Planning, Learning and Intelligent Decision Making

Lecture 1

PADInt 2023

Sequential models

Weather in Lisbon

Weather in Lisbon

- We want to predict the weather in Lisbon
- Suppose that the weather is either **Sunny** or **Rainy**

Weather in Lisbon

- Two important observations:
  - The weather is not deterministic (there is some uncertainty associated with any prediction)
  - The weather is dynamic (it changes from one day to the other)

Representing uncertainty

What is probability?

- Has roots in games of chance
- Used to **measure** the likelihood of occurrence of events

Natural tool to model uncertainty

What is probability?

- Classical definition of probability of event A
  - N possible events
  - N_A ways by which A can occur

$$\mathbb{P}[A] = \frac{N_A}{N}$$

  - Examples:
    - Throw a die 10 times: 1, 2, 3, 2, 3, 5, 4, 6, 2, 1
    - What is the probability of drawing an even number in a die?

What is probability?

- Frequentist definition of probability of event A
  - Relative frequency of event A
  - N observed events
  - N_A times that event A occurred

$$\mathbb{P}[A] = \lim_{N\to\infty} \frac{N_A}{N}$$

  - Examples:
    - What is the probability of drawing a $\spadesuit$ in a card deck?

What is probability?

- Subjective definition of probability of event A
    - Degree of belief that event A may occur
- Example:
    - Probability that Benfica will win the championship

Formally, ...

- Probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where:
  - Ω is the sample space $\longleftarrow$ Things that can happen
  - $\mathcal{F}$ is the set of events $\longleftarrow$ Things we want to measure
  - $\mathbb{P}$ is a probability measure $\longleftarrow$ Way to measure them

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Formally, ...

- **Sample space Ω**
  - Space of possible outcomes
  - Each and every thing that may happen
  - Example:

    In a die throw, the possible outcomes are $\{1, 2, 3, 4, 5, 6\}$
  - Example:

    When drawing a card, the possible outcomes are $\{A\spadesuit, 2\spadesuit, ..., J\diamondsuit, Q\diamondsuit, K\diamondsuit\}$

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Formally, ...

- **Set of events,** $\mathcal{F}$
  - Subsets of Ω that we can "measure", i.e., assign a probability
  - Includes the empty set $\varnothing$ and the full set Ω
  - Examples of events (die throw):
    - Drawing an even number: $\{2, 4, 6\}$
    - Drawing a number larger than 3: $\{4, 5, 6\}$
    - Drawing no number: $\varnothing$

Formally, ...

- **Probability, $\mathbb{P}$**
  - "Measures" each event in $\mathcal{F}$
  - Axioms of probability:
    - $\mathbb{P}(A) \geq 0$
    - $\mathbb{P}(\Omega) = 1$
    - Given disjoint events A_1, ..., A_n

$$\mathbb{P}[A_1 \cup \ldots \cup A_n] = \sum_{i=1}^{n} \mathbb{P}[A_i]$$

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Conditional probability

- Conditional probability of event A given B:

- Probability of A occurring, given that B occurred

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

... assuming $\mathbb{P}(B) > 0$.

- Verifies all axioms of probability

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Conditional probability

- **Important properties:**

- Independent events: $\mathbb{P}[A \mid B] = \mathbb{P}[A]$

- Law of total probability: given disjoint events A_1, ..., A_n, with

$$A_1 \cup A_2 \cup \ldots \cup A_n = \Omega,$$

it holds that

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i] \cdot \mathbb{P}[A_i]$$

Conditional probability

- **Important properties:**

- Bayes theorem:

$$\underbrace{\mathbb{P}[A}_{\text{hypothesis}} \mid \underbrace{B]}_{\text{observations}} = \frac{\overbrace{\boxed{\mathbb{P}[B \mid A]}}^{\text{likelihood}}\overbrace{\boxed{\mathbb{P}[A]}}^{\text{prior}}}{\underbrace{\boxed{\mathbb{P}[B]}}_{\text{evidence}}}$$

Random variable

- Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
  - Sometimes Ω is a cumbersome set to work with
  - It would be more convenient to instead work in some other space, more mathematically convenient (for example, $\mathbb{R}$)
  - A **random variable** X is a map $X : \Omega \to E$, where E is some convenient space (usually $\mathbb{R}$)
  - We usually work with random variables, ignoring the underlying probability space

Random variable

- We write $\mathbb{P}[X = x]$ to represent

$$\mathbb{P}[X = x] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}]$$

- If the r.v. takes values in a discrete set, we call it a discrete random variable

- If the r.v. takes values in a continuous set, we call it a continuous random variable

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… back to the weather example…

Weather in Lisbon

- Two important observations:
  - The weather is not deterministic (there is some uncertainty associated with any prediction)
  - The weather is dynamic (it changes from one day to the other)

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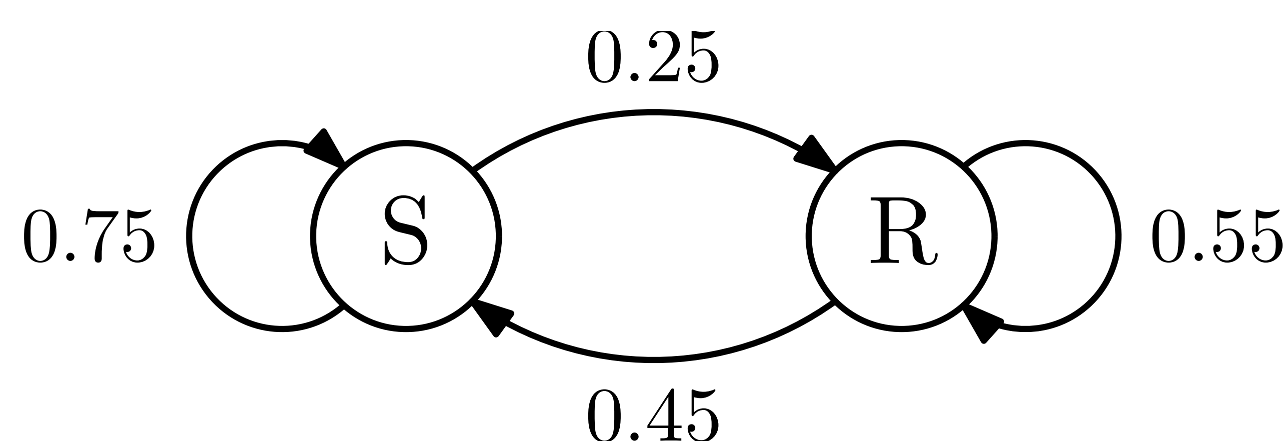
Weather in Lisbon

- Sunny day → Sunny day (prob. 75%) Uncertainty

Dynamics

- Rainy day → Rainy day (prob. 55%) Uncertainty

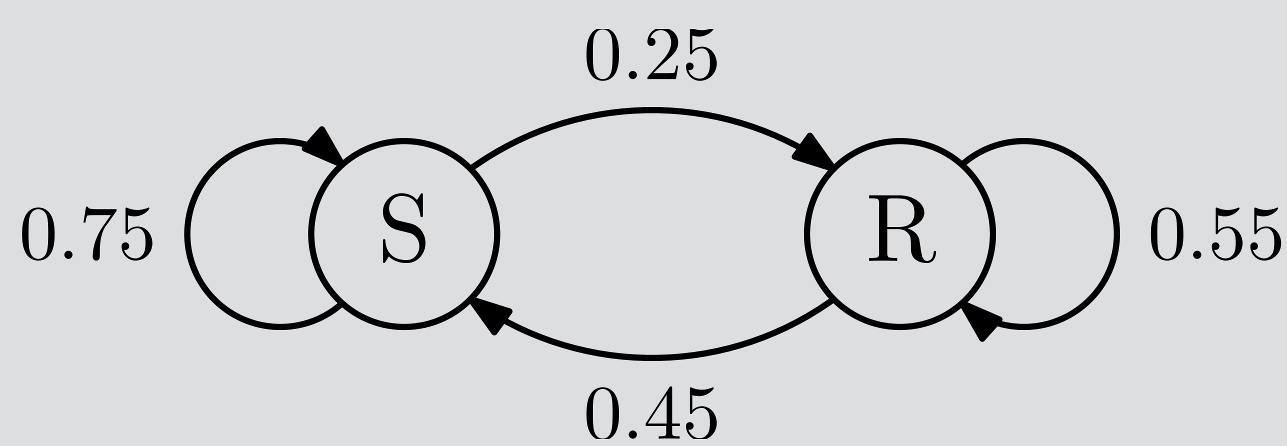
- Let's draw this:



Transition diagram

Predicting the weather...

- Today is **Sunny**
- What is the weather **tomorrow**?



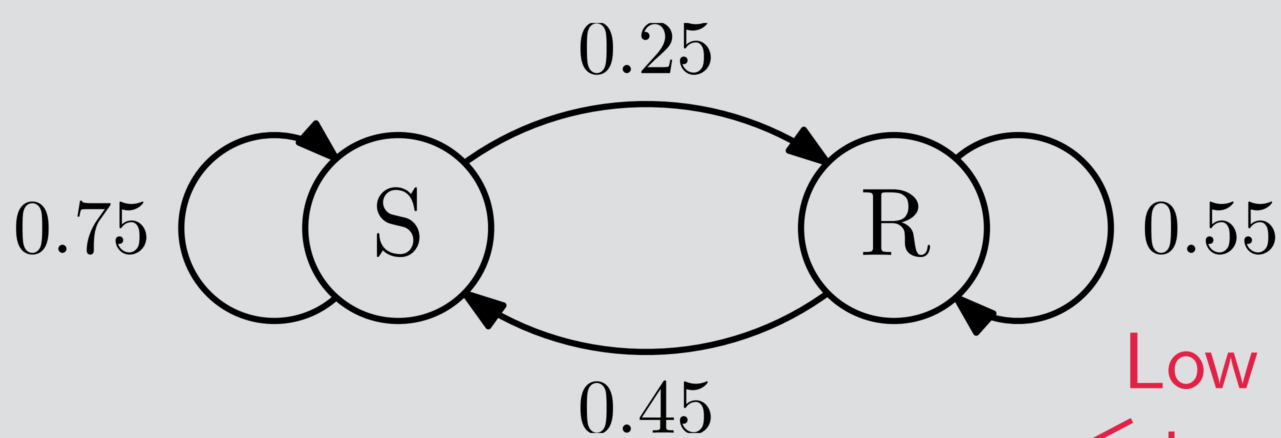
Conditional probability

- **Rainy**, with 25% probability → $\mathbb{P}[X_{\text{tomorrow}} = R \mid X_{\text{today}} = S]$
- **Sunny**, with 75% probability → $\mathbb{P}[X_{\text{tomorrow}} = S \mid X_{\text{today}} = S]$

Conditional probability

Predicting the weather...

- Today is **Sunny**
- What is the weather **tomorrow**?



Low case upright letters for r.v.s

- **Rainy**, with 25% probability $\mathbb{P}[\mathrm{x}_{\text{tomorrow}} = R \mid \mathrm{x}_{\text{today}} = S]$
- **Sunny**, with 75% probability $\mathbb{P}[\mathrm{x}_{\text{tomorrow}} = S \mid \mathrm{x}_{\text{today}} = S]$

Predicting the weather...

- Is this mathematically correct?

- x_0: weather today
- x_1: weather tomorrow
- We have:
  - $$\mathbb{P}\left[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S\right] = 75\%$$
    $$\mathbb{P}\left[\mathrm{x}_1 = R \mid \mathrm{x}_0 = S\right] = 25\%$$
  - $$\mathbb{P}\left[\mathrm{x}_1 = S \mid \mathrm{x}_0 = R\right] = 45\%$$
    $$\mathbb{P}\left[\mathrm{x}_1 = R \mid \mathrm{x}_0 = R\right] = 55\%$$

Predicting the weather...

- Today is **Sunny**
- What is the weather the day after tomorrow?

<table>
<tr><td>

- If the weather tomorrow is **Sunny**, then
  - **Rainy**, with 25% probability
  - **Sunny**, with 75% probability

</td><td>75% probability</td></tr>
<tr><td>

- If the weather tomorrow is **Rainy**, then
  - **Rainy**, with 55% probability
  - **Sunny**, with 45% probability

</td><td>25% probability</td></tr>
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Predicting the weather...

- Is this mathematically correct?

- x_0: weather today
- x_1: weather tomorrow
- x_2: weather day after tomorrow
  - $\mathbb{P}[x_2 = S \mid x_0 = S]$

$$
\begin{aligned}
&= \mathbb{P}[x_2 = S \mid x_1 = S]\mathbb{P}[x_1 = S \mid x_0 = S] \\
&\quad + \mathbb{P}[x_2 = S \mid x_1 = R]\mathbb{P}[x_1 = R \mid x_0 = S] \\
&= 0.75 \times 0.75 + 0.45 \times 0.25 \\
&= 67.5\%
\end{aligned}
$$

Total probability law!

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Assumptions

- The weather in one day is enough to predict the weather in the next day

- This prediction does not depend on the particular day

Is this always true?

- No.

