AMS 301
Ma, Pei
Set 6

Sec 5.1: 16abc, 30, 36
Sec 5.2: 8, 38, 54
Sec 5.3: 6, 9, 15, 21, 22
Sec 5.4: 2, 3ab, 7, 10, 11, 12, 48

Section 5.1
16.

(a) How many different outcomes are possible when a pair of dice, one red and one white, are rolled two successive times?

> Let $R = W = 1, 2, 3, 4, 5, 6$ where R is the set outcomes of rolling the red die and W is of the white die.
> Possible outcomes in one roll can be expressed as $R \times W$, and if we take the cardinality, $O = |R \times W| = |R| \times |W| = 6 * 6 = 36$, which is the number of possble outcomes in one roll.
> The number of outcomes in two successive rolls will be $O \times O$, and the number of possible outcomes in two rolls is $|O \times O| = 36 \times 36 = 1296$.

(b) What is the probability that each die shows the same value on the second roll as on the first roll?

> Since there are 36 possible outcomes for a single roll, there are 36 possible outcomes for two rolls to have the same values. So, the probability that each die shows the same value on the second roll as on the first roll is $\frac{36}{1296} = 0.46\%$.

(c) What is the probability that the sum of the two dice is the same on both rolls?

> The possible sums are 2 to 12 (11 different outcomes). The possible outcomes for each sum are as following, 1 to 6 for sums 2 to 7, and 5 to 1 for sums 8 to 12. So, if the sum of the two rolls are the same, the total outcomes for a sum x will be the number of outcomes for that sum squared. For example, a sum of 7 has 6 outcomes for one roll, so for two rolls to be equal, there are 6^2 outcomes. If we sum up the outcomes for each sum, $1^2 + 2^2 + ... + 6^2 + 5^2 + ... + 1^2 = 146$. So, the probability that the sum of the two dice is the same on both rolls is $146/1296 = 11.3\%$.

30. How many times is the digit 5 written when listing all numbers from 1 to 100,000? Comment on the answers **(a)** 4, **(b)** 5×10^4, and **(c)** $1 + 10 + 100 + 1000$.

(a) I don't know how they got 4. A singular number 55555 has five 5s, which is already greater than 4.
(b) This answer is correct.
(c) If we consider the number of numbers with 5 in the ones place, we can see by observation, that starting from 5, every 10 numbers will have a 5. Since there are 100,000 numbers, this means there are at least 10,000 numbers with at least one digit being 5.

36. If two different integers between 1 to 100 inclusive are chosen at random, what is the probability that the difference of the two numbers is 15?

The minimum integer is 1 in the pair (1, 16) and the maximum is 100 in the pair (85, 100). Then, we can see that there are 85 pairs. There are $\binom{100}{2} = \frac{100!}{98!2!} = 100 \cdot 99/2 = 4950$ total ways to pick two different integers. So, the probability that two different integers chosen have a difference of 15 is $85/4950 = 1.7\%$.

Section 5.2
8. There are nine white balls and four red balls in an urn. How many different ways are there to select a subset of six balls, assuming the 13 balls are different? What is the fraction of selections with four whites and two reds?

There are $\binom{13}{6} = \dfrac{13!}{6!(13-6)!} = 1716$ ways to select a subset of six balls, assuming the 13 balls are different.
There are $\binom{9}{4}\binom{4}{2} = \dfrac{9!4!}{4!(9-4)!2!(4-2)!} = 756$ selections with 4 whites and two reds.
So, the fraction of selections is $\dfrac{756}{1716} = 44.1\%$.

38. A student must answer five out of 10 questions on a test, including at least two of the first five questions. How many subsets of five questions can be answered?

We require two of the first five questions to be chosen, but that does not limit only two of the five to be chosen. So, we must consider the following cases, 2, 3, 4, or 5 of the first five questions get answered.
Case 2: Choose 2 of the first 5, then choose 3 of the last 5.
Case 3: Choose 3 of the first 5, then choose 2 of the last 5.
Case 4: Choose 4 of the first 5, then choose 1 of the last 5.
Case 5: Choose 5 of the first 5.
$\binom{5}{2}\binom{5}{3} + \binom{5}{3}\binom{5}{2} + \binom{5}{4}\binom{5}{1} + \binom{5}{5}\binom{5}{0} = 2 \cdot \dfrac{5!5!}{2!2!3!3!} + \cdot \dfrac{5!5!}{1!1!4!4!} + \cdot \dfrac{5!5!}{0!0!5!5!} = 200 + 25 + 1 = 226$ subsets.

54. How many arrangements of MATHEMATICS are there in which each consonant is adjacent to a vowel?

There are 4 vowels: A, E, A, I; and 7 consonants: M, T, H, M, T, C, S Let c denote consonant and v denote consonant. We can have the following forms:
cvc cvc cvc cv = $\{1, 2, 2, 2, 0\}$ (x2 for symmetry)
cvc cvc cvc vc = $\{1, 2, 2, 1, 1\}$ (x2 for symmetry)
cvc cvc vcc vc = $\{1, 2, 1, 2, 1\}$
And by symmetry and the rule that the ends in the set must be 1, we have 5 different forms.
The arrangements for the vowels is $\frac{4!}{2!} = 12$ since we have 2 A's, and the arrangements for the consonants is $dfrac7!2!2! = 1260$ since we have 2 M's and 2 T's.

So, there are a total of $5 \times 12 \times 1260 = 75600$ arrangements of MATHEMATICS where each consonant is adjacent to a vowel.

Section 5.3
6. If four identical dice are rolled, how many different outcomes can be recorded?

4 identical - xxxx : $\binom{6}{1} = 6$
3 identical 1 random - xxxy : $\binom{6}{2} = 15$
2 identical 2 different - xxyz : $\binom{6}{3} = 20$
2 pairs of 2 identical - xxyy : $\binom{6}{2} = 15$
all different - wxzy : $\binom{6}{4} = 15$

9. How many ways are there to pick a selection of coins from \$1 worth of identical pennies, \$1 worth of identical nickels, and \$1 worth of identical dimes if (a) You select a total of 9 coins? (b) You select a total of 16 coins?

There are 100 pennies, 20 nickels, and 10 dimes for selection.

15. How many 8-digit sequences are there involving exactly six different digits?

21. How many arrangements of the letters in MATHEMATICS are there in which TH appear together but the TH is not immediately followed by an E (not THE)?

22. How many arrangements of the letters in PEPPERMILL are there with (a) The M appearing to the left of all the vowels? (b) The first P appearing before the first L?