Midterm 3

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Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ both be continuous at the point $c \in \mathbb{R}$. Use the $\epsilon - \delta$ characterization of continuity to show that the product $f(x)\, g(x)$ is continuous at $c \in \mathbb{R}$.

Proof: We suppose directly that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ both be continuous at the point $c \in \mathbb{R}$, then given an $\epsilon > 0$, there exists a $\delta_1 > 0$ such that when $|x - c| < \delta_1$

$$|f(x) - f(c)| < \frac{\epsilon}{2N}$$

for some $N \in \mathbb{R}$ such that $|f(c)| \leq N$, and there exists a $\delta_2 > 0$ such that when $|x - c| < \delta_2$,

$$|g(x) - g(c)| < \frac{\epsilon}{2M}.$$

for some $M \in \mathbb{R}$ such that $|g(x)| + \epsilon < M$. Next we look closely at $|f(x)\, g(x) - f(c)\, g(c)|$ and begin to manipulate it.

$$\begin{aligned}
|f(x)\, g(x) - f(c)\, g(c)| &= |f(x)\, g(x) - f(c)\, g(x) + f(c)\, g(x) - f(c)\, g(c)| \\
&= |g(x)\,(f(x) - f(c)) + f(c)\,(g(x) - g(c))| \\
&\leq |g(x)|\, |f(x) - f(c)| + |f(c)|\, |g(x) - g(c)| \,.
\end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$, then when $|x - c| < \delta$ we get that

$$\begin{aligned}
|f(x)\, g(x) - f(c)\, g(c)| &\leq N\, |f(x) - f(c)| + M\, |g(x) - g(c)| \\
&= N \frac{\epsilon}{2N} + M \frac{\epsilon}{2M} \\
&= \epsilon.
\end{aligned}$$

Thus the product of two continuous functions at a point c is continuous.

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Exercise 2. Prove exactly one of the following theorems:

Theorem (Preservation of Compact Sets). Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, prove that $f(K)$ is compact as well.

Proof: Let (y_n) be an arbitrary sequence contained in $f(K)$. To show that $f(K)$ is a compact set, we must show that (y_n) contains a subsequence that converges to a point in $f(K)$. Since $(y_n) \subseteq f(K)$, there exists a sequence $(x_n) \in K$ such that $f(x_n) = y_n$. Since K is a compact set, the sequence (x_n) contains a subsequence (x_{n_k}) that converges to a point $m \in K$. Let (y_{n_k}) be the subsequence of (y_n) that corresponds to (x_{n_k}), i.e. $f(x_{n_k}) = y_{n_k}$. Since f is continuous, as $(x_{n_k}) \to m$, $f(x_{n_k}) \to f(m)$. In other words, $(y_{n_k}) \to f(m)$ as $(x_{n_k}) \to m$. Since $f(m) \in f(K)$, this shows that the subsequence (y_{n_k}) converges to a point in $f(K)$. Therefore, every sequence contained in the range of the function, contains a subsequence that converges to a point in the range. Hence, the range is compact.

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Exercise 3. Give an example of each of the following, or provide a short argument for why the request is impossible.

a) A continuous function defined on $[0,1]$ with range $(0,1)$.
Proof: The request is impossible according to the theorem: Preservation of Compact sets. Since the function is continuous and the domain $[0,1]$ is compact, the image must be compact; however, $(0,1)$ is not compact thus the request is impossible.

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b) A continuous function defined on $(0,1)$ with range $[0,1]$.
Proof: This exists. Consider the function $f(x) = \sin^2\left(\frac{3\pi}{2}x\right)$. When $x = \frac{1}{3}$, then

$$f\left(\frac{1}{3}\right) = \sin^2\left(\frac{3\pi}{2}\cdot\frac{1}{3}\right) = 1,$$

and when $x = \frac{2}{3}$,

$$f\left(\frac{2}{3}\right) = \sin^2\left(\frac{3\pi}{2}\cdot\frac{2}{3}\right) = 0.$$

Since $0 \leq \sin^2(y) \leq 1$ for all $y \in \mathbb{R}$,

$$f((0,1)) = [0,1].$$

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Exercise 4. Assume $f:(a,b)\to\mathbb{R}$ is differentiable at all $x\in(a,b)$. Assume that $f'(x)\neq 0$ for all $x\in(a,b)$ and that for some $c\in(a,b)$ we have $f'(c)>0$. Prove that $f'(x)>0$ for all $x\in(a,b)$.

