Homework 17 Section 4.2

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1,2,3,4,5

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Exercise 1. (Q1): Complete the following

a) Supply the details for how Corollary 4.2.4 part (ii) follows from the Sequential Criterion for Functional Limits in theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

Proof: Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two functions, $A \subseteq \mathbb{R}$, $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. According to the sequential criterion for functional limits, for any sequence $(a_n) \subseteq A$ satisfying $a_n \neq c$, and $(a_n) \to c$, it follows that $f(a_n) \to L$ and $g(a_n) \to M$. Thus, we can form a new sequence $f_n = f(a_n)$ and $g_n = g(a_n)$ such that $(f_n) \to L$ and $(g_n) \to M$. According to the Algebraic Limit Theorem, since (f_n) and (g_n) converge, $(f_n + g_n) \to L + M$. In other words, $\lim_{x \to c} (f(x) + g(x)) = L + M$. ∎

b) Now, write another proof for Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3

Proof: We suppose directly that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then given an $\epsilon > 0$, there exists an $\delta \in \mathbb{R} > 0$ such that whenever $0 < |x - c| < \delta$ it follows that

$$|f(x) - L| < \frac{\epsilon}{2},$$

and

$$|g(x) - M| < \frac{\epsilon}{2}.$$

Now we consider the term $|f(x) + g(x) - L - M|$ and manipulate it

$$\begin{aligned} |f(x) + g(x) - L - M| &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $\lim_{x \to c} (f(x) + g(x)) = L + M$. ∎

c) Repeat (a) and (b) for Corollary 4.2.4 part (iii)

a) We will first prove Corollary 4.2.4 part (iii) using the algebraic limit theorem, but we will be more brief this time since it is similar to part (a).

Proof: Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two functions, $A \subseteq \mathbb{R}$, $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. According to the sequential criterion for functional limits, for any sequence $(a_n) \subseteq A$ satisfying $a_n \neq c$, and $(a_n) \to c$, it follows that $f(a_n) \to L$ and $g(a_n) \to M$. Thus, we can form a new sequence $f_n = f(a_n)$ and $g_n = g(a_n)$ such that $(f_n) \to L$ and $(g_n) \to M$. According to the Algebraic Limit Theorem, since (f_n) and (g_n) converge, $(f_n g_n) \to LM$. In other words, $\lim_{x \to c} (f(x) g(x)) = LM$. ∎

a) We will now prove it using the definition of the limit of a function.

Proof: We suppose directly that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Since the functions converge, they are also bounded. Let $|f(x)| \leq F$ and $|g(x)| \leq G$ for all $x \in A$. Then given an $\epsilon > 0$, there exists a $\delta > 0$, such that whenever $0 < |x - c| < \delta$,

$$|f(x) - L| < \frac{\epsilon}{2(G+1)},$$

and

$$|g(x) - M| < \frac{\epsilon}{2(|L|+1)}.$$

It then follows that,

$$\begin{aligned}|f(x)g(x)-LM+Lg(x)-Lg(x)| &= |g(x)(f(x)-L)+L(g(x)-M)| \\ &\leq |g(x)(f(x)-L)|+|L||g(x)-M| \\ &\leq G|f(x)-L|+|L||g(x)-M| \\ &< \frac{G\epsilon}{2(G+1)}+\frac{|L|\epsilon}{2(|L|+1)} \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Therefore, if $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$, then $\lim_{x\to c}(f(x)g(x))=LM$. ■

Exercise 2. (Q2): For each stated limit, find the largest possible $\delta-$neighborhood that is a proper response to the given ϵ challenge.

a) $\lim_{x\to 3}(5x-6)=9$, where $\epsilon=1$.
   a) We begin with the definition of the limit and perform scratch work.

$$\begin{aligned}|5x-6-9| &< 1 \\ |5x-15| &< 1 \\ 5|x-3| &< 1 \\ |x-3| &< \frac{1}{5},\end{aligned}$$

   thus $\delta=\frac{1}{5}$.

b) $\lim_{x\to 4}\sqrt{x}=2$, where $\epsilon=1$.
   a) We begin with the definition of the limit and perform scratch work.

$$\begin{aligned}\left|\sqrt{x}-2\right| &< 1 \\ -1<\sqrt{x}-2 &< 1 \\ 1<\sqrt{x} &< 3 \\ 1<x &< 9 \\ -3<x-4 &< 5 \\ |x-4| &< 3,\end{aligned}$$

   thus $\delta=3$.

c) $\lim_{x\to\pi}[[x]]=3$, where $\epsilon=1$.
   a) When x is less than three, then the error is greater than 1, and when x is 4 or above, the error is greater than 1. Thus we get that $0<|x-3|<\delta=\pi-3$.

d) $\lim_{x\to\pi}[[x]]=3$, where $\epsilon=1$.
   a) For the same reason as the previous part, $\delta=\pi-3$. For any δ greater than this, the error would be 1.

