Exponential and Log Maps of Matrix Lie Groups

I. Introduction

In this document we will quickly explore groups, manifolds, the exponential maps, and logarithmic maps in order to see how they all fit together. We will begin with a broad view of each concept, but then focus our attention on the special orthogonal groups and the special Euclidean groups of 2,3-dimensions and the exponential and log maps for each group. This is not intended to be extensive or thorough by any means. For more information on any of the topics, I refer you to [1], [2], [6]

II. Matrix Lie Groups

To understand what a matrix Lie group is, we need a basic understanding of groups and manifolds.

A. Group

Definition 1. A **group** G is a set of elements combined with an operation $\bullet$ with the following properties

1) Closure
   a) For all $g_1, g_2 \in G$, the result of the operation $g_1 \bullet g_2 \in G$.
2) Associativity
   a) For all g_1, g_2 and g_3 in G, $(g_1 \bullet g_2) \bullet g_3 = g_1 \bullet (g_2 \bullet g_3)$.
3) Identity Element
   a) There exists an element $\mathbf{1} \in G$ that for all $g \in G$, the equation $\mathbf{1} \bullet g = g \bullet \mathbf{1} = g$ holds.
4) Inverse Element
   a) For each $g_1 \in G$ there exists an element $g_2 \in G$, such that $g_1 \bullet g_2 = g_2 \bullet g_1 = \mathbf{1}$, where $\mathbf{1}$ is the identity element.

The overarching matrix group is the **general linear group**. It is the set of all $n \times n$ invertible matrices together with the group operation of matrix multiplication and is denoted as $GL(n)$. All other matrix groups is a subgroup of a general linear group. There are four groups of interest that we will consider in this document:

1) Special orthogonal group of 2-dimensions $SO(2) := \left\{R \in \mathbb{R}^{2\times 2} | R^\top R = \mathbf{1} = I, \det(R) = 1\right\}$
2) Special orthogonal group of 3-dimensions $SO(3) := \left\{R \in \mathbb{R}^{3\times 3} | R^\top R = \mathbf{1} = I, \det(R) = 1\right\}$
3) Special Euclidean group of 2/3-dimesnions is the group of (Euclidean) isometries of an Euclidean space $\mathbb{E}^n$ comprised of all translations and rotations. It is isomorphic to the space $SO(n) \times \mathbb{R}^n$.

B. Manifold

A manifold M is a topological space S together with an atlas $\mathcal{A}$ whose charts $(\mathcal{U}_i, \phi_i), i \in \mathbb{R}^+$ are compatible and provide a bijective map $\phi_i : \mathcal{U}_i \in S \to \mathbb{E}^n$. Forgive me for the imprecise definition I'm about to give. If every map ϕ_i is a C^r- diffeomorphism where $r \in \{\mathbb{R}^+, \infty, \omega\}$, then M is a C^rdifferentiable manifold. In very simple terms, a manifold is a space that locally resembles Euclidean space. Because of this, the subsets $\mathcal{U}_i \in S$ can be mapped to Euclidean space. And if M is a differentiable manifold, then I can perform calculus in Euclidean space and map them to the tangent bundle of the manifold.

For a precise definition of a manifold, see [1].

C. Matrix Lie Group

A **matrix Lie group** is a matrix group that has the additional structure of a smooth manifold, i.e., there are smooth diffeomorphic maps from the matrix Lie group to Euclidean space.

III. Exponential and Log Maps

The matrix exponential is a surjective map whose inverse is the matrix logarithmic map. By restricting the domain of the matrix exponential to a subset, the mapping becomes locally diffeomorphic, and we can use it for our purposes. The log map is used to map elements of a matrix Lie group to Euclidean space, and the exponential map performs the inverse.

A. General Matrix Exponential

Let $A \in \mathbb{R}^{n\times n}$ be an $n \times n$ matrix with real components, then the matrix exponential is the map $\exp : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined using the Taylor series as

$$\exp\left(A\right) \coloneqq \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{1}$$

which is guaranteed to converge for all real, square matrices [6]. This means that the definition of the matrix exponential in 1works on the entire domain of real $n \times n$ matrices. There are subspaces of $\mathbb{R}^{n\times n}$ whose matrix exponential can have a simplified expression that's derived from the original definition. We will discuss these later on.

