Homework 30 Section 6.6

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Exercises: 1,2,5,10

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Exercise 1. (Q1): The derivation in Example 6.6.1 shows the Taylor series for $\arctan(x)$ is valid for all $x \in (-1, 1)$. Notice, however, that the series also converges when $x = 1$. Assuming that $\arctan(x)$ is continuous, explain why the value of the series at $x = 1$ must necessarily be $\arctan(1)$. What interesting identity do we get in this case?

We note that $\arctan(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} (-1)^n$. We want to show that the series

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} (-1)^n$$

converges at $x = 1$. When $x = 1$, we get

$$\sum_{n=0}^{\infty} \frac{1^{2n+1}}{2n+1} (-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

which converges by the alternating series test. Since the series converges at $x = 1$, according to Abel's Theorem, the series converges uniformly on the interval $[0, 1]$ which means that it converges to a continuous function that is continuous on the interval $[0, 1]$. Thus

$$\begin{aligned} \arctan(x) &= \lim_{x \to 1^{-1}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} (-1)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \end{aligned}$$

According to section 6.1, we get that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Exercise 2. (Q2): Starting from one of the previously generated series in this section, use manipulations similar to those in Example 6.6.1 to find Taylor series representations for each of the following functions. For precisely what values of x is each series representation valid?

a) $x \cos(x^2)$

  a) We know that $\cos(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!}$. Substituting x^2 for y and multiplying the series by x yields

$$x \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n}}{(2n)!}.$$

  Using the ratio test, we get that

$$\left| \frac{\frac{x^{5(n+1)}}{(2(n+1))!}}{\frac{x^{5n}}{(2n)!}} \right| = \left| \frac{x^5}{(2n+1)(2n+1)} \right|$$

  which is less than 1 as $n \to \infty$. Thus the series converges for all of $x \in \mathbb{R}$.

b) $\frac{x}{(1+4x^2)^2}$

  a) We know that $\frac{1}{(1-y)^2} = \sum_{n=0}^{\infty} (n+1) y^n$ which converges when $|y| < 1$. Substituting in $-4x^2$ for y and multiplying by x yields

$$x \sum_{n=0}^{\infty} (n+1) \left(-4x^2\right)^n = \sum_{n=0}^{\infty} (n+1) (-1)^n 4^n x^{2n+1}$$

  which converges when $\left|4x^2\right| < 1$ which is equivalent to when $|2x| < 1$.

c) $\log\left(1 + x^2\right)$

a) We know that $\log(1+y) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{y^n}{n}$ which converges when $y \in (-1,1]$. Substituting in x^2 for y yields

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$$

which converges for $|x| \leq 1$.

Exercise 3. (Q5): Complete the following.

a) Generate the Taylor coefficients for the exponential function $f(x) = e^x$, and then prove that the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.
Proof: The Taylor series coefficients are generated by

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Since $\frac{d}{dx} e^x = e^x$ we can reduce the coefficients to

$$a_n = \frac{1}{n!}$$

which means that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

For some $N \in \mathbb{N}$, we can calculate the error function

$$E_N = \frac{e^c}{(N+1)!} x^{N+1}$$

for some c between x and 0. Since e^c is finite and fixed, and that factorials grow much faster than exponentials, it follows that $E_N \to 0$. Thus the series converges uniformly to e^x. ■

b) Verify the formula $f'(x) = e^x$.
Proof: Taking the derivative of the power series representation of e^x we get

$$\begin{aligned} \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{n!} x^n &= \frac{d}{dx} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) \\ &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) \\ &= e^x \end{aligned}$$

■

c) Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying together the two series and collecting common powers of x.

a) By substituting

$$e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n.$$

and

$$\begin{aligned} e^{-x} \cdot e^x &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n \right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n \right) \\ &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right) \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - x - x^2 - \frac{x^3}{2!} + \frac{x^2}{2!} + \frac{x^3}{2!} + \ldots \\ &= 1 \end{aligned}$$

Exercise 4. (10): Consider $f(x) = \frac{1}{\sqrt{1-x}}$

a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $\left[0, \frac{1}{2}\right]$.

Proof: We first generate the Taylor series for f centered at zero by generating the Taylor series coefficients. Differentiating f we get

$$\begin{aligned}
f'(x) &= \frac{1}{2(1-x)^{\frac{3}{2}}} \\
f''(x) &= \frac{1\cdot 3}{3^2(1-x)^{\frac{5}{2}}} \\
f'''(x) &= \frac{1\cdot 3\cdot 5}{2^3(1-x)^{\frac{7}{2}}} \\
&\vdots \\
f^{(n)}(x) &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(1-x)^{2n+1}}
\end{aligned}$$

Evaluating each term at $x=0$, we get that the Taylor series is

$$f(x) = 1 + \frac{1}{2}x + \frac{3}{2^2 2!} + \frac{15}{2^3 3!}x^3 + \frac{15\cdot 7}{2^4 4!}x^4\cdots$$

Given an $N\in\mathbb{N}$ a $x\in\left[0,\frac{1}{2}\right]$ and a $c\in(0,x)$, the Lagrange's Remainder is

$$\begin{aligned}
E_N &= \frac{1\cdot 3\cdots(2(N+1)-1)}{2^{(N+1)}(N+1)!(1-c)^{(2N+1)/2}}x^{N+1} \\
&\leq \frac{1\cdot 3\cdots(2N+1)}{2^{(N+1)}(N+1)!}
\end{aligned}$$

Using the same procedure we get that

$$E_{N+1} \leq \frac{1\cdot 3\cdots(2N+1)(2N+3)}{2^{(N+2)}(N+2)!}.$$

Note that

$$\frac{E_{N+1}}{E_N} = \frac{(2N+3)}{2(N+2)} < 1.$$

Since $0<E_n$, is bounded and $E_n > E_{n+1}$ for all $n\in\mathbb{N}$, by the monotone convergence theorem, $(E_n)\to 1$. Thus the Taylor series for f converges centered at 0 and on $\left[0,\frac{1}{2}\right]$. ■

b) Use Cauchy's Remainder Theorem proved in Exercise 6.6.9 to show that the series representation of f holds on $[0,1)$.
Proof: Let $c\in(0,x)$. The remainder term is

$$\begin{aligned}
R_n &= \frac{(x-c)^N}{N!}xf^{(N+1)}(c) \\
&= \frac{(x-c)^N}{N!}x\cdot\frac{1\cdot 3\cdot 5\cdots(2N+1)}{2^n(1-c)^{(2N+3)/2}} \\
&= \frac{(2N+2)!}{2^{2N+2}(N+1)!N!}\cdot\frac{(x-c)^N x}{(1-c)^{(2N+3)/2}} \\
&= \frac{y_{N+1}x}{(1-c)^{\frac{3}{2}}}\cdot\left(\frac{x-c}{1-c}\right)^N(N+1),
\end{aligned}$$

with

$$y_n = \frac{(2n)!}{2^{2n}(n!)^2}.$$

The sequence (y_n) converges according to the ratio test. Thus

$$\frac{y_{N+1}x}{(1-c)^{\frac{3}{2}}}$$

converges to a constant and $x < 1$. Note that

$$\begin{aligned}\frac{x-c}{1-c} &= \frac{x-1+1-c}{1-c} \\ &= \frac{1-c}{1-c} + \frac{x-1}{1-c} \\ &= 1 + \frac{x-1}{1-c} \\ &< 1.\end{aligned}$$

Thus, as $n \to \infty$,

$$R_n = kr^n$$

where k is a constant and $|r| < 1$. Thus $R_n \to 0$ as $n \to \infty$ provided that $c \in [0, 1)$. ■