Multiagent decision making: Bayesian Games

AASMA 2023

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Outline

- **Introduction to Bayesian games**
- First definition
- Second definition
- Analyzing Bayesian games
- Exercises

Bayesian Games

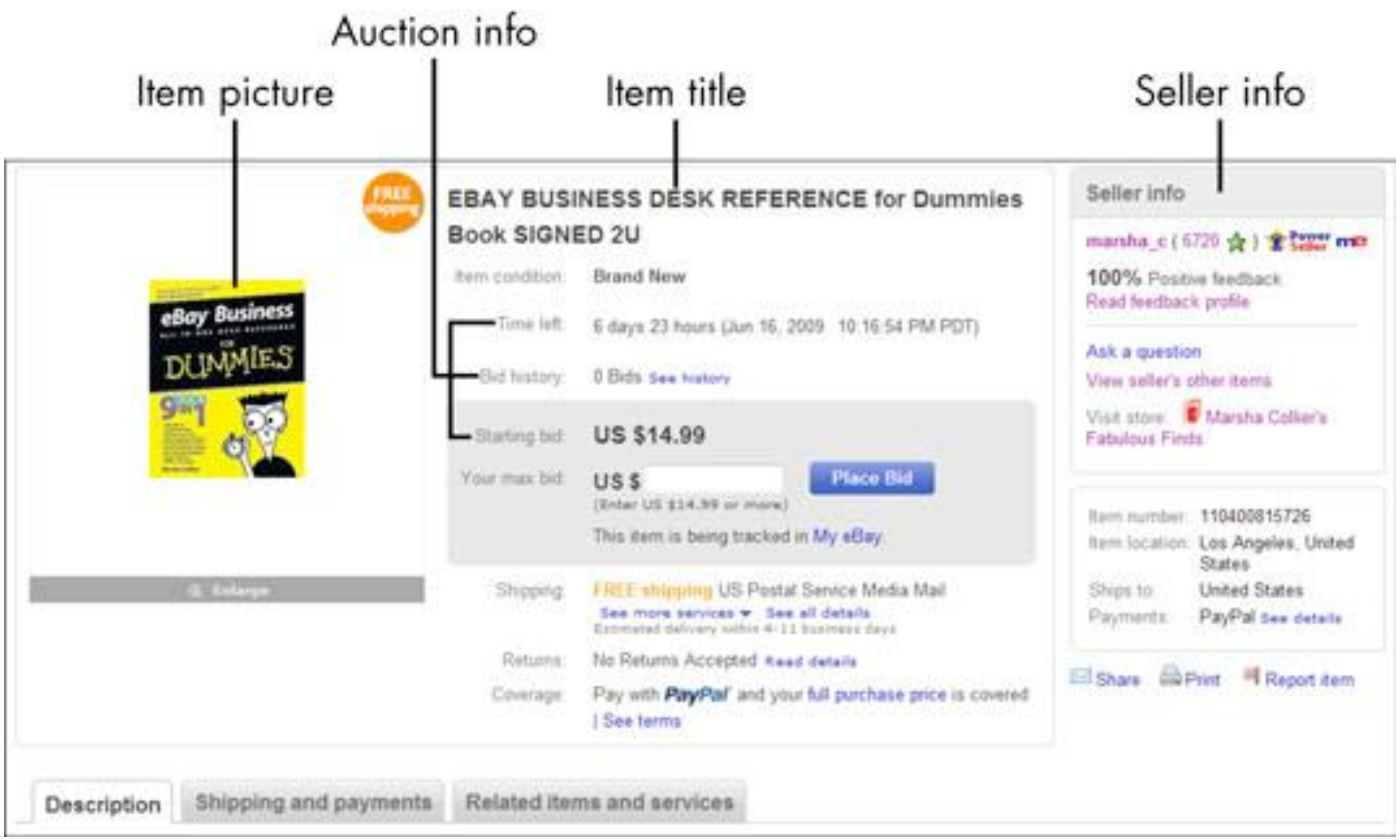
- So far, all of the games make the following **assumption**:
  - **All agents know what game is being played**
- More specifically, we assume to be common knowledge for all agents:
  - the **number of agents**
  - the **actions** available to each agent
  - the **payoff** associated with each joint action

Bayesian Games

What if the agents are uncertain about the game being played?

