Credit Risk : Modeling, Valuation and Hedging / T.R. Bielecki, M. Rutkowski.

2 authors:


Tomasz R. Bielecki
Illinois Institute of Technology


Marek Rutkowski
The University of Sydney

Tomasz R. Bielecki
Marek Rutkowski

Credit Risk: Modeling, Valuation, and Hedging

December 5, 2001

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Preface

Mathematical finance and financial engineering have been rapidly expanding fields of science over the past three decades. The main reason behind this phenomenon has been the success of sophisticated quantitative methodologies in helping professionals manage financial risks. It is expected that the newly developed credit derivatives industry will also benefit from the use of advanced mathematics. This industry has grown around the need to handle credit risk, which is one of the fundamental factors of financial risk. In recent years, we have witnessed a tremendous acceleration in research efforts aimed at better comprehending, modeling and hedging this kind of risk.

Although in the first chapter we provide a brief overview of issues related to credit risk, our goal was to introduce the basic concepts and related notation, rather than to describe the financial and economical aspects of this important sector of financial market. The interested reader may consult, for instance, Francis et al. (1999) or Nelken (1999) for a much more exhaustive description of the credit derivatives industry.

The main objective of this monograph is to present a comprehensive survey of the past developments in the area of credit risk research, as well as to put forth the most recent advancements in this field. An important aspect of this text is that it attempts to bridge the gap between the mathematical theory of credit risk and financial practice, which serves as the motivation for the mathematical modeling studied in this book. Mathematical developments are presented in a thorough manner and cover the structural (value-of-the-firm) and the reduced-form (intensity-based) approaches to credit risk modeling, applied both to single and to multiple defaults. In particular, this book offers a detailed study of various arbitrage-free models of defaultable term structures of interest rates with several rating grades.

This book is divided into three parts. Part I, consisting of Chapters 1-3, is mainly devoted to the classic *value-of-the-firm approach* to the valuation and hedging of corporate debt. The starting point is the modeling of the dynamics of the total value of the firm's assets (combined value of the firm's debt and equity) and the specification of the capital structure of the assets of the firm. For this reason, the name *structural approach* is frequently attributed to this approach. For the sake of brevity, we have chosen to follow the latter convention throughout this text.

Modern financial contracts, which are either traded between financial institutions or offered over-the-counter to investors, are typically rather complex and they involve risks of several kinds. One of them, commonly referred to as a *market risk* (such as, for instance, the interest rate risk) is relatively well understood nowadays. Both theoretical and practical methods dealing with this kind of risk are presented in detail, and at various levels of mathematical sophistication, in several textbooks and monographs. For this reason, we shall pay relatively little attention to the market risk involved in a given contract, and instead we shall focus on the credit risk component.

As mentioned already, Chapter 1 provides an introduction to the basic concepts that underlie the area of credit risk valuation and management. We introduce the terminology and notation related to defaultable claims, and we give an overview of basic market instruments associated with credit risk. We provide an introductory description of the three types of credit-risk sensitive instruments that are subsequently analyzed using mathematical tools presented later in the text. These instruments are: corporate bonds, vulnerable claims and credit derivatives. So far, most analyses of credit risk have been conducted with direct reference to corporate debt. In this context, the contract-selling party is typically referred to as the borrower or the obligor, and the purchasing party is usually termed the creditor or the lender. However, methodologies developed in order to value corporate debt are also applicable to vulnerable claims and credit derivatives.

To value and to hedge credit risk in a consistent way, one needs to develop a quantitative model. Existing academic models of credit risk fall into two broad categories: the *structural models* and the *reduced-form models*, also known as the *intensity-based models*. Our main purpose is to give a thorough analysis of both approaches and to provide a sound mathematical basis for credit risk modeling. It is essential to make a clear distinction between stochastic models of credit risk and the less sophisticated models developed by commercial companies for the purpose of measuring and managing the credit risk. The latter approaches are not covered in detail in this text.

The subsequent two chapters are devoted to the so-called *structural approach.* In Chapter 2, we offer a detailed study of the classic Merton (1974) approach and its variants due to, among others, Geske (1977), Mason and Bhattacharya (1981), Shimko et al. (1993), Zhou (1996), and Buffet (2000). This method is sometimes referred to as the *option-theoretic approach*, since it was directly inspired by the Black-Scholes-Merton methodology for valuation of financial options. Subsequently, in Chapter 3, a detailed study of the Black and Cox (1976) ideas is presented. We also discuss some generalizations of their approach that are due to, among others, Brennan and Schwartz (1977, 1980), Kim et al. (1993a), Nielsen et al. (1993), Longstaff and Schwartz (1995), Briys and de Varenne (1997), and Cathcart and El-Jahel (1998). Due to the way in which the default time is specified, the models worked out in the references quoted above are referred to as the *first-passage-time models.*

Within the framework of the structural approach, the default time is defined as the first crossing time of the value process through a default triggering barrier. Both the value process and the default triggering barrier are the model's primitives. Consequently, the main issue is the joint modeling of the firm's value and the barrier process that is usually specified in relation to the value of the firm's debt. Since the default time is defined in terms of the model's primitives, it is common to state that it is given *endogenously* within the model. Another important ingredient in both structural and reduced-form models is the amount of the promised cash flows recovered in case of default, typically specified in terms of the so-called *recovery rate* at default or, equivalently, in terms of the *loss-given-default.* Formally, it is thus possible to single out the *recovery risk* as a specific part of the credit risk; needless to say, the spread, the default and the recovery risks are intertwined both in practice and in most existing models of credit risk. Let us finally mention that econometric studies of recovery rates of corporate bond are rather scarce; the interested reader may consult, for instance, the studies by Altman and Kishore (1996) or Carty and Lieberman (1996).

The original Merton model focuses on the case of defaultable debt instruments with finite maturity, and it postulates that the default may occur only at the debt's maturity date. By contrast, the first-passage-time technique not only allows valuation of debt instruments with both a finite and an infinite maturity, but, more importantly, it allows for the default to arrive during the entire life-time of the reference debt instrument or entity.

The structural approach is attractive from the economic point of view as it directly links default events to the evolution of the firm's capital structure, and thus it refers to market fundamentals. Another appealing feature of this set-up is that the derivation of hedging strategies for defaultable claims is straightforward. An important aspect of this method is that it allows for a study of the optimal capital structure of the firm. In particular, one can study the most favorable timing for the decision to declare bankruptcy as a dynamic optimization problem. This line of research was originated by Black and Cox (1976), and it was subsequently continued by Anderson, Pan and Sundaresan (1992), Leland (1994), Anderson and Sundaresan (1996), Anderson, Sundaresan and Tychon (1996), Leland and Toft (1996), Mella-Barral and Tychon (1996), Fan and Sundaresan, (1997), Mella-Barral and Perraudin (1997), Ericsson (2000), Anderson and Sundaresan (2000).

Some authors use this methodology to forecast default events; however, this issue is not discussed in much detail in this text. Let us notice that the structural approach leads to modeling of default times in a way which does not provide any elements of surprise – in the sense that the resulting random times are predictable with respect to the underlying filtrations. This feature is the source of the observed discrepancy between the credit spreads for short maturities predicted by structural models and the market data.

In Part II, we provide a systematic exposition of technical tools that are needed for an alternative approach to credit risk modeling – the reduced-form approach that allows for modeling of unpredictable random times of defaults or other credit events. The main objective of Part II is to work out various mathematical results underlying the reduced-form approach. Much attention is paid to characterization of random times in terms of hazard functions, hazard processes, and martingale hazard processes, as well as to evaluating relevant (conditional) probabilities and (conditional) expectations in terms of these functions and processes. In this part, the reader will find various pertinent versions of Girsanov's theorem and the martingale representation theorem. Finally, we present a comprehensive study of the problems related to the modeling of several random times within the framework of the intensity-based approach.

The majority of results presented in this part were already known; however, it is not possible to quote all relevant references here. The following works deserve a special mention: Dellacherie (1970, 1972), Chou and Meyer (1975), Dellacherie and Meyer (1978a, 1978b), Davis (1976), Elliott (1977), Jeulin and Yor (1978), Mazziotto and Szpirglas (1979), Jeulin (1980), Brémaud (1981), Artzner and Delbaen (1995), Duffie et al. (1996), Duffie (1998b), Lando (1998), Kusuoka (1999), Elliott et al. (2000), Bélanger et al. (2001), and Israel et al. (2001). Let us emphasize that the exposition in Part II is adapted from papers by Jeanblanc and Rutkowski (2000a, 2000b, 2001).

Part III is dedicated to an investigation of diverse aspects of the *reduced-form approach,* also commonly referred to as the *intensity-based approach.* To the best of our knowledge, this approach was initiated by Pye (1974) and Litterman and Iben (1991), and then formalized independently by Lando (1994), Jarrow and Turnbull (1995), and Madan and Unal (1998). Further developments of this approach can be found in papers by, among others, Hull and White (1995), Das and Tufano (1996), Duffie et al. (1996), Schönbucher (1996), Lando (1997, 1998), Monkkonen (1997), Lotz (1998, 1999), and Collin-Dufresne and Solnik (2001).

In many respects, Part III, where we illustrate the developed theory through examples of real-life credit derivatives and we describe market methods related to risk management, is the most practical part of the book. In Chapter 8, we discuss the most fundamental issues regarding the intensity-based valuation and hedging of defaultable claims in case of single reference credit. From the mathematical perspective, the intensity-based modeling of random times hinges on the techniques of modeling random times developed in the reliability theory. The key concept in this methodology is the survival probability of a reference instrument or entity, or, more specifically, the hazard rate that represents the intensity of default. In the most simple version of the intensity-based approach, nothing is assumed about the factors generating this hazard rate. More sophisticated versions additionally include factor processes that possibly impact the dynamics of the credit spreads.

Important modeling aspects include: the choice of the underlying probability measure (real-world or risk-neutral – depending on the particular application), the goal of modeling (risk management or valuation of derivatives), and the source of intensities. In a typical case, the value of the firm is not included in the model; the specification of intensities is based either on the model's calibration to market data or on the estimation based on historical observations. In this sense, the default time is *exogenously* specified. It is worth noting that in the reduced-form approach the default time is not a predictable stopping time with respect to the underlying information flow. In contrast to the structural approach, the reduced-form methodology thus allows for an element of surprise, which is in this context a practically appealing feature. Also, there is no need to specify the priority structure of the firm's liabilities, as it is often the case within the structural approach. However, in the so-called hybrid approach, the value of the firm process, or some other processes representing the economic fundamentals, are used to model the hazard rate of default, and thus they are used indirectly to define the default time.

Chapters 9 and 10 deal with the case of several reference credit entities. The main goal is to value basket derivatives and to study default correlations. In case of conditionally independent random times, the closed-form solutions for typical basket derivatives are derived. We also give some formulae related to default correlations and conditional expectations. In a more general situation of mutually dependent intensities of default, we show that the problem of quasi-explicit valuation of defaultable bonds is solvable. This should be contrasted with the previous results obtained, in particular, by Kusuoka (1999) and Jarrow and Yu (2001), who seemed to suggest that the valuation problem is intractable through the standard approach, without certain additional restrictions.

In view of the important role played in the modeling of credit migrations by the methodologies based on the theory of Markov chains, in Chapter 11 we offer a presentation of the relevant aspects of this theory.

In Chapter 12, we examine various aspects of credit risk models with multiple ratings. Both in case of credit risk management and in case of valuation of credit derivatives, the possibility of migrations of underlying credit name between different rating grades may need to be accounted for. This reflects the basic feature of the real-life market of credit risk sensitive instruments (corporate bonds and loans). In practice, credit ratings are the natural attributes of credit names. Most authors were approaching the issue of modeling of the credit migrations from the Markovian perspective. Chapter 12 is mainly devoted to a methodical survey of Markov models developed by, among others, Das and Tufano (1996), Jarrow et al. (1997), Nakazato (1997), Duffie and Singleton (1998a), Arvanitis et al. (1998), Kijima (1998), Kijima and Komoribayashi (1998), Thomas et al. (1998), Lando (2000a), Lando and Skødeberg (2000), and Wei (2000).

The topics touched upon in Chapter 12 are continued and further developed in Chapter 13. Following, in particular, Bielecki and Rutkowski (1999, 2000a, 2000b, 2001a) and Schönbucher (2000), we present the most recent developments, which combine the HJM methodology of modeling of instantaneous forward rates with a conditionally Markov model of credit migrations. Probabilistic interpretation of the market price of interest rate risk and the market price of the credit risk is highlighted. The latter is used as the motivation for our mathematical developments, based on martingale methods combined with the analysis of random times and the theory of time-inhomogeneous conditionally Markov chains and jump processes.

