Exercise 20. (Q12): Assume $a \neq 0$. Prove that the geometric series

$$\sum_{k=0}^{\infty} ar^k$$

converges if and only if $|r| < 1$. In the case $|r| = 1$, show that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Proof: To prove the statement, we will break it into three cases. When $|r| \geq 1$ and $|r| < 1$.

Case 1. We suppose that $|r| \geq 1$, then $ar^k \not\to 0$ as $k \to \infty$. Thus by the divergence criteria, $\sum_{k=0}^{\infty} ar^k$ does not converge.

Case 2. We suppose that $|r| < 1$, then $r^k \to 0$ as $k \to \infty$. We next take the partial series

$$\sum_{k=0}^{m} ar^k,$$

with $m \in \mathbb{N}$ and multiply it by $(1-r)$ to get

$$\begin{aligned}\left(\sum_{k=0}^{m} ar^k\right)(1-r) &= a - ar + ar - ar^2 + \cdots + ar^m - ar^{m+1} \\ &= a - ar^{m+1} \\ &= a\left(1 - r^{m+1}\right).\end{aligned}$$

Since $|r| < 1$, $(1-r) \neq 0$, thus we can divide both sides by $(1-r)$ to get

$$\sum_{k=0}^{m} ar^k = \frac{a\left(1 - r^{m+1}\right)}{1-r}.$$

As we take the limit as $m \to \infty$, $r^m \to 0$, thus

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Therefore, the geometric series converges if and only if $|r| < 1$, and it converges to

$$\frac{a}{1-r}.$$

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