

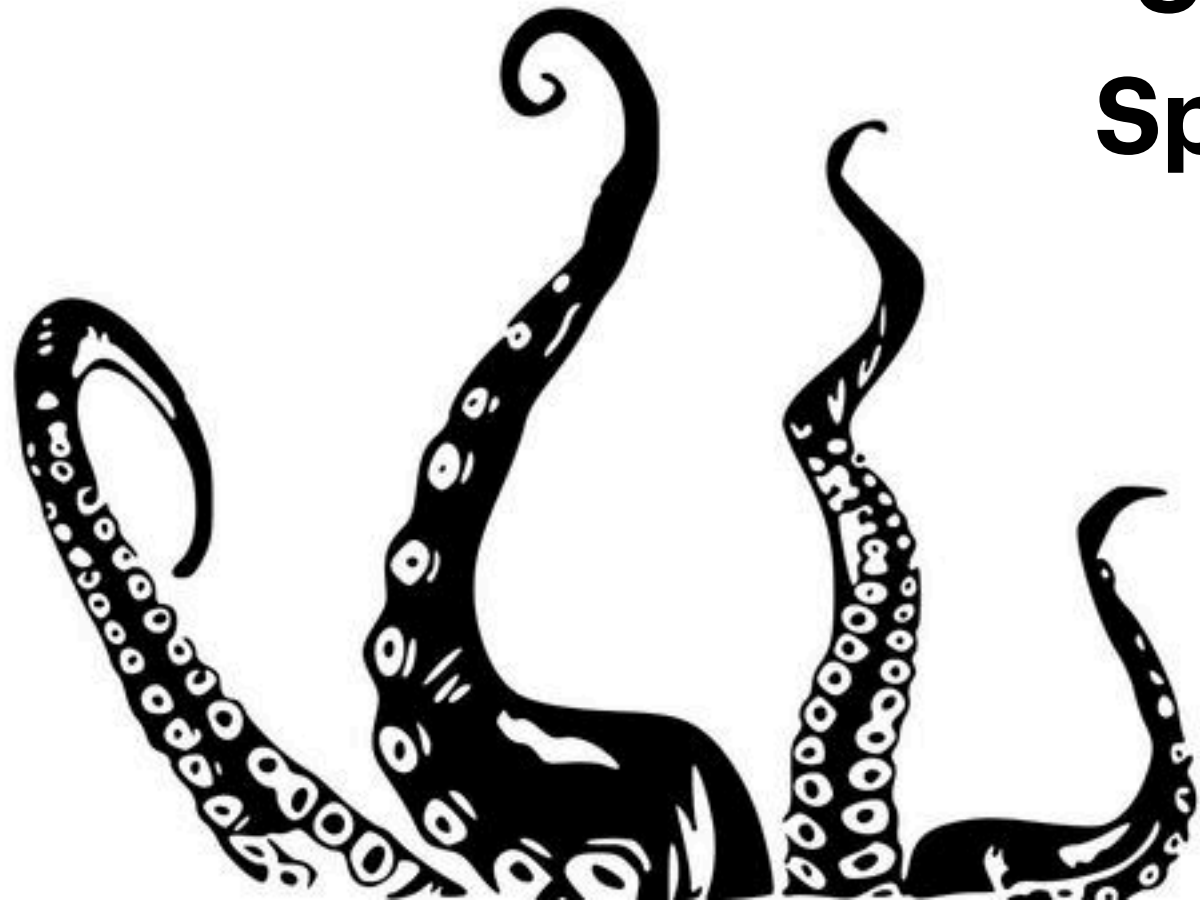
# Mathematics for Computer Graphics I

**Scalar Fields, Vector Spaces, Affine Spaces, Euclidean  
Spaces, Dot Product, Cross Product, Coordinate Frames**

# Mathematics for Computer Graphics I

**Scalar Fields, Vector Spaces, Affine Spaces, Euclidean  
Spaces, Dot Product, Cross Product, Coordinate Frames**

Carlos Martinho 2025



# Scalar Fields

**We need to represent size**

$$\forall \alpha, \beta, \gamma \in S$$

commutative:  $\alpha + \beta = \beta + \alpha$

$$\alpha \cdot \beta = \beta \cdot \alpha$$

associative:  $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$$

distributive:  $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$

identity:  $\alpha + 0 = 0 + \alpha = \alpha$

$$\alpha \cdot 1 = 1 \cdot \alpha = \alpha$$

inverse:  $\alpha + (-\alpha) = 0$

$$\alpha \cdot (\alpha^{-1}) = 1$$

# Vector Spaces

We need to represent movement

$$\forall \alpha, \beta \in S, \forall u, v, w \in V$$

$$u + v \in V$$

$$u + v = v + u$$

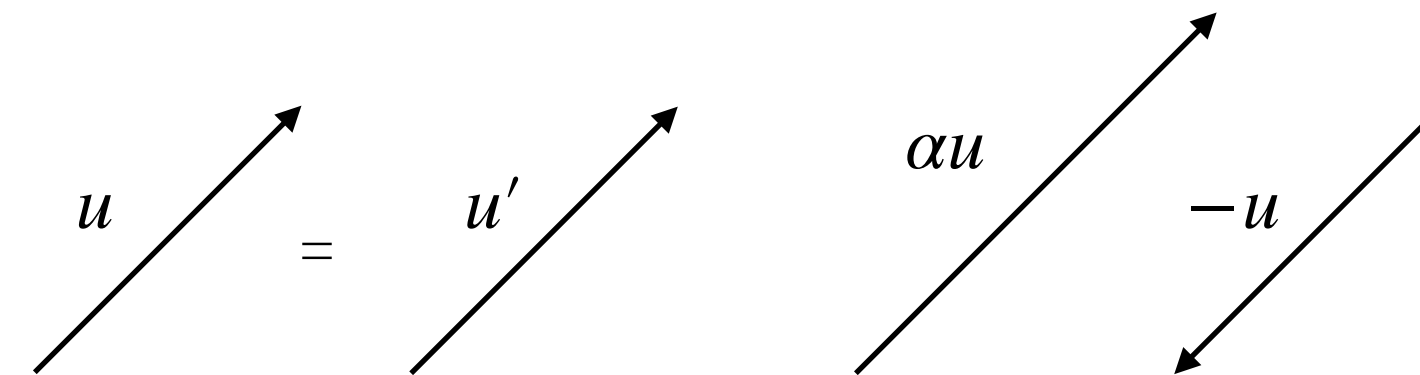
$$u + (v + w) = (u + v) + w$$

$$u + 0 = u$$

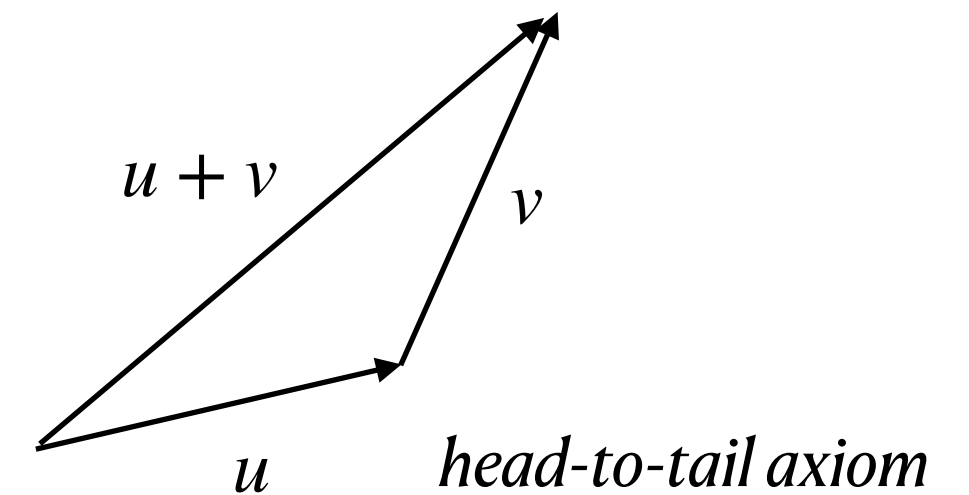
$$u + (-u) = 0$$

$$\alpha(u + v) = \alpha u + \alpha v$$

$$(\alpha + \beta)u = \alpha u + \beta u$$



Real numbers and geometric vectors



$$u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$$

$$v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$$

$$u + v = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$\alpha v = (\alpha v_1, \alpha v_2, \dots, \alpha v_n)$$

n-tuples of scalars (real or complex numbers)

# Vector Spaces

## Linear combination of vectors

If the only set of vectors  $v_i$  such as  $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$  is  $\alpha_1 = \dots = \alpha_n = 0$ , then the vectors are **linearly independent**.

The greatest number of independent vectors defines the dimension of the space.

Any set of  $n$  linearly independent vectors  $v_i$  or  $v'_i$  form a **basis** for a space of dimension  $n$ , and vectors can be expressed as  $v = \beta_1 v_1 + \dots + \beta_n v_n$  or  $v = \beta'_1 v'_1 + \dots + \beta'_n v'_n$ .

There is a  $n \times n$  matrix  $M_{n \times n}$  such as 
$$\begin{bmatrix} \beta'_1 \\ \vdots \\ \beta'_n \end{bmatrix} = M_{n \times n} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

# Affine Spaces

We need to represent position (points)

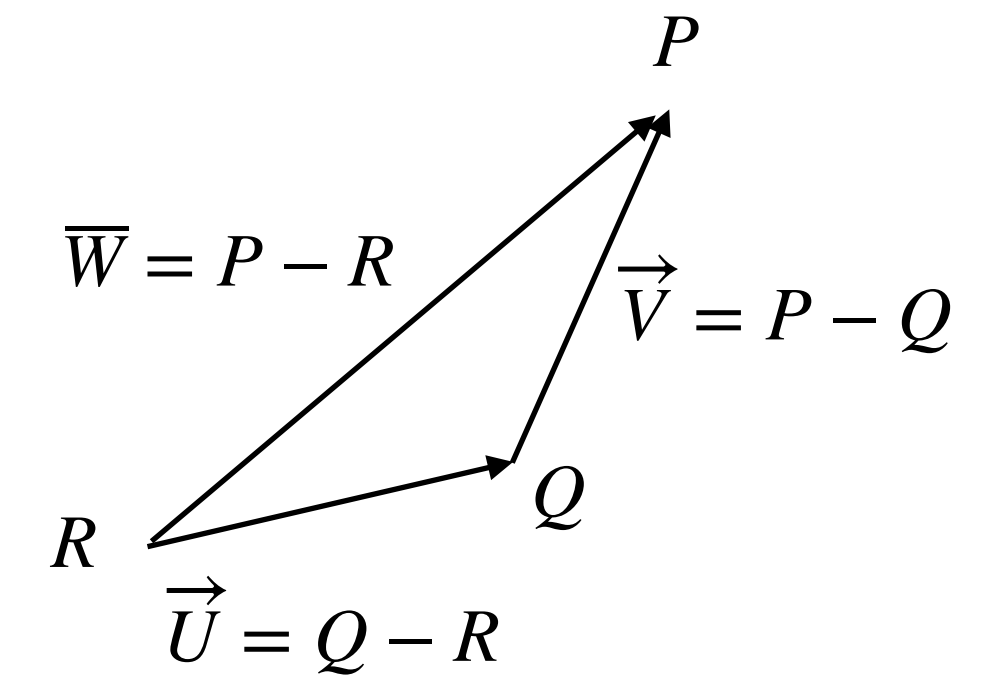
Vectors as point-point subtraction:  $\vec{V} = P - Q, P = Q + \vec{V}$

Parametric form:  $P = Q + \alpha V$

Affine spaces allow the creation of a **frame**: an origin  $P_0$  and a set of  $n$  vectors  $v_1, \dots, v_n$  defining a basis.

vector  $v = \alpha_1 v_1 + \dots + \alpha_n v_n$ , represented as  $(\alpha_1, \dots, \alpha_n)$

point  $P = P_0 + \beta_1 v_1 + \dots + \beta_n v_n$ , represented as  $(\beta_1, \dots, \beta_n)$



$$(P - Q) + (Q - R) = (P - R)$$

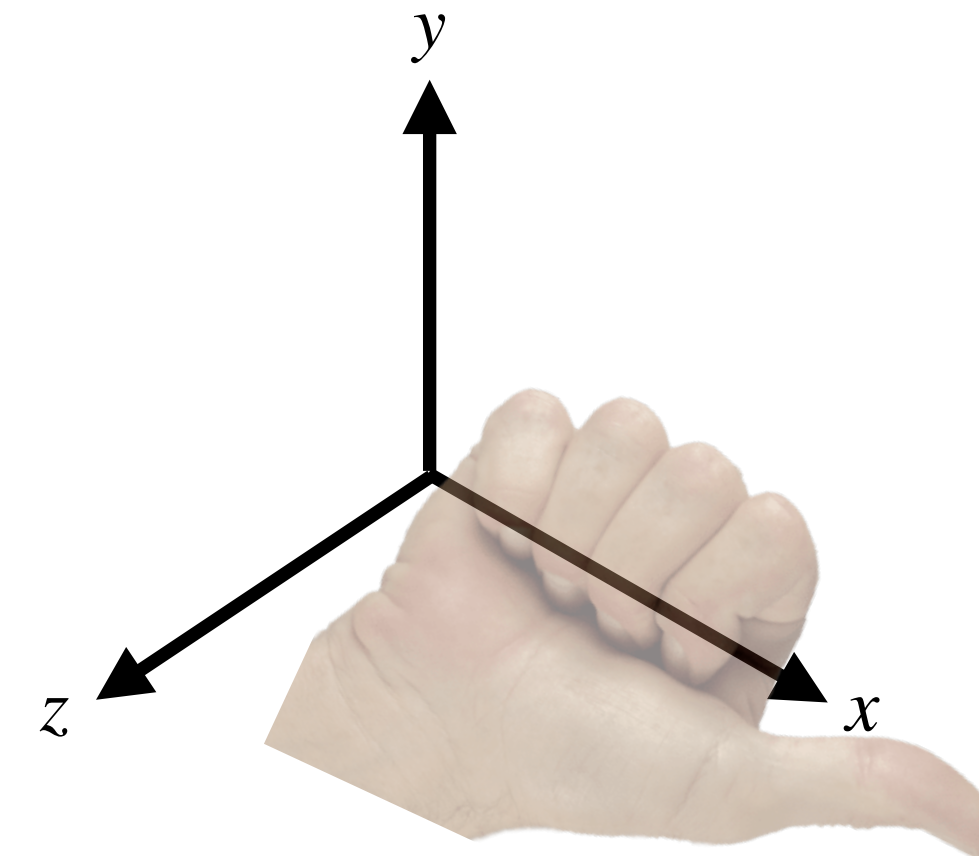
# Euclidean Spaces

We need to represent distance and angle

We will work in  $\mathbb{R}^3$  and use **right-handed coordinates**.

