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Multiagent decision making and Games in Extensive Form (Part 2)

Outline

- **Imperfect-information games in extensive form**
- Strategies and equilibria
- Mixed and Behavioral Strategies

Imperfect-information extensive-form games

- So far, we modeled agents that can **specify the actions they will take at every choice node**
  - This implies the **agents know the node they are**
  - And **all prior choices**, including those of other agents
- We called this **perfect-information games**!

Imperfect-information extensive-form games

- However, we might not want to **make such a strong assumption** about:
  - Our **agents**
  - And our **environment**

Imperfect-information extensive-form games

- In some situation, we might want to **model our agents** that:
  - **act with partial or no knowledge of the actions** taken by other agents
  - or even **agents with limited memory** of their own past actions

Imperfect-information extensive-form games

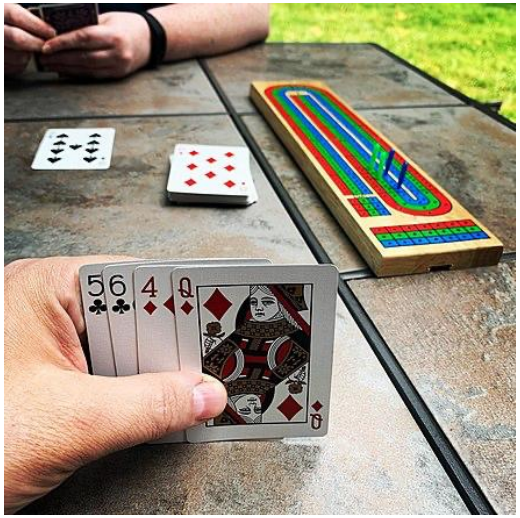
- However, we could **not model these type of agents with perfect-information agents**

- Hence, **imperfect-information games address this limitation**

Imperfect-information extensive-form games

- For example, the **following games involve hidden actions from other agents**:

Poker



Cribbage

Imperfect-information extensive-form games

- An imperfect-information game is an extensive-form game in which **each agent's choice nodes are partitioned into information sets**
- Intuitively, **if two choice nodes are in the same information set then the agent cannot distinguish between them**

Imperfect-information extensive-form games

- **Definition (Imperfect-information game):** *An imperfect-information game (in extensive form) is a tuple* $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$*, where:*

  - $(N, A, H, Z, \chi, \rho, \sigma, u)$ *is perfect-information extensive-form game; and*

  - $I = (I_1, \ldots, I_n)$*, where* $I_i = (I_{i,1}, \ldots, I_{i,ki})$ *is a set of equivalence classes on (i.e., a partition of)* $\{h \in H : \rho(h) = i\}$ *with the property that* $\chi(h) = \chi(h')$ *and* $\rho(h) = \rho(h')$ *whenever there exists a j for which* $h \in I_{i,j}$ *and* $h' \in I_{i,j}$.

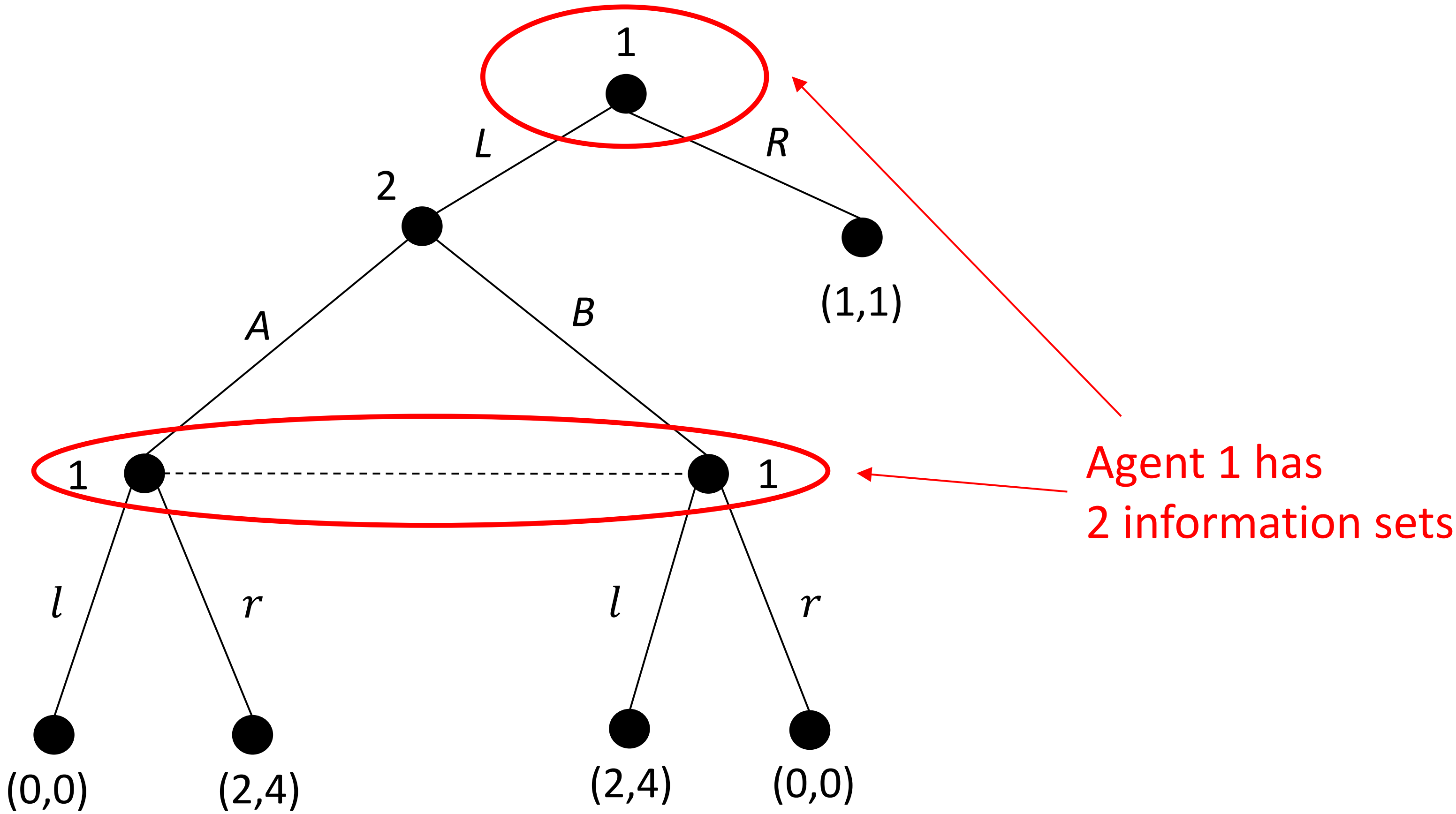
Imperfect-information extensive-form games

- **For the choice nodes to be truly indistinguishable**

  - We require that the **set of actions at each choice node in an information set be the same** (otherwise, the agent would be able to distinguish the nodes)

- Hence, if $I_{i,j} \in I$ is an equivalence class, **we can unambiguously use the notation $\chi(\boldsymbol{I_{i,j}})$** to denote the set of actions available to agent i at any node in information set $I_{i,j}$

Imperfect-information extensive-form games



Interpretation: Agent 1 does not know whether Agent 2 has chosen *A* or *B* when he has to decide between l or r

Outline

- Imperfect-information games in extensive form
- **Strategies and equilibria**
- Mixed and Behavioral Strategies

Strategies and Equilibria

- A **pure strategy** for an agent in an imperfect-information game is:
  - **one of the available actions in each information set** of that agent

