Midterm 1

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Exercise 1. (Q1.1): The Set A is countable.

A set is called countable if there exists a bijection $g : \mathbb{N} \to A$, or if A is finite. In other words, $|\mathbb{N}| \sim |A|$.

Exercise 2. (Q1.2): The set A is uncountable.
A set is called uncountable if it isn't countable.

Exercise 3. (Q1.3): The sequence (a_n) converges to A.

If for any $\epsilon \in \mathbb{R} > 0$, there exists an $N \in \mathbb{N}$, such that whenever $n > N$,

$$|a_n - A| < \epsilon,$$

then we say that the sequence (a_n) converges to A .

Exercise 4. (Q1.4): The sequence (b_n) is a Cauchy sequence.

A sequence (b_n) is a Cauchy sequence if given any $\epsilon \in \mathbb{R} > 0$, there exists an $N \in \mathbb{N}$, such that whenever $n, m \in \mathbb{N} > N$,

$$|b_n - b_m| < \epsilon.$$

Exercise 5. (Q1.5): The series $\sum_{k=1}^{\infty} a_k$ converges.

Let $s_n = \sum_{k=1}^{n} a_k$ denote the partial series. The series $\sum_{k=1}^{\infty} a_k$ is said to converge to L if given any $\epsilon \in \mathbb{R} > 0$, there exists and $N \in \mathbb{N}$, such that whenever $n \in \mathbb{N} > N$,

$$|s_n - L| < \epsilon.$$

Exercise 6. (Q2.1): Sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges.

Let $x_n = n$ and $y_n = -n$, then (x_n) and (y_n) both diverge; however, $x_n + y_n = 0$, thus their sum converges to 0.

Exercise 7. (Q2.2): Sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges.

This is not possible. Let L_1 denote the limit of $(x_n + y_n)$, then given some $\epsilon_1 > 0$, there exists an $N_1 \in \mathbb{N}$, such that whenever $n \in \mathbb{N} > N$,

$$|x_n + y_n - L_1| < \epsilon_1.$$

Now let L_2 denote the limit of (x_n), then given some $\epsilon_2 > 0$, there exists and $N_2 \in \mathbb{N}$ such that when $m \in \mathbb{N} > N$,

$$|x_m - L_2| < \epsilon_2.$$

Lastly, let $L_1 = L_2 + L_3$ for some $L_3 \in \mathbb{R}$ and $N = \max(N_1, N_2)$, then when $n > N$,

$$\begin{aligned}
|x_n + y_n - L_1| &< \epsilon_1 \\
|x_n + y_n - L_2 - L_3| &< \epsilon_1 \\
||x_n - L_2| - |y_n - L_3|| &< \epsilon_1 \\
|y_n - L_3| &< \epsilon_1 + \epsilon_2 \\
|y_n - L_3| &< \epsilon_3,
\end{aligned}$$

since (y_n) doesn't converge, there exists an ϵ_3 such that $|y_n - L_3| > \epsilon_3$. Thus this is a contradiction which shows that the request is impossible.

Exercise 8. (Q2.3): A convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges.

Let $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then (b_n) converges to 0, and $b_n \neq 0$ for any n. Then $\frac{1}{b_n} = n$, thus $\left(\frac{1}{b_n}\right)$ is not bounded and therefore doesn't converge.

Exercise 9. (Q2.4): An unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded.

This is not possible. Let $\lim b_n = L$, then given an $\epsilon \in \mathbb{R} > 0$, there exists an $N \in \mathbb{N}$, such that whenever $n > N$,

$$\begin{gathered} |b_n - L| < \epsilon \\ -\epsilon + L < b_n < \epsilon + L \\ -L - \epsilon < -b_n < -L + \epsilon \\ a_n - L - \epsilon < a_n - b_n < a_n - L + \epsilon, \end{gathered}$$

which shows that since (a_n) isn't bounded, neither can $(a_n - b_n)$ be bounded.

