Exercise 11. (Q3): Assume $\lim_{n\to\infty} a_n = 3$. Using the definition of a convergent sequence, prove that

$$\lim_{n\to\infty} \frac{a_n^2+1}{a_n-2} = 10.$$

Proof: We begin by manipulating the term

$$\begin{aligned}\left|\frac{a_n^2+1}{a_n-2} - 10\right| &= \left|\frac{a_n^2+1-10a_n+20}{a_n-2}\right| \\ &= \left|\frac{(a_n-7)(a_n-3)}{a_n-2}\right| \\ &= \frac{|a_n-7|\,|a_n-3|}{|a_n-2|}.\end{aligned}$$

Since we assume directly that $\lim_{n\to\infty} a_n = 3$, there exists an $N_1 \in \mathbb{N}$ such that

$$|a_{N_1} - 3| < \frac{1}{2},$$

which is equivalent to

$$\frac{1}{2} < a_{N_1} - 2 < \frac{3}{2}.$$

It is also equivalent to

$$\begin{aligned}-4-\frac{1}{2} < a_{N_1} - 7 < -4+\frac{1}{2} \\ -\frac{9}{2} < a_{N_1} - 7 < -\frac{7}{2}.\end{aligned}$$

Thus, for any $n > N_1$, we have that

$$\begin{aligned}\frac{|a_n-7|\,|a_n-3|}{|a_n-2|} &< \frac{\frac{7}{2}\,|a_n-3|}{\frac{1}{2}} \\ &= 7\,|a_n-3|\,.\end{aligned}$$

Once again, Since we assume directly that $\lim_{n\to\infty} a_n = 3$, given an $\epsilon > 0$, there exists an $N_2 \in \mathbb{N}$ such that whenever $m > N_2$,

$$|a_n - 3| < \frac{\epsilon}{7}.$$

Thus, given an $\epsilon > 0$, let $N = \max(N_1, N_2)$, then whenever $n \in \mathbb{N} > N$,

$$\begin{aligned}\left|\frac{a_n^2+1}{a_n-2} - 10\right| &< 7\,|a_n-3| \\ &< 7\frac{\epsilon}{7} \\ &= \epsilon.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{a_n^2+1}{a_n-2} = 10.$$

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