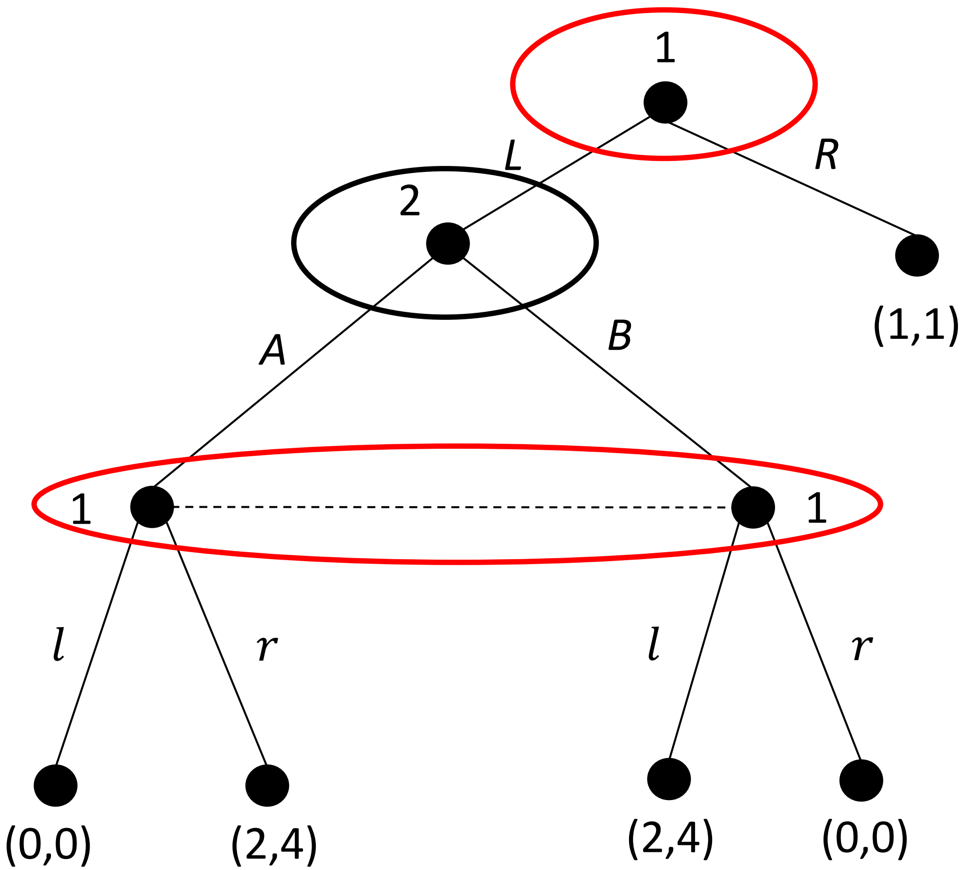
Strategies and Equilibria

- **Definition (Pure strategies):** *Let* $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ *be a imperfect-information extensive-form game. Then the pure strategies of agent* i *consist of the Cartesian product*

$$\Pi_{I_{i,j} \in I_i} \chi(I_{i,j}).$$

Action function

Imperfect-information extensive-form games



Pure Strategies:

S_1 = {(L,l), (L,r), (R,l), (R,r)}

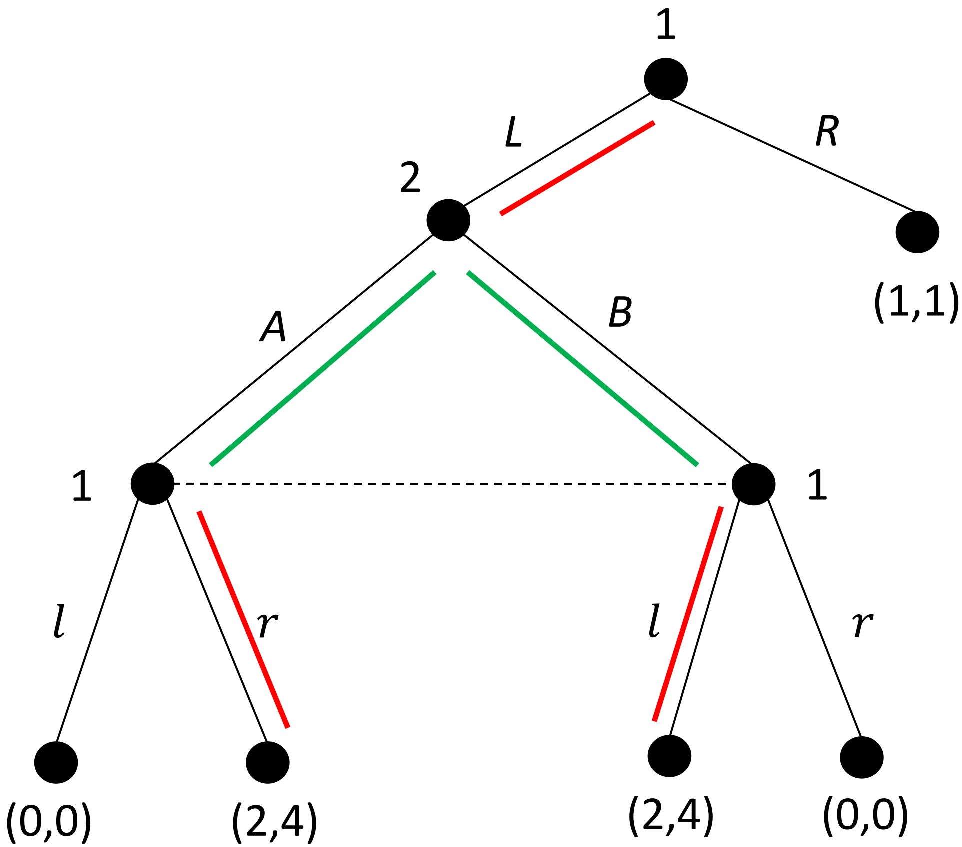
S_2 = {A,B}

Strategies and Equilibria

- **So how do we compute the Nash equilibria?**
  - The definition of best response is the same as we've seen so far!
  - The Nash Equilibrium (both pure and mixed) concept remains the same for imperfect-information extensive-form games

Strategies and Equilibria

- We convert an imperfect-information game to an equivalent normal-form game and compute the Nash equilibria



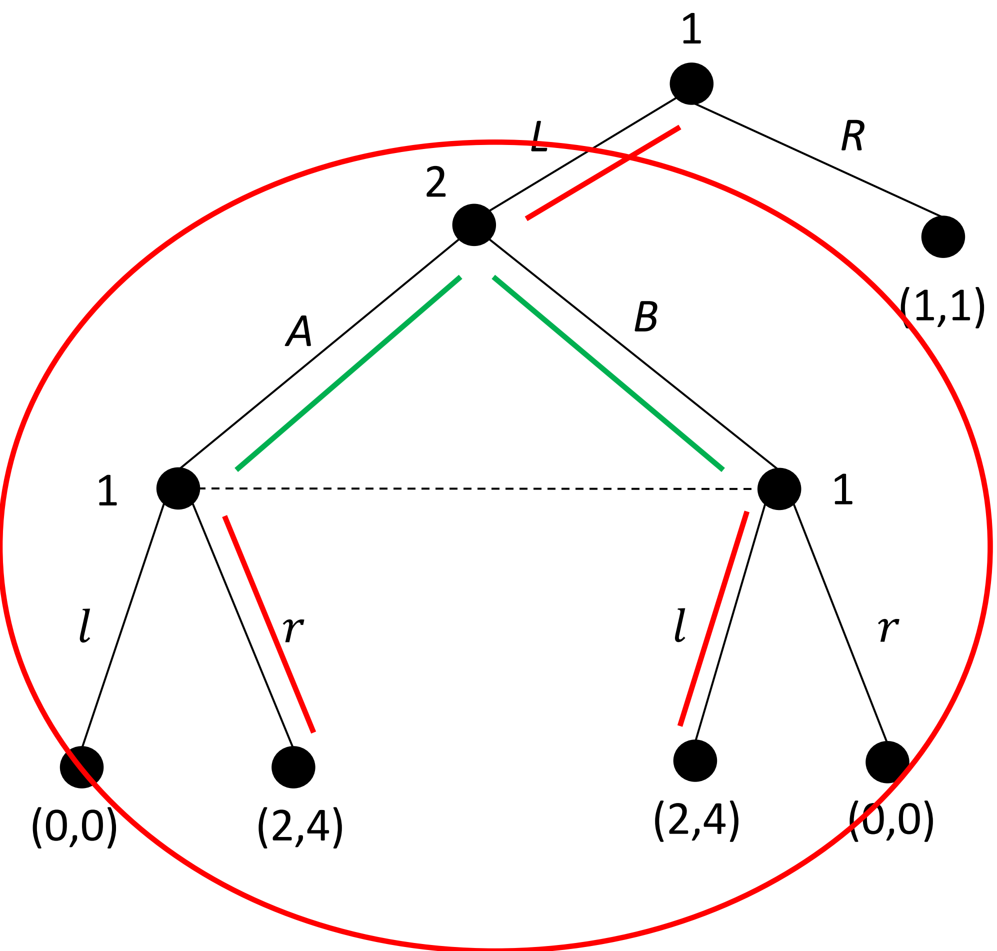
	A	***B***
(L,l)	0, 0	$\underline{2}$, $\underline{4}$
(L,r)	$\underline{2}$, $\underline{4}$	0, 0
(R,l)	1, $\underline{1}$	1, $\underline{1}$
(R,r)	1, $\underline{1}$	1, $\underline{1}$

Nash equilibria in this game:
{(L,l), B} and {(L,r), A}

Does this make sense?

