Final

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Exercise 1. Let M be a Riemannian manifold where, for any $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p to q. Prove that the Riemannian curvature tensor on M is identically zero.

Proof: Using the hint provided in the book. Consider a parameterized surface $f : U \subseteq \mathbb{R}^2 \to M$, where

$$U = \left\{(s,t) \in \mathbb{R}^2; -\epsilon < t < 1+\epsilon, -\epsilon < s < 1+\epsilon, \epsilon > 0\right\}$$

and $f(s,0) = f(0,0)$, for all s. f evaluated at a specific s is a curve from p to q, thus s selects the curve we are traveling on.

Let $V_0 \in T_{f(0,0)}M$ and define a field V along f by: $V(s,0) = V_0$ and, if $t \neq 0$, V(s,t) is the parallel transport of V_0 along the curve $t \to f(s,t)$. Then, from Lemma 4.1,

$$\frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V.$$

The first term $\frac{D}{\partial s}\frac{D}{\partial t}V = 0$ since the vector is being parallel transported along the curve. The second term $\frac{D}{\partial t}\frac{D}{\partial s}V = 0$ since parallel transport does not depend on the curve chosen according to the assumption made. Thus, $V(s,1)$ is the parallel transport of $V(0,1)$ along the curve $s \to f(s,1)$, hence $\frac{D}{\partial s}V(s,1) = 0$. Thus

$$R_{f(0,1)}\left(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1)\right)V(0,1) = 0.$$

Since the surface f and V are arbitrary, then for all $X, Y, Z \in \mathcal{X}(M)$ we have

$$R(X,Y)Z = 0.$$

Therefore, the curvature is zero. ■

Exercise 2. Consider $\mathbb{R}$ with the connection $\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \lambda$, for some $\lambda \in \mathbb{R}$. Let $c : [0,1] \to \mathbb{R}$ be a curve such that $\frac{dc}{dt} = \frac{\partial}{\partial x}$ for all $t \in [0,1]$. Show that the parallel transport along c is given by

$$P_{c(t),c(0)}\left(v\frac{\partial}{\partial x}\right) = ve^{-\lambda t}\frac{\partial}{\partial x}$$

for $v \in T_{c(t)}\mathbb{R}$.

Proof: According to the equation in Do Carmo, A vector that is parallel transported must satisfy the equation

$$\frac{dv^k}{dt} + \sum_{i,j}\Gamma^k_{ij}v^j\frac{dx_i}{dt} = \frac{Dv}{dt} = 0.$$

Since we are working in one dimension and $\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \lambda$, we can simplify the equation to

$$\frac{dv}{dt}\frac{\partial}{\partial x} = -\lambda v\frac{\partial}{\partial x}.$$

Solving the differential equation with initial conditions $v(0) = v$ gives us

$$P_{c(t),c(0)}\left(v\frac{\partial}{\partial x}\right) = ve^{-\lambda t}\frac{\partial}{\partial x}.$$

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Exercise 3. Consider $z = 6xy - x^2 - y^2$ in $\mathbb{R}^3$ with the induced metric. At the point $(0,0,0)$ the vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ form an orthonormal basis for the tangent space to the surface at the point. Find the matrix of the derivative of the Gauss map with respect to this basis. Also, find the Gaussian curvature, principal curvatures, and principal directions at this point.

Proof: We will use Matlab to help with the computations. Let $r : \mathbb{R}^2 \to \mathbb{R}^3$ be the parameterized surface $r(u,v) = \left(u, v, 6uv - u^2 - v^2\right)$. The differential of r is

$$dr = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 6v - 2u & 6u - 2v \end{bmatrix},$$

from which we can calculate the induced metric

$$I_p = g = dr'dr = \begin{bmatrix} (2u-6v)^2+1 & -(2u-6v)(6u-2v) \\ -(2u-6v)(6u-2v) & (6u-2v)^2+1 \end{bmatrix};$$

therefore, the coefficients of the first fundamental form are

$$\begin{aligned} E &= (2u-6v)^2+1 \\ F &= -(2u-6v)(6u-2v) \\ G &= (6u-2v)^2+1. \end{aligned}$$

The Gaus map is

$$N = \frac{dr_u \times dr_v}{\|dr_u \times dr_v\|} = \frac{1}{\sqrt{(2u-6v)^2+(2v-6u)^2+1}} \begin{bmatrix} 2u-6v \\ 2v-6u \\ 1 \end{bmatrix}.$$