As is well known, there are several alternative approaches to the modeling of the default-free term structure of interest rates, based on the short-term rate, instantaneous forward rates, or the so-called market rates (such as, LIBOR rates or swap rates). As we have mentioned above, a model of defaultable term structure based on the instantaneous forward rates is presented in Chapter 13. In Chapters 14 and 15, which in a sense complement the content of Chapter 13, various typical examples of defaultable forward contracts and the associated types of defaultable market rates are introduced. We conclude by presenting the BGM model of forward LIBOR rates, Jamshidian's model of forward swap rates, as well as some ideas related to the modeling of defaultable LIBOR and swap rates.

We hope that this book may serve as a valuable reference for the financial analysts and traders involved with credit derivatives. Some aspects of the text may also be useful for market practitioners involved with managing credit-risk sensitive portfolios. Graduate students and researchers in areas such as finance theory, mathematical finance, financial engineering and probability theory will also benefit from this book. Although it provides a comprehensive treatment of most issues relevant to the theory and practice of credit risk, some aspects are not examined at all or are treated only very succinctly; these include: liquidity risk, credit portfolio management and econometric studies.

Let us once more stress that the main purpose of models presented in this text is the valuation of credit-risk-sensitive financial derivatives. For this reason, we focus on the arbitrage-free (or *martingale*) approach to the modeling of credit risk. Although *hedging* appears in the title of this monograph, we were able to provide only a brief account of the theoretical results related to the problem of hedging against the credit risk. A complete and thorough treatment of this aspect would deserve a separate text.

On the technical side, readers are assumed to be familiar with graduate level probability theory, theory of stochastic processes, elements of stochastic analysis and PDEs. As already mentioned, a systematic exposition of mathematical techniques underlying the intensity-based approach is provided in Part II of the text.

For the mathematical background, including the most fundamental definitions and concepts from the theory of stochastic process and the stochastic analysis based on the Itô integral, the reader may consult, for instance, Dellacherie (1972), Elliott (1977), Dellacherie and Meyer (1978a), Brémaud (1981), Jacod and Shiryaev (1987), Ikeda and Watanabe (1989), Protter (1990), Karatzas and Shreve (1991), Revuz and Yor (1991), Williams (1991), He et al. (1992), Davis (1993), Krylov (1995), Neftci (1996), Øksendal (1998), Rolski et al. (1998), Rogers and Williams (2000) or Steele (2000). In particular, for the definition and properties of the standard Brownian motion, we refer to Chap. 1 in Itô and McKean (1965), Chap. 2 in Karatzas and Shreve (1991) or Chap. II in Krylov (1995).

Some acquaintance with arbitrage pricing theory and fundamentals on financial derivatives is also expected. For an exhaustive treatment of arbitrage pricing theory, modeling of the term structure of interest rates and other relevant aspects of financial engineering, we refer to the numerous monographs available; to mention a few: Baxter and Rennie (1996), Duffie (1996), Lamberton and Lapeyre (1996), Neftci (1996), Musiela and Rutkowski (1997a), Pliska (1997), Bingham and Kiesel (1998), Björk (1998), Karatzas and Shreve (1998), Shiryaev (1998), Elliott and Kopp (1999), Mel'nikov (1999), Hunt and Kennedy (2000), James and Webber (2000), Jarrow and Turnbull (2000a), Pelsser (2000), Brigo and Mercurio (2001), and Martinellini and Priaulet (2001). More specific issues related to credit risk derivatives and management of credit risk are discussed in Duffee and Zhou (1996), Das (1998a, 1998b), Caouette et al. (1998), Tavakoli (1998), Cossin and Pirotte (2000), Ammann (1999, 2001), and Duffie and Singleton (2001).

It is essential to stress that we make, without further mention, the common standard technical assumptions:

- all reference probability spaces are assumed to be complete (with respect to the reference probability measure),
- all filtrations satisfy the *usual conditions* of right-continuity and completeness (see Page 20 in Karatzas and Shreve (1991)),
- the sample paths of all stochastic processes are right-continuous functions, with finite left-limits, with probability one; in other words, all stochastic processes are assumed to be RCLL (i.e., càdlàg),
- all random variables and stochastic processes satisfy suitable integrability conditions, which ensure the existence of considered conditional expectations, deterministic or stochastic integrals, etc. For the sake of expositional simplicity, we frequently postulate the boundedness of relevant random variables and stochastic processes.

As a rule, we adopt the notation and terminology from the monograph by Musiela and Rutkowski (1997a). For the sake of the reader's convenience, an index of the most frequently used symbols is also provided. Although we have made an effort to use uniform notation throughout the text, in some places an ad hoc notation was also used.

We are very grateful to Monique Jeanblanc for her numerous helpful comments on the previous versions of our manuscript, which have led to several essential improvements in the text. The second-named author is happy to have opportunity to thank Monique Jeanblanc for fruitful and enjoyable collaboration.

During the process of writing, we have also profited from valuable remarks by, among others, John Fuqua, Marek Musiela, Ben Goldys, Ashay Kadam, Atsushi Kawai, Volker Läger, and Jochen Georg Sutor. They also discovered numerous typographical errors in the previous drafts; all remaining errors are, of course, ours. A large portion of the manuscript was completed during the one-year stay of Marek Rutkowski in Australia. He would like to thank the colleagues from the School of Mathematics at the University of New South Wales in Sydney for their hospitality.

Tomasz Bielecki gratefully acknowledges partial support obtained from the National Science Foundation under grant DMS-9971307 and from the State Committee for Scientific Research (Komitet Badań Naukowych) under grant PBZ-KBN-016/P03/1999.

Marek Rutkowski gratefully acknowledges partial support obtained from the State Committee for Scientific Research (Komitet Badań Naukowych) under grant PBZ-KBN-016/P03/1999.

We would like to express our gratitude to the staff of Springer-Verlag. We thank Catriona Byrne for her encouragement and relentless editorial supervision since our first conversation in the Banach Centre in Warsaw in June 1998, as well as Daniela Brandt and Susanne Denskus for their invaluable technical assistance. We are also grateful to Katarzyna Rutkowska who helped with the editing of the text.

Last, but not least, we would like to thank our wives. Tomasz Bielecki is grateful to Małgosia for her patience, and Marek Rutkowski thanks Ola for her support.

Chicago — Tomasz R. Bielecki
Warszawa — Marek Rutkowski

Table of Contents

Preface V

Part I. Structural Approach

1. Introduction to Credit Risk 3
1.1 Corporate Bonds 4
1.1.1 Recovery Rules 5
1.1.2 Safety Covenants 6
1.1.3 Credit Spreads 7
1.1.4 Credit Ratings 7
1.1.5 Corporate Coupon Bonds 8
1.1.6 Fixed and Floating Rate Notes 9
1.1.7 Bank Loans and Sovereign Debt 11
1.1.8 Cross Default 11
1.1.9 Default Correlations 11
1.2 Vulnerable Claims 12
1.2.1 Vulnerable Claims with Unilateral Default Risk 12
1.2.2 Vulnerable Claims with Bilateral Default Risk 13
1.2.3 Defaultable Interest Rate Contracts 14
1.3 Credit Derivatives 16
1.3.1 Default Swaps and Options 18
1.3.2 Total Rate of Return Swaps 21
1.3.3 Credit Linked Notes 22
1.3.4 Asset Swaps 24
1.3.5 First-to-Default Contracts 24
1.3.6 Credit Spread Swaps and Options 25
1.4 Quantitative Models of Credit Risk 26
1.4.1 Structural Models 26
1.4.2 Reduced-Form Models 27
1.4.3 Credit Risk Management 29
1.4.4 Liquidity Risk 30
1.4.5 Econometric Studies 30

2. Corporate Debt 31
2.1 Defaultable Claims 33
2.1.1 Risk-Neutral Valuation Formula 34
2.1.2 Self-Financing Trading Strategies 37
2.1.3 Martingale Measures 38
2.2 PDE Approach 40
2.2.1 PDE for the Value Function 44
2.2.2 Corporate Zero-Coupon Bonds 47
2.2.3 Corporate Coupon Bond 50
2.3 Merton's Approach to Corporate Debt 51
2.3.1 Merton's Model with Deterministic Interest Rates 51
2.3.2 Distance-to-Default 57
2.4 Extensions of Merton's Approach 58
2.4.1 Models with Stochastic Interest Rates 59
2.4.2 Discontinuous Value Process 60
2.4.3 Buffet's Approach 64

3. First-Passage-Time Models 65
3.1 Properties of First Passage Times 66
3.1.1 Probability Law of the First Passage Time 67
3.1.2 Joint Probability Law of Y and τ 69
3.2 Black and Cox Model 71
3.2.1 Corporate Zero-Coupon Bond 71
3.2.2 Corporate Coupon Bond 79
3.2.3 Corporate Consol Bond 81
3.3 Optimal Capital Structure 82
3.3.1 Black and Cox Approach 82
3.3.2 Leland's Approach 84
3.3.3 Leland and Toft Approach 86
3.3.4 Further Developments 88
3.4 Models with Stochastic Interest Rates 90
3.4.1 Kim, Ramaswamy and Sundaresan Approach 96
3.4.2 Longstaff and Schwartz Approach 98
3.4.3 Cathcart and El-Jahel Approach 103
3.4.4 Briys and de Varenne Approach 104
3.4.5 Saá-Requejo and Santa-Clara Approach 107
3.5 Further Developments 113
3.5.1 Convertible Bonds 113
3.5.2 Jump-Diffusion Models 113
3.5.3 Incomplete Accounting Data 113
3.6 Dependent Defaults: Structural Approach 114
3.6.1 Default Correlations: J.P. Morgan's Approach 116
3.6.2 Default Correlations: Zhou's Approach 117

Part II. Hazard Processes

4. Hazard Function of a Random Time 123
4.1 Conditional Expectations w.r.t. Natural Filtrations 123
4.2 Martingales Associated with a Continuous Hazard Function . . 127
4.3 Martingale Representation Theorem 131
4.4 Change of a Probability Measure 133
4.5 Martingale Characterization of the Hazard Function 137
4.6 Compensator of a Random Time 140

5. Hazard Process of a Random Time 141
5.1 Hazard Process Γ 141
5.1.1 Conditional Expectations 143
5.1.2 Semimartingale Representation of the Stopped Process 150
5.1.3 Martingales Associated with the Hazard Process Γ . . . 152
5.1.4 Stochastic Intensity of a Random Time 155
5.2 Martingale Representation Theorems 156
5.2.1 General Case 156
5.2.2 Case of a Brownian Filtration 159
5.3 Change of a Probability Measure 162

6. Martingale Hazard Process 165
6.1 Martingale Hazard Process Λ 165
6.1.1 Martingale Invariance Property 166
6.1.2 Evaluation of Λ: Special Case 167
6.1.3 Evaluation of Λ: General Case 169
6.1.4 Uniqueness of a Martingale Hazard Process Λ 172
6.2 Relationships Between Hazard Processes Γ and Λ 173
6.3 Martingale Representation Theorem 177
6.4 Case of the Martingale Invariance Property 179
6.4.1 Valuation of Defaultable Claims 180
6.4.2 Case of a Stopping Time 182
6.5 Random Time with a Given Hazard Process 183
6.6 Poisson Process and Conditional Poisson Process 186

7. Case of Several Random Times 197
7.1 Minimum of Several Random Times 197
7.1.1 Hazard Function 198
7.1.2 Martingale Hazard Process 198
7.1.3 Martingale Representation Theorem 200
7.2 Change of a Probability Measure 203
7.3 Kusuoka's Counter-Example 209
7.3.1 Validity of Condition (F.2) 216
7.3.2 Validity of Condition (M.1) 218

Part III. Reduced-Form Modeling

8. Intensity-Based Valuation of Defaultable Claims 221
8.1 Defaultable Claims 222
8.1.1 Risk-Neutral Valuation Formula 223
8.2 Valuation via the Hazard Process 225
8.2.1 Canonical Construction of a Default Time 227
8.2.2 Integral Representation of the Value Process 230
8.2.3 Case of a Deterministic Intensity 232
8.2.4 Implied Probabilities of Default 234
8.2.5 Exogenous Recovery Rules 236
8.3 Valuation via the Martingale Approach 239
8.3.1 Martingale Hypotheses 242
8.3.2 Endogenous Recovery Rules 243
8.4 Hedging of Defaultable Claims 246
8.5 General Reduced-Form Approach 250
8.6 Reduced-Form Models with State Variables 253
8.6.1 Lando's Approach 253
8.6.2 Duffie and Singleton Approach 255
8.6.3 Hybrid Methodologies 259
8.6.4 Credit Spread Models 264

