Homework 10

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Exercise 1. Show that the origin $(0,0,0)$ of the hyperboloid $r(u,v)=(u,v,auv)$ we have $K=-a^2$ and $H=0$.

The differential of the surface is

$$dr=\begin{bmatrix}1 & 0\\ 0 & 1\\ av & au\end{bmatrix},$$

from which we obtain the first fundamental form or the induced metric

$$\begin{aligned}I_p &= dr'dr\\ &= \begin{bmatrix}a^2v^2+1 & a^2uv\\ a^2uv & a^2u^2+1\end{bmatrix},\end{aligned}$$

the normal unit vector

$$\begin{aligned}N &= \frac{dr_u\times dr_v}{\|dr_u\times dr_v\|}\\ &= \frac{1}{\sqrt{a^2v^2+a^2u^2+1}}\begin{bmatrix}-av\\ -au\\ 1\end{bmatrix},\end{aligned}$$

and its derivative

$$dN=\frac{1}{\left(a^2v^2+a^2u^2+1\right)^{3/2}}\begin{bmatrix}a^3uv & -a\left(a^2u^2+1\right)\\ -a\left(a^2v^2+1\right) & a^3uv\\ -a^2u & -a^2v\end{bmatrix}.$$

Using the derivative of the normal unit vector and the differential of the surface, we compute the second fundamental form

$$\begin{aligned}II_p &= dr'dN\\ &= \frac{1}{\left(a^2v^2+a^2u^2+1\right)^{3/2}}\begin{bmatrix}0 & -\left(a^3u^2+a^3v^2+a\right)\\ -\left(a^3u^2+a^3v^2+a\right) & 0\end{bmatrix}.\end{aligned}$$

Using the first and second fundamental form we can calculate the Gaussian curvature and the mean curvature

$$\begin{aligned}K &= \frac{-\left(a^3u^2+a^3v^2+a\right)^2}{\left(a^2v^2+a^2u^2+1\right)^3\left(\left(a^2v^2+1\right)\left(a^2u^2+1\right)-a^2uv\right)}\\ H &= \frac{1}{2}\frac{2\left(a^3u^2+a^3v^2+a\right)a^2uv}{\left(a^2v^2+a^2u^2+1\right)^3\left(\left(a^2v^2+1\right)\left(a^2u^2+1\right)-a^2uv\right)}.\end{aligned}$$

Evaluating them at $p=(0,0,0)$ yields

$$\begin{aligned}K &= -a^2\\ H &= 0\end{aligned}$$

Exercise 2. Consider Enneper's surface

$$r(u,v)=\left(u-\frac{u^3}{3}+uv^2, v-\frac{v^3}{3}+vu^2, u^2-v^2\right)$$

and show that

1) the coefficients for the first fundamental form are $E=G=\left(1+u^2+v^2\right)^2$ and $F=0$,
2) the coefficients for the second fundamental form are $e=2$, $f=0$, and $g=-2$, and
3) the principle curvatures are

$$k_1=\frac{2}{\left(1+u^2+v^2\right)^2}\text{ and }k_2=-\frac{2}{\left(1+u^2+v^2\right)^2}.$$

The differential of the surface is

$$dr=\begin{bmatrix}-u^2+v^2+1 & 2uv\\ 2uv & u^2-v^2+1\\ 2u & -2v\end{bmatrix},$$

from which we obtain the first fundamental form or the induced metric

$$\begin{aligned} I_p &= dr' dr \\ &= \begin{bmatrix} \left(1+u^2+v^2\right)^2 & 0 \\ 0 & \left(1+u^2+v^2\right)^2 \end{bmatrix}, \end{aligned}$$

the normal unit vector

$$\begin{aligned} N &= \frac{dr_u \times dr_v}{\|dr_u \times dr_v\|} \\ &= \frac{1}{(1+u^2+v^2)} \begin{bmatrix} -2u \\ 2v \\ 1-u^2-v^2 \end{bmatrix}, \end{aligned}$$

and its derivative

$$dN = \frac{1}{(1+u^2+v^2)^2} \begin{bmatrix} -2\left(1+v^2-u^2\right) & 4uv \\ -4uv & 2\left(1+u^2-v^2\right) \\ -4u\left(u^2+v^2-2\right) & -4v\left(u^2+v^2-2\right) \end{bmatrix}.$$

Using the derivative of the normal unit vector and the differential of the surface, we compute the second fundamental form

$$\begin{aligned} II_p &= -dr' dN \\ &= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}. \end{aligned}$$

Using the first and second fundamental forms, we calculate the principal curvatures

$$\begin{aligned} k_1 &= \frac{2\left(1+u^2+v^2\right)^2}{(1+u^2+v^2)^4} \\ &= \frac{2}{(1+u^2+v^2)^2} \\ k_2 &= \frac{-2\left(1+u^2+v^2\right)^2}{(1+u^2+v^2)^4} \\ &= \frac{-2}{(1+u^2+v^2)^2}. \end{aligned}$$

Exercise 3. Using the theorem in the homework assignment, show that no neighborhood of a point in a sphere may be isometrically mapped into a plane.

