Homework 22 Section 5.2

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Exercises: 2,3,4,5,6

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Exercise 1. (Q2): Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbb{R}$.

a) Functions f and g not differentiable at zero but where fg is differentiable at zero.
   a) Possible. Let
$$f(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ \frac{1}{2} & \text{else} \end{cases},$$
   and
$$g(x) = \begin{cases} \frac{1}{2} & \text{if } x \geq 0 \\ 2 & \text{else} \end{cases},$$
   be two piecewise functions not continuous at 0, thus they are not differentiable at 0. However,
$$fg = 2,$$
   which is differentiable.

b) A function f not differentiable at zeros and a function g differentiable at zero where fg is differentiable at zero.
   a) Possible. Let
$$f(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ \frac{1}{2} & \text{else} \end{cases},$$
   and $g(x) = 0$, then $f(x)\, g(x) = 0$ which is differentiable everywhere.

c) A function f not differentiable at zero and a function g differentiable at zero where $f + g$ is differentiable at zero.
   a) Not possible. According to the Algebraic Differentiability Theorem, since g and $(f+g)$ are differentiable at zero, the function $((f+g)-g)$ is also differentiable; however,
$$((f+g)-g) = f,$$
   which f is not differentiable. This is a contradiction, and thus the request is impossible.

d) A function f differentiable at zero but not differentiable at any other point.
   a) Possible. Let f be define as
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}.$$
   This function is only continuous at $x = 0$, so it can't be differentiable anywhere else. To show that f is differentiable at 0, we take
   b)
$$f'(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases} = 0$$
   which shows that $f'(0) = 0$.

Exercise 2. (Q3): Complete the following.

a) Use definition 5.2.1 to produce the proper formula for the derivative $h(x) = \frac{1}{x}$,

$$\begin{aligned} h^{'}(c) &= \lim_{x\to c} \frac{\frac{1}{x} - \frac{1}{c}}{x-c} = \lim_{x\to c} \frac{\frac{c-x}{xc}}{x-c} \\ &= \lim_{x\to c} \frac{-(x-c)}{xc(x-c)} \\ &= \lim_{x\to c} -\frac{1}{xc} \\ &= -\frac{1}{c^2}. \end{aligned}$$

b) Combine the result in part (a) with the Chain Rule to supply a proof for part (iv) of Theorem 5.2.4
Proof: We suppose that f and g are functioned defined on an interval A and that both are differentiable at some point $c \in A$, $g(c) \neq 0$, and let $h(x) = \frac{1}{x}$ and $h'(x) = -\frac{1}{x^2}$ then

$$\begin{aligned} (f/g)' &= (fh(g))' \\ &= f'h(g) + fh'(g)g' \\ &= f'/g - f/g^2g' \\ &= \frac{f'g - fg'}{g^2}. \end{aligned}$$

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c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (iii).

Proof: We suppose that f and g are functioned defined on an interval A and that both are differentiable at some point $c \in A$ and $g(c) \neq 0$, then

$$\begin{aligned} (f/g)' &= \lim_{x\to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} \\ &= \lim_{x\to c} \frac{\frac{g(c)f(x)-g(x)f(c)}{g(x)g(c)}}{x-c} \\ &= \lim_{x\to c} \frac{g(c)f(x) - g(x)f(c)}{g(x)g(c)(x-c)} \\ &= \lim_{x\to c} \frac{g(c)f(x) - g(x)f(c) - g(c)f(c) + g(c)f(c)}{g(x)g(c)(x-c)} \\ &= \lim_{x\to c} \frac{g(c)(f(x) - f(c)) - f(c)(g(x) - g(c))}{g(x)g(c)(x-c)} \\ &= \lim_{x\to c} \frac{g(c)}{g(x)g(c)} \frac{(f(x) - f(c))}{(x-c)} - \lim_{x\to c} \frac{f(c)}{g(x)g(c)} \frac{(g(x) - g(c))}{(x-c)} \\ &= \lim_{x\to c} \frac{g(c)}{g(x)g(c)} f'(c) - \lim_{x\to c} \frac{f(c)}{g(x)g(c)} g'(c) \\ &= \frac{g(c)f'(c) - f(c)g'(c)}{g(c)^c}. \end{aligned}$$

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Exercise 3. (Q4): Follow these steps to provide a slightly modified proof of the Chain rule.

a) Show that a function $h : A \to \mathbb{R}$ is differentiable at $a \in A$ if and only if there exists a function $l : A \to \mathbb{R}$ which is continuous at a and satisfies

$$h(x) - h(a) = l(x)(x-a).$$

Proof: This is a biconditional statement so we must prove both ways.
$(\implies)$: We suppose directly that $h : A \to \mathbb{R}$ is differentiable at $a \in A$, then

$$h'(a) = \lim_{x\to a} \frac{h(x) - h(a)}{x-a},$$

exists and h is continuous at a. We can construct the function

$$l(x) = \frac{h(x) - h(a)}{x - a},$$

such that $h'(a) = \lim_{x \to c} l(x)$ which implies that $l(x)$ is continuous at a. We can then multiply $l(x)$ by $(x - a)$ to get

