Homework 7

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Exercise 1. Denote by (u,v) the cartesian coordinates of $\mathbb{R}^2$. Show that the function $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(u,v) = (f(v)\cos(u), f(v)\sin(u), g(v))$,

$$U = \left\{(u,v) \in \mathbb{R}^2 \mid u_0 < u < u_1; v_0 < v < v_1\right\},$$

where f and g are differentiable functions, with $f'(v)^2 + g'(v)^2 \neq 0$ and $f \neq 0$, is an immersion. The image $\varphi(U)$ is the surface generated by the rotation of the curve $(f(v), g(v))$ around the axis $0z$ and is called a surface of revolution S. The image by φ of the curves u=constant and v=constant are called meridians and parallels, respectively, of S.

1) Show that the induced metric in the coordinates (u,v) is given by $g_{11} = f^2$, $g_{12} = 0$, and $g_{22} = (f')^2 + (g')^2$.
2) Show that the local equations of a geodesic γ are

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0,$$

$$\frac{d^2v}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\left(\frac{dv}{dt}\right)^2 = 0.$$

3) Obtain the following geometric meaning of the equations above: the second equaiton is, except for meridians and parallels, equivalent to the fact that the "energy" $|\gamma'(t)|^2$ of a geodesic is constant along γ; the first equation signifies that if $\beta(t)$ is the oriented angle, $\beta(t) < \pi$, of γ with a parallel P intersecting γ at $\gamma(t)$, then

$$r\cos(t) = \text{const},$$

where r is the radius of the parallel P.
4) Use Clairaut's relation to show that a geodesic of the paraboloid

$$\left(f(v) = v, g(v) = v^2, 0 < v < \infty, -\epsilon < u < 2\pi + \epsilon\right),$$

which is not a meridian, intersects itself an infinite number of times.

Proof: To show that φ is an immersion, we must prove that $d\varphi$ is injective for all $p \in U$. The derivative of φ is

$$d\varphi = \begin{bmatrix} -f(v)\sin(u) & f'(v)\cos(u) \\ f(v)\cos(u) & f'(v)\sin(u) \\ 0 & g'(v) \end{bmatrix}.$$

Since the functions f, f', and g' never equal 0, and cos and sin are orthogonal functions, $\operatorname{rank}(d\varphi) = 2$ for all $p \in U$; therefore, $d\varphi$ is injective and thus φ is a local immersion. If $U = \mathbb{R}^2$, then φ is a global immersion.

The induced metric can be described by the matrix

$$\begin{aligned} d\varphi^\top d\varphi &= \begin{bmatrix} -f(v)\sin(u) & f(v)\cos(u) & 0 \\ f'(v)\cos(u) & f'(v)\sin(u) & g'(v) \end{bmatrix} \begin{bmatrix} -f(v)\sin(u) & f'(v)\cos(u) \\ f(v)\cos(u) & f'(v)\sin(u) \\ 0 & g'(v) \end{bmatrix} \\ &= \begin{bmatrix} f^2\sin^2 + f^2\cos^2 & -ff'\sin\cos + ff'\sin\cos \\ -ff'\sin\cos + ff'\sin\cos & (f')^2\cos^2 + (f')^2\sin^2 + (g')^2 \end{bmatrix} \\ &= \begin{bmatrix} f^2 & 0 \\ 0 & (f')^2 + (g')^2 \end{bmatrix}. \end{aligned}$$

We can use the induced metric to compute the local equations of geodesics using equation (1) on page 62 or **Do Carmo**. This equation requires the Christoffel symbols which are

