Homework 2

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Sections 1 & 2

III. Section

Exercise III.1. Determine whether the given sentence is a statement. If it is, indicate its truth value

a) The number 0 is an even integer.
   a) Statement, True
b) Let x=2
   a) Not a Statement
c) If 2 is an even integer
   a) Not a Statement
d) Either 2 is even or 4 is odd
   a) Compound Statement, True
e) George Washington had seven children
   a) Statement, False
f) There are 7254 different species of ants in the United States.
   a) Statement, False

Exercise III.2. Let P, Q, and R be statements. Construct a truth table showing the possible truth values fore each of the follow compound statements.

a) $(P \wedge Q) \implies P$

P	Q	$P \wedge Q$	$(P \wedge Q) \implies P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

b) $P \implies (P \vee Q)$

P	Q	$P \vee Q$	$P \implies (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

c) $\neg(P \implies Q) \wedge (\neg P)$

P	Q	$\neg P$	$P \implies Q$	$\neg(P \implies Q)$	$\neg(P \implies Q) \wedge (\neg P)$
T	F	F	F	T	F
T	T	F	T	F	F
F	F	T	T	F	F
F	T	T	T	F	F

d) $(P \vee Q) \wedge R$

P	Q	$P \vee Q$	R	$(P \vee Q) \wedge R$
T	T	T	T	T
T	T	T	F	F
T	F	T	T	T
T	F	T	F	F
F	T	T	T	T
F	T	T	F	F
F	F	F	T	F
F	F	F	F	F

e) $P \vee (Q \wedge R)$

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Exercise III.3. Use truth tables to prove the given logical equivalences

a) $\neg (P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg (P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

b) $P \implies Q \equiv (\neg P) \vee Q$

P	Q	$\neg P$	$P \implies Q$	$(\neg P) \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

c) $(P \vee Q) \implies R \equiv (P \implies R) \wedge (Q \implies R)$

P	Q	R	$P \vee Q$	$P \implies R$	$Q \implies R$	$(P \vee Q) \implies R$	$(P \implies R) \wedge (Q \implies R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Exercise III.4. Determine whether the two given compound statements are logically equivalent.

a) $\neg (P \implies Q)$ and $P \wedge (\neg Q)$

From part b) of Exercise 3 we know that $P \implies Q \equiv (\neg P) \vee Q$. Negating both sides yields

$$\begin{aligned} P \implies Q &\equiv (\neg P) \vee Q \\ \neg (P \implies Q) &\equiv \neg ((\neg P) \vee Q)\,, \quad \text{distribute the negatation} \\ &\equiv P \wedge (\neg Q)\,. \end{aligned}$$

Thus they are equivalent.

b) $(P \wedge Q) \implies R$ and $P \implies (\neg Q \vee R)$

Starting with $(P \wedge Q) \implies R$ we can get

$$\begin{aligned}(P \wedge Q) \implies R &\equiv (\neg (P \wedge Q)) \vee R, \quad \text{By part b) of exercises 3}\\ &\equiv (\neg P) \vee (\neg Q) \vee R, \quad \text{distribute the negation}\\ &\equiv P \implies (\neg Q \vee R), \quad \text{By part b) of exercises 3}\end{aligned}$$

Thus they are equivalent.

c) $P \implies (Q \vee R)$ and $(P \wedge \neg Q) \implies R$

Starting with $P \implies (Q \vee R)$ we can get

$$\begin{aligned}P \implies (Q \vee R) &\equiv (\neg P) \vee (Q \vee R)\\ &\equiv ((\neg P) \vee Q) \vee R\\ &\equiv \neg ((\neg P) \vee Q) \implies R\\ &\equiv P \wedge \neg Q \implies R\end{aligned}$$

Thus they are equivalent.

d) $(P \vee Q) \implies R$ and $(\neg R \wedge P) \implies (\neg Q)$

Starting with $(P \vee Q) \implies R$ we can get

$$\begin{aligned}(P \vee Q) \implies R &\equiv (\neg P \wedge \neg Q) \vee R\\ &\equiv (\neg P \vee R) \wedge (R \vee \neg Q)\\ &\equiv (\neg R \wedge P) \implies (R \vee \neg Q)\\ &\not\equiv (\neg R \wedge P) \implies (\neg Q)\end{aligned}$$

Thus they are not equivalent. This is easily shown when the compound statements are evaluated at $P = \text{T}$, $Q = \text{F}$, and $R = \text{F}$ which gives

$$\begin{aligned}(P \vee Q) \implies R &\neq (\neg R \wedge P) \implies (\neg Q)\\ (\text{T} \vee \text{F}) \implies \text{F} &\neq (\neg \text{F} \wedge \text{T}) \implies \neg \text{F}\\ \text{F} &\neq \text{T}.\end{aligned}$$

e) $P \iff Q$ and $(P \implies Q) \wedge (Q \implies P)$

Starting with

$$\begin{aligned}(P \implies Q) \wedge (Q \implies P) &\equiv (\neg P \vee Q) \wedge (\neg Q \vee P)\\ &\equiv (\neg P \wedge (\neg Q \vee P)) \vee (Q \wedge (\neg Q \vee P))\\ &\equiv (\neg P \wedge \neg Q) \vee (Q \wedge P)\\ &\equiv P \iff Q\end{aligned}$$

Thus they are equivalent.

