Homework 32 Section 7.3

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Exercises: 1,3,5,7,8

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Exercise 1. **(Q1)**: Consider the function

$$h(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 2 & \text{for } x = 1 \end{cases}$$

over the interval $[0,1]$.

a) Show that $L(f,P) = 1$ for every partition P of $[0,1]$.
Proof: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[0,1]$, then $x_n = 1$. Any closed subinterval of $[0,1]$ that does not include 1 will have an infimum of 1 since 1 is the only value $h(x)$ will take take on the subinterval. Any closed subinterval $[x_{n-1}, x_n]$ will include an element other than 1, thus $\inf [x_{n-1}, x_n] = 1$. Thus

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} m_k (x_k - x_{k-1}) \\ &= 1 \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= 1 (x_n - x_0) \\ &= 1. \end{aligned}$$

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b) Construct a partition P for which $U(f,P) < 1 + \frac{1}{10}$.
 a) Let $P = \left\{0, 1 - \frac{1}{20}, 1\right\}$, then

$$\begin{aligned} U(f,P) &= 1\left(1 - \frac{1}{20} - 0\right) + 2\left(1 - 1 + \frac{1}{20}\right) \\ &= 1 + \frac{2}{20} - \frac{1}{20} \\ &= 1 + \frac{1}{20} \end{aligned}$$

c) Given $\epsilon > 0$, construct a partition P_ϵ for which $U(f, P_\epsilon) < 1 + \epsilon$.
 a) Let $P_\epsilon = \left\{0, 1 - \frac{\epsilon}{2}, 1\right\}$, then

$$\begin{aligned} U(f,P) &= 1\left(1 - \frac{\epsilon}{2}\right) + 2\left(1 - 1 + \frac{\epsilon}{2}\right) \\ &= 1 + \frac{\epsilon}{2}. \end{aligned}$$

Exercise 2. **(Q3)**: Let

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{else} \end{cases}.$$

Show that f is integrable on $[0,1]$ and compute $\int_0^1 f$.

Proof: Given an $\epsilon > 0$, let $A = \left\{\frac{1}{n} : n \in \mathbb{N} \text{ and } \frac{1}{n} > \frac{\epsilon}{2}\right\}$, then A is finite. Let $M = |A|$, y be the smallest element of A, and let P be the partition defined as

$$P = \left\{0, \frac{\epsilon}{2}, y - \frac{\epsilon}{4M}, y + \frac{\epsilon}{4M}, \ldots, \frac{1}{3} - \frac{\epsilon}{4M}, \frac{1}{3} + \frac{\epsilon}{4M}, \frac{1}{2} - \frac{\epsilon}{4M}, \frac{1}{2} + \frac{\epsilon}{4M}, 1 - \frac{\epsilon}{2M}, 1\right\},$$

then

$$\begin{aligned}U(f,P) &= 1\left(\frac{\epsilon}{2}\right)+\sum_{k=2}^{M}\left(\frac{1}{k}+\frac{\epsilon}{4M}-\left(\frac{1}{k}-\frac{\epsilon}{4M}\right)\right)+\left(1-\left(1-\frac{\epsilon}{2M}\right)\right) \quad \text{all other terms are 0}\\ &= \frac{\epsilon}{2}+(M-1)\left(\frac{2\epsilon}{4M}\right)+\frac{\epsilon}{2M}\\ &= \frac{\epsilon}{2}+\frac{\epsilon}{2}\\ &= \epsilon.\end{aligned}$$

Since $L(f,P)=0$ for any partition, we get that

$$U(f,P)-L(f,P)<\epsilon.$$

Therefore, according to the integrability criterion, f is integrable. ■

Exercise 3. (Q5): Provide and example or give a reason why the request is impossible.