9. Conditionally Independent Defaults 265
9.1 Basket Credit Derivatives 266
9.1.1 Mutually Independent Default Times 267
9.1.2 Conditionally Independent Default Times 268
9.1.3 Valuation of the i^{th}-to-Default Contract 274
9.1.4 Vanilla Default Swaps of Basket Type 281
9.2 Default Correlations and Conditional Probabilities 284
9.2.1 Default Correlations 284
9.2.2 Conditional Probabilities 287

10. Dependent Defaults 293
10.1 Dependent Intensities 295
10.1.1 Kusuoka's Approach 295
10.1.2 Jarrow and Yu Approach 296
10.2 Martingale Approach to Basket Credit Derivatives 306
10.2.1 Valuation of the i^{th}-to-Default Claims 311

11. Markov Chains 313
11.1 Discrete-Time Markov Chains 314
11.1.1 Change of a Probability Measure 316
11.1.2 The Law of the Absorption Time 320
11.1.3 Discrete-Time Conditionally Markov Chains 322

11.2 Continuous-Time Markov Chains 324
11.2.1 Embedded Discrete-Time Markov Chain 329
11.2.2 Conditional Expectations 329
11.2.3 Probability Distribution of the Absorption Time 332
11.2.4 Martingales Associated with Transitions 333
11.2.5 Change of a Probability Measure 334
11.2.6 Identification of the Intensity Matrix 338
11.3 Continuous-Time Conditionally Markov Chains 340
11.3.1 Construction of a Conditionally Markov Chain 342
11.3.2 Conditional Markov Property 346
11.3.3 Associated Local Martingales 347
11.3.4 Forward Kolmogorov Equation 350

12. Markovian Models of Credit Migrations 351
12.1 JLT Markovian Model and its Extensions 352
12.1.1 JLT Model: Discrete-Time Case 354
12.1.2 JLT Model: Continuous-Time Case 362
12.1.3 Kijima and Komoribayashi Model 367
12.1.4 Das and Tufano Model 369
12.1.5 Thomas, Allen and Morkel-Kingsbury Model 371
12.2 Conditionally Markov Models 373
12.2.1 Lando's Approach 374
12.3 Correlated Migrations 376
12.3.1 Huge and Lando Approach 380

13. Heath-Jarrow-Morton Type Models 385
13.1 HJM Model with Default 386
13.1.1 Model's Assumptions 386
13.1.2 Default-Free Term Structure 388
13.1.3 Pre-Default Value of a Corporate Bond 390
13.1.4 Dynamics of Forward Credit Spreads 392
13.1.5 Default Time of a Corporate Bond 394
13.1.6 Case of Zero Recovery 397
13.1.7 Default-Free and Defaultable LIBOR Rates 398
13.1.8 Case of a Non-Zero Recovery Rate 400
13.1.9 Alternative Recovery Rules 403
13.2 HJM Model with Credit Migrations 405
13.2.1 Model's Assumption 405
13.2.2 Migration Process 407
13.2.3 Special Case 408
13.2.4 General Case 410
13.2.5 Alternative Recovery Schemes 413
13.2.6 Defaultable Coupon Bonds 415
13.2.7 Default Correlations 416
13.2.8 Market Prices of Interest Rate and Credit Risk 417

13.3 Applications to Credit Derivatives 421
13.3.1 Valuation of Credit Derivatives 421
13.3.2 Hedging of Credit Derivatives 422

14. Defaultable Market Rates 423
14.1 Interest Rate Contracts with Default Risk 424
14.1.1 Default-Free LIBOR and Swap Rates 424
14.1.2 Defaultable Spot LIBOR Rates 426
14.1.3 Defaultable Spot Swap Rates 427
14.1.4 FRAs with Unilateral Default Risk 428
14.1.5 Forward Swaps with Unilateral Default Risk 432
14.2 Multi-Period IRAs with Unilateral Default Risk 434
14.3 Multi-Period Defaultable Forward Nominal Rates 438
14.4 Defaultable Swaps with Unilateral Default Risk 441
14.4.1 Settlement of the 1st Kind 442
14.4.2 Settlement of the 2nd Kind 444
14.4.3 Settlement of the 3rd Kind 445
14.4.4 Market Conventions 446
14.5 Defaultable Swaps with Bilateral Default Risk 447
14.6 Defaultable Forward Swap Rates 449
14.6.1 Forward Swaps with Unilateral Default Risk 449
14.6.2 Forward Swaps with Bilateral Default Risk 450

15. Modeling of Market Rates 451
15.1 Models of Default-Free Market Rates 452
15.1.1 Modeling of Forward LIBOR Rates 452
15.1.2 Modeling of Forward Swap Rates 458
15.2 Modeling of Defaultable Forward LIBOR Rates 465
15.2.1 Lotz and Schlögl Approach 465
15.2.2 Schönbucher's Approach 469

References 479

Basic Notation 495

Subject Index 497

Part I

Structural Approach

1. Introduction to Credit Risk

A *default risk* is a possibility that a counterparty in a financial contract will not fulfill a contractual commitment to meet her/his obligations stated in the contract. If this actually happens, we say that the party defaults, or that the default event occurs. More generally, by a *credit risk* we mean the risk associated with any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads, and the default event. The *spread risk* is thus another components of credit risk. To facilitate the analysis of complex agreements, it is important to make a clear distinction between the *reference* (*credit*) *risk* and the *counterparty* (*credit*) *risk*. The first generic term refers to the situation when both parties of a contract are assumed to be default-free, but due to specific features of the contract the credit risk of some reference entity appears to play an essential role in the contract's settlement. In other words, the reference risk is that part of the contract's risk, which is associated with the third party; i.e., with the entity, which is not a party in a given agreement. In the present context, the third party is referred to as the reference entity of a given contract. *Credit derivatives* are recently developed financial instruments that allow market participants to isolate and trade the reference credit risk. The main goal of a credit derivative is to transfer the reference risk, either completely or partially, between the counterparties. In most cases, one of the parties can be seen as a buyer of an insurance against the reference risk. Such a party is called the seller of the reference risk; consequently, the party that bears the reference risk is referred to as its buyer.

Let us now focus on the counterparty risk. An important feature of all over-the-counter derivatives is that, unlike the exchange-traded contracts, they are not backed by the guarantee of a clearinghouse or an exchange, so that each counterparty is exposed to the default risk of the other party. In practice, parties are sometimes required to post collateral or mark to market periodically, though. The counterparty risk emerges in a clear way in such contracts as *vulnerable claims* and *defaultable swaps*. In both these cases, one needs to quantify the default risk of both parties in order to correctly assess the contract's value. Depending on whether the default risk of one or both parties is taken into account, we say that a contract involves the unilateral (one-sided) or the bilateral (two-sided) default risk.

1.1 Corporate Bonds

Corporate bonds are debt instruments issued by corporations. They are a part of the capital structure of the firm (just like the equity). By issuing bonds a corporation commits itself to make specified payments to the bondholders at some future dates, and the corporation charges a fee for this commitment. However, the corporation (or firm) may default on its commitment in which case the bondholders will not receive the promised payment in full, and thus they will suffer a financial loss. Of course, the occurrence of default, possibly caused by the firm's bankruptcy, is meaningful only during the lifetime of a particular bond – that is, during the time period between the bond's inception and its maturity.

A corporate bond is an example of a *defaultable claim* (a formal definition of a defaultable claim is provided in Sect. 2.1). We set the notional amount, or the face value, of the bond equal to L units of cash (e.g., U.S. dollars). For the moment, we shall concentrate on a discount bond – that is, we assume that the bond pays no coupons. We also fix the maturity date of the bond, denoted by T. Usually, the arbitrage price at time t of a T-maturity defaultable bond will be denoted by $D(t,T)$, in particular, $D(T,T) = L$ provided that default has not occurred prior to, or at the maturity date T. By contrast, the notation $B(t,T)$ is used to denote the arbitrage price at time t of a T-maturity default-free bond with the face value 1; hence necessarily $B(T,T) = 1$.

The *defaultable term structure* is the term structure of interest rates implied by the yields on the default prone *corporate bonds* or on the default prone *sovereign bonds*. A large portion of the credit risk literature is devoted to the modeling of a defaultable term structure as well as to pricing related credit derivatives. Much of the theory presented later in this book applies to both types of bonds. We shall use a generic term *defaultable bond* for any kind of bond with the possibility of default. By contrast, a *default-free bond* pays surely both the coupons and the face value to the at bondholders predetermined dates. Defaultable bonds are also known as *risky bonds,* and the default-free bonds are commonly referred to as *risk-free bonds* or *Treasury bonds.* Holders of all bonds are, of course, exposed to the market (interest rate) risk. The adjective risk-free refers to the presumed absence (or at least negligibility) of the credit risk in bonds of the highest credit quality.

The mathematical techniques presented in this text are applicable to the valuation of general corporate liabilities (or corporate debt). Corporate bonds are one example of such liabilities; corporate loans are another one. Most of the discussion that follows applies to corporate debt in general, although we choose to specify it to corporate bonds. Corporate bonds are characterized by various attributes that we shall now briefly describe. Let us observe that the features such as: recovery rules, safety covenants, credit ratings and default correlations are related not only to corporate bonds but indeed to general defaultable claims, as defined in Sect. 2.1.

1.1.1 Recovery Rules

We first provide a simplified description of recovery rules, which is suitable for formulating a mathematical model. In practice, the specific recovery rules will typically include clauses such as, for instance, priority payments upon default based on the debt's seniority (*seniority rules* or the *priority structure*). Generally speaking, *recovery schemes* (or *recovery covenants*, or *recovery rules*) determine the timing and the amount of *recovery payment* that is paid to creditors if the default occurs before the bond's maturity.

The recovery payment is frequently specified by the *recovery rate* δ; i.e., the fraction of the bond's face amount paid to the bondholders in case of default. The timing of the recovery payoff is, of course, another essential ingredient. If a fixed fraction of bond's face value is paid to the bondholders at time of default, usually denoted by τ in what follows, then the recovery scheme is referred to as the *fractional recovery of par value.* Assuming that the bond's face value equals 1 (that is, $L = 1$), we may represent it as the contingent claim $\tilde{D}^{\delta}(T,T)$, which settles at time T and equals

$$\tilde{D}^{\delta}(T,T) = \mathbb{1}_{\{\tau>T\}} + \delta B^{-1}(\tau,T)\mathbb{1}_{\{\tau\leq T\}}.$$

If, in case of default, the fixed fraction of bond's face value is repaid at maturity date T, the recovery scheme is termed the *fractional recovery of Treasury value.* Under this rule, the bond is formally equivalent to the payoff

$$D^{\delta}(T,T) = \mathbb{1}_{\{\tau>T\}} + \delta\mathbb{1}_{\{\tau\leq T\}},$$

and its value at time t is denoted as $D^{\delta}(t,T)$. It is clear that $\tilde{D}^{0}(t,T) = D^{0}(t,T)$. A still another convention – the *fractional recovery of market value* – postulates that at time of default the bondholders receive a fraction of the pre-default market value of a corporate bond. The equivalent contingent claim now takes the following form:

$$D(T,T) = \mathbb{1}_{\{\tau>T\}} + \delta D(\tau-,T)B^{-1}(\tau,T)\mathbb{1}_{\{\tau\leq T\}},$$

where $D(\tau-,T)$ stands for the value of the bond just before the default time. In financial literature, it is not uncommon to use the generic term *loss given default* (LGD, for short) to describe the likely loss of value in case of default (in principle, LGD thus equals 1 minus the recovery rate).

Let us now focus on a more abstract formulation of a recovery rule. In most works on credit risk, it is assumed that if a bond defaults during its lifetime then the recovery payment is made either at the default time τ, or at the maturity T of the bond. In the former case, the recovery payment is determined by the value Z_τ at default time of the *recovery process* Z. In the latter, the recovery payment is determined by the realization of the *recovery claim* $\tilde{X}$. Formally, the two cases described above correspond to the claims DCT^2 and DCT^1 of Sect. 2.1, where they are termed defaultable claims with *recovery at default* and *recovery at maturity,* respectively.

It should be stressed that the recovery process and/or the recovery value may be specified either exogenously or endogenously with respect to the current market value of the bond. In fact, the specification of the recovery rules may become quite intricate from the mathematical standpoint. As an example of an endogenous recovery rule, let us consider the situation when the recovery payment is paid at maturity T in the amount of V_T/L_T per unit of the bond's face value, provided that $V_T < L_T$ (otherwise, the firm's debt is paid back to the lenders in full). Here, V_T stands for the total value of the firm's assets at the bond's maturity, and L_T represents the total value of the firm's liabilities at this date. The (random) ratio V_T/L_T, commonly referred to as the *recovery ratio,* plays an essential role in the bond's valuation in most structural models of defaultable term structure of interest rates such as, for instance, Merton's (1974) model or the Black and Cox (1976) model.