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Exercise 5. If $f'(x) > 0$ for all $x \in (a, b)$, prove that $f(x)$ is strictly increasing on (a, b).

Proof: Let $m, n \in (a, b)$ such that $n > m$. According to the mean value theorem, there exists a $c \in (m, n)$ such that

$$f'(c) = \frac{f(n) - f(m)}{n - m}.$$

Since $c \in (a, b)$, we know that $f'(c) > 0$. Also, since $n > m$, we get that $n - m > 0$. Thus,

$$f(n) - f(m) > 0.$$

Since m, n are arbitrary points in (a, b) that satisfy the condition $n > m$, and $f(n) > f(m)$, $f(x)$ is strictly increasing on (a, b). ∎

Exercise 6. Let

$$f_n(x) = \frac{nx}{1+nx^2}.$$

a) Find the pointwise limit f of (f_n) for all $x \in (0, \infty)$.

We want to show that $f = \frac{1}{x}$ is the pointwise limit of (f_n).

Proof: Given an $\epsilon > 0$, let $N = \left(\frac{1}{\epsilon} - x\right)/x^3$, then whenever $n > N$, it follows that

$$\begin{aligned}|f_n(x) - f(x)| &= \left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| \\ &= \left|\frac{-1}{x(1+nx^2)}\right| \\ &< \frac{1}{x+Nx^3} \\ &= \frac{1}{x+\left(\frac{1}{\epsilon}-x\right)x^3/x^3} \\ &= \epsilon.\end{aligned}$$

Therefore, (f_n) converges pointwise to $f = \frac{1}{x}$. ■

b) Either prove that the converges is uniform or is not uniform on $(0, 1)$.

We want to show that f is not uniform on $(0, 1)$.

Proof: From part (a) we got that

$$|f_n(x) - f(x)| = \left|\frac{1}{x(1+nx^2)}\right|.$$

Let $\epsilon_0 > 0$. Given any $n \in N$, there exists a $x \in (0, 1)$ such that

$$\left|\frac{1}{x(1+nx^2)}\right| > \epsilon_0.$$

For example, let $x < \left|\sqrt{\frac{1}{\epsilon_0 n} - \frac{1}{n}}\right|$, then we get

$$\begin{aligned}\left|\frac{1}{x(1+nx^2)}\right| &> \frac{1}{1+nx^2} \\ &> \frac{1}{1+n\left(\frac{1}{\epsilon_0 n} - \frac{1}{n}\right)} \\ &= \epsilon_0.\end{aligned}$$

Therefore, the sequence does not converge uniformly on $(0, 1)$. ■

c) Either prove that the convergence is uniform or is not uniform on $(1, \infty)$.

We want to show that f is uniform on $(1, \infty)$.

Proof: Using some of the calculations from from part (a), then given an $\epsilon > 0$, let $N = \frac{1}{\epsilon}$. Then, whenever $n > N$

