Advanced Mathematical Economics

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Part III

Non-linear ordinary differential equations

Chapter 4

Non-linear differentiable ODE

Non-linear ordinary differential equations have the form

$$\dot{y} = f(y,t),\ y : \mathcal{T} \rightarrow \mathcal{Y} \subseteq \mathbb{R}^n,\ f : \mathcal{T} \times \mathcal{Y} \rightarrow \mathcal{Y}$$

where $f(.)$ is a non-linear function of y. Again we have an **autonomous ODE** if function $f(.)$ is time-independent, i.e., it is of type $f(y)$. Again, if y is of dimension one, i.e., if $n = 1$ we say it is a **scalar** ODE, if $n = 2$ it is a **planar** ODE, and if $n > 2$ it is a **higher-dimensional** ODE.

In general, the non-linear ODE that interests us depends on a parameter or a vector of parameters $\varphi \in \Phi$, takes the form

$$\dot{y} = f(y,\varphi),\ f : \mathcal{Y} \times \Phi \rightarrow \mathcal{Y} \tag{4.1}$$

In equation (4.1) function $f(.)$ can be exactly or qualitatively specified. Sometimes, in economic theory models we require that it has some slope and/or curvature properties.

We assume in this chapter that $f(.)$ **is continuous and differentiable** (i.e, all its derivatives are finite). In this case it can be proved that *a solution exists, and is unique.* For initial-value problems the solution is also continuous in time.

There are several new aspects introduced by non-linearity when comparing with linear ODE's. First, **most non-linear ODE's do not have a closed form solution**. If an equation has a closed form solution we can characterise completely its solution but if the solution is not known (or it has not a closed form solution in terms of known functions) we can still characterise the solution qualitatively. Second, differently from the linear case **there may be a difference between the local and the global properties of the solution** (i.e., the local behavior of the solution may be different at different points of points in $\mathcal{Y}$). The qualitative (or geometric) study of the solutions of non-linear equations is based upon finding topological equivalence with linear equations or to some non-linear equations with known solutions called *normal forms.* When there is topological equivalence with linear ODE's the local and global properties of a non-linear ODE are the same, but when the topological equivalence is with some non-linear normal form, the local and global properties are different.

Therefore, qualitatively specified ODE's can only have non-explicit solutions but exactly specified ODE's can have either explicit or non-explicit solutions. In all those cases, we usually need to characterize the qualitative properties of the solutions.

The qualitative (or geometrical) theory of ODE's explores that topological equivalence allowing for the characterization of the solution of non-lineae ODE's. It consists in the application of three important results:

- the **Grobmann-Hartman theorem**
- the **Poincaré-Bendixon theorem**
- several **bifurcation theorems**, stating the qualitative change of the solutions when some critical parameter or parameters vary.

For any type of non-linear ODE (explicit or not and with or without explicit solutions) we can characterise the **local dynamics** by:

(1) determining the existence and number of steady states (or time-independent solutions) or of other invariant solutions;

(2) determining the stability properties for every steady state by linearizing function $f(y)$ in the neighbourhood of every steady state: i.e., by approximating locally a linear equation of type

$$\dot{y} = D_y f(\bar{y}, \varphi)(y - \bar{y})$$

where $\bar{y} \in \{y : f(y, \varphi) = 0\}$ and $D_y f(\bar{y}, \varphi)$ is the Jacobian of $f(y,)$ evaluated at the steady state $\bar{y}$. In some cases, some **global dynamics** properties not existing in linear models (heteroclinic and homoclinic trajectories, limit cycles, for instance), can also be identified. We will see that, in order to get topological equivalence, this amounts to Taylor - expanding further, v.g [1],

$$\dot{y} = D_y f(\bar{y}, \varphi)(y - \bar{y}) + \frac{1}{2} D_y^2 f(\bar{y}, \varphi)(y - \bar{y})^2 + o((y - \bar{y})^2),$$

A **phase diagram** represents the geometry of the solution on the space $\mathcal{Y}$ for a given value of the parameter(s): it is characterized by the number of steady states, their local dynamics and other types of global trajectories.

[1]The rest, call it $R(y - \bar{y})$, being of order $o(y - \bar{y})^2$ in a weak sense, means that $\lim_{y \to \infty} \dfrac{R(y - \bar{y})}{(y - \bar{y})^2} = 0$.

(3) determining the existence of critical values of the parameters, or bifurcation points. Usually, not only the number and the magnitude of the steady states but also their dynamic properties depend on the value of the parameters. We say $(\bar{y}(\varphi_0), \varphi_0)$ is a **bifurcation** point if introducing a small quantitative change in φ the characteristics of the phase-diagram change qualitatively. There are, again, local but also global bifurcations. A **bifurcation diagram**, plotting $(\bar{y}(\varphi), \varphi)$ for all values $\varphi \in \Phi$, with a reference to the stability properties, is a useful device for conducting bifurcation analysis.

We start by presenting the normal forms for scalar and for some planar non-linear ODE's and next present the main results from the qualitative theory of ODE's.

4.1 Normal forms

Some equations are called normal forms: they are the simplest cases of a whole family of equations whose solution are known, and such other qualitatively similar equations can be transformed by topological equivalence. [2] into one of those forms.

4.1.1 Scalar ODE's

For the scalar case we have (see (Hale and Koçak, 1991, ch. 2)) two quadratic equations with a single bifurcation parameter (a), the Ricatti's equation $\dot{y} = a + y^2$ and a quadratic Bernoulli equation, $\dot{y} = ay + y^2$, and three cubic equations with one (a) or two bifurcation parameters (a and b): the cubic Bernoulli equation, $\dot{y} = ay - y^3$ and two Abel's equations, $\dot{y} = a + y - y^3$ and $\dot{y} = a + by - y^3$.

For each equation we present: (1) the closed form solution (in most cases) and characterize it; (2) the steady states; and (3) the bifurcation points. Those equations are usually associated to the particular bifurcations that they generate. We will present the relevant bifurcation diagrams.

We assume next that $y : \mathbb{R}_+ \to \mathcal{Y} \subseteq \mathbb{R}$, and

The Ricatti's equation: saddle-node or fold bifurcation The quadratic equation

$$\dot{y} = a + y^2 \tag{4.2}$$

[2] In heuristic terms, we say functions $f(y)$ and $g(x)$ are topologically equivalent if there exists a diffeomorphism, i.e., a smooth map h with a smooth inverse h^{-1}, such that if $y = h(x)$ then $h(g(x)) = f(h(x))$. This property may hold globally of locally. The last case is the intuition behind the Grobman-Hartmann theorem.

is called Ricatti's equation. It has an explicit solution[3] :

$$y(t) = \begin{cases} -\frac{1}{t+k}, & \text{if } a = 0 \\ \sqrt{a}\left(\tan\left(\sqrt{a}(k+t)\right)\right), & \text{if } a > 0 \\ -\sqrt{-a}\left(\tanh\left(\sqrt{-a}(k+t)\right)\right), & \text{if } a < 0 \end{cases}$$

where k is an arbitrary constant belonging to the domain of y, $\mathcal{Y}$.

The behavior of the solution is the following:

- if $a = 0$, the solution takes an infinite value at a finite time $t = -k$ [4], i.e., $\lim_{t\to -k} y(t) = \pm\infty$ and tends asymptotically to a steady state $\bar{y} = 0$, that is $\lim_{t\to\infty} y(t) = 0$ independently of the value of k;

- if $a > 0$ the solution takes infinite values for a periodic sequence of times $t \in \{-k, \pi - k, 2\pi - k, \ldots, n\pi - k, \ldots\}$,

$$\lim_{t\to n\pi - k} y(t) = \pm\infty, \text{ for } n \in \mathbb{N}$$

  and it has no steady state;

- if $a < 0$, the solution converges to

$$\lim_{t\to\infty} y(t) = \begin{cases} -\sqrt{-a}, & \text{if } k < \sqrt{-a} \text{ or } -\sqrt{-a} < k < \sqrt{-a} \\ +\infty, & \text{if } k > \sqrt{-a}. \end{cases}$$

Therefore the dynamic properties depend on the value of a:

- existence and number of steady states: $\bar{y} = \{y : \ a + y^2 = 0\}$: if $a > 0$ there are no steady states, if $a = 0$ there is one steady state $\bar{y} = 0$, and if $a < 0$ there are two steady states $\bar{y} \in \{-\sqrt{-a}, \sqrt{-a}\}$;

- local dynamics at a steady state: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable and if $a < 0$ steady state $\bar{y} = -\sqrt{-a}$ is asymptotically stable and steady state $\bar{y} = \sqrt{-a}$ is unstable. This is because $f_y(y) = 2y$ then $f_y(0) = 0$, $f_y(-\sqrt{a}) = -2 - \sqrt{a} < 0$, and $f_y(\sqrt{a}) = 2 - \sqrt{a} > 0$. If $a < 0$ the basin of attraction, or stable manifold associated to steady state $\bar{y} = -\sqrt{-a}$, is

$$\mathcal{W}^s_{-\sqrt{-a}} = \{y \in \mathcal{Y} : y < \sqrt{-a}\}.$$

  Comparing to the linear case, for the case in which the steady state is asymptotically stable, the stable manifold is a subset of $\mathcal{Y}$ not the whole $\mathcal{Y}$.

[3]See appendix section 4.A

[4]This is different to the linear case, v.g., $\dot{y} = y$ which, if $y(0) \neq 0$, whose solution $y(t) = y(0)e^t$ takes an infinite value only in infinite time.

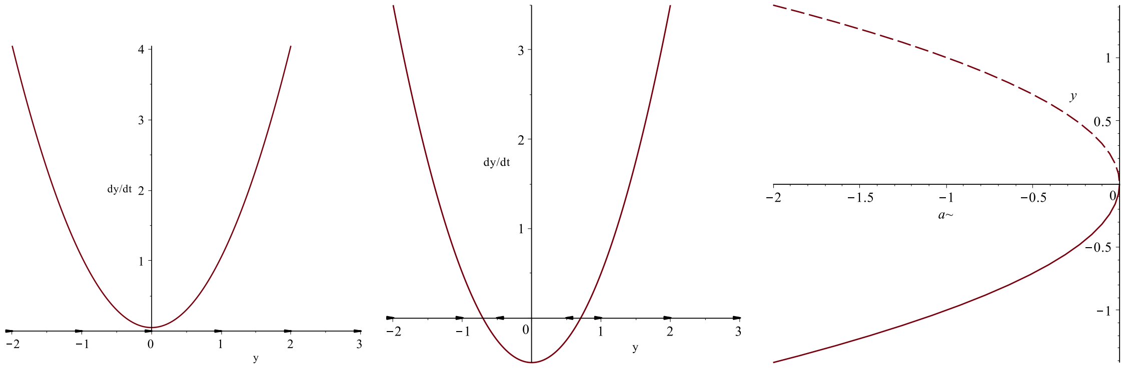


Figure 4.1: Phase diagrams for $a > 0$, and $a < 0$ and bifurcation diagram for equation (4.2)

There is a bifurcation point at $(y, a) = (0, 0)$, which is called **saddle-node bifurcation**. We find the bifurcation point by solving, jointly to (y, a) the system

$$\begin{cases} f(y,a) = 0 \\ f_y(y,a) = 0 \end{cases} \Leftrightarrow \begin{cases} a + y^2 = 0 \\ 2y = 0. \end{cases}$$

Figure 4.1 shows phase diagrams for the $a > 0$ (left sub-figure) and for the $a < 0$ (center sub-figure) cases and the ifurcation diagram (right sub-figure). In the bifurcation diagram we depict points (a, y) such that $a+y^2 = 0$, say $\bar{y}(a)$, and in full-line the subset of points such that $f'(\bar{y}(a)) < 0$ and in dashed-line the the subset of points such that $f'(\bar{y}(a)) > 0$. The first case corresponds to asymptotically state steady states and the second to unstable steady states. Observe that the curve does not lie in the positive quadrant for a which is the geometrical analogue to the non-existence of steady states. The saddle-node bifurcation point is at the origin $(0, 0)$.

Quadratic Bernoulli equation: transcritical bifurcation The equation

$$\dot{y} = ay + y^2 \tag{4.3}$$

is a particular case of the Bernoulli's equation $\dot{y} = ay + by^\eta$, for a real number η, and also has an explicit solution [5]:

$$y(t) = \begin{cases} \frac{1}{1/k - t}, & \text{if } a = 0 \\ \frac{a}{(1+a/k)e^{-at} - 1}, & \text{if } a \neq 0 \end{cases}$$

where k is an arbitrary element of $\mathcal{Y}$.

The behavior of the solution is the following:

[5]See appendix section 4.B for the explicit solution for the general Bernoulli ODE.

- if $a > 0$

$$\lim_{t\to\infty} y(t) = \begin{cases} -a, & \text{if } k < 0 \\ +\infty, & \text{if } k > 0 \end{cases}$$

- if $a = 0$, it behaves as the Ricatti's equation when $a = 0$
- if $a < 0$,

$$\lim_{t\to\infty} y(t) = \begin{cases} 0, & \text{if } k < -a \\ +\infty, & \text{if } k > -a \end{cases}$$

The dynamic properties depend on the value of a:

- existence and number of steady states: if $a = 0$ there is one steady state $\bar{y} = 0$ and if $a \neq 0$ there are two steady states $\bar{y} = \{0, -a\}$;
- local dynamics at the steady states: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable; if $a < 0$ steady state $\bar{y} = 0$ is asymptotically stable and steady state $\bar{y} = -a$ is unstable; and if $a > 0$ steady state $\bar{y} = 0$ is unstable and steady state $\bar{y} = -a$ is asymptotically stable. The stable manifolds associated to the asymptotically stable equilibrium points are: if $a < 0$

$$\mathcal{W}_0^s = \{y \in \mathcal{Y} : y < -a\}.$$

  and, if $a < 0$,

$$\mathcal{W}_{-a}^s = \{y \in \mathcal{Y} : y < 0\}.$$

There is a bifurcation point at $(y, a) = (0, 0)$, which is called **transcritical bifurcation**. Figure 4.2 shows two phase diagrams and the bifurcation diagram.

Bernoulli's cubic equation: subcritical pitchfork The equation

$$\dot{y} = ay - y^3 \tag{4.4}$$

is also a Bernoulli equation and also has also an explicit solution:

$$y(t) = \pm\sqrt{a}\left[1 - \left(1 - \frac{a}{k^2}\right)e^{-2at}\right]^{-1/2}$$

where k is an arbitrary element of $\mathcal{Y}$. The solution trajectories have the following properties for different values of the parameter a:

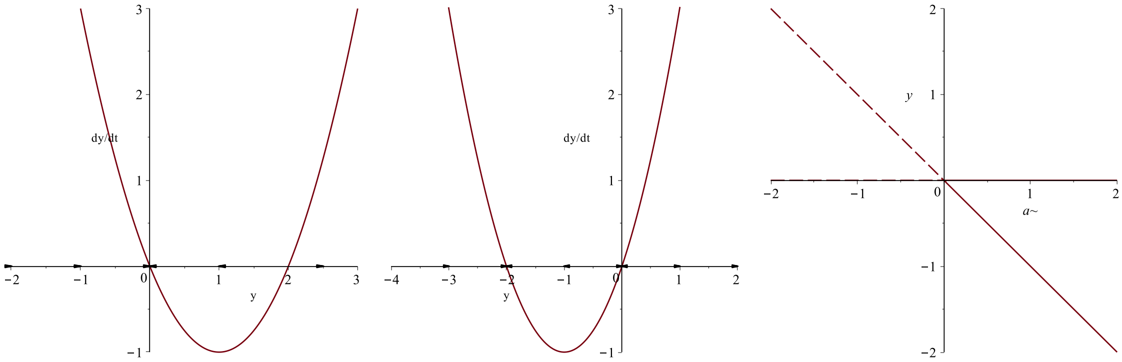


Figure 4.2: Phase diagrams for $a < 0$, and $a > 0$ and bifurcation diagram for equation (4.3)

- if $a \leq 0$, $\lim_{t\to\infty} y(t) = 0$
- if $a > 0$,

$$\lim_{t\to\infty} y(t) = \begin{cases} -\sqrt{a}, & \text{if } k < 0 \\ \sqrt{a}, & \text{if } 0 < k < \sqrt{a} \\ +\infty, & \text{if } k > \sqrt{a} \end{cases}$$

The dynamic properties depend on the value of a:

- existence and number of steady states: there is one steady state $\bar{y} = 0$ and if $a \leq 0$ and there are three steady states $\bar{y} = \{0, -\sqrt{a}, \sqrt{a}\}$ if $a > 0$;
- local dynamics at the steady states: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable; if $a < 0$ steady state $\bar{y} = 0$ is asymptotically stable; and if $a > 0$ steady state $\bar{y} = 0$ is unstable and the other two steady states $\bar{y} = -\sqrt{a}$ and $\bar{y} = \sqrt{a}$ are asymptotically stable.

There is a bifurcation point at $(y, a) = (0, 0)$, which is called **subcritical pitchfork**. Figure 4.3 shows two phase diagrams and the bifurcation diagram.

Exercise: Study the solution for equation $\dot{y} = ay + y^3$. Show that point $(y, a) = (0, 0)$ is also a bifurcation point called **supercritical pitchfork**.

Abel's equation: hysteresis The equation

$$\dot{y} = a + y - y^3 \tag{4.5}$$

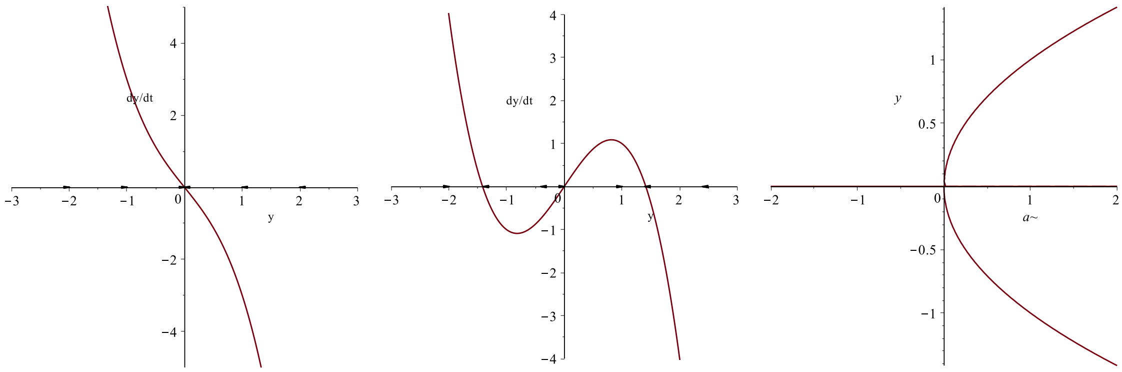


Figure 4.3: Phase diagrams for $a < 0$, and $a > 0$ and bifurcation diagram for equation (4.4)

is called an Abel equation of the first kind. Although closed form solutions have been found recently [6] they are too cumbersome to report. If $a = 0$ the Abel's equation reduces to a particular Bernoulli's equation (4.4) $\dot{y} = y - y^3$.

Equation (4.5) can have one, two or three equilibrium points, which are the real roots of the polynomial equation $f(y, a) \equiv a + y - y^3 = 0$.

We can determine bifurcation points in the space $\mathcal{Y} \times \Phi$ by solving for (y, a)

$$\begin{cases} f(y, a) = 0, \\ f_y(y, a) = 0. \end{cases}$$

Because

$$\begin{cases} a + y - y^3 = 0, \\ 1 - 3y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3(a + y) - 3y^3 = 0, \\ y - 3y^3 = 0, \end{cases} \Leftrightarrow \begin{cases} 3a + 2y = 0 \\ y = \pm\sqrt{1/3}, \end{cases}$$

we readily find that the ODE (4.5) has two critical points, called **hysteresis** points:

$$(y, a) = \left\{ \left(-\sqrt{\frac{1}{3}}, \frac{2}{3}\sqrt{\frac{1}{3}} \right), \left(\sqrt{\frac{1}{3}}, -\frac{2}{3}\sqrt{\frac{1}{3}} \right) \right\}.$$

By looking at figure 4.4 (to the right sub-figure) we see that:

- for $a > \frac{2}{3}\sqrt{\frac{1}{3}}$ or for $a < -\frac{2}{3}\sqrt{\frac{1}{3}}$ there is one asymptotically stable steady state
- for $a = \frac{2}{3}\sqrt{\frac{1}{3}}$ there are two steady states: one asymptotically stable equilibrium and a bifurcation point for $\bar{y} = \sqrt{\frac{1}{3}}$, for $a = -\frac{2}{3}\sqrt{\frac{1}{3}}$ there are two steady states: one asymptotically stable equilibrium and a bifurcation point for $\bar{y} = -\sqrt{\frac{1}{3}}$

[6] For known closed form solutions of ODEs see, Valentin F. Zaitsev (2003) or Zwillinger (1998).

