Homework 4

Mark Petersen

Sections 6 and 7

6. Direct Proofs

Exercise 6.1. Let $x \in \mathbb{R}$. Prove that if $x \neq 3$, then $x^2 - 2x + 3 \neq 0$. (Would this result be true if we took $x \in \mathbb{C}$).

Proof: We assume that $x \neq 3$. Let $x = 3+\delta$ where $\delta \in \mathbb{R}-\{0\}$. Substituting x into the polynomial yields

$$\begin{aligned} x^2 - 2x + 3 &\neq 0 \\ (3+\delta)^2 - 2(3+\delta) + 3 &\neq 0 \\ 9 + \delta^2 + 6\delta - 2\delta - 6 + 3 &\neq 0 \\ \delta^2 + 4\delta + 6 &\neq 0. \end{aligned}$$

The roots of the polynomial $\delta^2 + 4\delta + 6$ are complex, which means $\forall x \in \mathbb{R}, x^2 - 2x + 3 \neq 0$, thus the statement is trivially true. ■

Exercise 6.2. Let $n \in \mathbb{N}$. Prove that if $2 < n < 3$, then $7n + 3$ is odd.

Proof: We assume that $2 < n < 3$ where $n \in \mathbb{N}$. This premise is always false since there is no natural number in the open interval $(2, 3)$, thus the statement is vacuously true. ■

Exercise 6.3. Prove that if x is an odd integer, then x^2 is odd.

Proof: We assume that x is an odd integer. This implies that $\exists k \in \mathbb{Z}, x = 2k + 1$, which implies

$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2\left(2k^2 + 2k\right) + 1 \\ &= 2y + 1, \end{aligned}$$

where $y \in \mathbb{Z}$. Thus x^2 is odd. ■

Exercise 6.4. Prove that if x is an even integer, then $7x - 5$ is odd.

Proof: We assume that x is an even integer. This implies that $\exists k \in \mathbb{Z}, x = 2k$, which implies

$$\begin{aligned} 7x - 5 &= 7(2k) - 5 \\ &= 14k - 5 \\ &= 14k - 6 + 1 \\ &= 2(7k - 3) + 1 \\ &= 2y + 1, \end{aligned}$$

where $y \in \mathbb{Z}$. Thus $7x - 5$ is odd. ■

Exercise 6.5. Let $a, b, c \in \mathbb{Z}$. Prove that if a and c are odd, then $ab + bc$ is even.

Proof: We assume that a and c are odd. This implies that $a = 2k + 1$ and $c = 2j + 1$ for some $k, j \in \mathbb{Z}$, which implies

$$\begin{aligned} ab + bc &= (2k+1)\, b + b\,(2j+1) \\ &= b\,(2k+2j+2) \\ &= 2b\,(k+j+1) \\ &= 2y, \end{aligned}$$

where $y = b\,(k+j+1) \in \mathbb{Z}$. Thus $ab + bc$ is even. ■

Exercise 6.6. Let $n \in \mathbb{Z}$. Prove that if $|n| < 1$, then $3n - 2$ is an even integer.

Proof: Let $|n| < 1$ such that $n \in \mathbb{Z}$, then $n = 0$ since 0 in the only integer in the interval $(-1, 1)$. Since $n = 0$, the term $3n-2$ simplifies to -2 which can be written as $2\,(-1)$ which is an even integer. Thus $3n - 2$ is an even integer if the premise holds. ■

Exercise 6.7. Prove that every odd integer is a difference of two square integers. In other words, if x is odd integer, then $\exists y, z \in \mathbb{Z}, x = y^2 - z^2$.

Proof: By definition of being an odd integer, let $x = 2k + 1$ for some $k \in \mathbb{Z}$. Also let $y = k + 1$ and $z = k$. Substituting this into $x = y^2 - z^2$ yields

$$\begin{aligned} 2k+1 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k+1, \end{aligned}$$

which shows that every odd integer is a difference of two square integers. ■

7. Contrapositive

Exercise 7.1. Let $a \in \mathbb{Z}$. Prove that if $a^2 + 3$ is odd, then a is even.

Proof: We work this contrapositively. Assume that a is odd so that $\exists k \in \mathbb{Z}, a = 2k + 1$, and substitute the expression for a into $a^2 + 3$ to get

$$\begin{aligned} a^2 + 3 &= (2k+1)^2 + 3 \\ &= 4k^2 + 4k + 1 + 3 \\ &= 2\left(2k^2 + 2k + 2\right), \end{aligned}$$

which shows that it is even. Thus if $a^2 + 3$ is odd, then a is even. ■

Exercise 7.2. Prove the following: Let $x, y \in \mathbb{Z}$. If $xy + y^2$ is even, then x is odd or y is even.

Proof: We work this contrapositively by proving when x is even and y is odd, then $xy + y^2$ is odd. The integers x and y can be written as $x = 2k$ and $y = 2j+1$ for some $k, j \in \mathbb{Z}$. Substituting in these expressions for x and y into $xy + y^2$ yields

$$\begin{aligned} xy + y^2 &= 2k\,(2j+1) + (2j+1)^2 \\ &= 2kj + 2k + 4j^2 + 2j + 1 \\ &= 2\left(kj + k + 2j^2 + j\right) + 1 \\ &= 2n + 1, \end{aligned}$$

where $n = kj + k + 2j^2 + j \in \mathbb{Z}$. Thus showing that $xy + y^2$ is odd when x is even and y is odd. ■

Example 7.3. Let $s \in \mathbb{Z}$. Prove that s is odd if and only if s^3 is odd.