Bayesian Games

Bayesian Games



Bayesian Games

Bayesian Games

- **Bayesian games** (or games of incomplete information) allow us to:
  - Represent **agents' uncertainties about the game being played**
  - The uncertainty regarding the game is represented as a **probability distribution** over a set of possible games

Bayesian Games

- Two key assumptions in Bayesian games:
  - The **games differ only in their payoffs**
    - Hence, all possible games have the same number of agents and the same action set for each agent
  - The **beliefs of the different agents are posteriors**
    - We obtain the **posteriors** by conditioning a **common prior** on **individual private signals**

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Bayesian Games – First Definition

- A Bayesian game consists of:
  - A set of games that differ only in their payoffs
  - A common prior defined over the games
  - A partition structure over the games for each agent

Bayesian Games – First Definition

- **Definition** (Bayesian game: information sets): *A* **Bayesian game** *is a tuple* (N, G, P, I) *where:*

  - N *is a set of agents*

  - G *is a set of games with* N *agents each*
    - *such that, if* $g, g' \in G$ *then for each agent* $i \in N$ *the action sets in* g *is identical to the action sets in* g'

  - $P \in \Pi(G)$ *is a common prior over games*
    - *where* $\Pi(G)$ *is the set of all probability distributions over* G

  - $I = (I_1, \ldots, I_N)$ *is a tuple of partitions of* G*, one for each agent*

Bayesian Games – First Definition

There are four possible games that might be played

	$I_{2,1}$	$I_{2,2}$
$I_{1,1}$	MP $\begin{array}{cc} 2,0 & 0,2 \\ 0,2 & 2,0 \end{array}$ $p = 0.3$	PD $\begin{array}{cc} 2,2 & 0,3 \\ 3,0 & 1,1 \end{array}$ $p = 0.1$
$I_{1,2}$	Coord $\begin{array}{cc} 2,2 & 0,0 \\ 0,0 & 1,1 \end{array}$ $p = 0.2$	BoS $\begin{array}{cc} 2,1 & 0,0 \\ 0,0 & 1,2 \end{array}$ $p = 0.4$