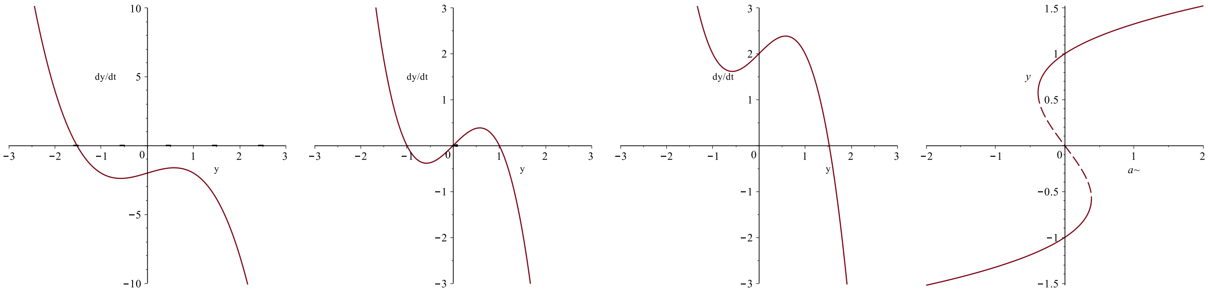


Figure 4.4: Phase diagrams for $a < \frac{2}{3}\sqrt{\frac{1}{3}}$, for $a > \frac{1}{3}$ and for an intermediate value and bifurcation diagram for equation (4.5)

- for $-\frac{2}{3}\sqrt{\frac{1}{3}} < a < \frac{2}{3}\sqrt{\frac{1}{3}}$ there are three steady states, two asymptotically stable (the extreme ones) and one unstable (the middle one)

Cubic equation: cusp The equation

$$\dot{y} = f(y, a, b) \equiv a + by - y^3 \tag{4.6}$$

is also an Abel equation of the first kind. Observe that we have now two parameters, a and b, that also allow for critical changes of its solution.

This ODE can have one, two or three equilibrium points, depending on the values of the parameters a and b. We can determine them by solving the cubic polynomial equation $a+by-y^3 = 0$ (see appendix section 4.C). Calling discriminant to

$$\Delta \equiv \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3; \tag{4.7}$$

it can be proven that: if $\Delta < 0$ there are three steady states, if $\Delta = 0$ there are two steady states, and if $\Delta > 0$ there is one steady state.

We can determine critical points by solving the system:

$$\begin{cases} f(y, a, b) = 0 \\ f_y(y, a, b) = 0. \end{cases} \tag{4.8}$$

Applying to equation (4.6) we have

$$\begin{cases} a + by - y^3 = 0, \\ b - 3y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3a + 2by = 0, \\ b - 3y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3a + 2by = 0 \\ 2b^2 + 9ay = 0, \end{cases} \Leftrightarrow \begin{cases} 27a^2 + 18aby = 0 \\ 4b^3 + 18aby = 0. \end{cases}$$

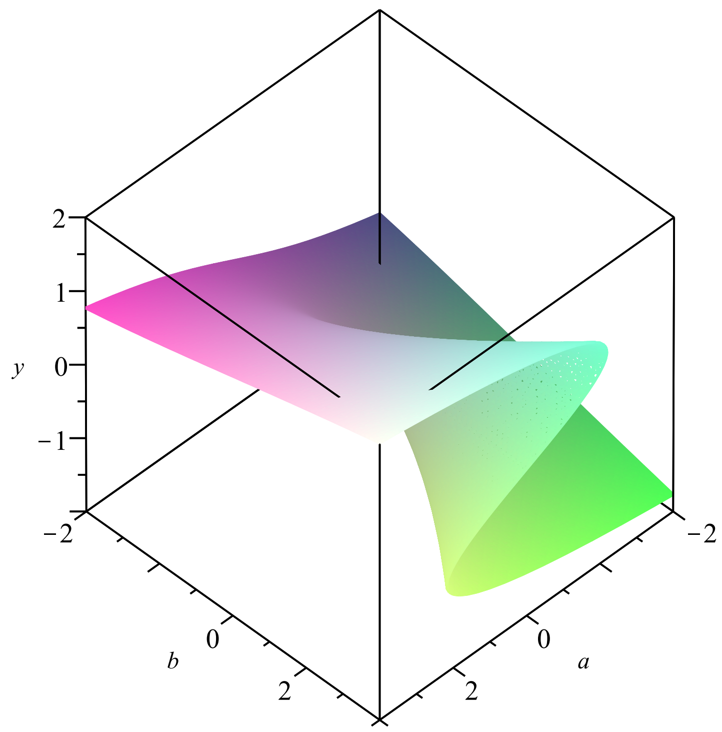


Figure 4.5: Bifurcation diagram for equation $\dot{y} = a + by - y^3$

The solutions to the system must verify

$$18aby = -12a^2 = -4b^3 \Leftrightarrow \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3 = 0$$

that is $\Delta = 0$. Function $f(y, a, b) = 0$ defines a surface in the surface in the three-dimensional space for (a, b, y) called **cusp** which is depicted in Figure 4.9 [7]. Because we have two parameters, the bifurcation loci, obtained from system (4.8) defines a line in the three-dimensional space (a, b, y). We can see how it changes by imagining horizontal slices in Figure 4.9 and project them in the (a, b)-plane. This would convince us that if $a = 0$ we would get the bifurcation diagram for the pitchfork, for equation (4.4), and if $a \neq 0$ and $b = 1$ we obtain the hysteresis diagram, for equation (4.5). This result would be natural because those two equations are a particular case of the cusp equation.

By solving the system

$$\begin{cases} f(y, a, b) = 0 \\ f_y(y, a, b) = 0 \\ f_b(y, a, b) = 0 \end{cases}$$

we find a bifurcation point $(y, a, b) = (0, 0, 0)$ corresponding to a bifurcation for a higher level of degeneracy (called bifurcation of co-dimension two).

4.1.2 Planar ODE's

Next we consider the planar ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi)$, in vector notation, $\mathbf{y} : \mathcal{T} \to \mathcal{Y} \subseteq \mathbb{R}^2$, depending on a vector of parameters, $\varphi \in \mathbb{R}^n$ for $n \geq 1$. Expanding, we have,

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \varphi) \\ \dot{y}_2 &= f_2(y_1, y_2, \varphi) \end{aligned} \tag{4.9}$$

There are a large number of normal forms that have been studied for planar ODEs (see Kuznetsov (2005).

In principle, we could consider combining all the previous scalar normal forms to have an idea of the number of possible cases, and extend the previous method to study the dynamics. That method consisted in finding critical points, corresponding to steady states and values of the parameters such that the derivatives of the steady variables would be equal to zero. However, for planar equation, to fully characterise the dynamics, we may have to study local dynamics in invariant orbits other than steady states. In general there are, at least, three types of **invariant orbits** that do not exist in planar linear models: homoclinic and heteroclinic orbits and limit cycles.

[7] This was one of the famous cases of catastrophe theory very popular in the 1980's see `https://en.wikipedia.org/wiki/Catastrophe_theory`.

In the next section we present a general method to finding bifurcation points associated to steady states. In the rest of this section we presents ODE's in which those invariant curves exists and are generic (in the sense that they hold for any values of a parameter, except for some particular values) and non-generic. The non-generic cases consist in one-parameter bifurcations for non-linear planar equations associated to heteroclinic and homoclinic orbits and limit cycles.

Heteroclinic orbits We say there is an **heteroclinic orbit** if, in a planar ODE in which there are at least two steady states, say $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$, and there are solutions $\mathbf{y}(t)$ that entirely lie in a curve joining $\bar{\mathbf{y}}^1$ to $\bar{\mathbf{y}}^2$ say $\text{Het}(\mathbf{y})$. Therefore, if $\mathbf{y}(0) \in \text{Het}(\mathbf{y})$ then $\mathbf{y}(t) \in \text{Het}(\mathbf{y})$ for $t > 0$ and either $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^1$ and $\lim_{t\to-\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^2$ or $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^2$ and $\lim_{t\to-\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^1$. Heteroclinics can exist if the stability type of the steady states are different or equal. In the first case, they existe between stable or unstable nodes and saddle points or between a stable and one unstable node. In the last case, the only possibility is if the two steady states are saddle points and we say we have a **saddle connection**.

Heteroclinic networks can also exist when there are more than two steady states which are connected.

Generic heteroclinic orbits Although there are several normal forms generating generic heteroclinic orbits, we focus next in the following case:

$$\begin{aligned} \dot{y}_1 &= a y_1 y_2 \\ \dot{y}_2 &= 1 + y_1^2 - y_2^2 \end{aligned} \tag{4.10}$$

where $a \neq 0$. This equation has two steady states: $\bar{\mathbf{y}}^1 = (0, -1)$ and $\bar{\mathbf{y}}^2 = (0, 1)$. Calling,

$$\mathbf{f}(\mathbf{y}) = \begin{pmatrix} a y_1 y_2 \\ 1 + y_1^2 - y_2^2 \end{pmatrix}$$

we have the Jacobian

$$J(\mathbf{y}) \equiv D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} a y_1 & a y_2 \\ 2y_1 & -2y_2 \end{pmatrix},$$

which has trace and determinant depending on the parameter a

$$\begin{aligned} \text{trace}\,(J(\mathbf{y})) &= (a-2)y_2 \\ \det\,(J(\mathbf{y})) &= -2a(y_1^2 + y_2^2). \end{aligned}$$

Then, remembering again that we assumed $a \neq 0$ and because, for any steady state $y_1^2 + y_2^2 > 0$ then $\det\,(J(\mathbf{y})) > 0$ if $a < 0$ and $\det\,(J(\mathbf{y})) < 0$ if $a > 0$.

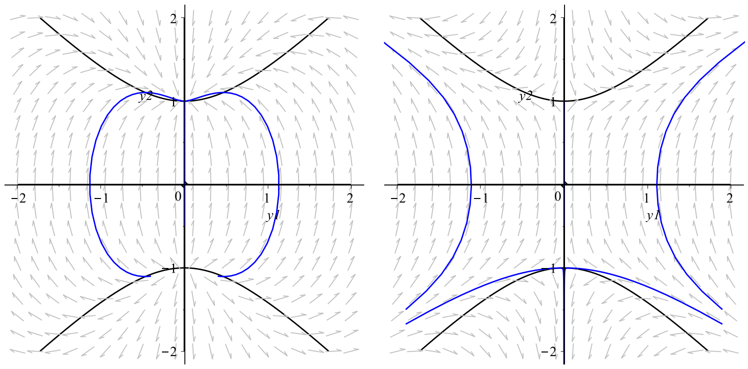


Figure 4.6: Phase diagrams for equation 4.10 for $a < 0$, and $a > 0$

Therefore, if $a < 0$, steady state $\bar{\mathbf{y}}^1$ is a stable node, because trace $\big(J(\bar{\mathbf{y}}^1)\big) < 0$ and det $\big(J(\bar{\mathbf{y}}^1)\big) > 0$, and steady state $\bar{\mathbf{y}}^2$ is an unstable node, because trace $\big(J(\bar{\mathbf{y}}^2)\big) > 0$ and det $\big(J(\bar{\mathbf{y}}^2)\big) > 0$. Then that for any any trajectory starting from any element of $\mathcal{Y}/\bar{\mathbf{y}}^2$ there is convergence to steady state $\bar{\mathbf{y}}^1$ (see the left subfigure in figure 4.6). If we denote $\text{Het}(\mathbf{y})$ the set points connecting $\bar{\mathbf{y}}^2$ to $\bar{\mathbf{y}}^1$ we readily see that $\text{Het}(\mathbf{y}) = \mathcal{Y}$, which means there are an infinite number of heteroclinic orbits, and that this set is coincident to the stable manifold $\mathcal{W}^s_{\bar{\mathbf{y}}^1}$.

However, if $a > 0$ both steady states, $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$, are saddle points, because det $\big(J(\bar{\mathbf{y}}^1)\big) =$ det $\big(J(\bar{\mathbf{y}}^2)\big) < 0$. In this case, there is one heteroclinic surface

$$\text{Het}(\mathbf{y}) = \{(y_1, y_2) : y_1 = 0,\ -1 \le y_2 \le 1\}$$

which is the locus of points connecting $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$ such that for any initial value $\mathbf{y}(0) \in \text{Het}(\mathbf{y})$ the solution will converge to $\bar{\mathbf{y}}^2$ (tee the right subfigure in figure 4.6). In this case $\text{Het}(\mathbf{y})$ is the set of points belonging to the intersection of the unstable manifold of $\bar{\mathbf{y}}^1$ and to the stable manifold of $\bar{\mathbf{y}}^2$: $\text{Het}(\mathbf{y}) = \mathcal{W}^u_{\bar{\mathbf{y}}^1} \cap \mathcal{W}^s_{\bar{\mathbf{y}}^2}$.

At last, we should notice that in both cases the heteroclinic orbits are generic, in the sense that they persist for a wide range of values for parameter a. This is not the c

Heteroclinic saddle connection bifurcation Assuming a related but slightly different normal form generates an heteroclinic bifurcation meaning we may have a bifurcation parameter that when it crosses a specific value heteroclinic orbits cease to exist. The following model is studied, for instance, in (Hale and Koçak, 1991, p.210).

$$\begin{aligned} \dot{y}_1 &= \lambda + 2y_1y_2 \\ \dot{y}_2 &= 1 + y_1^2 - y_2^2 \end{aligned} \tag{4.11}$$

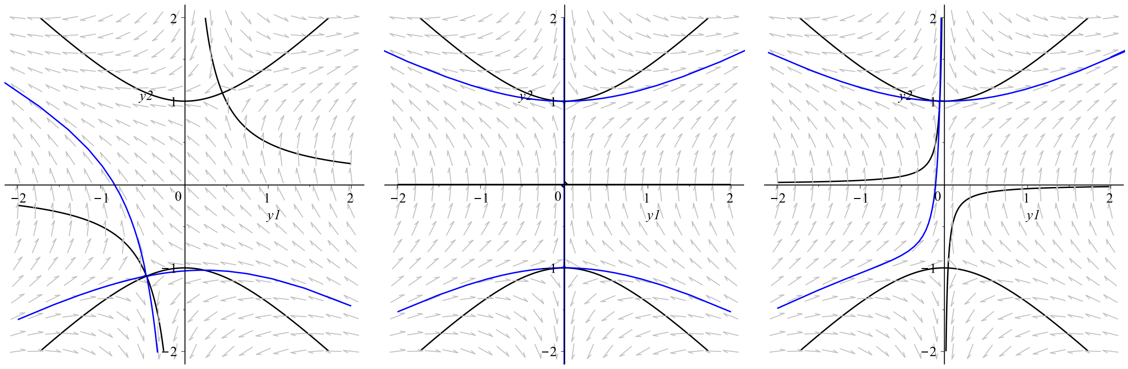


Figure 4.7: Phase diagrams for equation 4.11 for $\lambda < 0$, $\lambda = 0$, $\lambda > 0$

In this case we have, for $\lambda = 0$, an heteroclinic orbit, connecting the two steady states exists and we have the second case in the previous model. When λ is perturbed away from zero we will have only one steady state which is a saddle point. See Figure .

Homoclinic orbits We say there is an **homoclinic orbit** if, in a planar ODE, there is a subset of points $\text{Hom}(\mathbf{y})$ connecting the steady state with itself. This is only possible if the steady state $\bar{\mathbf{y}}$ is a saddle point in which the stable manifold contains a closed curve, that we call homoclinic curve. Because of this fact, homoclinic orbits exist jointly with periodic trajectories.

Again, homoclinic orbits can be generic or non-generic. Next we illustrate both cases.

Generic homoclinic orbits Consider the non-linear planar ODE depending on one parameter, a, of type

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 - a y_1^2. \end{aligned} \tag{4.12}$$

It has two steady states $\bar{\mathbf{y}}^1 = (0, 0)$ and $\bar{\mathbf{y}}^2 = (1/a, 0)$. The Jacobian

$$J(\mathbf{y}) \equiv D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ 1 - 2ay_1 & 0 \end{pmatrix}$$

has following the trace and the determinant

$$\begin{aligned} \text{trace}\,(J(\mathbf{y})) &= 0 \\ \det\,(J(\mathbf{y})) &= 2ay_1 - 1. \end{aligned}$$

that both depend on the parameter a. It is easy to see that steady state $\bar{\mathbf{y}}^1$ is always a saddle point, because $\det\big(J(\bar{\mathbf{y}}^1)\big) = -1 < 0$, and the steady state $\bar{\mathbf{y}}^2$ is always locally a center, because $\det\big(J(\bar{\mathbf{y}}^2)\big) = 1 > 0$ and $\text{trace}\big(J(\bar{\mathbf{y}}^2)\big) = 0$, for any value of a.

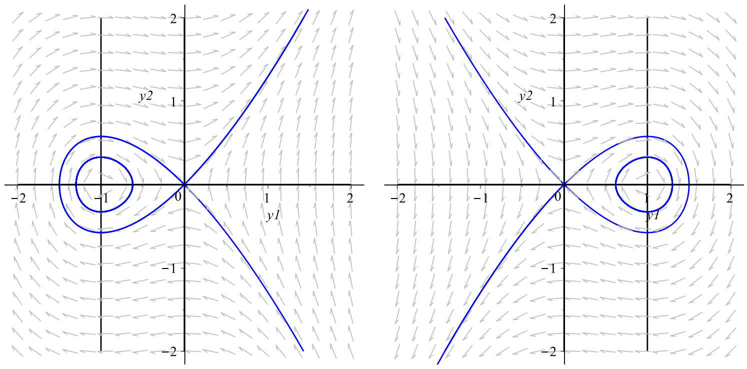


Figure 4.8: Phase diagrams for equation 4.12 for $a < 0$, and $a > 0$

Furthermore, we can prove that there is an invariant curve, such that solutions follow a potential or first integral curve which is constant.

In order to see this we introduce a **Lyapunov function** which is a differentiable function $H(\mathbf{y})$ such that the time derivative is $\dot{H} = D_{\mathbf{y}}H\dot{\mathbf{y}}$, that is $\dot{H} = H_{y_1}\dot{y}_1 + H_{y_2}\dot{y}_2$. A **first integral** is a set of points (y_1, y_2) such that $\dot{H} = 0$. In this case the orbits passing through those points allow for a conservation of energy in some sense and $H(\mathbf{y}(t)) =$ constant. For values such that $H(\mathbf{y}(t)) = 0$ that curve passes through a steady state.

For this case consider the function

$$H(y_1, y_2) = -\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{a}{3}y_1^3.$$

If we time-differentiate this Lyapunov function and substitute equations (4.13) we have

$$\begin{aligned}\dot{H} &= (ay_1 - 1)y_1\dot{y}_1 + y_2\dot{y}_2 = \\ &= (ay_1 - 1)y_1y_2 + y_2y_1(1 - y_1) = \\ &= 0.\end{aligned}$$

Then $\dot{H} = 0$, for any values of $\mathbf{y}$. We call homoclinic surface to the set of points such that there are homoclinic orbits. In our case, homoclinic orbits converge both for $t \to \infty$ and $t \to -\infty$ to point $\bar{\mathbf{y}}^1$. Therefore the homoclinic surface is the set of points

$$\text{Hom}(\bar{\mathbf{y}}^1) = \{(y_1, y_2) : H(y_1, y_2) = 0, \ \text{sign}(\bar{y}_1) = \text{sign}(a)\}$$

Figure 4.8 depicts phase diagrams for the case in which $a < 0$ (left sub-figure) and $a > 0$ (right sub-figure).

We see that the homoclinic trajectories are generic, i.e, they exist for different values of the parameters. This is not always the case as we show next.

