Robot Localization on $SE(2)$

Autonomous System's Project

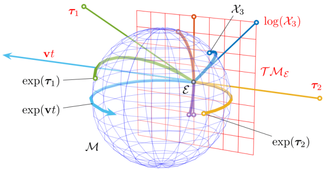
Jake Johnson and Mark Petersen

Brigham Young University

12/11/2019

Table of Contents

Lie Algebra



$$\mathcal{X} \in SE(2) \triangleq \begin{bmatrix} R_{2\times 2} & t_{2\times 1} \\ 0_{1\times 2} & 1_{1\times 1} \end{bmatrix}$$

$$\tau^\wedge \in \mathfrak{se}(2) \triangleq \begin{bmatrix} [\omega]_\times & \rho \\ 0 & 0 \end{bmatrix}$$

$$\tau \in \mathbb{R}^3 \triangleq \begin{bmatrix} \rho \\ \omega \end{bmatrix}$$

$$\wedge : \mathbb{R}^3 \to \mathfrak{se}(2) \triangleq (\tau)^\wedge$$

$$\vee : \mathfrak{se}(2) \to \mathbb{R}^3 \triangleq \left(\tau^\wedge\right)^\vee$$

$$\exp : \mathfrak{se}(2) \to SE(2) \triangleq \exp\left(\tau^\wedge\right)$$

$$\log : SE(2) \to \mathfrak{se}(2) \triangleq \log(\mathcal{X})$$

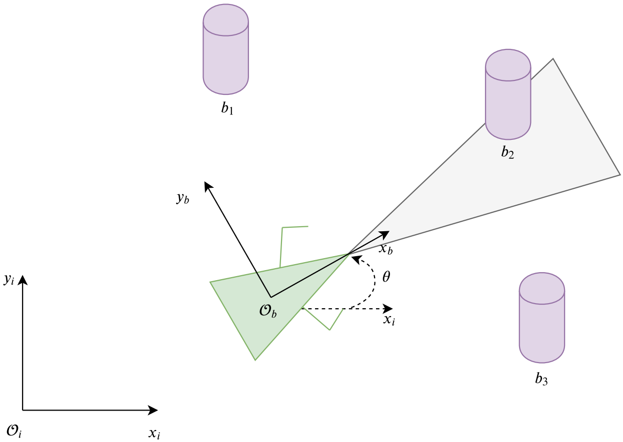
More Operators and Matrix Adjoint

$$
\begin{aligned}
\boxplus : SE(2) \times \mathbb{R}^3 \to \mathbb{R}^3; &\quad \mathcal{X}_1 = \mathcal{X}_2 \boxplus \tau \triangleq \mathcal{X}_2 \exp\left(\tau^\wedge\right) \in SE(2) \\
\boxminus : SE(2) \times SE(2) \to \mathbb{R}^3; &\quad \tau = \mathcal{X}_1 \boxminus \mathcal{X}_2 \triangleq \log\left(\mathcal{X}_2^{-1}\mathcal{X}_1\right)^\vee \in \mathbb{R}^3 \\
\oplus : G \times \mathfrak{g} \to \mathfrak{g}; &\quad \mathcal{X}_1 = \mathcal{X}_2 \oplus \tau^\wedge \triangleq \mathcal{X}_2 \exp\left(\tau^\wedge\right) \in SE(2) \\
\ominus : G \times G \to \mathfrak{g}; &\quad \mathcal{X}_1 \ominus \mathcal{X}_2 \triangleq \log\left(\mathcal{X}_2^{-1}\mathcal{X}_1\right) \in \mathfrak{se}(2)
\end{aligned}
$$

The matrix adjoint at $\mathcal{X}$ is

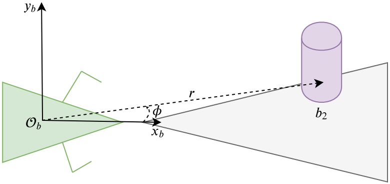
$$
\mathbf{Ad}_{\mathcal{X}} = \begin{bmatrix} R & -[1]_\times P \\ 0 & 1 \end{bmatrix}
$$

Project Description



- $B_j \triangleq \left[x^i_{j/i}, y^i_{j/i}\right]^\top$
- $x = \left[x^i_{r/i}, y^i_{r/i}, \theta^i_{r/i}\right]^\top$

Sensor Measurements



Real Measurements

- $r \sim \mathcal{N}\left(0, \sigma_r^2\right)$
- $\phi \sim \mathcal{N}\left(0, \sigma_\phi^2\right)$

