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Sections 24 and 25

24. Defining Functions

Exercise 24.1. Which of the following relations are functions from the set $A = \{1,2,3,4\}$ to the set $B = \{1,2,3,4,5\}$?

a) $f_1 = \{(1,3),(2,3),(3,3),(4,3)\}$
   This is a function since $\forall a \in A, \exists! b \in B, f(a) = b$.
b) $f_2 = \{(1,2),(2,3),(3,5),(4,6)\}$
   This is not a function since $f_2(4) = 6 \notin B$.
c) $f_3 = \{(1,2),(2,3),(2,4),(4,5)\}$
   This is not a function since $f_3(2) = 3, 4$
d) $f_4 = \{(1,2),(1,3),(2,3),(3,4),(4,1)\}$
   This is not a function since $f_4(1) = 2, 3$
e) $f_5 = \{(1,2),(2,3),(4,5)\}$
   This is not a function since $f_5(3)$ is not mapped to anything
f) $f_6 = \{(1,2),(1,2),(2,3),(3,4),(4,1)\}$
   This is a function since $\forall a \in A, \exists! b \in B, f(a) = b$.

Exercise 24.2. In some textbooks it is claimed that $f : \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = \frac{1}{x}$ is a function. Technically this is incorrect because the rule is not defined at $x = 0$, and so the largest available domains should be $\mathbb{R} - \{0\}$. Find the largest domain for the following functions (supposedly defined from $\mathbb{R}$ to $\mathbb{R}$)

a) $f(x) = \sin(x)$
   Domain is $\mathbb{R}$
b) $g(x) = \tan(x)$
   Domain is $\left\{x \in \mathbb{R} : x \neq \frac{\pi}{2} k \, \forall k \in \mathbb{Z} - \{0\}\right\}$
c) $h(x) = \ln(x)$
   Domain is $\{x \in \mathbb{R} : x > 0\}$
d) $p(x) = \sqrt{1 - x}$
   Domain is $\{x \in \mathbb{R} : x \geq 1\}$
e) $q(x) = \sqrt[3]{x}/(2 + x)$
   Domain is $\{x \in \mathbb{R} : x \geq 0\}$

Exercise 24.3. Let A be a finite set and let B be any set. Let $f : A \to B$ be a function. Considering f as a set of ordered pairs, prove that $|f| = |A|$.

Proof: We suppose directly that $f : A \to B$, then $\forall a \in A, \exists! b \in B, f(a) = b$. This means that $(a, b) \in f$. Since each element of a must be a first coordinated in an ordered pair of f, and that each element of a is only paired with one element of B, we get $|f| = |A|$. ∎

Exercise 24.4. For $a \in \mathbb{Z}$, denote the congruence class of a modulo 8 by $\bar{a}$, and the congruence class of a modulo 4 by $[a]$. Determine which of the following definitions give well-defined functions. For those that are well-defined, give a proof. For those that are not well defined, give an example to demonstrate this face.

a) Define $f : \mathbb{Z}_8 \to \mathbb{Z}_4$ by $f(\bar{a}) = [a]$.

This is well defined.

Proof: We suppose directly that $\bar{a} \in \mathbb{Z}_8$, then $8 \mid a - x$ for some $a, x \in \bar{a}$. Since $4 \mid 8$, we know that $4 \mid a - x$. Therefore $\bar{a} = [a]$. ■

b) Define $g : \mathbb{Z}_4 \to \mathbb{Z}_8$ by $g([a]) = \bar{a}$.

This is not well defined

Example: Let $13 \in [1]$ such that $4 \mid (13 - 1)$; however $8 \nmid (13 - 1)$, thus $13 \in \bar{1}$.

c) Define $h : \mathbb{Z}_4 \to \mathbb{Z}_8$ by $h([a]) \to \overline{2a}$

This is well defined

Proof: We suppose directly that $a, b \in [a]$ such that. Then $4 \mid a - b$, in other words $a - b = 4m$ for some $m \in \mathbb{Z}$. Multiplying both sides by 2 yields $2(a - b) = 2 \cdot 4m$ which is equivalent to $8 \mid 2(a - b)$. Thus if $a, b \in [a]$, then $2a, 2b \in \overline{2a}$. ■

d) Define $j : \mathbb{Z}_4 \to \mathbb{Z}_8$ by $j([a]) = \overline{3a}$.

