Multiagent decision making and Auctions

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Outline

- **Introduction to auctions**
- Canonical auctions
- Bidding in first-price auctions
- Bidding in second-price auctions

Introduction

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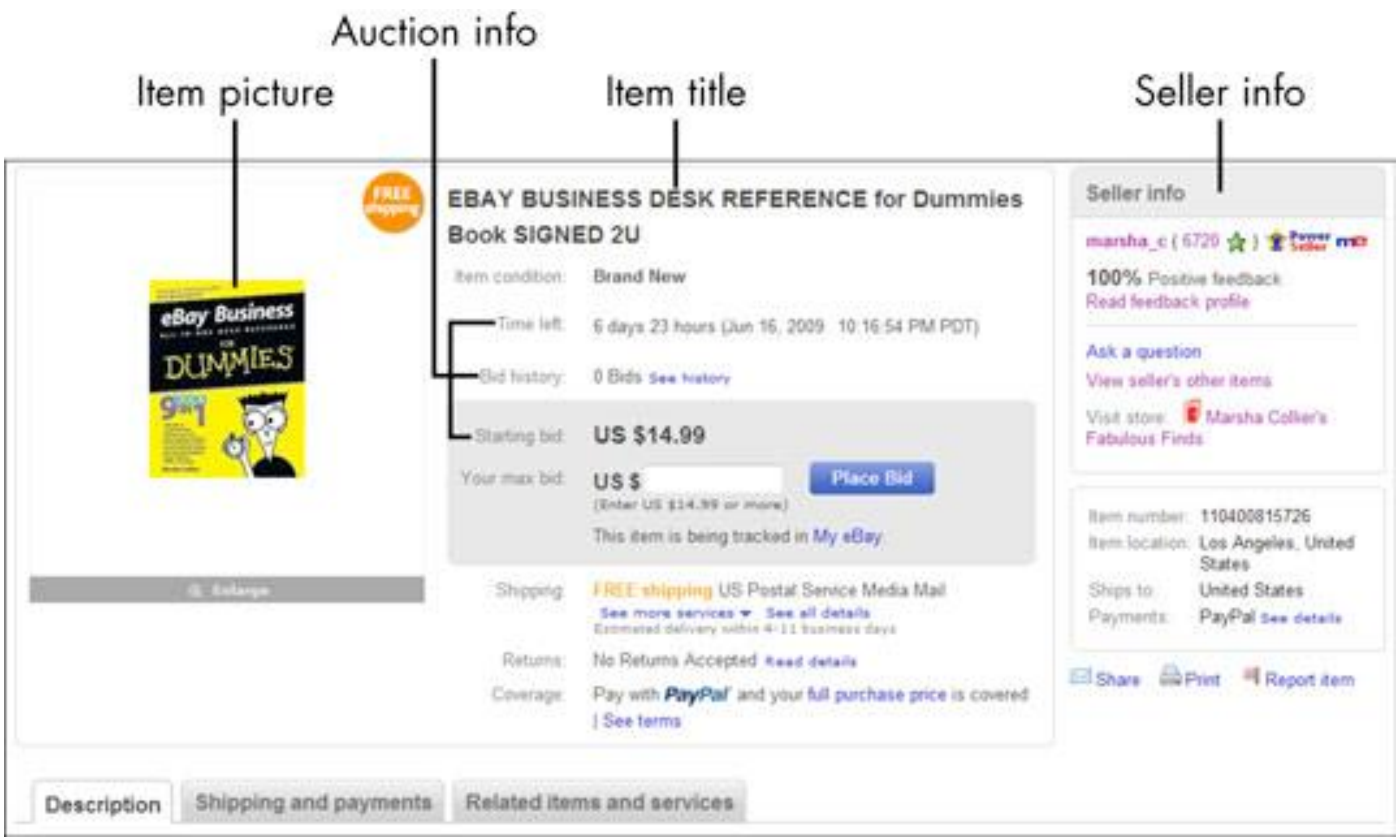

