Rotation Matrices

I. Introduction

Rotation matrices have always been a bane in my understanding of mechanical systems since the very beginning. I have heard terms like forward and backward transformations, passive and active rotations, Body-Centric vs Inertial representation without truly understanding them, and I have gotten confused with transforming the components of vectors verses the basis vectors of a frame. My aim at this document is to explore what rotation matrices really are by deriving them from a geometric perspective and only using coordinates when necessary. Along this journey I hope to clarify the terms like forward and backward transformations, passive and active rotations, and distinguish the difference between Body-Centric and Inertial representation.

Throughout this document I will work in Euclidean spaces since we can measure distances and angles in this space, and where the inner product and cross product are well defined. These tools are necessary since they allow us to explore rotation matrices from a geometric perspective. Euclidean spaces also allow us to construct reference frames. Reference frames can be defined using the right-hand or left-hand convention. All of the frames used in this document will be right-handed frames; however, everything done in this note can be derived for left-handed frames.

In order to start this journey, we need to equip you with the proper tools to tackle such a dangerous adventure. These tools will consist of a quick introduction to vectors, reference frames, definitions of the inner product and cross product, and some notation. Once equipped with these tools, you will begin your journey by exploring the forward and backward transformations and active and passive transformations of vectors using arbitrary reference frames in 2-Dimensional Euclidean space. Make no mistake, the first part of this trek is steep and treacherous, but you will be rewarded with a short hike on gentle terrain as we repeat this exploration using standard reference frames in 2-Dimensional Euclidean space.

But don't you dare feel at ease, because the next stretch is the intense exploration of forward and backward transformations and active and passive transformations of vectors in 3-Dimensional Euclidean space! And only once you have made it this far, we can discuss the difference between Body-Centric and Inertial representation of rotations. Finally at the end of our journey together, we will sit down, take a deep breath, and quickly summarize the main points. And then you can walk away knowing that you are a better person, that you have journeyed where only a few dare to go.

II. The Geometric Tools

To discuss rotation matrices from a geometric perspective, we need a basic understanding of vectors, reference frames, geometric concepts like the inner product and cross product, Euclidean space, several definitions, and some notation. Many of you will already have a solid understanding of this material and can comfortably skip to the next section; however, I still encourage you to read it because I might use different notation then what you are used to, and a refresher is always nice.

A. *Vectors Frames*

A vector is an object in a space that is invariant to changes in coordinates or frames and is often used to express displacement, i.e., a direction and magnitude. I will denote all vectors with bold font, e.g., $\mathbf{v}$,$\mathbf{e}$, $\mathbf{i}$, etc. Note that these are objects in a space, and they are not "tied down" by any coordinates. What makes vectors so amazing is that you can add them together (see figure 1a) , and multiply them by scalars (see figure 1b).

(a) (b)

These properties of vectors allows them to form a linear vector space. [1] A linear vector space S over the field of scalars $\mathbb{R}$ is a collection of vectors equipped with an additive operation $+$ and scalar multiplication operation $\cdot$ that satisfy the following properties:

1) S forms a group under addition, i.e., the following properties are satisfied
   a) For any $\mathbf{a}$ and $\mathbf{b}$ in S, $\mathbf{a}+\mathbf{b} \in S$. (The addition operation is closed)
   b) For any $\mathbf{a}$ and $\mathbf{b}$ in S, they commute under addition
   $$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}.$$
   c) There is an identity element in S, which we will denote as $\mathbf{0}$, such that for any $\mathbf{a} \in S$,
   $$\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}.$$
   d) For every element $\mathbf{a} \in S$ there is another element $\mathbf{b} \in S$ such that
   $$\mathbf{a}+\mathbf{b}=\mathbf{0}.$$
   The element $\mathbf{b}$ is the additive inverse of $\mathbf{a}$ and is usually denoted as $-\mathbf{a}$.
   e) The addition operation is associative; for any $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in S,
   $$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}).$$
2) For any $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in S$,
$$\begin{gathered}\alpha \mathbf{b} \in S \\ \alpha(\beta \mathbf{b})=(\alpha \beta) \mathbf{b} \\ (\alpha+\beta) \mathbf{b}=\alpha \mathbf{b}+\beta \mathbf{b} \\ \alpha(\mathbf{a}+\mathbf{b})=\alpha \mathbf{a}+\alpha \mathbf{b}\end{gathered}$$

Remark 1. Some of you might be tempted to think of vectors in $\mathbb{R}^n$, but stop that! It is true that all vector spaces are isomorphic to $\mathbb{R}^n$, but I have said nothing about atlases and coordinates. Keep these vectors simply as objects that can be added together and scaled.

I will denote linear vector spaces with the letter S. If it is necessary to specify the dimension of the vector space, I will use the notation S^n where $n \in \mathbb{R}$ denotes the dimension of the space.

B. Reference Frames

Before we can talk about frames, let me introduce several definitions that might be familiar to you.

Definition 2. Let $U \subset S$ be a finite subset of the vector space S. The elements of the subset are **linearly independent** if
$$\sum_{i=1}^{|U|} \alpha^i \mathbf{u}_i=0 \iff \alpha^i=0;\ \mathbf{u}_i \in U,\ \alpha^i \in \mathbb{R},$$
where $|U|$is the cardinality of the subset U.

Definition 3. A subset $U \subset S$ **spans** the set S if every element of S can be written as a linear combination of the elements of U , i.e.,

$$\mathbf{s} = \sum_{i=1}^{|U|} \alpha^i \mathbf{u}_i;\ \forall \mathbf{s} \in S,\ \mathbf{u}_i \in U.$$

We denote that a subset $U \subset S$ spans the space S using the notation $S = \text{span}\,(U)$.

The definition of a set spanning another set does not imply that the elements of the spanning set are linearly independent. If they are, then you have a basis of the original set.

Definition 4. A subset $U \subset S$ that is linearly independent and spans S is a **basis** of S. An element of the basis is called a **basis vector**.

To solidify the definitions, let's go over a quick example. Let $\mathbf{e}_1$, and $\mathbf{e}_2$ form a basis in the vector space S such that $\alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2 = 0 \iff \alpha^1 = \alpha^2 = 0,\ \alpha^1, \alpha^2 \in \mathbb{R}$ (*linear independence*) and $S^2 = \text{span}\,\{\mathbf{e}_1, \mathbf{e}_2\}$, then the set $U \coloneqq \{\mathbf{e}_1, \mathbf{e}_2\} \subset S^2$ is a basis of S^2. An example of this basis is depicted in figure 1. By visual inspection you can see that the two basis vectors are linearly independent, and that any element in S^2 can be represented as a linear combination of the basis vectors.

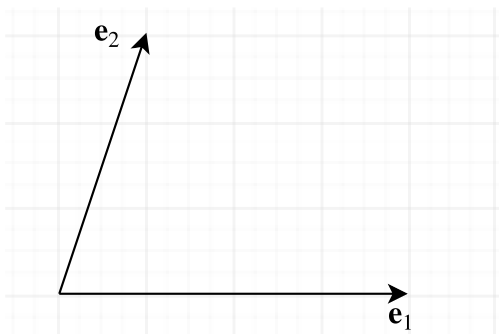


Figure 1

Now that we have an understanding of what a basis is, let's move onto reference frames.

Definition 5. A **reference frame** is a basis equipped with an origin denoted as $\mathcal{F} = (O, U)$ where O is point in the vector space S and $U \subset S$ is a basis of S.

A reference frame, or frame for short, allows us to measure a point relative to it's origin to create a distance vector. It also allows us to express vectors relative to it's basis using components. For example consider the scenario depicted in figure 2 which shows a frame $\mathcal{F}^a = (O, \{\mathbf{e}_1, \mathbf{e}_2\})$ a point p and a vector $\mathbf{v} = p - O$.

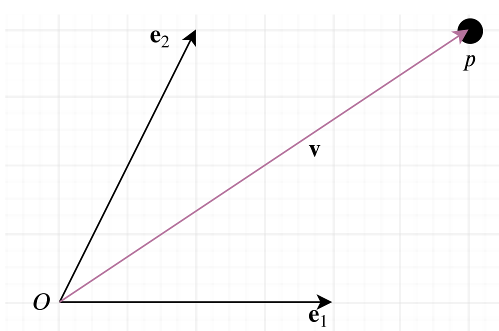


Figure 2

The frame, $\mathcal{F}^a$, allows me to represent the distance vector $\mathbf{v}$ as a linear combination of the basis vectors using components such as

$$\mathbf{v}^a = \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2$$

where $\alpha^i \in \mathbb{R}$ is the component of $\mathbf{v}$ with respect to (w.r.t.) to the basis $\mathbf{e}_i$, and the superscript a on $\mathbf{v}$ is notation used to indicate that is being expressed in $\mathcal{F}^a$. The vector $\mathbf{v}$ can also be represented using matrix notation

$$\mathbf{v}^a = \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix}$$

where the first entry is the component of $\mathbf{v}$ w.r.t. to $\mathbf{e}_1$ and the second entry is the component of $\mathbf{v}$ w.r.t. $\mathbf{e}_2$.

From figure 2, we can see that $\alpha^1 = \alpha^2 = 1$ so that

$$\mathbf{v}^a = \mathbf{e}_1 + \mathbf{e}_2$$

Remark 6. I want to take a moment to stress that I have not associated coordinates to the vectors, i.e., $\mathbf{e}_2 = 2i + 4j$ where i and j are the orthogonal coordinates of $\mathbb{R}^2$. I have left the basis vectors as objects somewhere in 2-dimensional Euclidean space, and I have represented the vector $\mathbf{v}$ using the reference frame $\mathcal{F}^a$.

Quick aside: I will often omit the superscript on a vector when it isn't needed.

C. Inner Product

The inner product is a positive definite bilinear map, $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$, that takes two elements of a Hilbert space H and maps them to an element in $\mathbb{R}$.

The inner product has four properties

1) *Commutative:*
   a) $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \overline{\langle \mathbf{v}_2, \mathbf{v}_1 \rangle}$ where the overbar indicates complex conjugation. We will only work in the realm of real vectors, so this property is simplified to $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle$.
2) *Distributive*
   a) $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle + \langle \mathbf{v}_2, \mathbf{v}_3 \rangle$.
3) Scalar Multiplication
   a) $\langle a\mathbf{v}_1, \mathbf{v}_2 \rangle = a \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ or $\langle a\mathbf{v}_1, b\mathbf{v}_2 \rangle = ab \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ where $a, b \in \mathbb{R}$.
4) *Positive Definite*
   a) $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ if $x \neq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

We can use the properties of the inner product to define the inner product of two vectors expressed in a reference frame. To show this let $\mathcal{F}^a = (O_a, \{\mathbf{e}_1, \mathbf{e}_2\})$ be an arbitrary reference frame in H^2 (2-dimensional Hilbert space), and let $\mathbf{v}_1 = \beta^1 \mathbf{e}_1 + \beta^2 \mathbf{e}_2$ and $\mathbf{v}_2 = \gamma^1 \mathbf{e}_1 + \gamma^2 \mathbf{e}_2$ be two vectors express in $\mathcal{F}^a$, then the inner product is defined as

$$\langle \mathbf{v}_1^a, \mathbf{v}_2^a \rangle = \sum_{k=1}^{2} \sum_{j=1}^{2} \langle \mathbf{e}_k, \mathbf{e}_j \rangle \beta^k \gamma^j$$

where $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \langle \mathbf{e}_2, \mathbf{e}_1 \rangle$ are not necessarily 0.

The inner product can be extended to any finite-dimension Hilbert space. For an n dimensional Hilbert space, H^n, the inner product is defined as

$$\langle \mathbf{v}_1^a, \mathbf{v}_2^a \rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} \langle \mathbf{e}_k, \mathbf{e}_j \rangle \beta^k \gamma^j \tag{1}$$

where $\mathbf{v}_1, \mathbf{v}_2 \in H$ and have components β^i (respectively γ^i) relative to the basis $\mathbf{e}_i$.

Definition 7. Two vectors $\mathbf{v}_1, \mathbf{v}_2 \in H$ are **orthogonal** if $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ and are **orthonormal** if $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$.

Definition 8. A **standard reference frame**, or standard frame, is a frame with an orthonormal basis.

The inner product between two vectors can be simplified if they are being expressed in a standard reference frame. To demonstrate this, let $\mathcal{F}^s = (O, \{\mathbf{e}_1, \ldots, \mathbf{e}_n\})$ be a standard reference frame in H^n with an orthonormal basis, and $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors expressed in $\mathcal{F}^s$ such that

$$\mathbf{v}_1^s = \sum_{k=1}^{n} \beta^k \mathbf{e}_k$$

and

$$\mathbf{v}_2^s = \sum_{j=1}^{n} \gamma^k \mathbf{e}_j,$$

then

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} \langle \mathbf{e}_k, \mathbf{e}_j \rangle \beta^k \gamma^j$$

which can be simplified using the orthonormal property

$$\langle \mathbf{e}_k, \mathbf{e}_j \rangle = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{else} \end{cases}, \; k, j \in \{1, \ldots, n\}$$

to

$$\begin{aligned} \langle \mathbf{v}_1^s, \mathbf{v}_2^s \rangle &= \sum_{k=1}^{n} \langle \mathbf{e}_k, \mathbf{e}_j \rangle \beta^k \gamma^k \\ &= \sum_{k=1}^{n} \beta^k \gamma^k. \end{aligned} \tag{2}$$

We often use the standard inner product, denoted $H \times H \mapsto H \cdot H$, since it is used to measure the magnitude of a vector and because it has the angle identity

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \, \|\mathbf{v}_2\| \cos\theta$$

where $\|\mathbf{v}_i\| = \sqrt{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}$; $\mathbf{v}_i \in H$, $i \in \{1, 2\}$ is the Euclidean norm, or magnitude, of the vector, θ is the angle between the vectors, and $H \cdot H$ denotes the standard inner product. From the angle identity of the standard inner product, we can see that the standard inner product is a measure of how much one vector is in the direction of another. If the vectors are unit length, the relation between the standard inner product of two vectors and the angle between the vectors is also simplified to

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \cos\theta. \tag{3}$$

D. *Cross Product*

The cross product is defined only in three dimensional vector space S^3 and is denoted as $(\cdot \times \cdot) : S^3 \times S^3 \to S^3$. The cross product maps two vectors in S to another vector in S that is orthogonal to both the first two vectors with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

Some common properties of the cross product are

1) *The self cross product of a vector is the zero vector*
   a) $\mathbf{v} \times \mathbf{v} = 0$
2) *Anticommutative*
   a) $\mathbf{v}_1 \times \mathbf{v}_2 = -(\mathbf{v}_2 \times \mathbf{v}_1)$
3) *Distributive*
   a) $\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_3$
4) *Scalar Multiplication*
   a) $(a\mathbf{v}_1) \times \mathbf{v}_2 = \mathbf{v}_1 \times (a\mathbf{v}_2) = a(\mathbf{v}_1 \times \mathbf{v}_2)$

If vectors are being expressed in a frame, their cross product is calculated as follows. Let $\mathcal{F}^a = (O_a, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$ be an arbitrary frame in S^3 whose basis is not necessarily orthogonal or unit length, and let

$$\begin{aligned} \mathbf{v}_1^a &= \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2 + \alpha^3 \mathbf{e}_3 \\ \mathbf{v}_2^a &= \beta^1 \mathbf{e}_1 + \beta^2 \mathbf{e}_2 + \beta^3 \mathbf{e}_3 \end{aligned}$$

be two vectors expressed in $\mathcal{F}^a$. the cross product of these two vectors is

$$\begin{aligned}\mathbf{v}_1 \times \mathbf{v}_2 &= \left(\alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2 + \alpha^3\mathbf{e}_3\right) \times \left(\beta^1\mathbf{e}_1 + \beta^2\mathbf{e}_2 + \beta^3\mathbf{e}_3\right)\\ &= \alpha^1\beta^1\left(\mathbf{e}_1 \times \mathbf{e}_1\right) + \alpha^1\beta^2\left(\mathbf{e}_1 \times \mathbf{e}_2\right) + \alpha^1\beta^3\left(\mathbf{e}_1 \times \mathbf{e}_3\right)\\ &+ \alpha^2\beta^1\left(\mathbf{e}_2 \times \mathbf{e}_1\right) + \alpha^2\beta^2\left(\mathbf{e}_2 \times \mathbf{e}_2\right) + \alpha^2\beta^3\left(\mathbf{e}_2 \times \mathbf{e}_3\right)\\ &+ \alpha^3\beta^1\left(\mathbf{e}_3 \times \mathbf{e}_1\right) + \alpha^3\beta^2\left(\mathbf{e}_3 \times \mathbf{e}_2\right) + \alpha^3\beta^3\left(\mathbf{e}_3 \times \mathbf{e}_3\right)\end{aligned}$$

which can be simplified to

$$\begin{aligned}\mathbf{v}_1 \times \mathbf{v}_2 &= \left(\alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2 + \alpha^3\mathbf{k}\right) \times \left(\beta^1\mathbf{e}_1 + \beta^2\mathbf{e}_2 + \beta^3\mathbf{e}_3\right)\\ &= \alpha^1\beta^1\left(0\right) + \alpha^1\beta^2\left(\mathbf{e}_1 \times \mathbf{e}_2\right) + \alpha^1\beta^3\left(\mathbf{e}_1 \times \mathbf{e}_3\right)\\ &+ \alpha^2\beta^1\left(\mathbf{e}_2 \times \mathbf{e}_1\right) + \alpha^2\beta^2\left(0\right) + \alpha^2\beta^3\left(\mathbf{e}_2 \times \mathbf{e}_3\right)\\ &+ \alpha^3\beta^1\left(\mathbf{e}_3 \times \mathbf{e}_1\right) + \alpha^3\beta^2\left(\mathbf{e}_3 \times \mathbf{e}_2\right) + \alpha^3\beta^3\left(0\right)\end{aligned}$$

This expression cannot be simplified further without knowing the cross product of the basis vectors of $\mathcal{F}^a$.

