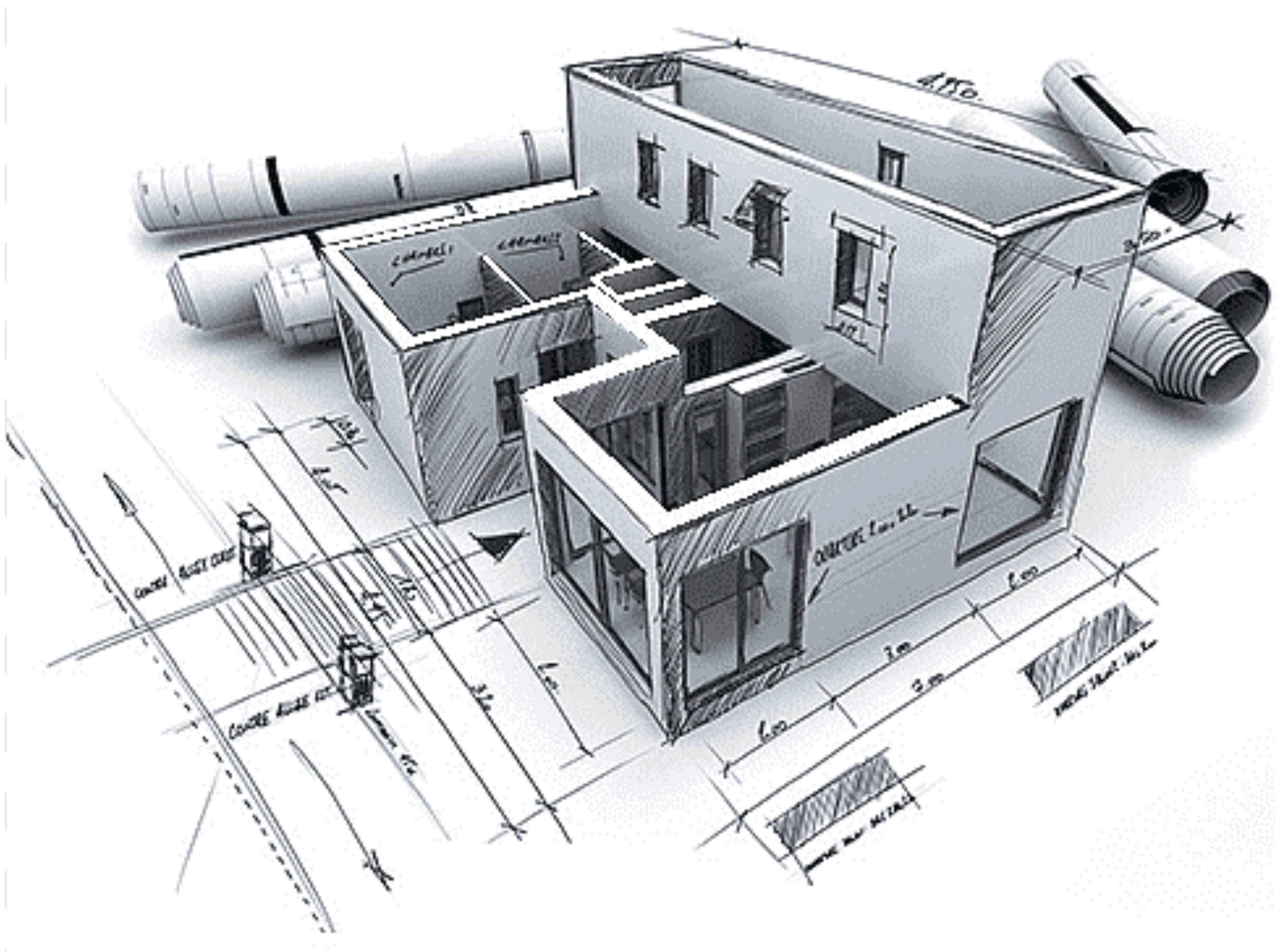
TÉCNICO LISBOA

Planning, Learning and Intelligent Decision Making

Lecture 2

PADInt 2023



MC examples (modeling)

Example: PageRank

- PageRank is the algorithm used by Google to rank a set of connected documents

- It simulates a "random bot" navigating the web of retrieved documents

TÉCNICO LISBOA

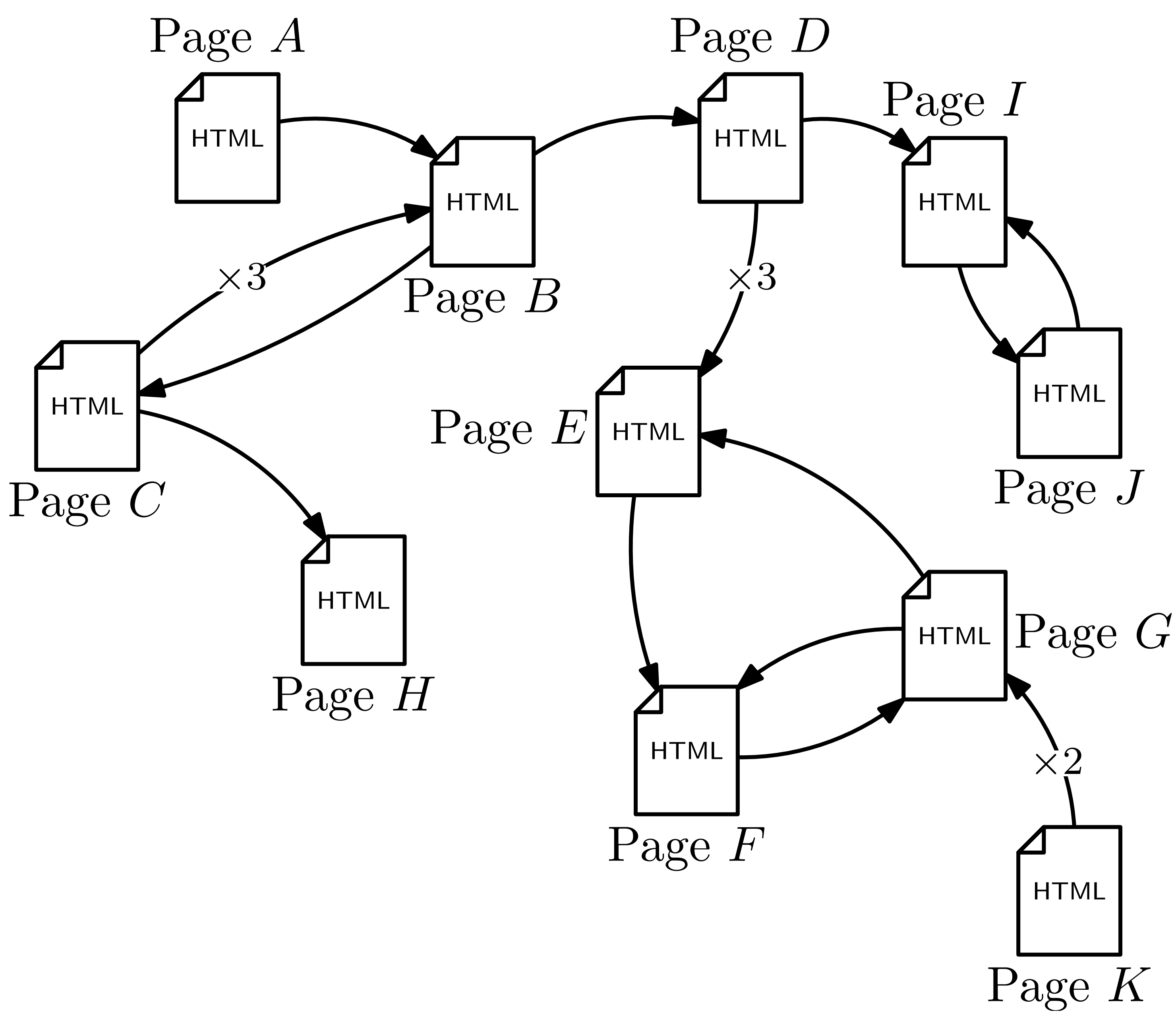
Example: PageRank

- Upon visiting a page, the bot randomly moves to one of the pages linked by the current one
- The rank corresponds to the "amount of time" that the bot spends on each page

TÉCNICO LISBOA

Example: PageRank

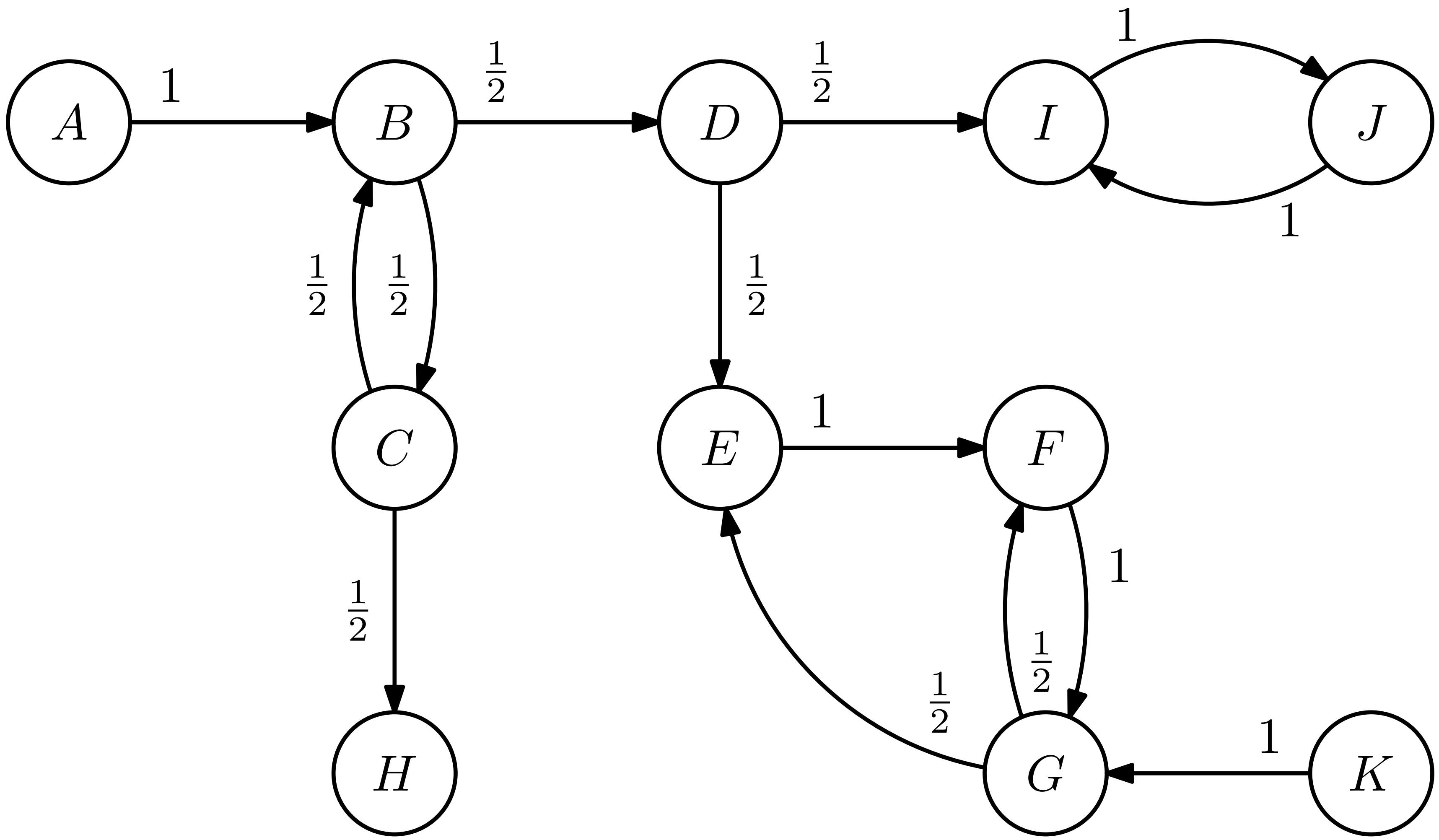
- Given the web of documents:



Example: PageRank

- States: $\mathcal{X} = \{A,\ B,\ C,\ D,\ E,\ F,\ G,\ H,\ I,\ J,\ K\}$

- Transitions:



Stability of MCs

What is stability?

- It is often important to understand how a MC behaves in the long run

  - **Stability** concerns the **long-term behavior** of the chain

TÉCNICO LISBOA

What is stability?

- We may want to know:
  - Depending on where the chain starts, can it **reach** any other state?
  - Is the behavior of the chain **cyclic**?
  - How **frequently** does the chain visit each state?

Irreducibility

- A state y can be reached from a state x if

$$\mathbf{P}^t(y \mid x) > 0$$

Positive probability of visiting y after visiting x

for some t

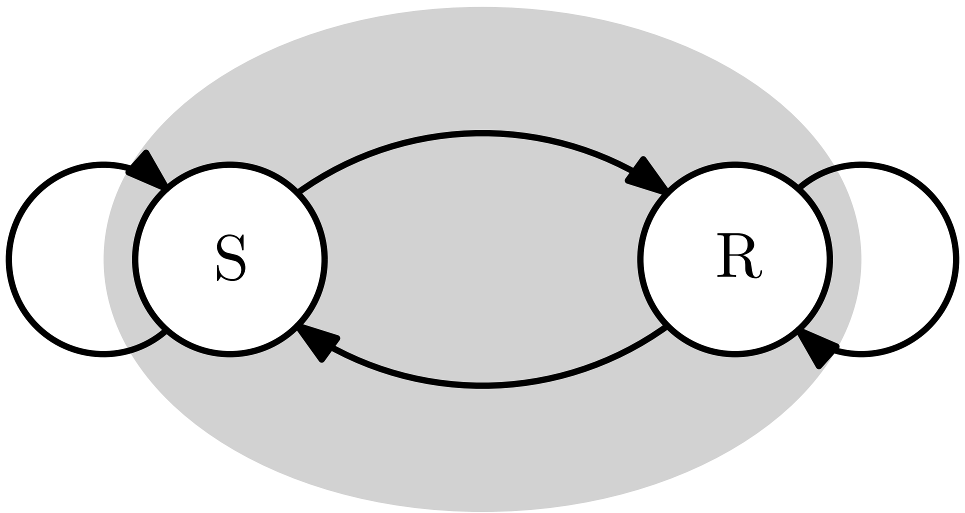
- A chain is **irreducible** if any state y can be reached from any other state x

Irreducibility

- We can split the state space of a chain in communicating classes (sets of states that are mutually reachable)

Irreducibility

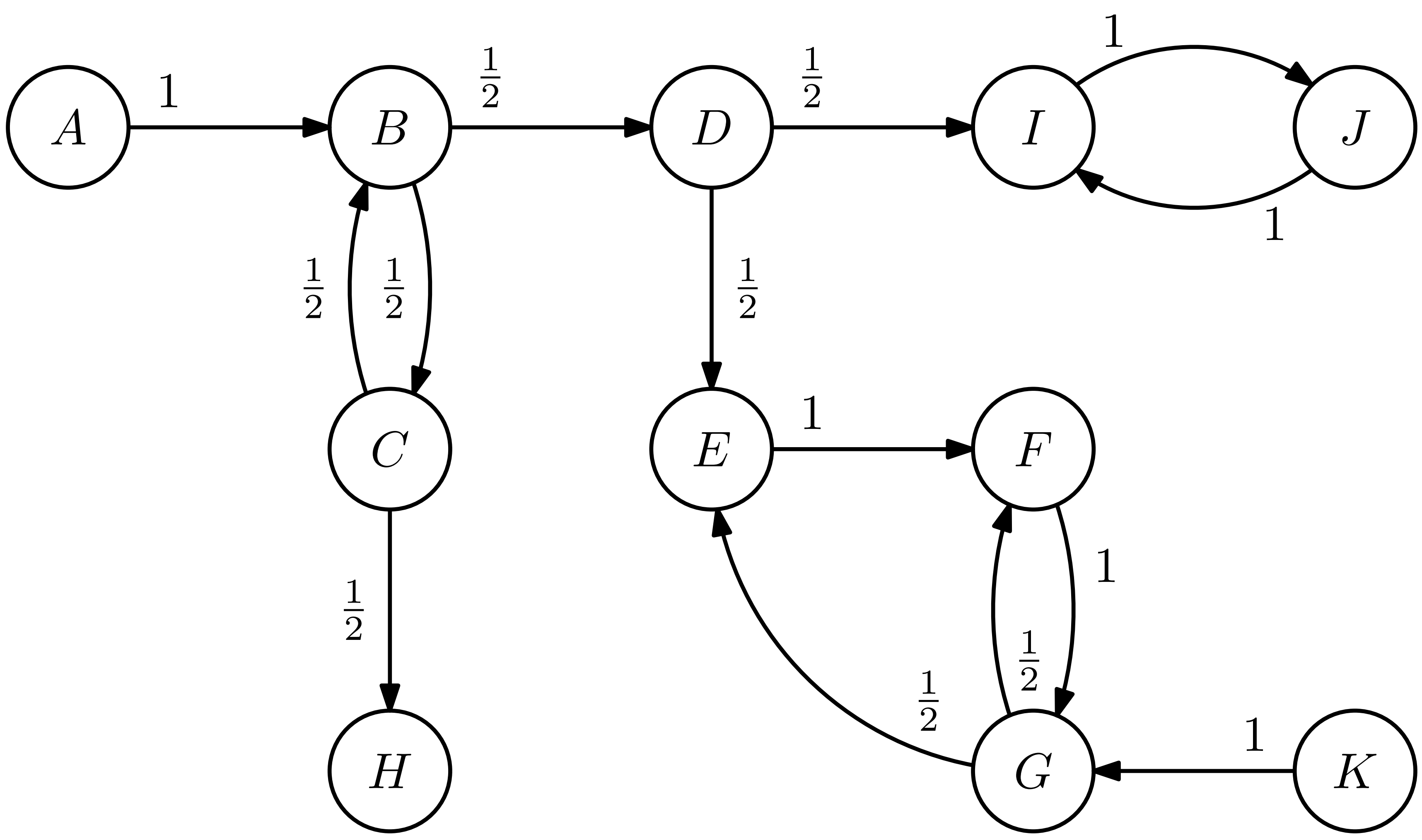
- In the weather example:



Single communicating class
$\Rightarrow$ irreducible chain

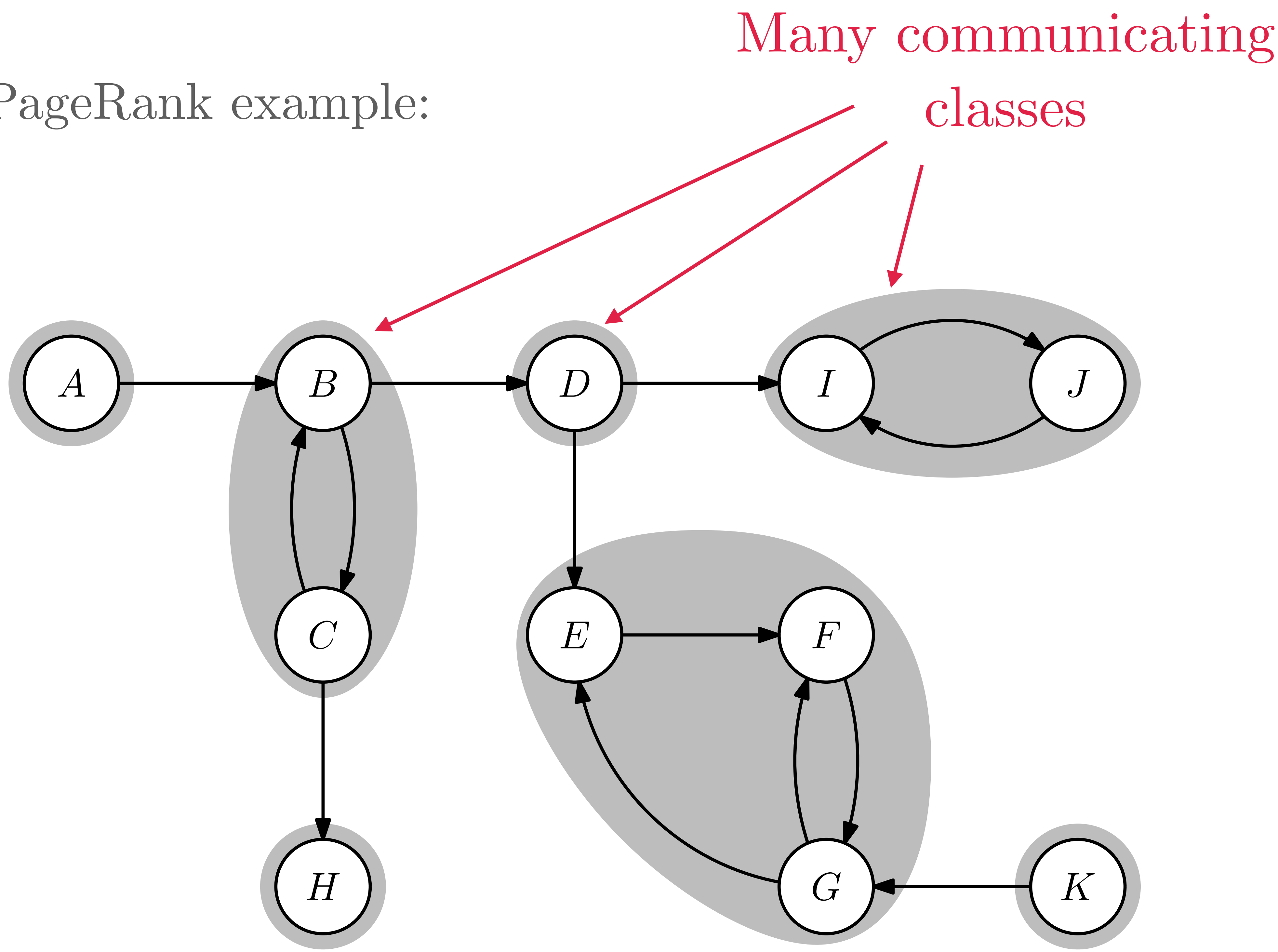
Irreducibility

- In the PageRank example:



Irreducibility

- In the PageRank example:



Aperiodicity

- The period of a state x is...
  - the **greatest common divider**...
  - of all time steps in which x can be visited...
  - if the chain departs from x

Aperiodicity

- Formally, the period of x is the number

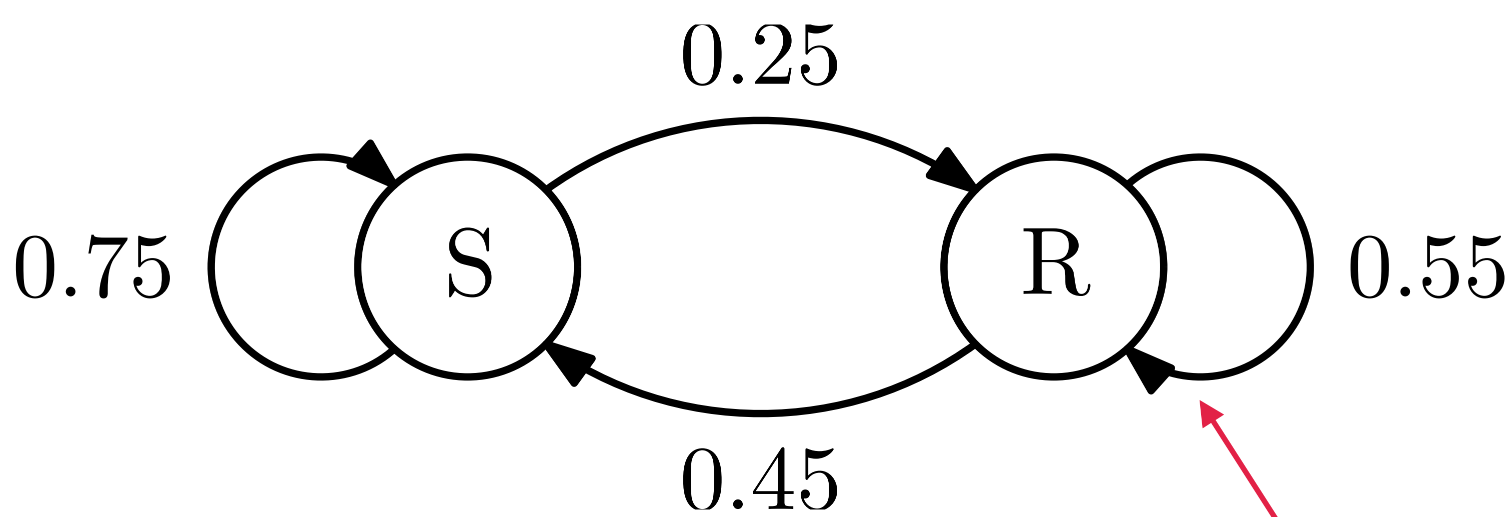
$$d_x = \gcd\left\{t \in \mathbb{N} \mid \mathbf{P}^t(x \mid x) > 0, t > 0\right\}$$

- A state x is aperiodic if $d_x = 1$
- A chain is aperiodic if all states are aperiodic, and periodic otherwise

TÉCNICO LISBOA

Aperiodicity

- In the weather example:

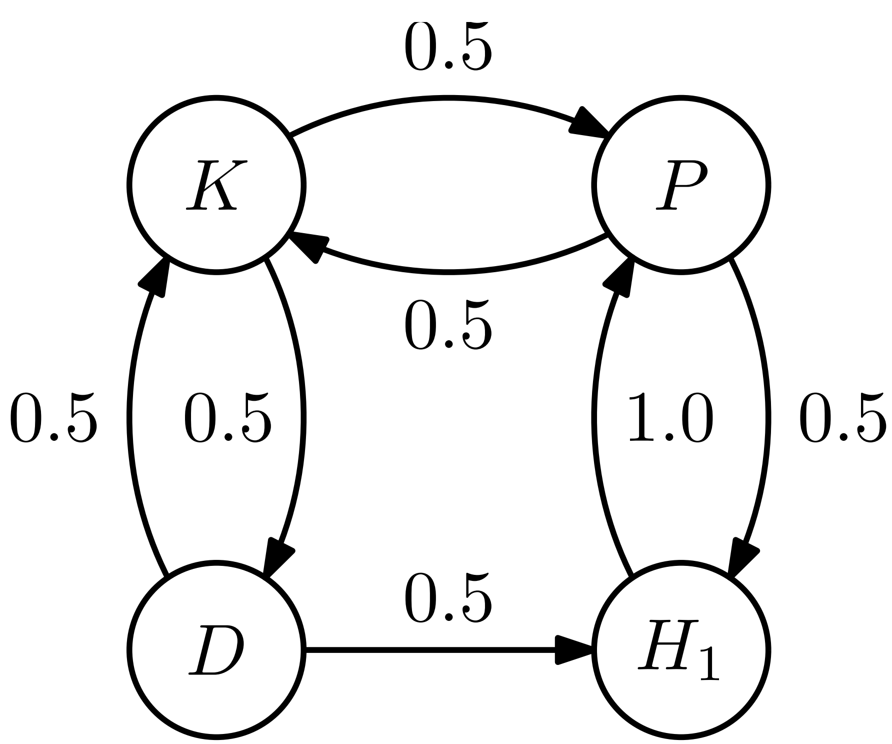


$\mathbf{P}(\mathrm{R} \mid \mathrm{R}) > 0$

$\Rightarrow$ aperiodic state

Aperiodicity

- In the robot example:



If the chain departs from P, when does it return?

↓

Several possibilities:

$t = 2$ (goes to H_1 and returns)

$t = 2$ (goes to K and returns)

$t = 4$ (goes to K, D and returns)

...

The period is $d = 2$

Stationary distribution

- Let μ be a distribution over $\mathcal{X}$

- In practical terms, μ is a row vector

$$\boldsymbol{\mu} = \begin{bmatrix} \mu(x_1) & \mu(x_2) & \dots & \mu(x_{|\mathcal{X}|}) \end{bmatrix}$$

Component $\mu(x)$ is the probability of x according to μ

where

$$\sum_{x \in \mathcal{X}} \mu(x) = 1$$

Stationary distribution

- μ can represent, for example,
    - ... the initial distribution for the chain;
    - ... the predicted distribution after t steps;
    - ... etc.

Stationary distribution

- The distribution μ is **stationary** if

$$\mu(x) = \sum_{y \in \mathcal{X}} \mu(y)\mathbf{P}(x \mid y)$$

- In other words, if the state at time t follows the distribution μ and μ is stationary, then the state at time $t+1$ also follows the distribution μ

- The stationary distribution corresponds to **stable behavior of the chain**

TÉCNICO LISBOA

Key stability results

An irreducible and aperiodic Markov chain **possesses a stationary distribution**.

Positive chain

For an irreducible and aperiodic Markov chain with stationary distribution μ^*,

$$\lim_{t\to\infty} \mu_0 \mathbf{P}^t = \mu^*$$

for any initial distribution μ_0.

Ergodic chain

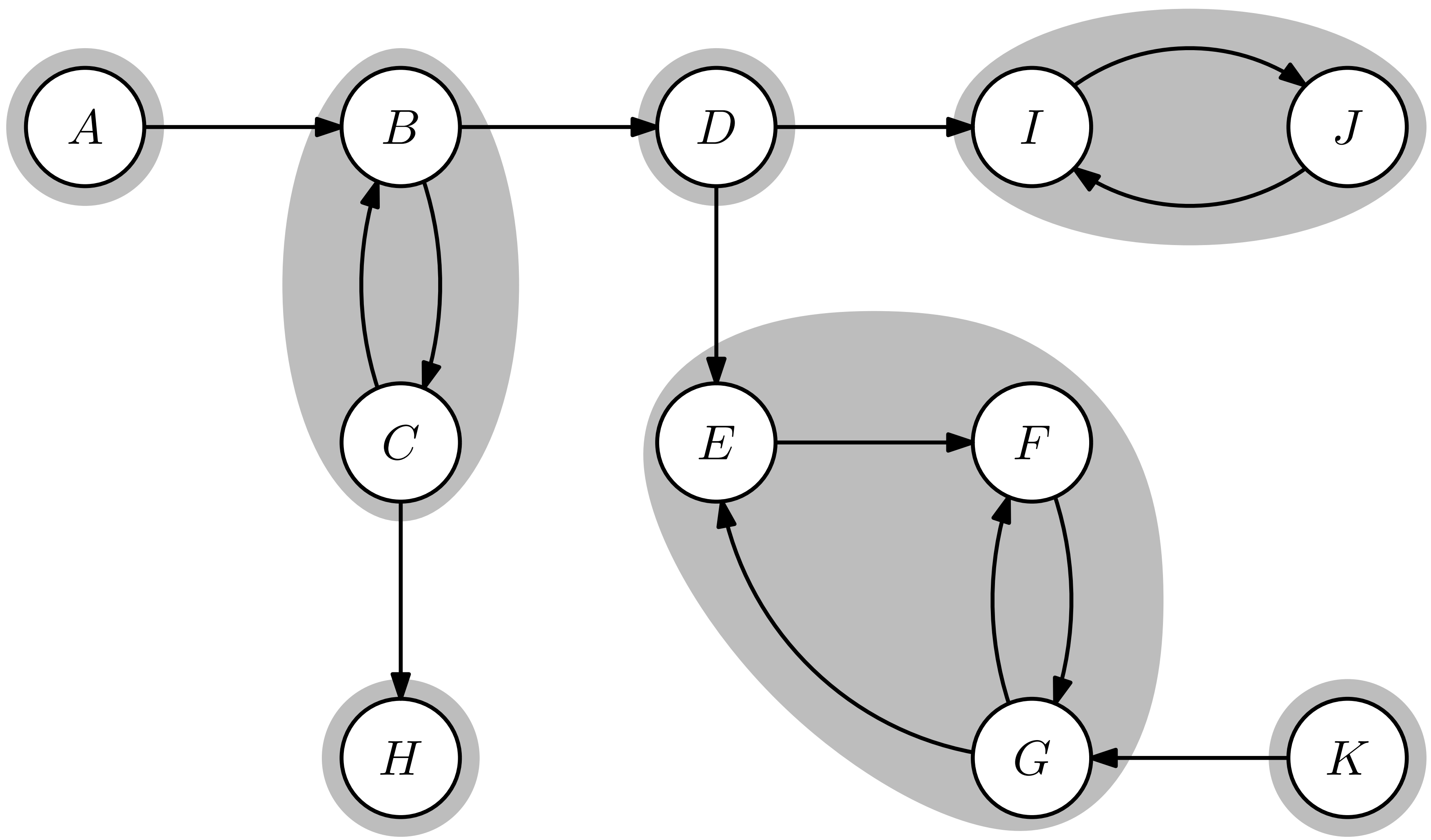
Summarizing...

- A positive chain possesses a stationary distribution μ:
    - If x_t is distributed according to μ, then so is x_{t+1}

- An ergodic chain eventually reaches the stationary distribution

Returning to PageRank

- The chain is **not** irreducible

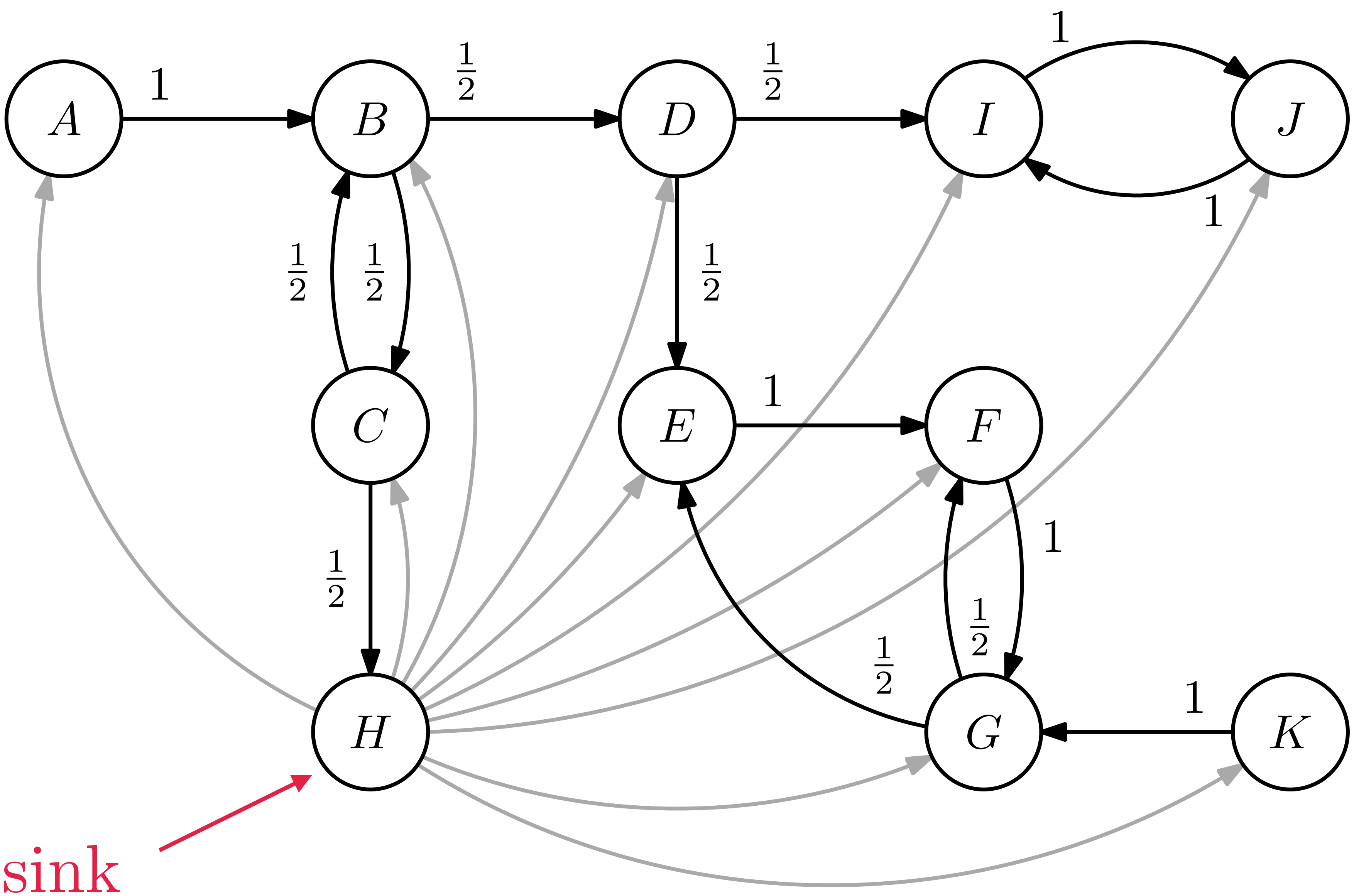


Returning to PageRank

- PageRank introduces two modifications to the chain:
  - Sinks link to all other nodes
  - There is a probability $1 - \gamma$ of "teleporting" to an arbitrary node

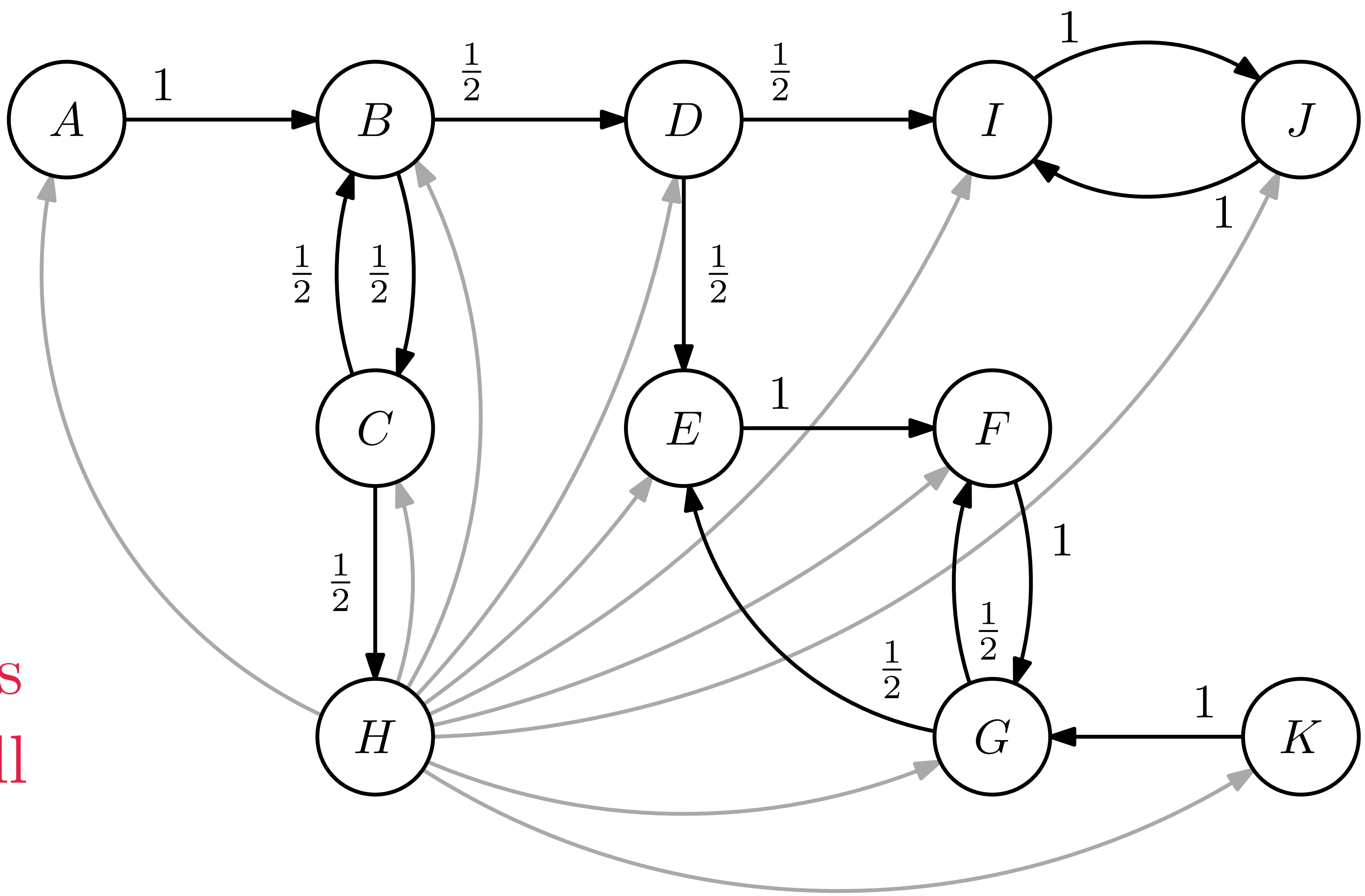
Returning to PageRank

- In our case:



Returning to PageRank

- In our case:



All nodes link to all nodes

Returning to PageRank

- Is the chain irreducible?
    - Yes
- Is the chain aperiodic?
    - Yes
- … then there is a stationary distribution