Exercise 3. (Q3): Review the definition of Thomae's function $t(x)$ from section 4.1.

a) Construct three different sequence (x_n), (y_n), and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.
   a) $x_n=1+\frac{1}{1+n}$, $y_n=1-\frac{1}{1+n}$, $z_n=1+\frac{1}{2+n}$ for all $n\in\mathbb{N}$.

b) Now, compute $\lim t(x_n)$, $\lim t(y_n)$, and $\lim t(z_n)$.
   a) Using the definition of the function $t:\mathbb{R}\to\mathbb{R}$, we get that

$$\begin{aligned}\lim t(x_n) &= \lim\left(\frac{1}{1+n}\right), \\ \lim t(y_n) &= \lim\left(\frac{1}{1+n}\right), \\ \lim t(z_n) &= \lim\left(\frac{1}{2+n}\right),\end{aligned}$$

   which all have limits of 0.

c) Make an educated conjecture for $\lim_{x\to 1} t(x)$, and use Definition 4.2.1 B to verify the claim. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbb{R} : t(x) \geq \epsilon\}$. Argue that all the points in this set are isolated.

a) **Conjecture**: The $\lim_{x\to 1} t(x) = 0$.

Proof: Our objective is to find a $\delta > 0$ given an $\epsilon > 0$, such that when $0 < |x-1| < \delta$,

$$|t(x)| < \epsilon.$$

When x is irrational $t(x) = 0 < \epsilon$, when $x = 0$, $t(x) = 1$, and when $x = \frac{m}{n} \in \mathbb{Q}\setminus\{0\}$ such that $n > 0$, $t(x) = \frac{1}{n}$. Thus we need to choose δ so that $x \neq 0$ and so that

$$\frac{1}{n} < \epsilon.$$

Manipulating the equation yields

$$\frac{1}{\epsilon} > n.$$

We then choose $\delta = \frac{1}{n}$. This gives

$$\begin{aligned} |x-1| &< \frac{1}{n} \\ -\frac{1}{n} < x-1 &< \frac{1}{n} \\ \frac{n-1}{n} < x &< \frac{n+1}{n} \end{aligned}$$

and $x \neq 1$. With these constraints, any value of x will have a denominator $m > n$, thus $|t(x)| < \frac{1}{n}$. This proves that

$$\lim_{x\to 1} t(x) = 0.$$

Given $\epsilon > 0$ and the set $A = \{x \in \mathbb{R} : t(x) \geq \epsilon\}$ then all points in A are isolate. For any $a \in A$, we get that $t(a) = \frac{1}{n}$ since a is a rational number. However, using a similar argument above, it can be shown that

$$\lim_{x\to a} t(x) = 0.$$

This shows that there is a $\delta-$neighborhood of a such that $V_\delta(a)$ does not contain any point in A, so every point in A must be isolate. ■

Exercise 4. (Q4): Consider the reasonable but erroneous claim that

$$\lim_{x\to 10} \frac{1}{[[x]]} = \frac{1}{10}.$$

a) Find the largest δ that represents a proper response to the challenge of $\epsilon = \frac{1}{2}$

a) We begin with scratch work.

$$\begin{aligned} \left|\frac{1}{[[x]]} - \frac{1}{10}\right| &< \frac{1}{2} \\ -\frac{4}{10} < \frac{1}{[[x]]} &< \frac{6}{10}, \end{aligned}$$

which means that

$$[[x]] < -\frac{10}{4} = -\frac{5}{2},$$

or

$$[[x]] > \frac{10}{6} = \frac{5}{3}.$$

Which implies that $x < -2$ or $x > 2$. Since we are taking the limit as $x \to 10$, we will focus on the inequality $x > 2$. Thus we have that $0 < |x-10| < 8$.

