MA-107 Summary

1 Supply and Demand

2 Recurrence equations

Equations relating values of y at discrete points in time, expressing value y_t of y at time t as a function of y_{t-1}, one unit of time before time t.

2.1 Solving recurrence equations

Solving recurrence equation refers to finding value of y_t as a function of t.

Problem
Solve the recurrence equation

$$y_t = ay_{t-1} + b$$

for $t \geq 0$ *given that* $y_0 = C$.

Solution
After making sure that the equation you're solving is in the form

$$y_t = ay_{t-1} + b$$

you proceed as follows:

1. *If* $a = 1$ *your solution is* $y_t = y_0 + bt$. *Otherwise:*
2. *Make sure that your recurrence equation is of the form* $y_t = ay_{t-1} + b$ *for* a, b *constants.*
3. *Find* $y^* = \dfrac{b}{1-a}$ *where* a, b *are constants from the equation above.*
4. *Solution is* $y(t) = y^* + (y_0 - y^*)a^t$

2.2 Long term behaviour

Describing behaviour of function y_t as $t \to \infty$.

Problem
Given $y_t = ay_{t-1} + b$ how does y_t behave as t tends to infinity?

Solution
*Assume recurrence equation $y_t = ay_{t-1} + b$ and time independent solution y^*, long term behaviour of y_t **depends on coefficient** $\mathbf{a}$:*

- *If $\mathbf{1} < \mathbf{a}$, then the function **increases unboundedly**($y_t \to +\infty$) if $y_0 > y^*$ and the function **decreases unboundedly** $y_t \to -\infty$ $y_0 < y^*$*
- *If $\mathbf{0} < \mathbf{a} < \mathbf{1}$, then the **function increases towards** y^* ($y_t \to y^*$) if $y_0 > y^*$ and the **function decreases towards** y^* ($y_t \to y^*$) if $y_0 < y^*$*
- *If $-\mathbf{1} < \mathbf{a} < \mathbf{0}$, then y_t **oscillates towards** y^*.*
- *If $-\mathbf{1} < \mathbf{a} < \mathbf{0}$, then y_t **oscillates unboundedly**.*

2.3 Investment schemes

Comparing return from different investments in terms of their present value.

Problem
Given access to a bank account with interest rate r accrued annually at the end of each year and an asset with value function $V(t)$ calculate present value of selling said asset after t years.

Solution
Present value $P(t)$ is given by the equation

$$P(t) = V(t)(1 + r)^{-t}$$

Problem
Given access to a bank account with continuously compounded interest with rate r and an asset with value function $V(t)$ calculate present value of selling said asset after t years.

Solution
Present value $P(t)$ is given by the equation

$$P(t) = V(t)e^{-tr}$$

Problem
Given access to a bank account with annually compounded interest with rate r calculate present value of:

1. *Receiving a sum M after n years.*
2. *Receiving a yearly salary of S for n years.*
3. *Selling an asset with value function $V(t)$ after n years.*

Solution
Present value P is given by the equation:

1. $P(n) = M(1+r)^{-n}$
2. $P(n) = S(r - \frac{r}{(1+r)^n})$
3. $P(n) = V(n)(1+r)^{-n}$

Problem
Given access to a bank account with continuously compounded interest with rate r calculate present value of:

1. *Receiving a sum M after time t.*
2. *Receiving a yearly salary of S for n years.*
3. *Selling an asset with value function $V(t)$ after time t.*

Solution
Present value P is given by the equation:

1. $P(t) = Me^{-rt}$
2. $P(n) = S(\frac{e^{rn} - 1}{e^{rn}(e^r - 1)})$
3. $P(t) = V(t)e^{-tr}$

3 Optimisation of functions in one variable

Identifying stationary points (minima and maxima) and economically significant values such as marginal cost, break-even point, starting point etc.

3.1 Differentiation

Mathematical theory allowing for economical analysis of function's behaviour.

Def (Lagrange's definition of a derivative)
Define the derivative of a function $f(x)$ with respect to x to be

$$f'(x) = lim_{h \to \infty} \frac{f(x+h) - f(x)}{h}$$

Standard derivatives include:

- $(e^x)' = e^x$
- $(ln(x))' = \frac{1}{x}$ for $x > 0$
- $(sin(x))' = cos(x)$
- $(cos(x))' = -sin(x)$
- $(x^n)' = nx^{n-1}$

Def (Product rule)
For a product of two functions f and g we have that:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Def (Quotient rule)
For a quotient of two functions f and g we have that:

$$(\frac{f}{g})' = \frac{f' \cdot g + f \cdot g'}{g \cdot g}$$

Def (Chain rule)
For a composition of two functions f and g we have that:

$$(f \circ g)' = f'(g(x))g'(x)$$

Problem
Describe nature of all stationary points of a function $f(x)$.