1.1.2 Safety Covenants

There are numerous ways in which default (or bankruptcy) occurs in market practice. Typically, bankruptcy implies that the firm's bondholders take control over the firm, and the firm undergoes a reorganization. In practice, an important part of the bankruptcy procedure is the bargaining process. For the sake of (relative) simplicity, this particular aspect is not taken into account in what follows.

Exogenous bankruptcy refers to the case when bankruptcy is specified in form of some protective covenants, such as positive net-worth covenant, or when bankruptcy is triggered at an exogenously specified asset value, for instance, the principal value of the debt.

The notion of an *endogenous bankruptcy* covers the situations when bankruptcy is declared by the firm's stockholders if the firm's value falls below certain pre-specified level. Within the framework of the optimal capital structure approach, this level is selected so that the firm's equity value is maximized, so that the value of firm's debt is minimized. We shall discuss some results related to the optimal capital structure in Sect. 3.3. For a more exhaustive analysis of the strategic debt service, we refer to the original papers by Leland (1994, 1998), Anderson and Sundaresan (1996, 2000), Leland and Toft (1996), Mella-Barral and Perraudin (1997), Ericsson and Reneby (1998), Mella-Barral (1999) or Ericsson (2000).

The mathematical concept of *safety covenants* associated with a corporate debt was introduced in literature dealing with the structural approach to credit risk in order to specify the default event. Generally speaking, a safety covenant is modeled as a *barrier process* (also called a *threshold process*), usually denoted as v in what follows. In most cases, the default event is triggered when the firm value process V falls below the barrier process v either prior, or at the maturity date T. For the purpose of this text, we choose to use the term *safety covenants* in order to describe any mechanism, which triggers default event before the maturity of the debt.

1.1.3 Credit Spreads

A *credit spread* measures the excess return on a corporate bond over the return on an equivalent Treasury bond, i.e., a bond, which is assumed to be free of the credit risk. Depending on the situation, a credit spread may be expressed, e.g., as the difference between respective yields to maturity, or as the difference between respective instantaneous forward rates. The generic term *term structure of credit spreads* will refer to the term structure of such differences. The determination of the credit spread is in fact the ultimate goal of most credit risk models. It is also the topic of several econometric studies. Some authors concentrate on direct modeling of the credit spread, rather than on its derivation from other fundamentals. Such an approach appears to be very convenient when one deals with these credit derivatives that have the credit spread as the underlying instrument.

A large credit spread of a corporate security over the comparable risk-free security is a widely accepted practical measure of the firm's financial distress. *Distressed securities* can thus be defined by directly referring to the high level of credit spreads yielded by some corporate securities, should they not default. A more narrow definition of a distressed security encompasses publicly held and traded debt or equity securities of firms that have defaulted or have filed for protection under the bankruptcy code. For a detailed analysis of the concept of a distressed security, we refer to Altman (1998).

1.1.4 Credit Ratings

A firm's *credit rating* is a measure of the firm's propensity to default. Credit ratings are typically identified with elements of a finite set, also referred to as the set of *credit classes* or *credit grades.* In some cases, the credit classes may correspond to credit ratings attributed by a commercial rating agency, such as Moody's Investors Service, Standard & Poor's Corporation, or Fitch IBCA, Duff & Phelps.[1] This does not mean, however, that in the theoretical approach credit ratings should necessarily be understood as being attributed by a commercial rating agency. First, many major financial institutions maintain their own credit rating systems, based on internally developed methodologies, and therefore known as the *internal ratings.* Second, the official credit ratings primarily reflect the likelihood of default, and thus do not necessarily provide the most adequate assessment of the debt's credit quality. Finally, the improvement (deterioration, resp.) of the firm's credit quality typically does not result in an immediate upgrade (downgrade, resp.) of its rating. For more information on existing rating systems, we refer to Altman (1997), Carty (1997), Crouhy at al. (2001) and Krahnen and Weber (2001). In this text, the generic term *credit rating* (or *credit quality*) is used to describe any classification of corporate debt that can be justified for specific purposes.

[1] The interested reader may consult, e.g., the Moody's Investors Service at *www.moodys.com* or Standard & Poor's Corporation at *www.standardpoors.com.*

1.1.5 Corporate Coupon Bonds

A clear-cut distinction needs to be made between the corporate coupon bond, which pays a coupon rate continuously in time, and a similar bond, which pays coupons at discrete time instants. The former one should be seen as a theoretical construct that is meant to facilitate the analysis of the latter. The corporate coupon bond with continuous coupon rate is widely used in financial literature, particularly in relation with the structural approach, in order to study quantitative and qualitative behavior of corporate debt.

Let us briefly describe a corporate coupon bond with coupon payments occurring at discrete time intervals. It should be stressed here that the coupon payments are only made prior to the default time. A coupon bond may thus be considered as a portfolio composed of the following securities:

– *defaultable coupons,* which sometimes are equivalent to defaultable zero-coupon bonds with zero recovery,[2]
– *defaultable face value,* which can be seen as a defaultable zero-coupon bond with, generally speaking, non-zero recovery.

Consider a corporate coupon bond with face value L, which matures at time $T = T_n$ and promises to pay (fixed or variable) coupons c_i at times $T_1 < T_2 < \cdots < T_n$. Assume, for instance, that the recovery payment is proportional to the face value and that it is made at maturity T, in case the default event occurs before or at the maturity date. Under this convention, the bond's cash flows are:

$$\sum_{i=1}^{n} c_i \mathbb{1}_{\{\tau > T_i\}} \mathbb{1}_{T_i}(t) + \big(L\mathbb{1}_{\{\tau > T\}} + \delta L \mathbb{1}_{\{\tau \le T\}}\big)\mathbb{1}_T(t), \tag{1.1}$$

where τ stands for the bond's default time and the variable t represents the running time. Notice that only the last term, which corresponds to the recovery payment at bond's maturity, depends on the choice of a particular recovery scheme (recall that the coupon payments are subject to the zero recovery). A corporate coupon bond described by (1.1) can also be formally represented as a single cash flow $D_c(T,T)$, which settles at the bond's maturity date T and is given by the following formula:

$$D_c(T,T) = \sum_{i=1}^{n} c_i B^{-1}(T_i,T)\mathbb{1}_{\{\tau > T_i\}} + L\mathbb{1}_{\{\tau > T\}} + \delta L \mathbb{1}_{\{\tau \le T\}}.$$

The arbitrage price at time $t < \tau$ of this contingent claim – that is, the value of a corporate coupon bond prior to default – is denoted by $D_c(t,T)$. We find it convenient to refer to $D_c(t,T)$ on the set $\{\tau > t\}$ as the bond's *pre-default value,* the random variable $D_c(\tau,T)$ is called the *post-default value* of a corporate coupon bond. A similar terminological convention applies to defaultable zero-coupon bonds and, indeed, to all kinds of defaultable claims.

[2] Formally, this equivalence holds if all the defaultable zero-coupon bonds, including the defaultable face value, have the same default time.

1.1.6 Fixed and Floating Rate Notes

If a debt contract stipulates that the coupon payments are fixed, we deal with a *fixed-coupon bond* (or, a *fixed-rate note*). Consider the two fixed-coupon bonds, a risk-free bond and a defaultable one, with otherwise identical covenants. If both bonds trade at par (i.e., their prices equal the face values), it is natural to expect that in order to compensate an investor for the default risk, the coupon rate of a corporate bond would be greater than that of a risk-free bond. This is indeed observed in the market practice, and the corresponding discrepancy is referred to as the *fixed-rate credit spread* over Treasury for a given corporate bond. As already noticed, the credit spread reflects the credit quality of the issuer, as perceived by the market – the financial market requires a higher risk premium for lower quality debt, so that the cost of capital for a debtor of lower credit quality is higher. Let us mention that the terms *credit risk* and *spread risk* are used interchangeably by market practitioners.

To quantify the fixed-rate credit spread, for $t = 0$, assume that $B(0, T_i)$ ($D(0, T_i)$, resp.) are known market prices of zero-coupon Treasury (corporate, resp.) bonds with unit face value. Then the spread equals $S := c' - c$, where the coupon rates c and c' can be easily found from the equalities

$$\sum_{i=1}^{n-1} cB(0, T_i) + (1 + c)B(0, T_n) = 1$$

and

$$\sum_{i=1}^{n-1} c'D(0, T_i) + (1 + c')D(0, T_n) = 1.$$

The last equality is based on an implicit assumption that the price of a corporate coupon bond equals the sum of its zero-coupon components, and this holds if all coupons default simultaneously and the recovery rate δ for the coupons equals zero. The credit spread varies with both the time t and the maturity $T = T_n$, thereby giving rise to a particular term structure of credit spreads. A corporate *floating rate note* (FRNs, for short) is another important example of a defaultable debt. In contrast to a fixed rate note, each coupon payment of an FRN is made according to the floating interest rate prevailing on this coupon's date (or more precisely, on the *reset date*). Consider an FRN specified as follows: the face value is L, the maturity is $T = T_n$, and the coupon payments are made at the dates $T_0 = 0 < T_1 < \cdots < T_n$. Let us denote by $L(T_i)$ the floating interest rate for the risk-free borrowing and lending over the accrual period $[T_i, T_{i+1}]$. On each coupon date T_i, the coupon payment of an FRN is made according to the credit-risk adjusted floating rate $\hat{L}(T_i) = L(T_i) + s,$ where $L(T_i)$ is the risk-free floating rate, and the non-negative constant s represents the bond-specific *floating-rate credit spread.* For otherwise comparable notes, the higher level of the credit spread s usually corresponds to the lower credit quality of the issuer.

Let us examine the credit spread over the risk-free floating rate of an FRN that trades at par. In view of our current convention concerning the recovery scheme (cf. expression (1.1)), the cash flows of such a corporate FRN can be formally represented as follows:

$$\sum_{i=0}^{n} \hat{L}(T_i) 1\!\!1_{\{\tau > T_i\}} 1\!\!1_{T_i}(t) + \big(1\!\!1_{\{\tau > T\}} + \delta 1\!\!1_{\{\tau \le T\}}\big) 1\!\!1_T(t),$$

where, without loss of generality, we have set $L = 1$. Alternatively, an FRN can be treated as a single cash flow, which settles at T, and equals

$$\mathrm{FRN}\,(T,T) = \sum_{i=0}^{n} \hat{L}(T_i) B^{-1}(T_i,T) 1\!\!1_{\{\tau > T_i\}} + 1\!\!1_{\{\tau > T\}} + \delta 1\!\!1_{\{\tau \le T\}}.$$

Recall that τ stands for the default time, and δ is the constant recovery rate. The arbitrage price at time $t < \tau$ of such a contingent claim will be denoted as FRN (t,T). Using simple no-arbitrage arguments, it is not difficult to show that if the floating rate $L(T_i)$ satisfies

$$L(T_i) = \frac{1}{(T_{i+1} - T_i)} \left(\frac{1}{B(T_i, T_{i+1})} - 1 \right)$$

then a risk-free floating-rate note necessarily trades at par at time 0. Assume that $\hat{L}(T_i) = L(T_i) + s$ and the FRN trades at par at time 0. Then the credit spread s at time 0 can be found from the equality

$$\sum_{i=0}^{n} \tilde{D}(0,T_i) + s \sum_{i=0}^{n} D^0(0,T_i) + D^\delta(0,T_n) = 1,$$

where $\tilde{D}(0,T_i)$ denotes the value at time 0 of the random payoff $L(T_i) 1\!\!1_{\{\tau > T_i\}}$, which settles at T_i, and $D^0(0,T_i)$, $i = 1,\ldots,n$ and $D^\delta(0,T_n)$ are the prices of zero-coupon bonds issued by the same entity as the FRN in question (by virtue of the *cross-default* covenant all these bonds default simultaneously). Notice that the level of the spread s depends, in particular, on the correlation between the risk-free rate $L(T_i)$ and the default event $\{\tau > T_i\}$. For the sake of computational simplicity, it is frequently assumed in financial modeling that these two random factors are mutually independent.

In the market practice, both *callable* and *putable* FRNs are common. The issuer of a callable FRN has the right to redeem the note before its maturity. The holder of a putable FRN has the right to force an early redemption. The changes in credit quality determine whether option exercise is advantageous. This means that typically a floating-rate note has also an embedded credit derivative, specifically, a call or put option on the value of a note.

Let us finally observe that the default risk of the bond's buyer has no relevance whatsoever to the value of a corporate bond. Therefore, a corporate (fixed- or floating-rate) bond may serve as a natural example of a credit-risk sensitive contract with unilateral default risk.

1.1.7 Bank Loans and Sovereign Debt

Apart from the market of corporate bonds, there exist two other important sections of the defaultable debt market, namely, the market of *syndicated bank loans* and the *sovereign debt* market.