Introduction

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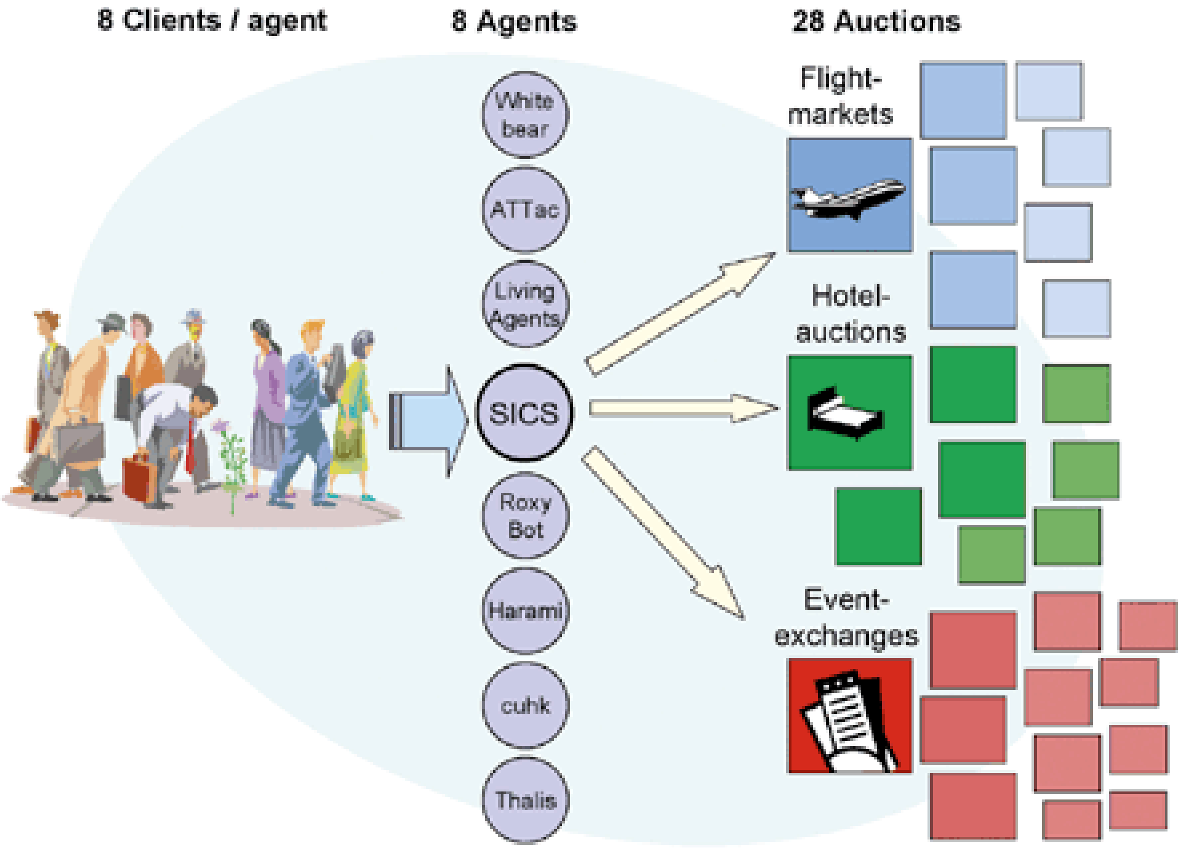


Auctions

- Auctions are a **mechanism for allocating resources among self-interested agents**
  - Normally scarce resources

- Widely used to:
  - Sell art
  - Sell public companies (privatization)
  - Sell or buy stocks
  - Sell used goods (e.g., eBay)
  - Procure parts
  - Etc.

Outline

- Introduction to auctions
- **Canonical auctions**
- Bidding in first-price auctions
- Bidding in second-price auctions

Canonical Auctions

- English auction
- Dutch auction
- First-price auction
- Second-price auction

English Auction

- An **English auction** is an open-outcry ascending auction that proceeds as follows:

  - The **auctioneer starts the bidding** at some starting price (reserve price)