Homoclinic or saddle-loop bifurcation This model is studied, for instance, in (Hale and Koçak, 1991, p.210) and (Kuznetsov, 2005, ch. 6.2). It is a non-linear ODE depending on one parameter, a, of type

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 + a\,y_2 - y_1^2. \end{aligned} \tag{4.13}$$

In this case, we have

$$\mathbf{f}(\mathbf{y}, a) = \begin{pmatrix} y_2 \\ y_1 + a\,y_2 - y_1^2. \end{pmatrix}.$$

The set of equilibrium point is $\bar{\mathbf{y}} = \{\mathbf{y} : \mathbf{f}(\mathbf{y}, a) = \mathbf{0}\}$. For equation (4.13) we have two equilibrium points,

$$\bar{\mathbf{y}}^1 = (\bar{y}_1^1, \bar{y}_2^1) = (0,0), \ \bar{\mathbf{y}}^2 = (\bar{y}_1^2, \bar{y}_2^2) = (1,0)$$

In order to determine the local dynamics we evaluate the Jacobian for any point $\mathbf{y} = (y_1, y_2)$,

$$D_{\mathbf{y}}\mathbf{f}(\mathbf{y}, a) = \begin{pmatrix} 0 & 1 \\ 1 - 2y_1 & a \end{pmatrix}.$$

The eigenvalues of the Jacobian are functions of the variables and of the parameter a,

$$\lambda_{\pm}(\mathbf{y}, a) = \frac{a}{2} \pm \left[\left(\frac{a}{2}\right)^2 + 1 - 2y_1 \right]^{\frac{1}{2}}.$$

If we evaluate the eigenvalues at the steady state $\bar{\mathbf{y}}^1 = (0,0)$, we find it is a saddle point, because the eigenvalues of the Jacobian $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^1)$ are

$$\lambda_{\pm}^1 \equiv \lambda_{\pm}(\bar{\mathbf{y}}^1, a) = \frac{a}{2} \pm \left[\left(\frac{a}{2}\right)^2 + 1 \right]^{\frac{1}{2}}$$

yielding $\lambda_-^1 < 0 < \lambda_+^1$. At the steady state $\bar{\mathbf{y}}^2 = (1,0)$ the eigenvalues of the Jacobian $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^2)$ are

$$\lambda_{\pm}^2 = \lambda_{\pm}(\bar{\mathbf{y}}^2, a) = \frac{a}{2} \pm \left[\left(\frac{a}{2}\right)^2 - 1 \right]^{\frac{1}{2}}$$

yielding $\mathrm{sign}(\mathrm{Re}(\lambda_{\pm}(\bar{\mathbf{y}}^2, a)) = \mathrm{sign}(a)$.

Therefore steady state $\bar{\mathbf{y}}^1$ is always a saddle point, and steady state $\bar{\mathbf{y}}^2$ is a stable node or a stable focus if $a < 0$, it is an unstable node or an unstable focus if $a > 0$, or it is a centre if $a = 0$.

When $a = 0$ another type of dynamics occurs. We introduce the following Lyapunov function

$$H(y_1, y_2) = -\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{3}y_1^3.$$

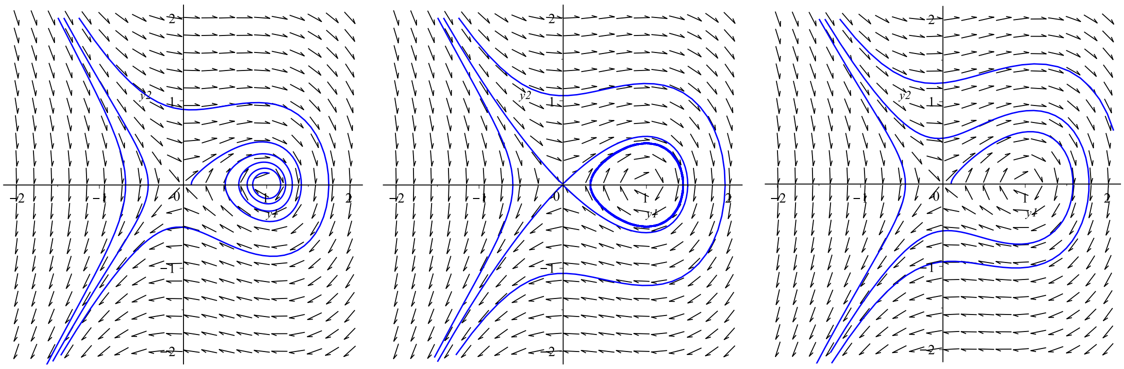


Figure 4.9: Phase diagrams for equation 4.13 for $a < 0$, $a = 0$, $a > 0$

and prove that it can only be a first integral if $a = 0$. To show this, if we time-differentiate this Lyapunov function and substitute equations (4.13) we have

$$\begin{aligned} \dot{H} &= (y_1 - 1)y_1\dot{y}_1 + y_2\dot{y}_2 = \\ &= (y_1 - 1)y_1y_2 + y_2y_1(1 - y_1) + ay_2^2 = \\ &= ay_2^2. \end{aligned}$$

Then $\dot{H} = 0$, for any values of $\mathbf{y}$, if and only if and only if $a = 0$.

In our case this generates an **homoclinic orbit** which is a trajectory that exits a steady state and returns to the same steady state. In this case, a homoclinic orbit exists if $a = 0$ and it does not exist if $a \neq 0$.

The next figure shows the phase diagrams for the cases $a < 0$, $a = 0$ and $a > 0$. If $a < 0$ there is a saddle point and a stable focus, if $a = 0$ there is a saddle point, an infinite number of centres surrounded by an homoclinic orbit. If $a > 0$ there is a saddle point and an unstable focus.

Planar equation: Andronov-Hopf bifurcation This model is studied, for instance, in (Hale and Koçak, 1991, p.212).

$$\begin{aligned} \dot{y}_1 &= y_2 + y_1(\lambda - y_1^2 - y_2^2) \\ \dot{y}_2 &= -y_1 + y_2(\lambda - y_1^2 - y_2^2) \end{aligned} \tag{4.14}$$

In figure 4.10 we see the following: if $\lambda \leq 0$ there will be only one steady state which is a stable focus, although the speed of convergence to the steady state increases very much when λ converges to zero, if $\lambda > 0$ a **limit circle** appears and the steady state becomes a unstable focus.

In order to determine the existence of a limit circle we transform the system in polar coordinates

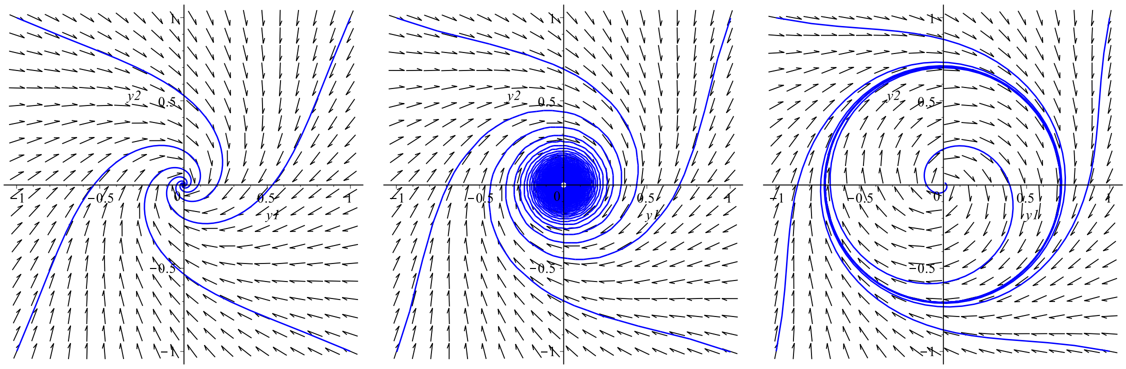


Figure 4.10: Phase diagrams for equation 4.14 for $\lambda < 0$, $\lambda = 0$, $\lambda > 0$

(see section 1.3) and get

$$\dot{r} = r(\lambda - r^2)$$
$$\dot{\theta} = -1$$

there is thus a periodic orbit with radius $\bar{r} = \sqrt{\lambda}$.

4.2 Qualitative theory of ODE

Next we present a short introduction to the qualitative (or geometrical) theory of ODE's.

We consider a generic ODE

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}),\ \mathbf{f} : \mathcal{Y} \to \mathcal{Y},\ \mathbf{y} : \mathcal{T} \to \mathcal{Y} \tag{4.15}$$

where $\mathbf{f} \in C^1(\mathcal{Y})$, i.e., $f(.)$ is continuously differentiable up to the first order.

The qualitative theory of ODEs consists in finding a (topological) equivalence between a non-linear (or even incompletely defined) function $\mathbf{f}(.)$ and a linear or a normal form ODE. This allows us to characterize the dynamics in the neighborhood of a steady state or of a periodic orbit or other invariant sets (homoclinic and heteroclinic orbits or limit cycles). If there are more than one invariant orbit or steady state we distinguish between local dynamics (in the neighborhood of a steady state or invariant orbit) from global dynamics (in all set $\mathcal{Y}$). If there is only one invariant set then local dynamics is qualitatively equivalent to global dynamics.

One important component of qualitative theory is **bifurcation analysis**, which consists in describing the change in the dynamics (that is, in the phase diagram) when one or more parameters take different values within its domain.

4.2.1 Local analysis

We study local dynamics of equation (4.15) by performing a local analysis close to an equilibrium point or a periodic orbit. There are three important results that form the basis of the local analysis: the Grobman-Hartmann, the manifold and the Poincaré-Bendixson theorems. The first two are related to using the knowledge on the solutions of an equivalent linearized ODE to study the local properties close to the a steady-state for a non-linear ODE and the third introduces a criterium for finding periodic orbits.

Equivalence with linear ODE's

Assume there is (at least) one equilibrium point $\bar{\mathbf{y}} \in \{\mathbf{y} \in \mathcal{Y} \subseteq \mathbb{R}^n : \mathbf{f}(\mathbf{y}) = \mathbf{0}\}$, for $n \geq 1$, and consider the Jacobian of $\mathbf{f}(.)$ evaluated at that equilibrium point

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \frac{\partial f_1(\bar{y})}{\partial y_1} & \dots & \frac{\partial f_1(\bar{y})}{\partial y_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n(\bar{y})}{\partial y_1} & \dots & \frac{\partial f_n(\bar{y})}{\partial y_n} \end{pmatrix}.$$

An equilibrium point is **hyperbolic** if the Jacobian J has no eigenvalues with zero real parts. An equilibrium point is **non-hyperbolic** if the Jacobian has at least one eigenvalue with zero real part.

Theorem 1 (**Grobman-Hartmann theorem**). *Let $\bar{\mathbf{y}}$ be an hyperbolic equilibrium point. Then there is a neighbourhood U of $\bar{\mathbf{y}}$ and a neighborhood U_0 of $\mathbf{y}(0)$ such that the ODE restricted to U is topologically equivalent to the variational equation*

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}}),\ \mathbf{y} - \bar{\mathbf{y}} \in U_0$$

Stability properties of $\bar{\mathbf{y}}$ are characterized from the eigenvalues of Jacobian matrix $J(\bar{\mathbf{y}}) = D_y\mathbf{f}(\bar{\mathbf{y}})$.

If all eigenvalues λ of the Jacobian matrix have negative real parts, $\mathrm{Re}(\lambda) < 0$, then $\bar{y}$ is asymptotically stable. If there is at least one eigenvalue λ such that $\mathrm{Re}(\lambda) > 0$ then $\bar{y}$ is unstable.

Example 1 Consider the scalar ODE

$$\dot{y} = f(y) \equiv y^{\alpha} - a \tag{4.16}$$

where a and α are two constants, with $a > 0$, and $y \in \mathbb{R}_+$. Then there is an unique steady state $\bar{y} = a^{\frac{1}{\alpha}}$. As

$$f_y(y) = \alpha y^{\alpha-1}$$

then

$$f_y(\bar{y}) = \alpha a^{\frac{\alpha-1}{\alpha}}.$$

Set $\lambda \equiv f_y(\bar{y})$. Therefore the steady state is hyperbolic if $\alpha \neq 0$ and it is non-hyperbolic if $\alpha = 0$. In addition, if $\alpha < 0$ the hyperbolic steady state $\bar{y}$ is asymptotically stable and if $\alpha > 0$ it is unstable.

If $\alpha \neq 0$ we can perform a first-order Taylor expansion of the ODE (4.16) in the neighborhood of the steady state

$$\dot{y} = \lambda(y - \bar{y}) + o((y - \bar{y}))$$

which means that the solution to (4.16) can be locally approximated by

$$y(t) = \bar{y} + (k - \bar{y})e^{\lambda t}$$

for any $k \in \mathbb{R}_+$. In particular, if we fix $y(0) = y_0$ then $k = y_0$.

Example 2 Consider the non-linear planar ODE

$$\begin{aligned} \dot{y}_1 &= y_1^{\alpha} - a,\ 0 < \alpha < 1,\ a \geq 0, \\ \dot{y}_2 &= y_1 - y_2 \end{aligned} \tag{4.17}$$

It has an unique steady state $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2) = (a^{\frac{1}{\alpha}}, a^{\frac{1}{\alpha}})$. The Jacobian evaluated at any point $\mathbf{y}$ is

$$J(\mathbf{y}) = D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} \alpha y_1^{\alpha-1} & 0 \\ 1 & -1 \end{pmatrix}.$$

If we approximate the system in a neighborhood of the steady state, $\bar{\mathbf{y}}$, we have the linear planar ODE

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})\,(\mathbf{y} - \bar{\mathbf{y}})$$

where $J(\bar{\mathbf{y}})$ is the Jacobian evaluated at the steady state,

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \alpha a^{\frac{\alpha-1}{\alpha}} & 0 \\ 1 & -1 \end{pmatrix}.$$

We already saw that the solution to this equation is

$$\mathbf{y}(t) = \mathbf{y} + \mathbf{P}\mathbf{e}^{J(\bar{\mathbf{y}})t}\mathbf{h}.$$

Because

$$\begin{aligned} \text{trace}(J(\bar{\mathbf{y}})) &= \alpha a^{\frac{\alpha-1}{\alpha}} - 1 \\ \det\left(J(\bar{\mathbf{y}})\right) &= -\alpha a^{\frac{\alpha-1}{\alpha}} \\ \Delta(J(\bar{\mathbf{y}})) &= \left(\frac{\alpha a^{\frac{\alpha-1}{\alpha}} + 1}{2}\right)^2 \end{aligned}$$

which implies that the eigenvalues of the Jacobian $J(\bar{\mathbf{y}})$ are

$$\lambda_{\pm} = \frac{\alpha a^{\frac{\alpha-1}{\alpha}} - 1}{2} \pm \frac{\alpha a^{\frac{\alpha-1}{\alpha}} + 1}{2}.$$

We can also find the eigenvector matrix of $J(\bar{\mathbf{y}})$,

$$\mathbf{P} = \left(\mathbf{P}^{+}\mathbf{P}^{-}\right) = \begin{pmatrix} 1+\lambda_{+} & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore, the approximate solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = h_{+} \begin{pmatrix} 1+\lambda_{+} \\ 1 \end{pmatrix} e^{\lambda_{+} t} + h_{-} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_{-} t}.$$

The eigenvalues of the Jacobian evaluated at the steady state are $\lambda_{+} = \alpha a^{\frac{\alpha-1}{\alpha}}$ and $\lambda_{-} = -1$. Therefore, the steady state is hyperbolic if $\alpha \neq 0$ and non-hyperbolic if $\alpha = 0$.

Furthermore, the steady state is a saddle point if $\alpha > 0$ and it is a stable node if $\alpha < 0$. In the first case the stable eigenspace is

$$\mathcal{E}^{s} = \{(y_1, y_2) : y_1 = \bar{y}_1\}$$

and, in the second case the stable eigenspace is the whole space, $\mathcal{E}^{s} = \mathcal{Y}$.

Local manifolds

Consider a neighbourhood $U \subset \mathcal{Y} \subseteq \mathbb{R}^{n}$ of $\bar{\mathbf{y}}$: the local stable manifold is the set

$$\mathcal{W}^{s}_{loc}(\bar{\mathbf{y}}) = \{\mathbf{k} \in U : \lim_{t\to\infty} \mathbf{y}(t,k) = \bar{y},\ y(t,\mathbf{k}) \in U, t \geq 0\}$$

the local unstable manifold is the set

$$\mathcal{W}^{u}_{loc}(\bar{\mathbf{y}}) = \{\mathbf{k} \in U : \lim_{t\to\infty} \mathbf{y}(-t,\mathbf{k}) = \bar{\mathbf{y}},\ \mathbf{y}(-t,\mathbf{k}) \in U, t \geq 0\}$$

The center manifold is denoted $\mathcal{W}^{c}_{loc}(\bar{\mathbf{y}})$. Let n_{-}, n_{+} and n_0 denote the number of eigenvalues of the Jacobian evaluated at steady state $\bar{\mathbf{y}}$ with negative, positive and zero real parts.

Theorem 2 (**Manifold Theorem**). *: suppose there is a steady state $\bar{\mathbf{y}}$ and $J(\bar{\mathbf{y}})$ is the Jacobian of the ODE* (4.15) *. Then there are local stable, unstable and center manifolds, $\mathcal{W}^{s}_{loc}(\bar{\mathbf{y}})$, $\mathcal{W}^{u}_{loc}(\bar{\mathbf{y}})$ and $\mathcal{W}^{c}_{loc}(\bar{\mathbf{y}})$, of dimensions n_{-}, n_{+} and n_0, respectively, such that $n = n_{-} + n_{+} + n_0$. The local manifolds are tangent to the local eigenspaces $\mathcal{E}^{s}$, $\mathcal{E}^{u}$, $\mathcal{E}^{c}$ of the (topologically) equivalent linearized ODE*

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}}).$$

The first two, eigenspaces $\mathcal{E}^s$ and $\mathcal{E}^u$, are unique, and $\mathcal{E}^c$need not be unique (see (Grass et al., 2008, ch.2)).

The eigenspaces are spanned by the eigenvectors of the Jacobian matrix $J(\bar{y})$ which are associated to the eigenvalues with negative, positive and zero real parts.

Example 2 Consider example 2 and let $\alpha > 0$ which implies that the steady state $\bar{\mathbf{y}}$ is a saddle point. Because the eigenvector associated to eigenvalue λ_- is $\mathbf{P}^- = (0,1)^\top$, then the stable eigenspace is

$$\mathcal{E}^s = \{(y_1, y_2) \in \mathbb{R}_+ : y_1 = \bar{y}_1 = a^{\frac{1}{\alpha}}\}.$$

The local stable manifold $\mathcal{W}^s_{loc}(\bar{\mathbf{y}})$ is tangent to $\mathcal{E}^s$ in a neighborhood of the steady state.

Periodic orbits

We saw that solution trajectories can converge or diverge not only as regards equilibrium points but also to periodic trajectories (see the Andronov-Hopf model).

The **Poincaré-Bendixson** theorem ((Hale and Koçak, 1991, p.367)) proves that if the limit set is bounded and it is not an equilibrium point it should be a periodic orbit.

In order to determine if there is a periodic orbit in a compact subset of $\mathcal{Y}$ the Bendixson criterium provides a method ((Hale and Koçak, 1991, p.373)):

Theorem 3 (**Bendixson-Dulac criterium**). *Let D be a compact region of $\mathcal{Y} \subseteq \mathbb{R}^n$ for $n \geq 2$. If,*

$$div(\mathbf{f}) = f_{1,y_1}(y_1, y_2) + f_{2,y_2}(y_1, y_2)$$

has constant sign, for $(y_1, y_2) \in D$, then $\dot{y} = f(y)$ has not a constant orbit lying entirely in D.

4.2.2 Global analysis

While local analysis consists in studying local dynamics in the neighbourhood of steady states or periodic orbits, this may not be enough to characterise the dynamics.

We already saw that there are orbits that are invariant and that cannot be determined by local methods, for instance heteroclinic and homoclinic orbits.

Homoclinic and heteroclinic orbits

There are methods to determine if there are homoclinic or heteroclinic orbits. They essentially consist in building a trapping area for the trajectories and proving there should exist trajectories that do not exit the ”trap”.

Global manifolds

There are global extensions of the local manifolds by continuation in time (in the opposite direction) of the local manifolds: $\mathcal{W}^s(\bar{y})$, $\mathcal{W}^u(\bar{y})$, $\mathcal{W}^c(\bar{y})$.

A trajectory $y(.)$ of the ODE is called a **stable path** of $\bar{y}$ if the orbit $\mathrm{Or}(y_0)$ is contained in the stable manifold $\mathrm{Or}(y_0) \subset \mathcal{W}^s(\bar{y})$ and $\lim_{t\to\infty} y(t, y_0) = \bar{y}$.

A trajectory $y(.)$ of the ODE is called a **unstable path** of $\bar{y}$ if the orbit $\mathrm{Or}(y_0)$ is contained in the stable manifold $\mathrm{Or}(y_0) \subset \mathcal{W}^u(\bar{y})$ and $\lim_{t\to\infty} y(-t, y_0) = \bar{y}$.

Dependence on parameters We already saw that the solution of linear ODE's, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, may depend on the values for the parameters in the coefficient matrix $\mathbf{A}$.

