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I have used MATLAB to faciltate calculating some of the components for the metric, Christoffel symbols, curvature symbols, etc. I have attached the pdf version of the code I used as part of my work.

Exercise 1. Prove part (i) of Proposition 2.2 on pg. 90 of do Carmo. The proposition states: let $f, g \in \mathcal{D}(M)$ and X_1, X_2, Y_1, Y_2. R is bilinear in $\mathcal{X}(M) \times \mathcal{X}(M)$, that is,

$$\begin{aligned} R(fX_1 + gX_2, Y_1) &= fR(X_1, Y_1) + gR(X_2, Y_1) \\ R(X_1, fY_1 + gY_2) &= fR(X_1, Y_1) + gR(X_1, Y_2). \end{aligned}$$

Proof: Proof let $Z \in \mathcal{X}(M)$. The first one can be shown as

$$\begin{aligned} R(fX_1 + gX_2, Y_1) Z &= \nabla_{Y_1} \nabla_{fX_1 + gX_2} Z - \nabla_{fX_1 + gX_2} \nabla_{Y_1} Z + \nabla_{[fX_1 + gX_2, Y_1]} Z \\ &= \nabla_{Y_1} (f\nabla_{X_1} + g\nabla_{X_2}) Z - (f\nabla_{X_1} + g\nabla_{X_2}) \nabla_{Y_1} Z + \nabla_{[fX_1, y_1]} Z + \nabla_{[gX_2, Y_1]} Z \\ &= (f\nabla_{Y_1}\nabla_{X_1} + \nabla_{Y_1}(f)\nabla_{X_1} + g\nabla_{Y_1}\nabla_{X_2} + \nabla_{Y_1}(g)\nabla_{X_2} - (f\nabla_{X_1} + g\nabla_{X_2})\nabla_{Y_1}) Z \\ &+ \left(\nabla_{f[X_1,Y_1] - Y_1(f)X_1} + \nabla_{g[X_2,Y_1] - Y_1(g)X_2}\right) Z \\ &= \left(f\nabla_{Y_1}\nabla_{X_1} + g\nabla_{Y_1}\nabla_{X_2} - (f\nabla_{X_1} + g\nabla_{X_2})\nabla_{Y_1} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_2,Y_2]}\right) Z \\ &+ (+\nabla_{Y_1}(f)\nabla_{X_1} + \nabla_{Y_1}(g)\nabla_{X_2} - \nabla_{Y_1}(f)\nabla_{X_1} - \nabla_{Y_1}(g)\nabla_{X_2}) Z \\ &= \left(f\nabla_{Y_1}\nabla_{X_1} + g\nabla_{Y_1}\nabla_{X_2} - (f\nabla_{X_1} + g\nabla_{X_2})\nabla_{Y_1} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_2,Y_2]}\right) Z \\ &= fR(X_1, Y_1) Z + gR(X_2, Y_1) Z. \end{aligned}$$

Similarly the second one is, omitting the vector field Z,

$$\begin{aligned} R(X_1, fY_1 + gY_2) &= \nabla_{fY_1 + gY_2} \nabla_{X_1} - \nabla_{X_1} \nabla_{fY_1 + gY_2} + \nabla_{[X_1, fY_1 + gY_2]} \\ &= (f\nabla_{Y_1} + g\nabla_{Y_2})\nabla_{X_1} - f\nabla_{X_1}\nabla_{Y_1} - g\nabla_{X_1}\nabla_{Y_2} - \nabla_{X_1}(f)\nabla_{Y_1} - \nabla_{X_1}(g)\nabla_{Y_2} \\ &+ f\nabla_{[X_1,Y_1]} + g\nabla_{[X_1,Y_2]} + \nabla_{X_1}(f)\nabla_{Y_1} + \nabla_{X_1}(g)\nabla_{Y_2} \\ &= (f\nabla_{Y_1} + g\nabla_{Y_2})\nabla_{X_1} - f\nabla_{X_1}\nabla_{Y_1} - g\nabla_{X_1}\nabla_{Y_2} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_1,Y_2]} \\ &= fR(X_1, Y_1) + gR(X_1, Y_2). \end{aligned}$$

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Exercise 2. Let S_r^2 be the sphere of radius r in $\mathbb{R}^3$ centered at the origin. Equip S_r^2 with the metric induced by Euclidean space. Consider the coordinate charts obtained by restricting the orthogonal projection of $\mathbb{R}^3$ to the coordinate planes.

