Exercise 12. (Q4): Assume (x_n) is a Cauchy sequence that satisfies $2 < x_n < 3$ for all $n \in \mathbb{N}$. By directly using the definition of a Cauchy sequence, show that

$$\left(\frac{x_n^2}{x_n - 1}\right)$$

is also a Cauchy sequence.

Proof: We assume directly that (x_n) is a Cauchy sequence that satisfies $2 < x_n < 3$ for all $n \in \mathbb{N}$, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$, such that whenever $n, m > N$,

$$(x_n - x_m) < \frac{\epsilon}{15}.$$

We next examine the term $\left(\frac{x_n^2}{x_n-1}\right)$, and begin to manipulate it.

$$\left|\frac{x_n^2}{x_n - 1} - \frac{x_m^2}{x_m - 1}\right| = \left|\frac{x_n^2\left(x_m - 1\right) - x_m^2\left(x_n - 1\right)}{\left(x_n - 1\right)\left(x_m - 1\right)}\right|.$$

Using the fact that $2 < x_n < 3$ for all $n \in \mathbb{N}$, we know that

$$\begin{aligned}
\left|\frac{x_n^2\left(x_m - 1\right) - x_m^2\left(x_n - 1\right)}{\left(x_n - 1\right)\left(x_m - 1\right)}\right| &\leq \left|\frac{x_n^2\left(x_m - 1\right) - x_m^2\left(x_n - 1\right)}{\left(2 - 1\right)\left(2 - 1\right)}\right| \\
&= \left|x_n^2\left(x_m - 1\right) - x_m^2\left(x_n - 1\right)\right| \\
&= \left|x_n^2 x_m - x_n^2 - x_m^2 x_n + x_m^2\right| \\
&= \left|x_n x_m\left(x_n - x_m\right) - x_n^2 + x_m^2\right| \\
&= \left|x_n x_m\left(x_n - x_m\right) - x_n^2 + x_m^2 - x_n x_m + x_n x_m\right| \\
&= \left|x_n x_m\left(x_n - x_m\right) - x_n\left(x_n - x_m\right) - x_m\left(x_n - x_m\right)\right| \\
&\leq \left|x_n x_m\left(x_n - x_m\right)\right| + \left|x_n\left(x_n - x_m\right)\right| + \left|x_m\left(x_n - x_m\right)\right| \\
&\leq 3 \cdot 3 \cdot \left|x_n - x_m\right| + 3 \cdot \left|x_n - x_m\right| + 3 \cdot \left|x_n - x_m\right| \\
&= 15\left|x_n - x_m\right| \\
&< 15\frac{\epsilon}{15} \\
&= \epsilon;
\end{aligned}$$

therefore, $\left(\frac{x_n^2}{x_n-1}\right)$ is a Cauchy sequence.