Homework 9

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Exercise 1. Let M be a Riemannian manifold with the property: given any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p to q. Prove that the curvature of M is identically zero, that is, for all $X, Y, Z \in \mathcal{X}(M)$, $R(X,Y)Z = 0$.

Proof: Usint the hint provided in the book. Consider a parameterized surface $f : U \subseteq \mathbb{R}^2 \to M$, where

$$U = \left\{(s,t) \in \mathbb{R}^2; -\epsilon < t < 1+\epsilon, -\epsilon < s < 1+\epsilon, \epsilon > 0\right\}$$

and $f(s,0) = f(0,0)$, for all s. Let $V_0 \in T_{f(0,0)}M$ and define a field V along f by: $V(s,0) = V_0$ and, if $t \neq 0$, V(s,t) is the parallel transport of V_0 along the curve $t \to f(s,t)$. Then, from Lemma 4.1,

$$\frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V.$$

Since parallel transport does not depend on the curve chosen, $V(s,1)$ is the parallel transport of $V(0,1)$ along the curve $s \to f(s,1)$, hence $\frac{D}{\partial s}V(s,1) = 0$. Thus

$$R_{f(0,1)}\left(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1)\right)V(0,1) = 0.$$

Since the surface f and V are arbitrary, then for all $X, Y, Z \in \mathcal{X}(M)$ we have

$$R(X,Y)Z = 0.$$

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Exercise 2. Compute the components R^l_{ijk} of the curvature tensor and the sectional curvature for

1) The cylinder, and
2) the hyperbolic upper half plane.

For part 1) we can parameterize the surface as $r(u,v) = (\cos(u), \sin(u), v)$. The differential is

$$dr = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 1 \end{bmatrix}.$$

Using the differential we compute the metric to be

$$g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From the metric we get that the components $R^l_{ijk} = 0$ for the cylinder.

For part 2) let $M \subset \mathbb{R}^2$ denote the upper half plane. Then for $(x,y) \in M$, we use the Riemannian metric defined as

$$g = \begin{bmatrix} \frac{1}{y^2} & \\ & \frac{1}{y^2} \end{bmatrix}.$$

Using the metric, we compute the Christoffel symbols and get

$$\begin{aligned} \Gamma^1_{12} &= \Gamma^1_{21} = -\frac{1}{y} \\ \Gamma^2_{11} &= \frac{1}{y} \\ \Gamma^2_{22} &= -\frac{1}{y} \\ \Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^1_{11} = \Gamma^1_{22} = 0. \end{aligned}$$

Using the Chrstoffel symbols, the components of $R^l_{ijk} = 0$ except for

$$\begin{aligned} R^2_{211} &= \frac{2}{y^2} \\ R^2_{121} &= -\frac{2}{y^2}. \end{aligned}$$

Exercise 3. Let $\omega \in \Omega^2\left(\mathbb{R}^3\right)$ be given by

$$\omega = e^{xz}dx \wedge dy - \sin(y)z^2 dy \wedge dz.$$

compute $d\omega$.

The first term $e^{xz}dx \wedge dy$ will be zero when taking the derivative w.r.t. x or y and the second term $\sin(y)z^2dy \wedge dz$will be zero when taking the derivative w.r.t. y or z, so we get

$$\begin{aligned} d\omega &= \frac{\partial e^{xz}dx \wedge dy}{\partial z} - \frac{\partial \sin(y)z^2 dy \wedge dz}{\partial x} \\ &= xe^{xz}dx \wedge dy \wedge dz - 0 \\ &= xe^{xz}dx \wedge dy \wedge dz \end{aligned}$$

Exercise 4. Let V be a finite dimensional vector space. Prove that there is a canonical isomorphism (basis-independent) between the space of bilinear maps of $V \otimes V^*$ and the space of linear maps from $V \to V$ denoted Hom(V,V).

