Homework 11 Section 2.7

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Exercises 1,2,3,4

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Exercise 1. (Q1): Proving the Alternating Series Test amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges.

a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.
Proof: We suppose directly that (s_n) is the alternating sequence $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_n$ such that

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \geq 0$$

and $(a_n) \to 0$.Then

$$\begin{aligned} |s_{n+1} - s_n| &= a_{n+1} \\ &\geq a_{n+1} - a_{n+2} \\ &\geq \sum_{k=n}^{m} (-1)^{k+1} a_{k+1} \\ &= |s_m - s_n|, \end{aligned}$$

with $m > n$. Since $(a_n) \to 0$, given an $\epsilon > 0$, there exists an $a_N < \epsilon$ with $N \in \mathbb{N}$. Thus

$$|s_N - s_{N-1}| = a_N < \epsilon,$$

which implies from our previous result that

$$|s_m - s_{N-1}| < \epsilon,$$

for all $m \geq N$. Therefore, (s_n) is a Cauchy sequence. ■

b) Supply another proof for this result using the Nested Interval Property.
Proof: We suppose directly that (s_n) is the alternating sequence $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_n$ such that

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \geq 0$$

and $(a_n) \to 0$. Let I_1 be the closed interval $[0, s_1]$ and $I_n = [s_{n-1}, s_n]$ for $n \in \mathbb{N} > 1$, then

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots .$$

According to the nested interval property $\cap_{k=1}^{\infty} I_n \neq \emptyset$. Since $(a_n) \to 0$, the length of the intervals I_n go to zero. Thus there exists a single element $x \in \cap_{k=1}^{\infty} I_n$. Also, since $(a_n) \to 0$, given an epsilon, there exists an a_N with $N \in \mathbb{N}$, such that

$$x - \epsilon < a_N < x + \epsilon,$$

thus

$$|a_N - x| < \epsilon.$$

Therefore, by the nested interval property (s_n) converges. ■

c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series test.
Proof: We suppose directly that (s_n) is the alternating sequence $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_n$ such that

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \geq 0$$

and $(a_n) \to 0$. Then the terms in (s_{2n}) have the order

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2k} \leq \cdots,$$

and the terms in (s_{2n+1}) have the order

$$s_1 \geq s_3 \geq s_5 \geq \cdots \geq s_{2k+1} \geq \cdots.$$

The sequence (s_n) is bounded by a_1 such that $|s_n| < a_1$ for all $n \in \mathbb{N}$, thus the subsequences (s_{2n}) and (s_{2n+1}) are also bounded. In addition, the subsequences (s_{2n}) and (s_{2n+1}) are monotone, thus according to the monotone convergence theorem, (s_{2n}) and (s_{2n+1}) both converge. Let A denote the limit of (s_{2n+1}) and B denote the limit of (s_{2n}). Since all the terms in (s_{2n+1}) are greater than or equal to the terms in (s_{2n}), by the order limit theorem we have that $A \geq B$. Since $(a_n) \to 0$, given an $\epsilon > 0$, there exists an a_N from (s_{2n}) such that

$$B - \frac{\epsilon}{2} < a_N < B + \frac{\epsilon}{2},$$

and

$$B - \frac{\epsilon}{2} < a_{N+1} < B + \frac{\epsilon}{2}.$$

Thus $A = B$. Therefore, (s_n) converges. ■

Exercise 2. (Q2): Decide whether each of the following series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$
   a) This limit converges by the comparison test. We note that $0 \leq \frac{1}{2^n+n} \leq \frac{1}{2^n}$, and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by the properties of the geometric series (i.e. $\left|\frac{1}{2}\right| < 1$), the series $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ also converges.

b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$
   a) This limit converges absolutely by the comparison test. We note that $0 \leq \left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$ converges. Thus $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges.

c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$
   a) This limit does not converge. The series is given by $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{2n}$. Since $\left(\frac{n+1}{2n}\right) \to \frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{2n}$ doesn't converge.

d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$
   Proof: The sequence $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$ can be written as

$$1 + \frac{1}{6} + \frac{1}{4} + \frac{1}{30} + \frac{1}{7} + \frac{1}{72}$$

whose sum is larger than $\sum_{k=1}^{\infty} \frac{1}{3k+1}$. In other words,

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots > \sum_{k=1}^{\infty} \frac{1}{3k+1}.$$

We note that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{3k+1} &\geq \sum_{k=1}^{\infty} \frac{1}{3k+3} \\ &> \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k}. \end{aligned}$$