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2}\left(\frac{\partial}{\partial u}f^2 + \frac{\partial}{\partial u}f^2 - \frac{\partial}{\partial u}f^2\right)\frac{1}{f^2} + \frac{1}{2}\left(\frac{\partial}{\partial u}0 + \frac{\partial}{\partial u}0 - \frac{\partial}{\partial v}f^2\right)0 \\ &= 0. \end{aligned}$$

$$\Gamma_{12}^{1}=\frac{1}{2}\left(\frac{\partial}{\partial u}0+\frac{\partial}{\partial v}f^{2}-\frac{\partial}{\partial u}0\right)\frac{1}{f^{2}}+\frac{1}{2}\left(\frac{\partial}{\partial u}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)+\frac{\partial}{\partial v}0-\frac{\partial}{\partial v}0\right)0$$
$$=\frac{ff'}{f^{2}}.$$

$$\Gamma_{22}^{1}=\frac{1}{2}\left(\frac{\partial}{\partial v}0+\frac{\partial}{\partial v}0-\frac{\partial}{\partial u}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)\right)\frac{1}{f^{2}}+\frac{1}{2}\left(\frac{\partial}{\partial v}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)+\frac{\partial}{\partial v}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)-\frac{\partial}{\partial v}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)\right)0$$
$$=0.$$

$$\Gamma_{11}^{2}=\frac{1}{2}\left(\frac{\partial}{\partial u}f^{2}+\frac{\partial}{\partial u}f^{2}-\frac{\partial}{\partial u}f^{2}\right)0+\frac{1}{2}\left(\frac{\partial}{\partial u}0+\frac{\partial}{\partial u}0-\frac{\partial}{\partial v}f^{2}\left(v\right)\right)\frac{1}{\left(f'\right)^{2}+\left(g'\right)^{2}}$$
$$=-\frac{ff'}{\left(f'\right)^{2}+\left(g'\right)^{2}}.$$

$$\Gamma_{12}^{2}=\frac{1}{2}\left(\frac{\partial}{\partial u}0+\frac{\partial}{\partial v}f^{2}\left(v\right)-\frac{\partial}{\partial u}0\right)0+\frac{1}{2}\left(\frac{\partial}{\partial u}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)+\frac{\partial}{\partial v}0-\frac{\partial}{\partial v}0\right)\frac{1}{\left(f'\right)^{2}+\left(g'\right)^{2}}$$
$$=0.$$

$$\Gamma_{22}^{2}=\frac{1}{2}\left(\frac{\partial}{\partial v}0+\frac{\partial}{\partial v}0-\frac{\partial}{\partial u}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)\right)0+\frac{1}{2}\left(\frac{\partial}{\partial v}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)+\frac{\partial}{\partial v}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)-\frac{\partial}{\partial v}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)\right)\frac{1}{\left(f'\right)^{2}+\left(g'\right)^{2}}$$
$$=\frac{f'f''+g'g''}{\left(f'\right)^{2}+\left(g'\right)^{2}}.$$

Using the Christoffel symbols we can now construct the local equations of geodesics

$$\frac{d^{2}u}{dt}+\sum_{i,j}\Gamma_{ij}^{1}\frac{dx_{i}}{dt}\frac{dx_{j}}{dt}=0$$
$$\frac{d^{2}u}{dt}+\frac{2ff'}{f^{2}}\frac{du}{dt}\frac{dv}{dt}=0 \tag{1}$$

$$\frac{d^{2}v}{dt}+\sum_{i,j}\Gamma_{ij}^{2}\frac{dx_{i}}{dt}\frac{dx_{j}}{dt}=0$$
$$\frac{d^{2}v}{dt}-\frac{ff'}{\left(f'\right)^{2}+\left(g'\right)^{2}}\left(\frac{du}{dt}\right)^{2}+\frac{f'f''+g'g''}{\left(f'\right)^{2}+\left(g'\right)^{2}}\left(\frac{dv}{dt}\right)^{2}=0. \tag{2}$$

