Mark Petersen EcEn 773 HW 2 1

6.2) Compute A^t and e^{At} for the following matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

A_1) $$A_1 A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda^t & t\lambda^{t-1} & 0 \\ 0 & \lambda^t & 0 \\ 0 & 0 & \lambda^t \end{bmatrix} \checkmark$$

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$$

$$\Rightarrow \begin{bmatrix} s-1 & -1 & 0 \\ 0 & s-1 & 0 \\ 0 & 0 & s-1 \end{bmatrix} \quad \text{cofactor} \begin{bmatrix} (s-1)^2 & 0 & 0 \\ (s-1) & (s-1)^2 & 0 \\ 0 & 0 & (s-1)^2 \end{bmatrix} \quad \text{adjugate} = \begin{bmatrix} (s-1)^2 & (s-1) & 0 \\ 0 & (s-1)^2 & 0 \\ 0 & 0 & (s-1)^2 \end{bmatrix}$$

$$\det(sI - A) = (s-1)^3$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} & 0 \\ 0 & \frac{1}{s-1} & 0 \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix} \qquad \mathcal{L}^{-1}(sI - A)^{-1} = e^t \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t) \checkmark$$

A_2) $$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 1 & a_{13}+1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{thus } A^t = \begin{bmatrix} 1 & 1 & t-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$$

$$\Rightarrow \begin{bmatrix} s-1 & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s-1 \end{bmatrix} \quad \text{cofactor} \begin{bmatrix} s(s-1) & 0 & 0 \\ s-1 & (s-1)^2 & 0 \\ 1 & s-1 & s(s-1) \end{bmatrix} \quad \text{adjugate} = \begin{bmatrix} s(s-1) & s-1 & 1 \\ 0 & (s-1)^2 & s-1 \\ 0 & 0 & s(s-1) \end{bmatrix}$$

$$\det(sI - A) = s(s-1)^2$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} & \frac{1}{s(s-1)^2} \\ 0 & \frac{1}{s} & \frac{1}{s(s-1)} \\ 0 & 0 & \frac{1}{(s-1)} \end{bmatrix} \qquad \frac{1}{s(s-1)} = \frac{1}{s} + \frac{-1}{s-1}$$

$$\frac{1}{s(s-1)^2} = \frac{1}{s} + \frac{-1}{(s-1)} + \frac{1}{(s-1)^2}$$

$s^2 - 2s + 1$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s} - \frac{1}{s-1} & \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s} & \frac{1}{s} - \frac{1}{s-1} \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}$$

$$\mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} e^t & 1-e^t & 1-e^t+te^t \\ 0 & 1 & 1-e^t \\ 0 & 0 & e^t \end{bmatrix} U(t)$$

A3) $A_3 = \begin{bmatrix} 2&0&0&0 \\ 2&2&0&0 \\ 0&0&3&3 \\ 0&0&0&3 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$ where $B_1 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ & $B_2 = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$

$A_3^k = \begin{bmatrix} B_1^k & 0 \\ 0 & B_2^k \end{bmatrix}$ & $e^{At} = \begin{bmatrix} e^{B_1 t} & 0 \\ 0 & e^{B_2 t} \end{bmatrix}$

$B_1^2 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 2\cdot 2 & 2^2 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda^k & 0 \\ k\lambda^{(k-1)} & \lambda^k \end{bmatrix} = \begin{bmatrix} 2^k & 0 \\ k\,2^{(k-1)} & 2^k \end{bmatrix}$

$B_2^k = \begin{bmatrix} \lambda^k & k\lambda^{(k-1)} \\ 0 & \lambda^k \end{bmatrix}$ From first part $= \begin{bmatrix} 3^k & k3^{(k-1)} \\ 0 & 3^k \end{bmatrix}$

$A^k = \begin{bmatrix} 2^k & 0 & 0 & 0 \\ k2^{(k-1)} & 2^k & 0 & 0 \\ 0 & 0 & 3^k & k3^{(k-1)} \\ 0 & 0 & 0 & 3^k \end{bmatrix}$ ✓

