Homework 9

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Sections 16 and 17

16. The Binomial Theorem

Exercise 16.1. Use the definition of the binomial coefficient to prove that for each integer $n \geq 0$,

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \text{and} \quad \binom{n}{1} = \binom{n}{n-1} = n.$$

a) We will first show that for each integer $n \geq 0$,

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Proof: We suppose directly that $n \in \mathbb{Z} \geq 0$. We will first show that $\binom{n}{0} = \binom{n}{n}$. According to the definition of the binomial coefficient

$$\begin{aligned}\binom{n}{0} &= \frac{n!}{0!\,(n-0)!} \\ &= \frac{n!}{(n-0)!0!} \\ &= \frac{n!}{n!\,(n-n)!} \\ &= \binom{n}{n},\end{aligned}$$

hence $\binom{n}{0} = \binom{n}{n}$. We now evaluate the expression to get

$$\begin{aligned}\binom{n}{0} &= \frac{n!}{0!\,(n-0)!} \\ &= \frac{n!}{n!} \\ &= \frac{1}{1}.\end{aligned}$$

Therefore the statement $\binom{n}{0} = \binom{n}{n} = 1$ is true. ■

b) We now show that for each integer $n \geq 0$,

$$\binom{n}{1} = \binom{n}{n-1} = n.$$

Proof: We suppose directly that $n \in \mathbb{Z} \geq 0$. We will first show that $\binom{n}{1} = \binom{n}{n-1}$. According to the definition of the binomial coefficient

$$\begin{aligned}\binom{n}{1} &= \frac{n!}{1!\,(n-1)!} \\ &= \frac{n!}{(n-1)!1!} \\ &= \binom{n}{n-1},\end{aligned}$$

hence $\binom{n}{1} = \binom{n}{n-1}$. We now evaluate the expression to get

$$\begin{aligned}\binom{n}{1} &= \frac{n!}{1!\,(n-1)!} \\ &= \frac{n\,((n-1)!)}{1!\,(n-1)!} \\ &= n,\end{aligned}$$

Therefore the statement $\binom{n}{1} = \binom{n}{n-1} = n$ is true for each $n \in \mathbb{Z} \geq 0$. ■

Exercise 16.2. Prove that for any $n, k \in \mathbb{Z}$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof: We suppose directly that $n, k \in \mathbb{Z}$. We have two cases to consider: case 1) $0 \leq k \leq n$ and case 2) $\neg(0 \leq k \leq n)$.

Case 1. Let $\neg(0 \leq k \leq n)$ be true, then $\binom{n}{k} = 0$ and according to the definition of the binomial coefficients. Noting that $\neg(0 \leq k \leq n)$ is equivalent to $k < 0$, $n < 0$, or $n < k$, we know that $\binom{n}{n-k} = 0$. Hence $\binom{n}{k} = \binom{n}{n-k} = 0$.

Case 2. Let $(0 \leq k \leq n)$ be true, then according to the definition of binomial coefficients we get

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!\,(n-k)!} \\ &= \frac{n!}{(n-k)!k!} \\ &= \frac{n!}{(n-k)!\,(n-n+k)!} \\ &= \binom{n}{n-k},\end{aligned}$$

hence $\binom{n}{k} = \binom{n}{n-k}$.

■

Since both cases hold true, we have shown that for any $n, k \in \mathbb{Z}$ that $\binom{n}{k} = \binom{n}{n-k}$.

Exercise 16.3. Let $n, h, k \in \mathbb{Z}$. Using the definition of the binomial coefficient, prove that
$$\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}.$$

Proof: We suppose directly that $n, h, k \in \mathbb{Z}$. We know that $\binom{n}{h} = 0$ when $n < 0$, $h < 0$ or $h > n$, $\binom{n}{k} = 0$ when $n < 0$, $k < 0$ or $k > n$, $\binom{n-h}{k} = 0$ when $k < 0$, $n - h < 0$ or $k + h > n$, and we know that $\binom{n-k}{h} = 0$ when $h < 0$, $n - k < 0$ or $h + k > n$ according to the definition of binomial coefficients. In order for $n < h + k$, the two statements $n < h$ and $n < k$ must be true. Therefore we really only have five cases to consider: $n < 0$, $n < h + k$, $k < 0$, $h < 0$ and $0 \le h + k \le n$ since at least one of these is always true.