We can extend this idea to non-linear ODE's of type

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi),\ \varphi \in \Phi \subseteq R^q$$

where φ is a vector of parameters of dimension $q \geq 1$

We can distinguish two types of parameter change:

- **perturbations** when parameter changes do not change the qualitative dynamics, i.e., they do not change the phase diagram. By qualitative change we mean change the number or the stability properties of steady states or other invariants. This is typically the case in economics when one performs comparative dynamics exercises
- **bifurcations** when a parameter change induces a qualitative change in the dynamics, i.e, the phase diagram.

4.2.3 Bifurcations

If a small variation of the parameter changes the phase diagram we say we have a bifurcation. As you saw, there are local (fixed points) and global bifurcations (heteroclinic connection, etc). Those bifurcations were associated to particular normal forms of both scalar and planar ODEs. This fact allows us to find classes of ODE's which are topologically equivalent to those we have already presented.

Bifurcations for scalar ODE's

Consider the scalar ODE

$$\dot{y} = f(y, \varphi),\ y, \varphi \in \mathbb{R}.$$

Fold bifurcation (see (Kuznetsov, 2005, ch. 3.3)): Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0,0)$, such that $f(0,0) = 0$, with $f_y(0,0) = 0$ and

$$f_{yy}(0,0) \neq 0,\ f_\varphi(0,0) \neq 0.$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi \pm y^2,$$

that is to the Ricatti's model (4.2).

Transcritical bifurcation: Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0,0)$, such that $f(0,0) = 0$, with $f_y(0,0) = 0$ and

$$f_{yy}(0,0) \neq 0,\ f_{\varphi y}(0,0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^2$$

that is to the Bernoulli model (4.3).

Pitchfork bifurcation: Let $f \in C^2(\mathbb{R})$ and consider $(\bar{y}, \varphi_0) = (0,0)$, such that $f(0,0) = 0$, with $f_y(0,0) = 0$ and

$$f_{yyy}(0,0) \neq 0,\ f_{\varphi y}(0,0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^3$$

that is to the Bernoulli model (4.4).

Bifurcations for planar ODE's

Consider the planar ODE

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi),\ \mathbf{y} \in \mathbb{R}^2, \varphi \in \mathbb{R}$$

Andronov-Hopf bifurcation (see (Kuznetsov, 2005, ch. 3.4)): Let $\mathbf{f} \in C^2(\mathbb{R})$ and consider $(\bar{\mathbf{y}}, \varphi_0) = (\mathbf{0}, 0)$ the Jacobian at $(\mathbf{0}, 0)$ has eigenvalues

$$\lambda_\pm = \eta(\varphi) \pm i\omega(\varphi)$$

where $\eta(0) = 0$ and $\omega(0) > 0$. If some additional conditions are satisfied then the ODE is locally topologically equivalent to

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

4.2.4 Comparative dynamics in economics

As mentioned, **comparative dynamics** exercises consist in introducing perturbation in a dynamic system: i.e., a small variation of the parameter that does not change the phase diagram. This kind of analysis only makes sense if the steady state is hyperbolic, that is if $\det(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)) \neq 0$ or $\text{trace}(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)) \neq 0$ if $\det(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)) > 0$.

In this case let the steady state be for a given value of the parameter $\varphi = \varphi_0$

$$\bar{\mathbf{y}}_0 = \{y \in \mathcal{Y} : \mathbf{f}(\mathbf{y}, \varphi_0) = \mathbf{0}\}.$$

If $\bar{\mathbf{y}}_0$ is a hyperbolic steady state, then we can expand the ODE into a linear ODE

$$\dot{\mathbf{y}} = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)(\mathbf{y} - \bar{\mathbf{y}}_0) + D_{\varphi}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)(\varphi - \varphi_0). \tag{4.18}$$

This equation can be solved as a linear ODE. Setting $\varphi = \varphi_0 + \delta_\varphi$ and because $\bar{\mathbf{y}} = \bar{\mathbf{y}}(\varphi)$ and $\bar{\mathbf{y}}_0 = \bar{\mathbf{y}}(\varphi_0)$ we have

$$D_{\varphi}\bar{\mathbf{y}}(\varphi_0) = \lim_{\delta_\varphi \to 0} \frac{\bar{\mathbf{y}}(\varphi_0 + \delta_\varphi) - \bar{\mathbf{y}}(\varphi_0)}{\delta_\varphi} = -D_{\mathbf{y}}^{-1}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)D_{\varphi}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)$$

which are called the **long-run multipliers** associated to a permanent change in φ. Solving the linearized system allows us to have a general solution to the problem of finding the **short-run** or **transition multipliers**, $d\mathbf{y}(t) \equiv \mathbf{y}(t) - \bar{\mathbf{y}}_0$ for a change in the parameter φ.

4.3 References

- (Hale and Koçak, 1991, Part I , III): very good introduction.
- (Guckenheimer and Holmes, 1990, ch. 1, 3, 6) Is a classic reference on the field.
- Kuznetsov (2005) Very complete presentation of bifurcations for planar systems.
- (Grass et al., 2008, ch.2) has a compact presentation of all the important results with some examples in economics and management science.

4.4 Application to economics

4.4.1 The optimality conditions for the Ramsey model

The Ramsey (1928) model (see also Cass (1965) and Koopmans (1965)) is the workhorse of modern macroeconomics and growth theory. It is a normative model (but can also be seen as a positive model if its behavior fits the data) on the optimal choice of consumption and where savings leads to the accumulation of capital, and therefore to future consumption. Therefore, the optimal trade-off between present and future consumption guides the accumulation of capital.

We will derive the optimality conditions when we study optimal control. In this section we assume that there are two primitives for the model related with technology and preferences: (1) the production function, $f(k)$ and (2) the elasticity of intertemporal substitution $\eta(c)$ and the rate of time preference ρ.

The first order conditions for an optimum take the form of two non-linear differential equations. Let k and c denote per-capita physical capital and consumption, respectively, and let the two variables be non-negative. That is $(k,c) \in \mathbb{R}^2_+$. The Ramsey model is the planar ODE

$$\dot{k} = f(k) - c \tag{4.19}$$

$$\dot{c} = \eta(c)\, c\left(f^{'}(k) - \rho\right), \tag{4.20}$$

supplemented with an initial value for k, $k(0) = k_0$ and the transversality condition $\lim_{t\to\infty} u^{'}(c)k(t)e^{-\rho t} = 0$, where $u(c)$ is the utility function from which we determine the elasticity of intertemporal substitution. For this section we will be concerned with trajectories that are bounded asymptotically, that is converging to a steady state.

The ODE system (also called modified Hamiltonian dynamic system MHDS) is non-linear when the two primitive functions are not completely specified, as is the case with system (4.19)-(4.20). Next we assume a smooth case and the following assumptions

1. preferences are specified by a constant elasticity of intertemporal substitution, $\eta(c) = \eta > 0$ is constant;

2. the rate of time preference is positive $\rho > 0$;

3. the production function is of the Inada type: it is positive for positive levels of capital, it is monotonously increasing and globally concave. Formally: $f(0) = 0$, $f(k) > 0$ for $k > 0$, $f^{'}(k) > 0$, $\lim_{k\to 0} f^{'}(k) = +\infty$, $\lim_{k\to +\infty} f^{'}(k) = 0$, and $f^{''}(k) < 0$ for all $k \in \mathbb{R}_+$.

Given the smoothness of the vector field, i.e, of functions $f_1(k,c) \equiv f(k) - c$ and $f_2(k,c) \equiv \eta c(f^{'}(k) - \rho)$, we know that a solution exists and it is unique. Therefore, in order to characterize the dynamics we can use the qualitative theory of ODE's presented previously in this section.

In particular we will

1. determine the existence and number of steady states
2. characterize them regarding hyperbolicity and local dynamics, performing, if necessary, a local bifurcation analysis
3. try to find other invariant trajectories of a global nature
4. conduct comparative dynamics analysis in the neigborhood of relevant hyperbolic steady states.

Steady states Any steady-state, $(\bar{k}, \bar{c})$, belongs to the set

$$\mathcal{E} = \{(k, c) \in \mathbb{R}^2_+ : \ \dot{k} = \dot{c} = 0\} = \{(0, 0), (k^*, c^*)\}$$

where $k^* = g(\rho)$, where $g(.) = (f^{'})^{-1}(.)$ and $c^* = f(k^*) = f(g(\rho))$.

To prove the existence and uniqueness of a positive steady state level for k we use the Inada and global concavity properties of the production function: first, $\dot{c} = 0$ if there is a value k that solves the equation $f^{'}(k) = \rho$; second, because $\rho > 0$ is finite and $f^{'}(k) \in (0, \infty)$ then there is at least one value for k that solves that equation; at last, because the function $f(.)$ is globally strictly concave then $f^{'}(k)$ is monotonously decreasing which implies that the solution is unique.

Characterizing the steady states In order to characterize the steady states, we find the Jacobian of system (4.19)-(4.20), is

$$D_{(k,c)}\mathbf{F}(k, c) = \begin{pmatrix} f^{'}(k) & -1 \\ \eta c f^{''}(k) & \eta\left(f^{'}(k) - \rho\right). \end{pmatrix} \tag{4.21}$$

The eigenvalues of $D_{(k,c)}\mathbf{F}(k, c)$ evaluated at steady state $(\bar{k}, \bar{c}) = (0, 0)$ are

$$\lambda^0_s = \eta(f^{'}(0) - \rho) = +\infty, \ \lambda^0_u = f^{'}(0) = +\infty$$

then this equilibrium point is not non-hyperbolic and this steady state is an unstable node.

For steady state $(\bar{k}, \bar{c}) = (k^*, c^*)$, the trace and the determinant of the Jacobian are

$$\text{trace}(D_{(k,c)}\mathbf{F}(k^*, c^*)) = \rho > 0, \ \det(D_{(k,c)}F(k^*, c^*)) = \eta c^* f^{''}(k^*) < 0$$

and the eigenvalues are

$$\lambda^*_s = \frac{\rho}{2} - \left(\left(\frac{\rho}{2}\right)^2 - \eta c^* f^{''}(k^*)\right)^{\frac{1}{2}} < 0, \ \lambda^*_u = \frac{\rho}{2} + \left(\left(\frac{\rho}{2}\right)^2 - \eta c^* f^{''}(k^*)\right)^{\frac{1}{2}} < 0$$

satisfy the relationships

$$\lambda_s^* + \lambda_u^* = \rho, \ \lambda_s^*\lambda_u^* = \eta c^* f^{''}(k^*) < 0.$$

The steady state (k^*, c^*) is also hyperbolic and it is a saddle-point. The intuition behind this property is transparent when we look at the expression for the determinant: the mechanism generating stability is related to the existence of decreasing marginal returns in producion. Because capital accumulation is equal to savings, and savings sustains future increases in consumption by increasing production, the existence of decreasing marginal returns implies that the marginal increase in production will tend to zero thus stopping the incentives for future capital accumulation.

As the Jacobians of system (4.19)-(4.20), evaluated at every steady state, does not have eigenvalues with zero real parts both steady states are hyperbolic and there are no local bifurcation points.

In addition, from the Grobman-Hartmann theorem the system (4.19)-(4.20) can be approximated by a (topologically equivalent) linear system in the neighborhood of every steady state.

Let us consider the steady state (k^*, c^*). As the Jacobian in this case is

$$D_{(k,c)}\mathbf{F}(k^*, c^*) = \begin{pmatrix} \rho & -1 \\ \eta c^* f^{''}(k^*) & 0 \end{pmatrix}$$

we can consider the **variational system**

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \eta c^* f^{''}(k^*) & 0 \end{pmatrix} \begin{pmatrix} k - k^* \\ c - c^* \end{pmatrix}$$

as giving the approximated dynamics in the neighborhood of the steady state (k^*, c^*).

Because

$$D_{(k,c)}\mathbf{F}(k^*, c^*) - \lambda_s^*\mathbf{I}_2 = \begin{pmatrix} \rho - \lambda_s^* & -1 \\ \eta c^* f^{''}(k^*) & -\lambda_s^* \end{pmatrix} = \begin{pmatrix} \lambda_u^* & -1 \\ \lambda_s^*\lambda_u^* & -\lambda_s^* \end{pmatrix}$$

we get the eigenvector associated to λ_s^*

$$\mathbf{P}_s^* = (1, \lambda_u^*)^\top.$$

This implies that the stable eigenspace of the linearized ODE,

$$\mathcal{E}^s = \{(k, c) \in N^* : c = \lambda_u^* k\}$$

gives the locus of points in the domain, wich are tangent to the local stable manifold for the original ODE (4.19)-(4.20)

$$\mathcal{W}_{loc}^s = \{(k, c) \in N^* : \lim_{t\to\infty} (k(t), c(t)) = (k^*, c^*)\}$$

where $N^* = \{(k, c) \in \mathbb{R}_+^2 : ||(k, c) - (k^*, c^*)|| < \delta\}$ for a small δ.

Global invariants We can prove that there is an heteroclinic orbit connecting steady states $(0,0)$ and (k^*, c^*). Furthermore, the points in that orbit belong to the stable manifold

$$\mathcal{W}^s = \{(k,c) \in \mathbb{R}^2_+ : \lim_{t\to\infty}(k(t), c(t)) = (k^*, c^*)\},$$

and take the form $c = h(k)$. Although we cannot determine explicitly the function $h(.)$ we can prove that it exists.

We already know that the steady state $(0,0)$ is an unstable node, which means that any small deviation will set a diverging path, and, because steady state (k^*, c^*) is a saddle point there is one unique path converging to it. There is an heteroclinic orbit if this path starts from from $(0,0)$. In order to prove this is the case we can consider a "trapping area" $T = \{(k,c) : c \le f(k),\ 0 \le k \le k^*\}$, where the isoclines $\dot{k} = 0$ and $\dot{c} = 0$ define the boundaries $S_1 = \{(k,c) : c = f(k), 0 \le k \le k^*\}$ and $S_2 = \{(k,c) : 0 \le c \le c^*,\ k = k^*\}$. We can see that all the trajectories coming from inside will exit T: first, the trajectories that cross S_1 will exit T because $\dot{k}|_{S_1} = 0$ and $\dot{c}|_{S_1} = \eta c(f^{'}(k) - \rho) = \eta f(k)(f^{'}(k) - f^{'}(k^*)) > 0$ because $f^{'}(k) > f^{'}(k^*)$ for $k < k^*$, second all trajectories that cross S_2 will exit T because $\dot{k}|_{S_2} = f(k^*) - c = c^* - c > 0$ and $\dot{c}|_{S_2} = 0$.

Add figure

Comparative dynamics Let us consider the steady state (k^*, c^*). As we saw that it is an hyperbolic point, small perturbations by a parameter will not change the local dynamic properties of the steady state, only its quantitative level. Therefore, we can perform a comparative dynamics exercise in its neighborhood.

Assume we start at a steady state and introduce a small change in ρ. As the steady state is a function of ϕ, this means that, after the change, the steady state will move and the initial point is not a steady state. That is we can see it as an arbitrary initial point out of the (new) steady state. From hyperbolicity, the new steady state is still a saddle point, which means that the small perturbation will generate unbounded orbits unless there is a "jump" to the new stable manifold associated to the new steady state. This is the intuition behind the comparative dynamics exercise in most perfect foresight macro models (see Blanchard and Khan (1980) and Buiter (1984)) that we illustrate next. We basically assume that variable k is continuous in time (it is pre-determined) and that c is piecewise continuous in time (it is non-predetermined).

Formally, as we also saw that it is a function of the rate of time preference, let us introduce a permanent change in its value from ρ to $\rho + d\rho$. This will introduce a time-dependent change in the two variables, from (k^*, c^*) to $(k(t), c(t))$ where $k(t) = k^* + dk(t)$ and $c(t) = c^* + dc(t)$. In order to find $(dk(t), dc(t))$ we make a first-order Taylor expansion on (k, c) generated by $d\rho$ to get

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = D_{(k,c)}\mathbf{F}(k^*, c^*) \begin{pmatrix} dk(t) \\ dc(t) \end{pmatrix} + D_\rho \mathbf{F}(k^*, c^*) d\rho \tag{4.22}$$

where $D_\rho \mathbf{F}(k^*, c^*) = (0, -\eta c^*)^\top$. This is a linear planar non-homogeneous ODE.

From $\dot{k} = \dot{c} = 0$ we can find the long-run multipliers

$$\begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} = \begin{pmatrix} \dfrac{dk^*}{d\rho} \\ \dfrac{dc^*}{d\rho} \end{pmatrix} = -D^{-1}_{(k,c)} \mathbf{F}(k^*, c^*) D_\rho \mathbf{F}(k^*, c^*),$$

that is

$$\begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \eta c^* f''(k^*) & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \eta c^* \end{pmatrix}.$$

Then $\partial_\rho k^* = \dfrac{1}{f''(k^*)} < 0$ and $\partial_\rho c^* = \rho \partial_\rho k^* < 0$. A permanent unanticipated change in ρ will reduce the long run capital stock and consumption level.

We are only interested in the trajectories that converge to the new steady state after a perturbation, $k^* + \partial_\rho k^* d\rho$ and $c^* + \partial_\rho c^* d\rho$, that is a saddle point. In order to make sure this is the case, we solve the variational system for the saddle path to get

$$\begin{pmatrix} \partial_\rho k(t) \\ \partial_\rho c(t) \end{pmatrix} = \begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} + x \begin{pmatrix} 1 \\ \lambda^*_u \end{pmatrix} e^{\lambda^*_s t},$$

where x is a positive arbitrary element. If we assume that the variable k is pre-determined, that is it can only be changed in a continuous way from the initial steady state value k^*, we set $\partial_\rho k(0) = 0$. Then, from

$$\partial_\rho k(0) = \partial_\rho k^* + x = 0 \Rightarrow x = -\partial_\rho k^*$$

At last we obtain the **short-run multipliers**

$$\begin{aligned} \partial_\rho k(t) &= \partial_\rho k^* \left(1 - e^{\lambda^*_s t}\right) \\ \partial_\rho c(t) &= \partial_\rho k^* \left(\rho - \lambda_u e^{\lambda^*_s t}\right) \end{aligned}$$

for $t \in [0, \infty)$. In particular we get the impact multipliers, for $t = 0$

$$\begin{aligned} \partial_\rho k(0) &= 0 \\ \partial_\rho c(0) &= \partial_\rho k^* \lambda_s > 0 \end{aligned}$$

which quantify the "jump" to the new stable eigenspace, and the long-run multipliers

$$\begin{aligned} \lim_{t \to \infty} \partial_\rho k(t) &= \partial_\rho k^* < 0 \\ \lim_{t \to \infty} \partial_\rho c(t) &= \rho \partial_\rho k^* = \partial_\rho c^* < 0. \end{aligned}$$

Therefore, on impact consumption increases, which reduces capital accumulation, which reduces again consumption through time. The process stops because the reduction in the per-capita stock will increase marginal productivity which reduces the incentives for further reduction in consumption.

Add figure

Observe also that we should have a ”jump” to the stable manifold to have convergence towards the new steady state. As we have determined convergence to the steady state within the stable eigenspace of the variational system, the trajectory we have determined is qualitatively but not quantitatively exact.

4.A Solution of the Ricatti's equation (4.2)

Let us start with the case: $a = 0$. Separating variables, we have

$$\frac{dy}{y^2} = dt$$

integrating both sides

$$\int \frac{dy}{y^2} = dt \Leftrightarrow -\frac{1}{y} = t + k$$

Then we get the solution

$$y(t) = -\frac{1}{t+k}$$

Now let $a \neq 0$. By using the same method we have

$$\frac{dy}{a+y^2} = dt. \tag{4.23}$$

At this point it is convenient to note that

$$\frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}, \quad \frac{d\tanh^{-1}(x)}{dx} = \frac{1}{1-x^2},$$

where

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}, \quad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Then we should deal separately with the cases $a > 0$ and $a < 0$. If $a > 0$ integrating equation (4.23)

$$\int \frac{dy}{a+y^2} = dt \Leftrightarrow \frac{1}{\sqrt{a}} \int \frac{1}{1+x^2} dx = t + k \Leftrightarrow \frac{1}{\sqrt{a}} \tan^{-1}(x) = t + k$$

where we defined $x = y/\sqrt{a}$. Solving the last equation for x and mapping back to y we get

$$y(t) = \sqrt{a}\left(\tan\left(\sqrt{a}(t+k)\right)\right).$$

If $a < 0$ we integrate equation (4.23) by using a similar transformation, but instead with $x = y/\sqrt{-a}$ to get

$$\int \frac{dy}{a+y^2} = dt \Leftrightarrow -\frac{1}{\sqrt{-a}} \int \frac{1}{1-x^2} dx = t + k \Leftrightarrow -\frac{1}{\sqrt{-a}} \tanh^{-1}(x) = t + k.$$

Then

$$y(t) = -\sqrt{-a}\left(\tanh\left(\sqrt{-a}(t+k)\right)\right).$$

4.B Solution for a general Bernoulli equation

Consider the Bernoulli equation

$$\dot{y} = ay + by^{\eta},\ a \neq 0,\ b \neq 0 \tag{4.24}$$

where $y : \mathcal{T} \to \mathbb{R}$. We intoduce a first transformation $z(t) = y(t)^{1-\eta}$, which leads to a linear ODE

$$\dot{z} = (1-\eta)(az+b) \tag{4.25}$$

beause

$$\begin{aligned} \dot{z} &= (1-\eta)y^{-\eta}\dot{y} = \\ &= (1-\eta)(ay^{1-\eta}+b) = \\ &= (1-\eta)(az+b). \end{aligned}$$

To solve equation (4.25) we introduce a second transformation $w(t) = z(t) + \frac{b}{a}$. Observing that $\dot{w} = \dot{z}$ we obtain a homogeneous ODE $\dot{w} = a(1-\eta)w$ which has solution

$$w(t) = k_w e^{a(1-\eta)t}.$$

Then the solution to equation (4.25) is

$$z(t) = -\frac{b}{a} + (k_z + \frac{b}{a})e^{a(1-\eta)t}$$

because $k_w = k_z + \frac{b}{a}$.