**distance** = magnitude of a vector  
between two points

**angle** = angle between two vectors  
“starting” on a same point.

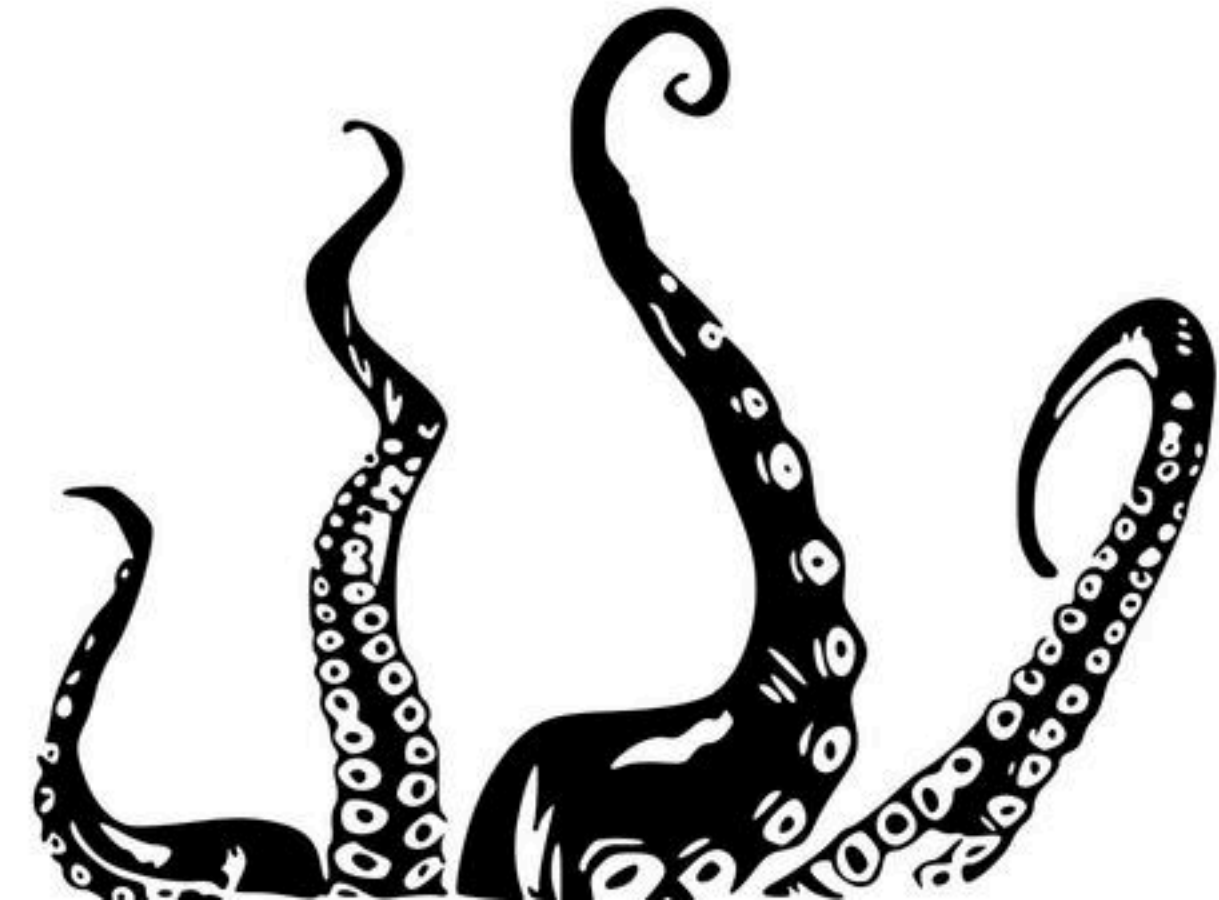
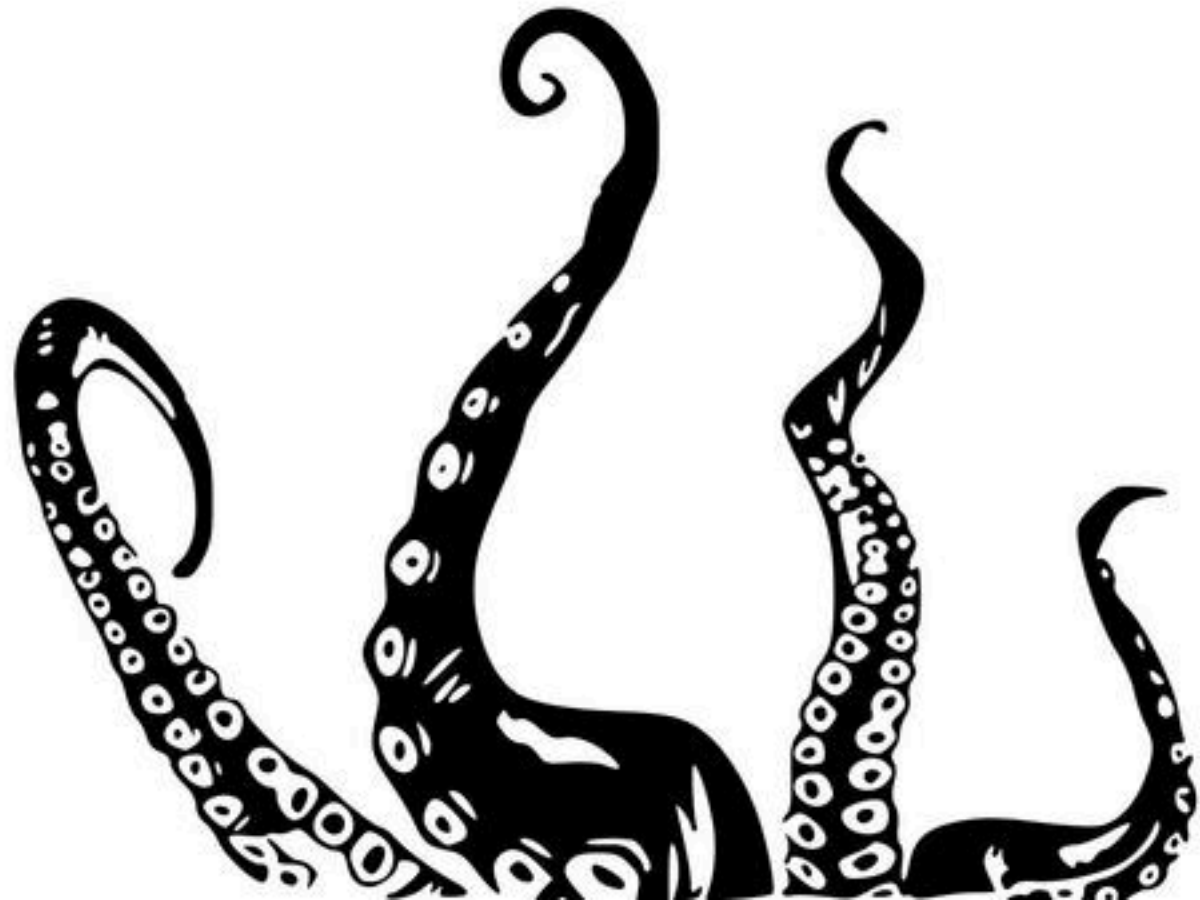


The **cartesian** system is an euclidean space with basis:

$$\vec{X} = (1,0,0), \vec{Y} = (0,1,0), \vec{Z} = (0,0,1)$$

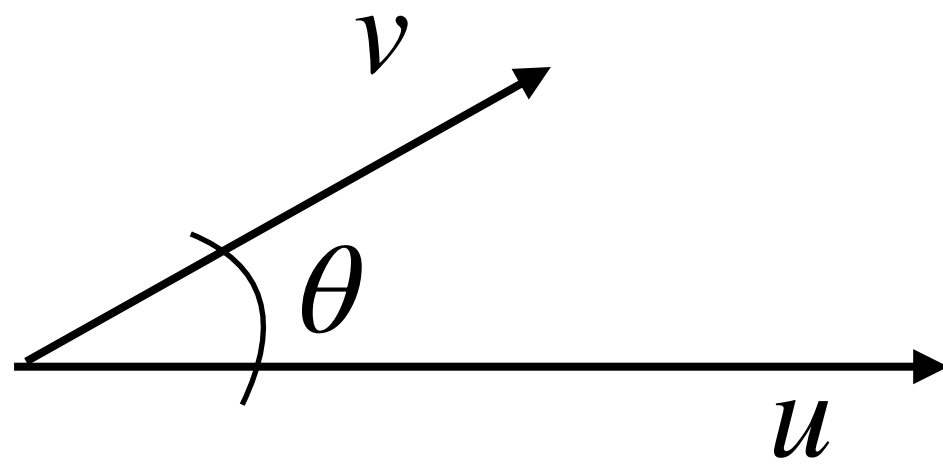
# Dot Product

a.k.a. inner product, scalar product



# Dot Product

**What about vector-vector multiplication?**



$$u \cdot v = ?$$

$$u \cdot v = |u| |v| \cos \theta$$

$$\text{if } |u| = |v| = 1, \text{ then } u \cdot v = \cos \theta$$

$$u \cdot v = v \cdot u$$

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w$$

$$u \cdot (v + w) = u \cdot v + u \cdot w$$

$$(\alpha u) \cdot v = \alpha(u \cdot v) = u \cdot (\alpha v)$$

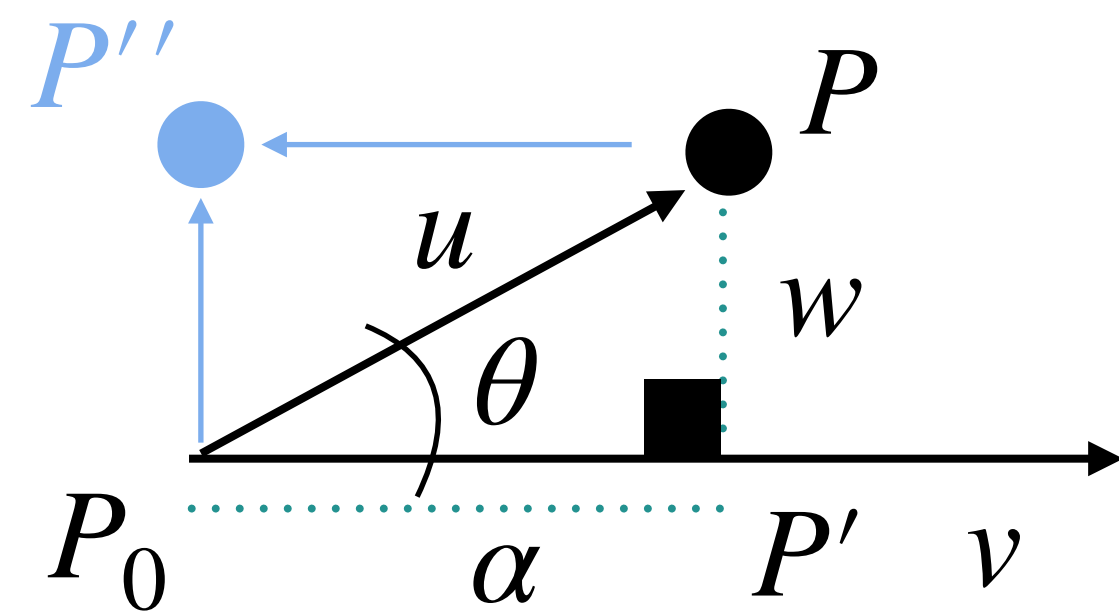
$$v \cdot v > 0 \text{ if } v \neq 0, 0 \cdot v = 0$$

$$|v| = \sqrt{v \cdot v}$$

$$\text{if } u \cdot v = 0 \text{ and } u, v \neq 0, \text{ then } u \perp v$$

# Dot Product

## Examples of use in CG



Angle between  $u$  and  $v$  :

$$u \cdot v = |u| |v| \cos \theta$$

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u| |v|}\right)$$

Shortest distance to a line from a point :

$$u = \alpha v + w \text{ and } v \cdot w = 0$$

$$u \cdot v = (\alpha v + w) \cdot v$$

$$u \cdot v = \alpha v \cdot v + w \cdot v = \alpha v \cdot v$$

$$\alpha = \frac{u \cdot v}{v \cdot v}$$

where  $\alpha v$  is the projection of  $u$  on  $v$

# Dot Product

**Efficient to compute in Cartesian coordinates.**

basis:  $\vec{X} = (1,0)$ ,  $\vec{Y} = (0,1)$

vectors:  $\vec{A} = (\alpha_x, \alpha_y) = \alpha_x \vec{X} + \alpha_y \vec{Y}$ ,  $\vec{B} = (\beta_x, \beta_y) = \beta_x \vec{X} + \beta_y \vec{Y}$

$$A \cdot B = (\alpha_x \vec{X} + \alpha_y \vec{Y}) \cdot (\beta_x \vec{X} + \beta_y \vec{Y})$$

$$A \cdot B = \alpha_x \beta_x (\vec{X} \cdot \vec{X}) + \alpha_y \beta_y (\vec{Y} \cdot \vec{Y}) + (\alpha_x \beta_y + \alpha_y \beta_x) (\vec{X} \cdot \vec{Y})$$

$$A \cdot B = \alpha_x \beta_x + \alpha_y \beta_y$$

$$A = \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} \quad B = \begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix} \quad A \cdot B = \begin{bmatrix} \alpha_x & \alpha_y \end{bmatrix} \begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix} = A^T B$$

# Quadrance

$$Q(v) = v_x^2 + v_y^2 + v_z^2, \text{ where } v = (v_x, v_y, v_z)$$

$$Q(v) + Q(w) = Q(w - v), \text{ where } v = (v_x, v_y, v_z) \text{ and } w = (w_x, w_y, w_z)$$

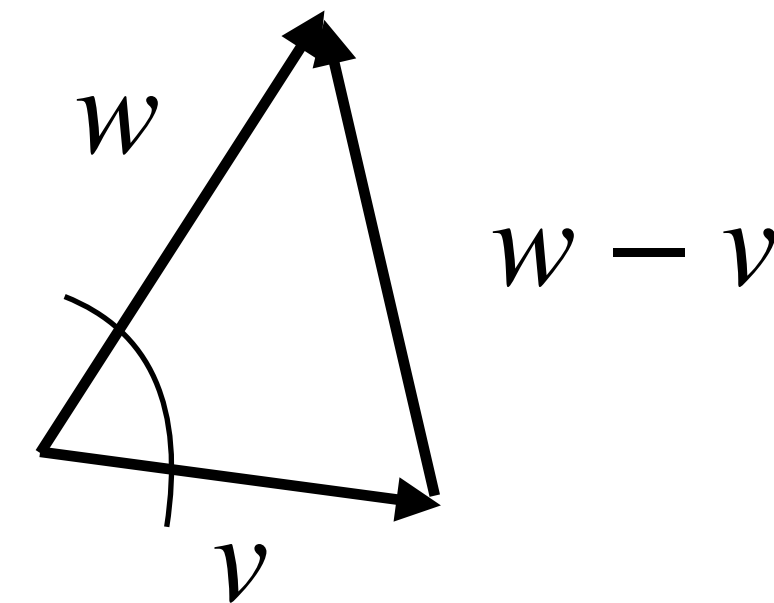
$$v_x^2 + v_y^2 + v_z^2 + w_x^2 + w_y^2 + w_z^2 = (w_x - v_x)^2 + (w_y - v_y)^2 + (w_z - v_z)^2$$

$$v_x^2 + v_y^2 + v_z^2 + w_x^2 + w_y^2 + w_z^2 = (w_x^2 - 2w_xv_x + v_x^2) + (w_y^2 - 2w_yv_y + v_y^2) + (w_z^2 - 2w_zv_z + v_z^2)$$

