Homework 9 Section 2.5

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Exercises 1,2,3,5,6

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Exercise 1. (Q1): Give an example of each of the following, or argue that such a request is impossible.

1) A sequence that has a subsequence that is bounded that is bounded but contains no subsequence that converges.
   a) This is impossible. Let (a_n) be a sequence and (a_{n_k}) be the bounded subsequence. According to the Bolzano-Weierstrass Theorem, since (a_{n_k}) is bounded, it has a subsequence that bounded which is a subsequence of the original sequence. Therefore, if a sequence contains a bounded subsequence, it must also contain a subsequence that converges.
2) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
   a) Let $(a_n) = \left\{\frac{1}{2}, 1+\frac{1}{2}, \frac{1}{3}, 1+\frac{1}{3}, \frac{1}{4}, 1+\frac{1}{4}, \cdots\right\}$ be the sequence that does not contain 0 or 1, but is bounded. Let (a_{n_k}) be the subsequence $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ that converges to 0, and let $\left(a_{n_j}\right)$ be the subsequence $\left\{1+\frac{1}{2}, 1+\frac{1}{3}, 1+\frac{1}{4}, \cdots\right\}$ that converges to 1.
3) A sequence that contains subsequences converging to every point in the infinite set
$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}.$$
   a) Let $\left(a_n^i\right)$ be the sequence defined by $\frac{1}{i}+\frac{1}{n}$ where $i \in \mathbb{N}$, and let (b_n) be the sequence defined as
$$\left\{a_1^1, a_1^2, a_1^3, \ldots a_2^1, a_2^2, a_2^3, \cdots a_3^1 a_3^2 a_3^3 \cdots\right\}.$$
   Since the infinite union of countable sets is still countable, the set $\{(b_n)\}$ is still countable. Therefore, it is a sequence with $\left(a_n^i\right)$ being the subsequences. (Forgive the different notation)
4) A sequence that contains subsequences converging to every point in the infinite set
$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\},$$
and no subsequences converging to points outside this set.
   a) This is impossible. Note that (b_n) contains the points
$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\},$$
   which converges to 0 and $0 \notin \{(b_n)\}$.

Exercise 2. (Q2): Decide whether the following propositions are true or false, providing a short justification for each conclusion.

1) If every proper subsequence of (x_n) converges, then (x_n) converges as well.
   a) True. Let (x_{n_k}) be the subsequence of (x_n) that contains all the terms of (x_n) except the first term, i.e. $n_k = k+1$. Since (x_{n_k}) converges, there exists some $K \in \mathbb{N}$ given some $\epsilon > 0$ such that for all $k > K$
$$|x_{n_k} - \lim x_{n_k}| < \epsilon,$$
   then
$$|x_{k+1} - \lim x_{n_k}| < \epsilon.$$
   Thus (x_n) converges.
2) If (x_n) contains a divergent subsequence, then (x_n) diverges.

   a) True. Consider the contrapositive: if (x_n) converges then it does not contain a divergent subsequence. This is true since every subsequence of a converging sequence converges; therefore, if (x_n) contains a divergent subsequence, then (x_n) diverges.

3) If (x_n) is bounded and diverges, then there exists two subsequences of (x_n)that converge to different limits.
   a) True. Since every bounded sequence contains a convergent subsequence we know that there exists a subsequence that converges. Let $\left(x^1_{n_k}\right)$ be a convergent subsequence that converges to L and contains all of the terms in (x_n) that converge to L. Since (x_n) diverges, there must be a subsequence that diverges. Let $\left(x^2_{n_k}\right)$ be a divergent subsequence that contains all of the terms in (x_n) except for the terms in $\left(x^1_{n_k}\right)$ such that $\left\{x^1_{n_k}\right\} \cup \left\{x^2_{n_k}\right\} = \{x_n\}$ and $\left\{x^1_{n_k}\right\} \cap \left\{x^2_{n_k}\right\} = \emptyset$. Since $\left(x^2_{n_k}\right)$ is a bounded subsequence, it must contain a convergent subsequence that converges to a number $B \neq L$. By induction, you could infer that if (x_n) is bounded and diverges, there exists an infinite number of subsequences of (x_n) that converge to distinct limits.

