Homework 2

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Exercise 1. Suppose $f : M \to N$ is a diffeomorphism. Prove that df_p is an isomorphism of the tangent space for all $p \in M$.

Proof: We suppose directly that $f : M \to N$ is a diffeomorphism. Then M and N are smooth manifold. Thus there exists a chart (U, φ) on M and a chart (V, ψ) on N such that $p \in U$ and $V \subseteq \varphi(U)$. Let $\gamma_i : (-\epsilon, \epsilon) \to M$ be a smooth curves on M with $\gamma_i(0) = p$. The tangent space at p is the collection of equivalence classes of smooth curves $\{[\gamma_i]\}$ with the equivalence relation $\gamma_j \sim \gamma_k$ being defined as $\tilde{\gamma}_j^{'}(0) = \tilde{\gamma}_k^{'}(0)$ where $\tilde{\gamma}_i \triangleq \varphi \circ \gamma_i$. An equivalence class of smooth curves can be equivalently represented in local coordinates as

$$\begin{aligned}
[\gamma_i] &\equiv \frac{d}{dt}(\varphi \circ \gamma_i)\Big|_0 \\
&= \sum_{k=0}^{n} v_{i,k} \frac{\partial}{\partial x_k} \\
&= v_i,
\end{aligned}$$

with n being the dimension of the manifolds M and N, $\frac{\partial}{\partial x_k}$ being the k^{th} basis vector in local coordinates and v_i being the vector in matrix notation. Working in local coordinates, we can pull back the vector space onto the tangent space T_pM. We can now add and scale the equivalence classes of smooth curves by adding and scaling their representation in local coordinates.

Since f is a diffeomorphism, it's differentiable and it's inverse is differentiable. This allows us to represent df_p in local coordinates. By definition,

$$df_p = \frac{\partial}{\partial x}\left(\psi \circ f \circ \varphi^{-1}\right)\Big|_{\varphi(p)},$$

which is a linear function and can be represented as a matrix which we will denote as A. Also, since f is a diffeomorphism, we can use it to push forward the tangent space T_pM to $T_{f(p)}N$ where $T_{f(p)}N = \{[f \circ \gamma_i]\}$. In local coordinates, this is represented as

$$\begin{aligned}
[f \circ \gamma_i] &= \frac{d}{dt}\left(\psi \circ f \circ \varphi^{-1} \circ \varphi \circ \gamma_i\right) \\
&= \frac{\partial}{\partial x}\left(\psi \circ f \circ \varphi^{-1}\right)\Big|_{\varphi(p)} \frac{d}{dt}(\varphi \circ \gamma_i)\Big|_0 \\
&= A v_i.
\end{aligned}$$

Finally, let $\alpha, \beta \in \mathbb{R}$. We can define a new vector $u = \alpha v_j + \beta v_k$ where v_j and v_k are the representation of an equivalence class of smooth curves in local coordinates. Then

$$\begin{aligned}
Au &= A(\alpha v_j + \beta v_k) \\
&= \alpha A v_j + \beta A v_k.
\end{aligned}$$

Therefore, A is an isomorphism which proves that df_p is an isomorphism. ∎

Exercise 2. Let $U \subset M$ be open. Show that $T_pU = T_pM$ for all $p \in U$.

Proof: We suppose directly that $U \subset M$ is open. Let $\gamma_i : (-\epsilon, \epsilon) \to M$ be a smooth curve such that $\gamma_i(0) = p$ and let $T_pM \triangleq \{[\gamma_i]\}$ with the traditional equivalence relation. By definition, T_pM is defined by all possible smooth curves γ_i. Since γ_i is a smooth curve, it maps an open interval to an open set in M that contains p. We can restrict the domain of γ_i so that the corresponding range is a subset of U. In other words $\gamma_i|_U (-\alpha, \alpha) \to U$. Since all the equivalence classes of smooth curves exist in some open set around p, for every γ_i a restricted counter part $\gamma_i|_U$ exists. Also, since the equivalence relationship is defined at the point $p \in U$, restricting the domain will not affect the equivalence relation. This shows that $T_pU \supseteq T_pM$. As mentioned before, since T_pM is defined by all possible smooth curves and $U \subset M$, there are no others that exist. Therefore $T_pU = T_pM$. ∎

Exercise 3. Prove that the tangent bundle of a product of manifolds is diffeomorphic to the product of the tangent bundles.