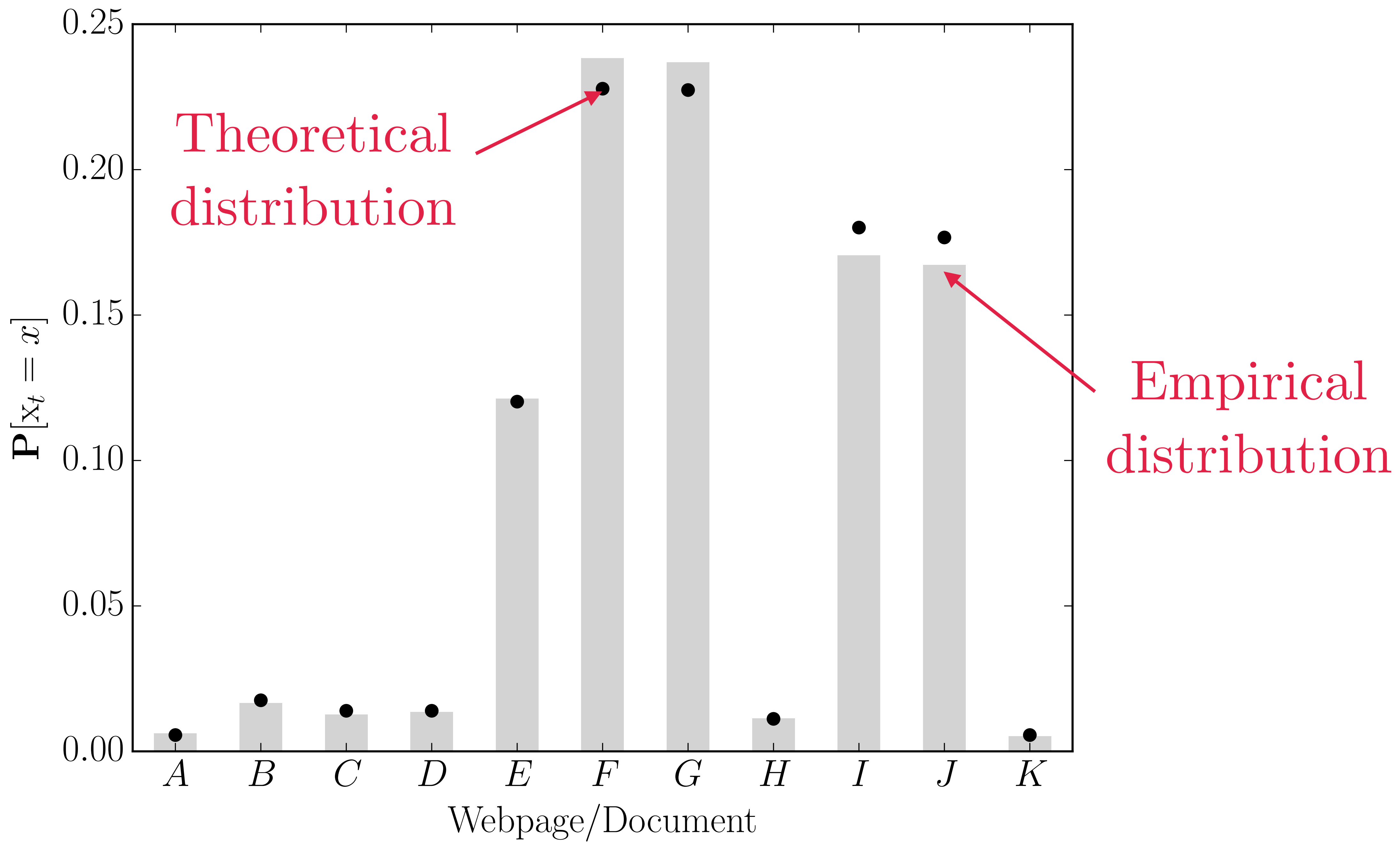
TÉCNICO LISBOA

Returning to PageRank

- The stationary distribution can be computed analytically (efficiently)

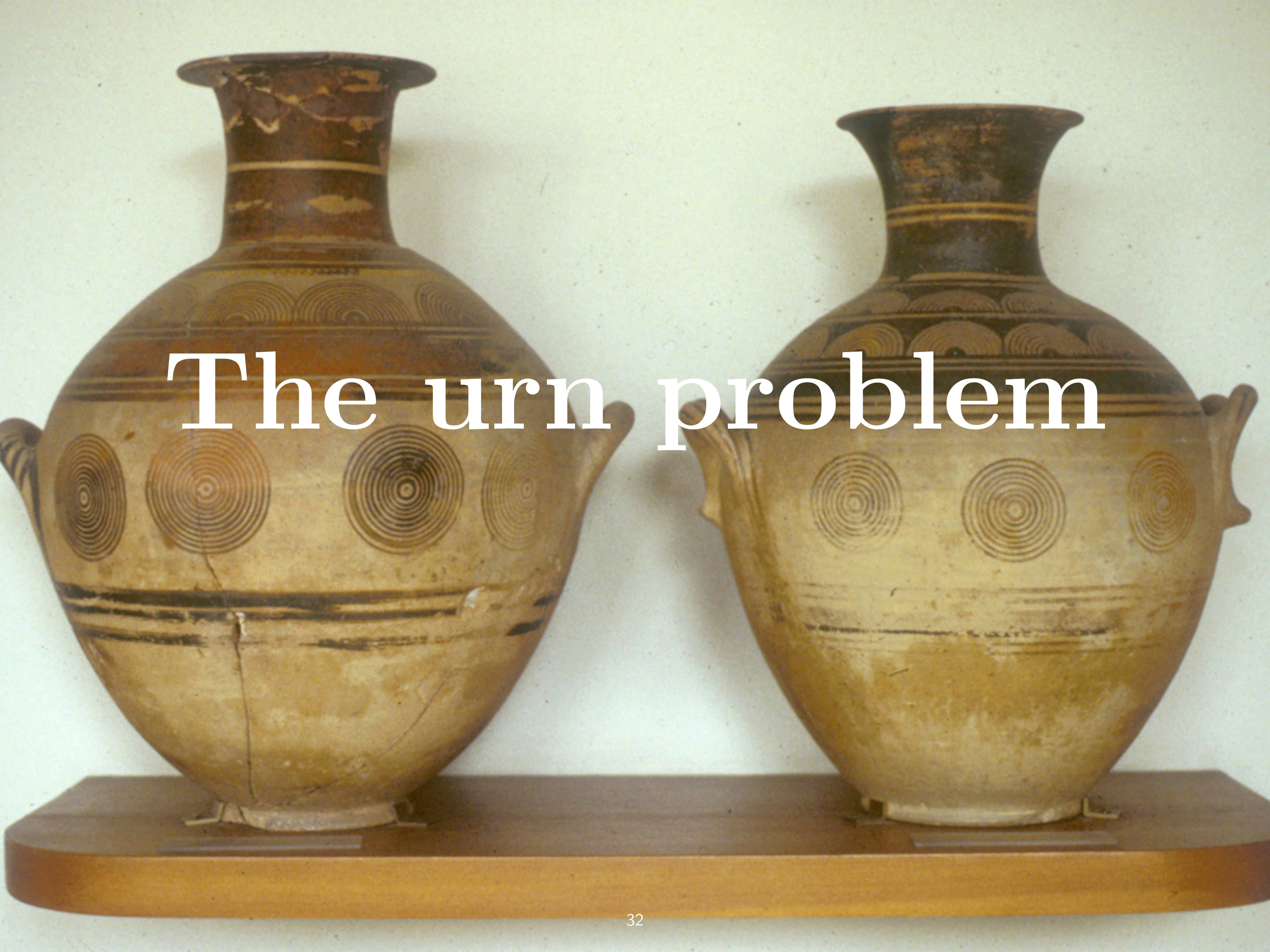
- It provides a convenient criterion for ranking the pages

Returning to PageRank



Sequential models (cont.)

The urn problem



The urn problem

- An oracle has available two urns
    - Each urn is filled with black and white balls
    - **Urn 1:** 25% white balls; 75% black balls
    - **Urn 2:** 25% black balls; 75% white balls

The urn problem

- At each step, the oracle draws a ball from one of the urns
- The ball is put back afterwards

The urn problem

- The current urn determines which urn the next ball will come from
  - The next ball will come from the same urn with 80% probability
  - The next ball will come from a different urn with 20% probability

TÉCNICO LISBOA

1. Is this a Markov chain?

- **Yes**
  - The urn-ball in the next step depends on the urn-ball in the current step

2. What are the states?

- All possible urn-ball pairs:
  - $\mathcal{X} = \{(1, w), (1, b), (2, w), (2, b)\}$

TÉCNICO LISBOA

3. Transition probabilities

- E.g., suppose that the initial urn-ball was $(1,\ w)$
- What is the next state?
  - $(1,\ w)$ with probability 0.2
  - $(1,\ b)$ with probability 0.6
  - $(2,\ w)$ with probability 0.15
  - $(2,\ b)$ with probability 0.05

80% probability

20% probability

TÉCNICO LISBOA

3. Transition probabilities?

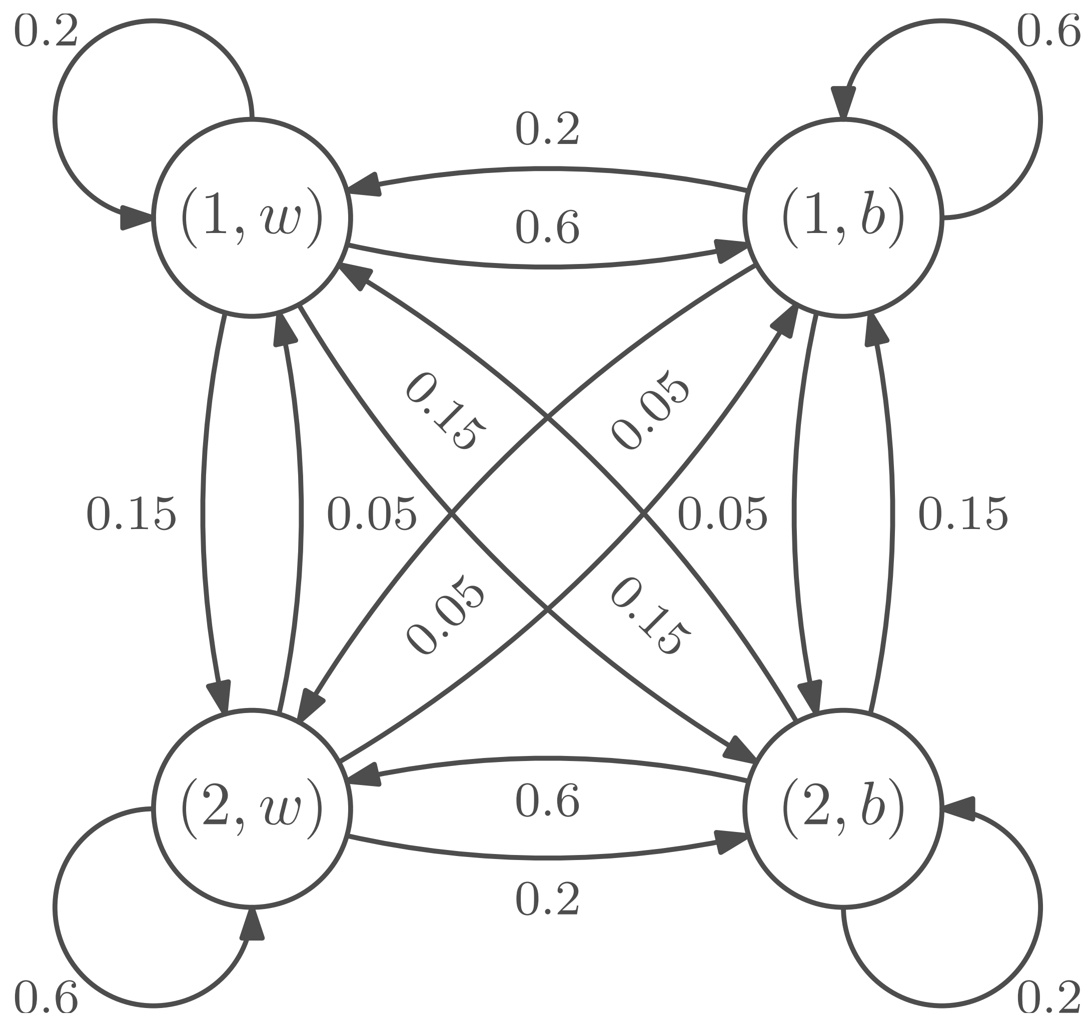
- We can represent the transition probabilities as a matrix:

$$
\begin{array}{c c}
 & \begin{array}{cccc} (1,\ w) & (1,\ b) & (2,\ w) & (2,\ b) \end{array} \\
\begin{array}{c} (1,\ w) \\ (1,\ b) \\ (2,\ w) \\ (2,\ b) \end{array} & \begin{bmatrix} 0.2 & 0.6 & 0.15 & 0.05 \\ 0.2 & 0.6 & 0.15 & 0.05 \\ 0.05 & 0.15 & 0.6 & 0.2 \\ 0.05 & 0.15 & 0.6 & 0.2 \end{bmatrix}
\end{array}
$$

TÉCNICO LISBOA

3. Transition probabilities?

- We can also represent the game with a transition diagram:



Can we make predictions?

- E.g., suppose that the initial urn-ball was $(1, b)$

- What is the next state?

$$\begin{bmatrix} 0.2 & 0.6 & 0.15 & 0.05 \\ \boxed{0.2 \quad 0.6 \quad 0.15 \quad 0.05} \\ 0.05 & 0.15 & 0.6 & 0.2 \\ 0.05 & 0.15 & 0.6 & 0.2 \end{bmatrix}$$

TÉCNICO LISBOA

Can we make predictions?

- E.g., suppose that the initial urn-ball was $(1,\ b)$

- What is the state at time $t = 3$?

$$\mathbf{P}^3 = \begin{bmatrix} 0.15 & 0.46 & 0.29 & 0.10 \\ \boxed{0.15 \quad 0.46 \quad 0.29 \quad 0.10} \\ 0.10 & 0.29 & 0.46 & 0.15 \\ 0.10 & 0.29 & 0.46 & 0.15 \end{bmatrix}$$

TÉCNICO LISBOA

Can we make predictions?

- E.g., suppose that the initial urn-ball was $(1,\ b)$
- What is the state at time $t = 3$?
  - $(1,\ w)$ with probability 0.15
  - $(1,\ b)$ with probability 0.46
  - $(2,\ w)$ with probability 0.29
  - $(2,\ b)$ with probability 0.10

And what if...?

The urn problem (revisited)

- At each step, the oracle draws a ball from one of the urns
- The ball is put back afterwards
- **The oracle reveals only the color of the ball**

Partial observability

↓

The state cannot be
fully observed

1. Is this a Markov chain?

- **No!**
  - The information available at each moment (ball color) is not enough to predict next ball color
  - Why?
  - The next distribution depends on the urn!

TÉCNICO LISBOA

Can we make predictions?

- Suppose that we know that:
  - At time $t = 0$, the ball was white and drawn from urn 1
  - At time $t = 1$, the ball was black
  - What is the state at time $t = 2$?

Can we make predictions?

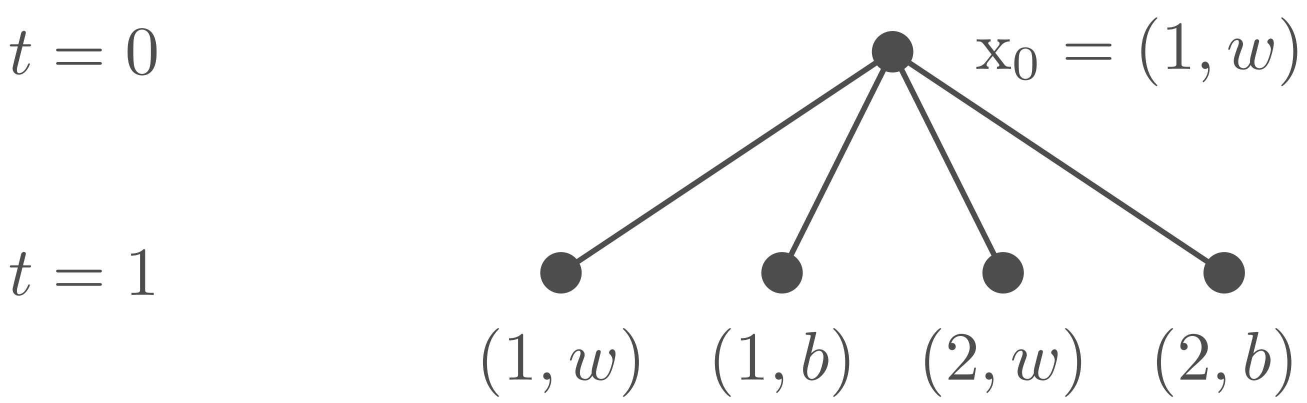
- We have:

$t = 0$

$\mathrm{x}_0 = (1, w)$

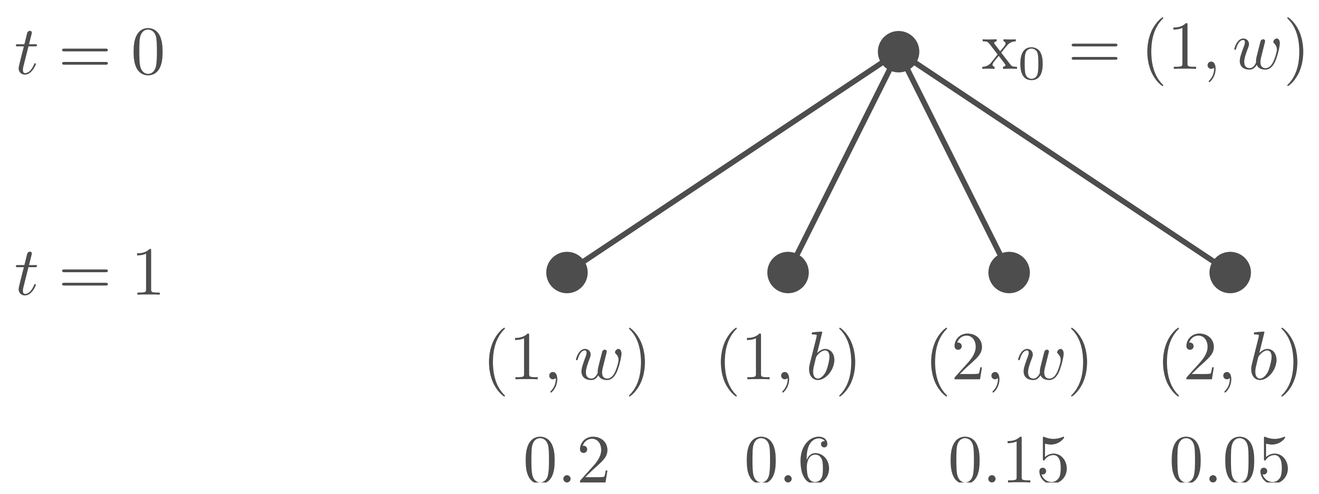
Can we make predictions?

- We have:



Can we make predictions?

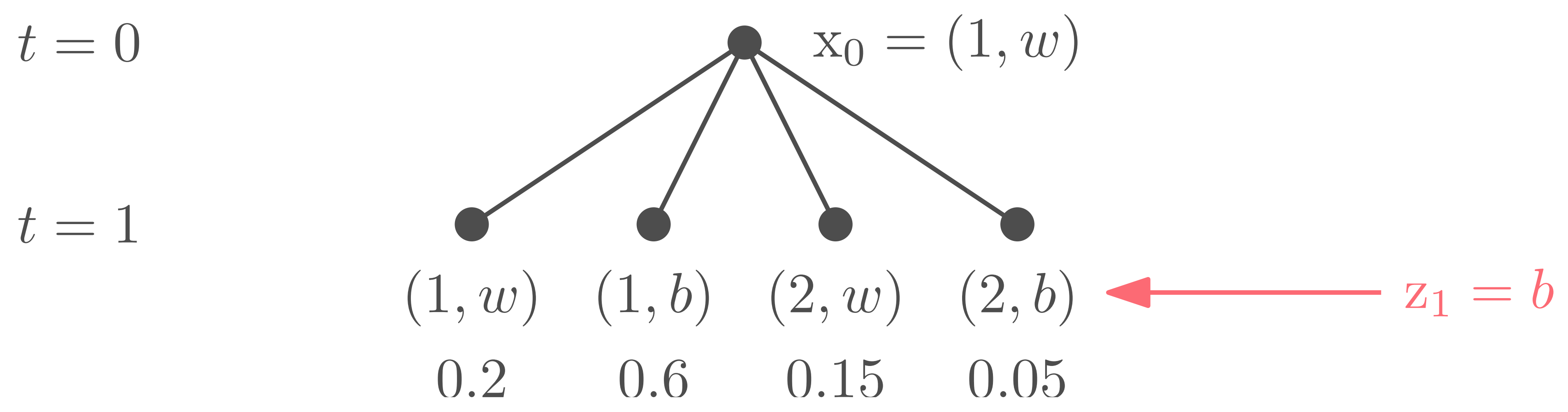
- We have:



TÉCNICO LISBOA

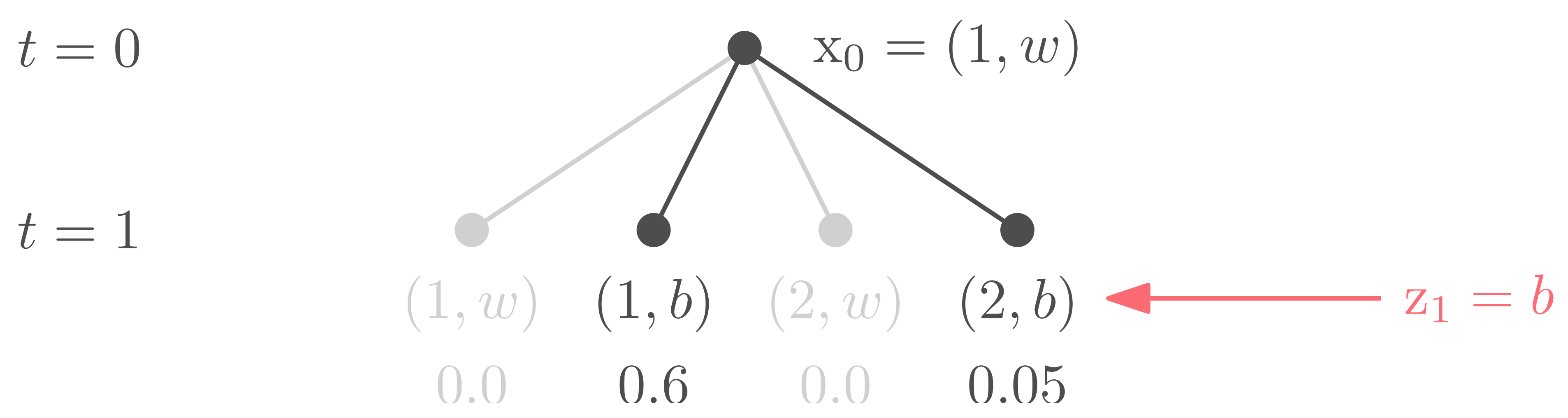
Can we make predictions?

