Homework 18 Section 4.3

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Exercises: 1,3,4,6,11

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Exercise 1. (Q1): Let $g(x) = \sqrt[3]{x}$.

a) Prove that g is continuous at $c = 0$.
Proof: Given an $\epsilon > 0$, let $\delta = \epsilon^3$ then it follows that

$$|x| < \delta = \epsilon^3$$

which implies that

$$\left|x^{(1/3)}\right| < \epsilon$$

which is equivalent to

$$|g(x) - g(0)| < \epsilon,$$

thus $g(x)$ is continuous at $c = 0$. ■

b) Prove that g is continuous at a point $c \neq 0$.
Proof: We suppose directly that $c \neq 0$. Given an $\epsilon > 0$, let $\delta = \min\left(|c|/2, \epsilon c^{2/3}\right)$Using the definition of a function being continuous at a point we start with

$$|g(x) - g(c)| = \left|x^{1/3} - c^{1/3}\right|.$$

Multiplying the term by a 1 we get

$$\left|x^{1/3} - c^{1/3}\right| \frac{\left|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}\right|}{\left|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}\right|} = \frac{|x - c|}{\left|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}\right|}.$$

Due to the constraint on x from $\delta \leq |c|/2$, the value of x and c must have the same sign. Thus we know that $x^{2/3} > 0$, $c^{2/3} > 0$ and $c^{1/3}x^{1/3} > 0$. Using this, we get

$$\begin{aligned}\frac{|x - c|}{\left|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}\right|} &\leq \frac{|x - c|}{c^{2/3}} \\ &< \frac{\epsilon c^{2/3}}{c^{2/3}} \\ &= \epsilon.\end{aligned}$$

Therefore, $g(x)$ is continuous on $\mathbb{R}$. ■

Exercise 2. (Q3): Complete the following

a) Supply a proof for Theorem 4.3.9 using the $\epsilon - \delta$ characterization of continuity.
Proof: We suppose directly that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ be continuous functions on their domains and $f(A) \subseteq B$. Let $c \in A$ be an arbitrary point. We want to show that $g \circ f$ is continuous at c. Given an $\epsilon > 0$, there exists an $\alpha > 0$ such that when

$$|y - f(c)| < \alpha,$$

$$|g(y) - g(f(c))| < \epsilon.$$

Also, since f is continuous on A, we know that there exists a δ such that when

$$|x - c| < \delta,$$

$$|f(x) - f(c)| < \alpha.$$

Since $f(x), f(c) \in B$, it follows that when

$$|f(x) - f(c)| < \alpha,$$

$$|g(f(x)) - g(f(c))| < \epsilon.$$

This shows that given an $\epsilon > 0$, there exists a $\delta \in \mathbb{R}$, such that when

$$|x - c| < \delta,$$

$$|g(f(x)) - g(f(c))| < \epsilon.$$

Thus $g \circ f$ is continuous on A. ■

b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).
Proof: We suppose directly that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ be continuous functions on their domains and $f(A) \subseteq B$. Let $c \in A$ be an arbitrary point. We want to show that $g \circ f$ is continuous at c. Let $k = f(c)$ and $l = g(k)$. Since g is continuous, there exists a sequence (x_n) such that as $(x_n) \to k$, $g(x_n) \to l$. Since f is continuous, there exists a sequence (y_n) such that as $(y_n) \to c$, $f(y_n) \to f(c) = k$. Then as $(f(y_n)) \to k$, $g(f(y_n)) \to l$. This shows that as $(y_n) \to c$, $g \circ f(y_n) \to l$. ■

Exercise 3. (Q4): Assume f and g are defined on all of $\mathbb{R}$ and that $\lim_{x\to p} f(x) = q$ and $\lim_{x\to q} g(x) = r$.

a) Give an example to show that it may not be true that

$$\lim_{x\to p} g(f(x)) = r.$$

a) Let $c \in \mathbb{R} > 0$, $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} q & \text{if } x \neq p \\ q + c & \text{if } x = p \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(x) = \begin{cases} r & \text{if } x \neq q \\ r + c & \text{if } x = q \end{cases},$$

then $\lim_{x\to p} f(x) = q$ and $\lim_{x\to q} g(x) = r$; however, $\lim_{x\to p} g \circ f$ is $r + c$ and not r.

b) Show that the result in (a) does follow if we assume f and g are continuous.
Proof: We suppose directly that f and g are continuous, then $f(p) = q$ and $g(q) = r$. Given an $\epsilon > 0$, there exists a $\alpha \geq 0$ such that when

$$|y - q| < \alpha$$

$$|g(y) - g(q)| < \delta,$$

$$|g(x) - q| < \alpha.$$

Also, there exists a $\delta \geq 0$, such that when

$$|x - p| < \delta$$

$$|f(x) - f(p)| < \alpha$$

Since $f(p) = q$ in this case, it follows that when

$$|x - p| < \delta,$$

$$|g(f(x)) - r| < \epsilon.$$

Which means when $\delta > 0$ and

$$0 < |x - p| < \delta,$$

$$|g(f(x)) - r| < \epsilon.$$

Which is equivalent to saying $\lim_{x\to p} g(f(x)) = r$. ■

c) Does the result in part (a) hold if we only assume f is continuous? How about if we only assume that g is continuous.

   a) The function f does not have to be continuous. For example, let $f(x)$ be defined as in part (a) and $g(x) = r$. f is not continuous at p but $g(x)$ is continuous at q, and $\lim_{x\to p} g \circ f = r$. The function g does need to be continuous at q since q can be in the image of f.