This is not well defined

Example: Let $5 \in [1]$ then $4 \mid 5 - 1$, Multiplying 5 and 3 by three gives $15 - 3 = 12$ which is not divisible by 8. Thus j is not a function.

Exercise 24.5. Define $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $f(\bar{a}) = \bar{a}^5$. Define $g : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $g(\bar{a}) = \bar{a}$. You may assume that f and g are both well-defined. Are f and g equal?

Yes

Proof: We suppose directly that $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $f(\bar{a}) = \bar{a}^5$, $g : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $g(\bar{a}) = \bar{a}$, and that they are both well defined. For two functions to be equal, they must have the same domain, co-domain, and same set of ordered pair. By inspection we can see that they have the same domain and co-domain. The relation g is the set

$$g = \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1}), (\bar{2}, \bar{2}), (\bar{3}, \bar{3}), (\bar{4}, \bar{4})\}.$$

For the relation f , let $\bar{a} = \bar{b}$ such that $5 \mid a - b$. Taking each a and b to the power of 5 yields $5 \mid a^5 - b^5$. According to Theorem 23.8, since $5 \mid a - b$, then $5 \mid a^5 - b^5$. thus $a^5, b^5 \in \bar{a}^5$. (I just proved that it is well defined again). Since it is well defined, we can take any representative of the class, take it to the power of 5 and see which new class it belongs to. This new class is also the class that all other possible representative will belong to. I will use the simple representatives

$$\begin{aligned}
0^5 &\mod 5 = 0 \\
1^5 &\mod 5 = 1 \\
2^5 &\mod 5 = 2 \\
3^5 &\mod 5 = 3 \\
4^5 &\mod 5 = 4,
\end{aligned}$$

thus the relation on f is

$$f = \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1}), (\bar{2}, \bar{2}), (\bar{3}, \bar{3}), (\bar{4}, \bar{4})\}$$

which is the same as g. Therefore, they are equal. ■

Exercise 24.6. Let A be a set, and let S_1, S_2 be subsets of A. Given an arbitrary $a \in A$, prove the following:

a) If $T = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, then $\chi_{T_1}(a) = \chi_{S_1}(a) + \chi_{S_2}(a)$

Proof: We suppose directly that $T = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. We have two cases to consider.

Case 1. Let $a \notin T$, then $a \notin S_1$ and $a \notin S_2$, thus $\chi_{T_1}(a) = 0$

Case 2. Let $a \in T$, then $a \in S_1$ or $a \in S_2$ since $S_1 \cap S_2 = \emptyset$. Without loss of generality, suppose $a \in S_1$, Then

$$\begin{aligned} \chi_{T_1}(a) &= \chi_{S_1}(a) \cdot \chi_{S_2}(a) \\ 1 &= 1 + 0, \end{aligned}$$

which is true.

Since the statement holds for both cases, the statement is true.

■

b) If $T = S_1 \cap S_2$, then $\chi_T(a) = \chi_{S_1}(a) \cdot \chi_{S_2}(a)$
Proof: We suppose directly that $T = S_1 \cap S_2$. We have four cases to consider:

Case 1. Let $a \in S_1$ and $a \in S_2$, then $a \in T$. Thus

$$\begin{aligned} \chi_{T_1}(a) &= \chi_{S_1}(a) \cdot \chi_{S_2}(a) \\ 1 &= 1 \cdot 1, \end{aligned}$$

which is true.

Case 2. Let $a \in S_1$ and $a \notin S_2$, then $a \notin T$. Thus

$$\begin{aligned} \chi_{T_1}(a) &= \chi_{S_1}(a) \cdot \chi_{S_2}(a) \\ 0 &= 1 \cdot 0 \\ 0 &= 0, \end{aligned}$$

which is true.

Case 3. Let $a \notin S_1$ and $a \in S_2$. This is similar to the previous case.