MP = Matching Pennies

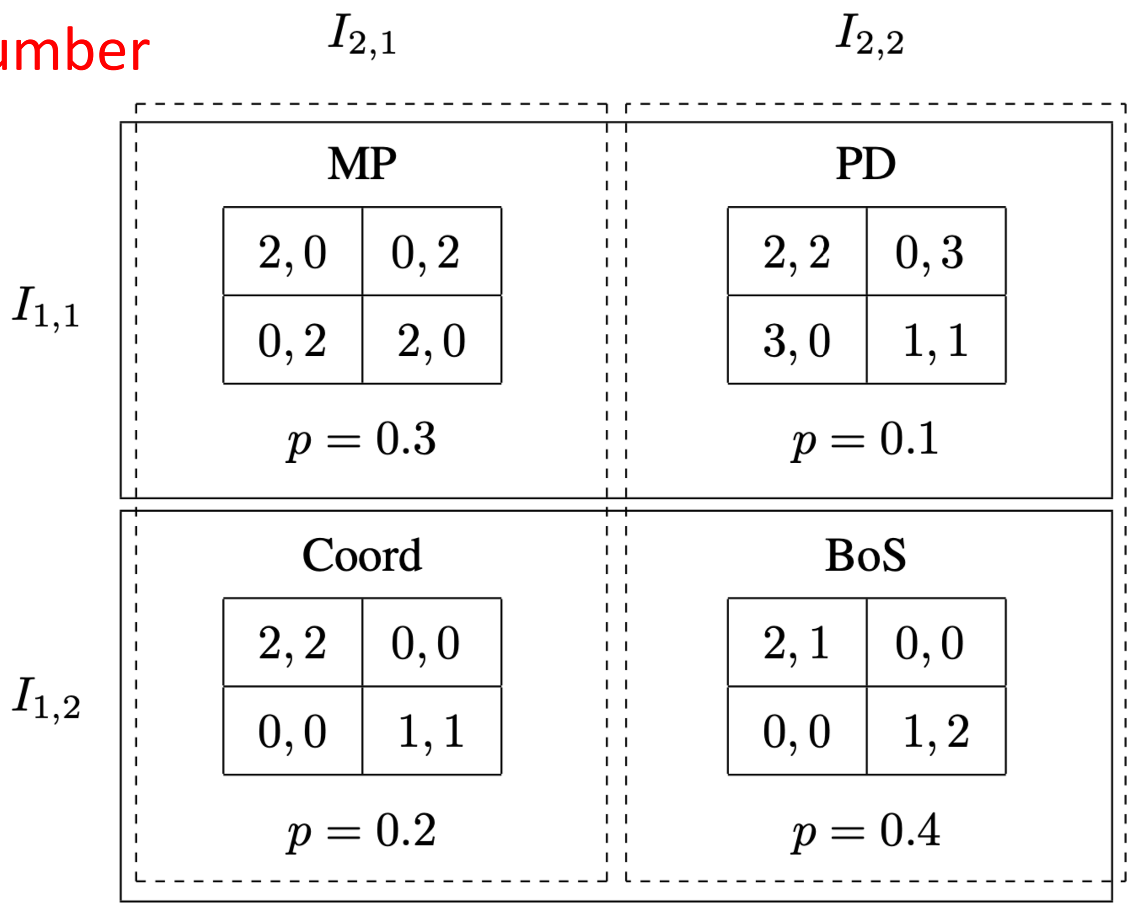
Coord = Coordination game

PD = Prisoner's dilemma

BoS = Battle of Sexes

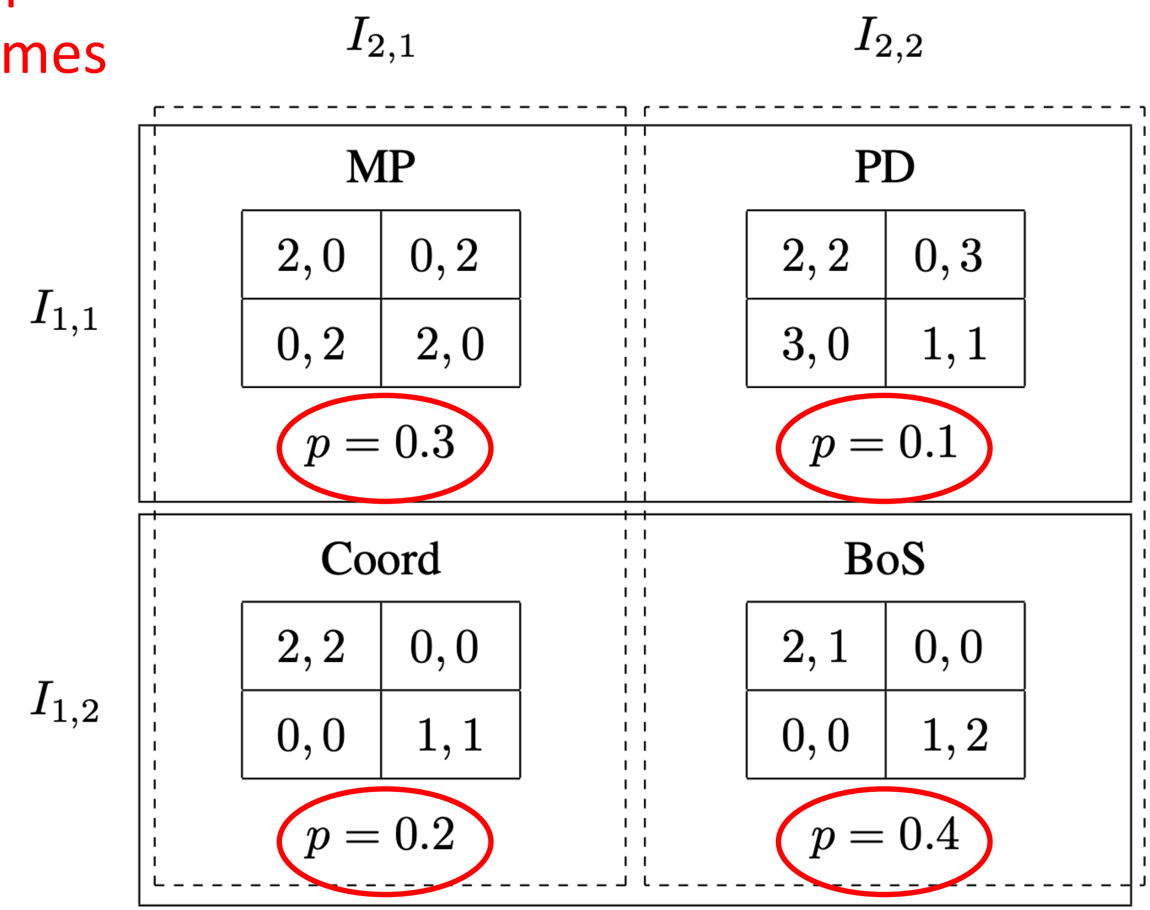
Bayesian Games – First Definition

All games have
the same number
of agents



Bayesian Games – First Definition

A common prior over the games



E.g., there is 0.4 chance that the Agents will play BoS

Bayesian Games – First Definition

The action sets are the same in all games

	$I_{2,1}$	$I_{2,2}$
$I_{1,1}$	MP	PD
$I_{1,2}$	Coord	BoS

MP

	L	R
U	$2, 0$	$0, 2$
D	$0, 2$	$2, 0$

$p = 0.3$

PD

	L	R
U	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

$p = 0.1$

Coord

	L	R
U	$2, 2$	$0, 0$
D	$0, 0$	$1, 1$

$p = 0.2$

BoS

	L	R
U	$2, 1$	$0, 0$
D	$0, 0$	$1, 2$

$p = 0.4$

Bayesian Games – First Definition

Information sets for agent 1 (indistinguishable sets)

