Homework 11

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Sections 20 and 21

20. Properties of Relations

Exercise 20.1. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 3), (2, 5), (2, 6), (3, 5), (4, 5), (4, 6)\}$

a) Give an example of elements $a, b \in A$ such that aRb.
   Since $(1, 1) \in R$ we have $a = 1$ and $b = 1$.
b) Give an example of elements $a, b \in A$ such that $a \not R b$
   Since $(5, 5) \notin R$ we have $a = 5$, $b = 5$.
c) For $a \in A$, let $S_a = \{x \in A : aRx\}$. Thus, S_a is the set of elements to which a relates. Write down the six Sets.
   $S_1 = \{1, 2, 3, 4\}$
   $S_2 = \{3, 5, 6\}$
   $S_3 = \{5\}$
   $S_4 = \{5, 6\}$
   $S_5 = S_6 = \emptyset$
d) For $a \in A$, let $T_a = \{x \in A : xRa\}$. Thus, T_a is the set of elements which relate to a. Write down the six sets $T_1, \ldots, T_6$.
   $T_1 = \{1\}$
   $T_2 = \{1\}$
   $T_3 = \{1, 2\}$
   $T_4 = \{1\}$
   $T_5 = \{2, 3, 4\}$
   $T_6 = \{2, 4\}$

Exercise 20.2. For the relations R from $\mathbb{R}$ to $\mathbb{R}$ defined below, write R as a set using set-builder notation, and graph R as a subset of $\mathbb{R}^2$. See fig 1 for the images.

a) Define R by xRy if $x \leq y$.
   $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq y\}$
b) Define R by xRy if $x \geq y$.
   $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq y\}$
c) Define R by xRy if $x > y$.
   $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x > y\}$.
d) Define R by xRy if $x = y$.
   $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y\}$.
e) Define R by xRy if $x \neq y$.
   $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq y\}$.

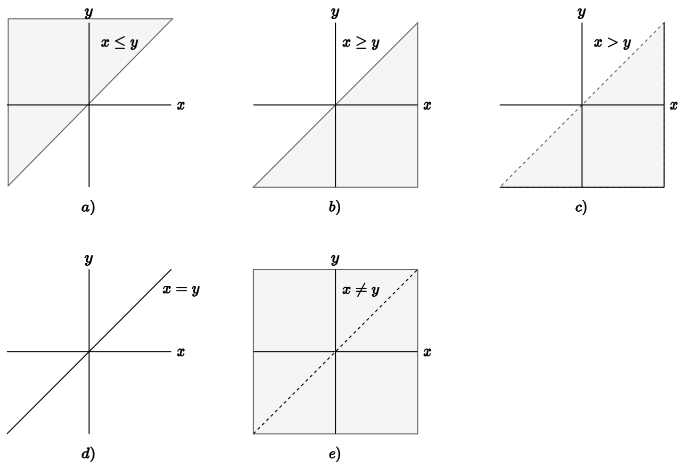

Figure 1. The shaded region indicates that the element is in the relation set. The dashed lines indicates that the element is not included in the relation set.

Exercise 20.3. Define a relation R on $\mathbb{R}$ by xRy is $xy < 0$.

a) Describe R as a set using set-builder notation.

   $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy < 0\}$.

b) Graph R as a subset of $\mathbb{R} \times \mathbb{R}$.

   See fig 2.

c) Determine whether R is reflexive, symmetric, transitive, and or antisymmetric.

   Reflexive: We want to verify that xRx for all $x \in \mathbb{R}$. We know that $x \cdot x = x^2 \geq 0$. Thus $(x, x) \notin R$. Therefore R is not reflexive.

   Symmetric: We suppose that $x, y \in \mathbb{R}$ and that $xy < 0$. Since scalar multiplication commutes, we have that $yx < 0$. Thus if $(x, y) \in R$, then $(y, x) \in R$. Therefore R is symmetric.

   Transitive: We suppose that $x, y, z \in \mathbb{R}$ and that xRy and yRz. Then $xy < 0$ and $yz < 0$. We quickly note that in order for the product of two real numbers to be less than 0, one must be negative and the other must be positive. So we know that x and y must have different signs and y and z must have different signs. This meas that x and z must have the same sign. Hence $xz > 0$. Therefore R is not transitive.