We finally get the solution for the Bernoulli equation (4.24)

$$y(t) = \left(-\frac{b}{a} + (k^{1-\eta} + \frac{b}{a})e^{a(1-\eta)t}\right)^{\frac{1}{1-\eta}} \tag{4.26}$$

4.C Solution to the cubic polynomial equation

Consider the (monic) cubic polynomial equation

$$y^3 - by - a = 0 \tag{4.27}$$

Write $y = u + v$. Then we get the equivalent representation

$$u^3 + v^3 + 3\left(uv - \frac{b}{3}\right) - a = 0$$

which holds if u and v solve simultaneously

$$\begin{cases} u^3 + v^3 = a \\ uv = \dfrac{b}{3} \end{cases} \Leftrightarrow \begin{cases} u^3u^3 + u^3v^3 - u^3a = 0 \\ u^3v^3 = \left(\dfrac{b}{3}\right)^3 \end{cases} \Leftrightarrow \begin{cases} u^6 - au^3 + \left(\dfrac{b}{3}\right)^3 = 0 \\ uv = \frac{b}{3}. \end{cases}$$

The first equation is a quadratic polynomial in u^3 which has roots

$$u^3 = \frac{a}{2} \pm \sqrt{\Delta}, \text{ where } \Delta \equiv \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3$$

where Δ is the discriminant in equation (4.7). We can take any solution of the previous equation and set $\theta \equiv \frac{a}{2} + \sqrt{\Delta}$.

At this stage it is useful to observe that the solutions of equation $x^3 = 1$ are

$$x_1 = 1,\ x_2 = \omega,\ x_3 = \omega^2.$$

where $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2}(1 - \sqrt{3}i)$ and $\omega^2 = e^{\frac{4\pi i}{3}} = -\frac{1}{2}(1 + \sqrt{3}i)$. Therefore $u^3 = \theta$ has also three solutions

$$u_1 = \theta^{\frac{1}{3}},\ u_2 = \omega\theta^{\frac{1}{3}},\ u_3 = \omega^2\theta^{\frac{1}{3}}$$

and because $v = \frac{b}{3u}$, we finally get the solutions to equation (4.27) are

$$y_j = w^{j-1}\theta^{\frac{1}{3}} + \frac{b}{3}\left(\omega^{j-1}\theta^{\frac{1}{3}}\right)^{-1},\ j = 1, 2, 3.$$

Chapter 5

Piecewise-smooth ODE

5.1 Introduction

Up until now we considered ODEs whose vector field $\mathbf{f}(y)$ was smooth, i.e, continuously differentiable.

Piecewise-smooth (PWS) dynamic systems are defined by vector fields that are not differentiable or are not continuous at some points. This implies that their solutions display some kind of discontinuity or some types of dynamics that do not exist in smooth ODEs. There are different types of discontinuities and the dynamic systems can be modelled by ODE, maps, DAE's, PDE's. In the next text we consider the case of ODEs.

PWS have been developed in several areas in applied mathematics, control theory, engineering, biology, in the last 20 years and were used to describe phenomena of friction, impact, switching, sliding, grazing.

Potential or existing application is economics are:

- for modelling endogenous dynamic regime switches;
- for solving dynamic optimization problems with side constraints;
- for characterizing solutions in dynamic games, possibly;
- for avoiding introducing ad-hoc non-linear functions in non-linear growth models;
- as a way of approximating non-linear dynamics through simpler, almost linear functions

Although "... the subject is huge that we cannot classify everything..." di Bernardo et al. (2008a) we present next the bare minimum to understand these systems. However, the approach we present next is not usually dealt with in economics textbooks.

5.1.1 A general overview

What are piecewise smooth dynamic systems ? How do their solutions differ from smooth DS ?

General definition:

1. Consider a domain $\mathcal{Y} \subseteq \mathbf{R}^n$ which is partitioned into a finite number of open sets $\mathcal{Y}_j,\ j = 1, \ldots, m$ such that
$$\mathcal{Y} = \cup_{j=1}^{m} \mathcal{Y}_j;$$

2. consider a dynamic system, represented by a differential equation
$$\dot{\mathbf{y}} = F(\mathbf{y})$$
where $\mathbf{y} : \mathcal{Y} \to \mathbf{R}_+$ is continuous or not of time $\mathbf{y}(t)$; such that
$$\mathbf{F}(\mathbf{y}) = \begin{cases} \mathbf{F}_1(\mathbf{y}) & \text{if } \mathbf{y} \in \mathcal{Y}_1 \\ \ldots & \\ \mathbf{F}_j(\mathbf{y}) & \text{if } \mathbf{y} \in \mathcal{Y}_j \\ \ldots & \\ \mathbf{F}_m(\mathbf{y}) & \text{if } \mathbf{y} \in \mathcal{Y}_m \end{cases}$$
where $\mathbf{F}_j,\ j = 1, \ldots, m$ are smooth functions.

We define the **switching boundaries** as intersection of the closure of two (or more) adjacent sets
$$\Sigma_{ij} = \overline{\mathcal{Y}}_i \cap \overline{\mathcal{Y}}_j,\ i \neq j = 1, \ldots, m.$$
Most of times, Σ_{ij} are defined by $n-1$-dimensional manifolds
$$\Sigma_{ij} = \{\mathbf{y} : H_{ij}(\mathbf{y}) = 0\},\ i \neq j = 1, \ldots, m$$
where every H_{ij} is smooth.

Specificities of PWS systems:

- If, given an initial point, $\mathbf{y}(0) = \mathbf{y}_0$, belongs to a particular area $\mathcal{Y}_j$, the solution trajectory lies inside the same set, and **does not collide** with any Σ_{ij}, then the solution behaves as in a smooth system as the ones studied in the previous chapters.

- However, if the solution trajectory **collides** with any particular Σ_{ij}, then several forms of interaction can occur, some of them giving birth to **new dynamic phenomena**: crossing, sliding, grazing, with single or multiple interactions;

- Then, some **new types of invariant sets** and associated local and global bifurcations may arise, which can be generally denominated as **discontinuity induced bifurcations**.

From now on we consider **the two-zone case**, i.e., we assume that $\mathcal{Y}$ is composed of only two sets, or consider a particular area comprising only two subsets, and a switching boundary between them.

That is assume that

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}) = \begin{cases} \mathbf{F}_1(\mathbf{y}), & \text{if } \mathbf{y} \in \mathcal{Y}_1 \\ \mathbf{F}_2(\mathbf{y}), & \text{if } \mathbf{y} \in \mathcal{Y}_2 \end{cases}$$

where the two zones and the switching boundary are implicitly defined by

$$\begin{aligned} \mathcal{Y}_1 &= \{\mathbf{y} : H(\mathbf{y}) \leq \mathbf{0}\} \\ \mathcal{Y}_2 &= \{\mathbf{y} : H(\mathbf{y}) > \mathbf{0}\} \\ \Sigma &= \{\mathbf{y} : H(\mathbf{y}) = 0\} \end{aligned}$$

where $D_{\mathbf{y}}H(\mathbf{y}) \neq \mathbf{0}$ for all $y \in \Sigma$.

Next we address the scalar non-smooth ODE (linear and non-linear), in section 5.2, next we study linear systems in section **??** and, at last, we present the general non-linear case, in section 5.3

5.2 Scalar equations

In this section we consider the non-smooth ODE, where y is an unknown function and φ is a known vector of parameters,

$$\dot{y} = \begin{cases} f_1(y,\varphi), & \text{if } h(y,\varphi) \leq 0 \\ f_2(y,\varphi), & \text{if } h(y,\varphi) > 0 \end{cases} \tag{5.1}$$

where $y : \mathbb{R}_+ \to \mathcal{Y} \subseteq \mathbb{R}$ is a not necessarily continuous function, $\varphi \in \Phi \subseteq \mathbb{R}^p$, $p \geq 1$, and $f_j : \mathcal{Y} \times \phi \to \mathbb{R}$ and $h : \mathcal{Y} \times \Phi \to \mathbb{R}$ are smooth functions. The switching boundary is

$$\Sigma = \{y : h(y,\varphi) = 0\}$$

where we assume that $h_y(y,\varphi) \neq 0$ for every pair $(y,\varphi) \in \mathcal{Y} \times \phi$.

We start with the linear case and then make some observations on the general, non-linear case.

5.2.1 The scalar linear case

We assume the special case of equation (5.1) where both the branch functions and the switching function are linear

$$f_j(y) = \lambda_j y + m_j,\ j = 1, 2 \tag{5.2}$$
$$h(y) = cy + n,\ c \neq 0 \tag{5.3}$$

that is

$$\dot{y} = f(y) = \begin{cases} \lambda_1 y + m_1, & \text{if } cy + n \leq 0 \\ \lambda_2 y + m_2, & \text{if } cy + n > 0. \end{cases}$$

Because of the assumption $c \neq 0$ then **switching boundary**

$$\Sigma = \{y \in \mathcal{Y} : cy + n = 0\} = \{\tilde{y}\}$$

has just one element $\tilde{y} = -n/c$ (i.e. a point in $\mathcal{Y}$). Therefore

$$\mathcal{Y}_1 = \{y \in \mathcal{Y} :\ cy + n \leq 0\},\ \mathcal{Y}_2 = \{y \in \mathcal{Y} :\ cy + n > 0\}.$$

Solutions

We should observe that although the differential equation for each zone is linear, the equation $\dot{y} = f(y)$ behaves as a non-linear system. However, equation (5.1) has explicit solutions.

General solutions The general solution for the equation in branch j is

$$\phi_j(k, t) = \begin{cases} \bar{y}_j + (k_j - \bar{y}_j)e^{\lambda_j t}, & \text{if } \lambda_j \neq 0 \\ k_j + m_j t & \text{if } \lambda_j = 0 \end{cases}$$

where $\bar{y}_j = -m_j/\lambda_j$ and $k_j \in \mathcal{Y}$ is an arbitrary element of the state space.

Hitting time Define the *hitting* time

$$\tilde{t} = \{t \geq 0 : \phi_j(k, t) = \tilde{y},\ k \in \mathcal{Y}_j\}$$

as the time of contact of a solution starting in branch j with the switching boundary Σ.

Solutions for initial value problems In an initial value problem let $y(0) = y_0 \in \mathcal{Y}_j$ be given.

Next we consider the case in which we have an initial point in branch 1, $y_0 \in \mathcal{Y}_1$, then several cases can occur:

1. If $\tilde{t} = \{t : \phi_1(y_0, t) = \tilde{y}\} < 0$, then the solution will remain in the branch $\mathcal{Y}_1$ behaves like the solution for a smooth ODE equation. Two cases are possible :

   (a) the solution is bounded and tends asymptotically to a steady state $\bar{y}_1 \in \mathcal{Y}_1$. In this case we say there is an **asymptotically stable regular equilibrium point** in branch $\mathcal{Y}_1$;

   (b) or the solution tends to $\pm\infty$ asymptotically. This case occurs in the following situations: if either there is no regular equilibrium point in branch $\mathcal{Y}_1$, or if there is a **unstable regular equilibrium point** and $\bar{y}_1 \neq y_0$, or if there is a **pseudo-equilibrium point** belonging to Σ which is unstable (we will see the next).

2. If $\lim_{t\to\infty} \phi_1(y(0), t) = \tilde{y}$, that is, the solution converges **asymptotically** to Σ, or $\tilde{t} = \infty$, in this case the solution converges to a **boundary equilibrium point**. It behaves also in a similar way as the solution for a smooth ODE equation.

3. If $\tilde{t} = \{t : \phi_1(y_0, t) = \tilde{y}\} > 0$ and is finite, then the solution **collides in finite time** with Σ in **finite** time. Two types of behavior may occur from that point on:

   (a) either the solution stops at Σ, in which case we say that there is a stable **pseudo-equilibrium**

   $$\phi(y_0, t) = \begin{cases} \phi_1(y_0, t), & \text{if } 0 \leq t < \tilde{t} \\ \tilde{y}, & \text{if } t \geq \tilde{t} \end{cases}$$

   (b) or the solution continues into the branch $\mathcal{Y}_2$,

   $$\phi(y_0, t) = \begin{cases} \phi_1(y(0), t), & \text{if } 0 \leq t < \tilde{t} \\ \tilde{y}, & \text{if } t = \tilde{t} \\ \phi_2(\tilde{y}, t - \tilde{t}), & \text{if } t > \tilde{t} \end{cases}$$

   In this case we could express the solution as $\phi_1(y_0, t) \circ \phi_2(\tilde{y}, t - \tilde{t})$. After hitting Σ, with speed

   $$\frac{d\phi_j(y_0, t)}{dt} \mid_{t=\tilde{t}} = \lambda_j (y_0 - y_j) e^{\lambda_j \tilde{t}}$$

   the solution can have two types of continuations: (a) either it converges to a steady state $\bar{y}_2 \in \mathcal{Y}_2$, or (b) will be unbounded in $\mathcal{Y}_2$.

Terminal value problems In economics, we are interested in the solutions of boundary-value problems. This is the case when we have a transversality condition. In this case, a similar reasoning can be made. There are two issues here: are solutions unique or not ? what is the behavior as regards the boundary (will it be crossed or not) ?

Classification of scalar non-smooth ODEs

Next we will see how we can qualitatively relate those different types of solutions to the parameters of the initial piecewise-smooth ODE and to two types of non-smooth equations, or canonical forms: piecewise-smooth continuous **PWSC** or **Filippov** equations.

Proposition 1 (**Canonical forms of non-smooth ODEs**). *Consider equation* (5.1) *where functions* $f_j(.)$ *and* $h(.)$ *take the form* (5.2) *and* (5.3) *and let* $c \neq 0$. *Then the following two cases are possible:*

1. *If* $(\lambda_1 - \lambda_2)n = c(m_1 - m_2)$ *and* $\lambda_1 \neq \lambda_2$ *then equation* (5.1) *is equivalent to the* ***piecewise-smooth continuous*** *(PWSC) equation*

$$\dot{y} = \begin{cases} \lambda_1 y + \varphi, & cy \leq 0 \\ \lambda_2 y + \varphi, & cy > 0 \end{cases} \tag{5.4}$$

   where $y - \tilde{y} \mapsto y$, *and* $\varphi = (m_1 c - \lambda_1 n)/c = (m_2 c - \lambda_2 n)/c$;

2. *If* $(\lambda_1 - \lambda_2)n \neq c(m_1 - m_2)$ *then equation* (5.1) *is equivalent to the* ***Filippov*** *equation*

$$\dot{y} = \begin{cases} \lambda_1 y + \varphi_1 & \text{if } cy \leq 0 \\ \lambda_2 y + \varphi_2 & \text{if } cy > 0 \end{cases} \tag{5.5}$$

   where $\varphi_1 \equiv (m_1 c - \lambda_1 n)/c$, *and* $\varphi_2 \equiv (m_2 c - \lambda_2 n)/c$

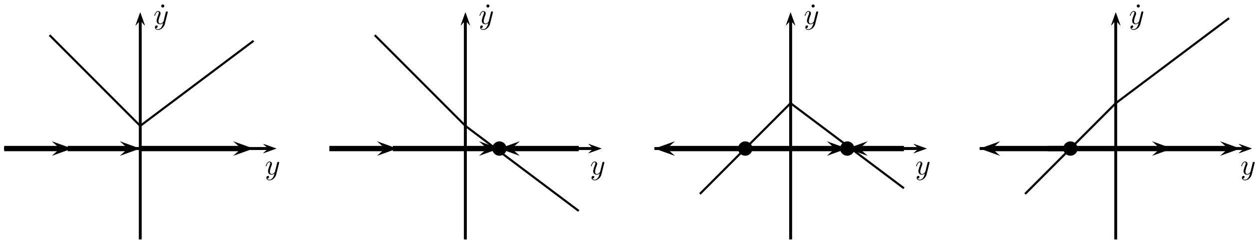
Proof. If we apply $y - \tilde{y} \mapsto y$. The rest of the proof is obvious. □

Next we characterize the solutions for PWSC and Filippov equations by determining the number of steady states, their stability properties and perform a bifurcation analysis, i.e, describe the types of phase diagrams that can occur when the parameters λ_1, λ_2, c and φ or φ_1 and φ_2 can take different values.

5.2.2 The linear scalar PWSC equation

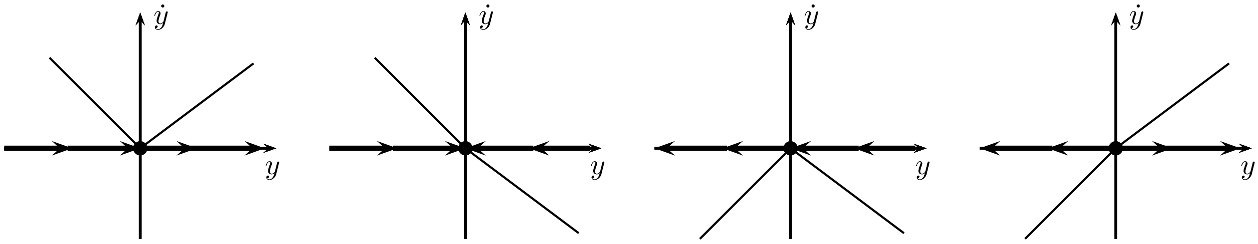
The PWSC equation has the following property: $f_1(\tilde{y}) = f_2(\tilde{y})$ but $f_{1,y}(\tilde{y}) \neq f_{2,y}(\tilde{y})$ [1] that is function $f(y)$ is **continuous but not differentiable** at the boundary Σ.

[1] When there is no room for confusion we denote $f_{j,y} = \frac{\partial f_j(y)}{\partial y}$.



$\lambda_1 < 0 < \lambda_2$ $\quad$ $\lambda_1 \leq 0,\ \lambda_2 < 0$ $\quad$ $\lambda_1 > 0 > \lambda_2$ $\quad$ $\lambda_1 > 0,\ \lambda_2 \geq 0$

Figure 5.1: PWSC equation for $\varphi > 0$: in all cases we have crossing trajectories. In the first case there is crossing and unbounded solutions, in the second there is crossing if $y_0 \in \mathcal{Y}_1$ and no crossing if $y_0 \in \mathcal{Y}_2$, in the third case there are two steady states and there is crossing only if $\bar{y}_1 < y_0 < 0$, and in the last case the crossing condition is the same.



$\lambda_1 \leq 0 \leq \lambda_2$ $\quad$ $\lambda_1 < 0,\ \lambda_2 < 0$ $\quad$ $\lambda_1 > 0 > \lambda_2$ $\quad$ $\lambda_1 > 0,\ \lambda_2 > 0$

Figure 5.2: PWSC equation for $\varphi = 0$: this is a degenerate case in which there are boundary equilibria bifurcations, i,e. $\bar{y} = \tilde{y} = 0$. We can have persistence (second and fourth cases) or a BEB (first and third cases).

For the scalar equation (5.4), as $c \neq 0$ then $\tilde{y} = 0$ and

$$\begin{cases} f_1(\tilde{y}) = f_2(\tilde{y}) & \text{because } \lambda_1\tilde{y} + \varphi = \lambda_2\tilde{y} + \varphi = 0 \\ f_{1,y}(\tilde{y}) \neq f_{2,y}(\tilde{y}) & \text{because } \lambda_1 \neq \lambda_2. \end{cases}$$

All the possible phase diagrams are shown in figures 5.1, 5.2 and 5.3. In the first two cases the steady states, when they exist, are **regular**, i.e, $\bar{y} \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ and in the case 5.2 they we have **boundary equilibrium bifurcations**, i.e, $\bar{y} = \tilde{y} = 0 \in \Sigma$.

