Exam 4, Final

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Exercise 1. Assume the sequence (a_n) satisfies $\lim_{n\to\infty} a_n = 2$. Directly use the definition of the limit of a sequence (and not theorems about sequences) to show that

$$\lim_{n\to\infty} \frac{1}{7-a_n^2} = \frac{1}{3}.$$

Proof: Since $\lim_{n\to\infty} a_n = 2$, given an $\epsilon > 0$, there exists an $N_1 \in \mathbb{R}$ such that when $n > N_1$,

$$|a_n - 2| < \frac{\epsilon}{2},$$

in addition, there exists an $N_2 \in \mathbb{R}$ such that when $n > N_2$,

$$|a_n - 2| = \frac{1}{2}$$

which implies

$$3.5 < a_n + 2 < 4.5$$

and

$$2.25 < a_n^2 < 6.25.$$

It follows that by choosing $N = \max(N_1, N_2)$, when $n > N$,

$$\begin{aligned}
\left|\frac{1}{7-a_n^2} - \frac{1}{3}\right| &= \left|\frac{3-7+a_n^2}{3\left(7-a_n^2\right)}\right| \\
&= \left|\frac{a_n^2-4}{3\left(7-a_n^2\right)}\right| \\
&= \left|\frac{\left(a_n-2\right)\left(a_n+2\right)}{3\left(7-a_n^2\right)}\right| \qquad \text{note that } \left(7-a_n^2\right) \neq 0 \text{ for our choice of } N. \\
&\leq \left|\frac{\left(a_n-2\right)\frac{9}{2}}{3\left(\frac{3}{4}\right)}\right| \\
&= \left|2\left(a_n-2\right)\right| \\
&< 2\frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{1}{7-a_n^2} = \frac{1}{3}$. ∎

Exercise 2. Assume the sequence (a_n) is a Cauchy sequence. By directly using the definition of a Cauchy sequence (and not theorems about Cauchy sequences), show that

$$\left(\frac{a_n^3}{a_n^2+1}\right)$$

is also a Cauchy sequence.

Before we prove this, we will first prove a lemma.

Lemma 1: Cauchy sequences are bounded.

Proof: Let (b_n) be a Cauchy sequences. Given $\epsilon = 1$, there exists an N such that $|x_n - x_m| < 1$ for all $n, m \geq N$. Hence, $|x_n| < |x_N| + 1$ for all $n \geq N$. It follows that

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N|\}$$

is a bound for the sequence (b_n), thus every Cauchy sequence is bounded. ∎

We now continue with the original problem.

Proof: Since (a_n) is a Cauchy sequence, given an $\epsilon > 0$, there exists an $N \in \mathbb{R}$ such that when ever $k > m > N$,

$$|a_k - a_m| < \frac{\epsilon}{M^4 + 3M^2},$$

where M is a bound on the sequence such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Note that if $M = 0$, then $a_n = 0$ for all $n \in \mathbb{N}$ and were done; otherwise, it follows that

$$\begin{aligned}
\left|\frac{a_k^3}{a_k^2+1} - \frac{a_m^3}{a_m^2+1}\right| &= \left|\frac{a_k^2 a_m^2 (a_k - a_m) + a_k^3 - a_m^3}{(a_k^2+1)(a_m^2+1)}\right| \\
&= \left|\frac{a_k^2 a_m^2 (a_k - a_m) + (a_k - a_m)(a_k^2 + a_k a_m + a_k^2)}{(a_k^2+1)(a_m^2+1)}\right| \\
&\leq \left|a_k^2\right| \left|a_m^2\right| |a_k - a_m| + \left|a_k^2 + a_k a_m + a_k^2\right| |a_k - a_m| \\
&\leq \left|a_k^2\right| \left|a_m^2\right| |a_k - a_m| + \left(a_k^2 + |a_k| |a_m| + a_k^2\right) |a_k - a_m| \\
&\leq \left(M^4 + 3M^2\right) |a_k - a_m| \\
&< \left(M^4 + 3M^2\right) \frac{\epsilon}{M^4 + 3M^2} \\
&= \epsilon.
\end{aligned}$$