A standard reference frame, $\mathcal{F}^s = (O_s, \{\mathbf{i}, \mathbf{j}, \mathbf{k}\})$, in S^3 has the cross product identities

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \mathbf{k}\\ \mathbf{j} \times \mathbf{k} &= \mathbf{i}\\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}\end{aligned}$$

which implies by the anitcommutative property

$$\begin{aligned}\mathbf{j} \times \mathbf{i} &= -\mathbf{k}\\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i}\\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j}\end{aligned}$$

and

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0.$$

These identities allow us to simplify the cross product of two vectors expressed in $\mathcal{F}^s$. To demonstrate this, let

$$\begin{aligned}\mathbf{v}_1 &= \alpha^1\mathbf{i} + \alpha^2\mathbf{j} + \alpha^3\mathbf{k}\\ \mathbf{v}_2 &= \beta^1\mathbf{i} + \beta^2\mathbf{j} + \beta^3\mathbf{k}\end{aligned}$$

be two vectors expressed in $\mathcal{F}^s$ then their cross product is

$$\begin{aligned}\mathbf{v}_1^s \times \mathbf{v}_2^s &= \left(\alpha^1\mathbf{i} + \alpha^2\mathbf{j} + \alpha^3\mathbf{k}\right) \times \left(\beta^1\mathbf{i} + \beta^2\mathbf{j} + \beta^3\mathbf{k}\right),\\ &= \alpha^1\beta^1\left(\mathbf{i} \times \mathbf{i}\right) + \alpha^1\beta^2\left(\mathbf{i} \times \mathbf{j}\right) + \alpha^1\beta^3\left(\mathbf{i} \times \mathbf{k}\right)\\ &+ \alpha^2\beta^1\left(\mathbf{j} \times \mathbf{i}\right) + \alpha^2\beta^2\left(\mathbf{j} \times \mathbf{j}\right) + \alpha^2\beta^3\left(\mathbf{j} \times \mathbf{k}\right)\\ &+ \alpha^3\beta^1\left(\mathbf{k} \times \mathbf{i}\right) + \alpha^3\beta^2\left(\mathbf{k} \times \mathbf{j}\right) + \alpha^3\beta^3\left(\mathbf{k} \times \mathbf{k}\right).\end{aligned}$$

Evaluating the cross products yields

$$\begin{aligned}\mathbf{v}_1 \times \mathbf{v}_2 &= \alpha^1\beta^1\left(0\right) + \alpha^1\beta^2\left(\mathbf{k}\right) + \alpha^1\beta^3\left(-\mathbf{j}\right)\\ &+ \alpha^2\beta^1\left(-\mathbf{k}\right) + \alpha^2\beta^2\left(0\right) + \alpha^2\beta^3\left(\mathbf{i}\right)\\ &+ \alpha^3\beta^1\left(\mathbf{j}\right) + \alpha^3\beta^2\left(-\mathbf{i}\right) + \alpha^3\beta^3\left(0\right).\end{aligned}$$

and collecting components gives

$$\mathbf{v}_1 \times \mathbf{v}_2 = \left(\alpha^2\beta^3 - \alpha^3\beta^2\right)\mathbf{i} + \left(\alpha^3\beta^1 - \alpha^1\beta^3\right)\mathbf{j} + \left(\alpha^1\beta^2 - \alpha^2\beta^1\right)\mathbf{k}.$$

Like the standard inner product, the cross product can be related to an angle. The formula for this relation is given as

$$\mathbf{v}_1 \times \mathbf{v}_2 = \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\sin\left(\theta\right)\mathbf{n};\ \mathbf{v}_i \in S^3$$

where $\|\mathbf{v}_i\|$, $i \in \{1, 2\}$ is the magnitude of the vector, θ is the angle between the vectors, and $\mathbf{n}$ is a unit vector orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$ whose direction is obtained using the right-hand rule.

E. Euclidean Space

For the rest of this journey together, we will be working in Euclidean Space. Euclidean space is an affine vector space equipped with the inner product and Euclidean norm. An affine vector space is a vector space whose origin can be translated to any point in space, i.e., vectors can translate in this space. An example of this is shown in figure 3 where $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$ and $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\})$ are two frames in a two dimensional affine vector space where the origin of $\mathcal{F}^1$ is denoted as O_1 and the origin of $\mathcal{F}^2$ is denoted as O_2. In a regular vector space, all vectors must originate from the same point; however, in affine vector space the origin of vectors can be translated from one point to any point in space. In figure 3, the vector $\mathbf{v}$ shows the translations of the vectors $\tilde{\mathbf{e}}_\mathbf{1}$ and $\tilde{\mathbf{e}}_2$ from O_1 to O_2.

Euclidean space is also the space where geometry is done. Just like any vector space, any Euclidean space is isomorphic to $\mathbb{R}^n$. I will denote Euclidean spaces as E, and if it is necessary to specify the dimension of the space, I will denote it as E^n where $n \in \mathbb{R}$ denotes the dimension.

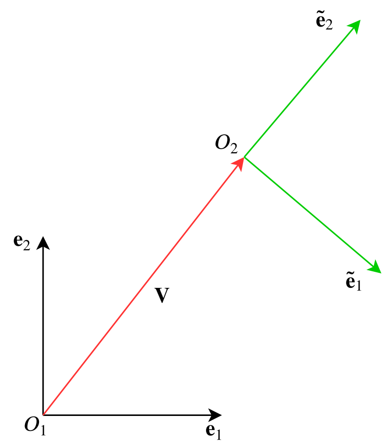


Figure 3

III. Transformations in E^2 with Arbitrary Reference Frames

The beginning of this journey is difficult but rewarding. In this section, we will look at the forward and backward transformation between arbitrary reference frames, and passive and active transformations of vectors expressed in arbitrary frames. What I mean by arbitrary, is that they do not have to be standard reference frame. In fact, in this section most of the frames will not be standard reference frames. I'm doing this because people get too comfortable working with standard frames that they forget the true essence of what is happening. And I am not your guide that will protect you from the dangers of this journey. I want you to get bruised and scraped; your mind to melt and fizz. And if a brain cell bursts, all the better!

A. Forward Transformation

It is often convenient to express a vector in one reference frame and then express it in a different reference frame. In order to do this we need to construct the **forward** and **backward** transformation between two reference frame. These transformations express the basis vectors in one frame w.r.t. the basis vectors in another frame. For example, consider the two frames arbitrary $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$ and $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\})$ with coincident origins in E^2 as depicted in figure 4. The forward transformation can either express the basis of $\mathcal{F}^1$ in $\mathcal{F}^2$, or it can express the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$. The backward transformation is just the inverse of the forward transformations, i.e., if we define the forward transformation as the transformation that expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$, then the backward transformation expresses the basis of $\mathcal{F}^1$ in $\mathcal{F}^2$. In this document, we will always define the forward transformation as the transformation that expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$.

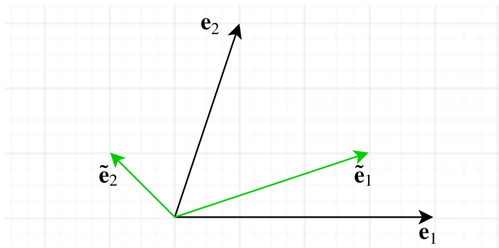


Figure 4

Let's construct the forward transformation between the frames $\mathcal{F}^1$ and $\mathcal{F}^2$ as depicted in figure 4. We do this by expressing each basis vector of $\mathcal{F}^2$ w.r.t. the basis vectors of $\mathcal{F}^1$. That means we need to find components for $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$ such that

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= \alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2,\\ \tilde{\mathbf{e}}_2 &= \beta^1\mathbf{e}_1 + \beta^2\mathbf{e}_2.\end{aligned}$$

The components can be found by inspection of figure 4 to be

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= \frac{2}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2,\\ \tilde{\mathbf{e}}_2 &- \frac{1}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2;\end{aligned}$$

however, I want to show you how the standard inner product can be used to find the components since this will be useful later on. Taking the inner product of $\tilde{\mathbf{e}}_1$ with each basis vector of $\mathcal{F}^1$ we get the system of equations

$$\begin{aligned}\tilde{\mathbf{e}}_1\cdot\mathbf{e}_1 &= \alpha^1\mathbf{e}_1\cdot\mathbf{e}_1 + \alpha^2\mathbf{e}_2\cdot\mathbf{e}_1\\ \tilde{\mathbf{e}}_1\cdot\mathbf{e}_2 &= \alpha^1\mathbf{e}_1\cdot\mathbf{e}_2 + \alpha^2\mathbf{e}_2\cdot\mathbf{e}_2,\end{aligned}$$

which can be represented in matrix notation

$$\begin{bmatrix}\tilde{\mathbf{e}}_1\cdot\mathbf{e}_1\\ \tilde{\mathbf{e}}_1\cdot\mathbf{e}_2\end{bmatrix} = \begin{bmatrix}\mathbf{e}_1\cdot\mathbf{e}_1 & \mathbf{e}_1\cdot\mathbf{e}_2\\ \mathbf{e}_2\cdot\mathbf{e}_1 & \mathbf{e}_2\cdot\mathbf{e}_2\end{bmatrix}\begin{bmatrix}\alpha^1\\ \alpha^2\end{bmatrix}, \tag{4}$$

where the components α^1 and α^2 are the unknowns and the inner products can be measured. To simplify this problem, I will introduce a standard reference frame $\mathcal{F}^3 = (O_3, \{\mathbf{i}, \mathbf{j}\})$ as depicted in figure 5.

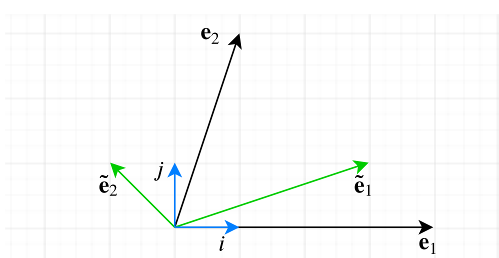


Figure 5

Using this new reference frame I can express each basis of $\mathcal{F}^1$ and $\mathcal{F}^2$ in $\mathcal{F}^3$ as such

$$\begin{aligned}\mathbf{e}_1 &= 4\mathbf{i} + 0\mathbf{j}\\ \mathbf{e}_2 &= 1\mathbf{i} + 3\mathbf{j}\\ \tilde{\mathbf{e}}_1 &= 3\mathbf{i} + 1\mathbf{j}\\ \tilde{\mathbf{e}}_2 &= -1\mathbf{i} + 1\mathbf{j},\end{aligned}$$

and easily take their inner products. The first inner product is given in detail as

$$\begin{aligned}\mathbf{e}_1\cdot\mathbf{e}_1 &= 4*4\,(\mathbf{i}\cdot\mathbf{i})+4*0\,(\mathbf{i}\cdot\mathbf{j})+0*4\,(\mathbf{j}\cdot\mathbf{i})+0*0\,(\mathbf{j}\cdot\mathbf{j})\\ &= 4*4\,(1)+4*0\,(0)+0*4\,(0)+0*0\,(1)\\ &= 16\end{aligned}$$

I will not write out the other inner products in detail, but all of them are given below.

$\mathbf{e}_1\cdot\mathbf{e}_1=16$	$\tilde{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1=10$	$\mathbf{e}_1\cdot\tilde{\mathbf{e}}_1=12$
$\mathbf{e}_2\cdot\mathbf{e}_1=4$	$\tilde{\mathbf{e}}_2\cdot\tilde{\mathbf{e}}_1=-2$	$\mathbf{e}_1\cdot\tilde{\mathbf{e}}_2=-4$
$\mathbf{e}_2\cdot\mathbf{e}_2=10$	$\tilde{\mathbf{e}}_2\cdot\tilde{\mathbf{e}}_2=2$	$\mathbf{e}_2\cdot\tilde{\mathbf{e}}_1=6$
		$\mathbf{e}_2\cdot\tilde{\mathbf{e}}_2=2$

Table I

Note that we omitted many of the inner products due to the symmetric property, i.e., $\mathbf{v}_1\cdot\mathbf{v}_2=\mathbf{v}_2\cdot\mathbf{v}_1$.

Substituting the inner products of table I into (4) yields

$$\begin{bmatrix}12\\6\end{bmatrix}=\begin{bmatrix}16 & 4\\4 & 10\end{bmatrix}\begin{bmatrix}\alpha^1\\\alpha^2\end{bmatrix},$$

and solving for the components gives $\alpha^1=\frac{2}{3}$ and $\alpha^2=\frac{1}{3}$.

The same process is done to express the basis $\tilde{\mathbf{e}}_2$ in $\mathcal{F}^1$. We begin with the desired component representation

$$\tilde{\mathbf{e}}_2=\beta^1\mathbf{e}_1+\beta^2\mathbf{e}_2.$$

Taking the inner product of $\tilde{\mathbf{e}}_2$ with each basis of $\mathcal{F}^1$ we get the system of equations

$$\begin{aligned}\tilde{\mathbf{e}}_2\cdot\mathbf{e}_1 &= \beta^1\mathbf{e}_1\cdot\mathbf{e}_1+\beta^2\mathbf{e}_2\cdot\mathbf{e}_1\\ \tilde{\mathbf{e}}_2\cdot\mathbf{e}_2 &= \beta^1\mathbf{e}_1\cdot\mathbf{e}_2+\beta^2\mathbf{e}_2\cdot\mathbf{e}_2,\end{aligned}$$

which can be represented in matrix notation

$$\begin{bmatrix}\tilde{\mathbf{e}}_2\cdot\mathbf{e}_1\\\tilde{\mathbf{e}}_2\cdot\mathbf{e}_2\end{bmatrix}=\begin{bmatrix}\mathbf{e}_1\cdot\mathbf{e}_1 & \mathbf{e}_1\cdot\mathbf{e}_2\\\mathbf{e}_2\cdot\mathbf{e}_1 & \mathbf{e}_2\cdot\mathbf{e}_2\end{bmatrix}\begin{bmatrix}\beta^1\\\beta^2\end{bmatrix}. \tag{5}$$

Substituting in the numerical values for the inner products yields

$$\begin{bmatrix}-4\\2\end{bmatrix}=\begin{bmatrix}16 & 4\\4 & 10\end{bmatrix}\begin{bmatrix}\beta^1\\\beta^2\end{bmatrix},$$

and solving for the components gives $\beta^1=-\frac{1}{3}$ and $\beta^2=\frac{1}{3}$ which we did by inspection above. Now that we have the components, we can construct the forward transformation

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= \frac{2}{3}\mathbf{e}_1+\frac{1}{3}\mathbf{e}_2,\\ \tilde{\mathbf{e}}_2 &= -\frac{1}{3}\mathbf{e}_1+\frac{1}{3}\mathbf{e}_2.\end{aligned}$$

We can represent represent the forward transformation using matrix notation as

$$\begin{aligned}F_2^1 &= \begin{bmatrix}\alpha^1 & \beta^1\\\alpha^2 & \beta^2\end{bmatrix}\\ &= \begin{bmatrix}\frac{2}{3} & -\frac{1}{3}\\\frac{1}{3} & \frac{1}{3}\end{bmatrix}.\end{aligned}$$

The subscript and superscript indicates that the forward transformation expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$.