   Antisymmetric: We suppose that $x, y \in \mathbb{R}$ such that $xy < 0$ and $yx < 0$. As we have previously stated, in order or the product of two real numbers to be less than 0, they must have opposite signs. Hence $x \neq y$. Therefore R is not antisymmetric.

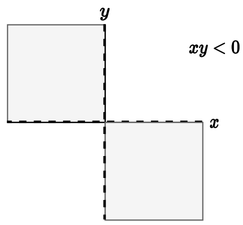

Figure 2. The shaded region indicates that the element is in the relation set. A depiction of the relation $xy < 0$. Note that $x \neq 0$ and $y \neq 0$ as indicated by the dashed lines.

Exercise 20.4. Define a relation R on $\mathbb{R}$ by xRy if $x - y \in \mathbb{Z}$.

a) Define R as a set using set-builder notation.
   $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$.
b) Graph R as a subset of $\mathbb{R} \times \mathbb{R}$.
   See fig 3.
c) Determine whether R is reflexive, symmetric, transitive, and/or antisymmetric.
   Reflexive: Let $x \in \mathbb{R}$, and we suppose that xRx is true. Then $x - x = 0 \notin \mathbb{Z}$. This is a contradiction. Thus R is not reflexive.
   Symmetric: We suppose that $x, y \in \mathbb{R}$ and that $x-y \in \mathbb{Z}$. Since $x-y \in \mathbb{Z}$ we know that $x - y = m$ for some $m \in \mathbb{Z}$. We can multiply both sides by -1 to get $y - x = -m$. Since $-m \in \mathbb{Z}$, $y - x \in \mathbb{Z}$. Therefore $xRy \implies yRx$, and R is symmetric.
   Transitive: We suppose that $x, y, z \in \mathbb{R}$ and that $x-y \in \mathbb{Z}$ and $y-z \in \mathbb{Z}$. Let $a, b \in \mathbb{Z}$ such that $x - y = a$ and $y - z = b$. Solving for y in the second equation yields $y = b + z$, and substituting this into the first equation yields $x - (b + z) = a$, which can be written as $x - z = b + a$. Since the sum of two integers is another integer, we get $x - z \in \mathbb{Z}$. Therefore R is transitive.
   Antisymmetric: Let $x, y \in \mathbb{R}$. We suppose that xRy and yRx. Let $x = 5$ and $y = 3$ and note that they satisfy xRy and yRx; however, $x \neq y$. This shows that $\exists x, y \in \mathbb{R}, xRy \wedge yRx, x \neq y$. Therefore R is not antisymmetric.

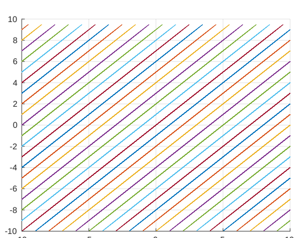

Figure 3.

Exercise 20.5. Define a relation R on $\mathbb{Z}$ by aRb if $a - b$ is even.

a) Describe R as a set using set-builder notation.
   $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b \text{ is even}\}$

b) Prove that R is reflexive, symmetric, and transitive.

Reflexive: Let $a \in \mathbb{Z}$, then $a - a = 0$ and 0 is even. Therefore R is reflexive.

Symmetric: Let $a, b \in \mathbb{Z}$. We assume directly aRb. So $a - b = 2m$ for some $m \in \mathbb{Z}$. We can multiply both sides by -1 to get $b - a = -2m$. Since $-2m$ is even, we know that $b - a$ is even. Hence $aRb \implies bRa$. Therefore R is symmetric.

Transitive: Let $a, b, c \in \mathbb{Z}$. We assume directly aRb and bRc. So $a - b = 2m$ and $b - c = 2k$ for some $m, k \in \mathbb{Z}$. Solving for b in the second equation yields $b = 2k + c$. Substituting this in for b in the first equation gives $a - (2k + c) = 2m$ which can be written as $a - c = 2(m + k)$ which is even. Hence $aRb \wedge bRc \implies aRc$. Therefore R is transitive.

c) Prove that R is not antisymmetric.

