Homework 23 Section 5.2

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Exercises: 7,8,9,11,12

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Exercise 1. **(Q7)**: Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if x=0} \end{cases}.$$

Find a particular (potentially noninterger value) for a so that

a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0,1]$.

  a) Differentiating $x^a \sin(1/x)$ gives us

  $$ax^{a-1}\sin(1/x) - x^{a-2}\cos(1/x),$$

  which gets large on the interval $[0,1]$ for any $a < 2$. Recall from exercise (5.2.5) that x^a is differentiable as long as $a > 1$. Since $\sin(1/x)$ is differentiable everywhere, the function

  $$x^a \sin(1/x)$$

  is differentiable as long as $a > 1$. Thus, let $a \in (1,2)$, then $g_a(x)$ is differentiable but unbounded on $[0,1]$.

b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at 0.

  a) In order to have g_a to be differentiable on $\mathbb{R}$ with g_a' continuous, we need $a > 2$. Recall

  $$g_a'(x) = ax^{a-1}\sin(1/x) - x^{a-2}\cos(1/x).$$

  For g_a' to be differentiable, we need $ax^{a-1}\sin(1/x)$ and $x^{a-2}\cos(1/x)$ to be differentiable at 0.

  $$\left(ax^{a-1}\sin(1/x)\right)' = a(a-1)x^{a-2}\sin(1/x) - ax^{a-3}\cos(1/x)$$

  which is not differentiable at $a < 3$. So let $a = 2.5$.

c) g_a is differentiable on $\mathbb{R}$ with g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

  a) In order for g_a' to be differentiable on $\mathbb{R}$, we need $a > 4$, since

  $$\left(x^{a-2}\cos(1/x)\right)' = (a-2)x^{a-3}\cos(1/x) + x^{a-4}\sin(1/x).$$

  Thus, let $a = 5$, then

  $$g_a'(x) = \left(9x^3 - x\right)\sin(1/x) - \left(x^3 + 3x^2\right)\cos(1/x)$$

Exercise 2. **(Q8)**: Review the definition of uniform continuity. Given a differentiable function $f : A \to \mathbb{R}$, let's say that f is uniformly differentiable on A if, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \epsilon \quad \text{whenever } 0 < |x-y| < \delta.$$

a) Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$? How about $g(x) = x^3$?

  Proof: Let $f(x) = x^2$, then given an $\epsilon > 0$, let $\delta = \epsilon$ such that whenever $0 < |x-y| < \delta$, it follows that

  $$\begin{aligned} \left|\frac{x^2 - y^2}{x - y} - 2y\right| &= \left|\frac{(x+y)(x-y)}{x-y} - 2y\right| \\ &= |x - y| \\ &< \delta \\ &= \epsilon, \end{aligned}$$

  thus $f(x)$ is uniformly differentiable on $\mathbb{R}$.

Let $g(x) = x^3$, then given an $\epsilon > 0$, let $\delta = ?$ such that whenever $0 < |x - y| < \delta$, it follows that

$$\begin{aligned}\left|\frac{x^3 - y^3}{x - y} - 3y^2\right| &= \left|\frac{(x - y)\left(x^2 + xy + y^2\right)}{x - y} - 3y^2\right| \\ &= \left|x^2 + xy - 2y^2\right| \\ &= \left|(x - y)(x + 2y)\right|,\end{aligned}$$

since the domain is not bounded, δ will be a function of x, y and ϵ. Thus it is not uniformly differentiable on $\mathbb{R}$. ■

b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.
Proof: We assume directly that f is uniformly differentiable on an interval A, then given an $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - y| < \delta$,

$$\left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \epsilon/2.$$

It follows that

$$\begin{aligned}|f'(x) - f'(y)| &= \left|f'(x) - \frac{f(x) - f(y)}{x - y} + \frac{f(x) - f(y)}{x - y} - f'(y)\right| \\ &\le \left|f'(x) - \frac{f(x) - f(y)}{x - y}\right| + \left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon,\end{aligned}$$

thus the derivative is continuous on A. ■

c) Is there a theorem analogous to Theorem 4.4.7 for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable.
Disproof: We want to show by example that functions that are differentiable on a closed interval $[a, b]$ are not necessarily uniformly differentiable. Since uniformly differentiable implies that the derivative is continuous, we just need to find a function whose derivative exists on the interval but is not continuous. Using the findings from exercise 7, let

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

be defined on the closed interval $[-2, 2]$, then $f(x)$ is differentiable, but the derivative is not continuous. Hence the statement is false. ■

Exercise 3. (Q9): Decide whether each conjecture is true or false. Provide an argument for those that are not true and a counterexample for each one that is false.