$e^{At} = \begin{bmatrix} e^{B_1 t} & 0 \\ 0 & e^{B_2 t} \end{bmatrix}$

$B_1 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ $\quad \mathcal{L}^{-1}(sI-B_1) = \mathcal{L}^{-1} \begin{bmatrix} s-2 & 0 \\ 2 & s-2 \end{bmatrix} \frac{1}{(s-2)^2} = \begin{bmatrix} e^{2t} & 0 \\ 2te^{2t} & e^{2t} \end{bmatrix}$

$B_2 = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$ $\quad \mathcal{L}^{-1}(sI-B_2) = \begin{bmatrix} e^{3t} & 3e^{3t}t \\ 0 & e^{3t} \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 2te^{2t} & 0 & 0 & 0 \\ 0 & 0 & e^{3t} & 3te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$

7.2 2nd Edition)

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

a) compute the characteristic & minimum polynomial of A.

$$\det(\lambda I - A) = \begin{vmatrix} s-1 & 1 & 0 \\ 0 & s-2 & 0 \\ 0 & 0 & s-2 \end{vmatrix} = (s-1)(s-2)^2 \Rightarrow (s-1)(s^2-4s+4)$$

$$\Rightarrow s^3 - 4s^2 + 4s - s^2 + 4s - 4 = s^3 - 5s^2 + 8s - 4 \checkmark$$

$$A^3 - 5A^2 + 8A - 4I = 0I \checkmark$$

b) Is this matrix diagonalizable? If so diagonalize it, otherwise compute its Jordan normal form.

$$\lambda = 1 \quad (\lambda I - A) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \quad \begin{matrix} x_2 = 0 \\ x_2 = 0 \\ -x_2 - x_3 = 0 \end{matrix} \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 2 \quad (A - \lambda I) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{matrix} -x_1 + x_2 = 0 \\ x_3 = \text{anything} \\ x_2 = 0 \end{matrix} \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(A - \lambda I) v_3 = v_2 \quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{matrix} -x_1 + x_2 = 0 \quad x_1 = -1 \\ x_3 = \text{anything} \\ x_2 = 1 \end{matrix} \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \checkmark$$

c) Compute e^{At}

$$e^{At} = V e^{Jt} V^{-1} = V \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} V^{-1}$$

7.

8.2) For a given matrix A, construct vectors for which (8.2) holds for each of the three norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$.

(8.2) For every matrix A $\exists$ a vector $x^* \in \mathbb{R}^n$ for which $\|A\|_p = \dfrac{\|Ax^*\|_p}{\|x^*\|_p}$

$\|A\|_2 = \sigma_{max}[A]$ where $\sigma_{max}[A]$ denotes the larges singular value of A

let v_1 correspond to the left eigen (svd) of σ_{max}

✓ $\|A\|_2 = \|Ax^*\|_2 / \|x^*\|_2 = \|U\Sigma V^T v_1\| / \|v_1\| = \|U\Sigma e_1\|_2 = \|U e_1 \sigma_{max}\| = \|Ue_1\|_2 \|\sigma_{max}\| = \|\sigma_{max}\|$

$\|A\|_1 = \max\limits_{1\le j\le n} \sum\limits_{i=1}^{n} |a_{ij}|$

let x^* be $e_i \in \mathbb{R}^n$ where i indicates the non-zero element = to one.

✓ $\|A\|_1 = \|Ax^*\|_1 / \|x^*\|_1 = \|Ae_i\| / \|e_i\|_1 = \max\limits_{1\le j\le n} \sum\limits_{i=1}^{n} |a_{ij}|$

$\|A\|_\infty = \max\limits_{1\le i\le m} \sum\limits_{j=1}^{n} |a_{ij}|$ or max row sum

let x^* be $v \in \mathbb{R}^n$ where v is filled elements (α) of the same value whose sign is the same as the elements in the max abs row.

