Homework 26 Section 6.2

Mark Petersen

Exercises: 3,7,9,12

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Exercise 1. (Q3): For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1+x^n}, \quad \text{and} \quad h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x \leq 1/n \end{cases}.$$

Answer the following questions for the sequences (g_n) and (h_n):

a) Find the point-wise limit on $[0, \infty)$.
   a) (g_n): The point wise limit for (g_n) is dependent on the domain.

$$\lim_{x\to\infty} g_n(x<1) = \lim_{x\to\infty} \frac{x}{1+x^n} = x$$
$$\lim_{x\to\infty} g_n(x=1) = \lim_{x\to\infty} \frac{1}{1+1^n} = \frac{1}{2}$$
$$\lim_{x\to\infty} g_n(x>1) = \lim_{x\to\infty} \frac{1}{1+x^n} = 0$$

   In summary

$$g(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

   b) (h_n) : The point wise limit for (h_n) is dependent on the domain.

$$\lim_{x\to\infty} h_n(x>0) = \lim_{x\to\infty} 1 = 1$$
$$\lim_{x\to\infty} h_n(x=0) = nx = 0.$$

   In summary

$$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.
   a) Since $h(x)$ and $g(x)$ are not continuous on the interval $[0, \infty)$, there cannot be sequence of functions that converge uniformly to $h(x)$ and $g(x)$. Otherwise it would contradict the continuous limit theorem.

c) Choose smaller sets over which the convergence is uniform and supply an argument to show that this is indeed the case.
   a) (g_n) converges uniformly to $g(x)$ on the interval $A = [0, 1)$.
   Proof: We will first show that (g_n) converges uniformly to $g(a)$ on A, and then for B.
   (A) : Given an $\epsilon > 0$, let $N =?$, then whenever $n > N$ we get

$$\left|\frac{x}{1+x^n} - x\right| = \left|\frac{-x^n}{1+x^n}\right|$$
$$\leq \frac{1}{2}|x^n|.$$

   Since $|x| < 1$, we can find an $n \geq N$ such that $|x^n| < \epsilon$ for all $n \geq N$, which shows convergence. ∎

a) (h_n) converges uniformly on the interval $A = [a, 0)$ where $a > 0$. Given an $a > n$, due to the Archimedean property, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < a$. Thus by choosing $N = \frac{1}{a}$, whenever $n > N$, we get

$$|1 - 1| = 0,$$

which is less than any $\epsilon > 0$.

Exercise 2. (Q7): Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f\left(x + \frac{1}{n}\right)$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous.

Proof: We suppose directly that f is uniformly continuous, then given and $\epsilon_1 > 0$, there exists a $\delta > 0$ such that for all $x, c \in \mathbb{R}$ whenever $|x - c| < \delta_1$

$$|f(x) - f(c)| < \epsilon_1.$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$, we can pick and N such that $\frac{1}{n} < \delta_1$ whenever $n > N$. This implies that

$$|f_n(x) - f(x)| < \epsilon_1$$

whenever $n > N$. Thus $f_n \to f$ uniformly.

To show that this proposition fails if f is only assumed to be continuous, let $f(x) = x^2$ and $f_n(x) = f\left(x + \frac{1}{n}\right)$, then

$$\begin{aligned} |f_n(x) - f(x)| &= \left|x^2 + 2\frac{x}{n} + \frac{1}{n^2} - x^2\right| \\ &= \left|2\frac{x}{n} + \frac{1}{n^2}\right|. \end{aligned}$$

Since x can be arbitrarily large, there does not exist a finite n such that

$$\left|2\frac{x}{n} + \frac{1}{n^2}\right| < \epsilon$$

for all x because surely there is an $x \in \mathbb{R}$ such that $\frac{x}{n} > \epsilon$ for any $n \in \mathbb{N}$. ■

Exercise 3. (Q9): Assume (f_n) and (g_n) are uniformly convergent sequence of functions.

a) Show that $(f_n + g_n)$ is uniformly convergent sequence of functions.
Proof: We suppose directly that (f_n) and (g_n) are uniformly convergent. Then given an $\epsilon > 0$, there exists an N such that whenever $n > N$

$$|f_n(x) - f| < \epsilon/2,$$

and

$$|g_n(x) - g(x)| < \epsilon/2$$

for all x in the domain. Now consider the term

$$\begin{aligned} |f_n(x) - f + g_n(x) - g(x)| &\le |f_n(x) - f| + |g_n(x) - g(x)| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

thus the sum of two uniformly convergent sequence is a uniformly convergent sequence. ■

b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

a) Let

$$f_n(x) = g_n(x) = x + \frac{1}{n},$$

then

$$f_n(x)\, g_n(x) = x^2 + 2\frac{x}{n} + \frac{1}{n^2},$$

which point wise converges to x^2, but notice that

$$\left|f_n(x)\,g_n(x) - x^2\right| = 2\frac{x}{n} + \frac{1}{n^2}.$$

Because of the term $\frac{x}{n}$, there is no N such that whenever $n > N$, $\left|2\frac{x}{n} + \frac{1}{n^2}\right| < \epsilon$ for all x. Thus $(f_n g_n)$ may not converge uniformly.