Case 4. Let $a \notin S_1$ and $a \notin S_2$, then $a \notin T$. Thus

$$\begin{aligned} \chi_{T_1}(a) &= \chi_{S_1}(a) \cdot \chi_{S_2}(a) \\ 0 &= 0 \cdot 0, \end{aligned}$$

which is true

Since all cases hold, the statement is true.

■

c) If $T = S_1 \cup S_2$, then $\chi_T(a) = \chi_{S_1}(a) + \chi_{S_2}(a) - \chi_{S_1}(a) \cdot \chi_{S_2}(a)$.
Proof: We suppose directly that $T = S_1 \cup S_2$. We have four cases to consider:

Case 1. Let $a \in S_1$ and $a \in S_2$, then $a \in T$. Thus

$$\begin{aligned} \chi_{T_1}(a) &= \chi_{S_1}(a) + \chi_{S_2}(a) - \chi_{S_1}(a) \cdot \chi_{S_2}(a) \\ 1 &= 1 + 1 - 1 \cdot 1 \\ &= 1, \end{aligned}$$

which is true.

Case 2. Let $a \in S_1$ and $a \notin S_2$, then $a \in T$. Thus

$$\begin{aligned} \chi_{T_1}(a) &= \chi_{S_1}(a) + \chi_{S_2}(a) - \chi_{S_1}(a) \cdot \chi_{S_2}(a) \\ 1 &= 1 + 0 - 1 \cdot 0 \\ 0 &= 1, \end{aligned}$$

which is true.

Case 3. Let $a \notin S_1$ and $a \in S_2$. This is similar to the previous case.

Case 4. Let $a \notin S_1$ and $a \notin S_2$, then $a \notin T$. Thus

$$\begin{aligned} \chi_{T_1}(a) &= \chi_{S_1}(a) + \chi_{S_2}(a) - \chi_{S_1}(a) \cdot \chi_{S_2}(a) \\ 0 &= 0 + 0 - 0 \cdot 0 \\ &= 0, \end{aligned}$$

which is true.

■

Since all cases hold, the statement is true.

Exercise 24.7. Let A and B be sets. Let I be an indexing set, and let $P = \{P_i : i \in I\}$be an arbitrary partition of A. For each $i \in I$, let $f_i : P_i \to B$ be a function. Prove that the relation

$$f = \bigcup_{i \in I} f_i$$

is a function from $A \to B$ and that the rule for f (as a piecewise defined function) is

$$f(a) = f_i(a) \quad \text{if a} \in P_i \text{ for some } i \in I.$$

Proof: We can consider the function f_i as a subset of $P_i \times B$ which is a subset of $A \times B$. Thus $f = \cup_{i \in I} f_i$ is also a subset of $A \times B$, so f is a relation from A to B. To see that f is a function, it suffices to check that each element of A is the first coordinate of exactly one ordered pair in f.

Let $a \in A$. Since the partition of P has $|P|$ disjoint, nonempty sets that cover A, there are $|P|$ cases. Without loss in generality, suppose $a \in P_i$. Since $a \in P_i$ it can't belong to any other element of the partition, and since $f_i : P_i \to B$ we have that a is the first coordinate of the ordered pair $(a, f_i(a)) \in P_i \times B \subseteq A \times B$ and cannot be the first coordinate of any other ordered pair in $P_i \times B$. Thus a is the first coordinate for one and only one ordered pair in f and the rule for f is

$$f(a) = f_i(a) \quad \text{if a} \in P_i \text{ for some } i \in I.$$

■

25. Injective and Surjective Functions

Exercise 25.1. Let $A = \{1,2,3\}$ and $B = \{x,y\}$. List all functions from $A \to B$, and for each function state whether it is injective, surjective, bijective or none of the above. Then do the same for all functions $B \to A$

$A \to B$

a) $f_1 = \{(1,x),(2,x),(3,x)\}$ None
b) $f_2 = \{(1,y),(2,y),(3,y)\}$ None
c) $f_3 = \{(1,x),(2,x),(3,y)\}$ Surjective
d) $f_4 = \{(1,x),(2,y),(3,x)\}$ Surjective
e) $f_5 = \{(1,x),(2,y),(3,y)\}$ Surjective
f) $f_6 = \{(1,y),(2,x),(3,x)\}$ Surjective
g) $f_7 = \{(1,y),(2,x),(3,y)\}$ Surjective
h) $f_8 = \{(1,y),(2,y),(3,x)\}$ Surjective