Exercise 10. (Q2.5): Two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Let $a_n = \frac{1}{n^3}$ and $b_n = n$ for all $n \in \mathbb{N}$, then $a_n b_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Thus $(a_n b_n)$ and (a_n) converge, but (b_n) doesn't.

Exercise 11. **(Q3)**: Assume $\lim_{n\to\infty} a_n = 3$. Using the definition of a convergent sequence, prove that

$$\lim_{n\to\infty} \frac{a_n^2+1}{a_n-2} = 10.$$

Proof: We begin by manipulating the term

$$\begin{aligned}\left|\frac{a_n^2+1}{a_n-2} - 10\right| &= \left|\frac{a_n^2+1-10a_n+20}{a_n-2}\right| \\ &= \left|\frac{(a_n-7)(a_n-3)}{a_n-2}\right| \\ &= \frac{|a_n-7|\,|a_n-3|}{|a_n-2|}.\end{aligned}$$

Since we assume directly that $\lim_{n\to\infty} a_n = 3$, there exists an $N_1 \in \mathbb{N}$ such that

$$|a_{N_1} - 3| < \frac{1}{2},$$

which is equivalent to

$$\frac{1}{2} < a_{N_1} - 2 < \frac{3}{2}.$$

It is also equivalent to

$$\begin{aligned}-4 - \frac{1}{2} < a_{N_1} - 7 < -4 + \frac{1}{2} \\ -\frac{9}{2} < a_{N_1} - 7 < -\frac{7}{2}.\end{aligned}$$

Thus, for any $n > N_1$, we have that

$$\begin{aligned}\frac{|a_n-7|\,|a_n-3|}{|a_n-2|} &< \frac{\frac{7}{2}\,|a_n-3|}{\frac{1}{2}} \\ &= 7\,|a_n-3|\,.\end{aligned}$$

Once again, Since we assume directly that $\lim_{n\to\infty} a_n = 3$, given an $\epsilon > 0$, there exists an $N_2 \in \mathbb{N}$ such that whenever $m > N_2$,

$$|a_n - 3| < \frac{\epsilon}{7}.$$

Thus, given an $\epsilon > 0$, let $N = \max(N_1, N_2)$, then whenever $n \in \mathbb{N} > N$,

$$\begin{aligned}\left|\frac{a_n^2+1}{a_n-2} - 10\right| &< 7\,|a_n-3| \\ &< 7\frac{\epsilon}{7} \\ &= \epsilon.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{a_n^2+1}{a_n-2} = 10.$$

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Exercise 12. (Q4): Assume (x_n) is a Cauchy sequence that satisfies $2 < x_n < 3$ for all $n \in \mathbb{N}$. By directly using the definition of a Cauchy sequence, show that

$$\left(\frac{x_n^2}{x_n - 1}\right)$$

is also a Cauchy sequence.

Proof: We assume directly that (x_n) is a Cauchy sequence that satisfies $2 < x_n < 3$ for all $n \in \mathbb{N}$, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$, such that whenever $n, m > N$,

$$(x_n - x_m) < \frac{\epsilon}{15}.$$

We next examine the term $\left(\frac{x_n^2}{x_n-1}\right)$, and begin to manipulate it.

$$\left|\frac{x_n^2}{x_n - 1} - \frac{x_m^2}{x_m - 1}\right| = \left|\frac{x_n^2\left(x_m - 1\right) - x_m^2\left(x_n - 1\right)}{\left(x_n - 1\right)\left(x_m - 1\right)}\right|.$$

