# Homework 8 Section 2.4

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Exercises 2,3(a),5(a),7,8

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**Exercise 1. (Q2)**: Consider the recursively defined sequence $y_1 = 1$,

$$y_{n+1} = 3 - y_n,$$

and set $y = \lim y_n$. Because $(y_n)$ and $(y_{n+1})$ have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for $y$, we conclude $\lim y_n = 3/2$. What is wrong with this argument?

1) The sequence $(y_n)$ is $\{1, 2, 1, 2, 1, 2, 1, 2, \cdots\}$, and doesn't have a limit. Therefore, $y$ cannot be the limit of $(y_n)$ since the limit doesn't exist.

**Exercise 2. (Q3(a))**: Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

*Proof:* Consider the recursively defined sequence $a_1 = \sqrt{2}$,

$$a_{n+1} = \sqrt{2 + a_n}.$$

We want to show that $(a_n) \to 2$. We do this by first showing that the sequence has converges by showing that it has an upper bound and that sequence continually increases.

To show that $(a_n)$ has an upper bound, we want to show that the open sentence

$$Q(n) : a_n < 2$$

for all $n \in \mathbb{N}$. We work this by induction.

**Base Case**: We first verify $Q(1)$ and $Q(2)$.

$$a_1 = \sqrt{2} < 2$$
$$a_2 = \sqrt{2+\sqrt{2}} < \sqrt{2+2} = 2.$$

**Induction Step**: We assume that $Q(k)$ is true for some $k \in \mathbb{N}$ and we want to show that $Q(k+1)$ is true. Well,

$$a_{k+1} = \sqrt{2 + a_k},$$

and since $a_k < 2$, we have

$$\begin{aligned} a_{k+1} &< \sqrt{2+2} \\ &= 2, \end{aligned}$$

thus 2 is an upper bound of the sequence $(a_n)$.

Since $a_1 < a_2 < a_3 < \cdots < a_n < a_{n+1}$, the sequence is monotonically increasing. Since $(a_n)$ is monotonic and bounded, the sequence converges. Thus a limit exists.

We now know that $(a_n)$ converges. Let $(a_n) = L$, then

$$\begin{aligned} \lim(a_{n+1}) &= \sqrt{2 + \lim(a_n)} \\ L &= \sqrt{2 + L}, \end{aligned}$$

which implies

$$L^2 - L - 2 = 0.$$

This gives $L = -1, 2$. Since $a_n > 0$ for all $n$, $L = 2$. Therefore, $(a_n) = 2$. ∎

**Exercise 3. (Q5(a))**: Calculating square roots. Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Show that $x_n^2$ is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

1) We will first show that $x_n^2$ is always greater than or equal to 2.
   *Proof:* We want to prove the open sentence

   $$Q(n) : x_n^2 \geq 2$$

   for all $n \in \mathbb{N}$ with $x_1 = 2$. Since $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$,

   $$\begin{aligned} x_{n+1}^2 &= \left(\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right)^2 \\ &= \frac{x_n^2}{4} + 1 + \frac{1}{x_n^2}. \end{aligned}$$

   **Base Case**: We first verify $Q(1)$ and $Q(2)$.

   $$\begin{aligned} x_1^2 &= 4 \geq 2 \\ x_2^2 &= \frac{4}{4} + 1 + \frac{1}{4} = 2.25 \geq 2. \end{aligned}$$

   **Induction Step**: Let $k \in \mathbb{N}$. We assume that $Q(k)$ is true, and we want to show that $Q(k+1)$ is true.

   $$\begin{aligned} x_{k+1}^2 &= \frac{x_k^2}{4} + 1 + \frac{1}{x_k^2} \\ &\geq \frac{2}{4} + 1 + \frac{1}{2} \\ &= 2, \end{aligned}$$

   therefore, $Q(n)$ is true. ∎
2) Prove that $x_n - x_{n+1} \geq 0$.
   *Proof:* We suppose directly that $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$, then

   $$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= x_n - \frac{x_n}{2} - \frac{1}{x_n} \\ &= \frac{x_n}{2} - \frac{1}{x_n} \\ &= \frac{x_n^2 - 2}{2x_n}, \end{aligned}$$

   since $x_n^2 \geq 2$ and $x_n > 0$ for all $n \in \mathbb{N}$,

   $$\frac{x_n^2 - 2}{2x_n} \geq 0,$$

   which means that $x_n - x_{n+1} \geq 0$. ∎
3) Conclude that $\lim x_n = \sqrt{2}$.

