Homework Section 1.3

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Exercises 1,4,7,8,906/22/2020

Exercise 1. a) Write a formal definition in the style of Definition 1.3.2 for the infimum or greatest lower bound of a set. b) Now state and prove a version of Lemma 1.3.8 for greatest lower bounds.

1) A real number s is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:
   a) s is a lower bound for A;
   b) if b is a lower bound for A, then $s \geq b$.
2) Assume $s \in \mathbb{R}$ is an lower bound for a set $A \subseteq \mathbb{R}$. Then, $s = \inf(A)$ if and only if, for every choice of $\epsilon > 0$, there exists and element $a \in A$ satisfying $s + \epsilon > a$.
   Proof: Since this is a bijection, we must show both ways.
   $(\implies)$: We assume directly that $s = \inf(A)$, then $s + \epsilon > s$. So there must exists an element $s < a < s + \epsilon$ such that $a \in A$. If $s + \epsilon \neq A$, then let a be any element in A. If $s + \epsilon \in A$, then let $a = s + \epsilon/2$. Thus regardless of ϵ, we can find an a such that $s + \epsilon > a$.
   $(\impliedby)$: We assume contrapositively that s is a lower bound and that $s \neq \inf(A)$, then $s - \inf(A) > 0$. Let $\epsilon = \frac{s - \inf(A)}{2}$, then $s + \epsilon < \inf(A)$, hence $s + \epsilon < a$ for all $a \in A$. Therefore, for every $a \in A$, there exists and $\epsilon > 0$, such that $s + \epsilon \leq a$. Since the contrapositive is true, the original implication must be true.
   Since we have proven both ways, we have proven the biconditional statement is true. ■

Exercise 2. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

1) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\cup_{k=1}^n A_k)$.
   a) **Conjecture:** $\sup(A_1 \cup A_2) = \max(\sup(A_1), \sup(A_2))$.
      Proof: We will show that $\sup(A_1 \cup A_2) = \max(\sup(A_1), \sup(A_2))$ by proving the inequalities $\sup(A_1 \cup A_2) \geq \max(\sup(A_1), \sup(A_2))$ and $\sup(A_1 \cup A_2) \leq \max(\sup(A_1), \sup(A_2))$.
      $(\geq)$: Let $x = \sup(A_1 \cup A_2)$, then for all $a \in A_1 \cup A_2$, we have that $a \leq x$. Since $A_1 \subseteq A_1 \cup A_2$, then for all $b \in A_1$ we have that $b \leq x$. Similarly, since $A_2 \subseteq A_1 \cup A_2$, then for all $c \in A_2$, it must be that $c \leq x$. Thus x is an upper bound for A_1 and A_2. This implies that $\sup(A_1) \leq x$ and $\sup(A_2) \leq x$. Thus $\max(\sup(A_1), \sup(A_2)) \leq x = \sup(A_1 \cup A_2)$.
      $(\leq)$: Let $x = \max(\sup(A_1), \sup(A_2))$, then x is an upper bound for A_1 and A_2. In other words, for all $b \in A_1$, $b \leq x$, and for all $c \in A_2, c \leq x$. Since $A_1 \cup A_2 = \{h : h \in A_1 \text{ or } h \in A_2\}$, we know that for all $d \in A_1 \cup A_2$, $d \leq x$. Thus x is an upper bound for $A_1 \cup A_2$. This means that $\sup(A_1 \cup A_2) \leq x$. Thus $\sup(A_1 \cup A_2) \leq \max(\sup(A_1), \sup(A_2))$.
      Since both inequalities hold, we have that $\sup(A_1 \cup A_2) = \max(\sup(A_1), \sup(A_2))$. ■
   b) **Conjecture:** $\sup(\cup_{k=1}^n A_k) = \max(\sup(A_1), \sup(A_2), \ldots, \sup(A_n))$.
      Proof: Let $n, k \in \mathbb{N}$ and $A_k \subseteq \mathbb{R}$ be bounded from above. We want to show that the open sentence

$$P(n) : \sup(\cup_{k=1}^n A_k) = \max(\sup(A_1), \sup(A_2), \ldots, \sup(A_n))$$

      is true for all $n \in \mathbb{N}$. We work this by induction.
      Base Case: $P(1)$ is trivial since $\sup(A_1) = \max(\sup(A_1))$. $P(2)$ has been proven above.
      Induction Step: Let $m \in \mathbb{N}$. We assume that $P(m)$ is true and we want to show that $P(m+1)$ is true which is the statement

$$\sup\left(\cup_{i=1}^{m+1} A_k\right) = \max(\sup(A_1), \sup(A_2), \ldots, \sup(A_{m+1})).$$

