Homework 27 Section 6.3

Mark Petersen

Exercises: 2,3,4,5

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Exercise 1. **(Q2)**: Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbb{R}$.
Proof: The pointwise limit of (h_n) is $|x|$. To show that it's convergence is uniform we do the following

$$\begin{aligned}
|h_n - |x|| &= \left|\sqrt{x^2 + \frac{1}{n}} - |x|\right| \\
&= \left|\sqrt{x^2 + \frac{1}{n}} - |x|\right| \frac{\left(\sqrt{x^2 + \frac{1}{n}} + |x|\right)}{\left(\sqrt{x^2 + \frac{1}{n}} + |x|\right)} \\
&= \frac{x^2 + \frac{1}{n} - x^2}{\sqrt{x^2 + \frac{1}{n}} + |x|} \\
&= \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + |x|} \\
&\leq \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} \\
&= \frac{1}{\sqrt{n}}.
\end{aligned}$$

Using the Archimedean property, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\frac{1}{\sqrt{N}} < \epsilon$. Therefore (h_n) converges uniformly. ■

b) Not that each (h_n) is differentiable. Show $g(x) = \lim h_n^{'}(x)$ exists for all x, and explain how we can be certain that the convergence is not uniform on any neighborhood of zero.
Proof: The derivative of the sequence h_n is

$$\begin{aligned}
h_n'(x) &= 2x\left(x^2 + \frac{1}{n}\right)^{-\frac{1}{2}} \\
&= \frac{2x}{\sqrt{x^2 + \frac{1}{n}}},
\end{aligned}$$

which converges pointwise to

$$\frac{2x}{x}.$$

The converges cannot be uniform on any neighborhood of zero since the derivative of $|x|$ does not exists at 0. ■

Exercise 2. **(Q3)**: Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}.$$

a) Find the points on $\mathbb{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbb{R}$. What is the limit function?
Proof: In order to find the points on $\mathbb{R}$ where each $f_n(x)$ attains its max and min value, we take derivative, set it equal to zero, and solve for x.

$$f_n' = \frac{1-nx^2}{(1+nx^2)^2}.$$

Which shows that $f_n'(x) = 0$ when $x = \pm\frac{1}{\sqrt{n}}$. Since $f_n(-\infty) = f_n(\infty) = 0$, we know that f_n obtains a maximum/minumum at $x = \pm\frac{1}{\sqrt{n}}$. This means that f_n is bounded and that the bound is

$$f_n\left(\pm\frac{1}{\sqrt{n}}\right) = \frac{\pm\frac{1}{\sqrt{n}}}{1+n\left(\frac{1}{\sqrt{n}}\right)^2} = \pm\frac{1}{2\sqrt{n}}.$$

Note that $f_n \to 0$ as $n \to \infty$, thus

$$|f_n - 0| \leq \frac{1}{2\sqrt{n}}.$$

Since we can choose an N given an $\epsilon > 0$ such that $\frac{1}{2\sqrt{N}} < \epsilon$, the sequence (f_n) converges uniformly to $f = 0$. ■

b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n^{'}(x)$.

  a) The derivative $f_n'(x)$ was computed in part (a). The pointwise limit of $\left(f_n^{'}\right)$ is 0. As shown in part (a), $f_n'(x) = 0$ when $x = \pm\frac{1}{\sqrt{n}}$.

Exercise 3. (Q4): Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $h_n \to 0$ uniformly on $\mathbb{R}$ but that the sequence of derivatives $\left(h_n^{'}\right)$ diverges for every $x \in \mathbb{R}$.

Proof: Given and $\epsilon > 0$, let $N = \frac{1}{\epsilon^2}$, then whenever $n > N$,

$$\begin{aligned} |h_n - 0| &= \left|\frac{\sin(nx)}{\sqrt{n}}\right| \\ &\leq \frac{1}{\sqrt{n}} \\ &< \epsilon, \end{aligned}$$

therefore, $h_n(x) \to 0$ uniformly on $\mathbb{R}$. Taking the derivative we get

$$h_n^{'}(x) = \cos(nx)\sqrt{n}$$

which diverges as $\cos(nx)$ oscillates except when $x = \pi k$ for $k = \{0, 1, 2, 3, \ldots\}$. ■

Exercise 4. (Q5): Let

$$g_n(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

  a)
$$\lim g_n(x) = \frac{1}{2}x,$$
  so $g(x) = \frac{1}{2}x$ and $g'(x) = \frac{1}{2}$.

b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. use Theorem 6.3.3 to conclude $g'(x) = \lim g_n^{'}(x)$.

a) $g_n^{'}(n) = \frac{1}{2} + \frac{x}{n}$. Given an $\epsilon > 0$, let $N = \frac{M}{\epsilon}$, then whenever $n > N$, it follows that

$$\begin{aligned} |g_n' - g'| &= \left|\frac{1}{2} + \frac{x}{n} - \frac{1}{2}\right| \\ &= \left|\frac{x}{n}\right| \\ &\leq \frac{M}{n} \\ &< \epsilon \end{aligned}$$

Therefore $g_n' \to \frac{1}{2}$ uniformly. According to Theorem 6.3.3 $g'(x) = \lim g_n^{'}(x)$.

c) Repeat parts (a) and (b) for the sequence $f_n(x) = \left(nx^2 + 1\right)/(2n + x)$.

a) We first compute $f(x)$ by algebraically taking the limit as $n \to \infty$ and then find $f'(x)$.

$$f_n = \frac{nx^2}{(2n + x)} + \frac{1}{2n + x}.$$

Taking the limit as $n \to \infty$ yields

$$f(x) = \frac{1}{2}x^2.$$

Taking the derivative gives

$$f'(x) = x.$$

b) We now compute $f_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (f') converges uniformly on every interval $[-M, M]$. use Theorem 6.3.3 to conclude $f'(x) = \lim f_n^{'}(x)$. The derivative is $f_n'(x) = \frac{4n^2x + nx^2 - 1}{(2n+x)^2}$. Note that

$$\begin{aligned} |f_n' - f'| &= \left|\frac{4n^2x + nx^2 - 1}{(2n + x)^2} - x\right| \\ &= \left|\frac{4n^2x + nx^2 - 1 - 4n^2x - 4nx^2 - x^3}{(2n + x)^2}\right| \\ &= \left|\frac{-3nx^2 - x^3 - 1}{(2n + x)^2}\right| \\ &\leq \left|\frac{3nM^2 + M^3 + 1}{4n^2}\right| \end{aligned}$$

which tends to 0 as $n \to \infty$. Thus, given an $\epsilon > 0$, there exists an N such that whenever $n > N$

$$|f_n' - f'| < \epsilon.$$

Since $f_n' \to x$ uniformly and $f_n \to f$ pointwise on the interval $[-M, N]$, According to Theorem 6.3.3 $f'(x) = \lim f_n^{'}(x)$.