To prove part (3) we note that by definition $\gamma\left(t\right)$ is a geodesic, thus

$$\begin{aligned}
0&=\frac{d}{dt}\left\langle\frac{d\gamma}{dt},\frac{d\gamma}{dt}\right\rangle\\
&=\frac{d}{dt}\left(\left(\frac{du}{dt}\right)^{2}f^{2}+\left(\frac{dv}{dt}\right)^{2}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)\right)\\
&=2\frac{d^{2}u}{dt^{2}}\frac{du}{dt}f^{2}+2\left(\frac{du}{dt}\right)^{2}ff'\frac{dv}{dt}+2\frac{d^{2}v}{dt^{2}}\frac{dv}{dt}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)+2\left(\frac{dv}{dt}\right)^{2}\left(f'f''\frac{dv}{dt}+g'g''\frac{dv}{dt}\right)\\
&=2\left(\frac{d^{2}u}{dt^{2}}f^{2}+2ff'\frac{du}{dt}\frac{dv}{dt}-ff'\frac{du}{dt}\frac{dv}{dt}\right)\frac{du}{dt}+2\left(\frac{d^{2}v}{dt^{2}}\frac{dv}{dt}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)+\left(\frac{dv}{dt}\right)^{2}\left(f'f''\frac{dv}{dt}+g'g''\frac{dv}{dt}\right)\right),
\end{aligned}$$

we can simplify this using equation (1) to get

$$\begin{aligned}
0&=-2ff'\left(\frac{du}{dt}\right)^{2}\frac{dv}{dt}+2\left(\frac{d^{2}v}{dt^{2}}\frac{dv}{dt}\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)+\left(\frac{dv}{dt}\right)^{2}\left(f'f''\frac{dv}{dt}+g'g''\frac{dv}{dt}\right)\right)\\
&=2\left(\left(f'\right)^{2}+\left(g'\right)^{2}\right)\left(\frac{d^{2}v}{dt}-\frac{ff'}{\left(f'\right)^{2}+\left(g'\right)^{2}}\left(\frac{du}{dt}\right)^{2}+\frac{f'f''+g'g''}{\left(f'\right)^{2}+\left(g'\right)^{2}}\left(\frac{dv}{dt}\right)^{2}\right)\frac{dv}{dt}\\
&=\frac{d^{2}v}{dt}-\frac{ff'}{\left(f'\right)^{2}+\left(g'\right)^{2}}\left(\frac{du}{dt}\right)^{2}+\frac{f'f''+g'g''}{\left(f'\right)^{2}+\left(g'\right)^{2}}\left(\frac{dv}{dt}\right)^{2}.
\end{aligned}$$

Therefore, we see that equaiton (2) is equivalent to constant energy along a geodesic.

We now proceed to show that $r\cos\beta = \text{const}$ where β is the angel between the geodesic $\gamma(t) = (u(t), v(t))$ and the parallel. We can define the parallel as $h(t) = (f(v_0)\cos(u(t)), f(v_0)\sin(u(t)), g(v_0))$ where $v_0 \in \mathbb{R}$ is a constant. Taking the derivative of $h(t)$ yields

$$\frac{dh}{dt} = \left(-f(v_0)\sin(u(t))\frac{du}{dt}, f(v_0)\cos(u(t))\frac{du}{dt}, 0\right).$$

We can map the derivative of γ from $\mathbb{R}^2$ to $\mathbb{R}^3$ using $d\varphi$ to get

$$\begin{aligned} d\varphi \circ \frac{d\gamma}{dt} &= \begin{bmatrix} -f(v)\sin(u) & f'(v)\cos(u) \\ f(v)\cos(u) & f'(v)\sin(u) \\ 0 & g'(v) \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} \\ &= \begin{bmatrix} -f(v)\sin(u)\frac{du}{dt} + f'(v)\cos(u)\frac{dv}{dt} \\ f(v)\cos(u)\frac{du}{dt} + f'(v)\sin(u)\frac{dv}{dt} \\ g'(v)\frac{dv}{dt}. \end{bmatrix} \end{aligned}$$

Calculating the angle between $\frac{dh}{dt}$ and $d\varphi \circ \frac{d\gamma}{dt}$ at the point of their intersection is

$$\cos\beta(t) = \frac{\left\langle \frac{dh}{dt}, d\varphi \circ \frac{d\gamma}{dt} \right\rangle}{\left\|\frac{dh}{dt}\right\| \left\|d\varphi \circ \frac{d\gamma}{dt}\right\|}.$$