Equilibrium points, stability and local bifurcations

For equation (5.4) steady states, or equilibrium points, are the only invariants. Differently from

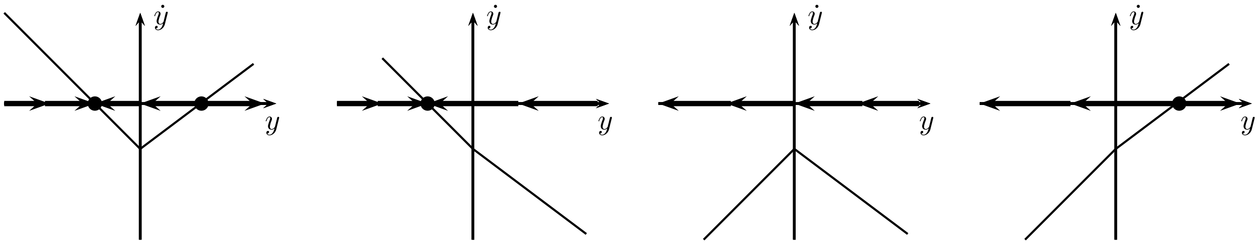


Figure 5.3: PWSC equation for $\varphi < 0$. This case is similar to the cases in Figure 5.1 but with orbits running in the opposite direction.

smooth ODE's we can have multiplicity of steady states.

We have **regular equilibria** when $\varphi \neq 0$, that is, steady states are located in the interior of $\mathcal{Y}_1$, $\mathcal{Y}_2$ or both, or steady states may not exist. We have the following cases:

1. there are two regular equilibria, $\bar{y}_1 < 0$ and $\bar{y}_2 < 0$ if

$$\begin{cases} \varphi > 0 \text{ and } \lambda_1 > 0 > \lambda_2 : \bar{y}_1 \text{ unstable, } \bar{y}_2 \text{ stable} \\ \varphi < 0 \text{ and } \lambda_1 < 0 < \lambda_2 : \bar{y}_1 \text{stable, } \bar{y}_2 \text{ unstable} \end{cases}$$

2. there is an unique regular equilibria if

$$\begin{cases} \varphi > 0 \text{ and } \lambda_1 \leq 0, \lambda_2 < 0 : \ \bar{y}_2 \text{ stable} \\ \varphi > 0\ \lambda_1 > 0, \lambda_2 \geq 0 : \ \bar{y}_1 \text{ unstable} \\ \varphi < 0 \text{ and } \lambda_1 < 0, \lambda_2 \leq 0 : \ \bar{y}_1 \text{ stable} \\ \varphi < 0\ \lambda_1 \geq 0, \lambda_2 > 0 : \ \bar{y}_2 \text{ unstable} \end{cases}$$

3. there is an infinite number of regular equilibria if

$$\varphi = 0 \text{ and } \lambda_1 = 0 \text{ or (exclusive) } \lambda_2 = 0$$

4. there are no regular equilibria if

$$\varphi > 0 \text{ and } \lambda_1 < 0 < \lambda_2$$

$$\varphi < 0 \text{ and } \lambda_1 \geq 0,\ \lambda_2 \geq 0$$

We say we have a **boundary equilibria** if $\bar{y} = 0 \in \Sigma$. This case occurs only if $\varphi = 0$.

There is a (co-dim 1) **boundary equilibrium bifurcation** at $y = 0$ for

$$\varphi = 0 \text{ and } \lambda_1 \neq 0 \text{ and } \lambda_2 \neq 0$$

1. there is **persistence** (or border-crossing) if $\varphi = 0$ and $\lambda_1 < 0$ and $\lambda_2 < 0$ or $\lambda_1 > 0$ and $\lambda_2 > 0$. There is
$$\begin{cases} \lambda_1 < 0,\ \lambda_2 < 0: \text{ stable-stable transition} \\ \lambda_1 > 0,\ \lambda_2 > 0: \text{ unstable-unstable transition} \end{cases}$$

2. there is a **non-smooth fold** if $\varphi = 0$ and $\lambda_1 < 0 < \lambda_2$ or $\lambda_1 > 0 > \lambda_2$

We can have an

Proposition 2. *Unfolding of the BEB*
Consider a BEB point $(y^*, \varphi^*) = (0, 0)$, for a small perturbation of the parameter φ, there is:

1. *persistence if* $\lambda_1 \lambda_2 > 0$
2. *non-smooth fold if* $\lambda_1 \lambda_2 < 0$

Leine's approach: differential inclusions There is an alternative approach (see Leine and van de Wouw (2008)) that allows for a compact alternative way of finding which take of boundary-equilibrium bifurcation exists. We define a generalized vector field by

$$\partial \mathcal{F}(y) = \begin{cases} \lambda_1, & \text{if } y < 0 \\ \{\alpha\lambda_1 + (1-\alpha)\lambda_2 : \ 0 \le \alpha \le 1\}, & \text{if } y = 0 \\ \lambda_2 & \text{if } y < 0 \end{cases}$$

where $\partial\mathcal{F}(0)$ is a weighted average of the coefficients of the two branches with weights α and $1-\alpha$ summing to one.

Proposition 3. *Unfolding of the BEB*

1. *persistence if* $0 \notin \partial\mathcal{F}(0)$
2. *non-smooth fold if* $0 \in \partial\mathcal{F}(0)$

That is, there is a non-smooth fold if there is one $\alpha \in (0, 1)$ such that $\alpha\lambda_1 + (1-\alpha)\lambda_2 = 0$.
We can obtain a generic condition for **local stability**:

Proposition 4. *If there is at least one steady state, $\bar{y}_j$ it is asymptotically stable if $\lambda_j < 0$*

5.2.3 The linear scalar Filippov equation

The Filippov equation (5.5) has the following property: $f_1(\tilde{y}) \neq f_2(\tilde{y})$ and, in general, also $f_{1,y}(\tilde{y}) \neq f_{2,y}(\tilde{y})$ that is function $f(y)$ is **neither continuous nor differentiable** at the boundary Σ.

For the scalar equation (5.5), because $c \neq 0$, then $\tilde{y} = 0$ and

$$\begin{cases} f_1(\tilde{y}) \neq f_2(\tilde{y}) & \text{because } \lambda_1 \tilde{y} + \varphi_1 \neq \lambda_2 \tilde{y} + \varphi_2 \\ f_{1,y}(\tilde{y}) \neq f_{2,y}(\tilde{y}) & \text{because } \lambda_1 \neq \lambda_2. \end{cases}$$

From those assumptions we obtain dynamics which do not occur in PWSC systems. In particular, there are pseudo-equilibria.

A **pseudo-equilibrium** is a point $\tilde{y} \in \Sigma$ which is not a boundary equilibrium for both branches of function $f(y)$ but it is a steady state of the system. It satisfies the following conditions: (1) $f_1(\tilde{y}) \neq 0$ and $f_2(\tilde{y}) \neq 0$; (2) there is a convex combination of $f_1(y)$ and $f_2(y)$, $f_s(y) = (1-\eta)f_1(y) + \eta f_2(y)$ with $0 < \eta < 1$, [2] such that $f_s(\tilde{y}) = 0$. Then a pseudo-equilibrium is a kind of "hidden" equilibrium point in Σ.

As with a regular equilibrium point, a pseudo-equilibrium point can also be stable or unstable.

The number of phase diagrams is now much larger than for the PWSC equation as shown in figures 5.4 to 5.11. Looking at those phase diagrams we can see find:

1. the number regular steady states: there are no regular steady states, there is a single steady state or two steady states (as in the PWSC system with $\varphi \neq 0$) if $\text{sign}(\varphi_1) = \text{sign}(\varphi_2)$ and $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$ (see figures 5.4 and 5.11);

2. if boundary equilibria exist: they can be isolated, as in the PWSC case with $\varphi = 0$, or can coexist with another regular equilibrium, differently from the PWSC case with $\varphi = 0$, if there is one φ_j that is equal to zero (see figures 5.5, 5.7, 5.8 and 5.10)

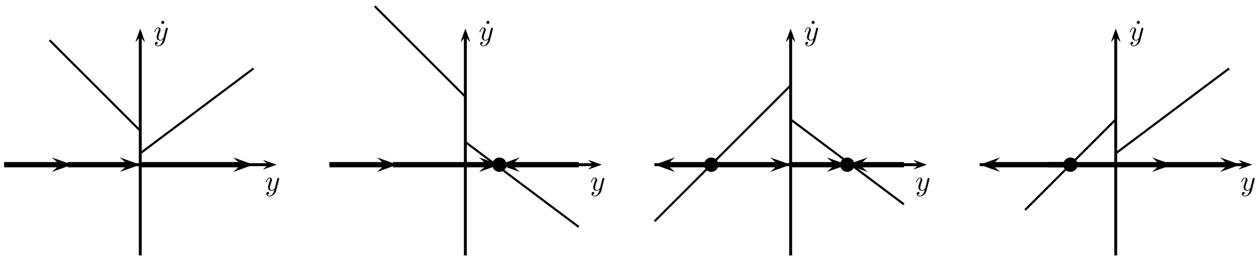
3. there is a pseudo-equilibrium, isolated or jointly with a regular equilibrium, if $\text{sign}(\varphi_1) \neq \text{sign}(\varphi_2)$ (see figures 5.6 and 5.9). This case cannot occur with PWSC equations.

Equilibria and stability for the case in which $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$.

We have the following cases

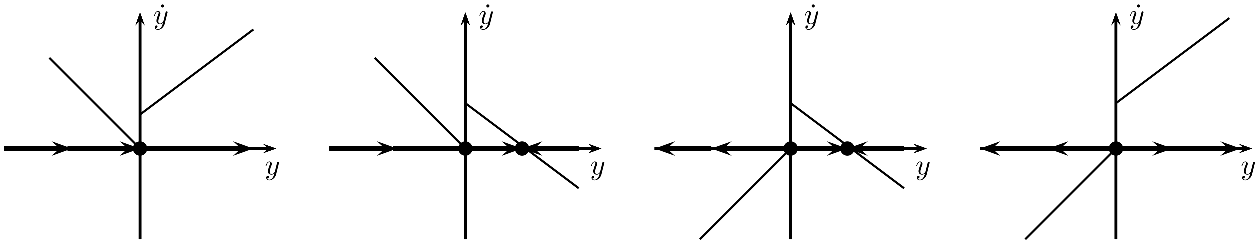
1. Regular equilibria

[2] Using the results in section 5.3 we find the weight $\eta = \dfrac{\varphi_1}{\varphi_1 - \varphi_2}$.



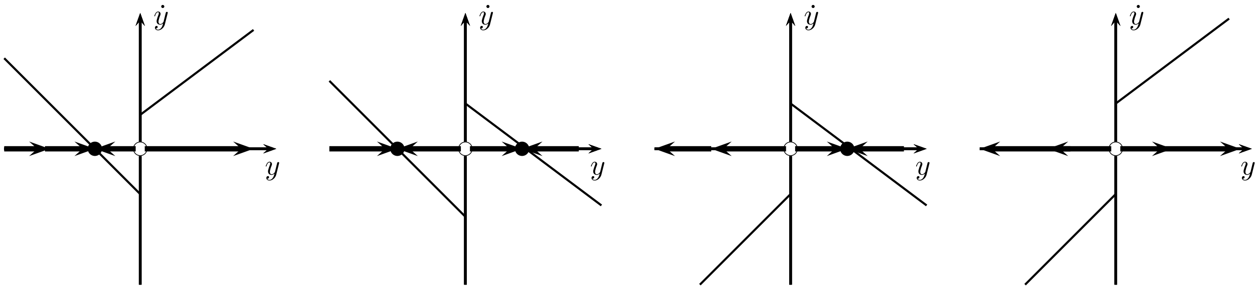
$\lambda_1 \leq 0 \leq \lambda_2$ $\lambda_1 \leq 0,\ \lambda_2 < 0$ $\lambda_1 > 0 > \lambda_2$ $\lambda_1 > 0,\ \lambda_2 \geq 0$

Figure 5.4: Case 1: $\varphi_1 > 0$ $\varphi_2 > 0$



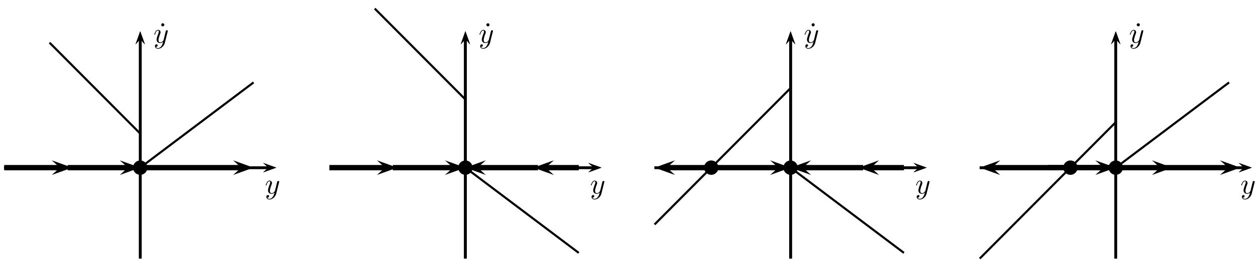
$\lambda_1 < 0 \leq \lambda_2$ $\lambda_1 < 0,\ \lambda_2 < 0$ $\lambda_1 > 0 > \lambda_2$ $\lambda_1 > 0,\ \lambda_2 \geq 0$

Figure 5.5: Case 2: $\varphi_1 = 0$ $\varphi_2 > 0$



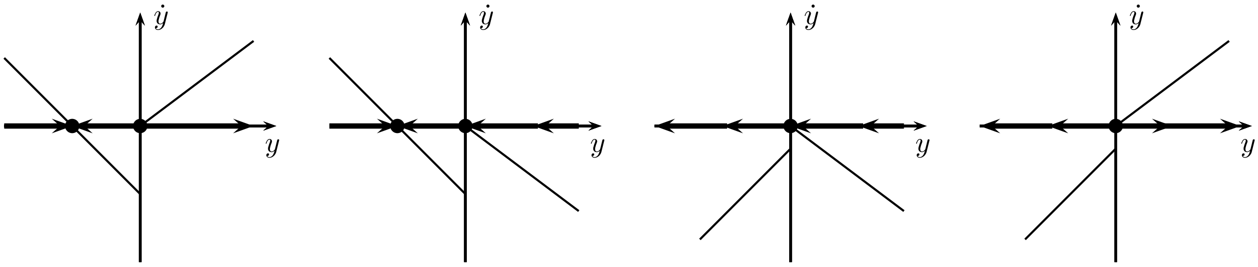
$\lambda_1 < 0 \leq \lambda_2$ $\lambda_1 < 0,\ \lambda_2 < 0$ $\lambda_1 > 0 > \lambda_2$ $\lambda_1 > 0,\ \lambda_2 \geq 0$

Figure 5.6: Case 3: $\varphi_1 < 0$ $\varphi_2 > 0$ existence of pseudo-equilibria



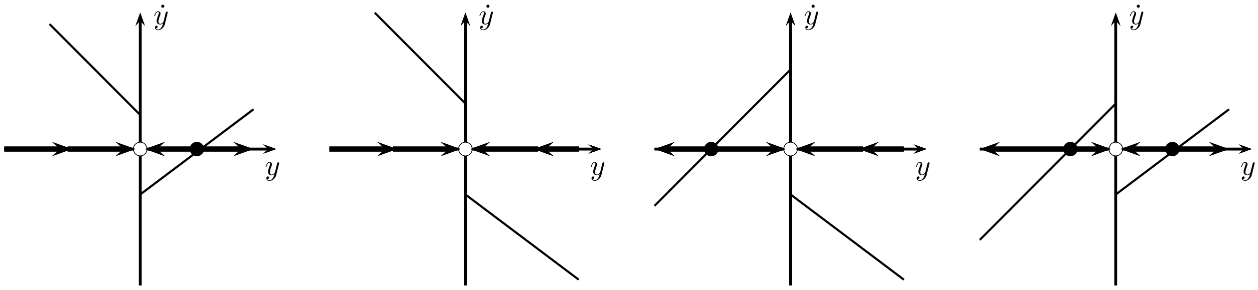
$\lambda_1 \leq 0 < \lambda_2$ $\lambda_1 \leq 0,\ \lambda_2 < 0$ $\lambda_1 > 0 > \lambda_2$ $\lambda_1 > 0,\ \lambda_2 \geq 0$

Figure 5.7: Case 4: $\varphi_1 > 0$, $\varphi_2 = 0$



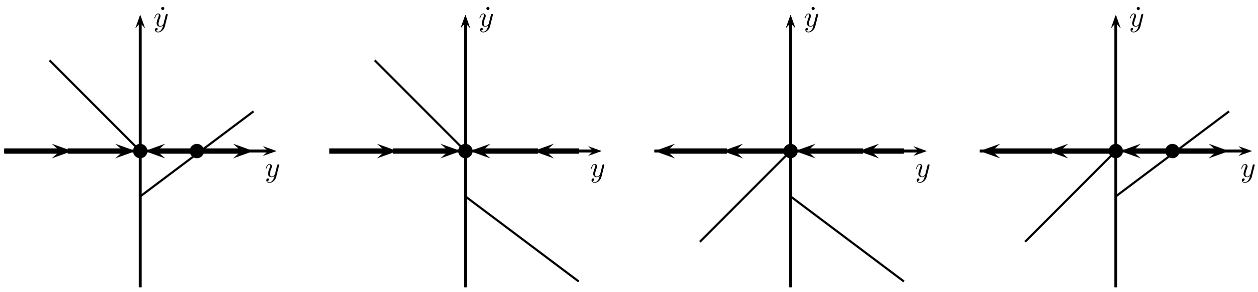
$\lambda_1 < 0 < \lambda_2$ $\quad$ $\lambda_1 < 0,\ \lambda_2 < 0$ $\quad$ $\lambda_1 \geq 0 > \lambda_2$ $\quad$ $\lambda_1 > 0,\ \lambda_2 \geq 0$

Figure 5.8: Case 5: $\varphi_1 < 0$, $\varphi_2 = 0$ Boundary equilibrium bifurcations



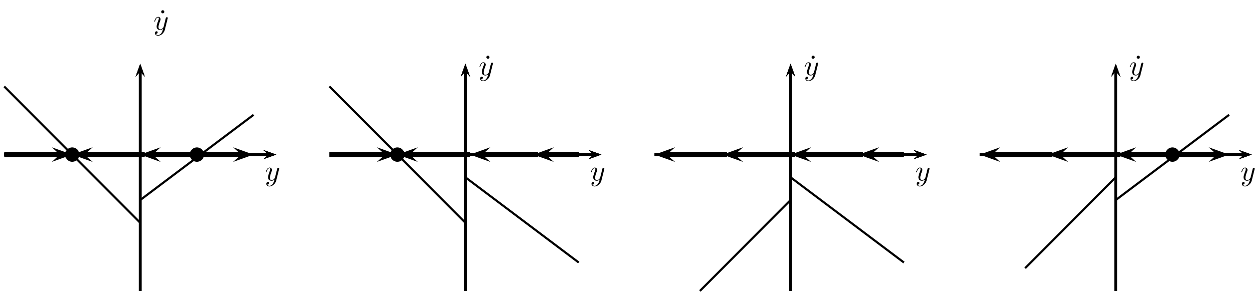
$\lambda_1 \leq 0 < \lambda_2$ $\quad$ $\lambda_1 \leq 0,\ \lambda_2 \leq 0$ $\quad$ $\lambda_1 > 0 \geq \lambda_2$ $\quad$ $\lambda_1 > 0,\ \lambda_2 > 0$

Figure 5.9: Case 6: $\varphi_1 > 0$, $\varphi_2 < 0$ pseudo-equilibria



$\lambda_1 < 0 < \lambda_2$ $\quad$ $\lambda_1 < 0,\ \lambda_2 \leq 0$ $\quad$ $\lambda_1 > 0 \geq \lambda_2$ $\quad$ $\lambda_1 > 0,\ \lambda_2 > 0$

Figure 5.10: Case 7: $\varphi_1 = 0$, $\varphi_2 < 0$



$\lambda_1 < 0 < \lambda_2$ $\quad$ $\lambda_1 < 0,\ \lambda_2 \leq 0$ $\quad$ $\lambda_1 \geq 0 \geq \lambda_2$ $\quad$ $\lambda_1 \geq 0,\ \lambda_2 > 0$