Exercise 4. (Q6): Provide an example of each or explain why the request is impossible.

a) Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x)+g(x)$ are continuous at 0.

   a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{else} \end{cases},$$

   and

$$g(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{else} \end{cases},$$

   then $g(x)f(x) = 0$ and $g(x)+f(x) = 1$ which are both continuous at 0.

b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)+g(x)$ is continuous at 0.

   a) This request is impossible.

   Proof: We suppose by contradiction that $f(x)$ is continuous at 0, $g(x)$ is not continuous at 0, and $f(x)+g(x)$ is continuous at 0, then by the algebraic limit theorem,

$$(f+g)(x) - f(x)$$

   must be continuous; however,

$$g(x) = (f+g)(x) - f(x),$$

   which is not a continuous function, thus this is a contradiction. ■

c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.

   a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$ and let $g(x)$ be any bounded function not continuous at 0. Then $f(x)g(x) = f(x)$ which is continuous.

d) A function $f(x)$ not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0.

   a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 2 & \text{if } x \neq 0 \\ \frac{1}{2} & \text{else} \end{cases},$$

   then $f(x)$ is not continuous at 0, but $f(x) + \frac{1}{f(x)} = 2.5$ which is continuous everywhere.

e) A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0.

   a) The request is impossible.

   Proof: Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = \sqrt[3]{x}$. Since $g(x)$ is continuous at 0 and $f(x)^3$ is continuous at zero, then the composite function $g \circ [f(x)]^3 = f(x)$ must be continuous, thus the request is impossible. ■

Exercise 5. (Q11): (Contraction mapping Theorem). Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c\,|x-y|$$

for all $x, y \in \mathbb{R}$

a) Show that f is continuous on $\mathbb{R}$.

   Proof: We supposed directly that $|f(x) - f(y)| \leq c\,|x-y|$. Given an ϵ, let $\delta = \frac{\epsilon}{c}$, then when

$$|x-y| < \delta,$$

it follows that

$$\begin{aligned} c\left|x-y\right| &< c\delta \\ &= \frac{c\epsilon}{c} \\ &= \epsilon, \end{aligned}$$

and since $\left|f(x)-f(y)\right| \leq c\left|x-y\right|$, it follows that $\left|f(x)-f(y)\right| \leq \epsilon$. Thus f is continuous. ■

b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$\left(y_1, f(y_1), f(f(y_1)), \ldots\right).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

Proof: For any two elements y_n and y_m of the sequence such that $n > m \in \mathbb{N}$,

$$\begin{aligned} \left|y_n - y_m\right| &\leq c\left|y_{n-1} - y_{m-1}\right| \\ &\leq c^2\left|y_{n-2} - y_{m-2}\right| \\ &\leq c^m\left|y_{n-m} - y_1\right|. \end{aligned}$$

Note that for a given interval $g = n - m$, the term $\left|y_{n-m} - y_1\right|$ is fixed, so let $\left|y_{n-m} - y_1\right| = M$. Thus we get

$$\left|y_n - y_m\right| \leq c^m\left|y_{n-m} - y_1\right|.$$

Since as $m \to \infty$, $c^m \to 0$, as $n \to \infty$ as the same rate as $m \to \infty$ (i.e. so that g is constant), given an $\epsilon > 0$, we can find an $N \in \mathbb{N}$ such that when $m > N$

$$c^m M < \epsilon$$

which means that

$$\left|y_n - y_m\right| < \epsilon.$$

Thus, (y_n) is a Cauchy sequence. Of course this value of N is dependent on the value of g and ϵ. ■

c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

Proof: Since $\lim y_n = y$, then $f(\lim y_n) = \lim y_n$. It follows that $f(y) = y$. To show that it is unique, suppose that $x \neq y$ is another limit point, then

$$\left|f(x) - f(y)\right| = \left|x - y\right| \leq c\left|x - y\right|,$$

since $c < 1$, the inequality $\left|x-y\right| \leq c\left|x-y\right|$ is not true, thus there is only one limit point. ■

d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $\left(x, f(x), f(f(x)), \ldots\right)$ converges to y defined in part (b).

Proof: Let (x_n) be the sequence $\left(x, f(x), f(f(x)), \ldots\right)$. We know that $(y_n) \to y$. Since $\left|f(x) - f(y)\right| \leq c\left|x-y\right|$, it follows that for some $n \in \mathbb{N} > 0$

$$\left|x_n - y_n\right| \leq c^n\left|x - y\right|.$$

We can fix $\left|x-y\right| = M$. Since as $n \to \infty$, $c^n \to 0$, given an $\epsilon > 0$, we can find an N, such that when $n > N$,

$$\left|x_n - y_n\right| \leq c^n M < \epsilon.$$

Thus (x_n) and (y_n) must converge to the same point. We already know that $(y_n) \to y$, thus $(x_n) \to y$. This shows that if x is any arbitrary point in $\mathbb{R}$, then the sequence $\left(x, f(x), f(f(x)), \ldots\right)$ converges to y. ■