# HWK 1

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**Exercise 1.** Let $M_{1,\ldots,}M_k$ be topological manifolds of dimensions $n_1,\ldots,n_k$ respectively. Prove that $M_1\times\cdots\times M_k$ is a topological manifold of dimension $n_1+\cdots+n_k$. Use this fact to show that the n-torus $\mathbb{T}^n=S^1_1\times\cdots\times S^1_n$ is an n-dimensional topological manifold.

*Proof:* We will prove this by induction. We assume that for some $m\in\mathbb{N}$ that $\mathcal{M}=M_1\times\cdots\times M_m$ is a topological manifold of dimension $N=n_1+\cdots+n_m$ and we want to show that $\mathcal{M}\times M_{m+1}$ is a topological manifold with dimention $N+n_{m+1}$. By definition, $\mathcal{M}$ and $M_{m+1}$ are Hausdorff and second countable topological spaces with $N$ respectively $n_{m+1}$ dimensional coordinate charts.

Since $\mathcal{M}$ and $M_{m+1}$ are Hausdorff then for every $p_{\mathcal{M}},q_{\mathcal{M}}\in\mathcal{M}$ and $p_{M_{m+1}},q_{M_{m+1}}\in M_{m+1}$ there exists open sets $U_{\mathcal{M}}\subseteq\mathcal{M}$, $V_{\mathcal{M}}\subseteq\mathcal{M}$, $U_{M_{m+1}}\subseteq M_{m+1}$ and $V_{M_{m+1}}\subseteq M_{m+1}$ with $p_{\mathcal{M}}\in U_{\mathcal{M}}$, $q_{\mathcal{M}}\in V_{\mathcal{M}}$, $p_{M_{m+1}}\in U_{M_{m+1}}$, $q_{M_{m+1}}\in V_{M_{m+1}}$ such that $U_{\mathcal{M}}\cap V_{\mathcal{M}}=\emptyset$ and $U_{M_{m+1}}\cap V_{M_{m+1}}\neq\emptyset$. Therefore, for all $p,q\in\mathcal{M}\times M_{m+1}$ where $p=\left(p_{\mathcal{M}},p_{M_{m+1}}\right)$ and $q=\left(q_{\mathcal{M}},q_{M_{m+1}}\right)$ there exists open sets $U=U_{\mathcal{M}}\times U_{M_{m+1}}$ and $V=V_{\mathcal{M}}\times V_{M_{m+1}}$ such that $U\cap V=\emptyset$. Thus $\mathcal{M}\times M_{m+1}$ is Hausdorff.

Since $\mathcal{M}$ and $M_{m+1}$ are second countable, they each have a basis with countable cardinality. Therefore, there exists contable bases $\mathcal{B}_{\mathcal{M}}$ and $\mathcal{B}_{M_{m+1}}$ such that for every open sets $U_{\mathcal{M}}\subseteq\mathcal{M}$ and $U_{M_{m+1}}\subseteq M_{m+1}$ and points $p_{\mathcal{M}}\in U_{\mathcal{M}}$ and $p_{M_{m+1}}\in U_{M_{m+1}}$ there exists a $B_{\mathcal{M}}\subseteq\mathcal{B}_{\mathcal{M}}$ and $B_{M_{m+1}}\subseteq\mathcal{B}_{M_{m+1}}$ such that $p_{\mathcal{M}}\in B_{\mathcal{M}}\subseteq U_{\mathcal{M}}$ and $p_{M_{m+1}}\in\mathcal{B}_{M_{m+1}}\subseteq U_{M_{m+1}}$. This implies that for every point $p=\left(p_{\mathcal{M}},p_{M_{m+1}}\right)\in U=U_{\mathcal{M}}\times U_{M_{m+1}}$, there exists a $B\subseteq\mathcal{B}_{\mathcal{M}}\times\mathcal{B}_{M_{m+1}}$ such that $p\in B\subseteq U$. Since the direct product of two countable sets is countable, the set $B$ is countable; thus $B$ is a countable basis for $\mathcal{M}\times M_{m+1}$.

