Homework 5

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Exercise 1. Compute the first fundamental form for the following surfaces

1) $r(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$
2) $r(u,v) = \left(au\cos v, bu\sin v, u^2\right)$

Each surface is a surface in $\mathbb{R}^3$ parameterized by $r(u,v)$ which is an immersion. The first fundamental form allows us pull back the standard inner product defined on $\mathbb{R}^3$ to the surface S. Let $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$, and $z = \gamma'(0)$. Then, we can measure the length of z using the standard inner product defined on $\mathbb{R}^3$ as

$$\begin{aligned}
\left\langle \frac{d}{dt}(r\circ\gamma)\Big|_{t=0}, \frac{d}{dt}(r\circ\gamma)\Big|_{t=0}\right\rangle &= \left\langle \frac{\partial r}{\partial x}\frac{d\gamma}{dt}\Big|_{t=0}, \frac{\partial r}{\partial x}\frac{d\gamma}{dt}\Big|_{t=0}\right\rangle \\
&= \left(\frac{\partial r}{\partial x}\frac{d\gamma}{dt}\Big|_{t=0}\right)^\top \left(\frac{\partial r}{\partial x}\frac{d\gamma}{dt}\Big|_{t=0}\right) \\
&= \frac{d\gamma}{dt}\Big|_{t=0}^\top \left(\frac{\partial r}{\partial x}^\top \frac{\partial r}{\partial x}\right)\frac{d\gamma}{dt}\Big|_{t=0}
\end{aligned}$$

where the fundamental form is given by $\left(\frac{\partial r}{\partial x}^\top \frac{\partial r}{\partial x}\right)$.

In the first problem, we have

$$\frac{\partial r}{\partial x} = \begin{bmatrix} a\cos u\cos v & -a\sin u\sin v \\ b\cos u\sin v & b\sin u\cos v \\ -c\sin u & 0 \end{bmatrix},$$

thus

$$\begin{aligned}
\left(\frac{\partial r}{\partial x}^\top \frac{\partial r}{\partial x}\right) &= \begin{bmatrix} a\cos u\cos v & b\cos u\sin v & -c\sin u \\ -a\sin u\sin v & b\sin u\cos v & 0 \end{bmatrix}\begin{bmatrix} a\cos u\cos v & -a\sin u\sin v \\ b\cos u\sin v & b\sin u\cos v \\ -c\sin u & 0 \end{bmatrix} \\
&= \begin{bmatrix} a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u & -a^2\cos u\sin u\sin v\cos v + b^2\sin u\cos u\sin v\cos u \\ -a^2\cos u\sin u\sin v\cos v + b^2\sin u\cos u\sin v\cos u & a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v \end{bmatrix},
\end{aligned}$$

and the fundamental form is

$$\begin{aligned}
E &= a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u \\
F &= -a^2\cos u\sin u\sin v\cos v + b^2\sin u\cos u\sin v\cos u \\
G &= a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v.
\end{aligned}$$

In the second problem we have $r(u,v) = \left(au\cos v, bu\sin v, u^2\right)$

$$\frac{\partial r}{\partial x} = \begin{bmatrix} a\cos v & -au\sin v \\ b\sin v & bu\cos v \\ 2u & 0 \end{bmatrix},$$

thus

$$\begin{aligned}
\left(\frac{\partial r}{\partial x}^\top \frac{\partial r}{\partial x}\right) &= \begin{bmatrix} a\cos v & b\sin v & 2u \\ -au\sin v & bu\cos v & 0 \end{bmatrix}\begin{bmatrix} a\cos v & -au\sin v \\ b\sin v & bu\cos v \\ 2u & 0 \end{bmatrix} \\
&= \begin{bmatrix} a^2\cos^2 v + b^2\sin^2 v + 4u^2 & -a^2u\cos v\sin v + b^2u\cos v\sin v \\ -a^2u\cos v\sin v + b^2u\cos v\sin v & a^2u^2\sin^2 v + b^2u^2\cos^2 v \end{bmatrix},
\end{aligned}$$

and the fundamental form is

$$\begin{aligned}
E &= a^2\cos^2 v + b^2\sin^2 v + 4u^2 \\
F &= -a^2u\cos v\sin v + b^2u\cos v\sin v \\
G &= a^2u^2\sin^2 v + b^2u^2\cos^2 v
\end{aligned}$$

