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I have used MATLAB to faciltate calculating some of the components for the metric, Christoffel symbols, curvature symbols, etc. I have attached the pdf version of the code I used as part of my work.

Exercise 1. Prove part (i) of Proposition 2.2 on pg. 90 of do Carmo. The proposition states: let $f, g \in \mathcal{D}(M)$ and X_1, X_2, Y_1, Y_2. R is bilinear in $\mathcal{X}(M) \times \mathcal{X}(M)$, that is,

$$R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1)$$
$$R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2).$$

Proof: Proof let $Z \in \mathcal{X}(M)$. The first one can be shown as

$$\begin{aligned}
R(fX_1 + gX_2, Y_1) Z &= \nabla_{Y_1} \nabla_{fX_1+gX_2} Z - \nabla_{fX_1+gX_2} \nabla_{Y_1} Z + \nabla_{[fX_1+gX_2,Y_1]} Z \\
&= \nabla_{Y_1} (f\nabla_{X_1} + g\nabla_{X_2}) Z - (f\nabla_{X_1} + g\nabla_{X_2}) \nabla_{Y_1} Z + \nabla_{[fX_1,y_1]} Z + \nabla_{[gX_2,Y_1]} Z \\
&= (f\nabla_{Y_1}\nabla_{X_1} + \nabla_{Y_1}(f)\nabla_{X_1} + g\nabla_{Y_1}\nabla_{X_2} + \nabla_{Y_1}(g)\nabla_{X_2} - (f\nabla_{X_1} + g\nabla_{X_2})\nabla_{Y_1}) Z \\
&+ \left(\nabla_{f[X_1,Y_1]-Y_1(f)X_1} + \nabla_{g[X_2,Y_1]-Y_1(g)X_2}\right) Z \\
&= \left(f\nabla_{Y_1}\nabla_{X_1} + g\nabla_{Y_1}\nabla_{X_2} - (f\nabla_{X_1} + g\nabla_{X_2})\nabla_{Y_1} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_2,Y_2]}\right) Z \\
&+ (+\nabla_{Y_1}(f)\nabla_{X_1} + \nabla_{Y_1}(g)\nabla_{X_2} - \nabla_{Y_1}(f)\nabla_{X_1} - \nabla_{Y_1}(g)\nabla_{X_2}) Z \\
&= \left(f\nabla_{Y_1}\nabla_{X_1} + g\nabla_{Y_1}\nabla_{X_2} - (f\nabla_{X_1} + g\nabla_{X_2})\nabla_{Y_1} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_2,Y_2]}\right) Z \\
&= fR(X_1, Y_1) Z + gR(X_2, Y_1) Z.
\end{aligned}$$

Similarly the second one is, omitting the vector field Z,

$$\begin{aligned}
R(X_1, fY_1 + gY_2) &= \nabla_{fY_1+gY_2} \nabla_{X_1} - \nabla_{X_1} \nabla_{fY_1+gY_2} + \nabla_{[X_1,fY_1+gY_2]} \\
&= (f\nabla_{Y_1} + g\nabla_{Y_2})\nabla_{X_1} - f\nabla_{X_1}\nabla_{Y_1} - g\nabla_{X_1}\nabla_{Y_2} - \nabla_{X_1}(f)\nabla_{Y_1} - \nabla_{X_1}(g)\nabla_{Y_2} \\
&+ f\nabla_{[X_1,Y_1]} + g\nabla_{[X_1,Y_2]} + \nabla_{X_1}(f)\nabla_{Y_1} + \nabla_{X_1}(g)\nabla_{Y_2} \\
&= (f\nabla_{Y_1} + g\nabla_{Y_2})\nabla_{X_1} - f\nabla_{X_1}\nabla_{Y_1} - g\nabla_{X_1}\nabla_{Y_2} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_1,Y_2]} \\
&= fR(X_1, Y_1) + gR(X_1, Y_2).
\end{aligned}$$

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Exercise 2. Let S_r^2 be the sphere of radius r in $\mathbb{R}^3$ centered at the origin. Equip S_r^2 with the metric induced by Euclidean space. Consider the coordinate charts obtained by restricting the orthogonal projection of $\mathbb{R}^3$ to the coordinate planes.

