Homework 6 Section 2.2

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Exercises 2,4,5,6,7

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Exercise 1. **(Q2)**: Prove, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

1) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$
   Proof: Given $\epsilon > 0$, $N \in \mathbb{N}$ such that $N = \left(\frac{3}{25\epsilon} - \frac{4}{5}\right)$ and $n \in \mathbb{N} > N$, then

$$\begin{aligned} \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \left|\frac{5(2n+1) - 2(5n+4)}{5(5n+4)}\right| \\ &= \left|\frac{-3}{25n+20}\right| \\ &< \left|\frac{-3}{25N+20}\right| \\ &= \left|\frac{-3}{25\left(\frac{3}{25\epsilon} - \frac{4}{5}\right) + 20}\right| \\ &= \left|\frac{-3}{\frac{3}{\epsilon} - 20 + 20}\right| \\ &= \epsilon, \end{aligned}$$

   thus $\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \epsilon$. Therefore, $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$. ■

2) $\lim \frac{2n^2}{n^3+3} = 0$.
   a) Given $\epsilon > 0$, $N \in \mathbb{N}$ such that $N = \frac{2}{\epsilon}$ and $n \in \mathbb{N} > N$, then

$$\begin{aligned} \left|\frac{2n^2}{n^3+3}\right| &< \left|\frac{2n^2}{n^3}\right| \\ &= \frac{2}{n} \\ &< \frac{2}{\frac{2}{\epsilon}} \\ &= \epsilon, \end{aligned}$$

   thus $\left|\frac{2n^2}{3n^3+3}\right| < \epsilon$. Therefore, $\lim \frac{2n^2}{n^3+3} = 0$.

3) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.
   a) Given $\epsilon > 0$, $N \in \mathbb{N}$ such that $N = \frac{1}{\epsilon^3}$ and $n \in \mathbb{N} > N$, then

$$\begin{aligned} \left|\frac{\sin\left(n^2\right)}{\sqrt[3]{n}}\right| &\leq \left|\frac{1}{\sqrt[3]{n}}\right| \\ &< \frac{1}{\sqrt[3]{N}} \\ &= \frac{1}{\frac{1}{\sqrt[3]{\epsilon^3}}} \\ &= \epsilon, \end{aligned}$$

thus $\left|\frac{\sin(n^2)}{\sqrt[3]{n}}\right| < \epsilon$. Therefore, $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Exercise 2. (Q4): Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

1) A sequence with an infinite number of ones that does not converge to one.
   a) Let a_n be the infinite sequence $(1, 0, 1, 0, 1, 0, \cdots)$. If $a_m = 1$, then $a_{m+1} = 0$, for all $m \in \mathbb{N}$. Thus for $\epsilon = 0.5$, there does exists a $k > N \in \mathbb{N}$ for any N such that
   $$|a_k - 1| > \epsilon.$$
2) A sequence with an infinite number of ones that converges to a limit not equal to one.
   a) This sequence is impossible. Suppose, by contrary, that a_n is a converging sequence with an infinite number of ones and converges to $a \neq 1$. Then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \in \mathbb{N} > N$, $|a_n - a| < \epsilon$. Since there are an infinite number of ones, for any n, which is finite, there is an $a_m = 1$ such that $m > 1$. This means that $|1 - a|$ must be less than ϵ as well for all N. But since limits are unique, this in only possible if $1 = a$. This is a contradiction, therefore the sequence a_n is impossible.
3) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.
   a) Let a_n be the sequence
   $$(1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \cdots),$$
   where there are k consecutive ones followed by a zero for every $k \in \mathbb{N}$. Since the sequence is infinite, for any $m \in \mathbb{N}$, there is an a_ℓ and a_j such that $\ell > j > m$ and $|a_\ell - a_j| = 1$. Thus the sequence doesn't converge.

Exercise 3. (Q5): Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$, find $\lim a_n$ and verify it with the definition of convergence.