1) Compute the components of the Riemann curvature R_{ijk}^s in these coordinates.
2) Use this to compute the sectional curvature $K(\sigma)$ at a point $p \in S_r^2$.
3) Prove that $K(\sigma)$ is constant.

We will use the charts $\left\{\left(U_i^\pm, \varphi_i^\pm\right)\right\}$ defined as

$$\begin{aligned} U_j^\pm &= \left\{(x_1, x_2, x_3) \in S_r^2 : \pm x_j > 0\right\} \\ \varphi_1^\pm(x_1, x_2, x_3) &= (x_2, x_3) \\ \left(\varphi_1^\pm\right)^{-1}(y_1, y_2) &= \left(\pm\sqrt{r^2 - y_1^2 - y_2^2}, y_1, y_2\right). \end{aligned}$$

Since S_r^2 is embedded in $\mathbb{R}^3$, the derivative of $\left(\varphi_j^\pm\right)^{-1}$ is injective.

$$d\left(\varphi_1^\pm\right)^{-1} = \begin{bmatrix} -\pm\frac{y_1}{\sqrt{r^2 - y_1^2 - y_2^2}} & -\pm\frac{y_2}{\sqrt{r^2 - y_1^2 - y_2^2}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $v_1, v_2 \in T_p S_r^2$ where $p \in \varphi_1^\pm$, then using the induced metric we have

$$\left\langle d\left(\varphi_1^\pm\right)^{-1} v_1, d\left(\varphi_1^\pm\right)^{-1} v_2 \right\rangle.$$

Thus the induced metric g in matrix form is

$$\left(d\left(\varphi_1^{\pm}\right)^{-1}\right)^{\top} d\left(\varphi_1^{\pm}\right)^{-1} = \begin{bmatrix} -\pm\frac{y_1}{\sqrt{r^2-y_1^2-y_2^2}} & 1 & 0 \\ -\pm\frac{y_2}{\sqrt{r^2-y_1^2-y_2^2}} & 0 & 1 \end{bmatrix} \begin{bmatrix} -\pm\frac{y_1}{\sqrt{r^2-y_1^2-y_2^2}} & -\pm\frac{y_2}{\sqrt{r^2-y_1^2-y_2^2}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{y_1^2}{(r^2-y_1^2-y_2^2)}+1 & \frac{y_1y_2}{(r^2-y_1^2-y_2^2)} \\ \frac{y_1y_2}{(r^2-y_1^2-y_2^2)} & \frac{y_2^2}{(r^2-y_1^2-y_2^2)}+1 \end{bmatrix}.$$

The computation of the Christoffel symobls is tedius, so we employed MATLAB to compute them. The Christoffel symbols are

$$\Gamma_{11}^1 = \frac{y_1\left(r^2-y_2^2\right)}{\alpha}$$
$$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{y_1^2 y_2}{\alpha}$$
$$\Gamma_{22}^1 = \frac{y_1\left(r^2-y_1^2\right)}{\alpha}$$
$$\Gamma_{11}^2 = \frac{y_2\left(r^2-y_2^2\right)}{\alpha}$$
$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{y_1 y_2^2}{\alpha}$$
$$\Gamma_{22}^2 = \frac{y_2\left(r^2-y_1^2\right)}{\alpha}$$
$$\alpha = r^2\left(r^2-y_1^2-y_2^2\right).$$