Therefore $\left(\frac{a_n^3}{a_n^2+1}\right)$ is a Cauchy sequence. ∎

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. Assume $\lim_{x\to c} f(x) = A \neq 0$ and $\lim_{x\to c} g(x) = b \neq 0$. Directly use the $\epsilon - \delta$ definitions of the limit to show that

$$\lim_{x\to c} \frac{1}{f(x)\,g(x)} = \frac{1}{AB}.$$

Proof: Given an $\epsilon > 0$, there exists a $\delta_1 \in \mathbb{R}$ such that whenever $0 < |x - c| < \delta_1$,

$$|f(x) - A| < \frac{|A|}{2},$$

which implies

$$\frac{|A|}{2} < |f(x)| < \left|\frac{3A}{2}\right|.$$

There also exists a $\delta_2 \in \mathbb{R}$ such that whenever $0 < |x - c| < \delta_2$,

$$|g(x) - B| < \frac{B}{2},$$

which implies

$$\frac{|B|}{2} < |g(x)| < \frac{3|B|}{2}.$$

When these conditions are met, it ensures that $g(x) \neq 0$ and $f(x) \neq 0$. Lastly, there exists a $\delta_3 \in \mathbb{R}$ such that whenever $0 < |x - c| < \delta_3$.

$$|f(x) - A| < \frac{\epsilon\,|A^2 B|}{4},$$

and

$$|g(x) - B| < \frac{\epsilon\,|B^2 A|}{8}.$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$, it follows that

$$\begin{aligned}
\left|\frac{1}{f(x)\,g(x)} - \frac{1}{AB}\right| &= \left|\frac{AB - f(x)\,g(x)}{ABf(x)\,g(x)}\right| \\
&= \left|\frac{AB - f(x)\,g(x) + Ag(x) - Ag(x)}{ABf(x)\,g(x)}\right| \\
&= \left|\frac{A(B - g(x)) + g(x)(A - f(x))}{ABf(x)\,g(x)}\right| \\
&\leq \frac{|A|\,|g(x) - B|}{|ABf(x)\,g(x)|} + \frac{|g(x)|\,|f(x) - A|}{|ABf(x)\,g(x)|} \\
&= \frac{|g(x) - B|}{|Bf(x)\,g(x)|} + \frac{|f(x) - A|}{|ABf(x)|} \\
&\leq \frac{|g(x) - B|}{\left|B\frac{A}{2}\frac{B}{2}\right|} + \frac{|f(x) - A|}{\left|AB\frac{A}{2}\right|} \\
&= 4\frac{|g(x) - B|}{|B^2 A|} + \frac{2\,|f(x) - A|}{|A^2 B|} \\
&< 4\frac{\epsilon\,|B^2 A|}{8\,|B^2 A|} + 2\frac{\epsilon\,|A^2 B|}{|A^2 B|\,4} \\
&= \epsilon.
\end{aligned}$$

Therefore $\lim_{x\to c} \frac{1}{f(x)g(x)} = \frac{1}{AB}$. ■

Exercise 4. Suppose a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at the point $x_0 \neq 0$. Prove that the series converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.

Proof: Since the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at the point $x_0 \neq 0$, given an $\epsilon > 0$, there exists an $N \in \mathbb{R}$ such that when $n > m \geq N$,

$$\sum_{k=m}^{n} \left|a_k x_0^k\right| < \epsilon.$$

This is equivalent to

$$\sum_{k=m}^{n} |a_k| \, |x_0|^k < \epsilon.$$

Note that for any $x \in [-c, c]$,

$$\left|\sum_{k=m}^{n} a_k x^k\right| \leq \sum_{k=m}^{n} |a_k| \, |x|^k \leq \sum_{k=m}^{n} |a_k| \, |x_0|^k < \epsilon,$$

thus according to the Cauchy Criterion for Uniform Convergence of series, the series converges uniformly. ∎

Exercise 5. If the series $\sum_{n=0}^{\infty} a_n x^n$ converges on the open interval $(-R, R)$, show that the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges on $(-R, R)$.

Proof: We suppose directly that the series $\sum_{n=0}^{\infty} a_n x^n$ converges on the open interval $(-R, R)$. There are two cases to consider:

Case 1. Let $x = 0$, then the differentiated series is 0 everywhere and thus converges on $(-R, R)$.