  - **Bidders** then **shout out ascending prices**

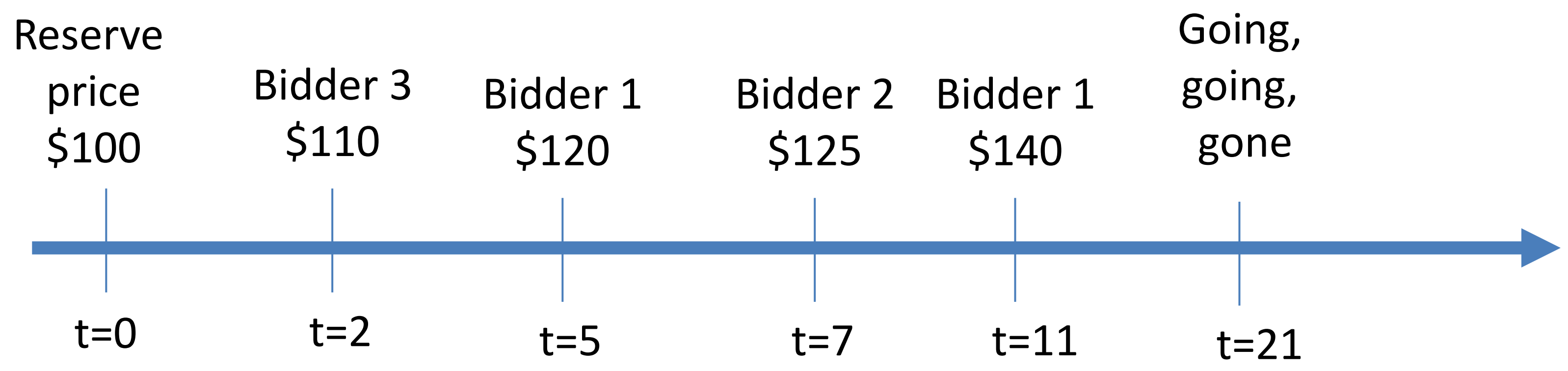
  - At any given moment, the **highest bidder** is considered to have the **standing bid**

English Auction

- An **English auction** is an open-outcry ascending auction that proceeds as follows:
  - The **standing bid becomes the winner** if **no competing bidder challenges the standing bid** within a given time frame
    - And the **item is sold to the highest bidder** at a **price equal to their bid**

English Auction

- **Example**



English Auction

- Interesting properties/facts of the **English auctions**:
  - Every bidder knows the number of bidders in the auction
  - The bids are public
  - Bidders can submit several bids
  - This type of auction is commonly used for selling art, antiques, wine, etc.

Dutch Auction

- A **Dutch auction** is an open-outcry descending auction that proceeds as follows:
  - The auctioneer starts a clock at some high asking price
  - The price lowers at each time step until a bidder accepts the current ask price (by shouting)

Dutch Auction

- **Example:**
  - A farmer wants to sell a basket of apples and uses a Dutch auction
  - The starting bid is $150
  - If nobody accepts the initial bid, the farmer (auctioneer) successively reduces the price in increments of $10 after 5 seconds:
    - t = 0 : ask price = $150
    - t = 5 : ask price = $140
    - t = 10 : ask price = $130
    - …

Dutch Auction

- **Example**:
  - A particular bidder is the first to shout out that he wants to buy the item when the price reaches $40
    - Note that the bidder feels that price is acceptable and that someone else might bid soon
    - The bidder pays $40 for the basket of apples

Dutch Auction

- Interesting properties/facts of the **Dutch auctions**:
  - Every bidder knows the number of bidders in the auction
  - The bid is public (and only one bid is submitted)
  - This type of auction is commonly used for selling flowers, fresh produce, tobacco, etc

First-Price Auction

- A **first-price sealed-bid auction** proceeds as follows:
  - All bidders submit sealed bids simultaneously
    - No bidder knows the bids of the other bidders
  - The highest bidder wins and pays the submitted price

First-Price Auction

- **Example**:
  - Bidder 1 submits a sealed bid of $100
  - Bidder 2 submits a sealed bid of $80
  - Bidder 3 submits a sealed bid of $95
  - Bidder 1 wins and pays $100 for the item

First-Price Auction

- Interesting properties/facts of the **first-price auctions**:
  - Every bidder knows the number of bidders in the auction
  - The bids are private
  - Bidders can only submit one bid
  - This type of auction is commonly used for privatization of public companies, selling concessions, etc

Second-Price Auction

- A **second-price sealed-bid auction** proceeds as follows:
  - All bidders submit sealed bids simultaneously
    - No bidder knows the bids of the other bidders
  - The highest bidder wins and pays the second highest bid

Second-Price Auction

- **Example**:
  - Bidder 1 submits a sealed bid of \$100
  - Bidder 2 submits a sealed bid of \$80
  - Bidder 3 submits a sealed bid of \$95
  - Bidder 1 wins and pays \$95 for the item (and not \$100!)

