Integration On Manifolds Example

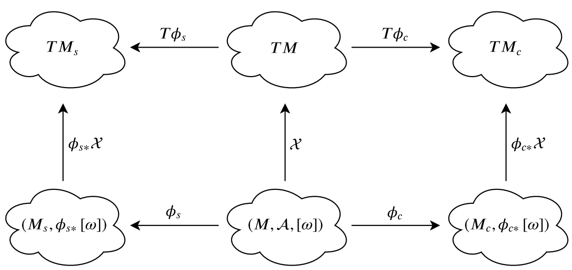

Figure 1.

I. Introduction

The purpose of this note is to show an example of integration on manifolds. Let $(M, \mathcal{A}, [\omega])$ be an oriented manifold where $M := \left\{(m_1, m_2, m_3) \in \mathbb{R}^3\right\}$, $\mathcal{A}$ is an atlas and $[\omega]$ an equivalence class of volume form with positive orientation. The atlas $\mathcal{A}$ contains the charts (U_s, ϕ_s) and (U_c, ϕ_c) where $U_s, U_c = M$, $\phi_s : M \to M_s$ and $\phi_c : M \to M_c$. M_s denotes $\mathbb{R}^3$ in spherical coordinates and M_c denotes $\mathbb{R}^3$ in Cartesian coordinates. The spaces TM, TM_s, TM_c are the associated tangent spaces of M, M_s, M_c. The maps $T\phi_s : TM \to TM_s$ and $T\phi_c : TM \to TM_c$ are called the tangent of ϕ_s and ϕ_c. The vector fields on M are denoted by $\mathcal{X}$, whose push forward is denoted $\phi_{s*}\mathcal{X}$ and $\phi_{c*}\mathcal{X}$. Lastly the push forward of the equivalence class of positive volume form $[\omega]$ is denoted $\phi_{s*}[\omega]$ and $\phi_{c*}[\omega]$. This scenario is depicted in fig 1.

II. Atlas

In this section we proceed to define the charts (U_s, ϕ_s) and (U_c, ϕ_c). In order to integrate on manifolds, the charts need to be diffeomorphic so we can construct $T\phi_s$ and $T\phi_c$, and the charts need to be positively oriented for integration to be well defined[1].

Recall that the tangent space of a manifold consists of all of the points on the manifold and all of the vectors tangent at every point. Since $\dim(M) = 3$ the tangents space will have dimension 6 such that $TM = \mathbb{R}^3 \times \mathbb{R}^3$. Every element in TM is a tuple of the form $(m, \dot{m})$ where $m \in M$ and $\dot{m}$ is a tangent vector at point m. The tangent maps $T\phi_s$ and $T\phi_c$ are tuples of the form $(\phi_s, \mathbf{d}\phi_s)$ and $(\phi_c, \mathbf{d}\phi_c)$ where the boldface $\mathbf{d}$ denotes the differential of the map, and is the part that maps tangent vectors.

A. First Chart

Let the first chart (U_c, ϕ_c) have the coordinates $(x(M), y(M), z(M))$ where

$$\begin{aligned} x &: M \to \mathbb{R};\ (m_1, m_2, m_3) \mapsto m_1 \\ y &: M \to \mathbb{R};\ (m_1, m_2, m_3) \mapsto m_2 \\ z &: M \to \mathbb{R};\ (m_1, m_2, m_3) \mapsto m_3 \end{aligned}$$

which we will simply denote as (x, y, z) with the mappings implied. The inverse mapping $\phi_c^{-1} : M_c \to M$ is

$$\begin{aligned} m_1 &: M_c \to \mathbb{R};\ (x, y, z) \mapsto x \\ m_2 &: M_c \to \mathbb{R};\ (x, y, z) \mapsto y \\ m_3 &: M_c \to \mathbb{R};\ (x, y, z) \mapsto z \end{aligned}$$

The tangent map $T\phi_c : TM \to TM_c$ is the tuple $(\phi_c, \mathbf{d}\phi_c)$ where $\mathbf{d}\phi_c$ is the differential of ϕ_c which is calculated by taking the partial derivative of ϕ_c w.r.t. m_1, m_2, and m_3.

$$\begin{aligned} dx &= \frac{\partial x}{\partial m_1} dm_1 + \frac{\partial x}{\partial m_2} dm_2 + \frac{\partial x}{\partial m_3} dm_3 \\ &= dm_1 \\ dy &= \frac{\partial y}{\partial m_1} dm_1 + \frac{\partial y}{\partial m_2} dm_2 + \frac{\partial y}{\partial m_3} dm_3 \\ &= dm_2 \\ dz &= \frac{\partial z}{\partial m_1} dm_1 + \frac{\partial z}{\partial m_2} dm_2 + \frac{\partial z}{\partial m_3} dm_3 \\ &= dm_3 \end{aligned}$$