a) If f' exists on an interval and is not constant, then f' must take on some irrational values.
Proof: True! Since f' is not constant but exists, f is continuous and there must exist an a, b in the interval such that $a < b$ (or $b > a$) and $f'(a) \neq f'(b)$. According to Darboux' theorem, given any $\alpha \in (f'(a), f'(b))$, there exists a $c \in (a, b)$ such that $f'(c) = \alpha$. Due to the density of the real numbers, we can pick α to be an irrational number. ■

b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a $\delta-$neighborhood $V_\delta(c)$ around c is which $f'(x) > 0$ for all $x \in V_\delta(c)$.
Disproof: Consider the function

$$f(x) = \begin{cases} x + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases},$$

then

$$f'(x) = \begin{cases} \frac{1}{4} + 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ \frac{1}{4} & \text{else} \end{cases}.$$

Thus, $f'(0) > 0$. Now consider the $\epsilon-$neighborhood, $V_\epsilon(0)$. We need to show that there exists an $x \in V_\epsilon(0)$ such that $f'(x) < 0$. Consider the sequence $x_n = \frac{1}{2\pi n}$. Note that $(x_n) \to 0$ as $n \to 0$. So, there will always exists some $N \in \mathbb{N}$ such that $x_N \in V_\epsilon(0)$. Well, $f'(x_N) = \frac{1}{4} + 2\frac{1}{2\pi N}\sin(2\pi N) - \cos(2\pi N) = -\frac{3}{4}$. Thus, for every $V_\epsilon(0)$, there exists a $x \in V_\epsilon(0)$ such that $f'(x) < 0$. ■

c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.
Disproof: False, since there is no guarantee that f' is continuous, the statement won't necessarily hold. For example, consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

which is differentiable on the interval $[-2, 2]$. Now, take the sequence $x_n = \frac{1}{2\pi n}$. The sequence $(x_n) \to 0$ as $n \to \infty$, but

$$\begin{aligned} \lim_{n\to\infty} f'(x_n) &= \lim_{n\to\infty} 2x_n \sin(1/x_n) - \cos(1/x_n) \\ &= \lim_{n\to\infty} (-\cos(2\pi n)) \\ &= -1, \end{aligned}$$

which is not $f(0) = 0$. Thus the statement is false. ■

Example 4. (Q11): Assume that g is differentiable of $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$.

a) Show that there exists a point $x \in (a, b)$ where $g(a) > g(x)$, and a point $y \in (a, b)$ where $g(y) < g(b)$.
Proof: By definition

$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a}.$$

Since $g'(a) < 0$,

$$\lim_{x\to a} \frac{g(x) - g(a)}{x - a} < 0.$$

Since $x > a$, the denominator must be positive. This implies that $g(x) < g(a)$ as $x \to a$. Thus there exists an x arbitrarily close to a such that $g(a) > g(x)$. The proof is similar to show that there exists a $y \in (a, b)$ such that $g(y) < g(b)$. ■

b) Now complete the proof of Darboux's Theorem started earlier. (The first part of the proof is taken from the book)
Proof: We first simplify matters by defining a new function $g(x) = f(x) - \alpha x$ on $[a, b]$. Notice that g is differentiable on $[a, b]$ with $g'(x) = f'(x) - \alpha$. In terms of g, our hypothesis states that $g'(a) < 0 < g'(b)$, and we hope to show that $g'(c) = 0$ for some $c \in (a, b)$. According to part (1), the function g has a minimum on the interval (a, b), thus for some $c \in (a, b)$, $g'(c) = 0$. This implies that

$$\begin{aligned} g'(c) &= f'(c) - \alpha = 0 \\ &\implies f'(c) = \alpha, \end{aligned}$$

so there exists a $c \in (a, b)$ such that $f'(c) = \alpha$. ■

Example 5. (Q12): If $f : [a, b] \to \mathbb{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In exercise 4.5.8 we saw that if f is continuous on $[a, b]$, then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a, b]$ with $f'(x) \neq 0$ for all $x \in [a, b]$. Show that f^{-1} is differentiable with

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x).$$

Proof: Using the definition of the derivative

$$\left(f^{-1}\right)' = \lim_{z\to y} \frac{f^{-1}(z) - f^{-1}(y)}{z - y},$$

where $z \in f([a,b])$. This is possible since f is continuous, thus $f([a,b])$ is a closed interval. Then there exists an $k \in [a,b]$ such that $f(k) = z$. Using this we get

$$\begin{aligned}\lim_{z \to y} \frac{f^{-1}(z) - f^{-1}(y)}{z - y} &= \lim_{k \to x} \frac{f^{-1}(f(k)) - f^{-1}(f(x))}{f(k) - f(x)} \\ &= \lim_{k \to x} \frac{k - x}{f(k) - f(x)} \\ &= \frac{1}{f'(x)}.\end{aligned}$$

■