Math 565 - Homework 10

Due Monday March 29, 2021

As a reminder if we let $r(u,v)$ be a parameterized surface and

$$\alpha(t) = r(u(t), v(t))$$

where $t \in (-\epsilon, \epsilon)$ a smooth parameterized curve with $p = \alpha(0) = r(u_0, v_0)$ and $\alpha'(0) = r_u u' + r_v v'$, then first fundamental form is

$$I_p(\alpha'(0)) = E(u')^2 + 2Fu'v' + G(v')^2$$

where $E(u_0, v_0) = \langle r_u, r_u \rangle_p$, $F(u_0, v_0) = \langle r_u, r_v \rangle_p$, and $G(u_0, v_0) = \langle r_v, r_v \rangle_p$.

For the derivative of the Gauss map we let $dN(\alpha'(0)) = N'(u(t), v(t)) = N_u u' + N_v v'$ where $N_u, N_v \in T_pS$ defined by $N_u = dN_p r_u$ and $N_v = dN_p r_v$. The second fundamental form for a surface is related to the shape operator and given by

$$II_p(\alpha'(0)) = -\langle dN(\alpha'(0)), \alpha'(0) \rangle$$
$$= -\langle N_u u' + N_v v', r_u u' + r_v v' \rangle = e(u')^2 + 2fu'v' + g(v')^2$$

where $e = -\langle N_u, r_u \rangle = \langle N, r_{uu} \rangle$,

$$f = -\langle N_v, r_u \rangle = \langle N, r_{uv} \rangle = \langle N, r_{vu} \rangle = -\langle N_u, r_v \rangle,$$

and $g = -\langle N_v, r_v \rangle = \langle N, r_{vv} \rangle$.

From the Weingarten equation we can derive that the Gaussian curvature (which is also the sectional curvature) is

$$K = \frac{eg - f^2}{EG - F^2},$$

and the mean curvature is

$$H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}.$$

Furthermore, if

$$dN_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then

$$a_{11} = \frac{fF - eG}{EG - F^2},$$

$$a_{12} = \frac{gF - fG}{EG - F^2},$$

$$a_{21} = \frac{eF - fE}{EG - F^2},$$

$$a_{22} = \frac{fF - gE}{EG - F^2}.$$

1. Show that at the origin $(0,0,0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.

2. Consider Enneper's surface

$$r(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$$

and show that

(a) the coefficients for the first fundamental form are $E = G = (1 + u^2 + v^2)^2$ and $F = 0$,

(b) the coefficients for the second fundamental form are $e = 2$, $f = 0$, and $g = -2$, and

(c) the principle curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2} \text{ and } k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

The following theorem is very important in the classification of surfaces.

Theorem 0.1 *(Bonnet) Let E, F, G, e, f, g be differentiable functions defined in an open set $V \subset \mathbb{R}^2$ with $E > 0$ and $G > 0$. Assume that the given functions satisfy the Gauss equation,*

$$(\Gamma^2_{12})_u + (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}$$

$$= -E\frac{eg - f^2}{EG - F^2} = -EK$$

and the Mainardi-Codazzi equations,

$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}$$

and

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12},$$

and that $EG - F^2 > 0$. Then, for all $q \in V$ there exists a neighborhood $U \subset V$ of q and a diffeomorphism $r : U \to r(U) \subset \mathbb{R}^3$ such that the regular surface $r(U) \subset \mathbb{R}^3$ has E, F, G and e, f, g as the coefficients of the first and second fundamental forms, respectively. Furthermore, if U is connected and if $\bar{r} : U \to \bar{r}(U) \subset \mathbb{R}^3$ is another diffeomorphism satisfying the same conditions, then there exist a translation T and a proper linear orthogonal transformation O in $\mathbb{R}^3$ such that $\bar{r} = T \circ O \circ R$.

3. Using the above theorem show that no neighborhood of a point in a sphere may be isometrically mapped into a plane.

4. Using the above results show that there exists no surface $r(u, v)$ such that $E = G = 1$, $F = 0$ and $e = 1$, $g = -1$, and $f = 0$.

5. Complete problems 1(a)-(c) and 8 (a)-(b) on p. 139 of Do Carmo.