Homework 24 Section 5.3

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Exercises: 1,2,3,4

07/22/2020

Exercise 1. **(Q1)**: Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

for all $x \neq y$ in A.

a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.
Proof: Since f' is continuous, then $f'([a, b])$ is compact and thus bounded by some $M \geq 0$. Using the mean value theorem, there exists a $c \in [a, b]$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Since $|f'(c)| \leq M$, it follows that

$$\left|\frac{f(b) - f(a)}{b - a}\right| \leq M.$$

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b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?
Proof: Yes! Let $M = 1$, then $|f'(x)| < M$ for all $x \in [a, b]$. It follows that

$$|f(b) - f(a)| < M|b - a|,$$

which implies

$$|f(b) - f(a)| \leq c|b - a|$$

for some $0 < c < M$. By the contraction mapping theorem, f is contractive on this set. ■

Exercise 2. **(Q2)**: Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Proof: Since $f'(x) \neq 0$ on A, then (according to the mean value theorem) for all $a, b \in A$ such that $a \neq b$,

$$\frac{f(b) - f(a)}{b - a} = z$$

with $z \neq 0$, thus $f(b) \neq f(a)$. Since a and b are arbitrary, this shows that f is injective.

An example to show that the converse need not be true. Let $A = [0, 1]$ and $f = x^2$, then f is injective on A but $f'(0) = 0$. ■

Exercise 3. **(Q3)**: Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$
Proof: Consider the continuous function $f(x) = h(x) - x$. Note that $f(0) = 1$ and $f(3) = -1$, then by the intermediate value property, there exists a $d \in (0, 3)$ such that $f(d) = 0$. This implies that $h(d) - d = 0$. Thus, there exists a $d \in [0, 3]$ such that $h(d) = d$. ■

b) Argue that at some point c we have $h'(c) = \frac{1}{3}$.

Proof: Using the mean value theorem, there is a $c \in [0,3]$ such that

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{1}{3}.$$

■

c) Argue that $h'(x) = \frac{1}{4}$ at some point in the domain.
Proof: Since $h(1) = 2$ and $h(3) = 2$, there is a point $c \in (1,3)$ such that $h'(c) = 0$ according to the MVT. Also, since $h(0) = 1$ and $h(1) = 2$, there exists a point $b \in (0,1)$ such that $h'(b) = 1$. Now, since $b < c$ and $h'(b) > h'(c)$, according to Darboux' theorem, there exists a $k \in (b,c)$ such that $h'(k) = \frac{1}{4}$ since $h'(c) < \frac{1}{4} < h'(b)$. ■

Exercise 4. (Q4): Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \to 0$ and $x_n \neq 0$.

a) If $f(x_n) = 0$ for all $n \in \mathbb{N}$, show $f(0) = 0$ and $f'(0) = 0$.
Proof: Since f is differentiable on A, then f is continuous. Also, since $(x_n) \to 0$, then $f(x_n) \to f(0)$ which shows that $f(0) = 0$. Now since $f(x_n) = f(x_{n+1}) = 0$, there exists a $y_n \in (x_n, x_{n+1})$ such that $f'(y_n) = 0$ an $y_n \to 0$ as $x_n \to 0$. This implies that $\lim f'(y_n) = 0$ which proves that $f'(0) = 0$. ■

b) Add the assumption that f is twice-differentiable at zero and show that $f''(0) = 0$ as well.
Proof: This proof is similar to the one above. We know that a sequence (y_n) exists such that $f'(y_n) = 0$ for all $n \in \mathbb{N}$ and that f' is continuous. Applying the mean mean value theorem, there exists a $z_n \in (y_n, y_{n+1})$ such that $f''(z_n) = \frac{f'(y_{n+1}) - f'(y_n)}{y_{n+1} - y_n} = 0$ and $(z_n) \to 0$ as $(y_n) \to 0$. This implies that $\lim f''(z_n) = 0$ which proves that $f''(0) = 0$. ■