Case 2. Let $x \neq 0$. Since the series $\sum_{n=0}^{\infty} a_n x^n$ converges on the open interval $(-R, R)$, given an $\epsilon > 0$, there exists an $N \in \mathbb{R}$ such that whenever $n > m > N$,

$$\left| \sum_{k=m}^{n} a_k x^k \right| < \frac{\epsilon}{n} x.$$

Let $b_n = n x^{-1}$, then by Abel's lemma

$$\left| \sum_{k=m}^{n} a_k x^k b_k \right| < \frac{\epsilon}{n} x b_n.$$

Expanding it out gives

$$\left| \sum_{k=m}^{n} k a_k x^k x^{-1} \right| < \frac{\epsilon}{n} x n x^{-1}$$
$$\left| \sum_{k=m}^{n} k a_k x^{k-1} \right| < \epsilon,$$

Thus the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges on $(-R, R)$

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Exercise 6. Prove the Sequential Criterion for Integrability: A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if an only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Proof: Since this is a biconditional statement we must prove both ways.

$(\implies)$: We assume directly that $f : [a, b] \to \mathbb{R}$ is a bounded function that is integrable on $[a, b]$, then according to the integrability Criterion, for every $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Let (ϵ_n) be a monotonic decreasing sequence such that $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > 0$ and so that $\lim_{n\to\infty} \epsilon_n = 0$, then there exists a partition P_{ϵ_n} for every ϵ_n such that

$$U(f, P_{\epsilon_n}) - L(f, P_{\epsilon_n}) < \epsilon_n.$$

We can then construct the sequence of partitions (P_n) such that $P_n = P_{\epsilon_n}$. Hence

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty} [U(f, P_{\epsilon_n}) - L(f, P_{\epsilon_n})] = 0.$$

Therefore there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

$(\impliedby)$: We assume directly that there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

By the definition of a limit, given an $\epsilon > 0$ there exists an $N \in \mathbb{R}$ such that when $n > N$

$$|U(f, P_n) - L(f, P_n)| < \epsilon.$$

Since $U(f, P_n) \geq L(f, P_n)$ for any partition, we can drop the absolute value sign to get

$$U(f, P_n) - L(f, P_n) < \epsilon.$$

This shows that for every $\epsilon > 0$, there exists a partition P_n such that

$$U(f, P_n) - L(f, P_n) < \epsilon.$$

Therefore, by the integrability criterion, f is integrable on $[a, b]$. And in this case, since

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$$

we get

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

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Exercise 7. Assume f is integrable of $[a,b]$. Let $c \in (a,b)$. Define $g : [a,b] \to \mathbb{R}$ by

$$g\left(x\right) = \begin{cases} f\left(x\right) & \text{if } x \in [a,b] \text{ and } x \neq c \\ d & \text{if } x = c \end{cases}.$$

Prove that g is integrable on $[a,b]$ and $\int_a^b g = \int_a^b f$.

Proof: Since f is integrable on $[a,b]$, given an $\epsilon > 0$, there exists a partition P_ϵ such that

$$U\left(f, P_\epsilon\right) - L\left(f, P_\epsilon\right) < \frac{\epsilon}{2}.$$

Let $P = P_\epsilon \cup \left\{ c - \frac{\epsilon}{|d|4}, c + \frac{\epsilon}{|d|4} \right\}$ be a refinement, we can write

$$\begin{aligned} U\left(g,P\right) - L\left(g,P\right) &= \sum_{k=1}^{n} \left(N_k - n_k\right) \Delta x_k \\ &= \sum_{k=1}^{\ell} \left(M_k - m_k\right) \Delta x_k + \sum_{k=\ell+2}^{n} \left(M_k - m_k\right) \Delta x_k + \left(N_{\ell+1} - n_{\ell+1}\right) \Delta x_{\ell+1} \end{aligned}$$

where M_k is the supremem of f on the interval Δx_k, m_k is the infimum of f on the interval Δx_k, $\Delta x_{\ell+1}$ is the interval that contains c and N_k and $n_{\mathbf{k}}$ is the infimum and supremem of g on the interval Δx_k. By the construction of the partition P, we know that $\Delta x_{\ell+1} \leq \frac{\epsilon}{2|d|}$. If $n_{\ell+1} < d < N_{\ell+1}$, then

$$\begin{aligned} U\left(g,P\right) - L\left(g,P\right) = U\left(f,P\right) - L\left(f,P\right) < \frac{\epsilon}{2}. \\ < \epsilon \end{aligned}$$