Case 1. We suppose directly that $n, h, k \in \mathbb{Z}$ and $n < 0$, then
$$\begin{aligned}\binom{n}{h}\binom{n-h}{k} &= 0 \cdot \binom{n-h}{k} \\ &= 0\end{aligned}$$
and
$$\begin{aligned}\binom{n}{k}\binom{n-k}{h} &= 0 \cdot \binom{n-k}{h} \\ &= 0\end{aligned}$$
hence we have shown that $\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}$ when $n < 0$.

Case 2. We suppose directly that $n, h, k \in \mathbb{Z}$ and $h < 0$, then
$$\begin{aligned}\binom{n}{h}\binom{n-h}{k} &= 0 \cdot \binom{n-h}{k} \\ &= 0\end{aligned}$$
and
$$\begin{aligned}\binom{n}{k}\binom{n-k}{h} &= \binom{n}{k} \cdot 0 \\ &= 0\end{aligned}$$
hence we have shown that $\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}$ when $h < 0$.

Case 3. We suppose directly that $n, h, k \in \mathbb{Z}$ and $k < 0$, then
$$\begin{aligned}\binom{n}{h}\binom{n-h}{k} &= \binom{n}{h} \cdot 0 \\ &= 0\end{aligned}$$
and
$$\begin{aligned}\binom{n}{k}\binom{n-k}{h} &= 0 \cdot \binom{n-k}{h} \\ &= 0\end{aligned}$$
hence we have shown that $\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}$ when $k < 0$.

Case 4. We suppose directly that $n, h, k \in \mathbb{Z}$ and $n < h + k$, then

$$\begin{aligned}\binom{n}{h}\binom{n-h}{k} &= 0 \cdot 0 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\binom{n}{k}\binom{n-k}{h} &= 0 \cdot 0 \\ &= 0\end{aligned}$$

hence we have shown that $\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}$ when $n < h + k$.

Case 5. We suppose directly that $n, h, k \in \mathbb{Z}$ and that $0 \le k + h \le n$, then

$$\begin{aligned}\binom{n}{h}\binom{n-h}{k} &= \frac{n!}{h!\,(n-h)!} \cdot \frac{(n-h)!}{k!\,(n-h-k)!} \\ &= \frac{n!}{h!k!\,(n-h-k)!} \\ &= \frac{n!\,(n-k)!}{h!k!\,(n-h-k)!\,(n-k)!} \\ &= \frac{n!}{k!\,(n-k)!} \cdot \frac{(n-k)!}{h!\,(n-k-h)!} \\ &= \binom{n}{k}\binom{n-k}{h},\end{aligned}$$

hence we have shown that $\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}$ when $0 \le k + h \le n$.

Since all cases hold, we have shown that the statement $\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}$ is true for any $n, h, k \in \mathbb{Z}$.

■

Exercise 16.4. Prove that for any integer $n \ge 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof: We wish to show that the open sentence

$$P(n) : \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

is true for any integer $n \ge 0$. We show this by induction.

Base Case: We verify that $P(0)$ is true.

$$\begin{aligned}\sum_{k=0}^{0} \binom{0}{k} &= \frac{0!}{0!\,(0-0)!} \\ &= \frac{1}{1} \\ &= 2^0.\end{aligned}$$

Induction Step: We suppose that $m \ge 0$, and that $P(m)$ is true which is the statement