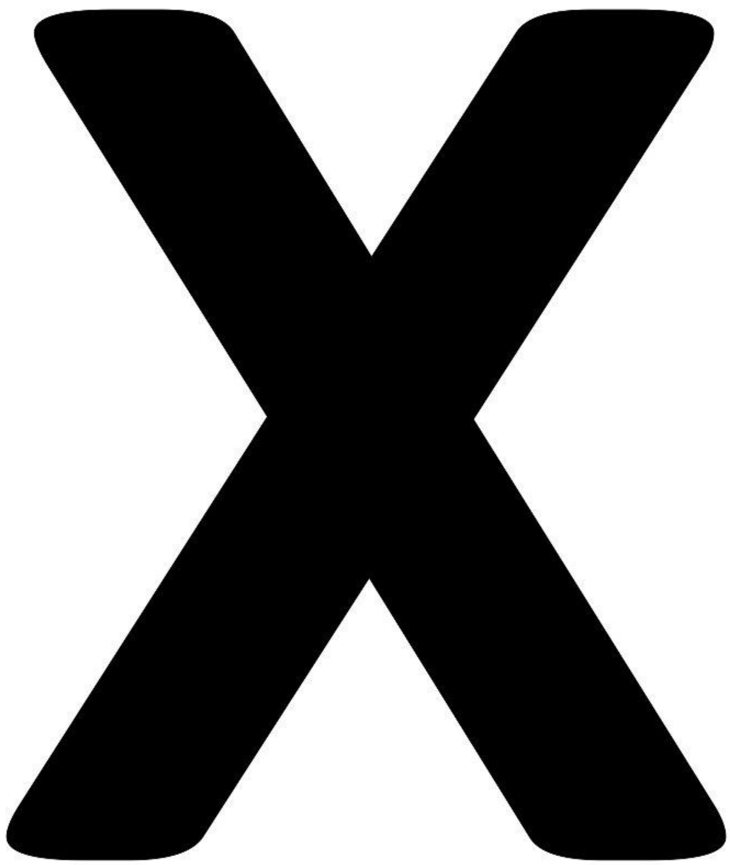
$$\begin{aligned}|f_n(x) - f(x)| &= \left|\frac{1}{x(1+nx^2)}\right| \\ &< \left|\frac{1}{(1+n)}\right| \\ &< \frac{1}{n} \\ &< \frac{1}{N} \\ &= \epsilon.\end{aligned}$$

Therefore, the sequence converges uniformly on $(1, \infty)$. ■

Exercise 7. Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Assume that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon$$

whenever $m, n \geq N$ and $x \in A$. Prove that the sequence (f_n) converges uniformly.



Exercise 8. Let f be a function defined on an open interval $A \subseteq \mathbb{R}$. Show that if f is differentiable at $c \in A$, then f is continuous at c.

Proof: We assume directly that f is differentiable at c, then given $x \in A$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Using the algebraic limit theorem, it follows that

$$\begin{aligned}
\lim_{x \to c} f(x) - f(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}(x - c) \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot 0 \\
&= 0,
\end{aligned}$$

thus

$$\lim_{x \to c} f(x) = f(c)$$

which shows that f is continuous at c. ■

Exercise 9. Assume $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. For each $n \in \mathbb{N}$, set

$$f_n(x) = f\left(x + \frac{1}{n}\right).$$

Show that (f_n) converges uniformly to f on $\mathbb{R}$.

Proof: Since $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, given an $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|m - c| < \delta$ for any $m, c \in \mathbb{R}$ we get that

$$|f(m) - f(c)| < \epsilon.$$

Since $\left(x + \frac{1}{n}\right) \to x$ as $n \to \infty$, given any $\delta > 0$, there exists a $N > 0$ such that whenever $n > N$, $\left|x + \frac{1}{n} - x\right| < \delta$ and

$$\left|f\left(x + \frac{1}{n}\right) - f(x)\right| < \epsilon$$

for all $x \in \mathbb{R}$. Replacing $f\left(x + \frac{1}{n}\right)$ with $f_n(x)$ yields

$$|f_n(x) - f(x)| < \epsilon,$$

therefore, (f_n) converges uniformly to f on $\mathbb{R}$.

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Exercise 10. Prove the following theorem:

Continuous Limit Theorem. Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converge uniformly on A to the function f. If f_n is continuous at $c \in A$, then f is continuous at c.

Proof: We suppose directly that (f_n) converges uniformly on A to the function f and that f_n is continuous at $c \in A$. Thus, given an $\epsilon > 0$, there exists an $N \in \mathbb{R}$ such that whenever $n > N$

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in A$. Also, since $f_n(x)$ is continuous at c, given an $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$,

$$|f_n(x) - f_n(c)| < \frac{\epsilon}{3}.$$

Using the facts above, it follows that given the ϵ defined above, when $n > N$ and $|x - c| < \delta$

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\ &= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Therefore, f is continuous at c.

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Exercise 11. Consider the sequence of functions

$$h_n = \sqrt{x^2 + \frac{1}{n}}.$$

a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbb{R}$.

We want to show that the pointwise limit is $h(x) = \sqrt{x^2}$ and that (h_n) converges uniformly to it.