B. General Logarithmic Map

A matrix $X \in \mathbb{C}^{n\times n}$ is a logarithm of $A \in \mathbb{C}^{n\times n}$ if $\exp\left(X\right) = A$ [5], and the logarithmic map is the map $\log : \mathbb{C}^{n\times n} \to \{X \in \mathbb{C}^{n\times n} | \mathbb{C}^{n\times n} \ni A = \exp\left(X\right)\}$. There are many definitions of this map, but we will only discuss two of them in this subsection. The first one is the map defined by the classical infinite series or power series and is defined as

$$\log\left(A\right) \coloneqq \sum_{k=1}^{\infty} \left(-1\right)^{k+1} \frac{\left(A - I\right)^k}{k} \tag{2}$$

which is only guaranteed to converges if $\|A - I\| < 1$ [6]. This specific definition of the matrix log has a very restrictive convergence criteria that greatly reduces the domain.

The next definition of the logarithmic map is known as the principle matrix logarithm and is well discussed in [5]. We will briefly present the algorithm here. "For $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$, the principal logarithm is the unique logarithm whose eigenvalues have imaginary parts lying in the interval $(-\pi, \pi)$"[5]. The principle algorithm is found by first computing $A^{\frac{1}{2^s}}$, for an integer s large enough so that $A^{\frac{1}{2^s}}$ is close to the identity. Then $\log\left(A^{\frac{1}{2^s}}\right)$ is approximated by $r_m\left(A^{\frac{1}{2^s}} - I\right)$, where r_m is an m/m Pade approximate to the function $\log\left(1 + x\right)$, and finally we can form the approximation

$$\log\left(A\right) \approx 2^s r_m\left(A^{\frac{1}{2^s}} - I\right) \tag{3}$$

. This approximation exploits the identity

$$\log\left(A\right) = 2^s \log\left(A^{\frac{1}{2^s}}\right).$$

The principle matrix logarithm (PML) is a great improvement to the power series logarithm since it is a locally diffeomorphic map whose domain is $\{A \in \mathbb{C}^{n\times n} | -1 \notin \{\text{eig}\left(A\right)\}\}$ and co-domain is $\{X \in \mathbb{C}^{n\times n} | \text{eig}\left(X\right) \in \left(-\pi i, \pi i\right)\}$.

Just like with the matrix exponential, there are subspaces of $\mathbb{C}^{n\times n}$ whose matrix logarithm can have a simplified expression that's derived from the definition of a the matrix exponential. We will discuss these later.

IV. SO(3)

$SO\left(3\right)$ is a matrix Lie group which isn't also a Euclidean space. Therefore it lacks the structure necessary to do calculus, i.e., take derivatives and perform integrations. However, since it is a manifold, we can construct charts $\left(\mathcal{U}_i, \phi_i\right)$ that maps subsets $\mathcal{U}_i \in SO\left(3\right)$ to $\mathbb{E}^3$ using diffeomorphic maps. Once in Euclidean space, we can take derivatives and map these derivatives to the tangent space of $SO\left(3\right)$ using the pull back function.

We will only construct one chart that covers most of $SO\left(3\right)$ because it is the only one ever used in literature. To construct the chart, we will first look at the inverse map ϕ^{-1}, the matrix exponential, since the map ϕ is easily derived from ϕ^{-1}. Next, we will derive ϕ from ϕ^{-1} and define the subsets of $SO\left(3\right)$ and $\mathbb{E}^3$ for which the map is diffeomorphic. Lastly, we will define the pull back function.

A. Matrix Exponential

The matrix exponential is the map ϕ^{-1} defined in subsection III-A for arbitrary square, complex matrices. However, the elements of $SO(3)$ have a unique structure that allows us to simplify the matrix exponential. We will not derive the simplified matrix exponential, but only present it.

Let $U_{\mathfrak{so}(3)} := \left\{X \in \mathbb{R}^{3\times 3} | X^\top + X = 0,\, \|X\| < \pi\right\}$ be a subset of the skew symmetric matrices $\mathfrak{so}(3) := \left\{X \in \mathbb{R}^{3\times 3} | X^\top + X = 0\right\}$. The condition $\|X\| < \pi$ is equivalent to the condition $\mathrm{eig}(X) \in (-\pi i, \pi i)$ for skew symmetric matrices since the eigenvalues of a 3-dimensional skew symmetric matrix are $0, \pm\|X\| i$. Using the subset $U_{\mathfrak{so}(3)}$ we restrict the domain and co-domain of the matrix exponential so that the map $\exp : U_{\mathfrak{so}(3)} \subset \mathfrak{so}(3) \to U_{so(3)} \subset SO(3)$ is locally diffeomorphic. Of course to show that the map is a local diffeomorphism, we need to show that the inverse of the matrix exponential exists and is differentiable in $U_{so(3)}$. The subset $U_{so(3)}$ will be defined in the next subsection.