B. Second Chart

Let the second chart (U_s, ϕ_c) have the coordinates $(r(M), \theta(M), \phi(M))$ where

$$r : M \to \mathbb{R};\ (m_1, m_2, m_3) \mapsto \sqrt{m_1^2 + m_2^2 + m_3^2}$$

$$\theta : M \to \mathbb{R};\ (m_1, m_2, m_3) \mapsto \arccos\left(\frac{m_3}{\sqrt{m_1^2 + m_2^2 + m_3^2}}\right)$$

$$\phi : M \to \mathbb{R};\ (m_1, m_2, m_3) \mapsto \arctan\left(\frac{m_2}{m_1}\right)$$

which we will simply denote as (r, θ, ϕ). The inverse mapping $\phi_s^{-1} : M_s \to M$ is

$$\begin{aligned} m_1 &: M_s \to \mathbb{R};\ (r, \theta, \phi) \mapsto r\sin(\phi)\cos(\theta) \\ m_2 &: M_s \to \mathbb{R};\ (r, \theta, \phi) \mapsto r\sin(\phi)\sin(\theta) \\ m_3 &: M_s \to \mathbb{R};\ (r, \theta, \phi) \mapsto r\cos(\phi) \end{aligned}$$

The tangent map $T\phi_s : TM \to TM_c$ is the tuple $(\phi_s, \mathbf{d}\phi_s)$ where $\mathbf{d}\phi_s$ is the differential of ϕ_s which is calculated by taking the partial derivative of ϕ_s w.r.t. m_1, m_2, and m_3.

$$\begin{aligned}
dr &= \frac{\partial r}{\partial m_1} dm_1 + \frac{\partial r}{\partial m_2} dm_2 + \frac{\partial r}{\partial m_3} dm_3 \\
&= \frac{m_1}{r} dm_1 + \frac{m_2}{r} dm_2 + \frac{m_3}{r} dm_3 \\
d\theta &= \frac{\partial \theta}{\partial m_1} dm_1 + \frac{\partial \theta}{\partial m_2} dm_2 + \frac{\partial \theta}{\partial m_3} dm_3 \\
&= \frac{m_1 m_3}{r\sqrt{m_1^2 + m_2^2}} dm_1 + \frac{m_2 m_3}{r\sqrt{m_1^2 + m_2^2}} dm_2 - \frac{\sqrt{m_1^2 + m_2^2}}{r^2} dm_3 \\
d\phi &= \frac{\partial \phi}{\partial m_1} dm_1 + \frac{\partial \phi}{\partial m_2} dm_2 + \frac{\partial \phi}{\partial m_3} dm_3 \\
&= \frac{m_2}{m_1^2 + m_2^2} dm_1 - \frac{m_1}{m_1^2 + m_2^2} dm_2
\end{aligned}$$

The inverse tangent map $T\phi_s^{-1} : TM_c \to TM$ is the tuple $\left(\phi_s^{-1}, \mathbf{d}\phi_s^{-1}\right)$ where $\mathbf{d}\phi_s^{-1}$ is the differential of ϕ_s^{-1}which is calculated by taking the partial derivative of ϕ_s^{-1} w.r.t. r, θ, and ϕ.

$$\begin{aligned}
dm_1 &= \frac{\partial m_1}{\partial r} dr + \frac{\partial m_1}{\partial \theta} d\theta + \frac{\partial m_1}{\partial \phi} d\phi \\
&= \cos\theta \sin\phi dr - r \sin\theta \sin\phi d\theta + r\cos\theta\cos\phi d\phi \\
dm_2 &= \frac{\partial m_2}{\partial r} dr + \frac{\partial m_2}{\partial \theta} d\theta + \frac{\partial m_2}{\partial \phi} d\phi \\
&= \sin\theta \sin\phi dr + r\cos\theta \sin\phi d\theta + r\cos\theta\cos\phi d\phi \\
dm_3 &= \frac{\partial m_3}{\partial r} dr + \frac{\partial m_3}{\partial \theta} d\theta + \frac{\partial m_3}{\partial \phi} d\phi \\
&= \cos\phi dr - r\sin\phi d\phi
\end{aligned}$$

C. *Representative Chart*

The charts on a manifold by definition must be bijective. This allows us to create maps between M_c and M_s over the intersection of $\phi_c^{-1}(M_c) \cap \phi_s^{-1}(M_s)$. In this example that intersection is all of M, and the two maps are defined as

$$\begin{aligned}
&\phi_s \circ \phi_c^{-1} : M_c \to M_s \\
&\qquad (x, y, z) \mapsto \left(r, \arccos\left(\frac{z^2}{r}\right), \arctan\left(\frac{y}{z}\right)\right)
\end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and

$$\begin{aligned}
&\phi_c \circ \phi_s^{-1} : M_s \to M_c \\
&\qquad (r, \theta, \phi) \mapsto (r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\phi)).
\end{aligned}$$

