Homework 19 Section 4.4

Mark Petersen

Exercises 3,4,5,6,9

07/15/2020

Exercise 1. (Q3): Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the set $[1,\infty)$ but not on the set $(0,1]$.

Proof: We will first show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the set $[1,\infty)$. Let $c \in [1,\infty)$. Given an $\epsilon > 0$, let $\delta = \frac{\epsilon}{2}$, then using the definition of continuous we get

$$\begin{aligned}\left|\frac{1}{x^2} - \frac{1}{c^2}\right| &= \frac{\left|c^2 - x^2\right|}{x^2c^2} \\ &= \frac{|c+x|\,|c-x|}{x^2c^2}.\end{aligned}$$

Let $b = \max(c,x)$ and $a = \min(c,x)$, then

$$\begin{aligned}\frac{|c+x|\,|c-x|}{x^2c^2} &\leq \frac{|c-x|\,2b}{ba^2} \\ &= \frac{|c-x|\,2}{a^2} \\ &\leq |c-x|\,2 \\ &< \frac{\epsilon 2}{2} \\ &= \epsilon.\end{aligned}$$

Since δ only depends on ϵ, the function $f(x)$ is uniformly continuous on the set $[1,\infty)$.

Next we want to show that $f(x)$ is not uniformly continuous on the set $(0,1]$. We show this by contradiction. Suppose that given an $\epsilon > 0$, there exists a $\delta = g(\epsilon) > 0$, where g is only a function of ϵ,such that when

$$|c-x| < \delta,$$

then

$$\left|\frac{1}{x^2} - \frac{1}{c^2}\right| < \epsilon.$$

Manipulating the last term we get

$$\begin{aligned}\left|\frac{1}{x^2} - \frac{1}{c^2}\right| &= \frac{|c+x|\,|c-x|}{x^2c^2} \\ &\leq \frac{2\,|c-x|}{x^2c^2} \\ &< \frac{2\delta}{x^2c^2} \\ &= \frac{2g(\epsilon)}{x^2c^2}.\end{aligned}$$

Since the terms x^2 and c^2 can be arbitrarily close to 0 and $g(\epsilon)$ is independent of x and c, the term $\frac{2g(\epsilon)}{x^2c^2}$ can become arbitrarily large, which means that it can be greater than ϵ. Thus, this contradicts that $f(x)$ is uniformly continuous on the set $(0,1]$; therefore, it is not uniformly continuous on that set. ∎

Exercise 2. (Q4): Decide whether each of the following statements is true or false, justifying each conclusion.

a) If f is continuous on $[a,b]$ with $f(x) > 0$ for all $x \in [a,b]$, then $\frac{1}{f}$ is bounded on $[a,b]$.
   a) True

Proof: According to the algebraic limit theorem, since $f(x) \neq 0$ for all x, the function $\frac{1}{f}$ is continuous. Also, since $[a,b]$ is compact, the set $1/f([a,b])$ is also compact according to the theorem of preservation of compact sets. Thus $\frac{1}{f}$ is bounded on $[a,b]$. ■

b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

a) True

Proof: We suppose directly that f is uniformly continuous on a bounded set A. Since A is bounded, the closure of A, $\overline{A}$, is bounded and thus compact. Thus $f\left(\overline{A}\right)$ is compact since f is continuous. Since $A \subseteq \overline{A}$, then $f(A) \subseteq f\left(\overline{A}\right)$. Since $f\left(\overline{A}\right)$ is compact, it is bounded. This implies that $f(A)$ is also bounded. ■

c) If f is defined on $\mathbb{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbb{R}$.

a) False,

Disproof: To disprove this, we will show a counter example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if } x \geq 0 \\ 1 & \text{else} \end{cases}.$$

The image $f(K) \subseteq \{-1, 1\}$ which is always compact; however, f is discontinuous at $x = 0$. ■

Exercise 3. (Q5): Assume that g is defined on an open interval (a,c) and it is known to be uniformly continuous on $(a,b]$ and $[b,c)$, where $a < b < c$. Prove that g is uniformly continuous on (a,c).