In homework 8 we computed that the Gaussian curvature of the sphere was nonzero constant and thus equal at every point. Therefore, the curvature R is also the same at every point according to the Bonnet theorem. In order for a neighborhood of a point on the sphere to be isometric to the plane, the point on the sphere and the plane would need to have the same curvature. Since the curvature of a plane is zero and the curvature of a sphere is nonzero constant at every point, there is no such isometry.

Exercise 4. Using the above results show that there exists no surface $r(u,v)$ such that $E = G = 1$, $F = 0$ and $e = 1$, $g = -1$, and $f = 0$.

Proof: Since E, F, G are zero, their partial derivatives are zero. This means that the Christoffel symbols are also zero. Therefore,

$$\Gamma^2_{12} + \Gamma^2_{11} + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} = 0.$$

According to the Gauss equation, this implies that

$$-E\frac{eg-f^2}{EG-F^2} = 0;$$

however

$$-E\frac{eg-f^2}{EG-F^2} = -1\frac{1\cdot 1-0^2}{1\cdot 1-0^2} = -1$$

which is a contradiction; thus there exists no such surface $r(u,v)$. ∎

Exercise 5. Let M_1 and M_2 be Riemannian manifolds, and consider the product $M_1 \times M_2$, with the product metric. Let ∇^1 be the Riemannian connection of M_1 and let ∇^2 be the Riemannian connection of M_2.

1) Show that the Riemannian connection ∇ of $M_1 \times M_2$ is given by $\nabla_{Y_1+Y_2}(X_1+X_2) = \nabla^1_{Y_1} X_1 + \nabla^2_{Y_2} X_2$, $X_1, Y_1 \in \mathcal{X}(M_1)$ and $X_2, Y_2 \in \mathcal{X}(M_2)$.
2) For every $p \in M_1$, the set $(M_2)_p = \{(p,q) \in M_1 \times M_2 \mid q \in M_2\}$ is a submanifold of $M_1 \times M_2$, naturally diffeomorphic to M_2. Prove that $(M_2)_p$ is a totally geodesic submanifold of $M_1 \times M_2$.
3) Let $\sigma(x,y) \subset T_{(p,q)}(M_1 \times M_2)$ be a plane such that $x \in T_p M_1$ and $y \in T_q M_2$. Show that $K(\sigma) = 0$.

Proof: We suppose directly that (M_1, g^1) and (M_2, g^2) are Riemannian manifolds with connections ∇^1 and ∇^2 and dimension n^1 and n^2, and that the Riemannian manifold $M_1 \times M_2$ has the product metric g defined as

$$g(u,v) = g^1(d\pi_1 u, d\pi_1 v) + g^2(d\pi_2 u, d\pi_2 v)$$

where $\pi_i : M_1 \times M_2 \to M_i$ and $u, v \in TM_1 \times M_2$. Let $X, Y \in TM_1 \times M_2$ such that $d\pi_1 X = X_1$, $d\pi_2 X = X_2$, $d\pi_1 Y = Y_1$ and $d\pi_2 Y = Y_2$; in other words $X = X_1 + X_2$ and $Y = Y_1 + Y_2$. In local coordinates, the vector fields can be expressed as $X = x_i e_i$ and $Y = y_i e_i$. The connection ∇ in local coordinates can be written as

$$\nabla_X Y = \sum_{k=1}^{n^1+n^2} \left(\sum_{i,j} x_i y_j \Gamma^k_{ij} + \sum_i x_i \frac{\partial y_k}{\partial e_i} \right) e_k.$$