- We have:



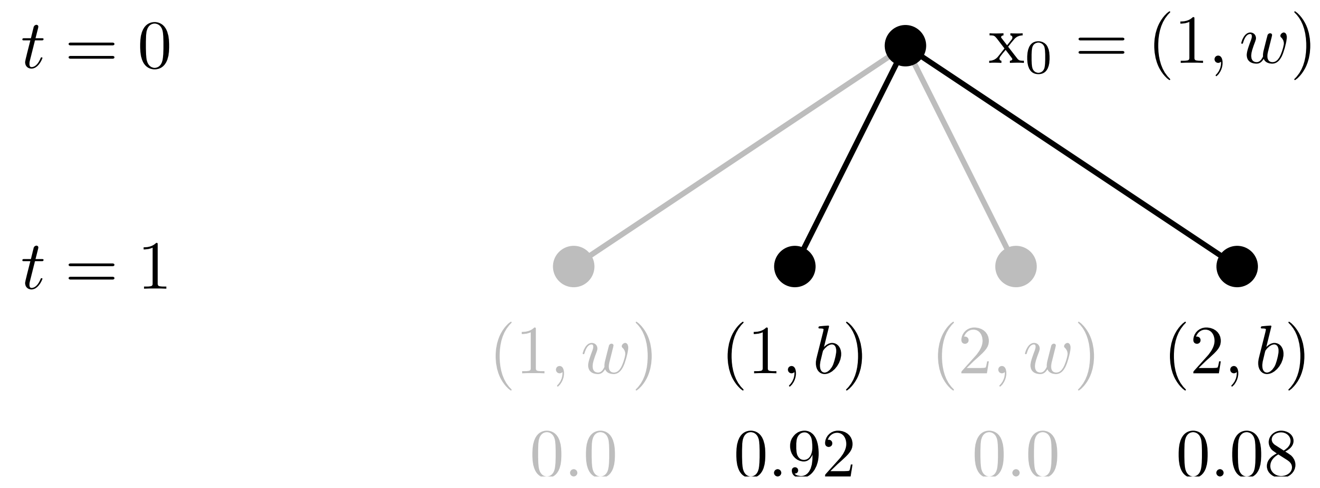
Can we make predictions?

- We have:



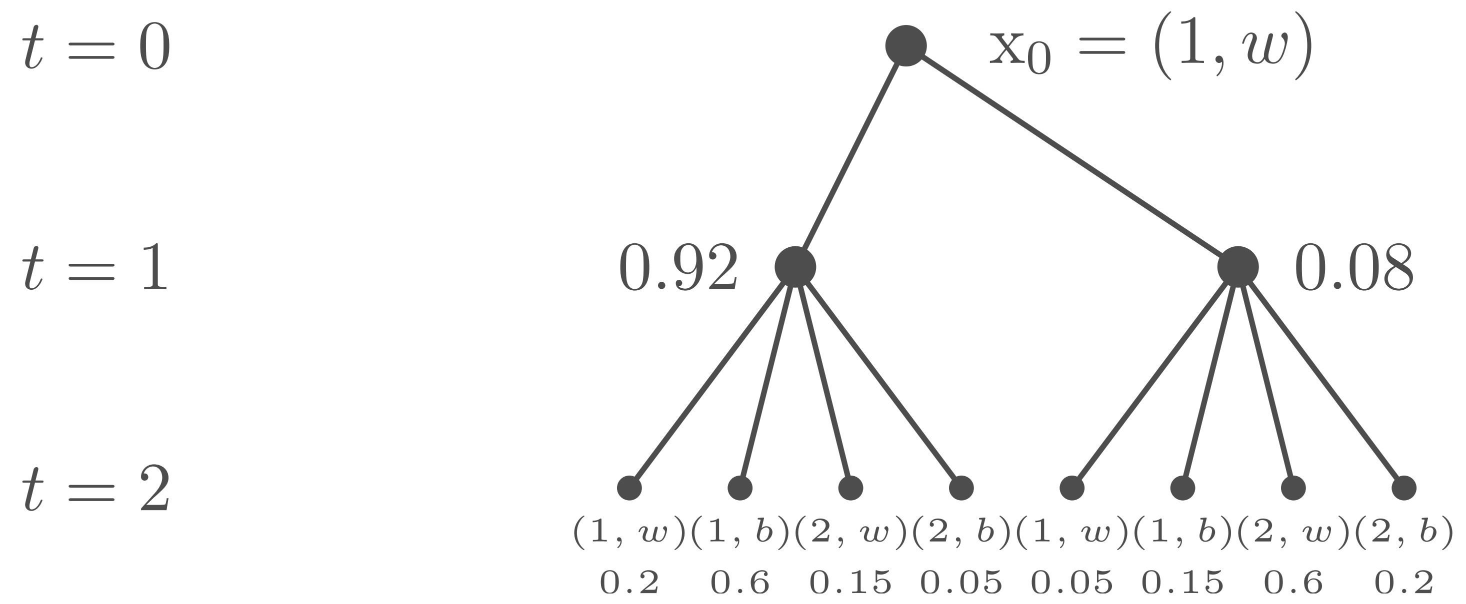
Can we make predictions?

- We have:



Can we make predictions?

- We have:

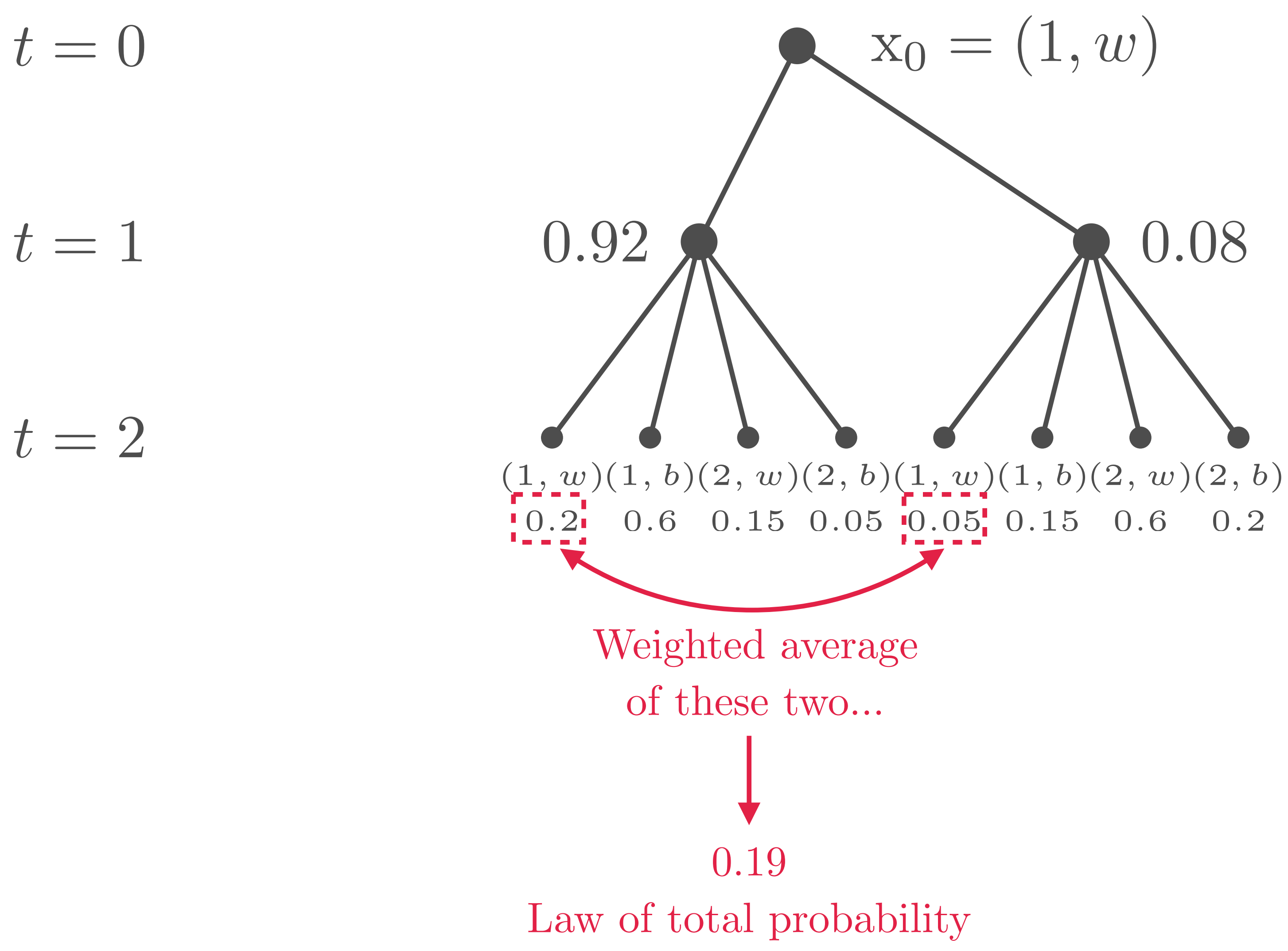


What is the probability of $(1, w)$?

TÉCNICO LISBOA

Can we make predictions?

- We have:



Can we make predictions?

- In formal terms:

Observation

$$\mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

Can we make predictions?

- In formal terms:

$$\mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

$$= \quad \mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_1 = x, \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

Intermediate state

Can we make predictions?

- In formal terms:

$$\mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

$$= \quad \mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_1 = x, \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]\mathbb{P}[\mathrm{x}_1 = x \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

Probability of
intermediate state

Can we make predictions?

- In formal terms:

$$\mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

$$= \sum_{x \in \mathcal{X}} \mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_1 = x, \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]\mathbb{P}[\mathrm{x}_1 = x \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

Weighted mean
of probabilities
(total probability law)

Bayes theorem

TÉCNICO LISBOA

Can we make predictions?

- In formal terms:

The observation depends only on x_1

Transition probabilities

$$\mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

$$= \sum_{x \in \mathcal{X}} \mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_1 = x, \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b] \frac{\mathbb{P}[\mathrm{z}_1 = b \mid \mathrm{x}_1 = x, \mathrm{x}_0 = (1, w)]\mathbb{P}[\mathrm{x}_1 = x \mid \mathrm{x}_0 = (1, w)]}{\mathbb{P}[\mathrm{z}_1 = b \mid \mathrm{x}_0 = (1, w)]}$$

We need only x_1 to predict x_2

Normalization factor

TÉCNICO LISBOA

Can we make predictions?

- In formal terms:

$$\mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

$$= \frac{1}{\rho} \sum_{x \in \mathcal{X}} \mathbf{P}((1, w) \mid x)\mathbb{P}[\mathrm{z}_1 = b \mid \mathrm{x}_1 = x]\mathbf{P}(x \mid (1, w))$$

If $\mathrm{x}_1 = (1,\ w)$ or $\mathrm{x}_1 = (2,\ w)$
this probability is 0,
otherwise it's 1

Can we make predictions?

- In formal terms:

$$\mathbb{P}[\mathrm{x}_2 = (1, w) \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b]$$

$$= \frac{1}{\rho}\big(\mathbf{P}((1, w) \mid (1, b))\mathbf{P}((1, b) \mid (1, w)) + \mathbf{P}((1, w) \mid (2, b))\mathbf{P}((2, b) \mid (1, w))\big)$$

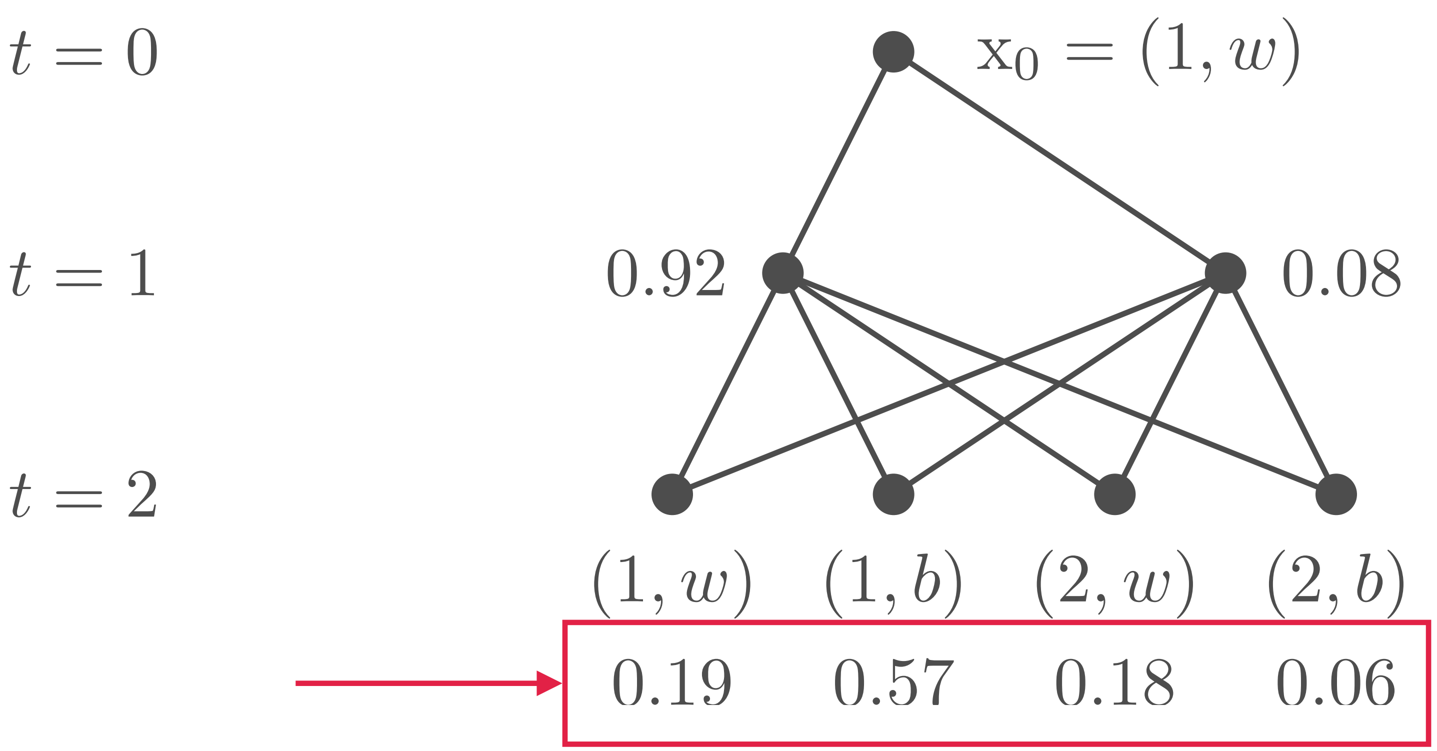
- More generally:

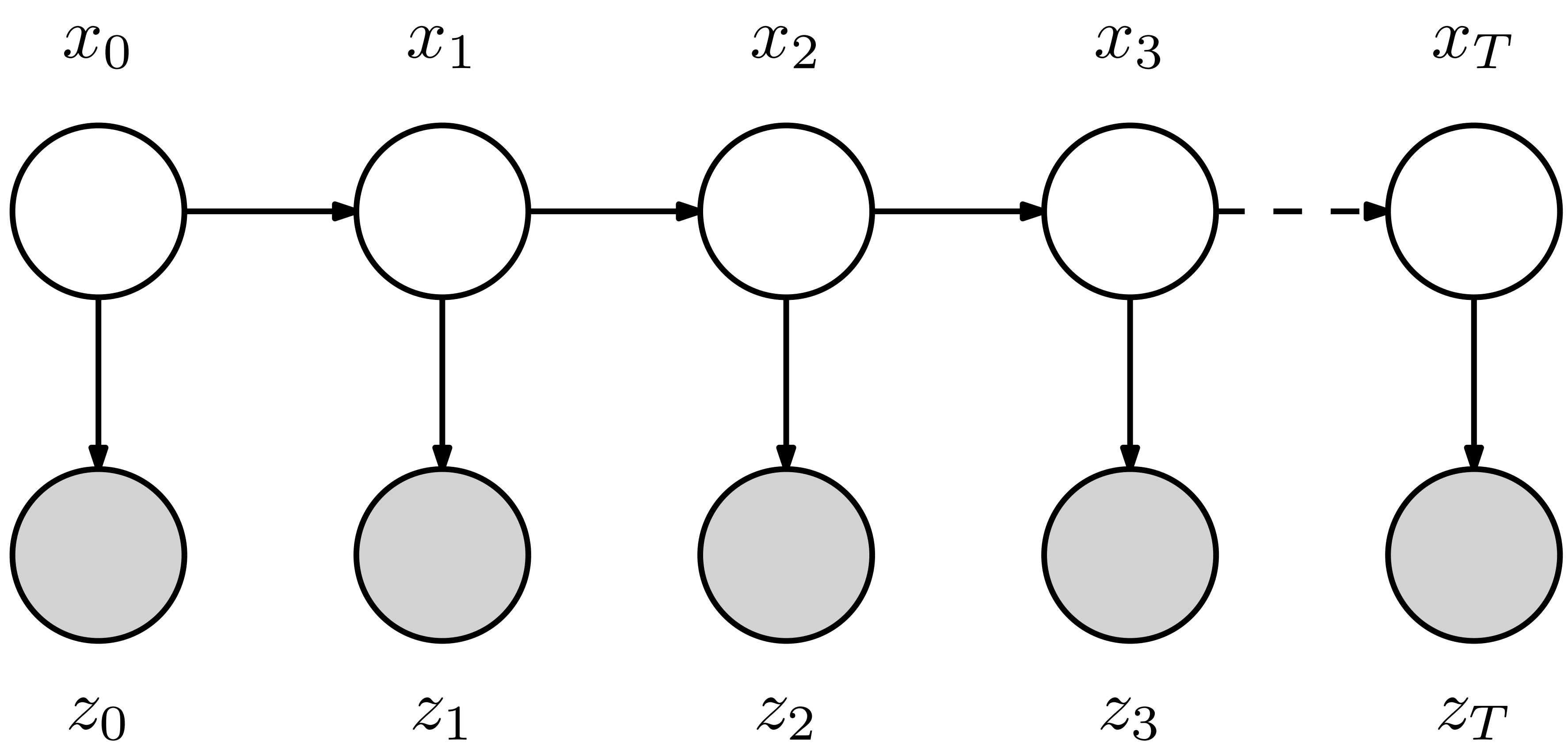
$$\mathbb{P}[\mathrm{x}_2 = x \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b] = \frac{1}{\rho}\big(\mathbf{P}(x \mid (1, b))\mathbf{P}((1, b) \mid (1, w)) + \mathbf{P}(x \mid (2, b))\mathbf{P}((2, b) \mid (1, w))\big)$$

TÉCNICO LISBOA

Can we make predictions?

