In[30]:=
```
(* OPtimal Control (Pontryiagin)
   Discrete Time

    PB: 23.12.2015
*)
```

In[31]:=
```
(* 1.2.4 *)
```

In[32]:=
```
ClearAll
ClearAll[u, y, ψ]
F[u_, y_] := y - (2 - u)^2
H[u_, y_, ψ_] := y - (2 - u)^2 + ψ (y - u) / 2
ss1 := Solve[D[H[u, y, ψ], u] == 0, u]
uf[y_, ψ_] := Evaluate[u /. ss1]
p[y_, ψ_] = D[H[u, y, ψ], y]
```

Out[32]= ClearAll

Out[38]= $1+\frac{\psi}{2}$

In[39]:=
```
ClearAll
mhsol = RSolve[{y[t + 1] == (y[t] - uf[y[t], ψ[t]]) / 2,
   ψ[t] == p[y[t + 1], ψ[t + 1]]}, {ψ[t], y[t]}, t]
ys[t_] := Evaluate[y[t] /. mhsol]
psis[t_] := Evaluate[ψ[t] /. mhsol]
us[t_] := uf[ys[t], psis[t]]
ITC = Simplify[Solve[ {ys[0] == 0, ys[4] == 45 / 2}, {C[1], C[2]}]]
```

Out[39]= ClearAll

Out[40]= $\left\{\left\{y[t]\to -\frac{3}{2}+2^{-t}\,C[1]+\frac{1}{3}\,2^{-2-t}\left(-1+2^{2t}\right)C[2],\ \psi[t]\to 2+2^{t}\,C[2]\right\}\right\}$

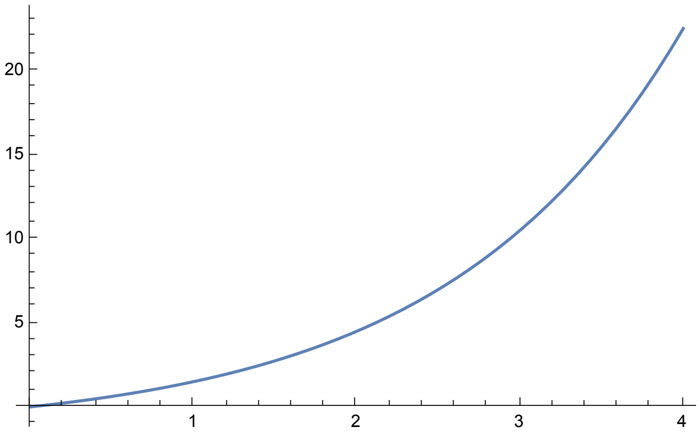
Out[44]= $\left\{\left\{C[1]\to \frac{3}{2},\ C[2]\to 18\right\}\right\}$

```
In[45]:= y[t_] = Factor[Evaluate[ ys[t]  /. C[1] → 3/2 /. C[2] → 18]]
         u[t_] = Factor[Evaluate[ us[t]  /. C[1] → 3/2 /. C[2] → 18]]
         Plot[y[t], {t, 0, 4}]
```

Out[45]= $\left\{\frac{3}{2}\left(-1+2^{t}\right)\right\}$

Out[46]= $\left\{\left\{-\frac{3}{2}\left(-1+3\times 2^{t}\right)\right\}\right\}$

Out[47]=




```
In[48]:= (* 1.2.15  *)
         ClearAll
         ClearAll[Iv, K, η]
         F[t_, Iv_, K_] = (1 + r) ^ (-t) (p K - Iv (1 + (1 / 2) Iv))
         h[Iv_, K_, η_] = p K - Iv (1 + (1 / 2) Iv) + η (Iv + (1 - δ) K)
         ct = Solve[D[h[Iv, K, η], Iv] == 0, Iv]
         IvF[K_, η_]  = -1 + η
         ELC[K_, η_] = D[h[Iv, K, η], K]
         Assumption[δ > 0 && δ < 1]
```

Out[48]= ClearAll

Out[50]= $\left(-\left(1+\frac{\mathrm{Iv}}{2}\right)\mathrm{Iv}+\mathrm{K}\,\mathrm{p}\right)(1+\mathrm{r})^{-t}$

Out[51]= $-\left(1+\frac{\mathrm{Iv}}{2}\right)\mathrm{Iv}+\mathrm{K}\,\mathrm{p}+(\mathrm{Iv}+\mathrm{K}\,(1-\delta))\,\eta$