Syndicated bank loans. Syndicated bank loans (SBLs) are primarily large, high grade commercial loans. In recent years, a considerable growth has been observed in the secondary trading on the market of syndicated bank loans. This has been paralleled by the emergence of bank loan ratings. In many respects the syndicated bank loans are similar to corporate bonds, and thus investors are now considering SBLs as substitutes or complements to corporate bonds. We refer to a recent article by Altman and Suggitt (2000), who present a thorough empirical analysis of the default rates on the market of syndicated bank loans.

Sovereign debt. As an important sector of the sovereign debt market, let us mention the so-called *Brady bonds.* Brady bonds were issued by several less developed countries. They are primarily denominated in U.S. dollars and traded in the global bond markets. Typically, they contain various forms of credit guarantees and protections, so that it is rather hard to isolate the country-specific credit spread that is embedded in yields on Brady bonds.

1.1.8 Cross Default

Due to the complexity of debt indentures, the definitions of the *cross-default event* existing in various accessible sources are rather cumbersome, and thus are subject to differing interpretations. The definition that we want to adopt reflects the fact that the cross-default covenant basically corresponds to provisions in loan agreements or bond indentures, which trigger an event of default if the counterparty (borrower or issuer) defaults on another obligation. Such a description agrees with the definition provided by the International Finance and Commodities Institute, which states that the cross default is: "A provision of a loan or swap agreement stating that any default on another loan or swap will be considered a default on the issue with the cross-default provision. The purpose of this provision is to protect a creditor or counterparty from actions favoring another creditor."

1.1.9 Default Correlations

Consider two different defaultable claims whose lifetimes intersect, formally defined as the two random variables on a common probability space. For instance, abstract corporate bonds with the same maturity date are considered to be different defaultable claims if they have different initial credit ratings, or if their recovery covenants differ. Consider also a period of time contained within the lifetimes of the two claims. Let X stand for the random variable, which takes value 1, if the first claim defaults during the specified period of time, and takes value 0 otherwise; an analogous random variable associated with the second claim is denoted by Y. By convention, the *default correlation* between the two defaultable claims is defined as the correlation coefficient between the random variables X and Y. Default correlations are an important building block of credit risk measurement and management methodologies for credit-risk sensitive portfolios, mentioned in Sect. 1.4. From the theoretical perspective, the issue of modeling correlated defaults was addressed by, among others, Duffie and Singleton (1999), Davis and Lo (1999, 2001), Jarrow et al. (1999), Kijima and Muromachi (2000), Frey and McNeal (2000), Kijima (2000), Embrechts et al. (2001) Jarrow and Yu (2001), and Zhou (2001). We refer also to Sect. 3.6 and 12.3 for an exhaustive discussion of this important topic.

1.2 Vulnerable Claims

Vulnerable claims are contingent agreements that are traded over-the-counter between default-prone parties; each side of the contract is thus exposed to the counterparty risk of the other party. The default risk of a counterparty (or of both parties) is thus an important component of financial risk embedded in a vulnerable claim; it should necessarily be taken it into account in valuation and hedging procedures for vulnerable claims. On the other hand, the underlying (reference) assets are assumed to be insensitive to credit risk. *Credit derivatives* are recently developed financial instruments, which allow for a secluded trading in the reference credit risk. In contrast to vulnerable claims, in which the counterparty risk appears as a nuisance or a side effect, credit derivatives are tailored as highly specialized and effective devices to handle or transfer the reference credit risk. Since credit derivatives are offered over-the-counter, a credit derivative typically represents also a vulnerable claim, though, unless the counterparty risk is negligible.

1.2.1 Vulnerable Claims with Unilateral Default Risk

The classic example of a vulnerable contingent claim with unilateral default risk is a European *vulnerable option* – that is, an option contract in which the option writer may default on his obligations . In other words, this is an option whose payoff at maturity depends on whether a default event, associated with the option's writer, has occurred before or on the maturity date, or not. The default risk of the holder of the option is manifestly not relevant.

Consider a vulnerable European call option on a default-free U-maturity zero-coupon bond – that is, a vulnerable claim with no reference risk. Let $T < U$ be the option expiration date, $B(T, U)$ be the price of the underlying bond, and K be the option's strike price. The payoff C_T at exercise date

T of a European call option written on a default-free zero-coupon bond of maturity U is equal to $C_T = (B(T,U) - K)^+$. Let $\mathcal{D} = \{\tilde{\tau} \leq T\}$ denote the event that the call writer defaults either before or on the exercise date T, where $\tilde{\tau}$ is the default time of the call writer. If default occurs, only a fraction $\tilde{\delta}$ of the call's intrinsic value is redeemed by the call owner. Thus, the payoff at the settlement date of this option may be written as

$$C_T^d = C_T \mathbb{1}_{\{\tilde{\tau} > T\}} + \tilde{\delta} C_T \mathbb{1}_{\{\tilde{\tau} \leq T\}}.$$

It is essential to distinguish between the above vulnerable option on a default-free bond, and a standard (non-vulnerable) option on a defaultable bond, with the payoff at expiry $D_T = \big(D(T,U) - K\big)^+$, where $D(T,U)$ represents the price of a U-maturity corporate bond at time T (cf. Sect. 1.1). Finally, we may also consider a vulnerable option on a defaultable bond, with the payoff at maturity given by the formula

$$\tilde{C}_T^d = D_T \mathbb{1}_{\{\tilde{\tau} > T\}} + \tilde{\delta} D_T \mathbb{1}_{\{\tilde{\tau} \leq T\}}.$$

The last option may serve as a simple example of a *hybrid derivative*; its valuation will involve both the reference and the counterparty risks.

1.2.2 Vulnerable Claims with Bilateral Default Risk

Vulnerable claims with *bilateral* (or *two-sided*) default risk are these contracts in which both counterparties are susceptible to default risk. The prime example are here swap agreements between two default-prone entities, known as *defaultable swaps*. Despite the similarity of names, a defaultable swap should not be confused with a default swap, which is in fact a form of insurance against the reference risk (the latter kind of contracts is explained in Sect. 1.3.1 below). In contrast to default-free swaps, alternative settlement rules in case of default may largely influence the valuation of defaultable swaps. In addition, we also need to specify the debt's seniority. Typically, it is assumed that swaps are subordinate to debt in bankruptcy. We shall follow this convention here. Thus, it is natural to assume that if the party that is in default on its original debt is due to make a swap payment, it will default also on the swap contract. If, on the other hand, the party in default is due to receive a swap payment, two alternative settlement rules can be examined: (i) the swap payment is received, or (ii) the swap payment is withheld. If the latter rule is adopted a swap becomes valueless in case of default. In the former case, the swap payment at default has option-like features, and thus the total value of a swap contract depends, in particular, on the value of the embedded option. Various aspects of defaultable swaps were analyzed by several authors, to mention a few: Cooper and Mello (1991), Rendleman (1992), Abken (1993), Duffie and Huang (1996), Huge and Lando (1999), Li (1998), Laurent (2000), Lotz and Schlögl (2000), and Schönbucher (2000b). We shall now introduce some basic notions related to default-prone interest rate contracts; for more details, see Chap. 14.

1.2.3 Defaultable Interest Rate Contracts

Let us compare some basic types of spot default-free interest rate agreements and swaps with the corresponding contracts that are susceptible to the default risk of a counterparty. A more detailed study of selected types of single- and multi-period defaultable interest rate contracts is postponed to Chap. 14.

We start by describing the basic type of a (default-free) spot *interest rate agreement* (or a *credit agreement*) with the notional amount L and a nominal interest rate κ, for the accrual period $[T, U]$. We shall refer to T as the *reset date* and to U as the *settlement date.* An interest rate agreement can be described as a financial contract between two parties, a *receiver* and a *payer,* which is subject to the following covenants:

– at time T the receiver passes the notional amount L to the payer,
– he receives from the payer the accrued amount $L(1+\kappa(U-T))$ at time U.

It is apparent that the covenants of this agreement assume that the payer (of the fixed rate κ) is certain to deliver the promised payment to the receiver at time U. The following features of default-free interest rate agreements are worth stressing. First, the actual timing of the payments is not essential. For instance, the covenants of the agreement introduced above might be equally well restated as follows:

– at the settlement date U, the receiver passes to the payer the notional amount $LB^{-1}(T, U)$ discounted from time T to time U,
– he receives at time U from the payer the accrued amount $L(1+\kappa(U-T))$

or, equivalently,

– the receiver passes at time T to the payer the notional amount L,
– at the reset date T, the receiver collects from the payer the accrued amount $L(1+\kappa(U-T))B(T, U)$ discounted from time U to time T.

The above equivalences are, of course, valid from the perspective of the inception time T only. Second, the covenants of the interest rate agreement described above invoke exchange of principal payments. Thereby, the agreement is in fact equivalent to a loan subject to a fixed interest rate κ, where the receiver is the lending party, and the payer is the borrowing party. Such an agreement gives rise to the concept of an abstract (default-free) *spot LIBOR rate* $L(T)$ – that is, the level of the fixed rate κ, which makes the contract have a value of zero at the inception date T. One easily checks that

$$L(T) = \frac{1}{(U-T)} \left(\frac{1}{B(T, U)} - 1 \right).$$

More generally, if the contract's inception date t precedes the reset date T, the corresponding *forward LIBOR rate* $L(t, T)$ equals

$$L(t, T) = \frac{1}{(U-T)} \left(\frac{B(t, T)}{B(t, U)} - 1 \right).$$

Of course, both $L(T)$ and $L(t, T)$ depend on U; for the sake of brevity, though, this is not reflected in our notation.

We turn our attention to the corresponding *defaultable interest rate agreement* (or, equivalently, a *defaultable credit agreement*) in which only the payer party is prone to default. The contract is subject to the following covenants (we assume that none of the parties has gone bankrupt before the date T):

– at time T, the receiver passes to the payer the notional amount L,
– if the payer does not default in the time interval $(T, U]$, then at the settlement date U he pays to the receiver the accrued amount $L(1 + \kappa(U - T))$,
– if the payer defaults in $(T, U]$, then he pays to the receiver at time U the reduced amount $\delta L(1 + \kappa(U - T))$, where δ is the recovery rate.

Essentially, we deal here with a loan in which the debtor (the borrower, or the payer in the present context) may default on his obligation to repay the debt. An abstract defaultable spot LIBOR rate is the interest rate associated with such a loan (see Chap. 14 for details. In Chap. 14, we study also defaultable credit agreements in which the receiver is the only default-prone party, as well as contracts with bilateral default risk – that is, agreements in which both parties are prone to default).

The basic type of a spot *default-free interest rate swap* is the spot fixed-for-floating swap for the accrual period $[T, U]$, settled in arrears, with the spot default-free LIBOR rate $L(T)$ being the reference floating rate. The covenants of such an agreement, entered into at the reset date T, may be summarized as follows: there are two parties to the agreement, where one party is the payer of the fixed rate κ and the other is the payer of the floating rate $L(T)$; the parties agree to exchange at the settlement date U the nominal interest payments based on the notional amount L; thus, if $L = 1$, the net cash flow at the contract's settlement date U to one of the parties, to the payer of the fixed rate κ say, is equal to $(L(T) - \kappa)(U - T)$ (of course, the other party receives the negative of this cash flow).

The value of the fixed rate κ, which makes this cash flow have a value zero at the inception date T, is called the (default-free) *spot swap rate*. It is evident that in the default-free environment the spot swap rate and the spot LIBOR rate coincide. Essentially, in the default-free environment, the loan agreements and the interest rate swaps are equivalent.

As an example of a defaultable counterpart of the default-free spot swap introduced above, consider the following agreement (again, we set $L = 1$):

– the receiver passes at the settlement date U to the payer the full floating amount due: $L(T)(U - T)$,
– if the payer does not default in the time period $(T, U]$, the receiver collects at the settlement date U the full fixed amount due: $\kappa(U - T)$,
– if the payer defaults in $(T, U]$, the receiver gets at time U the reduced amount: $\delta\kappa(U - T)$.

The fixed rate κ that makes the above contract valueless at the inception time T, is termed the *defaultable spot swap rate*. As shown in Chap. 14, it typically differs from the defaultable spot LIBOR rate. In other words, in the presence of a counterparty risk, the loan and the swap contract are not equivalent to each other.

1.3 Credit Derivatives

Credit derivatives are privately negotiated derivative securities that are linked to a credit-sensitive asset (index) as the underlying asset (index). More specifically, the reference security of a credit derivative can be any financial instrument that is subject to risk of default (or, more generally, to the credit risk). For example, an actively-traded corporate or sovereign bond, or a portfolio of these bonds, may serve as an underlying asset or index for such a derivative. A credit derivative can also have a loan (or a portfolio of loans) as the underlying reference credit. It is clear that a credit derivative derives its value from the price – and thus from the credit quality – of the underlying default prone credit instrument.