- We have:





Hidden Markov models

Hidden Markov model

- Model for **sequential process with partial observability**

- Process evolves in **discrete time steps** (as in Markov chains)

- **State of the process** at time step t: x_t

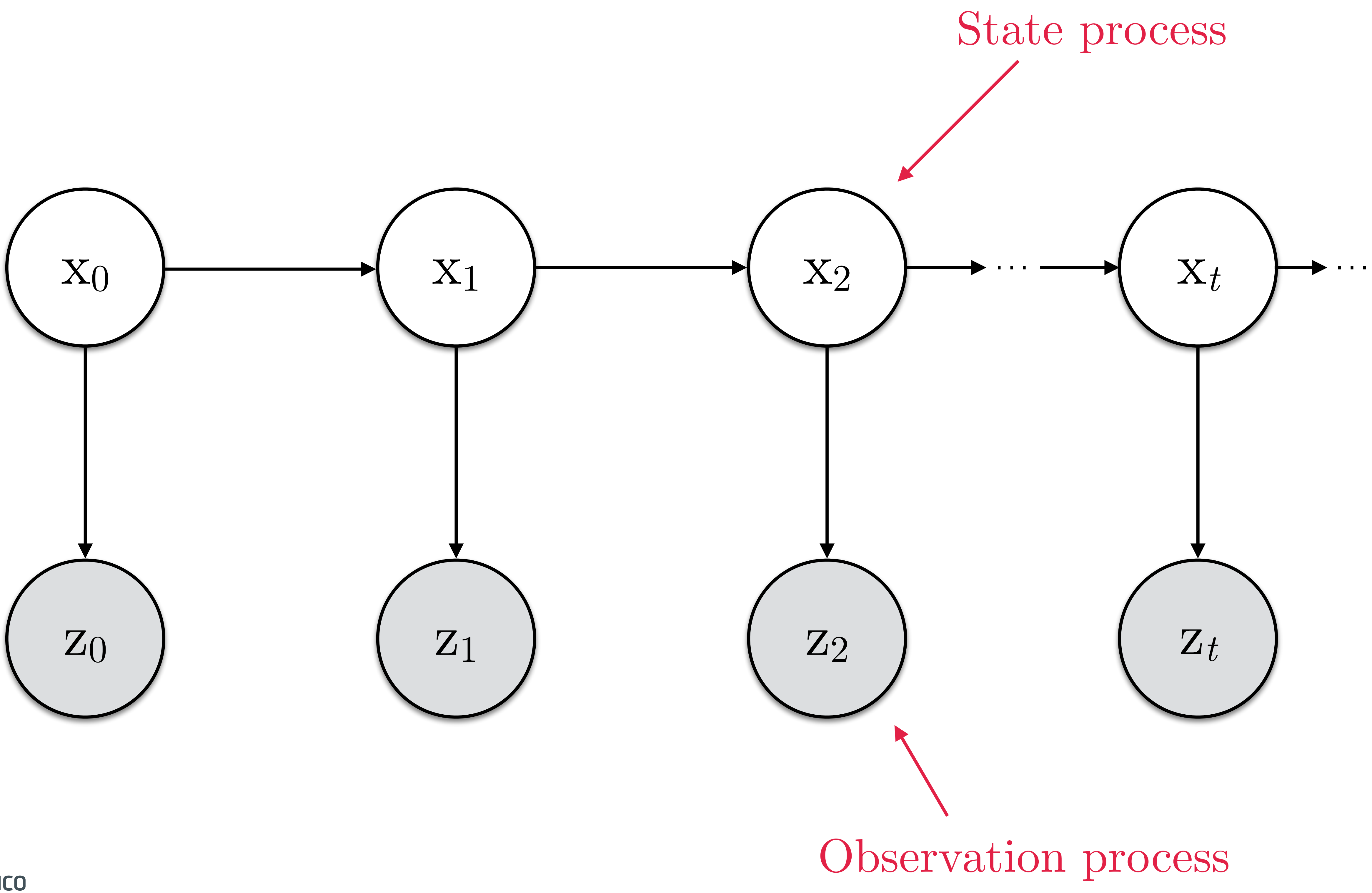
- **Observation** at time step t: z_t

Quantity of interest

Observable

TÉCNICO LISBOA

Hidden Markov model



TÉCNICO LISBOA

Hidden Markov model

Markov state

The state at instant t is enough to predict the state at instant $t+1$:

$$\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}, \mathbf{z}_{0:t} = \boldsymbol{z}_{0:t}\right] = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x_t\right]$$

Depends only on the last state

Hidden Markov model

State-dependent observations

The state at instant t is enough to predict the observation at instant t:

$$\mathbb{P}\left[\mathrm{z}_t = z \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}, \mathbf{z}_{0:t-1} = \boldsymbol{z}_{0:t-1}\right] = \mathbb{P}\left[\mathrm{z}_t = z \mid \mathrm{x}_t = x_t\right]$$

Depends only on the last state

Hidden Markov model

Markov state

The state at instant t is enough to predict the state at instant $t+1$:

$$\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}, \mathbf{z}_{0:t} = \boldsymbol{z}_{0:t}\right] = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x_t\right]$$

State-dependent observations

The state at instant t is enough to predict the observation at instant t:

$$\mathbb{P}\left[\mathrm{z}_t = z \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}, \mathbf{z}_{0:t-1} = \boldsymbol{z}_{0:t-1}\right] = \mathbb{P}\left[\mathrm{z}_t = z \mid \mathrm{x}_t = x_t\right]$$

Hidden Markov model

- Other assumptions (for most of this course):
  - There is only a **finite number** of possible states
  - $\mathcal{X}$ is the set of possible states (**state space**)
  - There is only a **finite number** of observations
  - $\mathcal{Z}$ is the set of possible observations (**observation space**)

TÉCNICO LISBOA

Hidden Markov model

- Other assumptions:

  - The transition probabilities $\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x\right]$ do not depend on t

  - The observation probabilities $\mathbb{P}\left[\mathrm{z}_t = z \mid \mathrm{x}_t = x\right]$ do not depend on t

TÉCNICO LISBOA

Transition probability matrix

- As with Markov chains, we store the transition probabilities in a **matrix P**

$$[\mathbf{P}]_{xy} = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x\right]$$

Observation probability matrix

- Similarly, we store the observation probabilities in a **matrix O**

$$[\mathbf{O}]_{xz} = \mathbb{P}\left[\mathrm{z}_t = z \mid \mathrm{x}_t = x\right]$$

Number in row x column z is the probability of observing z in x

TÉCNICO LISBOA

Example

- **The urn problem**

$$
\begin{array}{c}
\\ (1,\ w) \\ (1,\ b) \\ (2,\ w) \\ (2,\ b)
\end{array}
\begin{array}{c}
\begin{array}{cccc} (1,\ w) & (1,\ b) & (2,\ w) & (2,\ b) \end{array} \\
\begin{bmatrix}
0.2 & 0.6 & 0.15 & 0.05 \\
0.2 & 0.6 & 0.15 & 0.05 \\
0.05 & 0.15 & 0.6 & 0.2 \\
0.05 & 0.15 & 0.6 & 0.2
\end{bmatrix}
\end{array}
\qquad
\begin{array}{c}
\\ (1,\ w) \\ (1,\ b) \\ (2,\ w) \\ (2,\ b)
\end{array}
\begin{array}{c}
\begin{array}{cc} w & b \end{array} \\
\begin{bmatrix}
\ & \ \\
\ & \ \\
\ & \ \\
\ & \
\end{bmatrix}
\end{array}
$$

TÉCNICO LISBOA

Example

- **The urn problem**

$$\begin{array}{c} \\ (1,\ w) \\ (1,\ b) \\ (2,\ w) \\ (2,\ b) \end{array}
\begin{array}{c} \begin{array}{cccc} (1,\ w) & (1,\ b) & (2,\ w) & (2,\ b) \end{array} \\ \begin{bmatrix} 0.2 & 0.6 & 0.15 & 0.05 \\ 0.2 & 0.6 & 0.15 & 0.05 \\ 0.05 & 0.15 & 0.6 & 0.2 \\ 0.05 & 0.15 & 0.6 & 0.2 \end{bmatrix} \end{array}
\qquad
\begin{array}{c} \\ (1,\ w) \\ (1,\ b) \\ (2,\ w) \\ (2,\ b) \end{array}
\begin{array}{c} \begin{array}{cc} w & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ & \\ & \\ & \end{bmatrix} \end{array}$$

Example

- **The urn problem**

$$\begin{array}{c|cccc|}
 & (1,\ w) & (1,\ b) & (2,\ w) & (2,\ b) \\
\hline
(1,\ w) & 0.2 & 0.6 & 0.15 & 0.05 \\
(1,\ b) & 0.2 & 0.6 & 0.15 & 0.05 \\
(2,\ w) & 0.05 & 0.15 & 0.6 & 0.2 \\
(2,\ b) & 0.05 & 0.15 & 0.6 & 0.2
\end{array}
\qquad
\begin{array}{c|cc|}
 & w & b \\
\hline
(1,\ w) & 1 & 0 \\
(1,\ b) & 0 & 1 \\
(2,\ w) & & \\
(2,\ b) & &
\end{array}$$

Example

- **The urn problem**

$$
\begin{array}{c}
 \\
(1,\ w) \\
(1,\ b) \\
(2,\ w) \\
(2,\ b)
\end{array}
\begin{array}{c}
\begin{array}{cccc}
(1,\ w) & (1,\ b) & (2,\ w) & (2,\ b)
\end{array} \\
\begin{bmatrix}
0.2 & 0.6 & 0.15 & 0.05 \\
0.2 & 0.6 & 0.15 & 0.05 \\
0.05 & 0.15 & 0.6 & 0.2 \\
0.05 & 0.15 & 0.6 & 0.2
\end{bmatrix}
\end{array}
\qquad
\begin{array}{c}
 \\
(1,\ w) \\
(1,\ b) \\
(2,\ w) \\
(2,\ b)
\end{array}
\begin{array}{c}
\begin{array}{cc}
w & b
\end{array} \\
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\end{array}
$$

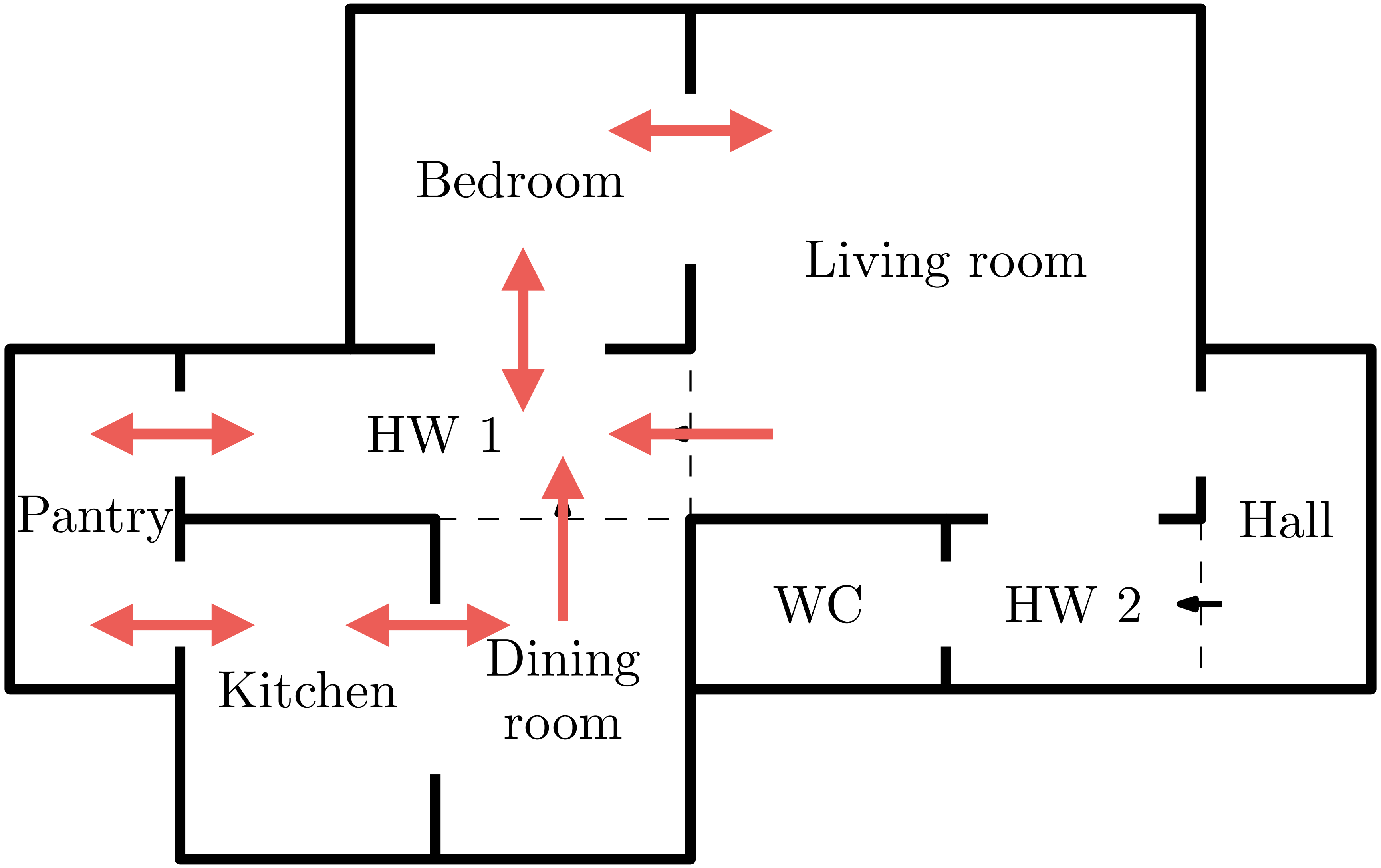
Summarizing...

- A HMM can be represented compactly as a tuple $(\mathcal{X}, \mathcal{Z}, \mathbf{P}, \mathbf{O})$
    - $\mathcal{X}$ is the set of possible states
    - $\mathcal{Z}$ is the set of possible observations
    - $\mathbf{P}$ is the transition probability matrix
    - $\mathbf{O}$ is the observation probability matrix

Examples

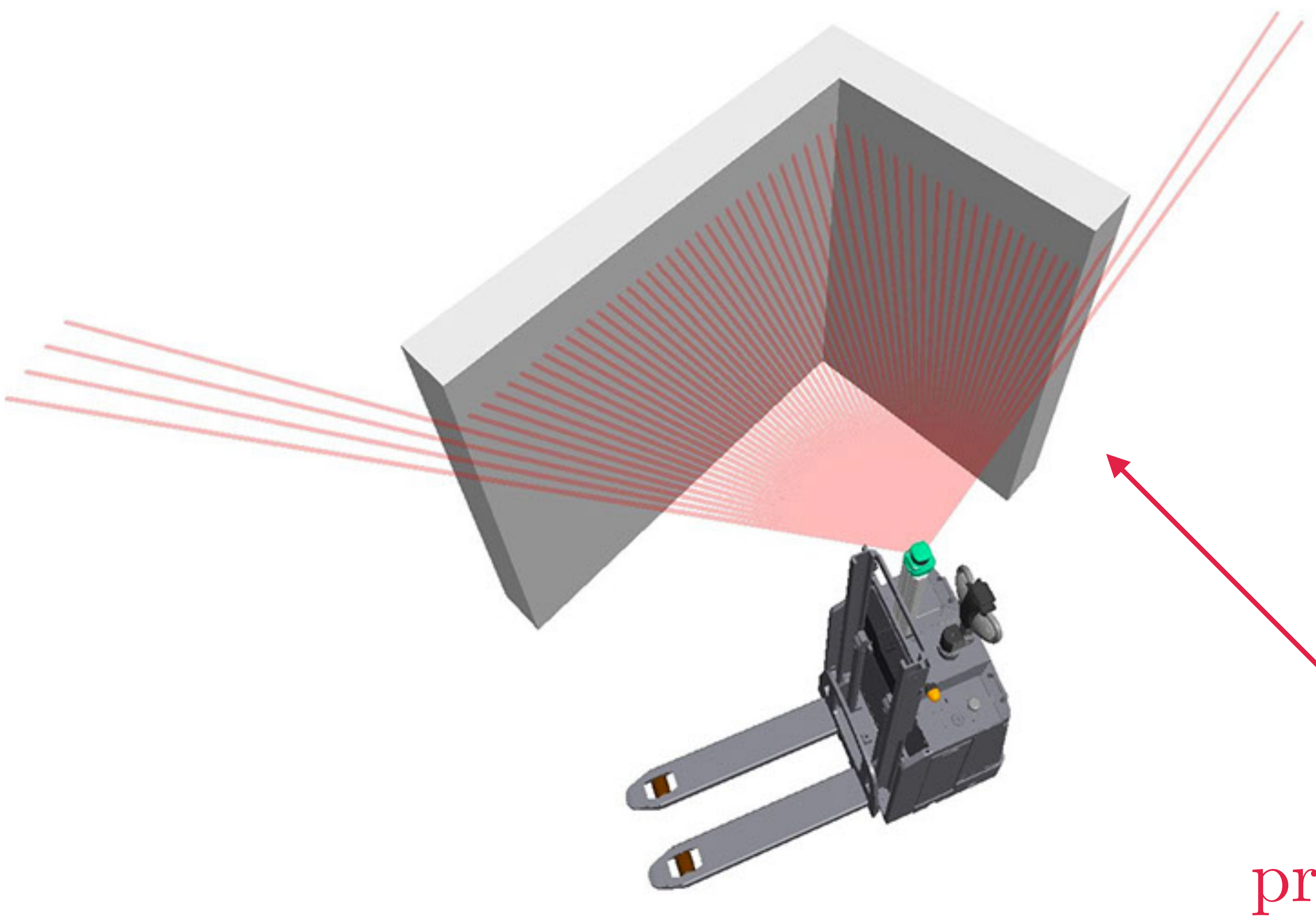
1. Household robot

- Remember?



1. Household robot

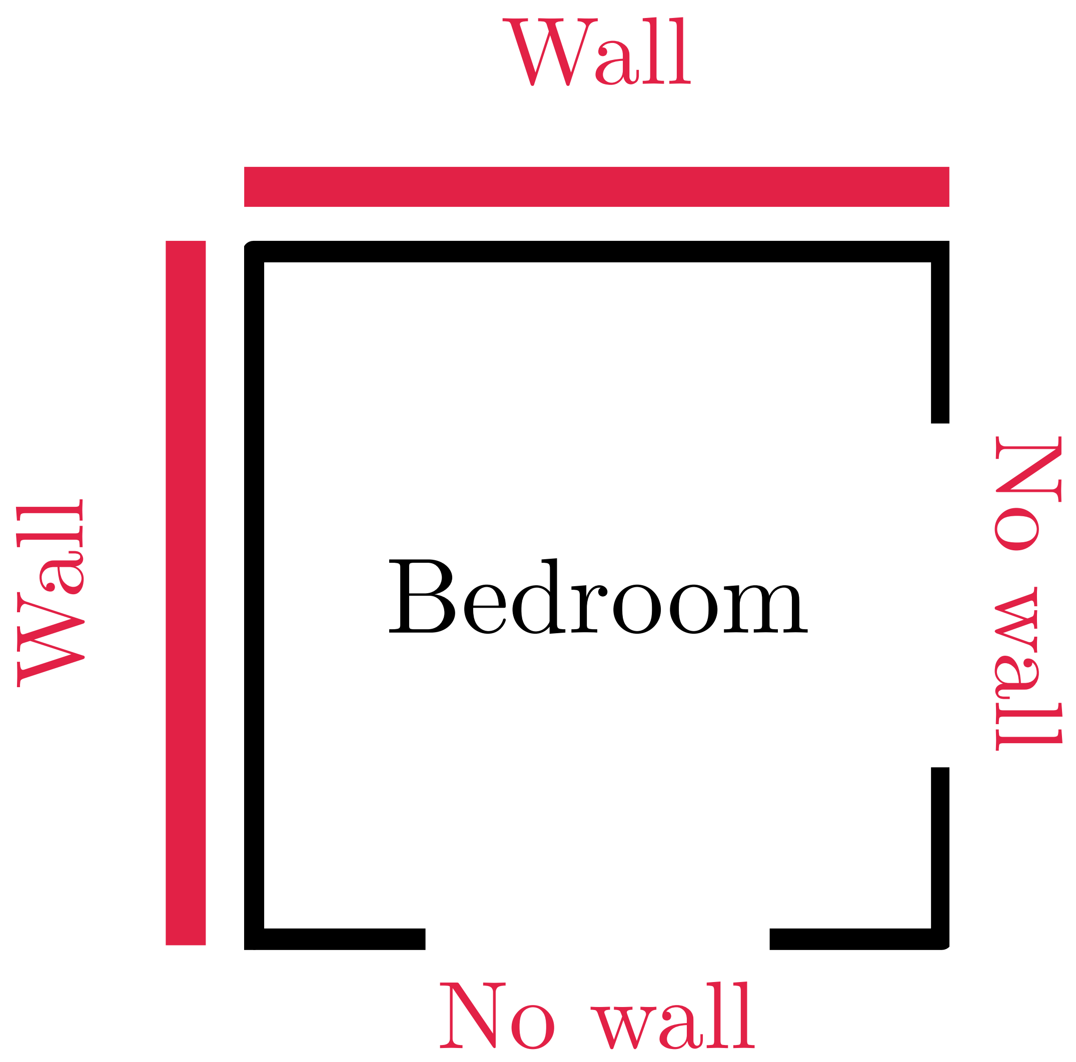
- Robot navigates using a laser



Laser detects presence or absence of walls

1. Household robot

- For example:



1. Household robot

- However, laser is not perfect
  - It fails to detect existing walls with 5% probability
  - It detects non-existing walls with 10% probability (in some situations with 20% probability)
  - Detection of a wall independent of adjacent walls

TÉCNICO LISBOA

Is this an HMM?

- State verifies the Markov property?
  - **Yes** - the position of the robot at time $t + 1$ depends only of position of the robot at time t
- Observation depends only on state?
  - **Yes** - the wall detections depends only on robot position

It is an HMM!

What are the states?