c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$does converge uniformly.
Proof: We suppose directly that (f_n) and (g_n) are uniformly convergent. Then given an $\epsilon > 0$, there exists an N such that whenever $n > N$

$$|f_n(x) - f| < \frac{\epsilon}{2M},$$

and

$$|g_n(x) - g(x)| < \frac{\epsilon}{2M}$$

for all x in the domain. Since $f_n \to f$ uniformly, $|f| \leq M$ as well. Using this, we get

$$\begin{aligned}
|f_n g_n - fg| &= |f_n g_n - fg - fg_n + fg_n| \\
&\leq |g_n|\,|f_n - f| + |f|\,|g_n - g| \\
&\leq M\,|f_n - f| + M\,|g_n - g| \\
&\leq M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} \\
&= \epsilon,
\end{aligned}$$

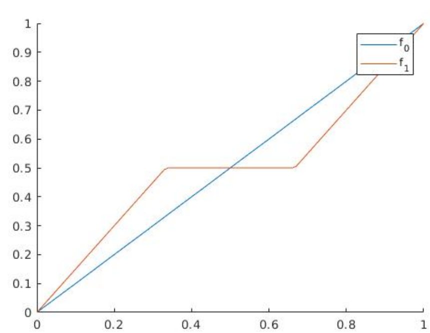
thus $(f_n g_n)$ converges uniformly to fg when they are bounded. ■

Exercise 4. (Q12): Review the construction of the Cantor set $C \subseteq [0,1]$ from section 3.1. This exercise makes use of results and notation from this discussion.

a) Define $f_0(x) = x$ for all $x \in [0,1]$. Now, let

$$f_1(x) = \begin{cases} (3/2)\,x & \text{for } 0 \leq x \leq 1/3 \\ (1/2) & \text{for } 1/3 < x < 2/3 \\ (3/2)\,x - 1/2 & \text{for } 2/3 \leq x \leq 1 \end{cases}$$

Sketch f_0 and f_1 over $[0,1]$ and observe that f_1 is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0,1] \setminus C_1$.



a)

b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, let

$$f_2(x) = \begin{cases} (1/2)\,f_1(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)\,f_1(3x-2) + \frac{1}{2} & \text{for } 2/3 \leq x \leq 1 \end{cases}$$

If we continue this process, show that the resulting sequence (f_n) converges uniformly on $[0,1]$.

a) The function f_2 can be written as

$$f_2(x) = \begin{cases} 9/4x & \text{for } 0 \le x \le 1/9 \\ 1/4 & \text{for } 1/9 < x < 2/9 \\ 9/4x - 1/4 & \text{for } 2/9 \le x \le 1/3 \\ (1/2) & \text{for } 1/3 < x < 2/3 \\ (9/4)x - 1 & \text{for } 2/3 \le x \le 7/9 \\ 3/4 & \text{for } 7/9 < x < 8/9 \\ (9/4)x - \frac{5}{4} & \text{for } 8/9 \le x \le 1 \end{cases}$$

b) By the construction of the sequence of functions, it can be seen that

$$|f_m - f_n| \le |f_{n+1} - f_n|$$

for any $m, n \in \mathbb{N}$ such that $m > n$. Thus $|f_{n+1} - f_n|$ serves as an upper bound. We then have three cases to consider:

Case 1. Assume that $(0 \le x \le 1/9)$, then

$$\begin{aligned} |f_m - f_n| \le |f_{n+1} - f_n| &= |(1/2)\, f_n(3x) - (1,2)\, f_{n-1}(3x)| \\ &= \frac{1}{2}\left|f_n(3x) - f_{n-1}(3x)\right|. \end{aligned}$$

Case 2. Assume that $(1/3 < x < 2/3)$, then

$$|f_m - f_n| = \frac{1}{2} - \frac{1}{2} = 0.$$

Case 3. Assume that $(2/3 \le x \le 1)$, then

$$\begin{aligned} |f_m - f_n| \le |f_{n+1} - f_n| &= \left|(1/2)\, f_n(3x-2) + \frac{1}{2} - (1,2)\, f_{n-1}(3x-2) - \frac{1}{2}\right| \\ &= \frac{1}{2}\left|f_n(3x-2) - f_{n-1}(3x-2)\right|. \end{aligned}$$

Thus we see that for each case

$$|f_m - f_n| \le \frac{1}{2}|f_n - f_{n-1}| \le \frac{1}{2^n}|f_1 - f_0|.$$

Since $|f_1 - f_0|$ is bounded, we can choose N such that

$$\frac{1}{2^N}|f_1 - f_0| < \epsilon$$

for any $\epsilon > 0$. This proves that (f_n) converges uniformly.

c) Let $f = \lim f_n$. Prove that f is a continuous, increasing function on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0,1] \setminus C$.

Proof: From part (2), we proved that (f_n) converges uniformly, thus f is continuous. Since $f_n(0) = 0$ and $f_n(1) = 1$ for all $n \in \mathbb{N}$, $f(0) = 0$ and $f(1) = 1$ by the sequence convergence property. Now we need to show that $f'(x) = 0$ for all $x \in [0,1] \setminus C$. Note that $A = [0,1] \setminus C$ is a union of open intervals. Let A_i denote the i^{th} open interval such that $A = \cup_{i \in \mathbb{N}} A_i$. By the construction of f, $f(A_i) = c_i$ where c_i is a constant. In other words, $f(A_i)$ is flat. Thus for any $x, y \in A_i$

$$\lim_{x \to y} \frac{f(x) - f(y)}{x - y} = \frac{c_i - c_i}{x - y} = 0.$$

Therefore, $f'(x) = 0$ for all $x \in A$. ∎