Homework 6

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Exercise 1. Let M be a Riemannian manifold. Consider the mapping

$$P = P_{c,t_o,t} : T_{c(t_0)}M \to T_{c(t)}M$$

defined by: $P_{c,t_o,t}(v), v \in T_{c(t_o)}M$, is the vector obtained by parallel transporting the vector v along the curve c. Show that P is an isometry and that, if M is oriented, P preserves the orientation.

Proof: Let $v_{t_0}, w_{t_0} \in T_{c(t_o)}M$ and $g_p : T_pM \times T_pM \to \mathbb{R}$ be a compatible Riemannian metric. Then

$$g_{c(t_0)}(v_{t_0}, w_{t_0}) = a$$

is equal to a constant a. Let $v(t) = P_{c,t_o,t}(v_{t_0})$ and $w(t) = P_{c,t_o,t}(w_{t_0})$. Taking the derivative of the metric applied to the vector fields yields

$$\begin{aligned}
\frac{d}{dt} g_{c(t)}(v(t), w(t)) &= g_{c(t)}\left(\frac{Dv(t)}{dt}, w(t)\right) + g_{c(t)}\left(v(t), \frac{Dw(t)}{dt}\right) \\
&= g_{c(t)}(0, w(t)) + g_{c(t)}(v(t), 0) \\
&= 0,
\end{aligned}$$

thus, the quantity $g_{c(t)}(v(t), w(t))$ is constant along the curve $c(t)$. This means that the length and angles of vectors that are parallel transported remain the same, which means that $P_{c,t_o,t}$ is an isometry. Since the angles between vectors stay the same, P preserves orientation. ■

Exercise 2. Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in M$ and let $c : I \to M$ be an integral curve of X through p, i.e. $c(t_0) = p$ and $\frac{dc}{dt} = X(c(t))$. Prove that the Riemannian connection of M is

$$(\nabla_X Y)(p) = \left.\frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(t)))\right)\right|_{t=t_0},$$

where $P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ is the parallel transport along c, from t_0 to t (this shows how the connection can be reobtained from the concept of parallelism).

Proof: Let X_i form a basis of vector fields in local coordinates that are parallel transport along c. We can write the vector field Y along c in local coordinates as

$$Y(t) = \sum y_i(t) X_i(t),$$

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$$\begin{aligned}
(\nabla_X Y)(p) = \frac{DY(t_0)}{dt} &= \sum \left.\frac{y_i(t)}{dt}\right|_{t=t_0} X_i(t_0) + y_i(t)\frac{DX_i}{dt} \\
&= \sum \left.\frac{y_i(t)}{dt}\right|_{t=t_0} X_i(t_0).
\end{aligned}$$

The term $\frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(t)))\right)\big|_{t=t_0}$ can be expanded out as

$$\begin{aligned}
\left.\frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(t)))\right)\right|_{t=t_0} &= \left.\frac{d}{dt}\left(P^{-1}_{c,t_0,t}\left(\sum y_i(t) X_i(t)\right)\right)\right|_{t=t_0} \\
&= \left.\frac{d}{dt}\sum y_i(t) P^{-1}_{c,t_0,t}(X_i(t))\right|_{t=t_0} \\
&= \left.\frac{d}{dt}\sum y_i(t) X_i(t_0)\right|_{t=t_0} \\
&= \sum \left.\frac{y_i(t)}{dt}\right|_{t=t_0} X_i(t_0).
\end{aligned}$$

■

Exercise 3. Let $M^2 \subset \mathbb{R}^3$ be a surface in $\mathbb{R}^3$ with the induced Riemannian metric. Let $c : I \to M$ be a differentiable curve on M and let V be a vector field tangent to M along c; V can be thought of as a smooth function $V : I \to \mathbb{R}^3$, with $V(t) \in T_{c(t)}M$.

1) Show that V is parallel if and only if $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}M \subset \mathbb{R}^3$ where $\frac{dV}{dt}$ is the usual derivative of $V : I \to \mathbb{R}^3$.
2) If $S^2 \subset \mathbb{R}^3$ is the unit sphere of $\mathbb{R}^3$, show that the velocity field along great circles, parametrized by arc length, is a parallel field. A similar argument holds for $S^n \subset \mathbb{R}^{n+1}$.

Proof: The first statement is biconditional and so we must prove both ways.