Figure 5.11: Case 8: $\varphi_1 < 0$, $\varphi_2 < 0$

(a) Two regular equilibria: $\bar{y}_1 < 0$ and $\bar{y}_2 > 0$;

$$\begin{cases} \varphi_1 > 0, \varphi_2 > 0, \lambda_1 > 0 > \lambda_2 : \ \bar{y}_1 \text{ unstable } \bar{y}_2 \text{ stable} \\ \varphi_1 < 0, \varphi_2 < 0, \lambda_1 < 0 < \lambda_2 : \ \bar{y}_1 \text{ stable } \bar{y}_2 \text{ unstable} \end{cases}$$

(b) One regular equilibria: $\bar{y}_1 < 0$ or $\bar{y}_2 > 0$;

$$\begin{cases} \varphi_1 > 0, \varphi_2 > 0, \lambda_1 \leq 0, \lambda_2 < 0 : \ \bar{y}_2 \text{ stable} \\ \varphi_1 > 0, \varphi_2 > 0, \lambda_1 > 0, \lambda_2 \geq 0 : \ \bar{y}_1 \text{ unstable} \\ \varphi_1 < 0, \varphi_2 < 0, \lambda_1 < 0, \lambda_2 \leq 0 : \ \bar{y}_1 \text{ stable} \\ \varphi_1 < 0, \varphi_2 < 0, \lambda_1 \geq 0, \lambda_2 > 0 : \ \bar{y}_2 \text{ unstable} \end{cases}$$

2. Regular and pseudo-equilibria

(a) Two regular equilibria, $\bar{y}_1 < 0$ and $\bar{y}_2 > 0$ and one pseudo-equilibrium $\tilde{y} = 0$

$$\begin{cases} \varphi_1 < 0, \varphi_2 > 0, \lambda_1 < 0, \lambda_2 < 0 : \ \bar{y}_1 \text{ stable } \bar{y}_2 \text{ stable, } \tilde{y} \text{ unstable,} \\ \varphi_1 > 0, \varphi_2 < 0, \lambda_1 > 0, \lambda_2 > 0 : \ \bar{y}_1 \text{ unstable } \bar{y}_2 \text{ unstable, } \tilde{y} \text{ stable} \end{cases}$$

(b) One regular equilibria, $\bar{y}_1 < 0$ or $\bar{y}_2 > 0$ and one pseudo-equilibrium $\tilde{y} = 0$

$$\begin{cases} \varphi_1 < 0, \varphi_2 > 0, \lambda_1 > 0 > \lambda_2 : \ \bar{y}_2 \text{ stable, } \tilde{y} \text{ unstable,} \\ \varphi_1 < 0, \varphi_2 > 0, \lambda_1 < 0 < \lambda_2 : \ \bar{y}_1 \text{ stable, } \tilde{y} \text{ unstable,} \\ \varphi_1 > 0, \varphi_2 < 0, \lambda_1 \leq 0 < \lambda_2 : \ \bar{y}_2 \text{ unstable } \tilde{y} \text{ stable,} \\ \varphi_1 > 0, \varphi_2 < 0, \lambda_1 > 0 \geq \lambda_2 : \ \bar{y}_1 \text{ unstable } \tilde{y} \text{ stable,} \end{cases}$$

We have **Boundary equilibrium bifurcations** when at least one $\varphi_1 = 0$ or $\varphi_2 = 0$ [3]

There are 16 different phase diagrams (in figures 5.5, 5.7, 5.9 and 5.10) displaying bifurcations of several types :

1. persistence between regular equilibria and pseudo-equilibria:

   (a) from one regular equilibrium to one pseudo-equilibrium : v.g., $\varphi_1 = 0$, $\varphi_2 > 0$; $\lambda_1 > 0$, $\lambda_2 \geq 0$

[3] When $\varphi_1 = \varphi_2 = 0$ we have a PWSC case in which $\varphi = 0$.

(b) from two regular equilibrium to one regular equilibrium and one pseudo-equilibrium: v.g., $\varphi_1 = 0$, $\varphi_2 > 0$; $\lambda_1 > 0 > \lambda_2$

2. non-smooth folds:

   (a) from no regular equilibrium to two equilibria (one regular and one pseudo-equilibrium): v.g., $\varphi_1 = 0$, $\varphi_2 > 0$ and $\lambda_1 < 0 < \lambda_2$

   (b) from one regular equilibrium to three equilibria (two regular and one pseudo-equilibrium): v.g., $\varphi_1 = 0$, $\varphi_2 > 0$; $\lambda_1 < 0$, $\lambda_2 < 0$

5.2.4 The scalar non-linear case

Now consider a non-linear scalar non-smooth ODE

$$\dot{y} = \begin{cases} f_1(y,\varphi), & y \leq 0 \\ f_2(y,\varphi), & y > 0 \end{cases}$$

where $f_1(y,\varphi) = f_2(y,\varphi) = 0$ for $y \in \Sigma$ if we have a PWSC system of $f_1(y,\varphi) \neq f_2(y,\varphi) = 0$ for $y \in \Sigma$ for the Filippov case.

Next we define the set of regular equilibrium points (or steady states) at every sub-range

$$\mathcal{E}_j(\varphi) = \{y \in \mathcal{Y}_j : f_j(y,\varphi) = 0\}.$$

In this case:

1. all that we have said about equilibrium, their number and their stability properties and bifurcations, for the linear case hold, but in addition;
2. If $f_{j,y} = 0$ for $y \in \mathcal{E}_j$ we have a **regular (or smooth) fold bifurcation**;
3. If $f_{j,y} = 0$ for $y \in \Sigma$ we have a **tangent point** ;

An example: existence of a regular fold bifurcation for a Filippov equation

$$\dot{y} = \begin{cases} \lambda_1 y + \varphi_1, & \text{if } y \leq 0 \\ -y^2 + \lambda_2 y + \varphi_2, & \text{if } y > 0 \end{cases}. \tag{5.6}$$

Figure 5.12 displays phase diagrams assuming that $\varphi_1 > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$ for several values of φ_2.

5.3 General two-zone systems

In this section we introduce some concepts from the PWS systems theory for the two-zone case. Let $\mathbf{y} \in \mathcal{Y} \subseteq \mathbb{R}^n$

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}) = \begin{cases} \mathbf{F}_1(\mathbf{y}), & \text{if } \mathbf{y} \in \mathcal{Y}_1 \\ \mathbf{F}_2(\mathbf{y}), & \text{if } \mathbf{y} \in \mathcal{Y}_2 \end{cases}$$

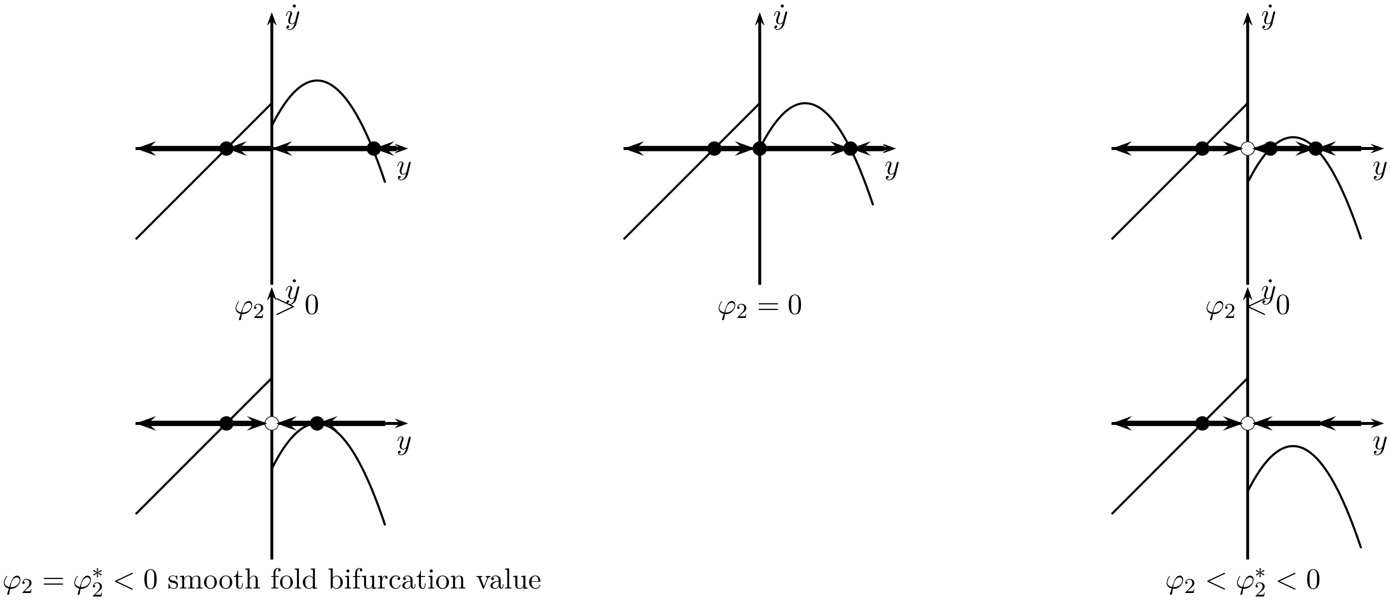


Figure 5.12: Phase diagrams for equation (5.6)

where the two zones and the switching boundary are implicitly defined by

$$\begin{aligned} \mathcal{Y}_1 &= \{\mathbf{y} : \mathbf{H}(\mathbf{y}) \le 0\} \\ \mathcal{Y}_2 &= \{\mathbf{y} : \mathbf{H}(\mathbf{y}) > 0\} \\ \Sigma &= \{\mathbf{y} : \mathbf{H}(\mathbf{y}) = 0\} \end{aligned}$$

where $\det\left(D_{\mathbf{y}}\mathbf{H}(\mathbf{y})\right) \neq 0$ for all $\mathbf{y} \in \Sigma$.

Next we present a qualitative approach to the solutions of the PWS system, called the Filippov's convex method and an introduction to the bifurcation analysis of those systems

5.3.1 Filippov's convex method

As $F(y)$ may not be defined if $y \in \Sigma$ the following formalism of differential inclusions may be used

$$\dot{y} \in \mathcal{F}(y) = \begin{cases} F_1(y), & \text{if } y \in \mathcal{Y}_1 \\ \overline{\text{co}}\{F_1(y), F_2(y)\}, & \text{if } y \in \Sigma \\ F_2(y), & \text{if } y \in \mathcal{Y}_2 \end{cases}$$

where

$$\overline{\text{co}}\{F_1(y), F_2(y)\} = \{(1-\alpha)F_1(y) + \alpha F_2(y),\ 0 \le \alpha \le 1\}$$

is the convex hull containing F_1 and F_2. This is called the *Filippov convex method*, and the solution to $\dot{y} \in \mathcal{F}(y)$ is called the solution in the sense of Filippov.

We can apply the same method as for the scalar case for determining a **generalized Jacobian**

$$\partial\mathcal{F}(y) = \begin{cases} F_{1,y}(y), & \text{if } y \in \mathcal{Y}_1 \\ \overline{\text{co}}\{F_{1,y}(y), D_y F_2(y)\}, & \text{if } y \in \Sigma \\ D_y F_2(y), & \text{if } y \in \mathcal{Y}_2 \end{cases}$$

where

$$\overline{\text{co}}\{F_{1,y}(y), D_y F_2(y)\} = \{(1-\alpha) D_y F_1 + \alpha F_{2,y},\ 0 \le \alpha \le 1\}.$$

If we evaluate the generalized Jacobian at $y \in \Sigma$ we can also determine the associated generalized eigenvalues

$$\{\lambda : \det(\partial\mathcal{F}(y) - \lambda I) = 0, y \in \mathcal{Y}\}.$$

This formalism is sufficiently general to encompass the two types of PWC ODE's, if we assume that y, is a continuous function of time[4]:

1. **PWSC continuous** ODE's where

   $$F_1(y) = F_2(y),\ D_y F_1(y) \neq D_y F_2(y),\ \text{if } y \in \Sigma$$

   then $\mathcal{F}(y)$ is single-valued and $\partial\mathcal{F}(y)$ is set-valued at Σ

2. **Filippov** ODE's where

   $$F_1(y) \neq F_2(y),\ \text{if } y \in \Sigma$$

   $\mathcal{F}(y)$ is set-valued at Σ

Solutions The theory of differential inclusions (see Leine and van de Wouw (2008)) guarantees existence, however, in some cases uniqueness may not hold.

The following types of solutions may occur

1. **standard solutions** if given an initial value $y_0 \in \mathcal{Y}_j$ the solution trajectory lies in the same zone $\mathcal{Y}_j$ the solution can be characterised with the methods already well know for smooth dynamical systems;

2. **non-standard solutions**: the solution collides with Σ then several cases can occur

[4]Tthat is, we are excluding impact of hybrid ODE's

A solution for the PWS is obtained by concatenating the two types of solution if there is a contact with Σ. Next we see which types of contact solution paths can have with the switching boundary.

Dynamic behavior at Σ

If $y \in \Sigma$ consider the projections of the vector fields F_1 and F_2 over the normal of the surface manifold $H(y) = 0$, which are given by the Lie derivatives

$$\mathcal{L}_{F_j}(H(y)) = D_y H(y) F_j(y), \text{ for } y \in \Sigma$$

where $D_y H(\tilde{y})$ is the vector normal to Σ.

Let

$$\sigma(y) \equiv \mathcal{L}_{F_1}(H).\mathcal{L}_{F_2}(H)$$

There are two main types of contact with Σ:

- **transversal crossing** (see figure 5.13): if

$$\mathcal{L}_{F_1}(H).\mathcal{L}_{F_2}(H) > 0, \ y \in \Sigma$$

the flow will exit $\mathcal{Y}_1$ and enter $\mathcal{Y}_2$ if $\mathcal{L}_{F_1}(H) > 0$ and $\mathcal{L}_{F_2}(H) > 0$, and will exit $\mathcal{Y}_2$ and enter $\mathcal{Y}_1$ if $\mathcal{L}_{F_1}(H) < 0$ and $\mathcal{L}_{F_2}(H) < 0$. We determine the **crossing set**, i.e, the subset of Σ such that trajectories starting in one of its neighborhood will cross it,

$$\Sigma_c = \{y \in \Sigma : \sigma(y) > 0\}$$

which is in general an open set;

- **sliding mode**: if

$$\mathcal{L}_{F_1}(H).\mathcal{L}_{F_2}(H) \leq 0, \ y \in \Sigma$$

then points starting close to Σ, both from $\mathcal{Y}_1$ and $\mathcal{Y}_2$, will approach Σ and will slide through it. Therefore, we can define the **sliding set**

$$\Sigma_s = \{y \in \Sigma : \sigma(y) \leq 0\}$$

is the union of closed sliding subsets and of isolated sliding points

- **tangent points**: if there is at least one $j \in \{1, 2\}$ such that

$$\mathcal{L}_{F_j}(H) = 0, \ y \in \Sigma_s$$

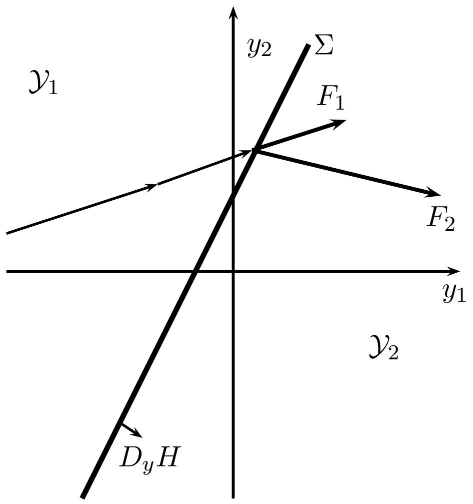


Figure 5.13: Transversal Crossing with $\mathcal{L}_{F_1}(H) > 0$ and $\mathcal{L}_{F_2}(H) > 0$

the flow is tangent to Σ from side $\mathcal{Y}_j$. If

$$\mathcal{L}_{F_j}(H) = 0,\ j = 1, 2,\ y \in \Sigma_s$$

and

$$\mathcal{L}_{F_1 - F_2}(H) = 0,\ y \in \Sigma$$

then $\tilde{y} \in \Sigma_s$ is a **singular tangent point** where both vectors F_1 and F_2 are tangent to Σ, or one or both vanish.

Sliding motion only exists for Filippov systems. We can classify them further according to their stability properties:

1. **attractive sliding mode** (see 5.14): if

$$\mathcal{L}_{F_1}(H) > 0,\ \text{and}\ \mathcal{L}_{F_2}(H) < 0,\ y \in \Sigma$$

   if a trajectory reaches Σ it does not leave Σ and the solution is unique in forward time. In this case Σ_s will be called a *stable sliding* segment;

2. **repulsive sliding mode** (see Figure 5.15): if

$$\mathcal{L}_{F_1}(H) < 0,\ \text{and}\ \mathcal{L}_{F_2}(H) > 0,\ y \in \Sigma$$

   if a trajectory reaches Σ but it can leave Σ and the solution is not unique in forward time. In this case Σ_s will be called a *unstable sliding* segment

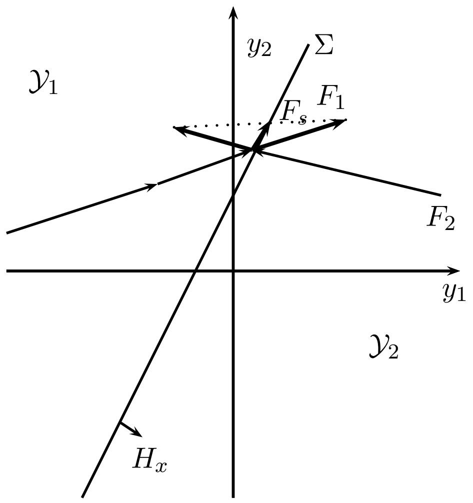


Figure 5.14: Attractive sliding mode with $\mathcal{L}_{F_1}(H) > 0$ and $\mathcal{L}_{F_2}(H) < 0$

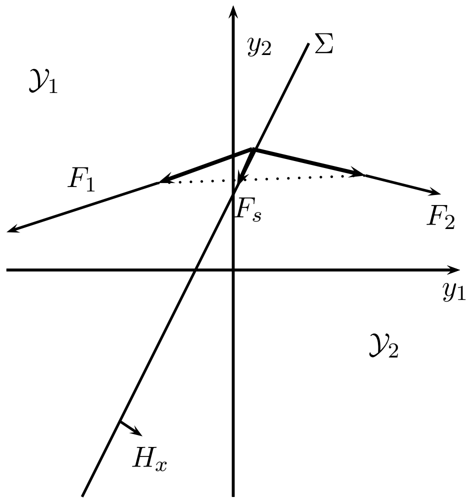


Figure 5.15: Repulsive sliding mode with $\mathcal{L}_{F_1}(H) < 0$ and $\mathcal{L}_{F_2}(H) > 0$

The next proposition gathers the previous results:

Proposition 5. *Assume that*

1. $\mathcal{L}_{F_1}(H).\mathcal{L}_{F_2}(H) < 0$*, and* $\mathcal{L}_{F_1}(H) > 0$ *and* $\mathcal{L}_{F_2}(H) < 0$ *(i.e, there is a non-empty attractive slidding set),*

2. $\mathcal{L}_{F_1-F_2}(H) \neq 0$.

If there is a function

$$\eta(y) = \frac{\mathcal{L}_{F_1}(H(y))}{\mathcal{L}_{F_1-F_2}(H(y))}$$

such that $0 < \eta(y) < 1$*, for* $y \in \Sigma$ *then there is a slidding motion and the slidding vector field can be defined by*

$$F_s(y) = (1-\eta(y))F_1(y) + \eta(y)F_2(y),\ y \in \Sigma.$$

and there is a differential equation

$$\dot{y} = F_s(y),\ y \in \Sigma$$

which has an unique solution in forward time, for a given initial state $y_0 \in \Sigma$.

In fact, we have the differential-algebraic equation (DAE)

$$\begin{cases} \dot{y} = F_s(y) \\ H(y) = 0 \end{cases}$$

and along the slidding vector field it holds that (at non-isolated sliding points)

$$\mathcal{L}_{F_s}(H) = 0.$$

A *pseudo-equilibrium* is an equilibrium point of the vector field F_s,

$$\tilde{y} = \{y \in \Sigma : F_s(y) = 0\}$$

and we can characterize the stability properties of the pseudo-equilibrium as for smooth ODEs.

5.3.2 Bifurcation analysis in the two zone case

Now consider

$$\dot{y} = \begin{cases} F_1(y,\varphi), & \text{if } H(y,\varphi) \leq 0 \\ F_2(y,\varphi), & \text{if } H(y,\varphi) > 0 \end{cases}$$

where $\varphi \in \mathbb{R}^p$ is a vector of parameters.