Second-Price Auction

- Interesting properties/facts of the **second-price auctions**:
  - Every bidder knows the number of bidders in the auction
  - The bids are private
  - Bidders can only submit one bid
  - Bidders can be invited/selected to the auction
  - It is used in digital ads tech (e.g., Google and Facebook)
  - It has very interesting theoretical results

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- Introduction to auctions
- Canonical auctions
- **Bidding in first-price auctions**
- Bidding in second-price auctions
- Revenue equivalence

Bidding in First-Price Auctions

- How do agents bid in a first-price auction?
  - An agent has an incentive to bid less than its true valuation
    - For instance, if the agent thinks the value of a good is \$10 then he might want to bid \$8
      - In a first price auction, if the agent bids \$8 and wins, then he pays \$8 and makes a profit equal to \$2 (i.e., profit = \$10 - \$8 = \$2)

Bidding in First-Price Auctions

- How do agents bid in a first-price auction?
  - The tradeoff in the bidding decision:
    - Probability of winning
      - lower bid → probability of winning is lower
      - higher bid → probability of winning is higher
    - Amount paid when winning
      - lower bid → higher profit
      - higher bid → lower profit

Bidding in First-Price Auctions

- How do agents bid in a first-price auction?
  - Bidders do not have a dominant strategy
    - strategy of player i depends on the strategy of other players

Bidding in First-Price Auctions

- How do agents bid in a first-price auction?
  - **Theorem**: In a first-price sealed-bid auction with:
    - Two risk-neutral bidders (i.e., agent 1 and agent 2)
    - The valuations v_1 and v_2 are i.i.d. and drawn from $U(0,1)$
    - Hence:
      - Agent 1 bids $\frac{1}{2}v_1$
      - Agent 2 bids $\frac{1}{2}v_2$
      - And $\left(\frac{1}{2}v_1, \frac{1}{2}v_2\right)$ is a Bayesian-Nash equilibrium

Bidding in First-Price Auctions

- **Proof:**
  - Let us assume bidder 2 bids $b_2 = \frac{1}{2} v_2$
    - Where v_2 is the value of the object from bidder 2's perspective
  - We now analyse bidder 1's optimal decision (best response) :
    - The optimal decision is is to maximize the expected profit:

$$\max_{b_1} \mathbb{E}[u_1]$$

Bidding in First-Price Auctions

- **Proof:**
  - We now analyse bidder 1's optimal decision (best response):
    - bidder 1 wins the auction when $b_2 < b_1$, hence $v_2 < 2b_1$ with profit $u_1 = v_1 - b_1$
    - bidder 1 looses the auction when $b_1 < b_2$, hence $v_2 > 2b_1$ with profit $u_1 = 0$
    - Hence, $\mathbb{E}[u_1] \ = P(win|b_1)(v_1 - b_1) + P(loose|b_1)0$

Bidding in First-Price Auctions

- **Proof:**
  - We now analyse bidder 1's optimal decision (best response):
    - $\mathbb{E}[u_1] \ = P(win|b_1)(v_1 - b_1) + P(loose|b_1)$ 0
    - $\mathbb{E}[u_1]$ = $P(win|b_1)(v_1 - b_1)$

Bidding in First-Price Auctions

- **Proof:**
  - We now analyse bidder 1's optimal decision (best response):
    - $\mathbb{E}[u_1] = P(win|b_1)(v_1 - b_1)$
    - $\mathbb{E}[u_1] = P(b_2 < b_1)(v_1 - b_1)$
  - Substituting $v_2 < 2b_1$ for $b_2 < b_1$:
    - $\mathbb{E}[u_1] = P(v_2 < 2b_1)(v_1 - b_1)$

Bidding in First-Price Auctions

- **Proof:**
  - We now analyse bidder 1's optimal decision (best response):
    - $\mathbb{E}[u_1] = P(v_2 < 2b_1)(v_1 - b_1)$
    - $\mathbb{E}[u_1] = F_{v2_{(}}2b_1)(v_1 - b_1)$

> **RECALL:**
> The ***cumulative distribution function*** **(CDF)** of a real-valued random variable X is the function given by:
>
> $$F_X(x) = P(X \leq x)$$