1) Compute the components of the Riemann curvature R_{ijk}^s in these coordinates.
2) Use this to compute the sectional curvature $K(\sigma)$ at a point $p \in S_r^2$.
3) Prove that $K(\sigma)$ is constant.

We will use the charts $\left\{\left(U_i^\pm, \varphi_i^\pm\right)\right\}$ defined as

$$U_j^\pm = \left\{(x_1, x_2, x_3) \in S_r^2 : \pm x_j > 0\right\}$$
$$\varphi_1^\pm(x_1, x_2, x_3) = (x_2, x_3)$$
$$\left(\varphi_1^\pm\right)^{-1}(y_1, y_2) = \left(\pm\sqrt{r^2 - y_1^2 - y_2^2}, y_1, y_2\right).$$

Since S_r^2 is embedded in $\mathbb{R}^3$, the derivative of $\left(\varphi_j^\pm\right)^{-1}$ is injective.

$$d\left(\varphi_1^\pm\right)^{-1} = \begin{bmatrix} -\pm\frac{y_1}{\sqrt{r^2-y_1^2-y_2^2}} & -\pm\frac{y_2}{\sqrt{r^2-y_1^2-y_2^2}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $v_1, v_2 \in T_pS_r^2$ where $p \in \varphi_1^\pm$, then using the induced metric we have

$$\left\langle d\left(\varphi_1^\pm\right)^{-1} v_1, d\left(\varphi_1^\pm\right)^{-1} v_2 \right\rangle.$$

Thus the induced metric g in matrix form is

$$\left(d\left(\varphi_1^{\pm}\right)^{-1}\right)^{\top} d\left(\varphi_1^{\pm}\right)^{-1} = \begin{bmatrix} -\pm\frac{y_1}{\sqrt{r^2-y_1^2-y_2^2}} & 1 & 0 \\ -\pm\frac{y_2}{\sqrt{r^2-y_1^2-y_2^2}} & 0 & 1 \end{bmatrix} \begin{bmatrix} -\pm\frac{y_1}{\sqrt{r^2-y_1^2-y_2^2}} & -\pm\frac{y_2}{\sqrt{r^2-y_1^2-y_2^2}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{y_1^2}{(r^2-y_1^2-y_2^2)}+1 & \frac{y_1y_2}{(r^2-y_1^2-y_2^2)} \\ \frac{y_1y_2}{(r^2-y_1^2-y_2^2)} & \frac{y_2^2}{(r^2-y_1^2-y_2^2)}+1 \end{bmatrix}.$$

The computation of the Christoffel symobls is tedius, so we employed MATLAB to compute them. The Christoffel symbols are

$$\begin{aligned} \Gamma_{11}^1 &= \frac{y_1\left(r^2-y_2^2\right)}{\alpha} \\ \Gamma_{12}^1 = \Gamma_{21}^1 &= \frac{y_1^2y_2}{\alpha} \\ \Gamma_{22}^1 &= \frac{y_1\left(r^2-y_1^2\right)}{\alpha} \\ \Gamma_{11}^2 &= \frac{y_2\left(r^2-y_2^2\right)}{\alpha} \\ \Gamma_{12}^2 = \Gamma_{21}^2 &= \frac{y_1y_2^2}{\alpha} \\ \Gamma_{22}^2 &= \frac{y_2\left(r^2-y_1^2\right)}{\alpha} \\ \alpha &= r^2\left(r^2-y_1^2-y_2^2\right). \end{aligned}$$

Once again using MATLAB, the coefficients of the curvature are

$$\begin{aligned} & R_{111}^1 = R_{221}^1 = R_{112}^1 = R_{222}^1 = R_{111}^2 = R_{221}^2 = R_{112}^2 = R_{222}^1 = 0 \\ & R_{121}^1 = \frac{-y_1y_2}{\alpha} \\ & R_{211}^1 = \frac{y_1y_2}{\alpha} \\ & R_{122}^1 = \frac{2y_1^2-r^2}{\alpha} \\ & R_{212}^1 = \frac{r^2-2y_1^2}{\alpha} \\ & R_{121}^2 = \frac{r^2-y_2^2}{\alpha} \\ & R_{211}^2 = \frac{y_2^2-r^2}{\alpha} \\ & R_{122}^2 = \frac{2y_1y_2}{\alpha} \\ & R_{212}^2 = \frac{-2y_1y^2}{\alpha} \end{aligned}$$

where α has been previously defined.

