Homework 3

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Exercise 1. Let $F:\mathbb{R}^{n+1}\to\mathbb{R}$ be C^r for some $r\geq 1$. Assume that $c\in\mathbb{R}$ such that $df_p\neq 0$ for all $p\in F^{-1}(c)$. Prove that $F^{-1}(c)$ is a smooth C^r manifold.

Proof: We assume directly that $F:\mathbb{R}^{n+1}\to\mathbb{R}$ be C^r for some $r\geq 1$. We assume also that for $c\in\mathbb{R}$, $df_p\neq 0$ for all $p\in F^{-1}(c)$. We note that $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ are smooth manifolds that have charts $\left(\mathbb{R}^{n+1},\varphi\right)$ and $(\mathbb{R},\psi)$. Since the codomain of F has rank 1, df_p has rank 1 for all $p\in F^{-1}(c)$. According to the implicit function theorem, for every $p=(x,y)$, where $x\in\mathbb{R}^n$ and $y\in\mathbb{R}$, there exists open neighborhoods V_x and V_y, where $x\in V_x$ and $y\in V_y$, and a smooth map $f_{V_x}:V_x\to V_y$ such that for every $x\in V_x$ we have $F\left(x,f_{V_x}(x)\right)=c$. We also note that since F is smooth the preimage is a Hausdorff and 2nd countable.

We can form charts on $F^{-1}(c)$ at p using the charts on $\mathbb{R}^{n+1}$. By restricting the domain of φ do $V_x\times V_y$ we get $\varphi|_{V_x\times V_y}(x,y)=\left(x,f_{V_x}(x)\right)$. With the domain restricted to V_x, the y component is known from the function f_{V_x}. Let $\pi_{V_x}:V_x\times V_y\to V_x$ be the projection function such that $\pi_{V_x}\circ\varphi|_{V_x\times V_y}(x,y)=x$ whose inverse is $\pi_{V_x}^{-1}:V_x\to V_x\times V_y$ defined as

$$\pi_{V_x}^{-1}=\left(x,f_{V_x}(x)\right).$$

Since f_{V_x} is smooth, the projection π_{V_x} and its inverse are smooth maps. We can define new charts on V_x which are $(V_x,\tilde{\varphi}_{V_x})$ with $\tilde{\varphi}_{V_x}=\varphi|_{V_x}$ where $\varphi|_{V_x}$ is the part of φ that only operates on V_x. Since $\varphi|_{V_x\times V_y}$ is smooth, the chart $(V_x,\tilde{\varphi}_{V_x})$ is smooth.

Now suppose that we have two charts $(V_x,\tilde{\varphi}_{V_x})$ and $(U_x,\tilde{\varphi}_{U_x})$ where $p\in V_x\times V_y$, $p\in U_x\times U_u$, $(V_x\times V_y)\cap(U_x\times U_y)\neq\emptyset$ and smooth maps f_{V_x} and f_{U_x} such that every $x\in V_x$ we have $F\left(x,f_{V_x}(x)\right)=c$ and for every $x\in U_x$ we have $F\left(x,f_{U_x}(x)\right)=c$. The existence of the smooth functions f_{V_x} and f_{U_x} are guaranteed by the implicit function theorem. We can construct the map between charts $\Phi:\tilde{\varphi}_{V_x}(V_x\cap U_x)\to\tilde{\varphi}_{U_x}(V_x\cap U_x)$ defined as

$$\Phi=\tilde{\varphi}_{V_x}\circ\pi_{V_x}\circ\pi_{U_x}^{-1}\circ\tilde{\varphi}_{U_x}^{-1}$$

which is smooth since f_{V_x} and f_{U_x} are smooth. Therefore, the preimage of F is Hausdorff, 2nd countable and has compatible smooth C^r charts; hence, it is a C^r manifold. ■

Exercise 2. Let M be a k-dimensional manifold and TS be the set of all points $(x,v)\in TM$ such that $|v|=1$. Prove that $S(M)$ is a $2k-1$-dimensional subbundle of TM called the sphere bundle of M.

Proof: We suppose directly that M is a k-dimensional smooth manifold and TS is the set previously described. Let $v=(v_{1:k^-},v_k)$ where $v_{1:k^-}=(v_1,\ldots,v_{k-1})$ and let $\Phi:\mathbb{R}^{k-1}\times\mathbb{R}\to\mathbb{R}$ be a smooth map defined as $\Phi(v)=\sqrt{1-\sum_{j=1}^{k-1}v_j^2}$. The partial derivative is

$$\frac{\partial\Phi}{\partial v}=\alpha\begin{bmatrix}-v_1 & \cdots & -v_{k-1} & 0\end{bmatrix},$$

with $\alpha=\left(1-\sum_{j=1}^{k-1}v_j^2\right)^{\frac{-1}{2}}$. Under the constraint $|v|=1$, the rank of $\frac{\partial\Phi}{\partial v}$ is always 1. Therefore, according to the implicit function theorem, there exists a neighborhood V_0 containing $v_{1:k^-}$, a neighborhood V_1 containing v_k and a smooth map $f:V_0\to V_1$ with constant rank 1. According to exercise 1, we then know that TS is a manifold. Let $F:TS\to TM$ defined as $F(x,v_{1:k^-})=F\left(x,(v_{1:k^-},f(v_{1:k^-}))\right)$. The function F is injective and constant rank; therefore, it is an immersion. The image of F is simply

$$N=\left\{F\left(x,(v_{1:k^-},1)\right):(x,v_{1:k^-})\in TS\right\}.$$

Note that N inherits a subspace topology from TM; therefore, for every open set $U\in TM$ the set $N\cap U$ is open in N. Let $V\times\{1\}$ be open in N, then $F^{-1}(V\times\{1\})=V$ which is open in TS; hence, F is also a homeomorphism. Therefore, F is an embedding. ■

Exercise 3. Let $G:\mathbb{R}^2\to\mathbb{R}^4$ be given by

$$G(x,y)=\left((r\cos y+1)\cos x,(r\cos y+1)\sin x,r\sin y\cos\frac{x}{2},r\sin y\sin\frac{x}{2}\right).$$

Show this gives an embedding of the Klein bottle.