$$0 = -2w_xv_x - 2w_yv_y - 2w_zv_z$$

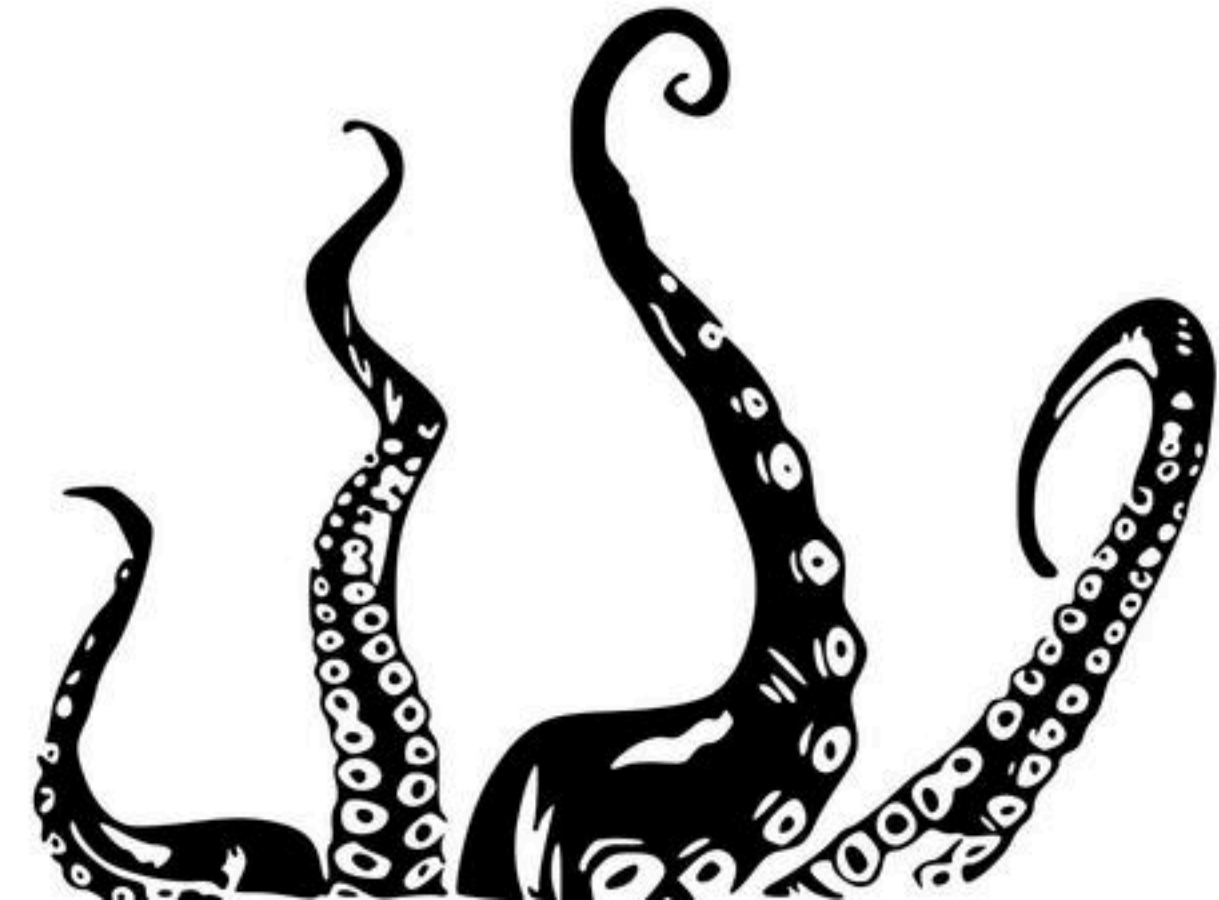
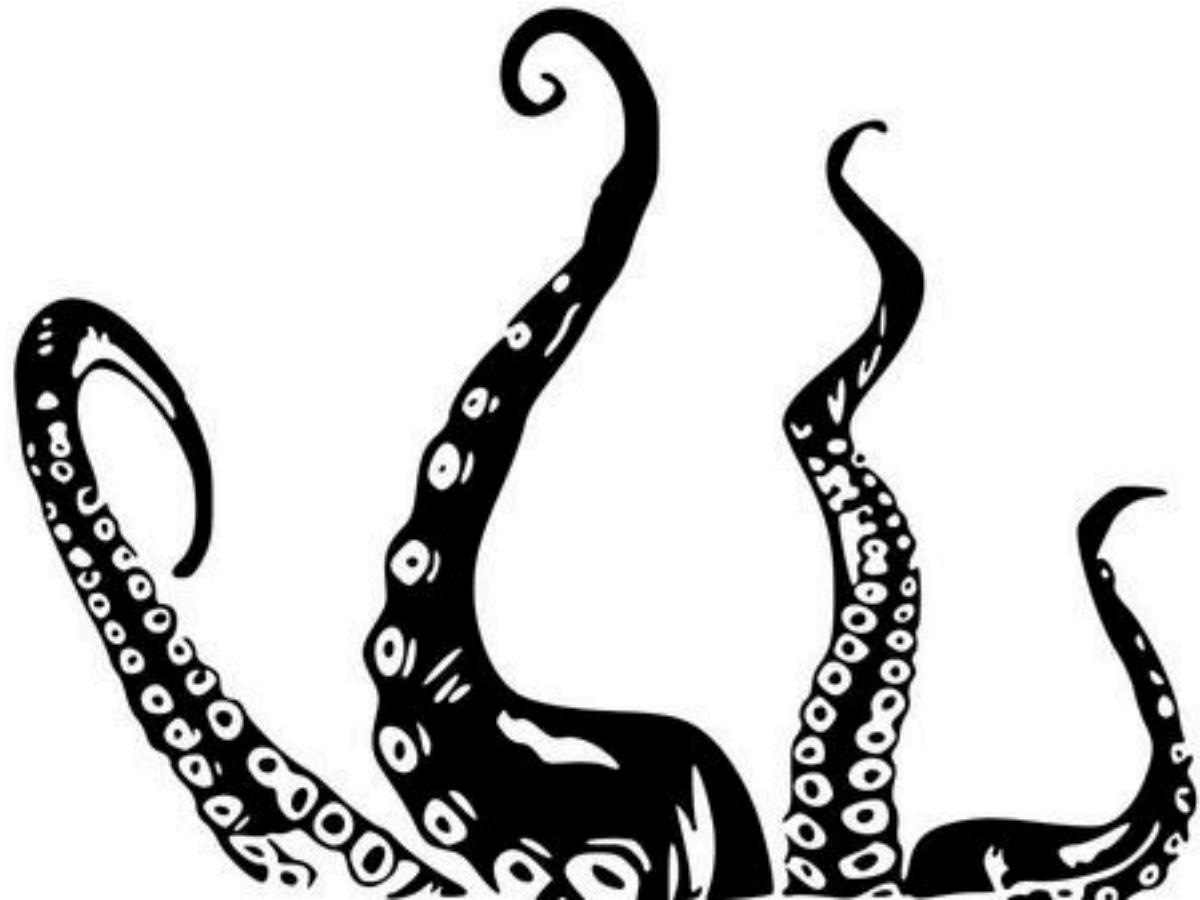
$$0 = w_xv_x + w_yv_y + w_zv_z$$

$$0 = w \cdot v$$



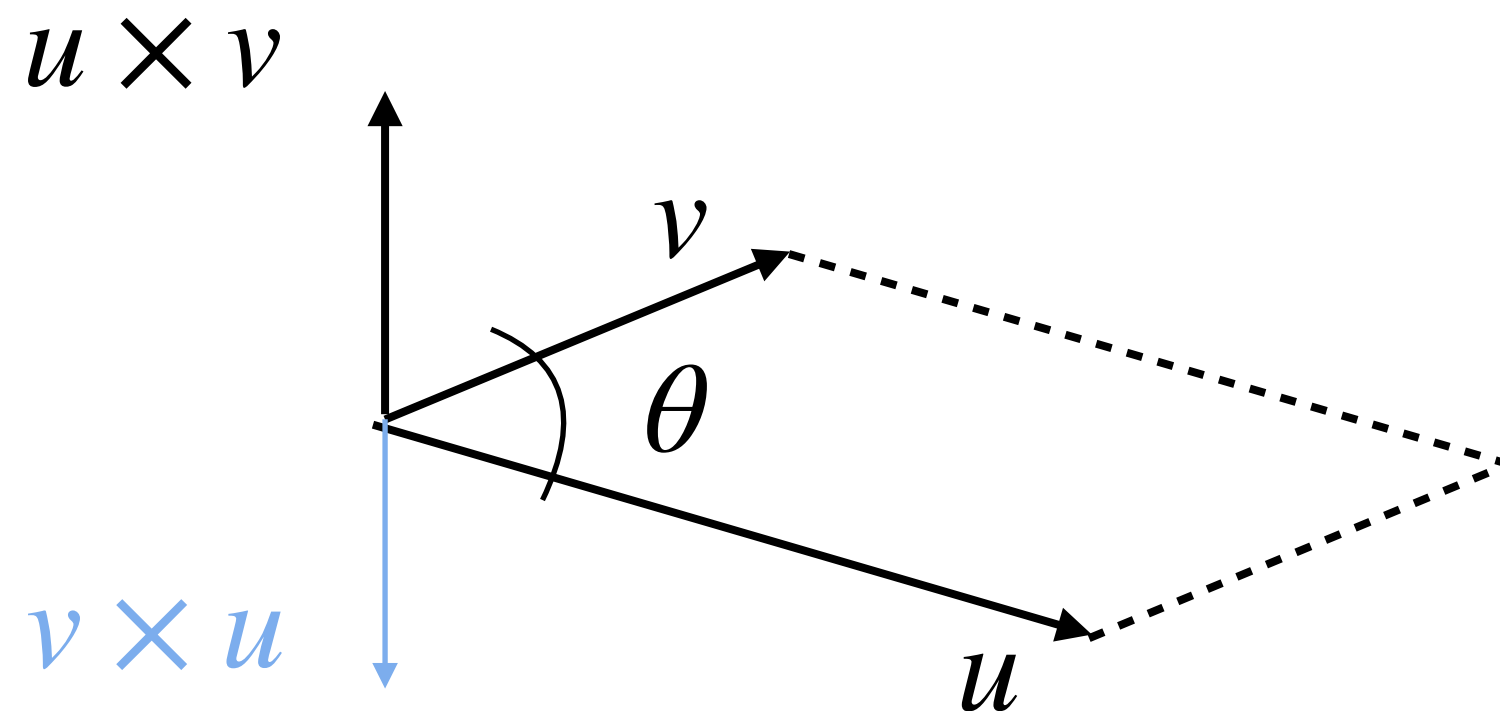
# Cross Product

**a.k.a. outer product, vector product**



# Cross Product

What about vector-vector multiplication?



$$|u \times v| = |u| |v| \sin \theta$$

$u \times v \perp u, v$  (right-hand rule)

$$u \times w = -w \times u \text{ (not commutative)}$$

$$u \times u = 0$$

$$u \times (v + w) = u \times v + u \times w$$

$$u \times (\alpha v) = \alpha(u \times v) = (\alpha u) \times v$$

In cartesian 3d space:

$$x \times y = z \quad y \times z = x \quad z \times x = y$$

$$y \times x = -z \quad z \times y = -x \quad x \times z = -y$$

# Cross Product

in Cartesian coordinates.

basis:  $\vec{X} = (1,0,0)$ ,  $\vec{Y} = (0,1,0)$ ,  $\vec{Z} = (0,0,1)$

vectors:  $\vec{A} = (\alpha_x, \alpha_y, \alpha_z) = \alpha_x \vec{X} + \alpha_y \vec{Y} + \alpha_z \vec{Z}$ ,  $\vec{B} = (\beta_x, \beta_y, \beta_z) = \beta_x \vec{X} + \beta_y \vec{Y} + \beta_z \vec{Z}$

$$\vec{A} \times \vec{B} = (\alpha_x \vec{X} + \alpha_y \vec{Y} + \alpha_z \vec{Z}) \times (\beta_x \vec{X} + \beta_y \vec{Y} + \beta_z \vec{Z})$$

$$\vec{A} \times \vec{B} = \alpha_x \beta_x (\vec{X} \times \vec{X}) + \alpha_x \beta_y (\vec{X} \times \vec{Y}) + \alpha_x \beta_z (\vec{X} \times \vec{Z})$$

$$+ \alpha_y \beta_x (\vec{Y} \times \vec{X}) + \alpha_y \beta_y (\vec{Y} \times \vec{Y}) + \alpha_y \beta_z (\vec{Y} \times \vec{Z})$$

$$+ \alpha_z \beta_x (\vec{Z} \times \vec{X}) + \alpha_z \beta_y (\vec{Z} \times \vec{Y}) + \alpha_z \beta_z (\vec{Z} \times \vec{Z})$$

$$\vec{A} \times \vec{B} = \alpha_x \beta_y \vec{Z} - \alpha_x \beta_z \vec{Y} - \alpha_y \beta_x \vec{Z} + \alpha_y \beta_z \vec{X} + \alpha_z \beta_x \vec{Y} - \alpha_z \beta_y \vec{X}$$

$$\vec{A} \times \vec{B} = (\alpha_y \beta_z - \alpha_z \beta_y) \vec{X} + (\alpha_z \beta_x - \alpha_x \beta_z) \vec{Y} + (\alpha_x \beta_y - \alpha_y \beta_x) \vec{Z}$$

# Cross Product

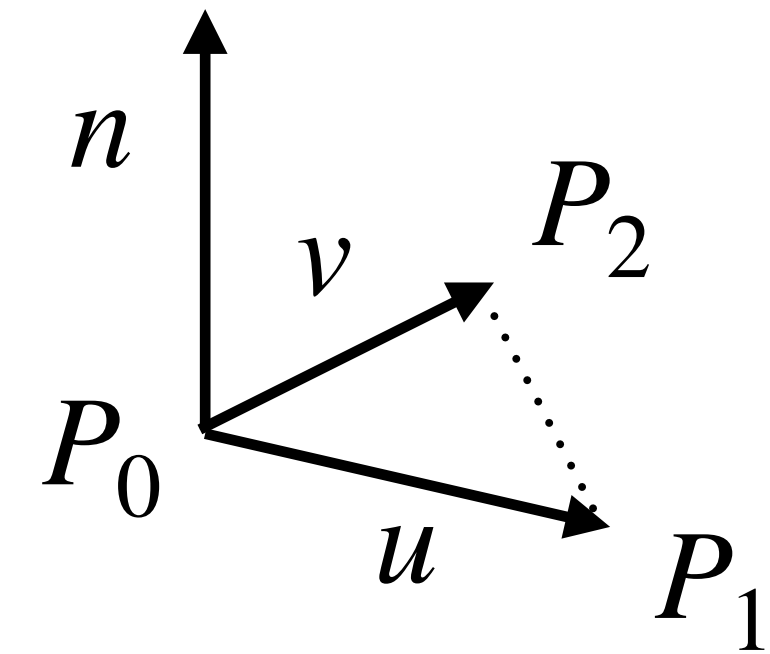
## Determinant and Matrix form

Determinant form:

$$a \times b = \begin{vmatrix} \vec{X} & \vec{Y} & \vec{Z} \\ \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \end{vmatrix} = \begin{bmatrix} \alpha_y \beta_z - \alpha_z \beta_y \\ \alpha_z \beta_x - \alpha_x \beta_z \\ \alpha_x \beta_y - \alpha_y \beta_x \end{bmatrix}$$

Dual Matrix:

$$a \times b = A^* b = \begin{bmatrix} 0 & -\alpha_z & \alpha_y \\ \alpha_z & 0 & -\alpha_x \\ -\alpha_y & \alpha_x & 0 \end{bmatrix} \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix}$$



Plane:

$$n \cdot (P - P_0) = 0$$

where  $n = u \times v$

and  $P, P_0$  are points on the plane

# Coordinate Frames

## Example of dot/cross product use in Computer Graphics

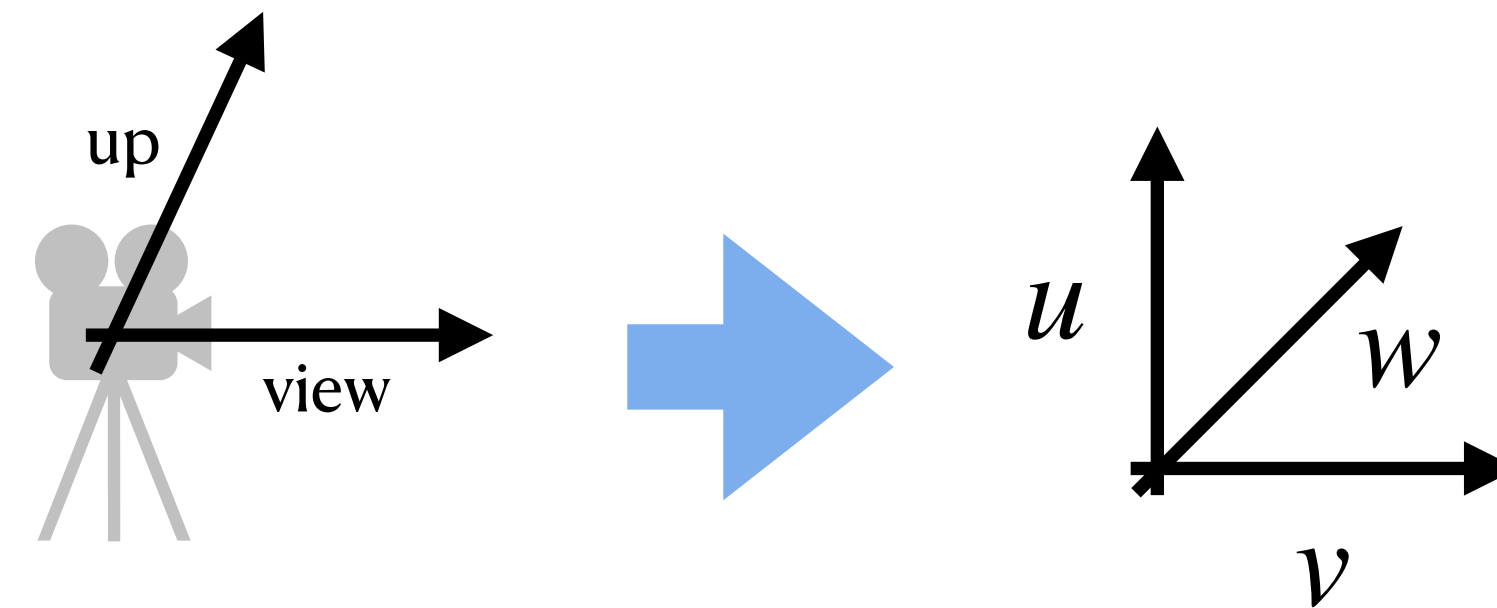
$(u, v, w)$  is a coordinate frame if:

- $|u| = |v| = |w| = 1$
- $u \cdot v = v \cdot w = u \cdot w = 0$
- $w = u \times v$

a point  $P$  can be represented by:

$$P = (P \cdot u)u + (P \cdot v)v + (P \cdot w)w$$

Create a coordinate frame for the **camera** from *view* and *up* vectors.