Now we proceed to compute the sectional curvature $K(\sigma)$ at point $p \in S_r^2$. Since S_r^2 is two dimensional, any two linearlly independent vectors in $T_pS_r^2$ will span $T_pS_r^2$. So we will use $v_1 = \frac{\partial}{\partial y_1}$ and $v_2 = \frac{\partial}{\partial y_2}$. The sectional curvature is then.

$$\begin{aligned} K(\sigma) &= \frac{(v_1, v_2, v_1, v_2)}{|v_1 \wedge v_2|^2} \\ &= \frac{R_{121}^1 g_{12} + R_{121}^2 g_{22}}{\langle v_1, v_1\rangle\langle v_2, v_2\rangle - \langle v_1, v_2\rangle^2} \\ &= \frac{\left(r^2-y_1^2-y_2^2\right)}{(2r^4-3r^2y_1^2-3r^2y_2^2+y_1^4+y_2^4+y_1^2y_2^2)}. \end{aligned}$$

We now proceed to prove that the sectional curvature is constant. According to Lemma 3.4 in do Carmo, the manifold M has constant sectional curvature equal to K_0 if an only if $R = K_0R'$ where R is the curvature of M and

$$\langle R'(X,Y,W), Z\rangle = \langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle X, Z\rangle,$$

for all $X, Y, W, Z \in T_pM$.

The curvature is

$$\begin{aligned}\left\langle R\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right)\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right\rangle &= \sum_\ell R^\ell_{121} g_{\ell 2}\\ &= \frac{1}{r^2 - y_1^2 - y_2^2},\end{aligned}$$

and

$$\begin{aligned}\left\langle R'\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_1}\right), \frac{\partial}{\partial y_2}\right\rangle &= \left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1}\right\rangle \left\langle \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_2}\right\rangle - \left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right\rangle \left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right\rangle\\ &= \frac{\left(2r^4 - 3r^2y_1^2 - 3r^2y_2^2 + y_1^4 + y_2^4 + y_1^2y_2^2\right)}{\left(r^2 - y_1^2 - y_2^2\right)^2}.\end{aligned}$$

From which we can see that $R = K_0 R'$. Therefore, M has constant sectional curvature.

Exercise 3. Recall the embeddings of the torus $T = \mathbb{R}/2\pi\mathbb{Z}$ is $\mathbb{R}^3$ and $\mathbb{R}^4$ given by the maps

$$\omega(\alpha, \beta) = \left(\left(\cos(\beta) + 4\right)\cos(\alpha), \left(\cos(\beta) + 4\right)\sin(\alpha), \sin(\beta)\right)$$

and

$$\psi(\alpha, \beta) = \left(\cos(\alpha), \sin(\alpha), \cos(\beta), \sin(\beta)\right)$$

respectively. Let T_3 be the torus equipped with the metric induced from $\mathbb{R}^3$ by the map ω, and let T_4 denote the torus equipped with the metric induced from $\mathbb{R}^4$ by the map ψ. Compute the components R^s_{ijk} of the curvature of T_3 and T_4 (in the coordinates induced by ω and ψ).

With the help of MATLAB, I get that the components R^s_{ijk} of T_3 are all zero except

$$\begin{aligned}R^1_{122} &= \frac{-\left(4\cos(\beta) + 1\right)}{\left(\cos(\beta) + 4\right)^2}\\ R^1_{212} &= -R^1_{122}\\ R^2_{121} &= 4\cos(\beta) + 2\cos(\beta)^2 - 1\\ R^2_{211} &= -R^2_{121}\end{aligned}$$

The components R^s_{ijk} of T_4 are all zero. This means that the torus in $\mathbb{R}^4$ with the chosen embedding is flat.

Exercise 4. For a parameterized surface S in $\mathbb{R}^3$ given by $r(u, v)$ we can find a unit normal at $p \in S$ by

$$N(p) = \frac{r_u \times r_v}{\|r_u \times r_v\|}.$$

The Gauss map is $N : S \to S^2$ defined by the equation above. The derivative of the map is $dN_p : T_pS \to T_{N(p)}S^2$. However, by construction we know that T_pS and $T_{N(P)}S^2$ have parallel tangent planes in $\mathbb{R}^3$ so we can think of dN_p as a map from $T_pS \to T_pS$. The idea is the following: For a parameterized curve $\alpha(t)$ in S such that $\alpha(0) = p$ we consider the curve $N(\alpha(t)) = N(t)$ in S^2. The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in T_pS. So dN_p measures how N "pulls away from" $N(p)$.