Since the first n^1 coefficients of X and Y are dependent on M_1 and the last n^2 coefficients of X and Y are only dependent on M_2, the partial derivative

$$\frac{\partial y_k}{\partial e_i} = 0$$

if k and i are not both less than or equal ton^1 or greater thann^1. We also note that since the Christoffel symbols are derived from the metric g. The fact that g doesn't have any cross terms between g^1 and g^2 means that the Christoffel symbols

$$\Gamma^k_{ij} = 0$$

when i, j and k are not all less than or equal ton^1 or greater thann^1. Therefore, we can simplify the connection to

$$\begin{aligned}
\nabla_X Y &= \sum_{k=1}^{n^1} \left(\sum_{i,j} x_i y_j \Gamma^k_{ij} + \sum_i x_i \frac{\partial y_k}{\partial e_i} \right) e_k + \sum_{k=n^1+1}^{n^1+n^2} \left(\sum_{i,j} x_i y_j \Gamma^k_{ij} + \sum_i x_i \frac{\partial y_k}{\partial e_i} \right) e_k \\
&= \nabla^1_{d\pi_1 X} d\pi_1 Y + \nabla^2_{d\pi_2 X} d\pi_2 Y \\
&= \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2.
\end{aligned}$$

The submanifold $(M_2)_p$ is totally geodesic if any geodesic on the submanifold $(M_2)_p$ with its induced Riemannian metric g is also a geodesic on the Riemannian manifold $M_1 \times M_2$. Let $\gamma^2 : I \to M_2$ be a geodesic on M_2 and let $\gamma : I \to (M_2)_p$ be a geodesic on $(M_2)_p$ defined as

$$\gamma(t) = \left(p, \gamma^2(t)\right).$$

Note that γ is also a curve on $M_1 \times M_2$. We want to show that if γ is a geodesic on $(M_2)_p$, then it is also a geodesic on $M_1 \times M_2$. Let $\gamma(t) = x^i(t) e_i$ in local coordinates. The differential equation of γ on $(M_2)_p$ is

$$\left(\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right) e_k = 0.$$

Since $\gamma(t) = \left(p, \gamma^2(t)\right)$, the first n^1 terms are zero, also since $\Gamma^k_{ij} = 0$ when i, j and k are not all less than or equal ton^1 or greater thann^1, this differential simplifies to

$$\nabla^2_{d\pi_2 \gamma'} d\pi_2 \gamma' = 0.$$

On $M_1 \times M_2$, we need $\nabla_{\gamma'} \gamma' = 0$. Expanding it out we get

$$\begin{aligned}
\nabla_{\gamma'} \gamma' &= \nabla^1_{d\pi_1 \gamma'} d\pi_1 \gamma' + \nabla^2_{d\pi_1 \gamma'} d\pi_1 \gamma' \\
&= \nabla^2_{d\pi_2 \gamma'} d\pi_2 \gamma' \\
&= 0,
\end{aligned}$$

since $d\pi_1 \gamma' = 0$. Therefore, if γ is a geodesic on $(M_2)_p$, then it is a geodesic on M_1.

Next we suppose that $\sigma(x,y) \subset T_{(p,q)}(M_1 \times M_2)$ is a plane such that $x \in T_p M_1$ and $y \in T_q M_2$. The sectional curvature is defined as

$$K(\sigma(x,y)) = \frac{g(R(x,y)x, y)}{\|x \wedge y\|}$$

where R is the curvature defined as

$$R(x,y)x = \nabla_y \nabla_x x - \nabla_x \nabla_y x + \nabla_{[x,y]} x.$$

Since $x \in T_pM_1$ and $y \in T_qM_2$ and by the definition of the connection: $\nabla_y x = 0$ and $\nabla_x y = 0$. From the symmetry property of the connection, we get $[x, y] = 0$. Lastly, since $\nabla_x x \in T_pM_1$, $\nabla_y \nabla_x x = 0$. Therefore $R(x, y)x = 0$ and the sectional curvature is also zero. ■

Exercise 6. (The Clifford torus). Consider the immersion $r : \mathbb{R}^2 \to \mathbb{R}^4$ defined as

$$r(\theta, \phi) = \frac{1}{\sqrt{2}}(\cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi)).$$

1) Show that the vectors

$$e_1 = (-\sin(\theta), \cos(\theta), 0, 0)$$
$$e_2 = (0, 0, -\sin(\phi), \cos(\phi))$$

form an orthonormal basis of the tangent space, and that the vectors

$$n_1 = \frac{1}{\sqrt{2}}(\cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi))$$
$$n_2 = \frac{1}{\sqrt{2}}(-\cos(\theta), -\sin(\theta), \cos(\phi), \sin(\phi))$$

form an orthonormal basis of the normal space.