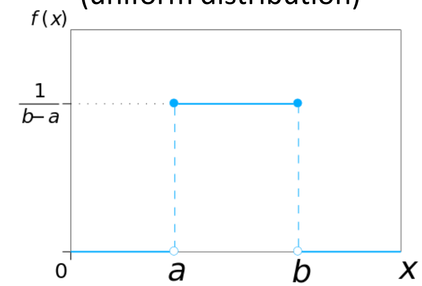
Bidding in First-Price Auctions

- **Proof:**
  - We now analyse bidder 1's optimal decision (best response):
    - $\mathbb{E}[u_1] = F_{v2_{(}}2b_1)(v_1 - b_1)$
    - $\mathbb{E}[u_1] = \int_0^{2b_1} dv_2 \ (v_1 - b_1)$

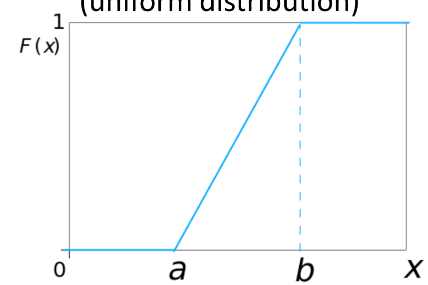
RECALL:
The CDF can be expressed as the integral of its probability density function:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

Probability density function (uniform distribution)



Cumulative distribution function (uniform distribution)



Bidding in First-Price Auctions

- **Proof:**
  - We now analyse bidder 1’s optimal decision (best response):
    - $\mathbb{E}[u_1] = \int_0^{2b_1} dv_2 \ (v_1 - b_1)$
    - $\mathbb{E}[u_1] = (v_2 + k)|_0^{2b_1}(v_1 - b_1)$
    - $\mathbb{E}[u_1] = 2b_1(v_1 - b_1) = 2v_1b_1 - 2b_1^2$

Bidding in First-Price Auctions

- **Proof:**
  - We now analyse bidder 1's optimal decision (best response):
    - Recall that we want to maximize the expected profit:

$$\max_{b_1} \mathbb{E}[u_1]$$

    - Hence, the first order condition is $\frac{\partial \mathbb{E}[u_1]}{\partial b_1} = 0$

Bidding in First-Price Auctions

- **Proof:**
  - We now analyse bidder 1's optimal decision (best response):
    - $\frac{\partial \mathbb{E}[u_1]}{\partial b_1} = \frac{\partial}{\partial b_1} 2v_1 b_1 - 2b_1^2 = 0$
    - $2v_1 - 4b_1 = 0$
    - $b_1 = \frac{v_1}{2}$

Bidding in First-Price Auctions

- **Proof:**

  - Let us assume bidder 1 bids $b_1 = \frac{1}{2} v_1$
    - Where v_1 is the value of the object from bidder 1's perspective

  - We now analyse bidder 2's optimal decision (best response) :
    - The optimal decision is is to maximize the expected profit:

$$\max_{b_2} \mathbb{E}[u_2]$$

    - And following the same steps in the previous slides:

$$b_2 = \frac{v_2}{2}$$

    - Hence, $\left(\frac{1}{2} v_1, \frac{1}{2} v_2\right)$ is a Bayesian-Nash equilibrium

Bidding in First-Price Auctions

- How do agents bid in a first-price auction?

  - **Theorem**: In a first-price sealed-bid auction with:
    - N risk-neutral bidders
    - The valuations v_i are i.i.d. and drawn from $U(0,1)$
    - Hence:
      - $\left(\frac{N-1}{N}v_1, \frac{N-1}{N}v_2, \dots, \frac{N-1}{N}v_N\right)$ is a Bayesian-Nash equilibrium

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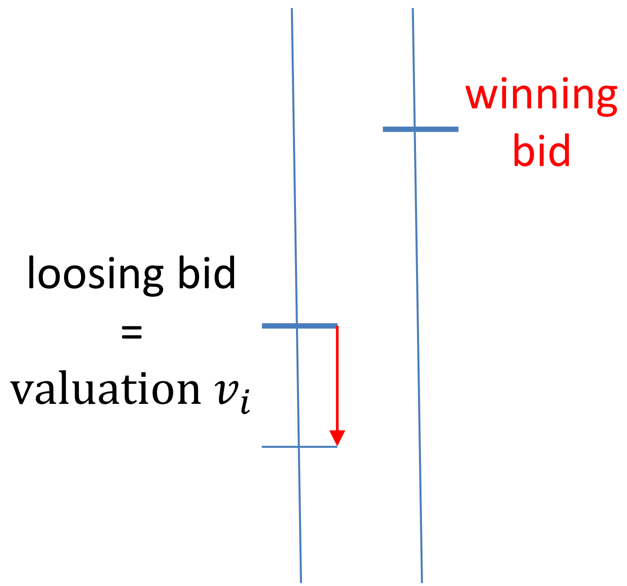
- Introduction to auctions
- Canonical auctions
- Bidding in first-price auctions
- **Bidding in second-price auctions**
- Revenue equivalence

