Homework 10 Section 2.6

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Exercises: 2,3,4,5

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Exercise 1. **(Q2)**: Give an example of each of the following, or ague that such a request is impossible.

1) A Cauchy sequence that is not monotone.
   a) Let $a_n = \frac{1}{n}\cos\left(\frac{\pi n}{2}\right)$, then $(a_n) = \left\{1, 0, -\frac{1}{3}, 0, \frac{1}{4}, \ldots\right\}$is not monotone, but it converges to 0.
2) A Cauchy sequence with an unbounded subsequence.
   a) This is not possible. A Cauchy sequence is a convergent sequence. Since every convergent sequence is bounded, then any subsequence is also bounded.
3) A divergent monotone sequence with a Cauchy subsequence.
   a) This is not possible.

   Proof: We suppose directly that (a_n) is a divergent monotone sequence and (a_{n_k}) is a Cauchy subsequence. Then given some epsilon $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $m, n > N$

   $$\left|a_{n_m} - a_{n_j}\right| < \epsilon.$$

   Since (a_n) is monotone, all of the terms between a_{n_m} and a_{n_j} also meet this criteria. In other words, let $x = n_m$ and $y = n_m$ then

   $$|a_x - a_y| < \epsilon$$

   and all the terms between these two terms meet the criteria. Thus (a_n) is a Cauchy sequence. This contradicts the original statement. Therefore it's not possible. ■
4) An unbounded sequence sequence containing a subsequence that is Cauchy.
   a) Let (a_n) be the sequence be the unbounded sequence $\{1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0 \cdots\}$, the subsequence (a_{n_k}) with terms $\{0, 0, 0, 0, 0, 0, 0, 0, 0, \cdots\}$ is bounded, and for all $x, y \in \mathbb{N}$

      $$\left|a_{n_x} - a_{n_y}\right| = 0 < \epsilon$$

      for any $\epsilon > 0$. Thus (a_{n_k}) is Cauchy.

Exercise 2. **(Q3)**: If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

1) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

   Proof: We suppose directly that (x_n) and (y_n) are Cauchy sequences, then given an epsilon $\epsilon > 0$, there exists and $N \in \mathbb{N}$ such that whenever $n, m \in \mathbb{N}$ are greater than N, the following inequalities hold

   $$|x_n - x_m| < \frac{\epsilon}{2}$$
   $$|y_n - y_m| < \frac{\epsilon}{2}.$$

   For the same n and m we have

   $$\begin{aligned} |x_n + y_n - x_m - y_m| &\le |x_n - x_m| + |y_n - y_m| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

   Therefore, if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ is a Cauchy sequence. ■

2) Give a direct argument that $(x_n y_n)$ is a Cauchy sequence.
   Proof: We suppose directly that (x_n) and (y_n) are Cauchy sequences, then (x_n) and (y_n) are bounded, $|x_n| \leq M_x$ and $|y_n| \leq M_y$. Let $M = \max(M_x, M_y)$. Given an epsilon $\epsilon > 0$, there exists and $N \in \mathbb{N}$ such that whenever $n, m \in \mathbb{N}$ are greater than N, the following inequalities hold

$$\begin{aligned} |x_n - x_m| &< \frac{\epsilon}{2M} \\ |y_n - y_m| &< \frac{\epsilon}{2M}. \end{aligned}$$

   For the same n and m we have

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_m y_m + x_n y_m - x_n y_m| \\ &= |x_n (y_n - y_m) + y_m (x_n - x_m)| \\ &\leq |M (y_n - y_m) + M (x_n - x_m)| \\ &\leq M |y_n - y_m| + M |x_n - x_m| \\ &= M \frac{\epsilon}{2M} + M \frac{\epsilon}{2M} \\ &= \epsilon. \end{aligned}$$

   Therefore, if (x_n) and (y_n) are Cauchy sequences, then $(x_n y_n)$ is a Cauchy sequence. ■

Exercise 3. (Q4): Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

1) $c_n = |a_n - b_n|$.
   Proof: We want to show that c_n is a Cauchy sequence. We suppose directly that (x_n) and (y_n) are Cauchy sequences, then given an epsilon $\epsilon > 0$, there exists and $N \in \mathbb{N}$ such that whenever $n, m \in \mathbb{N}$ are greater than N, the following inequalities hold

$$\begin{aligned} |x_n - x_m| &< \frac{\epsilon}{2} \\ |y_n - y_m| &< \frac{\epsilon}{2}. \end{aligned}$$

   It follows that

$$\begin{aligned} |c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\ &\leq |a_n - b_n - a_m + b_m| \\ &\leq |a_n - a_m| + |b_n - b_m| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

   Therefore, c_n is a Cauchy sequence. ■
2) $c_n = (-1)^n a_n$.
   Proof: We want to show that c_n is not a Cauchy sequence with a counterexample. Let $a_n = 1$, then (c_n) has terms $\{-1, 1, -1, 1, \ldots\}$ which does not converge. Therefore, c_n is not necessarily a Cauchy sequence. ■
3) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.
   Proof: We want to show that c_n is not a Cauchy sequence with a counterexample. Let a_n be a non monotonic Cauchy sequence that oscillates around an integer. For example, $a_n = 0.5 \exp(n) \cos(n\pi)$, then (a_n) will oscillate around 0, then (c_n) would have terms $\{0, -1, 0, -1, 0, -1, \ldots\}$ and doesn't converge. Therefore, c_n is not necessarily a Cauchy sequence. ■

Exercise 4. (Q5): Consider the following (invented) definition: A sequence (s_n) is pseudo-Cauchy if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$. Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

1) Pseudo-Cauchy sequences are bounded.

Disproof: Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the partial sum $s_n = \sum_{i=1} \frac{1}{i}^n$. The sequence (s_n) is known to not converge and to not be bounded. However, since $|s_{m+1} - s_m| = \frac{1}{m+1}$ for any $m \in \mathbb{N}$, given an $\epsilon > 0$, there exists an N such that if $m \geq N$ then

$$\frac{1}{m+1} < \epsilon.$$

This follows from the Archimedean property, thus the sequence (s_n) is Pseudo-Cauchy but not bounded. ■

2) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.
Proof: We suppose directly that (x_n) and (y_n) are pseudo-Cauchy sequences, then given an epsilon $\epsilon > 0$, there exists and $N \in \mathbb{N}$ such that whenever $n \in \mathbb{N}$ is greater than N, the following inequalities hold

$$\begin{aligned} |x_{n+1} - x_n| &< \frac{\epsilon}{2} \\ |y_{n+1} - y_n| &< \frac{\epsilon}{2}. \end{aligned}$$

It follow that

$$\begin{aligned} |x_{n+1} + y_{n+1} - x_n - y_n| &\leq |x_{n+1} - x_n| + |y_{n+1} - y_n| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, if (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well. ■