Once again using MATLAB, the coefficients of the curvature are

$$R_{111}^1 = R_{221}^1 = R_{112}^1 = R_{222}^1 = R_{111}^2 = R_{221}^2 = R_{112}^2 = R_{222}^1 = 0$$
$$R_{121}^1 = \frac{-y_1y_2}{\alpha}$$
$$R_{211}^1 = \frac{y_1y_2}{\alpha}$$
$$R_{122}^1 = \frac{2y_1^2-r^2}{\alpha}$$
$$R_{212}^1 = \frac{r^2-2y_1^2}{\alpha}$$
$$R_{121}^2 = \frac{r^2-y_2^2}{\alpha}$$
$$R_{211}^2 = \frac{y_2^2-r^2}{\alpha}$$
$$R_{122}^2 = \frac{2y_1y_2}{\alpha}$$
$$R_{212}^2 = \frac{-2y_1y^2}{\alpha}$$

where α has been previously defined.

Now we proceed to compute the sectional curvature $K(\sigma)$ at point $p \in S_r^2$. Since S_r^2 is two dimensional, any two linearlly independent vectors in $T_pS_r^2$ will span $T_pS_r^2$. So we will use $v_1 = \frac{\partial}{\partial y_1}$ and $v_2 = \frac{\partial}{\partial y_2}$. The sectional curvature is then.

$$K(\sigma) = \frac{(v_1, v_2, v_1, v_2)}{\left|v_1 \wedge v_2\right|^2}$$
$$= \frac{R_{121}^1 g_{12} + R_{121}^2 g_{22}}{\langle v_1, v_1\rangle \langle v_2, v_2\rangle - \langle v_1, v_2\rangle^2}$$
$$= \frac{\left(r^2-y_1^2-y_2^2\right)}{(2r^4-3r^2y_1^2-3r^2y_2^2+y_1^4+y_2^4+y_1^2y_2^2)}.$$

We now proceed to prove that the sectional curvature is constant. According to Lemma 3.4 in do Carmo, the manifold M has constant sectional curvature equal to K_0 if an only if $R = K_0R'$ where R is the curvature of M and

$$\langle R'(X, Y, W), Z\rangle = \langle X, W\rangle \langle Y, Z\rangle - \langle Y, W\rangle \langle X, Z\rangle,$$

for all $X, Y, W, Z \in T_pM$.

The curvature is

$$\begin{aligned}\left\langle R\left(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2}\right)\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2}\right\rangle &= \sum_{\ell} R^{\ell}_{121} g_{\ell 2}\\ &= \frac{1}{r^2-y_1^2-y_2^2},\end{aligned}$$

and

$$\begin{aligned}\left\langle R'\left(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\frac{\partial}{\partial y_1}\right),\frac{\partial}{\partial y_2}\right\rangle &= \left\langle\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_1}\right\rangle\left\langle\frac{\partial}{\partial y_2},\frac{\partial}{\partial y_2}\right\rangle - \left\langle\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2}\right\rangle\left\langle\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2}\right\rangle\\ &= \frac{\left(2r^4-3r^2y_1^2-3r^2y_2^2+y_1^4+y_2^4+y_1^2y_2^2\right)}{\left(r^2-y_1^2-y_2^2\right)^2}.\end{aligned}$$

From which we can see that $R = K_0 R'$. Therefore, M has constant sectional curvature.

Exercise 3. Recall the embeddings of the torus $T = \mathbb{R}/2\pi\mathbb{Z}$ is $\mathbb{R}^3$ and $\mathbb{R}^4$ given by the maps

$$\omega(\alpha,\beta) = \left((\cos(\beta)+4)\cos(\alpha),(\cos(\beta)+4)\sin(\alpha),\sin(\beta)\right)$$

and

$$\psi(\alpha,\beta) = (\cos(\alpha),\sin(\alpha),\cos(\beta),\sin(\beta))$$

respectively. Let T_3 be the torus equipped with the metric induced from $\mathbb{R}^3$ by the map ω, and let T_4 denote the torus equipped with the metric induced from $\mathbb{R}^4$ by the map ψ. Compute the components R^s_{ijk} of the curvature of T_3 and T_4 (in the coordinates induced by ω and ψ).

