AMS 301
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Set 7

Sec 5.4: 16, 19, 21, 22, 36, 38, 42c
Sec 6.1: 2b, 4bc, 6, 8, 10, 16
Sec 6.2: 2, 20, 22

Section 5.4

16. Consider the problem of distributing 10 distinct books among three different people with each person getting at least one book. Explain why the following solution strategy is wrong: first select a book to give to the first person in 10 ways; then select a book to give to the second person in nine ways; then select a book to give to the third person in eight ways; and finally distribute the remaining seven books in 7^3 ways.

> There are two issues with this method.
> First, to distribute the remaining 7 books, there would be 3^7 ways, as this is a cases of distributing 7 distinct objects to 3 different people.
> Second, it does not account for repeating cases where the first book each person gets is repeated when distributing the remaining 7 books. For example, lets say person 1 gets book 1. Then, person 2 and 3 gets books 2 and 3 respectively. The final distribution is that person 1 gets 1, 4, 5, and 6. It does not matter what persons 2 or 3 gets. This distribution gets repeated if person 1 gets book 4 as their first book, then gets books 1, 5, and 6 when distribution the rest of the 7 books, and we assume persons 2 and 3 gets the same books to illustrate the case of repeating cases.

19. How many ways are there to distribute three different teddy bears and nine identical lollipops to four children

(a) Without restriction?

> Distribute 3 distinct objects to 4 children, so there are $4^3 = 64$ distributions. Then, to distribute 9 identical objects to 4 different children, there are $C(9+4-1, 9) = \frac{12!}{3!9!} = 220$ distributions. So, there are a total of $64 \times 220 = 14080$ distributions.

(b) With no child getting two or more teddy bears?

> We can only give a child at most 1 teddy bear, which implies that there will always be one child, so there will be $4 * (3 * 2 * 1) = 24$ distributions. Then, there are a total of $24 \times 220 = 5280$ distributions.

(c) With each child getting three "goodies"?

We only need to distribute the distinct objects. So, there will be $4^3 = 64$ distributions. The way the lollipops are distributed is trivial since they're identical.

21. Suppose a coin is tossed 12 times and there are three heads and nine tails. How many such sequences are there in which there are at least five tails in a row?

Let can describe the scenarios with x being tails and | being heads. So, we can write xx|xxxxx|x|x. $x_1 + x_2 + x_3 + x_4 = 9$ where one $x_i \geq 5$ and that $x_i = z_i + 5$. $C(9 - (5) + 4 - 1, 4 - 1) = \frac{(9-5+4-1)!}{(7-4+1)!(4-1)!} = 35$. There are 4 sections that can have the sequence of 5 or more tails, so the total sequences is $35 \times 4 = 140$.

22. How many binary sequences of length 20 are there that

(a) Start with a run of 0s—that is, a consecutive sequence of (at least) one 0— then a run of 1s, then a run of 0s, then a run of 1s, and finally finish with a run of 0s?

Each sequence will look like 0_1_0_1_0_. We have to distribute the remaining 15 digits into 5 boxes. $C(15 + 5 - 1, 5 - 1) = 3876$ binary sequences.

(b) Repeat part (a) with the constraint that each run is of length at least 2.

Each sequence will look like 00_11_00_11_00_. We have to distribute the remaining 10 digits into 5 boxes. $C(10 + 5 - 1, 5 - 1) = 1001$ binary sequences.

36. How many election outcomes in the race for class president are there if there are five candidates and 40 students in the class and

(a) Every candidate receives at least two votes?

Distribute two votes to every candidate, and now there are 30 remaining undistributed votes. $C(30 + 5 - 1, 5 - 1) = 46376$ election outcomes.

(b) One candidate receives at most one vote and all the others receive at least two votes?

There are 5 possibilities of one candidate receiving at most one vote and the others receiving two. $5 \times C(31 + 5 - 1, 5 - 1) = 5 \times 52360 = 261800$

(c) No candidate receives a majority of the votes?

For a candidate to get major vote, they must get more than 20 votes. So, the distributions that a candidate gets majority of the votes is $5 \times C(40 - 21 +$

$5-1, 40-21) = 44275$ distributions. There are a total of $C(40+5-1, 40) = 135751$ distributions. So, the total of distributions where no candidate receives a majority of the votes is $135751 - 8855 = 91476$.

(d) Exactly three candidates tie for the most votes?

