Instructor: Todd Fisher

Math 565: Winter 2021
Final Exam

Instructions: The test is take home, open note, open text, no time limit. You may use the text for the class. You may not discuss solutions with others in the class. If you have any questions please let me know. The test must be turned in by **Tuesday, April 20th by 10am on gradescope**. For computational problems (computing the curvature tensor, Christoffel symbols, etc.) you may use a computer algebra system, but be sure to show the final answers of the computation, as well as any formulae used.

(1) Let M be a Riemannian manifold where, for any $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p to q. Prove that the Riemannian curvature tensor on M is identically zero. (*There is a hint provided on page 105 of do Carmo. You may use the hint, but be sure to justify all steps that don't follow directly from facts we've either proven in class or which are proven in do Carmo.*)

(2) Consider $\mathbb{R}$ with the connection $\nabla_{(\partial/\partial x)}(\partial/\partial x) = \lambda$, for some $\lambda \in \mathbb{R}$. Let $c : [0,1] \to \mathbb{R}$ be a curve such that $dc/dt = \partial/\partial x$ for all $t \in [0,1]$. Show that the parallel transport along c

$$P_{c(t),c(0)} : T_{c(0)}\mathbb{R} \to T_{c(t)}\mathbb{R}$$

is given by

$$P_{c(t),c(0)}(v\partial/\partial x) = ve^{-\lambda t}(\partial/\partial x)$$

for $v \in T_{c(t)}\mathbb{R}$. Note that $\lambda = 0$ gives the usual connection, and every $\lambda \neq 0$ determines a non-Euclidean parallelism on $\mathbb{R}$.

(3) Consider $z = 6xy - x^2 - y^2$ in $\mathbb{R}^3$ with the induced metric. At the point $(0,0,0)$ the vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ form an orthonormal basis for the tangent space to the surface at the point. Find the matrix of the derivative of the Gauss map with respect to this basis. Also, find the Gaussian curvature, principal curvatures, and principal directions at this point.

(4) Let $\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ and let $g_{ij} = \frac{\delta_{ij}}{x_n^2}$. Prove that $\mathbb{H}^n$ with this metric has constant sectional curvature -1.

(5) Show that if J is a Jacobi field such that $J(0) = 0$ along a geodesic γ that the component of J tangent to γ and normal to γ are both Jacobi fields.

(6) Complete problem 4 (a) on page 181 of Do Carmo