Homework 7 Section 2.3

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Exercises 2,3,5,8,10

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Exercise 1. (Q2) Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

1) $\left(\frac{2x_n - 1}{3}\right) \to 1$
Proof: We suppose directly that $(x_n) \to 2$, then given an $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that

$$|x_n - 2| < \frac{3\epsilon}{2},$$

which implies that

$$\begin{aligned} \frac{2}{3}\left|x_n - 2\right| &< \epsilon \\ \left|\frac{2x_n - 4}{3}\right| &< \epsilon \\ \left|\frac{2x_n - 1}{3} - 1\right| &< \epsilon, \end{aligned}$$

thus, $\left(\frac{2x_n - 1}{3}\right) \to 1$. ■

2) $\left(\frac{1}{x_n}\right) \to \frac{1}{2}$
Proof: We suppose directly that $(x_n) \to 2$, then given an $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that $|x_n - 2| < 2\epsilon$. Also, there exists an $N_2 \in \mathbb{N}$ such that $|x_n - 2| < 1$ in other words

$$\begin{aligned} -1 < x_n - 2 &< 1 \\ 1 < x_n &< 3. \end{aligned}$$

By selecting $N = \max(N_1, N_2)$ we get that

$$\begin{aligned} |x_n - 2| &< 2\epsilon \\ \frac{|x_n - 2|}{x_n} &< 2\epsilon \quad \text{Since } 1 < x_n < 3 \text{ when } N \geq N_2 \\ \frac{|x_n - 2|}{|2x_n|} &< \epsilon \\ \left|\frac{1}{x_n} - \frac{1}{2}\right| &< \epsilon, \end{aligned}$$

therefore, $\left(\frac{1}{x_n}\right) \to \frac{1}{2}$. ■

Exercise 2. (Q3): Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Proof: We suppose directly that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and that $\lim x_n = \lim z_n = l$, then given an $\epsilon > 0$, there exists an $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - l| < \epsilon$$

for all $n > N_1$ and

$$|z_n - 1| < \epsilon,$$

for all $n > N_2$.

In other words, for $n > \max(N_1, N_2)$ we get

$$-\epsilon < x_n - l < \epsilon$$
$$-\epsilon + l < x_n < \epsilon + l$$

and

$$-\epsilon < z_n - l < \epsilon$$
$$-\epsilon + l < z_n < \epsilon + l.$$

Since $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, we have that for $n > \max(N_1, N_2)$

$$-\epsilon + l < x_n \leq y_n \leq z_n < \epsilon + l$$
$$-\epsilon + l < y_n < \epsilon + l$$
$$|y_n - l| < \epsilon,$$

therefore, $\lim y_n = l$. ■

Exercise 3. (Q5): Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence

$$(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots).$$

Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Proof: This is a biconditional statement, so we must prove both implications.

$(\Longrightarrow)$: We assume directly that (z_n) converges to l, then given an $\epsilon_1, \epsilon_2 > 0$, there exists an $N_1, N_2 \in \mathbb{N}$ such that whenever $2n - 1 > N_1$ and $2n > N_2$, we have

$$|z_{2n-1} - l| < \epsilon_1$$

and

$$|z_{2n} - l| < \epsilon_2.$$

Since $x_m = z_{2m-1}$ and $y_m = z_{2m}$ for all $m \in \mathbb{N}$, we have that

$$|x_m - l| < \epsilon_1$$

and

$$|y_m - l| < \epsilon_2.$$

Therefore, if (z_n) converges to l, then (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

$(\Longleftarrow)$: We suppose directly that (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$. With no loss in generality, let their limit be denoted by L, then given an $\epsilon_1, \epsilon_2 > 0$, there exists an $N_1, N_2 \in \mathbb{N}$ such that whenever $j \in \mathbb{N} > N_1$ and $\ell \in \mathbb{N} > N_2$ we get

$$|x_j - L| < \epsilon_1$$
$$|y_\ell - L| < \epsilon_2.$$

Let $N = \max(N_1, N_2)$, and $\epsilon = \min(\epsilon_1, \epsilon_2)$, then for any $q \in \mathbb{N} > N$ we get

$$|x_q - L| < \epsilon$$
$$|y_q - L| < \epsilon.$$

Since $x_q = z_{2q-1}$ and $y_q = z_{2q}$ we have that

$$|z_{2q-1} - L| < \epsilon$$

and

$$|z_{2q} - L| < \epsilon$$

for all $q > N$. Let $N_f = \frac{N+1}{2}$, then for all $m \in \mathbb{N} > N_f$ we have

$$|z_m - L| < \epsilon,$$

therefore, if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$, then z_n converges. ■

Exercise 4. (Q8): Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

1) Show that $p(x_n) \to p(x)$.
   Proof: We suppose directly that $(x_n) \to x$. The polynomial $p(x_n)$ can be written as
   $$c_k x_n^k + c_{k-1} x_n^{k-1} + \cdots + c_1 x_n + c_0,$$
   and the limit of $p(x_n)$ is
   $$\lim_{n\to\infty} c_k x_n^k + c_{k-1} x_n^{k-1} + \cdots + c_1 x_n + c_0.$$
   According to the algebraic limit theorem, if $(x_n) \to x$, then
   $$\lim_{n\to\infty} c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0 = p(x).$$
   ■
2) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.
   a) Let x_n be the sequence $\frac{1}{x_n}$, then $(x_n) \to 0$. Also, let $f(z)$ be the piecewise function defined as
   $$f(z) = \begin{cases} 10 & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases},$$
   then $f(x) = f(0) = 0$ and $\lim f(x_n) = 10$ since $x_n > 0$ for all $n \in \mathbb{N}$.

Exercise 5. (Q10): Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

1) If $\lim (a_n - b_n) = 0$, then $\lim a_n = \lim b_n$
   Disproof: We suppose directly that $\lim (a_n - b_n) = 0$. Let $a_n = b_n = n$, then $a_n - b_n = 0$; therefore, $\lim (a_n - b_n) = 0$. However, the limits of a_n and b_n do not exist. ■
2) If $(b_n) \to b$, then $|b_n| \to |b|$.
   Proof: We suppose directly that $(b_n) \to b$, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \in \mathbb{N} > N$
   $$|b_n - b| < \epsilon,$$
   using the triangle inequality we get that
   $$||b_n| - |b|| < \epsilon.$$
   Therefore, $|b_n| \to |b|$ if $(b_n) \to b$. ■
3) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.
   Proof: We suppose directly that $\lim (a_n - b_n) = 0$ and $(a_n) \to a$, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \in \mathbb{N} > N$,
   $$|a_n - a| < \frac{\epsilon}{2}$$
   and
   $$|a_n - b_n| < \frac{\epsilon}{2}.$$
   Thus
   $$\begin{aligned} -\frac{\epsilon}{2} < a_n - b_n < \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} - \frac{\epsilon}{2} < a_n - a - b_n < \frac{\epsilon}{2} - a + \frac{\epsilon}{2} \\ a - \epsilon < b_n < -a + \epsilon \\ -\epsilon < b_n - a < \epsilon \\ |b_n - a| < \epsilon, \end{aligned}$$
   thus if $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$. ■
4) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then $(b_n) \to b$.

Proof: We suppose directly that $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \in \mathbb{N} > N$,

$$|a_n| < \epsilon,$$

hence

$$\begin{aligned} |b_n - b| &\leq a_n < \epsilon \\ |b_n - b| &< \epsilon, \end{aligned}$$

therefore, $(b_n) \to b$. ∎