Final Exam

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Exercise 1. Prove that $S \cap (T \cup R) = (S \cap T) \cup R \iff R \subseteq S$ for all sets S, T, R.

Proof: This is a biconditional statement so we must prove both ways.

$(\implies)$: We suppose directly that $S \cap (T \cup R) = (S \cap T) \cup R$. Then by distributing the union we get $S \cap (T \cup R) = (S \cup R) \cap (T \cup R)$. For this equality to hold, $S \cap (T \cup R) \subseteq (S \cup R) \cap (T \cup R)$ and $S \cap (T \cup R) \supseteq (S \cup R) \cap (T \cup R)$. We will see under which conditions these hold.

$(\subseteq)$: Let $x \in S \cap (T \cup R)$, then $x \in S$ and $x \in T$ or $x \in R$. Thus $x \in (S \cup R)$ and $x \in (T \cup R)$. Therefore $x \in (S \cup R) \cap (T \cup R)$. Thus $S \cap (T \cup R) \subseteq (S \cup R) \cap (T \cup R)$ for any sets S, T, R.

$(\supseteq)$: Let $x \in (S \cup R) \cap (T \cup R)$, then $x \in (S \cup R)$ and $x \in (T \cup R)$. In other words, $x \in S$ or $x \in R$ and $x \in T$ or $x \in R$. Assume that $x \notin S$, and that $x \in R$, then $x \in (S \cup R) \cap (T \cup R)$, but $x \notin S \cap (T \cup R)$. This can't be the case, since $S \cap (T \cup R) = (S \cap T) \cup R$. Hence, if $x \in R$, it must be that $x \in S$ in order for $x \in S \cap (T \cup R)$. Therefore $R \subseteq S$.

$(\impliedby)$: We suppose directly that $R \subseteq S$. The term $(S \cap T) \cup R$ can be expanded by distributing the union to get $(S \cup R) \cap (T \cup R)$. Since $R \subseteq S$, $S \cup R = S$. Thus $(S \cup R) \cap (T \cup R) = S \cap (T \cup R)$.

Since we have shown both implications, the statement $S \cap (T \cup R) = (S \cap T) \cup R \iff R \subseteq S$ for all sets S, T, R is true. ■

Exercise 2. Show that the function $f(x) = \frac{3x+1}{5x+2}$ is continuous at $x = 1$ by giving ϵ, δ proof of the limit as $x \to 1$.

Proof: Let $\epsilon \in \mathbb{R} > 0$, $\delta = \min\left(\frac{1}{5}, 42\epsilon\right)$, and $x \in \mathbb{R}$. We suppose directly that $0 < |x - 1| < \delta$. We first verify that the function is defined at 1.

$$\begin{aligned} f(1) &= \frac{3 \cdot 1 + 1}{5 \cdot 1 + 2} \\ &= \frac{4}{7}. \end{aligned}$$

Since $\delta = \min\left(\frac{1}{5}, 42\epsilon\right)$, $\delta \leq \frac{1}{5}$. Thus

$$\begin{aligned} |x - 1| &< \frac{1}{5} \\ -\frac{1}{5} < x - 1 &< \frac{1}{5} \\ -\frac{35}{5} < 35x - 35 &< \frac{35}{5} \\ -7 < 35x - 35 &< 7 \\ -7 + 49 < 35x - 35 + 49 &< 7 + 49 \\ 42 < 35x + 14 &< 56 \\ 42 < |35x + 14| &< 56. \end{aligned}$$

Hence

$$\begin{aligned}
\frac{|x-1|}{|35x+14|} &< \frac{\delta}{42} \\
\left|\frac{x-1}{35x+14}\right| &< \frac{42\epsilon}{42} \\
\left|\frac{21x+7-20x-8}{35x+14}\right| &< \epsilon \\
\left|\frac{7\left(3x+1\right)-4\left(5x+2\right)}{7\left(5x+1\right)}\right| &< \epsilon \\
\left|\frac{3x+1}{5x+1}-\frac{4}{7}\right| &< \epsilon \\
\left|f\left(x\right)-f\left(a\right)\right| &< \epsilon
\end{aligned}$$

Therefore, $\forall\epsilon \in \mathbb{R} > 0$, $\exists\delta \in \mathbb{R} > 0$, $x \in \mathbb{R}$ such that if $0 < |x-1| < \delta$, then $\left|f\left(x\right)-f\left(a\right)\right| < \epsilon$. ■

Exercise 3. Let $x \in \mathbb{Z}$. Prove that if $5x+7$ is even, then $3x+2$ is odd in three different ways: directly, contrapositively, and by contradiction.

