The Ramsey growth model

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A short history of the model

- Frank Ramsey (see `https://en.wikipedia.org/wiki/Frank_P._Ramsey`) made several important contributions in his short life (he died at 26) one of them Ramsey (1928)
- His contribution was only fully recognized in the early 60's (Cass (1965), Koopmans (1965)) as presenting a rigorous alternative to the ad-hoc aspects (dynamic inefficiency) of the Solow (1956) model (now we call it **exogenous growth theory**)
- It was rejoined again in the middle of the 1980's which saw the onset of **endogenous growth theory**
- It is also the founding rock of the DGE (dynamic general equilibrium theory) of macroeconomics

The Ramsey model

The basic idea

- output is a function of the capital stock and can be used for investment or for consumption (everything in per capita terms): this introduces a **intratemporal budget constraint**
- **savings** is determined by a **arbitrage between present and future consumption**: it balances two effects:
  - present consumption is a good thing, although its utility decreases with the amount consumed;
  - however, if people sacrifice present consumption to save and increase the capital stock they improve their prospects for more consumption in the future;
- this idea can be formalized by a **intertemporal optimization problem**

The Ramsey model

Assumptions

- Production:
  - closed economy producing a single composite good
  - production uses two factors: labor and physical capital
  - production technology: neoclassical (increasing, concave, Inada, CRTS)
- Reproducible factor:
  - physical capital (machines)
- Population:
  - exogenous and constant

The Ramsey model

Assumptions: cont

- Households: optimizing behavior
  - maximize an intertemporal utility functional with consumption as the control variable
  - subject to a budget constraint
  - labor is supplied inelastically
  - they have perfect foresight
- Equilibrium is Pareto optimal, therefore it is equivalent to a central planer problem

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The model: production technology

- in aggregate terms

$$Y(t) = F(A, K(t), L(t)) = AK(t)^{\alpha} L(t)^{1-\alpha},\ 0 < \alpha < 1$$

  where: A TFP productivity, K stock of capital, $L = N$ loabor input = population

- In per capita terms:

$$y(t) = Ak(t)^{\alpha}$$

  where $y = Y/N$ and $k = K/N$

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The model: preferences

Preferences: for the representative agent

- the intertemporal utility functional is

$$V[c] = \int_0^\infty u(c(t))e^{-\rho t}dt$$

- $c = C/N$ per capita consumption, $[c] = \big(c(t)\big)_{t\in[0,\infty)}$
- $\rho > 0$ is the rate of time preference
- the instantaneous utility function is

$$u(c) = \begin{cases} \dfrac{c^{1-\theta} - 1}{1-\theta}, & \text{if } \theta \in (0,\infty)/\{1\} \\ \ln(c), & \text{if } \theta = 1 \end{cases}$$

where $1/\theta$ is the elasticity of intertemporal substitution

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Versions

- We are assuming an **homogeneous agent** (or representative) economy
- There are two versions of the model
  - **centralized** version: maximization of social welfare given the budget constraint
  - **decentralized** (DGE) version: individual maximization of households an firms coordinated by market equilibrium
  - because there are no externalities the are **equivalent** (in the sense that generate the same allocations, of consumption and capital through time)

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The centralized version

- The central planner solves the problem

$$\max_{(c)_{t\geq 0}} \int_0^{\infty} \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

- subject to

$$\dot{k} = Ak(t)^{\alpha} - c(t) - \delta k(t),$$

- $k(0) = k_0$ given
- $\lim_{t\to\infty} h(t)k(t) \geq 0$ physical capital is asymptotically bounded ($h(t)$ is any discount factor)

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Solving by using the Pontriyagin's max principle