Proof: Let $M=[0,2\pi]\times[0,2\pi]\subsetneq\mathbb{R}^2$ be the subset of $\mathbb{R}^2$ on which the Klein bottle is defined. That is every $(0,y)\in M$ is identified with $(2\pi,y)$ and every $(x,2\pi)\in M$ is identified with $(2\pi-x,0)$. We suppose by contradiction that $G|_M$ is not injective, then there exists a $a=(x_1,y_1)\,,b=(x_2,y_2)\in M$ such that $a\neq b$ and $G(a)=G(b)$.

$$G(a)=\left((r\cos y_1+1)\cos x_1,(r\cos y_1+1)\sin x_1,r\sin y_1\cos\frac{x_1}{2},r\sin y_1\sin\frac{x_1}{2}\right)$$

and

$$G(b)=\left((r\cos y_2+1)\cos x_2,(r\cos y_2+1)\sin x_2,r\sin y_2\cos\frac{x_2}{2},r\sin y_2\sin\frac{x_2}{2}\right).$$

By assumption

$$\begin{aligned}(r\cos y_1+1)\cos x_1&=(r\cos y_2+1)\cos x_2\\(r\cos y_1+1)\sin x_1&=(r\cos y_2+1)\sin x_2\\r\sin y_1\cos\frac{x_1}{2}&=r\sin y_2\cos\frac{x_2}{2}\\r\sin y_1\sin\frac{x_1}{2}&=r\sin y_2\sin\frac{x_2}{2}.\end{aligned}$$

Note that the last term divided by the third term is

$$\frac{r\sin y_1\sin\frac{x_1}{2}}{r\sin y_1\cos\frac{x_1}{2}}=\frac{r\sin y_2\sin\frac{x_2}{2}}{r\sin y_2\cos\frac{x_2}{2}}$$

which simplifies to

$$\tan\left(\frac{x_1}{2}\right)=\tan\left(\frac{x_2}{2}\right),$$

which can only be possible if $x_1=x_2$. Thus $\cos x_1=\cos(x_2)$. Rearranging the first term and substituting in $\alpha=\cos(x_1)=\cos(x_2)$ we get that

$$(r\cos y_1+1)\,\alpha=(r\cos y_2+1)\,\alpha$$

which is simplified to

$$\cos y_1=\cos y_2,$$

which con only be possible if $y_1=y_2$. This is a contradiction; hence, $G|_M$ is injective.

Next we take the partial derivative of G

$$\frac{\partial G}{\partial(x,y)}=\begin{bmatrix}-(r\cos y+1)\sin x & -r\sin y\cos x\\(r\cos y+1)\cos x & -r\sin y\sin x\\-\frac{1}{2}r\sin y\sin\frac{x}{2} & r\cos y\cos\frac{x}{2}\\\frac{1}{2}r\sin y\cos\frac{x}{2} & r\cos y\sin\frac{x}{2}\end{bmatrix}.$$

For every $p\in M$, the partial derivative of G is rank 2; hence, G gives an immersion of the Klein bottle. Since $G|_M$ is a smooth injective immersion whose domain is compact, $G|_M$ is an embedding; therefore, G is an embedding of the Klein bottle. ■

Exercise 4. Show that $f:S^1\to\mathbb{R}$ given by $f(t)=(\sin(2t)\cos(t)\,,\sin(2t)\sin(t))$ is an immersion. Explain why $f\left(S^1\right)$ is not a submanifold of $\mathbb{R}^2$.

Proof: The partial derivative of f is

$$\frac{\partial f}{\partial t}=\begin{bmatrix}2\cos(2t)\cos(t)-\sin(2t)\sin(t) & 2\cos(2t)\sin(t)+\sin(2t)\cos(t)\end{bmatrix},$$

which can be simplified to

$$\frac{\partial f}{\partial t}=\begin{bmatrix}6\cos^3(t)-4\cos(t) & 2\cos(2t)\sin(t)+\sin(2t)\cos(t)\end{bmatrix}$$

For the partial derivative to be injective we need it to be rank 1 for all $t\in S^1$. Essentially, we must show that $\frac{\partial f}{\partial t}\neq0$ for all $t\in S^1$. We suppose by contradiction that there exists a $t\in S^1$ such that $\frac{\partial f}{\partial t}=0$, then there exists a $t\in S^1$ such that the first and second terms are zero. Setting the first term to zero and solving for t yields

$$t=\pm\arccos\left(\sqrt{\frac{2}{3}}\right),$$

plugging these values of t into the second term yields ≈0.6367 and ≈0.817 which are not zero; thus the partial derivative of f has rank 1 for all values of $t\in S^1$; hence f is an immersion. However, at $t=k\frac{\pi}{2}$ where $k\in\mathbb{Z}$, $f(t)=0$ so it is not injective and cannot be a homeomorphism. Thus f is not an embedding which means that $f\left(S^1\right)$ is not a submanifold of $\mathbb{R}^2$. ■