B. Backward Transformation

The backward transformation is the inverse of the forward transformation. Using the same process, we can express the basis of $\mathcal{F}^1$ in $\mathcal{F}^2$ as

$$\mathbf{e}_1 = \gamma^1\tilde{\mathbf{e}}_1 + \gamma^2\tilde{\mathbf{e}}_2$$
$$\mathbf{e}_2 = \chi^1\tilde{\mathbf{e}}_1 + \chi^2\tilde{\mathbf{e}}_2$$

where γ^i and χ^i, $i \in \{1, 2\}$ are the components. Taking the inner product of $\mathbf{e}_1$ with the basis vectors of $\mathcal{F}^2$ generates the system of equations

$$\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1 = \gamma^1\tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1 + \gamma^2\tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1$$
$$\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2 = \gamma^1\tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2 + \gamma^2\tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2$$

which can be represented in matrix notation

$$\begin{bmatrix}\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1 \\ \mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2\end{bmatrix} = \begin{bmatrix}\tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1 \\ \tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2 & \tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2\end{bmatrix}\begin{bmatrix}\gamma^1 \\ \gamma^2\end{bmatrix}.$$

Substituting in the numerical values for each inner product from table I yields

$$\begin{bmatrix}12 \\ -4\end{bmatrix} = \begin{bmatrix}10 & -2 \\ -2 & 2\end{bmatrix}\begin{bmatrix}\gamma^1 \\ \gamma^2\end{bmatrix}$$

,

and solving for the components gives $\gamma^1 = 1$ and $\gamma^2 = -1$. Using the same process to solve for χ^1 and χ^2 yields $\chi^1 = -1$ and $\chi^2 = 2$.

Now that we have the components of the backward transformation, we can write it as

$$\mathbf{e}_1 = 1\tilde{\mathbf{e}}_1 - 1\tilde{\mathbf{e}}_2$$
$$\mathbf{e}_2 = 1\tilde{\mathbf{e}}_1 + 2\tilde{\mathbf{e}}_2$$

which can be represented in matrix notation

$$\begin{aligned} B_1^2 &= \begin{bmatrix}\gamma^1 & \chi^1 \\ \gamma^2 & \chi^2\end{bmatrix} \\ &= \begin{bmatrix}1 & 1 \\ -1 & 2\end{bmatrix}. \end{aligned}$$

We can verify the forward and backward transformations using the fact that they are inverses of each other. This indicates that

$$F_2^1 B_1^2 = I.$$

What this means is that if we use the forward transformation to transform the basis vectors of $\mathcal{F}^1$ into the basis vectors of $\mathcal{F}^2$, and then use the backward transformation to transform the basis vectors of $\mathcal{F}^2$ to the basis vectors of $\mathcal{F}^1$, we will end up with the same basis vectors. Let's now verify that the forward and backward transformations that we calculated are indeed inverses

$$\begin{aligned} F_2^1 B_1^2 &= I \\ \begin{bmatrix}\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1 & 1 \\ -1 & 2\end{bmatrix} &= I \\ I &= I. \end{aligned}$$

C. Passive Transformation of a Vector

A passive transformation of a vector changes which frame you are representing the vector in. This procedure is also commonly known as a change of basis.

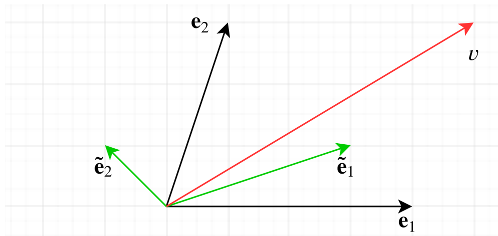


Figure 6

Consider two arbitrary frames $\mathcal{F}^1$ and $\mathcal{F}^2$ with coincident origins and a vector $\mathbf{v}$ in E^2 as depicted in figure 6. We have already calculated the forward and backward transformations in subsections III-A and III-B, so I will not repeat those steps here, but they will be given again.

$$F_2^1 := \begin{cases} \tilde{\mathbf{e}}_1 & = \frac{2}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 & = -\frac{1}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 \end{cases}, \text{ or } \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{6a}$$

$$B_1^2 := \begin{cases} \mathbf{e}_1 & = 1\tilde{\mathbf{e}}_1 - 1\tilde{\mathbf{e}}_2 \\ \mathbf{e}_2 & = 1\tilde{\mathbf{e}}_1 + 2\tilde{\mathbf{e}}_2 \end{cases}, \text{ or } \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \tag{7a}$$

The vector $\mathbf{v}$ can be represented in both coordinate frames as

$$\begin{aligned} \mathbf{v}^1 &= 1\mathbf{e}_1 + 1\mathbf{e}_2 \\ \mathbf{v}^2 &= 2\tilde{\mathbf{e}}_1 + 1\tilde{\mathbf{e}}_2, \end{aligned}$$

where the superscript denotes which frame it is being expressed in. You can use the same process outlined in the previous subsections III-A and III-B to compute the components using the inner product, but we will skip that process and grab the components using visual inspection.

To transform the components of $\mathbf{v}^1$ to the components of $\mathbf{v}^2$, we start with $\mathbf{v}^1$ and use the backward transform defined in (7) to transform $\mathbf{v}^1$to $\mathbf{v}^2$ as follows

$$\begin{aligned} \mathbf{v} &= 1\mathbf{e}_1 + 1\mathbf{e}_2 \\ &= 1\left(1\tilde{\mathbf{e}}_1 - 1\tilde{\mathbf{e}}_2\right) + 1\left(1\tilde{\mathbf{e}}_1 + 2\tilde{\mathbf{e}}_2\right) \\ &= (1+1)\,\tilde{\mathbf{e}}_1 + (-1+2)\,\tilde{\mathbf{e}}_2 \\ &= 2\tilde{\mathbf{e}}_1 + 1\tilde{\mathbf{e}}_2 \\ &= \mathbf{v}^2 \end{aligned}$$

which can be done using matrix notation

$$\begin{aligned} \mathbf{v}^2 &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= B_1^2 \mathbf{v}^1. \end{aligned}$$

Note that in order to transform the components of $\mathbf{v}^1$ to $\mathbf{v}^2$ we used the backward transformation. This implies that we can reverse the transformation using the forward transformation. To show this we begin the vector $\mathbf{v}^2$ and transform its components to $\mathbf{v}^1$ using the forward transformation defined in (6).

$$\begin{aligned}
\mathbf{v} &= 2\tilde{\mathbf{e}}_1 + 1\tilde{\mathbf{e}}_2 \\
&= 2\left(\frac{2}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2\right) + 1\left(-\frac{1}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2\right) \\
&= \left(2 * \frac{2}{3} - \frac{1}{3}\right)\mathbf{e}_1 + \left(2 * \frac{1}{3} + \frac{1}{3}\right)\mathbf{e}_2 \\
&= 1\mathbf{e}_1 + 1\mathbf{e}_2 \\
&= \mathbf{v}^1
\end{aligned}$$

which can be done using matrix notation

$$\begin{aligned}
\mathbf{v}^1 &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= F_2^1 \mathbf{v}^2.
\end{aligned}$$

The notation used to express this transform can be described as the transformation F_2^1 that transforms the vector $\mathbf{v}$ from $\mathcal{F}^2$ to $\mathcal{F}^1$. This notation is easy to remember because same superscripts and subscripts are crossed out, and the remaining superscript indicates which frame it is. For example

$$\begin{aligned}
\mathbf{v}^1 &= F_2^1 \mathbf{v}^2 \\
&= F_{\cancel{2}}^1 \mathbf{v}^{\cancel{2}} \\
&= \mathbf{v}^1.
\end{aligned}$$

Remark 9. We used the forward transformation to express the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$ and we used the backward transformation to transform the components of $\mathbf{v}^1$ to the components of $\mathbf{v}^2$. This shows that the components of a vector transform contravariently, or inversely, to the a change in basis; hence vectors are contravariant.

To build intuition of why vectors are contravariant, consider the simple example of measuring a person's height as depicted in figure 7. In this figure we have two frames $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1\})$ and $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1\})$ with coincident origin. Note that in the figure the frame's origins seems to be separated, but I did this for visual neatness.

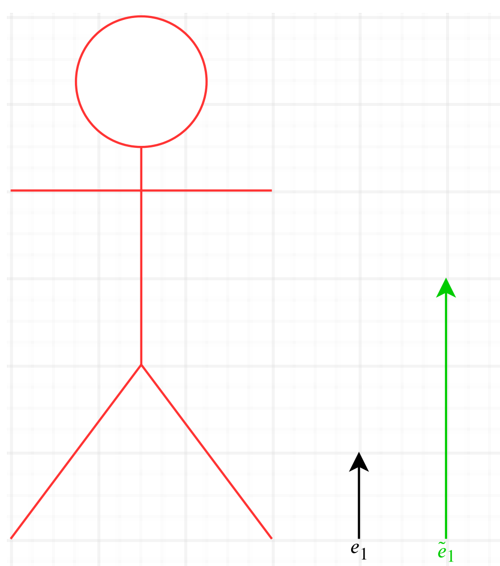


Figure 7

We can easily create the forward and backward transformations between the two frames as

$$\tilde{\mathbf{e}}_1 = 3\mathbf{e}_1$$
$$\mathbf{e}_1 = \frac{1}{3}\tilde{\mathbf{e}}_1.$$

We can easily represent the person's height relative to the two frames

$$\mathbf{h}^1 = 6\mathbf{e}_1$$
$$\mathbf{h}^2 = 2\tilde{\mathbf{e}}_1$$

where $\mathbf{h}$ is the vector representing the person's height. Since the basis vector $\tilde{\mathbf{e}}_1$ is geometrically 3 times larger than the basis vector $\mathbf{e}_1$, as I transform the component of $\mathbf{h}^1$ to $\mathbf{h}^2$, the component needs to get three times smaller. To illustrate this better, suppose $\mathbf{e}_1$ represents 1 foot and $\tilde{\mathbf{e}}_1$ represents 1 yard, then the person is 6 feet or 2 yards tall. As I convert between feet to yards, the component gets smaller, but my unit of measure gets bigger. Hopefully this explains why vector components are contravariant. If you are still confused, just sit down on a stump next to the path and reflect on what you have read. If you are still perplexed, I don't know what to say but good luck.

D. Active Transformation of a Vector

An **active transformation** is a transformation that maps the vector $\mathbf{v}$ to a new orientation $\mathbf{v}'$ expressed in the same frame $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$ as depicted in figure 8. In other words, an active transformation maps the components of $\mathbf{v}$ to the components of $\mathbf{v}'$. There are many ways you can construct an active transformation. The method described here will help you build intuition for active transformations in E^3.

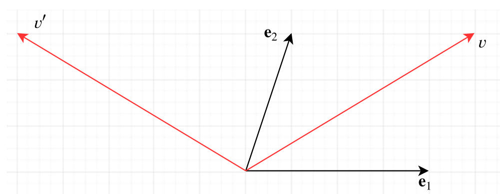


Figure 8

To help us derive the active transformation, let's construct another frame $\mathcal{F}^2 = \left(O_2, \left\{\mathbf{v}, \mathbf{v}^\perp\right\}\right)$ that has an orthogonal basis where each basis vector is the same length, and an origin that is coincident with the origin of $\mathcal{F}^1$. These constraints leave two possible choices for the configurations of the basis of $\mathcal{F}^2$. One possible choice of this new construction is shown in figure 9.

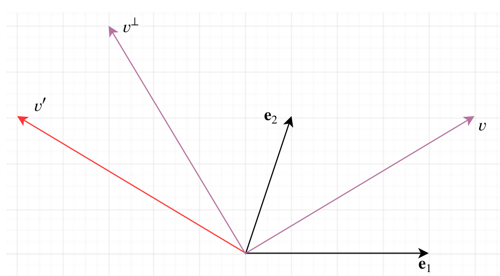


Figure 9

We can construct the forward transformation that expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$

$$\mathbf{v} = 1\mathbf{e}_1 + 1\mathbf{e}_2 \tag{8a}$$
$$\mathbf{v}^\perp = -\frac{7}{6}\mathbf{e}_1 + \frac{5}{3}\mathbf{e}_2 \tag{8b}$$

which can be represented in matrix notation as

$$F_2^1 = \begin{bmatrix} 1 & -\frac{7}{6} \\ 1 & \frac{5}{3} \end{bmatrix}.$$

Remark 10. We have defined the basis vector $\mathbf{v}^{\perp}$ such that it meets the criteria $\mathbf{v} \cdot \mathbf{v}^{\perp} = 0$ and $\left\|\mathbf{v}^{\perp}\right\| = \|\mathbf{v}\|$. These two constrains allows us to write the components of $\mathbf{v}^{\perp}$ as a function of the components of $\mathbf{v}$. I won't get into this derivation, but know that $\mathbf{v}^{\perp}$ can be expressed as a function of $\mathbf{v}$.

Now let's represent the vector $\mathbf{v}'$ in $\mathcal{F}^2$ as

$$\mathbf{v}' = \alpha^1 \mathbf{v} + \alpha^2 \mathbf{v}^{\perp}$$

The representation of $\mathbf{v}'$ expressed in $\mathcal{F}^2$ is

$$\begin{aligned}\mathbf{v}' &= \left(\mathbf{v}' \cdot \mathbf{v}\right) \mathbf{v} + \left(\mathbf{v}' \cdot \mathbf{v}^{\perp}\right) \mathbf{v}^{\perp} \\ &= -\frac{8}{17}\mathbf{v} + \frac{15}{17}\mathbf{v}^{\perp}\end{aligned}$$

Substituting in the forward transformation defined in (III-A) yields

$$\begin{aligned}\mathbf{v}' &= -\frac{8}{17}\mathbf{v} + \frac{15}{17}\mathbf{v}^{\perp} \\ &= -\frac{8}{17}\left(1\mathbf{e}_1 + 1\mathbf{e}_2\right) + \frac{15}{17}\left(-\frac{7}{6}\mathbf{e}_1 + \frac{5}{3}\mathbf{e}_2\right) \\ &= \left(-\frac{8}{17} - \frac{35}{34}\right)\mathbf{e}_1 + \left(-\frac{8}{17} + \frac{75}{51}\right)\mathbf{e}_2 \\ &= -1.5\mathbf{e}_1 + 1\mathbf{e}_2\end{aligned}$$

Honestly, the beauty of expressing the active transformation using this method cannot be fully appreciated until we move to standard frames which we will do next. The main take away from this subsection, is that an active transformation is a transformation between vectors in the same frame.

IV. Transformations in E^2 with Standard Reference Frames

Well done brave adventurer! You have made it through the worst of it and will now be rewarded for a short season. At this point in the journey we will transition from talking about transformations in E^2 with arbitrary reference frames to standard reference frames. Standard reference frames can be used to simplify the derivation of the forward and backward transformations and passive and active transformations. To show this we will derive each transformation again. But that is not all! In this section we will also derive the directional derivative of the forward and backward transformations and conclude with the introduction of the Lie group $SO(2)$.

A. *Simplifications when using Standard Reference Frames*

Up to this point we have worked with frames that have a nonorthonormal basis which can be difficult to work with. Using an orthonormal basis greatly simplifies finding the components of a vector relative to each basis vector. For example if I had a vector $\mathbf{v}$ in E^2 , and I wanted to express it in $\mathcal{F}^1 = (O, \{\mathbf{e}_1, \mathbf{e}_2\})$ such that

$$\mathbf{v} = \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2, \tag{9}$$

We can find the components by taking the inner product of $\mathbf{v}$ with each basis vector of $\mathcal{F}^1$ to form the system of equations

$$\begin{aligned}\left(\mathbf{v} \cdot \mathbf{e}_1\right) &= \alpha^1 \left(\mathbf{e}_1 \cdot \mathbf{e}_1\right) + \alpha^2 \left(\mathbf{e}_2 \cdot \mathbf{e}_1\right) \\ \left(\mathbf{v} \cdot \mathbf{e}_2\right) &= \alpha^1 \left(\mathbf{e}_1 \cdot \mathbf{e}_2\right) + \alpha^2 \left(\mathbf{e}_2 \cdot \mathbf{e}_2\right),\end{aligned}$$

which can be further reduced noting that $\left(\mathbf{e}_2 \cdot \mathbf{e}_1\right) = 0$ since the basis vectors are orthogonal. Using this reduction we get

$$\begin{aligned}\left(\mathbf{v} \cdot \mathbf{e}_1\right) &= \alpha^1 \left(\mathbf{e}_1 \cdot \mathbf{e}_1\right) \\ \left(\mathbf{v} \cdot \mathbf{e}_2\right) &= \alpha^2 \left(\mathbf{e}_2 \cdot \mathbf{e}_2\right)\end{aligned}$$

which can be simplified even more noting that $\left(\mathbf{e}_1 \cdot \mathbf{e}_1\right) = \left(\mathbf{e}_2 \cdot \mathbf{e}_2\right) = 1$ since the basis vectors are unit length. Using this simplification we get

$$\begin{aligned}\left(\mathbf{v} \cdot \mathbf{e}_1\right) &= \alpha^1 \\ \left(\mathbf{v} \cdot \mathbf{e}_2\right) &= \alpha^2\end{aligned}$$

which allows us to rewrite (9) as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\, \mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\, \mathbf{e}_2$$

which can also be expressed in terms of angles as

$$\mathbf{v} = \|\mathbf{v}\| \cos(\theta_1)\, \mathbf{e}_1 + \|\mathbf{v}\| \cos(\theta_2)\, \mathbf{e}_2.$$

Even though there are two angles θ_1 and θ_2 in the above equation, the angles are related to each other. This relation can be discovered by taking the inner product of both sides of the equation with $\mathbf{v}$ to get

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{v} &= \mathbf{v} \cdot (\|\mathbf{v}\| \cos(\theta_1)\, \mathbf{e}_1 + \|\mathbf{v}\| \cos(\theta_2)\, \mathbf{e}_2) \\
&= \|\mathbf{v}\| \left(\cos(\theta_1)\, (\mathbf{v} \cdot \mathbf{e}_1) + \cos(\theta_2)\, (\mathbf{v} \cdot \mathbf{e}_2)\right) \\
&= \|\mathbf{v}\| \left(\cos(\theta_1)\, \|\mathbf{v}\| \cos(\theta_1) + \cos(\theta_2)\, \|\mathbf{v}\| \cos(\theta_2)\right) \\
&= \|\mathbf{v}\|^2 \left(\cos^2(\theta_1) + \cos^2(\theta_2)\right)
\end{aligned}$$

which forces the constraint

$$\cos^2(\theta_1) + \cos^2(\theta_2) = 1$$

which implies

$$\cos(\theta_2) = \pm \sin(\theta_1)\,,$$

where the sign can be determined by the sign of the inner product.