- The current-value Hamiltonian is

$$H(c,k,q) = \frac{c^{1-\theta}-1}{1-\theta} + q(Ak^{\alpha} - c - \delta k)$$

- the optimality conditions are

$$\begin{aligned} \frac{\partial H}{\partial c} = 0 \quad &\Leftrightarrow \quad c^{-\theta}(t) = q(t),\ t \in [0,\infty) \\ \dot{q} = \rho q - \frac{\partial H}{\partial k} \quad &\Leftrightarrow \quad \dot{q} = q(t)\left(\rho + \delta - \alpha A k(t)^{\alpha-1}\right),\ t \in [0,\infty) \\ \lim_{t\to\infty} q(t)k(t)e^{-\rho t} \quad &= \quad 0 \end{aligned}$$

- the admissibility conditions

$$\begin{aligned} \dot{k} \quad &= \quad Ak(t)^{\alpha} - c(t) - \delta k(t),\ t \in [0,\infty) \\ k(0) \quad &= \quad k_0,\ t = 0 \end{aligned}$$

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The modified Hamiltonian dynamic system

- An optimum path $(c^*(t), k^*(t))_{t\in[0,+\infty)}$ is the solution of the (MHDS)

$$\begin{aligned}
\dot{c} &= \frac{c}{\theta}\left(r(k(t)) - \rho - \delta\right) \\
\dot{k} &= Ak(t)^{\alpha} - c(t) - \delta k(t) \\
0 &= \lim_{t\to\infty} c(t)^{-\theta} k(t) e^{-\rho t} \\
k(0) &= k_0 \text{ given}
\end{aligned}$$

- with the (gross) rate of return for capital

$$r(k) = \alpha A k^{\alpha-1}$$

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Steady states

- they are fixed points of the system

$$\frac{c^*}{\theta}\left(r(k^*)-\rho)\right)=0,$$
$$c^*=A(k^*)^\alpha-\delta k^*.$$

- there are three steady states

$$(c^*,k^*)=\{(0,0),(0,(A/\delta)^{1/(1-\alpha)}),(\bar{c},\bar{k})\}$$

for

$$\bar{k}=\left(\frac{\alpha A}{\delta+\rho}\right)^{1/(1-\alpha)},\ \bar{c}=\frac{\rho+\delta(1-\alpha)}{\alpha}\bar{k}$$

- the last one verifies the transversality condition (the second not: check)
- then steady state GDP levels

$$\boxed{\bar{y}=A\bar{k}^\alpha=\left[A\left(\frac{\alpha}{\delta+\rho}\right)^\alpha\right]^{1/(1-\alpha)}.} \tag{1}$$

The Ramsey model

Solving the Ramsey model

- In general the Ramsey **does not have an explicit solution** (also called exact or closed form)
- We can only find an exact solution for the case $\theta = \alpha$ (which is counterfactual)
- Analytical methods for finding the solution:
  - get a **linear approximate** system and force the solution to converge to the steady state;
  - use exact methods by **transforming** the MHDS into a known differential equation (only for that very special case)
- In all cases, **it is always a good idea to build the phase diagram**

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- there is no explicit solution
- we study dynamics of the approximate system in a neighbourhood of $(\bar{c}, \bar{k})$
- the linearised MHDS is

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{\bar{c}r'(\bar{k})}{\theta} \\ -1 & \rho \end{pmatrix} \begin{pmatrix} c(t) - \bar{c} \\ k(t) - \bar{k} \end{pmatrix}$$

- where $r' = (\alpha - 1)\alpha A k^{\alpha-2}|_{k=\bar{k}} = -\dfrac{(1-\alpha)\rho}{\bar{k}} < 0$
- and $\dfrac{\bar{c}r'(\bar{k})}{\theta} = -d \equiv -\dfrac{(1-\alpha)\,\rho\,\big(\rho + \delta(1-\alpha)\big)}{\alpha\theta} < 0$

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- the system is of type $\dot{x} = Jx$
- where the Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} 0 & -d \\ -1 & \rho \end{pmatrix}$$