*Proof:* Since $x_n - x_{n+1} \geq 0$, the sequence $(x_n)$ is monotonically decreasing. Also, since $x_n^2 \geq 2$ for all $n$, two serves as a lower bound. Let $x = \lim\, x_n = \lim\, x_{n+1}$, then

$$\begin{aligned} \lim\, x_{n+1} &= \lim\, \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ x &= \frac{1}{2}x + \frac{1}{x} \\ x^2 &= \frac{1}{2}x^2 + 1 \\ \frac{1}{2}x^2 &= 1 \\ x &= \sqrt{2}, \end{aligned}$$

thus the limit is $\sqrt{2}$. ■

**Exercise 4. (Q7)**: Let $(a_n)$ be a bounded sequence.

1) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.
   *Proof:* We suppose directly that the sequence $a_n$ is bounded, then there exists a number $M > 0$, such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. According to the axiom of completeness, any upper bounded set has a least upper bound, in other words, a supremem. Thus the $\sup\{a_k : k \geq n\}$ exists for all $k, n \in \mathbb{N}$. Since the sequence $(a_n)$ is bounded, the sequence $(y_n)$ is also bounded. The supremem of any subset if equal or less to the supremem of the original set, thus, $y_j \leq y_m$ for any $j, m \in \mathbb{N}$ such that $j < m$. Thus the sequence $(y_n)$ is monotonically decreasing and bounded; therefore, according to the monotone convergence theorem, the sequence $(y_n)$ converges. ■
2) The limit superior of $(a_n)$, or $\limsup\, a_n$ is defined by

   $$\limsup\, a_n = \lim\, y_n,$$

   where $(y_n)$ is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf\, a_n$ and briefly explain why it always exists for any bounded sequence.
   a) The limit inferior of $(a_n)$ is defined by

      $$\liminf\, a_n = \lim\, z_n,$$

      where $z_n = \inf\{a_k : k \geq n\}$. The reason why this limit exists is similar to that the previous portion of the exercise. The sequence $(z_n)$ is bounded an monotonically increases; therefore, it converges.
3) Prove that $\lim\,\inf\, a_n \leq \lim\,\sup\, a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.
   *Proof:* We suppose directly that $|a_n| \leq M$ for all $n$. We want to show that $\liminf\, a_n \leq \limsup\, a_n$. Let $y_n = \sup\{a_n : k \geq n\}$ and $z_n = \inf\{a_k : k \geq n\}$. Let $S_n = \{a_k : k \geq n\}$, then for all $s \in S_n$, $y_n \geq s$ and $z_n \leq s$. Hence, $y_n \geq s \geq z_n$ for all $n$. In other words, $y_n \geq z_n$. Thus, as $n \to \infty$, $y_n \geq z_n$ ■
   a) An example of a bounded sequence for which the inequality is strict is the sequence $a_n = \cos\left(\frac{2n}{10}\pi\right)$. $z_n = 1$ and $y_n = -1$ for all $n$. Thus $y_n < z_n$.
4) Show that $\liminf\, a_n = \limsup\, a_n$ if and only if $\lim\, a_n$ exists.
   *Proof:* This is a biconditional statement so we must prove both ways.
   $(\implies)$ : Let $z_n = \inf\{a_k : k \geq n\}$, $y_n = \sup\{a_k : k \geq n\}$. We suppose directly that $\liminf\, a_n = \limsup\, a_n$, then $\lim\, z_n = \lim\, y_n$. According to the order limit theorem and the squeeze theorem, since

   $$z_n \leq a_n \leq y_n$$

   for all $n$,

   $$\lim\, z_n \leq \lim\, a_n \leq \lim\, y_n.$$

   Since $\lim\, z_n = \lim\, y_n$, we have that $\lim\, a_n = \lim\, y_n$.