The way we distribution of the three candidates that get the most votes is $C(5, 3) = 10$.
$x_1 + x_1 + x_1 + x_2 x_3 = 3x_1 + x_2 + x_3 = 40$, where $x_1 > x_2 \geq x_3 \geq 0$. If we set $3x_1 \leq 40$, we get that $x_1 \leq 13$. We can set $x_2 = x_3$ to find the minimum x_1. $2x_2 = 40 - 3x_1$ If everyone tied, then each candidate would have 8 votes; this means the minimum x_1 must be is 9. Now, we can split each distribution of x_1 votes into cases.
Case 1: $x_1 = 9$
$x_2 + x_3 = 40 - 3 \times 9 = 13$, but $x_2, x_3 < 9$. Consider $(x_2, x_3) = (8, 5)$, then there are 4 distributions, where $x_2, x_3 \in (5, 8)$.
Case 2: $x_1 = 10$
$x_2 + x_3 = 10$ and $x_2, x_3 < 10$. There will be 9 distributions.
Case 3: $x_1 = 11$
$x_2 + x_3 = 7$, and $x_2, x_3 < 11$ There will be $C(7+2-1, 7) = 8$ distributions.
Case 4: $x_1 = 12$
$x_2 + x_3 = 4$, there are 5 distributions.
Case 5: $x_1 = 13$
$x_2 + x_3 = 1$, there are 2 distributions.
So, the total number of distributions where exactly three candidates tie for most votes is $10(4+9+8+5+2) = 280$.

38. How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 \leq 15$ with $x_i \geq -10$?

$z_i = x_i + 10 \rightarrow x_i = z_i - 10$. $z_i - 10 \geq -10 \rightarrow z_i \geq 0$. We get the equation, $z_1 + z_2 + z_3 + z_4 - 40 \leq 15$ or $z_1 + \cdots + z_4 =\leq 55$.
We can change this to $z_1 + \cdots + z_5 = 55$, where $z_5 \geq 0$. So, there will be $C(55+5-1, 55) = 455126$ integer solutions.

42. How many nonnegative integer solutions are there to $x_1 + x_2 + \cdots + x_5 = 20$
(c) With $x_1 = 2x_2$

$3x_2 + x_3 + x_4 + x_5 = 20 \rightarrow x_3 + x_4 + x_5 = 20 - 3x_2$
So, $x_2 \in (0, 6)$.
The total number of solutions is $\sum_{x_2=0}^{6} C((20-3x_2)+3-1, 3-1) = 672$ total solutions.

Section 6.1

2. Build a generating function for a_r, the number of integer solutions to the following equations: (b) $e_1 + e_2 + e_3 = r, 0 < e_i < 6$

$1 \le e_i \le 5 \to g(x) = (x + x^2 + x^3 + x^4 + x^5)^3$

4. Build a generating function for a_r, the number of distributions of r identical objects into

(b) Three different boxes with between three and six objects in each box

$e_1 + e_2 + e_3 = r, 3 \le e_i \le 6$
$g(x) = (x^3 + x^4 + x^5 + x^6)^3$

(c) Six different boxes with at least one object in each box

$e_1 + \cdots + e_6 = r, e_i \ge 1$

$$g(x) = (x + x^2 + \dots)^6 = \frac{x^6}{(1-x)^6}$$

6. Use a generating function for modeling the number of different selections of r hot dogs when there are four types of hot dogs.

Let e_1, e_2, e_3, e_4 be the number of each type of hot dog selected. Then, we can write the equation $e_1 + e_2 + e_3 + e_4 = r$, $e_i \ge 0$.
$g(x) = (1 + x + x^2 + \dots)^4$ The number of ways to select r hotdogs from 4 types is $C(r + 4 - 1, r)$.

8.

(a) Use a generating function for modeling the number of different election outcomes in an election for class president if 25 students are voting among four candidates. Which coefficient do we want?

In a general case, there are a total of r votes in the election amongst 4 candidates. So, we can write, $e_1 + e_2 + e_3 + e_4 = r$, where $e_i \ge 0$. Then, the generating function is $g(x) = (1 + x + x^2 + \dots)^4$
We want to coefficient for x^{25}.
$a_{25} = C(25 + 4 - 1, 4 - 1) = 3276$.

(b) Suppose each student who is a candidate votes for herself or himself. Now what is the generating function and the required coefficient?

$e_1 + e_2 + e_3 + e_4 = r$, where $e_i \geq 1$.
$g(x) = (x + x^2 + \dots)^4$
We want the coefficient for x^{21}. $a_{21} = C(21 + 4 - 1, 4 - 1) = 2024$

(c) Suppose no candidate receives a majority of the vote. Repeat part (a).