The derivative of the Gaus map is

$$dN = \begin{bmatrix} \left(-64v^2+192uv+2\right)/c_1 & -\left(192u^2-64vu+6\right)/c_1 \\ -\left(192v^2-64uv+6\right)/c_1 & \left(-64u^2+192uv+2\right)/c_1 \\ -(80u-48v)/c_2 & (48u-80v)/c_2 \end{bmatrix},$$

where

$$\begin{aligned} c_1 &= \left(40u^2-48uv+40v^2+1\right)^{\frac{3}{2}}, \\ c_2 &= 2\left((2u-6v)^2+(6u-2v)^2+1\right)^{\frac{3}{2}}. \end{aligned}$$

Using the derivative of the Gauss map, we can compute the second fundamental form

$$II_p = -\langle dN, dr\rangle = \begin{bmatrix} e & f \\ f & g \end{bmatrix},$$

where

$$\begin{aligned} e &= -\left(80u^2-96uv+80v^2+2\right)/c_3 \\ f &= \left(240u^2-288uv+240v^2+6\right)/c_3 \\ g &= -\left(80u^2-96uv+80v^2+2\right)/c_3 \\ c_3 &= \left(40u^2-48uv+40v^2+1\right)^{3/2}. \end{aligned}$$

Since we are interested in the curvature at the point $(0,0,0)$ we evaluate the first and second fundamental forms at that point to get

$$\begin{aligned} I_p(0,0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ II_p(0,0,0) &= \begin{bmatrix} -2 & 6 \\ 6 & -2 \end{bmatrix} \end{aligned}$$

Using the coefficients of the first and second fundamental form, we compute the Gaussian curvature

$$K = \frac{ef-f^2}{EG-F^2} = -36.$$

We can define the derivative of the Gauss map on the surface intrinsically as

$$dN_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where

$$\begin{aligned} a_{11} &= \frac{fF-eG}{EG-F^2} \\ a_{12} &= \frac{gF-fG}{EG-F^2} \\ a_{21} &= \frac{eF-fE}{EG-F^2} \\ a_{22} &= \frac{fF-gE}{EG-F^2}. \end{aligned}$$

Evaluating it at the point $(0,0,0)$ gives us

$$dN_p = \begin{bmatrix} 2 & -6 \\ -6 & 2 \end{bmatrix}.$$

The eigenvalues and eigenvectors of dN_p are the principle curvature and principle directions which are

$$\lambda = -4, 8$$
$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

■

Exercise 4. Let $\mathbb{H}^2 = \left\{(x_1 \ldots . x_n) \in \mathbb{R}^2 : x_n > 0\right\}$ and let $g_{ij} = \frac{\delta_{ij}}{x_n^2}$. Prove that $\mathbb{H}^n$ with this metric has constant sectional curvature -1.

Proof: Since we did exercise 1 on page 179 of Do Carmo I am going to use the results in this proof without proving them since I proved them previously. We can rewrite the metric as

$$g_{ij} = \frac{\delta_{ij}}{F^2},$$

where $F = x_n$. According to what we proved in exercise 1 on page 179, the metric g_{ij} has constant curvature K if and only if F is of the form

$$F = \sum G_i(x_i),$$

where

$$G_i(x_i) = a x_i^2 + b_i x_i + c_i.$$

By the definition of our metric

$$G_i(x_i) = \begin{cases} 0 & i \neq n \\ x_i & i = n \end{cases},$$

therefore, the metric has constant curvature. From the same exercise, it follows that the constant curvature K can calculated as

$$\sum \left(4c_i a - b_i^2\right) = K,$$

which is simplified to

$$-1.$$

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Exercise 5. Show that if J is a Jacobi field such that $J(0) = 0$ along a geodesic γ that the component of J tangent to γ and normal to γ are both Jacobi fields.

Proof: For simplicity, we assume that the geodesic has unit speed, i.e. $\|\gamma'\| = 1$. Let $\{E_i\}$ be an orthonormal frame along γ, then

$$J(t) = \sum J_i(t) E_i$$
$$\gamma'(t) = \gamma_i'(t) E_i.$$

The component of $J(t)$ along the geodesic is

$$\langle J, \gamma' \rangle \frac{\gamma'}{\|\gamma'\|} = \left(\sum J_j \gamma_j\right) \gamma_i' E_i = J^T(t).$$

