Foundations of Financial Economics *2020/21*
Problem set 4: two-period APT

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1. Consider a finance economy in which (assets in the columns)

$$S = (1, s)\,, \quad V = \begin{pmatrix} r+\epsilon & r(1-\epsilon) \\ r-\epsilon & r(1+\epsilon) \end{pmatrix}$$

   where $0 < r < 1$, $s > 0$ and ϵ can take any (real) value:

   (a) Under which conditions arbitrage opportunities exist ?

   (b) Can we have absence of arbitrage opportunities and market incompleteness ?

   (c) Compute the market price for an Arrow-Debreu contingent claim, under the assumption of absence of arbitrage opportunities. What is replicating transactions' strategy ? Explain.

2. Consider a finance economy in which

$$S = \left(1, \frac{1}{1+r}\right), \quad V = \begin{pmatrix} r+\epsilon & 1 \\ r-\epsilon & 1 \end{pmatrix}$$

   where $r > 0$ and ϵ can take any (real) value.

   (a) Under which conditions arbitrage opportunities exist ?

   (b) Compute the state prices, by assuming the appropriate existence conditions.

   (c) Under the conditions that you imposed in previous point, can there be market completeness ? How ?

   (d) Compute the market price for an Arrow-Debreu contingent claim, under the assumption of absence of arbitrage opportunities. What is replicating transactions' strategy ? Explain.

3. Assume an asset market represented by the pair of payoff prices (S, V) and consider the linear pricing relationship $S = qV$. Consider a simple case in which the number of assets is equal to the number of states of nature and both equal to 2:

(a) determine an equivalent expression involving a stochastic discount factor and asset returns;

(b) determine a sufficient condition for the existence of arbitrage in terms of the asset returns;

(c) assume that there are no arbitrage opportunities, determine conditions for the existence of completeness in terms of the asset returns;

(d) assume that there are no arbitrage opportunities, determine conditions for the existence of incompleteness in terms of the asset returns;

(e) what are the consequences of the existence of incompleteness on the expected value of asset returns, variances and the covariance with the stochastic discount factor ?

4. Consider the following return matrices ($K \times N$)

$$\begin{pmatrix} 1.1 & 1.2 \\ 1.02 & 1.01 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 1.111 & 1.122 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 1.111 & 1.122 \\ 0.99 & 0.95 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 0.99 & 1.02 \end{pmatrix}.$$

Characterize the asset market as regards the existence of arbitrage opportunities and completeness

5. Consider two (alternative) financial markets, A and B, characterized by the return matrices ($N \times K$)

$$R = \begin{pmatrix} 1 & 1+a & b \\ 1 & 1-a & b \end{pmatrix}, \text{ for financial market } A, \text{ and } R = \begin{pmatrix} 1 & 1 \\ 1+a & 1-a \\ 1-b & 1+b \end{pmatrix}, \text{ for financial market } B$$

where a and b can take any real value.

(a) Under which conditions there are no arbitrage opportunities and there is market completeness in the financial market A

(b) Under which conditions there are no arbitrage opportunities and there is completeness in the financial market B

6. Assume there is a financial market with two assets, one risky asset with return $1 + r$ and one riskless asset with return $1 + i$. Assume there are no arbitrage opportunities. Prove that the Sharpe index verifies

$$\left| \frac{E[r-i]}{\sigma[r]} \right| \leq \frac{\sigma[m]}{E[m]}$$

where m is the stochastic discount factor.

7. Assume there is a financial market with two assets, one risky asset and one riskless asset with prices and payoffs

$$S = \begin{pmatrix} \frac{1}{1+i} & s \end{pmatrix}, \ V = \begin{pmatrix} 1 & d_1 \\ 1 & d_2 \end{pmatrix},$$

where $i > 0$ and $0 < d_1 < d_2$. Introduce an european call option with exercise price $d_1 < p < d_2$. Prove that its price, if there are absence of arbitrage opportunities is

$$S_o = \frac{(r_1 - i)(d_2 - p)}{(1 + i)(r_1 - r_2)}$$

8. Consider a finance economy in which there are three assets with the vector of prices and payoff matrix given by

$$S = \left(p, 1, \frac{1}{1+r}\right), \quad V = \begin{pmatrix} r - \epsilon & r + \epsilon & 1 \\ r + \epsilon & r - \epsilon & 1 \end{pmatrix}$$

where we assume that $r > 0$, $p > 0$ and $\epsilon \geq 0$.

(a) Under which conditions there are no arbitrage opportunities ?

(b) Under the conditions that you imposed in the answer to the previous point, what would be the meaning of market completeness in this economy ? Determine the conditions for the existence of market completeness.

(c) Compute the price for the first asset, p, by building a replicating transactions' strategy. Explain.

9. Consider a two-period finance economy in which the information is given by a binomial tree with objective probabilities (π_1, π_2). The financial market is characterized by the return matrix in the (state $\times$ asset) form

$$\begin{pmatrix} 1 & R_1 \\ 1 & R_2 \end{pmatrix}.$$

(a) Provide conditions for the absence of arbitrage opportunities and for the existence of complete markets. Consider those conditions from now on.

(b) Deduce the Sharpe index (tip: start by proving that the covariance between the two random variables $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, adapted to the previous binomial tree, is $COV(X, Y) = \pi_1 \, \pi_2 \, (x_1 - x_2) \, (y_1 - y_2))$. What are the consequences of the conditions you have derived in (a) on the sign and magnitude of the Sharpe index.

(c) Find a relationship between the objective and the risk-neutral probabilities such that the expected risk premium is non-negative. Explain.