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m,$$

and we want to show that $P(m+1)$ is true. We begin with

$$\begin{aligned}
\sum_{k=0}^{m+1}\binom{m+1}{k} &= \sum_{k=0}^{m}\binom{m+1}{k}+\binom{m+1}{m+1}\\
&= \sum_{k=0}^{m}\left(\binom{m}{k}+\binom{m}{k-1}\right)+1, \quad \text{(Using Pascal's Equality)}\\
&= 2^m+1+\sum_{k=0}^{m}\binom{m}{k-1}, \quad \text{(Since } P(m) \text{ is true)}\\
&= 2^m+1+\sum_{k=0}^{m-1}\binom{m}{k}\\
&= 2^m+\sum_{k=0}^{m}\binom{m}{k}\\
&= 2^m+2^m\\
&= 2^{m+1}
\end{aligned}$$

hence $P(m+1)$ is true. Therefore the statement $P(n)$ is true for any integer $n \geq 0$. ■

Exercise 16.5. Prove that for any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0.$$

Proof: We wish to show that the statement

$$P(n): \sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$$

is true for any $n \in \mathbb{N}$. We work this by induction.

Base Case: We verify that $P(1)$ is true

$$\sum_{k=0}^{1}(-1)^k\binom{1}{k}=1-1=0.$$

Induction Step: We assume that $P(m)$ is true for any $m \in \mathbb{N}$, which is the statement

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}=0,$$

and we want to show that $P(m+1)$ is true. We begin with

$$\begin{aligned}
\sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} &= \sum_{k=0}^{m} (-1)^k \binom{m+1}{k} + (-1)^{m+1} \binom{m+1}{m+1} \\
&= \sum_{k=0}^{m} \left((-1)^k \binom{m}{k} + (-1)^k \binom{m}{k-1} \right) + (-1)^{m+1} 1, \quad \text{(Using Pascal's Equality)} \\
&= 0 + \sum_{k=0}^{m} (-1)^k \binom{m}{k-1} + (-1)^{m+1} 1, \quad \text{(Since } P(m) \text{ is true)} \\
&= \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} + (-1)^{m+1} 1 \\
&= \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} + (-1)^{m+1} 1 + (-1)^m 1 - (-1)^m \\
&= \sum_{k=0}^{m} (-1)^k \binom{m}{k} + (-1)^{m+1} 1 - (-1)^m \\
&= 0 + (-1)^m (-1) - (-1)^m \\
&= (-1)^m (1-1) \\
&= 0,
\end{aligned}$$

hence $P(m+1)$ is true. Therefore the statement $P(n)$ is true for any $n \in \mathbb{N}$. ■

Exercise 16.6. Determine the coefficient of x^5y^3 in the expansion of $(2x+3y)^8$.

According to the binomial theorem, if x and y are variables and n is a non-negative integer, then

$$(z+q)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} q^k.$$

The term x^5y^3 must correspond to

$$\begin{aligned}
\binom{8}{3} (2x)^{8-3} (3y)^3 &= 56 \cdot 32 \cdot 27 \left(x^5 y^3\right) \\
&= 48384 x^5 y^3.
\end{aligned}$$

Exercise 16.7. Use the definition of the binomial coefficient to prove that for any $n, k \in \mathbb{Z}$,

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

Proof: We show this directly assuming that $n, k \in \mathbb{Z}$. The binomial coefficient $\binom{n}{k}$ is 0 when $n < 0$, $k < 0$ or $n < k$, and the binomial coefficient $\binom{n-1}{k-1}$ is 0 when $n < 1$, $k < 1$ or $n < k$. This leads to three different cases: case 1) $n < 0$, $k < 1$, or $n < k$, case 2) $n = 0$, $k < 1$, or $n < k$ and case 3) $0 < k \leq n$

Case 1. Let $n < 0$, $k < 1$, or $n < k$, then

$$k \binom{n}{k} = 0$$

and

$$n \binom{n-1}{k-1} = 0,$$

hence
$$k\binom{n}{k} = n\binom{n-1}{k-1} = 0.$$

Thus this case is always true.

Case 2. Let $n = 0$, $k < 1$, or $n < k$. When $n = 0$, the binomial coefficient $\binom{n}{k}$ is not 0 when $k = 0$ and is 0 when $k < 0$. This means that
$$k\binom{n}{k} = 0,$$

In addition
$$n\binom{n-1}{k-1} = 0,$$

hence
$$k\binom{n}{k} = n\binom{n-1}{k-1} = 0.$$

Thus this case is always true.

