Homework 12

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Exercise 1. (Do Carmo p. 119 exercise 2) Let M be a Riemannian manifold, γ :$[0,1] \to M$ a geodesic, and J a Jacobi field along γ. Prove that there exists a parameterized surface $f\left(t,s\right)$, where $f\left(t,0\right)=\gamma\left(t\right)$. and the curves $t \to f\left(t,s\right)$ are geodesics, such that $J\left(t\right)=\frac{\partial f}{\partial s}\left(t,0\right)$.

Proof: Let $\lambda\left(s\right), s\in\left(-\epsilon,\epsilon\right)$ be a curve in M such that $\lambda\left(0\right)=\gamma\left(0\right)$, $\lambda'\left(0\right)=J\left(0\right)$. Also, choose a vector field $W\left(s\right)$ along λ such that $W\left(0\right)=\gamma'\left(0\right)$, $\frac{DW}{ds}=\left(0\right)=\frac{DJ(0)}{dt}$. Now we defined $f\left(s,t\right)=\exp_{\lambda(s)} tW\left(s\right)$. We note that

$$f\left(0,0\right)=\exp_{\lambda(0)} tW\left(0\right)=\exp_{\gamma(0)} t\gamma'\left(0\right).$$

We note that by holding $t=0$ constant, we see that f is moving along λ, i.e. f$(0,s)=\lambda\left(s\right)$; thus,

$$\frac{\partial f}{\partial s}\left(0,0\right)=\frac{d\lambda}{ds}\left(0\right)=J\left(0\right)$$

by construction. Note that at $s=0$, $\frac{\partial f}{dt}=W$. Since we can switch the order of differentiation,

$$\frac{D}{dt}\frac{\partial f}{\partial s}\left(0,0\right)=\frac{D}{ds}\frac{\partial f}{\partial t}\left(0,0\right)=\frac{DW}{ds}\left(0\right)=\frac{DJ}{dt}\left(0\right).$$

Since $f\left(t,s\right)$ is a parameterized surface,

$$\frac{D}{\partial s}\frac{D}{\partial t}\frac{Df}{\partial t}-\frac{D}{\partial t}\frac{D}{\partial s}\frac{Df}{\partial t}=R\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}.$$

Since $f\left(t,0\right)=\gamma\left(t\right)$ and $\gamma\left(t\right)$ is a geodesic,

$$\frac{D}{\partial s}\frac{D}{\partial t}\frac{Df}{\partial t}=0;$$

thus,

$$0=\frac{D}{\partial t}\frac{D}{\partial s}\frac{Df}{\partial t}+R\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}.$$

Since Jacobi fields are uniquely determined by their initial conditions, by putting $\frac{\partial f}{\partial s}\left(t,0\right)=J\left(t\right)$ we obtain the fact that J is a Jacobi field. ∎

Exercise 2. (Do Carmo p. 119 exercise 3) Let M be a Riemannian manifold with non-positive sectional curvature. Prove that, for all p, the conjugate locus $C\left(p\right)$ is empty.

Proof: Let $\gamma:[0,t]\to M$ be a geodesic such that $\gamma\left(0\right)=p$ and let $J\left(t\right)$ be a Jacobi field along γ that is not identically zero with $J\left(0\right)=0$ and $J'\left(0\right)\neq 0$, then

$$\begin{aligned}\left\langle J,J\right\rangle^{"} &= 2\left\langle J',J'\right\rangle+2\left\langle J^{"},J\right\rangle\\ &= 2\left\langle J',J'\right\rangle-2\left\langle R\left(\gamma',J\right),\gamma',J\right\rangle\\ &= 2\left\|J'\right\|^2-2K\left(\gamma',J\right)\left\|\gamma'\wedge J\right\|^2.\end{aligned}$$

Since the sectional curvature is non-positive, the magnitude of the Jacobi field is always increase; therefore, $J\left(t\right)$ is only zero at $t=0$. This implies that there is no point along γ that is a conjugate point to p. Thus, the conjugate locus is empty. ∎

Exercise 3. (Do Carmo p. 179 exercise 1 (a) - (c)) Consider, on a neighborhood in $\mathbb{R}^n$, $n>2$ the metric

$$g_{ij}=\frac{\delta_{ij}}{F^2}$$

where $F\neq 0$ is a function of $(x_1,\ldots,x_n)\in\mathbb{R}^n$. Denote by $F_i=\frac{\partial F}{\partial x_i}$, $F_{ij}=\frac{\partial^2 F}{\partial x_i\partial x_j}$, etc.

