Homework Section 1.2

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Exercises: 4,5,7,8,9

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Exercise 1.1. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\cup_{i=1}^{\infty} A_i = \mathbb{N}$.

Let $P = \{1, 2, 3, 5, 7, \ldots\}$ be the set of all prime numbers including 1, $A_i \subseteq \mathbb{N}$, be the subset of the natural numbers whose elements have a factorization consisting of i distinct elements of the set P, and 1 is never multiplied by a prime. For example, let $a \in A_k$, then $a = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ where $p_1, p_2, \ldots, p_k \in P$, $b_1, b_2, \ldots, b_k \in \mathbb{N}$, $p_1 < p_2 < \cdots < p_k$ and if $k \geq 2$, then $p_1, p_2, \ldots, p_k \neq 1$. We need to show that $A_i \cap A_j = \emptyset$ for all $i \neq j$, the collection is infinite, each set is infinite, and $\cup_{i=1}^{\infty} A_i = \mathbb{N}$.

Proof: Suppose directly that A_i is the set as defined above. Let $m \in A_m$, then $m = p_1^{b_1} p_2^{b_2} \cdots p_m^{b_m}$. In other words, it is composed of m distinct primes. Thus $m \notin A_n$, where $m \neq n$, since n is composed of n distinct primes. Thus $A_m \cap A_n = \emptyset$ for all $m \neq j$. Since there is an infinite number of primes, there is an infinite collection of sets A_i. Since the power of each prime can be any natural number, each set must contain an infinite number of elements. Since every natural number, other than 1, has a unique prime factorization, $\mathbb{N} \subseteq \cup_{i=1}^{\infty} A_i$, and since the product of primes is a natural number, $\cup_{i=1}^{\infty} A_i \subseteq \mathbb{N}$, thus $\cup_{i=1}^{\infty} A_i = \mathbb{N}$. Therefore, $A_i \cap A_j = \emptyset$ for all $i \neq j$, the collection is infinite, each set is infinite, and $\cup_{i=1}^{\infty} A_i = \mathbb{N}$. ∎

Exercise 1.2. Let A and B be subsets of $\mathbb{R}$.

a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
   a) If $x \in (A \cap B)^c$, then $x \notin (A \cap B)$. In other words, $x \notin A$ or $x \notin B$, which is equivalent to $x \in A^c \cup B^c$.

b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.
   Proof: We suppose directly that $x \in A^c \cup B^c$, then $x \in A^c$ or $x \in B^c$. In other words, $x \notin A$ or $x \notin B$. This means that $x \notin A \cap B$, which is equivalent to $x \in (A \cap B)^c$. Therefore, $(A \cap B)^c \supseteq A^c \cup B^c$. Combining the results of part a) and b) shows that $(A \cap B)^c = A^c \cup B^c$. ∎

c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.
   Proof: To show that two sets are equal, we must show that $(A \cup B)^c \subseteq A^c \cap B^c$ and $(A \cup B)^c \supseteq A^c \cap B^c$.
   $(\subseteq)$: We suppose directly that $x \in (A \cup B)^c$, then $x \notin A \cup B$. Which means that $x \in A^c$ and $x \in B^c$. Thus $x \in A^c \cap B^c$, and $(A \cup B)^c \subseteq A^c \cap B^c$.
   $(\supseteq)$: We suppose directly that $x \in A^c \cap B^c$, then $x \notin A$ and $x \notin B$, thus $x \notin A \cup B$, which is equivalent to $x \in (A \cup B)^c$. Hence, $(A \cup B)^c \supseteq A^c \cap B^c$.
   Since both inclusions hold, we have that $(A \cup B)^c = A^c \cap B^c$. ∎

Exercise 1.3. Given a function f and a subset A of its domain, let $f(A)$ represent the rang of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

a) Let $f(x) = x^2$. If $A = [0, 2]$ and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
  a) Since f is a continuous, non-decreasing function, $f(A) = [f(0), f(2)] = [0, 4]$, and $f(B) = [f(1), f(4)] = [1, 16]$.
  b) $f(A \cap B) = f([1, 2]) = [f(1), f(2)] = [1, 4] = [0, 4] \cap [1, 16] = f(A) \cap f(B)$, thus $f(A \cap B) = f(A) \cap f(B)$
  c) $f(A \cup B) = f([0, 4]) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B)$

b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
  a) Let $A = [-2, -1]$ and $B = [1, 2]$, then $f(A \cap B) = f(\emptyset) = \emptyset$, and $f(A) \cap f(B) = [4, 1] \cap [4, 1] = [4, 1]$.

