Homework 4

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Exercise 1. For the set $C^\infty(M)$ there exists an additive identity, addition is closed, there exists additive inverses, there is a multiplication identity, and multipilication is closed. These properties show that $C^\infty(M)$ is a ring. Prove that if $f, g \in C^\infty(M)$ and $Y, Z \in \mathcal{X}^\infty(M)$ we have

1) $f(Y+Z) = fY + fZ$
2) $(f+g)Y = fY + gY$
3) $(fg)Y = f(gY)$, and
4) $1Y = Y$

Proof: We suppose directly that $f, g \in C^\infty(M)$ and $Y, Z \in \mathcal{X}^\infty(M)$. Let (U, φ) be a smooth chart on M local coordinates $(x_1, \ldots, x_n)$, $p \in U$ and f, g, Y and Z be given in the local coordinates. The first property is

$$\begin{aligned}
f(Y+Z) &= f(p)\left(y_i(p)\frac{\partial}{\partial x_i} + z_i(p)\frac{\partial}{\partial x_i}\right) \\
&= f(p)\,y_i(p)\frac{\partial}{\partial x_i} + f(p)\,z_i(p)\frac{\partial}{\partial x_i} \\
&= fY + fZ.
\end{aligned}$$

The second property is

$$\begin{aligned}
(f+g)Y &= (f(p)+g(p))\,y_i(p)\frac{\partial}{\partial x_i} \\
&= f(p)\,y_i(p)\frac{\partial}{\partial x_i} + g(p)\,y_i(p)\frac{\partial}{\partial x_i} \\
&= fY + gY.
\end{aligned}$$

The third property is

$$\begin{aligned}
(fg)Y &= (f(p)\,g(p))\,y_i(p)\frac{\partial}{\partial x_i} \\
&= f(p)\,g(p)\,y_i(p)\frac{\partial}{\partial x_i} \\
&= f(p)\left(g(p)\,y_i(p)\frac{\partial}{\partial x_i}\right). \\
&= f(gY).
\end{aligned}$$

The last property is

$$\begin{aligned}
1Y &= 1(p)\,y_i(p)\frac{\partial}{\partial x_i} \\
&= y_i(p)\frac{\partial}{\partial x_i},
\end{aligned}$$

with $1(p)$ being the multiplicative identity. ■

Exercise 2. Let $X, Y, Z \in \mathcal{X}^\infty(M)$ be smooth vector fields, and $f, g \in C^\infty(M)$ be smooth functions. Prove the following:

1) Jacobi's identity: $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$
2) $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$

Proof: We suppose directly that $f, g \in C^\infty(M)$ and $X, Y, Z \in \mathcal{X}^\infty(M)$. Let (U, φ) be a smooth chart on M local coordinates $(x_1, \ldots, x_n)$, $p \in U$, and the functions and vector fields be given in local coordinates.

To prove the first one we will take it term by term

$$
\begin{aligned}
[[X,Y],Z] &= \left[x_i(p)\frac{\partial}{\partial x_i}\left(y_j(p)\frac{\partial}{\partial x_j}\right) - y_i(p)\frac{\partial}{\partial x_i}\left(x_j(p)\frac{\partial}{\partial x_j}\right), Z\right] \\
&= \left[x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} + x_i(p)\,y_i(p)\frac{\partial^2}{\partial x_i\partial x_j} - y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} - y_i(p)\,x_i(p)\frac{\partial^2}{\partial x_i\partial x_j}, Z\right] \\
&= \left[x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} - y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}, Z\right] \\
&= \left(x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} - y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}\right)z_k(p)\frac{\partial}{\partial x_k} - z_k(p)\frac{\partial}{\partial x_k}\left(x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} - y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}\right) \\
&= x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}\left(z_k(p)\frac{\partial}{\partial x_k}\right) - y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}\left(z_k(p)\frac{\partial}{\partial x_k}\right) - z_k(p)\frac{\partial}{\partial x_k}\left(x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}\right) \\
&\quad + z_k(p)\frac{\partial}{\partial x_k}\left(y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}\right) \\
&= x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\left.\frac{\partial z_k}{\partial x_j}\right|_p\frac{\partial}{\partial x_k} + x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p z_k(p)\frac{\partial^2}{\partial x_j\partial x_k} - y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\left.\frac{\partial z_k}{\partial x_j}\right|_p\frac{\partial}{\partial x_k} - y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p z_k(p)\frac{\partial^2}{\partial x_j\partial x_k} \\
&\quad - z_k(p)\left.\frac{\partial x_i}{\partial x_k}\right|_p\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} - z_k(p)\,x_i(p)\frac{\partial^2}{\partial x_k\partial x_j} + z_k(p)\frac{\partial y_i}{\partial x_k}\left|\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} + z_k(p)\,y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial^2}{\partial x_k\partial x_j} \\
&= x_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\left.\frac{\partial z_k}{\partial x_j}\right|_p\frac{\partial}{\partial x_k} - y_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\left.\frac{\partial z_k}{\partial x_j}\right|_p\frac{\partial}{\partial x_k} - z_k(p)\left.\frac{\partial x_i}{\partial x_k}\right|_p\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} + z_k(p)\left.\frac{\partial y_i}{\partial x_k}\right|_p\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}.
\end{aligned}
$$