Case 3. Let $0 < k \leq n$, then
$$\begin{aligned} k\binom{n}{k} &= k\frac{n!}{k!\,(n-k)!} \\ &= \frac{n\,((n-1)!)\,k}{k\,(k-1)!\,(n-1-k+1)!} \\ &= n\frac{((n-1)!)}{(k-1)!\,(n-1-k+1)!} \\ & n\binom{n-1}{k-1}, \end{aligned}$$

hence this case is true.

Since all possible cases are true, the statement $k\binom{n}{k} = n\binom{n-1}{k-1}$ is true for any $n, k \in \mathbb{Z}$.

■

Exercise 16.8. Prove that for $n \in \mathbb{N}$, the "middle" binomial coefficient
$$\binom{2n}{n}$$
is an even integer.

Proof: We wish to show that the open sentence
$$P(n) : \binom{2n}{n} \text{ is an even integer}$$
is true for any $n \in \mathbb{N}$. We work this by induction.

Base Case: We verify that $P(1)$ is true.
$$\binom{2}{1} = \frac{2!}{1!1!} = 2.$$

Induction Step: We assume that for $k \in \mathbb{N}$, $P(k)$ is true. We want to show that $P(k+1)$ is true. We begin with

$$\begin{aligned}
\binom{2(k+1)}{k+1} &= \binom{2k+2}{k+1} \\
&= \binom{2k+1}{k} + \binom{2k+1}{k+1}, \quad \text{(Using Pascal's triangle)} \\
&= \binom{2k}{k} + \binom{2k}{k-1} + \binom{2k}{k} + \binom{2k}{k+1} \quad \text{(Using Pascal's triangle on each term)} \\
&= 2\binom{2k}{k} + 2\binom{2k}{k-1}, \quad \text{(Using the fact } \binom{2k}{k+1} = \binom{2k}{k-1}\text{, proven in exercise 2} \\
&= 2\left(\binom{2k}{k} + \binom{2k}{k-1}\right),
\end{aligned}$$

which is an even integer since the binomial coefficient is always an integer. Therefore $P(n)$ is true for any $n \in \mathbb{N}$. ■

Exercise 16.9. Let $n, k \in \mathbb{Z}$.

a) Use induction to prove that for $n > 8$,

$$\binom{n}{k} < 2^{n-2}, \quad \text{for each } k \in \mathbb{Z}.$$

Proof: We want to show that the open sentence

$$P(n): \binom{n}{k} < 2^{n-2}$$

is true for $n \in \mathbb{Z} > 8$ and $k \in \mathbb{Z}$. We work this by induction.
Base Case: We will verify $P(9)$. We have two cases to consider: when $k < 0$ or $9 < k$ and when $0 \leq k \leq 9$.

Case 1. Let $k < 0$ or $9 < k$, then

$$\begin{aligned}\binom{9}{k} &= 0 \\ &< 2^m,\end{aligned}$$

for any $m \in \mathbb{N}$. Thus this case holds.

Case 2. Let $0 \leq k \leq 9$, then

$$\binom{9}{k} = \frac{9!}{k!\,(n-k)!},$$

which is largest when $k \in \{4, 5\}$ according to Pascal's triangle (Figure 16.5 in the book). Thus

$$\binom{9}{k} \leq 126$$

for all k. Using this information we note that

$$\begin{aligned}126 &< 2^{9-2} \\ &< 128,\end{aligned}$$

Hence this case holds.

Since both cases hold, $P(9)$ is true.