Solution *1. Calculate first derivative $f'(x)$ of f.*

*2. Find all values of x for which $f'(x) = 0$. These are the **stationary points**.*

3. Compute value of second derivative $f''(p)$ at every previously found stationary point p.

  - *If $f''(p) > 0$ then p is a (local) minimum.*
  - *If $f''(p) < 0$ then p is a (local) maximum.*
  - *If $f''(p) = 0$ then look at the sign of a $f'(x)$ for x on both sides if p.*
    - *If $f'(x)$ changes sign from + to - at p then we have a (local) maximum.*
    - *If $f'(x)$ changes sign from - to + at p then we have a (local) minimum.*
    - *If sign of $f'(x)$ does not change sign at p then p is an inflection point (neither min nor max).*

3.2 Cost related optimisation

Def (Profit function)
*Define **profit** Π of a firm as a function of produced quantity q to be:*

$$\Pi(q) = pq - C(q)$$

*where for a **small, efficient firm** we treat p as a constant since the amount it produces does not affect market price.*

Define indirect profit Π^* of a firm to be a profit function Π evaluated at its stationary point q^* ie $\Pi^*(q) = pq^* - C(q^*)$.

Def (Start-up and break-even points)
Given a supplier firm with cost and profit functions $C(q), \Pi(q)$ define:

- ***start-up point** q_s to be the largest quantity q produced that will bring the same profit as producing nothing.*
  It can be found as a solution to:

$$\Pi(q_s) = \Pi(0)$$

- ***break-even point** q_b to be the smallest quantity q produced that will cover the cost of production.*
  It can be found as a solution to:

$$\Pi(q_b) = 0$$

Problem (Small firm)
Given a small efficient firm with cost function C(q):

- *Write the profit and find a value of selling price p of the its product that maximises the profit for $q > 0$.*
- *Find its start-up point q_0 and break-even point q_b.*

Solution (Standard small firm procedure)

1. *Write profit function $\Pi(q)$ as defined above.*
2. *Differentiate $\Pi(q)$ to show that it has a stationary point when $p = C'(q)$.*
3. *Verify that it is a max by finding $\Pi''(q)$ for $p = C'(q)$.*
4. *Find q_0 by solving $\Pi^*(q_0) = \Pi(0)$*
5. *Find q_b by solving $\Pi^*(q_b) = 0$*

4 Optimisation of functions in two variables

4.1 Differentiation

Before when we had a function is one variable $f(x)$ we could write $f'(x)$ as its derivative with respect to x, namely $\dfrac{df(x)}{dx}$ or simply $\dfrac{df}{dx}$.

With $f(x, y)$ being a function in two variables notation $f'(x, y)$ doesn't make much sense any more. So we define first order partial derivatives with respect to x and y to be

$$f_x = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y = \frac{\partial f}{\partial y} \quad \text{respectively}$$

And second order derivatives are:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

Def (Chain rule)
Given function in two variables $F(x, y)$ and given that x and y are in turn functions of another variable t we can define $f(t) = F(x(t), y(t))$ and get its derivative with respect to t from chain rule as follows:

$$\frac{df}{dt} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}$$

Problem (Implicit differentiation)
Suppose we have a function $G(x, y)$ and that y depends on x and is being defined implicitly by $G(x, y) = C$. Find derivative of y w.r.t. x ie $\dfrac{dy}{dx}$

Solution
Define $g(x) = G(x, y(x))$. From chain rule above we have that:

$$\frac{dg}{dx} = \frac{\partial G}{\partial x}\frac{dx}{dx} + \frac{\partial G}{\partial y}\frac{dy}{dx}$$

From the implicit equation $G(x, y) = C$ we have that

$$\frac{dg}{dx} = \frac{\partial G}{\partial x} = \frac{\partial C}{\partial x} = 0$$

Putting the two equations together we have that

$$\frac{dg}{dx} = \frac{\partial G}{\partial x}\frac{dx}{dx} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0$$

Which after rearranging gives us:

$$\frac{dy}{dx} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}} = -\frac{G_x}{G_y}$$

G_x and G_y can be easily computed by differentiating G w.r.t. x and y

Def (Homogeneous function)
We say that a function $f(x, y)$ is homogeneous of degree r if for any λ we have:

$$f(\lambda x, \lambda y) = \lambda^r f(x, y)$$

Def (Euler's theorem)
If $f(x, y)$ is homogeneous of degree r then we have that:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = rf(x, y)$$