Strategies and Equilibria

- Let us look at this subgame and **convert it into a normal-form game** where both **agents select actions simultaneously**



Coordination game

	A	***B***
l	0, 0	$\underline{2}$, $\underline{4}$
r	$\underline{2}$, $\underline{4}$	0, 0

Pure strategy Nash equilibria:
{l, B} and {r, A}

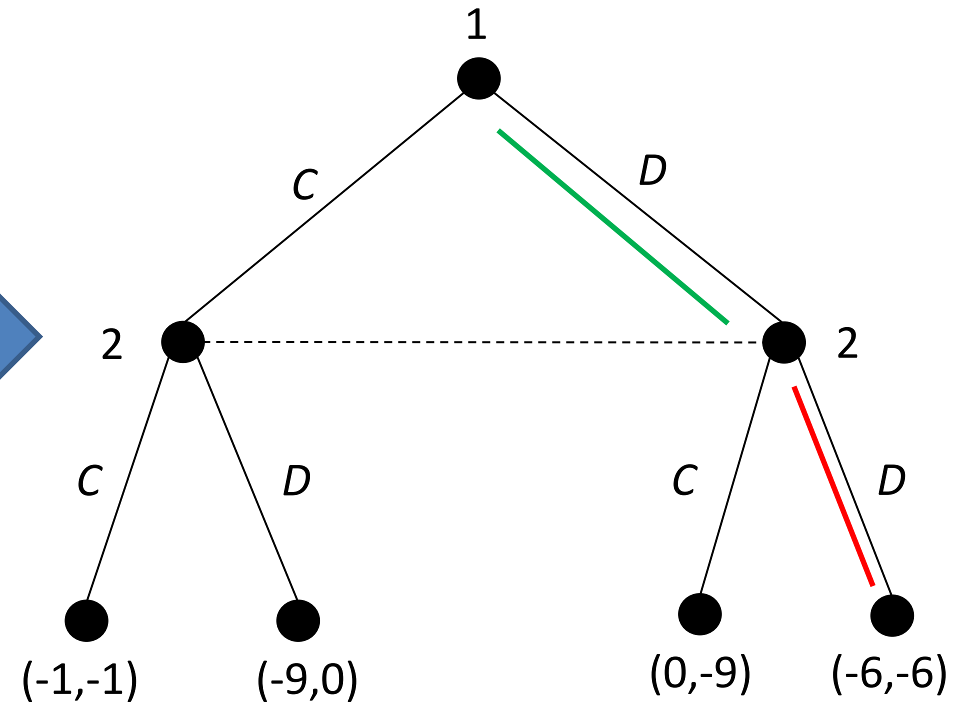
Mixed strategy Nash equilibrium:
$\{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$

Strategies and Equilibria

- Converting a normal-form game to an imperfect-information extensive-form game

	C	***D***
C	-1, -1	-9, 0
D	0, -9	-6, -6

Prisoner's Dilemma



Perfect-information games were not expressive enough to capture the Prisoner's Dilemma game (and many other ones)

Strategies and Equilibria

So these mappings are possible

Imperfect-information extensive-form game	→	Normal-form game
Normal-form game	→	Imperfect-information extensive-form game

Strategies and Equilibria

What happens if we apply each mapping in turn?

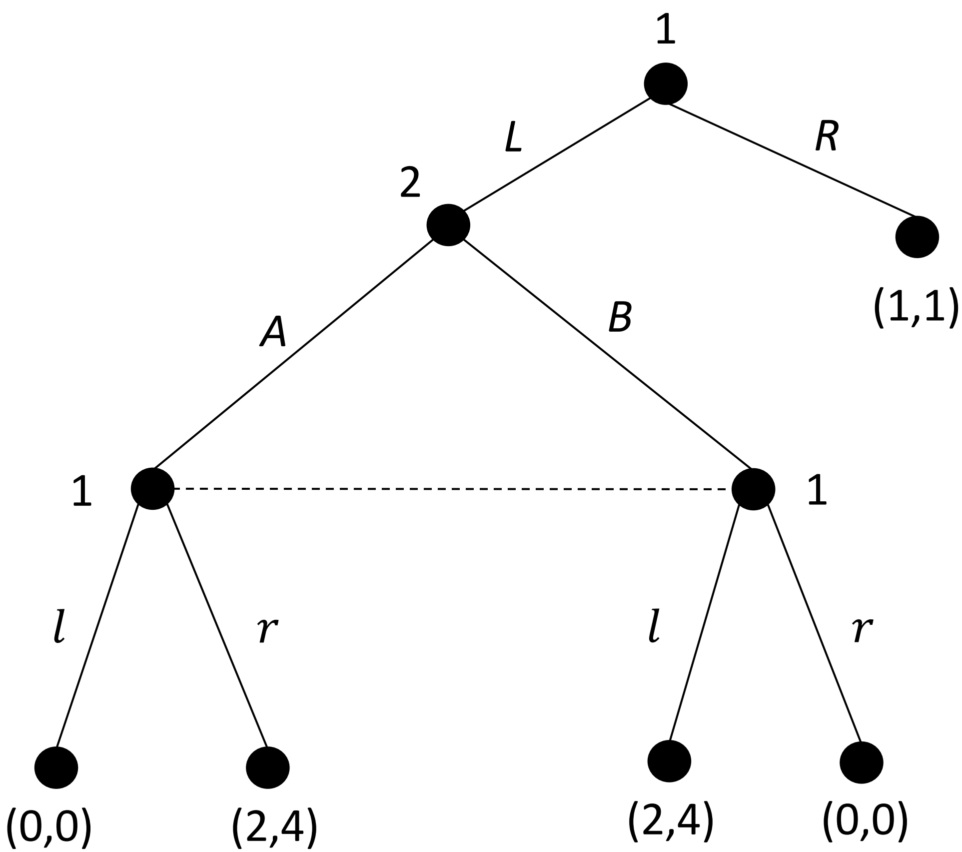
Imperfect-information extensive-form game → Normal-form game

Normal-form game → Imperfect-information extensive-form game

Do we end up with the same game?