With the help of MATLAB, I get that the components R^s_{ijk} of T_3 are all zero except

$$\begin{aligned}R^1_{122} &= \frac{-(4\cos(\beta)+1)}{(\cos(\beta)+4)^2}\\ R^1_{212} &= -R^1_{122}\\ R^2_{121} &= 4\cos(\beta)+2\cos(\beta)^2-1\\ R^2_{211} &= -R^2_{121}\end{aligned}$$

The components R^s_{ijk} of T_4 are all zero. This means that the torus in $\mathbb{R}^4$ with the chosen embedding is flat.

Exercise 4. For a parameterized surface S in $\mathbb{R}^3$ given by $r(u,v)$ we can find a unit normal at $p \in S$ by

$$N(p) = \frac{r_u \times r_v}{\|r_u \times r_v\|}.$$

The Gauss map is $N : S \to S^2$ defined by the equation above. The derivative of the map is $dN_p : T_pS \to T_{N(p)}S^2$. However, by construction we know that T_pS and $T_{N(P)}S^2$ have parallel tangent planes in $\mathbb{R}^3$ so we can think of dN_p as a map from $T_pS \to T_pS$. The idea is the following: For a parameterized curve $\alpha(t)$ in S such that $\alpha(0) = p$ we consider the curve $N(\alpha(t)) = N(t)$ in S^2. The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in T_pS. So dN_p measures how N "pulls away from" $N(p)$.

1) For a plane $ax + by + cz = d$ show that $dN \equiv 0$.
2) For the unit sphere with inward pointing normals show that $dN_p v = -v$.
3) Find dN_p for the cynlider with $r(u,v) = (\cos u, \sin u, v)$.
4) For the hyperbolic paraboloid $r(u,v) = \left(u, v, v^2-u^2\right)$ compute the unit normal vectors. At $p = (0,0,0)$ show that

$$dN_p\left(u'(0), v'(0), 0\right) = \left(2u'(0), -2v'(0), 0\right).$$

So the vectors $(1,0,0)$ and $(0,1,0)$ are eigenvectors dN_p with eigenvalues 2 and -2 repectfully.

For part 1) we can parameterize the plane as

$$r(x,y) = (ax, by, d-ax-by),$$

which implies that

$$r_x = \begin{bmatrix} a\\ 0\\ -a\end{bmatrix}, \quad r_y = \begin{bmatrix} 0\\ b\\ -b\end{bmatrix}.$$

Therefore

$$N(p) = \frac{r_x \times r_y}{\|r_x \times r_y\|} = \frac{ab}{\sqrt{3}\,|ab|}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Since $N(p)$ is a constant, its derivative $dN \equiv 0$. This is what we would expect since a plane is flat. We could've parameterized the plane differently, but the result is the same.

For part 2) we can parameterize the unit spere as

$$r(\phi,\theta) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$$

which implies that

$$r_\phi = \begin{bmatrix}\cos(\phi)\cos(\theta)\\ \cos(\phi)\sin(\theta)\\ -\sin(\phi)\end{bmatrix}, \quad r_\theta = \begin{bmatrix}-\sin(\phi)\sin(\theta)\\ \sin(\phi)\cos(\theta)\\ 0\end{bmatrix},$$

where

$$dr = [r_\phi, r_\theta] : T_pS \to T_p\mathbb{R}^3.$$

The metric of the first form $g1$ is

$$g1 = \begin{bmatrix}1 & 0\\ 0 & \sin^2(\phi)\end{bmatrix}.$$

The normal vector is

$$N(p) = \frac{r_\phi \times r_\theta}{\|r_\phi \times r_\theta\|} = \begin{bmatrix}\cos(\theta)\sin(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\phi)\end{bmatrix}.$$