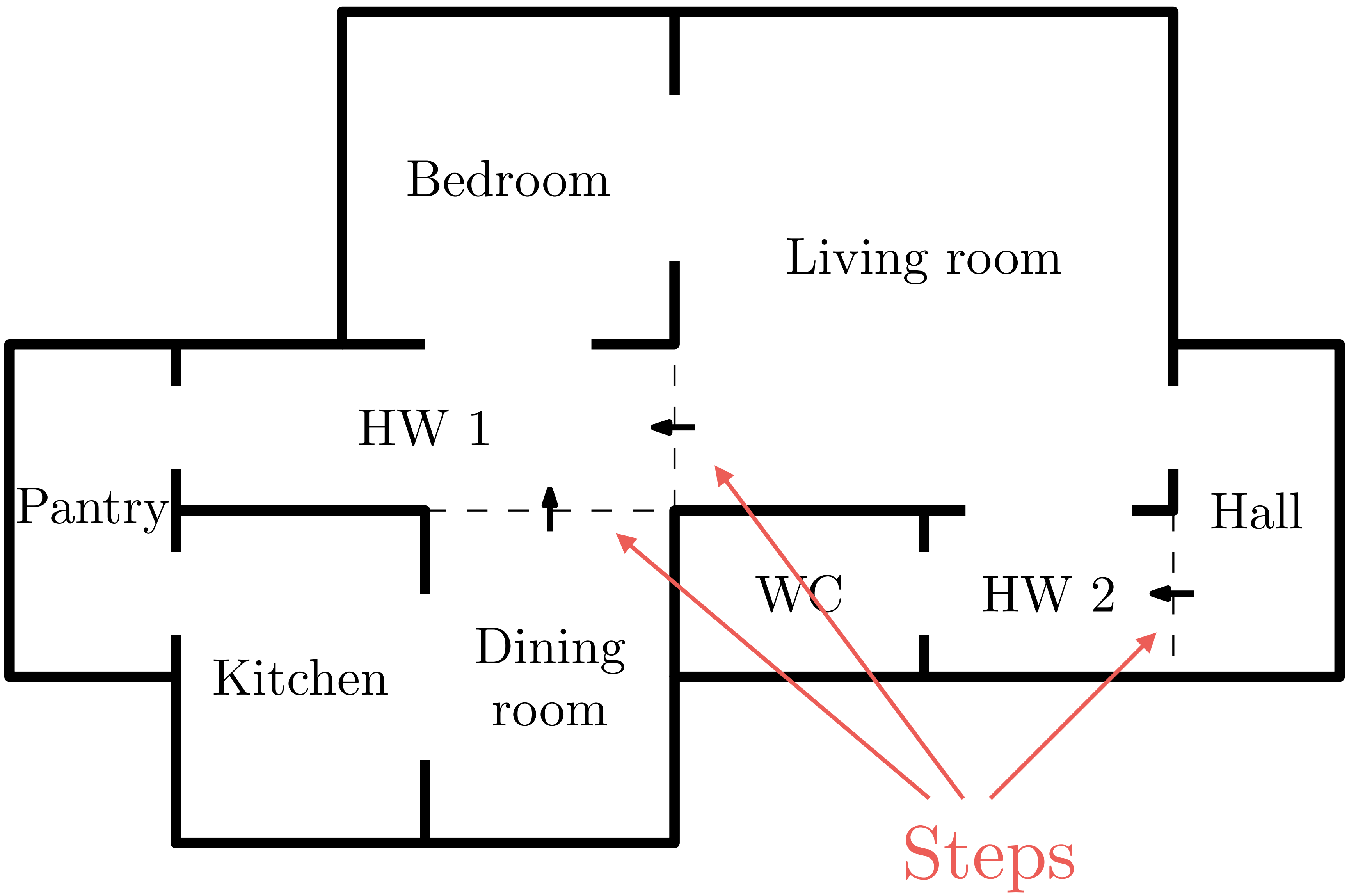
- … however, many times it's almost true 🙂

"Models have limitations;
Stupidity does not."

The household robot

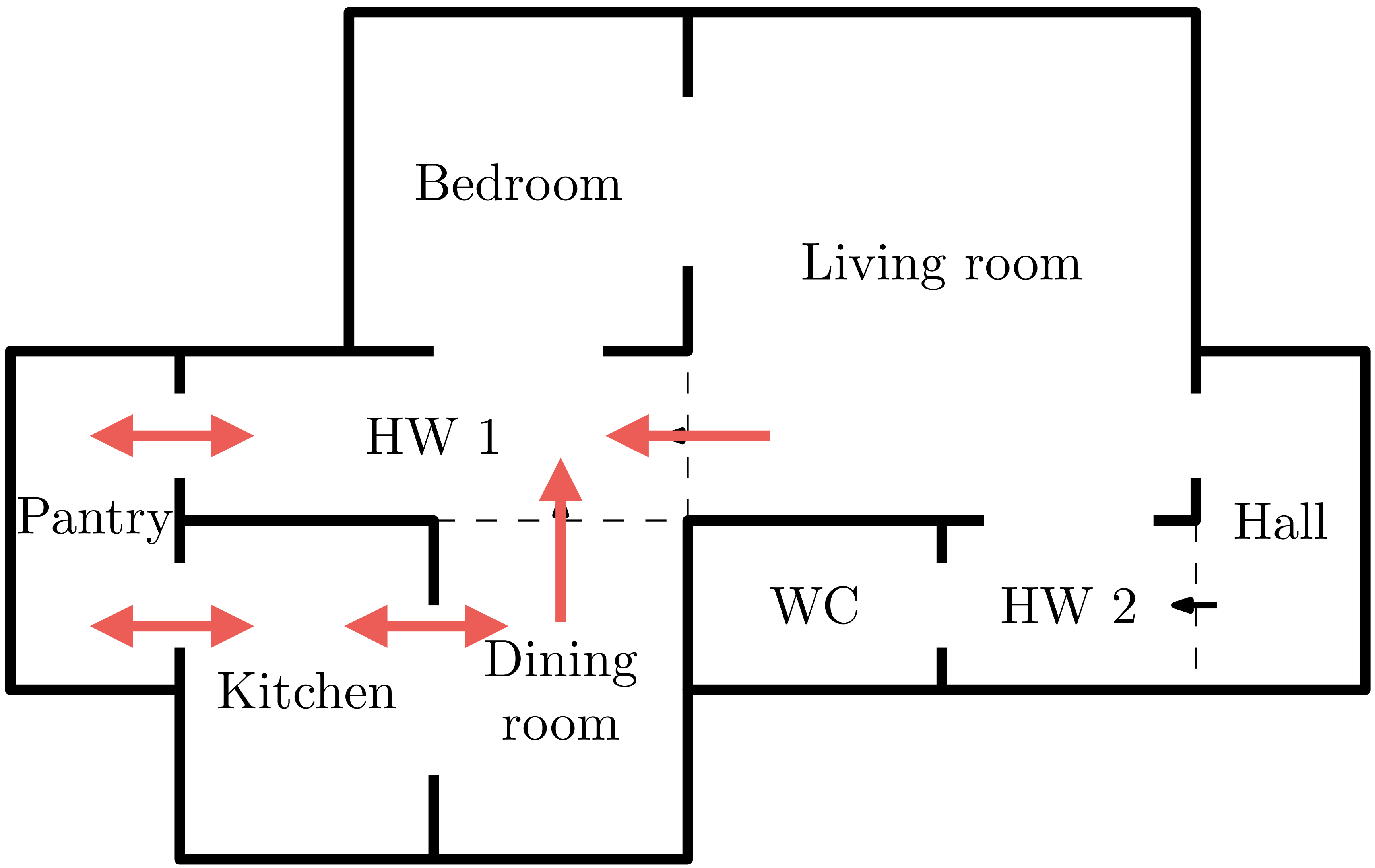
Household robot

- Consider the household



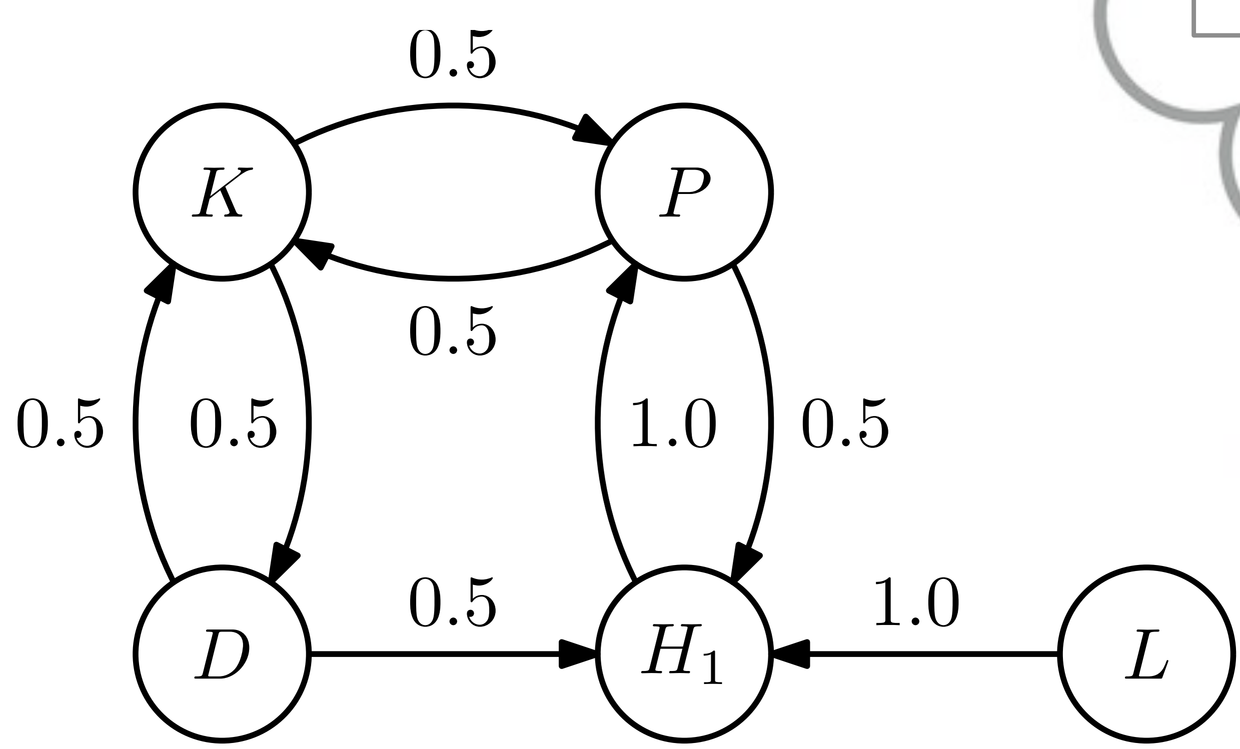
Household robot

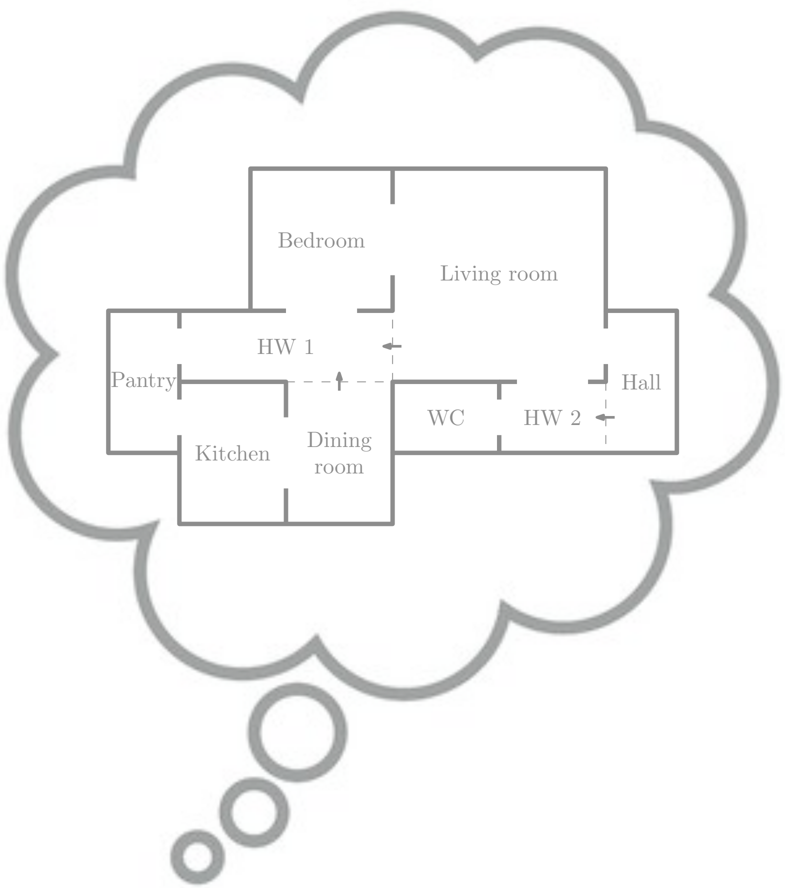
- Robot circulates between **Kitchen**, **Pantry**, **Dining room**, **Hallway 1** and **Living room**



Household robot

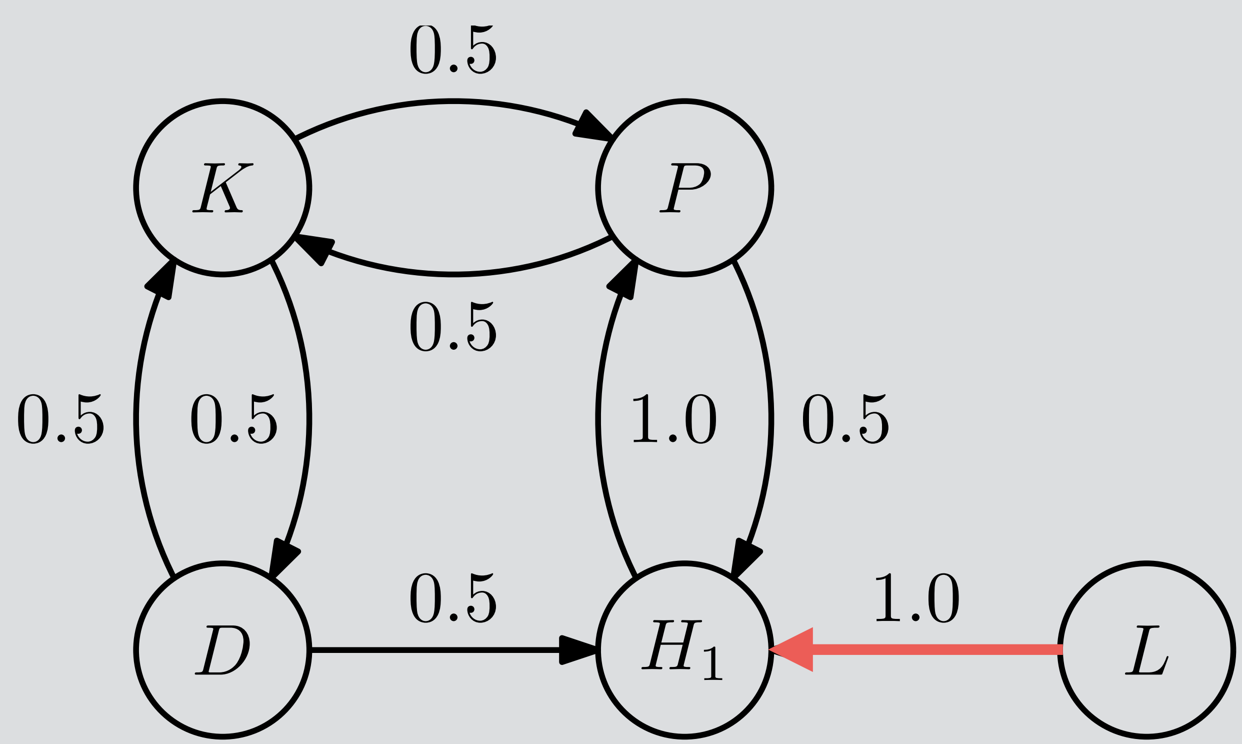
- Robot selects randomly the next room to visit
- Robot can't move back in steps
- Transition diagram:





Predicting the position...