- the solution is of type

$$x(t) = h_s \mathbf{V}^s e^{\lambda_s t} + h_u \mathbf{V}^u e^{\lambda_u t}$$

- where λ_j are the eigenvalues and $\mathbf{V}^j$ are the associated eigenvectors of J and h_s are arbitrary constants

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Case $\theta \neq \alpha$: approximate solution

- the eigenvalues of $\mathbf{J}$ are

$$\lambda_u = \frac{\rho}{2} + \left[\left(\frac{\rho}{2}\right)^2 + d\right]^{1/2} > \rho > 0$$

$$\lambda_s = \frac{\rho}{2} - \left[\left(\frac{\rho}{2}\right)^2 + d\right]^{1/2} < 0$$

- satisfying $\lambda_s + \lambda_u = \rho > 0$, $\lambda_s \lambda_u = -d$
- then $(\bar{c}, \bar{k})$ is a **saddle-point**

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Case $\theta \neq \alpha$: approximate solution

- the eigenvectors are determined as follows
- $\mathbf{V}^s$ solves the homogeneous system

$$(\mathbf{J} - \lambda_s \mathbf{I}_2)\,\mathbf{V}^s = \mathbf{0}$$

- that is

$$\begin{pmatrix} -\lambda_s & -d \\ -1 & \rho - \lambda_s \end{pmatrix} \begin{pmatrix} \mathbf{V}_1^s \\ \mathbf{V}_2^s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- the members of vector $\mathbf{V}^s$ should satisfy

$$\frac{\mathbf{V}_1^s}{\mathbf{V}_2^s} = -\frac{d}{\lambda_s} = \lambda_u \Rightarrow \mathbf{V}^s = \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix}$$

  (because $\rho - \lambda_s = \lambda_u$)
- for $\mathbf{V}^u$ we find (prove this)

$$\frac{\mathbf{V}_1^u}{\mathbf{V}_2^u} = -\frac{d}{\lambda_u} = \lambda_s \Rightarrow \mathbf{V}^u = \begin{pmatrix} \lambda_s \\ 1 \end{pmatrix}$$

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- Then the general solution is

$$\begin{pmatrix} c(t) - \bar{c} \\ k(t) - \bar{k} \end{pmatrix} = h_s \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix} e^{\lambda_s t} + h_u \begin{pmatrix} \lambda_s \\ 1 \end{pmatrix} e^{\lambda_u t}$$

- We determine h_s and h_u by forcing the general solution to satisfy the two remaining conditions

$$\lim_{t \to \infty} \frac{h(t)}{c(t)^{\theta}} e^{-\rho t} = 0, \text{ and } k(0) = k_0$$

  - the first condition holds if $\lim_{t\to\infty}(c(t) - \bar{c}) = \lim_{t\to\infty}(k(t) - \bar{k}) = 0$, i.e., they converge to the steady state, which is obtained by eliminating the effect of $e^{\lambda_u t}$ (wich converges to ∞) by setting $h_u = 0$
  - the second condition holds if

$$k(0) = \bar{k} + h_s - \bar{k} = k_0 \;\rightarrow h_s = k_0$$

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Case $\theta \neq \alpha$: approximate solution

- the approximate solution is, for $t \in [0, \infty)$

$$
\begin{aligned}
c(t) &= \bar{c} + \lambda_u (k_0 - \bar{k}) e^{\lambda_s t}, \\
k(t) &= \bar{k} + (k_0 - \bar{k}) e^{\lambda_s t}.
\end{aligned}
$$

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Case $\theta \neq \alpha$: approximate solution

- at $t = 0$ we have

$$\begin{pmatrix} c(0) \\ k(0) \end{pmatrix} = \begin{pmatrix} \bar{c} + \lambda_u (k_0 - \bar{k}) \\ k_0 \end{pmatrix}$$

  observe that λ_u gives the variation of consumption as $c(0) - \bar{c} = \lambda_u (k_0 - \bar{k})$ and the initial consumption is determined from **future data** ($\bar{c}$ and $\bar{k}$)