In summary, a vector $\mathbf{v}$ can be expressed in a standard frame as

$$\mathbf{v} = \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2$$

where

$$\begin{aligned}
\alpha^1 &= (\mathbf{v} \cdot \mathbf{e}_1) = \|\mathbf{v}\| \cos(\theta_1)\,, \\
\alpha^2 &= (\mathbf{v} \cdot \mathbf{e}_2) = \mathrm{sign}\,(\mathbf{v} \cdot \mathbf{e}_2) \sin(\theta_1)\,.
\end{aligned}$$

If $\mathbf{v}$ is unit length, the equations simplify to

$$\begin{aligned}
\alpha^1 &= (\mathbf{v} \cdot \mathbf{e}_1) = \cos(\theta_1) \\
\alpha^2 &= (\mathbf{v} \cdot \mathbf{e}_2) = \mathrm{sign}\,(\mathbf{v} \cdot \mathbf{e}_2) \sin(\theta_1)
\end{aligned}$$

If the vector $\mathbf{v}$ is in a higher dimensional space of dimension n being expressed in a standard frame we still have the identities

$$\alpha^i = (\mathbf{v} \cdot \mathbf{e}_i) = \|\mathbf{v}\| \cos(\theta_i)\,,\ i = \{1, \ldots, n\}$$

and the constraint

$$\sum \left(\alpha^i\right)^2 = \|\mathbf{v}\|\,,\ i = \{1, \ldots, n\}$$

B. Forward Transformation

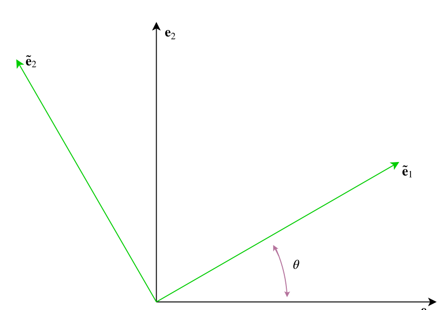


Figure 10

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$ and $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\})$ be standard frames with coincident origins in E^2 with θ denoting the angle between $\mathbf{e}_1$ and $\tilde{\mathbf{e}}_1$ as depicted in figure 10. The forward transformation can be easily derived from the inner product using the simplifications discussed in subsection IV-A. Using these simplifications, the forward transformation is

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 &= (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2\end{aligned}$$

or in matrix notation

$$F_2^1 = \begin{bmatrix} (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1) \\ (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) \end{bmatrix}.$$

We can calculate the inner products between basis vectors as we have previously done. You should be a master at by now, so we won't review it again. Let's now express the inner products in terms of angles using what we learned from subsection II-C. The forward transformation expressed in terms of angles is

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= \|\tilde{\mathbf{e}}_1\|\,\|\mathbf{e}_1\| \cos(\phi_{1,1})\,\mathbf{e}_1 + \|\tilde{\mathbf{e}}_1\|\,\|\mathbf{e}_2\| \cos(\phi_{1,2})\,\mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 &= \|\tilde{\mathbf{e}}_2\|\,\|\mathbf{e}_1\| \cos(\phi_{2,1})\,\mathbf{e}_1 + \|\tilde{\mathbf{e}}_2\|\,\|\mathbf{e}_2\| \cos(\phi_{2,2})\,\mathbf{e}_2\end{aligned}$$

where $\phi_{a,b},\ a, b \in \{1, 2\}$ denotes the angle between vectors $\tilde{\mathbf{e}}_a$ and $\mathbf{e}_b$. Since the frames have an orthonormal basis, we can simplify the transformation to

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= \cos(\theta)\,\mathbf{e}_1 + \cos\left(\frac{\pi}{2} \mp \theta\right)\mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 &= \cos\left(\frac{\pi}{2} \pm \theta\right)\mathbf{e}_1 + \cos(\theta)\,\mathbf{e}_2\end{aligned}$$

where θ is the angle between $\tilde{\mathbf{e}}_1$ and $\mathbf{e}_1$ and the sign of each function $\cos\left(\frac{\pi}{2} \pm \theta\right)$ is determined by the sign of the corresponding inner product as discussed in subsection IV-A. We can easily read off the sign of the inner products using figure 10 to get

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= \cos(\theta)\,\mathbf{e}_1 + \cos\left(\frac{\pi}{2} - \theta\right)\mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 &= \cos\left(\frac{\pi}{2} + \theta\right)\mathbf{e}_1 + \cos(\theta)\,\mathbf{e}_2.\end{aligned}$$

Using the identities $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$ and $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$, the final forward transformation parametrized by θ is

$$\tilde{\mathbf{e}}_1 = \cos(\theta)\,\mathbf{e}_1 + \sin(\theta)\,\mathbf{e}_2 \tag{10a}$$

$$\tilde{\mathbf{e}}_2 = -\sin(\theta)\,\mathbf{e}_1 + \cos(\theta)\,\mathbf{e}_2 \tag{10b}$$

or in matrix notation we have

$$F_2^1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

where the superscript and subscript indicates that the forward transformation expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$.

Remark 11. The angle θ has not yet been used as a generalized coordinate. This is because θ only describes the angle between the basis vectors $\mathbf{e}_1$ and $\tilde{\mathbf{e}}_1$. For example the forward transformation for the frames shown in figure 11 is

$$F_2^1 = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

which cannot be determined by θ alone and requires the sign of the inner products.

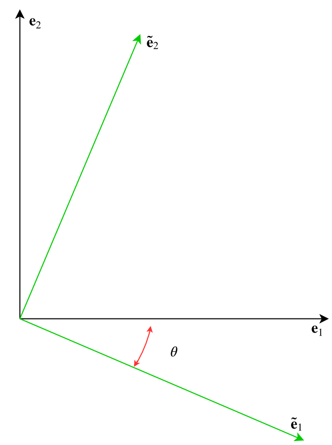


Figure 11

Now, if we say that θ is the angle measured from $\mathbf{e}_1$ to $\tilde{\mathbf{e}}_1$ such that a positive θ denotes a counter-clockwise rotation from $\mathbf{e}_1$ to $\tilde{\mathbf{e}}_1$, then θ is a generalized coordinate because I have given a reference for the coordinate θ such that I can parametrize the forward transformation using only θ. When I use angles as generalized coordinates, I will specify it explicitly.

C. Backward Transformation

In a similar manner we can compute the backward transformation of the frames shown in figure 10. The backward transformation expresses the basis of $\mathcal{F}^1$ in $\mathcal{F}^2$. To construct the backward transformations we follow the same procedure as done in the previous subsection; however, we will skip a few steps here. We begin by relating each basis vector in $\mathcal{F}^1$ to the basis of $\mathcal{F}^2$ using the inner product. This relation is

$$\mathbf{e}_1 = (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2$$
$$\mathbf{e}_2 = (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2$$

or in matrix notation

$$B_1^2 = \begin{bmatrix} (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1) & (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1) \\ (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2) & (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2) \end{bmatrix}$$

The backward transformation can also be expressed in terms of angles

$$\mathbf{e}_1 = \cos(\theta)\,\tilde{\mathbf{e}}_1 + \cos\left(\frac{\pi}{2} \pm \theta\right)\tilde{\mathbf{e}}_2$$
$$\mathbf{e}_2 = \cos\left(\frac{\pi}{2} \mp \theta\right)\tilde{\mathbf{e}}_1 + \cos(\theta)\,\tilde{\mathbf{e}}_2$$

where the signs for $\cos\left(\frac{\pi}{2} \mp \theta\right)$ and $\cos\left(\frac{\pi}{2} \pm \theta\right)$ are determined from the inner product to be

$$\mathbf{e}_1 = \cos(\theta)\,\tilde{\mathbf{e}}_1 + -\sin(\theta)\,\tilde{\mathbf{e}}_2$$
$$\mathbf{e}_2 = \sin(\theta)\,\tilde{\mathbf{e}}_1 + \cos(\theta)\,\tilde{\mathbf{e}}_2$$

which gives us the backward transformation in terms of the angle θ. The transformation can be represented using matrix notation as

$$B_1^2 = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

To verify the forward and backward transformations we must check that they are inverses of each other. It is easily verified that

$$\begin{aligned}
B^2_1 F^1_2 &= {F^1_2}^{-1} F^1_2 \\
&= {F^1_2}^{\top} F^1_2 \\
&= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & -\cos(\theta)\sin(\theta) + \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix} \\
&= I
\end{aligned}$$

The fact that ${F^1_2}^{-1} = {F^1_2}^{\top}$ means that the transformations are unitary.

Remark 12. Now that I have given you several examples of how to determine the signs of the functions $\cos\left(\frac{\pi}{2} \mp \theta\right)$ and $\cos\left(\frac{\pi}{2} \pm \theta\right)$ using the inner product, I will skip this step in the future.

D. *Passive Transformation of a Vector*

Once again, the passive transformation of a vector is the transformation of components of a vector w.r.t. one frame to be w.r.t. another frame. We will follow the same procedure in subsection III-C, and introduce the simplifications when using standard reference frames in E^2.

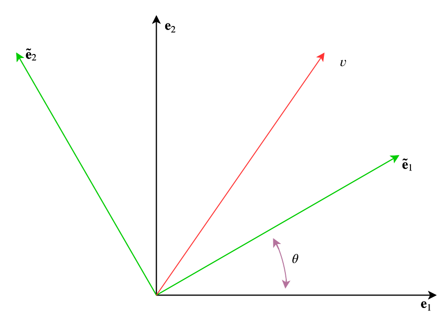


Figure 12

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$, and $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\})$ be standard frames with coincident origins in E^2 with θ denoting the angle between $\mathbf{e}_1$ and $\tilde{\mathbf{e}}_2$ and $\mathbf{v}$ being an arbitrary vector as shown in figure 12, and suppose we express $\mathbf{v}$ in $\mathcal{F}^1$ as

$$\mathbf{v} = \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2, \tag{11}$$

but we now wish to express it in $\mathcal{F}^2$. We can do this using the backward transformation calculated in subsection IV-C, which was

$$\begin{aligned}
\mathbf{e}_1 &= \cos(\theta)\, \tilde{\mathbf{e}}_1 + -\sin(\theta)\, \tilde{\mathbf{e}}_2 \\
\mathbf{e}_2 &= \sin(\theta)\, \tilde{\mathbf{e}}_1 + \cos(\theta)\, \tilde{\mathbf{e}}_2
\end{aligned}$$

or in matrix notation

$$B^2_1 = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Substituting the backward transformation into (11) yields

$$\begin{aligned}\mathbf{v} &= \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2 \\ &= \alpha^1 \left(\cos(\theta)\,\tilde{\mathbf{e}}_1 + -\sin(\theta)\,\tilde{\mathbf{e}}_2\right) + \alpha^2 \left(\sin(\theta)\,\tilde{\mathbf{e}}_1 + \cos(\theta)\,\tilde{\mathbf{e}}_2\right) \\ &= \left(\alpha^1 \cos(\theta) + \alpha^2 \sin(\theta)\right)\tilde{\mathbf{e}}_1 + \left(-\alpha^1 \sin(\theta) + \alpha^2 \cos(\theta)\right)\tilde{\mathbf{e}}_2\end{aligned}$$

which can be represented using matrix notation

$$\begin{aligned}\mathbf{v}^2 &= \begin{bmatrix}\cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}\alpha^1 \\ \alpha^2\end{bmatrix} \\ &= B_1^2 \begin{bmatrix}\alpha^1 \\ \alpha^2\end{bmatrix} \\ &= B\mathbf{v}^1.\end{aligned}$$

This should be so surprise after section III-C. To inverse the process, you would simply use the inverse transformation which is the forward transformation.

I purposely didn't calculate the components α^1 and α^2 so that you could easily identify the components in the derivation. Just for fun and for more exposure let's calculate them in terms of an angle ϕ which denotes the angle between the vectors $\mathbf{v}$ and $\mathbf{e}_1$.

$$\begin{aligned}\mathbf{v} &= \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2 \\ &= (\mathbf{v}\cdot\mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{v}\cdot\mathbf{e}_2)\,\mathbf{e}_2 \\ &= (\|\mathbf{v}\|\cos(\phi))\,\mathbf{e}_1 + (\|\mathbf{v}\|\sin(\phi))\,\mathbf{e}_2\end{aligned}$$

E. Active Transformation of a Vector

Once again, an active transformation is a transformation that maps the vector $\mathbf{v}$ to a new orientation $\mathbf{v}'$ expressed in the same frame. We will use the same procedure as outlined in subsection III-D and introduce the simplifications bestowed from using standard reference frames.

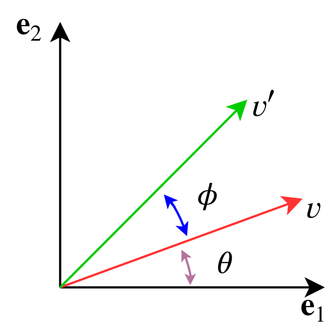


Figure 13

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$ be a standard frame in E^2, $\mathbf{v}$ a vector in E^2, $\mathbf{v}'$ the same vector $\mathbf{v}$ but transformed by the angle ϕ, and let θ denote the angle between the vectors $\mathbf{v}$ and $\mathbf{e}_1$. Suppose we already know the components of $\mathbf{v}$ relative to the frame $\mathcal{F}^1$ such that

$$\begin{aligned}\mathbf{v} &= \alpha^1 \mathbf{e}_1 + \alpha^2 \mathbf{e}_2 \\ &= (\mathbf{v}\cdot\mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{v}\cdot\mathbf{e}_2)\,\mathbf{e}_2 \\ &= \|\mathbf{v}\|\cos(\theta)\,\mathbf{e}_1 + \|\mathbf{v}\|\sin(\theta)\,\mathbf{e}_2.\end{aligned}$$

Our objective is to find the transformation that transforms the components of $\mathbf{v}$ to the components of $\mathbf{v}'$. To do this, we will introduce another frame $\mathcal{F}^2 = \left(O_2, \left\{\mathbf{v}, \mathbf{v}^\perp\right\}\right)$ whose origin is coincident with O_1, and whose basis vectors satisfy the constraints $\mathbf{v}\cdot\mathbf{v}^\perp = 0$ and $\|\mathbf{v}\| = \left\|\mathbf{v}^\perp\right\|$. This new scenario is depicted in figure 14.