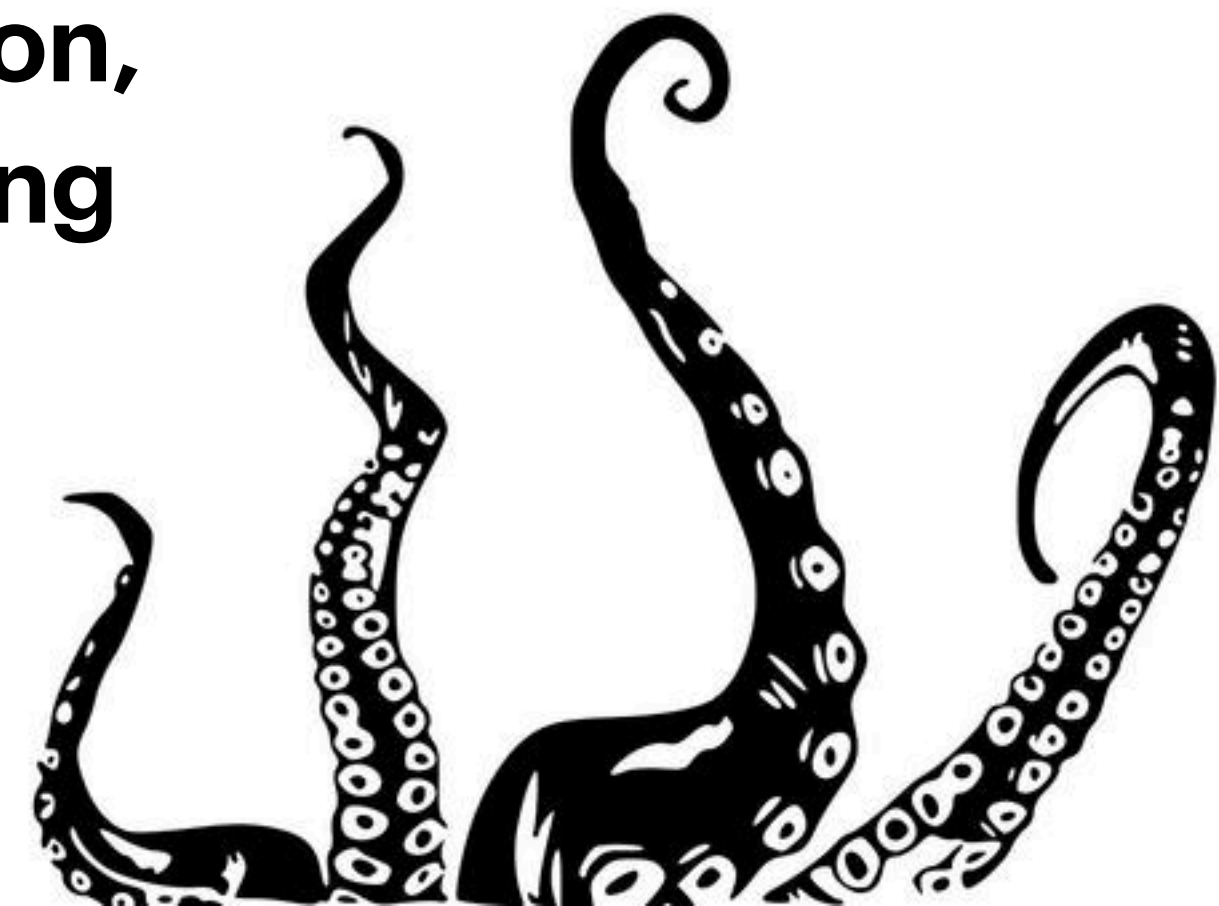
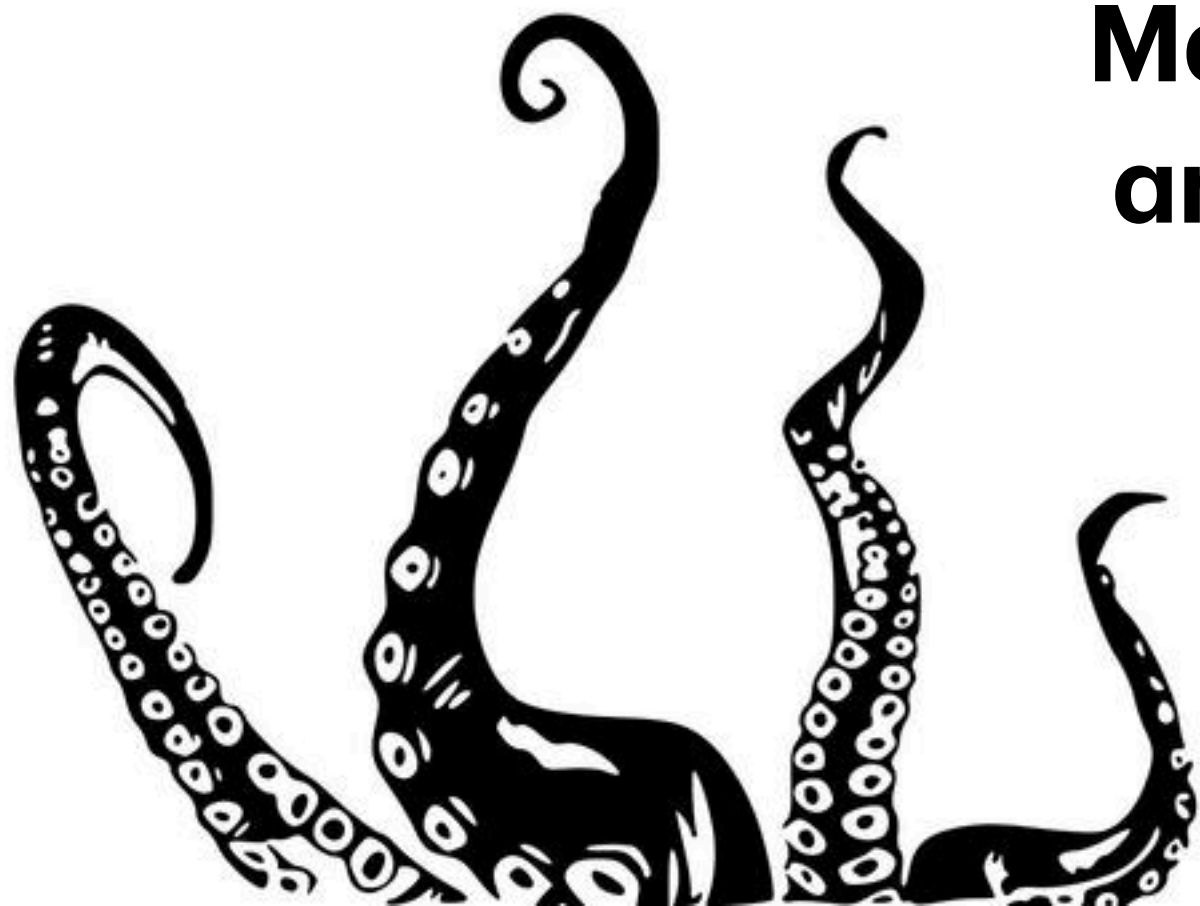


$$v = \frac{\text{view}}{|\text{view}|}, w = \frac{\text{up} \times v}{|\text{up} \times v|}, u = v \times w$$

# Mathematics for Computer Graphics II

**Matrices, Basic Transformations: Scale, Shear, Rotation,  
and Translation, Homogenous Coordinates, Combining  
Transformations, Transformation of Normals**

Carlos Martinho 2025



# Matrices

## Introduction

All transformations in the rendering pipeline are handled by matrices.

Matrices transform points (multiplying a matrix by a vector).

Matrices are arrays of numbers (vectors are 1-column matrices).

Matrix  $A_{m,n}$  ( $m$  rows and  $n$  columns) of elements  $a_{i,j}$  (element at row  $i$ , column  $j$ ):

$$A_{m,n} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

# Matrices

## Addition and Multiplication

Addition of matrices:

$$A_{m,n} + B_{m,n} = [a_{i,j}] + [b_{i,j}] = [a_{i,j} + b_{i,j}]$$

Multiplication by a scalar:

$$\alpha A_{m,n} = \alpha[a_{i,j}] = [\alpha a_{i,j}]$$

Multiplication by another matrix:

$$\begin{bmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 27 & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & 12 \end{bmatrix}$$

$m \times n$        $n \times k$

$m \times k$

$$\begin{bmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{bmatrix} = ?$$

# Matrices

## Properties

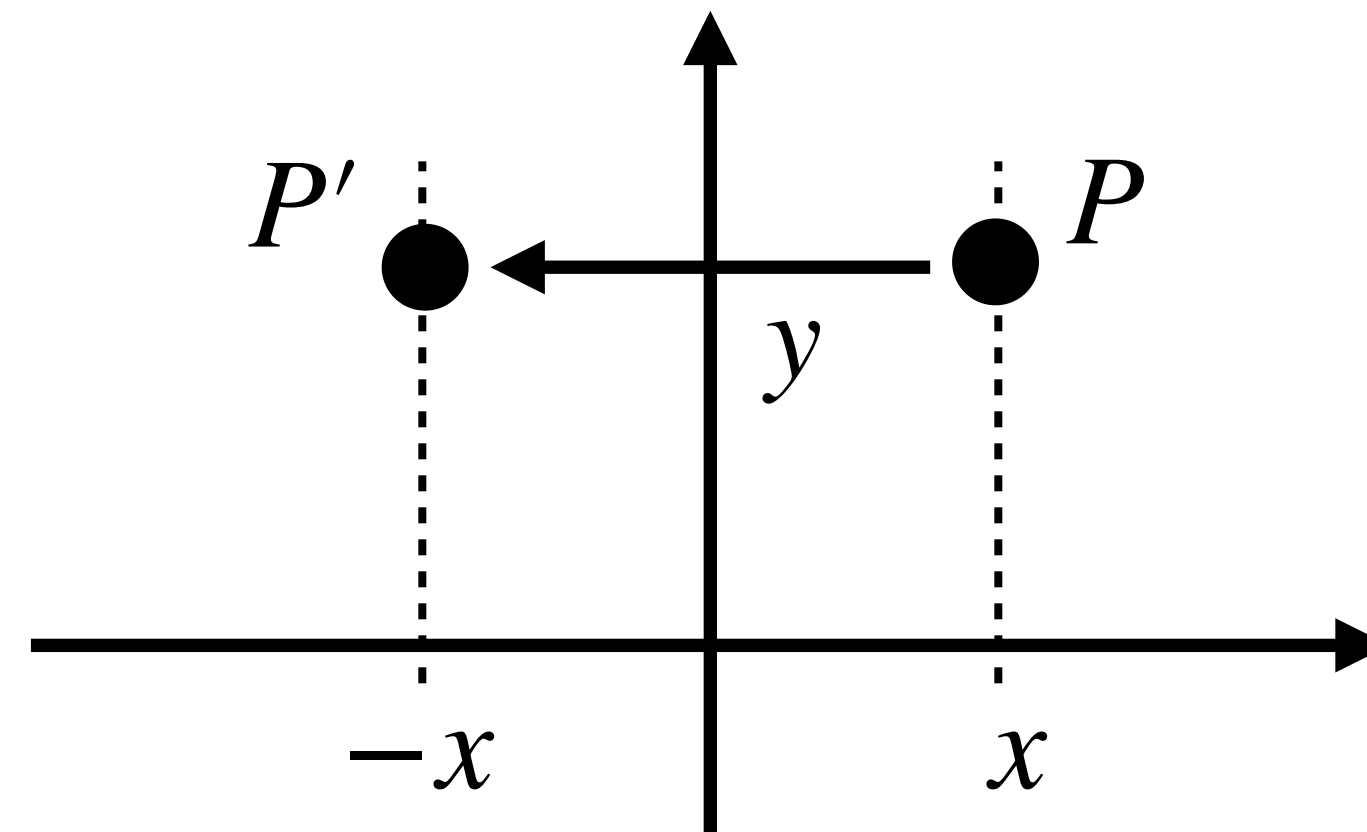
Not commutative:  $AB \neq BA$

Associative:  $(A + B) + C = A + (B + C)$  and  $(AB)C = A(BC)$

Distributive:  $A(B + C) = AB + AC$  and  $(A + B)C = AC + BC$

Matrix-Vector multiplication:  $P' = MP$

$$\begin{matrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} -x \\ y \end{bmatrix} \\ M & P & & P' \end{matrix}$$



# Matrices

## Transpose and Identity

Transpose:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \quad (AB)^T = B^T A^T$$

$m \times n \quad n \times k \quad k \times n \quad n \times m$

Additive identity:

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiplicative identity (identity matrix):

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad AA^{-1} = A^{-1}A = I, (AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

# Matrices

## Dot product and Cross product

$$a = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} \quad b = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix}$$

Cross Product:

$$A^* = \begin{bmatrix} 0 & -\alpha_z & \alpha_y \\ \alpha_z & 0 & -\alpha_x \\ -\alpha_y & \alpha_x & 0 \end{bmatrix}$$

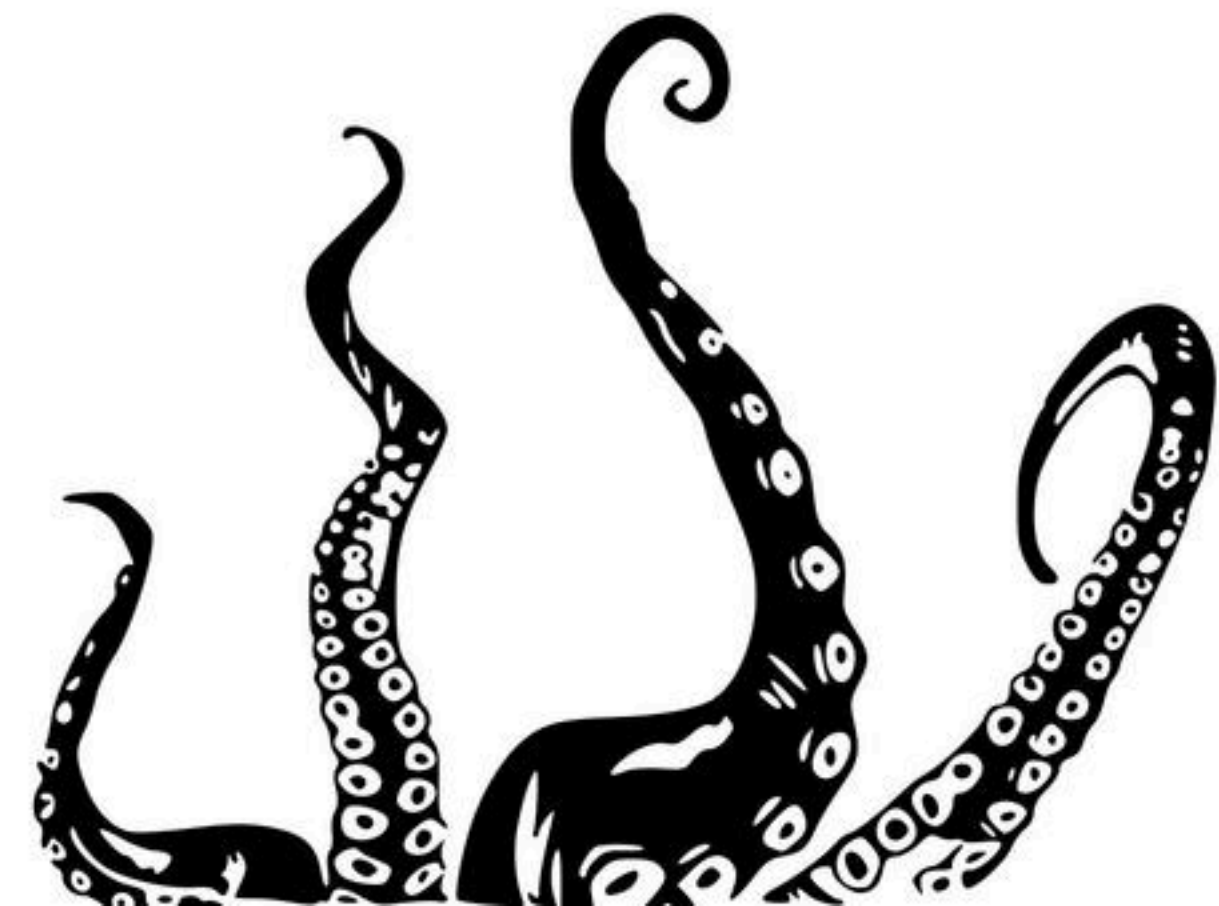
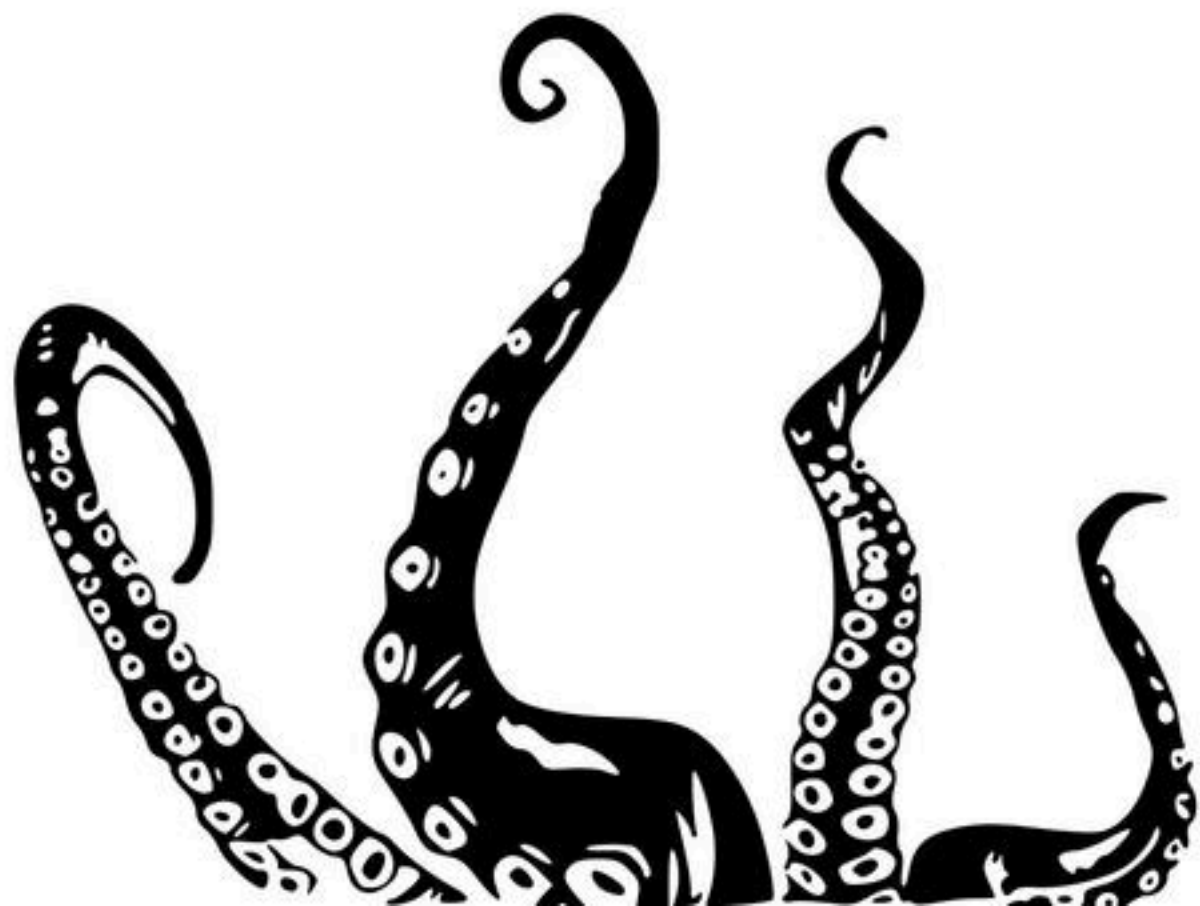
Dot Product:

$$A^T B = \begin{bmatrix} \alpha_x & \alpha_y & \alpha_z \end{bmatrix} \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix}$$

$$a \times b = A^* b = \begin{bmatrix} 0 & -\alpha_z & \alpha_y \\ \alpha_z & 0 & -\alpha_x \\ -\alpha_y & \alpha_x & 0 \end{bmatrix} \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix}$$

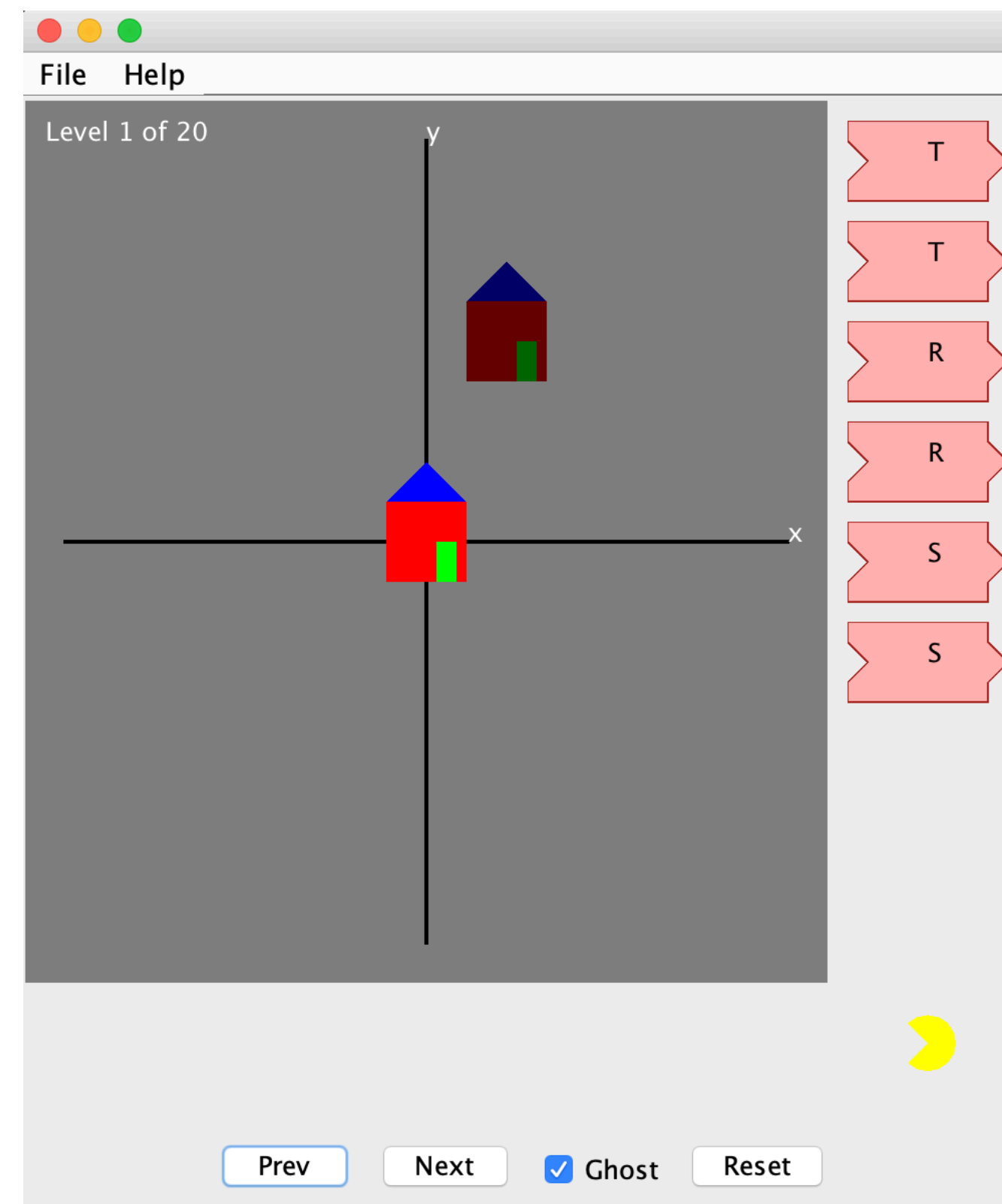
# Basic Transformations

**Scale, Shear, 2D and 3D Rotation,  
Combining Transformations**



# Transformation Game

(<http://www.cs.brown.edu/exploratories/>)

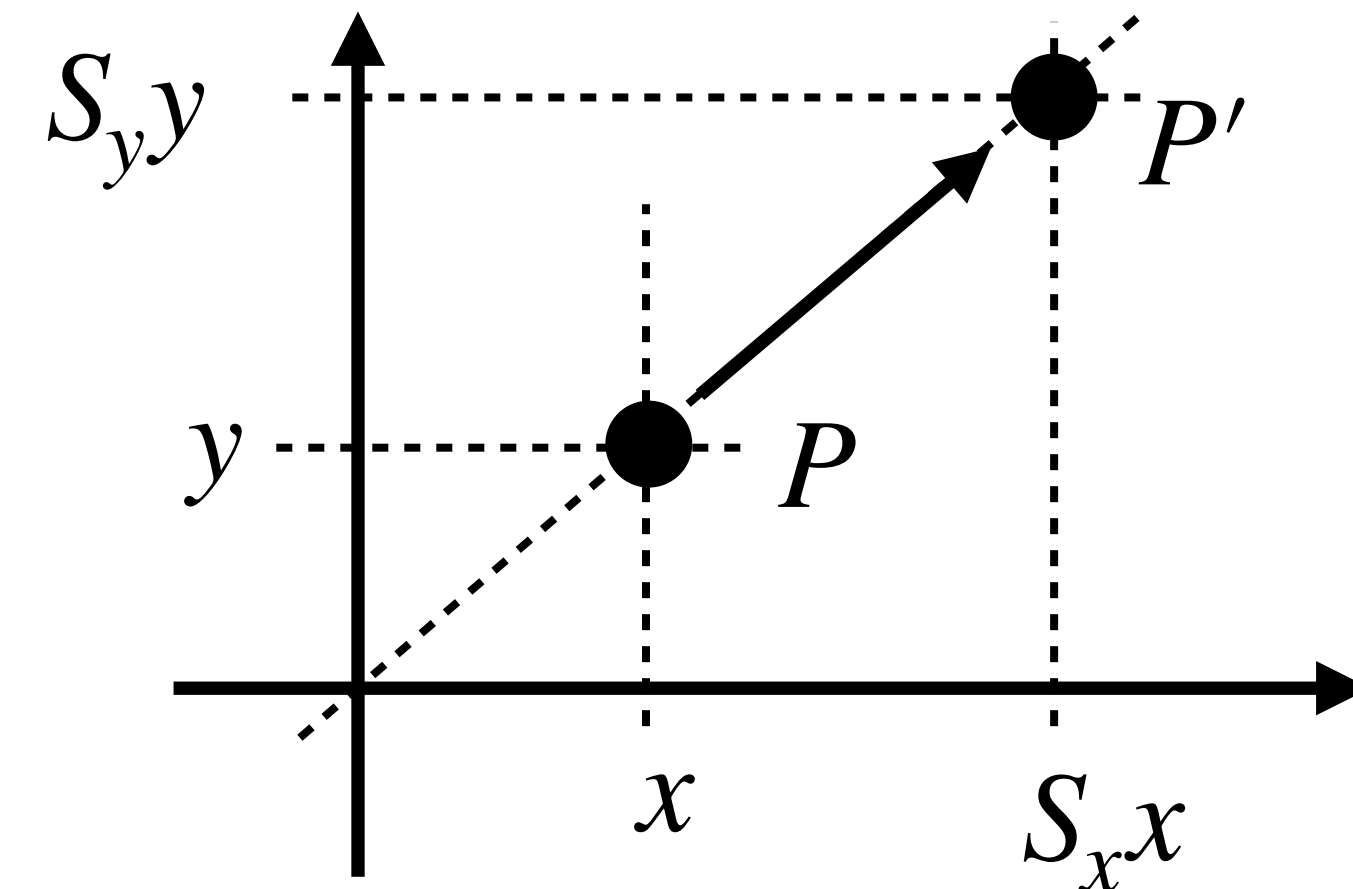


# Scale

$$S_{2D}(S_x, S_y) = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

$$S_{2D}^v = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} S_x x \\ S_y y \end{bmatrix}$$

$$S_{2D}^{-1}(S_x, S_y) = \begin{bmatrix} S_x^{-1} & 0 \\ 0 & S_y^{-1} \end{bmatrix}$$

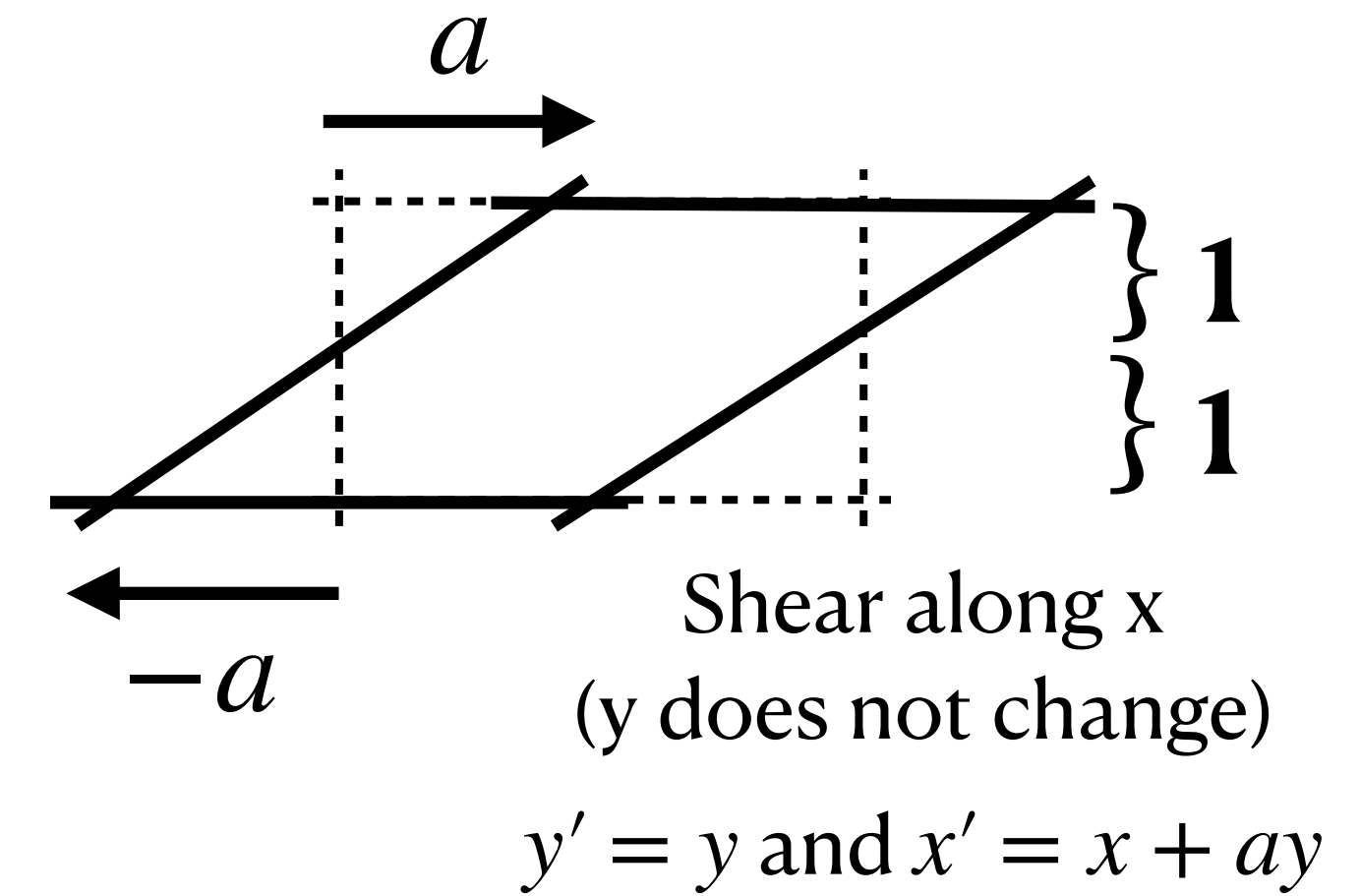


$$S_{3D} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_z \end{bmatrix}$$

# Shear

$$Sh_x(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, Sh_x v = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

$$Sh_x^{-1}(a) = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$$



$$Sh_y(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

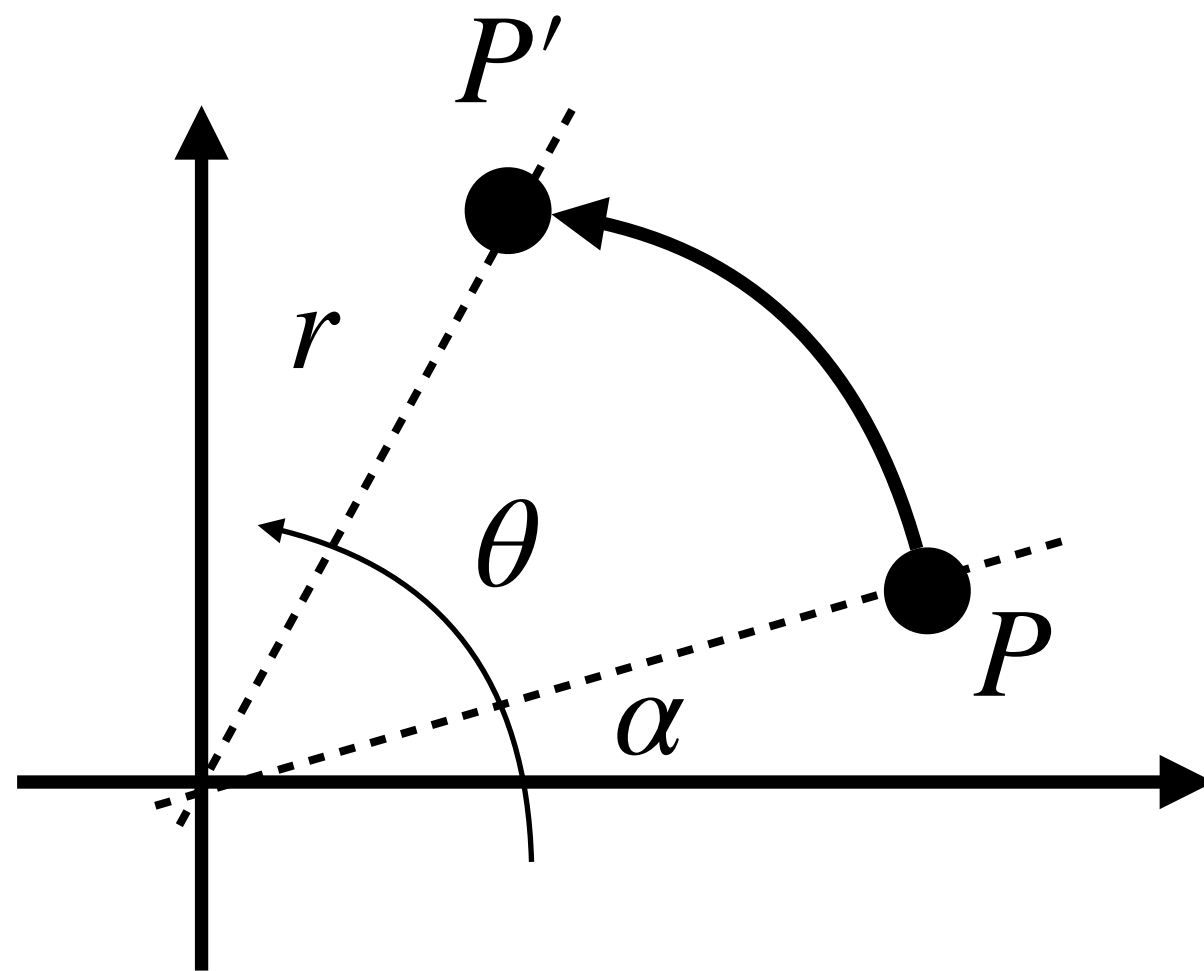
$$Sh_y v = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ax + y \end{bmatrix}$$

$$Sh_z v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{zx} & a_{zy} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ sh_{zx}x + sh_{zy}y + z \end{bmatrix}$$

etc.