1) $a_n = [[5/n]]$.
   Proof: We want to show that $\lim a_n = 0$. Given $\epsilon > 0$, let $N \in \mathbb{N} > 10$ and $n \in \mathbb{N} > N$. Then $[[5/n]] = [[0.5]] = 0$, thus
   $$|[[5/n]] - 0| = 0 < \epsilon.$$
   ■
2) $a_n = [[(12 + 4n)/3n]]$.
   Proof: We want to show that $\lim a_n = 1$. We do this by first considering the sequence $b_n = (12 + 4n)/3n$ and show that $\lim b_n = \frac{4}{3}$. Given $\epsilon > 0$, $N \in \mathbb{N} > \frac{4}{\epsilon}$, and $n \in \mathbb{N} > N$, then
   $$\begin{aligned}\left|\frac{12+4n}{3n} - \frac{4}{3}\right| &= \left|\frac{12+4n-4n}{3n}\right| \\ &= \frac{12}{3n} \\ &= \frac{4}{n} \\ &< \frac{4}{\frac{4}{\epsilon}} \\ &= \epsilon,\end{aligned}$$
   thus $\lim b_n = \frac{4}{3}$. Since $\lim b_n = \frac{4}{3}$,
   $$\begin{aligned}\lim a_n &= [[\lim b_n]] \\ &= \left[\left[\frac{4}{3}\right]\right] \\ &= 1.\end{aligned}$$
   ■

Exercise 4. (Q6): Prove theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

Proof: Let a_n be a sequence. We suppose, by contradiction, that $(a_n) \to a$ and $(a_n) \to b$ such that $a \neq b$. Then $|a-b| > 0$. Let $A = \left\{x \in a_n : |x-a| < \frac{|a-b|}{2}\right\}$, and $B = \left\{x \in a_n : |x-b| < \frac{|a-b|}{2}\right\}$, then $A \cap B = \emptyset$, in other words, the open sets A and B are disjoint. This means that there are some elements of a_n that cannot be arbitrarily close to both a and b. In other words, given an $\epsilon > 0$, there doesn't exists an $N \in \mathbb{N}$ that such whenever $n \in \mathbb{N} > N$

$$|a_n - a| < \epsilon$$

and

$$|a_n - b| < \epsilon$$

since the neighborhood around a and b are disjoint. The simple counterexample is when $\epsilon < \frac{|a-b|}{2}$. ■

Exercise 5. (Q7): Here are two useful definitions:

(i) A sequence (a_n) is eventually in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is frequently in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

1) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$.
   a) The sequence is frequently in the set, since if $(-1)^m \in \{1\}$, then $(-1)^{m+1} \notin \{1\}$.
2) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?
   a) The definition eventually is stronger, since eventually means that the sequence stays in the a set A after some $n \in \mathbb{N}$, and if it stays in the set A after some n, then for all $m \geq n$, $a_m \in A$. Thus it is frequently in the set A as well. The converse, however is not true. For example, consider the sequence mentioned in part a). It is frequently in the set $\{1\}$, but not eventually in the set since it keeps leaving.
3) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?
   a) A sequence (a_n) converges to a if, given any $\epsilon-$neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence m, such that (a_n) is eventually in the neighborhood $V_\epsilon(a)$.
4) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$.
   a) It is not eventually in the interval. Consider the counterexample

   $$(x_n) = \{2, 0, 2, 0, 2, 0, 2, 0, \cdots\},$$

   that contains an infinite number of 2's but alternates with 0. Thus it will always leave the interval $(1.9, 2.1)$.
   b) (x_n) is frequently in the interval $(1.9, 2.1)$.
   Proof: Suppose that an infinite number of terms of a sequence (x_n) are equal to 2, then for every $N \in \mathbb{N}$, $N < \infty$. Since there are an infinite number of 2s in the sequence, and $N < \infty$, there must be an $a_m = 2$, where $m \in \mathbb{N} > N$. Thus (x_n) is frequently in the interval $(1.9, 2.1)$ ■