Out[52]= $\{\{\mathrm{Iv}\to -1+\eta\}\}$

Out[53]= $-1+\eta$

Out[54]= $\mathrm{p}+(1-\delta)\,\eta$

Out[55]= Assumption[$\delta > 0$ && $\delta < 1$]

```
In[56]:= (* mhds:=
          {K[t+1]==IvF[K[t],η[t]]+(1-δ)K[t], η[t]==(1+r)^(-1) ELC[K[t+1],η[t+1]]} *)
```

```
In[57]:= mhds := {K[t + 1] == η[t] - 1 + (1 - δ) K[t], (1 + r) η[t] == p + (1 - δ) η[t + 1]}
```

In[58]:= **mhdssol = Assuming[δ > 0 && δ < 1 && r > 0 && t ≥ 0, RSolve[{mhds}, {K[t], η[t]}, t]]**

Out[58]= $$\left\{\left\{K[t] \to -\frac{-p+r+\delta}{\delta\,(r+\delta)} + (1-\delta)^t\, C[1] + \frac{\left(-(1-\delta)^t + \left(-\frac{1+r}{-1+\delta}\right)^t\right)(-1+\delta)\, C[2]}{-r-2\,\delta+\delta^2},\right.\right.$$
$$\left.\left.\eta[t] \to \frac{p}{r+\delta} + \left(\frac{-1-r}{-1+\delta}\right)^t C[2]\right\}\right\}$$

In[59]:= **Ks[t_] = Evaluate[K[t] /. mhdssol]**
etas[t_] = Evaluate[η[t] /. mhdssol]

Out[59]= $$\left\{-\frac{-p+r+\delta}{\delta\,(r+\delta)} + (1-\delta)^t\, C[1] + \frac{\left(-(1-\delta)^t + \left(-\frac{1+r}{-1+\delta}\right)^t\right)(-1+\delta)\, C[2]}{-r-2\,\delta+\delta^2}\right\}$$

Out[60]= $$\left\{\frac{p}{r+\delta} + \left(\frac{-1-r}{-1+\delta}\right)^t C[2]\right\}$$

In[61]:= **ITC = Simplify[Solve[{Ks[0] == ϕ, Ks[T] == ϕ}, {C[1], C[2]}]]**

Out[61]= $$\left\{\left\{C[1] \to \frac{1-\frac{p}{r+\delta}}{\delta} + \phi,\ C[2] \to \frac{\left(-1+(1-\delta)^T\right)(-r+(-2+\delta)\,\delta)\,(-p+(r+\delta)\,(1+\delta\,\phi))}{\left(-\left(\frac{1+r}{1-\delta}\right)^T + (1-\delta)^T\right)(-1+\delta)\,\delta\,(r+\delta)}\right\}\right\}$$

In[62]:= **K[p_, r_, δ_, ϕ_, t_, T_] =**
$$\mathbf{Simplify}\left[\mathbf{Evaluate}\left[K[t]\ /.\ \ mhdssol\ /.\ C[1] \to \frac{1-\frac{p}{r+\delta}}{\delta} + \phi\ /.\ C[2] \to \frac{\left(-1+(1-\delta)^T\right)(-r+(-2+\delta)\,\delta)\,(-p+(r+\delta)\,(1+\delta\,\phi))}{\left(-\left(\frac{1+r}{1-\delta}\right)^T + (1-\delta)^T\right)(-1+\delta)\,\delta\,(r+\delta)}\right]\right]$$

Out[62]= $$\left\{-\frac{-p+r+\delta}{\delta\,(r+\delta)} + (1-\delta)^t\left(\frac{1-\frac{p}{r+\delta}}{\delta} + \phi\right) + \frac{\left(\left(\frac{1+r}{1-\delta}\right)^t - (1-\delta)^t\right)\left(-1+(1-\delta)^T\right)(-p+(r+\delta)\,(1+\delta\,\phi))}{\left(-\left(\frac{1+r}{1-\delta}\right)^T + (1-\delta)^T\right)\delta\,(r+\delta)}\right\}$$