Using the fact that $2 < x_n < 3$ for all $n \in \mathbb{N}$, we know that

$$\begin{aligned}
\left|\frac{x_n^2\left(x_m - 1\right) - x_m^2\left(x_n - 1\right)}{\left(x_n - 1\right)\left(x_m - 1\right)}\right| &\leq \left|\frac{x_n^2\left(x_m - 1\right) - x_m^2\left(x_n - 1\right)}{\left(2 - 1\right)\left(2 - 1\right)}\right| \\
&= \left|x_n^2\left(x_m - 1\right) - x_m^2\left(x_n - 1\right)\right| \\
&= \left|x_n^2 x_m - x_n^2 - x_m^2 x_n + x_m^2\right| \\
&= \left|x_n x_m\left(x_n - x_m\right) - x_n^2 + x_m^2\right| \\
&= \left|x_n x_m\left(x_n - x_m\right) - x_n^2 + x_m^2 - x_n x_m + x_n x_m\right| \\
&= \left|x_n x_m\left(x_n - x_m\right) - x_n\left(x_n - x_m\right) - x_m\left(x_n - x_m\right)\right| \\
&\leq \left|x_n x_m\left(x_n - x_m\right)\right| + \left|x_n\left(x_n - x_m\right)\right| + \left|x_m\left(x_n - x_m\right)\right| \\
&\leq 3 \cdot 3 \cdot \left|x_n - x_m\right| + 3 \cdot \left|x_n - x_m\right| + 3 \cdot \left|x_n - x_m\right| \\
&= 15\left|x_n - x_m\right| \\
&< 15\frac{\epsilon}{15} \\
&= \epsilon;
\end{aligned}$$

therefore, $\left(\frac{x_n^2}{x_n-1}\right)$ is a Cauchy sequence.

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Exercise 13. (Q5): Prove that the open interval $(0,1)$ is uncountable by using Cantor's diagonalization method.

Proof: We suppose, by contradiction, that $(0,1)$ is countable, then there exists a bijection $f : \mathbb{N} \to (0,1)$. For all $n \in \mathbb{N}$, let

$$f(n) = 0.b_{n1}b_{n2}b_{n3}b_{n4}\cdots$$

with b_{ij} being the j^{th} decimal digit of the value $f(i)$. Any number in the interval $(0,1)$ can be written as

$$0.d_1d_2d_3d_4\cdots$$

where d_j is the j^{th} decimal digit. Let $a \in (0,1)$ be the number whose j^{th} decimal digit is defined by

$$d_j = \begin{cases} 3 & \text{if } b_{jj} \neq 3 \\ 7 & \text{else} \end{cases},$$

then $a \neq f(1)$ since $d_1 \neq b_{11}$, $a \neq f(2)$, since $d_2 \neq b_{22}$, $a \neq f(3)$ since $d_3 \neq b_{33}$, etc. Thus a is not in the image of f. Which means that f is not a bijection. This is a contradiction. Therefore, the open interval $(0,1)$ is uncountable. ■

Exercise 14. (Q6): Let (a_n) be a convergent sequence and assume that $\lim_{n\to\infty} a_n = A \neq 0$. By directly using the definition of the limit, prove that

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}.$$

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Exercise 15. (Q7): Let (x_n) be the sequence defined recursively by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for $n \geq 1$. Prove that $\lim x_n = L$ exists and find the value of L.

Proof: In order to show that the limit exists, we will show that (x_n) is a bounded, monotonic sequence. First we will prove that $x_n \geq 2 - \sqrt{3}$ for all $n \in \mathbb{N}$. We do this by induction.

Base Case: $x_1 = 3$, then

$$\begin{aligned} x_2 &= \frac{1}{4-3} \\ &= 1 \\ &\geq 2 - \sqrt{3}. \end{aligned}$$

Induction Step: Let $k \in \mathbb{N}$, we suppose directly that $x_k \geq 2 - \sqrt{3}$, then

$$\begin{aligned} x_{k+1} &= \frac{1}{4 - x_k} \\ &\geq \frac{1}{4 - 2 + \sqrt{3}} \\ &= \frac{1}{2 + \sqrt{3}} \\ &= \frac{2 - \sqrt{3}}{\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right)} \\ &= 2 - \sqrt{3}, \end{aligned}$$

thus $x_{k+1} \geq 2 - \sqrt{3}$; therefore, $x_n \geq 2 - \sqrt{3}$ for all $n \in \mathbb{N}$.