If $d \geq N_{\ell+1}$ or $d \leq n_{\ell+1}$, then

$$\left(N_{\ell+1} - n_{\ell+1}\right) \leq \left(M_{\ell+1} - m_{\ell+1}\right) + |d|,$$

hence

$$\begin{aligned} U\left(g,P\right) - L\left(g,P\right) &\leq U\left(f,P\right) - L\left(f,P\right) + |d| \Delta x_{\ell+1} \\ &< \frac{\epsilon}{2} + |d| \frac{\epsilon}{2|d|} \\ &= \epsilon, \end{aligned}$$

Thus $g\left(x\right)$ is integrable. We note that

$$\begin{aligned} U\left(g,P\right) &\leq U\left(f,P\right) + |d| \Delta x_{\ell+1} \\ &= U\left(f,P\right) + |d| \frac{\epsilon}{2|d|} \\ &= U\left(f,P\right) + \epsilon. \end{aligned}$$

Since ϵ can be arbitrarily small, we get

$$U\left(g,P\right) = U\left(f,P\right),$$

thus

$$\int_a^b g = \int_a^b f$$

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Exercise 8. Prove the Integrable Limit Theorem: Assume that $f_n \to f$ uniformly on $[a, b]$ and that each f_n is integrable. Then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof: We will first show that f is integrable. Since f_n is integrable, given an $\epsilon > 0$, there exists a partition P_ϵ such that

$$U\left(f_n, P_\epsilon\right) - L\left(f_n, P_\epsilon\right) < \frac{\epsilon}{2}.$$

Also, since $f_n \to f$ uniformly, given the same ϵ as above, there exists an $N \in \mathbb{R}$ such that when $n > N$

$$\left|f_n\left(x\right) - f\left(x\right)\right| < \frac{\epsilon}{b-a}$$

for all $x \in [a, b]$. This implies that

$$f_n\left(x\right) - \frac{\epsilon}{2\left(b-a\right)} < f\left(x\right) < f_n\left(x\right) + \frac{\epsilon}{4\left(b-a\right)}.$$

By definition

$$U\left(f_n, P_\epsilon\right) - L\left(f_n, P_\epsilon\right) = \sum_{k=1}^{n} \left(M_k - m_k\right) \Delta x_k,$$

where M_k is the supremem of f_n on the interval Δx_k and m_k is the infimum of f_n on the interval Δx_k. Hence

$$\begin{aligned} U\left(f, P_\epsilon\right) - L\left(f, P_\epsilon\right) &\leq \sum_{k=1}^{n} \left(M_k + \frac{\epsilon}{4\left(b-a\right)} - m_k - \frac{\epsilon}{4\left(b-a\right)}\right) \Delta x_k \\ &= \sum_{k=1}^{n} \left(M_k - m_k\right) \Delta x_k + \frac{\epsilon}{2\left(b-a\right)} \left(b-a\right) \\ &= U\left(f_n, P_\epsilon\right) - L\left(f_n, P_\epsilon\right) + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, by the integrability criterion, f is integrable on the interval $[a, b]$. Now that we have shown that f is integrable, we wish to show that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$. Using the properties of the integral, we assert that for any f_n,

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b \left(f_n - f\right)\right| \leq \int_a^b \left|f_n - f\right|.$$

Let $\epsilon > 0$, and since $f_n \to f$ uniformly, there exists an $N \in \mathbb{R}$ such that whenever $n > N$

$$\left|f_n\left(x\right) - f\left(x\right)\right| < \frac{\epsilon}{b-a}$$

for all $x \in [a, b]$. Thus, for $n > N$ we get that

$$\begin{aligned}\int_a^b |f_n - f| &\leq \int_a^b \frac{\epsilon}{b-a} \\ &= \epsilon,\end{aligned}$$

thus

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

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Exercise 9. Prove the Fundamental Theorem of Calculus, Part 1: If f:$[a,b] \to \mathbb{R}$ is integrable, and $F:[a,b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a,b]$,then

$$\int_a^b f = F(b) - F(a).$$

This proof is in the textbook, so I will utilize it since its allowed.