In[63]:= **η[p_, r_, δ_, ϕ_, t_, T_] =**
$$\mathbf{Simplify}\left[\mathbf{Evaluate}\left[\eta[t]\ /.\ \ mhdssol\ /.\ C[1] \to \frac{1-\frac{p}{r+\delta}}{\delta} + \phi\ /.\ C[2] \to \frac{\left(-1+(1-\delta)^T\right)(-r+(-2+\delta)\,\delta)\,(-p+(r+\delta)\,(1+\delta\,\phi))}{\left(-\left(\frac{1+r}{1-\delta}\right)^T + (1-\delta)^T\right)(-1+\delta)\,\delta\,(r+\delta)}\right]\right]$$

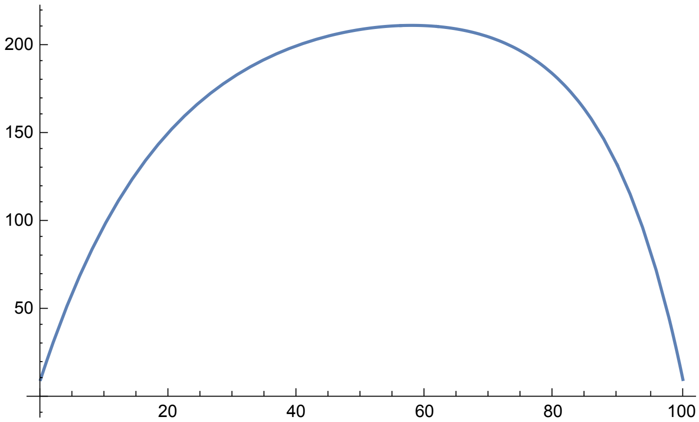
Out[63]= $$\left\{\frac{p + \frac{\left(-1+(1-\delta)^T\right)\left(\frac{1+r}{1-\delta}\right)^t(-r+(-2+\delta)\,\delta)\,(-p+(r+\delta)\,(1+\delta\,\phi))}{\left(-\left(\frac{1+r}{1-\delta}\right)^T+(1-\delta)^T\right)(-1+\delta)\,\delta}}{r+\delta}\right\}$$

In[64]:=
```
Simplify[K[p, r, δ, ϕ, 0, T]]
Simplify[K[p, r, δ, ϕ, T, T]]
Plot[K[1, 0.03, 0.05, 10, t, 100], {t, 0, 100}]
```

Out[64]= {ϕ}

Out[65]= {ϕ}

Out[66]=




```
(* Checking solutions *)
```

In[67]:=
```
Simplify[K[p, r, δ, ϕ, t, T] - (p - r - δ) / (δ (r + δ)) - (ϕ - (p - r - δ) / (δ (r + δ))) *
    ((((1 + r) / (1 - δ)) ^T - 1) (1 - δ) ^t + (1 - (1 - δ) ^T) ((1 + r) / (1 - δ)) ^t) /
     (((1 + r) / (1 - δ)) ^T - (1 - δ) ^T)]
Simplify[η[p, r, δ, ϕ, t, T] - p / (r + δ) -
   ((((1 + r) / (1 - δ) - (1 - δ)) (ϕ - (p - r - δ) / (δ (r + δ))) (1 - (1 - δ) ^T)
      ((1 + r) / (1 - δ)) ^t) / (((1 + r) / (1 - δ)) ^T - (1 - δ) ^T)]
```

Out[67]= {0}

Out[68]= {0}

In[69]:=
```
(*1.2.16 *)
ClearAll
ClearAll[Z, η, x]
F[t_, C_, Z_] = β^t Log[C^α Z^(1 - α)]
h[C_, Z_, η_] = Log[C^α Z^(1 - α)] + η (δ (Z - C))
ct = Solve[D[h[C, Z, η], C] == 0, C]
CF[Z_, η_] = α/(δ η)
ELC[Z_, η_] = D[h[C, Z, η], Z]
Assumption[α > 0 && α < 1]
```

Out[69]= ClearAll

Out[71]= $\beta^{t} \operatorname{Log}\left[C^{\alpha} Z^{1-\alpha}\right]$

Out[72]= $(-C + Z)\ \delta\ \eta + \operatorname{Log}\left[C^{\alpha} Z^{1-\alpha}\right]$

Out[73]= $\left\{\left\{C \to \frac{\alpha}{\delta\ \eta}\right\}\right\}$

Out[74]= $\frac{\alpha}{\delta\ \eta}$

Out[75]= $\frac{1-\alpha}{Z} + \delta\ \eta$

Out[76]= Assumption[$\alpha > 0$ && $\alpha < 1$]