2) Use the fact that

$$\langle S_{n_k}(e_i), e_j\rangle = -\langle \bar{\nabla}_{e_i} n_k, e_j\rangle = \langle \bar{\nabla}_{e_i} e_j, n_k\rangle,$$

where $\bar{\nabla}$ is the covariant derivative (that is the usual derivative of $\mathbb{R}^4$), and $i, j, k = 1, 2$ to establish that the matrices of S_{n_1} and S_{n_2} with respect to the basis $\{e_1, e_2\}$ are

$$s_{n_1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$s_{n_2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proof: We begin by showing part 1). Since $\mathbb{R}^4$ is 4 dimensions, it's tangent space is also 4 dimensions; thus, a basis is composed of four vectors. Now, the differential of the map r is

$$dr = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sin(\theta) & 0 \\ \cos(\theta) & 0 \\ 0 & -\sin(\phi) \\ 0 & \cos(\phi) \end{bmatrix}.$$

From the differential dr, we get that the vectors

$$E_1 = \frac{1}{\sqrt{2}}(-\sin(\theta), \cos(\theta), 0, 0)$$
$$E_2 = \frac{1}{\sqrt{2}}(0, 0, -\sin(\phi), \cos(\phi))$$

span the tangent plane and that they are orthogonal to each other. Making each one orthonormal gives us the orthonormal basis vectors e_1 and e_2. Now we proceed to show that n_1 and n_2 are orthonormal vectors that span the normal space. It is easily shown that $\sqrt{\langle n_1, n_1\rangle} = 1$, and $\sqrt{\langle n_2, n_2\rangle} = 1$ which indicates that they are unit vectors. Next we perform the calculations to show that they are all orthogonal to each other

$$\langle e_1, e_2\rangle = 0$$
$$\langle e_1, n_1\rangle = 0$$
$$\langle e_1, n_2\rangle = 0$$
$$\langle e_2, n_1\rangle = 0$$
$$\langle e_2, n_2\rangle = 0$$
$$\langle n_1, n_2\rangle = 0.$$

Since each vector in $\{e_1, e_2, n_1, n_2\}$ is orthonormal, they must span the space $T\mathbb{R}^4$. Now since $\{e_1, e_2\}$ span the tangent space, $\{n_1, n_2\}$ must span the normal space.

We now proceed to show part 2). The shape operator in $\mathbb{R}^4$ can be simplified to

$$\begin{aligned} s_{n_k} e_i &= -\left(\bar{\nabla}_{e_i} n_k\right)^\top \\ &= a^\ell \frac{\partial}{\partial x_\ell}\left(b^j\right)\frac{\partial}{\partial x_j} \end{aligned}$$

where $e_i = a^\ell \frac{\partial}{\partial x_\ell}$ and $n_k = b^j \frac{\partial}{\partial x_j}$ since all of the Christoffel symbols are zero. We now calculate the different combinations by noting that from the parameterization of the surface

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}\cos(\theta) \\ x_2 &= \frac{1}{\sqrt{2}}\sin(\theta) \\ x_3 &= \frac{1}{\sqrt{2}}\cos(\phi) \\ x_4 &= \frac{1}{\sqrt{2}}\sin(\phi) \end{aligned}$$

which allows us to write the surface as

$$r(\theta, \phi) = (x_1, x_2, x_3, x_4).$$

Using this, we can easily see that

$$\frac{\partial}{\partial x_i}(n_1) = \frac{\partial}{\partial x_i}$$

and

$$\frac{\partial}{\partial x_i}(n_2) = \begin{cases} -\frac{\partial}{\partial x_i} & i = 1,2 \\ \frac{\partial}{\partial x_i} & i = 3,4 \end{cases};$$

thus

$$\begin{aligned} \nabla_{e_1} n_1 &= e_1 \\ \nabla_{e_2} n_1 &= e_2 \\ \nabla_{e_1} n_2 &= -e_1 \\ \nabla_{e_2} n_2 &= e_2 \end{aligned}$$

which gives us

$$\begin{aligned} s_{n_1} e_1 &= -e_1 \\ s_{n_1} e_2 &= -e_2 \\ s_{n_2} e_1 &= e_1 \\ s_{n_2} e_2 &= -e_2; \end{aligned}$$

therefore, the matrices of S_{n_1} and S_{n_2} with respect to the basis $\{e_1, e_2\}$ are

$$\begin{aligned} s_{n_1} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ s_{n_2} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

$\blacksquare$