Proof: Let $A : V \to V$ be a linear map, $f \in T^1_1(V)$ be a bilinear map and g be a metric on V. To the linear transformation A we associate a bilinear map $(v, \varphi) \mapsto \langle Av, \varphi\rangle_g$ on $V \times V^*$ to $\mathbb{R}$. The metric g is unique and independent of basis, so the association is independent of basis. Let $dx_1, \ldots, dx_n$ be a basis for V, $\partial x_1, \ldots, \partial x_n$ be a basis for V^*, then in local coordinates $A = a^j_i dx^i \otimes \partial x_j$, $f = f^j_i dx^i \otimes \partial x_j$, and $g = g^{ik}\partial x_i \otimes \partial x_j$ be a basis for g. Since g is positive definite, the coefficients g_{ij} in matrix form are invertible. We can now associated to A to f using g as

$$f^j_i dx^i \otimes \partial x_j = g^{jk}a^i_k.$$

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Exercise 5. Show that the restriction of $\sigma = x^1dx^2 - x^2dx^1 + x^3dx^4 - x^4dx^3$ from $\mathbb{R}^4$ to the sphere S^3 is never zero on S^3.

Proof: The manifold S^3 is defined as

$$S^3 = \left\{ \left(x^1, x^2, x^3, x^4\right) \in \mathbb{R}^4 \mid \sqrt{\left(\sum_{i=1}^{4} \left(x^i\right)^2\right)} = 1 \right\};$$

therefore, the vectors in T_pS mapped into $T_p\mathbb{R}^4$ have the constraint

$$\sum_{i=1}^{4} \dot{x}^i x^i = 0.$$

I can rotate my sphere using a rotation R such that the rotated point of p is

$$Rp = (1, 0, 0, 0).$$

At this point, the constraint on the velocity is

$$\dot{x}^1 \cdot 1 + \dot{x}^2 \cdot 0 + \dot{x}^3 \cdot 0 + \dot{x}^4 \cdot 0 = 0$$

thus

$$\dot{x}^1 = 0$$

and

$$\dot{x}^2 \text{ and/or } \dot{x}^3 \text{ and or } \dot{x}^4 \neq 0.$$

Let $\dot{x} \in T_p\mathbb{R}^4$ defined as $\dot{x} = \left[\dot{x}^1, \dot{x}^2, \dot{x}^3, \dot{x}^4\right]^\top$ with the above constraint. The restriction can be written as a tensor in matrix notation as

$$\sigma = \begin{bmatrix} -x^2 & x^1 & -x^4 & x^3 \end{bmatrix},$$

then

$$\sigma\left(\dot{x}\right) = -\dot{x}^1 x^2 + \dot{x}^2 x^1 - \dot{x}^3 x^4 + \dot{x}^4 x^3.$$

At the rotated point of p, the restriction simplifies to

$$\sigma\left(\dot{x}\right) = \dot{x}^2.$$

This component of the velocity can be zero with the velocity vector not being zero; therefore $\sigma\left(\dot{x}\right)$ can be zero with the satisfied constraints. I have shown that the statement is false. So I don't think I completely understand the statement I am trying to prove. ■

Exercise 6. As in problem (1) of Homework 4, prove that the set of all smooth covector fields on M is a $C^\infty(M)$ module over the functions $C^\infty(M)$.

Proof: Let $x \in M$, $v \in TM$, $a, b \in C^\infty(M)$ and $f, g \in \mathcal{X}^*(M)$. For the set of all smooth covector fields on M to be a module, it must have the the following properties. ■

1) $a\left(f_x(v) + g_x(v)\right) = af_x(v) + ag_x(v)$
2) $(a+b) f_x(v) = af_x(v) + bf_x(v)$
3) $(ab) f_x(v) = a\left(bf_x(v)\right)$
4) $1f_x(v) = f_x(v)$

where the subscript x denotes the covector field evaluated at x.

In local coordinates we can write $f = f^i \partial x_i$, $g = g^i \partial x_i$, thus

1) $a\left(f^i \partial x_i + g^i \partial x_i\right) = af^i \partial x_i + ag^i \partial x_i$
2) $(a+b) f^i \partial x_i = af^i \partial x_i + bf^i \partial x_i$
3) $(ab) f^i \partial x_i = a\left(bf^i \partial x_i\right)$
4) $1f^i \partial x_i = f^i \partial x_i$

Therefore, the set of all smooth covector fields on M is a $C^\infty(M)$ module.