Proof: Let $a, b \in \mathbb{Z}$. We suppose directly that $aRb \wedge bRa$. So $a - b = 2m$ and $b - a = -2m$ for some $m \in \mathbb{Z}$. This gives the constraint $a - b = a - b$ which is satisfied for any value of $a, b \in \mathbb{Z}$. For example let $a = 5$ and $b = 7$. Then $a - b = -2$ and $b - a = 2$ which are both even numbers, but $a \neq b$. Hence $(aRb \wedge bRa \not\Longrightarrow R)$. Therefore R is not antisymmetric. ∎

d) For which integers b is it the case that $1Rb$.

Since $1 - b$ must be even, we can say $1 - b = 2m$ for some $m \in \mathbb{Z}$. Solving for b yields $b = 2(-m) + 1$. Thus we see that b is an odd integer. So any odd integer satisfies $1Rb$.

Exercise 20.6. For each part, give an example of a relation R on the set $A = \{1, 2, 3\}$ with the specified properties. Write R as a set of ordered pairs.

a) R is reflexive, symmetric, and transitive

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$.

b) R is reflexive and symmetric, but not transitive.

$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3)\}$.

c) R is reflexive and transitive, but not symmetric.

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$.

d) R is reflexive, not symmetric, and not transitive.

$R = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$.

e) R is not reflexive, but is symmetric and transitive.

$R = \{(1,1), (1,2), (2,1), (2,2)\}$.

f) R is not reflexive and not transitive, but is symmetric.

$R = \{(1,2), (2,1)\}$.

g) R is not reflexive and not symmetric, but is transitive.

$R = \{(1,2), (1,3), (2,3)\}$.

h) R is not reflexive, not symmetric, and not transitive.

$R = \{(1,2), (2,3)\}$.

Exercise 20.7. Let $A = \{1, 2, 3, 4, 5\}$ and let $S = \{\{1, 2, 3\}, \{3, 4\}, \{5\}\}$. Define

$$R = \{(a, b) \in A \times A : \text{for some } X \in S, \text{ both } a \in X \text{ and } b \in X\}.$$

a) Write out the elements of R.

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (4,3), (4,4), (5,5)\}$

b) Is R reflexive? Symmetric? Transitive?

Reflexive: To prove reflexive, we need that $\forall a \in A$, aRa. From the listing we can see that $(a, a) \in R$ for all $a \in A$. Thus R is reflexive.

Symmetric: Suppose directly that $a, b \in A$ such that aRb. In order for $(a, b) \in R$, a and b must be elements of the same element of S by how R is defined. Then by how R is defined we also know that $(b, a) \in R$ since b and a are elements of the same element of S. Therefore R is symmetric.

Transitive: We disprove this. Let $a, b, c \in A$ such that $a = 1$, $b = 3$, and $c = 4$. From the listing of R we see that $aRb \wedge bRc$; however, $a \not{R} c$. This is a counterexample. Therefore R is not transitive.

Exercise 20.8. Let $A = \{1, 2, 3, 4, 5\}$ and let $S = \{\{1, 2\}, \{4, 5\}\}$. Define

$$R = \{(a, b) \in A \times A : \text{for some } X \in S, \text{ both } a \in X \text{ and } b \in X\}.$$

a) Write out the elements of R.

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (4, 4), (4, 5), (5, 4), (5, 5)\}$

b) Is R reflexive, symmetric or transitive?

Reflexive: Disproof. The element $(3, 3) \notin R$ with $3 \in A$. Hence $3 \not{R} 3$. Therefore, R is not reflexive.

Symmetric: Suppose directly that $a, b \in A$ such that aRb. In order for $(a, b) \in R$, a and b must be elements of the same element of S by how R is defined. Then by how R is defined we also know that $(b, a) \in R$ since b and a are elements of the same element of S. Therefore R is symmetric.