Proof: Let P be a partition of $[a,b]$ and apply the Mean Value Theorem to F on the subinterval $[x_k, x_{k+1}]$ where $x_k, x_{k+1} \in P$. We get

$$\begin{aligned} F(x_{k+1}) - F(x_k) &= F'(t_{k+1})(x_{k+1} - x_k) \\ &= f(t_{k+1}) \Delta x_{k+1} \end{aligned}$$

where $t_{k+1} \in (x_k, x_{k+1})$. Let M_k be the supremem of f on the interval Δx_k and m_k be the infimum of f on the interval Δx_k, then $m_k < t_k < M_k$ and thus

$$L(f,P) \leq \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) \leq U(f,P).$$

Since $\sum_{k=1}^{n} F(x_k) - F(x_{k-1})$ is a telescoping sum, we can simplify the expression to

$$L(f,P) \leq F(b) - F(a) \leq U(f,P).$$

Since $F(b) - F(a)$ is independent of partitions and f is integrable, we get that

$$L(f) \leq F(b) - F(a) \leq U(f).$$

Therefore, $\int_a^b f = L(f) = U(f) = F(b) - F(a)$. ■

Exercise 10. Prove the Fundamental Theorem of Calculus, Part 2: Let $g : [a, b] \to \mathbb{R}$ be integrable for $x \in [a, b]$, define

$$G(x) = \int_a^x g.$$

Then G is continuous on $[a, b]$. If g is continuous at some point $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.

This proof is in the textbook, so I will utilize it.

Proof: Let $x, y \in [a, b]$ such that $x > y$, then

$$\begin{aligned} |G(x) - G(y)| &= \left| \int_a^x g - \int_a^y g \right| \\ &= \left| \int_y^x g \right|. \end{aligned}$$

According to the properties of the integral,

$$\left| \int_y^x g \right| \leq \int_y^x |g| .$$

Since g is integrable on $[a, b]$, it is also bounded on $[a, b]$. Let $M \in \mathbb{R}$ such that $|g(k)| \leq M$ for all $k \in [a, b]$, then

$$\int_y^x |g| \leq M(x - y),$$

which implies that

$$|G(x) - G(y)| \leq M(x - y).$$

Thus G is Lipschitz and hence it is uniformly continuous on $[a, b]$.

Now we assume that g is continuous at $c \in [a, b]$ and we want to show that G is differentiable at c and that $G'(c) = g(c)$. Using the definition of the derivative of G,

$$\begin{aligned} G'(c) &= \lim_{x \to c} \frac{G(x) - G(c)}{x - c} = \lim_{x \to c} \frac{1}{x - c} \left(\int_a^x g - \int_a^c g \right) \\ &= \lim_{x \to c} \frac{1}{x - c} \int_c^x g. \end{aligned}$$

We would like to show that $\lim_{x \to c} \frac{1}{x-c} \int_c^x g = g(c)$. Since g is assumed continuous, we can pick a $\delta > 0$, such that when $0 < |x - c| < \delta$

$$|g(t) - g(c)| < \epsilon.$$

To take advantage of this, we can write the constant $g(c)$ as

$$g(c) = \frac{1}{x - c} \int_c^x g(c)\, dt.$$

Now, using the definition of the limit, given an $\epsilon > 0$, let $\delta > 0$, then when $0 < |x - c| < \delta$,

$$\begin{aligned} \left| \frac{1}{x - c} \left(\int_c^x g \right) - g(c) \right| &= \left| \frac{1}{x - c} \left(\int_c^x g(x)\, dt - g(c)\, dt \right) \right|. \\ &\leq \left| \frac{1}{x - c} \int_c^x \epsilon dt \right| \\ &= \epsilon. \end{aligned}$$

Therefore, if g is continuous at some point $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$. ■