Midterm 2

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Exercise 1. Prove that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous at every $c > 0$.

Proof: We will show that f is continuous on $[0, 1]$ and $[1, \infty)$, and thereby continuous on $[0, \infty)$.

$[0, 1]$: Given an $\epsilon_1 > 0$, let $\delta_1 = 2\epsilon_1$, then when $|x - c| < \delta_1$, we get by manipulation

$$\begin{aligned} |x - c| &= \left|\sqrt{x} - \sqrt{c}\right| \left|\sqrt{x} + \sqrt{c}\right| \\ &\leq \left|\sqrt{x} - \sqrt{c}\right| |2| , \end{aligned}$$

which means that

$$\begin{aligned} 2\left|\sqrt{x} - \sqrt{c}\right| &< \delta_1 \\ 2\left|\sqrt{x} - \sqrt{c}\right| &< 2\epsilon_1 \\ \left|\sqrt{x} - \sqrt{c}\right| &< \epsilon_1, \end{aligned}$$

which shows that f is continuous (in fact uniformly continuous) on $[0, 1]$.

$[1, \infty)$: Given an $\epsilon_2 > 0$, let $\delta_2 = \epsilon_2$, then when $|x - c| < \delta_2$, we get by manipulation

$$\begin{aligned} \left|\sqrt{x} - \sqrt{c}\right| &= \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \\ &< \frac{|x - c|}{\sqrt{c}} \\ &< |x - c| \\ &< \delta_2 \\ &= \epsilon_2, \end{aligned}$$

which shows that f is continuous (in fact uniformly continuous) on $[1, \infty)$. Since f is uniformly continuous on $[0, 1]$ and $[1, \infty)$, it is uniformly continuous on $[0, \infty)$, and thus continuous on $[0, \infty)$.

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Exercise 2. By directly using the definition of the limit of a sequence (without using any theorems about limits) show that if $\lim_{n\to\infty} x_n = 3$, then $\lim_{n\to\infty} \frac{x_n+2}{x_n} = \frac{5}{3}$.

Proof: We suppose directly that $\lim_{n\to\infty} x_n = 3$, then given an $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$, such that whenever $n > N_1$, $|x_n - 3| < \epsilon$, which implies that there exists an $N_2 \in \mathbb{N}$ such that whenever $n > N_2$, $|x_n - 3| < 1$. This is equivalent to

$$2 < x_n < 4.$$

By choosing $N = \max(N_1, N_2)$, whenever $n > N$ we get

$$\begin{aligned}
\left|\frac{x_n+2}{x_n} - \frac{5}{3}\right| &= \left|\frac{3x_n + 6 - 5x_n}{3x_n}\right| \\
&= 2\left|\frac{x_n - 3}{3x_n}\right| \\
&\leq 2\frac{|x_n - 3|}{3 \cdot 2} \quad \text{Since } N \leq N_2 \\
&< |x_n - 3| \\
&< \epsilon \quad \text{Since } N \leq N_1.
\end{aligned}$$

Thus if $\lim_{n\to\infty} x_n = 3$, then $\lim_{n\to\infty} \frac{x_n+2}{x_n} = \frac{5}{3}$.

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Exercise 3. By directly using the definition of a Cauchy sequence (without using theorems about Cauchy sequences) show that if (x_n) is a Cauchy sequence satisfying $-5 < x_n < -2$, then

$$\left(\frac{x_n^2}{1+x_n}\right)$$

is also a Cauchy sequence.

Proof: We suppose directly that (x_n) is a Cauchy sequence satisfying $-5 < x_n < -2$, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n, m > N$, $|x_n - x_m| < \frac{\epsilon}{35}$. Now to show that $\left(\frac{x_n^2}{1+x_n}\right)$ is a Cauchy sequence, we use the definition of a Cauchy sequence.

$$\begin{aligned}
\left|\frac{x_n^2}{1+x_n} - \frac{x_m^2}{1+x_m}\right| &= \left|\frac{x_n^2 + x_n^2 x_m - x_m^2 - x_m^2 x_n}{(1+x_n)(1+x_n)}\right| \\
&= \left|\frac{(x_n - x_m)(x_n + x_m) - x_n x_m (x_n - x_m)}{(1+x_n)(1+x_m)}\right| \\
&\leq \left|\frac{|(x_n - x_m)(x_n + x_m)| + |x_n x_m (x_n + x_m)|}{(1+x_n)(1+x_m)}\right| \qquad \text{Since } 2<|x|<5 \\
&\leq 35\,|x_n - x_m| \\
&< \frac{35\epsilon}{35} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\left(\frac{x_n^2}{1+x_n}\right)$ is a Cauchy sequence. ■

