Homework 5 Section 1.6

Mark Petersen

Exercises 3,4,5,6,7

06/26/2020

Exercise 1. (Q3): Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

1) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbb{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.
   a) The set of rational numbers between 0 and 1 is $S = \left\{ \frac{m}{n} \in \mathbb{Q} \, : \, 0 < \frac{m}{n} < 1 \right\}$. Since there is an irrational number between every two real numbers, we know that $S \subsetneq (0,1)$. This means that we cannot define a real rational number $x \in S$ with the decimal expansion $x = 0.b_1 b_2 b_3 \cdots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

   or any variant of the rule, since there is no guarantee that this rule will ensure that $x \in S$ since x could become an irrational number.
2) Some numbers have two different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating $9's$. for instance, $\frac{1}{2}$ can be written as 0.5 or as 0.4999... Doesn't this cause some problems?
   a) In the proof of Theorem 1.5.6, we assume that the function $f : \mathbb{N} \to (0,1)$ is bijective, so the image of f doesn't contain any repeating elements, and by using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

   we ensure that we do not construct a number with repeating $9's$ or that terminates in a 0 in order to avoid repeating numbers.

Exercise 2. (Q4): Let S be the set consisting of all sequences of $0's$ and $1's$. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely

$$S = \left\{ (a_1, a_2, a_2, \ldots) \, : \, a_n = 0 \text{ or } 1 \right\}.$$

Give a rigorous argument showing that S is uncountable.

Proof: Let $f : (0,1) \to S$ be the map that sends $x \in (0,1)$ to it's binary representation with the decimals digits forming a tuple. For example, if $x = 0.8125$ then it's binary representation is 0.1101 and tuple form is $(1,1,0,1)$. In other words, $f(x) = (1,1,0,1)$. By showing that this map is injective, we show that $|(0,1)| \leq |S|$, and since $(0,1)$ is uncountable, then S must be uncountable. Let $a, b \in (0,1)$ and suppose, contrapositively, that $f(a) = f(b)$, then

$$f(a) = (a_1, a_2, a_2, \cdots) = f(b),$$

where $a_i \in \{0,1\}$, then a and b have the binary representation

$$0.a_1 a_2 a_3 \cdots,$$

and must be the same. Thus the function f is injective. Since f is injective $|(0,1)| \leq |S|$, therefore S is an uncountable set. ■

Exercise 3. (Q5): a) Let $A = \{a, b, c\}$. List the eight elements of $\mathcal{P}(A)$. b) If A is finite with n elements, show that $\mathcal{P}(A)$ has 2^n elements.

1)
$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

2) If A is finite with n elements, show that $\mathcal{P}(A)$ has 2^n elements.
Proof: A subset of A is any set U such that for all $u \in U$, $u \in A$. The cardinality of these subsets are $0 \leq |U| \leq n$. In other words, in order to construct each subset, we choose k elements from the n elements of A. The binomial coefficient can be used to calculate the number of unique subsets of A that can be constructed by choosing k elements from A, thus
$$\begin{aligned} |\mathcal{P}(A)| &= \sum_{k=0}^{n} \binom{n}{k} \\ &= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}. \end{aligned}$$
We now wish to show that the open sentence
$$Q(n) : \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} = 2^n$$
for all $n \geq 0$. We work this by induction.
Base Case: We first verify $Q(0)$ and $Q(1)$. Let $n = 0$, then
$$\begin{aligned} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} &= \sum_{k=0}^{0} \frac{n!}{k!\,(n-k)!} \\ &= \frac{0!}{0!\,(0-0)!} \\ &= 1 \\ &= 2^0. \end{aligned}$$
Let $n = 1$, then
$$\begin{aligned} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} &= \sum_{k=0}^{1} \frac{n!}{k!\,(n-k)!} \\ &= \frac{1!}{0!\,(1-0)!} + \frac{1!}{1!\,(0-1)!} \\ &= 2 \\ &= 2^1. \end{aligned}$$
Thus $Q(1)$ and $Q(2)$ are true.
Induction Step: Let $m \in \mathbb{Z} \geq 0$. We assume that $Q(m)$ is true, and we want to show that $Q(m+1)$ is true.
$$\begin{aligned} \sum_{k=0}^{m+1} \frac{(m+1)!}{k!\,(m+1-k)!} &= \sum_{k=0}^{m+1} \frac{(m)!}{k!\,(m-k)!} + \sum_{k=1}^{m+1} \frac{(m)!}{(k-1)!\,(m-k-1)!} \\ &= \sum_{k=0}^{m} \frac{(m)!}{k!\,(m-k)!} + \frac{(m)!}{m!\,(m-m)!} + \sum_{k=1}^{m+1} \frac{(m)!}{(k-1)!\,(m-k-1)!} \\ &= 2^m + \sum_{k=0}^{m} \frac{(m)!}{k!\,(m-k)!} \\ &= 2^m + 2^m \\ &= 2^{m+1}, \end{aligned}$$
thus $Q(m+1)$ is true. Therefore, the open sentence $Q(n)$ is true. Hence $|\mathcal{P}(A)| = 2^n$. ■

Exercise 4. (Q6): Three problems

1) Using the particular set $A = \{a, b, c\}$, exhibit two different injective mappings from A into $\mathcal{P}(A)$.
   a) Let $f_1 : A \to \mathcal{P}(A)$ defined as
$$f_1(a) = \{a\}$$
$$f_1(b) = \{b\}$$
$$f_1(c) = \{c\}.$$
   b) Let $f_2 : A \to \mathcal{P}(A)$ defined as
$$f_2(a) = \{c\}$$
$$f_2(b) = \{b\}$$
$$f_2(c) = \{a\}.$$
2) Letting $C = \{1, 2, 3, 4\}$, produce an example of an injective map $g : C \to \mathcal{P}(\mathcal{C})$.
   a) Let $g : C \to \mathcal{P}(C)$ be defined as
$$g(x) = \{x\},$$
   where $x \in C$.
3) Explain why, in parts (a) and (b), it is impossible to construct mappings that are onto.
   a) Let S be an arbitrary finite set with cardinality n, then $|\mathcal{P}(S)| = 2^n$. Thus there are always more elements in $\mathcal{P}(S)$, then in S. Thus an injection exists, but not a surjection.

Exercise 5. (Q7): Return to the particular functions constructed in Exercise 1.6.6 and construct the subset B that results using the preceding rule.

1) $f_1 : A \to \mathcal{P}(A)$ was defined as
$$f_1(a) = \{a\}$$
$$f_1(b) = \{b\}$$
$$f_1(c) = \{c\}.$$
   So $B = \emptyset$.
2) $f_2 : A \to \mathcal{P}(A)$ was defined as
$$f_2(a) = \{c\}$$
$$f_2(b) = \{b\}$$
$$f_2(c) = \{a\}.$$
   So $B = \{a, c\}$.
3) $g : C \to \mathcal{P}(C)$ was defined as
$$g(x) = \{x\},$$
   where $x \in C$. So $B = \emptyset$.