c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$.
Proof: We suppose directly that $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus $g(x) \in g(A)$ and $g(x) \in g(B)$. In other words, $g(x) \in g(A) \cap g(B)$. Therefore, $g(A \cap B) \subseteq g(A) \cap g(B)$. ■

d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$.
Conjecture: Let $g : \mathbb{R} \to \mathbb{R}$, and $A, B \subseteq \mathbb{R}$, then $g(A \cup B) \subseteq g(A) \cup g(B)$.
Proof: We suppose directly that $g : \mathbb{R} \to \mathbb{R}$, and $A, B \subseteq \mathbb{R}$. Let $x \in A \cup B$, then $x \in A$ and/or $x \in B$. This means that $g(x) \in g(A)$ and/or $g(x) \in g(B)$, thus $g(x) \in g(A) \cup g(B)$. Therefore, $g(A \cup B) \subseteq g(A) \cup g(B)$. ■

Exercise 1.4. Here are two important definitions related to a function $f : A \to B$. The function f is injective if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is surjective if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$. Give an example of each or state that the request is impossible.

a) $f : \mathbb{N} \to \mathbb{N}$ is injective but not surjective.
  a) Define f as $f(x) = x + 1$.

b) $f : \mathbb{N} \to \mathbb{N}$ is surjective but not injective.
  a) Define f as
$$f(x) = \begin{cases} x & \text{if } x < 20 \\ x - 1 & \text{else} \end{cases}$$

c) $f : \mathbb{N} \to \mathbb{N}$ that is bijective.
  a) $f = id_A$. That is the identity map. It is defined as $f(x) = x$.

Exercise 1.5. Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the preimage of B.

a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
  a) Since the domain D is arbitrary, I cannot explicitly find the preimages of A and B, thus I will leave them generic. $f^{-1}(A) = \{x \in D \,:\, f(x) \in A\}$, and $f^{-1}(B) = \{x \in D \,:\, f(x) \in B\}$. Under the assumption that $D \subseteq \mathbb{R}$, then $f^{-1}(A) = [0, 2]$ and $f^{-1}(B) = [-1, 1]$.
  b) We want to show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. We will show this for the function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and arbitrary sets $A, B \subseteq \mathbb{R}$.
  Proof: Since this is an equality statement, we must show inclusion both ways.

$(\subseteq)$: Let $x \in f^{-1}(A \cap B)$, then $f(x) \in A$ and $f(x) \in B$, thus $f(x) \in A \cap B$, which means that $x \in f^{-1}(A \cap B)$. Thus $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.
$(\supseteq)$: Let $y \in f^{-1}(A) \cap f^{-1}(B)$, then $y \in f^{-1}(A)$ and $y \in f^{-1}(B)$, thus there exists an $a \in A$ and $b \in B$ such that $f(y) = a$ and $f(y) = b$. Since a function maps elements in the domain to a unique element in the codomain, it must be that $f(y) = f(y)$, which means that $a = b$. Thus $a \in A$ and $a \in B$. Hence, $y \in f^{-1}(A \cap B)$. This means that $f^{-1}(A \cap B) \supseteq f^{-1}(A) \cap f^{-1}(B)$.
Since inclusion holds both ways, we have that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. ■

c) We want to show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
Proof: Since this is an equality statement, we must show inclusion both ways.
$(\subseteq)$: Let $y \in f^{-1}(A \cup B)$, then $f(y) \in A \cup B$. This means that $f(y) \in A$ and/or $f(y) \in B$. Thus $y \in f^{-1}(A)$ and/or $y \in f^{-1}(B)$. Thus $y \in f^{-1}(A) \subseteq f^{-1}(B)$. Therefore, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
$(\supseteq)$: Let $y \in f^{-1}(A) \cup f^{-1}(B)$, then $f(y) \in A$ and/or $f(y) \in B$. In other words, $f(y) \in A \cup B$, which means that $y \in f^{-1}(A \cup B)$. Therefore, $f^{-1}(A \cup B) \supseteq f^{-1}(A) \cup f^{-1}(B)$.
Since both inclusions hold, we have that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. ■

d) Using the proofs done above. In the case that $f^{-1}(A) = [0, 2]$ and $f^{-1}(B) = [-1, 1]$. We have that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) = [0, 1]$. And $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = [-2, 2]$.

b) The good behavior of preimages demonstrated in part a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap (B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets A, $B \subseteq \mathbb{R}$.
Proof: With no loss in generality, see the proof in part a) with the domain $D = \mathbb{R}$, and codomains $A, B \subseteq \mathbb{R}$. ■