However, $N(p)$ is outward pointing. To get inward pointing we negate it to get

$$N'(p) = -N(p) = \begin{bmatrix}-\cos(\theta)\sin(\phi)\\ -\sin(\theta)\sin(\phi)\\ -\cos(\phi)\end{bmatrix}.$$

Taking the partial derivative, we get the differential

$$dN'(p) = \begin{bmatrix}-\cos(\phi)\cos(\theta) & \sin(\phi)\sin(\theta)\\ -\cos(\phi)\sin(\theta) & -\sin(\phi)\cos(\theta)\\ \sin(\phi) & 0\end{bmatrix}.$$

We can now compute the metric of the second form

$$\begin{aligned} g2 &= \langle dN', dr\rangle \\ &= \begin{bmatrix}-1 & 0\\ 0 & -\sin^2(\phi)\end{bmatrix}.\end{aligned}$$

Let $v = v_i \frac{\partial}{\partial x_i}$, we want to show that $dr(v) = -dN'(p)(v)$. Since dr and $dN'(p)$ are linear maps we see that $dr = -dN'(p)$; thus

$$dr(v) = -dN'(p)(v).$$

For part 3), let the surface be parameterized by $r(u,v) = (\cos(u), \sin(u), v)$. The differential is

$$dr = \begin{bmatrix}-\sin(u) & 0\\ \cos(u) & 0\\ 0 & 1\end{bmatrix},$$

The metric of the first form $g1$ is

$$g1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

and

$$N(p) = \begin{bmatrix}\cos(u)\\ \sin(u)\\ 0\end{bmatrix}.$$

Computing the differential of $N(p)$ we get

$$dN(p) = \begin{bmatrix}-\sin(u) & 0\\ \cos(u) & 0\\ 0 & 0\end{bmatrix}.$$

We can now compute the metric of the second form

$$\begin{aligned} g2 &= \langle dN', dr \rangle \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

This shows that there is no curve along the z axis which is what we would expect for a cylinder that is oriented along the z-axis.

For part 4), let the hyperbolid paraboloid be parameterized by

$$r(u, v) = \left(u, v, v^2 - u^2\right).$$

The differential is

$$dr = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2u & 2v \end{bmatrix}.$$

The metric of the first form $g1$ is

$$g1 = \begin{bmatrix} 4u^2 + 1 & -4uv \\ -4uv & 4v^2 + 1 \end{bmatrix}.$$

The unit normal vector is

$$N(p) = \frac{1}{\left(4u^2 + 4v^2 + 1\right)^{1/2}} \begin{bmatrix} 2u \\ -2v \\ 1 \end{bmatrix}.$$

The differential of $N(p)$ is

$$dN(p) = \frac{1}{\left(4u^2 + 4v^2 + 1\right)^{3/2}} \begin{bmatrix} 2\left(4v^2 + 1\right) & -8uv \\ 8uv & -2\left(3u^2 + 1\right) \\ -4u & -4v \end{bmatrix}.$$

Evaulating $dN(p)$ at $p_0 = (0, 0, 0)$ yields

$$dN(p_0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}.$$

We can now compute the metric of the second form

$$\begin{aligned} g2 &= \langle dN', dr \rangle \\ &= \frac{8u^2 + 8v^2 + 2}{\left(4u^2 + 4v^2 + 1\right)^{(3/2)}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Using the first and second form of the metric we can calculate the Gaussian curvature at $p_0 = (0, 0)$

$$\begin{aligned} K &= g1^{-1} g2\big|_{p_0} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}. \end{aligned}$$

From the Gaussian curvature we can see that the eigenvalues and eigenvectors are $\lambda \pm 2$ and $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

Exercise 5. The eigenvalues of dN_p give the maximum and the minimum curvature of curves at p. These are called the principle curvatures of S at p. What are the principle curvatures for parts a), b), c) and d) above.

