AMS 301
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Set 6

Sec 5.1: 16abc, 30, 36
Sec 5.2: 8, 38, 54
Sec 5.3: 6, 9, 15, 21, 22
Sec 5.4: 2, 3ab, 7, 10, 11, 12, 48

Section 5.1
16.

(a) How many different outcomes are possible when a pair of dice, one red and one white, are rolled two successive times?

> Let $R = W = 1, 2, 3, 4, 5, 6$ where R is the set outcomes of rolling the red die and W is of the white die.
> Possible outcomes in one roll can be expressed as $R \times W$, and if we take the cardinality, $O = |R \times W| = |R| \times |W| = 6 * 6 = 36$, which is the number of possble outcomes in one roll.
> The number of outcomes in two successive rolls will be $O \times O$, and the number of possible outcomes in two rolls is $|O \times O| = 36 \times 36 = 1296$.

(b) What is the probability that each die shows the same value on the second roll as on the first roll?

> Since there are 36 possible outcomes for a single roll, there are 36 possible outcomes for two rolls to have the same values. So, the probability that each die shows the same value on the second roll as on the first roll is $\frac{36}{1296} = 0.46\%$.

(c) What is the probability that the sum of the two dice is the same on both rolls?

> The possible sums are 2 to 12 (11 different outcomes). The possible outcomes for each sum are as following, 1 to 6 for sums 2 to 7, and 5 to 1 for sums 8 to 12. So, if the sum of the two rolls are the same, the total outcomes for a sum x will be the number of outcomes for that sum squared. For example, a sum of 7 has 6 outcomes for one roll, so for two rolls to be equal, there are 6^2 outcomes. If we sum up the outcomes for each sum, $1^2 + 2^2 + ... + 6^2 + 5^2 + ... + 1^2 = 146$. So, the probability that the sum of the two dice is the same on both rolls is $146/1296 = 11.3\%$.

30. How many times is the digit 5 written when listing all numbers from 1 to 100,000? Comment on the answers **(a)** 4, **(b)** 5×10^4, and **(c)** $1 + 10 + 100 + 1000$.

(a) I don't know how they got 4. A singular number 55555 has five 5s, which is already greater than 4.
(b) This answer is correct.
(c) If we consider the number of numbers with 5 in the ones place, we can see by observation, that starting from 5, every 10 numbers will have a 5. Since there are 100,000 numbers, this means there are at least 10,000 numbers with at least one digit being 5.

36. If two different integers between 1 to 100 inclusive are chosen at random, what is the probability that the difference of the two numbers is 15?

The minimum integer is 1 in the pair (1, 16) and the maximum is 100 in the pair (85, 100). Then, we can see that there are 85 pairs. There are $\binom{100}{2} = \frac{100!}{98!2!} = 100 \cdot 99/2 = 4950$ total ways to pick two different integers. So, the probability that two different integers chosen have a difference of 15 is $85/4950 = 1.7\%$.

Section 5.2
8. There are nine white balls and four red balls in an urn. How many different ways are there to select a subset of six balls, assuming the 13 balls are different? What is the fraction of selections with four whites and two reds?

There are $\binom{13}{6} = \dfrac{13!}{6!(13-6)!} = 1716$ ways to select a subset of six balls, assuming the 13 balls are different.
There are $\binom{9}{4}\binom{4}{2} = \dfrac{9!4!}{4!(9-4)!2!(4-2)!} = 756$ selections with 4 whites and two reds.
So, the fraction of selections is $\dfrac{756}{1716} = 44.1\%$.

38. A student must answer five out of 10 questions on a test, including at least two of the first five questions. How many subsets of five questions can be answered?

We require two of the first five questions to be chosen, but that does not limit only two of the five to be chosen. So, we must consider the following cases, 2, 3, 4, or 5 of the first five questions get answered.
Case 2: Choose 2 of the first 5, then choose 3 of the last 5.
Case 3: Choose 3 of the first 5, then choose 2 of the last 5.
Case 4: Choose 4 of the first 5, then choose 1 of the last 5.
Case 5: Choose 5 of the first 5.
$\begin{pmatrix}5\\2\end{pmatrix}\begin{pmatrix}5\\3\end{pmatrix} + \begin{pmatrix}5\\3\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix}\begin{pmatrix}5\\1\end{pmatrix} + \begin{pmatrix}5\\5\end{pmatrix}\begin{pmatrix}5\\0\end{pmatrix} = 2 \cdot \dfrac{5!5!}{2!2!3!3!} + \cdot \dfrac{5!5!}{1!1!4!4!} + \cdot \dfrac{5!5!}{0!0!5!5!} =$
$200 + 25 + 1 = 226$ subsets.

54. How many arrangements of MATHEMATICS are there in which each consonant is

adjacent to a vowel?

