# Homework 13 Section 3.3

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Exercises 2,3,5,6,7,8

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**Exercise 1. (Q2)**: Let

$$A = \left\{ (-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots \right\}$$

and

$$B = \{x \in \mathbb{Q} : 0 < x < 1\}.$$

Answer the following for each set.

a) What are the limit points.
  a) The limit points for the set $A$ are $-1$ and $1$ since given an $\epsilon > 0$, there exists an $N = \frac{2}{\epsilon}$, such that whenever $n \in \mathbb{N} > N$, $(-1)^n + \frac{2}{n} \in V_\epsilon(1)$ when $n$ is even, else $(-1)^n + \frac{2}{n} \in V_\epsilon(-1)$.
  b) The limit points for the set $B$ is the closed interval $S = [0, 1]$. This follows from the density of the real numbers.

b) Is the set open? Closed?
  a) The set $A$ is not open since there are gaps in the set, and it is not closed since it doesn't contain all of it's limit points.
  b) The set $B$ is not open, since it doesn't contain any of the irrational numbers, thus there are holes in the set. The set $B$ is not closed since it doesn't contain all of it's limit points.

c) Does the set contain any isolated points?
  a) All of the elements of $A$ are isolated points since none of them are limit points.
  b) There are no isolated points in $B$ since all of the elements are limit points.

d) Find the closure of the set.
  a) The closure of set $A$ is $A \cup \{1, -1\}$.
  b) The closure of the set $B$ is the closed interval $[0, 1]$.

**Exercise 2. (Q3)**: Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no $\epsilon-$neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

a) $\mathbb{Q}$
  a) The rational number is neither closed or open. Given any rational number $q$ and $\epsilon-$neighborhood $V_\epsilon(q)$, there is an irrational number $i \in V_\epsilon(q)$ due to the density of $\mathbb{R}$. Thus $V_\epsilon(q) \not\subseteq \mathbb{Q}$, and hence $\mathbb{Q}$ is not open. Given any irrational number $i$ and $\epsilon-$neighborhood $V_\epsilon(i)$, there is a rational number $q \in V_\epsilon(i)$. Therefore, all the irrational numbers are limit points of $\mathbb{Q}$. Thus $\mathbb{Q}$ is not closed.

b) $\mathbb{N}$
  a) The natural number is not open but it is closed. Given any rational number $n$ and $\epsilon-$neighborhood $V_\epsilon(n)$, there is an irrational number $i \in V_\epsilon(n)$ due to the density of $\mathbb{R}$. Thus $V_\epsilon(n) \not\subseteq \mathbb{N}$, and hence $\mathbb{N}$ is not open. Since there are no limit points of $\mathbb{N}$, $\mathbb{N}$ is closed vacuously.

c) $\{x \in \mathbb{R} : x \neq 0\}$
  a) The set is open, but not closed since the limit point $0$ is not contained in the set.

d) $A = \left\{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$
  a) The set $A$ is neither open or closed. To show that it isn't open, take any element in $A$ with any $\epsilon-$neighborhood. Since there will be at least one irrational number in the neighborhood, the neighborhood is not a

subset of $A$, and thus $A$ is not open. The set $A$ is not closed, since the limit $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is not an element of $A$.

e) $A = \left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \,:\, n \in \mathbb{N}\right\}$

a) The set $A$ is not open but it is closed. To show that it isn't open, take any element in $A$ with any $\epsilon-$neighborhood. Since there will be at least one irrational number in the neighborhood, the neighborhood is not a subset of $A$, and thus $A$ is not open. Since there is not limit point of $A$, the set is closed.

**Exercise 3. (Q5)**: Prove Theorem 3.2.8.

*Proof:* We want to prove that a set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in $F$ has a limit that is also an element of $F$. This is a biconditional statement so we must prove both ways.

$(\implies)$ : We assume directly that $F$ is closed, then it contains all of it's limit points. According to theorem 3.2.5, every possible sequence $(a_n)$ contained in $F$ must have a limit in $F$. Since every convergent sequence is a Cauchy sequence, every Cauchy sequence contained in $F$ has it's limit in $F$.

$(\impliedby)$ : We assume directly that every Cauchy sequence contained in $F$ has a limit that is also an element of $F$. According to theorem 3.2.5, a point is a limit point if and only if it is the limit of some sequence contained in $F$. Thus every limit point is contained in $F$; therefore, $F$ is closed.