Exercise 4. By directly using the $\epsilon - \delta$ definition of the limit show that

$$\lim_{x\to 2}\frac{1}{x^3}=\frac{1}{8}.$$

Proof: Given an $\epsilon > 0$, let $\delta = \min\left(1, \frac{8}{19}\epsilon\right)$, then when $0 < |x-2| < \delta$, it follows that $|x-2| < 1$ which is equivalent to

$$1 < x < 3.$$

Using the definition of the limit, we get

$$\begin{aligned}
\left|\frac{1}{x^3}-\frac{1}{8}\right| &= \left|\frac{x^3-8}{8x^3}\right| \\
&= \left|\frac{(x-2)\left(x^2+2x+4\right)}{8x^3}\right| \\
&\leq \frac{|x-2|\left|3^2+2\cdot 3+4\right|}{8\cdot 1^3} \\
&= \frac{19}{8}|x-2| \\
&< \frac{19}{8}\frac{8\epsilon}{19} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\lim_{x\to 2}\frac{1}{x^3}=\frac{1}{8}$. ■

Exercise 5. By directly using the $\epsilon - \delta$ definition of the limit show that

$$\lim_{x \to 2} \frac{x}{x^2 - 1} = \frac{2}{3}.$$

Proof: Given an $\epsilon > 0$, let $\delta = \min\left(\frac{1}{2}, \frac{5}{4}\epsilon\right)$, then when $0 < |x - 2| < \delta$, it follows that $|x - 2| < \frac{1}{2}$ which implies

$$\frac{1}{2} < x - 1 < \frac{3}{2},$$
$$\frac{5}{2} < x + 1 < \frac{7}{2},$$

and

$$4 < 2x + 1 < 6.$$

Using the definition of the limit, we get

$$\begin{aligned}
\left| \frac{x}{x^2 - 1} - \frac{2}{3} \right| &= \left| \frac{3x - 2x^2 + 2}{3\left(x^2 - 1\right)} \right| \\
&= \left| \frac{(x - 2)(2x + 1)}{3(x - 1)(x - 2)} \right| \\
&\leq \frac{|x - 2|\, 6}{3 \cdot \frac{1}{2} \cdot \frac{5}{2}} \\
&= \frac{4}{5}|x - 2| \\
&< \frac{4}{5} \cdot \frac{5}{4}\epsilon \\
&= \epsilon.
\end{aligned}$$

Therefore, $\lim_{x \to 2} \frac{x}{x^2 - 1} = \frac{2}{3}$.

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Exercise 6. Prove exactly one of the following theorem:
I am choosing the nested compact interval property.

Proof: We suppose directly that K_i is a compact set with $i \in \mathbb{N}$, and that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$

is a nested sequence of nonempty compact set. Since K_i is compact, any sequence in K_i contains a subsequence that converges to a point in that set. Since all of the sets are nonempty, we can construct a sequence (x_n) with $x_n \in K_n$. Since $(x_n) \subseteq K_1$, it must contain a subsequence (x_{n_k}) that converges to a point $c \in K_1$. Since (x_n) is contained in any K_i except for a finite number of terms at the beginning of the sequence, (we will refer to this sequence as (x_i) such that $(x_i) \subseteq (x_n)$), then (x_i) must have a convergent subsequence (x_{i_k}) that converges to the point $c \in K_i$. Since the sequence (x_i) contains an infinite number of terms and i is arbitrary, it must be that the limit point c is an element of every set K_i. Thus

$$\cap_{n=1}^{\infty} K_n$$

is not empty since it contains at least the point c. ■

Exercise 7. Assume that $f : (a, c) \to \mathbb{R}$ is uniformly continuous on each of the intervals $(a, b]$ and $[b, c)$, where $a < b < c$. Show that f is uniformly continuous on the interval (a, c).

