Midterm

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Exercise 1. Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^{n+1} \{0\}$ be a smooth map, with $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}^{n+1} \backslash \{0\}$. Prove that the map

$$\begin{aligned} P_* : \mathbb{RP}^n &\to \mathbb{RP}^k \\ [x] &\mapsto [P(x)] \end{aligned}$$

is well defined and smooth.

Proof: The projective space $\mathbb{RP}^n$ consists of an equivalence class of points defined by the relation $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{R} \backslash \{0\}$. Thus for an equivalence class $[x]$, it can be equally represented as $[\lambda x]$. According the the function P_*

$$\begin{aligned} P_*([\lambda x]) &= \left[\lambda^d P(x)\right] \\ &= \left[\lambda^d y\right] \\ &= [y], \end{aligned}$$

which shows that P_* maps an equivalence class $[x]$ to an equivalence class $[y]$ regardless of the representation of $[x]$; therefore, it is well defined. By definition of the function P_* it maps an equivalence class $[x]$ using the smooth function P to the equivalence class $[P(x)]$. Since the equivalence class is represented using $x \in \mathbb{R}^{n+1} \backslash \{0\}$, and P is a smooth map on $\mathbb{R}^{n+1} \backslash \{0\}$, P_* is smooth. ■

Exercise 2. If $S^1 = \left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\right\}$, prove that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

Proof: Let $\gamma : (-\epsilon, \epsilon) : S^1$ denote a smooth curve on S^1 such that $\gamma(0) = p$, (U, φ) be a chart on S^1 and $\tilde{\gamma} = \varphi \circ \gamma$ denote the smooth curve in local coordinates. The tangent space $T_p M$ is defined as the set of equivalence class of curves such that $\tilde{\gamma}_1^{'}(0) = \tilde{\gamma}_2^{'}(0)$. As stated in class, this is independent of the chart used, so we can define the equivalence class in any chart. The derivative of the curve $\tilde{\gamma}$ at p is

$$\left.\frac{d\tilde{\gamma}}{dt}\right|_{t=0} = \alpha \frac{\partial}{\partial x},$$

where $\alpha \in \mathbb{R}$. Thus, all of the curves in the same equivalence class can be identified by α in the chart (U, φ). Let $[\gamma]$ represent the equivalence class whose derivative at $t = 0$ in (U, φ) is

$$\alpha \frac{\partial}{\partial x},$$

we can then create a map $f : T_p S^1 \to \{p\} \times \mathbb{R}$ defined as

$$f(p, [\gamma]) = (p, \alpha).$$

which shows that $T_p S^1$ is diffeomorphic to $\{p\} \times \mathbb{R}$ since the map is invertible and smooth in both directions. The tangent bundle is the disjoint union of the tangent spaces; therefore, TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

Note: I know you mentioned in class that for U a neighborhood of p, TU is diffeomorphic to $U \times \mathbb{R}$; but this is not always true on the global scale. I don't understand it well enough to know when it holds globally and when it doesn't. ■

Exercise 3. Let M be a smooth compact manifold (without boundary), of dimension n, and $F : M \to \mathbb{R}^n$ a smooth map. Prove that F cannot be a submersion, and it cannot be an immersion.

Proof: Since M is a smooth compact manifold (without boundary) of dimension n, it is not a subset of $\mathbb{R}^n$ and it can be covered by a finite number of charts denoted (U_i, φ_i). We denote the differential of F as

$$dF = \frac{\partial}{\partial x}\left(F \circ \varphi^{-1}\right).$$

F is a submersion if $\forall p \in M$, dF is surjective. Since M and $\mathbb{R}^n$ are of the same dimension, this requires that the rank of $dF = 2$ for all $p \in M$. F is an immersion if $\forall p \in M$, dF is injective. This also requires that the rank of $dF = 2$ for all $p \in M$. Now since F is smooth, it is continuous and thus $F(M)$ is compact which means it is closed and bounded. Let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve such that $F \circ \gamma$ is a curve that moves straight towards a boundary in $F(M)$ and $\frac{d\gamma}{dt} =$ constant . Then there is a $t \in (-\epsilon, \epsilon)$ such that $\frac{dF \circ \gamma}{dt} = 0$ since $F \circ \gamma$ must stay inside $F(M)$. At this point $\gamma(p)$, the differential dF is not rank 2; therefore, F cannot be a submersion nor an immersion. ■

Exercise 4. Let M be a smooth manifold with Riemannian metric g. Let $\gamma : [a,b] \to M$ be a smooth curve, and $\tau : [c,d] \to [a,b]$ a diffeomorphism. Prove that the length of the curve γ is the same as the length of the curve $\gamma \circ \tau : [c,d] \to M$.