a) **Directly**
Proof: We suppose directly that $5x+7$ is even, then $5x+7 = 2k$ for some $k \in \mathbb{Z}$. Thus

$$\begin{aligned}
5x+7 &= 2k \\
5x+7-2x-5 &= 2k-2x-5 \\
3x+2 &= 2\left(k-x-3\right)+1.
\end{aligned}$$

Hence $3x+2$ is odd. ■

b) **Contrapositively**
Proof: We assume contrapositively that $3x+2$ is even. Then $3x+2 = 2n$ for some $n \in \mathbb{Z}$. Thus

$$\begin{aligned}
3x+2 &= 2n \\
3x+2+2x+5 &= 2n+2x+5 \\
5x+7 &= 2\left(n+x+2\right)+1,
\end{aligned}$$

which shows that $5x+7$ is odd. Therefore, if $5x+7$ is even, then $3x+2$ is odd. ■

c) **Contradiction**
Proof: We assume by contradiction that $5x+7$ is even and $3x+2$ is even. Then $5x+7 = 2k$ for some $k \in \mathbb{Z}$ and $3x+2 = 2n$ for some $n \in \mathbb{Z}$. Thus

$$\begin{aligned}
5x+7+3x+2 &= 2k+2n \\
8x+9 &= 2k+2n \\
2\left(4x+4\right)+1 &= 2\left(k+n\right),
\end{aligned}$$

which is a contradiction since the left hand side is odd and the right hand side is even. Therefore, if $5x+7$ is even, then $3x+2$ is odd. ■

Exercise 4. Let $A = (0,1) \cup (2,5) \cup \{7, 10, \pi\}$ and $B = (8, 13)$. Show that A has the same cardinality as B.

Proof: To show that $|A| = |B|$, we can show that there exists an injective map $f : A \to B$ and an injective map $g : B \to A$. We also quickly note that $A = (0,1) \cup (2,5) \cup \{7, 10\}$ since $\pi \in (2,5)$.

$f : A \to B$: Let $A_1 = (0,1)$, $A_2 = (2,5)$, $A_3 = \{7\}$, $A_4 = \{10\}$, $B_1 = (8,9)$, $B_2 = (9,12)$, $B_3 = \{9\}$ and $B_4 = \{12.9999\}$. We can define the injective functions

$$\begin{aligned} f_1 &: A_1 \to B_1; x \mapsto x+8 \\ f_2 &: A_2 \to B_2; x \mapsto x+7 \\ f_3 &: A_3 \to B_3; x \mapsto 9 \\ f_4 &: A_4 \to B_4; x \mapsto 12.9999. \end{aligned}$$

It is easily seen that the functions are injective. Since $\{A_i\}_{i\in\{1,2,3,4\}}$ forms a partition of A, and since all of the functions f_i are injective with disjoint codomains, we can glue the functions together to form one injective function $f : A \to B$ defined as $f(x) = f_i(x)$ when $x \in A_i$.

$g : B \to A$. Let $A_1 = (0,1) \subseteq A$, and define g as the injection $g(x) = \frac{x-8}{5}$. It's injective since if

$$\begin{aligned} f(b_1) &= f(b_2) \\ \frac{b_1 - 8}{5} &= \frac{b_2 - 8}{5} \\ b_1 &= b_2 \end{aligned}$$

for all $b_1, b_2 \in B$.

Since both function f and g are injective, $|A| = |B|$. ■

Exercise 5. let $x_1 = 1$ and $x_{n+1} = \sqrt{1+3x_n}$. Prove that $x_n \le 4$ for all $n \in \mathbb{N}$.

Proof: We want to prove that the open sentence

$$P(n): \text{ Given } x_1 = 1 \text{and } x_{n+1} = \sqrt{1+3x_n},\ x_n \le 4,$$

for all $n \in \mathbb{N}$. We work this by induction.

Base Case: We verify $P(1)$ and $P(2)$. Since $x_1 = 1$ and $x_2 = 2$, we see that $x_1 \le 4$ and $x_2 \le 4$. Thus $P(1)$ and $P(2)$ are true.

Inductive Step: Let $k \in \mathbb{N}$. We suppose that $P(k)$ is true, and we want to show that $P(k+1)$ is true, which is the statement $x_{k+1} \le 4$. Since $P(k)$ is true, we know that $x_k \le 4$. We now solve for x_{k+1}.

$$\begin{aligned} x_{k+1} &= \sqrt{1+3x_k} \\ &\le \sqrt{1+3\cdot 4} \\ &= \sqrt{13} \\ &< 4, \end{aligned}$$

thus $P(k+1)$ is true. Therefore $P(n)$ is true for all $n \in \mathbb{N}$. ■