Homework 29 Section 6.5

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Exercises: 1,2,6,7

07/31/2020

Exercise 1. (Q1): Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

a) Is g defined on $(-1,1)$? Is it continuous on this set? Is g defined on $(-1,1]$? Is it continuous on this set? What happens on $[-1,1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.
Proof: The function $g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} x^n$. When $x = 1$, then $g(x)$ converges by the alternating series test. Thus $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} x^n$ converges uniformly on $(-1,1]$, thus $g(x)$ is continuous on $(-1,1]$. When $x = -1$, then the series becomes

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} (-1)^n = -1 \sum_{n=1}^{\infty} \frac{1}{n}$$

which does not converge. The power series for $g(x)$ cannot converge for any other point, because if it did then this point c would satisfy the condition $|-1| < |c|$ which would imply that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} x^n$ converges at $x = -1$ which would be a contradiction. ■

b) For what values of x is $g'(x)$ defined? Find a formula for g'.
   a) g' is defined on the open interval $(-1,1)$ and can be given by the formula

$$g'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}$$

Exercise 2. (Q2): Find suitable coefficients (a_n) so that the resulting power series

$$\sum a_n x^n$$

has the given properties, or explain why such a request is impossible.

a) Converges for every value $x \in \mathbb{R}$.
   a) Let $a_n = 0$.
b) Diverges for every value of $x \in \mathbb{R}$.
   a) This request is impossible since when $x = 0$, the series $\sum a_n x = 0$.
c) Converges absolutely for all $x \in [-1,1]$ and diverges off of this set.
   a) Let $a_n = \frac{1}{n^2}$. Then this converges on the open interval $[-1,1]$ by the comparison test and the alternating series test. When $|x| > 1$, then

$$\lim_{n\to\infty} \frac{x^n}{n^2} = \infty$$

   using L'Hospitals rule. Thus it diverges.
d) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.
   a) Such a request is impossible. Since it converges absolutely at $x = 1$, it must also converge absolutely at $x = -1$.

$$\sum |a_n(-1)| = \sum |a_n 1|$$

   which converges absolutely.
e) Converges conditionally at both $x = -1$ and $x = 1$.
   a) Let

$$a_n = \begin{cases} \frac{(-1)^{n/2}}{n} & \text{if } n \text{ is even} \\ 0 & \text{else} \end{cases},$$

   then we get the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n}$$

which converges conditionally at $x = -1, 1$ but not absolutely.

Exercise 3. (Q6): Previous work on geometric series justifies the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad \text{for all } |x| < 1.$$

Use the results about power series proved in this section to find values for

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

and

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

The discussion in Section 6.1 may be helpful.

Since the geometric series converges for all $|x| < 1$, it's derivative also converges. Taking the derivative we get

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 +\right)$$

which results in

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Let $x = \frac{1}{2}$, then we get

$$\frac{1}{\left(1-\frac{1}{2}\right)^2} = 1 + 2 \cdot \frac{1}{2} + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots.$$

Multiplying both sides by $\frac{1}{2}$ yields

$$\begin{aligned}
\frac{1}{2}\left(\frac{1}{\left(1-\frac{1}{2}\right)^2}\right) &= \frac{1}{2}\left(1 + 2 \cdot \frac{1}{2} + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots\right) \\
&= \left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots\right) \\
&= \sum_{n=1}^{\infty} \frac{n}{2^n}
\end{aligned}$$

Thus, the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges to $\frac{1}{2}\left(\frac{1}{\left(1-\frac{1}{2}\right)^2}\right) = 2$.

Taking the derivative again we get

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) &= \frac{d}{dx}\left(\sum_{n=1}^{\infty} n x^{n-1}\right) \\
2\frac{1}{(1-x)^3} &= \sum_{n=1}^{\infty} n(n-1)x^{n-2} \\
&= \sum_{n=1}^{\infty} n^2 x^{n-2} - \sum_{n=1}^{\infty} n x^{n-2} \\
&= x^{-2}\sum n^2 x^n - x^{-2}\sum n x^n.
\end{aligned}$$

Let $x = \frac{1}{2}$, then we get

$$\begin{aligned}
2\frac{1}{\left(\frac{1}{2}\right)^3} &= x^{-2}\sum n^2 x^n - x^{-2}\sum n x^n \\
&= 4\sum n^2 \frac{1}{2^n} - 4\sum \frac{n}{2^n} \\
&= 4\sum n^2 \frac{1}{2^n} - \frac{1}{4}\frac{1}{2}\left(\frac{1}{\left(1-\frac{1}{2}\right)^2}\right) \\
&= 4\sum \frac{n^2}{2^n} - 4 \cdot 2
\end{aligned}$$

Solving for the series $\sum \frac{n^2}{2^n}$ gives

$$\sum \frac{n^2}{2^n} = (16+8)/4$$
$$= 6$$

Exercise 4. (Q7): Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and assume

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists.

a) Show that if $L \neq 0$, then the series converges for all x in $(-1/L, 1/L)$.
Proof: Since $|x| < \frac{1}{L}$ we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$< L\frac{1}{L}$$
$$= 1$$

By the ratio test, the series converges absolutely. ■

b) Show that if $L = 0$, then the series converges for all $x \in \mathbb{R}$.
Proof: Using the same process as above, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x|$$
$$< L|x|$$
$$= 0.$$

By the ratio test, the series converges absolutely. ■

c) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} s_n \quad \text{where} \quad s_n = \sup\left\{ \left| \frac{a_{k+1}}{a_k} : k \geq n \right| \right\}.$$

Proof: Let $\epsilon > 0$, for all $L' + \epsilon$, there exists an N such that whenever $n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} \right| < L' + \epsilon.$$

By letting $x \in \left(-\frac{1}{L'}, \frac{1}{L}\right)$ and choosing $\delta > 0$ such that $|x| < \frac{1}{L'} - \delta$, it follows that

$$\left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| < (L' + \epsilon)\left(\frac{1}{L'} - \delta\right)$$
$$= 1 + \epsilon\left(\frac{1}{L'} - \delta\right) - \delta L'.$$

We can choose

$$\epsilon < \frac{\delta L'}{\left(\frac{1}{L'} - \delta\right)}$$

so that

$$\left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| < 1 + \frac{\delta L'}{\left(\frac{1}{L'} - \delta\right)}\left(\frac{1}{L'} - \delta\right) - \delta L'$$
$$= 1$$

which shows that the series converges absolutely by the ratio test. ■