Multiagent decision making and Games in Normal Form (Part 2)

Outline

- **Application of a Nash Equilibrium: Duopoly**
  - **Cournot Model**
- Mixed Strategy
- Exercise
- Final remarks - Mixed Strategy

Cournot Model – Application of a NE

- Some of the **earliest applications** of game theory
- The model focuses on the analyses of an **oligopoly**
- The model was created by **Antoine Augustin Cournot** in **1838**
  - **An application of a Nash equilibrium**
  - **A century before John Nash's equilibrium (1950)**

Cournot Model

- **Features** of the Cournot model:

  - There is **more than one firm** in the market
    - The number of firms is fixed
    - We will focus on the scenario with 2 firms (duopoly)

  - All firms produce a **homogeneous product**
    - In other words, there is no product differentiation

  - Firms **do not cooperate**
    - In other words, there is no collusion (or coalition)

Cournot Model

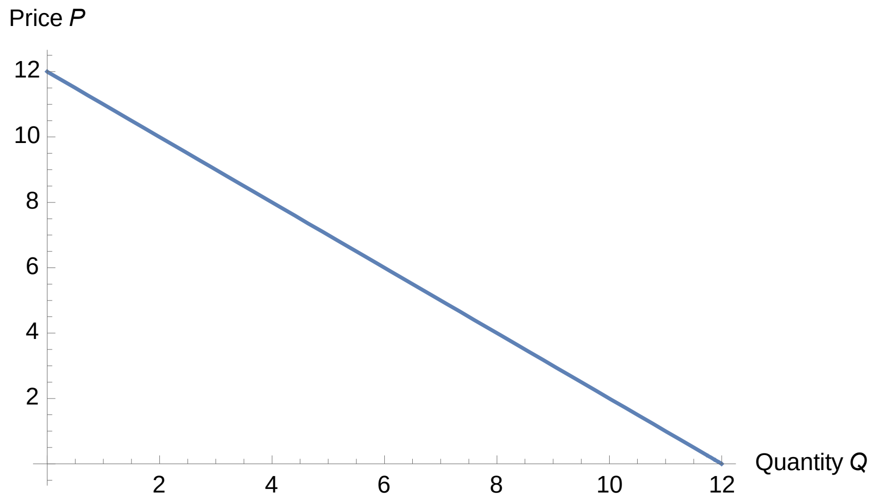
- **Features** of the Cournot model:
  - Firms have **market power**
    - In other words, each firm's output decision affects the product’s market price
  - Firms **choose quantities simultaneously**
    - One-shot game
    - Firms **compete in quantities**
  - The **firms are economically rational and act strategically**
    - Seeking to maximize profit given their competitors' decisions

Cournot Model

- Let q_1 and q_2 denote **the quantities** (of a homogeneous product) produced by firms 1 and 2, respectively

- Let $P(Q) = a - Q$ be the **market-clearing price** when the aggregate quantity on the market is $Q = q_1 + q_2$.

  - More precisely:
    - $P(Q) = a - Q$ for $Q < a$
    - $P(Q) = 0$, for $Q \geq a$



Cournot Model

- Assume that the **total cost to firm $\boldsymbol{i}$** of producing quantity q_i is $C_i(q_i) = cq_i$
  - there are **no fixed costs** and the **marginal cost is constant at $\boldsymbol{c}$**, where we assume $c < a$