Strategies and Equilibria

- Imperfect-information game to an equivalent normal-form game

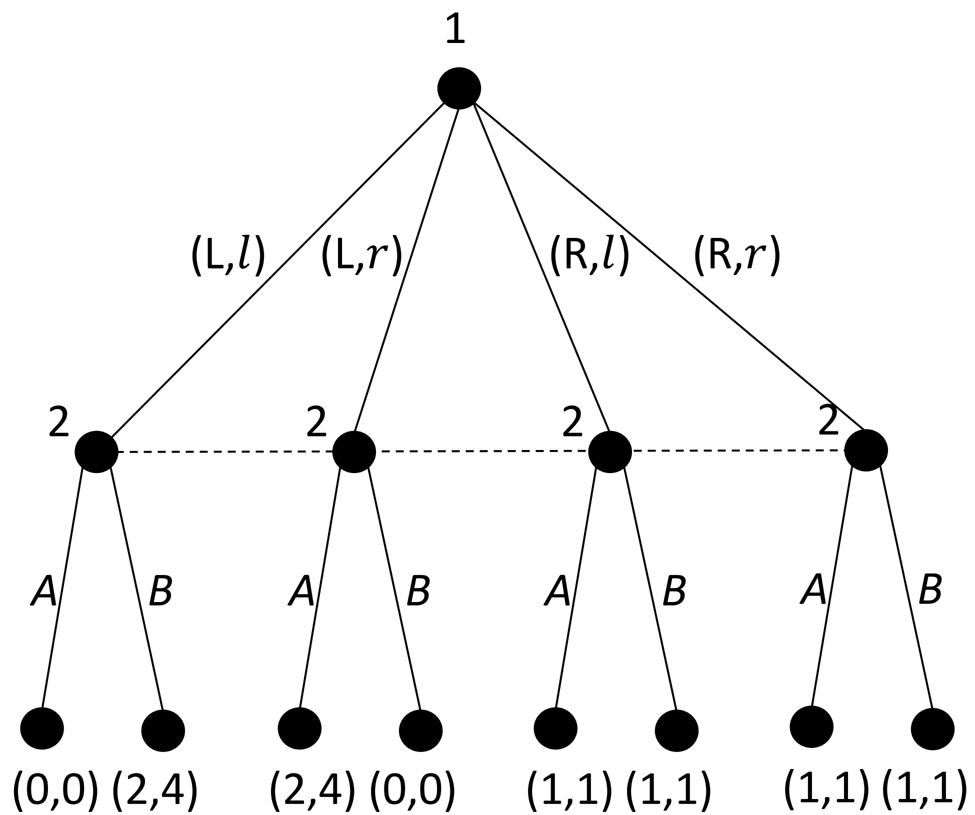


	A	***B***
(L,l)	0, 0	2, 4
(L,r)	2, 4	0, 0
(R,l)	1, 1	1, 1
(R,r)	1, 1	1, 1

Strategies and Equilibria

- Normal-form game to an equivalent imperfect-information game

	A	***B***
(L,l)	0, 0	2, 4
(L,r)	2, 4	0, 0
(R,l)	1, 1	1, 1
(R,r)	1, 1	1, 1



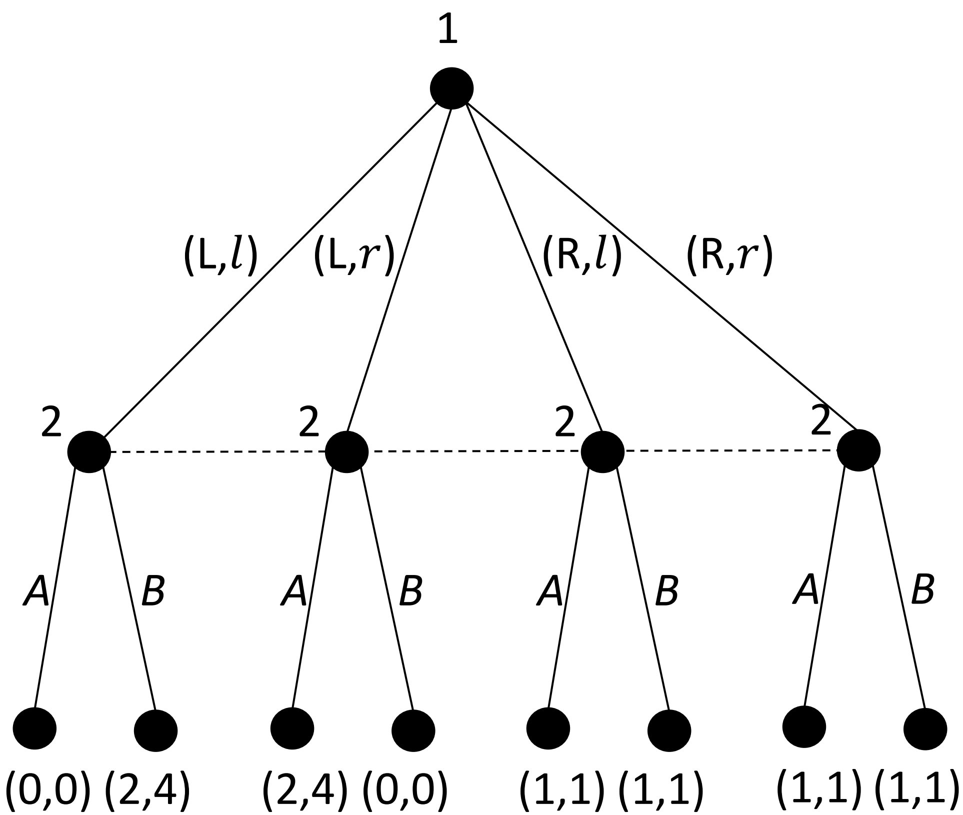
Not the same game!

Strategies and Equilibria

- What happens if we apply each mapping in turn?
  - We might not end up with the same game
  - However, **we do get one with the same strategy space and equilibria!**

Strategies and Equilibria

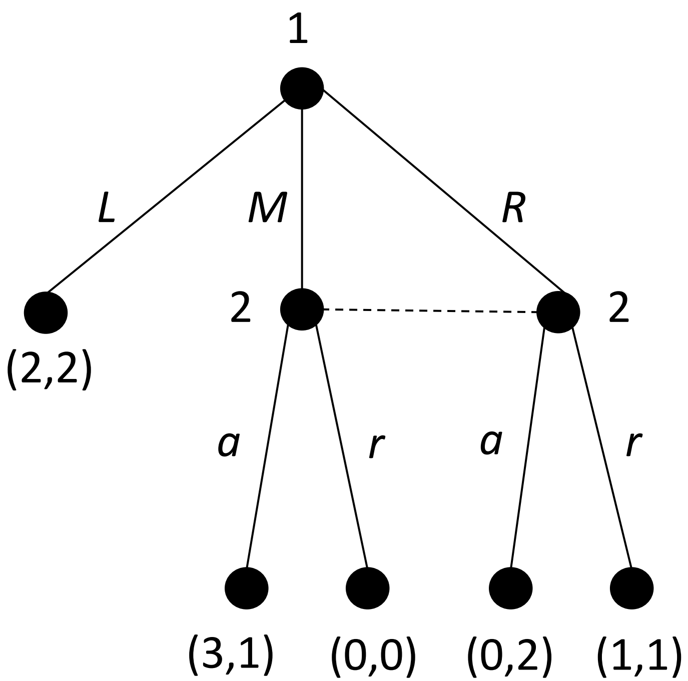
- **We do get one with the same strategy space and equilibria!**



	A	***B***
(L,l)	0, 0	$\underline{2}$, $\underline{4}$
(L,r)	$\underline{2}$, $\underline{4}$	0, 0
(R,l)	1, $\underline{1}$	1, $\underline{1}$
(R,r)	1, $\underline{1}$	1, $\underline{1}$

We end up with strategically equivalent games!

Exercise



1 - Present the pure strategies

2 - Convert this game into an equivalent normal-form game

3 - Find the Nash equilibria

Outline

- Imperfect-information games in extensive form
- Strategies and equilibria
- **Mixed and Behavioral Strategies**

Mixed and Behavioral Strategies

- **Two meaningful different kinds of randomized strategies** in imperfect-information extensive-form games (and perfect-information game too)
  - Mixed strategies
  - Behavioral strategies

Mixed and Behavioral Strategies

- **Mixed strategies**: randomize over pure strategies
- **Behavioral strategy**: randomize every time an information set is encountered

Mixed and Behavioral Strategies

- **Definition (mixed strategy):** *Let* $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ *be a imperfect-information extensive-form game. Then a mixed strategy* s_i *is any distribution over of agent* i*'s pure strategies*

$$s_i \in \Delta(A^{I_i})$$

Mixed and Behavioral Strategies

- **Definition (behavioral strategy):** *A behavioral strategy b_i is a mapping from an agent's information sets to a distribution over the actions at that information set, which is sampled independently each time the agent arrives at the information set:*

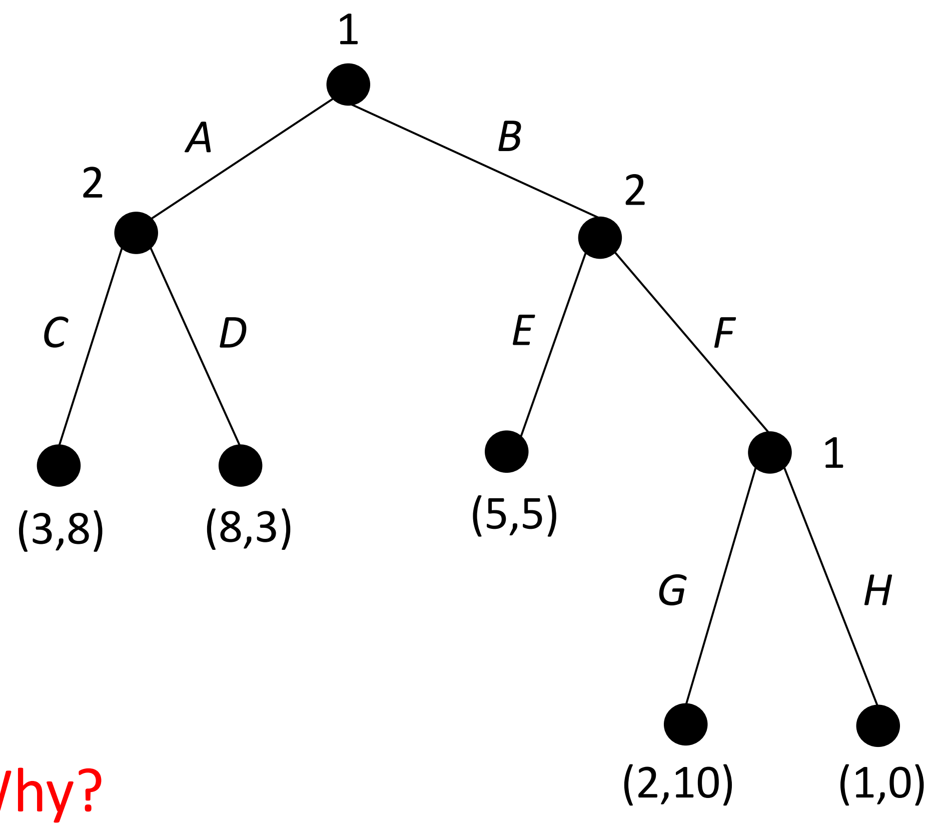
$$b_i \in [\Delta\big(\chi(I)\big)]_{I \in I_i}$$

Mixed and Behavioral Strategies

For example:

Mixed strategy:
[0.6 : (*A*,*G*), 0.4 : (*B*,*H*)]

Behavioral strategy:
([0.6 : *A*, 0.4 : *B*],
[0.6 : *G*, 0.4 : *H*])



Are the strategies equivalent? Why?

