Midterm 1

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Exercise 1. Prove or disprove the following statements. For these statements we assume that n, a, and b are integers.

1) If $n \mid (ab)$ and $n \nmid a$, then $n \mid b$
Proof: We prove this by contradiction by assuming directly $n \mid ab$ and $n \nmid a$, and $n \nmid b$. This means that $ab = nk$, $a \neq nm$ and that $b \neq n\ell$ for some $k, m, \ell \in \mathbb{Z}$. By multiplying a and b we get that $ab \neq n^2 m\ell \neq n\,(nm\ell)$ which is a contradiction since we assumed $ab = nk$. Hence if $n \mid (ab)$ and $n \nmid a$, then $n \mid b$. ■
2) If $a \mid n$ and $b \mid n$, then $(ab) \mid n$
Disproof: Let $n = 12$, $a = 6$, and $b = 4$ then $a \mid n$ and $b \mid n$, however $ab \nmid n$ since $12 \neq 24k$ for some $k \in \mathbb{Z}$. ■
3) There exists irrational numbers a, b such that a^b is rational.
Proof: We show this considering two cases or possibilities. We use the irrational number $\sqrt{2}$, which was proven to be irrational on a homework assignment. ■

Case 1. Let $a = b = \sqrt{2}$, then if $a^b = \sqrt{2}^{\sqrt{2}}$ is rational we are done, otherwise consider the next case.
Case 2. If $\sqrt{2}^{\sqrt{2}}$ is irrational, let $a = \sqrt{2}^{\sqrt{2}}$ and let $b = \sqrt{2}$, then

$$\begin{aligned} a^b &= \sqrt{2}^{\sqrt{2}^{\sqrt{2}}} \\ &= \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}}, \text{ using the property of powers} \\ &= \sqrt{2}^{2} \\ &= 2, \end{aligned}$$

which is rational.

Hence we have shown that there exists irrational numbers a, b such that a^b is rational.

Exercise 2. Let $s = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ and answer the following questions.

1) How many elements does S have?
   a) S has three elements, the empty set, the set containing the empty set, and the set containing the set that contains the empty set.
2) Which of the following are elements of S and which are subsets $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$?
   a) The empty set $\emptyset$ is an element of S as shown in the listing of S, and the empty set is a subset of every set so it must be a subset of S.
   b) The set $\{\emptyset\}$ is an element of S as shown in the listing of S, and it is a subset because the set $\{\emptyset\}$ only contains the empty set which is an element of S.
   c) The set $\{\emptyset, \{\emptyset\}\}$ is not an element of S since it is not shown in the listing of S; however, it is a subset since both $\emptyset$ and $\{\emptyset\}$ are elements of S.
3) Calculate the following, $\{\emptyset\} \cap S$ and $\emptyset \cup S$
   a) $\{\emptyset\} \cap S = \{\emptyset\}$ since the empty set is the only element they have in common.
   b) $\emptyset \cup S = S$ since the empty set doesn't have any elements. The union of the empty set with any set A must be A.

Exercise 3. Prove the following statement in three ways, directly, contrapositively and by contradiction.
Thm. For all integers n if $n + 3$ is even, then $3n + 4$ is odd.

1) **Directly**
Proof: We suppose directly that $n + 3$ is even. This means that $n + 3 = 2k$ for some $k \in \mathbb{Z}$. Solving for n we get $n = 2k - 3$. Substituting this into $3n + 4$ yields

$$\begin{aligned} 3n + 4 &= 3\,(2k - 3) + 4 \\ &= 3\,(2k) - 9 + 4 \\ &= 3\,(2k) - 5 \\ &= 3\,(2k - 2) + 1, \end{aligned}$$

which is odd. Hence if $n + 3$ is even, then $3n + 4$ is odd. ■
2) **Contrapositively**

Proof: We suppose contrapositively that $3n+4$ is even. This means that $3n+4=2k$ for some $k\in\mathbb{Z}$. Solving for $3n$ we get

$$\begin{aligned}3n &= 2k-4\\ &= 2\left(k-2\right),\end{aligned}$$

which is positive. According to the corollary 7, since 3 is odd (can be written as $2+1$), n must be even in order for their product to be even. This means that $n=2m$ for some $m\in\mathbb{Z}$. Substituting $2m$ for n in $n+3$ yields

$$\begin{aligned}n+3 &= 2m+3\\ &= 2\left(m+1\right)+1,\end{aligned}$$

which is odd. Hence if $3n+4$ is even, then $n+3$ is odd. ■

3) **Contradiction**

Proof: We assume that $n+3$ is even and that $3n+4$ is even. We have previously shown that in order for $3n+4$ to be even, n must be even. This means that $n=2m$ for some $m\in\mathbb{Z}$. Substituting this into $n+3$ yields

$$\begin{aligned}n+3 &= 2m+3\\ &= 2\left(m+1\right)+1,\end{aligned}$$

which is odd. This is a contradiction since we assumed that $n+3$ is even. Hence the compound statement $n+3$ is even and $3n+4$ is even is false. ■