4) If (x_n) is monotone and contains a convergent subsequence, the (x_n) converges.
   a) True. Let (x_{n_k}) be the convergent subsequence, then $|x_{n_k}| \leq M$ for all $k \in \mathbb{N}$ and for some $M > 0$. Let $n_1 = z$ and $n_m = y$, then $x_z = x_{n_1}$ and $x_y = x_{n_m}$ for some $m \in \mathbb{N}$. Since (x_n) is a monotone sequence any terms of the sequence between x_{n_1} and x_{n_m} must also be bounded. Since m can be arbitrarily large and $x_k \geq k$, every term of the sequence (x_n) must be bounded. Thus (x_n) is bounded. Since (x_n) is bounded and monotone, it converges.

Exercise 3. (Q3): Do the following

1) Prove that if an infinite series converges, then the associate property holds. Assume $a_1 + a_2 + a_3 + a_4 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + a_3 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

   leads to a series that also converges to L.
   Proof: We suppose directly that $a_1 + a_2 + a_3 + a_4 + \cdots$ converges to a limit L, in other words $(s_n) \to L$. Let

$$\begin{aligned} b_{n_1} &= (a_1 + a_2 + a_3 + \cdots + a_{n_1}) \\ b_{n_2} &= (a_1 + a_2 + a_3 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) \\ \vdots &= \vdots, \end{aligned}$$

   so that $b_{n_k} = s_{n=n_k}$. Since (s_n) converges, given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$

$$\begin{aligned} |s_n - L| &< \epsilon \\ |b_{n_k=n} - L| &< \epsilon, \end{aligned}$$

   hence the associate property holds. ■

2) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in part (1) apply to this example?
   a) The proof in part (1) holds only when the infinite series converges. The example discussed at the end of Section 2.1 doesn't converge, so the proof doesn't pertain to it.

Exercise 4. (Q5): Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Proof: We suppose, by contradiction, that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$, that (a_n) is bounded, and that (a_n) does not converge. Since (a_n) is bounded, it must contain at least one subsequence $\left(a^1_{n_k}\right)$ that converges according to the Bolzano-Weierstrass Theorem. Let $\left(a^1_{n_k}\right)$ contain all the terms of (a_n) that converge to a, and let $\left(a^2_{n_k}\right)$ be the subsequence that contains all of the other terms. The subsequence $\left(a^2_{n_k}\right)$ must diverge and contain an infinite number of terms since (a_n) diverges. Since $\left(a^2_{n_k}\right)$ is bounded, it must contain a subsequence that converges according to the Bolzaon-Weierstrass Theorem. Let this subsequence be denoted as $\left(a^3_{n_k}\right)$. By construction, the subsequence $\left(a^3_{n_k}\right)$ does not converge to a, but to some other number. This contradicts our assumption that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Therefore, the original statement must be true. ■

Exercise 5. (Q6): Use a similar strategy to the one in Example 2.5.3 to show that $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit.

Proof: Let a_n be the sequence defined as $a_n = b^{\frac{1}{2}}$. We assume that $b \geq 0$, then (a_n) is $\left\{b, b^{\frac{1}{2}}, b^{\frac{1}{3}}, \cdots\right\}$. The sequence is monotonically decreasing if $b \geq 1$ or monotonically increasing if $b < 1$. Also, since $b \geq 0$, the sequence (a_n) is bounded between 0 and b, thus (a_n) converges. Using the algebraic limit theorem, since $\lim \frac{1}{n} = 0$ the $\lim b^{\frac{1}{n}} = b^0$. If $b \neq 0$, then $\lim b^{1/n} = b$, else it is 0. ■