In[77]:=

In[78]:=
```
mhds = {Z[t + 1] == δ (Z[t] - CF[Z[t], η[t]]), η[t] == β ELC[Z[t + 1], η[t + 1]]}
```

Out[78]= $\left\{Z[1+t] == \delta\left(Z[t] - \frac{\alpha}{\delta\ \eta[t]}\right),\ \eta[t] == \beta\left(\frac{1-\alpha}{Z[1+t]} + \delta\ \eta[1+t]\right)\right\}$

In[79]:=

In[80]:=
```
mhdssol = Assuming[α > 0 && α < 1 && δ > 0 && t ≥ 0,
   RSolve[{Z[t + 1] == δ (Z[t] - CF[Z[t], η[t]]),
     η[t] == β ELC[Z[t + 1], η[t + 1]]}, {Z[t], η[t]}, t]]
```

Out[80]= $\operatorname{RSolve}\left[\left\{Z[1+t] == \delta\left(Z[t] - \frac{\alpha}{\delta\ \eta[t]}\right),\ \eta[t] == \beta\left(\frac{1-\alpha}{Z[1+t]} + \delta\ \eta[1+t]\right)\right\},\ \{Z[t],\ \eta[t]\},\ t\right]$

In[81]:=
```
mdx = RSolve[x[t + 1] == (-α - β (1 - α) + δ x[t]) / (β δ), {x[t]}, t]
```

Out[81]= $\left\{\left\{x[t] \to \frac{\left(1 - \left(\frac{1}{\beta}\right)^{t}\right)(-\alpha - \beta + \alpha\beta)}{(-1+\beta)\ \delta} + \left(\frac{1}{\beta}\right)^{-1+t} C[1]\right\}\right\}$

In[82]:=
```
x1[t_] := x[t] /. mdx
TC = Evaluate[Assuming[β > 0 && β < 1, Limit[Simplify[β^t x1[t]], t → Infinity]]]
TCC = Solve[TC == 0 , C[1]]
```

Out[83]= $\left\{\frac{\alpha - \alpha\beta + \beta\ (1 + (-1+\beta)\ \delta\ C[1])}{(-1+\beta)\ \delta}\right\}$

Out[84]= $\left\{\left\{C[1] \to \frac{-\alpha - \beta + \alpha\beta}{(-1+\beta)\ \beta\ \delta}\right\}\right\}$

In[85]:= ```
Xc = Simplify[Evaluate[x[t] /. mdx /. C[1] → (-α - β + α β)/((-1 + β) β δ)]]
```

Out[85]= $\left\{\frac{\alpha+\beta-\alpha\,\beta}{\delta-\beta\,\delta}\right\}$

In[86]:= ```
mdZ = RSolve[{Z[t + 1] == δ (Z[t] - (α (δ - β δ) Z[t])/((α + β - α β) δ))}, {Z[t]}, t]
```

Out[86]= $\left\{\left\{Z[t]\to\left(\frac{\beta\,\delta}{\alpha+\beta-\alpha\,\beta}\right)^{-1+t}C[1]\right\}\right\}$

In[87]:=

In[88]:= ```
Solve[((β δ)/(α + β - α β))^-1 C[1] == ϕ, C[1]]
```

Out[88]= $\left\{\left\{C[1]\to\frac{\beta\,\delta\,\phi}{\alpha+\beta-\alpha\,\beta}\right\}\right\}$

In[89]:= ```
Z[t_] = Simplify[Evaluate[Z[t] /. mdZ /. C[1] → (β δ ϕ)/(α + β - α β)]]
eta[t_] = ((α + β - α β)/(δ - β δ)) / Z[t]
Cons[t_] = Simplify[α/(δ eta[t])]
```

Out[89]= $\left\{\left(\frac{\beta\,\delta}{\alpha+\beta-\alpha\,\beta}\right)^{t}\phi\right\}$

Out[90]= $\left\{\frac{(\alpha+\beta-\alpha\,\beta)\left(\frac{\beta\,\delta}{\alpha+\beta-\alpha\,\beta}\right)^{-t}}{(\delta-\beta\,\delta)\,\phi}\right\}$

Out[91]= $\left\{\frac{\alpha\,(-1+\beta)\left(\frac{\beta\,\delta}{\alpha+\beta-\alpha\,\beta}\right)^{t}\phi}{\alpha\,(-1+\beta)-\beta}\right\}$