1) For a plane $ax + by + cz = d$ show that $dN \equiv 0$.
2) For the unit sphere with inward pointing normals show that $dN_p v = -v$.
3) Find dN_p for the cynlider with $r(u, v) = (\cos u, \sin u, v)$.
4) For the hyperbolic paraboloid $r(u, v) = \left(u, v, v^2 - u^2\right)$ compute the unit normal vectors. At $p = (0, 0, 0)$ show that

$$dN_p\left(u'(0), v'(0), 0\right) = \left(2u'(0), -2v'(0), 0\right).$$

So the vectors $(1, 0, 0)$ and $(0, 1, 0)$ are eigenvectors dN_p with eigenvalues 2 and -2 repectfully.

For part 1) we can parameterize the plane as

$$r(x, y) = (ax, by, d - ax - by),$$

which implies that

$$r_x = \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix}, \quad r_y = \begin{bmatrix} 0 \\ b \\ -b \end{bmatrix}.$$

Therefore

$$N(p) = \frac{r_x \times r_y}{\|r_x \times r_y\|} = \frac{ab}{\sqrt{3}\,|ab|}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Since $N(p)$ is a constant, its derivative $dN \equiv 0$. This is what we would expect since a plane is flat. We could've parameterized the plane differently, but the result is the same.

For part 2) we can parameterize the unit spere as

$$r(\phi,\theta) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$$

which implies that

$$r_\phi = \begin{bmatrix}\cos(\phi)\cos(\theta)\\ \cos(\phi)\sin(\theta)\\ -\sin(\phi)\end{bmatrix}, \quad r_\theta = \begin{bmatrix}-\sin(\phi)\sin(\theta)\\ \sin(\phi)\cos(\theta)\\ 0\end{bmatrix},$$

where

$$dr = [r_\phi, r_\theta] : T_pS \to T_p\mathbb{R}^3.$$

The metric of the first form $g1$ is

$$g1 = \begin{bmatrix}1 & 0\\ 0 & \sin^2(\phi)\end{bmatrix}.$$

The normal vector is

$$N(p) = \frac{r_\phi \times r_\theta}{\|r_\phi \times r_\theta\|} = \begin{bmatrix}\cos(\theta)\sin(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\phi)\end{bmatrix}.$$

However, $N(p)$ is outward pointing. To get inward pointing we negate it to get

$$N'(p) = -N(p) = \begin{bmatrix}-\cos(\theta)\sin(\phi)\\ -\sin(\theta)\sin(\phi)\\ -\cos(\phi)\end{bmatrix}.$$

Taking the partial derivative, we get the differential

$$dN'(p) = \begin{bmatrix}-\cos(\phi)\cos(\theta) & \sin(\phi)\sin(\theta)\\ -\cos(\phi)\sin(\theta) & -\sin(\phi)\cos(\theta)\\ \sin(\phi) & 0\end{bmatrix}.$$

We can now compute the metric of the second form

$$\begin{aligned} g2 &= \langle dN', dr\rangle \\ &= \begin{bmatrix}-1 & 0\\ 0 & -\sin^2(\phi)\end{bmatrix}. \end{aligned}$$

Let $v = v_i \frac{\partial}{\partial x_i}$, we want to show that $dr(v) = -dN'(p)(v)$. Since dr and $dN'(p)$ are linear maps we see that $dr = -dN'(p)$; thus

$$dr(v) = -dN'(p)(v).$$

For part 3), let the surface be parameterized by $r(u,v) = (\cos(u), \sin(u), v)$. The differential is

$$dr = \begin{bmatrix}-\sin(u) & 0\\ \cos(u) & 0\\ 0 & 1\end{bmatrix},$$

The metric of the first form $g1$ is

$$g1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

and

$$N(p) = \begin{bmatrix}\cos(u)\\ \sin(u)\\ 0\end{bmatrix}.$$

Computing the differential of $N(p)$ we get

$$dN(p) = \begin{bmatrix}-\sin(u) & 0\\ \cos(u) & 0\\ 0 & 0\end{bmatrix}.$$

We can now compute the metric of the second form

$$\begin{aligned} g2 &= \langle dN', dr \rangle \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

This shows that there is no curve along the z axis which is what we would expect for a cylinder that is oriented along the z-axis.