# 2D Rotation

(commutative)



$$R(x + y) = R(x) + R(y)$$

$$P = (P_x, P_y) = (r \cos \alpha, r \sin \alpha)$$

$$P' = (P'_x, P'_y) = (r \cos(\alpha + \theta), r \sin(\alpha + \theta))$$

Trigonometric identities:

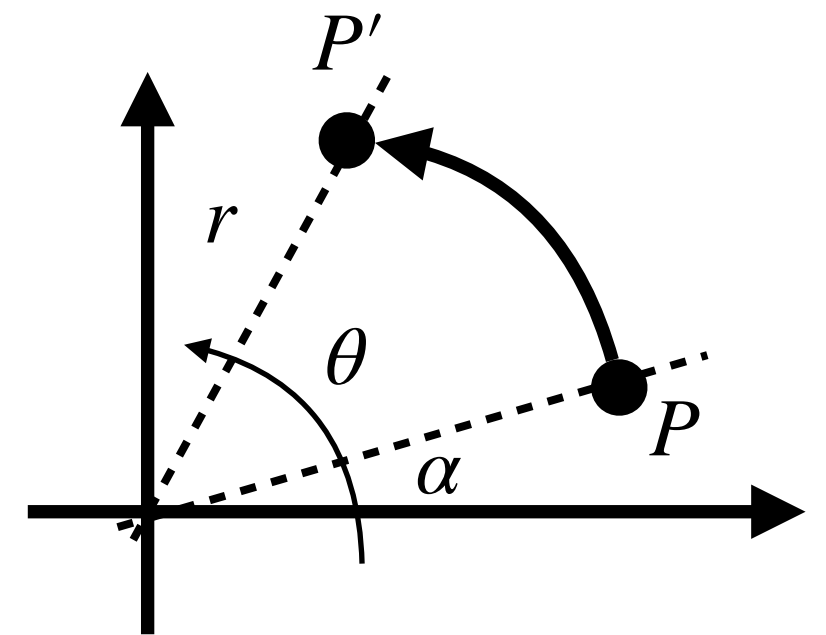
$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

# 2D Rotation

$$\begin{aligned} P' &= (r \cos(\alpha + \theta), r \sin(\alpha + \theta)) \\ &= (r \cos \alpha \cos \theta - r \sin \alpha \sin \theta, r \sin \alpha \cos \theta + r \cos \alpha \sin \theta) \\ &= (\cos \theta P_x - \sin \theta P_y, \sin \theta P_x + \cos \theta P_y) \end{aligned}$$

$$R(\theta)P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} P'_x \\ P'_y \end{bmatrix}$$



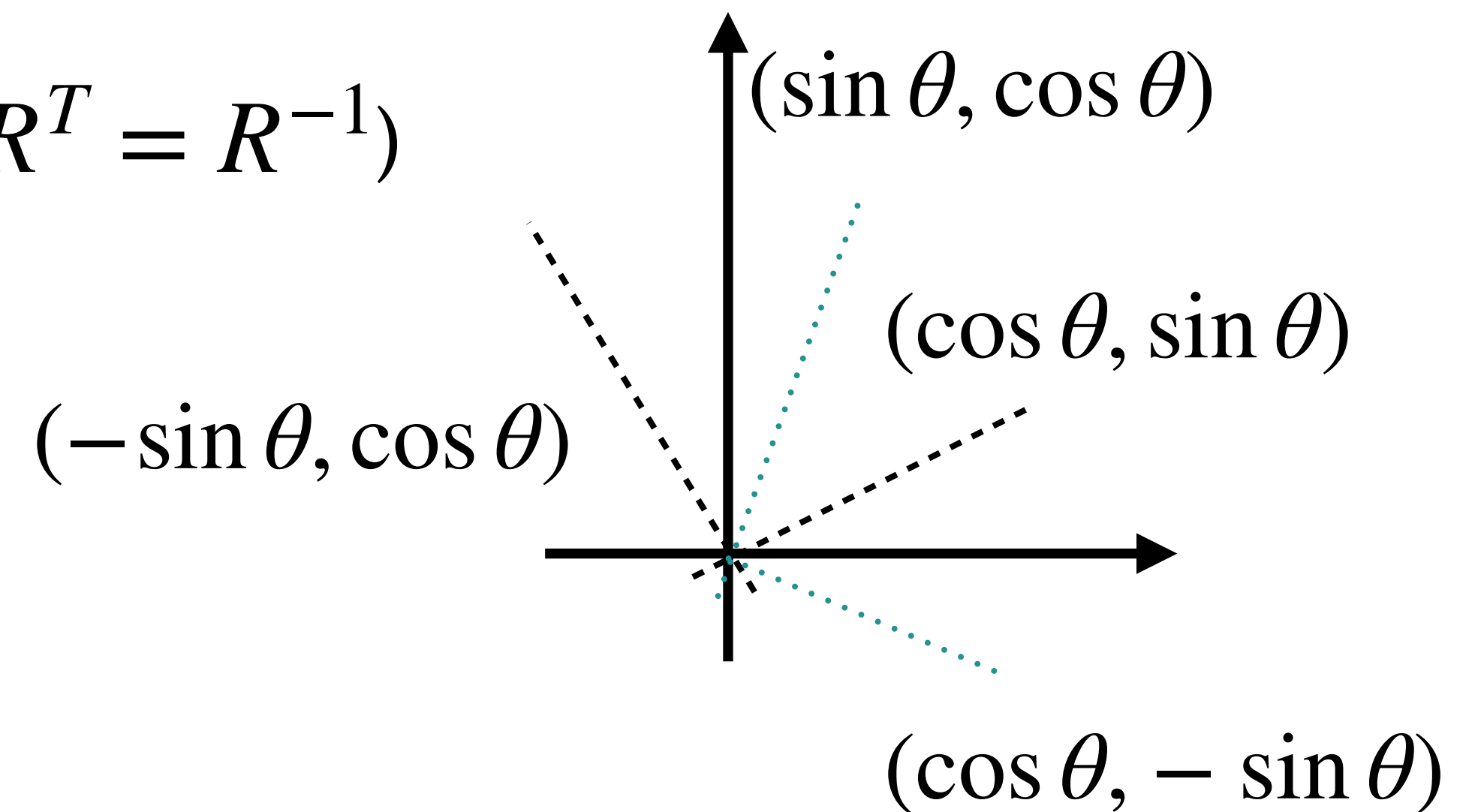
$$R^{-1}(\theta) = R(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R^T(\theta)$$

# 2D Rotation

Trigonometric identity:  $\cos^2 \theta + \sin^2 \theta = 1$

$$R(\theta)R^T(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$R$  is an orthogonal matrix:  $R^T R = R R^T = I$  ( $R^T = R^{-1}$ )



# Combining Transformations

Scale uniformly by a factor of 2 then rotate  $45^\circ$ ?

Multiplying matrices representing transformations is still a transformation matrix that represents the composition of the individual transformations.

$$x_2 = S(x_1), x_3 = R(x_2) = R(S(x_1)) = (RS)(x_1), x_3 \neq (SR)(x_1)$$

How to invert a combination of transforms?

$$M = M_1 M_2 M_3, M^{-1} = M_3^{-1} M_2^{-1} M_1^{-1}$$

$$\underline{M^{-1} M = M_3^{-1} M_2^{-1} M_1^{-1} M_1 M_2 M_3 = I}$$

# 3D Rotation

Using the cartesian axes

$$R_z = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_z^{-1} ?$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}, R_x^{-1} ?$$

$$R_y = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, R_y^{-1} ?$$

Matrices are orthogonal and linear.

Remember:

$(u, v, w)$  is a coordinate frame if

$$|u| = |v| = |w| = 1$$

$$u \cdot v = v \cdot w = u \cdot w = 0$$

$$w = u \times v$$

Think of the rows as unit vectors in a new coordinate frame...

# 3D Rotation

## Geometric Interpretation

Think of the rows of the rotation matrix as unit vectors of a new coordinate frame.

$$R_{uvw}P = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} u \cdot P \\ v \cdot P \\ w \cdot P \end{bmatrix}$$

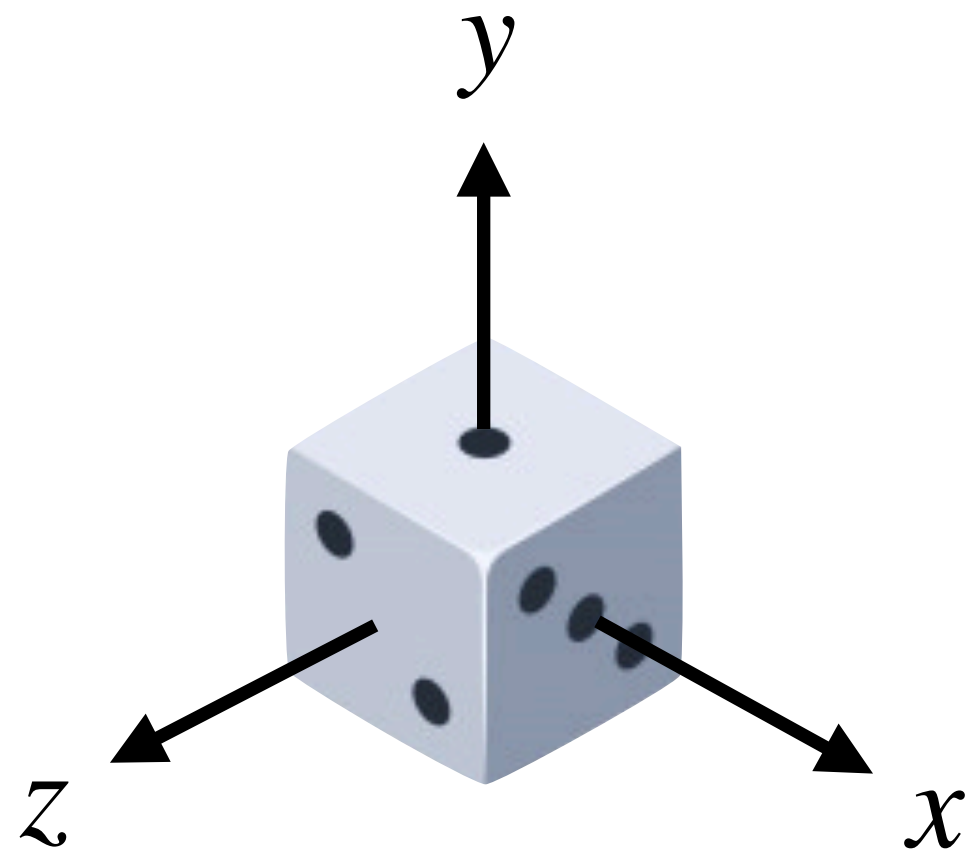
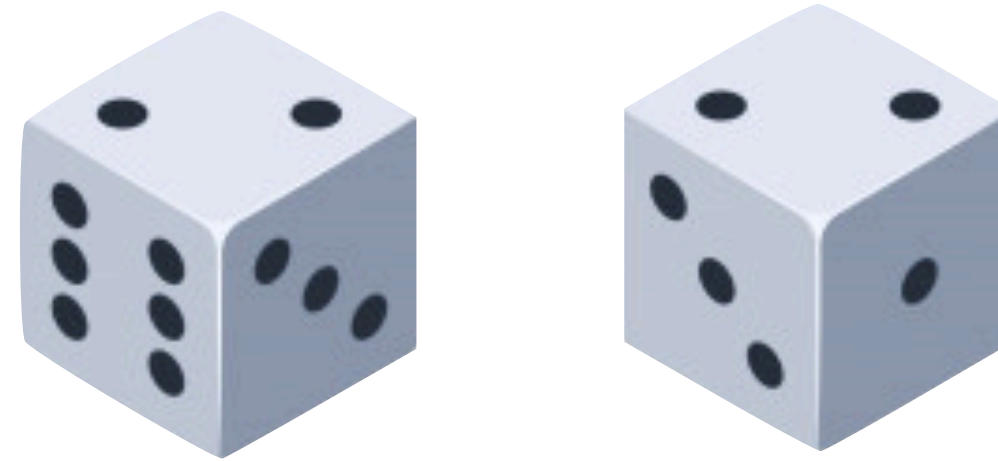
We are projecting the vector in the new coordinate frame, e.g.  $u$  will become (1,0,0):

$$R_{uvw}u = \begin{bmatrix} u \cdot u = 1 \\ v \cdot u = 0 \\ w \cdot u = 0 \end{bmatrix}, \text{ similarly for the other dimensions.}$$

# 3D Rotation

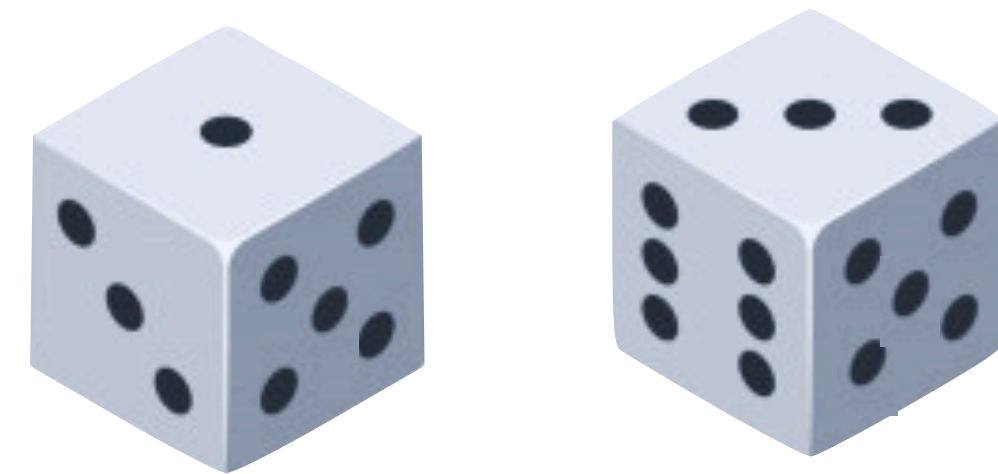
**NOT commutative (unlike 2D rotation)**

Rotate about x then y:



D6

Rotate about y then x:



# Arbitrary 3D Rotation

about an axis  $a$  by an angle of  $\theta$

Assuming  $|a| = 1$

Components of  $v$  :

- parallel to  $a$  :  $v_{||} = (a \cdot v)a$  (**projection**)