$\|A\|_\infty = \|Ax^*\|_\infty / \|x^*\|_\infty = \|Av\|_\infty / \|v\|_\infty$

$$= \left\| \begin{matrix} \alpha a_{11} + \alpha a_{12} + \dots + \alpha a_{1n} \\ \alpha a_{21} + \alpha a_{22} + \dots + \alpha a_{2n} \\ \alpha a_{m1} + \alpha a_{m2} + \dots + \alpha a_{mn} \end{matrix} \right\| \Big/ \alpha = \|\alpha\| \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}| \Big/ \|\alpha\| = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|$$

8.4) Consider a linear system with a state-transition matrix $\Phi(t,\tau)$ for which

$$\Phi(t,0) = \begin{bmatrix} e^{t}\cos 2t & e^{-2t}\sin 2t \\ -e^{t}\sin 2t & e^{-2t}\cos 2t \end{bmatrix}$$

See Matlab

c) Compute the eigenvalues of $A(t)$

$$\det(A-sI) = \det\begin{bmatrix} 3\cos(\phi)/2 - 1/2 - s & 2-3\sin(\phi)/2 \\ -3\sin(\phi)/2 - 2 & -1/2 - 3\cos(\phi)/2 - s \end{bmatrix} \quad \text{where } \phi = 4t - 4t_0$$

$$= (3\cos(\phi)/2 - 1/2 - s)(-1/2 - 3\cos\phi/2 - s) - (2 - 3\sin(\phi)/2)(-3\sin(\phi)/2 - 2)$$

$$= -9\cos^2\phi/4 - \cancel{3\cos\phi/4} - s\cos\phi/2 + 1/4 + \cancel{3\cos\phi/4} + 1/2 s + 1/2 s + s\cancel{\cos}\phi/2 + s^2$$
$$[+3\cos(\phi) + 4 - 9\sin^2\phi/4 - 3\sin(\phi)]$$

$$= -9/4(c_\phi^2 + s_\phi^2) + 1/4 + s + s^2 + 4 - 3s_\phi + 3c_\phi$$

$$= s^2 + s + 2 - 3s_\phi + 3c_\phi$$

$$\lambda = \frac{-1 \pm \sqrt{1-4(2-3s_\phi+3c_\phi)^2}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1-4(4-6s_\phi+6c_\phi-6s_\phi+9s_\phi^2-9s_\phi c_\phi+6c_\phi-9s_\phi c_\phi+9c_\phi^2}}{2}$$

$$\lambda = -\frac{1}{2} \pm \sqrt{1-4(13-12s_\phi+12c_\phi-18c_\phi s_\phi)}$$

See Matlab

using Matlab for the rest

$$\lambda = -1/2 - \tfrac{1}{2}\sqrt{7}\,j \quad \text{and} \quad -1/2 + 1/2\sqrt{7}\,j$$

d) Classify this system in terms of Lyapunov stability

Looking at the state transition matrix it looks unstable.

3.5) verify that $(e^{At})' = e^{A't}$

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \Rightarrow (e^{At})' = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k\right)' = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A^k)' = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A')^k = e^{A't}$$

8.6) Consider the Continuous-time LTI system

$$\dot{x} = Ax \qquad \lambda \in \mathbb{R}^n$$

and suppose that there exists a positive constant u and positive-definite matrices $P, Q \in \mathbb{R}^n$ for the lyapunov equation

$$A'P + PA + 2uP = -Q$$

Show that all eigenvalues of A have real parts less than $-u$.

For clearification Let $B = A + UI$ then

$$B'P_b + P_bB = -Q_b \Rightarrow (A' + uI)P_b + P_b(A + uI) = -Q_b \Rightarrow A'P_b + P_bA + 2uP_b = -Q_b$$

Thus according to Theorem 8.2 all of the eigen values of B have strictly negative real parts. $\text{Eig}(B) < 0$

$B = A + UI = V^{-1}\Lambda V + UI$ wher V & Λ are the eigen vecters and eigen values of A, then

$B = V^{-1}\Lambda V + UV^{-1}IV = V^{-1}(\Lambda + UI)V$ where $\Lambda + UI$ are the eigen values of B thus $\Lambda + UI < 0I$ and $\Lambda < -UI$

