Homework 20 Section 4.5

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Exercises 1,2,3,4

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Exercise 1. (Q1): Show how the Intermediate Value Theorem follows as a corollary to Theorem 4.5.2.

Proof: Suppose directly that $f : A \to \mathbb{R}$ is a continuous function that maps the connected set A to the real numbers. Since every connected subset of the real numbers is an interval, we can construct the closed subset $C = [a, b] \subseteq A$ which is a compact and connected set. Now we also suppose directly that L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$. Since A is a connected set, the image $f(A)$ is connected, i.e. it is an interval. Thus, if $f(a), f(b) \in f(A)$, then $L \in f(A)$. So there must exists a $c \in A$, such that $f(c) = L$. Since f is continuous, given an $\epsilon > 0$, there exists a $\delta > 0$, such that whenever

$$|a - b| < \delta,$$

then

$$|f(a) - f(b)| < \epsilon.$$

Since $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then

$$|L - f(a)|, |L - f(b)| < \epsilon,$$

which implies

$$|a - c|, |a - c| < \delta,$$

thus $c \in (a, b)$. ■

Exercise 2. (Q2): Provide an example of each of the following, or explain why the request is impossible.

a) A continuous function defined on an open interval with range equal to a closed interval.
   a) Possible. Let f be defined as $f(x) = 0$, then the range is $\{0\}$ which is a closed interval regardless of the domain.
b) A continuous function defined on a closed interval with range equal to an open interval.
   a) Possible. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x$. Since $\mathbb{R}$ is both open an closed, the range is equal to an open interval.
c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.
   a) Possible. Let $f : \mathbb{R} \to [0, \infty)$ be defined as $f(x) = x^2$.
d) A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.
   a) Not possible. A continuous function preserves connectedness. $\mathbb{R}$ is connected, but $\mathbb{Q}$ is not since it is not an interval, thus a function does not exist.

Exercise 3. (Q3): A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property, then f is continuous on $[a, b]$.

Proof: We suppose directly that f is increasing on $B = [a, b]$ and it satisfies the intermediate value property, then given an $x, y, c \in B$, if $f(x) < f(c) < f(y)$, then

$$c \in (x, y).$$

This implies that

$$|f(x) - f(c)| < |f(y) - f(x)|,$$

and

$$|f(y) - f(c)| < |f(x) - f(c)|.$$

Now let's fix y. Given an $\epsilon > 0$, we can construct the interval

$$S = \{x \in B : |f(x) - f(y)| < \epsilon\},$$

and let $\delta = \min(|\sup(S) - y|, |y - \inf(S)|)$, then for any $x \in S$ such that

$$|x - y| < \delta,$$

$$|f(x) - f(y)| < \epsilon.$$

Therefore, the function f is continuous. ■

Exercise 4. (Q4): Let g be continuous on an interval A and let F be the set of points where g fails to be injective; that is,

$$F = \{x \in A : f(x) = f(y) \text{ for some } y \neq x \text{ and } y \in A\}$$

Show that F is either empty or uncountable.

Proof: We suppose directly that g is continuous on an interval A. If g is an injection or if $|A| \leq 1$, then F is obviously empty. Now we consider the case where g is not injective and $|A| > 1$. If $|A| > 1$ then A is an uncountable set since it is an interval. Since g is an injection, there exists at least two points $x, y \in A$ such that $x < y$ $g(x) = g(y)$. We can then form the uncountable closed subset $B = [x, y]$. Let $c \in B$ such that $g(c) \neq g(x)$ (if such a c does not exist, then $B \subseteq F$ which implies F is uncountable). since B is a closed interval and g in continuous, $g(B)$ is closed and connected. Without loss in generality, let $g(c) > g(x)$, then $[g(x), g(c)] = [g(y), g(c)] \subseteq g(B)$. Due to the intermediate value theorem, for any $L \in [g(x), g(c)]$ there exists a $z \in (x, c)$ and $k \in (c, y)$ such that $f(z) = f(k) = L$. ■