- Possible positions of the robot:
    - $\mathcal{X} = \{K, P, D, H_1, B, L\}$

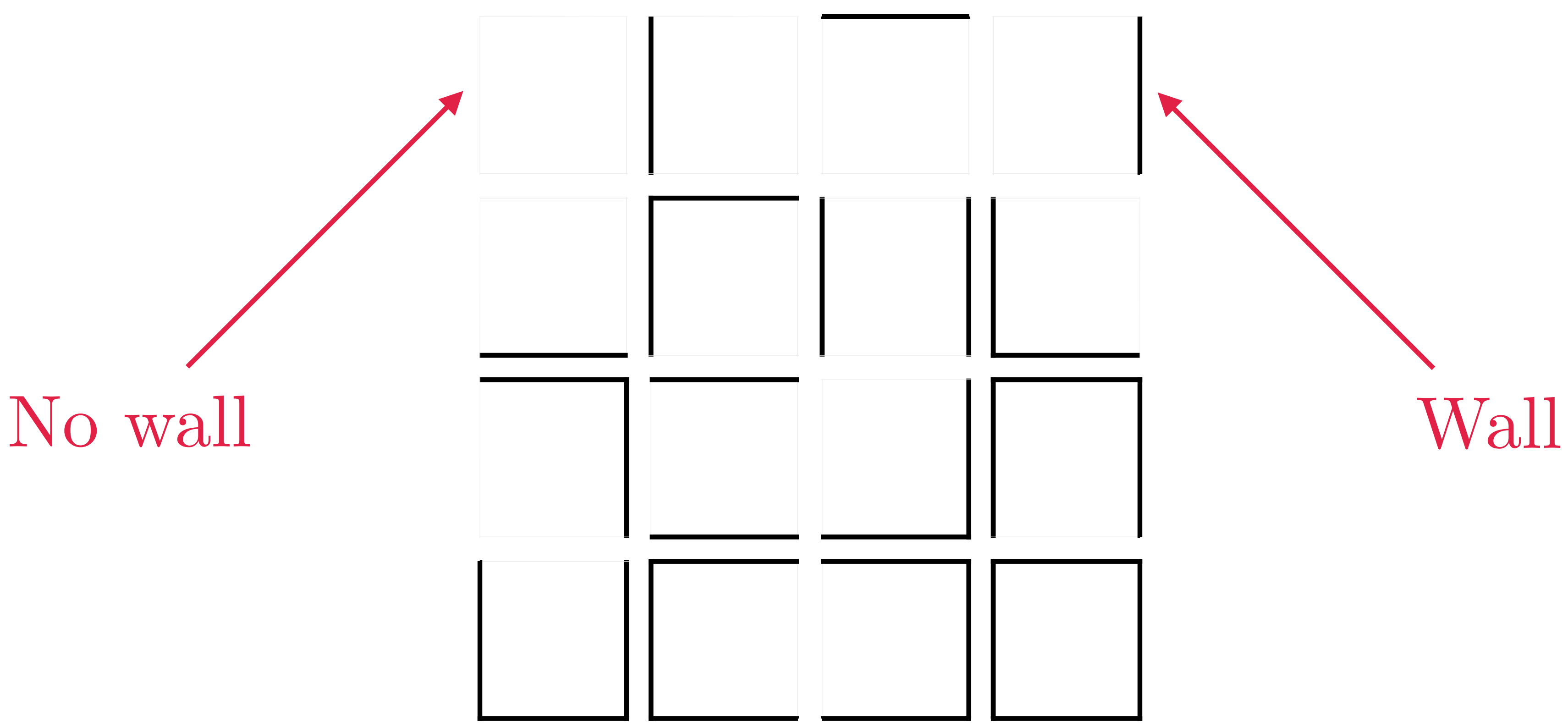
What are the observations?

- What does the robot "see"?

TÉCNICO LISBOA

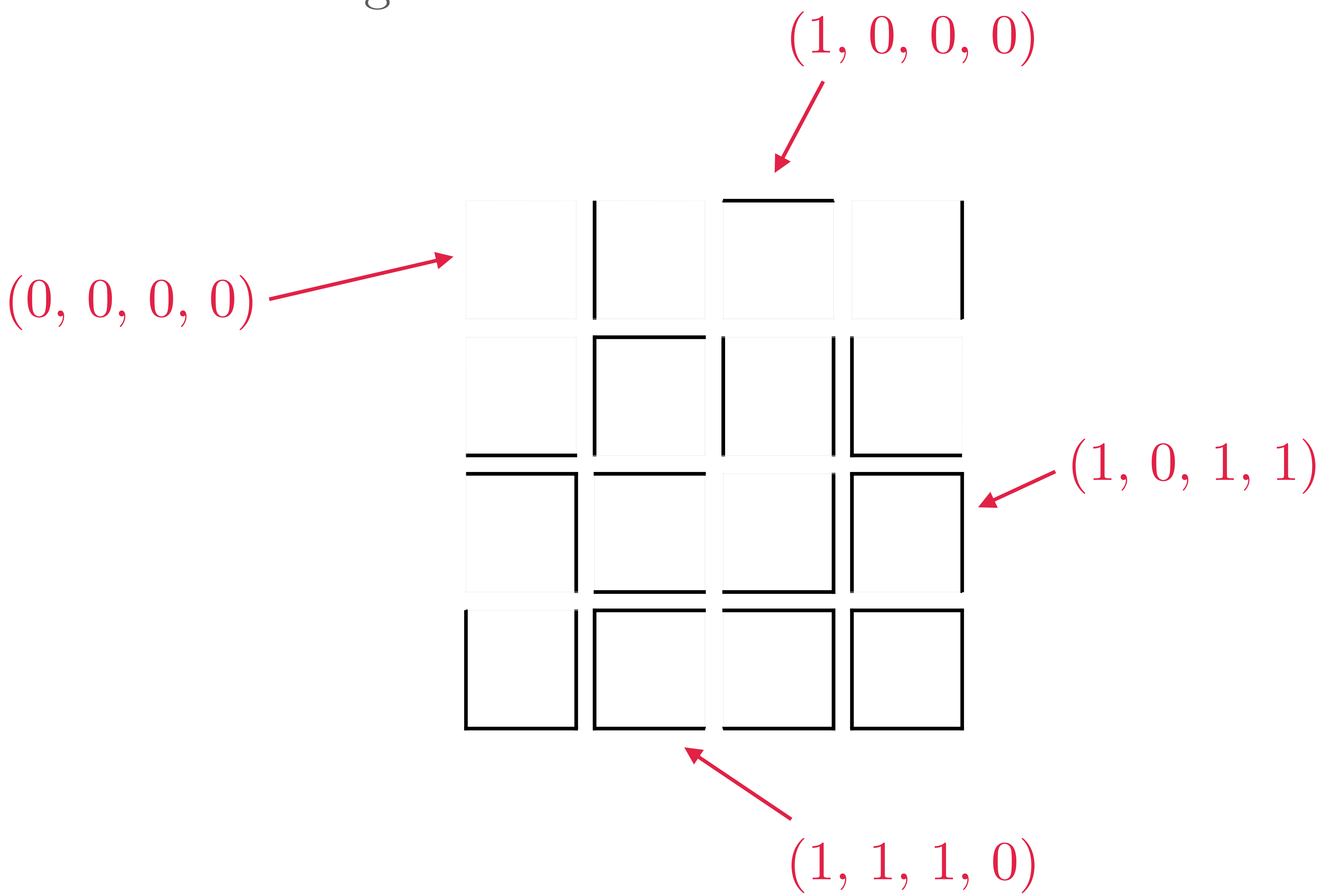
What are the observations?

- Possible wall configurations:



What are the observations?

- Possible wall configurations:



What are the observations?

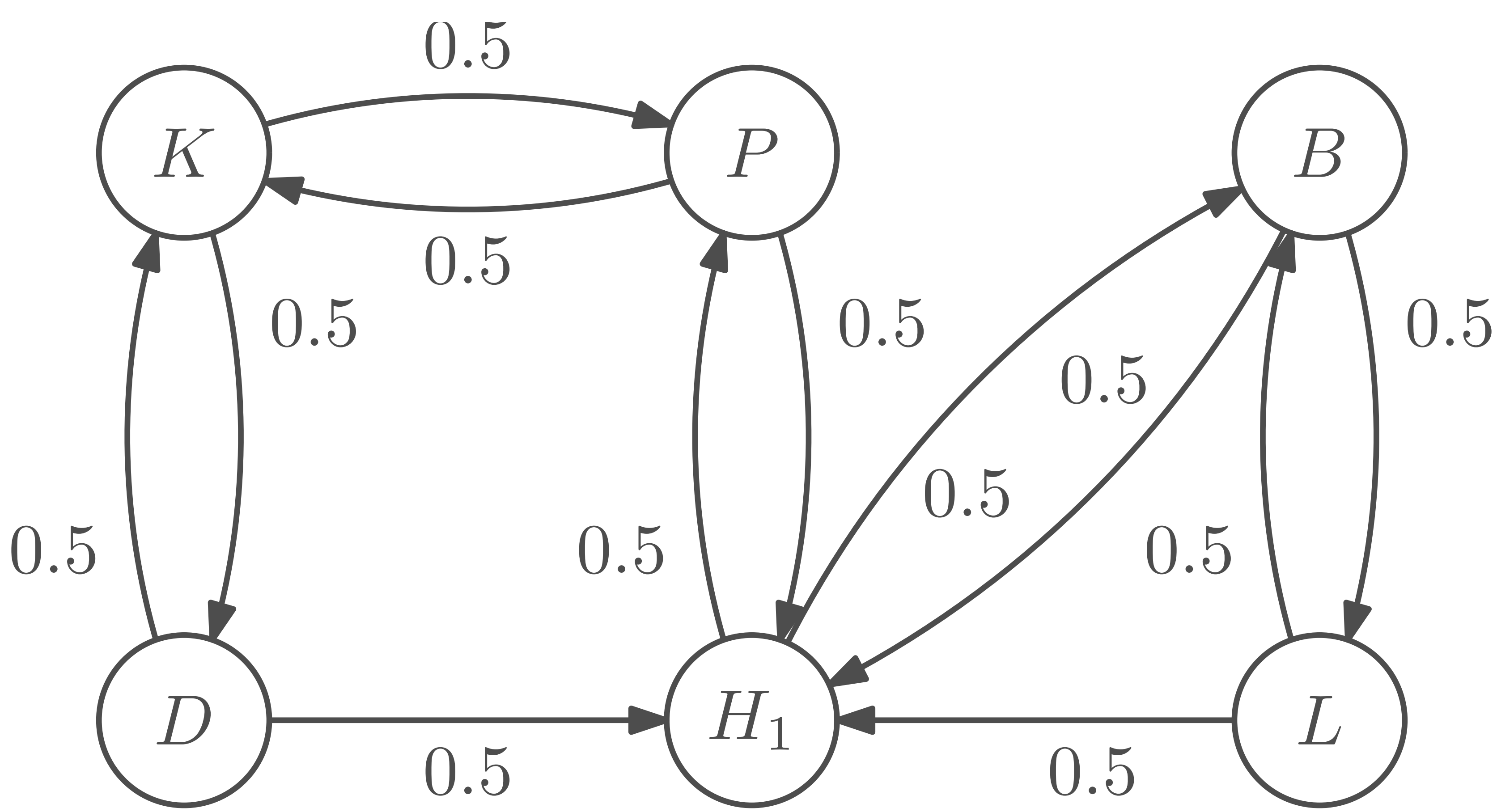
- Possible wall configurations:

- $\mathcal{Z} = \{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,0,1,1),$
  $(0,1,0,0),(0,1,0,1),(0,1,1,0),(0,1,1,1),$
  $(1,0,0,0),(1,0,0,1),(1,0,1,0),(1,0,1,1),$
  $(1,1,0,0),(1,1,0,1),(1,1,1,0),(1,1,1,1)\}$

(U, D, L, R)

Transition probabilities

- We can represent the underlying state transitions using a transition diagram:



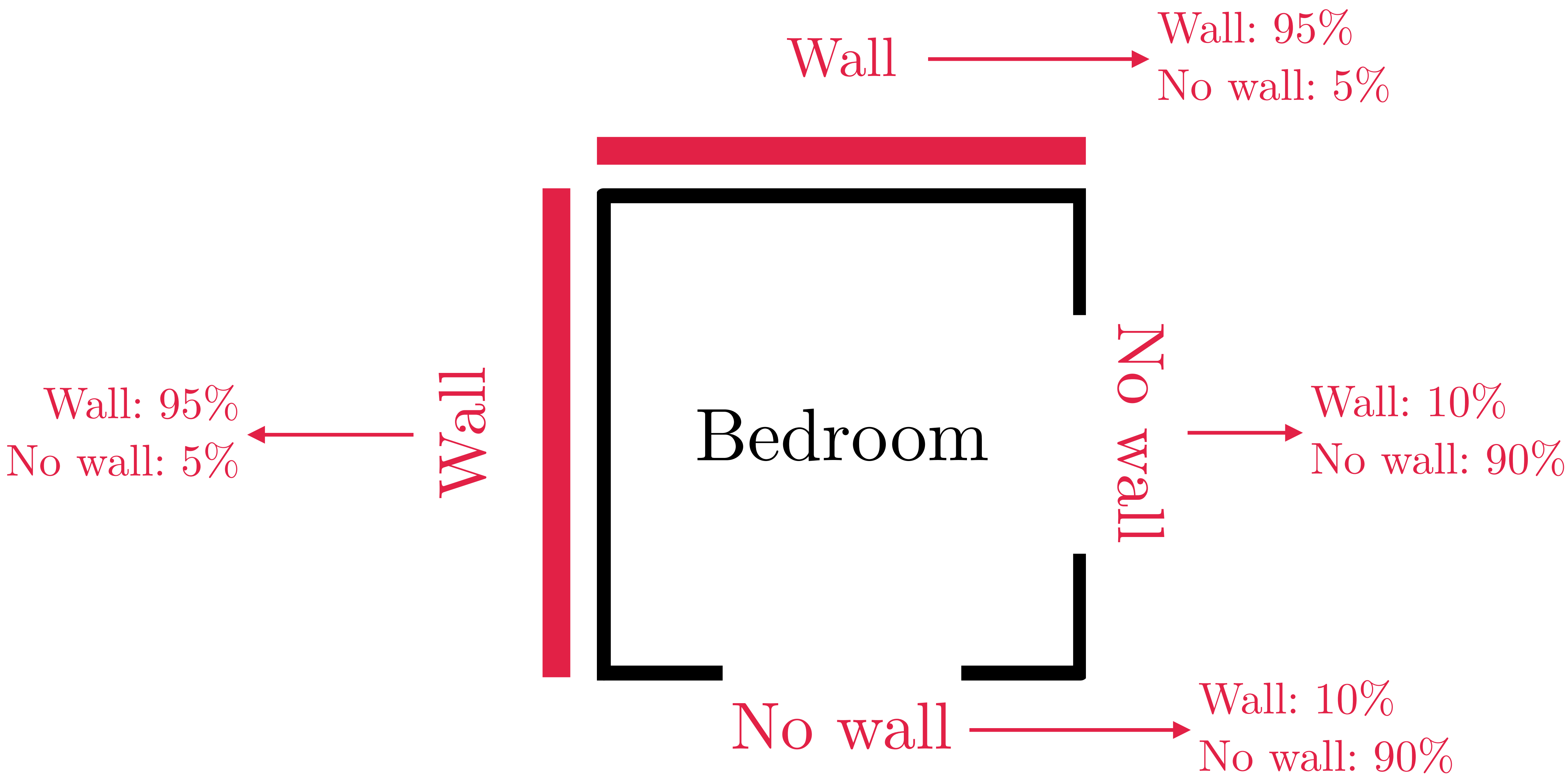
Transition probabilities

- ... or as a matrix:

$$\mathbf{P} = \begin{bmatrix} 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 \end{bmatrix}$$

Observation probabilities

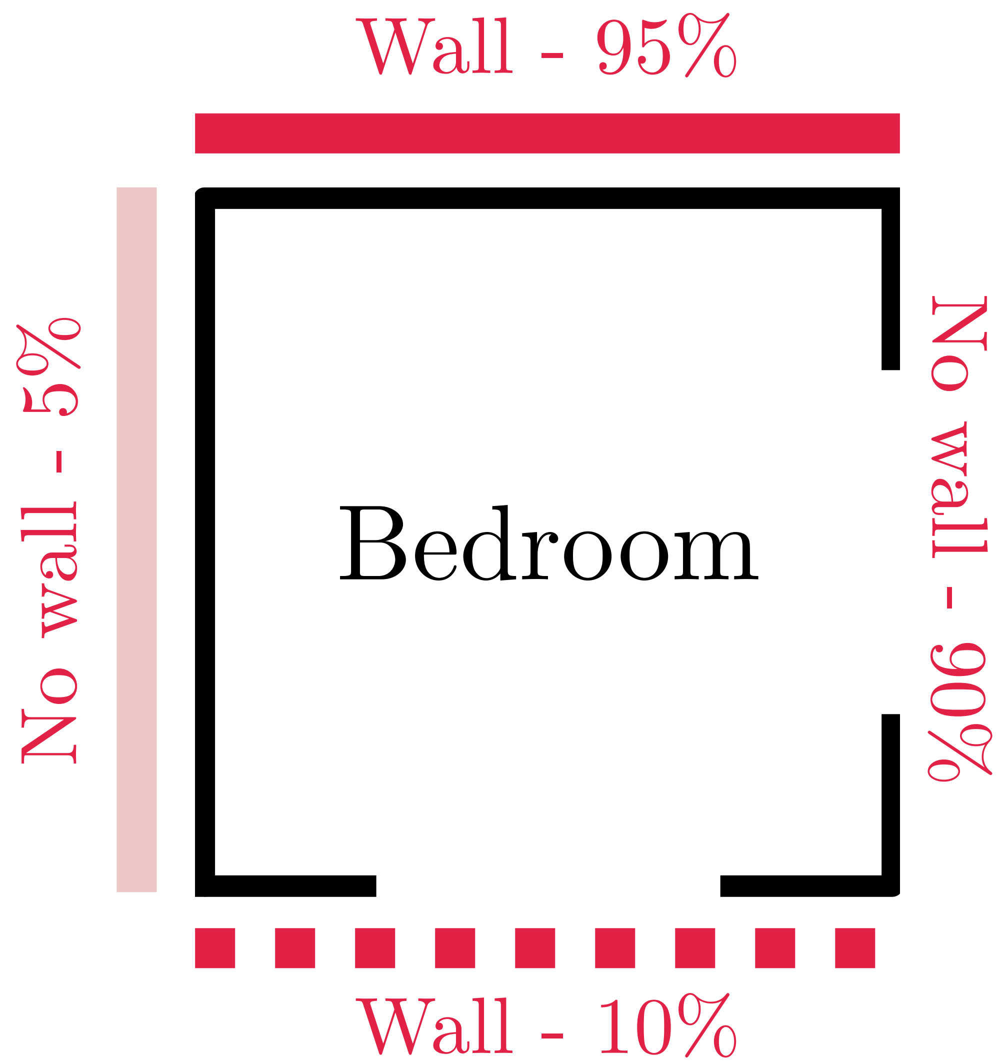
- The observation probabilities can be computed for each wall independently:



Observation probabilities

- For example:

$$\mathbb{P}[\mathrm{z}_t = (\underline{1}, \underline{1}, \underline{0}, \underline{0}) \mid \mathrm{x}_t = B]$$



Result: 0.0042

Observation probabilities

- This yields a large matrix:

$$\mathbf{O} = \begin{bmatrix} 0.0018 & 0.0004 & 0.0342 & \dots & 0.0009 & 0.0722 & 0.0180 \\ 0.0001 & 0.0021 & 0.0021 & \dots & 0.0045 & 0.0045 & 0.0857 \\ 0.0020 & 0.0002 & 0.0384 & \dots & 0.0005 & 0.0812 & 0.0090 \\ 0.5184 & 0.0576 & 0.1296 & \dots & 0.0016 & 0.0036 & 0.0004 \\ 0.0020 & 0.0385 & 0.0002 & \dots & 0.0812 & 0.0004 & 0.0090 \\ 0.0324 & 0.0036 & 0.0081 & \dots & 0.0076 & 0.0171 & 0.0019 \end{bmatrix}.$$

TÉCNICO LISBOA

HMM problems

HMM problems

- **Estimation:**
    - Estimate states from sequences of observations
- **~~Inference:~~**
    - Infer an HMM model from data

Estimation

- **Filtering:**

  Forward algorithm

  - Given a sequence of observations, estimate the final state

- **Smoothing:**

  Viterbi algorithm

  - Given a sequence of observations, estimate the sequence of states

- **Prediction:**

  - Given a sequence of observations, predict future states

Filtering

Filtering

- We are given a sequence of observations $\boldsymbol{z}_{0:T}$

- We want to estimate

What is the final state?

$$\mathbb{P}_{\mu_0}\left[\mathrm{x}_T = x \mid \mathbf{z}_{0:t} = \boldsymbol{z}_{0:T}\right]$$

where μ_0 is the initial distribution, i.e.,

$$\mu_0(x) = \mathbb{P}\left[\mathrm{x}_0 = x\right]$$

Filtering

- For simplicity, write

$$\mu_{T|0:T}(x) = \mathbb{P}_{\mu_0}\left[\mathrm{x}_T = x \mid \mathbf{z}_{0:t} = \boldsymbol{z}_{0:T}\right]$$

Distribution at time T

Given observations $0{:}T$

Forward mapping

Forward mapping

Given a sequence of observations $\boldsymbol{z}_{0:t}$, the forward mapping $\alpha_t : \mathcal{X} \to \mathbb{R}$ is defined for each t as

$$\alpha_t(x) = \mathbb{P}_{\mu_0}\left[\mathrm{x}_t = x, \mathbf{z}_{0:t} = \boldsymbol{z}_{0:t}\right]$$

Joint probability

How likely is it that I end up in x having observed $\boldsymbol{z}_{0:t}$?

So what?

