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MSc in Computer Science and Engineering

Planning, Learning and Decision Making

Homework 1. Markov chains

Consider the kids' game Insey-Winsey-Spider.

The single player has several levels to climb (corresponding to steps in a ladder starting at level 0) and wants to reach the top level.

At each turn, the player rolls a die. After that, the player has the possibility to climb the number of steps given by the die. However, the player will only go up if it is a sunny day; if it is a rainy day, then the player goes back to level 0. To determine the weather, the player turns an arrow to see where it lands (let's assume that the rainy part corresponds to 20% of the area).

To simplify, consider that there are 6 steps on the ladder (0-5) and we are using a two-sided die (so the player can go up 1 or 2 steps). Assume also that the two-sided die is fair.

After reaching level 5, the game continues, but it is no longer possible to move up. On level 4, either a 1 or 2 on the die will make it jump to level 5 (no need to get the correct value).

Exercise

(a) Write down a Markov chain model that describes the movement of the player.

(b) What is the minimum number of steps to reach level 5? And what is the probability of that happening (make your solution explicit by showing all the steps).

(c) If we play many steps, what is the probability of observing the game at level 0 and at level 5 (make your solution explicit by showing all the steps)?

(d) Is the Markov chain ergodic?

Solution:

(a) The Markov chain is specified as a pair $(\mathcal{X}, \boldsymbol{P})$, where the states correspond to positions in the ladder. We have that

$$\mathcal{X} = \{0, 1, 2, 3, 4, 5\},$$

where the states are labeled after the corresponding level. The transition probabilities can be extracted from the provided description of the game rules.

If we are in state l we can go either 1 or 2 steps up if it is sunny, or to 0 if it is rainy

$$\begin{aligned} x \to &\quad x+1, \text{ with prob. } 0.5 \times 0.8 \\ x \to &\quad x+2, \text{ with prob. } 0.5 \times 0.8 \\ x \to &\qquad 0, \text{ with prob. } 0.2 \end{aligned}$$

$$\boldsymbol{P} = \begin{bmatrix} .2 & .4 & .4 & 0 & 0 & 0 \\ .2 & 0 & .4 & .4 & 0 & 0 \\ .2 & 0 & 0 & .4 & .4 & 0 \\ .2 & 0 & 0 & 0 & .4 & .4 \\ .2 & 0 & 0 & 0 & 0 & .8 \\ .2 & 0 & 0 & 0 & 0 & .8 \end{bmatrix}.$$

In numpy we can implement it as follows.

```
import numpy as np

P = np.array([[.2 , .4 , .4 , 0 , 0 , 0],
[.2 , 0 , .4 , .4 , 0 , 0],
[.2 , 0 , 0 , .4 , .4 , 0],
[.2 , 0 , 0 , 0 , .4 , .4],
[.2 , 0 , 0 , 0 ,0  , .8],
[.2 , 0 , 0 , 0 , 0 , .8]])
```

(b) We can compute the transition for t steps until the probability of being in state 5 is not 0, at that point we have the number of steps and the probability.

$$\mu_0 P = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\end{bmatrix} P = \begin{bmatrix}.2 & .4 & .4 & 0 & 0 & 0\end{bmatrix} = \mu_1$$
$$\mu_1 P = \begin{bmatrix}0.2 & 0.08 & 0.24 & 0.32 & 0.16 & 0.\end{bmatrix} = \mu_2$$
$$\mu_2 P = \begin{bmatrix}0.2 & 0.08 & 0.112 & 0.128 & 0.224 & 0.256\end{bmatrix} = \mu_3$$

For the previous calculation, we can see that after 3 steps we have a probability of 0.256 to be in state 5.

In numpy we can implement it as follows.

```
u = np.array([1,0,0,0,0,0])

print(u @ P)
print(u @ P @ P)
print(u @ P @ P @ P)

[0.2 0.4 0.4 0.  0.  0. ]
[0.2  0.08 0.24 0.32 0.16 0.  ]
[0.2   0.08  0.112 0.128 0.224 0.256]
```

(c) We can compute the stationary distribution

$$\mu_\infty = \mu_0 P^\infty = \begin{bmatrix}0.2 & 0.08 & 0.112 & 0.0768 & 0.07552 & 0.45568\end{bmatrix}$$

and observe that we have a probability of 0.2 of being in state 0 and of 0.45568 of being in state 5.

We can also do this by computing the eigenvectors of the matrix P.

$\mu P = \mu$

In numpy:

```
w, v = np.linalg.eig(P.T)
# don't forget the transpose to compute the left-side eigenvectors

# Eigenvector associated with eigenvalue 1 is v[0, :]
# don't forget the transpose
u = v.T[0,:]
```

```
# Normalize:
u = np.real(u / u.sum())
print(u)
```

(d) The chain is aperiodic and irreducible, therefore it's ergodic.

- Irreducibility: Notice that from any state there is always a 20% chance (rainy day) to jump directly to state 0. And from state 0 (under sunny conditions) you can eventually climb to any higher level. Thus every state communicates with every other state, i.e. there is a single communicating class.
- Aperiodicity: It can be proven that all states in the same communicating class have the same period. For our Markov chain, state 0 has a self-loop with p(0,0) = 0, which makes it aperiodic. Since all states belong to the same communicating class, the entire chain is aperiodic.

Also, from the previous question, we know that the Markov chain has a stationary distribution. Now, to prove that it converges to the stationary distribution regardless of the initial distribution, we compute P^∞ by multiplying the matrix P by itself repeatedly until it stabilizes. When P^∞ is reached, all its rows are identical and equal to μ^*. Therefore, whatever the initial distribution $\mu P^\infty = \mu^*$, since essentially we will always be doing a weighted average over the same vector.

$$P^\infty = \begin{bmatrix} 0.2 & 0.08 & 0.112 & 0.0768 & 0.07552 & 0.45568 \\ 0.2 & 0.08 & 0.112 & 0.0768 & 0.07552 & 0.45568 \\ 0.2 & 0.08 & 0.112 & 0.0768 & 0.07552 & 0.45568 \\ 0.2 & 0.08 & 0.112 & 0.0768 & 0.07552 & 0.45568 \\ 0.2 & 0.08 & 0.112 & 0.0768 & 0.07552 & 0.45568 \\ 0.2 & 0.08 & 0.112 & 0.0768 & 0.07552 & 0.45568 \end{bmatrix}$$

Yet another way to prove this is by Wielandt's theorem, which implies that a Markov chain with n states is ergodic if and only if all elements of P^m are positive for $m = (n-1)^2 + 1$.