Bidding in Second-Price Auctions

- How do agents bid in a second-price auction?
  - **Theorem**: In a second-price sealed-bid auction with:
    - N risk-neutral bidders
    - The valuations v_i
    - Hence:
      - $(v_1, v_2, \dots, v_N)$ is a Nash equilibrium

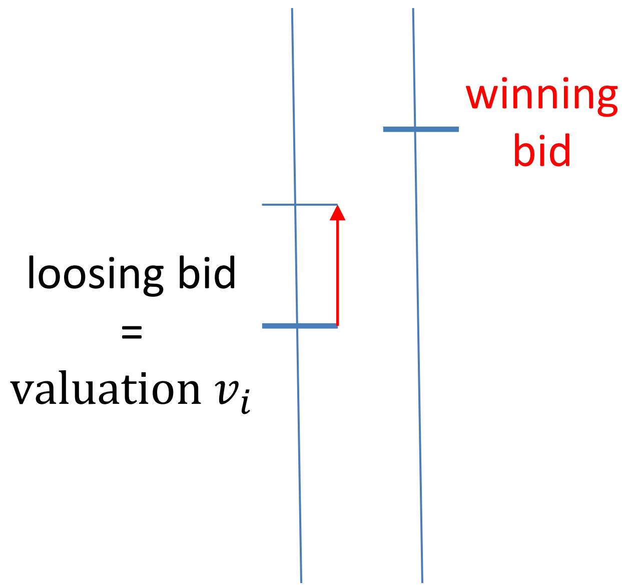
Bidding in Second-Price Auctions

- Consider the **losers' strategies**:
  - The **profit/payoff of a loosing bid is equal to zero in the NE** (i.e., $b_i = v_i$)
  - **Reducing their bids does not change the profit** because they **still loose and do not pay anything**
    - profit/payoff is still equal to zero



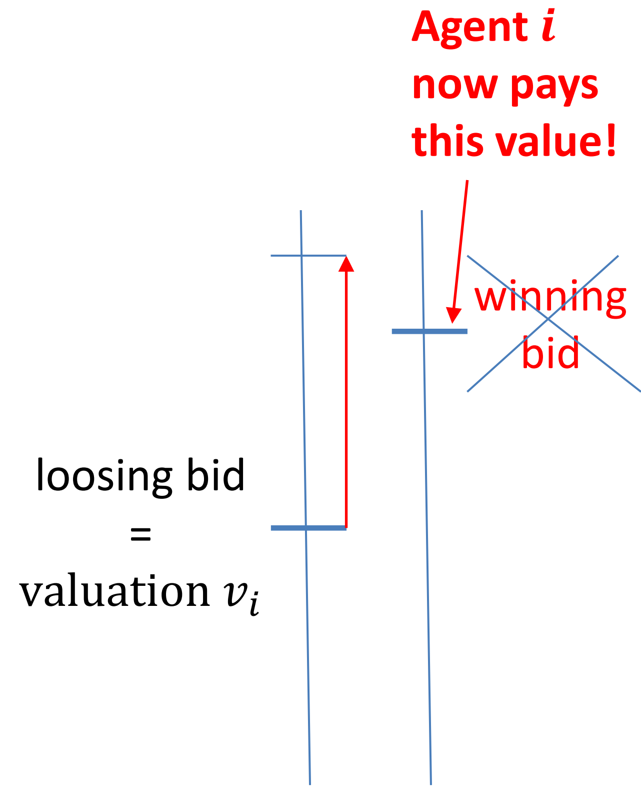
Bidding in Second-Price Auctions

- Consider the **losers' strategies**:
  - Increasing their bids above their valuation may or may not change the profit:
    - **If they still loose with the increased bid, they still do not pay anything**
      - profit/payoff is still equal to zero