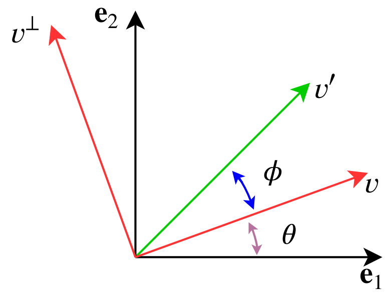


Figure 14

We can express the new basis vector $\mathbf{v}^{\perp}$ in $\mathcal{F}^1$ as

$$\begin{aligned}\mathbf{v}^{\perp} &= \beta^1\mathbf{e}_1 + \beta^2\mathbf{e}_2\\ &= \left(\mathbf{v}^{\perp}\cdot\mathbf{e}_1\right)\mathbf{e}_1 + \left(\mathbf{v}^{\perp}\cdot\mathbf{e}_2\right)\mathbf{e}_2\\ &= -\left\|\mathbf{v}\right\|\sin\left(\theta\right)\mathbf{e}_1 + \left\|\mathbf{v}\right\|\cos\left(\theta\right)\mathbf{e}_2.\end{aligned}$$

Note that the components of $\mathbf{v}^{\perp}$ can be written as a function of the components of $\mathbf{v}$, i.e.,

$$\begin{aligned}\mathbf{v}^{\perp} &= -\left(\mathbf{v}\cdot\mathbf{e}_2\right)\mathbf{e}_1 + \left(\mathbf{v}\cdot\mathbf{e}_1\right)\mathbf{e}_2\\ &= -\alpha^2\mathbf{e}_1 + \alpha^1\mathbf{e}_2.\end{aligned}$$

We can now form the forward transformation

$$\begin{aligned}\mathbf{v} &= \alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2\\ \mathbf{v}^{\perp} &= -\alpha^2\mathbf{e}_1 + \alpha^1\mathbf{e}_2.\end{aligned}$$

In the next step we leave the expression of the forward transformation in terms of α^1 and α^2 for clarity. We now express the vector $\mathbf{v}'$ in $\mathcal{F}^2$ in terms of the angle ϕ as

$$\mathbf{v}' = \left(\mathbf{v}'\cdot\frac{\mathbf{v}}{\left\|\mathbf{v}\right\|}\right)\frac{\mathbf{v}}{\left\|\mathbf{v}\right\|} + \left(\mathbf{v}'\cdot\frac{\mathbf{v}^{\perp}}{\left\|\mathbf{v}^{\perp}\right\|}\right)\frac{\mathbf{v}^{\perp}}{\left\|\mathbf{v}^{\perp}\right\|},$$

where the basis vectors of $\mathcal{F}^2$ are normalized in order to treat it like a standard frame. The above expression is simplified to

$$\begin{aligned}\mathbf{v}' &= \left(\left\|\mathbf{v}'\right\|\cos\left(\phi\right)\right)\frac{\mathbf{v}}{\left\|\mathbf{v}\right\|} + \left(\left\|\mathbf{v}'\right\|\sin\left(\phi\right)\right)\frac{\mathbf{v}^{\perp}}{\left\|\mathbf{v}^{\perp}\right\|}\\ &= \cos\left(\phi\right)\mathbf{v} + \sin\left(\phi\right)\mathbf{v}^{\perp}.\end{aligned}$$

Substituting in the forward transformation to express $\mathbf{v}'$ in $\mathcal{F}^1$ yields

$$\begin{aligned}\mathbf{v}' &= \cos\left(\phi\right)\mathbf{v} + \sin\left(\phi\right)\mathbf{v}^{\perp}\\ &= \cos\left(\phi\right)\left(\alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2\right) + \sin\left(\phi\right)\left(-\alpha^2\mathbf{e}_1 + \alpha^1\mathbf{e}_2\right)\\ &= \left(\cos\left(\phi\right)\alpha^1 - \sin\left(\phi\right)\alpha^2\right)\mathbf{e}_1 + \left(\sin\left(\phi\right)\alpha^1 + \cos\left(\phi\right)\alpha^2\right)\mathbf{e}_2\end{aligned}$$

or in matrix notation is

$$\underbrace{\begin{bmatrix}\cos\left(\phi\right) & -\sin\left(\phi\right)\\ \sin\left(\phi\right) & \cos\left(\phi\right)\end{bmatrix}}_{A}\begin{bmatrix}\alpha^1\\ \alpha^2\end{bmatrix}$$

where A is the active transformation that transforms the components of $\mathbf{v}$ to the components of $\mathbf{v}'$.

Remark 13. You may notice that the transformation A looks like a forward transformation, and you might be tempted to think of it as a forward transformation, but it isn't. They are fundamentally different and express two different things. A forward transformation is between frames, and an active transformation is between vectors expressed in the same frame.

My dear fellow adventurer, I find it to be an appropriate time to discuss active transformations in the presence of coordinates. I only do this because you will come across it in other literature, and I want you to prepared to face it with full confidence and glory! Take this opportunity to notice the subtle differences.

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$ be a standard frame in E^2, $\mathbf{v}$ a vector in E^2, $\mathbf{v}'$ the same vector $\mathbf{v}$ but rotated by the angle $t\omega$, θ be a generalized coordinate that denotes the angle from $\mathbf{e}_1$ to $\mathbf{v}$ where a positive θ denotes a counter-clockwise rotation, ω is the derivative of θ w.r.t. time, and let t denote a time period as depicted in figure 15.

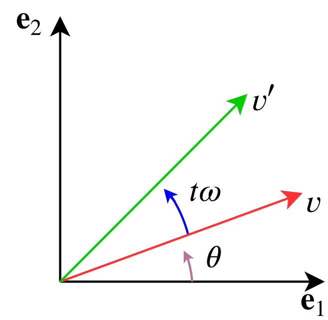


Figure 15

Once again the objective is to derive a transformation that maps the components of $\mathbf{v}$ to the components of $\mathbf{v}'$. I will show you how to udo this using a different method than the one employed in the first part of this subsection. To begin we express the vector $\mathbf{v}$ in $\mathcal{F}^1$ as

$$\begin{aligned}\mathbf{v} &= \alpha^1\mathbf{e}_1 + \alpha^2\mathbf{e}_2\\ &= (\mathbf{v}\cdot\mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{v}\cdot\mathbf{e}_2)\,\mathbf{e}_2\\ &= \|\mathbf{v}\|\cos(\theta)\,\mathbf{e}_1 + \|\mathbf{v}\|\cos\left(\frac{\pi}{2}\mp\theta\right)\mathbf{e}_2.\end{aligned}$$

Since θ is a generalized coordinate, I do not need to look at the sign of the inner product $(\mathbf{v}\cdot\mathbf{e}_2)$ to determine the sign in the function $\cos\left(\frac{\pi}{2}\mp\theta\right)$. I get it for free since θ is no longer ambiguous and is well defined. Thus

$$\mathbf{v} = \|\mathbf{v}\|\cos(\theta)\,\mathbf{e}_1 + \|\mathbf{v}\|\sin(\theta)\,\mathbf{e}_2.$$

At $t > 0$ the vector $\mathbf{v}'$ in $\mathcal{F}^1$ is

$$\begin{aligned}\mathbf{v}' &= \|\mathbf{v}'\|\cos(\theta + t\omega)\,\mathbf{e}_1 + \|\mathbf{v}'\|\sin(\theta + t\omega)\,\mathbf{e}_2\\ &= \|\mathbf{v}\|\cos(\theta + t\omega)\,\mathbf{e}_1 + \|\mathbf{v}\|\sin(\theta + t\omega)\,\mathbf{e}_2.\end{aligned} \tag{12}$$

Using trig identities, $\mathbf{v}'$ in $\mathcal{F}^1$ can be written as

$$\mathbf{v}' = \|\mathbf{v}\|\left(\cos(\theta)\cos(t\omega) - \sin(\theta)\sin(t\omega)\right)\mathbf{e}_1 + \|\mathbf{v}\|\left(\sin(\theta)\cos(t\omega) + \cos(\theta)\sin(t\omega)\right)\mathbf{e}_2,$$

and in matrix notation

$$\mathbf{v}' = \underbrace{\begin{bmatrix}\cos(t\omega) & -\sin(t\omega)\\ \sin(t\omega) & \cos(t\omega)\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\|\mathbf{v}\|\cos(\theta)\\ \|\mathbf{v}\|\sin(\theta)\end{bmatrix}}_{(\alpha^1,\alpha^2)}$$

where A is the active transformation that transforms the components of $\mathbf{v}$ to the components of $\mathbf{v}'$.

For many of you, this new method to derive the active transformation might seem more intuitive; however, I could only derive it this way since θ is a generalize coordinate which tells me how to add the angles θ and $t\omega$ in (12) without the use of inner products.

F. Concatenation of Transformations

So far we have only been working with two reference frames which is sufficient for some simple cases, but we need to be equipped with the means to handle many frames. By showing you how to construct the forward transformation between three standard frames, you will know how to construct the forward and backward transformations between any number of frames.

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$, $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\})$ and $\mathcal{F}^3 = (O_3, \{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\})$ be standard frames with coincident origins in E^2 with θ denoting the angle between $\mathbf{e}_1$ and $\tilde{\mathbf{e}}_2$ and ϕ denoting the angle between $\bar{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_1$ as depicted in figure 16. Since we have three frames, we have multiple forward transformations. We will define the first forward transformation F^1_2 as the transformation that expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$, the second forward transformation F^2_3 as the transformation that expresses the basis of $\mathcal{F}^3$ in $\mathcal{F}^2$, and the third forward transformation F^1_3 as the transformation that expresses the basis of $\mathcal{F}^3$ in $\mathcal{F}^1$.

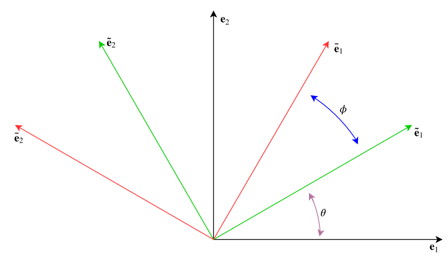


Figure 16

The first forward transformation is

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2\\ \tilde{\mathbf{e}}_2 &= (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2\end{aligned}$$

or in matrix notation

$$F^1_2 = \begin{bmatrix}(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\\ (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\end{bmatrix}.$$

The second forward transformation is

$$\begin{aligned}\bar{\mathbf{e}}_1 &= (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2\\ \bar{\mathbf{e}}_2 &= (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2\end{aligned}$$

or in matrix notation

$$F^2_3 = \begin{bmatrix}(\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\\ (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\end{bmatrix}.$$

The third forward transformation is

$$\begin{aligned}\bar{\mathbf{e}}_1 &= (\bar{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\bar{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2\\ \bar{\mathbf{e}}_2 &= (\bar{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\bar{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2\end{aligned}$$

or in matrix notation

$$F^1_3 = \begin{bmatrix}(\bar{\mathbf{e}}_1 \cdot \mathbf{e}_1) & (\bar{\mathbf{e}}_2 \cdot \mathbf{e}_1)\\ (\bar{\mathbf{e}}_1 \cdot \mathbf{e}_2) & (\bar{\mathbf{e}}_2 \cdot \mathbf{e}_2)\end{bmatrix}.$$

I know you are probably yawning at this point because you have seen this so many times, but restrain your yawn because we are now ready to derive the relationship between the forward transformations. We do this by substituting in the forward transformation F_3^2 into the forward transformation F_3^1 and they simplify using the properties of inner products. We first recall that the forward transformation F_3^1 is

$$\begin{aligned}\bar{\mathbf{e}}_1 &= (\bar{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\bar{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= (\bar{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\bar{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2\end{aligned}$$

and then substitute F_3^2 into the inner products of F_3^1 to get

$$\begin{aligned}\bar{\mathbf{e}}_1 &= (((\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2) \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (((\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2) \cdot \mathbf{e}_2)\,\mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= (((\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2) \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (((\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2) \cdot \mathbf{e}_2)\,\mathbf{e}_2\end{aligned}$$

Distributing the inner product shows the relationship

$$\begin{aligned}\bar{\mathbf{e}}_1 &= ((\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1)\,(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2)\,(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1))\,\mathbf{e}_1 + ((\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1)\,(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2)\,(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2))\,\mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= ((\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\,(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\,(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1))\,\mathbf{e}_1 + ((\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\,(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\,(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2))\,\mathbf{e}_2\end{aligned}$$

or in matrix notation is

$$F_3^1 = \begin{bmatrix} (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1)\,(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2)\,(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\,(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\,(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1) \\ (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1)\,(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2)\,(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\,(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\,(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) \end{bmatrix}$$

which can be factored out to be

$$F_3^1 = F_2^1 F_3^2 = \begin{bmatrix} (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1) \\ (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) \end{bmatrix} \begin{bmatrix} (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1) \\ (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2) \end{bmatrix}. \tag{13}$$

Note the order of operations when constructing F_3^1 from F_2^1 and F_3^2. This order of operation should help the notation make sense. When you multiply transformations together, the common superscripts and subscripts are crossed off, and the subscript and superscript that are not crossed off get carried over as shown below;

$$F_3^1 = F_{\cancel{2}}^1 F_3^{\cancel{2}}.$$

This notation works with vectors. For example, let $\mathbf{v}^3$ denote the vector $\mathbf{v}$ expressed in $\mathcal{F}^3$ and $\mathbf{v}^1$ denote the same vector expressed in $\mathcal{F}^1$. The notation for the transformation from $\mathbf{v}^3$ to $\mathbf{v}^1$ is

$$\mathbf{v}^1 = F_3^1 \mathbf{v}^3$$

or more expressively it is

$$\begin{aligned}\mathbf{v}^1 &= F_2^1 F_3^2 \mathbf{v}^3 \\ &= F_2^1 \mathbf{v}^2 \\ &= \mathbf{v}^1.\end{aligned}$$

The forward transformations can also be represented using angles. I will not go through this derivation, but I will show you the end result for F_3^1

$$F_3^1 = F_3^2 F_2^1 = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

An interesting thing you can see easily from this representations is that

$$\begin{aligned}F_3^1 &= F_3^2 F_2^1 = F_3^2 F_2^1 \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}\end{aligned}$$

which means that these types of transformations commute. This is not the case for higher dimensional transformations, and so to be consistent, I will only use the convention shown in 13.

Since we have the forward transformations, we already have the backward transformations. They are

$$\begin{aligned}B_1^2 &= \left(F_2^1\right)^{-1} \\ B_2^3 &= \left(F_3^2\right)^{-1}\end{aligned}$$

$$
\begin{aligned}
B_1^3 &= \left(F_3^1\right)^{-1} \\
&= \left(F_2^1 F_3^2\right)^{-1} \\
&= \left(F_3^2\right)^{-1}\left(F_2^1\right)^{-1} \\
&= B_2^3 B_1^2
\end{aligned}
$$

Note that when relating the backward transformation B_1^3 to the backward transformations B_3^2 and B_1^2, the order of operation is reversed.

G. Derivative of the Forward and Backward Transformations

So far we have been working with static frames which is interesting, but limited. What if you have one frame rotating w.r.t. another frame and you want to how the transformation is changing? This is just the derivative of a transformation. To tackle this beastly problem we will utilize coordinates in order to perform calculus.

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2\})$ and $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\})$ be standard frames with coincident origins in E^2 where θ is the generalized coordinate defined as the angle from $\mathbf{e}_1$ to $\tilde{\mathbf{e}}_1$ where a positive θ corresponds to a counter-clockwise rotation of $\mathcal{F}^2$, ω is the derivative of θ denoting the angular velocity, and t denotes a time interval. This scenario is depicted in figure 17.