1) Show that a necessary and sufficient condition for the metric to have constant curvature K is

$$\begin{cases}F_{ij}=0, & i\neq j\\ F\left(F_{jj}+F_{ii}\right)=K+\sum_{i=1}^{n}\left(F_i\right)^2\end{cases}.$$

2) Use part 1 to prove that the metric g_{ij} has constant curvature K if and only if

$$F=\sum G_i\left(x_i\right),$$

where

$$G_i(x_i) = ax_i^2 + b_i x_i + c_i$$

and

$$\sum_{i=1}^{n} \left(4c_i a - b_i^2\right) = K.$$

3) Put $a = K/4$, $b_i = 0$, $c_i = \frac{1}{n}$ and obtain the formula of Riemann

$$g_{ij} = \frac{\delta_{ij}}{\left(1 + \frac{K}{4}\sum x_i^2\right)^2}$$

for a metric of constant curvature K. If $K < 0$ the metric g_{ij} is defined in a ball of radius $\sqrt{\frac{4}{-K}}$.

Proof: We being with part 1) and we follow the procedure in section 8.3 of the book.

Let $g^{ij} = F^2\delta_{ij}$ denote the inverse matrix of g_{ij}. Note that

$$\frac{\partial g_{ik}}{\partial x_j} = -\delta_{ik}\frac{2F_j}{F^3}.$$

Using the metric, we can calculate the Christoffel symbols and get

$$\Gamma_{ij}^{k} = -\delta_{jk}\frac{F_i}{F} - \delta_{ki}\frac{F_j}{F} + \delta_{ij}\frac{F_k}{F}.$$

Therefore, if all three indices are distinct, $\Gamma_{ij}^{k} = 0$, while if at least two indices are equal, we have

$$\Gamma_{ij}^{i} = -\frac{F_j}{F}, \quad \Gamma_{ii}^{j} = \frac{F_j}{F}, \quad \Gamma_{ij}^{j} = -\frac{F_i}{F}, \quad \Gamma_{ii}^{i} = -\frac{F_i}{F}.$$

For sectional curvature, we need to calculate the coefficient R_{ijij} which is

$$\begin{aligned}
R_{ijij} &= \sum_{\ell} R_{iji}^{\ell} g_{\ell j} = R_{iji}^{j} g_{jj} = R_{iji}^{j}\frac{1}{F^2} \\
&= \frac{1}{F^2}\left\{\sum_{\ell}\Gamma_{ii}^{\ell}\Gamma_{j\ell}^{i} - \sum_{\ell}\Gamma_{ji}^{\ell}\Gamma_{i\ell}^{j} + \frac{\partial}{\partial x_j}\Gamma_{ii}^{j} - \frac{\partial}{\partial x_i}\Gamma_{ji}^{j}\right\} \\
&= \frac{1}{F^2}\left\{-\sum_{\ell}\frac{F_\ell^2}{F^2} + \frac{F_i^2}{F^2} + \frac{F_j^2}{F^2} - \frac{F_i^2}{F^2} - \frac{F_j^2}{F^2} + \frac{F_{jj}^2}{F} + \frac{F_{ii}^2}{F}\right\} \\
&= \frac{1}{F^2}\left\{-\sum_{\ell}\frac{F_\ell^2}{F^2} + \frac{F_{jj}}{F} + \frac{F_{ii}}{F}\right\} \\
&= \frac{1}{F^4}\left\{-\sum_{\ell}F_\ell^2 + FF_{jj} + FF_{ii}\right\}.
\end{aligned}$$

The sectional curvature with respect to the plane generated by $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ is

$$\begin{aligned}
K_{ij} &= \frac{R_{ijij}}{g_{ii}g_{jj}} = R_{ijij}F^4 \\
&= -\sum_{\ell}F_\ell^2 + F\left(F_{jj} + F_{ii}\right).
\end{aligned}$$

Part 2 is biconditional so we prove both ways.