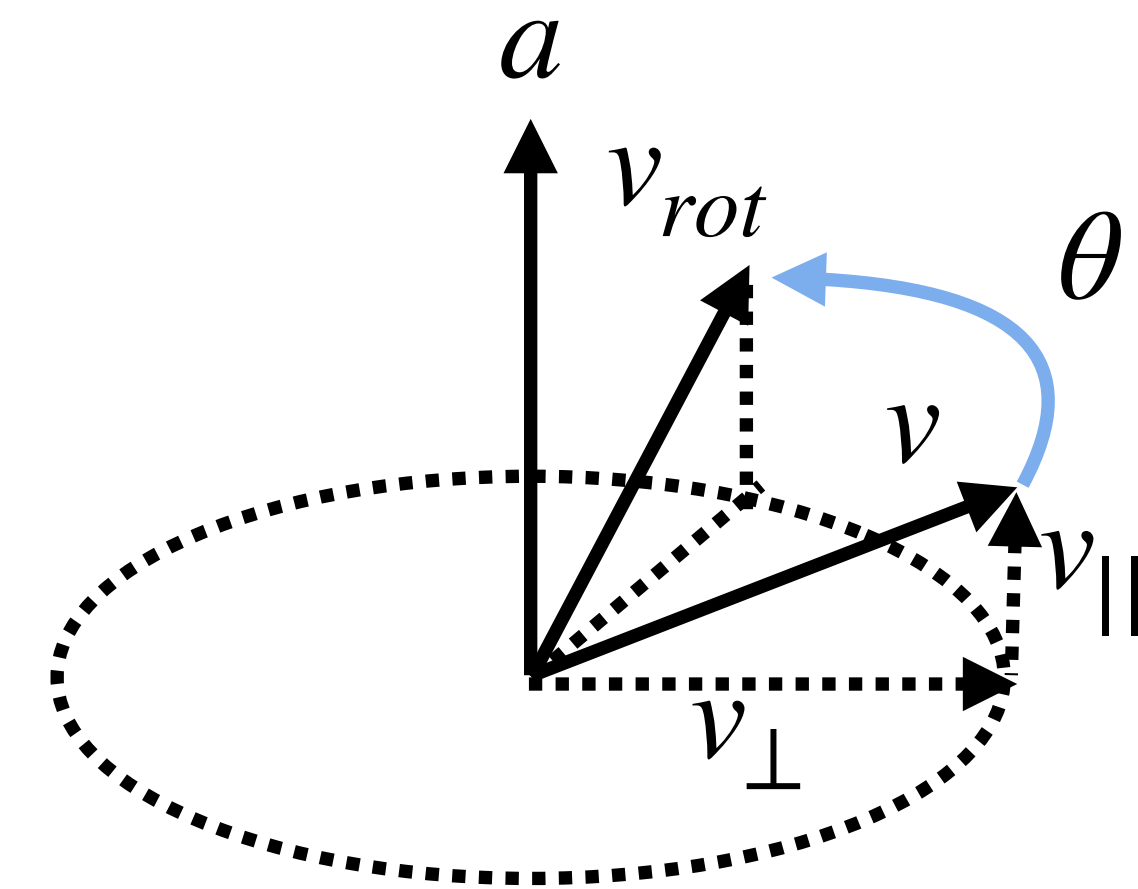
- orthogonal to  $a$  :  $v_{\perp} = v - v_{||} = v - (a \cdot v)a$  (**rejection**)

Define  $c = a \times v$

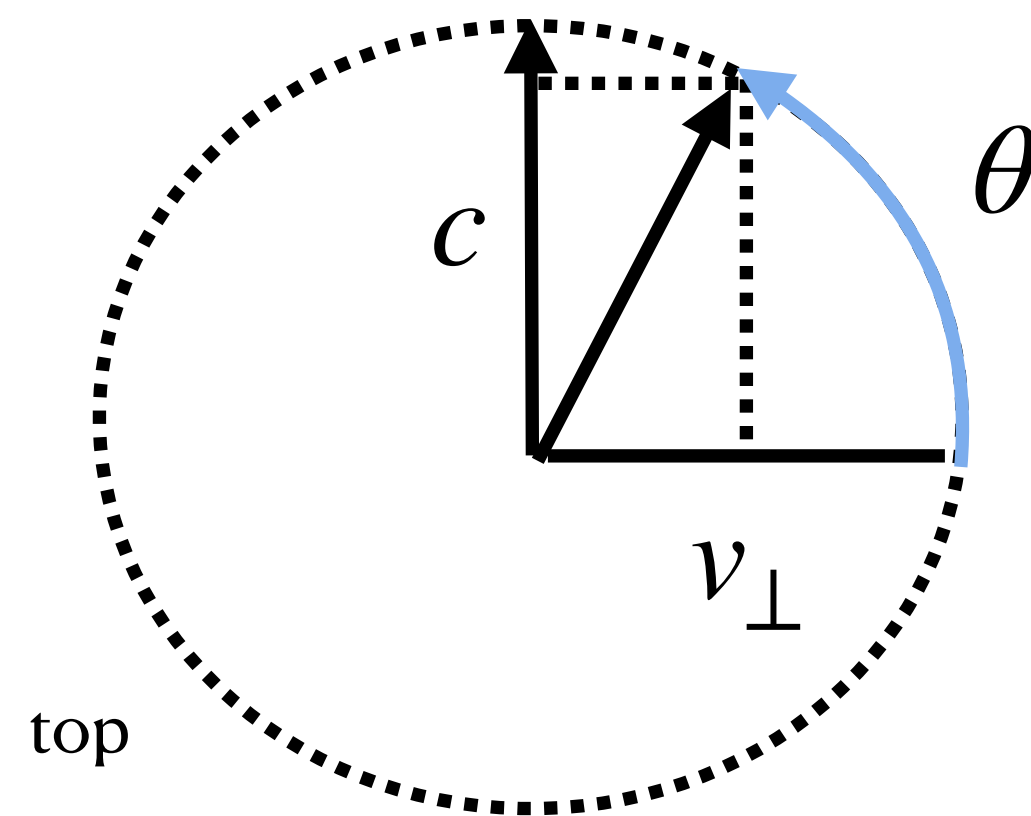
After the rotation, we have:

$$\begin{aligned} v_{\perp rot} &= v_{\perp} \cos \theta + c \sin \theta \\ &= (v - (a \cdot v)a) \cos \theta + (a \times v) \sin \theta \end{aligned}$$

$$v_{|| rot} = v_{||} = (a \cdot v)a \quad (\textbf{unchanged})$$



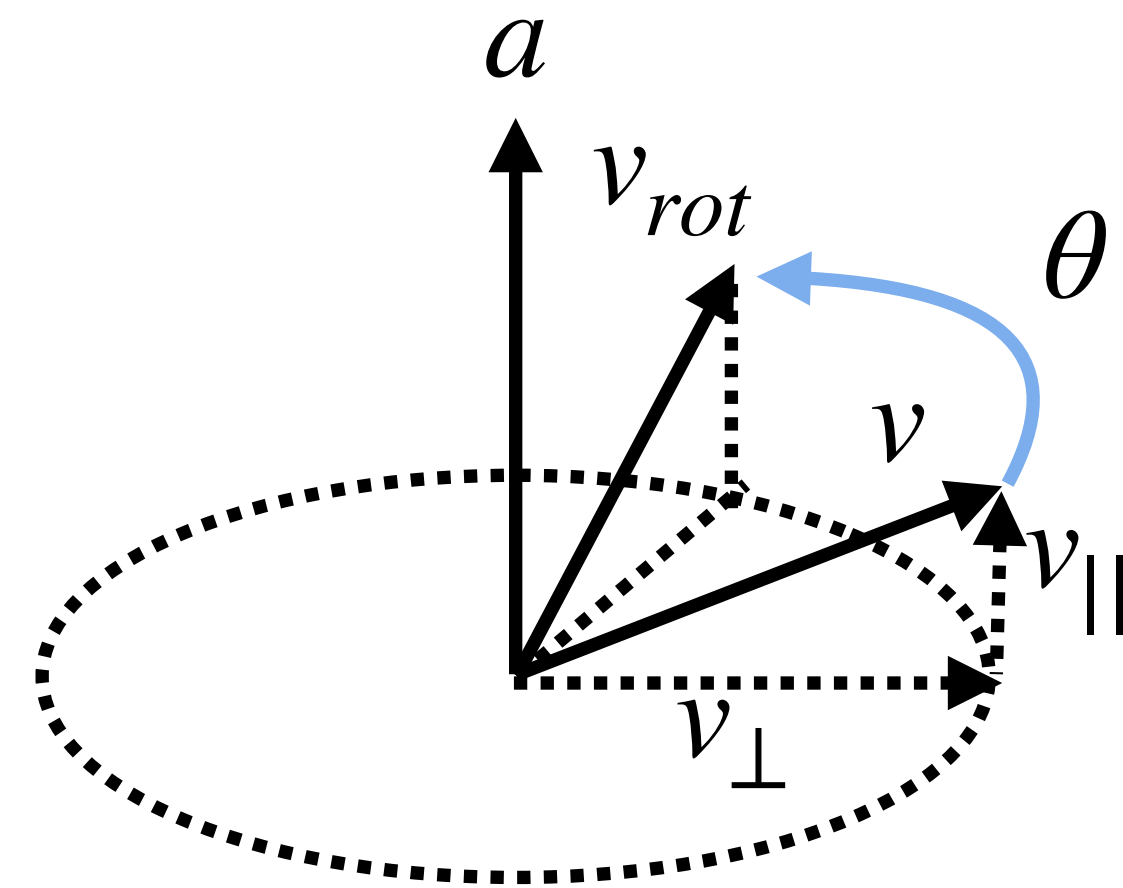
side



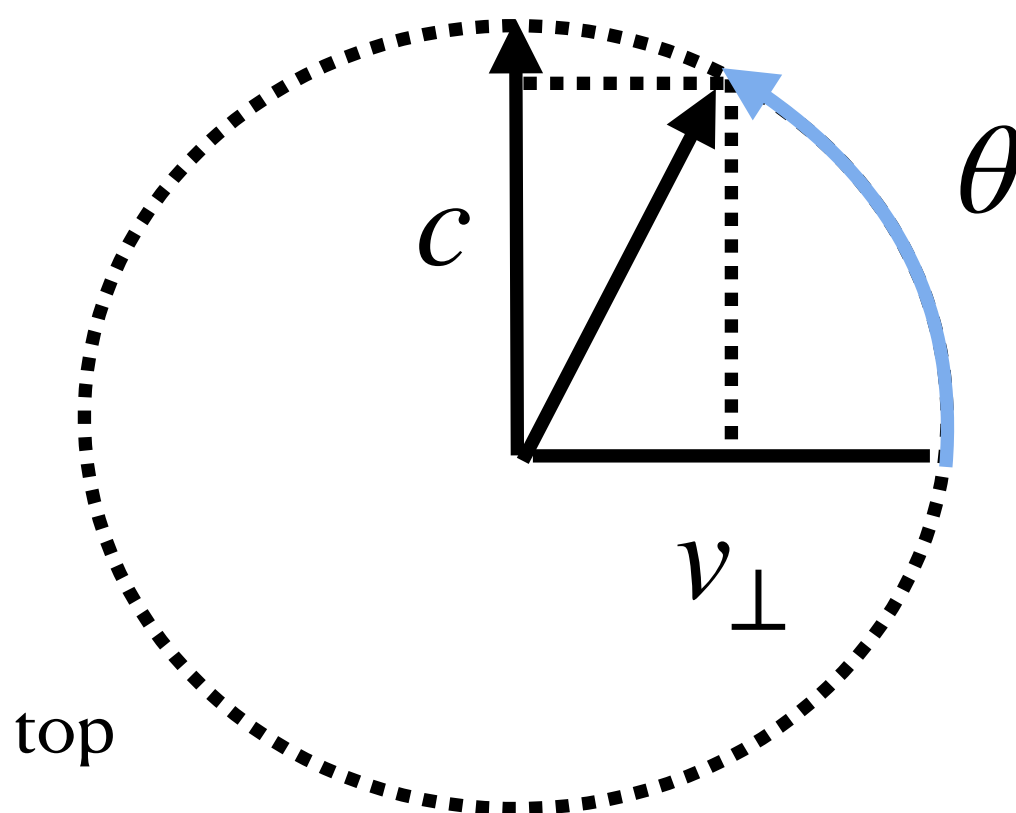
top

# Arbitrary 3D Rotation

## Rodrigues' Rotation Formula



side



top

$$\begin{aligned}
 v_{rot} &= v_{\perp rot} + v_{\parallel rot} \\
 &= (v - (a \cdot v)a)\cos \theta + (a \times v)\sin \theta + (a \cdot v)a \\
 &= v \cos \theta - (a \cdot v)a \cos \theta + (a \times v)\sin \theta + (a \cdot v)a \\
 &= v \cos \theta + (a \times v)\sin \theta + a(a \cdot v)(1 - \cos \theta)
 \end{aligned}$$

Rodrigues' Rotation Formula (~1850)

# Arbitrary 3D Rotation

## Matrix notation of Rodrigues Rotation Formula

$$a = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}, v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \text{ dual matrix } A = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \quad (Av = a \times v)$$

$$\begin{aligned} v_{rot} &= v \cos \theta + (a \times v) \sin \theta + a(a \cdot v)(1 - \cos \theta) \\ &= v - v + v \cos \theta + (a \times v) \sin \theta + a(a \cdot v)(1 - \cos \theta) \\ &= v - v(1 - \cos \theta) + (a \times v) \sin \theta + a(a \cdot v)(1 - \cos \theta) \\ &= v + (Av) \sin \theta + (a(a \cdot v) - v)(1 - \cos \theta) \end{aligned}$$

# Arbitrary 3D Rotation

## Matrix notation of Rodrigues Rotation Formula

Using vector triple product:

$$i \times (j \times k) = j(i \cdot k) - k(i \cdot j)$$

with  $i = j = a$  and  $k = v$ :

$$a(a \cdot v) - v(a \cdot a) = a \times (a \times v)$$

$$a(a \cdot v) - v = A^2 v$$

$$v_{rot} = v + (Av)\sin\theta + (a(a \cdot v) - v)(1 - \cos\theta)$$

$$= v + (\sin\theta)Av + (1 - \cos\theta)A^2v$$

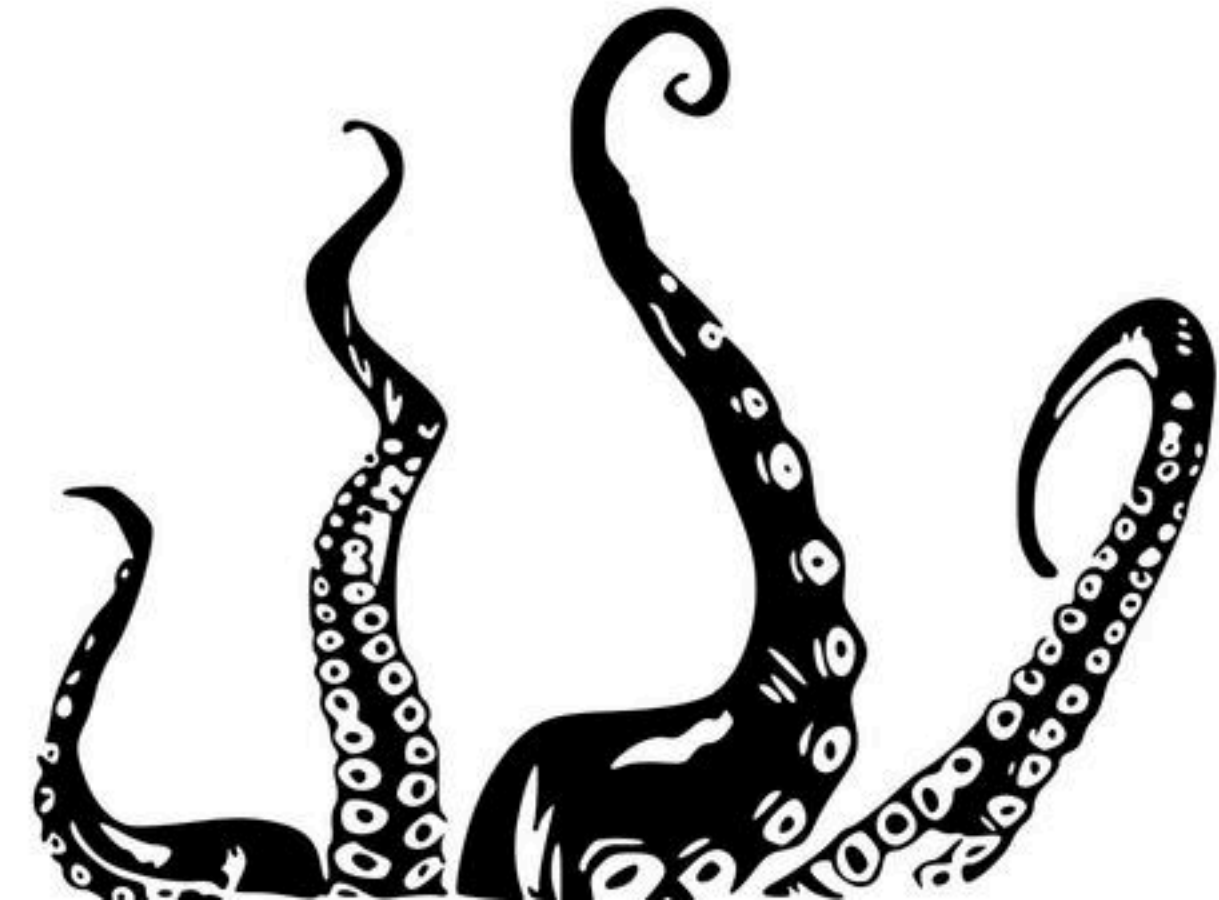
$$= Rv$$

$$R = I + (\sin\theta)A + (1 - \cos\theta)A^2$$

$$A^2 = \begin{bmatrix} -a_y^2 - a_z^2 & a_x a_y & a_x a_z \\ a_x a_y & -a_x^2 - a_z^2 & a_y a_z \\ a_x a_z & a_y a_z & -a_x^2 - a_y^2 \end{bmatrix}$$

# Homogeneous Coordinates

**Translation, Combining Transformations**



# Translation

Why leave the easiest for the end?