Since the harmonic series diverges and

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots > \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k},$$

it must be that the original series diverges. ■

e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

Proof: We note the inequality

$$1-\frac{1}{2^2}+\frac{1}{3}-\frac{1}{4^2}+\frac{1}{5}-\frac{1}{6^2}+\frac{1}{7}-\frac{1}{8^2}+\cdots>1-\frac{1}{3^2}+\frac{1}{3}-\frac{1}{5^2}+\frac{1}{5}-\frac{1}{7^2}+\frac{1}{7}-\frac{1}{9^2}+\cdots.$$

The series on the right side can be written as

$$1+\sum_{k=1}^{\infty}\frac{2k}{(2k+1)^2}$$

by noting that

$$\frac{1}{n}-\frac{1}{n^2}=\frac{n-1}{n^2}.$$

Then we note that

$$\begin{aligned}1+\sum_{k=1}^{\infty}\frac{2k}{(2k+1)^2} &\geq 1+\sum_{k=1}^{\infty}\frac{2k+2}{(2k+2)^2}\\ &=1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k+1}\\ &>1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}.\end{aligned}$$

Since the harmonic series diverges, $1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. And since $1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ is less than the original series, the original series must diverge. ■

Exercise 3. (Q3): This question has two parts.

a) Provide the details for the proof of the Comparison Test using the Cauchy Criterion for Series.
Proof: We assume directly that (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. For part (i) of the theorem we also assume that $\sum_{k=1}^{\infty} b_k$ converges. Then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n > m \in \mathbb{N}$, we have that

$$\left|\sum_{k=1}^{n} b_k-\sum_{k=1}^{m} b_m\right|<\epsilon.$$

The left hand side can be simplified as

$$\left|b_{m+1}+b_{m+2}+\cdots+b_n\right|.$$

Since $0 \leq a_k \leq b_k$, we have that

$$\epsilon>\left|b_{m+1}+b_{m+2}+\cdots+b_n\right| \geq \left|a_{m+1}+a_{m+2}+\cdots+a_n\right|,$$

thus

$$\left|\sum_{k=1}^{n} a_k-\sum_{k=1}^{m} a_m\right|<\epsilon.$$

For part (ii) this is simply the contrapositive of part (i), which we have already proven. ■

b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.
Proof: We assume directly that (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Let $s_n = \sum_{k=1}^{n} b_k$ and $r_n = \sum_{k=1}^{n} a_k$. For part (i) of the theorem we also assume that (s_n) converges. Since all of the terms in the sequence (b_k) and (a_k) are positive, the sequences (s_n) and (r_n) are monotonic. Also, since (s_n) converges, the sequence is bounded, which means that (r_n) is also bounded. Since (r_n) is bounded and monotonic, it converges. ■

Exercise 4. (Q4): Give an example of each or explain why the request is impossible referencing the proper theorems.

a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.
  a) Let $x_n = y_n = \frac{1}{n}$, then $\sum x_n$ and $\sum y_n$ are the harmonic series which diverges. However, $\sum x_n y_n = \sum \frac{1}{n^2}$ which converges.

b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

a) Let x_n be the sequence $(-1)^{n+1} \frac{1}{n}$, whose series converges according to the alternating series test, and let $y_n = (-1)^{n+1}$ so that (y_n) is a bounded sequence. The product

$$x_n y_n = \frac{(-1)^{n+1} (-1)^{n+1}}{n} = \frac{1}{n},$$

thus

$$\sum x_n y_n = \sum \frac{1}{n},$$

which is the harmonic series and diverges.

c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

a) This is impossible. Note that

$$\begin{aligned} \sum y_n &= \sum (x_n + y_n - x_n) \\ &= \sum (x_n + y_n) - \sum x_n. \end{aligned}$$

Since $\sum (x_n + y_n)$ and $\sum x_n$ are convergent, according to the Algebraic Limit theorem, $\sum y_n$ converges. This is a contradiction, thus there is no example.

d) A sequence (x_n) satisfying $0 \le x_n \le \frac{1}{n}$ where $\sum (-1)^n x_n$ diverges.

a) Let x_n have the terms

$$\left(1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, \dots\right),$$

then

$$\begin{aligned} \sum (-1)^n x_n &= -\left(1 + \sum_{n=1}^{\infty} \frac{1}{2n+1}\right) \\ &< -\left(1 + \sum_{n=1}^{\infty} \frac{1}{2n}\right). \end{aligned}$$

Since $\sum \frac{1}{2n}$ diverges, $\sum (-1)^n x_n$ diverges.