Since $\frac{D\gamma'}{dt} = 0$ and $R(\gamma', \gamma')\gamma' = 0$, $\frac{DJ^\top}{dt} = 0$ and $R\left(\gamma', J^\top\right)\gamma' = 0$, thus $J^\top(t)$ is a Jacobi field. The component of J normal to γ is

$$J - J^\top = J^N.$$

Since Jacobi fields form a vector space, J^N must also be a Jacobi field. We can show this differently. The Jacobi field J is a solution to

$$\frac{D^2 J}{dt} + R(\gamma', J)\gamma' = 0,$$

thus the solution to J^N is

$$\frac{D^2 J}{dt} + R(\gamma', J)\gamma' - \frac{D^2 J^\top}{dt} - R\left(\gamma', J^\top\right)\gamma' = 0.$$

Since $\frac{D^2 J^\top}{dt} = 0$ and $R\left(\gamma', J^\top\right)\gamma' = 0$, we get that

$$\frac{D^2 J}{dt} + R\left(\gamma', J\right)\gamma' = 0,$$

thus J^N is also a solution. ■

Exercise 6. Identify $\mathbb{R}^4$ with C^2 by letting (x_1, x_2, x_3, x_4) correspond to $(x_1 + ix_2, x_3 + ix_4)$. Let

$$S^3 = \left\{(z_1, z_2) \in C^2 \,:\, |z_1| + |z_2| = 1\right\},$$

and let $h : S^3 \to S^3$ be given by

$$h\left(z_1, z_2\right) = \left(e^{\frac{2\pi i}{q}} z_1, e^{\frac{2\pi i r}{q}} z_2\right), \quad (z_1, z_2) \in S^3$$

where q and r are relatively prime integers, $q > 2$. Show that $G = \left\{id, h, \ldots h^{q-1}\right\}$ is a group of isometries of the sphere S^3, with the usual metric, which operates in a totally discontinuous manner.

Proof: We begin by showing that it is a group with the operator being the concatenation . By construction of the set G, $id \in G$. The function h^k can be written as

$$h^k\left(z_1, z_2\right) = \left(e^{\frac{k2\pi i}{q}} z_1, e^{\frac{k2\pi i r}{q}} z_2\right),$$

where $k = \{0, \ldots, q-1\}$. From this we can easily see that $h^{(q-k)\mathrm{mod} q} \circ h^k = id$. Thus the inverse exists. Lastly, we can see that $id \circ h^k = h^k \circ id = h^k$. Therefore, G is a group. Next, we proceed to show that h^k acts totally discontinuously. Let $x = (z_1, z_2) \in S^3$. We construct the subsets $U_1, U_2 \subset S^3$ as follows. Let U_i be a neighborhood of z_i such that

$$h^1\left(U_1, U_2\right) \cap U_1 \times U_2 = \emptyset$$

and

$$h^{q-1}\left(U_1, U_2\right) \cap U_1 \times U_2 = \emptyset.$$

If such a set exists, then it follows that

$$h^m\left(U_1, U_2\right) \cap U_1 \times U_2 = \emptyset$$

for all $m = \{1, \ldots, q-1\}$. Thus we just need to construct U_1 and U_2. Let

$$U_1 = \left\{\left(e^{ni} z_1, z_2\right) : -\frac{\pi}{q} < n < \frac{\pi}{q}\right\}$$
$$U_2 = \left\{\left(z_1, e^{ni} z_2\right) : -\frac{\pi r}{q} < n < \frac{\pi r}{q}\right\}.$$

In other words

$$U_1 = \left(e^{-\frac{\pi}{q} i} z_1, e^{\frac{\pi}{q} i} z_1\right) \times z_2$$

and

$$U_2 = z_1 \times \left(e^{-\frac{\pi r}{q} i} z_2, e^{\frac{\pi r i}{q}} z_2\right).$$

Applying h^1 we get

$$h^1\left(U_1, U_2\right) = U_1' \times U_2'$$

where

$$U_1' = \left(e^{\frac{\pi}{q} i} z_1, e^{\frac{3\pi}{q} i} z_1\right) \times z_2$$
$$U_2' = z_1 \times \left(e^{\frac{\pi r}{q} i} z_2, e^{\frac{3\pi r i}{q}} z_2\right).$$

From this we can see that

$$h^1\left(U_1, U_2\right) \cap U_1 \times U_2 = \emptyset.$$

We can similarly show that

$$h^{q-1}\left(U_1, U_2\right) \cap U_1 \times U_2 = \emptyset.$$

Thus, G is a totally discontinuous group. Since the functions h^k are simply rotating z_i around a circle and rotation preserve length, we know that G is also a group of isometries. In summary, G is a totally discontinuous group of isometries. ■