Following the patter we get

$$
[[Y,Z],X] = y_i(p)\left.\frac{\partial z_j}{\partial x_i}\right|_p\left.\frac{\partial x_k}{\partial x_j}\right|_p\frac{\partial}{\partial x_k} - z_i(p)\left.\frac{\partial y_j}{\partial x_i}\right|_p\left.\frac{\partial x_k}{\partial x_j}\right|_p\frac{\partial}{\partial x_k} - x_k(p)\left.\frac{\partial y_i}{\partial x_k}\right|_p\left.\frac{\partial z_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} + x_k(p)\left.\frac{\partial z_i}{\partial x_k}\right|_p\left.\frac{\partial y_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}.
$$

$$
[[Z,X],Y] = z_i(p)\left.\frac{\partial x_j}{\partial x_i}\right|_p\left.\frac{\partial y_k}{\partial x_j}\right|_p\frac{\partial}{\partial x_k} - x_i(p)\left.\frac{\partial z_j}{\partial x_i}\right|_p\left.\frac{\partial y_k}{\partial x_j}\right|_p\frac{\partial}{\partial x_k} - y_k(p)\left.\frac{\partial z_i}{\partial x_k}\right|_p\left.\frac{\partial x_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} + y_k(p)\left.\frac{\partial x_i}{\partial x_k}\right|_p\left.\frac{\partial z_j}{\partial x_i}\right|_p\frac{\partial}{\partial x_j}.
$$

Adding the three terms together and adjusting indices we can see that the terms cancel. Thus $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

To prove the second we will expand out the term

$$
\begin{aligned}
[fX, gY] &= f(p)\,x_i(p)\frac{\partial}{\partial x_i}\left(g(p)\,y_j(p)\frac{\partial}{\partial x_j}\right) - g(p)\,y_i(p)\frac{\partial}{\partial x_i}\left(f(p)\,x_j\frac{\partial}{\partial x_j}\right) \\
&= f(p)\,x_i(p)\left(y_j(p)\left.\frac{\partial g}{\partial x_i}\right|_p\frac{\partial}{\partial x_j} + g(p)\frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial x_j}\right) - g(p)\,y_i(p)\left(\left.\frac{\partial f}{\partial x_i}\right|_p x_j\frac{\partial}{\partial x_j} + f(p)\frac{\partial x_j}{\partial x_i}\frac{\partial}{\partial x_j}\right) \\
&= fX(g)\,Y + fgXY - gY(f)\,X - gfYX \\
&= fg[X,Y] + fX(g)\,Y - gY(f)\,X.
\end{aligned}
$$

■

Exercise 3. Compute $[V,W]$ on $\mathbb{R}^3$ for the following pairs of vector fields:

1) $V = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}$ and $W = \frac{\partial}{\partial y}$.
2) $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $W = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$

