Math 565 - Homework 8

Due Monday March 15, 2021

1. Prove part (i) of Proposition 2.2 on pg. 90 of do Carmo.

2. Let S_r^2 be the sphere of radius r in $\mathbb{R}^3$ centered at the origin. Equip S_r^2 with the metric induced by Euclidean space. Consider the coordinate charts obtained by restricting the orthogonal projection of $\mathbb{R}^3$ to the coordinate planes (similar to the charts $(U_j^{\pm}, \varphi_j^{\pm})$ we've used on S^2 in the past).

   (a) Compute the components of the Riemann curvature R_{ijk}^s in these coordinates.

   (b) Use this to compute the sectional curvature $K(\sigma)$ at a point $p \in S_r^2$.

   (c) Prove that $K(\sigma)$ is constant.

3. Recall the embeddings of the torus $T = \mathbb{R}/2\pi\mathbb{Z}$ in $\mathbb{R}^3$ and $\mathbb{R}^4$ given by the maps

$$\omega(\alpha, \beta) = ((\cos(\beta) + 4) \cdot \cos(\alpha), (\cos(\beta) + 4) \cdot \sin(\alpha), \sin(\beta))$$

   and

$$\psi(\alpha, \beta) = (\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta))$$

   respectively. Let T_3 be the torus equipped with the metric induced from $\mathbb{R}^3$ by the map ω, and let T_4 denote the torus equipped with the metric induced from $\mathbb{R}^4$ by the map ψ. Compute the components R_{ijk}^s of the curvature of T_3 and T_4 (in the coordinates induced by ω and ψ).

4. For a parameterized surface S in $\mathbb{R}^3$ given by $r(u,v)$ we can find a unit normal at $p \in S$ by
$$N(p) = \frac{r_u \times r_v}{\|r_u \times r_v\|}.$$
The Gauss map is $N : S \to S^2$ defined by equation above. The derivative of the map is $dN_p : T_pS \to T_{N(p)}S^2$. However, by construction we know that T_pS and $T_{N(p)}S^2$ have parallel tangent planes in $\mathbb{R}^3$ so we can think of dN_p as a map from $T_pS \to T_pS$. The idea is the following: For a parameterized curve $\alpha(t)$ in S such that $\alpha(0) = p$ we consider the curve $N(\alpha(t)) = N(t)$ in S^2. The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in T_pS. So dN_p measures how N "pulls away from" $N(p)$.

   (a) For a plane $ax + by + cz = d$ show that $dN \equiv 0$.

   (b) For the unit sphere with inward pointing normals show that $dN_pv = -v$.

   (c) Find dN_p for the cylinder with $r(u,v) = (\cos u, \sin u, v)$.

   (d) For the hyperbolic paraboloid $r(u,v) = (u, v, v^2 - u^2)$ compute the unit normal vectors. At $p = (0,0,0)$ show that $dN_p(u'(0), v'(0), 0) = (2u'(0), -2v'(0), 0)$. So the vectors $(1,0,0)$ and $(0,1,0)$ are eigenvectors of dN_p with eigenvalues 2 and -2, respecitvely.

5. The eigenvalues of dN_p give the maximum and minimum curvature of curves at p. These are called the principle curvatures of S at p. What are the principle curvatures for (a), (b), (c) above and (d) at (0,0,0).

6. Let S be a parameterized surface in $\mathbb{R}^3$, $p \in S$, and $dN_p : T_pS \to T_pS$ be the Gauss map. The Gaussian curvature of S at p is $\det(dN_p)$. A point in S is

   (a) elliptic if $\det(dN_p) > 0$,

   (b) hyperbolic if $\det(dN_p) < 0$,

   (c) parabolic if $\det(dN_p) = 0$, but $dN_p \neq 0$, and

   (d) planar if $dN_p = 0$.

   Classify the curvature of the plane, sphere, cylinder, and the point $(0,0,0)$ on the hyperbolic parapoloid.