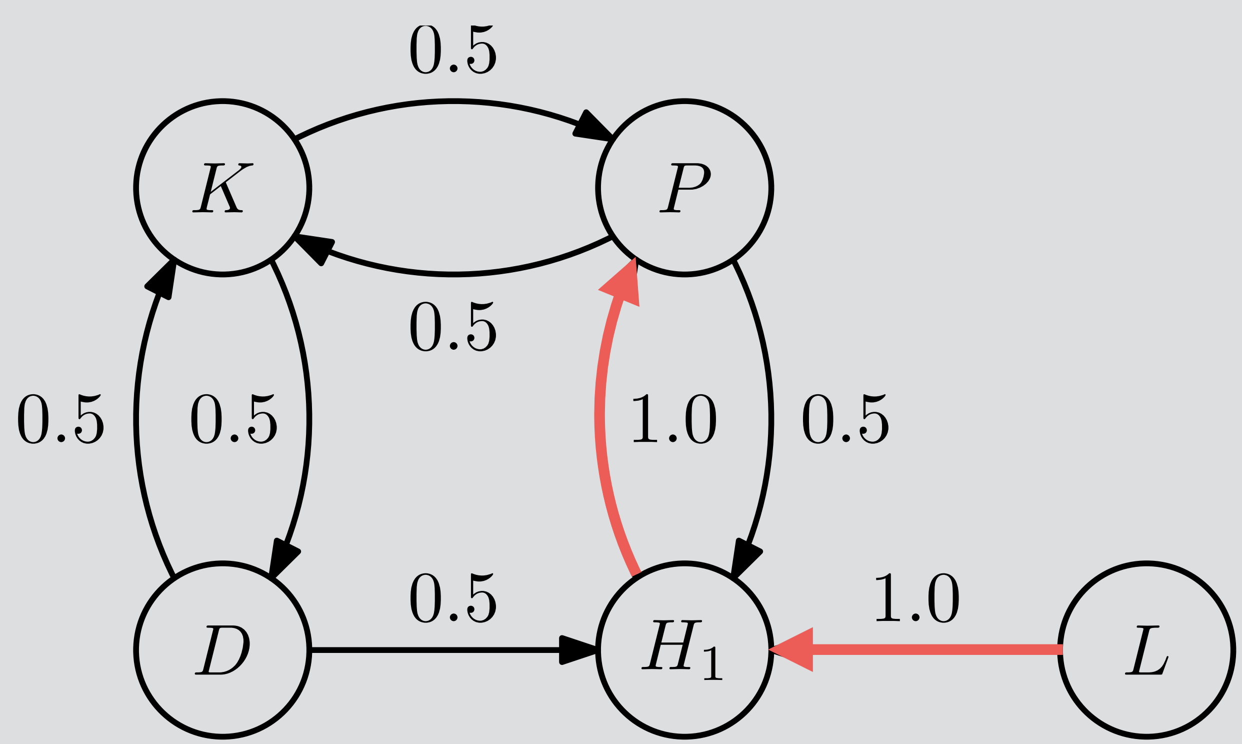
- Robot is in the **Living room**
- Where will it be the next time step?



- **Hallway 1**, with 100% probability

Predicting the position...

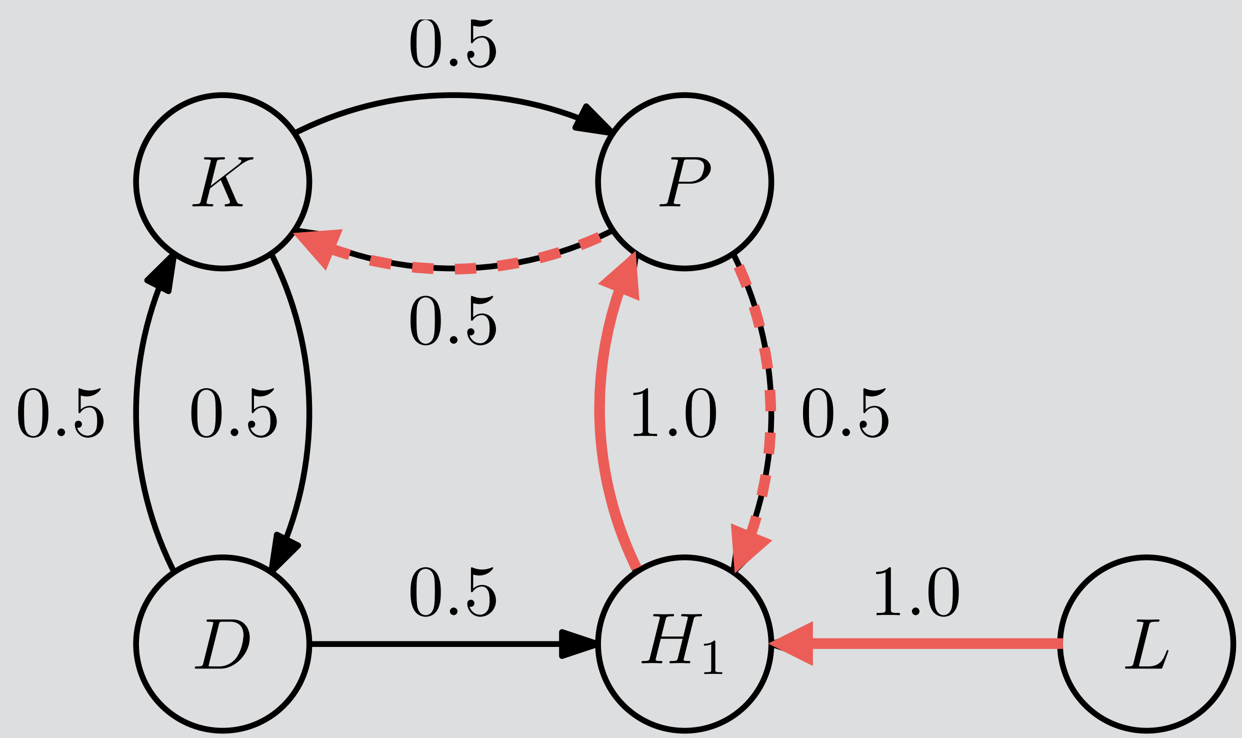
- Robot is in the **Living room**
- Where will it be in **two** time steps?



- **Pantry**, with 100% probability

Predicting the position...

- Robot is in the **Living room**
- Where will it be in **three** time steps?



- **Kitchen**, with 50% prob.; **Hallway 1**, with 50% prob.

Predicting the position...

- Robot is in the **Living room**
- Where will it be 5 time steps from now?

Not practical to list all possibilities...

... but you can have fun at home 😜

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Assumptions

- The position of the robot in an instant is enough to predict its position in the next instant

- This prediction does not depend on the particular instant

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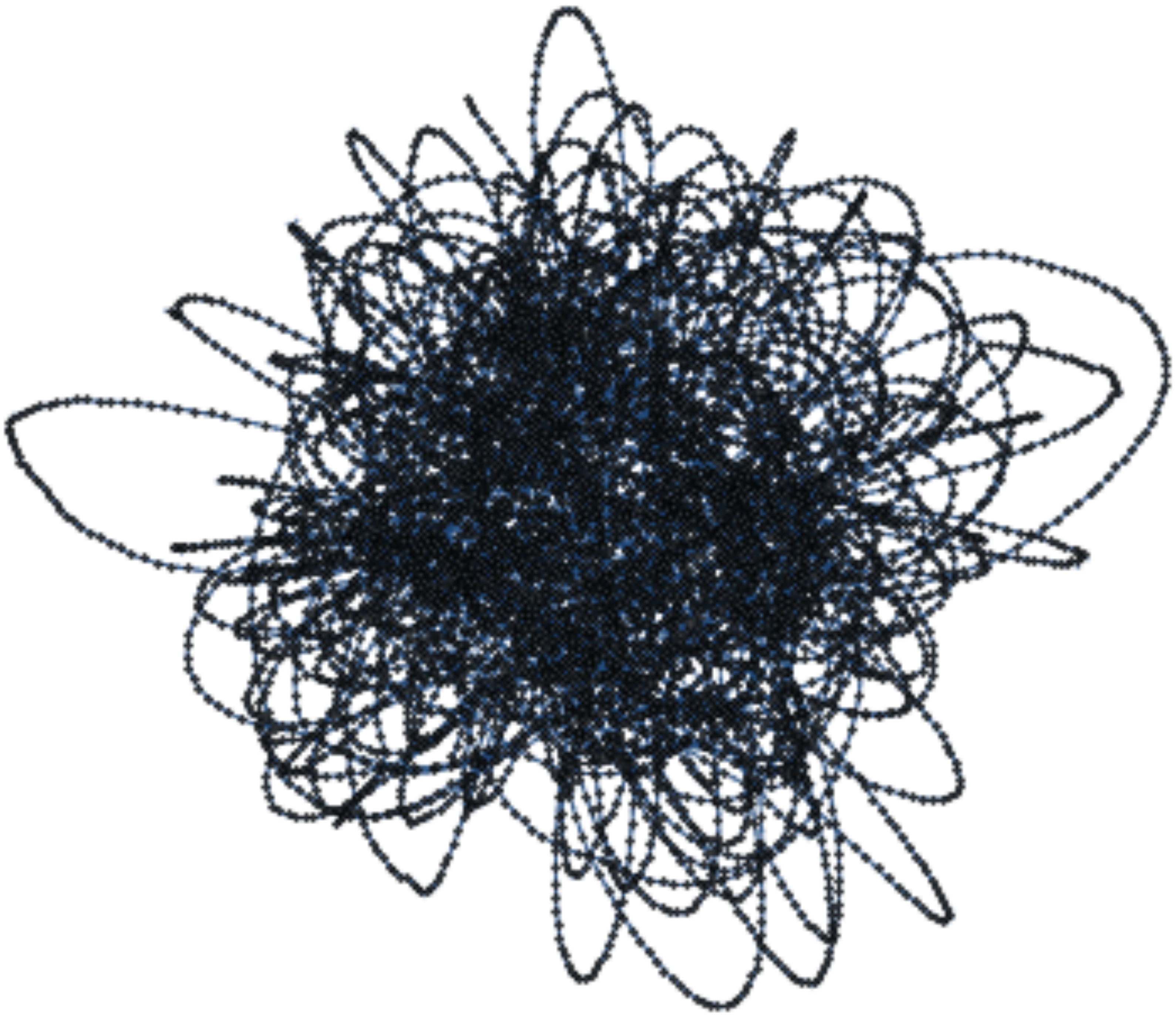

Markov Chains

Markov chain

- Model for **sequential process**
- Process evolves in **discrete time steps** (hence the chain)
- **State of the process** at time step t: x_t

Quantity of interest
(weather, robot
position, etc.)

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Markov chain

Key Property: Markov property

The state at instant t is enough to predict the state at instant $t + 1$:

$$\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}\right] = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x_t\right]$$

Depends only on the last state

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Markov chain

- Other assumptions (for this most of this course):

  - There is only a **finite number** of possible states

  - $\mathcal{X}$ is the set of possible states (**state space**)

Finite chain

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Markov chain

- Other assumptions (for most of this course):

  - The probabilities $\boxed{\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x\right]}$ do not depend on t

Transition probability
from x to y

Homogeneous chain

Transition probability matrix

- Computationally, it is easier to store the transition probabilities in a **matrix P**

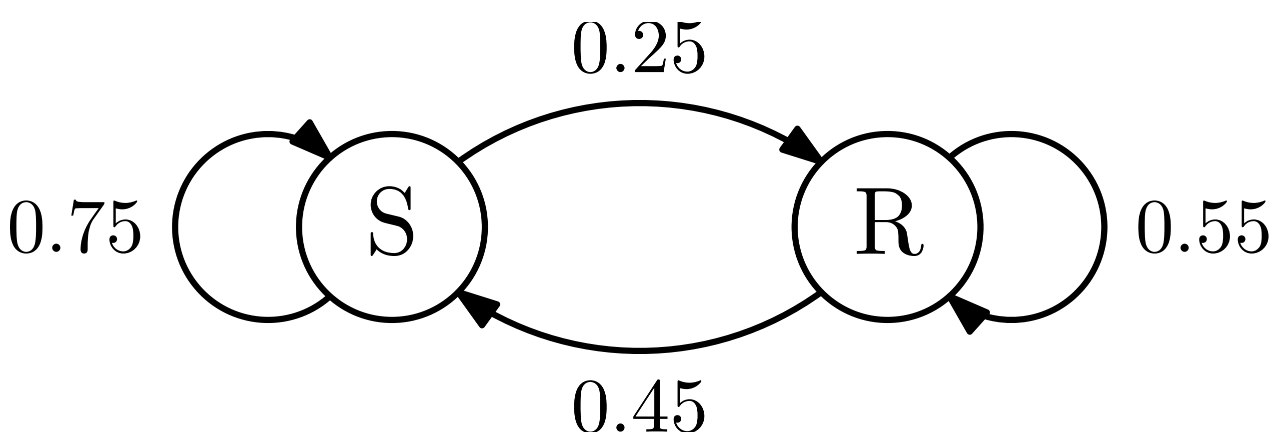
$$[\mathbf{P}]_{xy} = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x\right]$$

Number in row x column y is the probability of "moving" from x to y

Transition probability matrix

- **Example:** the weather in Lisbon

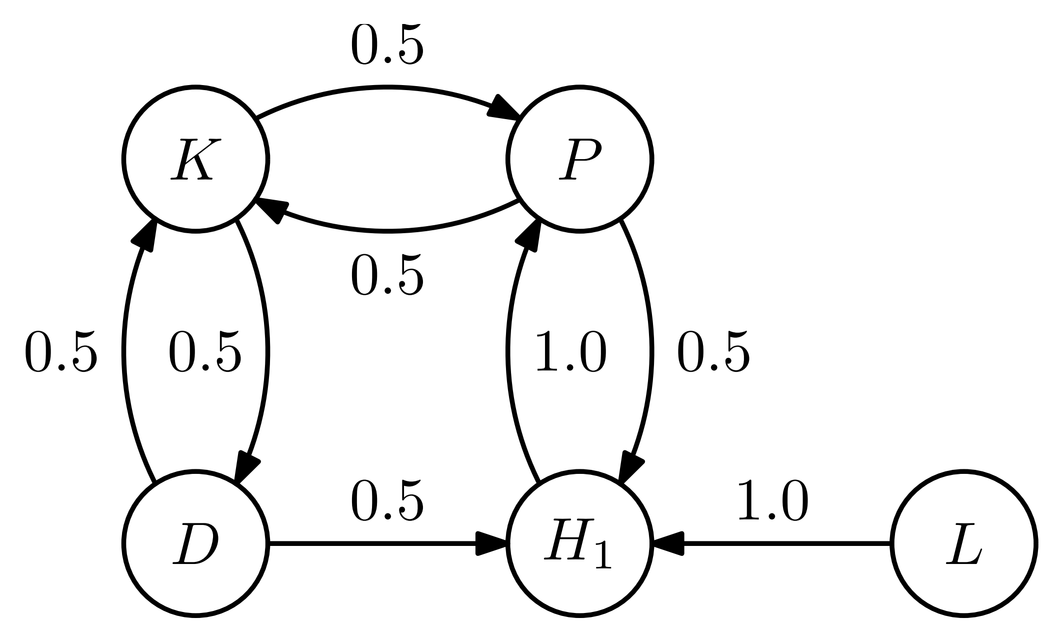
$$\begin{array}{c c} & \begin{array}{cc} \mathrm{S} & \mathrm{R} \end{array} \\ \begin{array}{c} \mathrm{S} \\ \mathrm{R} \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.45 & 0.55 \end{bmatrix} \end{array}$$