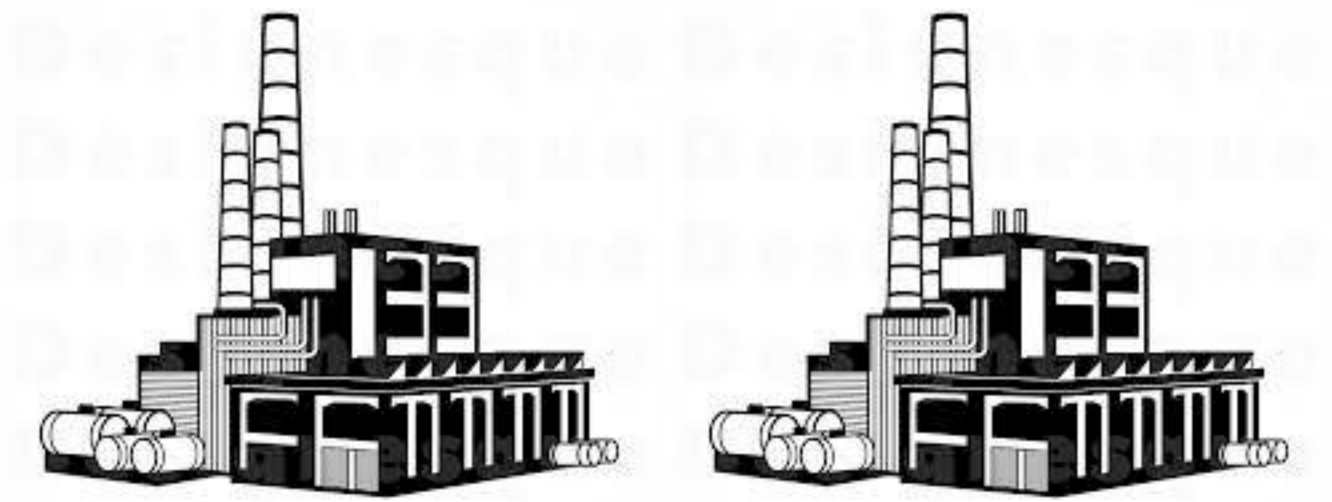
- Assume that the **firms choose their quantities simultaneously**

Cournot Model

- Normal-form game:
  - How many agents?
  - What are the action sets?
  - What are the payoffs?

Cournot Model

- **How many agents?**
  - There are (of course) two agents in any duopoly game
    - i.e., two firms



Cournot Model

- **What are the action sets?**
  - the **actions available** to each firm are **the different quantities** it might produce
    - We will assume that **output** is **continuously divisible**
    - Naturally, **negative outputs are not feasible**.
    - Thus, each firm's **action set** can be represented as $A_i = [0, \infty)$ (the nonnegative real numbers)
      - A typical action a_i is a quantity choice $q_i \geq 0$

Cournot Model

- **What are the payoffs?**
  - The firm's payoff is its profit:

$$\pi_i(q_i, q_j) = q_i[P(q_i + q_j) - c] = q_i[a - (q_i + q_j) - c]$$

Cournot Model

- **How do we find the Nash equilibrium?**
  - The quantity pair (q_1^*, q_2^*) is a Nash equilibrium if, for each firm i, q_i^* solves

$$\max_{0 \le q_i \le \infty} \pi_i\left(q_i, q_j^*\right) = \max_{0 \le q_i \le \infty} q_i\left[a - (q_i + q_j^*) - c\right]$$

Cournot Model

- How do we find the Nash equilibrium?

  - Let us start with firm 1 and solve $\max_{0 \le q_1 \le \infty} q_1[a - (q_1 + q_2^*) - c]$:

$$\frac{\partial}{\partial q_1} q_1[a - (q_1 + q_2^*) - c] = 0$$

$$a - 2q_1^* - q_2^* - c = 0$$

$$q_1^* = \frac{1}{2}(a - q_2^* - c)$$

Cournot Model

- How do we find the Nash equilibrium?

  - And now for firm 2, we can solve $\max_{0 \leq q_2 \leq \infty} q_2[a-(q_2+q_1^*)-c]$:

$$\frac{\partial}{\partial q_2} q_2[a-(q_2+q_1^*)-c]=0$$

$$a-2q_2^*-q_1^*-c=0$$

$$q_2^*=\frac{1}{2}(a-q_1^*-c)$$

Cournot Model

- How do we find the Nash equilibrium?
  - We now need to solve the following pair of equations:

$$q_1^* = \frac{1}{2}(a - q_2^* - c)$$
$$q_2^* = \frac{1}{2}(a - q_1^* - c)$$

  - Which yields:

$$q_1^* = q_2^* = \frac{1}{3}(a - c)$$

Cournot Model

- How do we find the Nash equilibrium?
  - In fact, these are the best response functions:

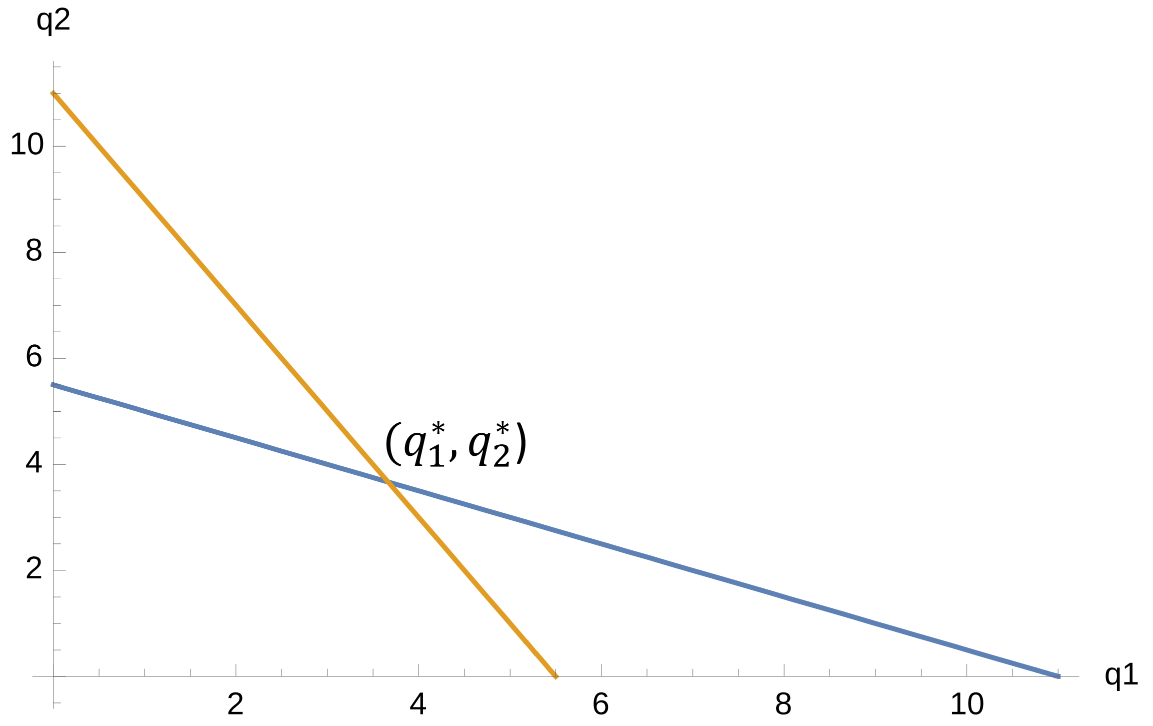
$$b_1(q_2) = \frac{1}{2}(a - q_2 - c)$$
$$b_2(q_1) = \frac{1}{2}(a - q_1 - c)$$

  - The Nash equilibrium is the point where the best response functions intersect:

$$q_1^* = q_2^* = \frac{1}{3}(a - c)$$

Cournot Model

- The figure shows the two best response functions and the point (q_1^*, q_2^*) where they intersect (i.e., the Nash equilibrium)



Outline

- Exercises
- Application of a Nash Equilibrium: Duopoly
  - Cournot Model
- **Mixed Strategy**
- Exercise
- Final remarks - Mixed Strategy

Mixed Strategy

- Before we start defining mixed strategy, let us look at a game called **Matching Pennies**:
  - Two agents
  - Each agent's action set is $\{Head, Tail\}$

Mixed Strategy

- **Matching Pennies**:
  - Information about the payoffs:
    - Imagine that each agent has a penny and must choose whether to display it with heads or tails facing up
    - If the **two pennies match** (i.e., both are heads up or both are tails up) then **agent 2 wins** agent 1's penny
    - If the **pennies do not match** then **agent 1 wins** agent 2's penny

Mixed Strategy

- The payoff matrix of the **Matching Pennies**:

This is a zero-sum game!

Agent 2

		Heads	***Tails***
Agent 1	***Heads***	-1, 1	1, -1
	Tails	1, -1	-1, 1

What is the Nash equilibrium?