Transitive: We suppose directly that $a, b, c \in A$, aRb, and bRc. Since the elements of S are disjoint sets and by the definition of R, if aRb and aRc, then a, b, c are all elements from the same elements of S, and that a, b, c are not elements of any other element of S. Since a, b, c must be elements of the same element of S, then by definition of R, $(a, c) \in R$ since $a \in X$ and $c \in X$ for some $X \in S$. Hence aRc. Therefore R is transitive.

21. Equivalence Relations

Exercise 21.1. Let $A = \{1, 2, 3\}$ an let R be the relation of A given by

$$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3)\}.$$

Is R reflexive, symmetric, transitive, antisymmetric? Is R an equivalence relation?

a) **Reflexive:**
Proof: We suppose directly that $a \in A$. From the listing of R we see that $(a, a) \in R$ for all $a \in A$. Therefore, R is reflexive. ■

b) **Symmetric**:
Proof: We suppose directly that $a, b \in A$. From the listing of R we see that $(a, b) \in R$ and $(b, a) \in R$ for all $a, b \in R$. Therefore, R is symmetric. ■

c) **Transitive**:
Disproof.: We want to show that for every $a, b, c \in A$, if $aRb \wedge bRc$, then aRc. However, note that $(1, 2), (2, 3) \in R$, but $(1, 3) \notin R$. This is a counterexample. Therefore R is not transitive. ■

Since R is not transitive, it cannot be an equivalence relation.

Exercise 21.2. Give an example of an equivalence relation R on the set $A = \{1, 2, 3, 4, 5\}$ that has exactly two equivalence classes. Explicitly write out the relation R as a set of ordered pairs.

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}.$$

Since $[1] = [2] = [3]$ and $[4] = [5]$, there are only two equivalence classes.

Exercise 21.3. Let R be an equivalence relation on the set $A = \{1, 2, 3, 4, 5\}$. Assume that $1R3$ and $3R4$. Given these conditions, which ordered pairs must belong to R? Let $T \subseteq R$ that contains the elements of R that meet the conditions given. Then

$$T = \{(1,1), (1,3), (1,4)\, (2,2), (3,1), (3,3), (3,4), (4,1), (4,3)\, (4,4), (5,5)\}.$$

Exercise 21.4. Let $A = \{1, 2, 3, 4, 5\}$ and let $S = \{\{1, 2\}, \{3, 4\}, \{5\}\}$. Define

$$R = \{(a, b) \in A \times A \,:\, \text{for some } X \in S, \text{ both } a \in X \text{ and } b \in X\}.$$

We have seen that R is an equivalence relation. What are the equivalence classes.

a) $[1] = \{1, 2\}$
b) $[2] = \{1, 2\}$
c) $[3] = \{3, 4\}$
d) $[4] = \{3, 4\}$
e) $[5] = \{5\}$

Since $[1] = [2]$ and $[3] = [4]$, we have three equivalence classes. $[1], [3], [5]$.

Exercise 21.5. Let $A = \mathbb{R} - \{0\}$. Define a relation $\sim$ on A by $a \sim b$ if $ab > 0$.

a) Prove that $\sim$ is an equivalence relation on A.
Proof: We want to show that $\sim$ is an equivalence relation on A, by showing that $\sim$ is reflexive, symmetric, and transitive.
Reflexive: Suppose directly that $a \in A$. Then $a \cdot a = a^2 > 0$. Hence $a \sim a$, and $\sim$ is reflexive.
Symmetric: Suppose directly that $a, b \in A$ and $ab > 0$. Then, since multiplication commutes, $ab = ba > 0$. Hence $a \sim b \implies b \sim a$. Therefore $\sim$ is symmetric.

Transitive: Suppose directly that $a, b, c \in A$ and $ab > 0$ and $bc > 0$. Let $bc = m$ for some $m \in \mathbb{R} > 0$. Solving for b we get $b = \frac{m}{c} \in \mathbb{R}$ since $c \in \mathbb{R} - \{0\}$. Substituting the expression for b into the first equation yields $a\left(\frac{m}{c}\right) > 0$. Multiplying both sides of the inequality by $\frac{c^2}{m}$ gives $ac > 0$ since $c^2 > 0$. Hence $(a \sim b) \wedge (b \sim c) \implies a \sim c$. Therefore the relation $\sim$ is transitive.
Since the relation is reflexive, symmetric, and transitive it is an equivalence relation on A. ■

b) Determine the equivalence classes of $\sim$.