$B \to A$

a) $h_1 = \{(x,1),(y,1)\}$ None
b) $h_2 = \{(x,2),(y,2)\}$ None
c) $h_3 = \{(x,3),(y,3)\}$ None
d) $h_4 = \{(x,1),(y,2)\}$ Injective
e) $h_5 = \{(x,1),(y,3)\}$ Injective
f) $h_6 = \{(x,2),(y,1)\}$ Injective
g) $h_7 = \{(x,3),(y,1)\}$ Injective
h) $h_8 = \{(x,2),(y,3)\}$ Injective
i) $h_9 = \{(x,3),(y,2)\}$ Injective

Exercise 25.2. For each of the following, determine (with proof) whether the function is injective and/or surjective

a) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 2n+1$
Proof: We wish to show that f is injective and not surjective. We first show that it is injective.
Let $k, m \in \mathbb{Z}$, then. We assume contrapositively that $f(k) = f(m)$, thus

$$\begin{aligned} 2k+1 &= 2m+1 \\ 2k &= 2m \\ k &= m \end{aligned}$$

Thus the function is injective.
To show that it isn't surjective we will prove the negation, that $\exists b, \forall a, f(a) \neq b$. Since the function $f(n)$ maps integers to odd numbers, then any b that is even satisfies what we are trying to prove.
Thus the function f is injective and not surjective. ■

b) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2 + 2x + 2$.
Proof: We wish to show that f is neither injective or surjective. We first show that it is not injective.
We assume by negation that $a_1, a_2 \in \mathbb{R}, a_1 \neq a_2, g(a_1) = g(a_2)$. This can be shown with an example. Let $a_1 = -2$ and $a_2 = 0$, then $g(a_1) = g(a_2) = 2$.
To show that it is not surjective we will prove the negation, that $\exists b \in \mathbb{R}, \forall a \in \mathbb{R}, g(a) \neq b$. By solving for a we get $a^2 + 2a + 2 = b$, which can be written as

$$a^2 + 2a + 2 - b = 0$$

which has roots

$$r_1, r_2 = -1 \pm \sqrt{1+b-2},$$

with r_1, r_2 being the roots of the polynomial. When $b < 1$, the roots are complex. This means that for any $b < 1$, there is not an $a \in \mathbb{R}$ such that $f(a) = b$. Therefore, the function is not surjective. ■

c) Define $h : \mathbb{Z} \to \mathbb{Z}$ by $h(n) = n + 3$.
Proof: We wish to show that h is injective and surjective. We will show that it is injective first.
Assume contrapositively that $h(m) = h(k)$ for some $m, k \in \mathbb{Z}$, then

$$\begin{aligned} m + 3 &= k + 3 \\ m &= k, \end{aligned}$$

thus h is injective.
To show that it is surjective, let $h(a) = b$ for some $a, b \in \mathbb{Z}$, then

$$\begin{aligned} b &= a + 3 \\ b - 3 &= a, \end{aligned}$$

plugging in a into the function h yields

$$\begin{aligned} h(a) &= (b-3) + 3 \\ &= b \end{aligned}$$

which shows that we can reach any $b \in \mathbb{Z}$. Therefore, the function is surjective. ■

Exercise 25.3. Define $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $f(\bar{a}) = \overline{2a+3}$.

a) Prove that f is well defined.
Proof: We suppose directly that $\bar{a} = \bar{b} \in \mathbb{Z}_5$, then $5 \mid a - b$. Hence $5 \mid 2(a-b)$ which means $5 \mid 2a - 2b$. Also $5 \mid 2a + 3 - 2b - 3$ which equals

$$5 \mid (2a+3) - (2b+3).$$

This is equivalent to

$$2a + 3 \equiv 2b + 3 \mod 5;$$

therefore the function f is well defined. ■

b) If f injective? Surjective? Give proofs. (Hint: You cannot divide by $\bar{2}$, but you can multiply by $\bar{3}$)
Proof: We wish to prove that it is both surjective and injective. We show this directly using a representative from each equivalence class and applying it to the function

$$\begin{aligned} f(0) &= 3 \\ f(1) &= 0 \\ f(2) &= 2 \\ f(3) &= 4 \\ f(4) &= 1, \end{aligned}$$

which shows that it is both injective and surjective. Hence bijective. ■

Exercise 25.4. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is

a) neither injective nor surjective
Let $f : \mathbb{R} \to \mathbb{R}$; $a \mapsto 0$