There are 4 vowels: A, E, A, I; and 7 consonants: M, T, H, M, T, C, S Let c denote consonant and v denote consonant. We can have the following forms:
cvc cvc cvc cv = $\{1, 2, 2, 2, 0\}$ (x2 for symmetry)
cvc cvc cvc vc = $\{1, 2, 2, 1, 1\}$ (x2 for symmetry)
cvc cvc vcc vc = $\{1, 2, 1, 2, 1\}$
And by symmetry and the rule that the ends in the set must be 1, we have 5 different forms.
The arrangements for the vowels is $\frac{4!}{2!} = 12$ since we have 2 A's, and the arrangements for the consonants is $\frac{7!}{2!2!} = 1260$ since we have 2 M's and 2 T's.

So, there are a total of $5 \times 12 \times 1260 = 75600$ arrangements of MATHEMATICS where each consonant is adjacent to a vowel.

Section 5.3
6. If four identical dice are rolled, how many different outcomes can be recorded?

There are 6 options and we need a combination of 4 numbers.
So the number of different outcomes is $\binom{6+4-1}{4} = 126$.

9. How many ways are there to pick a selection of coins from \$1 worth of identical pennies, \$1 worth of identical nickels, and \$1 worth of identical dimes if
(a) You select a total of 9 coins?

There are 100 pennies, 20 nickels, and 10 dimes for selection.
$p + n + d = 9; \quad p \leq 100, n \leq 20, d \leq 10.$
$\binom{9+3-1}{9} = \frac{11!}{9!2!} = 55$ selections.

(b) You select a total of 16 coins?

There are 100 pennies, 20 nickels, and 10 dimes for selection.
Assume, we have an unlimited amount of each coin. Then, there are $\binom{16+3-1}{16} = 153$ different selections.
Now, if we consider the restriction of 10 dimes, we have to remove the additional selections where there are more than 10 dimes, hence $p + n + (d - 11) = 5$. $\binom{5+3-1}{5} = 21$. So, there are 21 selections where there are more than 10 dimes, which means there are 153-21=132 selections of 16 coins.

Comment on the answers:

(a) $\binom{16+3-1}{16}$
this solution does not consider the restriction.

(b) $\sum_{k=0}^{10} \binom{16}{k}\binom{(16-k)+2-1}{(16-k)}$
This answer does not make any sense. The sum appears to consider the possibilities if there were k dimes.

(c) $\binom{16+3-1}{16} - \sum_{k=11}^{16} \binom{16}{k}\binom{(16-k)+2-1}{(16-k)}$
$< \binom{18}{16} - \binom{16}{11}\binom{6}{5} = -26055$. This solution is negative, which doesn't make sense. There cannot be a negative amount of selections.

15. How many 8-digit sequences are there involving exactly six different digits?

There are 10 different digits. You can have a triplet of a digit, or a two pairs of digits for repetitions.
For a triplet digit, there are $\binom{10}{6} = 210$ sets of digits, 6 choices for the repeating digit, and $\frac{8!}{3!} = 6720$ different sequences.
For the two pairs, there are $\binom{10}{6} = 210$ sets of digits, $\binom{6}{2} = 15$ repeating pairs, and $\frac{8!}{2!2!} = 10080$ sequences.
So there are a total of $210 \times 6 \times 6720 + 210 \times 15 \times 10080 = 42487200$ different sequences.

21. How many arrangements of the letters in MATHEMATICS are there in which TH appear together but the TH is not immediately followed by an E (not THE)?

Consider "THE" as one character. Note that there are 2 copies of M, two copies of A, and two copies of T. So there are 2 ways to arrange "THE". There are 11 letters, but if "THE" is considered one letter, then there are $11-3+1 = 9$ letters. So, there must be $\frac{9!}{2!2!} = 90720$ arrangements where "THE" are put together (We divide by 2!2! for the two copies of M and A).
Consider "TH" as one character, and the copy of T. Following the same logic, there are 10 letter, so there must be $10!/2 = 907200$ arrangements.
So, there are $907200 - 90720 = 816480$ arrangements of the letters in MATHEMATICS which TH appear together, but not followed by an E.

22. How many arrangements of the letters in PEPPERMILL are there with

(a) The M appearing to the left of all the vowels?

There are 3 P's, 2 E's, 2 L's.
There are 10 letters, which 3 are vowels: E, E, I.
There are a total of $\frac{10!}{3!2!2!} = 151200$ arrangements without any conditions.
If we consider all the orders of M, E, E, I, then M will be left of the vowels one fourth of the time. So, there will be $151200/4 = 37800$ arrangements where M appear to the left of all the vowels.

(b) The first P appearing before the first L?

There are 3 P's, 2 L's, so P must appear before the first L three fifths of the time. Then, there will be $151200 \times \frac{3}{5} = 90720$ arrangements.

Section 5.4
2. How many ways are there to distribute 18 different toys among four children?

