Homework 4 Section 1.5

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Exercises 1,2,3,7

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Exercise 1. **(Q1)**: Finish the following proof for Theorem 1.5.7.

Assume B is a countable set. Thus, there exists $f : \mathbb{N} \to B$, which is $1-1$ and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a bijection g.

a) Given $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$, let

$$n_{m+1} = \min\{n \in \mathbb{N} : f(n) \in A - \{f(n_1), f(n_2), \ldots, f(n_m)\}\}$$

with $m \in \mathbb{N} > 1$. We can then set $g(k) = f(n_k)$ with $k \in \mathbb{N}$.

Exercise 2. **(Q2)**: Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable:

Assume, by contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\cap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\cap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies $\mathbb{Q}$ must therefore be uncountable.

The flaw is that the construction only states that there is no rational number in the intersection $\cap_{n=1}^{\infty} I_n$, it doesn't guarantee that there isn't an irrational number in the intersection $\cap_{n=1}^{\infty} I_n$. So this doesn't contradict NIP. Thus the proof given is false.

Exercise 3. **(Q3)**: Use the following outline to supply proofs for the statements in theorem 1.5.8.

a) First, prove statement (i) for two countable sets, A_1 and A_2.
Proof: We suppose that A_1 and A_2 are countable sets. Let $B_2 = A_2 \backslash A_1$ so that $A_1 \cup A_2 = A_1 \cup B_2$. There are three cases to consider.

Case 1. If A_1 and B_2 are finite, then their union is finite, and thus countable.

Case 2. With no loss in generality, if A_1 is finite and B_2 is countably infinite, then the elements in the set $A_1 \cup B_2$ can be arranged as

$$\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots\},$$

with $|A_1| = n$. We could then construct a bijective function $f : \mathbb{N} \to A_1 \cup B_2$ defined as

$$\begin{aligned} f(1) &= a_1 \\ f(2) &= a_2 \\ &\vdots \\ f(n) &= a_n \\ f(n+1) &= b_1 \\ f(n+2) &= b_2 \\ &\vdots \end{aligned}$$

thus the set $A_1 \cup B_2$ is countable.

Case 3. If A_1 and B_2 are countably infinite sets, then the elements in $A_1 \cup B_2$ can be arranged as

$$A_1 \cup B_2 = \{a_1, b_1, a_2, b_2 \cdots\}.$$

We can construct the bijective function $g : \mathbb{N} \to A_1 \cup B_2$ defined as

$$\begin{aligned} g(1) &= a_1 \\ g(2) &= b_1 \\ g(3) &= a_2 \\ g(4) &= b_2 \\ &\vdots, \end{aligned}$$

which shows that the union of two infinite sets is countably infinite.

Since all three cases hold, the union of two countable sets is countable.

■

a) We can show by induction that the union of a finite many countable sets is also countable.

Proof: Let $m \in \mathbb{N}$. We suppose directly that the sets $A_1, A_2, \ldots, A_m$ are countable sets. We want to show that the union $\cup_{k=1}^{m} A_m$ is countable. We work this by induction.
Base Case: Let $m = 1$. Then $\cup_{k=1}^{m} A_k = A_1$ which is trivially countable. Let $m = 1$, then $\cup_{k=1}^{m} A_k = A_1 \cup A_2$ is countable as proven in the first part of this exercise.
Inductive Step: We assume that $\cup_{k=1}^{x} A_k$ is countable, and we want to show that $\cup_{k=1}^{x+1} A_k$ is countable. Let $B = \cup_{k=1}^{x} A_k$, which is a countable set. Then $B \cup A_{x+1}$ is also countable since the union of two countable sets is countable. Thus $\cup_{k=1}^{x+1} A_k$ is countable.
Therefore, for any $m \in \mathbb{N}$, the union $\cup_{k=1}^{m} A_m$ is countable. ■

b) Explain why induction cannot be used to prove part (ii) of Theorem 1.5.8 from part (i).

a) Induction is a method used to prove that an open sentence $P(n)$ is true for all $n \in \mathbb{N}$. It is not designed to handle the infinite case as in the infinite union. Therefore, induction cannot be used to prove part (ii).

c) Show how arranging $\mathbb{N}$ into the two-dimensional array

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof of Theorem 1.5.8 (ii).
The two dimensional array shows how we can construct a countably infinite number of disjoint subsets of $\mathbb{N}$. Let $B_k \subseteq \mathbb{N}$ be the set containing all of the elements in the k^{th} column of the two dimensional array, then $|B_k| = |\mathbb{N}|$ and $\cup_{j=1}^{\infty} B_j = \mathbb{N}$. Thus $|\cup_{j=1}^{\infty} B_j| = |\mathbb{N}|$. Now, given an infinite number of disjoint, countably infinite sets, A_m, there exists a bijection $f_i : B_i \to A_i$, thus $|B_i| = |A_i|$. So $|\cup_{j=1}^{\infty} B_j| = |\cup_{j=1}^{\infty} A_j|$. Thus $|\cup_{j=1}^{\infty} A_j| = |\mathbb{N}|$. Therefore, the union of an infinite number of countably infinite sets is a countably infinite set. If some of the sets in the infinite union are finite or not disjoint, then the infinite union will have less elements, which means that it's still a countable set.

Exercise 4. (Q7): Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $s = \{(x, y) : 0 < x, y < 1\}$.

a) Find an injective function that maps $(0, 1)$ into, but not necessarily onto, S.

a) Let $f : (0, 1) \to S$ be defined as $f(x) = (x, 1)$. This is an injective function.

Proof: We assume, by contrapositive, that $f(x_1) = f(x_2)$, then

$$\begin{aligned} f(x_1) &= f(x_2) \\ (x_1, 1) &= (x_2, 1), \end{aligned}$$

thus $x_1 = x_2$. Therefore, f is injective. ■

b) Use the fact that every real number has a decimal expansion to produce a 1-1 function that maps S into $(0, 1)$.

Proof: We wish to show that there exists an injective function $h : S \to (0,1)$. Let $(x,y) \in S$, then x and y can have the decimal expansion

$$x = 0.x_1x_2x_3\cdots$$
$$y = 0.y_1y_2y_3\cdots ,$$

where x_i and y_i is the i^{th} digit in the decimal expansion of x and respectively y. Let $h : S \to \mathbb{R}$ be the injective function defined as

$$g\left((x,y)\right) = 0.x_1y_1x_2y_2x_3y_3\cdots .$$

To show that h is injective we assume, by contrapositive, that $g\left((x,y)\right) = g\left((a,b)\right)$, then

$$g\left((x,y)\right) = g\left((a,b)\right)$$
$$0.x_1y_1x_2y_2x_3y_3\cdots = 0.a_1b_1a_2b_2a_3b_3\cdots ,$$

which shows that $x_i = a_i$ and $y_i = b_i$ for all i in the sequence. Thus $(x,y) = (a,b)$, and therefore, the function g is an injection.

However, the function isn't onto. Consider the the number $z \in (0,1)$ that has the decimal expansion

$$z = 0.x_19x_29x_39\cdots ,$$

then $g\left((x, 0.999\cdots)\right) = z$. However, $0.9999 = 1 \notin (0,1)$, thus $(x, 0.99) \notin S$. This shows that there exists an element $z \in (0,1)$ such that there is no $(x,y) \in S$ such that $g\left(x,y\right) = z$. Hence, the function is not onto. ■

Since we have constructed the injective functions $f : (0,1) \to S$ and $g : S \to (0,1)$, then according to the Schroder-Bernstein Theorem, $|(0,1)| = |S|$.