We next want to show that $x_n \leq 3$ for all $n \in \mathbb{N}$. We work this by induction.

Base Case: $x_1 = 3$, then

$$\begin{aligned} x_2 &= \frac{1}{4-3} \\ &= 1 \\ &\leq 3. \end{aligned}$$

Induction Step: Let $k \in \mathbb{N}$, we suppose directly that $x_k \leq 3$, then

$$\begin{aligned} x_{k+1} &= \frac{1}{4 - x_k} \\ &\leq \frac{1}{4 - 3} \\ &= 1, \end{aligned}$$

thus $x_{k+1} \leq 3$; therefore, $x_k \leq 3$ for all $k \in \mathbb{N}$. We now know that (x_n) is bounded such that $2 - \sqrt{3} \leq x_n \leq 3$ for all $n \in \mathbb{N}$.

Next we show that $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{4 - x_n} \\ &= \frac{x_n\left(4 - x_n\right) - 1}{4 - x_n} \\ &= \frac{-\left(x_n - 2 + \sqrt{3}\right)\left(x_n - 2 - \sqrt{3}\right)}{4 - x_n} \\ &\geq 0, \end{aligned}$$

hence $x_n \geq x_{n+1}$. Since (x_n) monotonic and bounded, it has a limit. Let L denote the limit of (x_n), then as $n \to \infty$ we have

$$L = \frac{1}{4-L}$$
$$L^2 - 4L + 1 = 0,$$

which has roots $2 \pm \sqrt{3}$. Since $2 + \sqrt{3} > 3$, it must be that $L = 2 - \sqrt{3}$. ■

Exercise 16. (Q8): Prove that the real numbers are uncountable with a proof that relies on the Nested Interval Theorem.

Proof: We suppose, by contradiction, that the real numbers are countable, thus there exists a bijection $f : \mathbb{N} \to \mathbb{R}$. We can construct nested closed intervals in the following manner. Let I_1 be the closed interval such that $I_1 \subseteq \mathbb{R}$ and $f(1) \notin I_1$. Then let I_2 be the closed interval such that $I_2 \subseteq \mathbb{R}$ and $f(2) \notin I_2$. Due to the density of $\mathbb{R}$, we can repeat this process recursively such that $I_n \subseteq I_{n+1}$ and $f(n) \notin I_n$. We then form the intersection $\cap_{n=1}^{\infty} I_n$ which is not empty according to the nested interval theorem. Since none of the elements in $\cap_{n=1}^{\infty} I_n$ are in the image of f, f is not surjective and hence not a bijection. This contradicts our assumption, thus $\mathbb{R}$ is uncountable. ■

Exercise 17. (Q9): Assume that the sequence (x_n) is a convergent sequence and $\lim_{n\to\infty} x_n = L$. Prove that (x_n) is also a Cauchy sequence.

Proof: We suppose directly that (x_n) is a convergent sequence and $\lim_{n\to\infty} x_n = L$, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n, m \in \mathbb{N} > N$,

$$|x_n - L| < \frac{\epsilon}{2},$$

and

$$|x_m - L| < \frac{\epsilon}{2}.$$

Adding the two together, we get

$$\begin{aligned} |x_n - L| + |x_m - L| &\geq |x_n - x_m + L - L| \\ &= |x_n - x_m|, \end{aligned}$$

thus

$$\begin{aligned} |x_n - x_m| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

thus the sequence (x_n) is also a Cauchy sequence. ■

Exercise 18. **(Q10)**: Prove that if the series $\sum_{k=1}^{\infty} |a_n|$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges.