$(\implies)$: We suppose directly that V is parallel along the curve $c(t)$, and let g denote the induced metric. By definition $\frac{DV}{dt} = 0$ which is the orthogonally projected vector of $\frac{dv}{dt}$ onto $T_{c(t)}M$.

$(\impliedby)$: We suppose directly that $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}M$, then the orthogonal projection of $\frac{dV}{dt}$ onto $T_{c(t)}M$ is zero, and thus $\frac{DV}{dt} = 0$. This implies that V is parallel along c.

For the second statement, we suppose directly that $S^2 \subset \mathbb{R}^3$. Let $\gamma : I \to S^2$ be an integral curve along a great circle defined as $\gamma(t) = (u(t), v(t))$, and let $c : I \to S^2$ be the rotated curve of γ such that c is the curve along the equator. Since the rotation doesn't affect the velocity of γ, γ is a parallel curve if and only if c is a parallel curve.

We can parameterize the surface by

$$r(u, v) = (\sin u(t) \cos v(t), \sin u(t) \sin v(t), \cos u(t)),$$

then

$$c(t) = (\cos(\alpha t), \sin(\alpha t), 0),$$

where $\alpha \in \mathbb{R}$ which is the arc length.

The derivative of $c(t)$ is

$$\frac{dc}{dt} = V(t) = (-\alpha \sin(\alpha t), \alpha \cos(\alpha t), 0).$$

which is a vector field along the great circle. Taking the derivative again yields

$$\frac{dV}{dt} = \left(-\alpha^2 \cos(\alpha t), -\alpha^2 \sin(\alpha t), 0\right).$$

We know that $c(t)$is orthogonal to $T_{c(t)}M$. We can project $\frac{DV}{dt}$ onto $T_{c(t)}M$ using the projection theorem

$$\begin{aligned}
\left(I - \frac{cc\top}{|c|^2}\right)\frac{dV}{dt} &= \begin{bmatrix} \sin^2(\alpha t) & -\cos(\alpha t)\sin(\alpha t) & 0 \\ -\cos(\alpha t)\sin(\alpha t) & \cos^2(\alpha t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\alpha^2 \cos(\alpha t) \\ -\alpha^2 \sin(\alpha t) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\alpha^2 \sin^2(\alpha t)\cos(\alpha t) - \alpha^2 \cos(\alpha t)\sin^2(\alpha t) \\ \alpha^2 \cos^2(\alpha t)\sin(\alpha t) - \alpha^2 \cos^2(\alpha t)\sin(\alpha t) \\ 0 \end{bmatrix} \\
&= 0.
\end{aligned}$$

Therefore, the velocity field along a great circle is a parallel field. ∎

Exercise 4. Consider the upper half-plane

$$\mathbb{R}^2_+ = \left\{(x, y) \in \mathbb{R}^2 \,|\, y > 0\right\}$$

with the metric given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = g_{21} = 0$

1) Show that the Christoffel symbols of the Riemannian connection are: $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{11} = \frac{1}{y}$, $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$.
2) Let $v_0 = (0, 1)$ be a tangent vector at the point $(0, 1)$ of $\mathbb{R}^2_+$, (v_0 is a unit vector on the y-axis with origin at $(0, 1)$). Let $v(t)$ be the parallel transport of v_0 along the curve $x = t$, $y = 1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

We can calculate the Christoffel symbols using the equation

$$\Gamma^m_{ij} = \frac{1}{2}\sum_k \left\{\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right\} g^{km},$$

where g^{km} is the inverse metric of g_{km} with coefficients $g^{11} = g^{22} = y^2$, $g^{12} = g^{21} = 0$.

