Homework 12 Section 2.7

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Exercises 5,6,7,8

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Exercise 1. (Q5): Now that we have proved the basic facts about geometric series, supply a proof for Corollary 2.4.7.

Proof: We want to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. Since this is a biconditional statement, we must prove both implications.

($\implies$): We assume directly that $\sum_{n=1}^{\infty} \frac{1}{n^p}$, then $\left(\frac{1}{n^p}\right) \to 0$ as $n \to 0$. If $p \leq 0$, the sequence $\left(\frac{1}{n^p}\right) \not\to 0$ as $n \to 0$. Thus p cannot be less than or equal to 0 according to the divergence test. If $0 < p \leq 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Since the harmonic series diverges, then by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ also diverges when $0 < p \leq 1$. If $p > 1$, then $\left(\frac{1}{n^p}\right)$ is decreasing and satisfies $\frac{1}{n^p} \geq 0$ for all $n \in \mathbb{N}$. Therefore, we can use the Cauchy Condensation Test. According to this test, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if the series $\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}}$ converges. Manipulating this series, we get

$$\sum_{n=0}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=0}^{\infty} 2^{(1-p)n}.$$

This series is a geometric series which only converges if $p > 1$, and since $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges p must be greater than 1. Thus, if $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, then $p > 1$.

($\impliedby$) : We assume that $p > 1$, then by the Cauchy Condensation Test shown above, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Therefore, by proving both implications, we have shown that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. ■

Exercise 2. (Q6): Let's say that a series subverges if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series.

a) If (a_n) is bounded, then $\sum a_n$ subverges.
   a) False, if (a_n) is bounded, then the sequence of partial sum (s_n) is not necessarily bounded. For example, if (a_n) is a sequence of 1s, then (s_n) is not bounded. Since it is not bounded, it doesn't have to contain a subsequence that converges. Just consider the example give, if (a_n) is a sequence of 1s, then (s_n) is monotonically increasing and not bounded, thus there is not subsequence that converges.

b) All convergent series are subvergent.
   a) True, one subsequence of the convergent series is the entire series. Since the entire series converges, this particular subsequence converges. Hence, all convergent series are subvergent.

c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.
   a) True. Let $x_n = \sum_{k=1}^{n} |a_n|$, $s_n = \sum_{k=1}^{n} a_n$. Let (x_{n_k}) denote the subsergent sequence of (x_n), and (s_{n_k}) denote the corresponding subsequence, then $-x_{n_k} \leq s_{n_k} \leq x_{n_k}$ for all k. Since x_{n_k} is a convergent subsequence, it is bounded. Let M denote this bound such that $x_{n_k} \leq M$ for all k. This means that

   $$-M \leq -x_{n_k} \leq s_{n_k} \leq x_{n_k} \leq M,$$

   thus the subsequence (s_{n_k}) is bounded. Since (s_{n_k}) is bounded, by the Bolzano-Weierstrass Theorem, it contains a convergent subsequence. Therefore, (s_n) contains a convergent subsequence and is thus subvergent.

d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

a) False. Let (a_n) be the sequence with the terms

$$(1, -1, 2, -2, 3, -3, \ldots),$$

ans let $s_n = \sum_{k=1}^{n} a_n$, then $s_{2n} = 0$, and is thus a convergent subsequence; however, since (a_n) is unbounded, it doesn't contain a convergent subsequence.

Exercise 3. (Q7): Do the following:

a) Show that if $a_n > 0$ and $\lim (na_n) = \ell$ with $\ell \neq 0$, then the series $\sum a_n$ diverges.
Proof: We suppose directly that $a_n > 0$ and $\lim (na_n) = \ell$ with $\ell \neq 0$. According to the order limit theorem, since $na_n > 0$, then $\ell > 0$. Since (na_n) converges, given and $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n \in \mathbb{N} > N$

$$|na_n - \ell| < \epsilon$$
$$-\epsilon + \ell < na_n < \ell + \epsilon$$
$$\frac{-\epsilon + \ell}{n} < a_n < \frac{\ell + \epsilon}{n}.$$

The value of ϵ can be chosen such that $-\epsilon + \ell > 0$, then we have

$$0 < \frac{\ell - \epsilon}{n} < a_n$$

for all $n > N$. Using the comparison test, since $\sum \frac{\ell - \epsilon}{n}$ diverges, the series $\sum a_n$ also diverges. ■

b) Assume $a_n > 0$ and $\lim \left(n^2 a_n\right)$ exists. Show that $\sum a_n$ converges.
Proof: We suppose directly that $a_n > 0$ and that $\lim \left(n^2 a_n\right)$ exists. Let ℓ denote the limit of $\left(n^2 a_n\right)$. Note that $n^2 a_n > 0$ for all $n \in \mathbb{N}$; thus, according to the order limit theorem, $\ell \geq 0$. Since $\left(n^2 a_n\right)$ converges, there exists an $N \in \mathbb{N}$ such that whenever $n \in \mathbb{N} > N$

$$\left|n^2 a_n - \ell\right| < \frac{\ell}{2},$$

which can be manipulated to yield

$$\frac{\ell}{2n^2} < a_n < \frac{3\ell}{2n^2}.$$

By the comparison test, since $\sum \frac{3\ell}{2n^2}$ converges according to the p-test, the series $\sum a_n$ also converges. ■

Exercise 4. (Q8): Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ converges absolutely.

a) True. Since $\sum a_n$ converges absolutely, given an epsilon $1 > \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \in \mathbb{N} > N$

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon.$$

Since this inequality must hold for when $0 < \epsilon < 1$, the individual terms $|a_{m+1}|, |a_{m+2}|, \ldots, |a_n|$ must be less than 1. Thus

$$|a_{m+1}| < a_{m+1}^2,$$
$$|a_{m+2}| < a_{m+1}^2,$$
$$\vdots$$
$$|a_n| < a_n^2,$$

Hence

$$a_{m+1}^2 + a_{m+2}^2 + \cdots + a_n^2 < \epsilon.$$

Therefore, given any $\epsilon >$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \in \mathbb{N} > N$

$$\sum_{k=m+1}^{n} a_k^2 < \epsilon,$$

thus $\sum a_n^2$ converges absolutely.

b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

   a) False, let $a_n = \frac{(-1)^{n+1}}{n^{0.1}}$, then by the alternating series test, $\sum a_n$ converges. Also, let $b_n = \frac{(-1)^{n+1}}{n^{0.9}}$, then (b_n) converges. Note that

$$a_n b_n = \frac{(-1)^{n+1}(-1)^{n+1}}{n^{0.1}n^{0.9}} = \frac{1}{n},$$

   thus

$$\sum a_n b_n = \sum \frac{1}{n},$$

   which is the harmonic series and does not converge. If (b_n) were a monotonic bounded series, then by Abel's test, $\sum a_n b_n$ would converge.

c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

   a) True. Suppose by contrapositive that $\sum n^2 a_n$ converges, then the series $\left(n^2 a_n\right) \to 0$ and $n \to \infty$. Thus, there exists and $N \in \mathbb{N}$ such that when $n \in \mathbb{N} > N$,

$$\begin{aligned} \left|n^2 a_n\right| &< 1 \\ |a_n| &< \frac{1}{n^2}. \end{aligned}$$

   By the comparison test, $\sum a_n$ converges absolutely, thus the original statement must be true.