Homework 3

Mark Petersen

5. Multiple Equations

Exercise 5.1. Write the negation of the following statements and open sentences. In each case, the domain of each variable x, ϵ, and δ is the set of real numbers. (Write any quantifiers and logical connectives using English)

a) $x > 2$ and $x < 3$
   $x \leq 2$ or $x \geq 3$
b) $x > 3 \implies x > 2$
   $x > 3$ and $x \leq 2$
c) $(x > 3) \wedge (x \neq 4) \implies x^2 \neq 16$
   $(x > 3) \wedge (x \neq 4) \wedge x^2 = 16$
d) $3 < x < 4 \implies 9 < x^2 < 16$
   $(3 < x < 4) \wedge \big(x^2 \leq 9 \vee x^2 \geq 16\big)$
e) $x = 2 \vee x = 3 \implies x^2 - 5x + 6 = 0$
   $(x = 2 \vee x = 3) \wedge \big(x^2 - 5x + 2 \neq 0\big)$
f) $\forall x \in \mathbb{R}, x^2 + 2x > 0$
   $\exists x \in \mathbb{R}, x^2 + 2x \leq 0$
g) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y > x^2$
   $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y \leq x^2$
h) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y > x^2$
   $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y \leq x^2$
i) $\forall \epsilon > 0, \forall x \in \mathbb{R}, \exists \delta > 0, 0 < |x - 2| < \delta \implies \big|x^2 - 4\big| < \epsilon$
   $\exists \epsilon > 0, \exists x \in \mathbb{R}, \forall \delta > 0,(0 < |x - 2| < \delta) \wedge \big(\big|x^2 - 4\big| \geq \epsilon\big)$

Exercise 5.2. For each pair of statements, decide if they have the same truth value.

a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0$ and $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 0$
   They are not the same. For the first statement it is true since $y = -x$. For the second statement, it is never true since a single value of y cannot satisfy the open sentence for all values of x.
b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 0$ and $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 0$
   The two statements have the same truth value for since $y = 0$ satisfies them both.
c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \neq 0$ and $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy \neq 0$
   The two statements have the same truth value since they are both false because when $x = 0, \forall y \in \mathbb{R}, xy = 0$.
d) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y + x^2 > 0$ and $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y + x^2 > 0$
   The two statements have the same truth value simply let $y > 0$.

Exercise 5.3. Answer the following two problems.

a) Give an example of a set of real numbers that has an upper bound, but does not have a greatest element.
   $[0, 0.1)$
b) Can there be a set that has a greatest element, but does not have an upper bound? Explain?
   No, if there is a greatest element, then there must be an element in the set which is also an upper bound.

Exercise 5.4. Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

a) Does S have an upper bound? If so, give an upper bound
   Yes, 1.

b) Does S have a greatest element. If so, what is it?
   Yes, 1.
c) Does S have a lower bound, f so, give a lower bound
   Yes, 0.
d) Does S have a least element? Is so, what is it?
   No.

Exercise 5.5. Let $S = (0, 1)$ be the open interval of real numbers between 0 and 1.

a) Does S have an upper bound?
   Yes, 1.
b) Does S have a greatest element?
   No
c) Does S have a lower bound?
   Yes, 0
d) Does S have a least element?
   No

Exercise 5.6. Let S be a set of real numbers, and let $x \in S$.

a) Write in symbolic logic the negation of the statement "x is the greatest element of S."
   $\neg (\forall y \in S, x \geq y) \equiv \exists y \in S, x < y$
b) Write in symbolic logic the negation of the statement "x is an upper bound for S."
   $\neg (\forall y \in S, x \geq y) \equiv \exists y \in S, x < y$
c) Write in symbolic logic the negation of the statement "S has an upper bound."
   $\forall \omega \in S, \exists y \in \mathbb{R}, y \geq \omega$

LaTex

The Cartesian product (or simply the product) $A \times B$ of two sets A and B is the set consisting of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B. In other words,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

For example, if $A = \{x, y\}$ and $B = \{1, 2, 3\}$, then

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\};$$

while

$$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

Since, for example, $(x, 1) \in A \times B$ and $(x, 1) \notin B \times A$, these two sets do not contain the same elements; so $A \times B \neq B \times A$. If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

For the sets A and B just mentioned, $|A| = 2$ and $|B| = 3$; while $|A \times B| = |B \times A| = 6$. Indeed, for all finite sets A and B,

$$|A \times B| = |A| \cdot |B|.$$