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_1} g_{11} + \frac{\partial}{\partial x_1} g_{11} - \frac{\partial}{\partial x_1} g_{11}\right\} g^{11} + \frac{1}{2}\left\{\frac{\partial}{\partial x_1} g_{12} + \frac{\partial}{\partial x_1} g_{21} - \frac{\partial}{\partial x_2} g_{11}\right\} g^{21} \\
&= \frac{1}{2}\left\{\frac{\partial}{\partial x_1}\frac{1}{y^2} + \frac{\partial}{\partial x_1}\frac{1}{y^2} - \frac{\partial}{\partial x_1}\frac{1}{y^2}\right\} y^2 + \frac{1}{2}\left\{\frac{\partial}{\partial x_1} 0 + \frac{\partial}{\partial x_1} 0 - \frac{\partial}{\partial x_2}\frac{1}{y^2}\right\} 0 \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\Gamma_{12}^{2} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_1}g_{21}+\frac{\partial}{\partial x_2}g_{11}-\frac{\partial}{\partial x_1}g_{12}\right\}g^{12}+\frac{1}{2}\left\{\frac{\partial}{\partial x_1}g_{22}+\frac{\partial}{\partial x_2}g_{21}-\frac{\partial}{\partial x_2}g_{12}\right\}g^{22}\\
&= \frac{1}{2}\left\{\frac{\partial}{\partial x_1}0+\frac{\partial}{\partial x_2}\frac{1}{y^2}-\frac{\partial}{\partial x_1}0\right\}0+\frac{1}{2}\left\{\frac{\partial}{\partial x_1}\frac{1}{y^2}+\frac{\partial}{\partial x_2}0-\frac{\partial}{\partial x_2}0\right\}y^2\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{22}^{1} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_2}g_{21}+\frac{\partial}{\partial x_2}g_{12}-\frac{\partial}{\partial x_1}g_{22}\right\}g^{11}+\frac{1}{2}\left\{\frac{\partial}{\partial x_2}g_{22}+\frac{\partial}{\partial x_2}g_{22}-\frac{\partial}{\partial x_2}g_{22}\right\}g^{21}\\
&= \frac{1}{2}\left\{\frac{\partial}{\partial x_2}0+\frac{\partial}{\partial x_2}0-\frac{\partial}{\partial x_1}\frac{1}{y^2}\right\}y^2+\frac{1}{2}\left\{\frac{\partial}{\partial x_2}\frac{1}{y^2}+\frac{\partial}{\partial x_2}\frac{1}{y^2}-\frac{\partial}{\partial x_2}\frac{1}{y^2}\right\}0\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{11}^{2} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_1}g_{11}+\frac{\partial}{\partial x_1}g_{11}-\frac{\partial}{\partial x_1}g_{11}\right\}g^{12}+\frac{1}{2}\left\{\frac{\partial}{\partial x_1}g_{12}+\frac{\partial}{\partial x_1}g_{21}-\frac{\partial}{\partial x_2}g_{11}\right\}g^{22}\\
&= \frac{1}{2}\left\{\frac{\partial}{\partial x_1}\frac{1}{y^2}+\frac{\partial}{\partial x_1}\frac{1}{y^2}-\frac{\partial}{\partial x_1}\frac{1}{y^2}\right\}0+\frac{1}{2}\left\{\frac{\partial}{\partial x_1}0+\frac{\partial}{\partial x_1}0-\frac{\partial}{\partial x_2}\frac{1}{y^2}\right\}y^2\\
&= \frac{1}{2}\left(-\left(-2\frac{1}{y^3}\right)\right)y^2\\
&= \frac{1}{y}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{12}^{1} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_1}g_{21}+\frac{\partial}{\partial x_2}g_{11}-\frac{\partial}{\partial x_1}g_{12}\right\}g^{11}+\frac{1}{2}\left\{\frac{\partial}{\partial x_1}g_{22}+\frac{\partial}{\partial x_2}g_{21}-\frac{\partial}{\partial x_2}g_{12}\right\}g^{21}\\
&= \frac{1}{2}\left\{\frac{\partial}{\partial x_1}0+\frac{\partial}{\partial x_2}\frac{1}{y^2}-\frac{\partial}{\partial x_1}0\right\}y^2+\frac{1}{2}\left\{\frac{\partial}{\partial x_1}\frac{1}{y^2}+\frac{\partial}{\partial x_2}0-\frac{\partial}{\partial x_2}0\right\}0\\
&= \frac{1}{2}\left(-2\frac{1}{y^3}\right)y^2\\
&= -\frac{1}{y}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{22}^{2} &= \frac{1}{2}\left\{\frac{\partial}{\partial x_2}g_{21}+\frac{\partial}{\partial x_2}g_{12}-\frac{\partial}{\partial x_1}g_{22}\right\}g^{12}+\frac{1}{2}\left\{\frac{\partial}{\partial x_2}g_{22}+\frac{\partial}{\partial x_2}g_{22}-\frac{\partial}{\partial x_2}g_{22}\right\}g^{22}\\
&= \frac{1}{2}\left\{\frac{\partial}{\partial x_2}0+\frac{\partial}{\partial x_2}0-\frac{\partial}{\partial x_1}\frac{1}{y^2}\right\}0+\frac{1}{2}\left\{\frac{\partial}{\partial x_2}\frac{1}{y^2}+\frac{\partial}{\partial x_2}\frac{1}{y^2}-\frac{\partial}{\partial x_2}\frac{1}{y^2}\right\}y^2\\
&= \frac{1}{2}\left(-2\frac{1}{y^3}-2\frac{1}{y^3}+2\frac{1}{y^3}\right)y^2\\
&= -\frac{1}{y}
\end{aligned}
$$