Proof: Let $x \in (a, b]$ and $y \in [b, c)$. We assume directly that $f : (a, c) \to \mathbb{R}$ is uniformly continuous on each of the intervals $(a, b]$ and $[b, c)$, then given an $\epsilon > 0$, there exists a δ_1, δ_2 such that when

$$|x - b| < \delta_1,$$

$$|f(x) - f(b)| < \frac{\epsilon}{2},$$

and when

$$|b - y| < \delta_2,$$

then

$$|f(b) - f(y)| < \frac{\epsilon}{2}.$$

Thus, given any $k, m \in (a, c)$, by letting $\delta = \min(\delta_1, \delta_2)$ we have three cases to consider.

Case 1. Suppose $k, m \in (a, b]$, then it is uniformly continuous by our assumption.

Case 2. Suppose $k, m \in [b, c)$, then it is uniformly continuous by our assumption.

Case 3. Suppose, without loss in generality, that $k \in (a, b]$ and $m \in [b, c)$, then when $|k - m| < \delta$,

$$|k - b| < \delta_1 < \delta$$

and

$$|b - m| < \delta_2 < \delta$$

thus

$$|f(k) - f(b)| < \frac{\epsilon}{2},$$

and

$$|f(b) - f(m)| < \frac{\epsilon}{2}.$$

Adding them together and using the triangle inequality we get that

$$|f(k) - f(m)| < \epsilon.$$

Therefore, f is uniformly continuous on (a, b).

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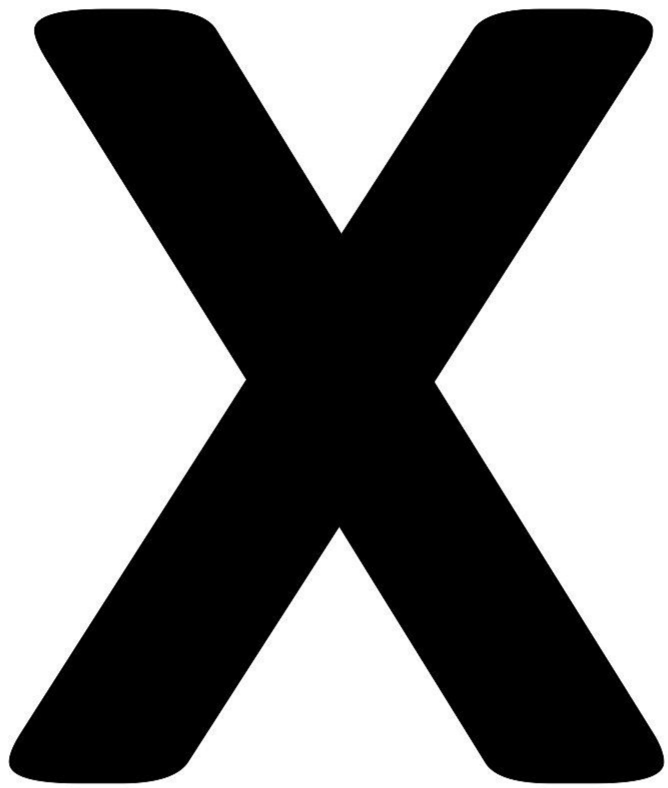
Exercise 8. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by...



Exercise 9. Let $F \subsetneq \mathbb{R}$ be a nonempty closed set. Let $F^c = \{x \in \mathbb{R} : x \notin F\}$ be the complement of F. Show that F^c is open.

Exercise 10. Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the set $[1, \infty)$, but not on the set $(0, 1]$.

Proof: We will first show that f is uniformly continuous on the set $[1, \infty)$. Given an $\epsilon > 0$, let $\delta = \epsilon/2$, then when $|x - c| < \delta$, it follows that

$$\begin{aligned}
\left|\frac{1}{x^2} - \frac{1}{c^2}\right| &= \left|\frac{x^2 - c^2}{x^2 c^2}\right| \\
&= \left|\frac{|x-c|\,|x+c|}{x^2c^2}\right| \\
&\leq \frac{|x-c|\,|x|}{|x^2c^2|} + \frac{|x-c|\,|c|}{|x^2c^2|} \\
&= \frac{|x-c|}{|xc^2|} + \frac{|x-c|}{|x^2c|} \\
&\leq |x-c| + |x-c| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Now we will show that f is not uniformly continuous on the set $(0, 1]$ using sequences. Let $a_n = \frac{1}{\sqrt{n+1}}$ and $b_n = \frac{1}{\sqrt{n}}$, then