Proof: Given an $\epsilon > 0$, let $N = \epsilon^2$, then whenever $n > N$, it follows that

$$\begin{aligned}
|h_n(x) - h(x)| &= \left|\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}\right| \\
&= \left|\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}\right| \frac{\left|\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}\right|}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} \\
&= \left|\frac{x^2 + \frac{1}{n} - x^2}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}}\right| \\
&= \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} \\
&\leq \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} \\
&= \frac{1}{\sqrt{n}} \\
&< \epsilon,
\end{aligned}$$

Therefore, (h_n) converges uniformly to h on all of $\mathbb{R}$. ■

b) Compute $h_n'(x)$ and find $g(x) = \lim_{n\to\infty} h_n'(x)$ for $x \in \mathbb{R}$. Explain how we can be certain that the convergence is not uniform in any neighborhood of zero.

Taking the derivative of h_n yields

$$h_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}.$$

Taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \frac{x}{\sqrt{x^2 + \frac{1}{n}}} = \frac{1}{x}.$$

Thus $g(x) = \frac{1}{x}$.

Since (h_n) converges to h, if h_n' converged uniformly to $g(x)$, then according the the limit of derivative theorem (something like this), $h' = g$ for all x. However; $h'(x)$ does not exist at 0 since it's the absolute value function; therefore, h_n' cannot converge uniformly to $g(x)$. Otherwise, it would be a contradiction.

Exercise 12. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the following three conditions:
(i) f is twice-differentiable on all of $\mathbb{R}$.
(ii) $f'(0) = -5$ and $f(0) = 3$.
(iii) $|f''(x)| \leq 2$.
Show that

a) $|f'(x)| \leq 2|x| + 5$ for all $x \in \mathbb{R}$.
Proof: We assume directly that f is twice-differentiable on all of $\mathbb{R}$, $|f''(x)| \leq 2$ and that $f'(0) = -5$. With no loss in generality, we also suppose that $x > 0$ According to the mean value theorem, there exists a $c \in (0, x)$ such that

$$f''(c) = \frac{f'(x) - f'(0)}{x - 0}.$$

Taking the absolute value of both sides yields

$$|f''(c)| = \frac{|f'(x) - f'(0)|}{|x - 0|},$$

thus

$$\begin{aligned} 2 &\geq \frac{|f'(x) - (-5)|}{|x|} \\ &\geq \frac{|f'(x) + 5|}{|x|}, \end{aligned}$$

which implies that

$$\begin{aligned} -2|x| \leq f'(x) + 5 \leq 2|x| \\ -2|x| - 5 \leq f'(x) \leq 2|x| - 5. \end{aligned}$$

Thus

$$|f'(x)| \leq 2|x| + 5.$$

The case for when $x < 0$ is similar. When $x = 0$, we simply get

$$|f'(0)| = 5 \leq 5.$$

Thus for all $x \in \mathbb{R}$, $|f'(x)| \leq 2|x| + 5$. ■

b) $|f(x)| \leq 2|x|^2 + 5|x| + 3$
Proof: We assume directly that f is twice differentiable on all of $\mathbb{R}$, $f(0) = 3$ and that $|f'(y)| \leq 2|y| + 5$. With no loss in generality we suppose that $x > 0$. According tot he mean value theorem, there exists a $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$

Taking the absolute value of both sides yields

$$|f'(c)| = \left| \frac{f(x) - f(0)}{x} \right|,$$

which is simplified to

$$|f'(c)| = \left| \frac{f(x) - 3}{x} \right|.$$

Since $|f'(c)| \leq 2|c| + 5$, we get

$$\frac{|f(x) - 3|}{|x|} \leq 2|c| + 5.$$

Since $|c| < |x|$ we get

$$\begin{aligned} |f(x) - 3| &\leq 2|c||x| + 5|x| \\ &\leq 2|x|^2 + f|x|. \end{aligned}$$

Adding 3 to both sides and using the triangle inequality gives

$$\begin{aligned} |f(x) - 3| + 3 &\leq 2|x|^2 + f|x| + 3 \\ |f(x) - 3 + 3| \leq |f(x) - 3| + 3 &\leq 2|x|^2 + f|x| + 3 \\ |f(x)| &\leq 2|x|^2 + f|x| + 3. \end{aligned}$$

The case for when $x < 0$ is similar. When $x = 0$, we simply get that $|f(0)| = 3 \leq 3$. Therefore, for all $x \in \mathbb{R}$, $|f(x)| \leq 2|x|^2 + 5|x| + 3$. ■