Mixed Strategy

- The payoff matrix of the **Matching Pennies**:

		Agent 2	
		Heads	***Tails***
Agent 1	***Heads***	-1, 1	1, -1
	Tails	1, -1	-1, 1

No joint action can satisfy NE…

Mixed Strategy

- The **distinguishing feature** of Matching Pennies is that **each agent would like to outguess the other**
- Versions of this game also arise in **poker, baseball,** and other settings
  - In poker, the analogous question is how often to bluff:
    - if **agent i is known never to bluff** then i's opponents will fold whenever i bids aggressively, thereby making it worthwhile for i to bluff on occasion
    - on the other hand, **bluffing too often** is *also a* losing strategy

Mixed Strategy

- In any game in which **each agent would like to outguess** the other(s):
  - **there is no Nash equilibrium** (if we use the previous definition)
  - This happens because **the solution to such a game necessarily involves uncertainty** about what the agents will do

Mixed Strategy

- We now introduce the notion of a ***mixed strategy***
  - We will interpret in terms of one agent's uncertainty about what another agent will do
- Formally:
  - **Definition**: *A* **mixed strategy** *for an agent i is a probability distribution over his actions $a_i \in A_i$*

Mixed Strategy

- We will hereafter refer to the actions in A_i as agents i's ***pure strategies***

- For example, in Matching Pennies:

  - A_i consists of the two pure strategies: $Heads$ and $Tails$

  - Hence, a **mixed strategy for agent *i*** is the probability distribution $(q, 1-q)$, where:
    - q is the probability of choosing $Heads$
    - $1 - q$ *is* the probability of choosing $Tails$
    - $0 \leq q \leq 1.$

  - The mixed strategy (0,1) is simply the pure strategy $Tails$; likewise, the mixed strategy (1,0) is the pure strategy $Heads$

Mixed Strategy

- Recall the previous definition of Nash equilibrium:
  - NE guarantees that **each agent's pure strategy is a best response to the other agents' pure strategies**
- To **extend** the previous definition to **include mixed strategies**:
  - we simply require that **each agent's *mixed* strategy be a best response to the other agents' *mixed* strategies**

Mixed Strategy

- This **extended definition subsumes the earlier one** because:
  - Any pure strategy can be represented as the mixed strategy that puts zero probability on all of the agent's other pure strategies
- **Computing agent i's best response** to a mixed strategy by agent j:
  - We can interpret agent j's mixed strategy as representing agent i's uncertainty about what agent j will do

Mixed Strategy

- Let us return to the **Matching Pennies**:

Agent 2

		Heads	***Tails***
Agent 1	***Heads***	-1, 1	1, -1
	Tails	1, -1	-1, 1

What is the mixed strategy Nash equilibrium?

Mixed Strategy

- Let us now guess that **both agents randomize**
- Suppose that agent 1 believes that agent 2 will choose
  - $Heads$ with probability q
  - $Tails$ with probability $1 - q$
- Consequently, **agent 1 believes that agent 2 will play the mixed strategy $(\boldsymbol{q}, \mathbf{1} - \boldsymbol{q})$**

Mixed Strategy

- Given this belief, **agent 1's expected payoff for choosing** $\boldsymbol{Heads}$ given agent 2's mixed strategy $(q, 1-q)$:

$$EU_1(Heads) = q\ (\mathbf{-1})\ +\ (1\ -\ q)\ \mathbf{1}\ =\ 1 -\ 2q$$

And **agent 1's expected payoff for choosing** $\boldsymbol{Tails}$ given agent 2's mixed strategy $(q, 1-q)$:

$$EU_1(Tails) = q\ \mathbf{1}\ +\ (1\ -\ q)(\mathbf{-1}) =\ 2q - 1$$

Agent 2

		Heads	***Tails***
Agent 1	***Heads***	-1, 1	1, -1
	Tails	1, -1	-1, 1

Mixed Strategy

- Let us first analyze $\boldsymbol{EU_1(Heads)} > \boldsymbol{EU_1(Tails)}$
  - Note that $1 - 2q > 2q - 1$ if and only if $q < 1/2$
  - Hence, **agent 1's best pure-strategy response is $\boldsymbol{Heads}$** if $q < 1/2$
- Let us now analyze $\boldsymbol{EU_1(Heads)} < \boldsymbol{EU_1(Tails)}$
  - Note that $1 - 2q < 2q - 1$ if and only if $q > 1/2$
  - Hence, **agent 1's best pure-strategy response is $\boldsymbol{Tails}$** if $q > 1/2$