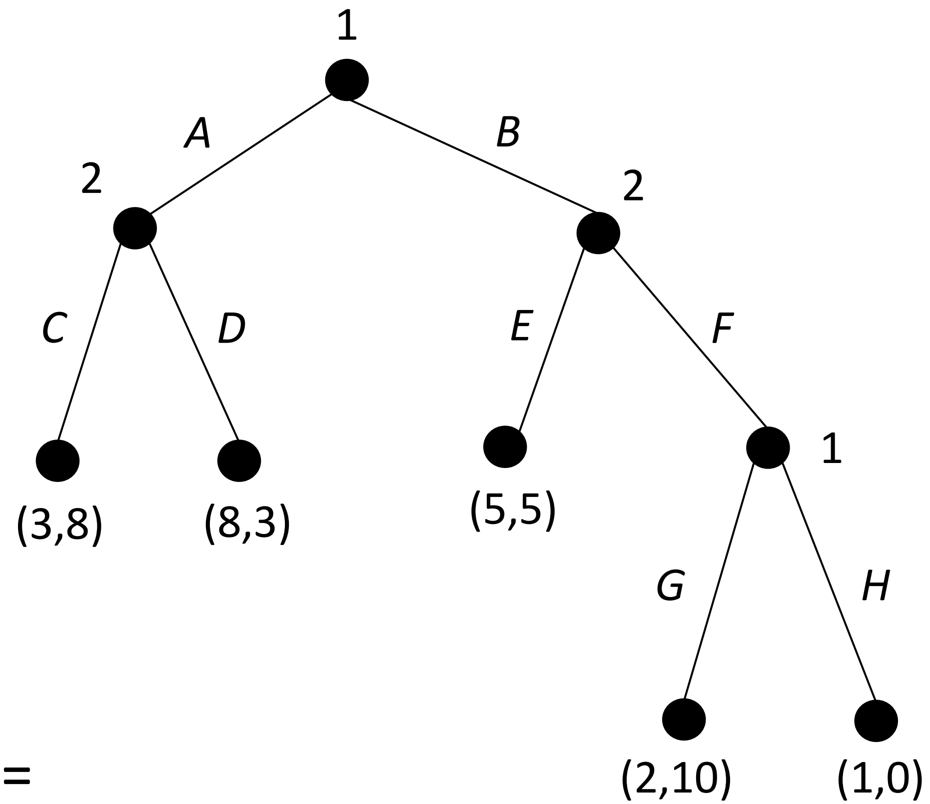
Mixed and Behavioral Strategies

Are the strategies equivalent? Why?

Mixed strategy:
[0.6 : (*A*,*G*), 0.4 : (*B*,*H*)] **NO**

Behavioral strategy:
([0.6 : *A*, 0.4 : *B*],
[0.6 : *G*, 0.4 : *H*])

Mixed strategy:
[0.6 x 0.6 : (*A*,*G*), 0.6 x 0.4 : (*A*,*H*),
0.4 x 0.6 : (*B*,*G*), 0.4 x 0.4 : (*B*,*H*)] =
[0.36 : (*A*,*G*), 0.24 : (*A*,*H*),
0.24 : (*B*,*G*), 0.16 : (*B*,*H*)] **YES**



Mixed and Behavioral Strategies

- Although mixed strategy and behavioral strategy are defined differently
- In perfect-information games, **mixed strategy and behavioral strategy can emulate each other** [Kuhn, 1953]

Mixed and Behavioral Strategies

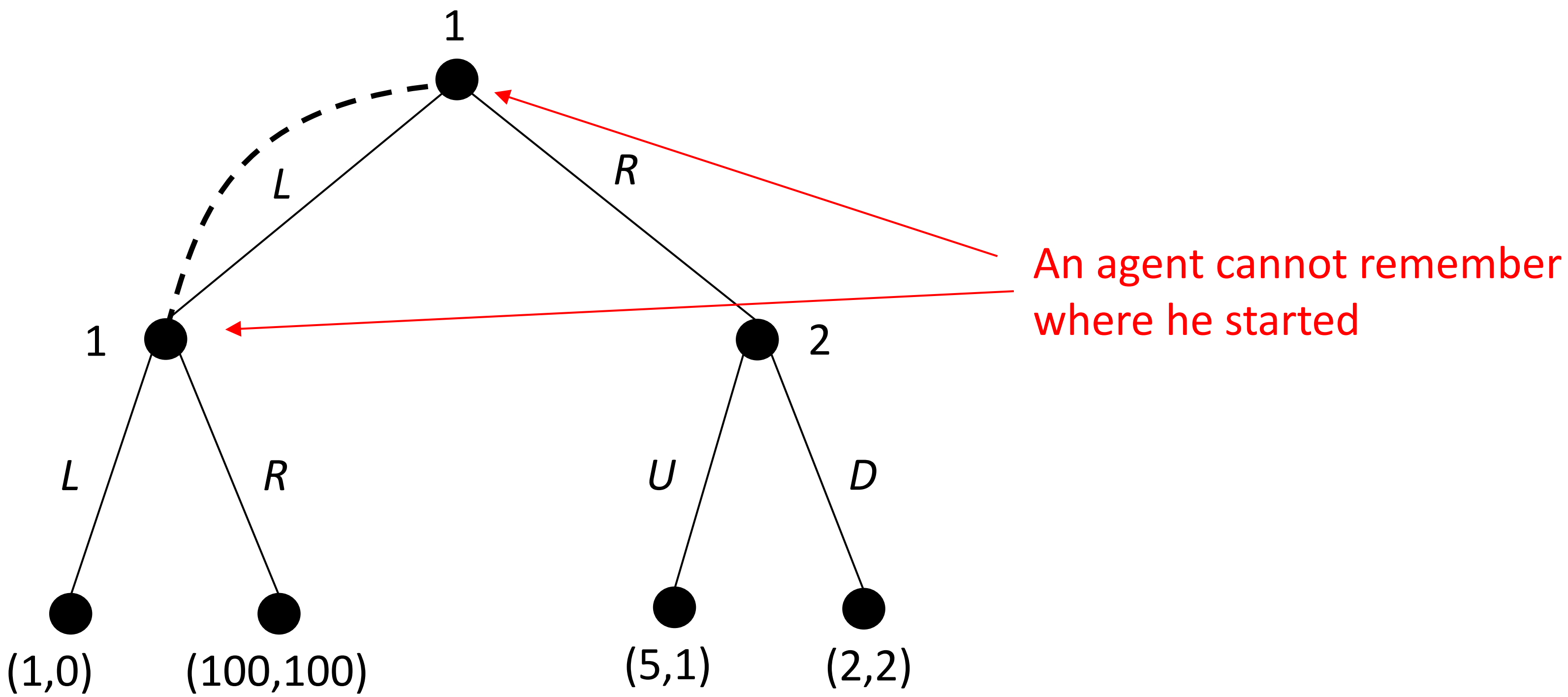
- This **can also apply to imperfect-information games,** but they need to have **perfect recall**

- **Theorem** [Kuhn, 1953]
  - In a **game of perfect recall**, any mixed strategy of a given agent can be replaced by an equivalent behavioural strategy, and any behavioural strategy can be replaced by an equivalent mixed strategy

Mixed and Behavioral Strategies

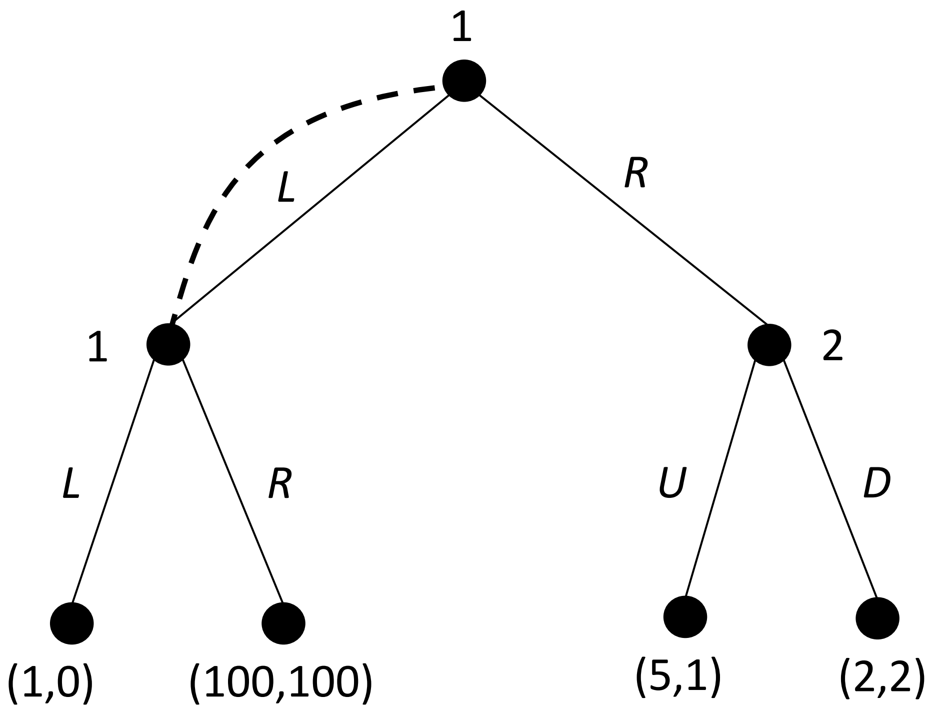
- Intuitively, a game of **perfect recall** needs to have **agents that can have full recollection of the experience** in the game
  - The agents **know all the information sets they visited previously**
  - The agents **know all the actions they have taken**

