Homework 15 Section 3.3

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Exercise 1,2,3,4,5,8

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Exercise 1. (Q1): Show that if K is compact and nonempty, then $\sup(K)$ and $\inf(K)$ exist and are elements of K.

Proof: We assume directly that K is compact and nonempty, then K is closed, bounded and has at least one element according to the Heine-Borel Thm. Since K is bounded, according to the axiom of completeness, it has a supremem and an infimum. Let $s = \sup(K)$, then for every $\epsilon > 0$, there is an element $x \in K$ such that $s - \epsilon < x$. This means that s is a limit point of K. Since K is closed, $s \in K$. And in a similar way it can be shown that $\inf(K) \in K$. ■

Exercise 2. (Q2): Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

a) $\mathbb{N}$
   a) The set of natural numbers is not compact. Let $a_n = 2n \in \mathbb{N}$, then (a_n) is a divergent, unbounded sequence and thus does not contain any subsequence that converges.

b) $\mathbb{Q} \cap [0,1]$
   a) The set is compact. The rational number is not a closed set since every irrational number in the interval $[0,1]$ is a limit point of $\mathbb{Q} \cap [0,1]$. Thus $\mathbb{Q} \cap [0,1]$ does not contain all of its limit points and cannot be closed.

c) The Cantor set
   a) The Cantor set is a subset of $[0,1]$ and is thus bounded. According to the Bolzano-Weierstrass theorem, every bounded sequence contains a convergent subsequence. The construction of the Cantor set consists of the intersection of an infinite number of closed intervals, thus the Cantor set is also closed. Since the set is closed, all convergent sequences converge to a limit point contained in the set. Thus the Cantor set is compact.

d) $A = \left\{1 + 1/2^2 + 1/3^3 + \cdots + 1/n^2 \,:\, \mathbb{N}\right\}$
   a) Not compact. This set A is the set of partial sums $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Since (s_n) converges, the set A is bounded. However, this set contains no limit point. To see this, let $m \in \mathbb{N}$ and $n = m + 1$, then

$$s_n - s_m = \frac{1}{n^2}.$$

   Due to the density of $\mathbb{R}$, we can find an $0 < \epsilon < \frac{1}{n^2}$ such that the set $V_\epsilon(s_m) \cap A$ only contains s_m. Thus, the set A does not contain any limit point. Therefore, any convergent subsequence converges to a point not in A. Hence, A is not compact.

e) $A = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$
   a) Compact. The set A is bounded. The only limit point of A is 1 which is in A. Thus, every sequence has a convergent subsequence which will converge to 1. Therefore, A is compact.

Exercise 3. (Q3): Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Proof: Let A be a closed and bounded set. Then all of it's limit points are contained in A. According to the theorem about limit points, all sequences contained in A that have a limit, converge to a limit point in A. Lastly, since A is bounded, according to the Bolzano-Weierstrass, every sequence contains a subsequence that converges, and these subsequences will converge to a limit point in A. Thus A is compact. ■

Exercise 4. (Q4): Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both or neither.

a) $K \cap F$
   a) Definitely compact and closed. Since K is closed and bounded and F is closed, the intersection $K \cap F$ is closed and its bounded since $K \cap F \subseteq K$.

b) $\overline{F^c \cup K^c}$
   a) Definitely closed. Since K is bounded, K^c is not bounded. Thus $F^c \cup K^c$ is not bounded. The closure of any set is closed, thus $\overline{F^c \cup K^c}$ is closed and not bounded.

c) $K \backslash F$
   a) This is neither definitely compact or closed. For example, let $K = [0, 1]$ and $F = \{0.5\}$, then $K \backslash F = [0, 0.5) \cup (0.5, 1]$.

d) $\overline{K \cap F^c}$
   a) Definitely compact and closed. Since $K \cap F^c \subseteq K$, it must be bounded. And the closure of a set is always closed. thus $\overline{K \cap F^c}$ is closed and bounded.

Exercise 5. (Q5): Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

a) The arbitrary intersection of compact sets is compact.
   a) True, A compact set is closed an bounded. The arbitrary intersection of closed sets is closed. Since all of the sets are bounded, the intersection of all of the sets must be bounded. Thus, arbitrary intersection of compact sets is compact.

b) The arbitrary union of compact sets is compact.
   a) False. Let $I_n = [0, n]$ where $n \in \mathbb{N}$, then I_n is compact. However, $\cup_{k=1}^{\infty} I_k = [0, \infty)$ which is not bounded and thus not compact.

c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.
   a) False, let K be a closed interval and A be an open interval contained in K, then $A \cap K = A$ which is open.

d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\cap_{n=1}^{\infty} F_n \neq \emptyset$.
   a) False. Let $F_n = [n, \infty)$, then $\cap_{n=1}^{\infty} F_n = \emptyset$.

Exercise 6. (Q8): Let K and L be nonempty compact sets, and define

$$d = \inf \{|x - y| \, : \, x \in K \text{ and } y \in L\} .$$

This turns out to be a reasonable definition for the distance between K and L.

a) If K and L are disjoint, show that $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.
   Proof: We suppose, by contradiction, that K and L are disjoint, nonempty compact sets and that $d = 0$. This means that given an $\epsilon > 0$, there exists some $x \in K$ and $y \in L$ such that $|x - y| < \epsilon$. Let $\epsilon_n = \frac{1}{n}$ and the corresponding elements be $x_n \in K$ and $y_n \in L$ such that $|x_n - y_n| < \epsilon_n$. Then we can construct the sequences (x_n) and (y_n). Since K and L are bounded, (x_n) and (y_n) must have subsequences that converge to a point; in fact, they must converge to the same point since as $n \to \infty$, $|x_n - y_n| \to 0$ since $\epsilon_n \to 0$. This means that K must contain a limit point of L, but this is a contradiction since K and L are disjoint. ■

b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.
   Proof: Let K be the set of all natural numbers and $L = \left\{n + \frac{1}{n} \, : \, n \in \mathbb{N}\right\}$, then $d = \inf \left\{\frac{1}{n} \, : \, n \in \mathbb{N}\right\} = 0$. ■