Transition probability matrix

- **Example:** the household robot

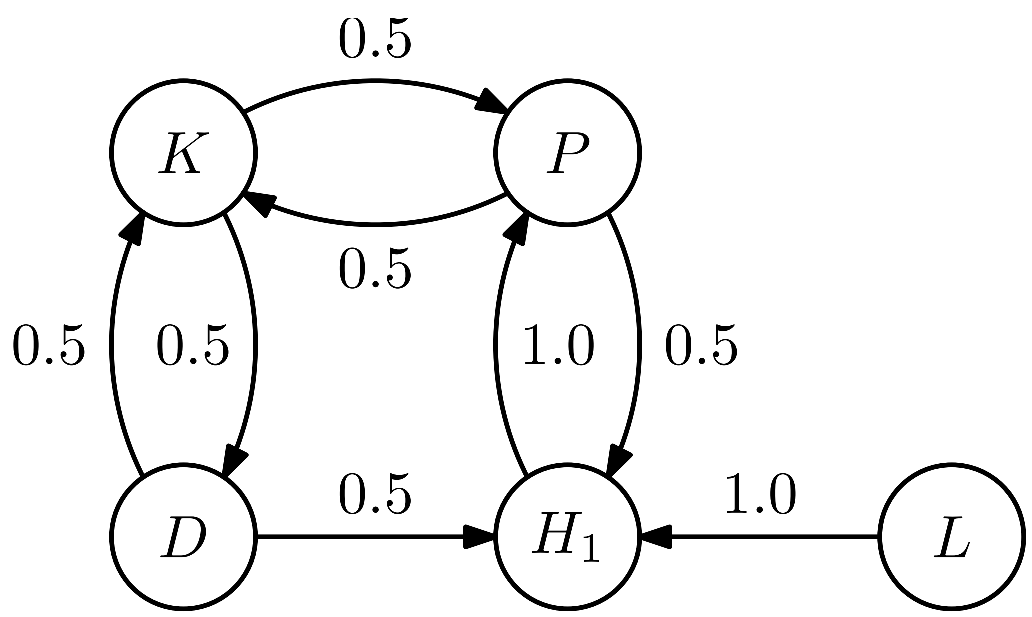
$$
\begin{array}{c|c}
 & \begin{array}{ccccc} \mathrm{L} & \mathrm{K} & \mathrm{P} & \mathrm{D} & \mathrm{H} \end{array} \\
\begin{array}{c} \mathrm{L} \\ \mathrm{K} \\ \mathrm{P} \\ \mathrm{D} \\ \mathrm{H} \end{array} & \begin{bmatrix} \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; \end{bmatrix}
\end{array}
$$



Transition probability matrix

- **Example:** the household robot

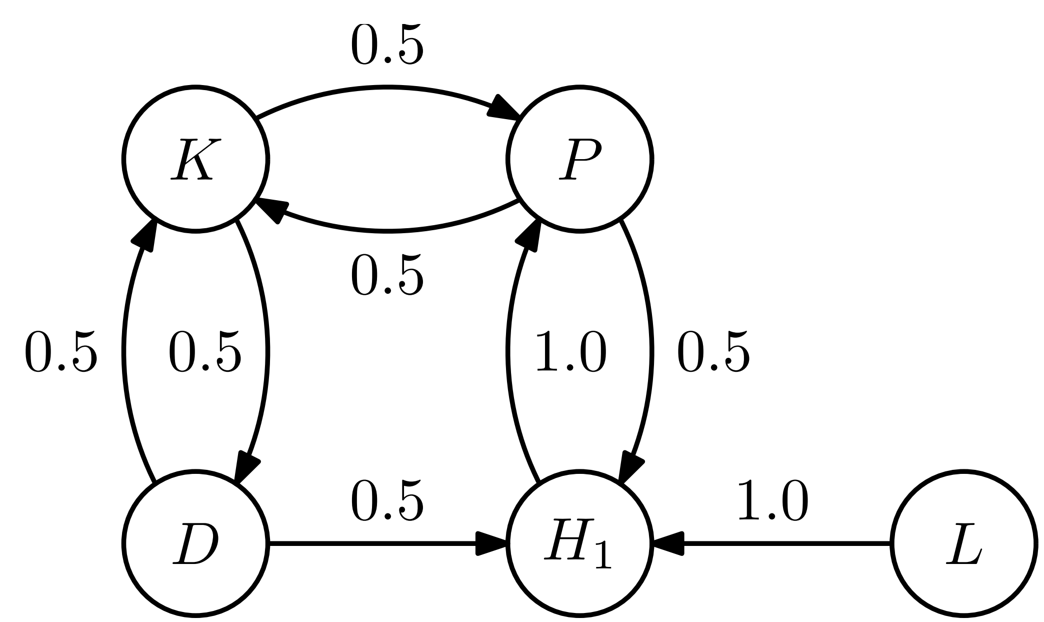
$$
\begin{array}{c}
\\ \text{L} \\ \text{K} \\ \text{P} \\ \text{D} \\ \text{H}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{L} & \text{K} & \text{P} & \text{D} & \text{H} \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}
\end{array}
$$



Transition probability matrix

- **Example:** the household robot

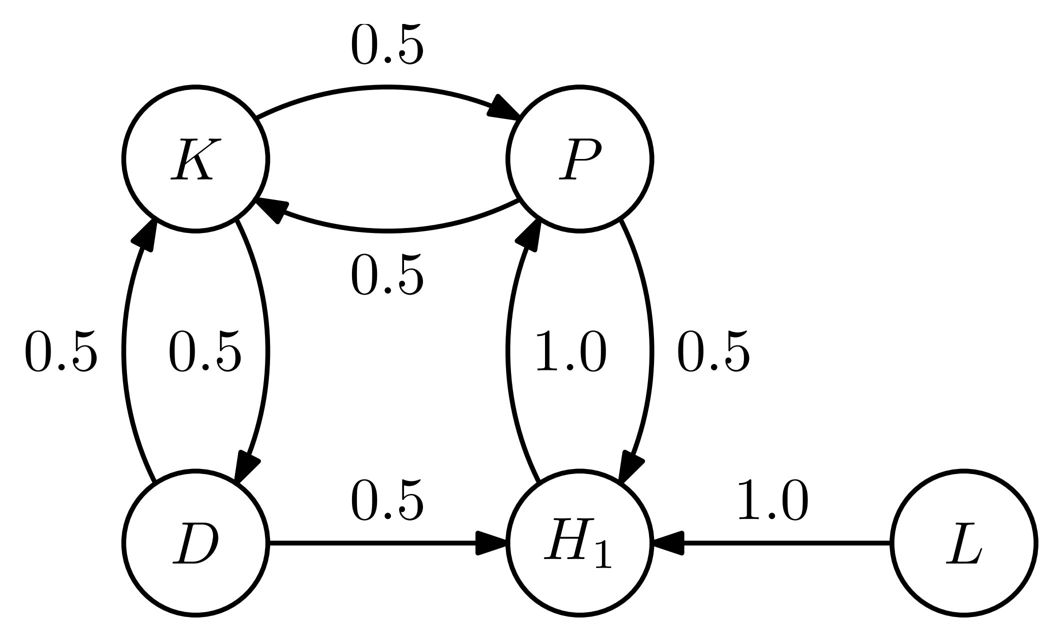
$$
\begin{array}{c}
\\ \text{L} \\ \text{K} \\ \text{P} \\ \text{D} \\ \text{H}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{L} & \text{K} & \text{P} & \text{D} & \text{H} \end{array} \\
\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0.5 & 0.5 & 0 \\
 & & & & \\
 & & & & \\
 & & & &
\end{array}\right]
\end{array}
$$



Transition probability matrix

- **Example:** the household robot

$$
\begin{array}{c|ccccc|}
 & \text{L} & \text{K} & \text{P} & \text{D} & \text{H} \\
\hline
\text{L} & 0 & 0 & 0 & 0 & 1 \\
\text{K} & 0 & 0 & 0.5 & 0.5 & 0 \\
\text{P} & 0 & 0.5 & 0 & 0 & 0.5 \\
\text{D} & 0 & 0.5 & 0 & 0 & 0.5 \\
\text{H} & 0 & 0 & 1 & 0 & 0
\end{array}
$$



Markov chain

- Compact representation/specification of a MC
  - A Markov chain can be represented as a pair $(\mathcal{X}, \mathbf{P})$
    - $\mathcal{X}$ is the set of possible states
    - $\mathbf{P}$ is the transition probability matrix
- We write $\mathbf{P}(y \mid x)$ to denote the element xy of matrix $\mathbf{P}$

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Predictions with MCs

Predicting the weather...

- Today is **Sunny**
- What is the weather **tomorrow**?

- **Rainy**, with 25% probability
- **Sunny**, with 75% probability

$$\begin{array}{c} \\ \mathrm{S} \\ \\ \mathrm{R} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{S} & \mathrm{R} \end{array} \\ \begin{bmatrix} \boxed{0.75 \quad 0.25} \\ 0.45 \quad 0.55 \end{bmatrix} \end{array}$$

Predicting the weather...

- Today is **Rainy**
- What is the weather **tomorrow**?

- **Rainy**, with 55% probability
- **Sunny**, with 45% probability

$$\begin{array}{c c} & \begin{array}{cc} S & R \end{array} \\ \begin{array}{c} S \\ R \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ \boxed{0.45} & \boxed{0.55} \end{bmatrix} \end{array}$$

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Predicting the weather...

- Today is **Sunny**
- How likely is the weather **Sunny** the day after tomorrow?

- $\mathbb{P}[x_2 = S \mid x_0 = S]$ ← Total probability law!

Predicting the weather...

- Today is **Sunny**
- How likely is the weather **Sunny** the day after tomorrow?

- $\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_0 = S]$

$$= \mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = S]\mathbb{P}[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S]$$

$$\underbrace{\begin{bmatrix} 0.75 & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_0 \text{ to } x_1} \underbrace{\begin{bmatrix} 0.75 & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_1 \text{ to } x_2}$$

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Predicting the weather...

- Today is **Sunny**
- How likely is the weather **Sunny** the day after tomorrow?

- $\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_0 = S]$

$$= \boxed{\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = S]}\,\mathbb{P}[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S]$$

$$\underbrace{\begin{bmatrix} 0.75 & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_0 \text{ to } x_1} \quad \underbrace{\begin{bmatrix} \boxed{0.75} & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_1 \text{ to } x_2}$$

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Predicting the weather...

- Today is **Sunny**
- How likely is the weather **Sunny** the day after tomorrow?

- $\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_0 = S]$

$$= \mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = S]\boxed{\mathbb{P}[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S]}$$

$$\underbrace{\begin{bmatrix} \boxed{0.75} & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_0 \text{ to } x_1} \quad \underbrace{\begin{bmatrix} \boxed{0.75} & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_1 \text{ to } x_2}$$

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Predicting the weather...

- Today is **Sunny**

- How likely is the weather **Sunny** the day after tomorrow?

- $\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_0 = S]$

  $= \mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = S]\mathbb{P}[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S]$

$$\underbrace{\begin{bmatrix} \boxed{0.75} & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_0 \text{ to } x_1} \underbrace{\begin{bmatrix} \boxed{0.75} & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_1 \text{ to } x_2}$$

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Predicting the weather...

- Today is **Sunny**

- How likely is the weather **Sunny** the day after tomorrow?

$$\begin{aligned}
&\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_0 = S] \\
&\quad = \mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = S]\mathbb{P}[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S] \\
&\qquad + \boxed{\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = R]}\mathbb{P}[\mathrm{x}_1 = R \mid \mathrm{x}_0 = S]
\end{aligned}$$

$$\underbrace{\begin{bmatrix} \boxed{0.75} & 0.25 \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_0 \text{ to } x_1} \quad \underbrace{\begin{bmatrix} \boxed{0.75} & 0.25 \\ \boxed{0.45} & 0.55 \end{bmatrix}}_{\text{From } x_1 \text{ to } x_2}$$

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Predicting the weather...

- Today is **Sunny**

- How likely is the weather **Sunny** the day after tomorrow?

- $\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_0 = S]$

$$= \mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = S]\mathbb{P}[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S]$$
$$+\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = R]\boxed{\mathbb{P}[\mathrm{x}_1 = R \mid \mathrm{x}_0 = S]}$$

$$\underbrace{\begin{bmatrix} 0.75 & \boxed{0.25} \\ 0.45 & 0.55 \end{bmatrix}}_{\text{From } x_0 \text{ to } x_1} \quad \underbrace{\begin{bmatrix} 0.75 & 0.25 \\ \boxed{0.45} & 0.55 \end{bmatrix}}_{\text{From } x_1 \text{ to } x_2}$$

Predicting the weather...

- Today is **Sunny**
- How likely is the weather **Sunny** the day after tomorrow?

- $\mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_0 = S]$

$$
\begin{aligned}
&= \mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = S]\mathbb{P}[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S] \\
&\quad + \mathbb{P}[\mathrm{x}_2 = S \mid \mathrm{x}_1 = R]\mathbb{P}[\mathrm{x}_1 = R \mid \mathrm{x}_0 = S] \\
&= 0.75 \times 0.75 + 0.45 \times 0.25 \\
&= 67.5\%
\end{aligned}
$$

$$
\underset{\text{From } x_0 \text{ to } x_1}{\begin{bmatrix} \boxed{0.75} & \boxed{0.25} \\ 0.45 & 0.55 \end{bmatrix}}
\qquad
\underset{\text{From } x_1 \text{ to } x_2}{\begin{bmatrix} \boxed{0.75} & 0.25 \\ \boxed{0.45} & 0.55 \end{bmatrix}}
$$

Transition probabilities

- We observe that

$$[\mathbf{P} \times \mathbf{P}]_{xy} = \mathbb{P}\left[\mathrm{x}_{t+2} = y \mid \mathrm{x}_t = x\right]$$

- In general,

$$[\mathbf{P}^k]_{xy} = \mathbb{P}\left[\mathrm{x}_{t+k} = y \mid \mathrm{x}_t = x\right]$$

k-step transition probabilities

Predicting the position...