Mixed Strategy

- **What happens when $q = \frac{1}{2}$?**
  - **This is Agent 1's best response when Agent 2 makes him indifferent between $Heads$ or $Tails$**
    - i.e., $EU_1(Heads) = EU_1(Tails)$
  - **In other words, Agent 1 will only best respond with a mixed strategy when Agent 2 makes him indifferent**

Mixed Strategy

- Suppose that agent 2 believes that agent 1 will choose
  - $Heads$ with probability r
  - $Tails$ with probability $1 - r$
- Consequently, **agent 2 believes that agent 1 will play the mixed strategy** $(\boldsymbol{r}, \mathbf{1} - \boldsymbol{r})$

Mixed Strategy

- Given this belief, **agent 2's expected payoff for choosing $\boldsymbol{Heads}$** given agent 1's mixed strategy $(r, 1-r)$:

$$EU_2(Heads) = r\ 1\ +\ (1-r)\ (-1)\ \ = 2r - 1$$

And **agent 2's expected payoff for choosing $\boldsymbol{Tails}$** given agent 1's mixed strategy $(r, 1-r)$:

$$EU_2(Tails) = r\ (-\mathbf{1}) +\ (1-r)\mathbf{1} = 1 - 2r$$

		Agent 2	
		Heads	***Tails***
Agent 1	***Heads***	-1, 1	1, -1
	Tails	1, -1	-1, 1

Mixed Strategy

- Let us first analyze $\boldsymbol{EU_2(Heads)} > \boldsymbol{EU_2(Tails)}$
  - Note that $2r - 1 > 1 - 2r$ if and only if $r > 1/2$, thus:
  - **Agent 2's best pure-strategy response is** $\boldsymbol{Heads}$ if $r > 1/2$
- Let us now analyze $\boldsymbol{EU_2(Heads)} < \boldsymbol{EU_2(Tails)}$
  - Note that $2r - 1 < 1 - 2r$ if and only if $r < 1/2$, thus:
  - **Agent 2's best pure-strategy response is** $\boldsymbol{Tails}$ if $r < 1/2$

Mixed Strategy

- **What happens when $r = \frac{1}{2}$?**
  - **This is Agent 2's best response when Agent 1 makes him indifferent between $Heads$ or $Tails$**
    - i.e., $EU_1(Heads)$ = $EU_1(Tails)$
  - **In other words, Agent 2 will only best respond with a mixed strategy when Agent 1 makes him indifferent**

Mixed Strategy

- **So how do we find the mixed strategy Nash equilibrium?**
  - If agent 1 best-responds with a mixed strategy, then agent 2 must make him indifferent between $Heads$ and $Tails$:

$$EU_1(Heads) = EU_1(Tails)$$
$$1 - 2q = 2q - 1$$
$$\boldsymbol{q = \frac{1}{2}}$$

  - If agent 2 best-responds with a mixed strategy, then agent 1 must make her indifferent between $Heads$ and $Tails$:

$$EU_2(Heads) = EU_2(Tails)$$
$$2r - 1 = 1 - 2r$$
$$\boldsymbol{r = \frac{1}{2}}$$

Mixed Strategy

- **So how do we find the mixed strategy Nash equilibrium?**
  - Thus, the mixed strategies $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)$ are a Nash equilibrium

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Exercise

- **Battle of the sexes:**

  - A man and woman want to get together for an evening of entertainment, but they have no means of communication

  - They can either go to the ballet or the fight
    - The man prefers going to the fight
    - The woman prefers going to the ballet
    - But they both prefer being together than being alone

Exercise

- How many agents?
- What are the action sets?
- What are the payoffs?
- **Is there a pure strategy Nash equilibria?**
- **Is there a mixed strategy Nash equilibrium?**

Outline

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Final remarks - Mixed Strategy

- What does it mean to play a mixed strategy?

  - Randomize to **confuse** your opponent
    - E.g., the matching pennies
      - An equilibrium only exists if we are confused about each other

  - Randomize when **uncertain** about the other's action
    - E.g., the battle of sexes

Final remarks - Mixed Strategy

- What does it mean to play a mixed strategy?

  - Mixed strategies are a concise description of what might happen in a **repeated play**
    - Count of pure strategies in the limit

  - Mixed strategies describe **population dynamics**
    - 2 agents chosen from a population, all having deterministic strategies
    - Mixed strategies gives the probability of getting each pure strategies

Thank You

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