We compute this quantity by looking at each component.

$$\begin{aligned} \left\|\frac{dh}{dt}\right\| &= \left\langle \frac{dh}{dt}, \frac{dh}{dt} \right\rangle^{\frac{1}{2}} \\ &= \left| f(v_0)\frac{du}{dt} \right|, \end{aligned}$$

$$\begin{aligned} \left\langle \frac{dh}{dt}, d\varphi \circ \frac{d\gamma}{dt} \right\rangle &= f(v_0) f(v_0)\sin^2(u)\left(\frac{du}{dt}\right)^2 - f(v_0) f'(v_0)\sin(u)\cos(u)\frac{du}{dt}\frac{dv}{dt} \\ &\quad + f(v_0) f(v)\cos^2(u)\left(\frac{du}{dt}\right)^2 + f(v_0) f'(v_0)\cos(u)\sin(u)\frac{du}{dt}\frac{dv}{dt} \\ &= f^2(v_0)\left(\frac{du}{dt}\right)^2, \end{aligned}$$

$$\begin{aligned} \left\|d\varphi \circ \frac{d\gamma}{dt}\right\| &= \left\langle d\varphi \circ \frac{d\gamma}{dt}, d\varphi \circ \frac{d\gamma}{dt} \right\rangle^{\frac{1}{2}} \\ &= \left(f(v_0)^2 \left(\frac{du}{dt}\right)^2 + (f'(v_0))^2 + (g'(v_0))^2 \left(\frac{dv}{dt}\right)^2 \right)^{\frac{1}{2}} \\ &= \alpha. \end{aligned}$$

$\alpha \in \mathbb{R}$ is a constant since γ is a geodesic. Thus we have,

$$\begin{aligned} \cos\beta(t) &= \frac{f^2(v_0)\left(\frac{du}{dt}\right)^2}{\left|f(v_0)\frac{du}{dt}\right| \alpha}, \\ &= \frac{f(v_0)\frac{du}{dt}}{\alpha} \end{aligned}$$

where $|\gamma'(t)| = \alpha \in \mathbb{R}$ is a constant. Multiplying both sides by $f(v_0)$, we get

$$f(v_0)\cos(\beta(t)) = \frac{f^2(v_0)\frac{du}{dt}}{\alpha}.$$

We can show that the term $f^2(v_0)\frac{du}{dt}$ is a constant, by taking its derivative w.r.t. to time to get

$$\frac{df^2(v_0)\frac{du}{dt}}{dt} = f^2\frac{d^2u}{dt^2} + 2ff'\frac{du}{dt}\frac{dv}{dt} = 0.$$

Therefore,

$$f(v)\cos(\beta(t)) = \text{const}.$$

To prove the last part, let $f(v) = v$ and $g(v) = v^2$ for $0 < v < \infty$ and $-\epsilon < u < 2\pi + \epsilon$, then the angle according to Clairaut's relation we have

$$v \cos(\beta(t)) = \text{const.}$$

which shows that v cannot tend towards 0 since $\cos(\beta(t))$ cannot tend towards zero. This means if the geodesic curve γ is heading down the parabaloid, it must eventually come back up. Note that as $v \to \infty$, $\cos(\beta) \to 0$, thus the angle is getting smaller. So, as the radius v increases, the geodesic approaches a parallel and wraps itself around alot. Yeah, I don't know how to prove this. ■

Exercise 2. It is possible to introduce a Riemannian metric in the tangent bundle TM of a Riemannian manifold M in the following manner. Let $(p, v) \in TM$ and V, W be tangent vectors in TM at (p, v). Choose curves in TM

$$\alpha : t \to (p(t), v(t)), \quad \beta : s \to (q(s), w(s)),$$

with $p(0) = q(0) = p$, $v(0) = w(0) = v$, and $V = \alpha'(0)$, $W = \beta'(0)$. Define an inner product on TM by

$$\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\right\rangle_p$$

where $d\pi$ is the differential of $\pi : TM \to M$.