How to move  $x$  by +7 units?

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+7 \\ y \\ z \end{bmatrix}, T = \begin{bmatrix} ? & ? & ? \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & \frac{7}{z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ?$$

Wrong: matrices must remain constant and not depend on a point.

# Homogeneous Coordinates

Add a 4th coordinate  $w$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + 7 \\ y \\ z \\ 1 \end{bmatrix}$$

$$P = [x \quad y \quad z \quad w]^T$$

$w = 0$  represents a point at infinity (vectors),  $w > 0$  is a physical point

inhomogeneous point  $P = [x \quad y \quad z \quad w]^T = \left[ \frac{x}{w} \quad \frac{y}{w} \quad \frac{z}{w} \quad 1 \right]^T$

# Homogeneous Coordinates

## Vectors and 4x4 Transforms

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}, \text{ as would be expected.}$$

Scaling becomes

$$\begin{bmatrix} S_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix}$$

Rotation becomes

$$\begin{bmatrix} R_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix}$$

Translation

$$\begin{bmatrix} I_{3 \times 3} & T_{3 \times 1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Combining Transformations

Typically, we take an object, scale it, rotate it, then translate it.

First rotate, then translate:

$$M = TR = \begin{bmatrix} I_{3 \times 3} & T_{3 \times 1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{3 \times 3} & T_{3 \times 1} \\ 0 & 1 \end{bmatrix}$$

First translate, then rotate:

$$M = RT = \begin{bmatrix} R_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_{3 \times 3} & T_{3 \times 1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{3 \times 3} & R_{3 \times 3}T_{3 \times 1} \\ 0 & 1 \end{bmatrix}$$

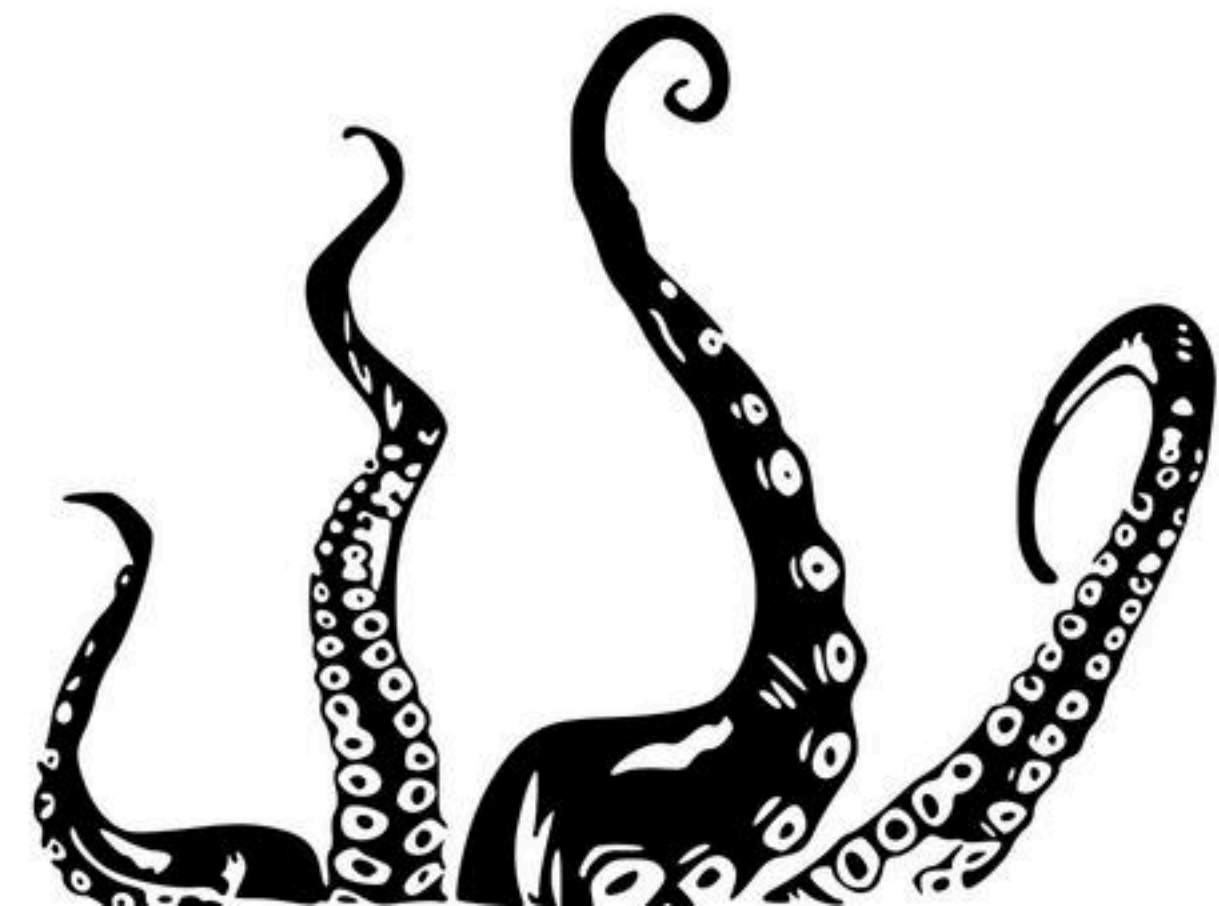
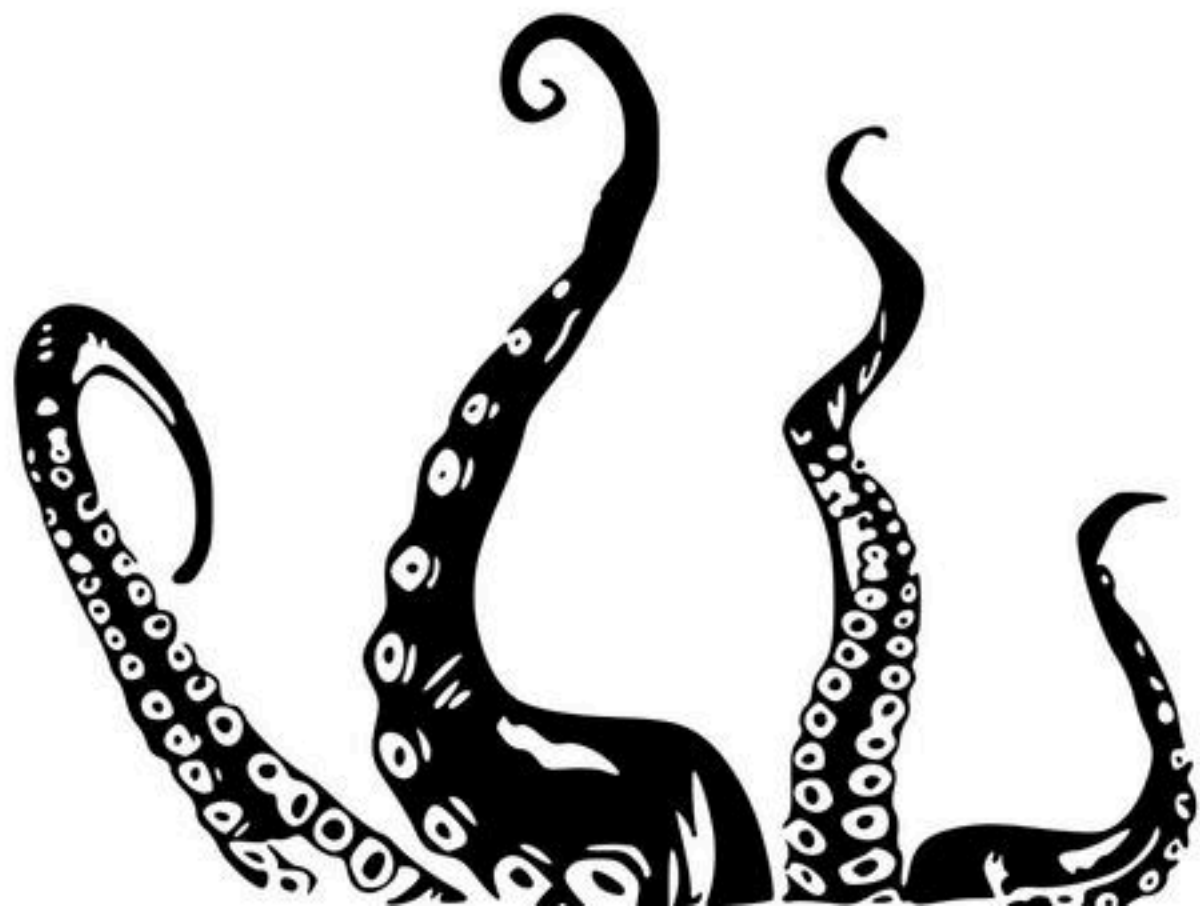
# Homogeneous Coordinates

## Advantages

- Homogeneous coordinates are a unified framework for transformations
- Can concatenate all transformations in a  $4 \times 4$  matrix
- Using homogeneous coordinates in perspective viewing requires 1 division
- Simpler formulae that does not blow up (e.g. intersecting parallel lines)
- Standard in Computer Graphics SW & HW

# Matrix Inversion

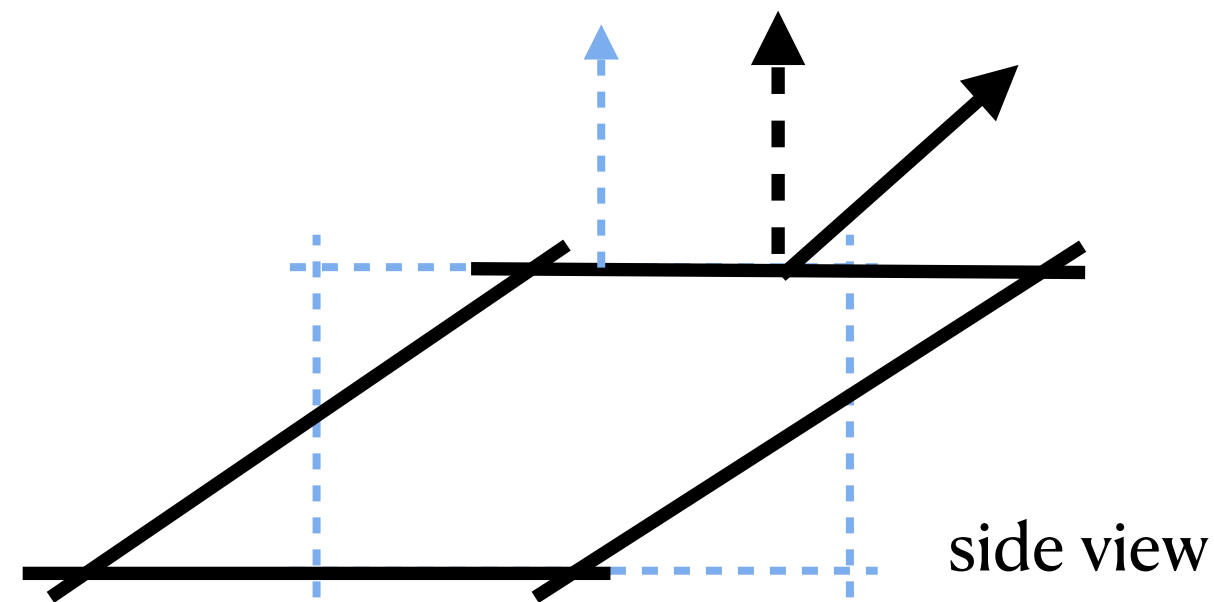
Transforming Normals



# Transforming Normals

## Problem:

Tangents are points on the surface and they transform with the object.



incorrect: shear applied to normal

## Normal matrix Q:

$$t \rightarrow Mt, n \rightarrow Qn$$

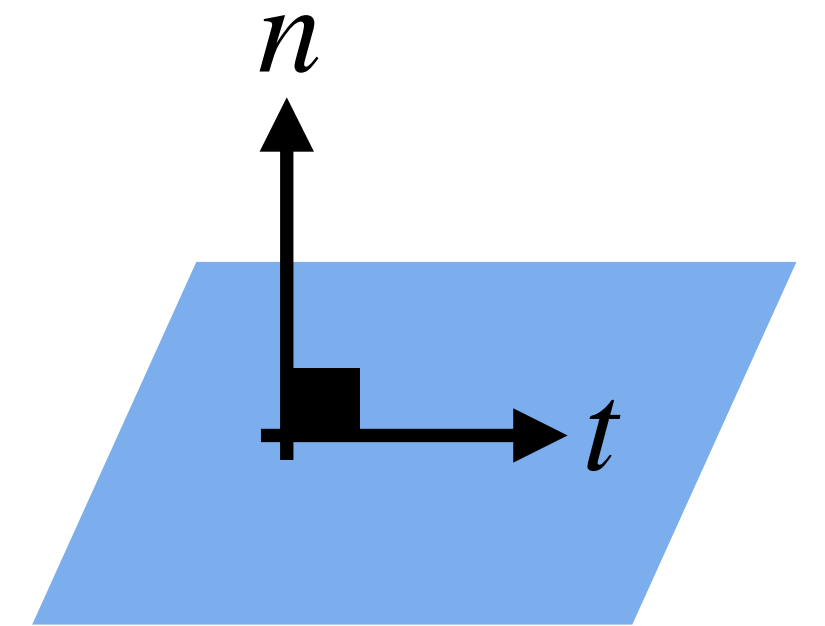
$$n^T t = 0$$

$$(Qn)^T (Mt) = 0$$

$$(n^T Q^T)(Mt) = 0$$

$$n^T (Q^T M)t = 0 \implies Q^T M = I$$

$$Q = (M^{-1})^T$$



# Matrix Inversion

**2x2**

**Determinant  $|A|$ :**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, |A| = ad - bc, \text{ if } |A| = 0 \text{ then } A \text{ is not invertible.}$$

**Inverse  $A^{-1}$ :**

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

# Matrix Inversion

## 3x3 through an example

**Determinant  $|M|$ :**

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

$$\begin{vmatrix} + & - & + \\ 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix} \quad \begin{vmatrix} + & - & + \\ 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix} \quad \begin{vmatrix} + & - & + \\ 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix}$$

$$\begin{aligned} |M| &= +1 \times (1 \times 0 - 4 \times 6) \\ &\quad -2 \times (0 \times 0 - 5 \times 4) \\ &\quad +3 \times (0 \times 6 - 1 \times 5) \\ &= -24 + 40 - 15 = 1 \end{aligned}$$

# Matrix Inversion

**determinant of 3x3 matrix**

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$\begin{aligned} |M| &= m_{11}(m_{22}m_{33} - m_{23}m_{32}) \\ &\quad - m_{12}(m_{21}m_{33} - m_{23}m_{31}) \\ &\quad + m_{13}(m_{21}m_{32} - m_{22}m_{31}) \end{aligned}$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24, \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24, \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18, \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24, \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18, \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5,$$

$$\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} = -20$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24, \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18, \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5,$$

$$\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} = -20, \begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} = -15$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24, \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18, \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5,$$

$$\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} = -20, \begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} = -15, \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24, \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18, \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5,$$

$$\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} = -20, \begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} = -15, \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4,$$

$$\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = -5$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24, \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18, \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5,$$

$$\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} = -20, \begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} = -15, \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4,$$

$$\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = -5, \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = -4$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

1. **Determinant  $|M|$**

$$|M| = 1 \neq 0$$

2. **Transpose  $M^T$**

$$M^T = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$$

3. **Determinant of minor matrices**

$$\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = -24, \begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18, \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5,$$

$$\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} = -20, \begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} = -15, \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4,$$

$$\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = -5, \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = -4, \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

# Matrix Inversion

inverse of 3x3 matrix

## 4. Matrix of cofactors / Adjugate / Adjoint

$$\begin{bmatrix} -24 & -18 & 5 \\ -20 & -15 & 4 \\ -5 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} = Adj(M)$$

$$5. \quad M^{-1} = \frac{1}{|M|} Adj(M) = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

# Thank You

**for your attention!**

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