Majority vote is if a candidate obtains 13 or more votes. Then, $e_i \leq 12$ if no candidate recieves a majority of the votes.
$g(x) = (1 + x + x^2 + \dots + x^{12})^4 = (\dfrac{1 - x^{13}}{1 - x})^4$
Suppose a candidate gets majority vote, then we have $e_1' + e_2 + e_3 + e_4 = 12$ where $e_1 = e_1' + 13$. So, the number of distributions where the candidate obtains majority vote is $C(12+4-1, 4-1) = 455$ The coefficient of x^25 is $3276 - 4 \times 455 = 1456$.

10. Given one each of u types of candy, two each of v types of candy, and three each of w types of candy, find a generating function for the number of ways to select r candies.

$g(x) = (1 + x)^u (1 + x + x^2)^v (1 + x + x^2 + x^3)^w$

16. Find a generating function for the number of integers between 0 and 999,999 whose sum of digits is r.

$e_1 + e_2 + e_3 + e^4 + e^5 + e^6 = r$, $0 \leq e_i \leq 9$
$g(x) = (1 + x + x^2 + \dots + x^9)^6 = \dfrac{(1 - x^{10})^6}{(1 - x)^6}$

Section 6.2
2. Find the coefficient of x^r in $(x^5 + x^6 + x^7 + \dots)^8$.

$(x^5 + x^6 + x^7 + \dots) = [(x^5)(1 + x + x^2 + \dots)]^8 = x^{40} \dfrac{1}{(1 - x)^8}$
$\dfrac{1}{(1 - x)^n} = 1 + C(1 + n - 1, 1)x + C(2 + n - 1, 2)x^2 + \dots + C(m + n - 1, r)x^m + \dots$
$\dfrac{x^{40}}{(1 - x)^8} = x^{40} + C(1 + 8 - 1, 1)x^{41} + C(2 + 8 - 1, 2)x^{42} + \dots + C((r - 40) + 8 - 1, r)x^{m+40} + \dots$
The coefficient for x^r for $r \geq 40$ is $C((r - 40) + 8 - 1, (r - 40))$.

20. How many ways are there to paint the 10 identical rooms in a hotel with five colors if at most three rooms can be painted green, at most three painted blue, at most three red, and no constraint is laid on the other two colors, black and white?

Max 3 green, 3 blue, 3 red, no constraint on black white.
$e_1 + e_2 + e_3 + e_4 + e_5 = r$, where $e_1, e_2, e_3 \leq 3$.
$g(x) = (1 + x + x^2 + x^3)^3 \cdot (1 + x + x^2 + \dots)^2 = \left(\frac{(1-x^4)^3}{(1-x)^5}\right) = \sum g_i x^i$
$(1 - x^4)^3 = 1 - C(3,1)x^4 + C(3,2)x^8 + C(3,3)x^{12} = \sum a_i x^i$
$\frac{1}{(1-x)^5} = 1 + \cdots + \binom{2+5-1}{2} x^2 + \cdots + \binom{6+5-1}{6} x^6 + \cdots + \binom{10+5-1}{10} x^{10} + \cdots = \sum b_i x^i$
$g_{10} = (a_0 b_{10} + a_4 b_6 + a_8 b_2) = 1 \cdot \binom{10+5-1}{10} - \binom{3}{1} \cdot \binom{6+5-1}{6} + \binom{3}{2} \cdot \binom{2+5-1}{2} = 416$

22. How many ways are there to get a sum of 25 when 10 distinct dice are rolled?

$\sum_{i=1}^{10} e_i = r, 1 \leq e_i \leq 6$
$g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^{10} = x^{10}(1 + x + \cdots + x^5)^{10} = \frac{x^{10}(1-x^6)^{10}}{(1-x)^{10}}$
We need coefficient of x^{15} in $\frac{(1-x^6)^{10}}{(1-x)^{10}}$.
$(1 - x^6) = 1 - C(10,1)x^6 + C(10,2)x^{12} \pm \cdots = \sum a_i x^i$
$\frac{1}{(1-x)^{10}} =$
$1 + \cdots + C(3+10-1,3)x^3 + \cdots + C(9+10-1,9)x^9 + \cdots + C(15+10-1,15)x^{15} + \cdots =$
$\sum b_i x^i$
$g_1 5 = (a_0 b_{15} + a_6 b_9 + a_{12} b_3) = 1 \cdot \binom{15+10-1}{15} + \binom{10}{1}\binom{9+10-1}{9} + \binom{10}{2}\binom{3+10-1}{3} = 831204$ ways.