1) Prove that the inner product is well defined and introduces a Riemannian metric on TM.
2) A vector at $(p, v) \in TM$ that is orthogonal (for the metric above) to the fiber $\pi^{-1}(p) \approx T_pM$ is called a horizontal vector. A curve
$$t \to (p(t), v(t))$$
in TM is horizontal if its tangent vector is horizontal for all t. Prove that the curve
$$t \to (p(t), v(t))$$
is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in M.
3) Prove that the geodesic field is a horizonal vector field.
4) Prove that the trajectories of the geodesic field are geodesics on TM in the metric above.
5) A vector at $(p, v) \in TM$ is called vertical if it is tangent to the fiber $\pi^{-1}(p) \approx T_pM$. Show that
$$\begin{aligned}\langle W, W\rangle_{(p,v)} &= \langle d\pi(W), d\pi(W)\rangle_p \quad \text{if W is horizontal,}\\ \langle W, W\rangle_{(p,v)} &= \langle W, W\rangle_p \quad \text{if W is vertical,}\end{aligned}$$

Proof: We will prove them in order. ■

1) Since the original Riemannian metric is smooth, bilinear and symmetric, the defined inner product is also smooth, biliear and symmetric. All that is left to check is that it is positive definite. Let $\gamma : I \to TM$ be smooth curves as defined above, and let (U, φ) be a chart on M which can be lifted to a chart on TM. In local coordinates, the curve can be defined as
$$\tilde{\gamma} \triangleq \varphi \circ \gamma = (p_1, \ldots, p_n, v_1, \ldots v_n)$$
where $\pi(\tilde{\gamma}) = (p_1, \ldots, p_n)$. The tangent vector of $\tilde{\gamma}$ is
$$\tilde{\gamma}' = \left(p_1' \frac{\partial}{\partial x_1}, \ldots, p_n' \frac{\partial}{\partial x_n}, v_1' \frac{\partial}{\partial x_1}, \ldots v_n \frac{\partial}{\partial x_n}\right).$$
The covariant in local coordinates is
$$\frac{Dv}{dt} = v_i' \frac{\partial}{\partial x_i} + v_i p_j' \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$
Therefore
$$\begin{aligned}\langle \alpha', \beta'\rangle &= \langle p', q'\rangle + \left\langle \frac{Dv}{dt}, \frac{Dw}{dt}\right\rangle \\ &= p_i' q_j' g_{ij} + \left(v_k' + v_i p_j' \Gamma_{ij}^k\right)\left(w_\ell' + w_i q_j' \Gamma_{ij}^\ell\right) g_{k\ell}.\end{aligned}$$
Note that if $p_i' = 0$, $q_i' = 0$ or $p_i' = q_i' = 0$ for all $i = 1, \ldots, n$ then it simplifies to
$$\left(v_k'\right)\left(w_\ell' + w_i q_j' \Gamma_{ij}^\ell\right) g_{k\ell}$$
$$\left(v_k' + v_i p_j' \Gamma_{ij}^k\right)\left(w_\ell'\right) g_{k\ell}$$
and
$$v_k' w_\ell' g_{k\ell}.$$

From this we can easily see that it is positive definite.

2) This proof is biconditional so we must prove both ways.

   a) ($\implies$): We suppose directly that $\beta = (p, w(t))$ is horizontal to $\alpha(p(t), v(t))$, then

   $$\langle \alpha', \beta' \rangle = 0.$$

   We can reduces the inner product in local coordinates to

   $$\langle \alpha', \beta' \rangle = \left(v'_k + v_i p'_j \Gamma^k_{ij}\right)\left(w'_\ell\right) g_{k\ell},$$

   which can only be zero if $\left(v'_k + v_i p'_j \Gamma^k_{ij}\right) = 0$, thus $\frac{Dv}{dt} = 0$ implies that $v(t)$ is parallel to $p(t)$.