Using the Christoffel symbols, we can then create the differential equations for the parallel vector whose ode must satisfy the equation

$$0=\frac{dv^k}{dt}+\sum_{i,j}\Gamma_{ij}^{k}v^j\frac{dx_i}{dt},\quad k=1,2$$

with initial conditions $v(t_0)=(0,1)$, and that moves along the curve $\gamma(t)=(t,1)$. The parallel vector field can be written as $v(t)=a(t)\frac{\partial}{\partial x_1}+b(t)\frac{\partial}{\partial x_2}$, thus the ode must satisfy

$$
\begin{aligned}
0 &= \left(\frac{da}{dt}+\Gamma_{11}^{1}a(t)\frac{dx_1}{dt}+\Gamma_{12}^{1}b(t)\frac{dx_1}{dt}+\Gamma_{21}^{1}a(t)\frac{dx_2}{dt}+\Gamma_{22}^{1}b(t)\frac{dx_2}{dt}\right)\frac{\partial}{\partial x_1}\\
&= \left(\frac{da}{dt}-\frac{1}{y}b(t)\right)\frac{\partial}{\partial x_1}
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \left(\frac{db}{dt}+\Gamma_{11}^{2}a(t)\frac{dx_1}{dt}+\Gamma_{12}^{2}b(t)\frac{dx_1}{dt}+\Gamma_{21}^{2}a(t)\frac{dx_2}{dt}+\Gamma_{22}^{2}b(t)\frac{dx_2}{dt}\right)\frac{\partial}{\partial x_2}\\
&= \left(\frac{db}{dt}+\frac{1}{y}a(t)\right)\frac{\partial}{\partial x_2}
\end{aligned}
$$

Using the hint in the book, we get that

$$a(t) = \cos\left(\frac{\pi}{2} - t\right)$$
$$b(t) = \sin\left(\frac{\pi}{2} - t\right),$$

thus

$$v(t) = \cos\left(\frac{\pi}{2} - t\right)\frac{\partial}{\partial x_1} + \sin\left(\frac{\pi}{2} - t\right)\frac{\partial}{\partial x_2}.$$

The angle between $v(t)$ and the y axis is calculated as

$$\begin{aligned}\cos^{-1}\frac{\left\langle v(t), \frac{\partial}{\partial x_2}\right\rangle}{\sqrt{\langle v(t)\, v(t)\rangle}\sqrt{\left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\right\rangle}} &= \cos^{-1}\left\{\frac{1}{\sqrt{\left(\cos^2\left(\frac{\pi}{2} - t\right) + \sin^2\left(\frac{\pi}{2} - t\right)\right)y}}\left(\cos\left(\frac{\pi}{2} - t\right)\left\langle\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\rangle + \sin\left(\frac{\pi}{2} - t\right)\left\langle\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\right\rangle\right.\right. \\ &= \cos^{-1}\left(\sin\left(\frac{\pi}{2} - t\right)\frac{1}{y^3}\right) \\ &= \cos^{-1}\left(\sin\left(\frac{\pi}{2} - t\right)\right) \\ &= \cos^{-1}(\cos(t)) \\ &= t.\end{aligned}$$

Exercise 5. Let M be a smooth path-connected manifold with Riemannian metric g. Recall that $\ell_a^b(\gamma)$ denotes the length of a smooth curve $\gamma : [a,b] \to M$. For any $p, q \in M$, let

$$d(p,q) = \inf\left\{\ell_a^b(\gamma) \mid \gamma : [a,b] \to M \text{ is piecewise smooth with } \gamma(a) = p \text{ and } \gamma(b) = q\right\}.$$

Prove that the pair (M, d) is a metric space.

