Homework 31 Section 7.2

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Exercises 2,3,4,5,7

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Exercise 1. (Q2): Consider $f(x) = \frac{1}{x}$ over the interval $[1, 4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

a) Compute $L(f, P)$, $U(f, P)$ and $U(f, P) - L(f, P)$.

  a) Note that $f(x)$ is a decreasing function over the interval $[1, 4]$. Let $x_0 = 1$, $x_1 = 3/2$, $x_2 = 2$ and $x_3 = 4$, then

$$\begin{aligned} m_1 &= \frac{2}{3} \\ m_2 &= \frac{1}{2} \\ m_3 &= \frac{1}{4} \\ M_1 &= 1 \\ M_2 &= \frac{2}{3} \\ M_3 &= \frac{1}{2} \end{aligned}$$

  and

$$\begin{aligned} L(f, P) &= \frac{2}{3}\left(\frac{3}{2} - 1\right) + \frac{1}{2}\left(2 - \frac{3}{2}\right) + \frac{1}{4}(4 - 2) \\ &= \frac{13}{12}, \end{aligned}$$

$$\begin{aligned} U(f, P) &= 1\left(\frac{3}{2} - 1\right) + \frac{2}{3}\left(2 - \frac{3}{2}\right) + \frac{1}{2}(4 - 2) \\ &= \frac{11}{6} \end{aligned}$$

$$U(f, P) - L(f, P) = \frac{9}{12}$$

b) What happens to the value of $U(f, P) - L(f, P)$ when we add the point 3 to the partition?

  a) When we add 3 to the partition

$$\begin{aligned} L(f, P) &= \frac{2}{3}\left(\frac{3}{2} - 1\right) + \frac{1}{2}\left(2 - \frac{3}{2}\right) + \frac{1}{3}(3 - 2) + \frac{1}{4}(4 - 3) \\ &= \frac{14}{12}, \end{aligned}$$

$$\begin{aligned} U(f, P) &= 1\left(\frac{3}{2} - 1\right) + \frac{2}{3}\left(2 - \frac{3}{2}\right) + \frac{1}{2}(3 - 2) + \frac{1}{3}(4 - 3) \\ &= \frac{10}{6}, \end{aligned}$$

$$U(f, P) - L(f, P) = \frac{6}{12}$$

  which indicates that the difference got smaller.

c) Find a partition P' of $[1, 4]$ for which $U(f, P') - L(f, P') < 2/5$

a) Let $P' = \left\{1, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \frac{8}{4}, \ldots, 4\right\}$, then

$$L(f, P) = \sum_{n=1}^{12} \left(\frac{1}{4+n} \right),$$
$$U(f, P) = \sum_{n=1}^{12} \left(\frac{1}{4+n-1} \right),$$

and

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{n=1}^{12} \left(\frac{1}{4+n-1} \right) - \sum_{n=1}^{12} \left(\frac{1}{4+n} \right) \\ &= \frac{1}{4} \end{aligned}$$

Exercise 2. (Q3): (Sequential Criterion for Integrability). Complete the following

a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} \left[U(f, P_n) - L(f, P_n) \right] = 0,$$

and in this case $\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$.
Proof: This is a biconditional statement so we must prove both ways.
($\Longrightarrow$) : We assume directly that f is bounded and integrable on $[a, b]$, then according to the integrability criterion, given an $\epsilon_n > 0$, there exists a partition P_{ϵ_n} of $[a, b]$ such that

$$U(f, P_{\epsilon_n}) - L(f, P_{\epsilon_n}) < \epsilon.$$

We can then construct a sequence consisting of ϵs such that $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > 0$ with corresponding partitions P_{ϵ_1}, P_{ϵ_2}, etc. We can then define the sequence $P_n = P_{\epsilon_n}$, then as $n \to \infty$, $\epsilon_n \to 0$ thus

$$\lim_{n\to\infty} \left[U(f, P_n) - L(f, P_n) \right] = 0.$$

($\Longleftarrow$) : We assume directly that there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} \left[U(f, P_n) - L(f, P_n) \right] = 0.$$

By the definition of a limit, given an $\epsilon > 0$, there exists an $N \in \mathbb{R}$ such that whenever $n > N$ it holds that

$$\left[U(f, P_n) - L(f, P_n) \right] < \epsilon,$$

since for every ϵ, there exists a partition in the sequence (P_n) such that

$$\left[U(f, P_n) - L(f, P_n) \right] < \epsilon.$$

According to the Integrability Criterion, the function is then integrable. ■

b) For each n, let P_n be the partition of $[0, 1]$ into n equal subintervals. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ if $f(x) = x$. The formula $1 + 2 + 3 + \cdots + n = n(n+1)/2$ will be useful.