Since $\mathcal{M}$ and $M_{m+1}$ are topological manifolds, then for all $p_{\mathcal{M}}\in\mathcal{M}$ and $p_{M_{m+1}}\in M_{m+1}$ there exists an $N$ and $n_{m+1}$ dimensional coordinate charts $(U_{\mathcal{M}},\varphi_{\mathcal{M}})$, $\left(U_{M_{m+1}},\varphi_{M_{m+1}}\right)$. We can take the direct product of the charts to create another coordinate chart on $\mathcal{M}\times M_{m+1}$. Let $p=\left(p_{\mathcal{M}},p_{M_{m+1}}\right)\in\mathcal{M}\times M_{m+1}$, then there exists a $N+n_{m+1}$ dimensional coordinate chart with domain $U=U_{\mathcal{M}}\times U_{M_{m+1}}$ and mapping $\varphi=\left(\varphi_{\mathcal{M}},\varphi_{M_{m+1}}\right):U\to\mathbb{R}^{N+n_{m+1}}$ defined as

$$\varphi(p)=\left(\varphi_{\mathcal{M}}(p_{\mathcal{M}}),\varphi_{M_{m+1}}\left(p_{M_{m+1}}\right)\right).$$

We have shown that $\mathcal{M}\times M_{m+1}$ is a Hausdorff, second countable topological space such that for all $p\in\mathcal{M}\times M_{m+1}$ there exists an $N+n_{m+1}$ dimensional coordinate chart. Therefore $\mathcal{M}\times M_{m+1}$ is an $N+n_{m+1}$ dimensional topological manifold. Thus by induction, $M_1\times\cdots\times M_k$ is a topological manifold of dimension $n_1+\cdots+n_k$.

We can use this fact to show that the n-torus $\mathbb{T}^n=S^1_1\times\cdots\times S^1_n$ is an n-dimensional topological manifold. Since $S^1$ is a topological manifold of dimension 1, then $\mathbb{T}^n$ is a topological manifold of dimension $n$. ■

**Exercise 2.** Show that if $M$ is a smooth manifold and $f:M\to\mathbb{R}^k$ is a smooth function. Then $f\circ\varphi^{-1}$ is smooth for all charts $(U,\varphi)$ of $M$.

*Proof:* We suppose directly that $M$ is a smooth manifold and $f:M\to\mathbb{R}^k$ is a smooth function. By definition, since $f$ is smooth, then for every $p\in M$, there exists a chart $(U_i,\varphi_i)$ on $M$ containing $p$ and a chart $(V_i,\psi_i)$ on $\mathbb{R}^k$ containing $f(p)$ such that $f(U_i)\subseteq V_i$ and $\psi_i\circ f\circ\varphi_i^{-1}:\varphi_i(U_i)\to\psi_i(V_i)$ is smooth. Since $\mathbb{R}^k$ is Euclidean, the maximal atlas is a single chart $\left(\mathbb{R}^k,I\right)$ with the function being the identity function. Therefore, we can simplify the map $\psi_i\circ f\circ\varphi_i^{-1}:\varphi_i(U_i)\to\psi_i(V_i)$ to $f\circ\varphi_i^{-1}:\varphi_i(U_i)\to V_i$. This means that $f\circ\varphi_i^{-1}$ is smooth for every chart. ■

**Exercise 3.** Let $M_1,\ldots,M_k$ and $N$ be smooth manifolds. Show that $f:N\to M_1\times\cdots\times M_k$ is smooth if an only if each component function is smooth $(\pi_i\circ f:N\to M_i)$.