- asymptotically (i.e., in the long run)

$$\lim_{t \to \infty} \begin{pmatrix} c(t) \\ k(t) \end{pmatrix} = \begin{pmatrix} \bar{c} \\ \bar{k} \end{pmatrix} = \begin{pmatrix} \frac{\rho + \delta(1-\alpha)}{\alpha} \bar{k} \\ \bar{k} \end{pmatrix}$$

  the solution converges to the steady state (this means that the transversality condition is satisfied)

- **the saddle path dynamics implies that the solution is unique**

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Case $\theta \neq \alpha$: phase diagrams for $\theta < \alpha$ and $\theta > \alpha$

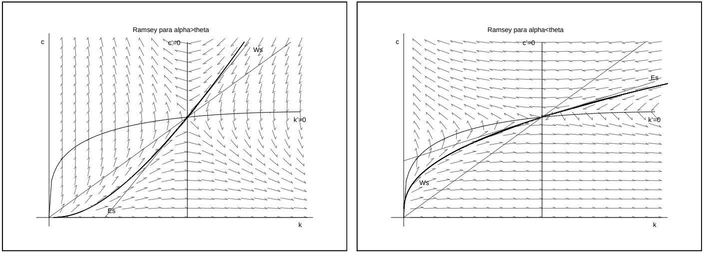

Figure: Exact (dark) and approximate (light) solutions

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Case $\theta = \alpha$: exact solution

- there is an explicit solution:

$$
\begin{aligned}
c(t) &= \frac{\delta + \rho(1-\alpha)}{\alpha} k(t), \\
r(t) &= \frac{r(0)(\delta+\rho)}{r(0) + (\delta + \rho - r(0))\, e^{-[(1-\alpha)(\delta+\rho)/\alpha]t}},
\end{aligned}
$$

  with $k(t) = (\alpha A / r(t))^{1/(1-\alpha)}$
- given $k(0)$ we get explicitly

$$
c(0) = \frac{\delta + \rho(1-\alpha)}{\alpha} k(0)
$$

- convergences asymptotically to the steady state,

$$
\begin{aligned}
\lim_{t\to\infty} c(t) &= \bar{c} \\
\lim_{t\to\infty} r(t) &= \bar{r} = \delta + \rho \\
\lim_{t\to\infty} k(t) &= \bar{k}
\end{aligned}
$$

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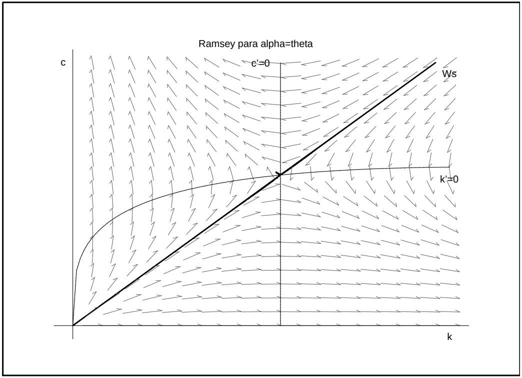
Case $\theta = \alpha$: phase diagram



Ramsey model

Properties of the solution paths

1. if $k(0) \neq \bar{k}$ then $\lim_{t\to\infty} k(t) = \bar{k}$,
2. given any initial value for k, $k(0)$, there is only a value for c, $c(0)$ which is determined endogenously such that $\lim_{t\to\infty} c(t) = \bar{c}$;
3. **the solution is determinate, i.e, unique**: this is the only solution for the ode system such that the transversality condition holds;
4. the saddle path is asymptotically tangent to the straight line
$$c(t) - \bar{c} = \lambda_u(k(t) - \bar{k})$$
5. the **approximate** per-capita output path is
$$y(t) = \left[\bar{y}^{1/\alpha} + (y(0)^{1/\alpha} - \bar{y}^{1/\alpha})e^{\lambda_s t}\right]^{\alpha} \tag{2}$$

the model only displays transitional dynamics as $\lambda_s < 0$.