Proof: We assume directly that $\sum_{k=1}^{\infty} |a_n|$ converges, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$, such that whenever $n > m \in \mathbb{N} > N$,

$$\sum_{k=m}^{n} |a_k| < \epsilon.$$

Well,

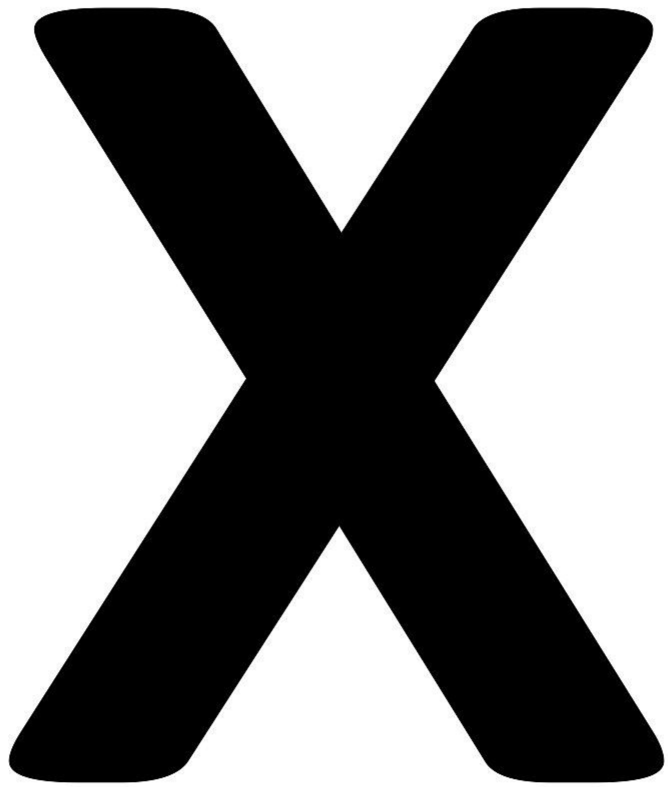
$$\sum_{k=m}^{n} |a_k| \geq \left| \sum_{k=m}^{n} a_k \right|,$$

thus

$$\left| \sum_{k=m}^{n} a_k \right| < \epsilon,$$

which is the Cauchy convergent series condition for the series $\sum_{k=1}^{\infty} a_k$, therefore, $\sum_{k=1}^{\infty} a_k$ converges if $\sum_{k=1}^{\infty} |a_n|$ converges. ■

Exercise 19. (Q11): Let A be a nonempty set and let $\mathcal{P}(A)$ denote the power set of A. Show that there does not exist a surjective function $g : A \to \mathcal{P}(A)$.



Exercise 20. (Q12): Assume $a \neq 0$. Prove that the geometric series

$$\sum_{k=0}^{\infty} ar^k$$

converges if and only if $|r| < 1$. In the case $|r| = 1$, show that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Proof: To prove the statement, we will break it into three cases. When $|r| \geq 1$ and $|r| < 1$.

Case 1. We suppose that $|r| \geq 1$, then $ar^k \not\to 0$ as $k \to \infty$. Thus by the divergence criteria, $\sum_{k=0}^{\infty} ar^k$ does not converge.

Case 2. We suppose that $|r| < 1$, then $r^k \to 0$ as $k \to \infty$. We next take the partial series

$$\sum_{k=0}^{m} ar^k,$$

with $m \in \mathbb{N}$ and multiply it by $(1-r)$ to get

$$\begin{aligned}
\left(\sum_{k=0}^{m} ar^k\right)(1-r) &= a - ar + ar - ar^2 + \cdots + ar^m - ar^{m+1} \\
&= a - ar^{m+1} \\
&= a\left(1 - r^{m+1}\right).
\end{aligned}$$

Since $|r| < 1$, $(1-r) \neq 0$, thus we can divide both sides by $(1-r)$ to get

$$\sum_{k=0}^{m} ar^k = \frac{a\left(1 - r^{m+1}\right)}{1-r}.$$

As we take the limit as $m \to \infty$, $r^m \to 0$, thus

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Therefore, the geometric series converges if and only if $|r| < 1$, and it converges to

$$\frac{a}{1-r}.$$

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