For part a) we got that $dN_p \equiv 0$, so that maximum and minimum eigen values are zero as they should be since a plane is flat.

For part b) we calculated the first and second form of the metric to be

$$g1 = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(\phi) \end{bmatrix}, \quad g2 = \begin{bmatrix} -1 & 0 \\ 0 & -\sin^2(\phi) \end{bmatrix},$$

Using the first and second form of the metric we can calculate the Gaussian curvature at $p_0 = (0, 0)$

$$\begin{aligned} K &= g1^{-1} g2\big|_{p_0} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

thus we get eigenvalues $\lambda_{1,2} = -1$ with eigenvectors $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

For part c) we calculated the first and second form of the metric to be

$$g1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad g2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

Using the first and second form of the metric we can calculate the Gaussian curvature at $p_0 = (0,0)$

$$\begin{aligned} K &= g1^{-1} g2\big|_{p_0} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

thus we get eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with eigenvectors $v_1 = (1,0)$ and $v_2 = (0,1)$.

Part d) was done in the previous problem which yielded eigenvalues and eigenvectors of $\lambda \pm 2$ and $v_1 = (1,0)$ and $v_2 = (0,1)$.

Exercise 6. Let S be a parameterized surface in $\mathbb{R}^3$, $p \in S$, and $dN_p : T_pS \to T_pS$ be the Gauss map. The Gaussian curvature of S at p is $\det(dN_p)$. A point in S is

1) elliptic if $\det(dN_p) > 0$,
2) hyperbolic if $\det(dN_p) < 0$,
3) parabolic if $\det(dN_p) = 0$, but $dN_p \neq 0$,and
4) planar if $dN_p = 0$.

Classify the curvature of the plane, sphere, cylindar, and the point $(0,0,0)$ on the hyperbolic paraboloid.

In part a) of exercise 4 we found that for the plane $dN_p = 0$, thus it is planar.

In part b) of exercise 5 we found that the Gaussian curvature of the sphere is

$$K = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $\det(dN_{p_0}) = 1$, thus the sphere is elliptic.

In part c) of exercise 5 we found that the Gaussian curvature of the cylinder is

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

thus $\det(dN_{p_0}) = 0$. Therefore, the cynlinder is parabolic.

In part d) of exercise 5 we found that the Gaussian curvature of the hyperbolic plane is

$$K = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

thus $\det(dN_{p_0}) = -4$. Therefore the hyperbolic plane is hyperbolic.

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Problem 2

```
syms y1 y2 r 'real'
y = [y1,y2];
dphi = [-y1/sqrt(r^2-y1^2-y2^2), -y2/sqrt(r^2-y1^2-y2^2); 1,0; 0,1];

% commpute metric
g = simplify(dphi.'*dphi);
gi = simplify(inv(g));
```

compute christoffel symbols

```
L = sym(zeros(2,2,2));
for mm=1:2
   for ii = 1:2
       for jj=1:2
           for kk = 1:2
               L(ii,jj,mm) = L(ii,jj,mm) + sym(1/2)*(diff(g(jj,kk),
 y(ii)) + diff(g(kk,ii),y(jj)) - diff(g(ii,jj),y(kk)))*gi(kk,mm);
           end
       end
   end
end

L = simplify(L)
```

compute curvature

```
R = sym(zeros(2,2,2,2));

for ss=1:2
   for ii = 1:2
       for jj=1:2
           for kk = 1:2
               for ll = 1:2
                   R(ii,jj,kk,ss) = L(ii,kk,ll)*L(jj,ll,ss)
 - L(jj,kk,ll)*L(ii,ll,ss) + diff(L(ii,kk,ss),y(jj)) -
 diff(L(jj,kk,ss),y(ii));
               end
           end
       end
```

```
    end
end

R = simplify(R)
```