Proof: We wish to show that f is neither injective nor surjective by proving their negations. We begin with showing that it is not injective.
Suppose that there exists an $\exists a_1, a_2 \in \mathbb{R}, a_1 \neq a_2, f(a_1) = f(a_2)$. Since every $a \in \mathbb{R}$ is mapped to 0, then it is not one-to-one and thus not injective.
To show that it is not surjective, suppose that $a, b \in \mathbb{R}$ and let $b \neq 0$. Then there is no $a \in \mathbb{R}$ such that $f(a) = b$ since f maps everything to 0.
Therefore, f is neither injective nor surjective. ■

b) injective but not surjective

Let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(a) = \begin{cases} a+1 & \text{if } a \geq 0 \\ a & \text{if } a < 0 \end{cases}$$

Proof: We wish to show that g is injective but not surjective. We begin by first showing that it is injective.
Suppose contrapositively that $a_1, a_2 \in \mathbb{R}$ and that $g(a_1) = g(a_2)$. We have four cases to consider.

Case 1. Let $a_1, a_2 \geq 0$, then

$$\begin{aligned} g(a_1) &= g(a_2) \\ a_1 + 1 &= a_2 + 1 \\ a_1 &= a_2 \end{aligned}$$

Case 2. Let $a_1 \geq 0$ and $a_2 < 0$, then

$$\begin{aligned} g(a_1) &= g(a_2) \\ a + 1 &= a, \end{aligned}$$

which is a contradiction, and thus can never happen.

Case 3. Let $a_2 \geq 0$ and $a_1 < 0$. This is similar to the previous case.

Case 4. Let $a_1, a_2 < 0$, then

$$\begin{aligned} g(a_1) &= g(a_2) \\ a_1 &= a_2, \end{aligned}$$

which is true.

Since all four cases hold, the function g is injective.
To show that it isn't surjective, we prove the negative of surjective. We do this with an example. Let 0 be an element of the codomain. We have two cases to consider.

Case 1. Let $a < 0$ be an element of the domain. Then $g(a) = a$ which will always be less than 0, and can never equal 0.

Case 2. Let $a \geq 0$, then $g(a) = a + 1$ which is always greater than 0, and can never be 0.

Thus the element 0 is not an image of any element a of the domain. Therefore, g is not surjective.

■

c) surjective but not injective

Let $h : \mathbb{R} \to \mathbb{R}$ be defined as

$$h(a) = \begin{cases} a & \text{if } a < 0 \\ a-1 & \text{if } a \geq 0 \end{cases}$$

Proof: We wish to show that h is surjective but not injective. We will first show that it is surjective.

We assume that b is an element of the codomain and a is an element of the domain. We have two cases to consider.

Case 1. Let $b < 0$, and let $a = b$, then $h(a) = b$.

Case 2. Let $b \geq 0$, and let $a = b + 1$, then $h(a) = b$.

Therefore, the map h is surjective since the range of h is all of $\mathbb{R}$.

To show that the map h is not injective, we give an example. Let $a_1 = -0.5$ and $a_2 = 5$ be elements of the domain, then $h(a_1) = h(a_2) = -0.5$. Thus there exists elements a_1 and a_2 in the domain such that $h(a_1) = h(a_2)$ but $a_1 \neq a_2$. Therefore, h is not injective. ■

d) both injective and surjective

Let $j : \mathbb{R} \to \mathbb{R}$ be the identity map.

Proof: We wish to show that j is both injective and surjective. We begin by showing that it is injective.

