Homework 14 Section 3.3

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Exercises 9,10,11,12,14

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Exercise 1. **(Q9)**: A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.5. The general argument is similar.

a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c$$

and

$$(\cap_{\lambda\in\Lambda} E_\lambda)^c = \cup_{\lambda\in\Lambda} E_\lambda^c.$$

Proof: We will first show that $(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c$. Since this is an equality, we must show that $(\cup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \cap_{\lambda\in\Lambda} E_\lambda^c$ and $(\cup_{\lambda\in\Lambda} E_\lambda)^c \supseteq \cap_{\lambda\in\Lambda} E_\lambda^c$.
$(\subseteq)$: Let $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$, then $x \notin \cup_{\lambda\in\Lambda} E_\lambda$. In other words, $x \in E_\lambda^c$ for all $\lambda \in \Lambda$. Therefore, $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$.
$(\supseteq)$: Let $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$, then $x \in E_\lambda^c$ for all $\lambda \in \Lambda$. This indicates that $x \notin E_\lambda$ for any $\lambda \in \Lambda$. Thus $x \notin \cup_{\lambda\in\Lambda} E_\lambda$, and so $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$.
Since both inclusions hold, we have that $(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c$.
We next show that $(\cap_{\lambda\in\Lambda} E_\lambda)^c = \cup_{\lambda\in\Lambda} E_\lambda^c$. Since this is an equality statement, we must show that $(\cap_{\lambda\in\Lambda} E_\lambda)^c \subseteq \cup_{\lambda\in\Lambda} E_\lambda^c$ and $(\cap_{\lambda\in\Lambda} E_\lambda)^c \supseteq \cup_{\lambda\in\Lambda} E_\lambda^c$.
$(\subseteq)$: Let $x \in (\cap_{\lambda\in\Lambda} E_\lambda)^c$, then $x \notin \cap_{\lambda\in\Lambda} E_\lambda$. In other words, $x \notin E_k$ for some $k \in \Lambda$. Thus $x \in E_k^c$, and so $x \in \cup_{\lambda\in\Lambda} E_\lambda^c$.
$(\supseteq)$: Let $x \in \cup_{\lambda\in\Lambda} E_\lambda^c$, then $x \in E_\lambda^c$ for all $\lambda \in \Lambda$. Thus $x \notin E_\lambda$ for all Λ. Which implies that $x \notin \cap_{\lambda\in\Lambda} E_\lambda$. Therefore, $x \in (\cap_{\lambda\in\Lambda} E_\lambda)^c$.
Since both inclusions hold, we have that $(\cap_{\lambda\in\Lambda} E_\lambda)^c = \cup_{\lambda\in\Lambda} E_\lambda^c$. ■

b) Now, provide the details for the proof of Theorem 3.2.14.
Proof: From Theorem 3.2.3 we know that (i) The union of an arbitrary collection of open sets is open, and (ii) The intersection of a finite collection of open sets is open. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of open and let $O = \cup_{\lambda\in\Lambda} O_\lambda$. Taking the complements of both sides gives

$$\begin{aligned} O^c &= (\cup_{\lambda\in\Lambda} O_\lambda)^c \\ &= \cap_{\lambda\in\Lambda} O_\lambda^c, \end{aligned}$$

thus, the intersection of a arbitrary collection of closed sets is closed. Now let $\{O_1, O_2, \ldots, O_n\}$ be a finite collection of open sets and $O = \cap_{k=1}^n O_k$. Taking the complement of both sides yields

$$\begin{aligned} O^c &= (\cap_{k=1}^n O_k)^c \\ &= \cup_{k=1}^n O_k^c; \end{aligned}$$

thus, the finite union of closed sets is closed. ■

Exercise 2. **(Q10)**: Only one of the following three descriptions can be realized. Provide an example that illustrates the viable description, and explain why the other two cannot exist.

a) A countable set contained in $[0, 1]$ with no limit points.
   a) This cannot exist. Since the set is bounded, by the Bolzano-Weierstrass theorem, there exists a limit point in the set.
b) A countable set contained in $[0, 1]$ with no isolated points.
   a) This can exist. Let $A = \{x \in \mathbb{Q} : x \in [0, 1]\}$. Since the rational numbers don't have any isolated points, A won't.

