MA-107 Summary

1 Supply and Demand

2 Recurrence equations

Equations relating values of y at discrete points in time, expressing value y_t of y at time t as a function of y_{t-1}, one unit of time before time t.

2.1 Solving recurrence equations

Solving recurrence equation refers to finding value of y_t as a function of t.

Problem
Solve the recurrence equation

$$y_t = ay_{t-1} + b$$

for $t \geq 0$ *given that* $y_0 = C$.

Solution
After making sure that the equation you're solving is in the form

$$y_t = ay_{t-1} + b$$

you proceed as follows:

1. *If* $a = 1$ *your solution is* $y_t = y_0 + bt$. *Otherwise:*
2. *Make sure that your recurrence equation is of the form* $y_t = ay_{t-1} + b$ *for* a, b *constants.*
3. *Find* $y^* = \dfrac{b}{1-a}$ *where* a, b *are constants from the equation above.*
4. *Solution is* $y(t) = y^* + (y_0 - y^*)a^t$

2.2 Long term behaviour

Describing behaviour of function y_t as $t \to \infty$.

Problem
Given $y_t = ay_{t-1} + b$ *how does* y_t *behave as* t *tends to infinity?*

Solution
Assume recurrence equation $y_t = ay_{t-1} + b$ *and time independent solution* y^**, long term behaviour of* y_t ***depends on coefficient*** $\mathbf{a}$*:*

- *If* $\mathbf{1} < \mathbf{a}$*, then the function* ***increases unboundedly****(*$y_t \to +\infty$*) if* $y_0 > y^*$ *and the function* ***decreases unboundedly*** $y_t \to -\infty$ $y_0 < y^*$
- *If* $\mathbf{0} < \mathbf{a} < \mathbf{1}$*, then the* ***function increases towards*** y^* *(*$y_t \to y^*$*) if* $y_0 > y^*$ *and the* ***function decreases towards*** y^* *(*$y_t \to y^*$*) if* $y_0 < y^*$
- *If* $-\mathbf{1} < \mathbf{a} < \mathbf{0}$*, then* y_t ***oscillates towards*** y^**.*
- *If* $-\mathbf{1} < \mathbf{a} < \mathbf{0}$*, then* y_t ***oscillates unboundedly****.*

2.3 Investment schemes

Comparing return from different investments in terms of their present value.

Problem
Given access to a bank account with interest rate r *accrued annually at the end of each year and an asset with value function* $V(t)$ *calculate present value of selling said asset after* t *years.*

Solution
Present value $P(t)$ *is given by the equation*

$$P(t) = V(t)(1 + r)^{-t}$$

Problem
Given access to a bank account with continuously compounded interest with rate r *and an asset with value function* $V(t)$ *calculate present value of selling said asset after* t *years.*

Solution
Present value $P(t)$ *is given by the equation*

$$P(t) = V(t)e^{-tr}$$

Problem
Given access to a bank account with annually compounded interest with rate r *calculate present value of:*

1. *Receiving a sum M after n years.*
2. *Receiving a yearly salary of S for n years.*
3. *Selling an asset with value function $V(t)$ after n years.*

Solution
Present value P is given by the equation:

1. $P(n) = M(1+r)^{-n}$
2. $P(n) = S(r - \frac{r}{(1+r)^n})$
3. $P(n) = V(n)(1+r)^{-n}$

Problem
Given access to a bank account with continuously compounded interest with rate r calculate present value of:

1. *Receiving a sum M after time t.*
2. *Receiving a yearly salary of S for n years.*
3. *Selling an asset with value function $V(t)$ after time t.*

Solution
Present value P is given by the equation:

1. $P(t) = Me^{-rt}$
2. $P(n) = S(\frac{e^{rn} - 1}{e^{rn}(e^r - 1)})$
3. $P(t) = V(t)e^{-tr}$

3 Optimisation of functions in one variable

Identifying stationary points (minima and maxima) and economically significant values such as marginal cost, break-even point, starting point etc.

3.1 Differentiation

Mathematical theory allowing for economical analysis of function's behaviour.

Def (Lagrange's definition of a derivative)
Define the derivative of a function $f(x)$ with respect to x to be

$$f'(x) = lim_{h \to \infty} \frac{f(x+h) - f(x)}{h}$$

Standard derivatives include:

- $(e^x)' = e^x$
- $(ln(x))' = \frac{1}{x}$ for $x > 0$
- $(sin(x))' = cos(x)$
- $(cos(x))' = -sin(x)$
- $(x^n)' = nx^{n-1}$

Def (Product rule)
For a product of two functions f and g we have that:

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Def (Quotient rule)
For a quotient of two functions f and g we have that:

$$(\frac{f}{g})' = \frac{f' \cdot g + f \cdot g'}{g \cdot g}$$

Def (Chain rule)
For a composition of two functions f and g we have that:

$$(f \circ g)' = f'(g(x))g'(x)$$

Problem
Describe nature of all stationary points of a function $f(x)$.

Solution 1. *Calculate first derivative $f'(x)$ of f.*

2. *Find all values of x for which $f'(x) = 0$. These are the **stationary points**.*

3. *Compute value of second derivative $f''(p)$ at every previously found stationary point p.*
   - *If $f''(p) > 0$ then p is a (local) minimum.*
   - *If $f''(p) < 0$ then p is a (local) maximum.*
   - *If $f''(p) = 0$ then look at the sign of a $f'(x)$ for x on both sides if p.*
     - *If $f'(x)$ changes sign from + to - at p then we have a (local) maximum.*
     - *If $f'(x)$ changes sign from - to + at p then we have a (local) minimum.*
     - *If sign of $f'(x)$ does not change sign at p then p is an inflection point (neither min nor max).*

3.2 Cost related optimisation

Def (Profit function)
*Define **profit** Π of a firm as a function of produced quantity q to be:*

$$\Pi(q) = pq - C(q)$$

*where for a **small, efficient firm** we treat p as a constant since the amount it produces does not affect market price.*

Define indirect profit Π^* of a firm to be a profit function Π evaluated at its stationary point q^* ie $\Pi^*(q) = pq^* - C(q^*)$.

Def (Start-up and break-even points)
Given a supplier firm with cost and profit functions $C(q), \Pi(q)$ define:

- ***start-up point*** *q_s to be the largest quantity q produced that will bring the same profit as producing nothing.*
  It can be found as a solution to:

  $$\Pi(q_s) = \Pi(0)$$

- ***break-even point*** *q_b to be the smallest quantity q produced that will cover the cost of production.*
  It can be found as a solution to:

  $$\Pi(q_b) = 0$$

Problem (Small firm)
Given a small efficient firm with cost function C(q):

- *Write the profit and find a value of selling price p of the its product that maximises the profit for $q > 0$.*
- *Find its start-up point q_0 and break-even point q_b.*

Solution (Standard small firm procedure)

1. *Write profit function $\Pi(q)$ as defined above.*
2. *Differentiate $\Pi(q)$ to show that it has a stationary point when $p = C'(q)$.*
3. *Verify that it is a max by finding $\Pi''(q)$ for $p = C'(q)$.*
4. *Find q_0 by solving $\Pi^*(q_0) = \Pi(0)$*
5. *Find q_b by solving $\Pi^*(q_b) = 0$*