Proof: Let (U,φ) be a chart on M, $\tilde{\gamma}$ be the local representation of γ and $\tilde{\tau}$ be the local representation of $\gamma \circ \tau$ defined as

$$\tilde{\tau} = \tilde{\gamma} \circ \tau.$$

Since τ is a diffeomorphism, it is a differentiable and a homeomorphism. This means that τ is a curve that only moves "forward" from a to b. The length of γ is defined as

$$\ell(\gamma) = \int_a^b g\left(\left.\frac{d\tilde{\gamma}}{dt}\right|_t, \left.\frac{d\tilde{\gamma}}{dt}\right|_t \right)^{1/2} dt,$$

and the length of $\tilde{\tau}$ is

$$\begin{aligned}\ell(\gamma \circ \tau) &= \int_c^d g\left(\left.\frac{\partial\tilde{\gamma}}{\partial x}\right|_{\tau(t)} \frac{d\tau}{dt}, \left.\frac{\partial\tilde{\gamma}}{\partial x}\right|_{\tau(t)} \frac{d\tau}{dt} \right)^{1/2} dt \\ &= \int_c^d g\left(\left.\frac{\partial\tilde{\gamma}}{\partial x}\right|_{\tau(t)}, \left.\frac{\partial\tilde{\gamma}}{\partial x}\right|_{\tau(t)} \right)^{1/2} \left|\frac{d\tau}{dt}\right| dt,\end{aligned}$$

where $\frac{d\tau}{dt}$ is a scalar and we are able to pull it out of the inner product. Notice that for every $\beta \in [c,d]$

$$\left. g\left(\left.\frac{\partial\tilde{\gamma}}{\partial x}\right|_{\tau(t)}, \left.\frac{\partial\tilde{\gamma}}{\partial x}\right|_{\tau(t)} \right)^{1/2} \right|_{t=\beta} = \left. g\left(\frac{d\tilde{\gamma}}{dt}, \frac{d\tilde{\gamma}}{dt} \right)^{1/2} \right|_{t=\tau(\beta)},$$

which represents the height of the function that is being integrated and $\left|\frac{d\tau}{dt}\right|$ is the change of the infinitesimal width of the integral. This is similar to the change of basis where $\left|\frac{d\tau}{dt}\right|$ is the Jacobian acting as the change in volume. Therefore,

$$\ell(\gamma) = \ell(\gamma \circ \tau).$$

■

Exercise 5. Let M be a smooth manifold of dimension n, and let $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ be a collection of coordinate charts which cover M. Suppose that for each $\lambda \in \Lambda$, there are n^2 functions $g^\lambda_{i,j} : U_\lambda \to \mathbb{R}$, where $1 \le i, j \le n$. State a set of necessary and sufficient conditions on the functions $\left\{g^\lambda_{i,j}\right\}$ so that there exists a Riemannian metric g on M whose coordinate description under the chart $(U_\lambda, \varphi_\lambda)$ is given by the n^2 functions $\left\{g^\lambda_{i,j}\right\}$.

A Riemannian metric is an assignment g of an inner product $g(p) = \langle , \rangle_p$ on each tangent space T_pM such that for any coordinate chart (U,φ) on M, $g_{i,j}(p)$ is a smooth function on U for each $1 \le i, j \le n$. An inner product is symmetric, bi-linear, and positive definite.

Let $v, w \in T_pM$ which can be represented as $v = v_i \frac{\partial}{\partial x_i}$ and $w = w_i \frac{\partial}{\partial x\, i}$ in local coordinates. We define the metric g^λ as

$$g^\lambda(v,w) = \sum_{i,j} g^\lambda_{ij} v_i w_j$$

which we can represent in matrix notation

$$g^\lambda(v,w) = [w_1, \ldots, w_n] \begin{bmatrix} g^\lambda_{11} & \cdots & g^\lambda_{1n} \\ \vdots & \ddots & \vdots \\ g^\lambda_{n1} & \cdots & g^\lambda_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

and we will denote as $\left[g^\lambda_{ij}\right]$. By this definition $\left[g^\lambda_{ij}\right]$ is bilinear. We then impose the conditions that $g^\lambda_{ij} = g^\lambda_{ji}$ and that g^λ_{ij} is a smooth function on U_λ such that $\left[g^\lambda_{ij}\right]$ is a positive definite matrix. Lastly, we need the metric to be independent of coordinate chart. Let (U_1, φ_1) and (U_2, φ_2) be coordinate charts on M where $1, 2 \in \Lambda$ and $U_1 \cap U_2 \ne \emptyset$. We can express the vectors v and w in either chart, and we want the measure of their length and angle to be independent of the chart we use. In other words, we want