compute sectional curvature

```
v1 = sym([1;0]);
v2 = sym([0;1]);
r = simplify((R(1,2,1,1)*g(1,2) + R(1,2,1,2)*g(2,2)));
K=r/ (v1'*g*v1 + v2'*g*v2-(v1'*g*v2)^2);

K = simplify(K)
```

simplified curvatuve

```
rp=(v1'*g*v1 + v2'*g*v2-(v1'*g*v2)^2);
rp = simplify(rp)
```

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Problem 3

```
syms a b 'real'
y = [a,b];
w = [ (cos(b)+4)*cos(a); (cos(b)+4)*sin(a); sin(b)];
psi = [cos(a);sin(a);cos(b);sin(b)];
dw = [diff(w,a), diff(w,b)];
dpsi = [diff(psi,a), diff(psi,b)];

% compute metric
gw = simplify(dw'*dw);
gpsi = simplify(dpsi'*dpsi);
```

compute christoffel symbols

```
Lw = computeChristoffel(gw,y);
Lpsi = computeChristoffel(gpsi,y);
```

compute curvature

```
Rw = computeCurvature(Lw,y);
Rpsi = computeCurvature(Lpsi,y);
```

```
function L = computeChristoffel(g,y)
    gi = simplify(inv(g));
    L = sym(zeros(2,2,2));

    for mm=1:2
       for ii = 1:2
           for jj=1:2
               for kk = 1:2
                  L(ii,jj,mm) = L(ii,jj,mm) + sym(1/2)*(diff(g(jj,kk),
 y(ii)) + diff(g(kk,ii),y(jj)) - diff(g(ii,jj),y(kk)))*gi(kk,mm);
               end
           end
       end
    end

    L = simplify(L);
```

```
end

function R = computeCurvature(L,y)

    R = sym(zeros(2,2,2,2));

    for ss=1:2
       for ii = 1:2
           for jj=1:2
               for kk = 1:2
                   for ll = 1:2
                      R(ii,jj,kk,ss) = L(ii,kk,ll)*L(jj,ll,ss)
 - L(jj,kk,ll)*L(ii,ll,ss) + diff(L(ii,kk,ss),y(jj)) -
 diff(L(jj,kk,ss),y(ii));
                   end
               end
           end
       end
    end

    R = simplify(R);

end
```

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Problem 4

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sphere

```
syms phi th 'real'
r = [sin(phi)*cos(th); sin(phi)*sin(th);cos(phi)];
dr = [diff(r,phi),diff(r,th)];
g1 = simplify(dr'*dr);                              % metric of
 first form
N = [-cos(th)*sin(phi);-sin(th)*sin(phi);-cos(phi)];
dN = simplify([diff(N,phi),diff(N,th)]);
g2 = simplify(dr'*dN);                              % metric of
 second form
K = simplify(inv(g1)*g2);                           % curvature
K = subs(K,phi,0);
K = subs(K,th,0);
```

cylinder

```
syms u v 'real'
r = [cos(u);sin(u);v];
dr = [diff(r,u),diff(r,v)];
g1 = simplify(dr'*dr);
N = [cos(u);sin(u); 0];
dN = simplify([diff(N,u),diff(N,v)]);
g2 = simplify(dr'*dN);
```

hyperboloid

```
syms u v 'real'
r = [u;v;v^2-u^2];
dr = [diff(r,u),diff(r,v)];
g1 = simplify(dr'*dr);
N = [2*u;-2*v;1]/(4*u^2+4*v^2 +1)^(1/2);
dN = simplify([diff(N,u),diff(N,v)]);
g2 = simplify(dr'*dN);
K = simplify(inv(g1)*g2);                           % curvature
K = subs(K,u,0);
K = subs(K,v,0);
```

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