Homework 3 Section 1.4

Mark Petersen

Exercises 1,3,4,5,8

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Exercise 1. (Q1): Recall that $\mathbb{I}$ stands for the set of irrational numbers.

a) Show that if $a, b \in \mathbb{Q}$, then ab and $a+b$ are elements of $\mathbb{Q}$.
Proof: We suppose directly that $a, b \in \mathbb{Q}$, then $a = \frac{m}{n}$ and $b = \frac{k}{j}$ for some $m, n \in \mathbb{Z}$ and $n, j \in \mathbb{Z} - \{0\}$. The product of a and b is

$$\begin{aligned} ab &= \frac{m}{n} \cdot \frac{k}{j} \\ &= \frac{mk}{nj}. \end{aligned}$$

Since the product of two integers is an integer $mk \in \mathbb{Z}$ and $nj \in \mathbb{Z} - \{0\}$. Thus $\frac{mk}{nj} \in \mathbb{Q}$, which means that $ab \in \mathbb{Q}$. The sum of a and b is

$$\begin{aligned} a + b &= \frac{m}{n} + \frac{k}{j} \\ &= \frac{mj + kn}{nj}. \end{aligned}$$

Since the product and sums of integers is an integer, $mj + kn \in \mathbb{Z}$ and $nj \in \mathbb{Z} - \{0\}$. Thus $\frac{mj+kn}{nj} \in \mathbb{Q}$, which means that $a + b \in \mathbb{Q}$. ■

b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.
Proof: We suppose, by contradiction, that $a \in \mathbb{Q}$, $t \in \mathbb{I}$, $a + t \in \mathbb{Q}$, and $at \in \mathbb{Q}$, then $a + t = m$ and $at \in n$ for some $m, n \in \mathbb{Q}$. Solving for t in both terms yields $t = m - a$ and $t = \frac{n}{a}$. Since the sum and product of two rational numbers is rational and since $a \neq 0$, $m - a \in \mathbb{Q}$ and $\frac{n}{a} \in \mathbb{Q}$. This implies that $t \in \mathbb{Q}$. This can't be the case since $t \in \mathbb{I}$, thus this is a contradiction. Therefore, if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$. ■

c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?
Proof: We want to show that $\mathbb{I}$ is not closed under addition and multiplication. Let $s, t \in \mathbb{I}$ such that $s = \sqrt{2}$ and $t = -\sqrt{2}$, then $s + t = 0$ which is a rational number and $st = -2$ which is a rational number. This shows that $\mathbb{I}$ is not closed under addition and multiplication. ■

Exercise 2. (Q3): Prove that $\cap_{n=1}^{\infty} (0, 1/n) = \emptyset$.

Proof: Let $A_n = (0, 1/n)$ be an open interval with $n \in \mathbb{N}$. The set $X = \cap_{n=1}^{\infty} A_n$ cannot contain any non-positive real numbers or any positive real number greater than 1 since there are no non-positive real numbers or any positive real number greater than 1 in any of the sets A_n. The only numbers left to consider are the numbers in the interval $(0, 1)$. Let $m \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $m > k$, then $0 < \frac{1}{m} < \frac{1}{k}$, thus $\frac{1}{m} \in A_k$. According to the Archimedean property, given any real number $x \in \mathbb{R} > 0$, there exists a natural number j such that $0 < \frac{1}{j} < x$. Hence, there exists a natural number ℓ such that $0 < \frac{1}{\ell} < \frac{1}{m}$; therefore, $\frac{1}{m} \notin A_\ell$. This shows that for every positive real number m, there exists an integer ℓ such that $\frac{1}{m} \notin A_\ell$. Therefore, $\cap_{n=1}^{\infty} (0, 1/n) = \emptyset$. ■

Exercise 3. (Q4): Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show that $\sup(T) = b$.

Proof: We suppose directly that $a < b$ are real numbers and that the set $T = \mathbb{Q} \cap [a, b]$. We have two cases to consider.

Case 1. Let $b \in \mathbb{Q}$, then $b \in T$. Using the fact that $a < b$, we have that for all $c \in T$, $c \leq b$. Since $b \in T$, any upper bound of T must be greater than or equal to b. Thus b is the supremum.

Case 2. Let $b \in \mathbb{I}$, then $b \notin T$ and is an irrational number. According to the density of $\mathbb{Q}$, there is a rational number between any two non equal real numbers. Let $\epsilon \in \mathbb{R} > 0$, then there exists an $r \in T$, such that $b - \epsilon < r < b$. Thus, by lemma 1.3.8, $b = \sup(T)$.

Since both cases hold, $\sup(T) = b$.

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Exercise 4. (Q5): Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{(2)}$.

Proof: We suppose directly that $a < b$ are real numbers, then $a - \sqrt{2} < b - \sqrt{2}$ which are real numbers. According to theorem 1.4.3, there exists a rational number r such that

$$a - \sqrt{2} < r < b - \sqrt{2}.$$

Adding $\sqrt{2}$ to all terms yields

$$a < r + \sqrt{2} < b.$$

According to part (a), the sum of a rational number and an irrational number is irrational. Since $\sqrt{2} \in \mathbb{I}$, $r + \sqrt{2} \in \mathbb{I}$, thus there exists an irrational number between any two, non-equal real numbers. ■

Exercise 5. (Q8): Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

a) Two sets A and B with $A \cap B = \emptyset$, $\sup(A) = \sup(B)$, $\sup(A) \notin A$ and $\sup(B) \notin B$.
   a) Let $A = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\}$ and $B = \left\{-\frac{\sqrt{2}}{n} : n \in \mathbb{N}\right\}$. The set A contains only rational numbers, and the set B contains only irrational numbers, thus $A \cap B = \emptyset$. The $\sup(A) = \sup(B) = 0$, and $0 \notin A$ and $0 \notin B$. Thus, this example satisfies all of the conditions.
b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $T = \cap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.
   a) Let $J_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, then 0 is the only element in every interval. Thus $T = \{0\}$ which is a finite set.
c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\cap_{n=1}^{\infty} L_n = \emptyset$.
   a) Let $L_n = [n, \infty)$. According to the Archimedean Property, given any real number x, there exists a natural number y such that $x < y$. This means that for every x, there exists a set L_y such that $x \notin y$. Thus $\cap_{n=1}^{\infty} L_n = \emptyset$.
d) A sequence of closed bounded intervals I_1, I_2, $I_{3,\ldots}$ with the property that $\cap_{n=1}^{N} \neq \emptyset$ for all $N \in \mathbb{N}$, but $\cap_{n=1}^{\infty} I_n = \emptyset$.
   a) Given $N \in \mathbb{N}$, let $I_n = [n, n+N]$, then $\cap_{n=1}^{N} = \{N\}$, but $\cap_{n=1}^{\infty} I_n = \emptyset$.