$$g^1(v,w) = g^2(v,w)$$

for all $v, w \in T_pM$. We can show this relation in matrix notation as

$$[w^1_1, \ldots, w^1_n] \begin{bmatrix} g^1_{11} & \cdots & g^1_{1n} \\ \vdots & \ddots & \vdots \\ g^1_{n1} & \cdots & g^1_{nn} \end{bmatrix} \begin{bmatrix} v^1_1 \\ \vdots \\ v^1_n \end{bmatrix} = [w^2_1, \ldots, w^2_n] \begin{bmatrix} g^2_{11} & \cdots & g^2_{1n} \\ \vdots & \ddots & \vdots \\ g^2_{n1} & \cdots & g^2_{nn} \end{bmatrix} \begin{bmatrix} v^2_1 \\ \vdots \\ v^2_n \end{bmatrix}$$

where the superscripts are used to denote the chart they are expressed in. Using the differential $d\left(\varphi_2 \circ \varphi_1^{-1}\right)$ we can map the vectors from the first chart to the second chart and map them back such that

$$g^1\left(d\left(\varphi_2 \circ \varphi_1^{-1}\right)^{-1} d\left(\varphi_2 \circ \varphi_1^{-1}\right) v^1, d\left(\varphi_2 \circ \varphi_1^{-1}\right)^{-1} d\left(\varphi_2 \circ \varphi_1^{-1}\right) v^2\right) = g^1(v, w).$$

Using this relation we get the last condition

$$\left(d\left(\varphi_2 \circ \varphi_1^{-1}\right)^{-1}\right)^{\top} \left[g_{ij}^1\right] d\left(\varphi_2 \circ \varphi_1^{-1}\right)^{-1} = \left[g_{ij}^2\right].$$

Exercise 6. Define the torus $\mathbb{T}^2$ as the quotient

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{R}^2/\left\{(x,y) \sim (x+n, y+m) \text{ for } m, n \in \mathbb{Z}\right\}.$$

Notice that any open square of the form $(a, a+1) \times (b, b+1) \subset \mathbb{R}^2$ induces a coordinate chart $(U_{a,b}, \varphi_{a,b})$ on $\mathbb{T}^2$ in a natural way. Notice that the map $\omega : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\omega(\alpha, \beta)\left((\cos(2\pi\beta) + 4)\cos(2\pi\alpha), (\cos(2\pi\beta) + 4)\sin(2\pi\alpha), \sin(2\pi\beta)\right)$$

induces a well defined map $\tilde{\omega} : \mathbb{T}^2 \to \mathbb{R}^3$, which is a diffeomorphism onto its image $T_2 = \tilde{\omega}\left(\mathbb{T}^2\right)$. Equip T_2 with the metric induced by the standard metric on $\mathbb{R}^3$. If S is the open square $(0,1) \times (0,1)$ and $S' = \omega(S) \subset T_2$, then the restriction of ω^{-1} to S' is a coordinate chart $\omega^{-1} : S' \to \mathbb{R}^2$ on T_2. Compute the coordinate representation of the metric on T_2 in this chart, as well as the associated Christoffel symbols.

We begin by computing the derivative of ω and representing it in matrix notation.

$$\frac{\partial \omega}{\partial x} = \begin{bmatrix} -2\pi(\cos(2\pi\beta) + 4)\sin(2\pi\alpha) & -2\pi\sin(2\pi\beta)\cos(2\pi\alpha) \\ 2\pi(\cos(2\pi\beta) + 4)\cos(2\pi\alpha) & -2\pi\sin(2\pi\beta)\sin(2\pi\alpha) \\ 0 & 2\pi\cos(2\pi\beta) \end{bmatrix}.$$

Let $v, w \in T\omega^{-1}(S')$, then the induced metric is

$$\begin{aligned}
\left\langle \frac{\partial \omega}{\partial x} v, \frac{\partial \omega}{\partial x} w \right\rangle &= \begin{bmatrix} -2\pi(\cos(2\pi\beta) + 4)\sin(2\pi\alpha) & 2\pi(\cos(2\pi\beta) + 4)\cos(2\pi\alpha) & 0 \\ -2\pi\sin(2\pi\beta)\cos(2\pi\alpha) & -2\pi\sin(2\pi\beta)\sin(2\pi\alpha) & 2\pi\cos(2\pi\beta) \end{bmatrix} \cdot \\
&\quad \begin{bmatrix} -2\pi(\cos(2\pi\beta) + 4)\sin(2\pi\alpha) & -2\pi\sin(2\pi\beta)\cos(2\pi\alpha) \\ 2\pi(\cos(2\pi\beta) + 4)\cos(2\pi\alpha) & -2\pi\sin(2\pi\beta)\sin(2\pi\alpha) \\ 0 & 2\pi\cos(2\pi\beta) \end{bmatrix} \\
&= \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} \\
&= \begin{bmatrix} 4\pi^2(\cos(2\pi\beta) + 4)^2 & \\ 0 & 4\pi^2 \end{bmatrix}.
\end{aligned}$$