$I_{2,1}$ $I_{2,2}$

$I_{1,1}$

MP

$2,0$	$0,2$
$0,2$	$2,0$

$p = 0.3$

PD

$2,2$	$0,3$
$3,0$	$1,1$

$p = 0.1$

$I_{1,2}$

Coord

$2,2$	$0,0$
$0,0$	$1,1$

$p = 0.2$

BoS

$2,1$	$0,0$
$0,0$	$1,2$

$p = 0.4$

E.g., agent 1 is going to find out that he is in this equivalence class

(i.e., either playing Coord or BoS, but not MP or PD)

E.g., if nature decides that the game being played is BoS

Bayesian Games – First Definition

Information sets for agent 2 (indistinguishable sets)

	$I_{2,1}$	$I_{2,2}$
$I_{1,1}$	**MP**	**PD**
$I_{1,2}$	**Coord**	**BoS**

MP

2, 0	0, 2
0, 2	2, 0

$p = 0.3$

PD

2, 2	0, 3
3, 0	1, 1

$p = 0.1$

Coord

2, 2	0, 0
0, 0	1, 1

$p = 0.2$

BoS

2, 1	0, 0
0, 0	1, 2

$p = 0.4$

E.g., agent 2 is going to find out that she is in this equivalence class

(i.e., either playing PD or BoS, but not MP or Coord)