The first agreements for a secluded transfer of credit risk were only signed in the early 1990s. It is thus worthwhile to mention that financial arrangements with features similar to credit derivatives – such as the *letter of credit* or the *bond insurance* – were largely used by commercial banks much earlier. Under a letter of credit, an issuer pays a bank an annual fee in exchange for the bank's promise to make debt payments on behalf of the issuer, should the issuer fail to do so. Under a bond insurance, an issuer pays an insurer to guarantee the performance on a bond (for more details, see, e.g., Fabozzi (2000)). However, in contrast to credit derivatives, these more traditional credit risk protections are not tradeable separately from the underlying obligation.

Credit derivatives can be structured in a large variety of ways; they are typically complex agreements, customized to the specific needs of an investor. Due to the rapidly growing demand, in the past few years the credit derivatives were the worldwide fastest-growing derivative products. As estimated by J.P. Morgan, in April 2000 the total nominal value of outstanding credit derivatives has passed the 1,000 billion U.S. dollar line. The common feature of all credit derivatives is the fact that they allow for the transfer of the credit risk from one counterparty to another; they thus constitute a natural and convenient tool to control the credit risk exposure. The overall risk an investor is concerned with involves two components: market risk and asset-specific credit risk. In contrast to 'standard' interest-rate sensitive derivatives, credit derivatives allow to isolate the firm-specific credit risk from the overall market risk. They also provide a way to synthesize assets that are otherwise not available to a particular investor (in this case, an investor 'buys' – rather than 'sells' – a specific credit risk). For an extensive analysis of economical reasons that support the use of these products, as well as expositions of various aspects of credit derivatives markets, we refer to Duffee and Zhou (1996), Das (1998a, 1998b), Tavakoli (1998), Francis et al. (1999) and Nelken (1999).

We shall now focus for a moment on credit derivatives associated with the defaultable term structure. Similarly as in the case of derivative securities associated with the risk-free term structure, we may formally distinguish three main types of agreements: forward contracts, swaps, and options.

A *forward contract* commits the buyer to purchasing a specified instrument at a specified future date at a price predetermined at contract inception. In a forward contract, the default risk is normally borne by the long party. If a credit event occurs, the transaction is marked to market and unwound. Forward contracts can also be transacted in spread form – that is, the agreement can be based on the specified bond's spread over a benchmark asset. In market practice, the most popular credit-sensitive swap contracts are: *total rate of return swap, asset swap* and *default swap* (explained in some detail in Sect. 1.3.1–1.3.2). *Credit options* are typically embedded in complex credit-sensitive agreements, though the over-the-counter traded credit options, such as: *options on an asset swap* or *credit spread options,* are also available.

Credit derivatives can also be classified into three groups. The first class consists of these credit derivatives that are linked exclusively to the default event; that is, it includes contracts with the payoff determined by the default event, as opposed to the changes in the credit quality of the underlying instrument. *Default swaps, default options,* and *first-to-default swaps* may serve as typical examples of such *default products.* The second group includes the so-called *spread products* – that is, credit derivatives whose payoff is primarily determined by the changes in the credit quality of the underlying security, e.g., *credit spread swaps, credit spread options* and *credit linked notes.* One may distinguish here the subclass of credit derivatives that are linked directly to rating upgrades or downgrades. Finally, the last class encompasses derivatives that allow to transfer the total risk of an asset between two parties. As an example of a *synthetic securitization*, we may cite the *total rate of return swap.* Let us observe that, due to the complexity of traded credit derivatives, the above classification is not definitive. Unfortunately, the terminological conventions relative to these contracts are not yet fully uniform, though a continuous standardization of OTC credit derivatives should be acknowledged. In 1999, the International Swaps and Derivatives Association (ISDA) published the ISDA 1999 *Credit Derivatives Definitions* providing a basic framework for documenting privately negotiated credit derivative transactions, and developed a standard documentation for credit derivatives – the ISDA 1999 *Master Agreement.* Through the ISDA standard documentation, the comparability and objectivity of credit derivatives has increased considerably.

In 2000, *Risk* conducted a survey aiming at determining the scope and the size of credit derivatives market. The following numbers represent percentage of the credit derivatives business generated by particular credit derivative products (in terms of notional outstanding): credit default swaps – 45%, synthetic securitizations – 26%, asset swaps – 12%, credit linked notes and asset repackaging – 9%, basket default swaps – 5%, credit spread options – 3%. The total notional amount of outstanding contracts of the participants in the *Risk* survey was around $810 billion. For more details, see Patel (2001).

1.3.1 Default Swaps and Options

Default swaps and *default options* may be considered as some sort of debt insurance contracts; that is why they are also known as *default insurance* or *default protection* (they are also sometimes called *credit default swaps/options*). In these agreements, periodic fixed payments (in the case of a default swap) or an upfront fee (in the case of a default option) from the protection buyer is exchanged for the promise of some specified payment from the protection seller to be made only if a particular, pre-specified *credit event* occurs. If a credit event occurs during the lifetime of the default swap/option, the seller pays the buyer an amount to cover the loss, and the contract then terminates. If no credit event has occurred prior to maturity of the contract, both sides end their obligations to each other. Let us stress that we deal here with a protection against the reference credit risk; the default risk of counterparties is thus neglected; in most practical cases it has only a minor impact on the valuation and hedging of a default swap (Hull and White (2001) and Lando (2000b) examine default swaps in the presence of the counterparty risk; see Sect. 12.3.1). The most important covenants of a default swap/option are:
– the specification of the credit event that is formally referred to as the 'default' (in practice, it may include: bankruptcy, insolvency, or payment default, a stipulated price decline for the reference asset, a rating downgrade of the reference entity, etc.),
– the contingent default payment, which may be structured in a number of ways; for instance, it may be linked to the price movement of the reference asset, or it can be set at a predetermined level (e.g., a fixed percentage of the notional amount of the transaction),

and, in the case of the default swap:
– the specification of periodic payments, which largely depend on the credit quality of the reference asset.

Default swaps/options are usually settled in cash; the agreement may also provide for physical delivery, though. For example, it may involve payment at par by the seller of the contract in exchange for the delivery of the defaulted reference asset.

Standard default swaps/options. To describe the cash flows in these contracts, let us consider a default swap/option with the maturity date T. As the reference asset, we first take a defaultable zero-coupon bond with the face value L and maturity date $U \geq T$. The contingent payment is triggered at the default time τ of the reference entity. In case of the default option, the protection buyer pays a lump sum (commonly referred to as the *premium*) at the contract's inception, and the payoff at default time is given as

$$\big(L - D(\tau, U)\big)\mathbb{1}_{\{\tau \leq T\}} = \big(L - D(\tau, U)\big)^{+}\mathbb{1}_{\{\tau \leq T\}}. \tag{1.2}$$

It is not surprising that such a contract is also commonly known as the *default put option*. Since the long party – i.e., the protection buyer – pays an upfront fee, the buyer's credit quality has no relevance in this kind of contract.

In case of the default swap, the protection buyer pays an *annuity*, also called *credit swap premium,* at times T_i, $i = 1, \dots, m$ prior to default or maturity date, whichever comes first, and the payoff at default is as given above. From the buyer's perspective, the cash flows of the default swap can be represented as follows:

$$\big(L - D(\tau, U)\big) 1\!\!1_{\{\tau \le T\}} 1\!\!1_{\{\tau\}}(t) - \sum_{i=1}^{m} \kappa 1\!\!1_{\{\min(\tau,T) > T_i\}} 1\!\!1_{\{T_i\}}(t), \tag{1.3}$$

where κ denotes the annuity amount. Since we are dealing here with a contract with bilateral default risk and a reference risk, the credit quality of both counterparties should be taken into account.

Let us emphasize once more that descriptions of cash flows of a default swap/option given above are valid only under an implicit assumption that the counterparty default risk can be disregarded.

Remarks. Alternative covenants for the recovery payment may be considered, for instance, a contract may stipulate that the recovery payment equals

$$\big(LB(\tau, U) - D(\tau, U)\big) 1\!\!1_{\{\tau \le T\}},$$

or

$$\big(D(\tau-, U) - D(\tau, U)\big) 1\!\!1_{\{\tau \le T\}},$$

where $D(\tau, U)$ represents the value of the defaultable bond immediately after the bond has defaulted – that is, its post-default value. Thus, in the former case, the protection buyer is compensated for the loss of value of the defaultable bond relative to the value of the Treasury bond at time τ. In the latter case, he is compensated for the loss in the value of the defaultable bond immediately after the default, relative to the value of the bond immediately before the default.

The generic expressions (1.2)–(1.3) can be given a more explicit form, as soon as a particular recovery scheme for the underlying bond is adopted. For instance, in the case of fractional recovery of Treasury value, with the fixed recovery rate δ, formulae (1.2)–(1.3) become: $L(1 - \delta B(\tau, U)) 1\!\!1_{\{\tau \le T\}}$ and

$$L\big(1 - \delta B(\tau, U)\big) 1\!\!1_{\{\tau \le T\}} 1\!\!1_{\{\tau\}}(t) - \sum_{i=1}^{m} \kappa 1\!\!1_{\{\min(\tau,T) > t_i\}} 1\!\!1_{\{t_i\}}(t),$$

respectively. If we postulate the fractional recovery of par value instead, then we obtain the following expressions: $L\big(1 - \delta\big) 1\!\!1_{\{\tau \le T\}}$ and

$$L\big(1 - \delta\big) 1\!\!1_{\{\tau \le T\}} 1\!\!1_{\{\tau\}}(t) - \sum_{i=1}^{m} \kappa 1\!\!1_{\{\min(\tau,T) > t_i\}} 1\!\!1_{\{t_i\}}(t),$$

respectively. Finally, when the fractional recovery of market value covenants are in place, then formulae (1.2)–(1.3) become $(L - \delta D(\tau-, U)) 1\!\!1_{\{\tau \le T\}}$ and

$$\big(L - \delta D(\tau-, U)\big)\mathbb{1}_{\{\tau \le T\}}\mathbb{1}_{\{\tau\}}(t) - \sum_{i=1}^{m} \kappa \mathbb{1}_{\{\min(\tau,T) > t_i\}}\mathbb{1}_{\{t_i\}}(t),$$

respectively. In practice, the typical default swap also requires the buyer to pay an accrued premium, based on the time since the previous annuity payment, if there is a default (expression (1.3) can be easily modified to account for this covenant). However, the accrued coupon payment is not covered by this protection. It is in fact more common for a default swap/option to have a corporate coupon bond (e.g., a floating-rate note), rather than a corporate zero-coupon bond, as the underlying asset. In such a case, formulae (1.2)–(1.3) become: $(L - \mathrm{FRN}(\tau, U))\mathbb{1}_{\{\tau \le T\}}$ and

$$\big(L - \mathrm{FRN}(\tau, U)\big)\mathbb{1}_{\{\tau \le T\}}\mathbb{1}_{\{\tau\}}(t) - \sum_{i=1}^{m} \kappa \mathbb{1}_{\{\tau > t_i\}}\mathbb{1}_{\{t_i\}}(t),$$

respectively, where $\mathrm{FRN}(\tau, U)$ denotes the market value of a corporate floating-rate note (FRN) just after the issuer's default. Since an FRN typically trades at a price close to par, the last two payoffs reflect more adequately than (1.2)–(1.3) the bond-specific default risk (as opposed to the market interest rate risk). If the default swap has an FRN as the underlying asset, the annuity κ is usually close to the credit spread s, between the floating rate on the note and the risk-free floating rate (or LIBOR rate).

Remarks. Let us note that FRNs may also serve as the underlying instruments of a *credit option.* If the investor takes a certain view regarding the credit quality at time $T < U$ of the firm issuing the underlying FRN, then he may purchase a call option with the payoff $(\mathrm{FRN}(T, U) - K)^+$, where K is the strike price and T the option's expiration date, or a put option with the payoff $(K - \mathrm{FRN}(T, U))^+$. The valuation of the credit call option described above requires modeling of the underlying corporate FRN.

Exotic default swaps/options. There are other variations of the standard default swap/option, referred to as *exotic default swaps/options.* These variations may regard the covenants determining the triggering of credit event and/or covenants determining the amount of insurance payoff. For instance, in a *digital default swap/option*, the payment to the long party at default is a pre-specified fixed amount. The *basket default swap/option* is a form of the first-to-default contract: the insurance payment takes place if the first one in a group of specified reference entities defaults (see Sect. 1.3.5 below). The *contingent default swap/option* is a contract in which the insurance payoff requires both the underlying credit event to occur, as well as an additional trigger, such as a credit event with respect to some other reference defaultable claim. Finally, the *dynamic default swap/option* is a variant of a default swap/option in which the notional amount, determining the amount of the insurance payoff, is the marked to market value of a designated portfolio of default swaps/options.