with

$$\begin{aligned}
g_{11} &= 4\pi^2(\cos(2\pi\beta) + 4)^2\sin(2\pi\alpha)^2 + 4\pi^2(\cos(2\pi\beta) + 4)^2\cos(2\pi\alpha)^2 \\
g_{22} &= 4\pi^2\sin^2(2\pi\beta)\cos^2(2\pi\alpha) + 4\pi^2\sin^2(2\pi\beta)\sin^2(2\pi\alpha) + 4\pi^2\cos^2(2\pi\beta).
\end{aligned}$$

Simplifying we get that the induced metric is

$$g = \begin{bmatrix} 4\pi^2(\cos(2\pi\beta) + 4)^2 & \\ 0 & 4\pi^2 \end{bmatrix}.$$

The Christoffel symbols are computed as

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right) g^{km}.$$

We note that when $k \neq m$ the term $g^{km} = g_{km} = 0$ and $\Gamma_{ij}^m = \Gamma_{ji}^m$. We will use this to simplify our computations

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{1}{2}\left(\frac{\partial}{\partial \alpha} 4\pi^2(\cos(2\pi\beta) + 4)^2 + \frac{\partial}{\partial \alpha} 4\pi^2(\cos(2\pi\beta) + 4)^2 - \frac{\partial}{\partial \alpha} 4\pi^2(\cos(2\pi\beta) + 4)^2\right) \frac{1}{4\pi^2(\cos(2\pi\beta) + 4)^2} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\Gamma^1_{12} &= \frac{1}{2}\left(\frac{\partial}{\partial\alpha}0 + \frac{\partial}{\partial\beta}4\pi^2\left(\cos\left(2\pi\beta\right)+4\right)^2 - \frac{\partial}{\partial\alpha}0\right)\frac{1}{4\pi^2\left(\cos\left(2\pi\beta\right)+4\right)^2} \\
&= \frac{-8\pi^3\left(\cos\left(2\pi\beta\right)+4\right)\sin\left(2\pi\beta\right)}{4\pi^2\left(\cos\left(2\pi\beta\right)+4\right)^2} \\
&= \frac{-2\pi\sin\left(2\pi\beta\right)}{\left(\cos\left(2\pi\beta\right)+4\right)}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}\left(\frac{\partial}{\partial\beta}0 + \frac{\partial}{\partial\beta}0 - \frac{\partial}{\partial\alpha}4\pi^2\right)\frac{1}{4\pi^2\left(\cos\left(2\pi\beta\right)+4\right)^2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{11} &= \frac{1}{2}\left(\frac{\partial}{\partial\alpha}0 + \frac{\partial}{\partial\alpha}0 - \frac{\partial}{\partial\beta}4\pi^2\left(\cos\left(2\pi\beta\right)+4\right)^2\right)\frac{1}{4\pi^2} \\
&= \frac{-8\pi^3\left(\cos\left(2\pi\beta\right)+4\right)\sin\left(2\pi\beta\right)}{4\pi^2} \\
&= -2\pi\left(\cos\left(2\pi\beta\right)+4\right)\sin\left(2\pi\beta\right)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2}\left(\frac{\partial}{\partial\alpha}4\pi^2 + \frac{\partial}{\partial\beta}0 - \frac{\partial}{\partial\beta}0\right)\frac{1}{4\pi^2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{22} &= \frac{1}{2}\left(\frac{\partial}{\partial\beta}4\pi^2 + \frac{\partial}{\partial\beta}4\pi^2 - \frac{\partial}{\partial\beta}4\pi^2\right)\frac{1}{4\pi^2} \\
&= 0
\end{aligned}
$$

In summary, the induced metric is

$$
g = \begin{bmatrix} 4\pi^2\left(\cos\left(2\pi\beta\right)+4\right)^2 & \\ 0 & 4\pi^2 \end{bmatrix},
$$

and the Christoffel symbols are $\Gamma^2_{22} = \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^1_{11}$ and

$$
\begin{aligned}
\Gamma^2_{11} &= -2\left(\cos\left(2\pi\beta\right)+4\right)\sin\left(2\pi\beta\right) \\
\Gamma^1_{12} &= \frac{-2\sin\left(2\pi\beta\right)}{\left(\cos\left(2\pi\beta\right)+4\right)}
\end{aligned}
$$