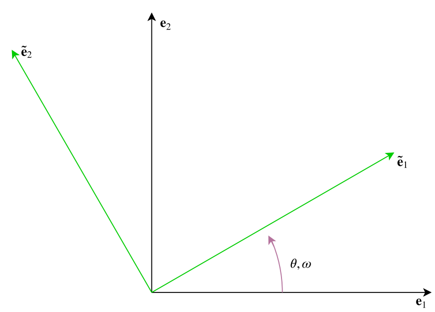


Figure 17

We want to calculate the derivative of the transformation along ω, this is done using the directional derivative. From subsection IV-B we know that

$$
\begin{aligned}
\tilde{\mathbf{e}}_1 &= \cos(\theta + t\omega)\, \mathbf{e}_1 + \sin(\theta + t\omega)\, \mathbf{e}_2 \\
\tilde{\mathbf{e}}_2 &= -\sin(\theta + t\omega)\, \mathbf{e}_1 + \cos(\theta + t\omega)\, \mathbf{e}_2
\end{aligned}
$$

Using these equations, we can compute the directional derivative as follows

$$
\begin{aligned}
\frac{d}{dt}\tilde{\mathbf{e}}_1 &= \frac{d}{dt}\left(\cos(\theta + t\omega)\, \mathbf{e}_1 + \sin(\theta + t\omega)\, \mathbf{e}_2\right)\Big|_{t=0} \\
\frac{d}{dt}\tilde{\mathbf{e}}_2 &= \frac{d}{dt}\left(-\sin(\theta + t\omega)\, \mathbf{e}_1 + \cos(\theta + t\omega)\, \mathbf{e}_2\right)\Big|_{t=0}
\end{aligned}
$$

evaluating this expression gives

$$
\begin{aligned}
\frac{d}{dt}\tilde{\mathbf{e}}_1 &= -\omega \sin(\theta)\, \mathbf{e}_1 + \omega \cos(\omega)\, \mathbf{e}_2 \\
\frac{d}{dt}\tilde{\mathbf{e}}_2 &= -\omega \cos(\theta)\, \mathbf{e}_1 + -\omega \sin(\theta)\, \mathbf{e}_2
\end{aligned}
$$

which can be represented in matrix notation as

$$\frac{d}{dt}F_2^1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$
$$= F_2^1 \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$

The derivative of the backwards transformation can be shown to be

$$\frac{d}{dt}B_1^2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} B_1^2$$

The derivative of the transformation can be expressed differently which will be helpful later on. We begin with (14) and separate the terms out to get

$$\frac{d}{dt}\tilde{\mathbf{e}}_1 = \frac{d}{dt}\left[(\cos(t\omega)\cos(\theta) - \sin(t\omega)\sin(\theta))\,\mathbf{e}_1 + (\cos(t\omega)\sin(\theta) + \sin(t\omega)\cos(\theta))\,\mathbf{e}_2\right]\Big|_{t=0}$$
$$\frac{d}{dt}\tilde{\mathbf{e}}_2 = \frac{d}{dt}\left[(-\sin(t\omega)\cos(\theta) - \cos(t\omega)\sin(\theta))\,\mathbf{e}_1 + (-\sin(t\omega)\sin(\theta) + \cos(t\omega)\cos(\theta))\,\mathbf{e}_2\right]\Big|_{t=0}$$

which can be represented in matrix form

$$\frac{d}{dt}\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(t\omega) & -\sin(t\omega) \\ \sin(t\omega) & \cos(t\omega) \end{bmatrix}\right)\Bigg|_{t=0}$$

Since θ doesn't depend on time, we can write the derivative as

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \left(\frac{d}{dt}\begin{bmatrix} \cos(t\omega) & -\sin(t\omega) \\ \sin(t\omega) & \cos(t\omega) \end{bmatrix}\Bigg|_{t=0}\right).$$

Evaluating the expression yields

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

which is the same result as above. However, this matrix representation allows us to represent the change in the direction of ω as a transformation between frames. This idea will come into play when finding the derivatives of transformations in E^3.

H. SO(2)

It is time for my confession. I haven't been telling you everything. The transformations that we have been discussing in this section are rotation matrices that belong to a group called the special orthogonal group of 2-dimensions denoted as $SO(2) \coloneqq \left\{R \in \mathbb{R}^{2\times 2} | R^\top R = I,\, \det(R) = 1\right\}$. This means that forward transformations can be written as $R_2^1 = F_2^1$ and backward transformations can be written as $R_1^2 = B_1^2$ such that $R_1^2 R_2^1 = I$ and $\left(R_1^2\right)^\top = R_2^1$. As a side note, all of the transformations that we have worked with have belonged to the general linear group of 2-dimensions. The definition of a group is

Definition 14. A **group** G is a set of elements combined with an operation $\bullet$ with the following properties

1) Closure
   a) For all $a, b \in G$, the result of the operation $a \bullet b \in G$.
2) Associativity
   a) For all a, b and c in G, $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.
3) Identity Element
   a) There exists an element $e \in G$ that for all $a \in G$, the equation $e \bullet a = a \bullet e = a$ holds.
4) Inverse Element
   a) For each $a \in G$ there exists an element $b \in G$, such that $a \bullet b = b \bullet a = e$, where e is the identity element.

The set of elements in $SO(2)$ is the set of all the matrices R such that $R^\top R = I$ and $\det(R) = 1$, and the operation of $SO(2)$ is the matrix multiplication. In the next section, we will be working with the group $SO(3)$ which is defined as $SO(3) \coloneqq \left\{R \in \mathbb{R}^{3\times 3} | R^\top R = I,\, \det(R) = 1\right\}$. Rotation matrices have the unique property that they preserver length, i.e., the inner product of the vector components do not change as they change basis.

V. Rotation Matrices in $SO(3)$

Every physical system inherently resides in E^3. Depending on what the system is, we can sometimes constrain it to reside in E^1 or E^2. In the case that the physical system is 1-dimensional, there is no orientation or need for a rotation matrix. In the case the physical system is 2-dimensional, it has orientation that can be represented using rotation matrices of $SO(2)$. In the case that the physical system is 3-dimensional, we represent the orientation using rotation matrices of $SO(3)$.

In this section we will discuss the forward and backward transformations of $SO(3)$, active and passive rotations of vectors in E^3, embedding $SO(2)$ into $SO(3)$, and the derivative of the elements of $SO(3)$. The pace will be a little fast because by now you should be familiar with the material.

A. Forward and Backward Transformation

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$ and $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\})$ be standard frames with coincident origins in E^3 as depicted in figure 18.

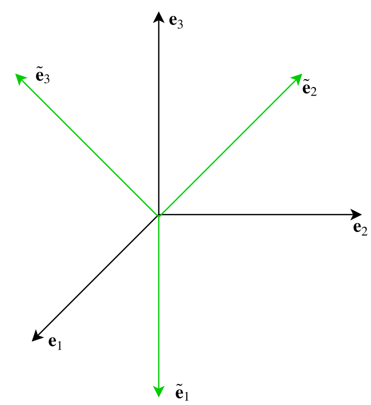


Figure 18

The forward transformation is defined using the inner product

$$
\begin{aligned}
\tilde{\mathbf{e}}_1 &= (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3)\,\mathbf{e}_3
\end{aligned}
$$

and in matrix notation we get

$$
R_2^1 = \begin{bmatrix} (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1) & (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1) \\ (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) & (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2) \\ (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3) & (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3) \end{bmatrix}
$$

Once we have the forward transformation, we can easily extract the backward transformation to be

$$
R_1^2 = \left(R_2^1\right)^\top .
$$

I am not using the previous notation F and B to denote the forward and backward transformations so that you can get use to them being denoted as R.

We can also represent the transformations using angles between basis vectors like we did with $SO(2)$ and get something like

$$
R_2^1 = \begin{bmatrix} \cos(\theta_{1,1}) & \cos(\theta_{2,1}) & \cos(\theta_{3,1}) \\ \cos(\theta_{1,2}) & \cos(\theta_{2,2}) & \cos(\theta_{3,2}) \\ \cos(\theta_{1,3}) & \cos(\theta_{2,3}) & \cos(\theta_{3,3}) \end{bmatrix}
$$

where $\theta_{i,j},\ i \in \{1,2,3\}$ is the angle between the basis vectors $\tilde{\mathbf{e}}_i$ and $\mathbf{e}_j,\ i,j \in \{1,2,3\}$. Unlike $SO(2)$, the elements of $SO(3)$ cannot be parameterized with one coordinate. This is due to the fact that $SO(3)$ has three degrees of freedom even though it has 9 variables. This reduction comes from the constraints derived from the properties $R^\top R = I$ and $\det(R) = 1$. These constraints are

$$\begin{aligned}
a_{11}^2 + a_{12}^2 + a_{13}^2 &= 1 \\
a_{21}^2 + a_{22}^2 + a_{23}^2 &= 1 \\
a_{31}^2 + a_{32}^2 + a_{33}^2 &= 1 \\
a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0 \\
a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} &= 0 \\
a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0 \\
\det(R) &= 1
\end{aligned}$$

where $a_{i,j} = (\tilde{\mathbf{e}}_i \cdot \mathbf{e}_j),\ i,j \in \{1,2,3\}$. The first 6 constraints ensures that the transformation is an element of the orthogonal group of three dimensions denotes as $O(3)$, and the last constraint ensures that it is an element of $SO(3)$. Even with these constraints, it is not easy to parameterize elements of $SO(3)$ using only three generalize coordinates. For this reason, it is best to leave it in its original definition of inner products and angles without any simplification.

There are methods that can be used to construct a rotation matrix such as axis-angle, Euler angles, Rodriguez parameterization, etc. However, these other forms of attitude representation are not diffeomorphic to $SO(3)$, and so care needs to be taken when using these other forms.

B. *Passive Rotations of Vectors in E^3*

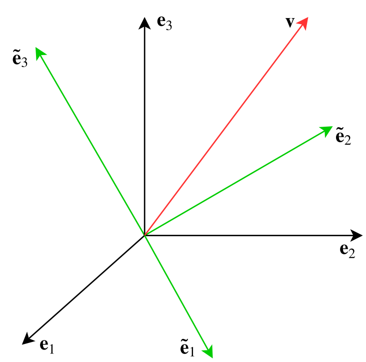


Figure 19

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$ and $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\})$ be standard frames with coincident origins in E^3 and vector $\mathbf{v}$ be a vector in E^3 as depicted in figure 19. We can express the vector $\mathbf{v}$ in $\mathcal{F}^1$ as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\mathbf{v} \cdot \mathbf{e}_3)\,\mathbf{e}_3.$$

Using the backward transformation defined in subsection V-A, we can transform the components of $\mathbf{v}$ to be expressed in $\mathcal{F}^2$ as follows

$$\begin{aligned}
\mathbf{v} &= (\mathbf{v} \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\mathbf{v} \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\
&= (\mathbf{v} \cdot \mathbf{e}_1)\left((\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2 + (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3\right) \\
&+ (\mathbf{v} \cdot \mathbf{e}_2)\left((\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2 + (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3\right) \\
&+ (\mathbf{v} \cdot \mathbf{e}_3)\left((\mathbf{e}_3 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\mathbf{e}_3 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2 + (\mathbf{e}_3 \cdot \tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3\right)
\end{aligned}$$

which can be simplified and represented using matrix notation as

$$\mathbf{v} = \underbrace{\begin{bmatrix} (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1) & (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1) & (\mathbf{e}_3 \cdot \tilde{\mathbf{e}}_1) \\ (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2) & (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2) & (\mathbf{e}_3 \cdot \tilde{\mathbf{e}}_2) \\ (\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_3) & (\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_3) & (\mathbf{e}_3 \cdot \tilde{\mathbf{e}}_3) \end{bmatrix}}_{R_1^2} \underbrace{\begin{bmatrix} (\mathbf{v} \cdot \mathbf{e}_1) \\ (\mathbf{v} \cdot \mathbf{e}_2) \\ (\mathbf{v} \cdot \mathbf{e}_3) \end{bmatrix}}_{\alpha}.$$

where the matrix R_1^2 is the backward transformation and the matrix α contains the components of $\mathbf{v}$ expressed in $\mathcal{F}^1$. Once again we see that the components of a vector transform contravariantly to the basis vectors.

C. *Active Rotations of Vectors in E^3*

The derivation of this rotation is somewhat intense, but I tried by best to prepare you for it in subsections III-D and IV-E. Hopefully these subsections will help us bare this section with a little more ease.

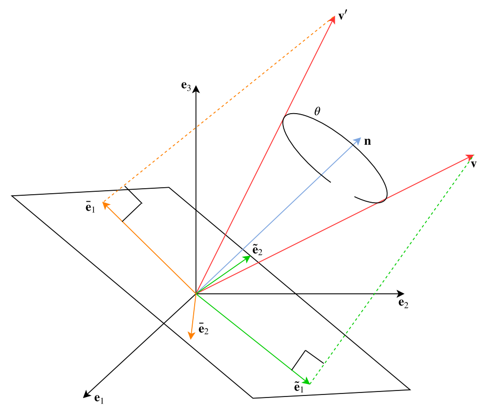


Figure 20

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$ be a standard frame in E^3, and let $\mathbf{v}$, $\mathbf{v}'$ and $\mathbf{n}$ denote vectors in E^3. The vector $\mathbf{v}'$ denotes the vector $\mathbf{v}$ after a rotation about the unit vector $\mathbf{n}$ by the generalize coordinate θ which is positive if the rotation about $\mathbf{n}$ is counter-clockwise. We have also constructed two additional *non-standard* frames $\mathcal{F}^2 = (O_1, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\})$ and $\mathcal{F}^3 = (O_3, \{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\})$ whose origins are coincident with O_1. We define the basis vectors of $\mathcal{F}^2$ and $\mathcal{F}^3$ as

$$\begin{aligned} \bar{\mathbf{e}}_3 &= \tilde{\mathbf{e}}_3 = \mathbf{n}, \\ \bar{\mathbf{e}}_2 &= \mathbf{n} \times \mathbf{v}', \\ \bar{\mathbf{e}}_1 &= -\mathbf{n} \times (\mathbf{n} \times \mathbf{v}'), \\ \tilde{\mathbf{e}}_2 &= \mathbf{n} \times \mathbf{v}', \\ \tilde{\mathbf{e}}_1 &= -\mathbf{n} \times (\mathbf{n} \times \mathbf{v}). \end{aligned}$$

where

$$\begin{aligned} \|\bar{\mathbf{e}}_3\| &= \|\tilde{\mathbf{e}}_3\| = 1, \\ \|\bar{\mathbf{e}}_2\| &= \|\mathbf{v}'\| \sin(\phi), \\ \|\bar{\mathbf{e}}_1\| &= \|\mathbf{v}'\| \sin(\phi), \\ \|\tilde{\mathbf{e}}_2\| &= \|\mathbf{v}\| \sin(\phi), \\ \|\tilde{\mathbf{e}}_1\| &= \|\mathbf{v}\| \sin(\phi), \end{aligned}$$

with ϕ being the angle between $\mathbf{v}$ and $\mathbf{n}$ which is the same angle between $\mathbf{v}'$ and $\mathbf{n}$. Note that the vector $\mathbf{v}$ resides in the plane spanned by $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_3$ and the vector $\mathbf{v}'$ resides in the plane spanned by $\bar{\mathbf{e}}_1$ and $\bar{\mathbf{e}}_2$. In fact we have that

$$\begin{aligned}\mathbf{v} &= (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} - \mathbf{n}\times(\mathbf{n}\times\mathbf{v}) && (15)\\ &= (\mathbf{v}\cdot\tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3 + \tilde{\mathbf{e}}_1 && (16)\end{aligned}$$

and

$$\begin{aligned}\mathbf{v}' &= (\mathbf{v}'\cdot\mathbf{n})\,\mathbf{n} - \mathbf{n}\times(\mathbf{n}\times\mathbf{v}')\,. && (17)\\ &= (\mathbf{v}'\cdot\bar{\mathbf{e}}_3)\,\bar{\mathbf{e}}_3 + \bar{\mathbf{e}}_1. && (18)\end{aligned}$$

We also note the following identities

$$\begin{aligned}(\mathbf{v}\cdot\tilde{\mathbf{e}}_1) &= (\mathbf{v}'\cdot\bar{\mathbf{e}}_1) && (19a)\\ (\mathbf{v}\cdot\tilde{\mathbf{e}}_2) &= (\mathbf{v}'\cdot\bar{\mathbf{e}}_2) && (19b)\\ (\mathbf{v}\cdot\tilde{\mathbf{e}}_3) &= (\mathbf{v}'\cdot\bar{\mathbf{e}}_3) && (19c)\\ (\mathbf{v}\cdot\mathbf{v}) &= (\mathbf{v}'\cdot\mathbf{v}') && (19d)\end{aligned}$$

where $(\mathbf{v}\cdot\tilde{\mathbf{e}}_2) = (\mathbf{v}'\cdot\bar{\mathbf{e}}_2) = 0$. Take a moment to really convince yourself of everything presented to you so far in this subsection; otherwise, you will get lost.

Recall that the objective of an active transformation is to transform the components of a vector $\mathbf{v}$ to the components of the vector $\mathbf{v}'$ expressed in $\mathcal{F}^1$. To do this, we will use the frames $\mathcal{F}^2$ and $\mathcal{F}^3$ to simplify the derivation. We will assume that we already have expressed $\mathbf{v}$ in $\mathcal{F}^1$ as $\mathbf{v} = \sum_{i=1}^{3} \alpha^i \mathbf{e}_i$. Now let's derive the transformation!