$$\lim_{n\to\infty} (a_n - b_n) = 0,$$

but

$$\begin{aligned}
\lim_{n\to\infty} |f(a_n) - f(b_n)| &= \lim_{n\to\infty} \left| \frac{1}{\frac{1}{(\sqrt{n+1})^2}} - \frac{1}{\frac{1}{(\sqrt{n})}^2} \right| \\
&= \lim_{n\to\infty} |n + 1 - n| \\
&= 1,
\end{aligned}$$

thus there exists an $\epsilon > 0$, such that for every $N \in \mathbb{N}$, whenever $n > N$

$$|f(a_n) - f(b_n)| > \epsilon.$$

This shows that f is not uniformly continuous on the set $(0, 1]$. ■

Exercise 11. For each of the following statement, circle True or False. No justification is needed.

For any set $A \subseteq \mathbb{R}$, $\left(\overline{A}\right)^c$ is open	True
A set A is closed if and only if $A = \overline{A}$	True
If A is a bounded set, then $s = \sup(A)$ is a limit point of A	False
An open set that contains every rational number must necessarily be all of $\mathbb{R}$	False
An arbitrary intersection of compact sets is compact	True
If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\cap_{n=1}^{\infty} F_n \neq \emptyset$	False
The notation $V_\delta(c)$ in the textbook denotes the interval $(c-\delta, c+\delta)$.	True
The Cantor set is compact.	True
Any finite set is compact.	True
The set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is compact.	False

Exercise 12. Complete the following.

a) Let $\{E_\lambda : \lambda \in \Lambda\}$ be any collection of sets. Prove the De Morgan Law

$$(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c.$$

Proof: This is an equivalence statement between sets. We must show inclusions both ways.
$(\subseteq)$: Let $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$, then $x \notin \cup_{\lambda\in\Lambda} E_\lambda$. Thus $x \notin E_\lambda$ for every $\lambda \in \Lambda$, which implies that $x \in E_\lambda^c$ for every λ. Hence, $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$. Therefore $(\cup_{\lambda\in\Lambda} E_\lambda)^c \subseteq \cap_{\lambda\in\Lambda} E_\lambda^c$.
$(\supseteq)$: Let $x \in \cap_{\lambda\in\Lambda} E_\lambda^c$, then $x \in E_\lambda^c$ for every $\lambda \in \Lambda$. Then $x \notin E_\lambda$ for every λ, and hence $x \notin \cup_{\lambda\in\Lambda} E_\lambda$. Hence, $x \in (\cup_{\lambda\in\Lambda} E_\lambda)^c$. Therefore $(\cup_{\lambda\in\Lambda} E_\lambda)^c \supseteq \cap_{\lambda\in\Lambda} E_\lambda^c$.
Since we have shown inclusion both ways, it must be that $(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c$. ■

b) Let $\{F_\lambda : \lambda \in \Lambda\}$ be a collection of closed sets. Prove that

$$\cap_{\lambda\in\Lambda} F_\lambda$$

is a closed set.
Proof: Let $\{E_\lambda : \lambda \in \Lambda\}$ be a collection of open sets. We want to show that $\cup_{\lambda\in\Lambda} E_\lambda$ is open. Well, let $x \in \cup_{\lambda\in\Lambda} E_\lambda$, then x must be an element of at least one E_λ and possibly more. For every E_λ such that $x \in E_\lambda$, there exists a neighborhood $V_\delta(x) \subseteq E_\lambda$ since E_λ is open. This implies that $V_\delta(x) \subseteq \cup_{\lambda\in\Lambda} E_\lambda$, hence $\cup_{\lambda\in\Lambda} E_\lambda$ is open. Since $\cup_{\lambda\in\Lambda} E_\lambda$ is open, according to De Morgan Law

$$(\cup_{\lambda\in\Lambda} E_\lambda)^c = \cap_{\lambda\in\Lambda} E_\lambda^c,$$

which is a closed set since the complement of an open set is closed. Also not that E_λ^c is also closed since each E_λ was assumed open. This proves that the intersection of an arbitrary collection of closes sets is a closed set. ■