Exercise 16. (Q8): Prove that the real numbers are uncountable with a proof that relies on the Nested Interval Theorem.

Proof: We suppose, by contradiction, that the real numbers are countable, thus there exists a bijection $f : \mathbb{N} \rightarrow \mathbb{R}$. We can construct nested closed intervals in the following manner. Let I_1 be the closed interval such that $I_1 \subseteq \mathbb{R}$ and $f(1) \notin I_1$. Then let I_2 be the closed interval such that $I_2 \subseteq \mathbb{R}$ and $f(2) \notin I_2$. Due to the density of $\mathbb{R}$, we can repeat this process recursively such that $I_n \subseteq I_{n+1}$ and $f(n) \notin I_n$. We then form the intersection $\cap_{n=1}^{\infty} I_n$ which is not empty according to the nested interval theorem. Since none of the elements in $\cap_{n=1}^{\infty} I_n$ are in the image of f, f is not surjective and hence not a bijection. This contradicts our assumption, thus $\mathbb{R}$ is uncountable. ■