Virtual Measurements

$$\begin{bmatrix} \tilde{x}^b_{j/b} \\ \tilde{y}^b_{j/b} \end{bmatrix} = \begin{bmatrix} r\cos\left(\phi\right) \\ r\sin\left(\phi\right) \end{bmatrix} \sim \mathcal{N}\left(0, Q\right)$$

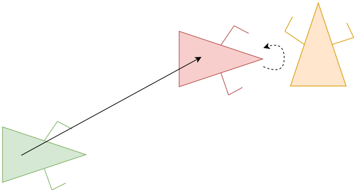
$$Q = M \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix} M^\top$$

$$M = \begin{bmatrix} \cos\left(\phi\right) & -r\sin\left(\phi\right) \\ \sin\left(\phi\right) & r\cos\left(\phi\right) \end{bmatrix}$$

Beacon

$$b^b_{j/b} = \left[\tilde{x}^b_{j/b}, \tilde{y}^b_{j/b}\right]^\top$$

Robot Kinematics



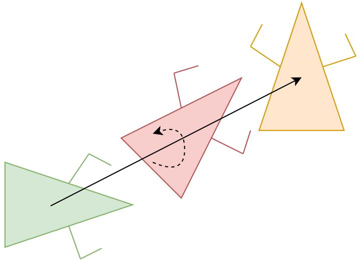
First Method: Independent rotation and translation

$$x_k = x_{k^-} + \Delta t \dot{x}_k$$

where

$$x = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Robot Kinematics Cont



Second Method: Simultaneous rotation and translation
Let $\mathcal{X} \in SE(2)$ then

$$\mathcal{X} \triangleq \begin{bmatrix} R_{2\times 2} & t_{2\times 1} \\ 0_{1\times 2} & 1_{1\times 1} \end{bmatrix}$$

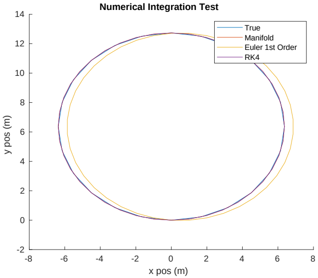
The derivative of $\mathcal{X}$ is

$$\dot{\mathcal{X}} = \mathcal{X} \underbrace{\begin{bmatrix} [\omega]_\times & \rho \\ 0 & 0 \end{bmatrix}}_{\tau^\wedge \in \mathfrak{se}(2)}$$

so that

$$\mathcal{X}_k = \mathcal{X}_{k^-} \exp\left(\delta\tau^\wedge\right)$$

Numerical Integration Comparison



The velocity in the body frame

$$\tau = \begin{bmatrix} 10 \\ 0 \\ \frac{\pi}{2} \end{bmatrix}$$

with time step $\delta = 0.1s$.

- Manifold:
  $\mathcal{X}_k = \mathcal{X}_{k-} \exp\left(\delta\tau^\wedge\right)$
- Euler First Order:
  $x_k = x_{k-} + \delta\dot{x}$
- Runge-Kutta 4:
  $x_k = x_{k-} + RK4\left(x_{k-}, \tau\right)$

Error State

The error state is a minimal representation used to show uncertainty.

- $\mathcal{X}$: True state
- $\hat{\mathcal{X}}$: Estimated state
- $\delta\mathcal{X}$: Error

The error is defined as

$$\delta\mathcal{X} = \mathcal{X} \boxminus \hat{\mathcal{X}} \sim \mathcal{N}(0, P)$$

Uncertainty Representation Euler

Euler Propagation

$$\dot{x} = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} + w_k, \; w_k \sim \mathcal{N}(0, Q)$$

$$x_k = x_{k^-} + \delta \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} + \delta w_k$$

$$E[x_k] = x_{k^-} + \delta \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$

$$\operatorname{cov}(x_k) = P_{k^-} + \delta^2 Q$$

Uncertainty Representation Manifold

Manifold Propagation

$$
\begin{aligned}
\delta\mathcal{X}_k &= F\delta\mathcal{X}_{k^-} + G w_k \\
\mathrm{E}\left[\delta\mathcal{X}_k\right] &= 0 \\
\mathrm{cov}(\delta\mathcal{X}_k) &= F P_{k^-} F^\top + G W G^\top
\end{aligned}
$$

where

$$
F = \mathbf{Ad}^{-1}_{\exp u_k} \quad G = J_r\left(u_k\right)
$$

Banana Distribution

https://www.youtube.com/watch?v=0WjAjHdGP78&list=PLj6HahJg1Z-y5etl0aWNjL0o51IpdFWkE&index=4

afds