Homework 11

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Exercise 1. Let $M \subset \mathbb{R}^3$ be compact, orientable, embedded 2-manifold with the induced metric.

1) Show that M cannot have $K \leq 0$ everywhere.
2) Show that M cannot have $K \geq 0$ everywhere unless $\mathcal{X}(M) > 0$.

Proof: We begin with part 1 and then prove part 2.

Part 1) Since M is compact, then it is complete as a metric space and geodesically complete. Thus given a point $p \in M$ at the origin, there exists a point $q \in M$ such that q is at a maximum distance away from p. At q, if the curvature is negative then there exists a point r next to q such that r is farther away from p, then q. Since q is the farthest point from p, the curvature at q cannot be negative. Now assume that the curvature at q is zero, then the surface at this point has a straight segment. Let r_1 and r_2 be points next to q on the straight segment such that r_1 and r_2 are on opposite sides. If r_1 is closer to p than q, then r_2 must be farther from p than q. This is a contradiction; thus, the curvature at q cannot be zero. This leaves us with the only option that the curvature at q is positive. Therefore, the Gaussian curvature K cannot be non positive everywhere.

Part 2) According to the Gauss-Bonnet Theorem

$$\int_M K dA = 2\pi \mathcal{X}(M),$$

where K is the sectional curvature. We suppose directly that $K \geq 0$ everywhere. Since M is compact and embedded in $\mathbb{R}^3$, there are no boundaries. This means M must have curvature and thus $K \neq 0$ everywhere. Thus $\int_M K dA > 0$. This means that $\mathcal{X}(M) > 0$. ■

Exercise 2. The (M, g) be a Riemannian 2-manifold. A curved polygon on M whose sides are geodesic segments is called a geodesic polygon. If g has everywhere nonpositive Gaussian curvature, prove that there are no geodesic polygons with exactly 1 or 2 vertices.

Proof: Let $\gamma(t)$ be the curve that makes up the geodesic polygon. According to the Gauss-Bonnet Formula

$$\int_\Omega K dA + \int_\gamma K_n dt + \sum \epsilon_i = 2\pi.$$

Since the edges are geodesics, $K_n = 0$ which simplifies the formula to

$$\int_\Omega K dA + \sum \epsilon_i = 2\pi.$$

Under the assumption that g has everywhere nonpositive Gaussian curvature, $K \leq 0$ which implies

$$\sum \epsilon_i \geq 2\pi.$$

Since each vertex is not a cusp, is has an angle between $(-\pi, \pi)$. Therefore there must be more than two vertices for the formula to hold. ■

Exercise 3. A geodesic triangle on a Riemannian 2-manifold (M, g) is a three-sided geodesic polygon. Prove that if M has constant Gaussian curvature K, show that the sum of the interior angles of a geodesic traingle γ is equal to $\pi + KA$, where A is the area of the region bounded by γ.

Proof: We suppose directly that γ is a geodesic triangle. According to the Gauss-Bonnet Formula

$$\int_\Omega K dA + \int_\gamma K_n dt + \sum \epsilon_i = 2\pi.$$

Since the triangle γ is geodesic $K_n = 0$. This allows us to simplify the formula to

$$\int_\Omega K dA + \sum \epsilon_i = 2\pi.$$

The interrior angles $\theta_i = \pi - \epsilon_i$. Substituting this in we get

$$\int_\Omega K dA - \sum \theta_i + 3\pi = 2\pi$$

which yields

$$\int_\Omega K dA + \pi = \sum \theta_i.$$

Since the curvature is constant, we get

$$KA + \pi = \sum \theta_i$$

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Exercise 4. An ideal triangle in the hyperbolic plane $\mathbb{H}^2$ is a region whose bounary consists of three geodesics, any two of which meet at a common point on the boundary of the disk (meaning both geodesics are vertical and would meet at infinity). Compute the area of an ideal triangle of your choice using the model of your coice and show it is π.

Proof: A hyperbolic space has constant negative sectional curvature of -1. Since the three edges of the triangle are geodesics and are verticle, they make a right angle on the boundary. Let γ_i and γ_j be two of the goedesics that meet at point p_{ij} such that γ_i is moving towards p_{ij} and γ_j is moving away from p_{ij} as would be the case for a geodesic polygon. Since both γ_i and γ_j make right angles at the boundary and are moving in a counter-clockwise direction, the exterior angle between the two is π. Therefore, using the Gauss-Bonnet Formula we get

$$\begin{aligned} A &= \sum \epsilon_i - 2\pi \\ &= 3\pi - 2\pi \\ &= \pi. \end{aligned}$$

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