In order for the product of two non zero real numbers to be greater than zero, they must have the same sign. Hence we have two equivalence classes. The set of all negative real numbers and the set of all positive real numbers.

Exercise 21.6. Let A be the set of humans with English names. Define a relation $\approx$ on A by $\alpha \approx \beta$ if α and β have the same first letter in their names.

a) Prove that $\approx$ is an equivalence relation on A.
Proof: We want to show that $\approx$ is an equivalence relation on A. We do this by showing that $\approx$ is reflexive, symmetric, and transitive.
Reflexive: Suppose directly $a \in A$. Then a starts with some letter γ, which is the same letter that a starts with since its the same word. Hence $a \approx a$. Therefore, $\approx$ is reflexive.
Symmetric: Suppose directly $a, b \in A$ and $a \approx b$. Then the first letter of b is the same as the first letter of a. Well, if they have the same first letter, then the first letter of a is the first letter of b. Hence $a \approx b$. Therefore $\approx$ is symmetric.
Transitive: Suppose directly that $a, b, c \in A$ and $(a \approx b \wedge b \approx c)$. Then a and b start with the same letter, and b and c start with the same letter. Hence a and c must start with the same letter, so that $a \approx b$. Therefore $\approx$ is transitive.
Since $\approx$ is reflexive, symmetric, and transitive, it is an equivalence relation. ■

b) Determine the equivalence classes of $\approx$.

Under the assumptions that every English first name starts with a letter and that for every letter in the alphabet there is at least one English first name that starts with it, this gives us a total of 26 equivalence classes. Let S denote the set containing every letter of the alphabet, then for every $\alpha \in S$, we define the equivalence classes as such

$$[\alpha] = \{x \in A \,:\, \alpha \text{ is the first letter of } x\}.$$

Exercise 21.7. Let A be a set. Let R be reflexive, symmetric, and antisymmetric relation on A. Prove that R is equality on A. (In other words, prove that for any $x, y \in A$, we have xRy if an only if $x = y$).

Proof: This is a biconditional statement. We must show both ways.
$(\implies)$: We suppose directly that R is a reflexive, symmetric and antisymmetric relation on A. Then we know that $\forall a, b \in A$, aRa, $aRb \implies bRa$, and $aRb \wedge bRa \implies a = b$. The reflexive property tells us that for all $a \in A$, $(a, a) \in R$. The symmetric property tells us that for all $a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$. The antisymmetric property tells us that for all $a, b \in A$ if $(a, b), (b, a) \in R$, then $(a, b) = (b, a) = (a, a) = (b, b)$. Due to the symmetric property, the antisymmetric property applies to every element in R. Hence $R = \{(a, b) \in A \times A \,:\, a = b\}$. Therefore if R is a reflexive, symmetric and antisymmetric relation on A, then the relation is the equality relation.

($\Longleftarrow$) : We suppose directly that R is the equality relation on A. Then R can be defined as $R = \{(a, b) \in A \times A \, : \, a = b\}$. We want to show that R is reflexive, symmetric and antisymmetric.

Reflexive: Let $a \in A$. By the definition of R we know that aRa, i.e. $(a, a) \in R$. Hence R is reflexive.

Symmetric: Let $a, b \in A$ and suppose that aRb. Then we know $a = b$ which is equivalent to $b = a$. Hence bRa. Therefore R is symmetric.

Antisymmetric: Let $a, b \in A$ and suppose that $aRb \wedge bRa$. Then $a = b$ and $b = a$. Hence $a = b$. Therefore R is antisymmetric.

Thus if R is the equality relation, then R is reflexive, symmetric, and antisymmetric.

Since we have shown both ways, we have proven that R is the equality relation on R if and only if R is reflexive, symmetric, and antisymmetric. ∎