The corresponding tangent maps are defined as

$$\begin{aligned}
\left(\phi_s \circ \phi_c^{-1}, \mathbf{d}\phi_s \circ \mathbf{d}\phi_c^{-1}\right) &: TM_c \to TM_s \\
\left(\phi_c \circ \phi_s^{-1}, \mathbf{d}\phi_c \circ \mathbf{d}\phi_s^{-1}\right) &: TM_s \to TM_c
\end{aligned}$$

where $\mathbf{d}\phi_s \circ \mathbf{d}\phi_c^{-1}$ in matrix form is

$$\begin{bmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r\sqrt{x^2+y^2}} & \frac{yz}{r\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{r^2} \\ \frac{x}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 \end{bmatrix},$$

and $\mathbf{d}\phi_c \circ \mathbf{d}\phi_s^{-1}$ in matrix form is

$$\begin{bmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\cos\theta\cos\phi \\ \cos\phi & & -r\sin\phi \end{bmatrix}.$$

Remark 1. The reason why the charts (U_s, ϕ_s) and (U_c, ϕ_c) had to be diffeomorphism was to enable us to find the tangent maps.

Remark 2. The differential of a function ϕ is related to the gradient of the function by $\nabla\phi = \mathbb{G}^\sharp(\mathbf{d}\phi)$ where $\mathbb{G}$ is the metric tensor. This is an important subtle difference.

III. Push-forward

Let the volume form ω on M be the standard volume form defined as $\omega := dm_1 \wedge dm_2 \wedge dm_3$. In order to integrate on M_c and M_s we need to push forward the volume form ω from M to M_c and M_s. This is really easy since we have the tangent maps $T\phi_s$ and $T\phi_c$.

$$\phi_{s*}\omega := \omega \circ T\phi_s^{-1}$$
$$\phi_{c*}\omega := \omega \circ T\phi_c^{-1}.$$

What these maps are showing is that if we have tangent vectors in TM_s and TM_c we can map them back to TM using the tangent maps and then use the volume form ω that is defined on M.

IV. Integration

Definition 3. Let $f \in C^\infty(M, \mathbb{R})$ then we call $\int_M f\omega$ the **integral of f with respect to ω[1].**

Theorem 4. ***Riesz Representation Theorem*** *Let (M, ω) be a volume manifold. Let $\mathcal{B}$ denote the Borel sets of M, the $\sigma - algebra$ generated by the open (or closed, or compact) subsets of M. Then there is a unique measure m_u on $\mathcal{B}$ such that for every continuous function of compact support*

$$\int_M f dm_\omega = \int_M f\omega.$$

This is saying that a volume form gives rise to a measure w.r.t. which functions can be integrated by the appropriate Legesgue integral. Honestly I don't fully understand this but according to [2] we have (And I copy this right from the book)

$$\int_D f(p)\,\nu(dp) = \int_{Dx} f(p)\,\nu\left(\frac{\partial p}{\partial x}\right) dx.$$

On the right side $p = p(x)$ is considered a function of x, D_x is the point-set in the space of coordinates x corresponding to $D \subset M$. We have to show that the value of the integral is independent of the coordinate system x. So suppose $\tilde{x}$ is another coordinate system defined on D. Since in matrix notation $\frac{\partial p}{\partial \tilde{x}} = \frac{\partial x}{\partial \tilde{x}} \frac{\partial p}{\partial x}$, one finds

$$\int_{D\tilde{x}} f(p)\,\nu\left(\frac{\partial p}{\partial \tilde{x}}\right) d\tilde{x} = \int_{D\tilde{x}} f(p)\,\nu\left(\frac{\partial p}{\partial x}\right) \left|\det \frac{\partial x}{\partial \tilde{x}}\right| d\tilde{x} = \int_{Dx} f(p)\,\nu\left(\frac{\partial p}{\partial x}\right) dx.$$

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Remark 5. The function ν is the volume form that we have denoted as ω.

A. Application

Let's apply this concept of integration to our problem. Lets start with the integral

$$\int_U f(m)\,\omega(dm)$$

where $m \in M$, f is a continuous function on M, U is the region of integration, and dm are the unit basis vectors of the tangent space at point m.

Let's push the integral onto M_c using the push forward. This is done by

$$\int_{\phi_c(U)} f\left(\phi^{-1}(p_c)\right) \omega\left(T\phi_c^{-1} dp_c\right) dc$$

where $U_c = \phi_c(U)$ is the image of U mapped by ϕ_c, $p_c \in M_c$, dp_c are the unit basis vectors of the tangent space at p_c, and dc is just a symbol denoting the small area of integration and should not be confused with the differential. The part that is most interesting to us is

$$\omega\left(T\phi_c^{-1} dp_c\right)$$

Since $T\phi_c^{-1}$ is the identity function, $\omega\left(T\phi_c^{-1} dp_c\right)$ evaluates to 1. I will stop here because I'm confused.

Remark 6. If f and ω have compact support, then $\phi_{s*}f$, $\phi_{s*}\omega$, $\phi_{c*}f$, and $\phi_{c*}\omega$ have compact support.

References

[1] Ralph Abraham, Jerrold Mardsen, and Tudor Ratiu. *Manifolds, Tensor Analysis, and Applications*. Springer-Verlag, New York, first edition, 1998.

[2] Wulf Rossmann. *Lie Groups: An Introduction Through Linear Groups*. Oxford University Press Inc., 2002.