Proof: We begin by showing that if s is odd then s^3 is odd. Since s is odd, then $\exists k \in \mathbb{Z}, s = 2k + 1$. Substituting this into s^3 gives

$$\begin{aligned} s^3 &= (2k+1)^3 \\ &= \left(4k^2 + 4k + 1\right)(2k+1) \\ &= 8k^3 + 4k^2 + 8k^2 + 4k + 2k + 1 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2\left(4k^2 + 6k^2 + 3k\right) + 1, \end{aligned}$$

which is odd. Next we show that if s^3 is odd, then s is odd. We work this contrapositively by proving that if s is even then s^3 is even. The integer s can be written as $s = 2j$ for some $j \in \mathbb{Z}$. Substituting this into the equation s^3 yields

$$\begin{aligned} s^3 &= (2j)^3 \\ &= 8j^3 \\ &= 2\left(4j^3\right) \end{aligned}$$

which shows that s^3 is an even number. Thus if s^3 is odd, then s is odd. ■

Exercise 7.4. Consider the following situation. A student is asked to prove the statement: "Given $x \in \mathbb{Z}$, if $2 \mid x$, then x is even." The student writes: "Assume, contrapositively, that x is even. Then $x = 2k$ for some $k \in \mathbb{Z}$. Hence $2 \mid x$." Identify what is wrong with this students proof and write a correct proof.

When proving the implication $x \in S, P(x) \implies Q(x)$ contrapositively, we are proving the implication $x \in S, \neg Q(x) \implies \neg P(x)$. The student didn't negate the premise $P(x)$ when writing the proof. This proof can easily be proven using a direct or contrapositive approach. We will prove it directly.

Proof: We assume that $2 \mid x$ which by definition means that $x = 2k$ for some $k \in \mathbb{Z}$ which is the definition of an integer being even. ■

Exercise 7.5. Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a \mid c$ and $b \mid d$ then $ab \mid cd$.

Proof: Since $a \mid c$ and $b \mid d$, c and d can be written as $c = ak$ and $d = bj$ for some $k, j \in \mathbb{Z}$. Using these definitions in the expression $ab \mid cd$ yields

$$\begin{aligned} ab \mid cd &= ab \mid akbj \\ &= ab \mid abkj. \end{aligned}$$

By definition of $ab \mid abjk$ we get $abjk = abm$ for some $m \in \mathbb{Z}$, which reduces to $jk = m$. Thus proving that if $a \mid c$ and $b \mid d$ then $ab \mid cd$. ■

Exercise 7.6. State the contrapositive of the implication in the previous exercise.

Let $a, b, c, d \in \mathbb{Z}$. If $ab \nmid cd$ then $a \nmid c$ or $b \nmid d$.

Exercise 7.7. Let $a \in \mathbb{Z}$. Prove that if $4 \nmid a^2$, then a is odd.

Proof: We prove this contrapositively by showing that if a is even, then $4 \mid a^2$. Assuming that a is even, we can write it as $a = 2k$ for some $k \in \mathbb{Z}$. Substituting this into the expression $4 \mid a^2$ yields

$$4 \mid a^2 = 4 \mid 4k^2,$$

which shows that 4 is a divisor of $4k^2$;therefore, 4 is a divisor of a^2. ■

Exercise 7.8. Prove the following implication two ways (directly and contrapositively): Given $x \in \mathbb{Z}$, then $5x - 1$ is even only if x is odd. In other words $5x - 1$ is even $\implies$ x is odd.

Proof: **(Direct)** We assume that $5x-1$ is even which means $5x-1=2j$ for some $j \in \mathbb{Z}$. Let $j=5k+2$ for some $k \in \mathbb{Z}$, then we get

$$\begin{aligned} 5x-1 &= 2j \\ &= 2\,(5k+2) \\ &= 10k+4, \end{aligned}$$

and solving for $5x$ gives us

$$5x = 10k+5.$$

Since $5 \mid 10k+5$, we can solve for x and get the solution

$$x = 2k+1,$$

which shows that x is an odd integer. ■

Proof: **(Contrapositively)** We prove this contrapositively by showing that if x is even then $5x-1$ is odd. Since x is even, it can be written as $x=2k$ for some $k \in \mathbb{Z}$. Substituting this into the term $5x-1$ yields

$$\begin{aligned} 5x-1 &= 5\,(2k)-1 \\ &= 10k-1 \\ &= 2\,(5k)-1 \\ &= 2\,(5k)-1-2+2 \\ &= 2\,(5k-1)+1 \\ &= 2y+1, \end{aligned}$$

where $y=5k-1 \in \mathbb{Z}$. Therefore, $5x-1$ is odd if x is even. ■

Remark 7.9. When I say we in the proofs, I really mean I. I am just use to saying we when writing papers.