Homework 25 Section 5.3

Mark Petersen

Exercises: 6,7,8,9

07/28/2020

Exercise 1. **(Q6)**: Complete the following:

a) Let $g : [0, a] \to \mathbb{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0, a]$. Show $|g(x)| \leq Mx$ for all $x \in [0, a]$.
Proof: Let $b, c \in [0, a]$ such that they satisfy the mean value theorem

$$g'(c) = \frac{g(0) - g(b)}{0 - b}.$$

Since $|g'(x)| \leq M$, we get

$$\begin{aligned} M &\geq \left| \frac{g(0) - g(b)}{0 - b} \right| \\ &\geq |g(b)| / |b|, \end{aligned}$$

which shows that $|g(b)| \leq Mb$ since $b \geq 0$. ■

b) Let $h : [0, a] \to \mathbb{R}$ be twice differentiable, $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$ for all $x \in [0, a]$. Show that $|h(x)| \leq Mx^2/2$.
Proof: Let $b, c, d, e \in [0, a]$ such that they satisfy the mean value theorem

$$h'(c) = \frac{h(0) - h(b)}{0 - b},$$

and

$$h''(d) = \frac{h'(0) - h'(e)}{0 - e}.$$

Since $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$, we get that

$$|h'(e)| \leq M|e|.$$

Let $g(x) = Mx^2/2$. Using the generalized MVT, we get

$$\frac{h'(c)}{g'(b)} = \frac{h(b) - h(0)}{g(b) - g(0)} = \frac{h(b)}{g(b)}.$$

Since $c \in (0, b)$, we get that $|h'(b)/g'(b)| \leq 1$, which shows that $h(b) \leq g(b)$ and implies that $h(b) \leq Mx^2/2$. ■

c) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

a) **Conjecture**: Let $f : [0, a] \to \mathbb{R}$ be three times differentiable, $f''(0) = f'(0) = f(0)$ and $|f''(x)| \leq M$ for all $x \in [0, a]$, then $|h(x)| \leq Mx^3/6$.

Proof: Let $b, c \in [0, a]$, then from part (b) we know that

$$|f'(x)| \leq Mx^2/2.$$

Let $h(x) = Mx^3/6$. According to the generalize MVT, we get

$$\frac{f'(c)}{h'(x)} = \frac{f(x)}{h(x)}.$$

Since $|f'(x)/h'(x)| \leq 1$, $|f(x)/h(x)| \leq 1$. This implies that

$$f(x) \leq Mx^3/6.$$

■

Exercise 2. (Q7): A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Proof: We suppose by contradiction that if f differentiable on an interval A with $f'(x) \neq 1$ for all $x \in A$, then there is more than one fixed point. Suppose that $a, b \in A$ are two fixed points, then according to the MVT, there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b} = 1.$$

This contradicts our assumption. Thus the original statement is true. ■

Exercise 3. (Q8): Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Proof: Let $h(x) = x$. By the generalized mean value theorem

$$\frac{f'(c)}{h'(c)} = \frac{f(x) - f(0)}{h(x) - h(0)},$$

which can be simplified to

$$f'(c) = \frac{f(x) - f(0)}{x}.$$

Taking the limit as $x \to 0$ yields

$$\lim_{c\to 0} f'(c) = \lim_{x\to 0} \frac{f'(x) - f(0)}{x - 0} = f'(0),$$

Thus $f'(0) = L$. ■

Exercise 4. (Q9): Assume f and g are as described in Theorem 5.3.6, but now add the assumption that f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Find a short proof for the $0/0$ case of L'Hospital's rule under this stronger hypothesis.

Proof: We suppose directly that f and g are continuous and differentiable on an interval, A, containing a, f' and g' are continuous at a with $g'(a) \neq 0$ and that $f(a) = g(a) = 0$. Then according the the generalized mean value theorem,

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

Taking the limit as $x \to a$ yields

$$\lim_{c\to a} \frac{f'(c)}{g'(c)} = \lim_{x\to a} \frac{f(x) - 0}{g(x) - 0}.$$

Since $f', g', g,$ and f are continuous at a, we get

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}.$$

■