Let's first construct the forward transformation that expresses the basis of $\mathcal{F}^3$ in $\mathcal{F}^2$ as such

$$\begin{aligned}\bar{\mathbf{e}}_1 &= (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)\,\frac{\tilde{\mathbf{e}}_1}{\|\tilde{\mathbf{e}}_1\|^2} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_2)\,\frac{\tilde{\mathbf{e}}_2}{\|\tilde{\mathbf{e}}_2\|^2} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3\\ \bar{\mathbf{e}}_2 &= (\bar{\mathbf{e}}_2\cdot\tilde{\mathbf{e}}_1)\,\frac{\tilde{\mathbf{e}}_1}{\|\tilde{\mathbf{e}}_1\|^2} + (\bar{\mathbf{e}}_2\cdot\tilde{\mathbf{e}}_2)\,\frac{\tilde{\mathbf{e}}_2}{\|\tilde{\mathbf{e}}_2\|^2} + (\bar{\mathbf{e}}_2\cdot\tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3\\ \bar{\mathbf{e}}_3 &= (\bar{\mathbf{e}}_3\cdot\tilde{\mathbf{e}}_1)\,\frac{\tilde{\mathbf{e}}_1}{\|\tilde{\mathbf{e}}_1\|^2} + (\bar{\mathbf{e}}_3\cdot\tilde{\mathbf{e}}_2)\,\frac{\tilde{\mathbf{e}}_2}{\|\tilde{\mathbf{e}}_2\|^2} + (\bar{\mathbf{e}}_3\cdot\tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3\end{aligned}$$

which can be simplified to

$$\begin{aligned}\bar{\mathbf{e}}_1 &= (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)\,\frac{\tilde{\mathbf{e}}_1}{\|\tilde{\mathbf{e}}_1\|^2} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_2)\,\frac{\tilde{\mathbf{e}}_2}{\|\tilde{\mathbf{e}}_2\|^2}\\ \bar{\mathbf{e}}_2 &= (\bar{\mathbf{e}}_2\cdot\tilde{\mathbf{e}}_1)\,\frac{\tilde{\mathbf{e}}_1}{\|\tilde{\mathbf{e}}_1\|^2} + (\bar{\mathbf{e}}_2\cdot\tilde{\mathbf{e}}_2)\,\frac{\tilde{\mathbf{e}}_2}{\|\tilde{\mathbf{e}}_2\|^2}\\ \bar{\mathbf{e}}_3 &= \tilde{\mathbf{e}}_3,\end{aligned}$$

since $(\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_3) = (\bar{\mathbf{e}}_2\cdot\tilde{\mathbf{e}}_3) = 0$ and $(\bar{\mathbf{e}}_3\cdot\tilde{\mathbf{e}}_3) = 1$.

Now let's express $\mathbf{v}'$ in $\mathcal{F}^3$ as

$$\mathbf{v}' = (\mathbf{v}'\cdot\bar{\mathbf{e}}_3)\,\bar{\mathbf{e}}_3 + \bar{\mathbf{e}}_1.$$

Using the identities in 19 we can express $\mathbf{v}'$ as

$$\mathbf{v}' = (\mathbf{v}\cdot\mathbf{n})\,\bar{\mathbf{e}}_3 + \bar{\mathbf{e}}_1.$$

Substituting in the forward transformation yields

$$\mathbf{v}' = (\mathbf{v}\cdot\mathbf{n})\,\tilde{\mathbf{e}}_3 + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)\,\frac{\tilde{\mathbf{e}}_1}{\|\tilde{\mathbf{e}}_1\|^2} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_2)\,\frac{\tilde{\mathbf{e}}_2}{\|\tilde{\mathbf{e}}_2\|^2}$$

which can be written as

$$\mathbf{v}' = (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)\,\frac{-\mathbf{n}\times(\mathbf{n}\times\mathbf{v})}{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_2)\,\frac{(\mathbf{n}\times\mathbf{v})}{\|\mathbf{n}\times\mathbf{v}\|^2}.$$

Using the relation in 16 allows us to write the above equation as

$$\mathbf{v}' = (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)\,\frac{(\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n})}{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_2)\,\frac{(\mathbf{n}\times\mathbf{v})}{\|\mathbf{n}\times\mathbf{v}\|^2}$$

which can be rewritten as

$$\mathbf{v}' = (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}\left(1-\frac{(\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)}{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2}\right) + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)\,\frac{\mathbf{v}}{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_2)\,\frac{(\mathbf{n}\times\mathbf{v})}{\|\mathbf{n}\times\mathbf{v}\|^2}.$$

which can be expressed in terms of angles

$$\begin{aligned}
\mathbf{v}' &= (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}\left(1-\frac{(\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)}{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2}\right) + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_1)\,\frac{\mathbf{v}}{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2} + (\bar{\mathbf{e}}_1\cdot\tilde{\mathbf{e}}_2)\,\frac{(\mathbf{n}\times\mathbf{v})}{\|\mathbf{n}\times\mathbf{v}\|^2} \\
&= (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}\left(1-\frac{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2\cos(\psi_1)}{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2}\right) + \frac{\mathbf{v}\,\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2\cos(\psi_1)}{\|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|^2} + \|\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\|\,\|\mathbf{n}\times\mathbf{v}\|\cos(\psi_2)\,\frac{(\mathbf{n}\times\mathbf{v})}{\|\mathbf{n}\times\mathbf{v}\|^2} \\
&= (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}\,(1-\cos(\psi_1)) + \mathbf{v}\cos(\psi_1) + \cos(\psi_2)\,(\mathbf{n}\times\mathbf{v})\,,
\end{aligned}$$

where ψ_1 is the angle between $\bar{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_1$ and ψ_2 is the angle between $\bar{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$. Note that we have not used any generalized coordinates yet.

If we use the generalized coordinate θ, we can express the transformation as

$$\mathbf{v}' = (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}\,(1-\cos(\theta)) + \mathbf{v}\cos(\theta) + \sin(\theta)\,(\mathbf{n}\times\mathbf{v}) \tag{20}$$

which is the Rodrigues rotation formula for rotating a vector by angle θ about the unit vector $\mathbf{n}$. This equation might seem fearsome, but remember that if the vectors $\mathbf{v}$ and $\mathbf{n}$ are represented in $\mathcal{F}^1$, then they can be expressed as

$$\mathbf{v} = \sum_{i=1}^{3}\alpha^i\mathbf{e}_i$$
$$\mathbf{n} = \sum_{i=1}^{3}\beta^i\mathbf{e}_i$$

or in matrix notation

$$\mathbf{v} = \underbrace{\begin{bmatrix}\alpha^1\\ \alpha^2\\ \alpha^2\end{bmatrix}}_{A}$$
$$\mathbf{n} = \underbrace{\begin{bmatrix}\beta^1\\ \beta^2\\ \beta^3\end{bmatrix}}_{B}$$

which allows us to write the Rodrigues rotation formula in matrix form as

$$\mathbf{v}' = (A\cdot B)\,B\,(1-\cos(\theta)) + A\cos(\theta) + \sin(\theta)\,[B]_\times\,A \tag{21}$$

where $[B]_\times$ is the skew symmetric matrix of B and $(A\cdot B) = \sum_{i=1}^{3}\alpha^i\beta^i$.

Remember that the objective was to form a rotation matrix to transform the components of $\mathbf{v}$ to the components of $\mathbf{v}'$, which we have not done. To do this, we will use the identity

$$(A\cdot B)\,B\,(1-\cos(\theta)) + A\cos(\theta) = \Big(I + [B]_\times^2\,(1-\cos(\theta))\Big)\,A$$

to write (21) as

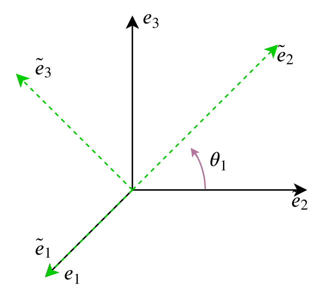
$$\begin{aligned}
\mathbf{v}' &= \Big(I + [B]_\times^2\,(1-\cos(\theta))\Big)\,A + \sin(\theta)\,[B]_\times\,A \\
&= \Big(I + \sin(\theta)\,[B]_\times + [B]_\times^2\,(1-\cos(\theta))\Big)\,A \\
&= RA
\end{aligned}$$

where the rotation matrix $R = \left(I + \sin(\theta)[B]_\times + [B]_\times^2 (1 - \cos(\theta))\right) = \exp\left(\theta [B]_\times\right)$. This identity would make more sense with an understanding of the Lie algebra of $SO(3)$, but that is beyond the scope of this document. Let it suffice that we have constructed an active rotation matrix from the Rodrigues rotation formula to rotate a vector by angle θ about a unit vector $\mathbf{n}$. There are other methods to construct an active rotation matrix, but this one is used extensively in robotics and is good to understand.

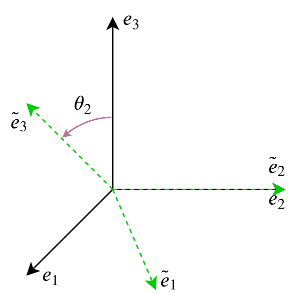
Remark 15. This active rotation matrix could have been derived purely from a geometric perspective instead of using the generalized coordinates θ if we kept track of the signs of $\cos(\psi_1)$ and $\cos(\psi_2)$ using the signs of their corresponding inner product. I didn't do that here because the Rodrigues rotation formula is typically used with a generalized coordinate that denotes the angle of rotation.

Remark 16. The Rodrigues rotation formula, also known as axis-angle, can also be used to represent a forward and backward transformation instead of using rotation matrices; however, the axis-angle representation is not diffeomorphic to $SO(3)$ and contains singularities. This is why I suggest not using the axis-angle representation to represent forward and backward transformations.

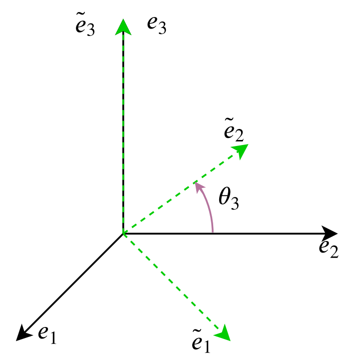
D. Embedding SO(2)



(a)



(b)



(c)

A simplified definition of embedding is the act of immersing a space inside of another space such that all of it's structure is preserved. A simple example of this is embedding $\mathbb{R}^2 := \{(x, y) \,|x \in \mathbb{R},\, y \in \mathbb{R}\}$ into $\mathbb{R}^3 := \left\{((x, y), z) \,|\, (x, y) \in \mathbb{R}^2,\, z = c \in \mathbb{R}\right\}$, which preserves all of it's structure. In a similar manner, $SO(2)$ can be embedded into $SO(3)$. There are an infinite number of ways to do this. I will only address the more common ones which are shown in figures 21a, 21b, and 21c. These figures describe a rotation about a basis vector of $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$ by $\theta_i,\, i \in \{1, 2, 3\}$ to get to $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\})$.

Let's derive the forward transformations for each scenario depicted in the figures. Figure 21a shows a rotation about $\mathbf{e}_1$ by angle θ_1. This implies that $\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1 = 1$ and

$$\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_j = \mathbf{e}_i \cdot \tilde{\mathbf{e}}_1 = 0,\, i, j \in \{2, 3\}$$

These constraints allows us to reduce the forward transformation

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\, \mathbf{e}_1 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2)\, \mathbf{e}_2 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3)\, \mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\, \mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\, \mathbf{e}_2 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3)\, \mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1)\, \mathbf{e}_1 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2)\, \mathbf{e}_2 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3)\, \mathbf{e}_3
\end{aligned}$$

to

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= 1\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= 0\mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\, \mathbf{e}_2 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3)\, \mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= 0\mathbf{e}_1 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2)\, \mathbf{e}_2 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3)\, \mathbf{e}_3
\end{aligned}$$

and we can represent the inner products as angles

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= 1\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_2 \\
\tilde{\mathbf{e}}_2 &= 0\mathbf{e}_1 + \cos(\theta_1)\,\mathbf{e}_2 + \sin(\theta_1)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= 0\mathbf{e}_1 - \sin(\theta_1)\,\mathbf{e}_2 + \cos(\theta_1)\,\mathbf{e}_3
\end{aligned}$$

which in matrix notation is

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$

Figure 21b shows a rotation about $\mathbf{e}_2$ by angle θ_2. This implies that $\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2 = 1$ and

$$\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_j = \mathbf{e}_i \cdot \tilde{\mathbf{e}}_2 = 0,\; i, j \in \{1, 3\}$$

These constraints allows us to reduce the forward transformation

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3)\,\mathbf{e}_3
\end{aligned}$$

to

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + 0\mathbf{e}_2 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= 0\mathbf{e}_1 + 1\mathbf{e}_2 + 0\mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + 0\mathbf{e}_2 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3)\,\mathbf{e}_3
\end{aligned}$$

and we can represent the inner products as angles

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= \cos(\theta_2)\,\mathbf{e}_1 + 0\mathbf{e}_2 - \sin(\theta_2)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= 0\mathbf{e}_1 + 1\mathbf{e}_2 + 0\mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= \sin(\theta_2)\,\mathbf{e}_1 + 0\mathbf{e}_2 + \cos(\theta_2)\,\mathbf{e}_3
\end{aligned}$$

which in matrix notation is

$$R_2^1 = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix}$$

Figure 21c shows a rotation about $\mathbf{e}_3$ by angle θ_3. This implies that $\mathbf{e}_3 \cdot \tilde{\mathbf{e}}_3 = 1$ and

$$\mathbf{e}_3 \cdot \tilde{\mathbf{e}}_j = \mathbf{e}_i \cdot \tilde{\mathbf{e}}_3 = 0,\; i, j \in \{1, 2\}$$

These constraints allows us to reduce the forward transformation

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3)\,\mathbf{e}_3
\end{aligned}$$

to

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + 0\mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + 0\mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= 0\mathbf{e}_1 + 0\mathbf{e}_2 + 1\mathbf{e}_3
\end{aligned}$$

and we can represent the inner products as angles

$$\begin{aligned}
\tilde{\mathbf{e}}_1 &= \cos(\theta_3)\,\mathbf{e}_1 + \sin(\theta_3)\,\mathbf{e}_2 + 0\mathbf{e}_3 \\
\tilde{\mathbf{e}}_2 &= -\sin(\theta_3)\,\mathbf{e}_1 + \cos(\theta_3)\,\mathbf{e}_2 + 0\mathbf{e}_3 \\
\tilde{\mathbf{e}}_3 &= 0\mathbf{e}_1 + 0\mathbf{e}_2 + 1\mathbf{e}_3
\end{aligned}$$

which in matrix notation is

$$R_2^1 = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These embeddings are used extensively. One such example is with Euler-angles which I will not get into. We will use these embeddings to derive the derivative of elements of $SO(3)$ later on.

E. Concatenations of Rotations

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$, $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\})$ and $\mathcal{F}^3 = (O_3, \{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\})$ be standard frames with coincident origins in E^3 with θ_1 being the angle of rotation between $\mathcal{F}^1$ and $\mathcal{F}^2$ about $\mathbf{e}_1$ and θ_2 being the angle of rotation between $\mathcal{F}^2$ and $\mathcal{F}^3$ about $\tilde{\mathbf{e}}_2$ as depicted in figure 21. If we have two forward transformations that express the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$ and the basis of $\mathcal{F}^3$ in $\mathcal{F}^2$, we can concatenate them to express the basis of $\mathcal{F}^3$ in $\mathcal{F}^1$.