Mixed and Behavioral Strategies



Game of imperfect recall

Mixed and Behavioral Strategies

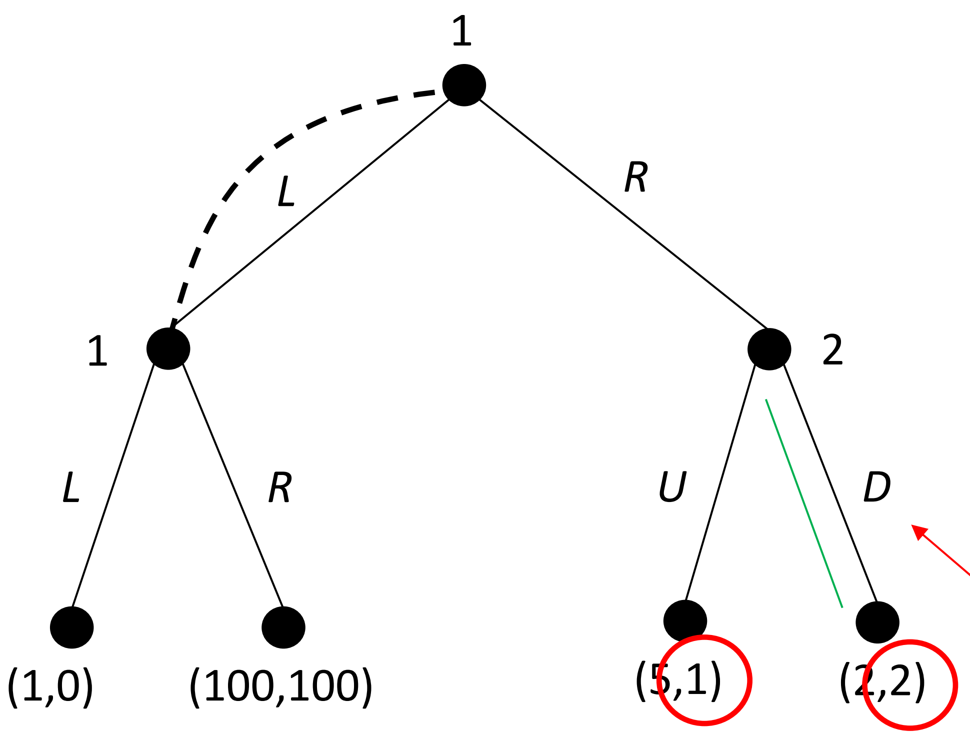


What are the pure strategies?

1: {L, R}
2: {U, D}

Game of imperfect recall

Mixed and Behavioral Strategies

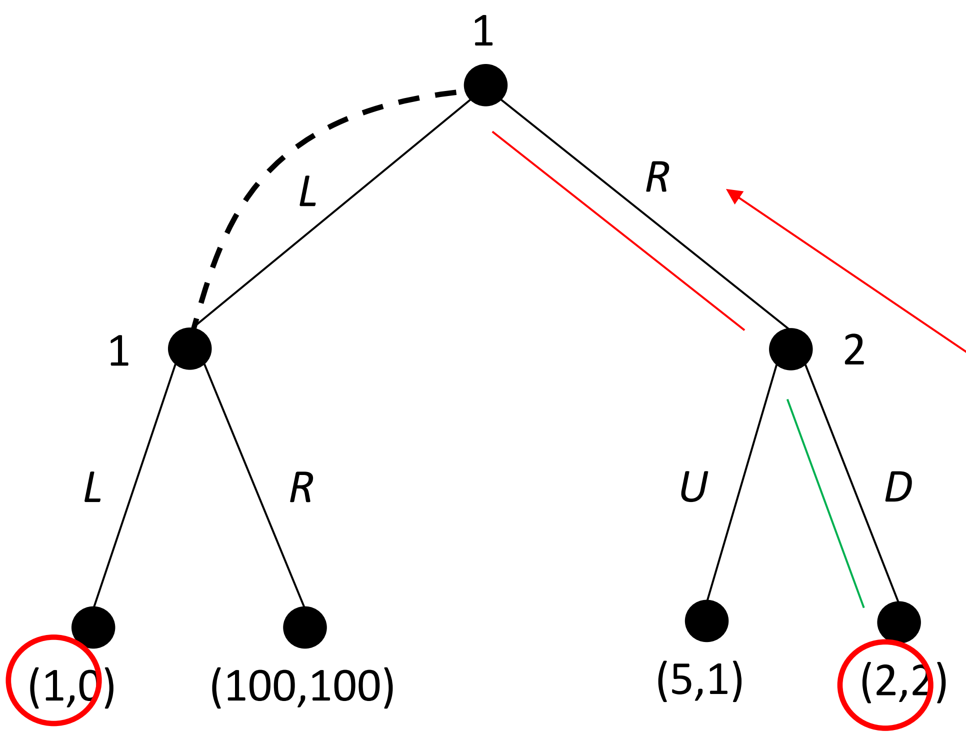


What is the mixed strategy Nash equilibrium?

Note that Agent 2 has a dominant strategy (D)

Game of imperfect recall

Mixed and Behavioral Strategies

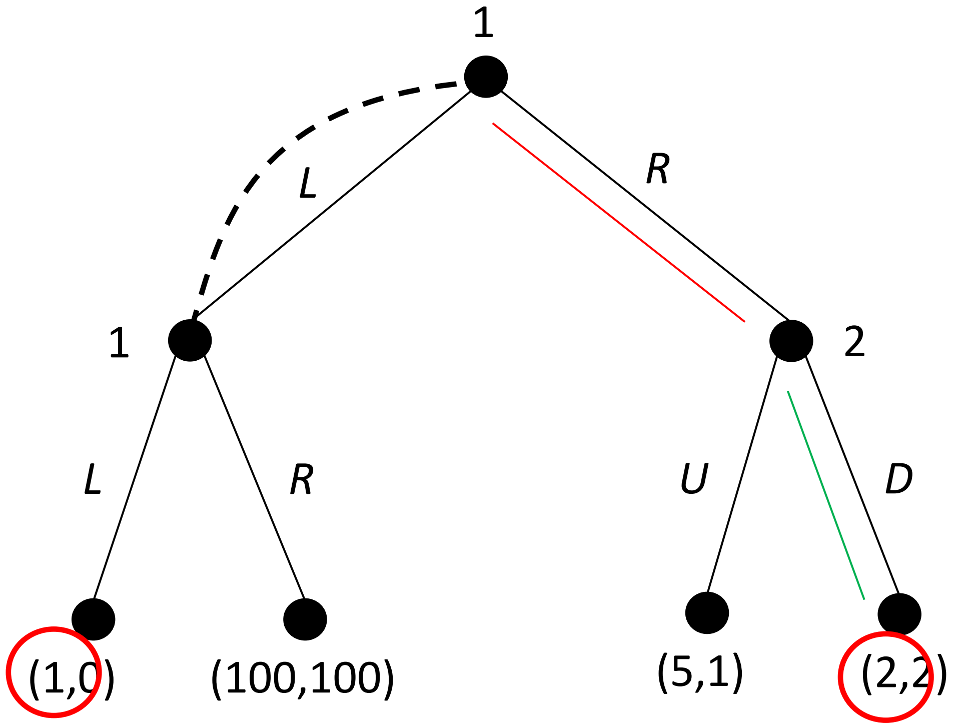


What is the mixed strategy Nash equilibrium?

Hence, agent 1's best response is *R*

Game of imperfect recall

Mixed and Behavioral Strategies



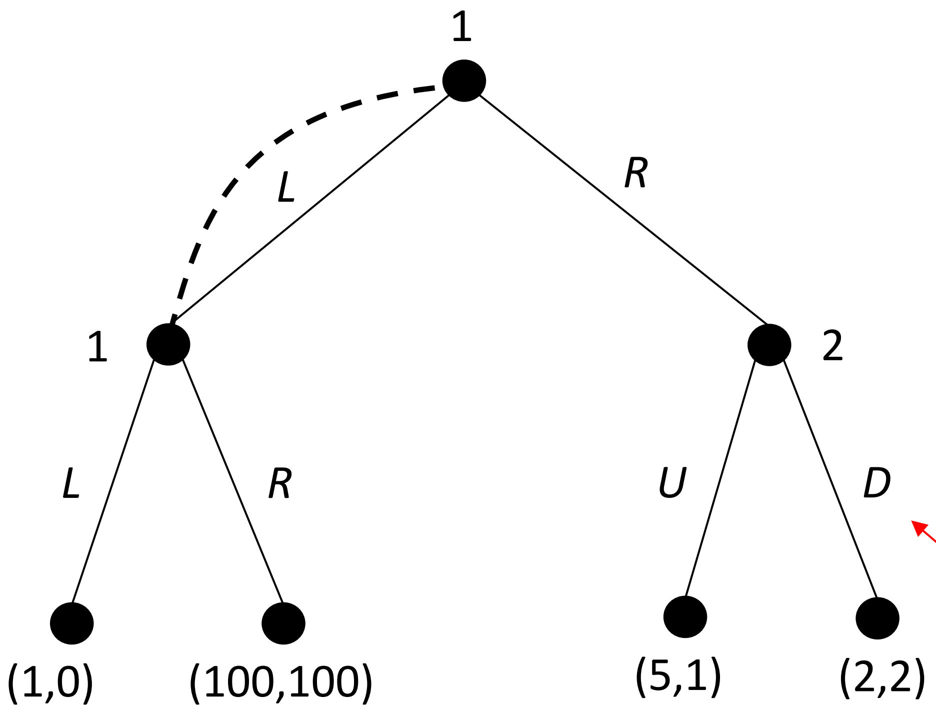
What is the mixed strategy Nash equilibrium?