c) A set with an uncountable number of isolated points.
   a) This cannot exist. Let A be the set and $B = \{x_\lambda : \lambda \in \Lambda\}$ be the set of all of the isolated points of A. Then for each x_λ, there exists an $\epsilon > 0$ such that $V_\epsilon(x_\lambda) \cap A = \{x_\lambda\}$. Due to the density or $\mathbb{R}$, there exists at least one rational number $q_\lambda \in V_\epsilon(x_\lambda)$ such that $q_\lambda \neq x_\lambda$. By taking one rational number within the set of each neighborhood, $V_\epsilon(x_\lambda)$, we can construct the set $C = \{q_\lambda : \lambda \in \Lambda\}$ and the bijection $f : C \to A$. Where $f(q_\lambda) = x_\lambda$. Since C is not an uncountable set, there cannot exist an uncountable number of isolate points.

Exercise 3. (Q11): Do the following.

a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
Proof: This is an equality of sets, so we must show inclusion both ways.
$(\implies)$: Suppose $y \in \overline{A \cup B}$, then $y \in A \cup B \cup L_{AB}$, with L_{AB} denoting the set of limit points of $A \cup B$. Let L_A and L_B denote the set of limit points of A (respectively B). Let $x \in L_{AB}$, then for an arbitrary $\epsilon-$neighborhood, there is an element a such that $a \in V_\epsilon(x) \cap (A \cup B)$ which is equivalent to

$$a \in (V_\epsilon(x) \cap A) \cup (V_\epsilon(x) \cap B),$$

thus x must be a limit point of A and/or B. Which means that $x \in L_A \cup L_B$. Using this fact, we get that

$$\begin{aligned} y &\in A \cup B \cup L_{AB} \\ &\in A \cup B \cup L_A \cup L_B \\ &\in \overline{A} \cup \overline{B}. \end{aligned}$$

Hence, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.
$(\impliedby)$: Suppose $y \in \overline{A} \cup \overline{B}$, then $y \in A \cup B \cup L_A \cup L_B$. Let $x \in L_A \cup L_B$, then given an arbitrary $\epsilon-$neighborhood, there is an element a such that $a \in (V_\epsilon(x) \cap A) \cup (V_\epsilon(x) \cap B)$ which is equivalent to

$$a \in V_\epsilon(x) \cap (A \cup B),$$

thus $x \in L_{AB}$, So

$$\begin{aligned} y &\in A \cup B \cup L_{AB} \\ &\in \overline{A \cup B}. \end{aligned}$$

Hence $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$. Since we have shown inclusions for both sides, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. ∎

b) Does this result about closures extend to infinite unions of sets?
   a) No. Consider the sets $A_i = \left\{\frac{1}{i}\right\}$ where $i \in \mathbb{N}$. Since each A_i has only one element, it doesn't contain any limit points. Thus $A_i = \overline{A}_i$, and therefore,

$$\cup_{i \in \mathbb{N}} \overline{A}_i = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}.$$

   Now consider the set $B = \cup_{i \in \mathbb{N}} A_i$, which has a limit point 0. Thus $\overline{B} = B \cup \{0\}$ which is not equivalent to $\cup_{i \in \mathbb{N}} \overline{A}_i$.

Exercise 4. (Q12): Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x : x \in A \text{ and } x < s\}$ and $\{x : x \in A \text{ and } x > s\}$ are uncountable. Show that B is nonempty and open.

Proof: Let $X_r = \{x : x \in A \text{ and } x < r\}$ and $Y_r = \{x : x \in A \text{ and } x > r\}$. Let T_l be the set of all $r \in \mathbb{R}$ such that X_r is countable and T_u be the set of all $r \in \mathbb{R}$ such that Y_r is countable. Next, let $t_l = \sup(T_l)$ and $t_u = \inf(T_u)$. Since X_{t_l} and Y_{t_u} are countable sets, their union is countable, thus $A \neq X_{t_l} \cup Y_{t_u}$ since A is uncountable. This means that there is still an uncountable many elements of A that are in the interval (t_l, t_u). Thus we see that $t_l < t_u$. Let $\epsilon > 0$, since $t_l = \sup(T_l)$ and $t_u = \inf(T_u)$, the sets $X_{t_l+\epsilon}$ and $X_{t_u-\epsilon}$ are uncountable. Therefore, let $b \in (t_l, t_u)$, then X_b and Y_b are uncountable, thus $b \in B$ which means that B is not empty. Since this is true for any $b \in (t_l, t_u)$, $B = (t_l, t_u)$. Which shows that B is open. ∎