Thus all the eigen values of A have real parts less than $-U$

8.7) Investigate whether or not the solutions to the following nonlinear systems converge to the given equilibrium point when they start close enough to it

a) The State Space system

$$\dot{x}_1 = -x_1 + x_1(x_1^2 + x_2^2) = f_1(x_1, x_2)$$
$$\dot{x}_2 = -x_2 + x_2(x_1^2 + x_2^2) = f_2(x_1, x_2)$$

whith equilibrium points $x_1 = x_2 = 0$

$$\partial f_1/\partial x_1 |_{eq} = -1 + x_1^2 + x_2^2 |_{eq} = -1$$
$$\partial f_1/\partial x_2 |_{eq} = 0 + 2x_1x_2 |_{eq} = 0$$
$$\partial f_2/\partial x_2 |_{eq} = 0 + 2x_1x_2 |_{eq} = 0$$
$$\partial f_2/\partial x_2 |_{eq} = -1 + x_1^2 + x_2^2 |_{eq} = -1$$

$\dot{x} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A} x$ thus $x(t) = e^{A}x(0)$ the eigen values of A are negative thus the system will converge to the equilibrium point when near it.

b) The second order system

$$\ddot{w} + g(w)\dot{w} + w = 0 \quad \Rightarrow \quad \ddot{w} = -g(w)\dot{w} - w = f(w, \dot{w})$$

with the equilibrium point $w = \dot{w} = 0$. Determine for which values of $g(0)$ we can guarantee convergence to the origin based on the local linearization

$$\frac{\partial f}{\partial w}\Big|_{eq} = -1$$

$$\frac{\partial f}{\partial \dot{w}}\Big|_{eq} = -g(w)\Big|_{eq} = -g(0)$$

$$\partial \ddot{w} = -g(0)\,\delta\dot{w} - \delta w \qquad \text{where } \delta = x(t) - x_{eq}$$

Let $r(w,\dot{w}) = f(w,\dot{w}) - \left(f(w_{eq}, \dot{w}_{eq}) - g(0)\delta\dot{w} - \delta w\right) = O(\|\delta w\|^2, \|\delta\dot{w}\|^2)$

basically r is the error in the taylor series approximation.

?

Which means that there is a constant c and a ball $\bar{B}$ around equilibrium for which

$$\|r\| \le c[\|\delta w\|^2 + \|\delta \dot{w}\|^2] \qquad \forall w, \dot{w} \in \bar{B}$$

Create the Lyapunov function

$$V(t) = \tfrac{1}{2}w(t)^2 + \tfrac{1}{2}\dot{w}(t)$$

$$\dot{V}(t) = \delta w(t)\,\delta\dot{w}(t) + \delta\dot{w}(t)\,f(x)$$

$$= \delta w(t)\,\delta\dot{w}(t) + \delta\dot{w}(t)\left[-g(0)\,\delta\dot{w} - \delta w + r\right]$$

$$= \delta w(t)\,\delta\dot{w}(t) - \delta\dot{w}(t)\,\delta w - \delta\dot{w}(t)^2 g(0) + \delta\dot{w}(t)\,r$$

$$= -\delta\dot{w}^2(t)\,g(0) + \delta\dot{w}(t)\,r$$

$$\le -\delta\dot{w}^2(t)\,g(0) + c\,\delta\dot{w}(t)\left[\|\delta w\|^2 + \|\delta\dot{w}\|^2\right]$$

$$\le -\|\delta\dot{w}(t)\|^2 g(0) + c\|\delta\dot{w}(t)\|\left[\|\delta w\|^2 + \|\delta\dot{w}\|^2\right]$$

$$\le -\|\delta\dot{w}\|^2\left[g(0) - c[\|\delta w\|^2 + \|\delta\dot{w}\|]/\|\delta\dot{w}\|\right]$$

we need $g(0) > c[\|\delta w\|^2 + \|\delta w\|]/\|\delta\dot{w}\|$

Or in state space form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -g(0) \end{bmatrix} x \qquad \text{eigen values are } \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -g(0)-\lambda \end{bmatrix} = -\lambda(-g(0)-\lambda) + 1$$

$$= \lambda^2 + g(0)\lambda + 1$$

$$\frac{-g(0) \pm \sqrt{g(0)^2 - 4}}{2}$$

$g(0)$ ~~st~~ $\quad -g(0) \pm \sqrt{g(0)^2 - 4} < 0$

$g(0) > 0$ The $Re\{\lambda s\}$ are strictly negative ✓

$g(0) < 0$ the $Re\{\lambda s\}$ are strictly positive ✓

$g(0) = 0$ The $R(\lambda s)$ are 0 and eigen values are complex ✓