Math 565 - Homework 11

Due Monday April 5, 2021

1. Let $M \subset \mathbb{R}^3$ be a compact, orientable, embedded 2-manifold with the induced metric.
   (a) Show that M cannot have $K \leq 0$ everywhere (**Hint:** look at a point where the distance from the origin takes a maximum.)
   (b) Show that M cannot have $K \geq 0$ everywhere unless $\chi(M) > 0$.
2. Let (M, g) be a Riemannian 2-manifold. A curved polygon on M whose sides are geodesic segments is called a *geodesic polygon.* If g has everywhere nonpositive Gaussian curvature, prove that there are no geodesic polygons with exactly 1 or 2 vertices. Give examples if the curvature hypothesis is not satisfied.
3. A *geodesic triangle* on a Riemannian 2-manifold (M, g) is a three-sided geodesic polygon. Prove that if M has constant Gaussian curvature K, show that the sum of the interior angles of a geodesic triangle γ is equal to $\pi + KA$, where A is the area of the region bounded by γ.
4. An *ideal triangle* in the hyperbolic plane $\mathbb{H}^2$ is a region whose boundary consists of three geodesics, any two of which meet at a common point on the boundary of the disk (in the Poincaré disk model) - or in the upper half-plane model they meet on the boundary or meet at infinity (meaning both geodesics are vertical and would meet at infinity). Compute the area of an ideal triangle of your choice using the model of your choice and show it is π. **Note:** It turns out that all ideal triangles have the same area.