The first one is

$$
\begin{aligned}
[V,W] &= \left(y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y}\right) - \left(\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}\right) \\
&= y\frac{\partial^2}{\partial z\partial y} - 2xy^2\frac{\partial^2}{\partial y\partial y} - \frac{\partial}{\partial z} - y\frac{\partial^2}{\partial y\partial z} + 4xy\frac{\partial}{\partial y} + 2xy^2\frac{\partial^2}{\partial y\partial y} \\
&= 4xy\frac{\partial}{\partial y} - \frac{\partial}{\partial z}.
\end{aligned}
$$

and the second one is

$$\begin{aligned}
[V,W] &= \left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)-\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\\
&= x\frac{\partial}{\partial z}+xy\frac{\partial^2}{\partial y\partial z}-xz\frac{\partial^2}{\partial y\partial y}-y^2\frac{\partial^2}{\partial x\partial z}+yz\frac{\partial^2}{\partial x\partial y}-yx\frac{\partial^2}{\partial z\partial y}+y^2\frac{\partial^2}{\partial z\partial x}+zx\frac{\partial^2}{\partial y\partial y}-z\frac{\partial}{\partial x}-zy\frac{\partial^2}{\partial y\partial x}\\
&= x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}.
\end{aligned}$$

Exercise 4. Let $f : M \to N$ be a diffeomorphism on connected oriented manifolds. Show that if $df_x : T_xM \to T_{f(x)}N$ preserves orientation at one point x, then f preserves orientation globally.

Proof: Let $\{(U_i,\varphi_i)\}$ denote the oriented charts on M and $\{(V_i,\psi_i)\}$ denote the oriented charts on N, at let $f : M \to N$ be a diffeomorphism and that $df_x : T_xM \to T_{f(x)}N$ preserves orientation at one point $x \in U_j$. Suppose by contradiction that f does not preserve orientation in U_j, then there exists a point $p \in U_j$ such that $df_p : T_pM \to T_{f(p)}N$ reverses orientation, i.e. $\det(df) < 0$. Since f is a diffeomorphism df is continuous, thus there must exists a point $q \in U_j$ such that $\det\left(\frac{\partial}{\partial x_i}\psi_j \circ f \circ \varphi_j^{-1}\Big|_{\varphi_j(q)}\right) = 0$. This is a contradiction since f is a diffeomorphism and thus always invertible. Therefore, f preserves orientation in U_j. Now suppose directly that $y \in U_j \cap U_k \neq \emptyset$ and $f(y) \in V_j$. We have shown that f preserves orientation in U_j, thus f preserves orientation at $y \in U_j$. Stitching the two charts together, we get

$$\begin{aligned}
\det\left(\frac{\partial}{\partial x_i}\psi_j \circ f \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_k^{-1}\Big|_{\varphi_k(y)}\right) &= \det\left(\frac{\partial}{\partial x_i}\psi_j \circ f \circ \varphi_j^{-1}\Big|_{\varphi_j(y)}\right)\det\left(\frac{\partial}{\partial x_i}\varphi_j \circ \varphi_k^{-1}\Big|_{\varphi_k(y)}\right)\\
&= \det\left(\frac{\partial}{\partial x_i}\psi_j \circ f \circ \varphi_j^{-1}\Big|_{\varphi_j(y)}\right)\alpha\\
&> 0
\end{aligned}$$

where $\alpha > 0$ since $(U_k,\varphi_k),(U_j,\varphi_j)$ are elements of the collection of oriented charts on M. Therefore f preserves orientation in U_k as well. Similarly, let $z \in U_j,\ f(z) \in V_j, V_k$. Stitching the two charts together we get

$$\begin{aligned}
\det\left(\frac{\partial}{\partial x_i}\psi_k \circ \psi_j^{-1} \circ \psi_j \circ f \circ \varphi_j^{-1}\Big|_{\varphi_j(z)}\right) &= \det\left(\frac{\partial}{\partial x_i}\psi_k \circ \psi_j^{-1}\Big|_{\psi_j(f(z))}\right)\det\left(\frac{\partial}{\partial x_i}\psi_j \circ f \circ \varphi_j^{-1}\Big|_{\varphi_j(z)}\right)\\
&= \beta\det\left(\frac{\partial}{\partial x_i}\psi_j \circ f \circ \varphi_j^{-1}\Big|_{\varphi_j(z)}\right)\\
&> 0,
\end{aligned}$$

where $\beta > 0$ since $(V_k,\psi_k)$and (V_j,ψ_k) are elements of the collection of oriented charts on N. Therefore, f preserves orientation in V_j. This process can be repeated for every chart in the collection of oriented charts which proves that f preserves orientation globally. ∎