Homework 28 Section 6.4

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Exercises: 2,3,4,5,6

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Exercise 1. (Q2): Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero
Proof: Let $s_k = \sum_{n=1}^{k} g_n$ denote the series of partial sums, then given an $\epsilon > 0$, there exists a N such that whenever $n > N$

$$|s_n(x) - g(x)| < \epsilon$$

for all x. This implies that

$$\left|\sum_{k=n+1}^{\infty} g_k(x)\right| < \epsilon.$$

Since ϵ can be made arbitrarily small, this implies that $(g_n) \to 0$ as $n \to \infty$ and thus converges uniformly. ■

b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.
Proof: True. Since $\sum_{n=1}^{\infty} g_n$ converges uniformly, we know by the Cauchy Criterion for uniform converges that

$$|f_{m+1}(x) + \cdots + f_n| \leq |g_{m+1}(x) + \cdots + g_n(x)| \leq \epsilon$$

, thus $\sum_{n=1}^{\infty} f_n$ converges. ■

c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist a constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.
Disproof: Let $f_n = (-1)^n \frac{1}{n}x$ where $x \in [0,1]$, then f_n converges uniformly on A; however $M_n \geq \frac{1}{n}$ which is the harmonic series and does not converge. ■

Exercise 2. (Q3): Complete the following

a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbb{R}$.
Proof: Let $g_n = \frac{\cos(2^n x)}{2^n}$, which is continuous by the Algebraic Continuity Theorem. By showing that $\sum g_n$ converges uniformly on $\mathbb{R}$, we prove that $g(x)$ is continuous by the Term-by-term continuity Theorem. We note that

$$|g_n| = \left|\frac{\cos(2^n x)}{2^n}\right| \leq \frac{1}{2^n},$$

where $\frac{1}{2^n}$ is a geometric series that converges to $\frac{1}{1-\frac{1}{2}} = 2$. Thus by the Weierstrass M-test, $\sum_{n=0}^{\infty} g_n$ converges uniformly, thus $g(x)$ is continuous. ■

b) The function g was cited in Section 5.4 as an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?

a) We first note that

$$\begin{aligned} g_n'(x) &= \frac{-\sin(2^n x)\, 2^n}{2^n} \\ &= -\sin(2^n x) \end{aligned}$$

which does not converge point wise on all of $\mathbb{R}$, thus it cannot converge uniformly. Therefore, theorem 6.4.3 cannot apply to it.

Exercise 3. (Q4): Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(1 + x^{2n})}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Let $g(x) = \sum_{n=0}^{\infty} g_n(x)$ with $g_n(x) = \frac{x^{2n}}{(1+x^{2n})}$. In order for $g(x)$ to converge, we need $(g_n) \to 0$ as $n \to \infty$. This only occurs when $|x| < 1$. When $|x| < 1$ we get the inequality

$$\begin{aligned} g_n(x) &\leq \frac{1}{1+x^{2n}} \\ &\leq \frac{1}{x^{2n}}, \end{aligned}$$

where $\frac{1}{x^{2n}}$ is a geometric series. Since $\sum \frac{1}{x^{2n}}$ converges, then by the Weierstrass M-test, the series $\sum g_n$ converges uniformly. Thus $g(x)$ is a continuous function.

Exercise 4. (Q5): Complete the following

a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1, 1]$.

Proof: Let $h(x) = \sum_{n=1}^{\infty} h_n(x)$ with $h_n(x) = \frac{x^n}{n^2}$. On the interval $[-1, 1]$,

$$|h_n(x)| \leq \frac{1}{n^2}.$$

This the series $\sum \frac{1}{n^2}$ converges, by the Weierstrass M-Test, the series $\sum_{n=1}^{\infty} h_n(x)$converges uniformly, and since $h_n(x)$ is continuous, the function $h(x)$ is continuous. ■

b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Proof: On the interval $A = [-1, 0)$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges according to the Alternating series test. On the set $B = \{0\}$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n} = 0$ which converges to 0. On the open interval $C = (-1, 1)$, then

$$\left| \frac{x^n}{n} \right| \leq |x^n|.$$

Since $\sum |x^n|$ is a geometric series and $|x| < 1$, the series converges. Then by the Weierstrass M-Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on the interval C and thus f is continuous on the interval C. ■

Exercise 5. (Q6): Let

$$f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \frac{1}{x+4} - \cdots$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

Proof: Let $f(x) = \sum_{n=0}^{\infty} f_n(x)$ with $f_n(x) = (-1)^{n+1} \frac{1}{x+n}$. This series converges by the alternating series test. The derivative of $f_n(x)$ is

$$f_n^{'}(x) = (-1)^n \frac{1}{(x+n)^2}.$$

Since $\left| f_n^{'}(x) \right| \leq \frac{1}{n^2}$ and the series $\sum \frac{1}{n^2}$ converges, then $\sum f_n^{'}(x)$ converges uniformly to some function $g'(x)$ according to the Weierstrass M-Test. Therefore, according to the term-by-term differentiability theorem, $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly to $f(x)$ and $f'(x) = g'(x)$. ■