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Case $\theta = \alpha$: GDP exact dynamics

- the **exact** per-capita output path is

$$y(t) = A\left[\frac{\alpha A k(0)^{\alpha-1}(\delta+\rho)}{\alpha A k(0)^{\alpha-1} + (\delta+\rho-\alpha A k(0)^{\alpha-1})\, e^{-[(1-\alpha)(\delta+\rho)/\alpha]t}}\right]^{\alpha},$$

- the solution converges asymptotically to the steady state

$$\lim_{t\to\infty} y(t) = \bar{y} = \left[A\left(\frac{\alpha}{\delta+\rho}\right)^{\alpha}\right]^{1/(1-\alpha)}$$

The Ramsey model

Growth implications

- there is **no long-run growth** $\bar{g} = 0$
- the **long-run level** $\bar{y}$ depends on $(A, \delta, \rho, \alpha)$: productivity, the rate of depreciation, the rate of time preference (impatience) and on the income shares (see equation (1));
- there is **only transitional dynamics**: the **speed** and the pattern of convergence depends on the relationship between the capital share, α, in income and the intertemporal elasticity of substitution θ (see equation (2)). This is because

$$\lambda_s = \frac{\rho}{2} - \left[\left(\frac{\rho}{2} \right)^2 + \frac{(1-\alpha)\,\rho\,\big(\rho + \delta(1-\alpha)\big)}{\alpha\theta} \right]^{\frac{1}{2}} < 0$$

the higher $|\lambda_s|$ is the faster the transition speed is.

The Neoclassical DGE model

Assumption

- Representative household: has initial financial wealth b and gets financial income (rb), and decides on consumption (c) and savings $(\dot{b})$;
- Households own firms with physical capital (k) which is only financed by bonds: thus $b = k$. Firms transform capital and labor into output (y)
- There are accounting restrictions.
- All markets are competitive
- Other assumptions: infinite-lived households with isoelastic utility and Cobb-Douglas production, function and no frictions.

The Neoclassical DGE model

- Household's problem: maximize discounted intertemporal utility subject to a financial constraint

$$\max_{c(.)} \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to: change in assets = income minus consumption

$$\dot{b} = r(t) b(t) + w(t) - c(t), \ t \geq 0$$

$$b(0) = b_0$$

$$\lim_{t \to \infty} e^{-\int_t^\infty r(s)\, ds} \geq 0$$

where $b =$bonds, $w =$wage

- Optimality conditions

$$\dot{c} = c(t) \frac{(r(t) - \rho)}{\theta}$$

$$\lim_{t \to \infty} e^{-\rho t} c(t)^{-\theta} b(t) = 0$$

The Neoclassical DGE model

- Firm's problem (price taker in all the markets): maximizes present value of profits

$$\max_{i} \int_0^\infty \left(Ak(t)^\alpha - w(t) - i(t)\right) e^{-\int_t^\infty r(s)\,ds} dt$$

subject to net investment = gross investment minus ddepreciation

$$\dot{k} = i - \delta k$$
$$k(0) = k_0$$

- F.o.c

$$r(t) = \alpha A k(t)^{\alpha-1} - \delta$$

The Neoclassical DGE model

- Micro-macro constraints:
  - Accounting identity $b(t) = k(t)$,
  - Then $\dot{b}(t) = \dot{k}(t)$,
  - Wage determination $w = y - rk = (1-\alpha)Ak^{\alpha}$,

- Then get the same dynamic system as in the Ramsey model

$$\dot{c} = c(t)\frac{(r(t) - \rho)}{\theta}$$
$$\dot{k} = Ak(t)^{\alpha} - c(t) - \delta k(t)$$

- Then the allocations of c and k are equal: we say that the **equilibrium is Pareto efficient**)

References

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