(*R*, *D*) or

(0, 1),(0, 1)

Game of imperfect recall

Mixed and Behavioral Strategies

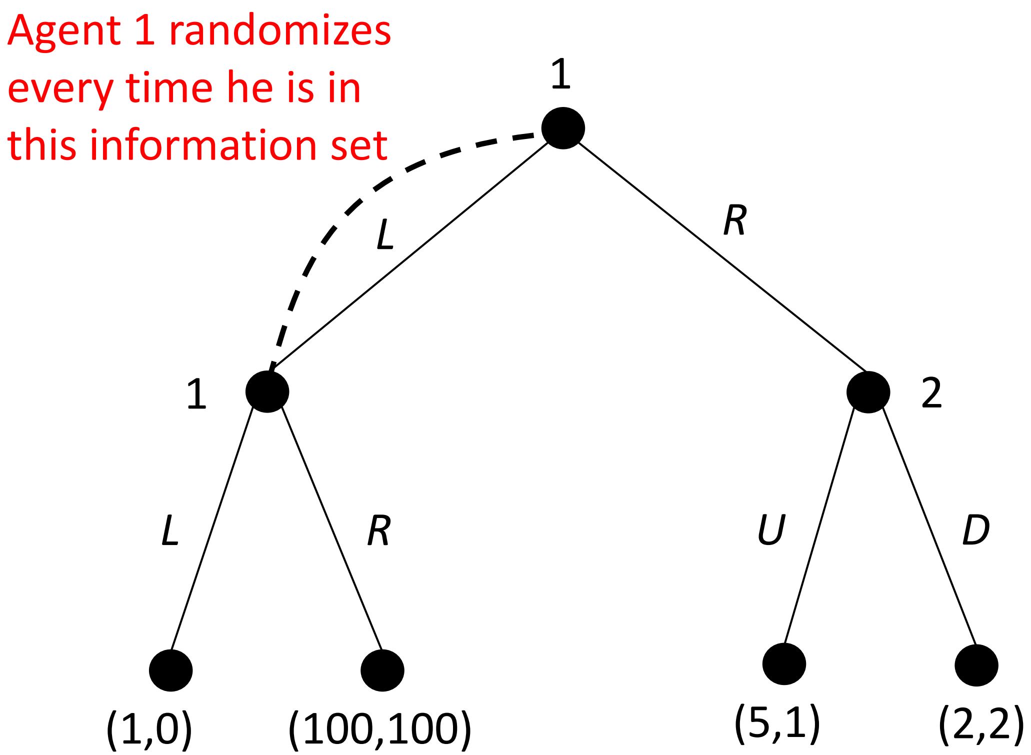


What is an equilibrium in behavioral strategy?

First note that Agent 2 still has a dominant strategy (D)

Game of imperfect recall

Mixed and Behavioral Strategies



What is an equilibrium in behavioral strategy?

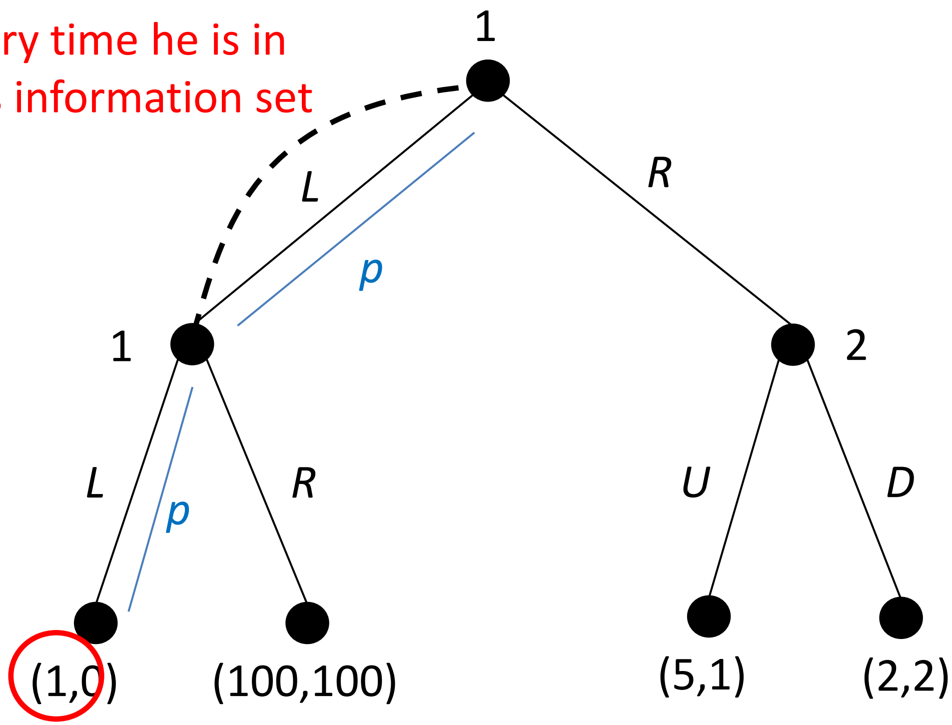
If Agent 1 uses a behavioral strategy $(p, 1 - p)$

What is agent 1's expected utility?

Game of imperfect recall

Mixed and Behavioral Strategies

Agent 1 randomizes every time he is in this information set



What is an equilibrium in behavioral strategy?

If Agent 1 uses a behavioral strategy (p, $1 - p$)

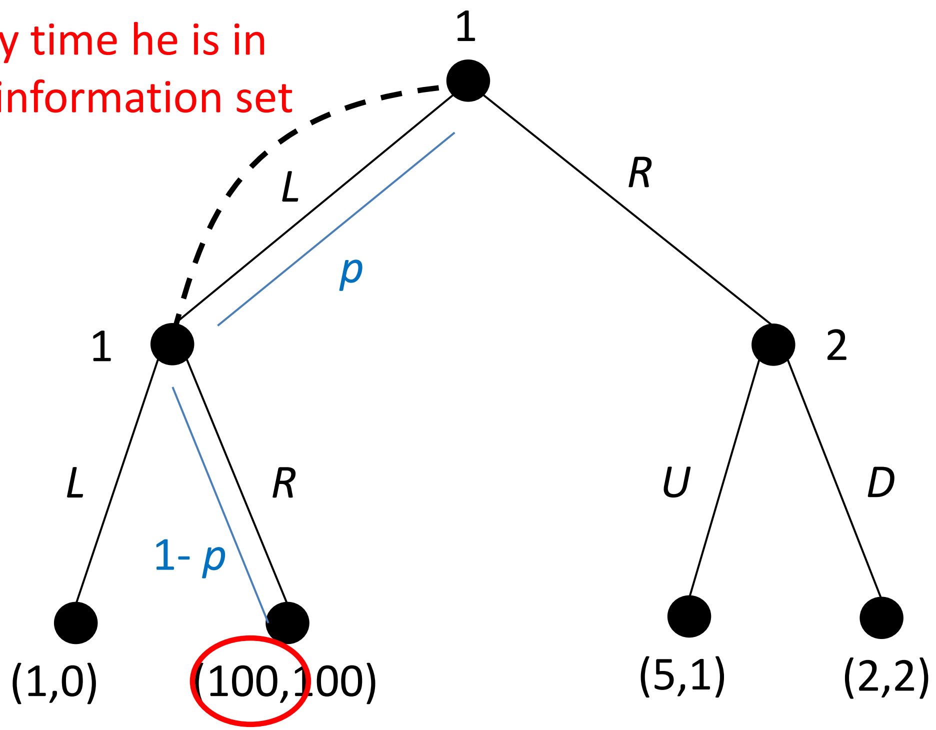
Agent 1's expected utility is:

$1 \times p \times p$

Game of imperfect recall

Mixed and Behavioral Strategies

Agent 1 randomizes every time he is in this information set



What is an equilibrium in behavioral strategy?