b) Find the largest δ that represents a proper response to $\epsilon = \frac{1}{50}$.

a) We begin with scratch work.

$$\begin{aligned}
\left|\frac{1}{[[x]]}-\frac{1}{10}\right| &< \frac{1}{50}\\
-\frac{1}{50}+\frac{1}{10} < \frac{1}{[[x]]} &< \frac{1}{50}+\frac{1}{10}\\
\frac{4}{50} < \frac{1}{[[x]]} &< \frac{6}{50}\\
\frac{2}{25} < \frac{1}{[[x]]} &< \frac{3}{25}\\
\frac{25}{2} > [[x]] &> \frac{25}{3},
\end{aligned}$$

which is equivalent to

$$13 > [[x]] > 8,$$

in other words

$$13 > x \geq 9$$

thus we get that $0 < |x-10| < 1$, thus $\delta = 1$.

c) Find the largest ϵ for which there is no suitable δ response possible.

a) Since the function $[[x]]$ rounds the value of x to the largest integer equal to or lower than x, using the smallest possible value of $\delta > 0$, we get that $0 < |x-10| < \delta$. Which is equivalent to $10-\delta < x < 10+\delta$ and $x \neq 10$. Thus, for the best possible value of δ, we get

$$\begin{aligned}
[[10-\delta]] \leq [[x]] &\leq [[10+\delta]]\\
9 \leq [[x]] &\leq 10.
\end{aligned}$$

Since $[[x]]$ can take on the value of 9, we must consider it when finding the epsilon neighborhood. Using the definition of the limit of a function, we calculate

$$\begin{aligned}
\left|\frac{1}{9}-\frac{1}{10}\right| &= \left|\frac{10-9}{90}\right|\\
&= \left|\frac{1}{90}\right|,
\end{aligned}$$

thus the smallest $\epsilon > 0$ is the next larger number to $\frac{1}{90}$.

Exercise 5. (Q5): Use definition 4.2.1 to supply a proper proof for the following limit statements.

a) $\lim_{x\to 2}(3x+4) = 10$.
Proof: Given an $\epsilon > 0$, we choose $\delta > \frac{\epsilon}{3}$, then when $0 < |x-1| < \delta$, we get

$$\begin{aligned}
|3x+4-10| &= |3x-6|\\
&= 3|x-2|\\
&< 3\frac{\epsilon}{3}\\
&= \epsilon.
\end{aligned}$$

Thus, $\lim_{x\to 2}(3x+4) = 10$. ■

b) $\lim_{x\to 0} x^3 = 0$
Proof: Given an $\epsilon > 0$, we choose $\delta = \min(\epsilon, 1)$, then

$$|x-0| < 1.$$

Using this, when $0 < |x| < \delta$, we get that

$$\begin{aligned}
\left|x^3-0\right| = \left|x^3\right| &\leq |x|, \text{ since } |x| < 1\\
&< \epsilon.
\end{aligned}$$

Thus $\lim_{x\to 0} x^3 = 0$. ■

c) $\lim_{x\to 2}\left(x^2+x-1\right) = 5$.

Proof: Given an $\epsilon > 0$, we choose $\delta = \min\left(\frac{\epsilon}{6}, 1\right)$, then when $0 < |x-2| < \delta$ we get

$$\begin{aligned}\left|x^2 + x - 1 - 5\right| &= \left|x^2 + x - 6\right| \\ &= |x-2|\,|x+3| \\ &= |x-2|\,|x-2+5| \\ &\le |x-2|\,(|x-2|+5) \quad \text{since } |x-2| < \delta \\ &< |x-2|\,(1+5) \\ &= 6\,|x-2| \\ &< \epsilon.\end{aligned}$$

Thus, $\lim_{x\to 2}\left(x^2 + x - 1\right) = 5.$ ■

d) $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$.
Proof: Given and $\epsilon > 0$, we choose $\delta = \min(\epsilon, 1)$, then when $0 < |x-3| < \delta$ we get

$$\begin{aligned}\left|\frac{1}{x} - \frac{1}{3}\right| &= \left|\frac{3-x}{3x}\right| \\ &< \frac{|x-3|}{3 \cdot 2} \\ &< |x-3| \\ &< \epsilon.\end{aligned}$$

Thus $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$. ■