$(\implies)$: We suppose directly that the metric g_{ij} has constant curvature and is of the form $\frac{\delta_{ij}}{F}$ where F is a function of $(x_1, \ldots, x_n) \in \mathbb{R}^n$. According to part 1) since K is constant it must follow that $F_{jj} = F_{ii}$ for all i, j. This implies that F is at most a second order polynomial and that $F_{ii} = 2a$ where $a \in \mathbb{R}^n$ is a the coefficient on the second order terms. Also, since $F_{ij} = 0$ for all $i \neq j$, there are no cross terms between x_i and x_j (i.e. we cannot have $x_i x_j$ where $i \neq j$). Therefore, the function F is of the form

$$F = \sum_i ax_i^2 + b_i x_i + c_i$$

where $F_{ii} = 2a$ and $F_i = 2ax_i + b_i$. Using the equation from part 1 we get

$$\begin{aligned} K &= -\sum F_i^2 + F\left(F_{ii} + F_{jj}\right) \\ &= -\sum \left(2ax_i + b_i\right)^2 + \left(\sum_i ax_i^2 + b_i x_i + c_i\right) 4a \\ &= -\sum \left(4a^2 x_i^2 + b_i^2 + 4ax_i b_i\right) + \left(\sum_i ax_i^2 + b_i x_i + c_i\right) 4a^2 \\ &= \sum \left(4c_i a - b_i^2\right). \end{aligned}$$

($\Longleftarrow$) : We suppose directly that $F = \sum_i ax_i^2 + b_i x_i + c_i$, then $F_i = 2ax_i + b_i$ and $F_{ii} = 2a$. It follows that

$$K = \sum \left(4c_i a - b_i^2\right),$$

and is thus constant.

Part c) Substituting in $a = K/4$, $b_i = 0$ and $c_i = \frac{1}{n}$ we get that

$$F = \sum \frac{K}{4} x_i^2 + 1,$$

therefore

$$g_{ij} = \frac{\delta_{ij}}{\left(\frac{K}{4} \sum x_i^2 + 1\right)^2}.$$

If $K < 0$, then the metric has a singularity at when

$$\frac{K}{4} \sum x_i^2 = -1.$$

Thus the metric is defined in a ball of radius

$$\sqrt{\frac{4}{-K}}.$$

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Exercise 4. (Do Carmo p. 179 exercise 6(a)) Let $\left(M^{n+1}, g\right)$ be a manifold with a Riemannian metric g and let ∇ be its Riemannian connection. We say an immersion $x : N^n \to M^{n+1}$ is totally umbilic if for all $p \in N$, the second fundamental form B of x satisfies

$$\langle B\left(X, Y\right), \eta \rangle\left(p\right) = \lambda\left(p\right) \langle X, Y \rangle$$

for a given unit field η normal to $x\left(N\right)$; here we are using $\langle , \rangle$ to denote the metric g on M and the metric induced by x on M. Show that if M^{n+1} has constant sectional curvature, λ does not depend on p.

Proof: Let $T, X, Y \in \mathcal{X}\left(N\right)$. From section 6.2 we get the identity $\langle B\left(X, Y\right), \eta \rangle = -\langle \nabla_X \eta, Y \rangle$. Thus the condition implies

$$\begin{aligned} -\langle \nabla_X \eta, Y \rangle &= \lambda \langle X, Y \rangle \\ -\langle \nabla_T \eta, Y \rangle &= \lambda \langle T, Y \rangle. \end{aligned}$$

Differentiating the first equation w.r.t. T and the second w.r.t. X we get

$$\begin{aligned} \langle \nabla_T \nabla_X \eta - \nabla_X \nabla_T \eta, Y \rangle &= -\left\langle T\left(\lambda\right) X - X\left(\lambda\right) T + \nabla_{[X,T]} \eta, Y \right\rangle \\ &= -\langle T\left(\lambda\right) X - X\left(\lambda\right) T, Y \rangle - \left\langle \nabla_{[X,T]} \eta, Y \right\rangle. \end{aligned}$$

By the definition of the curvature R we get

$$\begin{aligned} \langle \nabla_T \nabla_X \eta - \nabla_X \nabla_T \eta, Y \rangle &= \left\langle R\left(X, Y\right) \eta - \nabla_{[X,T]} \eta, Y \right\rangle \\ &= \langle R\left(X, Y\right) \eta, Y \rangle - \left\langle \nabla_{[X,T]} \eta, Y \right\rangle; \end{aligned}$$

This implies that

$$-\langle T\left(\lambda\right) X - X\left(\lambda\right) T, Y \rangle = \langle R\left(X, Y\right) \eta, Y \rangle.$$

Since the sectional curvature is constant, $\langle R\left(X, Y\right) \eta, Y \rangle = 0$ which means

$$\langle T\left(\lambda\right) X - X\left(\lambda\right) T, Y \rangle = 0.$$

Since T and X can be chosen linearly independently, this implies that $X\left(\lambda\right) = 0$, for all $X \in \mathcal{X}\left(N\right)$; therefore, $\lambda = \text{const}$.

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