E.g., if nature decides that the game being played is BoS

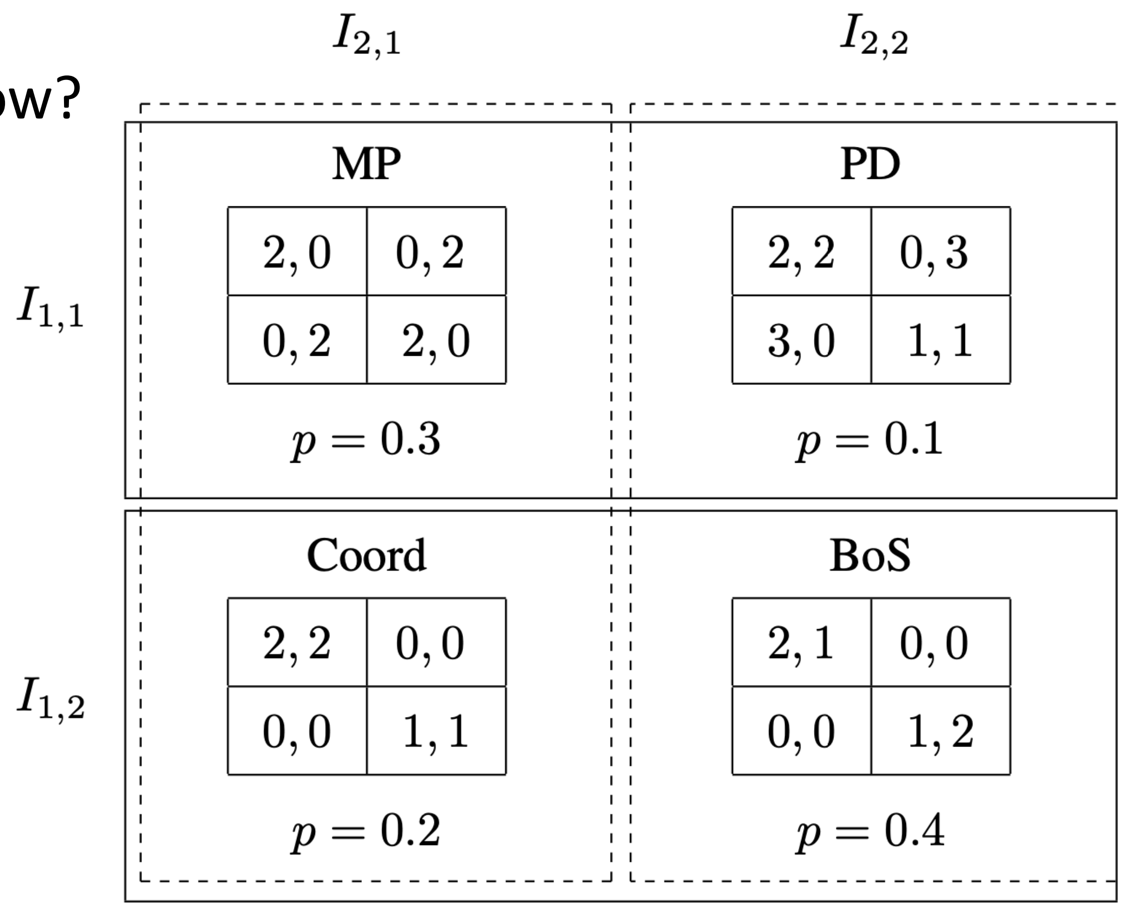
Bayesian Games – First Definition

- So what does this mean?
  - When the agents are **deciding what action to take**:
    - They decide **without fully knowing the game** that will be played
    - They have **to reason about the other agents** without knowing what the other agents are going to think

Bayesian Games – First Definition

So what do the agents know?

- They know everything about this setting:
  - The games
  - The common prior
  - The equivalence classes of all agents



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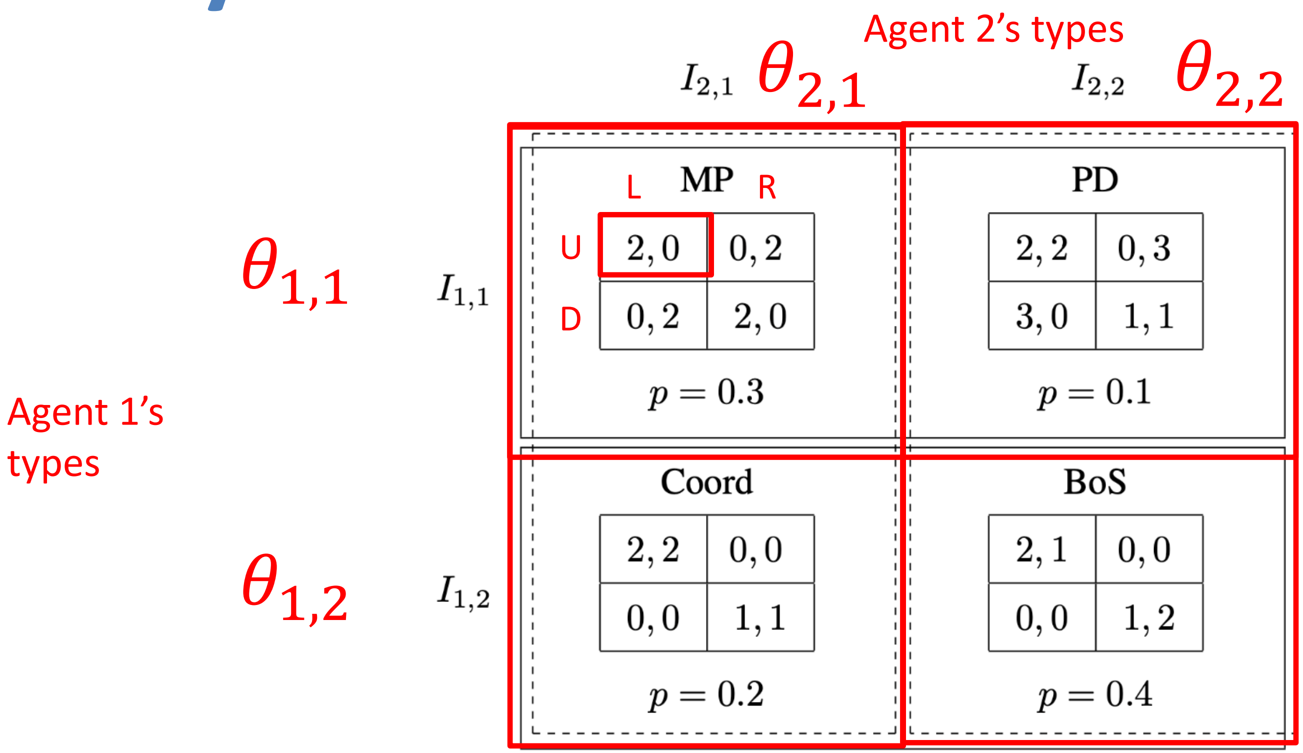
Bayesian Games – Second Definition