Exercise 5. (Q14): A dual notation to the closure of a set is the interior of a set. The interior of E is denoted E^o and is defined as

$$E^o = \{x \in E : \text{there exists } V_\epsilon(x) \subseteq E\}.$$

Results about closures and interiors possess a useful symmetry.

a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^o = E$.
Proof: We start by showing that E is closed if and only if $\overline{E} = E$. Since this is a biconditional statement, we must prove both ways.
$(\implies)$: Let E be closed, then E contains all of it's limit points. Let L denote the set of the limit points of E, then $L \subseteq E$. Hence $\overline{E} = E$.
$(\impliedby)$: Suppose that $\overline{E} = E$. Then all of the limit points of E must be contained in E. Thus E is closed.
Since both implications are true, E is closed if and only if $\overline{E} = E$.
Next we show that E is open if and only if $E^o = E$. Since this is a biconditional statement, we must prove both ways.
$(\implies)$: Let E be open, then for every $x \in E$, there exists $V_\epsilon(x) \subseteq E$. Therefore, every $x \in E$ is also an element of E^0. Since $E^o \subseteq E$, $E^o = E$.
$(\impliedby)$: Suppose that $E^o = E$, then there exists a $V_\epsilon(x) \subseteq E$ for every $x \in E$. By definition, E is open. ■

b) Show that $\overline{E}^c = (E^c)^o$, and similarly that $(E^o)^c = \overline{E^c}$.
Proof: We start by showing that $\overline{E}^c = (E^c)^o$. Since this is an equivalent statement between sets, we must show inclusion both ways.
$(\subseteq)$: Let $x \in \overline{E}^c$, then $x \notin \overline{E}$. In other words, $x \notin E \cup L$ where L is the set of the limit points of E. That means, for every $x \in E^c$, there exists an open $\epsilon-$neighborhood such that $V_\epsilon(x) \subseteq E^c$. This is because x is not a limit point of E so it cannot be arbitrarily close to an element of E. Therefore, $x \in (E^c)^o$.
$(\supseteq)$: Let $x \in (E^c)^o$, then for every $x \in E^c$, there exists an $\epsilon-$neighborhood such that $V_\epsilon(x) \subseteq E^c$. Since a neighborhood in entirely contained in E^c, it cannot be a limit point of E. Thus $x \notin E \cup L$ where L is the set of the limit points of E. In other words, $x \notin \overline{E}$, thus $x \in \overline{E}^c$.
Since both inclusions hold, $\overline{E}^c = (E^c)^o$.
Next we show that $(E^o)^c = \overline{E^c}$. Since this is an equivalent statement between sets, we must show inclusion both ways.
$(\subseteq)$: Let $x \in (E^o)^c$, then $x \notin E^o$. In other words, there does not exist an $\epsilon-$neighborhood such that $V_\epsilon(x) \subseteq E^o$. This means that x is either an isolated point of E or not in E. An isolate point of E is a limit point of it's complement since there is a point in E^c that is arbitrarily close to any isolated point of E. Thus $x \in E^c$ or $x \in L_c$ where L_c is the set of limit points of E^c. Therefore, $x \in \overline{E^c}$, which shows that $(E^o)^c \subseteq \overline{E^c}$.
$(\supseteq)$: Let $x \in \overline{E^c}$, then $x \in E^c \cup L_c$ where L_c is the set of limit points of E^c. Since $x \in E^c$ or there exists an $\epsilon-$neighborhood such that there is another element $a \in E^c$ such that $a \in V_\epsilon(x)$, x cannot be an interior point of E, thus $x \notin E^o$. Hence, $x \in (E^o)^c$. Therefore, $(E^o)^c \supseteq \overline{E^c}$.
Since both inclusions hold, $(E^o)^c = \overline{E^c}$. ■