a) Let $\Delta_x = \frac{1}{n}$, then

$$\begin{aligned} L(f, P_n) &= \sum_{k=1}^{n} \Delta_x \frac{k-1}{n} \\ &= \frac{n(n+1)}{2n^2} - \frac{1}{n^2}, \end{aligned}$$

and

$$\begin{aligned} U(f, P_n) &= \sum_{k=1}^{n} \Delta_x \frac{k}{n} \\ &= \frac{n(n+1)}{2n^2} \end{aligned}$$

c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Proof: Taking the limit as $n \to \infty$ gives

$$\begin{aligned}\lim_{n\to\infty}\left[U\left(f,P_n\right)-L\left(f,P_n\right)\right] &= \lim_{n\to\infty}\frac{n\left(n+1\right)}{2n^2}-\frac{n\left(n+1\right)}{2n^2}+\frac{1}{n^2}\\ &= \lim_{n\to\infty}\frac{1}{n^2}=0,\end{aligned}$$

thus f is integrable on $[0,1]$ and

$$\int_0^1 f=\lim_{n\to\infty}\frac{n\left(n+1\right)}{2n^2}=\frac{1}{2}.$$

■

Exercise 3. (Q4): Let g be bounded on $[a,b]$ and assume there exists a partition P with $L\left(g,P\right)=U\left(g,P\right)$. Describe g. Is it integrable? If so, what is the value of $\int_a^b g$?

g is integrable. Since $L\left(g,P\right)=U\left(g,P\right)$ any refinement Q of P must satisfy the inequality

$$L\left(g,P\right)\leq L\left(g,Q\right)\leq U\left(g,Q\right),U\left(g,P\right).$$

Since $L\left(g,P\right)=U\left(g,P\right)$, $L\left(g,P\right)=U\left(g,P\right)=L\left(g,Q\right)=U\left(g,Q\right)$, thus the upper integral of g is

$$U\left(f\right)=U\left(g,P\right)$$

and the lower integral of g is

$$L\left(f\right)=L\left(g,P\right).$$

Therefore, by the definition of Reimann integrability, g is integrable ans

$$\int_a^b g=U\left(g,P\right)=L\left(g,P\right).$$

Exercise 4. (Q5): Assume that, for each n, f_n is an integrable function on $[a,b]$. If $(f_n)\to f$ uniformly on $[a,b]$, prove that f is also integrable on this set.

Proof: Since $(f_n)\to f$ uniformly, given an $\epsilon>0$, there exists a $N\in\mathbb{R}$ such that whenever $n>N$,

$$\left|f_n\left(x\right)-f\left(x\right)\right|<\frac{\epsilon}{3\left(b-a\right)}$$

for all x. Since f_n is an integrable function on $[a,b]$, for every ϵ_n there exists a partition P_n such that

$$U\left(f_n,P_n\right)-L\left(f_n,P_n\right)<\frac{\epsilon}{3}.$$

By definition

$$U\left(f_n,P_n\right)=\sum_{k=1}^{m}M_k\left(x_k-x_{k-1}\right).$$

Using the fact that $(f_n)\to f$ uniformly

$$\begin{aligned}U\left(f,P_n\right) &\leq \sum_{k=1}^{m}\left(M_k+\frac{\epsilon}{3\left(b-a\right)}\right)\left(x_k-x_{k-1}\right)\\ &= U\left(f_n,P_n\right)+\sum_{k=1}^{m}\frac{\epsilon}{3\left(b-a\right)}\left(x_k-x_{k-1}\right)\\ &= U\left(f_n,P_n\right)+\frac{\epsilon}{3},\end{aligned}$$

and similarly

$$L\left(f,P_n\right)\geq L\left(f_n,P_n\right)-\frac{\epsilon}{3}.$$

Thus

$$\begin{aligned}U\left(f,P_n\right)-L\left(f,P_n\right) &\leq U\left(f_n,P_n\right)+\frac{\epsilon}{3}-L\left(f_n,P_n\right)+\frac{\epsilon}{3}\\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\\ &= \epsilon.\end{aligned}$$

Therefore, by the integrability criterion, f is integrable on $[a,b]$. ■

Exercise 5. (Q7): Let $f:[a,b]\to\mathbb{R}$ be increasing on the set $[a,b]$ (i.e., $f(x)\leq f(y)$ whenever $x<y$). Show that f is integrable on $[a,b]$.

Proof: Since f is increasing on the set $[a,b]$, the function is bounded. Let $M=\max\left(\left|f(a),|f(b)|\right|\right)$. Let $\epsilon>0$. For each $n\in\mathbb{N}$, let P_n be a partition of the interval $[a,b]$ into n equal length subintervals. Thus

$$\Delta_{x_n}=\frac{b-a}{n}.$$

We can choose n large enough such that

$$\left(f(b)-f(a)\right)\frac{(b-a)}{n}<\epsilon.$$

Then

$$\begin{aligned}
U(f,P_n)-L(f,P_n)&=\sum_{k=1}^{n}\left(M_k-m_k\right)\Delta_x\\
&=\frac{b-a}{n}\sum_{k=1}^{n}\left(M_k-m_k\right)\\
&=\frac{b-a}{n}\sum_{k=1}^{n}\left(f(x_k)-f(x_{k-1})\right)\\
&=\frac{b-a}{n}\left(f(b)-f(a)\right)\\
&<\epsilon.
\end{aligned}$$

Therefore, by the integrability criterion, the function is integrable. ∎