- Robot is in the **Living room**

- Where will it be 5 time steps from now?

- We compute $\mathbf{P}^5$

$$\mathbf{P}^5 = \begin{array}{c} \begin{array}{ccccc} \mathsf{L} & \mathsf{K} & \mathsf{P} & \mathsf{D} & \mathsf{H} \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.75 & 0.25 & 0 \end{bmatrix} \end{array}$$

- Kitchen: 50% prob.; Hallway 1: 50% probability.

Predicting the position...

- In Python...

fmelo — python — 80×24

```
>>> import numpy as np
>>> P = np.array([[0, 0, 0, 0, 1], [0, 0, .5, .5, 0], [0, .5, 0, 0, .5], [0, .5,
 0, 0, .5], [0, 0, 1, 0, 0]])
>>> np.linalg.matrix_power(P, 5)
array([[ 0.  ,  0.5 ,  0.  ,  0.  ,  0.5 ],
       [ 0.  ,  0.  ,  0.75,  0.25,  0.  ],
       [ 0.  ,  0.5 ,  0.  ,  0.  ,  0.5 ],
       [ 0.  ,  0.5 ,  0.  ,  0.  ,  0.5 ],
       [ 0.  ,  0.  ,  0.75,  0.25,  0.  ]])
>>>
```

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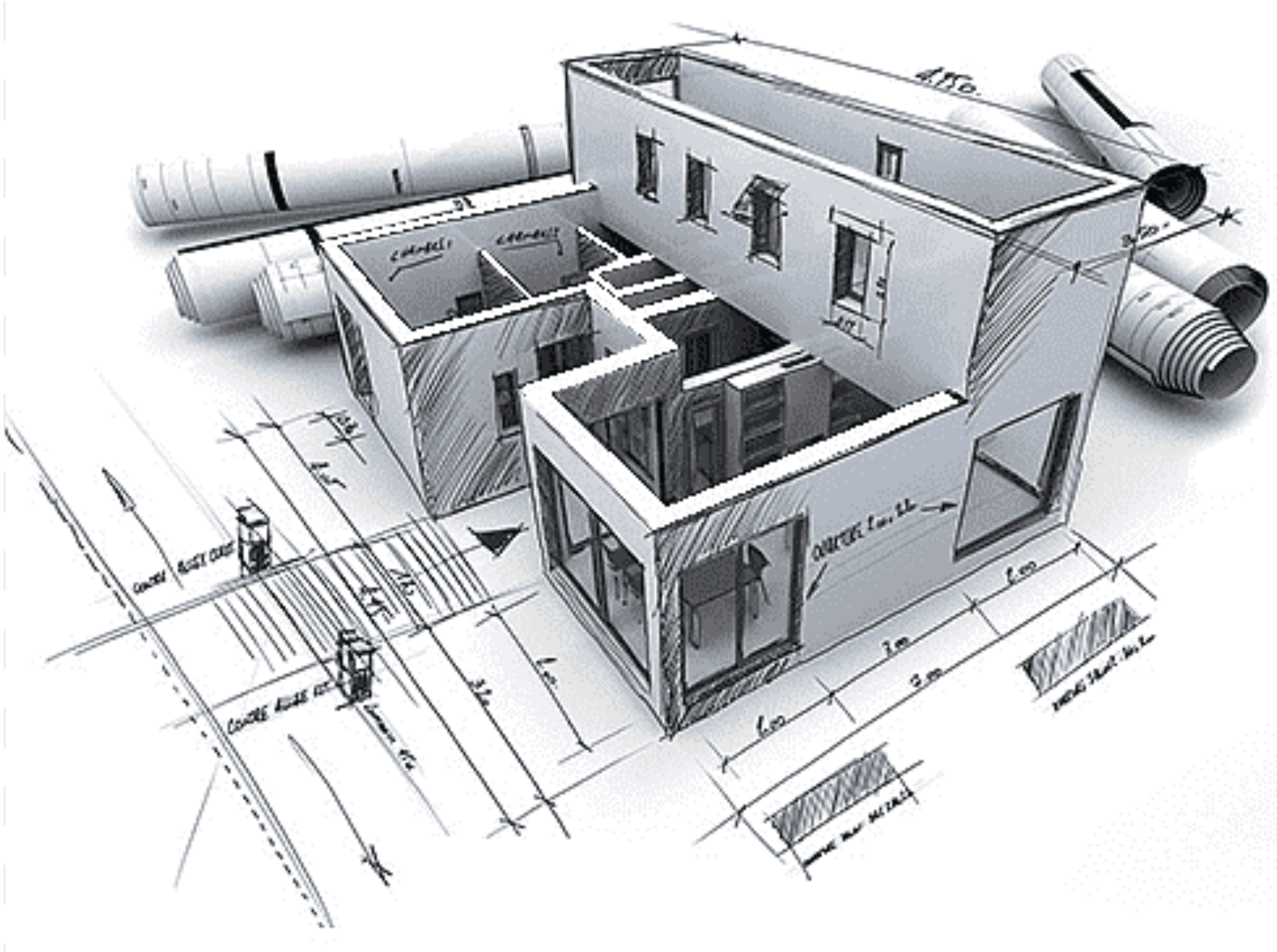


MC examples (modeling)

Example 1. The Gambler

- A gambler, Adam, enters a casino with M euros
- Adam decides to play a chance game:
    - At each round of the game, he bets 1 euro;
    - If Adam wins the bet, he receives his euro back plus one additional euro
    - If Adam loses the bet, he loses his euro
    - Adam wins with probability p

Example 1. The Gambler

- The game stops when:
  - Adam is out of money
  - Adam doubles his money

1. Is this a Markov chain?

- **Yes**

    - The money at step $t+1$ depends only on the money at step t

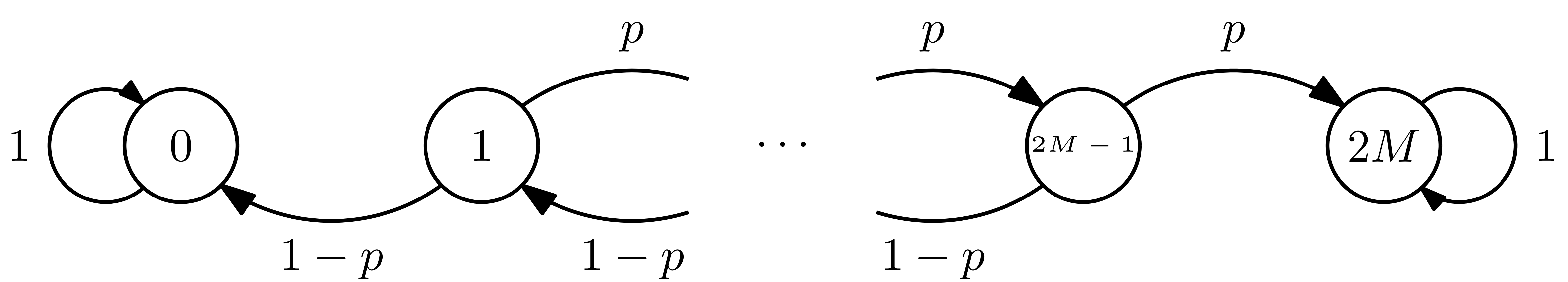
2. What are the states?

- The possible amounts of money:

    - $\mathcal{X} = \{0, 1, 2, ..., M, ..., 2M\}$

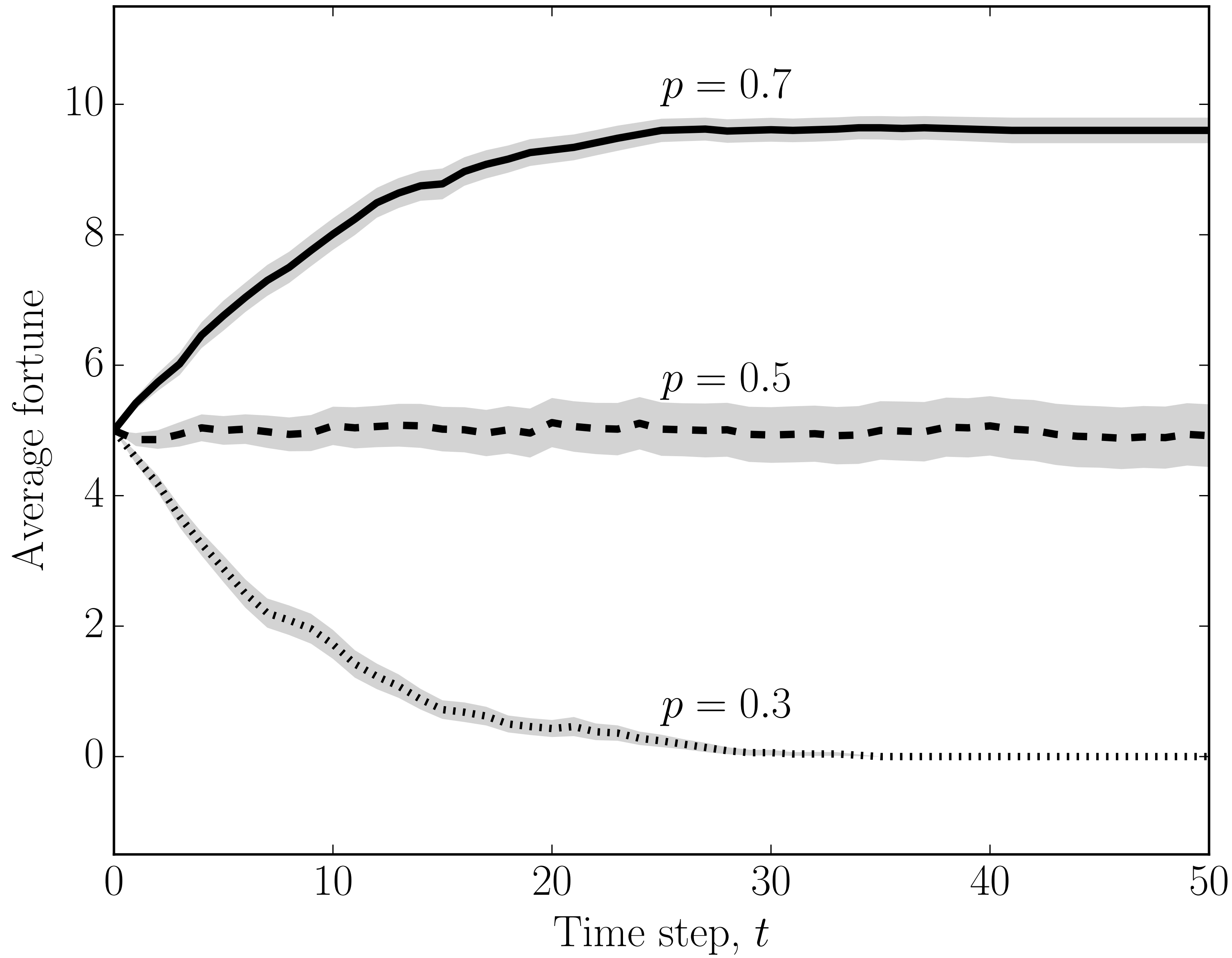
3. Transition probabilities?

- We can represent the game with a transition diagram:



What does it look like?

- We can check what happens to Adam's money for different values of p:



Example 2. PageRank

- PageRank is the algorithm used by Google to rank a set of connected documents

- It simulates a “random bot” navigating the web of retrieved documents

Example 2. PageRank

- Upon visiting a page, the bot randomly moves to one of the pages linked by the current one

- The rank corresponds to the "amount of time" that the bot spends on each page

1. Is this a Markov chain?

- **Yes**
    - The next page visited by the bot depends only on the current page (namely, its links)

2. What are the states?

- The web of pages

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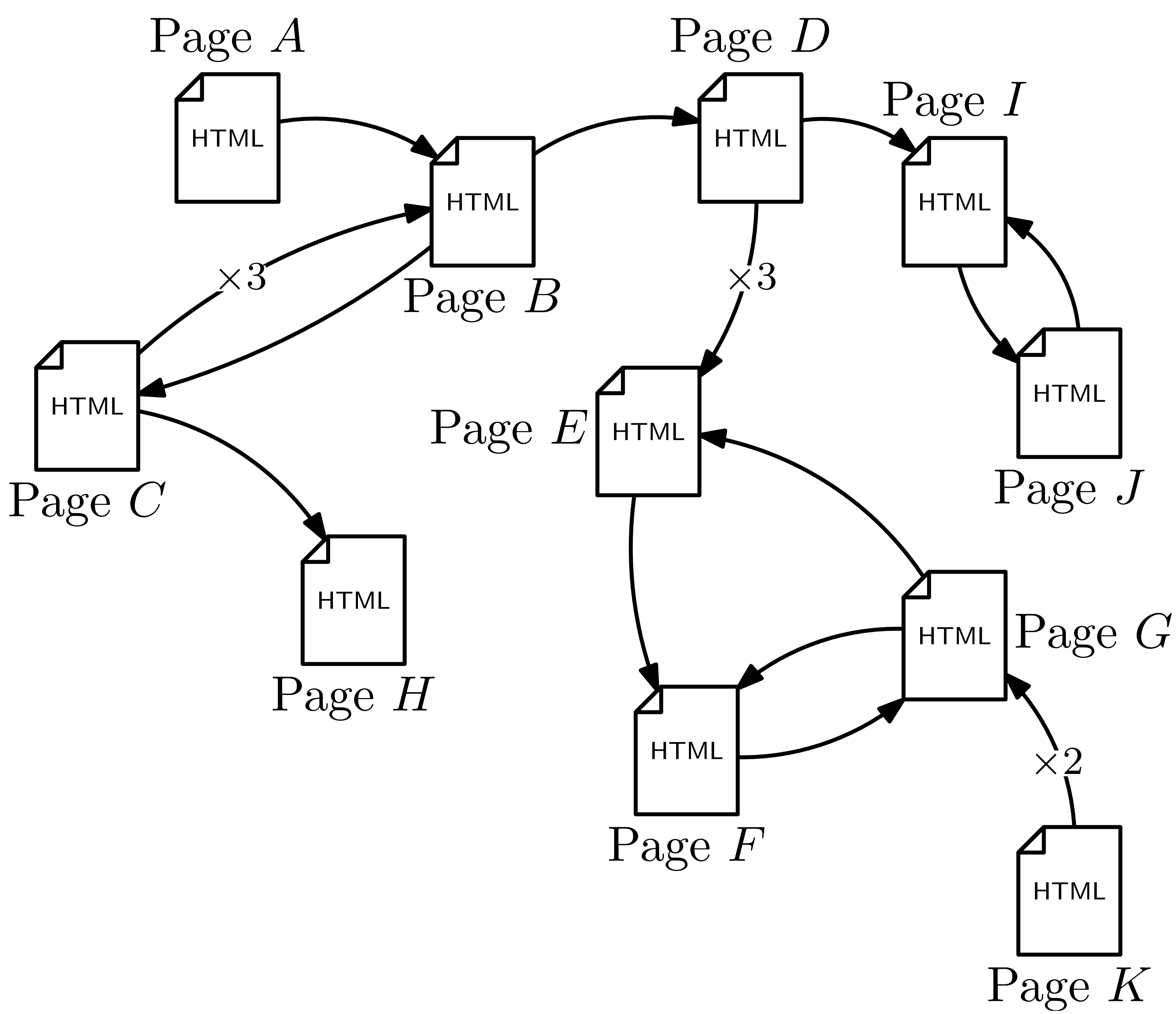
2. What are the states?