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Induction Step: Assume that $m \in \mathbb{Z} > 8$ and $P(m)$ is true. We want to show that $P(m+1)$ is true. We being with

$$\begin{aligned}\binom{m+1}{k} &= \binom{m}{k} + \binom{m}{k+1}, \quad \text{(Using Pascal's triangle equality)}\\ &< 2^{m-2} + 2^{m-2}, \quad \text{(Since we assume } P(m) \text{ to be true)}\\ &< 2^{m+1-2},\end{aligned}$$

hence the statement $P(m+1)$ is true. Therefore $P(n)$ is true for any $n \in \mathbb{Z} > 8$.

b) Use induction to prove that for $n > 7$,

$$\binom{n}{k} < (n-3)! \quad \text{for each } k \in \mathbb{Z}$$

Proof: We want to show that the open sentence

$$P(n): \binom{n}{k} < (n-3)!$$

is true for each $k \in \mathbb{Z}$ and $n \in \mathbb{N} > 7$. We work this by induction.

Base Case: We verify that $P(8)$ is true. We show this using two cases.

Case 1. Let $k < 0$ or $k > 8$, then

$$\begin{aligned}\binom{8}{k} &= 0\\ &< 120\\ &= (8-3)!,\end{aligned}$$

hence this case holds.

Case 2. Let $0 \leq k \leq 8$, then

$$\begin{aligned}\binom{8}{k} &\leq 70, \quad \text{according to Pascal's triangle (Figure 16.5 in the book)}\\ &< 120\\ &= (8-3)!\end{aligned}$$

hence this case holds.

Since both cases hold, $P(8)$ is true.

Induction Step: We assume that $m \in \mathbb{Z} > 7$ and that $P(m)$ is true. We want to show that $P(m+1)$ is true. We begin with

$$\begin{aligned}\binom{m+1}{k} &= \binom{m}{k} + \binom{m}{k+1}, \quad \text{(Using Pascal's triangle equality)}\\ &\leq (m-3)! + (m-3)!, \quad \text{(Since we assume } P(m) \text{ to be true)}\\ &= 2(m-3)!\\ &= 2\frac{(m+1-3)}{(m+1-3)}(m-3)!\\ &= 2\frac{(m+1-3)!}{(m+1-3)}\\ &\leq (m+1-3)!, \quad \text{(Since } m > 7 \text{, we know that } m+1-3 > 5 \text{, thus } 0 < \frac{2}{m+1-3} < 1).\end{aligned}$$

Hence $P(m+1)$ is true. Therefore $P(n)$ is true for $n \in \mathbb{N} > 7$ and $k \in \mathbb{Z}$.

■

17. Divisibility

Exercise 17.1. For the given values of n and d, compute the values of q and r guaranteed by the division algorithm.

a) Let $n = 17$, $d = 5$.
   $17 = 3 \cdot 5 + 2$. Thus $q = 3$ and $r = 2$.
b) Let $n = 17$, $d = -5$.
   $17 = (-3) \cdot (-5) + 2$. Thus $q = -3$ and $r = 2$.
c) Let $n = -17$, $d = 5$.
   $-17 = (-4) \cdot (5) + 3$. Thus $q = -4$ and $r = 3$.
d) Let $n = -17$, $d = -5$.
   $-17 = 4 \cdot (-5) + 3$. Thus $q = 4$ and $r = 3$.
e) Let $n = 256$, $d = 25$.
   $256 = 10 \cdot 25 + 6$. Thus $q = 10$ and $r = 6$.
f) Let $n = 256$, $d = -25$.
   $256 = (-10) \cdot (-25) + 6$. Thus $q = -10$ and $r = 6$.
g) Let $n = -256$, $d = 25$.
   $-256 = (-11) \cdot 25 + 19$. Thus $q = -11$ and $r = 19$.
h) Let $n = -256$, $d = -25$.
   $-256 = 11 \cdot (-25) + 19$. thus $q = 11$ and $r = 19$.

Exercise 17.2. Let a be an integer. Recall that a is even if there is some $k \in \mathbb{Z}$ such that $a = 2k$, and a is odd if there is some $\ell \in \mathbb{Z}$ such that $a = 2\ell + 1$. Prove the following statements, which we took for granted previously: Every integer is even or odd and no integer is both even and odd.