Since the matrix exponential defined in (1) is well defined for all $A \in \mathbb{C}^{n\times n}$, it is well defined for $U_{\mathfrak{so}(3)}$; however, the unique properties of $\mathfrak{so}(3)$ allows us to use a simplified matrix exponential known as the Rodrigues Formula which we now present. Any given $X \in U_{\mathfrak{so}(3)}$ can be written as $X = \theta\omega$ where $\theta = \|X\|$ and $\omega = \frac{X}{\|X\|}$, and the Rodrigues Formula is

$$\exp := I + \sin(\theta)\,\omega + (1 - \cos(\theta))\,\omega^2. \tag{4}$$

The Rodrigues Formula offers us several benefits. The first is that it is faster than computing an infinite series, and is still a smooth map. The other benefit is that it will help us derive an inverse map that is a simplified version of the matrix logarithm presented in subsection III-B.

B. Matrix Logarithm

The simplified matrix logarithm for $U_{so(3)} \subset SO(3)$ is derived using the Rodrigues formula. This derivation will help us define $U_{so(3)}$ for reasons you will see.

We will follow [4] in this derivation and fill in some of the steps that the author omitted. This derivation is done in several steps. The first step is to define a function $f_1 : U_{so(3)} \to [0, \pi)$ which will give us the angle θ in (4). The second step is to define a function $f_2 : U_{so(3)} \to \left\{X \in \mathbb{R}^{3\times 3} | X^\top + X = 0,\, \|X\| = 1\right\}$ which will give us ω in (4). Once we have θ and ω, we can reconstruct the skew symmetric matrix X.

To derive f_1 we start with the Rodrigues formula

$$R = I + \sin(\theta)\,\omega + (1 - \cos(\theta))\,\omega^2.$$

where $R \in SO(3)$, and take the trace, $Tr(\cdot)$, of R

$$Tr(R) = Tr\left(I + \sin(\theta)\,\omega + (1 - \cos(\theta))\,\omega^2\right).$$

Using the linearity property of the trace we get

$$Tr(R) = Tr(I) + Tr(\sin(\theta)\,\omega) + Tr\left((1 - \cos(\theta))\,\omega^2\right).$$

Since ω is a skew symmetric matrix, $Tr(\sin(\theta)\,\omega) = 0$. Using this property we can reduce the equation to

$$Tr(R) = 3 + Tr\left((1 - \cos(\theta))\,\omega^2\right).$$

Since ω is a skew symmetric matrix with unit norm, $Tr(\omega) = -2$. Using this property, we can reduce the equation to

$$Tr(R) = 1 + 2\cos(\theta).$$

We can then solve for θ

$$\theta = \arccos\left(\frac{Tr(R) - 1}{2}\right),$$

which gives us the map

$$f_1(R) := \arccos\left(\frac{Tr(R) - 1}{2}\right). \tag{5}$$

To derive f_2 we start with the Rodrigues formula

$$R = I + \sin(\theta)\,\omega + (1 - \cos(\theta))\,\omega^2,$$

and add the negative of it's transpose to each side to give us

$$\left(R - R^{\top}\right) = \left(I + \sin(\theta)\,\omega + (1 - \cos(\theta))\,\omega^2\right) - \left(I + \sin(\theta)\,\omega + (1 - \cos(\theta))\,\omega^2\right)^{\top}.$$

Since ω is a skew symmetric matrix, $\omega - \omega^{\top} = 2\omega$, and since ω^2 is a symmetric matrix, $\omega^2 - \left(\omega^2\right)^{\top} = 0$. Using these properties we get

$$\left(R - R^{\top}\right) = 2\sin(\theta)\,\omega. \tag{6}$$

which gives us the first part of the second map

$$f_{2,1}\left(R, f_1(R)\right) := \frac{\left(R - R^{\top}\right)}{2\sin\left(f_1(R)\right)}. \tag{7}$$

The map $f_{2,1}$ is not valid when $\theta = k\pi$, $k \in \mathbb{Z}$. Fortunately there exists a map for when $\theta = 0$. This map is found by substituting $\theta = 0$ into (4) to get

$$\begin{aligned} R &= I + \sin(0)\,\omega + (1 - \cos(0))\,\omega^2 \\ &= I. \end{aligned}$$

which gives us the second part of the map f_2

$$f_{2,1}(R) := I.$$

Note that if $\theta = 0$, then ω must be the zero matrix.

Unfortunately, there is not bijective map for when $\theta = \pm\pi$, only a multivalued map. This should make sense since

$$\exp(\theta\omega) = \exp\left((-\theta)\,\omega\right) = \exp\left(\theta\,(-\omega)\right) = \exp\left((-\theta)\,(-\omega)\right)$$

which means that the Rodrigues Formula is surjective when $\theta = \pm\pi$, and it's inverse must be multivalued. Therefore, we need to identify the subset of $SO(3)$ such that $f_1(R) = \pi$.