   b) ($\impliedby$) :We suppose directly that $v(t)$ is parallel to $p(t)$, then $\frac{Dv}{dt} = 0$, and let $\beta = \frac{d}{dt}(p, w(t))$ be a vector at $\alpha(0)$, then

   $$\langle \alpha, \beta \rangle = 0.$$

   Therefore, if $v(t)$ is parallel to $p(t)$, then α is a horizontal.

   c) Let $\gamma(t)$ be a Geodesic as defined by the metric. We define a curve in TM as $\bar{\gamma} = (\gamma, \gamma')$. By the definition of a geodesic, we get that $\frac{D\gamma'}{dt} = 0$. In other words, γ' is parallel to γ; therefore, it is horizontal.

   d) Let $\bar{\gamma}(t) = (\gamma(t), \gamma'(t))$ define a Geodesic field, the length of $\bar{\gamma}$ is

   $$\begin{aligned}\int \langle \bar{\gamma}, \bar{\gamma} \rangle &= \int \left(\langle \gamma', \gamma' \rangle + \left\langle \frac{D\gamma'}{dt}, \frac{D\gamma'}{dt} \right\rangle\right)^{\frac{1}{2}} dt \\ &= \int \langle \gamma', \gamma' \rangle \, dt\end{aligned}$$

   since $\int \langle \gamma', \gamma' \rangle \, dt = \ell(\gamma)$ is the minimum length between the two connecting points, $\bar{\gamma}$ is the minimum length for joining any two points in TM. Therefore, it is a Geodesic.

   e) Let $W = (p', v')$ be a horizontal vector, then $\frac{Dv}{dt} = 0$, thus the metric reduces to

   $$\begin{aligned}\langle W, W \rangle &= \langle p', p' \rangle + \left\langle \frac{Dv}{dt}, \frac{Dv}{dt} \right\rangle \\ &= \langle p', p' \rangle .\end{aligned}$$

   Now suppose that W is tangent, then $W = (0, v')$. In local coordinates, the metric reduces to

   $$\begin{aligned}\langle W, W \rangle &= \left(v'_k + v_i p'_j \Gamma^k_{ij}\right)\left(v'_\ell + v_i p'_j \Gamma^\ell_{ij}\right) g_{k\ell} \\ &= v'_k v'_\ell g_{k\ell} \\ &= \langle v', v' \rangle .\end{aligned}$$

Exercise 3. A subset A of a differentiable manifold M is contractible to a point $a \in A$ when the mapping id_A and $k_a : x \in A \to a \in A$ are homotopic with base point a. A is contractible if it is contractible to one of its points.

1) Show that a convex neighborhood in a Riemannian manifold M is a contractible subset w.r.t any of its points.
2) Let M be a differentiable manifold. Show that there exists a covering $\{U_\alpha\}$ of M with the following properties:
   a) U_α is open and contractible, for each α.
   b) If $U_{\alpha_1,\ldots,}U_{\alpha_n}$ are elements of the covering, then $\cap_1^n U_{\alpha_i}$ is contractible.

Proof: We begin by proving the first statement. We suppose directly that $A \subset M$ is convex neighborhood and $x, a \in A$. Since A is a convex neigborhood, there exists a geodesic $\gamma : I \to M$ that connects the points a and x such that $\gamma(I) \subset A$, $\gamma' = v$, $\gamma(0) = a$ and $\gamma(1) = x$ where $0, 1 \in I$. Since the geodesic is smooth, we can shrink the geodesic by assigning it a new tangent vector $v' = \delta v$ where $\delta = \mathbb{R}$. As $\delta \to 0$, $\gamma(1) \to \gamma(0)$. Therefore; any geodesic in A starting at a is homotopic with the identity mapping. This means that we can shrink A to the point a.

