(a\right) &= \frac{5a-2}{a-7} \\
&= \frac{5\left(\frac{2-7b}{5-b}\right)-2}{\frac{2-7b}{5-b}-7} \\
&= \frac{5\left(2-7b\right)-2\left(5-b\right)}{2-7b-7\left(5-b\right)} \\
&= \frac{-35b+10-10+2b}{2-7b-35+7b} \\
&= \frac{-33b}{-33} \\
&= b,
\end{aligned}$$

hence there exists a $a$ such that $f\left(a\right) = b$. Therefore, the function $f$ is surjective.
Since $f$ is a surjective and injective function, $f$ is bijective. ■

c) Find $f^{-1} : \mathbb{R} - \{5\} \to \mathbb{R} - \{7\}$ and prove that it is the inverse.
*Proof:* We suppose directly that $f$ is a bijective function and that $f\left(x\right) = \frac{5x-2}{x-7}$. Using the results from part $b$, we get that $f^{-1}\left(b\right) = \frac{2-7b}{5-b}$. To show that $f^{-1}$ is the inverse of $f$, we will show that $f^{-1}$ is well defined and that $f^{-1} \circ f = id_{\mathbb{R}-\{7\}}$.
**Well Defined**: Since $5 \notin \mathbb{R} - \{5\}$, $\left|f^{-1}\left(b\right)\right| \neq \infty$. In addition, we will show that $7 \notin f^{-1}\left(\mathbb{R} - \{5\}\right)$ by trying to solve for $b$ such that $f^{-1}\left(b\right) = 7$ as follows

$$\begin{aligned}
\frac{2-7b}{5-b} &= 7 \\
2-7b &= 35-7b \\
2 &= 35,
\end{aligned}$$

which is a contradiction, thus $7 \notin f^{-1}\left(\mathbb{R} - \{5\}\right)$. Lastly, since a real number divided by a real number that is not zero, produces another real number $f^{-1} : \mathbb{R} - \{5\} \to \mathbb{R} - \{7\}$, and $f^{-1}$ is well defined.

**Identity**: Let $a \in \mathbb{R} - \{7\}$, then for $f^{-1} \circ f = id_{\mathbb{R}-\{7\}}$, we must get $f^{-1} \circ f(a) = a$. We show this directly,

$$\begin{aligned} f^{-1} \circ f(a) &= \frac{2 - 7\left(\frac{5a-2}{a-7}\right)}{5 - \frac{5a-2}{a-7}} \\ &= \frac{2(a-7) - 7(5a-2)}{5(a-7) - 5a - 2} \\ &= \frac{2a - 14 - 35a + 14}{5a - 35 - 5a - 2} \\ &= \frac{-33a}{33} \\ &= a, \end{aligned}$$

thus $f^{-1}$ is the inverse of $f$. ■

**Exercise 0.4.** Complete 6 of the following 8 definitions

a) A function $f : A \to B$ is injective if for all $a_1, a_2 \in A$ such that $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.
b) A function $f : A \to B$ is surjective if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.
c) If $g : S \to \mathcal{P}(S)$ is a function, then the barber set of $g$ is the set $B = \{s \in S : s \notin g(s)\}$.
d) Two sets $A, B$ have the same cardinality if there exists a bijection $f : A \to B$. Or equivalently if there exists a bijection $f^{-1} : B \to A$.
e) We write $|S| < |T|$ for sets $S$ and $T$ to mean that the cardinality of $S$ is less than the cardinality of $T$. In other words, there exists an injection $f : S \to T$, but no surjection $g : S \to T$.
f) A set $A$ is countably infinite if $|A| = |\mathbb{N}|$. In other words, there exists a bijection $f : A \to \mathbb{N}$.

**Exercise 0.5.** Let $f : A \to B$ and $g : B \to C$ be functions. Prove or disprove the following.

a) If $g \circ f$ is surjective, then $g$ is surjective.
*Proof:* We suppose directly that $f : A \to B$ and $g : B \to C$ are functions, and that $g \circ f$ is surjective. Then for every $c \in C$, there exists an $a \in A$ such that $c = g \circ f(a)$. Well, $f(a) \in B$, so let $b = f(a)$, then

$$\begin{aligned} c &= g \circ f(a) \\ &= g(f(a)) \\ &= g(b), \end{aligned}$$

which shows that for every $c \in C$, there exists a $b \in B$ such that $c = g(b)$. Therefore, $g$ is surjective. ■
b) If $f$ is injective, then so is $g \circ f$.
*Disproof:* We disprove this with a counterexample. Let $A = \{1, 2\}$, $B = \{1, 2\}$, $C =\{1, 2\}$, $f(a) = a$, and $g(b) = 1$. Then $f$ is injective, but the composition is not since $g \circ f(1) = 1$ and $g \circ f(2) = 1$. Therefore, the statement is false. ■
c) If $g$ is surjective, then so is $g \circ f$.
*Disproof:* We disprove this with a counterexample. Let $A = \{1, 2\}$, $B = \{1, 2\}$, $C =\{1, 2\}$, $f(a) = 1$, and $g(b) = b$. Then $g(b)$ is surjective, but the composition is not since $g \circ f(1) = 1$ and $g \circ f(2) = 1$. Therefore, the statement is false. ■