- An **alternative definition** of Bayesian games
  - It is mathematically equivalent to the first definition
- This definition has a **different presentation**
  - It is based on **types -** a way of defining uncertainty directly over a game's payoff function

Bayesian Games – Second Definition

- **Definition** (Bayesian game: types): *A* **Bayesian game** *is a tuple* (N, A, θ, p, u) *where:*
  - N *is a set of agents*
  - $A = A_1 \times \ldots \times A_n$, *where* A_i *is the set of actions available to agent* i
  - $\Theta = \Theta_1 \times \ldots \times \Theta_n$, *where* Θ_i *is the type space of agent* i
  - $p: \Theta \longmapsto [0,1]$ *is a common prior over types*
  - $u = (u_1, \ldots, u_n)$, *where* $u_i: A \times \Theta \longmapsto \mathbb{R}$ *is the utility function for agent* i

Bayesian Games – Second Definition



a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

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Analyzing Bayesian Games

- We will reason about a Bayesian game using types (i.e., second definition)

- How?

  - **Bayesian Nash Equilibrium =** A plan of action for each player as a function of types that maximize each type's utility:

    - Expecting over the actions of the other players

    - Expecting over the types of the other players

Analyzing Bayesian Games

- **Given a Bayesian game** (N, A, Θ, p, u) with a finite set of agents, actions, and types

- We can **define strategies** as follows:

  - **Pure strategy**: $\alpha_i : \Theta_i \longmapsto A_i$
    - A mapping from **every type** (from agent i) **to an action** (from agent i)

Analyzing Bayesian Games

- Three standard notions of expected utility (depending on the timing of the decision)
  - ***ex-ante***
    - The agent knows nothing about anyone's actual type (including his own)
  - ***interim***
    - An agent knows her own type but not the types of the other agents
  - ***ex-post***
    - The agent knows all agents' types

Analyzing Bayesian Games

- **Given a Bayesian game** (N, A, θ, p, u) with a finite set of agents, actions, and types.

- Agent i's ***interim expected utility*** with respect to θ_i and a pure strategy profile α is:

$$EU_i(\alpha|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i)\, u_i(\alpha, \theta_i, \theta_{-i})$$

Analyzing Bayesian Games

- **Definition**: *A* Bayesian Nash equilibrium *is a pure strategy profile a that satisfies*

$$\alpha_i \in \underset{\alpha'_i}{\operatorname{argmax}} EU_i(\alpha'_i, \alpha_{-i} | \theta_i)$$

Best responses

for each i *and* $\theta_i \in \Theta_i$

- The above is defined based on the ***interim*** stage

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Exercise 1

- A sheriff faces an armed suspect and they must (simultaneously) decide whether to shoot the other or not

Exercise 1

- A sheriff faces an armed suspect and they must (simultaneously) decide whether to shoot the other or not, and:
  - The suspect is either a criminal with probability p or innocent with probability $1 - p$
  - The sheriff would rather shoot if the suspect shoots, but not if the suspect does not
  - The criminal would rather shoot even if the sheriff does not, as the criminal would be caught if it does not shoot
  - The innocent suspect would rather not shoot even if the sheriff shoots

Exercise 2

- Consider a game where a Firm is recruiting a Worker

Exercise 2

- A **worker** can be of **high ability**, in which case he would like to **work when he is hired**
- Or a **worker** can be of **low ability**, in which case he would rather **shirk** (i.e., to avoid work)
- The firm would want to hire the worker that will work but not the worker that will shirk
- The worker knows his ability level
- The firm does not know whether the worker is of high ability or low ability