      We can write the right hand side as

$$\max(\max(\sup(A_1), \sup(A_2), \ldots, \sup(A_m)), \sup(A_{m+1})).$$

Since $P(m)$ is true, we can reduce it to

$$\max\left(\sup\left(\cup_{j=1}^{m}A_j\right),\sup\left(A_{m+1}\right)\right)$$
$$\max\left(\sup\left(B\right),\sup\left(A_{m+1}\right)\right),$$

with $B=\cup_{j=1}^{m}A_j$ which is a subset of $\mathbb{R}$ that is bounded from above. This problem then reduces to the base case $P(2)$. Thus

$$\sup\left(B\cup A_{m+1}\right)=\max\left(\sup\left(B\right),\sup\left(A_{m+1}\right)\right)$$
$$\sup\left(\left(\cup_{i=1}^{m}A_j\right)\cup A_{m+1}\right)=\max\left(\sup\left(B\right),\sup\left(A_{m+1}\right)\right)$$
$$\sup\left(\cup_{i=1}^{m+1}A_k\right)=\max\left(\sup\left(B\right),\sup\left(A_{m+1}\right)\right),$$

hence

$$\sup\left(\cup_{i=1}^{m+1}A_k\right)=\max\left(\sup\left(A_1\right),\sup\left(A_2\right),\ldots,\sup\left(A_{m+1}\right)\right).$$

Since $P(m+1)$ is true, we have shown that $P(n)$ is true for all n. ■

2) Consider $\sup\left(\cup_{k=1}^{\infty}A_k\right)$. Does the formula in (a) extend to the infinite case?
   a) No. Consider the upper bounded sets $A_i=\{i\}$ with $i\in\mathbb{N}$, then each set is bounded from above by $i+1$. However, the set $\cup_{k=1}^{\infty}A_k$ isn't bounded from above. To show this, suppose $y=\sup\left(\cup_{k=1}^{\infty}A_k\right)$, then y is not an upper bound of A_{y+1}. But since $A_{y+1}\subseteq\cup_{k=1}^{\infty}A_k$, y cannot be a supremem. This is a contradiction, thus $\sup\left(\cup_{k=1}^{\infty}A_k\right)$ does not exist.

Exercise 3. Prove that if a is an upper bound for A, and if a is also an element of A, then $a=\sup(A)$.

Proof: Let T be the set containing all of the upper bounds of A. We suppose directly that a is an upper bound for A, and $a\in A$. Let $t\in T$, then for every $b\in A$, $b\leq t$. Since $a\in A$, then $a\leq t$ for every $t\in T$. Thus by definition, a is the least upper bound, or in other words, $a=\sup(A)$. ■

Exercise 4. Compute, without proofs, the suprema and infima (if they exist) of the following sets:

1) $\{m/n\,:\,m,n\in\mathbb{N}\text{ with }m<n\}$.
   a) suprema: 1
   b) infima: 0
2) $\{(-1)^m/n\,:\,m,n\in\mathbb{N}\}$.
   a) suprema: 1
   b) infima: -1
3) $\{n/(3n+1)\,:\,n\in\mathbb{N}\}$.
   a) suprema: $\frac{1}{3}$
   b) infima: $\frac{1}{4}$
4) $\{m/(m+n)\,:\,m,n\in\mathbb{N}\}$.
   a) suprema: 1
   b) infima: 0

Exercise 5. See problem description below.

1) If $\sup(A)<\sup(B)$, show that there exists an element $b\in B$ that is an upper bound for A.
   Proof: Let $a^*=\sup(A)$ and $b^*=\sup(b)$. We suppose directly that $a^*<b^*$. Then there must exists an $\epsilon>0$ such that $b^*-a^*>\epsilon>0$. By lemma 1.3.8, there exists a $b\in B$ for every $\delta>0$, such that $b^*-\delta<b$. Therefore, there exists a b such that $a^*<b^*-\epsilon<b<b^*$. Therefore, b is an upper bound for A. ■
2) Give an example to show that this is not always the case if we assume $\sup(A)\leq\sup(B)$.
   a) Let $A=[0,1]$ and $B=\left\{\frac{n}{1+n}\,:\,n\in\mathbb{N}\right\}$, then $\sup(A)=1$ and $\sup(B)=1$. Thus $\sup(A)\leq\sup(B)$. Since $1\notin B$, there exists no element $b\in B$ that is an upper bound for A since no element in B is equal to or greater than 1.