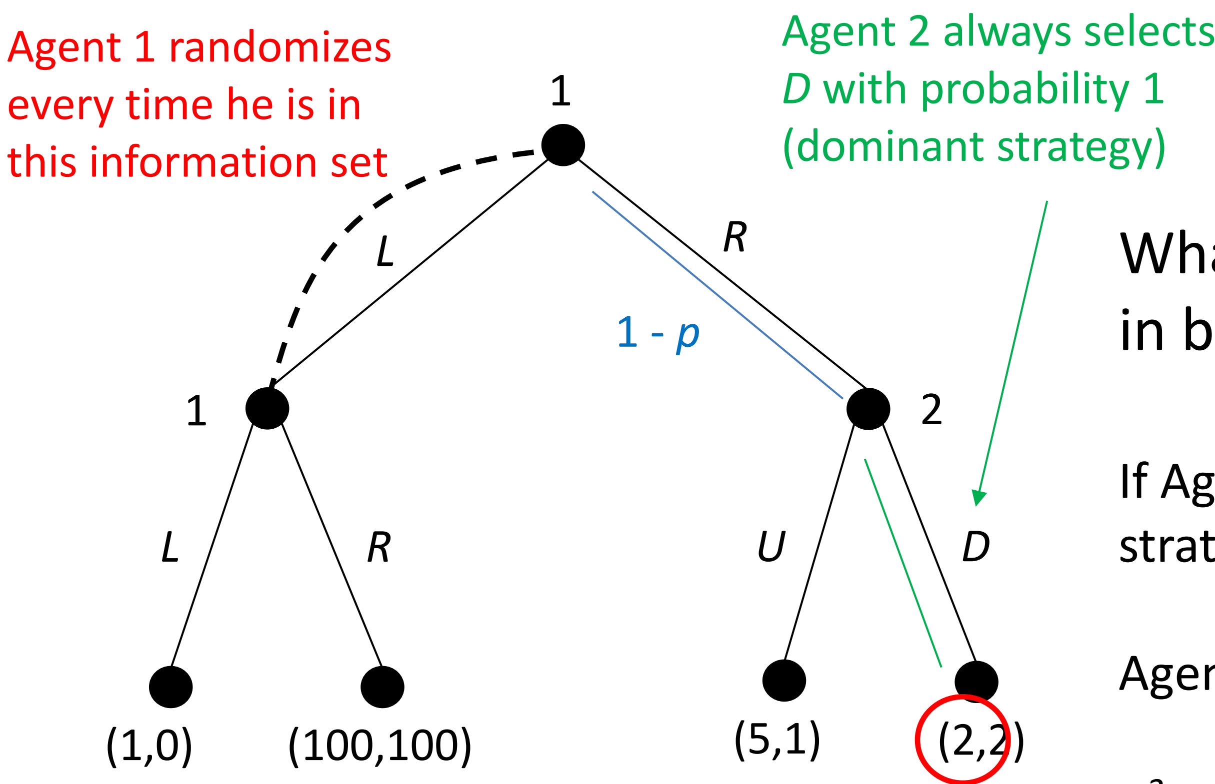
If Agent 1 uses a behavioral strategy $(p, 1 - p)$

Agent 1's expected utility is:

$p^2 + 100 \times p \times (1 - p)$

Game of imperfect recall

Mixed and Behavioral Strategies



What is an equilibrium in behavioral strategy?

If Agent 1 uses a behavioral strategy $(p, 1 – p)$

Agent 1’s expected utility is:

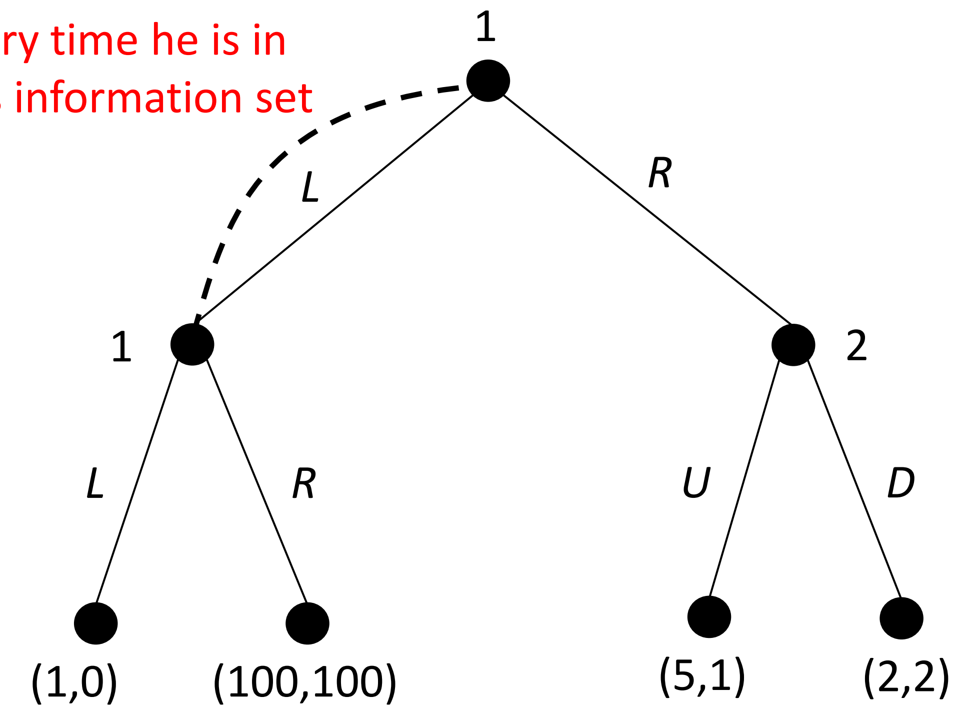
$p^2 + 100\, p\, (1 – p) + 2 \times (1 – p) =$

$-99\, p^2 + 98\, p + 2$

Game of imperfect recall

Mixed and Behavioral Strategies

Agent 1 randomizes every time he is in this information set



Game of imperfect recall

What is an equilibrium in behavioral strategy?

What is Agent 1's maximum expected utility?

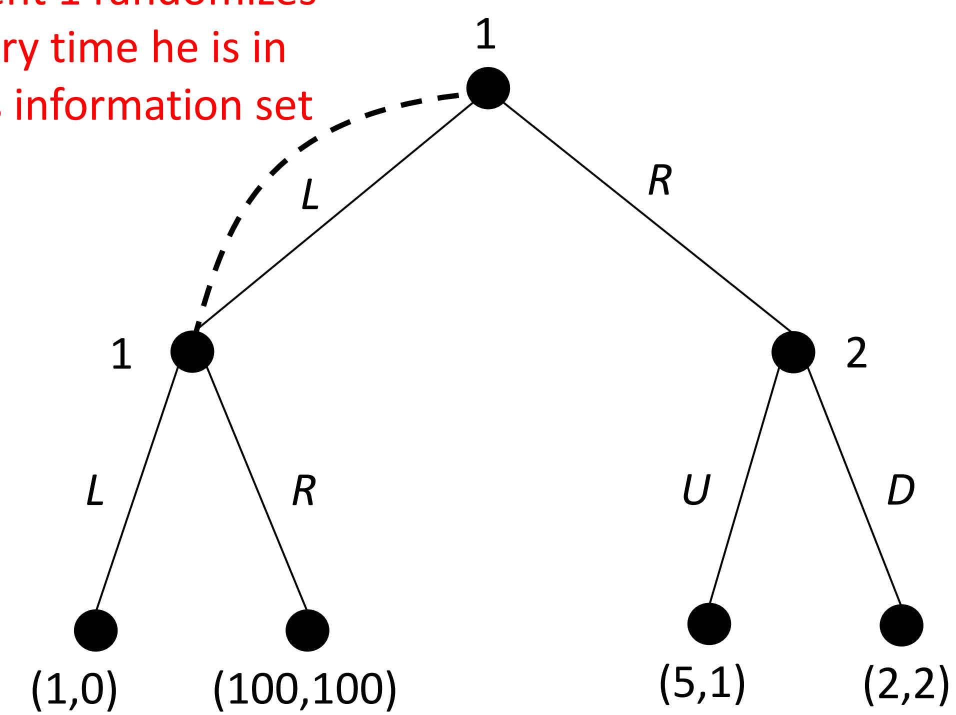
$$\frac{d}{dp}(-99\,p^2 + 98\,p + 2) = 0$$

$$-198\,p + 98 = 0$$

$$p = \frac{98}{198}$$

Mixed and Behavioral Strategies

Agent 1 randomizes every time he is in this information set



What is an equilibrium in behavioral strategy?

$$\left(\frac{98}{198}, \frac{100}{198}\right), (0, 1)$$

Thus, the mixed strategy equilibrium can be different from the behavioral strategy equilibrium in an imperfect recall game

Game of imperfect recall

Thank You

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