Proof: We suppose directly that $m \in \mathbb{Z}$. According to the division algorithm, given a number $d \in \mathbb{Z}$, we can represent m as $m = qd + r$ where q and r are unique integers and with $0 \leq r < |d|$. By letting $d = 2$, we can then write every integer m as $m = 2q + r$ with $0 \leq r < 2$. This means that $r \in \{0, 1\}$. This gives us two cases.

Case 1. Let $r = 0$, then $m = 2q$ and is even.
Case 2. Let $r = 1$, then $m = 2q + 1$ and is odd.

Hence every integer is either even or odd and cannot be both.

■

Exercise 17.3. Write out all the divisors of 60 in a list, and then all the divisors of 42 is a separate list. Write the common divisors in a third list, and find the GCD.

Let $T = \{x \in \mathbb{Z} : x \mid 60\}$, $S = \{x \in \mathbb{Z} : \ x \mid 42\}$, and $U = \{x \in \mathbb{Z} : \ x \mid 60 \text{ and } x \mid 42\}$. These sets can be written out as

$$T = \{-60, -30, -15, -12, -10, -6, -5, -3, -2, -1, 1, 2, 3, 5, 6, 10, 12, 15, 30, 60\}$$
$$S = \{-42, -21, -7, -6, -3, -2, -1, 1, 2, 3, 6, 7, 21, 42\}$$
$$U = \{-3, -2, -1, 1, 2, 3\}$$

We can easily see now that $\mathrm{GCD}\,(60, 42) = 3$.

Exercise 17.4. Use the Euclidean algorithm to compute the following GCDs.

a) $\mathrm{GCD}\,(60, 42)$

We use the Division algorithm to multiple times

$$\begin{aligned}60 &= 1 \cdot 42 + 18\\ 42 &= 2 \cdot 18 + 6\\ 18 &= 3 \cdot 6\end{aligned}$$

to get that GCD $(60, 42) = 6$

b) GCD $(667, 851)$

We use the Division algorithm to multiple times

$$\begin{aligned}851 &= 1 \cdot 667 + 184\\ 667 &= 3 \cdot 184 + 115\\ 184 &= 115 + 69\\ 115 &= 69 + 46\\ 69 &= 46 + 23\\ 49 &= 2 \cdot 23 + 0\end{aligned}$$

to get that GCD $(667, 851) = 23$

c) GCD $(1855, 2345)$

We use the Division algorithm to multiple times

$$\begin{aligned}2345 &= 1855 + 490\\ 1855 &= 3 \cdot 490 + 385\\ 490 &= 385 + 105\\ 385 &= 3 \cdot 105 + 70\\ 105 &= 70 + 35\\ 70 &= 2 \cdot 35\end{aligned}$$

to get that GCD $(1855, 2345) = 35$

d) GCD $(589, 437)$

We use the Division algorithm to multiple times

$$\begin{aligned}589 &= 437 + 152\\ 437 &= 2 \cdot 152 + 133\\ 152 &= 133 + 19\\ 133 &= 7 \cdot 19\end{aligned}$$

to get that GCD $(589, 437) = 19$

Exercise 17.5. Recall that the Fibonacci numbers are defined by the relations $F_1 = 1$, $F_2 = 1$, and for $n > 2$ the recursion $F_n = F_{n-1} + F_{n-2}$. Prove by induction that for each $n \in \mathbb{N}$ we have $GCD\left(F_{n+1}, F_n\right) = 1$.

Proof: We want to show that the open sentence

$$P(n) : \text{ GCD}\left(F_{n+1}, F_n\right) = 1$$

is true for each $n \in \mathbb{N}$. We work this by induction.

Base Case: We verify that $P(1)$ is true using the division algorithm

$$\begin{aligned}F_2 &= F_1 + 0\\ 1 &= 1 + 0,\end{aligned}$$

hence the GCD is 1 and $P(1)$ is true.