Exercise 4. Write a truth table for the statement $(P\vee Q)\implies(Q\wedge R)$

P	Q	R	$P\vee Q$	$Q\wedge R$	$(P\vee Q)\implies(Q\wedge R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	T	F	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	F	F	T

Exercise 5. For sets A, B, C prove that $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

Proof: We must show that $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$ and $A\cap(B\cup C)\supseteq(A\cap B)\cup(A\cap C)$.

$(\subseteq)$: We assume directly that $x\in A\cap(B\cup C)$. This means that $x\in A$ and $x\in B\cup C$. Since x must be in either B, C or both we have three cases.

Case 1. Assume that $x\in B$, then $x\in A$ and $x\in B$. In other words $x\in A\cap B$.

Case 2. Assume that $x\in C$, then $x\in A$ and $x\in C$. In other words $x\in A\cap C$.

Case 3. Assume that $x\in B$ and $x\in C$, then the two previous cases hold and $x\in A\cap B\cap C$.

Regardless of which cases are true, since one of them must be true we know that x is an element of $A\cap B$, $A\cap C$, or $A\cap B\cap C$. In other words $x\in(A\cap B)\cup(A\cap C)\cup(A\cap B\cap C)$. This can be simplified by noting that if $x\in A\cap B\cap C$ then it must be in $A\cap B$ and $A\cap C$. If x is in both $A\cap B$ and $A\cap C$, then it is also in $(A\cap B)\cup(A\cap C)$, thus the expression can be reduced to $x\in(A\cap B)\cup(A\cap C)$. Hence $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$

$(\supseteq)$: We assume directly that $x\in(A\cap B)\cup(A\cap C)$. Therefore x is in $A\cap B$ or $A\cap C$. We now consider the following three cases.

Case 1. Assume that $x\in A\cap B$. Therefore, $x\in A$ and $x\in B$.

Case 2. Assume that $x\in A\cap B$. Therefore, $x\in A$ and $x\in C$.

Case 3. Assume that $x\in A\cap B$ and $x\in A\cap C$. Therefore, $x\in A$, $x\in B$, and $x\in C$.

Regardless of the case, $x\in A$ and x is either and element of B, C or both. In other words, $x\in A\cap(B\cup C\cup B\cap C)$. If $x\in B\cap C$ then $x\in B$ and $x\in C$. This is stricter than $x\in B\cup C$, i.e. $B\cap C\subseteq B\cup C$, so we can reduce $x\in A\cap(B\cup C\cup B\cap C)$ to $x\in A\cap(B\cup C)$. Hence $A\cap(B\cup C)\supseteq(A\cap B)\cup(A\cap C)$.

Of course we could've applied De Morgan's law to simply get

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C),$$

but that's too easy.

■

Appendix

Lemma 6. *The product of two even numbers is even, the product of an even number and an odd number is even, and the product of two odd numbers is odd.*

Proof: We show this by proving all three cases. ■

Case 1. Let a and b be even, then they can be written as $a = 2k$ and $b = 2m$ for some $k, m \in \mathbb{Z}$. Their product is $ab = (2k)(2m) = 2(2km)$ which is even.

Case 2. Without loss in generality, let a be even and b be odd. Then they can be written as $a = 2k$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$. Their product is

$$ab = 2k(2m+1)$$
$$2(2mk+k),$$

which is even.

Case 3. Let a and b be odd, then they can be written as $a = 2k+1$ and $b = 2m+1$ for some $k, m \in \mathbb{Z}$. Their product is

$$ab = (2k+1)(2m+1)$$
$$2(2km+k+m)+1,$$

which is odd.

Corollary 7. *The product of an odd integer a with another integer b is even if and only if b is even.*

Proof: This proof follows directly from lemma 6. If b is odd, then ab is even. And if b is even, then ab is even. Hence, ab is even only if b is even. ■