- For example, for the documents:



2. What are the states?

- We have that
  - $\mathcal{X} = \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H,\ I,\ J,\ K\}$

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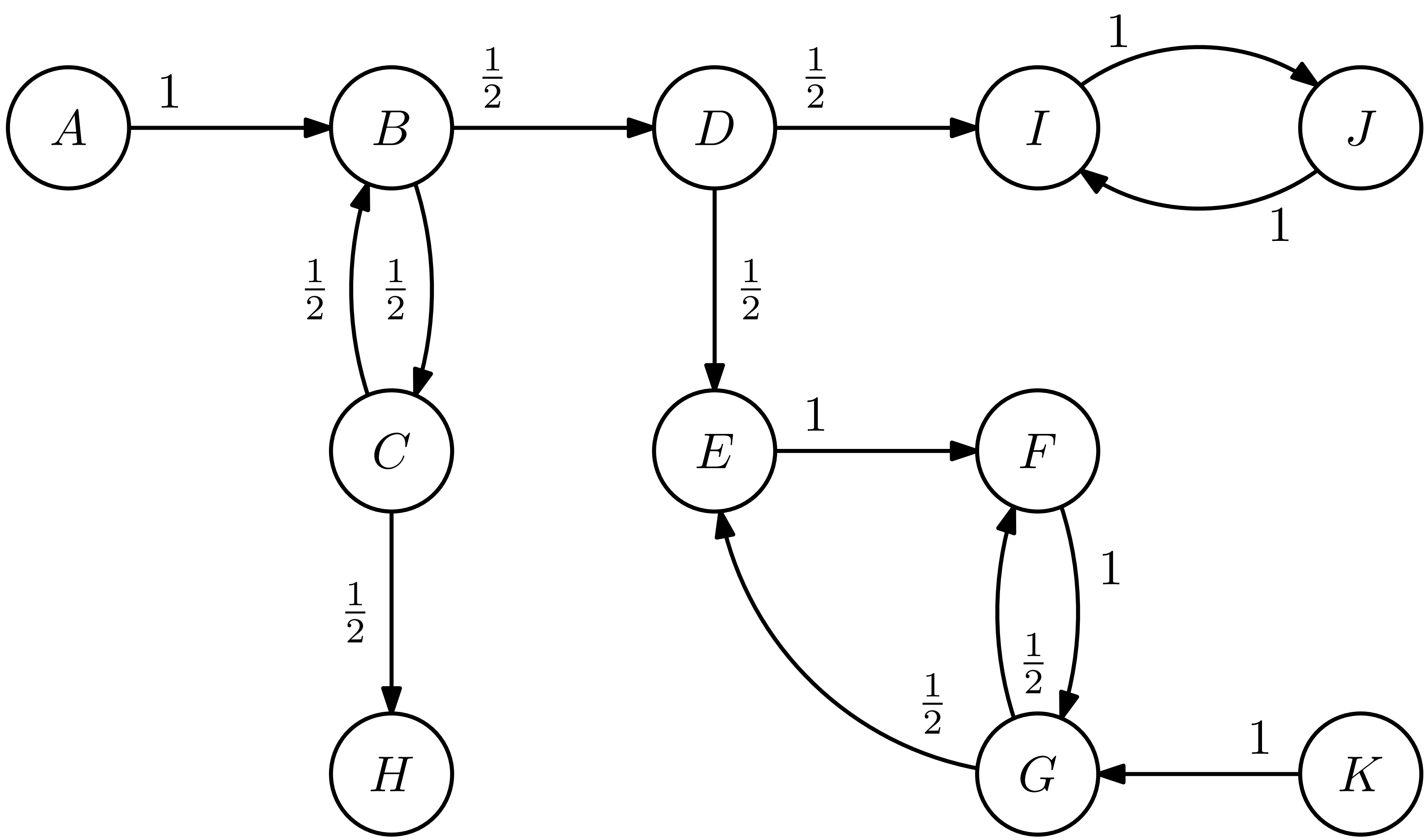
3. Transition probabilities?

- We can represent the motion of the bot as a transition diagram



... we'll revisit this diagram later.

Stability of MCs

What is stability?

- It is often important to understand how a MC behaves in the long run
  - **Stability** concerns the **long-term behavior** of the chain

What is stability?

- We may want to know:
  - Depending on where the chain starts, can it **reach** any other state?
  - Is the behavior of the chain **cyclic**?
  - How **frequently** does the chain visit each state?

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Irreducibility

- A state y can be reached from a state x if

$$\mathbf{P}^t(y \mid x) > 0$$

Positive probability of visiting y after visiting x

for some t

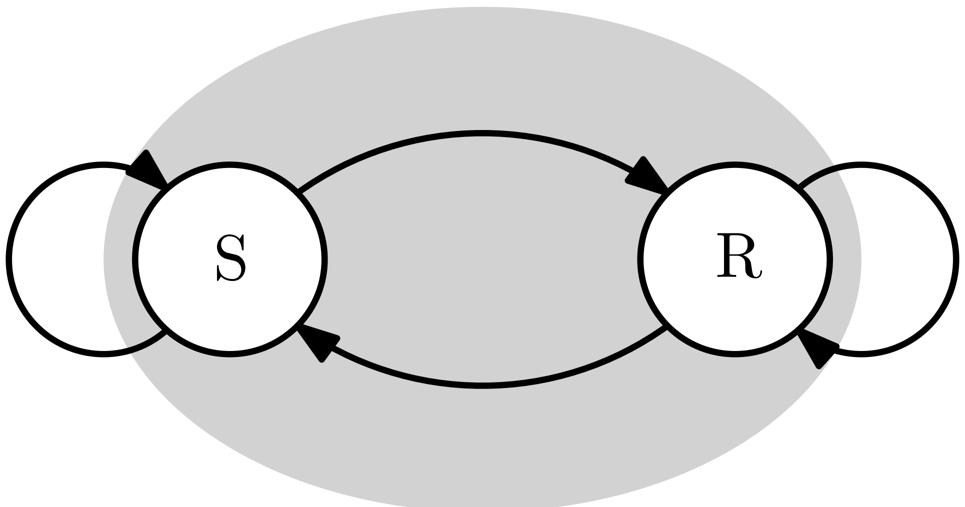
- A chain is **irreducible** if any state y can be reached from any other state x

Irreducibility

- We can split the state space of a chain in communicating classes (sets of states that are mutually reachable)

Irreducibility

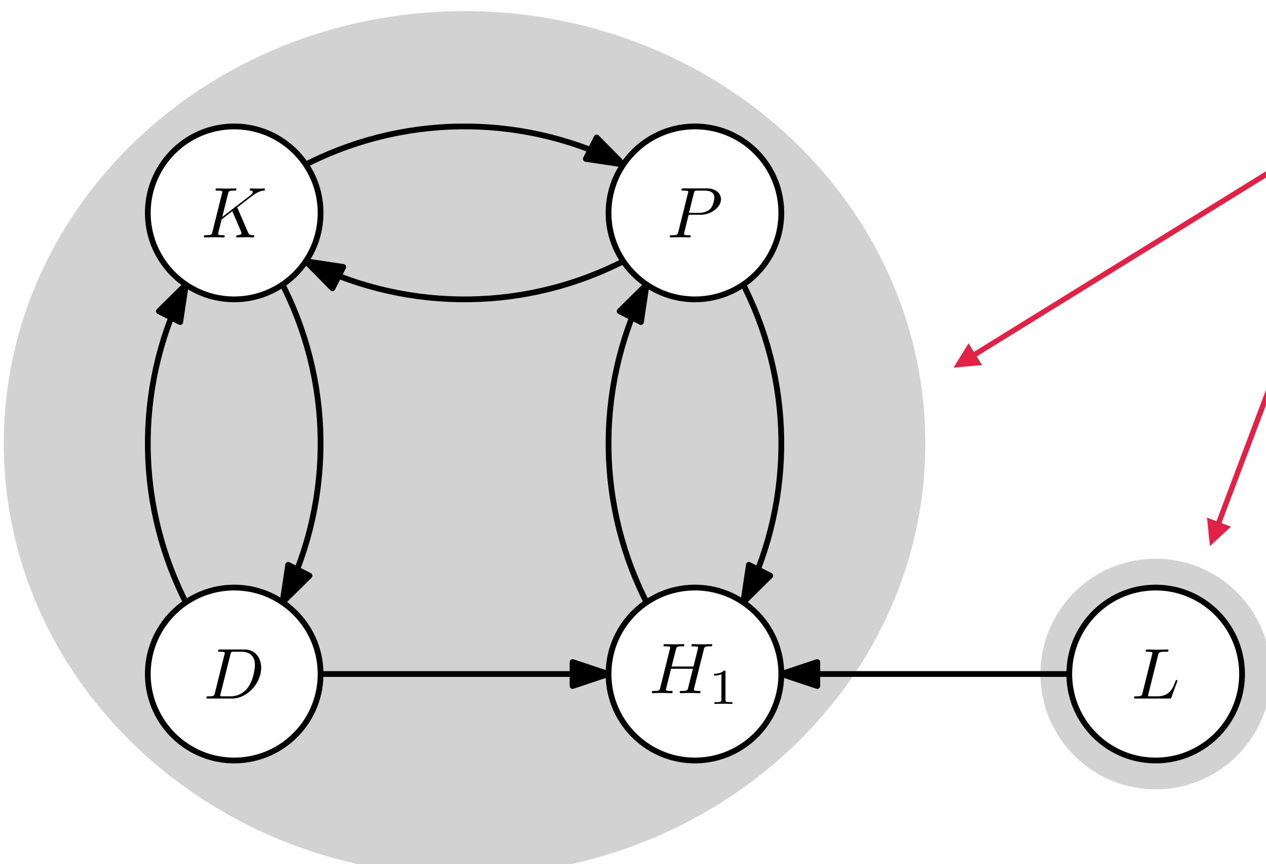
- In the weather example:



Single communicating class
$\Rightarrow$ irreducible chain

Irreducibility

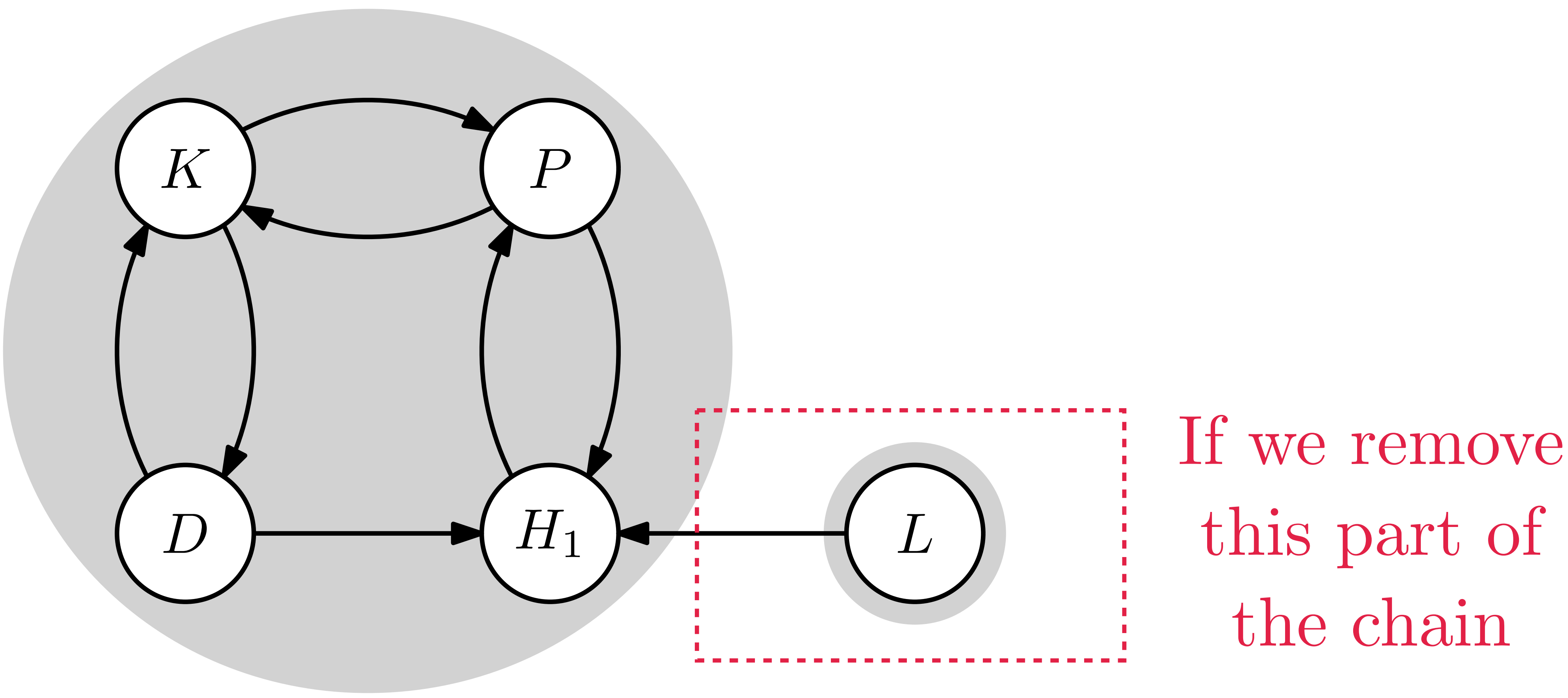
- In the robot example:



Two communicating classes
$\Rightarrow$ not irreducible

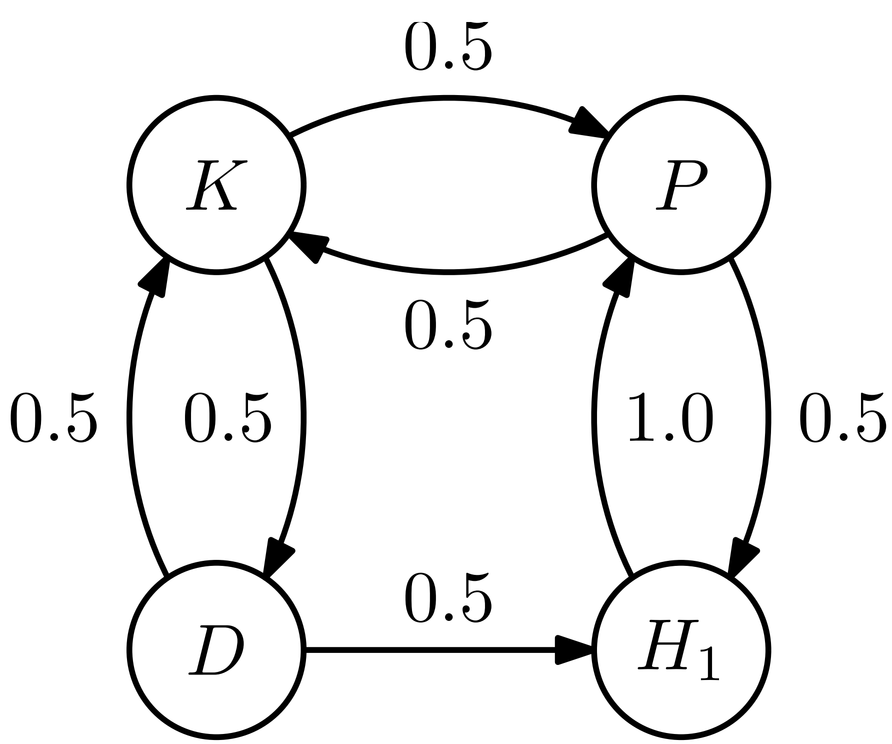
Irreducibility

- In the robot example:



Irreducibility

- In the robot example:



How many communicating classes?

Aperiodicity

- The period of a state x is...
    - the **greatest common divider**...
    - of all time steps in which x can be visited...
    - if the chain departs from x

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Aperiodicity

- Formally, the period of x is the number

$$d_x = \gcd\left\{t \in \mathbb{N} \mid \mathbf{P}^t(x \mid x) > 0, t > 0\right\}$$

- A state x is aperiodic if $d_x = 1$

- A chain is aperiodic if all states are aperiodic, and periodic otherwise

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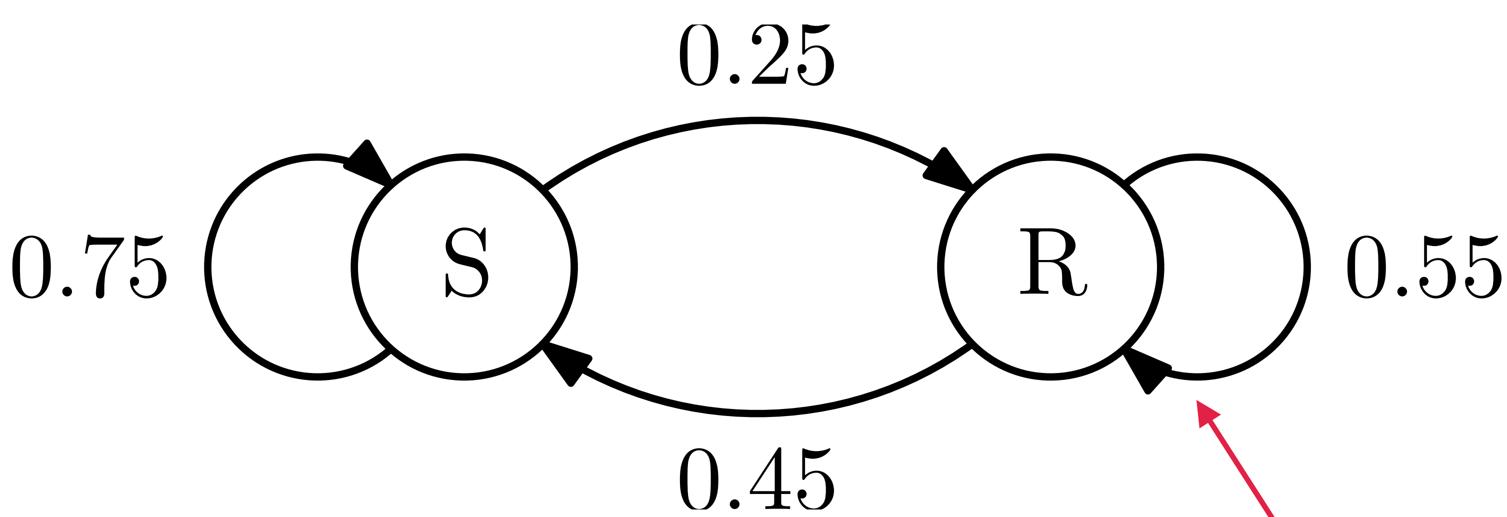
Aperiodicity

- In the weather example:

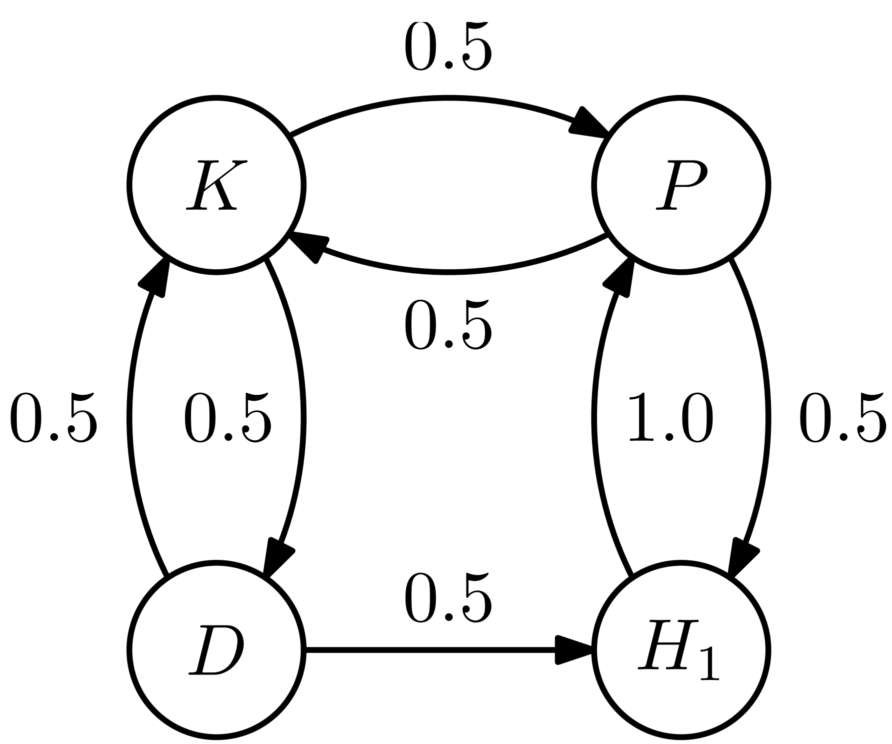


$\mathbf{P}(\mathrm{R} \mid \mathrm{R}) > 0$

$\Rightarrow$ aperiodic state

Aperiodicity

- In the robot example:



If the chain departs from P, when does it return?

↓

Several possibilities:

$t = 2$ (goes to H_1 and returns)

$t = 2$ (goes to K and returns)

$t = 4$ (goes to K, D and returns)

...

The period is $d = 2$

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Stationary distribution

- Let μ be a distribution over $\mathcal{X}$

- In practical terms, μ is a row vector

$$\boldsymbol{\mu} = \begin{bmatrix} \mu(x_1) & \mu(x_2) & \dots & \mu(x_{|\mathcal{X}|}) \end{bmatrix}$$

Component $\mu(x)$ is the probability of x according to μ

where

$$\sum_{x \in \mathcal{X}} \mu(x) = 1$$

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Stationary distribution

- μ can represent, for example,

  - ... the initial distribution for the chain;

  - ... the predicted distribution after t steps;

  - ... etc.

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Stationary distribution

- The distribution μ is **stationary** if

$$\mu(x) = \sum_{y \in \mathcal{X}} \mu(y)\mathbf{P}(x \mid y)$$

- In other words, if the state at time t follows the distribution μ and μ is stationary, then the state at time $t + 1$ also follows the distribution μ

- The stationary distribution corresponds to **stable behavior of the chain**

Key stability results

An irreducible and aperiodic Markov chain **possesses a stationary distribution**.

Positive chain

For an irreducible and aperiodic Markov chain with stationary distribution μ^*,

$$\lim_{t\to\infty} \mu_0 \mathbf{P}^t = \mu^*$$

for any initial distribution μ_0.

Ergodic chain

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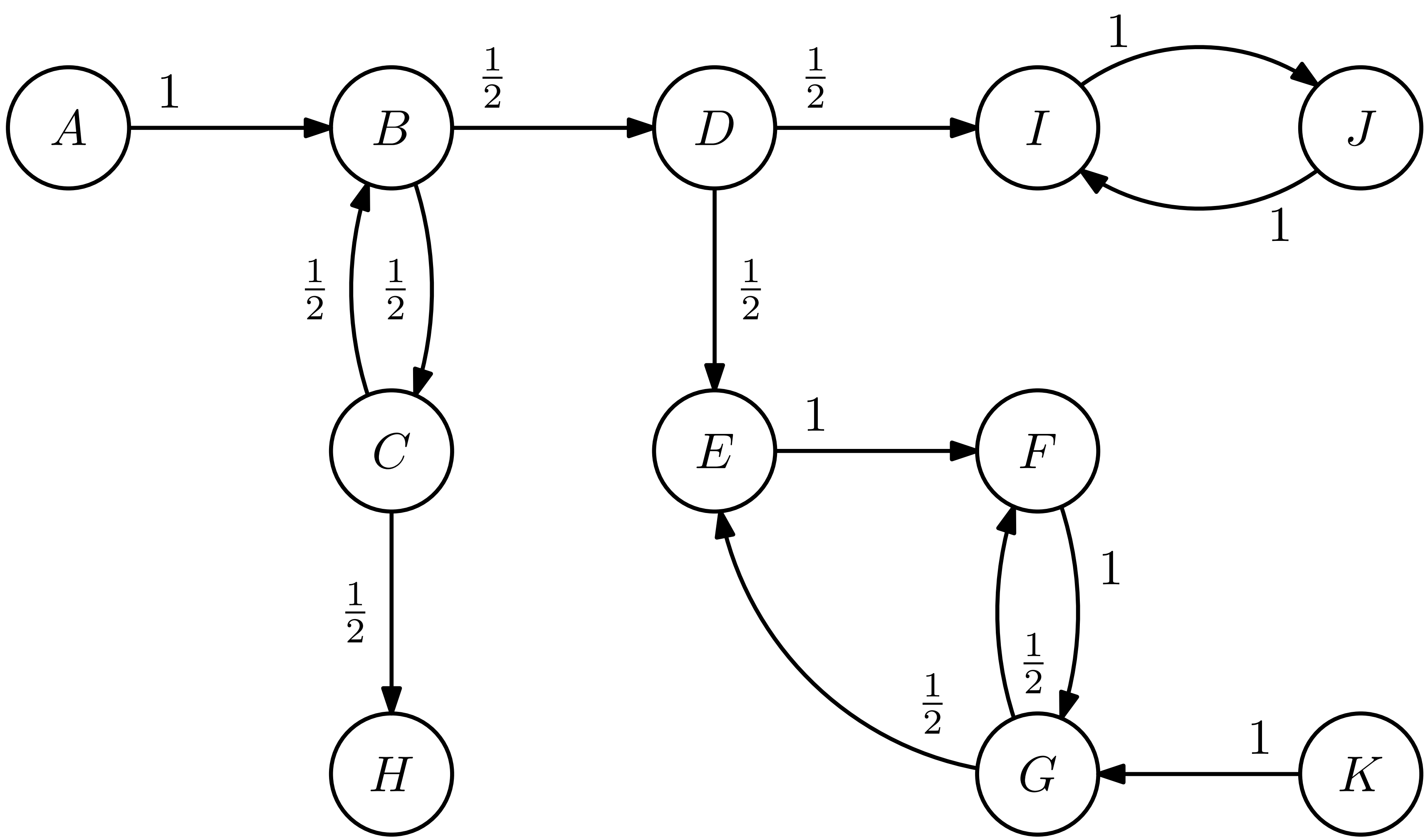
Summarizing...