Induction Case: We assume that $k \in \mathbb{N}$ and that $P(k)$ is true, which is the statement

$$\text{GCD}(F_{k+1}, F_k) = 1,$$

and we want to show that $P(k+1)$ is true, which is the statement

$$\text{GCD}(F_{k+2}, F_{k+1}) = 1.$$

To show that $P(k+1)$ we will use the division algorithm which states

$F_{k+2} = F_{k+1} + F_k, \quad$ (By definition of the division algorithm and the Fibonacci numbers)

From the division algorithm,and using the GCD switching theorem, we can see that the GCD$(F_{k+2}, F_{k+1}) =$ GCD (F_{k+1}, F_k), and since we assume $P(k)$ we know that

$$\begin{aligned} \text{GCD}(F_{k+2}, F_{k+1}) &= \text{GCD}(F_{k+1}, F_k) \\ &= 1, \end{aligned}$$

hence the statement $P(k+1)$ is true. Therefore the open sentence $P(n)$ is true for $n \in \mathbb{N}$. ■

Exercise 17.6. Let $n \in \mathbb{Z}$. Prove that GCD $(2n+1, 4n+3) = 1$.

Proof: We suppose directly that $k \in \mathbb{Z}$, and we want to show that GCD $(2k+1, 4k+3) = 1$. We show this using the division algorithm which states

$$4k + 3 = 2(2k+1) + 1,$$

thus, according to the GCD switching theorem, we see that GCD $(2k+1, 4k+3) =$ GCD $(2k+1, 1)$ which must be 1. Therefore if $n \in \mathbb{Z}$, then the open sentence GCD $(2n+1, 4n+3) = 1$ is true. ■

Exercise 17.7. Let $n \in \mathbb{Z}$. Prove that GCD $(6n+2, 12n+6) = 2$.

Proof: We suppose directly that $k \in \mathbb{Z}$, and we want to show that GCD $(6k+2, 12k+6) = 2$. We show this using the division algorithm in combination with the GCD switching theorem which states

$$\begin{aligned} 12k + 6 &= 2(6k+2) + 2 \\ 6k + 2 &= 2(3k+1) + 0 \\ 2 &= 1 \cdot 2 + 0, \end{aligned}$$

hence the GCD $(6k+2, 12k+6) = 2$. Therefore we have proven that the open sentence GCD $(6n+2, 12n+6) = 2$ is true for any integer n. ■

Exercise 17.8. Complete the proof of Theorem 17.13 as follows.

a) Using the fact that the theorem is true for nonnegative n and positive d, prove the theorem for arbitrary n and positive d.
   Proof: We assume directly that $n \in \mathbb{Z}$ and d is a positive integer. The statement in theorem 17.13 has already been proven for nonnegative n and positive d, therefore we only need to show for when $n < 0$. We can show this by writing the division algorithm as

   $$-m = qd + r,$$

   where $m = -n$ such that $m > 0$, $q, r \in \mathbb{Z}$, and $0 \leq r < |d|$. Multiplying the equation on both sides by -1 yields

   $$\begin{aligned} m &= -qd - r \\ &= (-q)\,d - r. \end{aligned}$$

If $r = 0$, then the equation reduces to

$$m = (-q)\, d,$$

and this statement has already been proven to be true. If $r \neq 0$, then we can manipulate the equation

$$\begin{aligned} m &= (-q)\, d - r + d - d \\ &= (-q)\,(d+1) + (d-r) \\ &= (-q)\,\ell + k, \end{aligned}$$

where $\ell = d + 1$ is a positive integer, and $k = d - r$ is a positive integer since $d > r > 0$. This also means that $0 \leq k < \ell$ which is in the form already proven. Therefore Theorem 17.13 is true for arbitrary n and positive d. ■

b) Using the fact that the theorem is true for positive d, prove the theorem for negative d.

Proof: We assume directly that $n \in \mathbb{N}$ and $d < 0$. Let $k = -d$ so that $k > 0$. The division algorithm can be written as

$$\begin{aligned} n &= qd + r \\ &= (-q)\, k + r, \end{aligned}$$

since we know that the theorem is true for positive d we know that the theorem applied to the equation

$$n = (-q)\, k + r$$

is true. Therefore, theorem is true for negative d. ■