Exercise 2. If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, show that the mapping g defined by

$$g(p_1, p_2) = ((X_1, X_2), (Y_1, Y_2)) = (g_1)(p_1)(X_1, Y_1) + (g_2)(p_2)(X_2, Y_2)$$

defines a Riemannian metric on $M_1 \times M_2$, where $X_1, Y_1 \in T_{p_1}M$ and $X_2, Y_2 \in T_{p_2}M_2$. Recall that $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2$.

Proof: By definition, a Riemannian metric is an inner product that is a smooth function. An inner product is a function on a vector space that must satisfy the following conditions for all vectors $X_i, Y_i \in T_{p_i}M_i$ and all scalars $a \in \mathbb{R}$ (assuming that the field is the real numbers). ■

1) $g_i(aX_i, Y_i) = ag_i(X_i, Y_i)$ linearity
2) $g_i(X_i, Y_i) = g_i(Y_i, X_i)$ conjugate symmetry
3) $g_i(X_i, X_i) > 0$ if $X_i \neq 0$.

Thus, we must show that $g(p_1, p_2)$ is a smooth function that satisfies these properties on $T_{(p_1,p_2)}(M_1 \times M_2)$.

Since the sum of two smooth functions is smooth, $g(p_1, p_2)$ is also smooth. For the linearity property

$$\begin{aligned} g(a(X_1, X_2), (Y_1, Y_2)) &= (g_1)(p_1)(aX_1, Y_1) + (g_2)(p_2)(aX_2, Y_2) \\ &= a((g_1)(p_1)(X_1, Y_1) + (g_2)(p_2)(X_2, Y_2)), \end{aligned}$$

where we used to property that g_i is linear to pull out the scalar a. For conjugate symmetry property

$$\begin{aligned} g((X_1, X_2), (Y_1, Y_2)) &= (g_1)(p_1)(Y_1, X_1) + (g_2)(p_2)(Y_2, X_2) \\ &= (g_1)(p_1)(X_1, Y_1) + (g_2)(p_2)(X_2, Y_2) \\ &= g((X_1, X_2), (Y_1, Y_2)), \end{aligned}$$

where we used the property that g_i is conjugate symmetric. For the positive definite property

$$g((X_1, X_2), (X_1, X_2)) = (g_1)(p_1)(X_1, X_1) + (g_2)(p_2)(X_2, X_2),$$

which is positive definite since each g_i is positive definite.

Exercise 3. If (M, g) is a Riemannian manifold, and $\left\{\left(U_i, \left(x_i^j\right)\right)\right\}$ is a covering of coordinate charts on M, prove that the functions

$$(g_i)_{k,\ell}(p) = \left\langle \frac{\partial}{\partial x_i^k}, \frac{\partial}{\partial x_i^\ell} \right\rangle_p$$

uniquely determine the Riemannian metric g on M.

Proof: To show that the functions g_i uniquely determine the Riemannian metric g on M, we must show that the length of a vector in one chart has the same length in another chart. Let g_a be the metric in (U_a, φ_a) and g_b be the metric (U_b, φ_b) as defined above with $U_a \cap U_b \neq \emptyset$. Also, let $v, u \in T_pM$. The vectors in the chart (U_a, φ_a) can be represented as $v_a^k \frac{\partial}{\partial x_a^k}$ and $u_a^k \frac{\partial}{\partial x_a^k}$, then the metric g_a acting on v and u is

$$(g_a)_{k,\ell}(p)(v, u) = \langle v, u \rangle = \sum v_a^k u_a^\ell \left\langle \frac{\partial}{\partial x_a^k}, \frac{\partial}{\partial x_a^\ell} \right\rangle_p.$$