a) A sequence $(f_n)\to f$ pointwise, where each f_n has at most a finite number of discontinuities but f is not integrable.
   a) Let $A=\{r_n\}_{n\in\mathbb{N}}$ be an enumeration of the rationals in $[0,1]$ that contains only n rational numbers, and let $f_n:[0,1]\to\mathbb{R}$ be defined as

$$f_n(x)=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{else}\end{cases}$$

   the for any n, f_n has a finite number of discontinuities, but

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{else}\end{cases}$$

   is not integrable since for any partition P, $U(f,P)=1$ and $L(f,P)=0$.
b) A sequence $(g_n)\to g$ uniformly where each g_n has at most a finite number of discontinuities and g is not integrable.
   a) Not possible. By uniform convergence g also has a finite number of continuities. On any compact set, g is integrable.
c) A sequence $(h_n)\to h$ uniformly where each h_n is not integrable but h is integrable.
   a) Let $h_n:[0,1]\to\mathbb{R}$ be define as

$$h_n(x)=\begin{cases}\frac{1}{n} & \text{if } x\in\mathbb{Q}\\ 0 & \text{else}\end{cases},$$

   then each h_n is not Riemann integrable. However, as $n\to\infty$, $h_n\to 0$ and $h=0$ is Riemann integrable.

Exercise 4. (Q7): Assume $f:[a,b]\to\mathbb{R}$ is integrable.

a) Show that if g satisfies $g(x)=f(x)$ for all but a finite number of points in $[a,b]$, then g is integrable as well.
   Proof: Let $M=\sup\{g(x)-g(y) : x,y\in[a,b]\}$ and N be the number of points at which $g\neq f$. Since f is integrable, given an $\epsilon>0$, there exists a partition P such that

$$U(f,P)-L(f,P)<\frac{\epsilon}{2}.$$

   We can construct a partition P such that the intervals $\Delta x_k<\frac{\epsilon}{2MN}$. Let Q be the collection of intervals where $g\neq f$ and R be the rest, then

$$\begin{aligned}U(g,P)-L(g,P) &= U(g,Q)-L(g,Q)+U(g,R)-U(g,R)\\ &< U(g,Q)-L(g,Q)+U(f,R)-U(f,R)\\ &< NM\frac{\epsilon}{2MN}+\frac{\epsilon}{2}\\ &= \epsilon,\end{aligned}$$

   thus g is also integrable. ■
b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.
   a) Let f and g be functions whose domain is $[0,1]$ and let $f(x)=0$ and

$$g(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{else}\end{cases}.$$

Since $\mathbb{Q}$ is a countable set, then g differs from f a countable number of times, but we already know that g is not integrable.

Exercise 5. (Q8): As in Exercise 7.3.6, let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rationals in $[0,1]$, but this time define

$$h_n(x) = \begin{cases} 1 & \text{if } r_n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq r_n \end{cases}$$

Show $H(x) = \sum_{n=1}^{\infty} h_n(x)/2^n$ is integrable on $[0,1]$ even though it had discontinuities at every rational point.

Proof: Since $h_n(x) \leq \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} h_n(x)/2^n$ converges uniformly. Then according to Cauchy criterion for uniform convergence, given an $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that

$$\left| \sum_{k=m}^{\infty} h_k(x) \right| < \frac{\epsilon}{2}$$

for all $x \in [0,1]$. Thus, for any partition Q,

$$U\left(\sum_{k=m}^{\infty} h_k(x), Q \right) < \frac{\epsilon}{2},$$

and

$$L\left(\sum_{k=m}^{\infty} h_k(x), Q \right) \geq 0,$$

thus

$$U\left(\sum_{k=m}^{\infty} h_k(x), Q \right) - L\left(\sum_{k=m}^{\infty} h_k(x), Q \right) < \frac{\epsilon}{2}.$$

Let $G(x) = \sum_{n=1}^{m-1} h_n(x)/2^n$. Since h_n is integrable, there exists a partition P_n such that

$$U(h_n, P_n) - L(h_n, P_n) < \frac{\epsilon}{2(m-1)}.$$

Let $P = \cup_{k=1}^{m-1} P_n$, then

$$\begin{aligned} U(G(x), P) - L(G(x), P) &= \sum_{n=1}^{m-1} U(h_n, P) - L(h_n, P) \\ &< \frac{\epsilon(m-1)}{2(m-1)} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

Thus

$$\begin{aligned} U(H(x), P) - L(H(x), P) &= U(G(x), P) - L(G(x), P) + U\left(\sum_{k=m}^{\infty} h_k(x), Q \right) - L\left(\sum_{k=m}^{\infty} h_k(x), Q \right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore $H(x)$ is integrable according to the integrability Criterion. ■