1.3.2 Total Rate of Return Swaps

The *total rate of return swap* – also known as the *total return swap* (*TROR* or *TRS,* for short) – is an agreement in which the total return on some reference entity (a basket of assets, an index, etc.) is exchanged for some other cash flows. One party, referred to as the *payer,* agrees to pay the total return of the reference entity (coupons plus or minus any change in the capital value) on a notional principal amount to another party, referred to as the *receiver.* In return, the *receiver* agrees to make periodic payments according to an agreed (fixed or floating) interest rate on the same notional amount. From the receiver's perspective, a total rate of return swap is thus similar to a synthetic purchase of the underlying entity.

If default of the reference entity occurs during the lifetime of a TROR, the contract terminates immediately; no further coupon or interest rate payment change hands. The receiver has the obligation to cover the change in value of the underlying asset by paying the payer the difference since the start of the swap, though. This means that the receiver accepts the price risk, including the credit risk, of the underlying reference security. Put another way, a TROR has an embedded default swap in which the payer is the protection buyer. The most relevant features of a TROR can be summarized as follows:

- no principal amounts are exchanged and no physical change of ownership occurs,
- the nominal principal of the swap may differ from the nominal value of the reference entity (a bond, a loan, etc.),
- the maturity of the total return swap agreement need not match that of the underlying reference entity,
- at the swap termination (i.e., either at its maturity, or upon default of the reference entity) a price settlement, based on the change in the value of the reference asset, is made.

We shall now give an example of a total return swap. As a reference asset, we take a corporate coupon bond with the promised coupons c_i at times T_i. We assume that the notional principal equals 1, and the maturity date of the swap is $U \leq T_n$, so that it expires before the bond's maturity date. In addition, suppose that the fixed *annuity payments* (*reference rate payments*) κ are made by the receiver at some scheduled dates $U_1 < U_2 < \cdots < U_m \leq U$. The receiver is entitled to all coupon payments during the lifetime of the contract (prior to default), as well as to the change in value of the underlying bond. From the receiver's perspective, the cash flows are:

$$\sum_{i=1}^{n} c_i \mathbb{1}_{\{\tilde{\tau} > T_i\}} \mathbb{1}_{\{T_i\}}(t) + \big(D_c(\tilde{\tau}, T) - D_c(0, T)\big) \mathbb{1}_{\{\tilde{\tau}\}}(t) - \kappa \sum_{i=1}^{m} \mathbb{1}_{\{\tilde{\tau} > U_i\}} \mathbb{1}_{\{U_i\}}(t),$$

where $\tilde{\tau} = \min(\tau, U) =: \tau \wedge U$. As before, τ stands for the default time of the bond, and $D_c(t,T)$ denotes the price of a defaultable coupon bond with unit face value at time t.

Let us observe that cash flows described above correspond to an implicit assumption that the counterparty risk can be neglected. In other words, the last expression embeds only the credit risk of the reference entity and the market risk, reflected here in the variations of the price $D_c(t,T)$ of a corporate coupon bond. More complex total return swaps can also incorporate put and call options (to establish caps and floors on the returns of the reference asset), as well as caps and floors on the floating reference rate.

1.3.3 Credit Linked Notes

A *credit linked note* (CLN) is a note paying an enhanced coupon to investors for bearing the credit risk of a reference entity. The buyer of the note funds the credit protection that the issuer of the note may thus sell to a third party; in exchange the note pays a higher-than-normal yield. Therefore, the buyer of a CLN can also be seen as a protection seller; the issuer of the CLN is thus the protection buyer. The basic credit linked note can be described by the following covenants:

– at the contract inception, an investor purchases a CLN by paying the issue price of the CLN in cash,
– during the life of the CLN, prior to default, she or he receives regular coupon payments,
– if there is no default by the reference entity during the lifetime of the CLN, the investor receives the full nominal value of the CLN,
– in case of default by the reference entity prior to maturity, the CLN terminates immediately, and the issuer of the CLN redeems the nominal value of the CLN in form of physical delivery of bonds issued by the reference entity (or the contract is settled in cash). No further coupon payments are distributed.

Since the investor makes an upfront fee, the default risk of the counterparty (the issuer of the CLN) is also essential – the enhanced coupon should compensate for both the reference credit risk and the unilateral default risk. Formally, a CLN may be seen as a combination of the default swap with the bond seen as a collateral. It protects the buyer of the default swap from the default risk of the protection seller – i.e., from the default risk of the buyer of a CLN. To summarize, these instruments involve a specific combination of a fixed-income instrument with an embedded credit derivative.

In the market practice, the CLNs are rather complex credit instruments. A credit linked note covenant may stipulate that the principal repayment is reduced to a certain level below par if the external corporate or sovereign debt defaults before the maturity of the note. The first-to-default CLN is linked to credit events from more than one reference entity. If there is a credit event, the note redeems early and there is no more exposure to other reference credit risks. As expected, the investor is compensated for bearing an additional credit risk by higher returns of the first-to-default CLNs.

CLNs are primarily issued by the special purpose corporations, also known as special purpose vehicles (SPVs for short). We shall henceforth refer to the issuer of the CLN as the investment bank. As an example of a credit linked note, let us examine a specific CLN, tied to a default swap associated with a reference corporate zero-coupon bond. We assume that the investment bank holds a high-rated collateral – a coupon bond with face value L, coupon rate c and maturity date T. At time $t' > 0$, the investment bank enters into a (reference) default swap with a third party. The default swap is relative to some reference credit, for instance, a low-rated corporate bond maturing at time T. In this default swap, the investment bank sells default protection in return for an annuity premium κ. Recall that the cash flows corresponding to a default swap were given in formula (1.3).

Here, the annuity payment dates t_i satisfy $t' < t_i$. At the same time t', the investment bank issues a CLN with the expiration date $T' < T$. An investor purchases the note at time t', for the par value L. If the reference credit does not experience a pre-specified credit event before the expiration date T' then the note pays coupon at the rate c' up until its expiration date, and the investor receives the par value back at the expiration date. Otherwise, the note pays the coupon at the rate c' only until the time when the reference credit experiences the specified credit event, and at this time the investor collects only some recovery portion of the par value.

The recovery portion of the par value is determined as follows: at the time when the reference credit experiences the specified credit event, the collateral coupon bond is liquidated, and the investor receives the proceeds only after the third party (i.e., the default swap counterparty) is paid the contingent payment. The investor thus bears credit risks of both the reference and collateral credits, and the enhanced coupon rate c' must compensate the investor for the two credit risks involved.

To describe the cash flows corresponding to the above structure, as seen from the perspective of the investment bank, we shall assume that the collateral bond promises to pay coupons on dates $T_1 < \cdots < T_k < T$, and that the CLN promises to pay coupons on dates $T'_1 < \cdots < T'_m < T'$, where $T'_1 > t'$. We denote the price process of the collateral bond as $D_c(t,T)$. As usual, τ stands for the random time of the underlying credit event, which in this case may be declared as the default event of the reference corporate bond (by assumption $\tau > t'$). From the perspective of the bank, the cash flows are

$$
\begin{aligned}
&L\mathbb{1}_{\{t'\}}(t) - \big(L - D(\tau,T)\big)\mathbb{1}_{\{\tau\le T\}}\mathbb{1}_{\{\tau\}}(t) + \sum_{i=1}^{n} \kappa \mathbb{1}_{\{\tau>t_i\}}\mathbb{1}_{\{t_i\}}(t) \\
&\quad + \sum_{j=1}^{k} cL\mathbb{1}_{\{\tau>T_j\}}\mathbb{1}_{\{T_j\}}(t) - \sum_{l=1}^{m} c'L\mathbb{1}_{\{\tau>T'_l\}}\mathbb{1}_{\{T_l\}}(t) - \\
&\quad - \Big(D_c(\tau,T) - \big(L - D(\tau,T)\big)\Big)^{+}\mathbb{1}_{\{\tau\le T\}}\mathbb{1}_{\{\tau\}}(t) - L\mathbb{1}_{\{\tau>T'\}}\mathbb{1}_{\{T'\}}(t),
\end{aligned}
$$

where $D(\tau,T)$ represents the post-default price of the reference bond.

1.3.4 Asset Swaps

Consider an investor that holds a corporate bond paying fixed rate coupon. Such an investor may be interested in entering into a swap, in which he will pay the fixed coupon and will receive a floating-rate coupon. The combination of the bond and the swap represents the so-called *asset swap.* It is apparent that a position in an asset swap is similar to holding a floating-rate note issued by the corporation of reference (in most cases, the floating rate is given as the LIBOR rate plus a spread, referred to as the *asset swap spread*). Typically, if there is default on the underlying bond, the swap obligation remains in force.

Asset swaps may serve as the underlying securities of *credit options.* A *call on an asset swap* is an option to buy the underlying bond for the predetermined strike price, and to enter at the same time the associated asset swap. The strike price and the strike spread are specified in option's contract (usually, the strike price equals the par value of the underlying bond). A holder of a *put on an asset swap* has the right to sell a corporate bond of reference for the strike price, and to simultaneously enter an asset swap as a receiver of the fixed coupon.

1.3.5 First-to-Default Contracts

In a *first-to-default contract,* the protection seller is exposed to the first entity within a portfolio (a basket) of credit risk sensitive instruments, which defaults (or, more generally, which experiences a pre-specified credit event). Such a contract is typically unwound immediately after the first credit event.

Consider a portfolio of n credit instruments. Let us denote by τ_i the time of the credit event of interest (e.g., default) associated with the i^{th} instrument. We denote by τ the time when the first of these credit events occurs – that is, $\tau = \min(\tau_1, \tau_2, \dots, \tau_n)$. We assume that $\tau_i \neq \tau_j$ for any $i \neq j$, and we consider a contingent claim with the settlement date T. If the first credit event happens no later than at time T, i.e., when $\tau \leq T$, a contingent payment is made. The amount of the contingent payment depends on the name (the instrument) that defaulted first. Denoting by A_i the contingent payment associated with the i^{th} instrument, we have that the amount received at time τ is A_i if $\tau = \tau_i$. If none of the names experiences the credit event prior to or at time T then a payment A is made at the maturity date T. The payoff associated with the abstract first-to-default contract can be formally represented as a single cash flow, which settles at time T, and equals

$$\mathbb{1}_{\{\tau \leq T\}} B^{-1}(\tau, T) \sum_{i=1}^{n} A_i \mathbb{1}_{\{\tau = \tau_i\}} + A \mathbb{1}_{\{\tau > T\}}.$$

First-to-default swaps – also known as *basket default swaps* – are default swaps linked to a portfolio of credit-sensitive securities. The first-to-default contracts are a special case of the i^{th}-to-default contracts. We refer to Chap. 9 and 10 for a detailed study of the latter ones.

1.3.6 Credit Spread Swaps and Options

Spread derivatives are credit derivatives that are constructed from market observable credit spread levels in such a way, that they allow for an easy transfer of credit price/spread risk between parties. The *credit spread swap,* also known as the *relative performance total return swap,* is a credit-risk sensitive agreement under which one party makes payments based on the yield-to-maturity of a specific issuer's debt, and the other party makes payments based on comparable Treasury yields (or some other benchmark rate). As expected, in practice the payments' specifications are done in a large variety of ways. For instance, under a *credit spread forward*[3] the payoff at the contract exercise date depends on the difference between the credit spread prevailing on the exercise date and some pre-specified strike level (or some benchmark rate). The payment may be executed in either direction, depending on whether it has a positive or a negative value.

To be more specific, let us assume that for a fee paid periodically, the investor receives (or makes) a payment at the contract exercise date depending on the difference between the credit spread on an underlying corporate discount bond and some specified strike. Here, the credit spread is measured as the difference between the yield of the underlying corporate discount bond and the yield of an equivalent Treasury bond. Formally, at the exercise time $T < U$ the investor receives the difference

$$D(T,U) - LB(T,U) - K,$$

where K is a predetermined strike. Recall that $D(t,U)$ is the price at time t of a U-maturity corporate discount bond with face value L, and $B(t,U)$ is the price of the U-maturity Treasury discount bond with face value 1. As a fee for this contract, the investor is required to pay annuities at times $t_1 < t_2 < \cdots < t_n < T$. Thus, the associated cash flows are:

$$\big(D(T,U) - LB(T,U) - K\big)\mathbb{1}_{\{T\}}(t) - \kappa \sum_{i=1}^{n} \mathbb{1}_{\{t_i\}}(t).$$

If the fee payment is made up front, the above contract is termed the credit spread option (such a contract is similar to the credit default option).

Credit spread options are option-like agreements whose payoff is associated with the yield difference of two credit-sensitive assets. For instance, the reference rate of the contract can be a spread of a corporate bond over a benchmark asset of comparable maturity. The option can be settled either in cash or through physical delivery of the underlying bond, at a price whose yield spread over the benchmark asset equals the strike spread. Options on credit spreads allow to isolate the firm-specific credit risk from the market risk. When preferable, the credit spread options can also incorporate interest rate risk by using a fixed strike or yield.