- Forward mapping has several useful properties

   1. We can compute $\mu_{T|0:T}$ from α_T:

$$\mu_{T|0:T}(x) = \frac{\alpha_T(x)}{\sum_{y \in \mathcal{X}} \alpha_T(y)}$$

It's what we want (up to a factor)!

So what?

- Forward mapping has several useful properties

1. We can compute $\mu_{T|0:T}$ from α_T

2. The forward mapping can be computed recursively:

$$\alpha_T(x) = \mathbf{O}(z_T \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y) \alpha_{T-1}(y)$$

Vectorizing the forward update

- Let's look at the update

$$\alpha_t(x) = \mathbf{O}(z_t \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y)\alpha_{t-1}(y)$$

using vectors:

$$\underset{\alpha_{t-1}}{\begin{bmatrix} \square & \square & \square & \cdots & \square \end{bmatrix}}$$

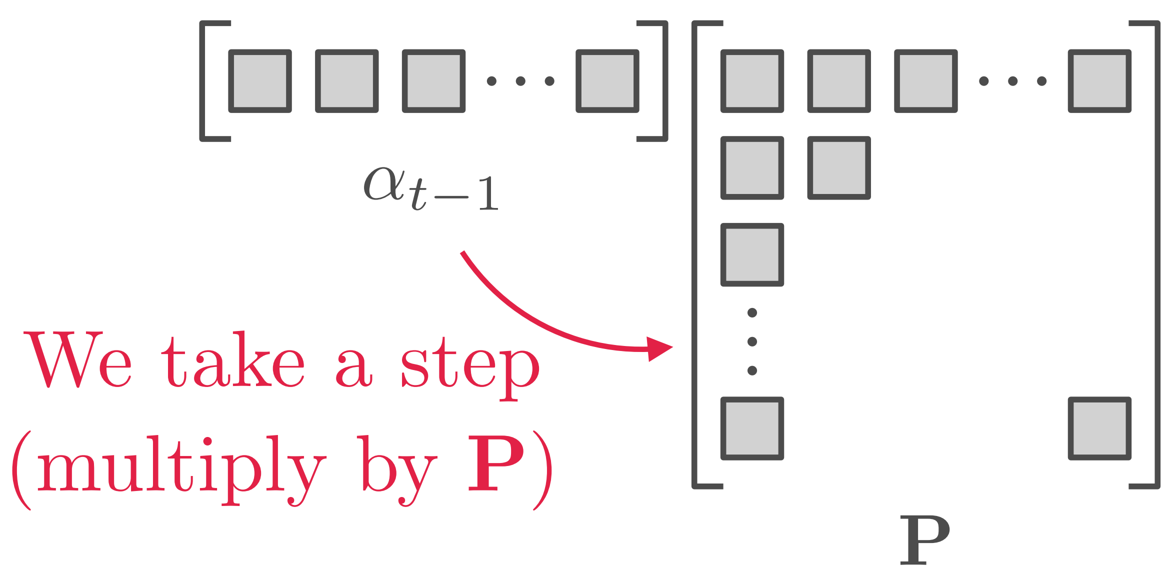
We start with our vector at time $t-1$ (assume row vector)

Vectorizing the forward update

- Let's look at the update

$$\alpha_t(x) = \mathbf{O}(z_t \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y)\alpha_{t-1}(y)$$

using vectors:



Vectorizing the forward update

- Let's look at the update

$$\alpha_t(x) = \mathbf{O}(z_t \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y)\alpha_{t-1}(y)$$

using vectors:

$$\begin{bmatrix} \square & \square & \square & \cdots & \square \end{bmatrix}$$

$$\alpha_{t-1}\mathbf{P}$$

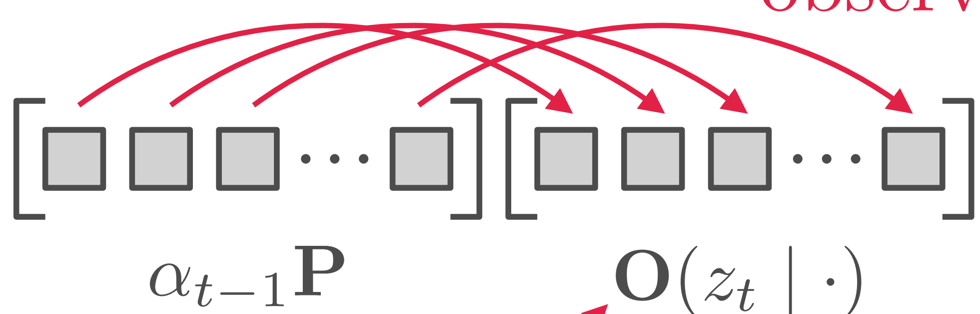
Vectorizing the forward update

- Let's look at the update

$$\alpha_t(x) = \mathbf{O}(z_t \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y)\alpha_{t-1}(y)$$

using vectors:

We multiply each value by the corresponding observation probability



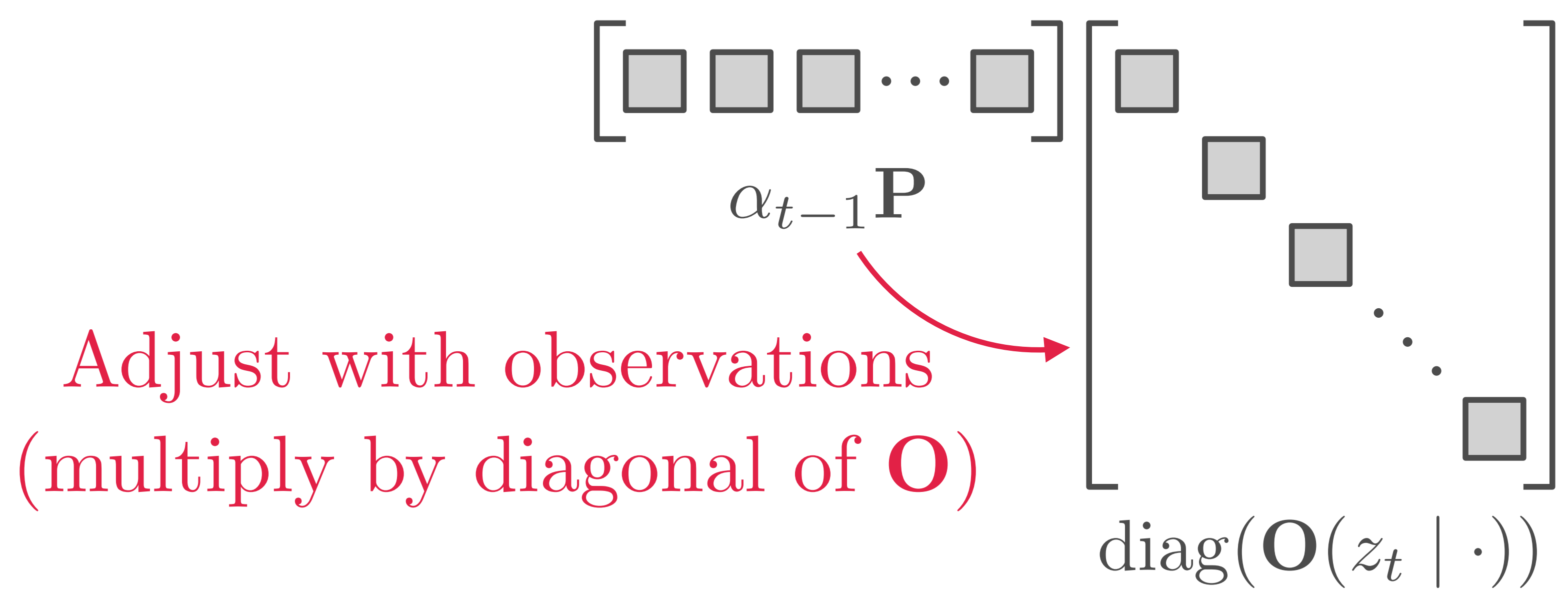
Probability of z_t in each state

Vectorizing the forward update

- Let's look at the update

$$\alpha_t(x) = \mathbf{O}(z_t \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y)\alpha_{t-1}(y)$$

using vectors:

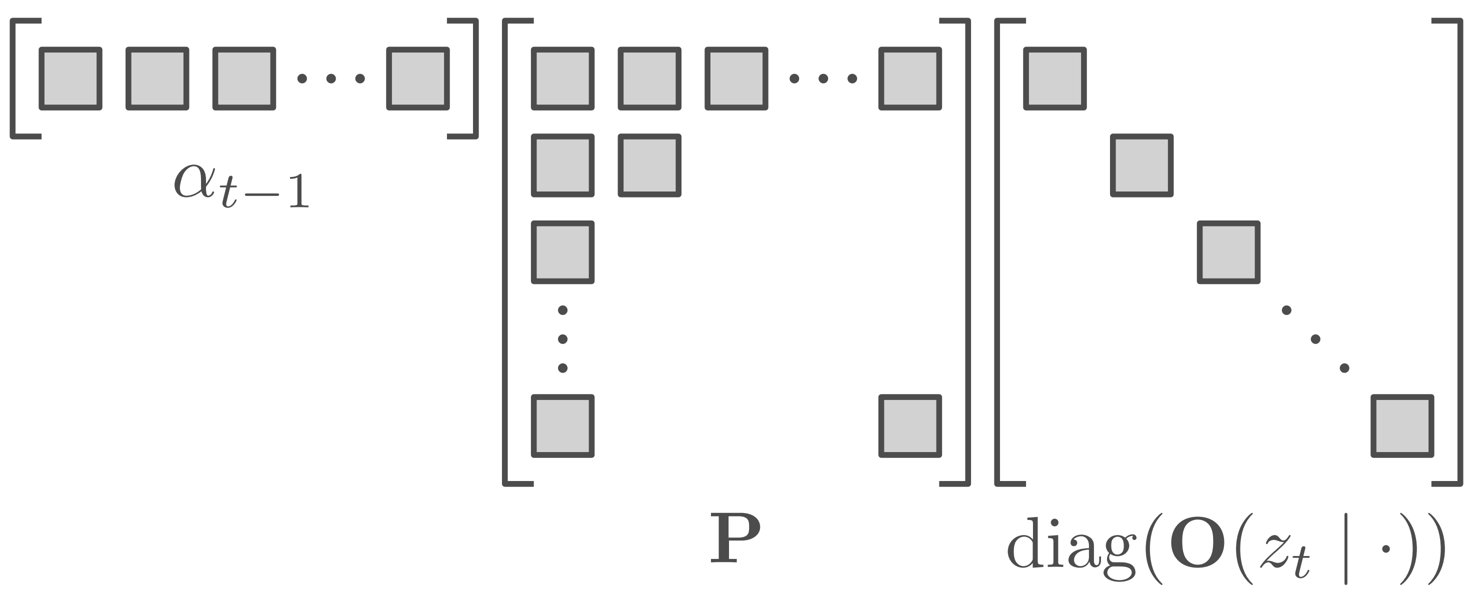


Vectorizing the forward update

- Let's look at the update

$$\alpha_t(x) = \mathbf{O}(z_t \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y)\alpha_{t-1}(y)$$

using vectors:

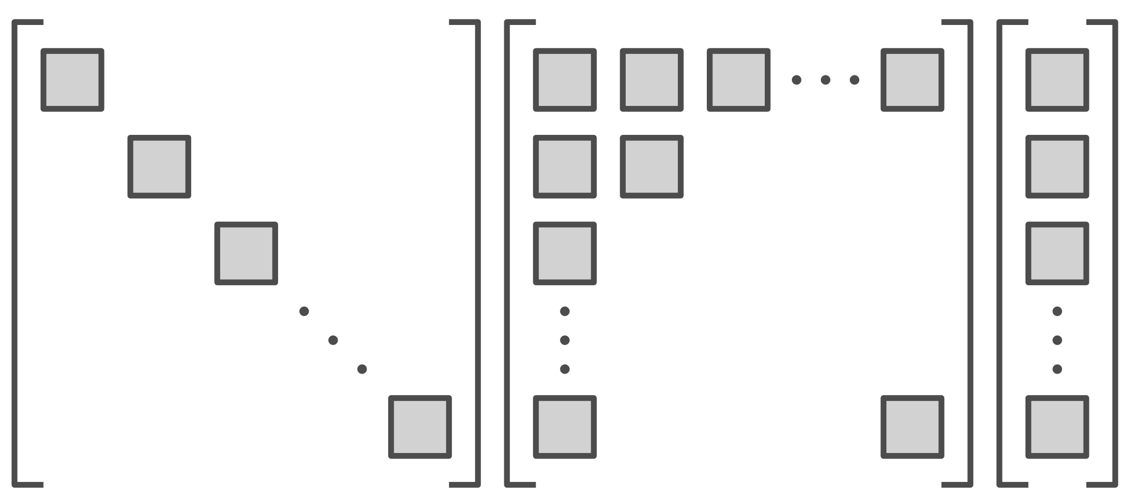


Vectorizing the forward update

- Let's look at the update

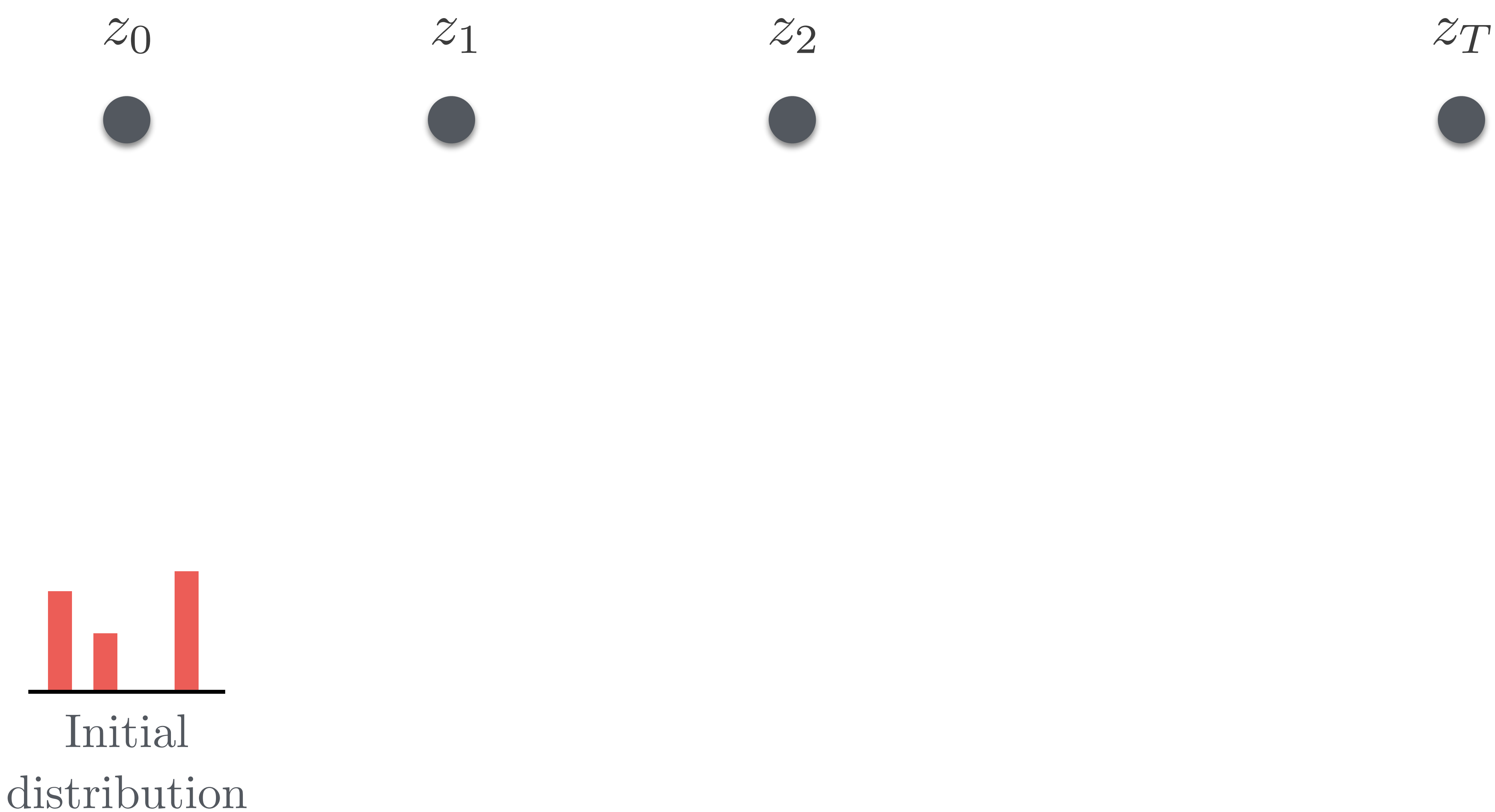
$$\alpha_t(x) = \mathbf{O}(z_t \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y)\alpha_{t-1}(y)$$

using a column vector:

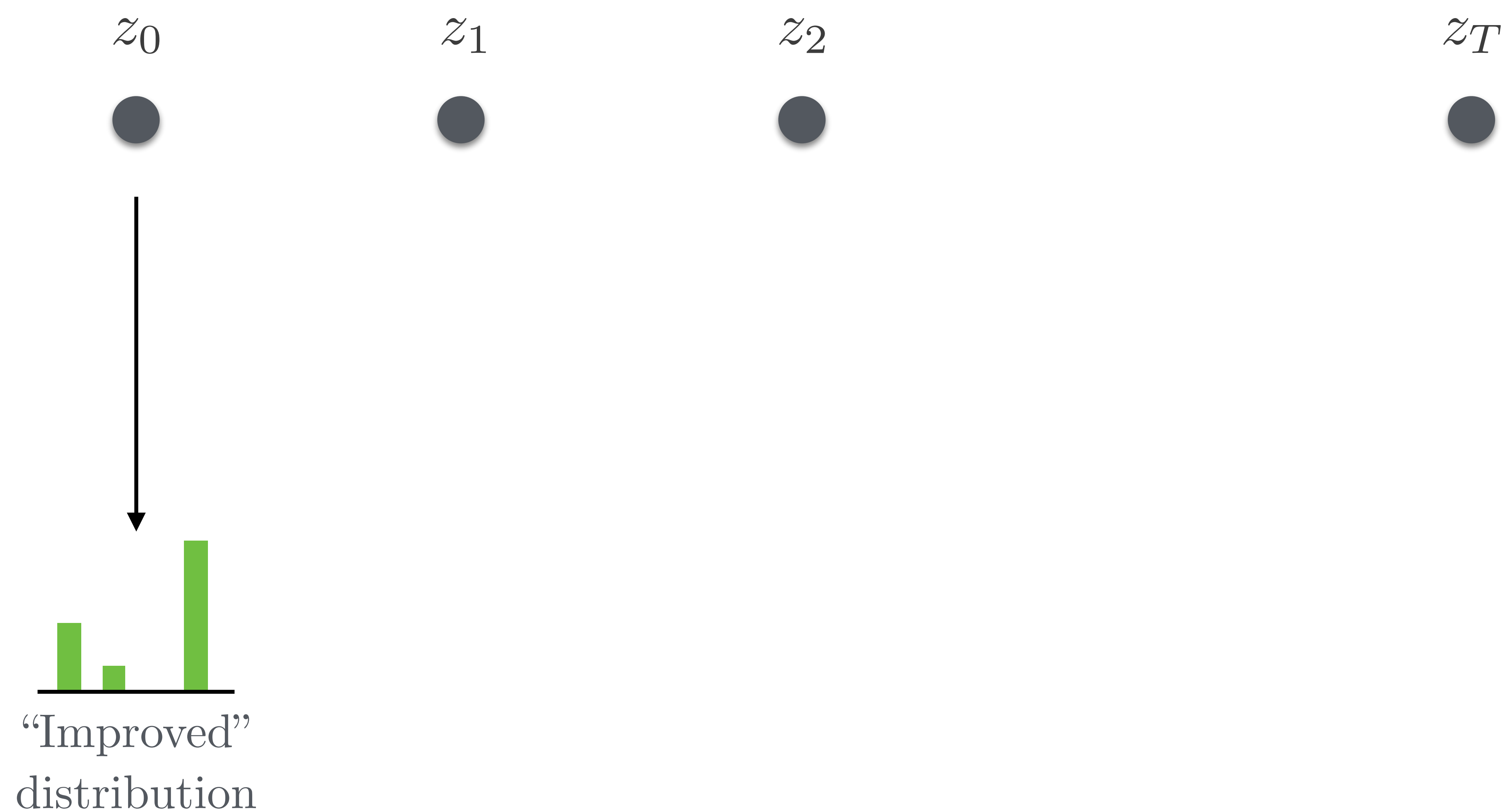


$\operatorname{diag}(\mathbf{O}(z_t \mid \cdot))$ $\quad \mathbf{P}^\top \quad$ α_{t-1}