For part 4), let the hyperbolid paraboloid be parameterized by

$$r(u, v) = \left(u, v, v^2 - u^2\right).$$

The differential is

$$dr = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2u & 2v \end{bmatrix}.$$

The metric of the first form $g1$ is

$$g1 = \begin{bmatrix} 4u^2 + 1 & -4uv \\ -4uv & 4v^2 + 1 \end{bmatrix}.$$

The unit normal vector is

$$N(p) = \frac{1}{\left(4u^2 + 4v^2 + 1\right)^{1/2}} \begin{bmatrix} 2u \\ -2v \\ 1 \end{bmatrix}.$$

The differential of $N(p)$ is

$$dN(p) = \frac{1}{\left(4u^2 + 4v^2 + 1\right)^{3/2}} \begin{bmatrix} 2\left(4v^2 + 1\right) & -8uv \\ 8uv & -2\left(3u^2 + 1\right) \\ -4u & -4v \end{bmatrix}.$$

Evaulating $dN(p)$ at $p_0 = (0, 0, 0)$ yields

$$dN(p_0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}.$$

We can now compute the metric of the second form

$$\begin{aligned} g2 &= \langle dN', dr \rangle \\ &= \frac{8u^2 + 8v^2 + 2}{\left(4u^2 + 4v^2 + 1\right)^{(3/2)}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Using the first and second form of the metric we can calculate the Gaussian curvature at $p_0 = (0, 0)$

$$\begin{aligned} K &= g1^{-1} g2\big|_{p_0} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}. \end{aligned}$$

From the Gaussian curvature we can see that the eigenvalues and eigenvectors are $\lambda \pm 2$ and $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

Exercise 5. The eigenvalues of dN_p give the maximum and the minimum curvature of curves at p. These are called the principle curvatures of S at p. What are the principle curvatures for parts a), b), c) and d) above.

For part a) we got that $dN_p \equiv 0$, so that maximum and minimum eigen values are zero as they should be since a plane is flat.

For part b) we calculated the first and second form of the metric to be

$$g1 = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(\phi) \end{bmatrix}, \quad g2 = \begin{bmatrix} -1 & 0 \\ 0 & -\sin^2(\phi) \end{bmatrix},$$

Using the first and second form of the metric we can calculate the Gaussian curvature at $p_0 = (0, 0)$

$$\begin{aligned} K &= g1^{-1} g2\big|_{p_0} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

thus we get eigenvalues $\lambda_{1,2} = -1$ with eigenvectors $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

For part c) we calculated the first and second form of the metric to be

$$g1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad g2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

Using the first and second form of the metric we can calculate the Gaussian curvature at $p_0 = (0, 0)$

$$\begin{aligned} K &= g1^{-1} g2\big|_{p_0} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

thus we get eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with eigenvectors $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

Part d) was done in the previous problem which yielded eigenvalues and eigenvectors of $\lambda \pm 2$ and $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

Exercise 6. Let S be a parameterized surface in $\mathbb{R}^3$, $p \in S$, and $dN_p : T_pS \to T_pS$ be the Gauss map. The Gaussian curvature of S at p is $\det(dN_p)$. A point in S is

1) elliptic if $\det(dN_p) > 0$,
2) hyperbolic if $\det(dN_p) < 0$,
3) parabolic if $\det(dN_p) = 0$, but $dN_p \neq 0$,and
4) planar if $dN_p = 0$.

Classify the curvature of the plane, sphere, cylindar, and the point $(0, 0, 0)$ on the hyperbolic paraboloid.

In part a) of exercise 4 we found that for the plane $dN_p = 0$, thus it is planar.

In part b) of exercise 5 we found that the Gaussian curvature of the sphere is

$$K = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $\det(dN_{p_0}) = 1$, thus the sphere is elliptic.

In part c) of exercise 5 we found that the Gaussian curvature of the cylinder is

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

thus $\det(dN_{p_0}) = 0$. Therefore, the cynlinder is parabolic.

In part d) of exercise 5 we found that the Gaussian curvature of the hyperbolic plane is

$$K = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

thus $\det(dN_{p_0}) = -4$. Therefore the hyperbolic plane is hyperbolic.