[3] Sect. 6.2.1.1 in Ammann (1999) discusses *credit forward contracts.* These contracts are similar to the credit spread forwards.

1.4 Quantitative Models of Credit Risk

Formally, by *credit event* we mean any random event whose occurrence affects the ability of the counterparty in a financial contract to fulfill a contractual commitment to meet his or her obligations stated in the contract. The default event is of course a credit event; other examples of credit events include, e.g., changes in credit quality of a corporate bond. Note that credit events may not be directly observed by the parties in a financial contract. We want to stress here that the ability of the counterparty in a financial contract to fulfill a contractual commitment to meet its obligations stated in the contract is not necessarily adversely affected by the occurrence of a credit event. For example, if the credit event occurs due to the increase in the credit quality of a corporate bond then, manifestly, the above mentioned ability is not adversely affected.

A vast majority of mathematical research devoted to the credit risk is concerned with the modeling of the random time when the default event occurs, i.e., the *default time.* Some approaches to the defaultable term structure allow for a possibility of intermediate credit events that are associated with changes in the credit quality of a corporate bond, which migrates between various rating classes. In this case, the modeling of random times of credit migration also becomes an important issue.

Another important problem arising in the quantitative modeling of the credit risk is the issue of mathematical modeling of the so-called *recovery rates.* As already mentioned, the recovery rate specifies the payment to the contract holder in case of default. Recovery payments together with the notional amount of the contract determine the potential cash flows associated with the contract. The main objective of the quantitative models of the credit risk is to provide ways to price and to hedge financial contracts that are sensitive to credit risk. Needless to say that any approach to pricing credit risk should aim at producing an internally consistent (that is, an arbitrage-free) financial model. As already noted, two competing methodologies have emerged in order to model the default/migration times and the recovery rates: the structural approach and the reduced-form approach.

1.4.1 Structural Models

Structural models are concerned with modeling and pricing credit risk that is specific to a particular corporate obligor (a firm). Credit events are triggered by movements of the firm's value relative to some (random or non-random) credit-event-triggering threshold (or barrier). Consequently, a major issue within this framework is the modeling of the evolution of the firm's value and of the firm's capital structure. For this reason, the structural approach is frequently referred to as the *firm value approach.* Through the modeling of credit events in terms of the value of the firm, the structural methodology links the credit events to the firm's economic fundamentals.

Most structural models are concerned with only one type of credit events, namely, the firm's default. The time of default is typically specified as the first moment when the value of the firm reaches a certain lower threshold, so that it is defined *endogenously* within the model. Such a default triggering mechanism has a natural interpretation as the *safety covenant,* which aims to protect the interests of bondholders against those of stockholders.

An alternative approach within the structural framework postulates that the bankruptcy decision is at the discretion of the stockholders; such an approach leads to the important problem of the specification of the optimal capital structure and the strategic debt service. Let us mention that the recovery rates are also frequently given endogenously within the model, as some function of the value of the firm. From the long list of works devoted to the structural approach, let us mention here: Merton (1974), Black and Cox (1976), Galai and Masulis (1976), Geske (1977), Brennan and Schwartz (1977, 1978, 1980), Pitts and Selby (1983), Cooper and Mello (1991), Rendleman (1992), Kim et al. (1993a), Nielsen et al. (1993), Leland (1994), Longstaff and Schwartz (1995), Anderson and Sundaresan (1996, 2000), Leland and Toft (1996), Mella-Barral and Tychon (1996), Briys and de Varenne (1997), Ericsson and Reneby (1998), and Ericsson (2000).

1.4.2 Reduced-Form Models

In this approach, the value of the firm's assets and its capital structure are not modeled at all, and the credit events are specified in terms of some exogenously specified jump process (as a rule, the recovery rates at default are also given exogenously). We can distinguish between the reduced-form models that are only concerned with the modeling of the default time, and that are henceforth referred to as the *intensity-based models,* and the reduced-form models with migrations between credit rating classes, called the *credit migration models.*

Intensity-based approach. The main emphasis in the intensity-based approach is put on the modeling of the random time of default, as well as evaluating conditional expectations under a risk-neutral probability of functionals of the default time and corresponding cash flows. Typically, the random default time is defined as the jump time of some one-jump process. As we shall see, a pivotal role in evaluating respective conditional expectations is played by the so-called *default intensity process.* Modeling of the intensity process, which is also known as the hazard rate process, is the starting point in the intensity approach. We need to emphasize here the crucial role of the conditioning information in the modeling of the intensity process. The intensity-based approach is examined in detail in the second part of the present text. The interested reader is also referred to the original papers by, among others, Pye (1974), Ramaswamy and Sundaresan (1986), Litterman and Iben (1991), Jarrow and Turnbull (1995), Duffie et al. (1996), Schönbucher (1996, 1998a, 1998b), Lando (1997, 1998), Monkkonen (1997), and Madan and Unal (1998).

Credit migrations. If migrations between credit ratings are not allowed, the model is referred to as the *single credit rating model,* though formally we always have another rating grade in the model, namely, the default state. Otherwise we deal with the *multiple credit ratings model.* The traditional intensity-based approach focuses on the pre-default value of a corporate bond in a single rating class model. More recent studies have extended the intensity-based approach to the case of multiple credit rating classes.

Assume that the credit quality of corporate debt is quantified and categorized into a finite number of disjoint *credit rating classes* (or *credit grades*). Each credit class is represented by an element of a finite set, denoted by $\mathcal{K}$. It is natural to distinguish a particular element K of the set $\mathcal{K} = \{1, \ldots, K\}$, which formally corresponds to the default event. As observed in practice, the credit quality of a given corporate debt changes over time. We shall refer to this feature by saying that the credit quality *migrates* between various credit classes. This migration is frequently modeled in terms of a (conditional) Markov chain, denoted by C, with finite state space $\mathcal{K}$ and either discrete or continuous time parameter. The process C is referred to as the *credit migration process.* In most cases, the multiple defaults are excluded, so that the default class represents the absorbing state for the chain C. The main issue in this approach is thus the modeling of the transition intensities matrix for the migration process, both under the risk-neutral and the real-world probabilities. The next step is the evaluation of conditional expectations under a risk neutral probability of certain functionals, typically related to the default time. Let us stress here the special role of the so-called *factor models* of credit migration. In these models, based on the theory of Cox processes, the intensities of default and/or migrations are specified as functions of both macro- and micro-economic factors. References dealing with the stochastic modeling of credit migrations include: Das and Tufano (1996), Jarrow et al. (1997), Duffie and Singleton (1998), Arvanitis et al. (1998), Kijima (1998), Kijima and Komoribayashi (1998), Thomas et al. (1998), Huge and Lando (1999), Bielecki and Rutkowski (2000), Lando (2000a) and Schönbucher (2000).

Defaultable term structure. A direct modeling of the defaultable term structure of interest rates is in fact not much different from the modeling of the default-free term structure. It is customary to start with some model of the term structure of interest rates for each credit rating class to be included in the overall model of the defaultable term structure. For example, one may prefer to do the modeling in terms of the instantaneous short-term rate (Vasicek (1977), Cox et al. (1985b)), or in terms of instantaneous forward rates (Heath et al. (1992)). Alternatively, one may adopt the framework of market rates such as: the forward LIBOR rates (Brace et al. (1997), Miltersen et al. (1997)) or the forward swap rates (Jamshidian (1997)). If migrations between rating classes are also modeled, one needs to describe the mechanism that governs the observed variations in credit quality. Otherwise, it is enough to specify the mechanism that triggers the default event.

1.4.3 Credit Risk Management

Hedging of defaultable claims. In most structural models of credit risk, the default time is predictable with respect to the information carried by traded primary (non-defaultable) securities. These models are typically complete with regard to both the market risk and the credit risk, and thus perfect replication of both default-free and defaultable claims is feasible. On the contrary, in reduced-form models, the default time tends to be an unpredictable stopping time, and perfect hedging of defaultable claims with the use of non-defaultable securities is impossible. To overcome this difficulty, one may either use some defaultable securities as hedging instruments (see Wong (1998), Bélanger et al. (2001), Blanchet-Scalliet and Jeanblanc (2001)), or one may apply other hedging and pricing principles, such as: the local risk minimization (see Lotz (1998)) or the optimal shortfall hedging (see Lotz (1999)), or the utility-based pricing (see Collin-Dufresne and Hugonnier (1999)).

Integration of risks. The market risk associated with a financial instrument is the risk resulting from adverse movements in the level or volatility of the value of this instrument. If a given instrument is sensitive to both market and credit risks, the two types of risk are intertwined, and they can not be easily disentangled. Most of the quantitative models of credit risk account for this feature by an appropriate integration of market and credit risks. In the structural approach, credit events are contingent on the movements of the firm value process and, in case of some models, on the dynamics of the value process of all firm's liabilities. This property provides thus an obvious link between the two risks involved. In the reduced-form approach, the intensities of default (or credit migrations) are sometimes postulated to depend on various financial factors such as, e.g., credit spreads or convenience yields. These factors are generally tied to the market risk associated with a given defaultable claim. Jarrow and Turnbull (2000b) provide an interesting discussion on the intersection of the market risk and the credit risk.

Portfolio management. Recently developed practical approaches to the active management of credit risk are typically concerned with calculating the probability distribution of losses in case of default for portfolios of credit-risk sensitive instruments. This probability distribution is assessed under the real-world (statistical) probability, so that these methods do not yield directly pricing models for credit derivatives. The credit risk measurement techniques that have recently gained a considerable prominence include:
– the KMV[4] Credit Monitor and Portfolio Manager (see Crosbie (1997)),
– J.P. Morgan's methodology CreditMetrics (see Gupton et al. (1997)),
– the CSFP[5] CreditRisk$^+$ (1997),
– Moody's methodologies Creditscore and RiskCalc,
– McKinsey's CreditPortfolioView (see Wilson (1997a, 1997b)).

[4] KMV Corporation was founded by S. Kealhofer, J. McQuown and O. Vasicek.
[5] CSFP is an acronym of Credit Suisse Financial Products [www.csfb.com].

We refer to Saunders (1999), Crouhy et al. (2000), Gordy (2000), Nyfeler (2000), and Cossin and Pirotte (2000) for up-to-date surveys and comparative studies of the credit risk measurement and management methodologies.

1.4.4 Liquidity Risk

It is not uncommon to argue that the credit spread inherent in many financial instruments could be more aptly explained as being caused by the presence of liquidity risk, rather than the credit risk. Such a view is supported by rather convincing financial arguments and/or related econometric studies, reported, for instance, in Amihud and Mendelson (1991), Boudoukh and Whitelaw (1991), Longstaff (1995) or Bangia et al. (1999). It seems reasonable to expect that in practice each of these risks may dominate the other. Unfortunately, both these kinds of financial risks are rather difficult to separate. It is also noteworthy that from the purely mathematical viewpoint, the modeling of liquidity risk is not much different from the modeling of credit risk, and, to the best of our knowledge, no original mathematical techniques were developed to deal with the former kind of risk. In Merton's framework, models that allow for a separate treatment of credit and liquidity risks were developed by Longstaff (1995) and Ericsson and Renault (2000). In the reduced-form approach, the liquidity effect is already included as a component of the total spread, and thus the credit risk premium and liquidity premium can not be easily separated. In this text, we have chosen to use the generic term credit risk to cover both kinds of risk, especially when dealing with the reduced-form approach. For an analysis of liquidity risk and further references, the reader may consult the recent paper by Longstaff (2001).

1.4.5 Econometric Studies

An essential step in the practical implementation of any mathematical model of credit risk relies on the model's calibration against the real-life data. A detailed presentation of empirical studies related to the credit/liquidity risk is beyond the scope of the present text, though. The interested reader is thus referred to original papers by Sarig and Warga (1989), Sun et al. (1993), Altman and Bencivenga (1995), Altman and Kishore (1996), Duffee (1996, 1998), Carty and Lieberman (1997), Duffie and Singleton (1997), Monkkonen (1997), Altman and Saunders (1998), Kiesel et al. (1999a, 1999b), Shumway (1999), Taurén (1999), Houweling et al. (1999), Altman and Suggit (2000), Christiansen (2000), Liu et al. (2000), Rachev et al. (2000), Bakshi et al. (2001), Collin-Dufresne et al. (2001) or Carey and Hrycay (2001). To the best of our knowledge, Jonkhart (1979) and Iben and Litterman (1991) were the first researchers who proposed to impute implied probabilities of default from the term structure of yield spreads between default-free and defaultable corporate securities (also see the recent paper by Delianidis and Geske (2001) in this regard).