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Exercise 13. (Q5): Prove that the open interval $(0, 1)$ is uncountable by using Cantor's diagonalization method.

Proof: We suppose, by contradiction, that $(0, 1)$ is countable, then there exists a bijection $f : \mathbb{N} \to (0, 1)$. For all $n \in \mathbb{N}$, let

$$f(n) = 0.b_{n1}b_{n2}b_{n3}b_{n4}\cdots$$

with b_{ij} being the j^{th} decimal digit of the value $f(i)$. Any number in the interval $(0, 1)$ can be written as

$$0.d_1d_2d_3d_4\cdots$$

where d_j is the j^{th} decimal digit. Let $a \in (0, 1)$ be the number whose j^{th} decimal digit is defined by

$$d_j = \begin{cases} 3 & \text{if } b_{jj} \neq 3 \\ 7 & \text{else} \end{cases},$$

then $a \neq f(1)$ since $d_1 \neq b_{11}$, $a \neq f(2)$, since $d_2 \neq b_{22}$, $a \neq f(3)$ since $d_3 \neq b_{33}$, etc. Thus a is not in the image of f. Which means that f is not a bijection. This is a contradiction. Therefore, the open interval $(0, 1)$ is uncountable. ∎