Let $M_1, \ldots, M_n$ be smooth manifolds of dimension $n_1, \ldots, n_n$, $\mathcal{M} = M_1 \times \cdots \times M_n$, $T_{p_i}M_i$ denote the tangent space at $p_i \in M_i$, $T_p\mathcal{M}$ denote the tangent space at $p \in \mathcal{M}$, TM_i denote the tangent bundle of M_i and $T\mathcal{M}$ denote the tangent bundle of $\mathcal{M}$. Let (U_i, φ_i) be a chart on M_i, (U, φ) be a chart on $\mathcal{M}$ and $\pi : \mathcal{M} \to M_1 \times \cdots \times M_n$, be defined as

$$\pi(p) = (\pi_1(\mathcal{M}), \ldots, \pi_n(\mathcal{M})),$$

where $\pi_i : \mathcal{M} \to M_i$ is the projection from $\mathcal{M}$ to M_i such that $\pi_i(p) = p_i$. Lastly, let $\gamma_j : (-\epsilon, \epsilon) \to \mathcal{M}$ be a smooth curve such that $\gamma_j(p) = 0$. The equivalence class of smooth curves under the typical relation construct $T_p\mathcal{M}$. The smooth curve γ_j can be pushed forward to the local coordinates of the manifold M_i using the composition of functions

$$f_i = \varphi_i \circ \pi_i \circ \varphi^{-1} \circ \varphi \circ \gamma_j.$$

Taking the derivative yields

$$\frac{d}{dt} f_i = \underbrace{\left(\frac{\partial}{\partial x_k} \varphi_i \circ \pi_i \circ \varphi^{-1} \right)\Big|_{\varphi(p)}}_{\Pi_{i,p}} \underbrace{\frac{d}{dt}\left(\varphi \circ \gamma_j\right)\Big|_0}_{v_j}.$$

$v_j \in T_p\mathcal{M}$ and is mapped to $T_{p_i}M_i$ through the function $\Pi_{i,p} : T_p\mathcal{M} \to T_iM$. Gluing these maps together, we get a final map $\Pi_p : T_pM \to T_{p_i}M_i \times \cdots \times T_{p_n}M_n$ defined as

$$\Pi_p(v_j) = (\Pi_i v_j, \ldots, \Pi_n v_j).$$

The map Π_p is linear and can be represented as a matrix. Since linear functions are smooth, Π_p is smooth. It is also invertible meaning that it is diffeomorphic. The vector v_j and the map Π_p are diffeomorphic under different charts. Let (V_i, ψ_i) be another chart on M_i such that $V_i \cap U_i \neq \emptyset$ and $p_i \in V_i$ and (V, ψ) be another chart on $\mathcal{M}$ such that $V \cap U \neq \emptyset$ and $p \in V$. Then the push forward of γ_j to the new local coordinates of the manifold M_i is

$$g_i = \psi_i \circ \varphi_i^{-1} \circ \varphi_i \circ \pi_i \circ \varphi^{-1} \circ \psi \circ \varphi^{-1} \circ \varphi \circ \gamma_j$$

and

$$\frac{d}{dt} g_i = \frac{\partial}{\partial x_l}\left(\psi_i \circ \varphi_i^{-1}\right)\big|_{\varphi_i \circ \pi_i(p)} \Pi_{i,p} \frac{\partial}{\partial x_q}\left(\psi \circ \varphi^{-1}\right)\Big|_{\varphi(p)} v_j,$$

which is smooth and linear. This shows that the projection π is a diffeomorphism, thus it's differential $(d\pi)$ is also a diffeomorphism between $T_p\mathcal{M}$ and $T_{p_1}M_1 \times \cdots \times T_{p_n}M_n$. Hence, we can construct a diffeomorphism between tangent bundles. Let $\Psi : T\mathcal{M} \to TM_1 \times \cdots \times TM_n$ be defined as

$$\Psi(p, v) \mapsto ((\pi_1(p), d\pi_1 v), \ldots, (\pi_n(p), d\pi_n v)).$$

Therefore, the tangent bundle of a product of manifolds is diffeomorphic to the product of the tangent bundles.