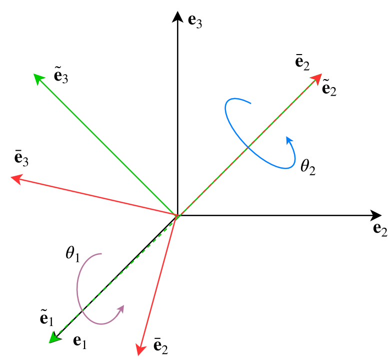


Figure 21

We begin with a geometric derivation to show to concatenate two rotations. To do this we construct the forward transformation that expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$ is

$$\tilde{\mathbf{e}}_1 = (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \tag{22a}$$
$$\tilde{\mathbf{e}}_2 = (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \tag{22b}$$
$$\tilde{\mathbf{e}}_3 = (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2)\,\mathbf{e}_2 + (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \tag{22c}$$

or in matrix notation is

$$R_2^1 = \begin{bmatrix} (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1) & (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1) \\ (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) & (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2) \\ (\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3) & (\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3) & (\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3) \end{bmatrix}$$

and the forward transformation that expresses the bases of $\mathcal{F}^3$ in $\mathcal{F}^2$ is

$$\bar{\mathbf{e}}_1 = (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2 + (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3 \tag{23a}$$
$$\bar{\mathbf{e}}_2 = (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2 + (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3 \tag{23b}$$
$$\bar{\mathbf{e}}_3 = (\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_1)\,\tilde{\mathbf{e}}_1 + (\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_2)\,\tilde{\mathbf{e}}_2 + (\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_3)\,\tilde{\mathbf{e}}_3 \tag{23c}$$

or in matrix notation is

$$R_3^2 = \begin{bmatrix} (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1) & (\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_1) \\ (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2) & (\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_2) \\ (\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_3) & (\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_3) & (\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_3) \end{bmatrix}$$

We can substitute (22) into (23) to get

$$\begin{aligned}
\bar{\mathbf{e}}_1 &= a_{11}\left(b_{11}\mathbf{e}_1 + b_{12}\mathbf{e}_2 + b_{13}\mathbf{e}_3\right) + a_{12}\left(b_{21}\mathbf{e}_1 + b_{22}\mathbf{e}_2 + b_{23}\mathbf{e}_3\right) + a_{13}\left(b_{31}\mathbf{e}_1 + b_{32}\mathbf{e}_2 + b_{33}\mathbf{e}_3\right)\\
\bar{\mathbf{e}}_2 &= a_{21}\left(b_{11}\mathbf{e}_1 + b_{12}\mathbf{e}_2 + b_{13}\mathbf{e}_3\right) + a_{22}\left(b_{21}\mathbf{e}_1 + b_{22}\mathbf{e}_2 + b_{23}\mathbf{e}_3\right) + a_{23}\left(b_{31}\mathbf{e}_1 + b_{32}\mathbf{e}_2 + b_{33}\mathbf{e}_3\right)\\
\bar{\mathbf{e}}_3 &= a_{31}\left(b_{11}\mathbf{e}_1 + b_{12}\mathbf{e}_2 + b_{13}\mathbf{e}_3\right) + a_{32}\left(b_{21}\mathbf{e}_1 + b_{22}\mathbf{e}_2 + b_{23}\mathbf{e}_3\right) + a_{33}\left(b_{31}\mathbf{e}_1 + b_{32}\mathbf{e}_2 + b_{33}\mathbf{e}_3\right)
\end{aligned}$$

where $a_{ij} := \left(\bar{\mathbf{e}}_i \cdot \tilde{\mathbf{e}}_j\right), i, j \{1,2,3\}$ and $b_{ij} := \left(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1\right), i, j \in \{1,2,3\}$. Rearranging the terms in the equation above yields

$$\begin{aligned}
\bar{\mathbf{e}}_1 &= \left(a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}\right)\mathbf{e}_1 + \left(a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}\right)\mathbf{e}_2 + \left(a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}\right)\mathbf{e}_3\\
\bar{\mathbf{e}}_2 &= \left(a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}\right)\mathbf{e}_1 + \left(a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}\right)\mathbf{e}_2 + \left(a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}\right)\mathbf{e}_3\\
\bar{\mathbf{e}}_3 &= \left(a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}\right)\mathbf{e}_1 + \left(a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}\right)\mathbf{e}_2 + \left(a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}\right)\mathbf{e}_3
\end{aligned}$$

or in matrix form

$$R_3^1 = \begin{bmatrix} \left(a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}\right) & \left(a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}\right) & \left(a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}\right)\\ \left(a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}\right) & \left(a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}\right) & \left(a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}\right)\\ \left(a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}\right) & \left(a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}\right) & \left(a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}\right) \end{bmatrix}$$

which can be expanded out to be

$$\begin{aligned}
R_3^1 &= \begin{bmatrix} b_{11} & b_{21} & b_{31}\\ b_{12} & b_{22} & b_{32}\\ b_{13} & b_{23} & b_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33} \end{bmatrix}\\
&= \begin{bmatrix} \left(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_1\right) & \left(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1\right) & \left(\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_1\right)\\ \left(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2\right) & \left(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2\right) & \left(\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_2\right)\\ \left(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_3\right) & \left(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3\right) & \left(\tilde{\mathbf{e}}_3 \cdot \mathbf{e}_3\right) \end{bmatrix} \begin{bmatrix} \left(\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1\right) & \left(\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_1\right) & \left(\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_1\right)\\ \left(\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2\right) & \left(\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2\right) & \left(\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_2\right)\\ \left(\bar{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_3\right) & \left(\bar{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_3\right) & \left(\bar{\mathbf{e}}_3 \cdot \tilde{\mathbf{e}}_3\right) \end{bmatrix}\\
&= R_2^1 R_3^2
\end{aligned}$$

Pay special attention on how the forward transformations are multiplied. Unlike in subsection IV-F where the order of multiplication didn't matter since elements of $SO(2)$ commute with the multiplication operator, the order matters in $SO(3)$. To make sense out of the order of concatenation, recall that a forward transformation R_3^1 will transform the components a vector expressed in $\mathcal{F}^3$ to be expressed in $\mathcal{F}^1$, and the operation is written as $R_2^1 R_3^2 \mathbf{v}^3$. The components of the vector $\mathbf{v}^3$ will first be transformed to be expressed in $\mathcal{F}^2$ which leaves the operation $R_2^1 \mathbf{v}^2$. Finally the components of the vector $\mathbf{v}^2$ will be transformed to be expressed in $\mathcal{F}^1$.

The derivation of the concatenation for backward transformations is simple now that we know how to concatenate forward transformations

$$\begin{aligned}
R_1^3 &= \left(R_3^1\right)^{-1}\\
&= \left(R_2^1 R_3^2\right)^{-1}\\
&= \left(R_3^2\right)^{-1}\left(R_2^1\right)^{-1}\\
&= R_2^3 R_1^2
\end{aligned}$$

Note that the order of multiplication for backward transformations is opposite to concatenating forward transformations.

Let us return to the example posed in figure (21). Using the embeddings of $SO(2)$ in $SO(3)$ we can easily express the forward transformations R_2^1 and R_3^2 using the generalized coordinates θ_1 and θ_2.

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos(\theta_1) & -\sin(\theta_1)\\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$

$$R_3^2 = \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2)\\ 0 & 1 & 0\\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix}$$

We can obtain the forward transformation R_3^1 by concatenating R_2^1 with R_3^2;

$$\begin{aligned} R_3^1 &= R_2^1 R_3^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ \sin(\theta_1)\sin(\theta_2) & \cos(\theta_1) & -\sin(\theta_1)\cos(\theta_2) \\ -\cos(\theta_1)\sin(\theta_2) & \sin(\theta_1) & \cos(\theta_1)\cos(\theta_2) \end{bmatrix} \end{aligned}$$

Since R_3^1 is a rotation about $\mathbf{e}_1$ by θ_1 followed by a rotation about $\tilde{\mathbf{e}}_2$ by θ^2. We can verify that this transformation makes sense by looking at $\bar{\mathbf{e}}_2$ in terms of the basis of $\mathcal{F}^1$. Since the second transformation is about the basis vector $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_2 \cdot \bar{\mathbf{e}}_2 = 1$. This means that $\bar{\mathbf{e}}_2$ is not affect by the second transformation, and that the transformation should only be parameterized by θ_1. This is the case since

$$\bar{\mathbf{e}}_2 = 0\mathbf{e}_1 + \cos(\theta_1)\,\mathbf{e}_2 + \sin(\theta_1)\,\mathbf{e}_3.$$

F. Derivatives of SO(3)

Let $\mathcal{F}^1 = (O_1, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$, $\mathcal{F}^2 = (O_2, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\})$ and $\mathcal{F}^3 = (O_3, \{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\})$ be standard frames in E^3, with ω_1 being the angular velocity of $\mathcal{F}^2$ about the basis vector $\tilde{\mathbf{e}}_1$ and $\mathcal{F}^3$ representing the orientation of $\mathcal{F}^2$ after a time period t as depicted in figure (22). Also, let R_2^1 denote the forward transformation that expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$, and let R_3^2 denote the forward transformation that expresses the basis of $\mathcal{F}^3$ in $\mathcal{F}^2$. We can calculate the derivative of R_2^1 by using the directional derivative and concatenations of forward transformations.

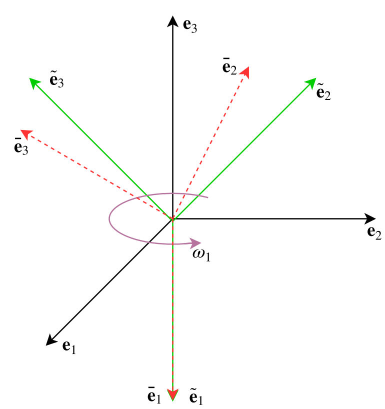


Figure 22

The derivative of R_3^1 is calculated as follows

$$
\begin{aligned}
\frac{d}{dt} R_3^1 &= \frac{d}{dt}\left(R_2^1 R_3^2\right)\Big|_{t=0} \\
&= R_2^1 \left(\frac{d}{dt} R_3^2\right)\Bigg|_{t=0} \\
&= R_2^1 \left(\frac{d}{dt}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t\omega_1) & -\sin(t\omega_1) \\ 0 & \sin(t\omega_1) & \cos(t\omega_1) \end{bmatrix}\right)\Bigg|_{t=0} \\
&= R_2^1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\omega_1 \sin(t\omega_1) & -\omega_1 \cos(t\omega_1) \\ 0 & \omega_1 \cos(t\omega_1) & -\omega_1 \sin(t\omega_1) \end{bmatrix}\Bigg|_{t=0} \\
&= R_2^1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_1 \\ 0 & \omega_1 & 0 \end{bmatrix}
\end{aligned}
$$

Now lets assume that $\mathcal{F}^2$ is rotating about each of it's basis vectors at angular rates ω_i, $i \in \{1,2,3\}$, and that $\mathcal{F}^3$ is the orientation of $\mathcal{F}^2$ after a time period t. We can use the same process as above to calculate the derivative of R_2^1 in the direction of R_3^2 where R_3^2 can be properly represented as

$$
\begin{aligned}
R_3^2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t\omega_1) & -\sin(t\omega_1) \\ 0 & \sin(t\omega_1) & \cos(t\omega_1) \end{bmatrix}\begin{bmatrix} \cos(t\omega_2) & 0 & \sin(t\omega_2) \\ 0 & 1 & 0 \\ -\sin(t\omega_2) & 0 & \cos(t\omega_2) \end{bmatrix}\begin{bmatrix} \cos(t\omega_3) & -\sin(t\omega_3) & 0 \\ \sin(t\omega_3) & \cos(t\omega_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\omega_2} c_{\omega_3} & -c_{\omega_2} s_{\omega_3} & s_{\omega_2} \\ s_{\omega_1} s_{\omega_2} c_{\omega_3} + c_{\omega_1} s_{\omega_3} & -s_{\omega_1} s_{\omega_2} s_{\omega_3} + c_{\omega_1} c_{\omega_3} & -s_{\omega_1} c_{\omega_2} \\ -c_{\omega_1} s_{\omega_2} c_{\omega_1} + s_{\omega_1} s_{\omega_3} & c_{\omega_1} s_{\omega_2} s_{\omega_3} + s_{\omega_1} c_{\omega_3} & c_{\omega_1} c_{\omega_2} \end{bmatrix}
\end{aligned}
$$

where $s_{\omega_i}$and c_{ω_i} denote $\sin(t\omega_i)$ and $\cos(t\omega_i)$ for $i \in \{1,2,3\}$. Note that the order of concatenation used to compose R_3^2 doesn't matter since we will be taking the limit as $t \to 0$; thus we are interested in the infinitesimal rotations about each basis vector. Now taking the derivative of R_3^1 is

$$
\begin{aligned}
\frac{d}{dt} R_2^1 &= \frac{d}{dt}\left(R_2^1 R_3^2\right)\Big|_{t=0} \\
&= R_2^1 \left(\frac{d}{dt} R_3^2\right)\Bigg|_{t=0} \\
&= R_2^1 \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \\
&= R_2^1 \left[\omega\right]_\times
\end{aligned}
$$

where $[\omega]_\times$ denotes the skew symmetric matrix of the instantaneous angular velocities.

We can verify that the derivative needs to be a skew symmetric matrix by taking the derivative of the identity

$$
\begin{aligned}
\frac{d}{dt} R^\top R &= I \\
\left(\frac{d}{dt} R^\top\right) R &= -R^\top \left(\frac{d}{dt} R\right) \\
[\omega]_\times^\top R^\top R &= -R^\top R \,[\omega]_\times \\
[\omega]_\times^\top &= -[\omega]_\times
\end{aligned}
$$

The derivative of the forward transformation can also be derived using the exponential matrix. Let

$$
R(t) = R_0 \exp\left(t\,[\omega]_\times\right)
$$

be a forward transformation where R_0 denotes the transformation at $t = 0$, and$[\omega]_\times$ denotes skew symmetric matrix representing the angular velocities about each bases of $\mathcal{F}^2$. The derivative of the forward transformation w.r.t. time is

$$\begin{aligned}\frac{d}{dt}R\left(t\right) &= \frac{d}{dt}\;R_0 \exp\left(t\left[\omega\right]_\times\right)\big|_{t=0} \\ &= R_0\left(\frac{d}{dt}\exp\left(t\left[\omega\right]_\times\right)\right)\Bigg|_{t=0} \\ &= R_0\left(\frac{d}{dt}\sum_{k=0}^{\infty}\left(t\left[\omega\right]_\times\right)^k\right)\Bigg|_{t=0} \\ &= R_0\left(\sum_{k=1}^{\infty}t^{k-1}\left(\left[\omega\right]_\times\right)^k\right)\Bigg|_{t=0} \\ &= R_0\left[\omega\right]_\times\end{aligned}$$

It can be easily verified that the derivative of the backward transformation is

$$\begin{aligned}\frac{d}{dt}R^\top &= \left[\omega\right]_\times^\top R^\top \\ &= -\left[\omega\right]_\times R^\top.\end{aligned}$$

VI. Common Notation

A part of what makes rotation matrices so confusing to me, is that when I see them in literature all I see is the letter R that denotes a rotation. If all I see is this R, I have no idea how it is defined. However, I have noticed a general trend with rotation matrix notation with mechanical systems which I will share here. In this section I will not define rotation matrices using the terminology of forward and backward transformation since you will probably never see them defined this way.

A. Simple Example

When modeling a mechanical system you always need to have an inertial reference frame which I will denote as $\mathcal{F}^i = (O_i, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$. You always have an inertial reference frame since it never moves, and it's the frame in which Newtonian mechanics is valid, i.e., there are no pseudo forces. All other frames that are part of the mechanical system are generally placed at the center of mass of a rigid body. If a rigid body is not connected to any other rigid body, or if it's position or orientation is not determined by any other body, then the rotation matrix typically expresses the rigid body's basis in $\mathcal{F}^i$. If a rigid body b is connected to another rigid body a that affects it's orientation or position, then the corresponding rotation matrix expresses the basis of the frame associated with body b in the frame associated with body a.

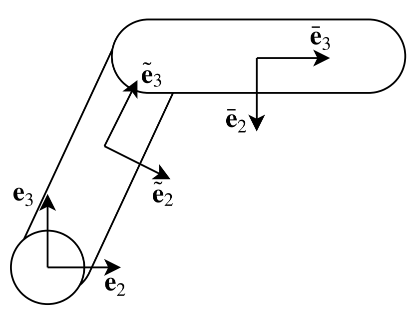


Figure 23

For example consider the simple two-link robot arm depicted in figure (23). The figure shows three frames. One frame is the inertial frame $\mathcal{F}^i = (O_i, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$ and is positioned at the base of the first link. The other frames are placed at the center of mass of each link and are denoted $\mathcal{F}^1 = (O_1, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\})$ and $\mathcal{F}^2 = (O_2, \{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\})$.

The first basis vector of each frame is pointing out of the figure so that each frame is right handed, and the third basis vector of each body frame points along the longer portion of the body.

Since there are three frames, there are three rotations that can be constructed. Note that the first rigid body's position and orientation is not affected by any other rigid bodies. Using common notation, we will define the first rotation R_1^i as the rotation that expresses the basis of $\mathcal{F}^1$ in $\mathcal{F}^i$. The second rigid body's position and orientation is affected by the first rigid body. Because of this, we will define the second rotation R_2^1 as the rotation that expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^1$. The final rotation is defined as $R_2^i = R_1^i R_2^1$ which expresses the basis of $\mathcal{F}^2$ in $\mathcal{F}^i$. In a way, you typically define a set of rotations so that any frame can be expressed in the inertial frame. This is not always the case, so be careful.

B. *Body-Centric vs Inertial Representation*

The body centric representation expresses the basis of a body frame $\mathcal{F}^b$ w.r.t. the basis of an inertial frame $\mathcal{F}^i$, and the inertial representation expresses the basis of $\mathcal{F}^i$ in $\mathcal{F}^b$; thus they are inverses of each other.

VII. Summary

We are finally at the end of our little journey. Before you leave, let's review the main points. Vectors are objects that are invariant to change of basis, and can be expresses w.r.t. any basis using components, and the components of a vector transform contravariantly to the basis when undergoing a change of basis. An active transformation denotes a rotation of a vector w.r.t. the same basis, and a passive transformation changes the components of a vector to express the vector w.r.t. another basis.

A transformation between bases is a transformation that expresses one basis w.r.t. another basis. The body centric representation expresses the basis of a body frame $\mathcal{F}^b$ w.r.t. the basis of an inertial frame $\mathcal{F}^i$, and the inertial representation expresses the basis of $\mathcal{F}^i$ in $\mathcal{F}^b$; thus they are inverses of each other. Lastly, every transformation was derived from geometric principles, and we only used coordinates when needed to take derivatives.

If you have any suggestions for this document, please let me know.

References

[1] Todd K Moon and Wynn C. Stirling. *Mathematical Methods and Algorithms for Signal Processing*. Prentice-Hall, Uppder Saddle River, New Jersey, 2000.