We can express the vector $v_a^k \frac{\partial}{\partial x_a^k}$ in the chart (U_b, φ_b) as

$$v_b^k \frac{\partial}{\partial x_a^k} = \left.\frac{\partial \varphi_b \circ \varphi_a^{-1}}{\partial x_a^k}\right|_p v_a^k \frac{\partial}{\partial x_a^k},$$

thus the metric g_b is

$$\begin{aligned} (g_b)_{k,\ell}(p)(v, u) = \langle v, u \rangle &= \sum v_a^k u_a^\ell \left\langle \left.\frac{\partial \varphi_b \circ \varphi_a^{-1}}{\partial x_a^k}\right|_p \frac{\partial}{\partial x_a^k}, \left.\frac{\partial \varphi_b \circ \varphi_a^{-1}}{\partial x_a^k}\right|_p \frac{\partial}{\partial x_a^\ell} \right\rangle_p \\ &= \sum v_b^k u_b^\ell \left\langle \frac{\partial}{\partial x_b^k}, \frac{\partial}{\partial x_b^\ell} \right\rangle_p. \end{aligned}$$

■

Exercise 4. Consider the Riemannian metric induced by $\mathbb{R}^4$ on the torus $S^1 \times S^1$, parameterized by $(\cos s, \sin s, \cos t, \sin t)$. Show that this induced metric is isometric to the torus $\mathbb{R}^2/\mathbb{Z}^2$ with the first fundamental form $E = G = 1$ and $F = 0$.

As done prevoiusly

$$\frac{\partial r}{\partial x} = \begin{bmatrix} -\sin s & 0 \\ \cos s & 0 \\ 0 & -\sin t \\ 0 & \cos t \end{bmatrix},$$

thus

$$\begin{aligned} \left(\frac{\partial r}{\partial x}^{\top} \frac{\partial r}{\partial x}\right) &= \begin{bmatrix} -\sin s & \cos s & 0 & 0 \\ 0 & 0 & -\sin t & \cos t \end{bmatrix} \begin{bmatrix} -\sin s & 0 \\ \cos s & 0 \\ 0 & -\sin t \\ 0 & \cos t \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

thus $E = G = 1$ and $F = 0$.

Exercise 5. Prove that the antipodal mapping $A : S^n \to S^n$ given by $A(p) = -p$ is an isometry of S^n. Use this fact to introduce a Riemannian metric on the real projective space $P^n(R)$ such that the natural projection $\pi : S^n \to P^n(R)$ is a local isometry.

Proof: Let g be a Riemannian metric on S^n and $\gamma : (-\epsilon, \epsilon) \to S^n$ be a curve on S^n such that $\gamma(0) = p$, then $A \circ \gamma = -\gamma$, thus $dA = -I$ where I is the identity matrix. Let $v, u \in T_p S^n$, then

$$\begin{aligned} g(v,u)_p &= g(-Iu, -Iv)_{A(p)} \\ &= g(u,v), \end{aligned}$$

thus A is an isometry of S^n. The real projective space $P^n(R)$ is the set of equivalence classes $\{[z_i]\}$ with the relation $z \sim y$ if $z = ty$ for some $t \in \mathbb{R}$ with $z, y \in \mathbb{R}^{n+1}$. Let $p \in S^n$ with components $p = (p_1, \ldots, p_n)$. The natural projection $\pi : S^n \to P^n(R)$ is defined as

$$\pi(p) = [p].$$

This is a surjective mapping such that if $\pi(p) = \pi(q)$, then $p = q$ or $p = A(q)$. Let us define the metric (h) on $P^n(R)$ as

$$\langle d\pi u, d\pi v \rangle \triangleq g(u,v).$$

In the latter case we get

$$\begin{aligned} g_{A(p)}(u,v) &= \left\langle d\pi_{A(p)} dA_p u, d\pi_{A(p)} dA_p v \right\rangle \\ &= \left\langle -d\pi_{A(p)} u, -d\pi_{A(p)} v \right\rangle \\ &= \langle -u, -v \rangle \\ &= g(-u, -v) \\ &= g(u,v). \end{aligned}$$

Since the projection π is a smooth map, we get that is is also a local isometry. ■