Forward algorithm

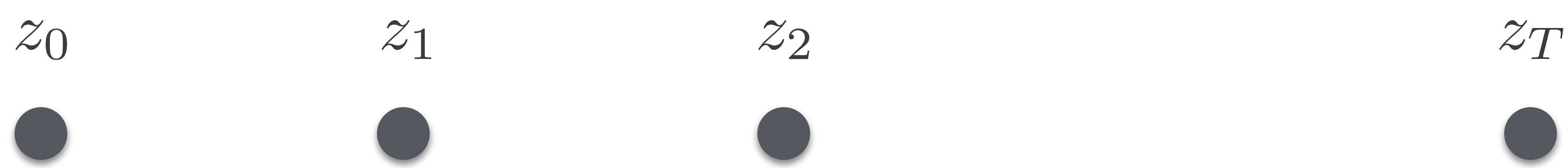


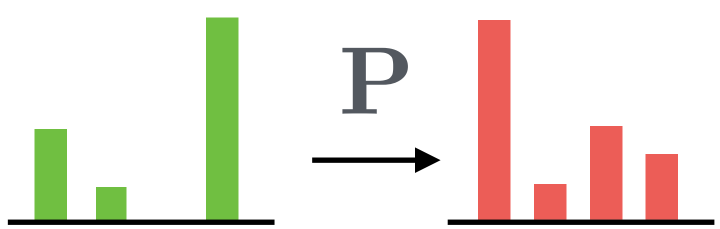
Forward algorithm



$$\mathrm{diag}(\mathbf{O}(z_0 \mid \cdot)\boldsymbol{\mu}_0^\top$$

Forward algorithm

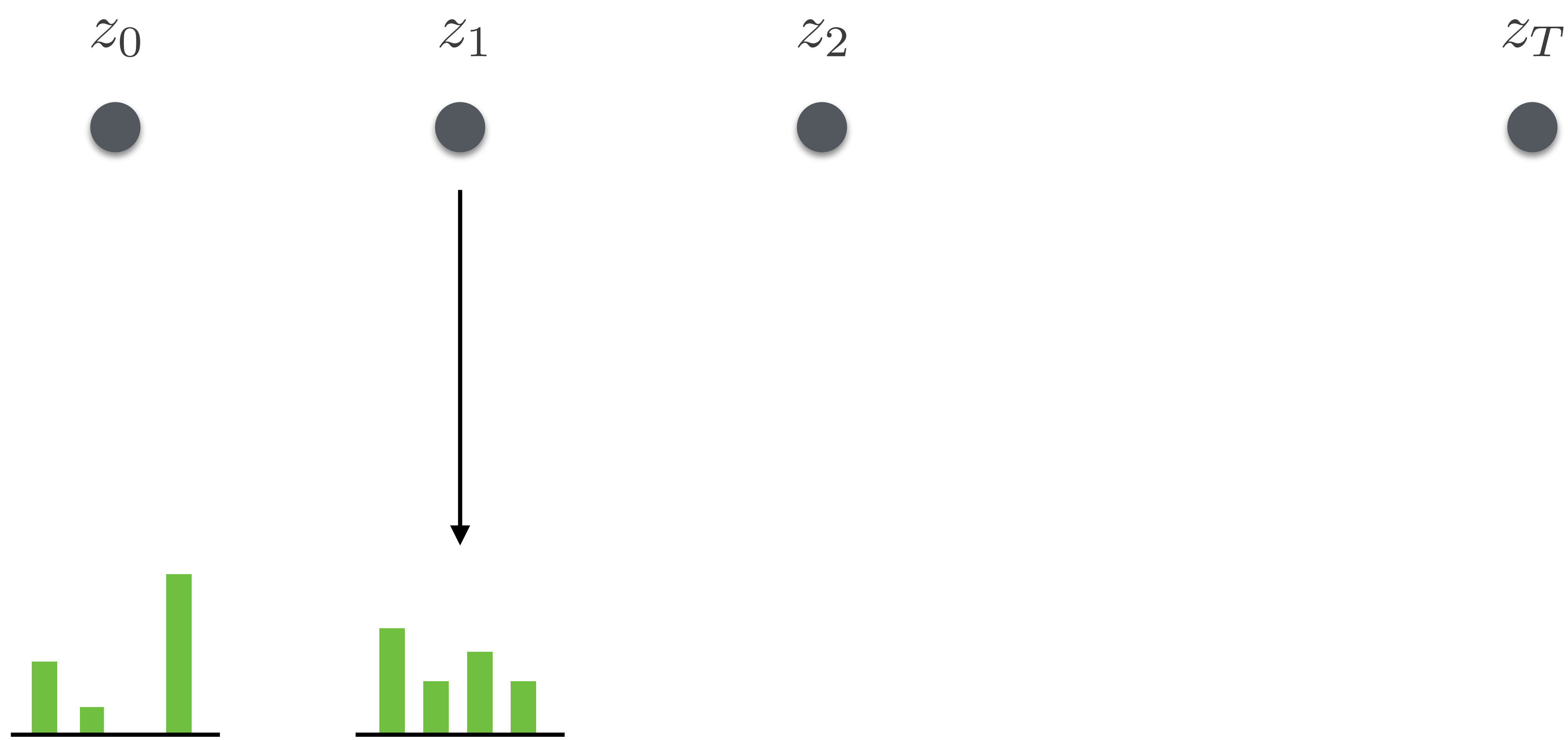




Predicted distribution

$$\mathbf{P}^\top \boldsymbol{\alpha}_0$$

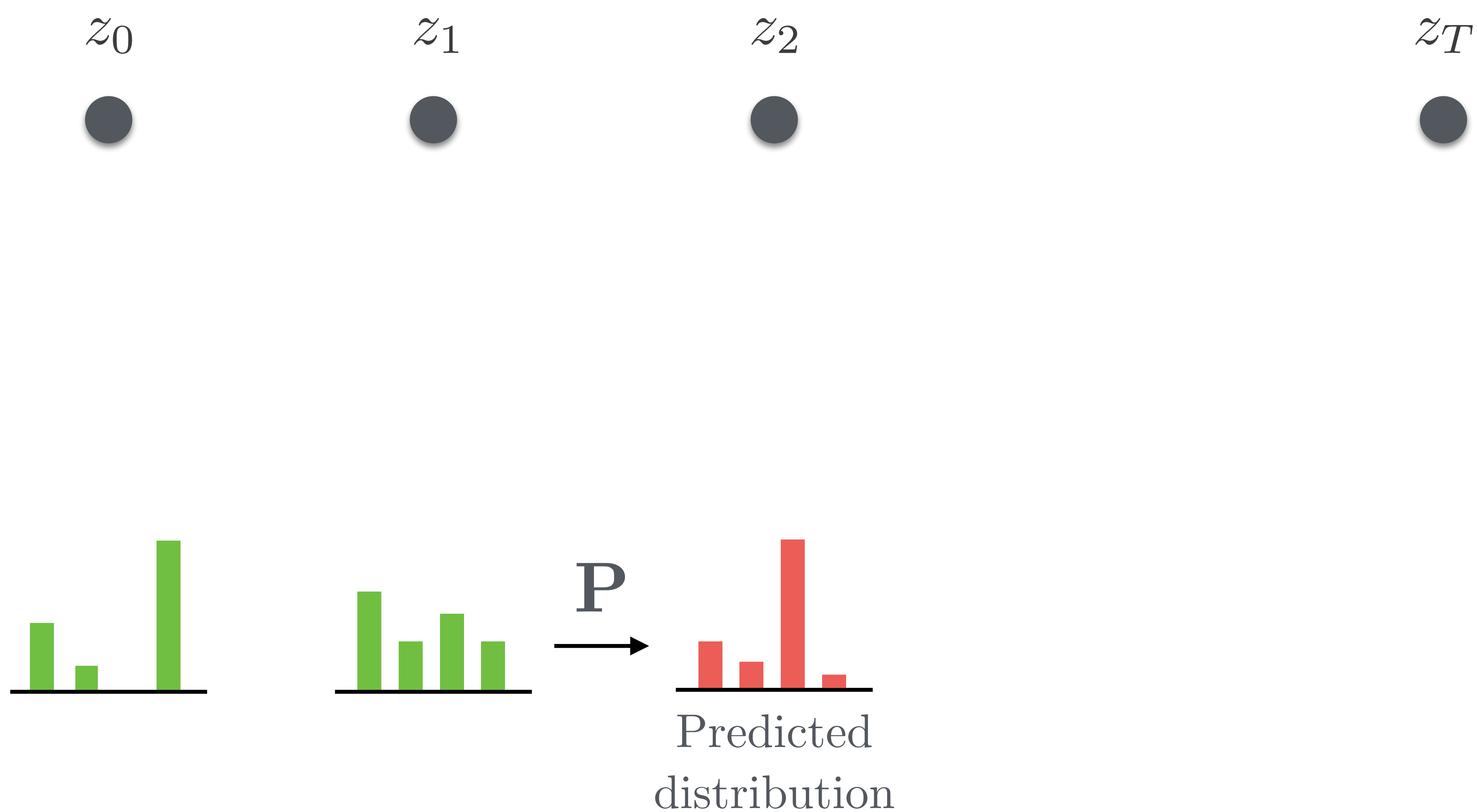
Forward algorithm



"Improved"
distribution

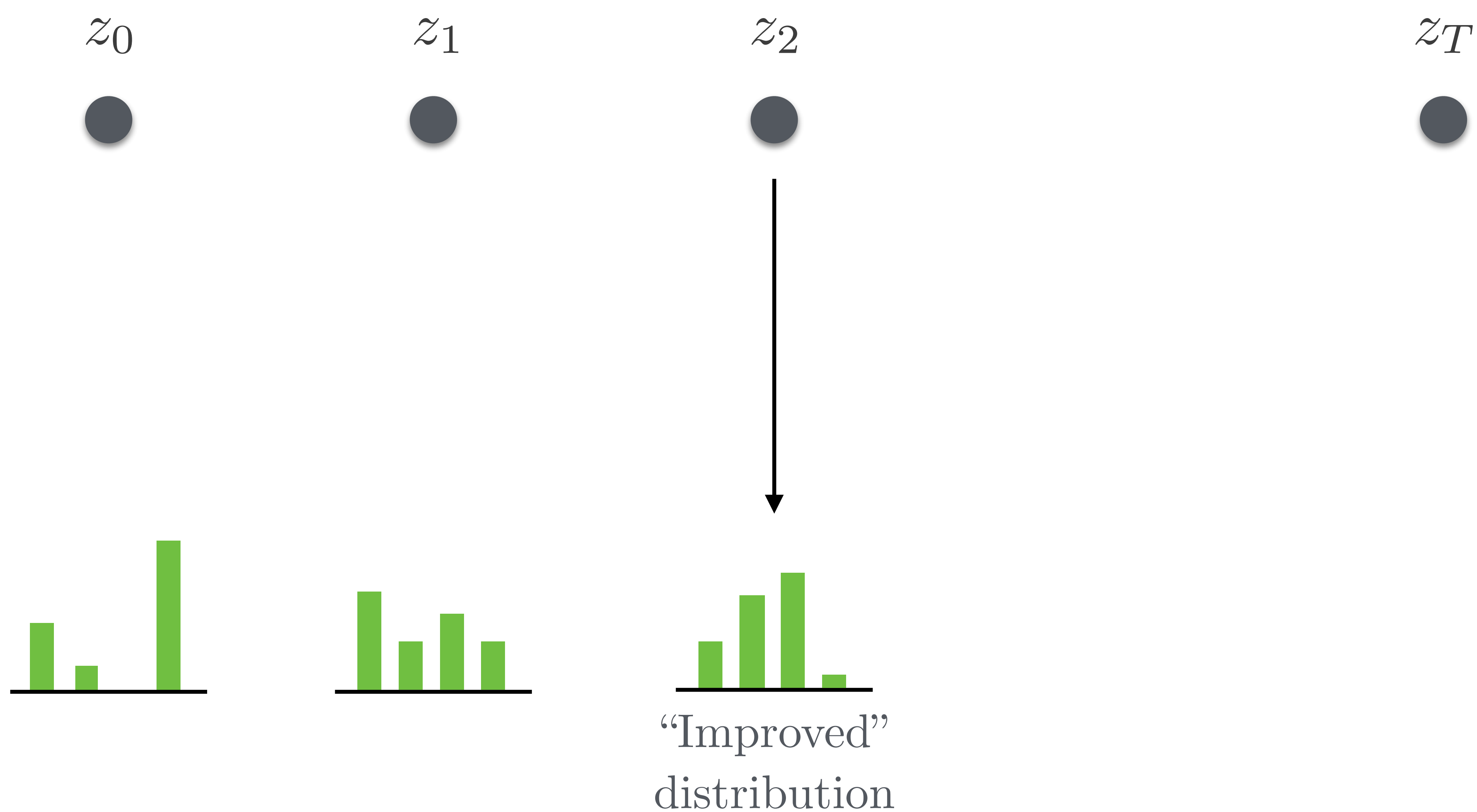
$$\mathrm{diag}(\mathbf{O}(z_1 \mid \cdot))\mathbf{P}^\top \boldsymbol{\alpha}_0$$

Forward algorithm



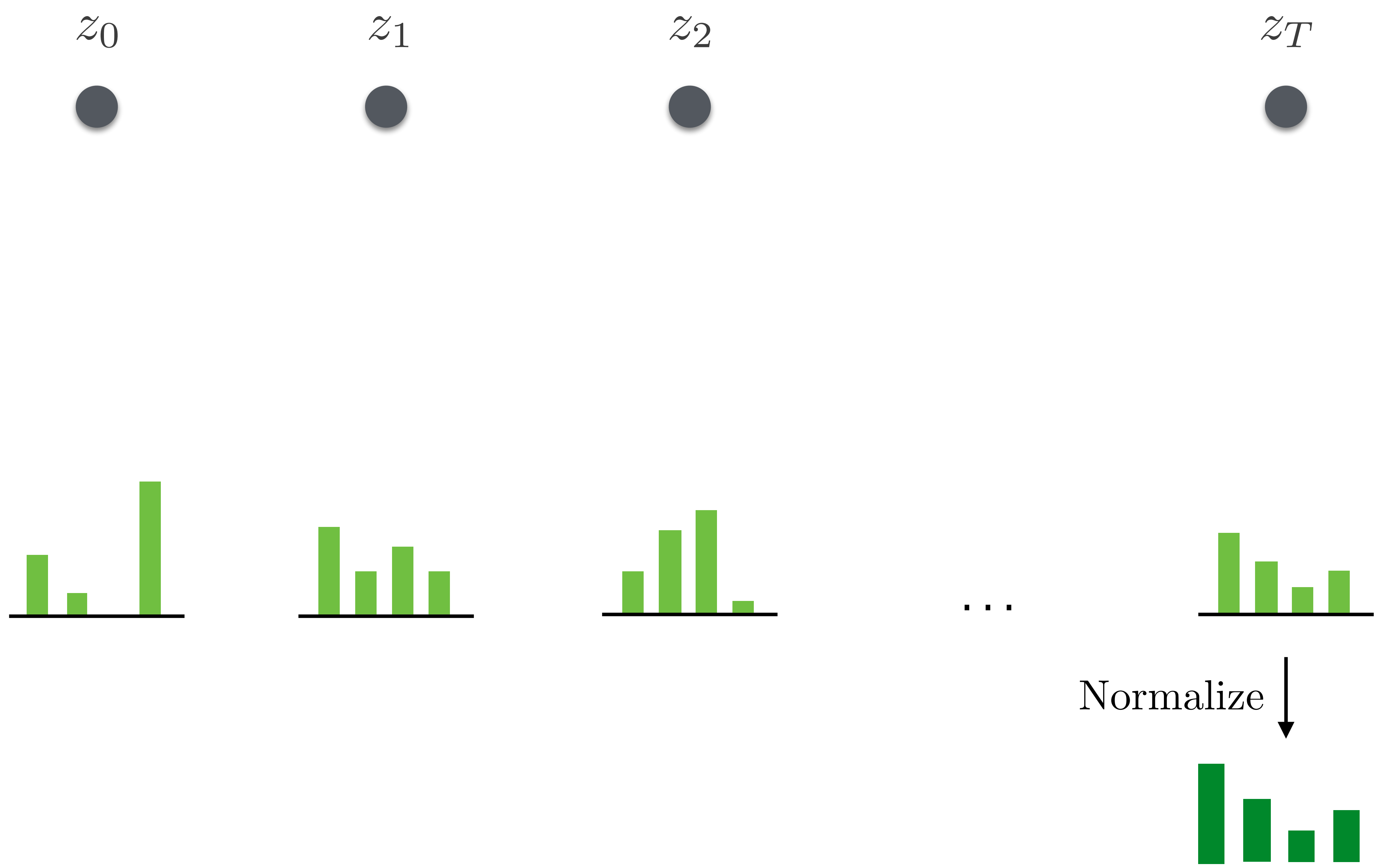
$$\mathbf{P}^\top \boldsymbol{\alpha}_1$$

Forward algorithm



$$\mathrm{diag}(\mathbf{O}(z_2 \mid \cdot))\mathbf{P}^\top \boldsymbol{\alpha}_1$$

Forward algorithm



Forward algorithm

Given observation sequence $\boldsymbol{z}_{0:T}$

1. Multiply initial distribution by $\mathbf{O}(z_0 \mid :)$

Improve
initial
distribution

Forward algorithm

Given observation sequence $\boldsymbol{z}_{0:T}$

1. Multiply initial distribution by $\mathbf{O}(z_0 \mid :)$
2. At each time step:
    a. Multiply current distribution by $\mathbf{P}$

Predict
1-step
move

Forward algorithm

Given observation sequence $\boldsymbol{z}_{0:T}$

1. Multiply initial distribution by $\mathbf{O}(z_0 \mid :)$
2. At each time step:
    a. Multiply current distribution by $\mathbf{P}$
    b. Multiply by $\mathbf{O}(z_t \mid :)$

Check prediction with observation

Forward algorithm

Given observation sequence $\boldsymbol{z}_{0:T}$

1. Multiply initial distribution by $\mathbf{O}(z_0 \mid :)$
2. At each time step:
    a. Multiply current distribution by $\mathbf{P}$
    b. Multiply by $\mathbf{O}(z_t \mid :)$
3. Normalize

Forward algorithm

Require: Observation sequence $\boldsymbol{z}_{0:T}$

1. Initialize $\boldsymbol{\alpha}_0 = \text{diag}(\mathbf{O}(z_0 \mid \cdot)\boldsymbol{\mu}_0^\top$

If we have an initial observation

2. for $t = 1, \ldots, T$ do

$$\boldsymbol{\alpha}_t = \text{diag}(\mathbf{O}(z_t \mid \cdot))\mathbf{P}^\top \boldsymbol{\alpha}_{t-1}$$

Here's our update!

3. end for

4. return $\boldsymbol{\mu}_{T|0:T} = \boldsymbol{\alpha}_T / \text{sum}(\boldsymbol{\alpha}_T)$

Example: The urn problem

- Suppose that

$$\boldsymbol{\mu}_0 = \begin{bmatrix} 0.125 & 0.375 & 0.375 & 0.125 \end{bmatrix}$$

- We observe the sequence of observations

$$\boldsymbol{z}_{0:2} = \{w, w, b\}$$

- What is the state at time $t = 2$?

Step 1: Initialize $\boldsymbol{\alpha}_0$

- $\boldsymbol{\alpha}_0 = \mathrm{diag}(\mathbf{O}(z_0 \mid \cdot)\boldsymbol{\mu}_0^\top$

$$\begin{array}{c} \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 1 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ 0 \\ 1 \end{matrix} \end{bmatrix} \\ z_0 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.125 \\ 0.375 \\ 0.375 \\ 0.125 \end{bmatrix} \rightarrow \begin{bmatrix} 0.125 \\ 0 \\ 0.375 \\ 0 \end{bmatrix}$$

Step 2: Compute $\boldsymbol{\alpha}_1$

- $\boldsymbol{\alpha}_1 = \operatorname{diag}(\mathbf{O}(z_1 \mid \cdot))\mathbf{P}^\top \boldsymbol{\alpha}_0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 & 0.05 & 0.05 \\ 0.6 & 0.6 & 0.15 & 0.15 \\ 0.15 & 0.15 & 0.6 & 0.6 \\ 0.05 & 0.05 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0.125 \\ 0 \\ 0.375 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.04 \\ 0 \\ 0.24 \\ 0 \end{bmatrix}$$

TÉCNICO LISBOA

Step 3: Compute $\boldsymbol{\alpha}_2$

- $\boldsymbol{\alpha}_2 = \mathrm{diag}(\mathbf{O}(z_2 \mid \cdot))\mathbf{P}^\top \boldsymbol{\alpha}_1$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0.2 & 0.2 & 0.05 & 0.05 \\ 0.6 & 0.6 & 0.15 & 0.15 \\ 0.15 & 0.15 & 0.6 & 0.6 \\ 0.05 & 0.05 & 0.2 & 0.2 \end{bmatrix}
\begin{bmatrix} 0.04 \\ 0 \\ 0.24 \\ 0 \end{bmatrix}
\rightarrow
\begin{bmatrix} 0 \\ 0.06 \\ 0 \\ 0.05 \end{bmatrix}
$$

Final step: Compute $\boldsymbol{\mu}_{2 \mid 0:2}$

- We finally get:

$$\begin{aligned}\boldsymbol{\mu}_{2|0:2} &= \boldsymbol{\alpha}_2/\mathrm{sum}(\boldsymbol{\alpha}_2)\\ &= \begin{bmatrix}0 & 0.552 & 0 & 0.448\end{bmatrix}\end{aligned}$$