Proof: A metric $d : M \times M \to \mathbb{R}$ must satisfy the following properties ■

1) $d(p,q) = d(q,p)$
2) $d(p,q) \geq 0$
3) $d(p,q) = 0$ if and only if $x = y$
4) For all points $p, q, z \in M$, $d(p,z) \leq d(p,q) + d(q,z)$.

The length of a smooth curve is defined as

$$\ell_a^b(\gamma) = \int_a^b g_{\gamma(t)}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)^{\frac{1}{2}} dt.$$

To prove the first property, suppose that γ is the curve from p to q with minimum length, then

$$\begin{aligned}d(p,q) &= \ell_a^b(\gamma) \\ &= \int_a^b g_{\gamma(t)}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)^{\frac{1}{2}} dt.\end{aligned}$$

We can construct another curve that goes from q to p as $\alpha(t) = \gamma(b - t)$ where $t \in [0, b - a]$, then

$$\begin{aligned}d(q,p) &= \ell_b^a(\alpha) \\ &= \int_0^{b-a} g_{\alpha(t)}\left(\frac{d\alpha}{dt}, \frac{d\alpha}{dt}\right)^{\frac{1}{2}} dt \\ &= \int_0^{b-a} g_{\gamma(b-t)}\left(-\frac{d\gamma}{dt}, -\frac{d\gamma}{dt}\right)^{\frac{1}{2}} dt \\ &= \int_0^{b-a} g_{\gamma(b-t)}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)^{\frac{1}{2}} dt.\end{aligned}$$

Since the integral is just a sum of positive numbers, the two integrals are equivalent.

The second and third property come from the fact that the Riemannian metric g is an inner product. Thus

$$g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) \geq 0,$$

and is only equal to zero when $\frac{d\gamma}{dt} = 0$. Thus the length of a curve is greater than or equal to zero, and is only equal to zero when

$$\int_a^b g_{\gamma(t)} \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right)^{\frac{1}{2}} dt = 0.$$

This implies that all of the velocities along the curve must be zero, thus $p = q$. Also, if $p = q$, then the shortest curve between them is the curve $\gamma(t) = p = q$ which is constant; thus $\frac{d\gamma}{dt} = 0$ and $\int_a^b g_{\gamma(t)} \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right)^{\frac{1}{2}} dt = 0$ which implies that $d(x, y) = 0$.

For the thrid property, let $\gamma_{i,j}$ be the curve from $i \in \{p, q, r\}$ to $j \in \{p, q, r\}$ such that $d(i, j) = \ell_a^b(\gamma_{i,j})$. We can construct a new curve $\beta : [a, 2b - a] \to M$ such that

$$\beta(t) = \begin{cases} \gamma_{p,q}(t) & \text{if } t \in [a, b] \\ \gamma_{q,z}(t - b) & \text{else} \end{cases}.$$

which is the curve that goes from p to q, then to z along the curves that are shortest distance from p to q and from q to p. The length of β is

$$\begin{aligned} \ell_a^{2b-a}(\beta) &= \ell_a^b(\gamma_{p,q}) + \ell_a^b(\gamma_{q,z}) \\ &= d(p, q) + d(q, z). \end{aligned}$$

By definition of the metric,

$$\begin{aligned} d(p, z) &= \ell_a^b(\gamma_{p,z}) \\ &\leq \ell_a^{2b-a}(\beta), \end{aligned}$$

since $\gamma_{p,z}$ is the shortest path between p and z and must be shorter than or equal to the curve β, thus $d(p, z) \leq d(p, q) + d(q, z)$.

Exercise 6. Let $F : M \to N$ be a smooth immersion, and let g_N be a Riemannian metric on N. Let (U, φ) and (V, ψ) be charts on M and N with $F(U) \subset V$,a nd let $(g_N)_{i,j}$ be the local coordinate description of g_N under the chart (V, ψ). If g_M is the metric induced on M from the immersion F, describe how the local coordinate description of g_M under (U, φ) denoted by $(g_M)_{i,j}$ are related to functions $(g_N)_{i,j}$.

Let $u_i \in TU$, then the differential $df : TU \to TM$, thus we can let $v_i = df u_i$. We can then use the metric g_N.

$$g_N(v_i, v_j) = g_N(df v_i, df v_j).$$

Let $[g_N]$ be the matrix form of g_N, then we can write the above equation in matrix notation as

$$v_j^\top df^\top [g_N] df v_i.$$

Thus $[g_M] = df^\top [g_N] df$.