$$\begin{aligned} l(x)(x-a) &= \frac{h(x) - h(a)}{(x-a)}(x-a) \\ &= h(x) - h(a). \end{aligned}$$

thus, if h is differentiable at $a \in A$, then $l(x)$ is continuous at a and $h(x) - h(a) = l(x)(x - a)$.
($\Longleftarrow$) : We suppose directly that $l : A \to \mathbb{R}$ is continuous at a and satisfies

$$h(x) - h(a) = l(x)(x - a).$$

it is trivial that $g(x) = x - a$ is also continuous at a. Since $g(x)$ and $l(x)$ are continuous at a, then $l(x) g(x)$ is continuous at a which implies that the $\lim_{x \to c} l(x)/g(x)$ exists and is

$$\lim_{x \to a} \frac{l(x)}{g(x)} = \lim_{x \to a} \frac{h(x) - h(a)}{(x - a)}$$

which means that h is differentiable at $a \in A$. ■

b) Use this criterion for differentiability (in both directions) to prove Theorem 5.2.5.
Proof: We suppose directly that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. We also suppose that f is differentiable at $c \in A$ and g is differentiable at $g(y(c))$. Since g is differentiable at $f(c)$, there exists a continuous function at $f(c)$, $l(y)$, such that

$$g(y) - g(f(c)) = l(y)(y - f(c)),$$

where $y \in B$. Since y is any arbitrary element of B, we can choose an element $f(t) \in f(A)$, with $t \in A$, in order to get

$$g(f(t)) - g(f(c)) = l(f(t))(f(t) - f(c)).$$

If $t \neq c$, then we can divide both sides by $t - c$ to get

$$\frac{g(f(t)) - g(f(c))}{t - c} = \frac{l(f(t))(f(t) - f(c))}{t - c},$$

taking the limit at $t \to c$ gives us

$$(g \circ f)'(c) = g'(f(c)) f'(c).$$

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Exercise 4. (Q5): Let

$$f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

a) For which values of a is f continuous at zero.
Proof: f is continuous for all values of $a > 0$. If $a > 0$, then

$$\lim_{x \to 0^+} x^a = 0,$$

however, if $a = 0$, then $\lim_{x \to 0^+} x^0 = 1$ which is not zero. And, if $a < 0$, then $\lim_{x \to 0^+} x^a$ does not exist. Thus f is continuous for all values of $a > 0$. ■

b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?
Proof: The function f is differentiable for all $a > 1$.

$$f' = \begin{cases} \lim_{x \to 0} \frac{0-0}{x-0} & \text{if } x \leq 0 \\ \lim_{x \to 0} \frac{x^a - 0}{x - 0} & \text{if } x > 0 \end{cases}.$$

Since $\lim_{x\to 0}\frac{0}{x}=1$ when $x\leq 0$, we only have left to verify when $x>0$.

$$\lim_{x\to 0}\frac{x^a}{x}=\lim_{x\to 0}x^{a-1}.$$

There are three cases to consider.

Case 1. Suppose that $a\leq 1$, then $\lim_{x\to 0}x^{a-1}$ doesn't exist since f_a is not continuous for all values of $a\leq 0$.

Case 2. Suppose $a>1$, then $x-a>0$ and

$$\lim_{x\to 0}x^{a-1}=0.$$

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c) For which values of a is f twice-differentiable?

Proof: It a similar way to the previous proof, it can be shown that if $a>2$, then f is twice differentiable. ■

Exercise 5. (Q6): Let g be defined on an interval A, and let $c\in A$.

a) Explain why $g'(c)$ in Definition 5.2.1 could have been given by

$$g'(c)=\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}.$$

a) According to definition 5.2.1,

$$g'(c)=\lim_{x\to c}\frac{g(x)-g(c)}{x-c}.$$

By using a change of variables $x=c+h$ we get

$$\begin{aligned}g'(c)&=\lim_{c+h\to c}\frac{g(c+h)-g(c)}{c+h-c}\\&=\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}.\end{aligned}$$

b) Assume A is open. If g is differentiable at $c\in A$, show

$$g'(c)=\lim_{h\to 0}\frac{g(c+h)-g(c-h)}{2h}.$$

Proof: We assume directly that A is open and that g is differentiable at $c\in A$. Then there exists a neighborhood $V_\epsilon(c)\in A$. Let h be chosen such that $c+h, c-h\in V_\epsilon(c)$. Thus we can take the limit

$$\lim_{h\to 0}\frac{g(c+h)-g(c-h)}{2h},$$

which is equivalent to

$$\begin{aligned}\lim_{h\to 0}\frac{g(c+h)-g(c-h)+g(c)-g(c)}{2h}&=\lim_{h\to 0}\frac{g(c+h)-g(c)}{2h}+\lim_{h\to 0}\frac{g(c)-g(c-h)}{2h}\\&=\frac{1}{2}\left(\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{h\to 0}\frac{g(c)-g(c-h)}{h}\right)\\&=\frac{1}{2}\left(g'(c)+g'(c)\right)\\&=g'(c).\end{aligned}$$

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