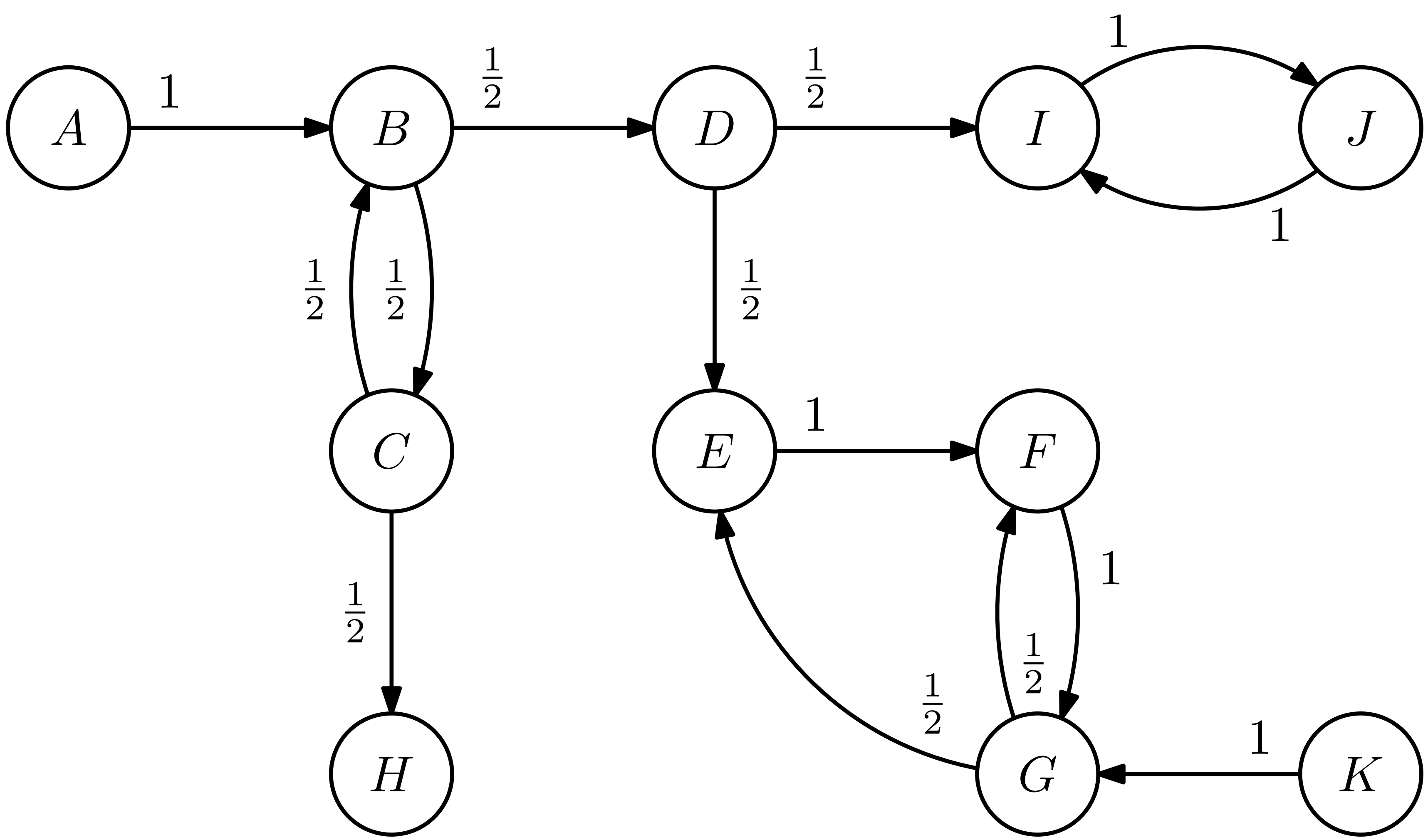
- A positive chain possesses a stationary distribution μ:
    - If x_t is distributed according to μ, then so is x_{t+1}
- An ergodic chain eventually reaches the stationary distribution

Returning to PageRank



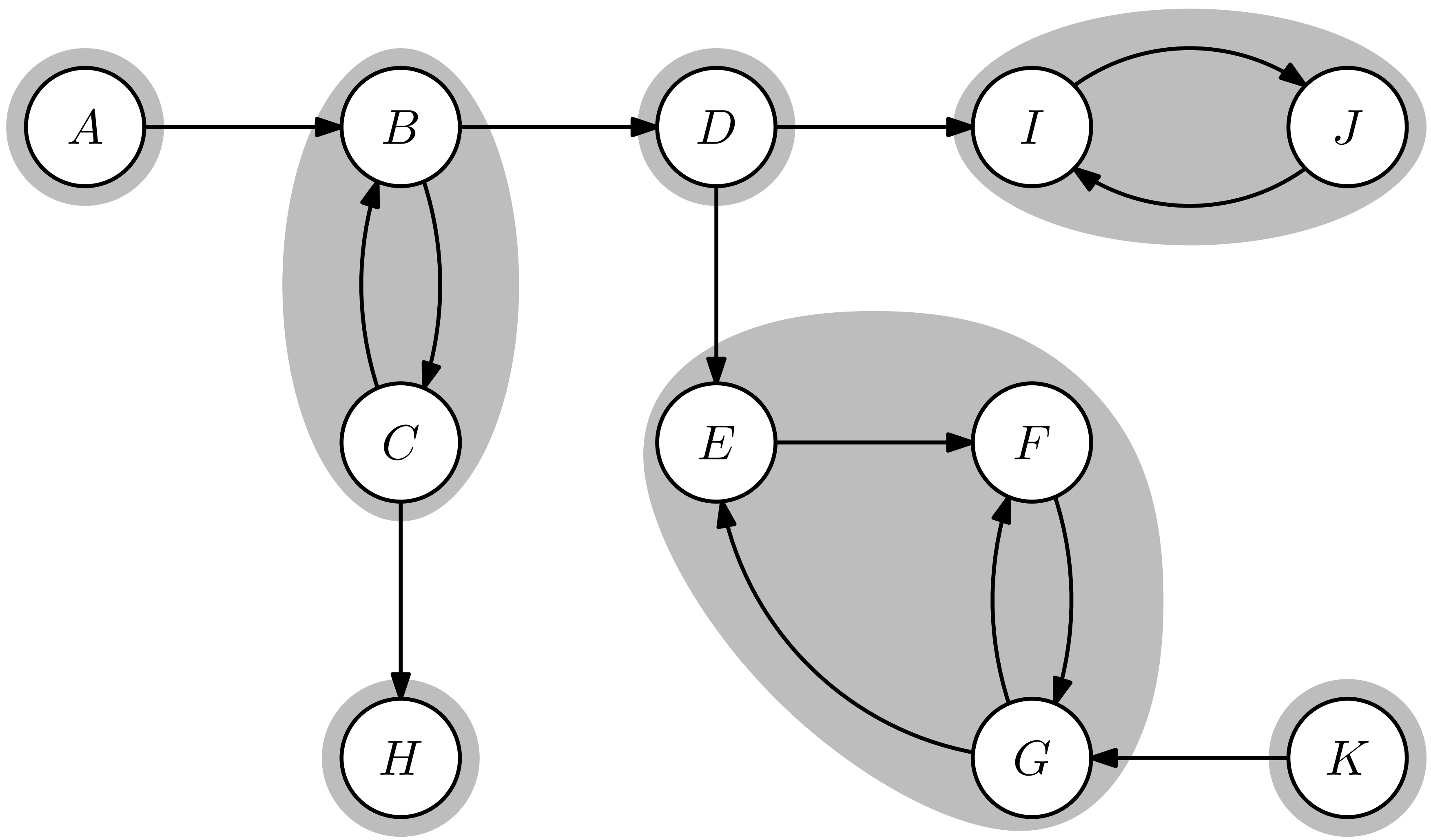
Returning to PageRank

- Is the chain irreducible?



Returning to PageRank

- Is the chain irreducible? No.

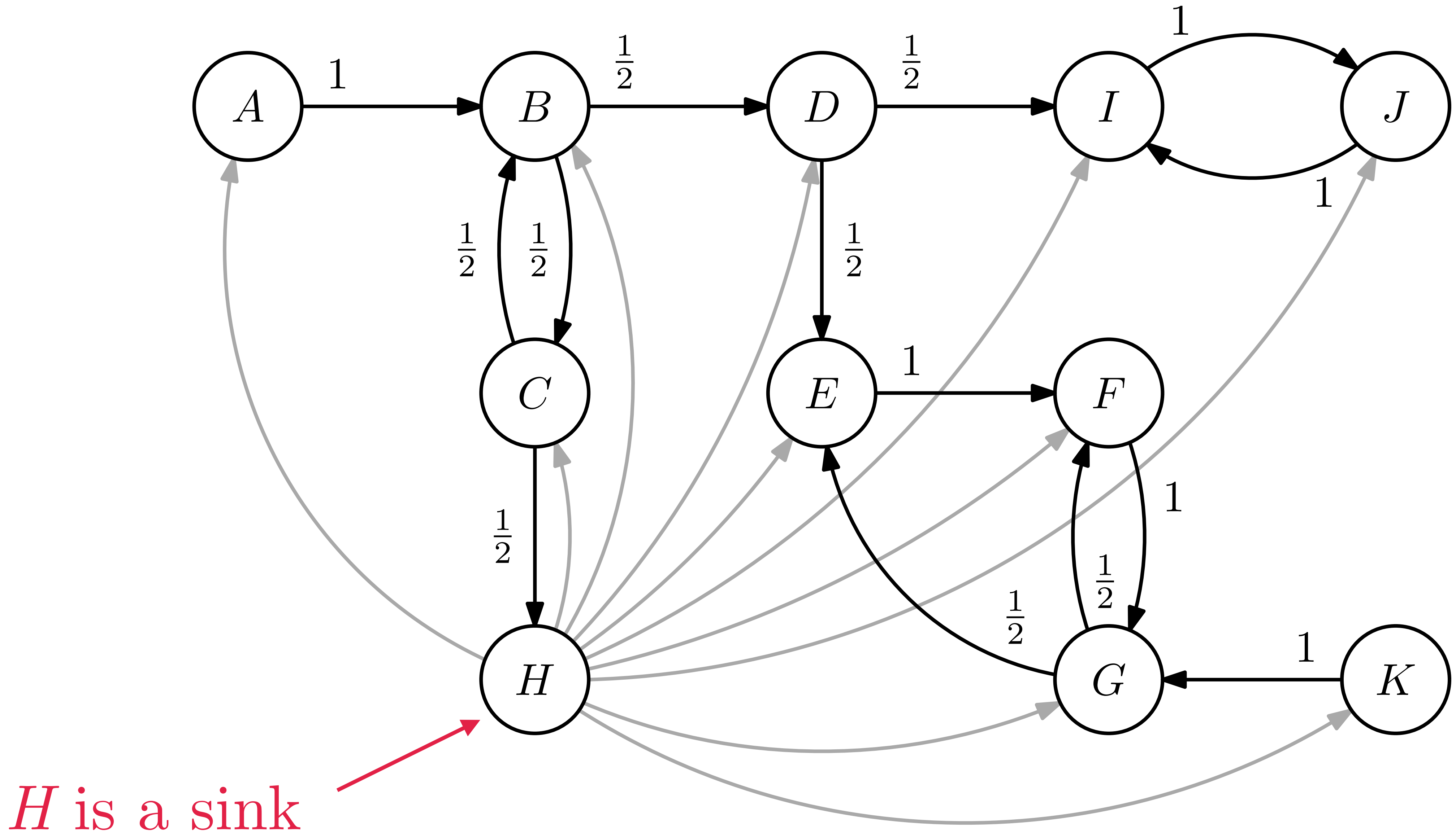


Returning to PageRank

- PageRank introduces two modifications to the chain:
  - Sinks link to all other nodes
  - There is a probability $1-\gamma$ of "teleporting" to an arbitrary node

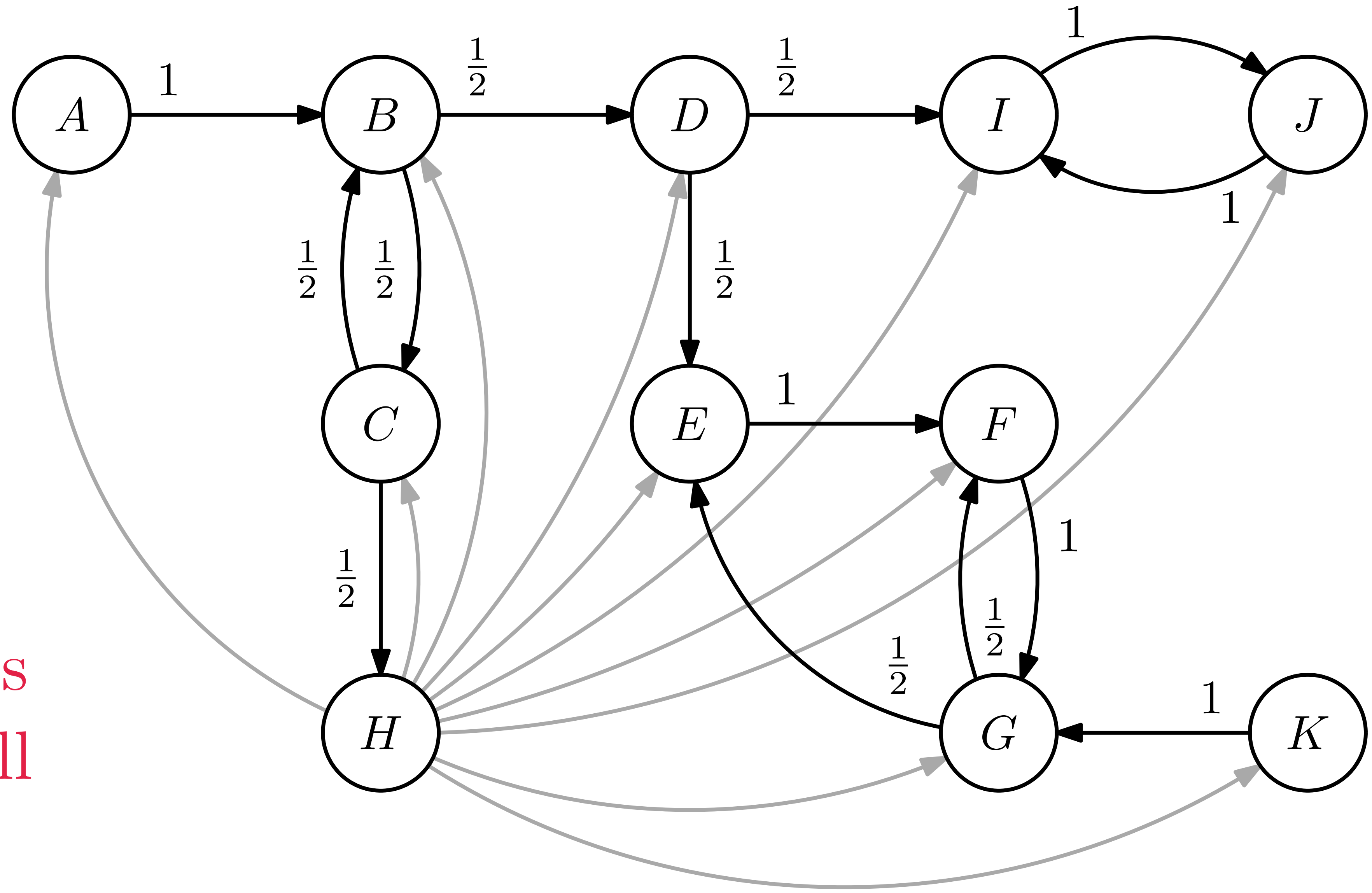
Returning to PageRank

- In our case:



Returning to PageRank

- In our case:



All nodes link to all nodes

Returning to PageRank

- Is the chain irreducible?
  - Yes
- Is the chain aperiodic?
  - Yes
- ... then there is a stationary distribution

Returning to PageRank