Exercise 2

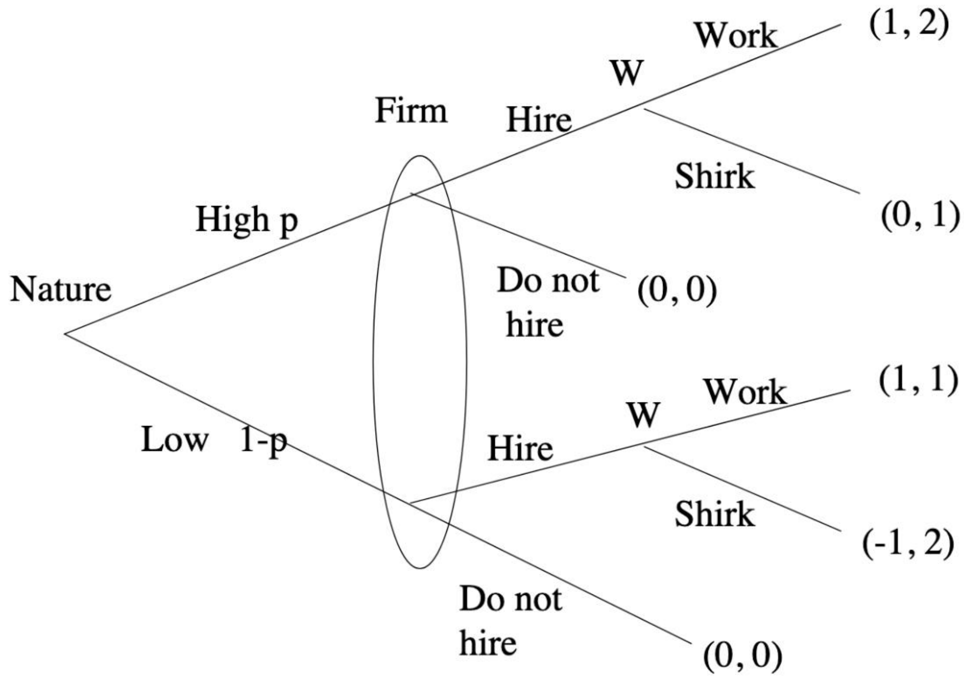
- The firm believes that the worker is of high ability with probability p and low ability with probability $1 - p$

- Most importantly, the firm knows that the worker knows his own ability level

Exercise 2

- To model this situation, we let Nature choose between a worker with **high** ability and **low** ability, with probabilities $\boldsymbol{p}$ and $\mathbf{1}-\boldsymbol{p}$, respectively.

- We then let the worker observe the choice of Nature, but we do not let the firm observe Nature's choice.



Exercise 2

- **Given a Bayesian game** (N, A, Θ, p, u) with a finite set of agents, actions, and types

- One can write the game in this exercise as a **Bayesian game** as follows:

  - $N = \{F, W\}$
  - $A_F = \{hire, dont\}, A_W = \{work, shirk\}$
  - $\Theta_F = \{t_f\}, \Theta_W = \{high, low\}$
  - $p(t_f, high) = p$
  - $p(t_f, low) = 1 - p$

F = firm
W = worker

Exercise 2

- One can write the game in this exercise as a **Bayesian game** as follows:

  - The utility function $u(a_F, a_W, \theta_F, \theta_W)$ is defined by the following tables:

$u(a_F, a_W, \theta_F = t_f, \theta_W = high)$

firm \ worker	***work***	***shirk***
hire	1, 2	0, 1
dont	0, 0	0, 0

$u(a_F, a_W, \theta_F = t_f, \theta_W = low)$

firm \ worker	***work***	***shirk***
hire	1, 1	-1, 2
dont	0, 0	0, 0

Exercise 2

- Consider:
  - $p = \frac{3}{4}$
  - the pure strategy profile $\alpha^* = (\alpha_F^*, \alpha_W^*)$ where
    - $\alpha_F^*(t_f) = hire$
    - $\alpha_W^*(high) = work$
    - $\alpha_W^*(low) = shirk$

  - Check if this pure strategy profile is a Bayes Nash equilibrium.

Thank You

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