Proof: Since g is uniformly continuous on $(a,b]$ and $[b,c)$, given an $\epsilon > 0$, we can find a $\delta_1 = \delta_2 = \epsilon/2$ such that

$$|x - b| < \delta_1,$$

and

$$|b - y| < \delta_2.$$

Adding the two together yields

$$\begin{aligned} |x - b| + |b - y| &< \delta_1 + \delta_2 \\ |x - b + b - y| &< \delta_1 + \delta_2 \\ |x - y| &< \delta_1 + \delta_2 \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, g is uniformly continuous on (a,c). ■

Exercise 4. (Q6): Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

a) A continuous function $f : (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

a) This is possible. Let f be defined as $f(x) = \frac{1}{n}$ and (x_n) be the sequence $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$, then $(f(x_n))$ is the sequence $\{2, 3, 4, 5, 6, 7, \ldots\}$ which is not a Cauchy sequence.

b) A uniformly continuous function $f : (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

a) This is impossible.

Proof: We suppose that $f : (0,1) \to \mathbb{R}$ is a uniformly continuous function, (x_n) is a Cauchy sequence and $f(x_n)$ is not a Cauchy sequence. Since (x_n) is a Cauchy sequence, given a $\delta > 0$, there exists an $M \in \mathbb{N}$, such that when $a, b > M$,

$$|x_a - x_b| < \delta.$$

Also, since f is uniformly continuous, given an $\epsilon > 0$, there exists a $\delta > 0$ as a function of ϵ only, such that when

$$|k - l| < \delta,$$

$$|f(k) - f(l)| < \epsilon.$$

In other words, we can find an M such that when

$$|x_a - x_b| < \delta,$$

$$|f(x_a) - f(x_b)| < \epsilon.$$

Thus, the sequence $f(x_a)$ is a Cauchy sequence. ■

c) A continuous function $f : [0, \infty) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

  a) This is impossible.
  Proof: Since $A = [0, \infty)$ is a closed set, any Cauchy sequence $(x_n) \subseteq A$ has it's limit point $l \in A$. Using the Characterizations of Continuity, since $(x_n) \to l$, then $f(x_n) \to f(l)$, which means that $f(x_n)$ is a Cauchy sequence. ■

Exercise 5. (Q9): (Lipschitz Functions). A function $f : A \to \mathbb{R}$ is called Lipschitz if there exists a bound $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \le M$$

for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.
Proof: We suppose directly that f is Lipschitz, then

$$\left|\frac{f(x) - f(y)}{x - y}\right| \le M,$$

which is equivalent to

$$|f(x) - f(y)| \le M|x - y|.$$

Given an $\epsilon > 0$, we can choose the distance $d = |x - y|$ such that $\epsilon > Md$, thus

$$|f(x) - f(y)| < \epsilon,$$

and

$$\begin{aligned} |x - y| &< \epsilon/M \\ &= \delta. \end{aligned}$$

Thus, by letting $\delta = \epsilon/M$, when

$$|x - y| < \delta,$$

then

$$|f(x) - f(y)| < \epsilon.$$

Thus $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A. ■

b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?
Disproof: We will disprove the converse with an example. Let $f : [0, \infty) \to \mathbb{R}$ be defined as $f(x) = \sqrt{x}$ which we know to be uniformly continuous. For it to be Lipschitz, we need that

$$\left|\sqrt{x} - \sqrt{y}\right| \le M|x - y|$$

for some $x, y \in \mathbb{R}$ and $M \in \mathbb{R}$. Well, $|x - y| = \left|\sqrt{x} + \sqrt{y}\right| \left|\sqrt{x} - \sqrt{y}\right|$. Using this identity in the previous equation yields

$$\begin{aligned} \left|\sqrt{x} - \sqrt{y}\right| &\le M\left|\sqrt{x} + \sqrt{y}\right| \left|\sqrt{x} - \sqrt{y}\right| \\ 1 &\le M\left|\sqrt{x} + \sqrt{y}\right|. \end{aligned}$$

For this inequality to hold for when x and y are arbitrarily small, M would have to be arbitrarily big, i.e. unbounded. Thus, no M exists. Therefore, not all uniformly continuous functions are Lipschitz. ■