For the second property, we suppose directly that M is a differentiable manifold, then M has a countable basis $\{V_i\}$. Let $p_i \in V_i$ be a point, then there exists a geodesic ball $B_\beta(p_i)$ which is strongly convex. If $V_i \subset B_\beta(i)$, then were done. Otherwise, we can pick a point $p_{i,k} \in B_\beta(p_i)$ and form another convex set $B_\beta(p_{i,k})$. We can repeat this process until $V_i \cup_{k=0}^n B_\beta(p_{i,k})$. Using this process, we can form the covering $\{U_\alpha\}$. For the second part, let $G = \cap_1^n U_{\alpha_i}$, then $G \subset U_{\alpha_j}$ where U_{α_j} is one of the open and contractible sets used in the intersection. The open set G may no longer be convex; however, since $G \subset U_{\alpha_j}$, there exists a minimizing geodesic between any two points in G that can contract to a single point. This geodesic may leave G at times, but will eventually contract to stay within G and eventually to one of the points. ∎

Exercise 4. Supoose that M and N are Riemannian manifolds and $f : M \to N$ is a diffeomorphism. Prove that f is distance preserving if and only if f is an isometry.

Proof: This is a biconditional statement so we must show both ways

($\Longrightarrow$) : We suppose directly that f is distance perserving. Let $\gamma : I \to M$ be a smooth curve with $\gamma(0) = p$. Then

$$\begin{aligned}\ell(\gamma) &= \ell(f \circ \gamma) \\ &= \int \left(\left\langle df\frac{d\gamma}{dt}, df\frac{d\gamma}{dt}\right\rangle\right)^{\frac{1}{2}} dt,\end{aligned}$$

which can only be true if

$$\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right\rangle = \left\langle df\frac{d\gamma}{dt}, df\frac{d\gamma}{dt}\right\rangle$$

for all time, therefore, f is an isometry.

($\Longleftarrow$) : We suppose directly that f is an isometry. Let γ be defined above. Then

$$\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right\rangle_p = \left\langle df\frac{d\gamma}{dt}, df\frac{d\gamma}{dt}\right\rangle_{f(p)}.$$

Since the metric is positive definite, we get that

$$\int \left(\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right\rangle\right)^{\frac{1}{2}} dt = \int \left(\left\langle df\frac{d\gamma}{dt}, df\frac{d\gamma}{dt}\right\rangle\right)^{\frac{1}{2}} dt,$$

therefore, f preserves length. ■

Exercise 5. Consider the upper half-plane $\mathbb{R}^2_+$ with the Lobatchevski metric

$$g_{11} = g_{22} = \frac{1}{y^2},\ g_{12} = 0.$$

Show that the line segment $x = 0$, $\epsilon \leq y \leq 1$ with $\epsilon > 0$ has length tending to ∞ as $\epsilon \to \infty$.

Proof: Let $\gamma(t) : [1, \epsilon] \to \mathbb{R}^2_+$ be defined as the curve $\gamma(t) = (0, t)$. The length of the curve is

$$\begin{aligned}\ell(\gamma) &= \lim_{\epsilon \to 0} \int_\epsilon^1 \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right\rangle^{\frac{1}{2}} dt \\ &= \lim_{\epsilon \to 0} \ln(x)\big|_\epsilon^1 \\ &= \ln(1) - \lim_{\epsilon \to 0} \ln(\epsilon) \\ &= -\lim_{\epsilon \to 0} \ln(\epsilon)\end{aligned}$$

which will go to infinity. ■

Exercise 6. A geodesic $\gamma : [0, \infty) \to M$ in a Riemannian manifold M is called a ray starting from $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$, for any $s \in (0, \infty)$. Assume that M is complete, non-compact, and let $p \in M$. Show that M contains a ray starting from p.

Proof: Since M is non-compact but complete, there exists a sequence of points y_i on M whose distance from p tends to infinity such that $d(y_i, p) \geq i$. Since M is complete, we can connect each y_i to p by a minimizing geodesic defined by $\gamma_i(s) = \exp(s\gamma_i'(0))$ with unit speed. The set of vectors $\{\gamma_i'\}$ for a unit sphere in T_pM is compact, thus the sequence must have some convergent subsequence with a limit say $\gamma(t) = \exp_p(tv)$. Then $\lim_{i\to\infty} d(\gamma(t), \gamma_i(t)) = 0$. Thus the ray is $\gamma(t)$, and it exists. ■