We suppose contrapositively that a_1, a_2 are elements of the domain and that $j(a_1) = j(a_2)$, then

$$\begin{aligned} j(a_1) &= j(a_2) \\ a_1 &= a_2, \end{aligned}$$

which is true. Therefore, it is surjective.

To show that it is surjective, let b be an element of the codomain and let $a = b$ be an element of the domain. Then $j(a) = b$ which shows that for every $b \in \mathbb{R}$ there exists and $a \in \mathbb{R}$ such that $j(a) = b$. ■

Exercise 25.5. Define $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{1\}$ by

$$f(x) = \frac{x-3}{x-2}.$$

a) Prove that f is a function from $\mathbb{R} - \{2\}$ to $\mathbb{R} - \{1\}$.

Proof: We suppose directly that $a \in \mathbb{R} - \{2\}$ then

$$f(a) = \frac{a-3}{a-2}$$

which is an element of $\mathbb{R}$. We need to verify that $f(a) \neq 1$ for all a. In order for $f(a) = 1$ we need that

$$\frac{a-3}{a-2} = 1,$$

solving for a gives us $a = a + 1$ which can never happen. Thus 1 is not in the image of $f(a)$. ■

b) Prove that f is injective.

Proof: We assume contrapositively that $a, b \in \mathbb{R} - \{2\}$ and that $f(a) = f(b)$. Thus

$$\frac{a-3}{a-2} = \frac{b-3}{b-2},$$

multiplying by the denominators on both sides yields

$$\begin{aligned} (a-3)(b-2) &= (b-3)(a-2) \\ ab - 2a - 3b + 6 &= ab - 3a - 2b + 6 \\ a &= b, \end{aligned}$$

thus it is injective. ■

c) Prove that f is surjective.

Proof: We assume directly that $c \in \mathbb{R} - \{1\}$ and that $a \in \mathbb{R} - \{2\}$. Solving for a gives us

$$\begin{aligned} \frac{a-3}{a-2} &= c \\ a-3 &= c\,(a-2) \\ a-3 &= ca-2c \\ a-ca &= 3-2c \\ a &= \frac{3-2c}{1-c}, \end{aligned}$$

plugging a into the function yields

$$\begin{aligned} f\,(a) &= \frac{\frac{3-2c}{1-c}-3}{\frac{3-2c}{1-c}-2} \\ &= \frac{3-2c-3\,(1-c)}{3-2c-2\,(1-c)} \\ &= \frac{c}{1} \\ &= c, \end{aligned}$$

thus for every $c \in \mathbb{R} - \{1\}$, there is an $a \in \mathbb{R} - \{2\}$ such that $f\,(a) = c$. Therefore f is surjective. ■

Exercise 25.6. Define $f : \mathbb{Z}^2 \to \mathbb{Z}$ by $f\,(m,n) = 3m - 2n$. Is f injective? Surjective?

Injective:

Proof: We wish to show that f is not injective. We suppose that there exists $a_1, a_2 \in \mathbb{Z}^2$ and $f\,(a_1) = f\,(a_2)$ such that $a_1 \neq a_2$. We show this by example. Let $a_1 = (0,0)$ and $a_2 = (2,3)$, then

$$\begin{aligned} f\,(a_1) &= f\,(a_2) \\ 3\cdot 0 - 2\cdot 0 &= 3\cdot 2 - 2\cdot 3 \\ 0 &= 0, \end{aligned}$$

thus it is not injective. ■

Surjective:

Proof: We wish to show that f is surjective. We suppose directly that $b \in \mathbb{Z}$ and that $a_1, a_2 \in \mathbb{Z}^2$. Since 3 and 2 are relatively prime, then there exists an a_1 such that $f\,(a) =_1 1$, in other words

$$1 = 3m - 2n,$$

multiplying both sides by b gives

$$b = 3\,(bm) - 2\,(bn)\,.$$

Thus, if $a_2 = (bm, bn)$, then $f\,(a_2) = b$. Therefore, for ever $b \in \mathbb{Z}$, there exists an $a \in \mathbb{Z}^2$ such that $f\,(a) = b$. ■

Exercise 25.7. Describe without proof the image of each of the following functions.