Bidding in Second-Price Auctions

- Consider the **losers' strategies**:
  - Increasing their bids above their valuation may or may not change the profit:
    - On the other hand, **a loser increasing his bid to a winning price gives him the good, but at a price higher than the maximum he was willing to pay**
      - profit/payoff is negative

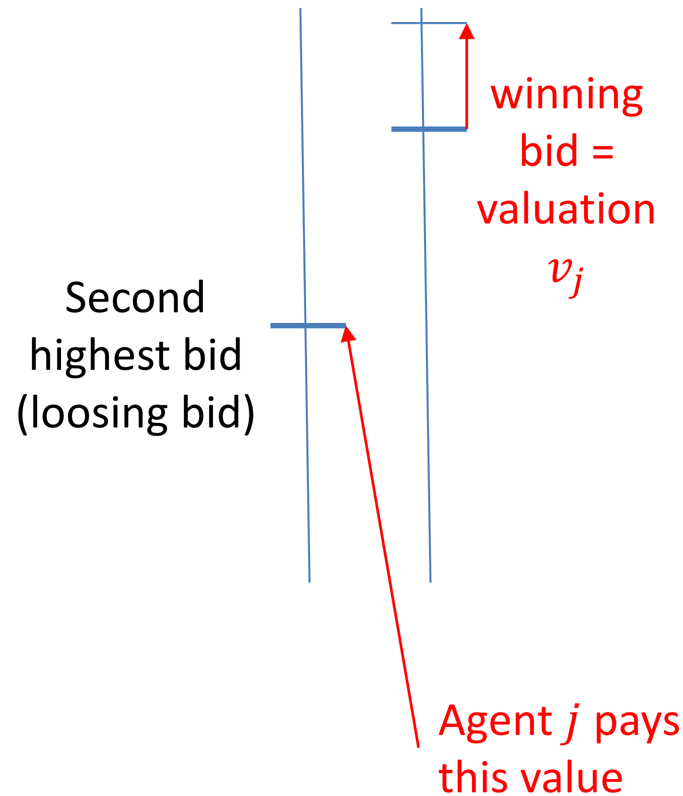


Bidding in Second-Price Auctions

- Consider the **losers' strategies**:
  - Consequently, losers have no incentives to deviate from the NE
    - i.e., losers have no incentives to change their bids

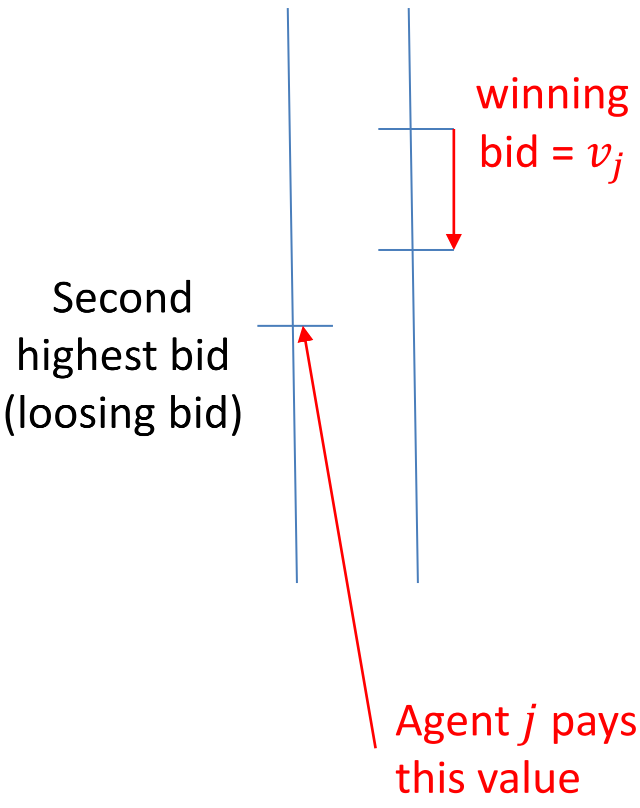
Bidding in Second-Price Auctions

- Consider the **winner's strategy**:
  - **Increasing his bid does not change anything**
    - He will still win and continue to pay the price of the second highest bid
      - profit/payoff is the same



