

AMS 301
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Set 7

Sec 7.1: 4, 6ab, 7, 11, 12, 15, 19, 28, 30
Sec 7.3: 1, 2, 3a

Section 7.1
4.

- (a) Find a recurrence relation for the number of ways to go n miles by foot walking at 2 miles per hour or jogging at 4 miles per hour or running at 8 miles per hour; at the end of each hour a choice is made of how to go the next hour.

$$a_{n-2} + a_{n-4} + a_{n-8} = a_n$$
$$a_0 = 1, a_i = 0 \text{ for } i \leq -1$$

- (b) How many ways are there to go 12 miles.

$$a_2 = a_0 + a_{-2} + a_{-6} = 1$$
$$a_4 = a_0 + a_2 + a_{-4} = 2$$
$$a_6 = a_4 + a_2 + a_{-2} = 2 + 1 = 3$$
$$a_8 = a_6 + a_4 + a_0 = 3 + 2 = 5$$
$$a_{10} = a_8 + a_6 + a_2 = 5 + 3 + 1 = 9$$
$$a_{12} = a_{10} + a_8 + a_4 = 9 + 5 + 2 = 16$$

6.

- (a) Find a recurrence relation for the number of n -digit binary sequences with no pair of consecutive 1s.

$$a_0 = 1, (\emptyset); a_1 = 2, (0, 1); a_2 = 3, (00, 01, 10); a_3 = 5, (000, 001, 010, 100, 101)$$
$$a_n = a_{n-1} + (n - 1);$$

(a_{n-1}) represent adding a 0 to the end of each term in a_{n-1} and $(n - 1)$ represent $n - 1$ possible terms in $n - 1$ digit binary sequence that ends with 1.

- (b) Repeat for n -digit ternary sequences.

$$a_0 = 1, (\emptyset); a_1 = 3, (0, 1, 2); a_2 = 8, (00, 01, 02, 10, 12, 20, 21, 22)$$
$$a_n = 2 \times a_{n-1} + (n - 1)$$

Follows similar logic to the previous question, but since we have the option of adding a 0 or 2 to the end of all $n - 1$ digit outcomes, we have $2 \times a_{n-1}$.

7. Find a recurrence relation for the number of pairs of rabbits after n months if (1) initially there is one pair of rabbits who were just born, and (2) every month each pair of rabbits that are over one month old have a pair of offspring (a male and a female).

$$a_0 = 1 \text{ pair; } a_1 = 1 \text{ pair}$$

$$a_n = a_{n-1} + a_{n-2}$$

11. Find a recurrence relation for an for the number of ways for an image to be reflected n times by internal faces of two adjacent panes of glass. The diagram below shows that $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$.

In the internal faces of two adjacent panes, there are 3 lines which the image can be reflected on. We can label them 0, 1, 2. It is possible to reflect under 0, above and under 1, and above 2. We assume the image to begin from above, making the only possible reflections to go from above.

For 0 reflections, there is only $a_0 = 1$ way for the image to be reflected, which is no ways.

For 1 reflection, the image can reflect against line 1 and 2 from above, which is $a_1 = 1[1, \text{above}] + 1[2, \text{above}] = 2$ ways.

For 2 reflections, we include the previous count, 2, which hits line 0 from below from line 2, and add 1 for the reflections which hits line 0 from below from line 1.

So, following similar logic, for n reflections, there are $a_n = a_{n-1} + a_{n-2}$ reflections.

12. Find a recurrence relation for the number of regions created by n mutually intersecting circles on a piece of paper (no three circles have a common intersection point).

Let a_n be the number of regions created by n mutually intersecting circles. $a_0 = 1$ region; $a_1 = 2$ region; $a_2 = 4$ intersections; $a_3 = 8$ intersections.

From a little bit of intuition, adding another intersection will create $2n$ the amount of regions and add an additional region. So, $a_n = a_{n-1} + 2(n-1)$ regions.

15. Find a recurrence relation for the amount of money in a savings account after n years if the interest rate is 6 percent and \$50 is added at the start of each year.

$$a_n = 1.06(a_{n-1} + 50) \text{ with initial balance } a_0.$$

19.

