Exercise 15. (Q7): Let (x_n) be the sequence defined recursively by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

for $n \geq 1$. Prove that $\lim x_n = L$ exists and find the value of L.

Proof: In order to show that the limit exists, we will show that (x_n) is a bounded, monotonic sequence. First we will prove that $x_n \geq 2 - \sqrt{3}$ for all $n \in \mathbb{N}$. We do this by induction.

Base Case: $x_1 = 3$, then

$$\begin{aligned} x_2 &= \frac{1}{4-3} \\ &= 1 \\ &\geq 2 - \sqrt{3}. \end{aligned}$$

Induction Step: Let $k \in \mathbb{N}$, we suppose directly that $x_k \geq 2 - \sqrt{3}$, then

$$\begin{aligned} x_{k+1} &= \frac{1}{4 - x_k} \\ &\geq \frac{1}{4 - 2 + \sqrt{3}} \\ &= \frac{1}{2 + \sqrt{3}} \\ &= \frac{2 - \sqrt{3}}{\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right)} \\ &= 2 - \sqrt{3}, \end{aligned}$$

thus $x_{k+1} \geq 2 - \sqrt{3}$; therefore, $x_n \geq 2 - \sqrt{3}$ for all $n \in \mathbb{N}$.

We next want to show that $x_n \leq 3$ for all $n \in \mathbb{N}$. We work this by induction.

Base Case: $x_1 = 3$, then

$$\begin{aligned} x_2 &= \frac{1}{4-3} \\ &= 1 \\ &\leq 3. \end{aligned}$$

Induction Step: Let $k \in \mathbb{N}$, we suppose directly that $x_k \leq 3$, then

$$\begin{aligned} x_{k+1} &= \frac{1}{4 - x_k} \\ &\leq \frac{1}{4-3} \\ &= 1, \end{aligned}$$

thus $x_{k+1} \leq 3$; therefore, $x_k \leq 3$ for all $k \in \mathbb{N}$. We now know that (x_n) is bounded such that $2 - \sqrt{3} \leq x_n \leq 3$ for all $n \in \mathbb{N}$.

Next we show that $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$.

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{4 - x_n} \\ &= \frac{x_n\left(4 - x_n\right) - 1}{4 - x_n} \\ &= \frac{-\left(x_n - 2 + \sqrt{3}\right)\left(x_n - 2 - \sqrt{3}\right)}{4 - x_n} \\ &\geq 0, \end{aligned}$$