Since we have proven both implications, the biconditional statement is true. ∎

**Exercise 4. (Q6)**: Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

a) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.
*Disproof:* Let $A$ be the open set that contains every rational number defined by $\mathbb{R} - \left\{\sqrt{2}\right\}$. Given any element $a \in A$, there exists a neighborhood $V_\epsilon(a)$ contained in $A$ due to the density of the set. For example, let $\epsilon = \frac{\left|a-\sqrt{2}\right|}{2}$, then $V_\epsilon(a) \subseteq A$ for all $a \in A$. Thus, this counterexample contradicts the original statement. ∎

b) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set".
*Disproof:* The empty set is a closed set. Any intersection of sets including the empty set results in the empty set, which is empty. Thus, this property doesn't remain true for any nested closed sets. ∎

c) Every nonempty open set contains a rational number.
*Proof:* Let $A$ be an arbitrary nonempty open set, there there exists an $a \in A$ and an open neighborhood $V_\epsilon(a)$ such that $V_\epsilon(a) \subseteq A$. Thus the open interval $(a-\epsilon, a+\epsilon)$ is a subset of $A$. Since there is a rational number between any two real numbers, there must be a rational number $q$ that is an element of $(a-\epsilon, a+\epsilon)$, and thus an element of $A$. Therefore, the statement is true. ∎

d) Every bounded infinite closed set contains a rational number.
*Disproof:* We will disprove the statement with a counter example. Consider the set $A = \left\{\sqrt{2} + \frac{\sqrt{2}}{n} \,:\, n \in \mathbb{N}\right\} \cup \left\{\sqrt{2}\right\}$. The set $A$ doesn't contain any rational number, it is bounded between $\sqrt{2}$ and $2\sqrt{2}$, it is infinite since the natural numbers is an infinite set, and it is closed since the limit point $\sqrt{2}$ is contained in $A$. ∎

e) The Cantor set is closed.
*Proof:* The Cantor set is constructed from the infinite intersection of closed sets. According to Theorem 3.2.14, the Cantor set must be closed. ∎

**Exercise 5. (Q7)**: Given $A \subseteq \mathbb{R}$, let $L$ be the set of all limit points of $A$.

a) Show that the set $L$ is closed.
*Proof:* We assume directly that $L$ is the set of all limit points of $A$. Let $x$ be a limit point of $L$, then given an $\epsilon_1-$neighborhood $V_{\epsilon_1}(x)$, there exists an $\ell \in V_{\epsilon_1}(x) \cap L$ such that $\ell \neq x$. Since $\ell$ is a limit point of $A$, then given an $\epsilon_2-$neighborhood, there exists an $a \in V_{\epsilon_2}(\ell) \cap A$ such that $a \neq \ell$. Let $\epsilon = \epsilon_1 + \epsilon_2$, then $a \in V_\epsilon(x) \cap A$ such that $a \neq x$. Thus $x$ is also a limit point of $A$ and must be in $L$. Therefore, all of the limit points of $L$ are contained in $L$, which means that $L$ is closed. ∎

b) Argue that if $x$ is a limit point of $A \cup L$, then $x$ is a limit point of $A$. Use this observation to furnish a proof for Theorem 3.2.12.
*Proof:* We assume directly that $L$ is the set containing all of the limit point of $A$, and is therefore closed according to part a). Since $L$ is closed, it contains all of it's limit points which must also be a limit point of $A$. Thus if $x$ is a limit point of $\overline{A} = A \cup L$, it must be a limit point of $A$. This means that all of the limit points of $\overline{A}$ are contained in $\overline{A}$. Now to show that $\overline{A}$ is the smallest closed set containing $A$, suppose, by

contradiction, that $\overline{B}$ is a smaller closed set that contains $A$, i.e. $\overline{B} \subsetneq \overline{A}$, then there is an element $x \in L$ that is not in $\overline{B}$, thus $\overline{B}$ doesn't contain all of the limit points of $A$ and cannot be closed. This is a contradiction, thus there exists no smaller closed set that contains $A$. ■

**Exercise 6. (Q8)**: Assume $A$ is an open set and $B$ is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

a) $\overline{A \cup B}$
   a) Definitely closed. By definition, the closure of a set is closed.

b) $A \backslash B = \{x \in A : x \notin B\}$
   a) Definitely open. The set $A \backslash B$ is equivalent to the set $A \cap B^c$. Since $B$ is closed, $B^c$ is open. The intersection of two open sets is open. Thus $A \backslash B$ is always open.

c) $(A^c \cup B)^c$
   a) Definitely open. Since $A$ is open, $A^c$ is closed, thus $A^c \cup B$ is closed. The complement of a closed set is open. Therefore, $(A^c \cup B)^c$ is open.

d) $(A \cap B) \cup (A^c \cap B)$
   a) Definitely closed. The set $(A \cap B) \cup (A^c \cap B)$ is equivalent to $B \cap (A \cup A^c)$, which is just $B$. Thus it is closed.

e) $\overline{A}^c \cap \overline{A^c}$
   a) Definitely open. This is because $\overline{A}^c \subseteq \overline{A^c}$, then $\overline{A}^c \cap \overline{A^c} = \overline{A}^c$. Since $\overline{A}^c$ is open, the set is open. To show that $\overline{A}^c \subseteq \overline{A^c}$, let $x \in \overline{A}^c$, then $x \notin \overline{A}$. In other words, $x \notin A \cup L$, with $L$ being the set containing all the limit points of $A$. This means that $x \in A^c \cap L^c$. Since $x \in A^c$, it must be in $\overline{A^c}$. Therefore, $\overline{A}^c \subseteq \overline{A^c}$.