*Proof:* Since this is a biconditional statement, we must show both ways. Let $\mathcal{M}=M_1\times\cdots\times M_k$

$(\Longrightarrow)$ : We assume directly that $f:N\to M_1\times\cdots\times M_k$ is smooth, then for every $p\in N$, there exists a chart $(U,\varphi)$ on $N$ containing $p$ and a chart $(V,\psi)$ on $\mathcal{M}$ containing $f(p)$ such that $f(U)\subseteq V$ and $\psi\circ f\circ\varphi^{-1}:\varphi(U)\to\psi(V)$ is smooth. The subset $V$ can be represented as $V=V_1\times\cdots\times V_k$ where $V_i\subseteq M_i$ and the mapping $\psi$ has a smooth inverse $\psi^{-1}$which can be represented as $\psi^{-1}=\left(\psi_1^{-1},\ldots,\psi_k^{-1}\right)$ where $\psi_i^{-1}:\psi(V)\to M_i$. Since $\psi\circ f\circ\varphi^{-1}:\varphi(U)\to\psi(V)$ is smooth, we know that

$$\psi\circ\psi^{-1}\circ\psi\circ f\circ\varphi^{-1}:\varphi(U)\to\psi(V)$$

is also smooth. Let $\Pi=\psi^{-1}\circ\psi\circ f:N\to M$ , then $\Pi$ is smooth and can be represented as $\Pi=(\Pi_i,\ldots,\Pi_k)$ where $\Pi_i=\psi_i^{-1}\circ\psi\circ f:N\to M_i$. Since $\Pi$ is smooth, then $\Pi_i$ is smooth. Therefore, $(\pi_i\circ f:N\to M_i)$ is smooth for each component.

$(\Longleftarrow)$ : We assume directly that each component function is smooth $(\pi_i\circ f:N\to M_i)$. Then for every $p\in N$, there exists a chart $(U,\varphi)$ on $N$ containing $p$ and a chart $(V_i,\psi_i)$ on $M_i$ containing $\pi_i\circ f(p)$ such that $\pi_i\circ f(U)\subseteq V_i$ and $\psi_i\circ\pi_i\circ f\circ\varphi^{-1}:\varphi(U)\to\psi_i(V_i)$ is smooth. From exercise 1, we showed that $M_1\times\cdots\times M_k$ is a topological manifold with charts $(V,\psi)$ where $V=V_i\times\cdots\times V_k$ and $\psi=(\psi_1,\ldots,\psi_k)$. Since each $\psi_i$ is smooth and compatible, then each $\psi$ is smooth and compatible; therefore, $\mathcal{M}$ is a smooth manifold. We can then take each component function to form a new function

$$\begin{aligned}\psi\circ f\circ\varphi^{-1}=\big(&\psi_1\circ\pi_1\circ f\circ\varphi^{-1},\ldots,\\&\psi_k\circ\pi_k\circ f\circ\varphi^{-1}\big):\varphi(U)\to\psi(V)\end{aligned}$$

which is smooth because each of its componentes is smooth.

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**Exercise 4.** Let $f : M \to N$ be a homeomorphism between topological manifolds. Assume that $N$ is a smooth manifold. Use $f$ to pullback the smooth structure on $N$ to a smooth structure on $M$.

Let $(U_i, \varphi_i)$ be the smooth charts on $N$ which constitute a maximal atlas, then any two charts $(U_j, \varphi_j)$ and $(U_k, \varphi_k)$ such that $U_j \cap U_k \neq \emptyset$ are compatible and the map $\varphi_j \circ \varphi_k^{-1} : \varphi_k(U_k) \to \varphi_j(U_j)$ is smooth. Since $f$ is a homeomorphism, its inverse is also a homeomorphism. This allows us to constuct the map

$$\varphi_j \circ f \circ f^{-1} \circ \varphi_k : \varphi_k(U_k) \to \varphi_j(U_j)$$

which is also smooth. Let $\varphi_i \circ f = \psi_i$, then the above map can be simplified to $\psi_j \circ \psi_k^{-1} : \varphi_k(U_k) \to \varphi_j(U_j)$ which is smooth. Since $f$ is a homeomorphism, $f^{-1}(U_i) = V_i$ is also an open set. We can then write the above map as

$$\psi_j \circ \psi_k^{-1} : \varphi_k \circ f(V_k) \to \varphi_j \circ f(V_j),$$

which is smooth. Thus we have the smooth charts $(V_i, \psi_i)$ on $M$ where $V_i = f^{-1}(U_i)$ and $\psi_i = \varphi_i \circ f$.