(a) Without restrictions?

4^{18} ways to distribute among the four children.

(b) If two children get seven toys and two children get two toys?

Pick 2 of the the 4 children to get seven toys, and the other two will get two toys. So, there are $\binom{4}{2} = 6$ ways.

3. In a bridge deal, what is the probability that:

(a) West has four spades, two hearts, four diamonds, and three clubs?

$C(13,4)C(13,2)C(13,4)C(13,3)/C(52,13) = 1.8\%$

(b) North and South have four spades, West has three spades, and East has two spades?

Ways to split the spades to is $\frac{13!}{4!4!3!2!}$.

Ways to split the non-spades is $\dfrac{39!}{9!9!10!11!}$

Total number of deals is $\dfrac{52!}{(13!)^4}$ The probability of the hands is (ways to split the spades) * (ways to split the non-spades) / (total number of deals) = 1.8%

7. How many ways are there to arrange the letters in VISITING with no pair of consecutive Is?

There are 3 I's and 5 other letters. There are $5! = 120$ arrangements of the other letters. We can place the I's in 5+1=6 spots, so $C(6, 3) = 20$ ways to place the I's. So, there are $120 \times 20 = 2400$ ways to arrange the letters with no pair of consecutive I's.

10. How many ways are there to arrange the 26 letters of the alphabet so that no pair of vowels appear consecutively (Y is considered a consonant)?

There are 21! ways to arrange the consonants. There are 21+1=22 spots to place the 5 vowels, so $C(22, 5)$ ways. There are 5! ways to order the vowels. So, there are $21! \times C(22, 5) \times 5!$ ways to arrange the 26 letters.

11. If you flip a coin 18 times and get 14 heads and four tails, what is the probability that there is no pair of consecutive tails?

If we have 4 tails, there will be 5 regions for heads, $\{r_1, t_1, r_2, t_2, r_3, t_3, r_4, t_4, r_5\}$. So, $r_1 + r_2 + r_3, +r_4 + r_5 = 14$ and $r_1, r_5 \geq 0; r_2, r_3, r_4 \geq 1$. So, we can let $r_i = z_i + 1$ for $i \in \{2, 3, 4\}$ and $z_1 = r_1, z_5 = r_5$.

$z_1 + z_2 + z_3 + z_4 + z_5 = 14 - 3 = 11$

So, there are $C(11 + 5 - 1, 5 - 1) = 1365$ arrangements where there are no pair of consecutive tails. There are $C(18, 4) = 3060$ arrangement without restriction. So, the probability that there $1365/3060 = 0.446 = 44.6\%$

12. How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 31$ with

(a) $x_i \geq 0$

$C(31 + 5 - 1, 31) = 52360$ solutions.

(b) $x_i > 0$

Let $x_i = z_i + 1$, then $z_1 + z_2 + z_3 + z_4 + z_5 = 31 - 5 = 26$.

So, there are $C(26 + 5 - 1, 5 - 1) = 27405$ solutions.

(c) $x_i \geq i (i = 1, 2, 3, 4, 5)$

> Let: $x_1 = z_1 + 1$, $x_2 = z_2 + 2$, $x_3 = z_3 + 3$, $x_4 = z_4 + 4$, $x_5 = z_5 + 5$
> $z_1 + z_2 + z_3 + z_4 + z_5 = 31 - (1 + 2 + 3 + 4 + 5)$
> So, there are $C(31 - (1 + 2 + 3 + 4 + 5) + (5 - 1), 5 - 1) = 4845$ solutions.

48. How many arrangements of the letters in INSTRUCTOR have all of the following properties simultaneously?

(a) The vowels appearing in alphabetical order

(b) At least 2 consonants between each vowel

(c) Begin or end with the 2 Ts (the Ts are consecutive)

> There are a total of 10 letters, 3 are vowels (I, U, O) and 7 are consonants.
> To satisfy the first condition, the order of the vowels must be:
> $r_1, I, r_2, O, r_3, U, r_4$, where r_i is a region of consonants.
> The second condition requires that region r_2 and r_3 to consist of two characters.
> $r_1, I, c_1, c_2, O, c_3, c_4, U, r_4$.
> The third condition requires either r_1 or r_4 to contain TT. So the forms are:
> $T, T, I, c_1, c_2, O, c_3, c_4, U, r_4$
> $r_1, I, c_1, c_2, O, c_3, c_4, U, T, T$
> This leaves the final regions to be 1 character long, or 0, with another consonant adjacant to TT, either cTT or TTc.
> So, there must be (TT are at r_1 or r_4) $\times$ (c is(n't) with the TT) $= 2 \times 2 = 4$ forms.
> Now, if we arrange the remaining consonants, there will be $5! = 120$ permutations.
> Hence, there will be $120 \times 4 = 480$ arrangements of the letters in INSTRUCTOR with the properties.