Exercise 17. (Q9): Assume that the sequence (x_n) is a convergent sequence and $\lim_{n\to\infty} x_n = L$. Prove that (x_n) is also a Cauchy sequence.

Proof: We suppose directly that (x_n) is a convergent sequence and $\lim_{n\to\infty} x_n = L$, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n, m \in \mathbb{N} > N$,

$$|x_n - L| < \frac{\epsilon}{2},$$

and

$$|x_m - L| < \frac{\epsilon}{2}.$$

Adding the two together, we get

$$\begin{aligned} |x_n - L| + |x_m - L| &\geq |x_n - x_m + L - L| \\ &= |x_n - x_m|, \end{aligned}$$

thus

$$\begin{aligned} |x_n - x_m| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

thus the sequence (x_n) is also a Cauchy sequence. ■