Exercise III.5. Let P, Q, and R be statements. Identify each of the following statements as a tautology, a contradiction, or neither.

a) $((P \implies Q) \wedge (\neg Q)) \implies (\neg P)$

We begin by manipulating the statement

$$\begin{aligned}&((P \implies Q) \wedge (\neg Q)) \implies (\neg P)\\ &\equiv ((\neg P \vee Q) \wedge (\neg Q)) \implies \neg P\\ &\equiv (\neg P \wedge \neg Q) \implies \neg P\\ &\equiv P \vee Q \vee (\neg P)\\ &\equiv \text{T}\end{aligned}$$

this is a tautology.

b) $((P \vee Q) \wedge (\neg P)) \implies Q$

We begin by manipulating the statement

$$\begin{aligned}&((P \vee Q) \wedge (\neg P)) \implies Q\\ &\equiv ((P \wedge \neg P) \vee (Q \wedge \neg P)) \implies Q\\ &\equiv (Q \wedge \neg P) \implies Q\\ &\equiv (\neg Q \vee P) \vee Q\\ &\equiv \text{T}\end{aligned}$$

this is a tautology.

c) $(P \implies Q) \implies (P \implies R)$

We begin by manipulating the statement

$$\begin{aligned}&(P \implies Q) \implies (P \implies R)\\ &\equiv (\neg P \vee Q) \implies (\neg P \vee R)\\ &\equiv (P \wedge \neg Q) \vee (\neg P \vee R)\\ &\equiv (P \vee (\neg P) \vee R) \wedge (\neg Q \vee \neg P \vee R)\\ &\equiv (\neg Q \vee \neg P \vee R)\end{aligned}$$

which is neither a tautology or a contradiction.

d) $((\neg Q) \implies (\neg P)) \wedge P \wedge (\neg Q)$

We begin by manipulating the statement

$$\begin{aligned}&((\neg Q) \implies (\neg P)) \wedge P \wedge (\neg Q)\\ &\equiv (Q \vee \neg P) \wedge P \wedge \neg Q\\ &\equiv (Q \wedge P \wedge \neg Q) \vee (\neg P \wedge P \wedge \neg Q)\\ &\equiv \text{F}\end{aligned}$$

this is a contradiction.

Exercise III.6. Let P and Q be statements

a) Prove that the compound statement $P \implies Q$ is not logically equivalent to $Q \implies P$.
Proof: Assume $P \implies Q \equiv Q \implies P$, and let $P = \text{T}$ and $Q = \text{F}$, then $P \implies Q$ is False but $Q \implies P$ is True which is not equivalent and thus the compound statements cannot be logically equivalent. ■

b) The statement $Q \implies P$ is called the converse of $P \implies Q$. give an example of statement P and Q for which $P \implies Q$ is true, but $Q \implies P$ is false. For the specific statements P and Q in your example, state their truth values.

We define the two statement

$$\begin{aligned}P:&\ \text{A certian shape is a square}\\ Q:&\ \text{A certain shape is a quadralateral}\end{aligned}$$

with the the truth table

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Exercise III.7. Explain why asking the grue the question "Is the left path the way home if and only if you are a truth teller?" can help you decide which path to take.

To explain this we can construct a truth table with the statements

$$L: \text{ The left plath is the way home}$$
$$TT: \text{ You are a truth teller}$$

L	TT	$L \iff TT$	What you will be told
T	T	T	T
T	F	F	T
F	T	F	F
F	F	T	F

According to the truth table, regardless of whether the grue is a truth teller or not, if the left path is the right path they both will answer True otherwise false. Thus you can easily deduce from their answer which path to take. However, this is all under the assumption that the grue can follow logic perfectly or is at least familiar with the bi-conditional statement.

IV. Section

Exercise IV.1. Let x be a variable with domain $\mathbb{Z}$. Define the open sentences

$$P(x) : x > 1$$
$$Q(x) : x^2 < 16$$
$$R(x) : x + 1 \text{ is even}$$

For each of the following compound sentences, describe the subset of $\mathbb{Z}$ (by listing its elements or using set-build notation) where that the open sentence is true.