a) $\sin\,(x)$

The image is $[-1, 1]$

b) e^x

The image is $(0, \infty)$

c) x^3

The image is $\mathbb{R}$

d) $\sqrt{|x|}$

The image is $[0, \infty)$

Exercise 25.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that we graph f in the $xy-$plane, with domain being the horizontal axis, and the codmain being the vertical axis. Prove the following.

a) (The vertical line test): Since f is a function, every vertical line intersects the graph of f at most once.
Proof: We suppose directly that f is a function, and that the graph of the function has coordinates $(a, f(a))$ for $a \in \mathbb{R}$. The function for a vertical line is $y = x + b$ for all $b, x \in \mathbb{R}$ which has coordinates $(x, x + b)$. Since the image of a vertical line at x spans all all of the y axis, there must exists a b such that $(a, f(a)) = (x, x + b)$ when $x = a$. And since every element of the domain is mapped to only one element of the codomain (for every a, there is only one $f(a)$), the vertical line can only intersect the graph at most once. In addition, since every element of the domain is mapped to the codomain, then every vertical line intersects the graph once. ■

b) (No holes in the domain): Since f is a function, every vertical line intersects the graph of f a least once.
Proof: This was shown in the previous proof. ■

c) (The horizontal line test) the function f is injective if and only if every horizontal line intersects the graph of f at most once.
Proof: Since this is a biconditional statement, we must show both ways.
$(\Longrightarrow)$: We assume directly that f is injective, then for every $a \in \mathbb{R}$ there is a distinct $f(a) \in \mathbb{R}$. This means that for the ordered pair $(a, f(a)) \in f$, a is only the left coordinate once and $f(a)$ is only the right coordinate once. The corresponding horizontal line is $x = f(a) + b$ where $b \in \mathbb{R}$ which has coordinates $(f(a) + b, f(a))$. Since the horizontal line at $f(a)$ spans the entire x-axis, the horizontal line intersect the graph of f at $(a, f(a))$. Since $f(a)$ is distinct, the corresponding a must also be distinct. Hence, if f is injective, then every horizontal line intersect he graph of f at most once.
$(\Longleftarrow)$: We assume directly that every horizontal line intersects the graph of f at most once. The equation for a horizontal line is $x = y + b$ where $b, y \in \mathbb{R}$, x represents the x-axis and y represents the y-axis. The horizontal line has coordinates $(y + b, y)$. Since the line intersects the graph of f at most once, then for every y (that corresponds to a line intersection with the graph of f), there is a unique $b* \in \mathbb{R}$ where the graph intersects the function. This point is $(y + b*, y)$. By letting $a = y + b*$ and $y = f(a)$ we get $(a, f(a))$. Thus for every $f(a)$ there is a distinct a. This is the contrapositive of the definition of what it means for a function to be injective. Hence the function f is injective. ■

d) (No holes in the codomain): The function f is surjective if an only if every horizontal line intersects the graph of f.
Proof: This is a biconditional statement, we must show both ways.
$(\Longrightarrow)$: We assume directly that f is surjective. This indicates that for every b in the codomain, there is an a in the domain such that $f(a) = b$. The function f has coordinates (a, b) where neither the left or right coordinate are distinct, but for every b in the codomain, there is at least one a in the domain, so the coordinates (a, b) exist. A horizontal line at y spans the entire x-axis and has the equation $x = y + c$ for some $c \in \mathbb{R}$. The equation has the coordinates $(y + c, y)$. By letting $b = y$ we get $(b + c, b)$. This shows that every horizontal line will intersect the graph of f since b spans

the y axis.
$(\Longleftarrow)$: We assume directly that every horizontal line intersects the graph of f. Let the horizontal line at y be $x = y + c$ for some $c \in \mathbb{R}$ which shows that the line spans the x-axis at the point y, and has coordinates $(y + c, y)$. Since every horizontal line intersects the graph of f, then there exists some a in the domain of f such that $(a, f(a) = b) = (y + c*, y)$ with $c* \in \mathbb{R}$, $(a, f(a)) \in f$ for all y. This shows that for every b in the domain, there exists an a in to codomain of f such that $f(a) = b$. Thus f is surjective. $\blacksquare$