( $\Longleftarrow$ ) : We suppose directly that $(a_n)$ converges to $L$. Every convergent sequence is bounded, so the sequence has a least upper bound and a greatest lower bound. Since $(a_n)$ converges to $L$, then given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$,

$$\begin{aligned} |a_n - L| &< \epsilon \\ -\epsilon < a_n - L &< \epsilon \\ L - \epsilon < a_n &< L + \epsilon. \end{aligned}$$

According to lemma 1.3.8, $L$ must be the limit supremum and the limit infimum of the sequence $(a_n)$. Therefore, if $\lim a_n$ exists, $\liminf a_n = \limsup a_n$. ■

**Exercise 5. (Q8)**: For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

1) $\sum_{n=1}^{\infty} \frac{1}{2^n}$
   a) We want to show that $s_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{2^n - 1}{2^n}$.
   *Proof:* We assume that $s_n = \sum_{k=1}^{n} \frac{1}{2^k}$, and we want to prove the open sentence

   $$Q(n) : s_n = \frac{2^n - 1}{2^n}$$

   is true for all $n \in \mathbb{N}$. We work this by induction.
   **Base Case**: We first verify $Q(1)$,

   $$s_1 = \frac{1}{2} = \frac{2^1 - 1}{2^1}.$$

   **Induction Step**: We assume that $Q(m)$ is true and we want to show that $Q(m+1)$ is true.

   $$\begin{aligned} s_{m+1} &= \sum_{k=1}^{m+1} \frac{1}{2^k} \\ &= \frac{2^m - 1}{2^m} + \frac{1}{2^{m+1}} \\ &= \frac{2(2^m - 1) + 1}{2^{m+1}} \\ &= \frac{2^{m+1} - 1}{2^{m+1}}, \end{aligned}$$

   thus $Q(m+1)$ is true; therefore, $Q(n)$ is true for all $n \in \mathbb{N}$. ■
   b) We want to show that $\lim s_n \to 1$.
   *Proof:* The value

   $$\begin{aligned} s_n &= \frac{2^n - 1}{2^n} \\ &= \frac{2^n}{2^n} - \frac{1}{2^n} \\ &= 1 - \frac{1}{2^n}. \end{aligned}$$

   Given an $\epsilon > 0$, let $N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}$, and choose $k \in \mathbb{N} > N$, then

   $$\begin{aligned} |s_k - 1| &= \left|1 - \frac{1}{2^k} - 1\right| \\ &= \left|\frac{1}{2^k}\right| \\ &< \frac{1}{2^N} \\ &< \epsilon. \end{aligned}$$

   Thus $(s_n) \to 1$. ■

2) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

a) We want to show that $s_k = \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n-1}$

*Proof:* The series $\sum_{k=1}^{n} \frac{1}{k(k+1)}$ can be expanded

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n-1}\right) \\ &= 1 - \frac{1}{n-1}.\end{aligned}$$

Thus $s(n) = 1 - \frac{1}{n-1}$. ■

b) We want to show that $(s_n) \to 1$.

*Proof:* Given an $\epsilon > 0$, let $N = \frac{1}{\epsilon} + 1$, then when $n \in \mathbb{N} > N$,

$$\begin{aligned}|s_n - 1| &= \left|1 - \frac{1}{n-1} - 1\right| \\ &= \frac{1}{n-1} \\ &< \frac{1}{N-1} \\ &= \frac{1}{\frac{1}{\epsilon} + 1 - 1} \\ &= \epsilon,\end{aligned}$$

thus $(s_n) \to 1$. ■

3) $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

a) We want to show that $s_n = \sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right) = \log(k+1)$.

*Proof:* The partial sum $s_n$ can be expanded out

$$\begin{aligned}\sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right) &= \sum_{k=1}^{n} (\log(k+1) - \log(k)) \\ &= (\log(2) - \log(1)) + (\log(3) - \log(2)) + \cdots + (\log(k+1) - \log(k)) \\ &= \log(k+1) - \log(1) \\ &= \log(k+1).\end{aligned}$$

Thus $s_n = \log(k+1)$. ■

b) The sequence $(s_n)$ does not converge since the natural logarithm is not bounded from above and it's argument is tending towards zero.