a) $P(x) \wedge Q(x)$

  $\{2, 3\}$

b) $Q(x) \wedge R(x)$

  $\{-3, -1, 1, 3\}$

c) $(Q(x) \vee \neg P(x)) \wedge \neg R(x)$

  This statement can be rewritten as $(Q(x) \wedge R(x)) \vee (\neg P(x) \wedge R(x))$. I am dropping the dependency on x for clarity. We already know the set for $Q \wedge R$ from part b). The set associated with $\neg P \wedge R$ is $\{1, -1, -3, \cdots\}$. Thus the subset is

$$\{1, -1, -3, \cdots\} \cup \{-3, -1, 1, 3\} = \{3, 1, -1, -3, \cdots\}.$$

c) $(P(x) \implies Q(x)) \implies R(x)$

  Dropping the dependency on x we can write and equivalent statement

$$\begin{aligned}(P(x) \implies Q(x)) \implies R(x) &\equiv (\neg P \vee Q) \implies R \\ &\equiv (P \wedge \neg Q) \vee R\end{aligned}$$

  The subset is

$$\{-4, -5, -6, \cdots\} \cup \{4, 5, 6, 7, 8 \cdots\} \cup \{-3, -1, 1, 3\}$$

Exercise IV.2. For each $x \in \{1, 2, 3, 4, 5, 6\}$, write down the truth value of

$$P(x) : \text{ If } x \text{ is an odd integer, then } \frac{3x+5}{2} \text{ is an odd integer,}$$

and then state whether you believe $\forall x \in \mathbb{Z}, P(x)$ is true or false.

The truth table for the open sentence is

x	1	2	3	4	5	6
$P(x)$	F	T	F	T	F	T

In order for the term $\frac{3x+5}{2} \in \mathbb{Z}$, the numerator must be even; otherwise, the term is a rational number. This implies that $\forall x \in \mathbb{Z}, P(x)$ is false, which we could've concluded from the truth table.

Exercise IV.3. For each $x \in \{1, 2, 3, 4, 5, 6\}$, write down the truth value of

$$Q(x) : \text{ If } x \text{ is an even integer, then } 3x + 5 \text{ is an odd integer,}$$

and then state whether you believe $\forall x \in \mathbb{Z}, Q(x)$ is true or false.

The corresponding truth table for the open sentence is

x	1	2	3	4	5	6
$Q(x)$	T	T	T	T	T	T

We can easily show that the statement $\forall x \in \mathbb{Z}, Q(x)$ is true. We know that an even integer multiplied by an odd integer is even, and we know that an even integer added to an odd integer is odd. Since 3 and 5 are odd, and if x is even then $3x + 5$ must be odd. In fact, this is bi-conditional .

Exercise IV.4. Let A and B be two subsets of a universal set U. Write a symbolic logic interpretation of the statement $A = B$. Explicitly write out any quantifiers involved in the statement.

$$\forall a \in A, \forall b \in B, b \in A, a \in B.$$

Exercise IV.5. Translate the following English sentences into symbolic logic. Explicitly write out any quantifiers involved in the statement.

a) There is an integer strictly between 4 and 6.
   $\exists x \in \mathbb{Z}, x \in (4, 6)$.
b) The square of any odd integer is odd.
   Let $S = \{\text{odd integers}\}$
   $$\forall x \in S, x^2 \in S.$$
c) If the square of an integer is odd, then the original integer is odd.
   Let $S = \{\text{odd integers}\}$
   $$\forall x \in \mathbb{Z}, x^2 \in S \implies x \in S.$$
d) If a real number is not rational, then it is not equal to 0.
   $\forall x \in \mathbb{R}, x \notin \mathbb{Q} \implies x \neq 0$
e) The sum of two rational numbers is rational.
   $\forall x, y \in \mathbb{Q}, x + y \in \mathbb{Q}$
f) The square of any real number is a nonnegative real number.
   $\forall x \in \mathbb{R}, x^2 \geq 0$
g) There is an integer solution to the equation $x^2 - 5x + 6 = 0$.
   $\exists x \in \mathbb{Z}, x^2 - 5x + 6 = 0$
h) Every real solution to $x^2 - 5x + 6 = 0$ is an integer.
   $\forall x \in \mathbb{R}, x^2 - 5x + 6 = 0 \implies x \in \mathbb{Z}$

Exercise IV.6. Translate the following symbolic logic statements into English

a) $\exists x \in \mathbb{R}, x = 2$.
   There exists a real number that is equal to 2.
b) $\forall x \in \mathbb{Z}, (x \text{ is even}) \iff \left(x^2 \text{ is even}\right)$
   An integer is even if and only if the square of the integer is even for all integers.
c) $\forall x \in \mathbb{R}, (x > 1) \implies \left(x^3 > 1\right)$
   For every real number, if the number is greater than 1, then the number cubed is greater than 1.
d) $\forall x \in \mathbb{R}, \left(x^2 - 2x + 1 = 0\right) \implies (x = 1)$
   For every real number, if it is a solution to $x^2 - 2x + 1 = 0$, then the number is 1.

e) $\exists x \in \mathbb{Q},\ 2x^3 - x^2 + 2x - 1 = 0$

There exists a rational solution to $2x^3 - x^2 + 2x - 1 = 0$.

f) $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, \exists z \in [0,1)\ ,\ x = y + z$

For any real number x, there is an integer y and a number z from the interval $[0,1)$ that satisfies the equation $x = y + z$.