This subset is easy to identify using the exponential property that the eigenvalues of a matrix A are the exponential of the eigenvalues of it's logarithm X where $A = \exp(X)$. We recall that $X = \theta\omega$ where has eigenvalues $0, \pm i$; therefore, X has eigenvalues $0, \pm\pi i$, and A must have eigenvalues $1, -1$ and -1. This means that the matrix logarithm we are deriving in this subsection is valid for $U_{so(3)} := \{R \in SO(3)\,|Tr(R) \neq -1\}$, which is the same valid domain for the matrix logarithm defined in subsection III-B.

Now that we have all of the pieces, we can put them together. The matrix logarithm derived from the Rodrigues formula is the map $\log : U_{so(3)} \to U_{\mathfrak{so}(3)}$ defined as

$$\log(R) := \begin{cases} I & R = I \\ f_1(R)\, f_{2,1}\left(R, f_1(R)\right) & R \neq I \end{cases}, \tag{8}$$

or more plainly written

$$\log(R) := \begin{cases} I & R = I \\ f_1(R)\,\frac{\left(R - R^{\top}\right)}{2\sin\left(f_1(R)\right)} & R \neq I \end{cases}.$$

C. Chart

We finally have the everything we need to construct the chart (U, ϕ) where the subset $U = U_{so(3)}$, ϕ is the matrix logarithm defined either by (III-B) or (8), and ϕ^{-1} is the matrix exponential defined either by (III-A) or (4). Since the chart doesn't cover all of $SO(3)$, we do not have an atlas. We could construct other charts, but they are typically not needed.

V. SE(3)

I will not go into a lot of detail with $SE(3)$, but rather only present the matrix exponential and matrix logarithm. For the interested reader, I refer you to [3] for the derivation of each map.

For $R \in SO(3)$ and $t \in \mathbb{R}^3$, and element of $C \in SE(3)$ is the homogenous matrix

$$C = \begin{bmatrix} R & t \\ 0_{1\times 3} & 1 \end{bmatrix},$$

whose inverse is

$$C^{-1} = \begin{bmatrix} R^\top & -R^\top t \\ 0_{1\times 3} & 1 \end{bmatrix}.$$

The set $SE(3)$ forms a matrix Lie group with the operator being defined as matrix multiplication.

$SE(3)$ has a corresponding Lie algebra $\mathfrak{se}(3) := \left\{ \begin{bmatrix} X & \rho \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n} | X^\top + X = 0,\, \rho \in \mathbb{R} \right\}$. There is a chart $\left(U_{SE(3)}, \phi\right)$ with a diffeomorphic map $\phi : U_{SE(3)} \to U_{\mathfrak{se}(3)}$ which we will discuss in this section.

A. Matrix Exponential

The matrix exponential is the map $\exp : \mathfrak{se}(3) \to SE(3)$ and is defined in (III-A). It has a simplified form that is similar to the Rodriguez formula and is defined for $c = \begin{bmatrix} X & \rho \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$ as

$$\exp(c) := \begin{bmatrix} \exp(X) & B(X)\rho \\ 0 & 1 \end{bmatrix},$$

where

$$B(X) := \begin{cases} I & X = 0 \\ I + \left(\frac{1-\cos(\|X\|)}{\|X\|}\right)\frac{X}{\|X\|} + \left(1 - \frac{\sin(\|X\|)}{\|X\|}\right)\frac{X^2}{\|X\|^2} & X \neq 0 \end{cases},$$

and $\exp(X)$ is already defined by (4).

B. Matrix Logarithm

The matrix logarithm is the map $\log : U_{SE(3)} \to U_{\mathfrak{se}(3)}$ where $U_{SE(3)} := \left\{ \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \in SE(3) \,|\, R \in U_{SO(3)},\, t \in \mathbb{R}^3 \right\}$ and $U_{\mathfrak{se}(3)} := \left\{ \begin{bmatrix} X & \rho \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3) \,|\, X \in U_{\mathfrak{so}(3)},\, \rho \in \mathbb{R}^3 \right\}$ and is defined in (3). It has a simplified form defined for $C = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \in U_{SE(3)}$ as

$$\log(C) := \begin{bmatrix} \log(R) & B^{-1}(\log(R))\, t \\ 0 & 0 \end{bmatrix},$$

where

$$B^{-1}(X) := \begin{cases} I & X = 0 \\ I - \frac{1}{2}X + \left(1 - \frac{\|X\|}{2}\cot\left(\frac{\|X\|}{2}\right)\right)\frac{X^2}{\|X\|^2} & X \neq 0 \end{cases},$$

and $\log(R)$ is already defined by (8).

References

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