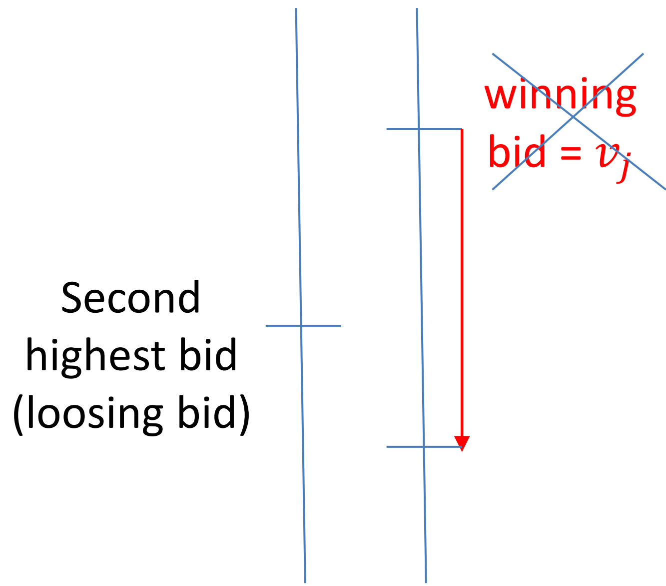
Bidding in Second-Price Auctions

- Consider the **winner's strategy**:
  - Decreasing the bid can only hurt him
    - If **the decreased bid stays above the second highest bid, he still wins and still pays the second highest bid**
      - profit/payoff is the same



Bidding in Second-Price Auctions

- Consider the **winner's strategy**:
  - Decreasing the bid can only hurt him
    - If **the decreased bid drops below the second highest bid, he now looses the auction**
      - profit/payoff is equal to zero



Bidding in Second-Price Auctions

- Consider the **winner's strategy**:
  - Consequently, the winner has no incentives to deviate from the NE
    - i.e., winner has no incentive to change his bid

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- **Revenue equivalence**

Revenue Equivalence

Which auction should an auctioneer choose?

Revenue Equivalence

Which auction should an auctioneer choose?

To some extent, it does not matter...

Revenue Equivalence

- **Theorem** (**Revenue Equivalence Theorem**): *Assume that each of n risk-neutral agents has an independent private valuation for a single good at auction, drawn from a common cumulative distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \overline{v}]$. Then any auction mechanism in which*

  - *the good will be allocated to the agent with the highest valuation; and*

  - *any agent with valuation $\underline{v}$ has an expected utility of zero;*

yields the same expected revenue, and hence results in any bidder with valuation v making the same expected payment.

Revenue Equivalence

- Can we use **Revenue Equivalence Theorem** with the first-price and second-price auctions?

YES!

- Why?

  - The first-price and second-price auctions are **symmetric games** and **every symmetric game has a symmetric equilibrium.** In addition, a symmetric equilibrium has the following property:

    higher bid ⇔ higher valuation

  - Hence, the good will be allocated to the agent with the highest valuation

  - And any agent with valuation $\underline{v}$ has an expected utility of zero

Revenue Equivalence

- We will use $\boldsymbol{k^{th}}$ **order statistic of a distribution** to analyze the revenue equivalence in first-price and second-price auctions

- $\boldsymbol{k^{th}}$ **order statistic of a distribution:** the expected value of the k^{th}-largest of n draws

- For n i.i.d. draws from a uniform distribution $[0, v_{max}]$, the k^{th} order statistic is:

$$\frac{n+1-k}{n+1} v_{max}$$

Revenue Equivalence

- **First-price auction**:
  - Recall that the winner pays the largest bid
  - However, following the **Revenue Equivalence Theorem**, the winning bidder in a first-price auction must bid his expected payment conditional on being the winner of a second-price auction

Revenue Equivalence

- The winning bidder in a first-price auction must bid his expected payment conditional on being the winner of a second-price auction

  - **If bidder i's valuation v_i is the highest**, there are then $n - 1$ other valuations drawn from the uniform distribution on $[0, v_i]$

  - Hence, the expected value of the second-highest valuation (bid) is the first-order statistic of $n - 1$ draws from $[0, v_i]$:

$$\frac{(n-1)+1-1}{(n-1)+1} v_i = \frac{n-1}{n} v_i$$

  - This provides a basis for our earlier claim about n-bidder first-price auctions

Revenue Equivalence

- This provides a basis for our earlier claim about n-bidder first-price auctions
  - However, we would still **have to check that this is an equilibrium**!
  - The revenue equivalence theorem does not say that every revenue-equivalent strategy profile is an equilibrium!

Thank You

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