- (a) Find a recurrence relation for the number of sequences of 1s, 3s, and 5s whose terms sum to n .

$$a_n = a_{n-1} + a_{n-3} + a_{n-5}, \text{ with initial condition } a_0 = 1.$$

- (b) Repeat part (a) with the added condition that no 5 can be followed by a 1.

Let $a_n = x_n + y_n$ where x_n is the number of valid sequences where the last term is not 5 and y_n is the number of valid sequences where the last term is 5.

For x_n , if the previous term ended with 1, then we have then the previous sequence must be x_{n-1} .

If the previous term ended with 3, then the previous sequence is $x_{n-3} + y_{n-3}$

So, we can write $x_n = x_{n-1} + (x_{n-3} + y_{n-3})$.

For y_n , it can follow any valid sequence summing to $n - 5$, $y_n = x_{n-5} + y_{n-5}$.

If we combine both terms,

$$a_n = x_{n-1} + (x_{n-3} + y_{n-3}) + (x_{n-5} + y_{n-5}) = x_{n-1} + a_{n-3} + a_{n-5}.$$

$$y_{n-1} = a_{n-6}$$

$$x_{n-1} = a_{n-1} - y_{n-1} \rightarrow x_{n-1} = a_{n-1} - a_{n-6}.$$

So, the final recurrence relation is $a_n = a_{n-1} + a_{n-3} + a_{n-5} - a_{n-6}$.

- (c) Repeat part (a) with the condition of no subsequence of 135.

The subsequence 135 has a sum of 9. So, for any sequence $0 \leq n < 9$, no sequence is excluded. We can follow the same formula as in part (a) $a_n = a_{n-1} + a_{n-3} + a_{n-5}$. Then, we must remove the only possible sequence with a sum of 9 that includes the subsequence 135, which is itself. This is done by $a_n = a_{n-1} + a_{n-3} + a_{n-5} - a_{n-9}$.

28. Find a recurrence relation for the number of ways to pick k objects with repetition from n types.

$a_{n,0} = 1$ for $n \geq 0$, there is 1 way to pick 0 items from n types.

$a_{0,k} = 0$, there are no ways to pick k item from 0 types.

$$a_{n,k} = a_{n-1,k} + a_{n,k-1}$$

$a_{n-1,k}$ is how many ways to pick if there was one less type. If we add another type, we can just pick that 0 times. Now, we need to consider $a_{n,k-1}$ which is where we pick at least 1 object of type n which is the new type.

30. Find a recurrence relation for $a_{n,k}$, the number of ways to order n doughnuts from k different types of doughnuts if two or four or six doughnuts must be chosen of each type.

$$a_{0,0} = 1$$

$a_{n,0} = 0$ for $n > 0$ since you cannot have donuts if you are picking from none.

$a_{n,k} = 0$ for $n < 2k$ since each type must contribute two doughnuts.

$$a_{n,k} = a_{n-2,k-1} + a_{n-4,k-1} + a_{n-6,k-1}$$

Section 7.3

1. If \$500 is invested in a savings account earning 8 percent a year, give a formula for the amount of money in the account after n years.

$$a_n = 500 \cdot 1.08^n$$

2. Find and solve a recurrence relation for the number of n -digit ternary sequences with no consecutive digits being equal.

$$a_n = 2a_{n-1} \text{ and } a_1 = 3. \quad a_n = 3 \cdot 2^{n-1}$$

3. Solve the following recurrence relations:

(a) $a_n = 3a_{n-1} + 4a_{n-2}, a_0 = a_1 = 1$

Get the characteristic equation: $r^2 - 3r - 4 = 0$, which we can solve and get $r = 4, -1$.

$$a_n = \alpha_1 \cdot 4^n + \alpha_2 \cdot (-1)^n.$$

$$a_0 = \alpha_1 + \alpha_2 = 1 \text{ and } a_1 = \alpha_1 \cdot 4^1 + \alpha_2 \cdot (-1)^1 = 1$$

We get $5\alpha_1 = 2 \rightarrow \alpha_1 = \frac{2}{5}$ and $\alpha_2 = \frac{3}{5}$.

So, the recurrence relation is equal to $a_n = \frac{2}{5} \cdot 4^n + \frac{3}{5} \cdot (-1)^n$.