The system may have several **invariant sets**: equilibria, limit cycles, homoclinic or heteroclinic trajectories.

Bifurcation theory studies the change in the dynamic properties of the system, in particular the change in the number and stability of the invariant sets, when the parameter φ changes. That it, it studies *structural stability*. In PWSC systems, a general classification of possible classes of dynamics is not simple because it may involve the change in the number and other features of the switching boundary.

According to the researchers in the area, the classification knowledge on the dynamics of PWSC related to the is still on its infancy. Possibly a complete characterization of all the possible types of dynamics may be impossible, given the virtually infinite number of possible cases. A group of researchers adopt a "pragmatic approach" (di Bernardo et al. (2008a))

Two main types of bifurcations may occur:

1. **regular or smooth bifurcations**, if the switching boundaries are not involved. That is, all the bifurcations which may occur in smooth ODE's (as in Kuznetsov (2005)) also hold in PWSC ODE's;

2. **discontinuity induced bifurcations**, if the switching boundaries are involved.

A **discontinuity-induced bifurcations** (DIB's) is said to occur at a parameter value φ at which the PWSC ceases to be structurally stable, for a small change in the parameter from that value. The following types of bifurcations had been discovered to occur in PWSC systems

1. **boundary equilibrium bifurcation** if an equilibrium point is located at the switching boundary, if pseudo-equilibria may appear or disappear or if a limit cycle is generated as a consequence of the existence of switching boundaries

2. **grazing bifurcation of limit cycles**: if a limit cycle or a flow becomes tangent to the switching boundary

3. **sliding-sticking bifurcations** may occur when there are limit cycles in Filippov systems

4. **bifurcations related to the existence of corners** when there are more than two switching boundaries

5. **particular types of global bifurcations**, v.g homoclinic-like connections between pseudo-equilibria.

Next we consider DIB of boundary equilibria.

5.3.3 DIB's branching from an equilibrium point

Types of equilibrium points for both PWSC and Filippov systems

Definition 1 (**regular equilibrium**). *y is a **regular equilibrium** if $y \in \mathcal{E}$, where $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$*

$$\mathcal{E}_j = \{y \in \mathcal{Y}_j : F_j(y, \varphi) = 0\}$$

Equivalently: $y \in \mathcal{E}$ if $F_1(y, \varphi) = 0$ and $H(y, \varphi) < 0$ or $F_2(y, \varphi) = 0$ and $H(y, \varphi) > 0$

Definition 2 (**virtual equilibrium**). *y is a **virtual equilibrium** if $y \in \mathcal{E}^v$, where $\mathcal{E}^v = \mathcal{E}_1^v \cup \mathcal{E}_2^v$*

$$\mathcal{E}_j^v = \{y \notin \mathcal{Y}_j : F_j(y, \varphi) = 0\}$$

Equivalently: $y \in \mathcal{E}$ if $F_1(y, \varphi) = 0$ and $H(y, \varphi) > 0$ or $F_2(y, \varphi) = 0$ and $H(y, \varphi) < 0$.

Definition 3 (**pseudo-equilibria**). *For Filippov systems there are equilibrium points for the sliding vector field, and which are not equilibria for the vector fields F_1 and F_2*

$$F_s(y, \varphi) = (1 - \eta)F_1(y, \varphi) + \eta F_2(y, \varphi) = 0.$$

*They are called **pseudo-equilibria**.*

Definition 4 (*regular pseudo-equilibrium*).

$$\mathcal{P} = \{x \in \Sigma : F_s(y, \varphi) = 0,\ \textit{and}\, 0 < \eta(x, \varphi) < 1\}$$

Definition 5 (*virtual pseudo-equilibrium*).

$$\mathcal{P} = \{y \in \Sigma : F_s(y, \varphi) = 0,\ \textit{and}\, 0 > \eta(y, \varphi)\ \textit{or}\, \eta(y, \varphi) > 1\}\,.$$

Definition 6 (*boundary equilibrium point*). *y is a boundary equilibrium point if $y \in \mathcal{B} = \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where*

$$\mathcal{B}_j = \{y \in \Sigma : F_j(y, \varphi) = 0\}, \; j = 1, 2.$$

it is an equilibrium which belongs to Σ and is in the boundary between sets of admissible and virtual equilibria or pseudo-equilibria

For the two types of systems we have:

1. for PWSC we have $F_1(y, \varphi) = F_2(y, \varphi)$ if $y \in \Sigma$, a boundary equilibrium belongs to the two branches of function $F(y)$, then $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$;

2. for Filippov systems we have $F_1(y, \varphi) \neq F_2(y, \varphi)$ if $y \in \Sigma$ then $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and every boundary equilibrium will belong to either $\mathcal{B}_1$ or $\mathcal{B}_2$

Definition 7 (*boundary equilibrium bifurcation point*). *if $\varphi = \varphi^*$ a boundary equilibrium bifurcation (of co-dim 1) is said to occur if there is an equilibrium point $y = \tilde{y} = y(\varphi^*)$ such that*

1. *$\tilde{y}$ is a boundary equilibrium: $\tilde{y} \in \mathcal{B}$*

2. *$\tilde{y}$ is a non-singular boundary equilibrium:* $\det(F_x(\tilde{y}, \varphi^*)) \neq 0$

3. *$\tilde{y}$ verifies a non-degeneracy condition as regards φ:*

$$\frac{dH(\tilde{y}, \varphi^*)}{d\varphi} = D_\varphi H(\tilde{y}, \varphi^*) - D_y H(\tilde{y}, \varphi^*)(D_y F_j)^{-1}(\tilde{y}, \varphi^*) D_\varphi F_j(\tilde{y}, \varphi^*) \neq 0$$

There are two generic types of **discontinuity induced bifurcations branching from BEB's**:

1. *Persistence (or border crossing)*: if there is a regular equilibrium in one side of Σ, which, upon variation of the parameter φ becomes a boundary equilibrium, at the bifurcation point, and, upon further variations, becomes a regular equilibrium in the other side, or a virtual pseudo-equilibrium in a Filippov system becomes regular;

2. *non-smooth fold*: if a at a bifurcation value of the parameter, two branches of equilibria collide with Σ, and coalesce becoming a boundary equilibrium and turning into virtual equilibria.

There are two (non-alternative) types of approaches for determining the type of bifurcation:

1. Leine and Nijmeijer (2004) approach We consider again the generalized jacobian of $F(y)$ and evaluate it at a BEB,

$$\partial\mathcal{F}(y,\varphi) = \overline{\text{co}}\{D_y F_1(y,\varphi), F_2(y,\varphi)\} = \{(1-\alpha)D_y F_1(y,\varphi)+\alpha D_y F_2(y,\varphi),\ 0 \leq \alpha \leq 1\},\ \text{if } y = \tilde{y}, \varphi = \varphi^*$$

A stronger version of their results is for an equilibrium or pseudo-equilibrium:

1. there is persistence if $0 \notin \det(\partial\mathcal{F}(\tilde{y},\varphi^*))$ if $y \in \Sigma$
2. there is a non-smooth fold if $0 \in \det(\partial\mathcal{F}(\tilde{y},\varphi^*))$ if $y \in \Sigma$, that is if the spectrum of $\partial\mathcal{F}(\tilde{y},\varphi^*)$ crosses the imaginary axis with a zero complex part;
3. there is a non-smooth Hopf bifurcation if the spectrum of $\partial\mathcal{F}(\tilde{y},\varphi^*)$ crosses the imaginary axis elsewhere.

2. The approach of di Bernardo et al. (2008a) The vector field on branch $\mathcal{Y}_2$ can be written as functions of the vector field on branch $\mathcal{Y}_1$

1. for PWSC systems,

$$F_1(y,\varphi) = F(y,\varphi)$$
$$F_2(y,\varphi) = F(y,\varphi) + G(y,\varphi)H(y,\varphi)$$

2. for Filippov systems

$$F_1(y,\varphi) = F(y,\varphi)$$
$$F_2(y,\varphi) = F(y,\varphi) + G(y,\varphi)$$

Consider a regular equilibrium in $\mathcal{Y}_1$, $y_1 \in \mathcal{E}_1$. Then

$$F(y_1,\varphi) = 0,\ H(y_1,\varphi) = z_1 < 0$$

Linearising about the BEB point $(\tilde{y},\varphi^*) = (0,0)$ we get

$$Ay_1 + M\varphi = 0,\ Cy_1 + N\varphi = z_1 < 0$$

where $A = D_y F_1(\tilde{y},\varphi^*)$, $M = F_{1,\varphi}(\tilde{y},\varphi^*)$, $C = H_x(\tilde{y},\varphi^*)$, $N = H_\varphi(\tilde{y},\varphi^*)$ As $\det A \neq 0$ (from the definition of BEB) then

$$y_1 = -A^{-1}M,\ z_1 = (N - CA^{-1}M)\varphi$$

Now, consider a regular equilibrium in $\mathcal{Y}_2$, $y_2 \in \mathcal{E}_2$. Then

$$F(y_2, \varphi) + G(y_2, \varphi)z_2 = 0,\ H(y_2, \varphi) = z_2 > 0$$

Linearising about the BEB point $(\tilde{y}, \varphi^*) = (0, 0)$ we get

$$Ay_2 + M\varphi + Bz_2 = 0,\ Cy_2 + N\varphi = z_2 > 0$$

where $B = G(\tilde{y}, \varphi^*)$. Then

$$z_2 = \frac{(N - CA^{-1}M)\varphi}{1 + CA^{-1}B} = z_2 = \frac{z_1}{1 + CA^{-1}B}.$$

Theorem (di Bernardo et al. (2008a)) *For a PWSC let* $(\tilde{y}, \varphi^*) = (0, 0)$ *be a boundary equilibrium bifurcation point and assume that*

$$\det(A) \neq 0, N - CA^{-1}M \neq 0,\ \textit{and}\ 1 + CA^{-1}B \neq 0$$

Then:

- *Persistence or border crossing if*

$$1 + CA^{-1}B < 0$$

- *Non-smooth fold if*

$$1 + CA^{-1}B > 0$$

Example Consider the following system for $(y_1, y_2) \in \mathbb{R}^2$:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \left\{ \mathbf{F}_1(y_1, y_2) = \begin{pmatrix} -y_1 - y_2 + 1 \\ y_1 - y_2 + 1 \end{pmatrix},\ \text{for } y_1 \leq 1,\ \mathbf{F}_2(y_1, y_2) = \begin{pmatrix} y_1 - y_2 - 1 \\ y_2 - 2 \end{pmatrix},\ \text{for } y_1 > 1 \right\} \tag{5.7}$$

Therefore

$$\Sigma = \{(y_1, y_2) : y_1 = 1\}$$

and $\mathcal{Y}_1 = \{(y_1, y_2) : y_1 < 1\}$ and $\mathcal{Y}_2 = \{(y_1, y_2) : y_1 > 1\}$.

We start by classifying the ODE. First, the two vector fields are different at points in Σ, i.e., $\{(1, y_2)\}$,

$$F_1(1, y_2) = \begin{pmatrix} -y_2 \\ 2 - y_2 \end{pmatrix} \neq F_2(1, y_2) = \begin{pmatrix} -y_2 \\ y_2 - 2 \end{pmatrix}.$$

Second because the ODE is linear in the interior of both branches $\mathcal{Y}_1$ and $\mathcal{Y}_2$ the Jacobians are constant, i.e,

$$D_{(y_1,y_2)}F_1(y_1,y_2) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},\ D_{(y_1,y_2)}F_2(y_1,y_2) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

This implies that the Jacobians evaluated at Σ are different,

$$D_{(y_1,y_2)}F_1(1,y_2) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \neq D_{(y_1,y_2)}F_2(1,y_2) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

We conclude that ODE is a Filippov equation (see the phase diagram in Figure 5.16). However, observe that the two isoclines

$$I_{y_1} = \left\{ (y_1,y_2) : \begin{cases} y_2 = 1 - y_1 & \text{for, } y_1 \leq 1 \\ y_2 = y_1 - 1 & \text{for, } y_1 > 1 \end{cases} \right\}$$

and

$$I_{y_2} = \left\{ (y_1,y_2) : \begin{cases} y_2 = 1 + y_1 & \text{for, } y_1 \leq 1 \\ y_2 = 2 & \text{for, } y_1 > 1 \end{cases} \right\}$$

are continuous at Σ.

There are two regular steady states and no boundary steady states nor pseudo-equilibria. (Check this). In order to study the behavior of the solutions close to Σ, we compute $\mathcal{L}_{F_j}(H) = D_yH(y)F_j(y)$ for $j = 1, 2$. As $D_yH = (1,0)^\top$ then

$$\mathcal{L}_{F_1}(H)(y) = -y_1 - y_2 + 1,\ \mathcal{L}_{F_2}(H)(y) = y_1 - y_2 - 1$$

evaluating for $\mathbf{y} = \tilde{\mathbf{y}} \in \Sigma$, where $\tilde{\mathbf{y}} = (1, y_2)$

$$\mathcal{L}_{F_1}(H)(\tilde{\mathbf{y}}) = \mathcal{L}_{F_2}(H)(\tilde{\mathbf{y}}) = -y_2$$

Then

$$\sigma(\tilde{\mathbf{y}}) = \mathcal{L}_{F_1}(H)(\tilde{\mathbf{y}})\mathcal{L}_{F_2}(H)(\tilde{\mathbf{y}}) = y_2^2 \geq 0$$

the sliding set is empty and there is crossing almost everywhere.

However, because at $y_2 = 0$ we have

$$\mathcal{L}_{F_1}(H)(1,0) = \mathcal{L}_{F_2}(H)(1,0) = 0$$

there are two trajectories tangent to Σ coming from both sides. In order to prove that point $\tilde{\mathbf{y}}_s = (1,0)$ is a singular tangency point, we compute $\mathcal{L}_{F_1-F_2}(H)(\tilde{\mathbf{y}}_s)$. Because

$$F_1(\mathbf{y}) - F_2(\mathbf{y}) = \begin{pmatrix} -2y_1 + 2 \\ y_1 - 2y_2 + 3 \end{pmatrix}$$

then $\mathcal{L}_{F_1-F_2}(H)(\mathbf{y}) = -2y_1 + 2$ and $\mathcal{L}_{F_1-F_2}(H)(\tilde{\mathbf{y}}_s) = 0$ which proves that point $\tilde{\mathbf{y}}_s$ is indeed is a singular tangency point.

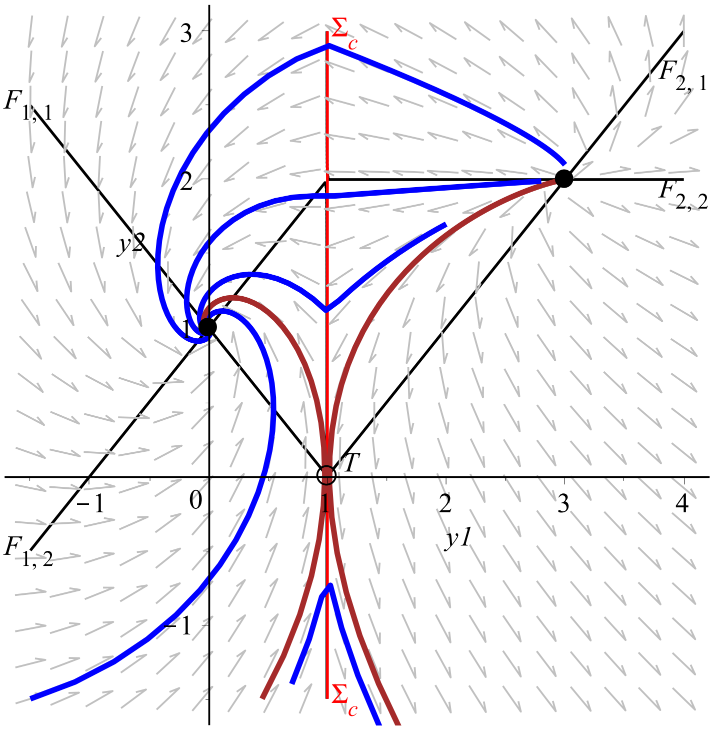


Figure 5.16: The ODE (5.7) has two regular steady states: a stable focus and an unstable node. It has two trajectories which are tangent to the switching surface Σ at the same point. Therefore $T = (1, 0)$ is a singular tangential point. It separates Σ into two crossing subsets. The trajectories crossing Σ at $y_2 > 0$ will converge to the regular steady state in $\mathcal{Y}_1$ and the trajectories crossing at $y_2 < 0$ will become unbounded within set $\mathcal{Y}_2$.

5.4 Application to the Ramsey model

We can take the example of the Ramsey model from chapter 4 and assume that the central planer does not want to allow consumption to fall bellow a minimum level c_{min}. In this case the model will be a PWS model

$$\dot{k} = \begin{cases} f(k) - c_{min} & \text{if } c = c_{min} \\ f(k) - c & \text{if } c > c_{min} \end{cases}$$

and

$$\dot{c} = \begin{cases} 0 & \text{if } c = c_{min} \\ \eta\, c\, \left(f'(k) - \rho\right) & \text{if } c > c_{min} \end{cases}$$

5.5 References

Literature:

- Seminal works comprise Feigin (1970) and Filippov (1988)
- This presentation was made possible by the existence of recent monographs Leine and Nijmeijer (2004), di Bernardo et al. (2008a) and Leine and van de Wouw (2008) and survey papers di Bernardo et al. (2008b).
- There were some applications in economics: in the "disequilibrium" model of Honkapohja and Ito (1983), growth model of van Marrewijk and Verbeek (1993), a Ramsey model Brito et al. (2013) and an endogeneous growth model Gil et al. (2013).

Chapter 6

ODE with singularities

6.1 Introduction

Up until now we considered the differential equation $\dot{y} = f(y)$ where $f(.)$ is a Lipschitz function (i.e, it is continuous and has **bounded** first derivatives).

Now we consider two types of equations

$$\epsilon\dot{y} = g(y) \tag{6.1}$$

where is a small parameter belonging to the interval $(-\bar{\epsilon}, \bar{\epsilon})$ where $\bar{\epsilon}$ is small (close to zero), and

$$a(y)\dot{y} = g(y) \tag{6.2}$$

where there is a value $y = y^s$ such that $a(y^s) = 0$. Or writing them together

$$\epsilon a(y)\,\dot{y} = g(y).$$

If we write the equation in the normal form

$$\dot{y} = f(y) = \frac{g(y)}{\epsilon a(y)}$$

we find that function $f(.)$ is not locally Lipschitz. Therefore the usual existence and uniqueness of solutions do not hold. We call *fast-slow singularity* to the singularity displayed in equation (6.1) and *impasse singularity* to the singularity occurring in equation (6.2).

Next we present, for the planar case, a general approach to both types of singularities and deal with them separately in the next sections.

Consider the equation

$$A(y, \varphi, \epsilon)\,\dot{y} = F(y, \varphi, \epsilon), \quad y \in \mathbb{R}^2 \tag{6.3}$$

where $y = (y_1, y_2)^\top \in \mathbb{Y} \subseteq \mathbb{R}^2$ and $\varphi \in \Phi \subseteq \mathbb{R}^n$ where n has a suitable dimension, $\epsilon \in \mathbb{E} \subset (-1, 1)$. Without loss of generality, we assume that matrix A and vector F are of the form

$$A(y, \varphi, \epsilon) \equiv \begin{pmatrix} 1 & 0 \\ 0 & \epsilon\, \delta(y, \varphi, \epsilon) \end{pmatrix}, \ F(y, \varphi, \epsilon) = \begin{pmatrix} f_1(y, \varphi, \epsilon) \\ f_2(y, \varphi, \epsilon) \end{pmatrix}.$$

We assume that functions $f_1(.)$, $f_2(.)$, and $\delta(.)$ are sufficiently smooth and define $\Omega = \mathbb{Y} \times \Phi$.

Equation (6.3) expands as

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \varphi, \epsilon), \\ \epsilon\, \delta(y_1, y_2, \varphi, \epsilon)\, \dot{y}_2 &= f_2(y_1, y_2, \varphi, \epsilon). \end{aligned} \tag{6.4}$$

The determinant of matrix A is

$$\det A(y, \varphi, \epsilon) = \epsilon\, \delta(y_1, y_2, \varphi, \epsilon).$$

A point $p \in \mathbb{Y} \times \mathbb{E}$ is said to be *regular* if $\det A(p) \neq 0$. We say we have a *singularity* at point p if $\det A(p) = 0$. The set of all singularities is called the **singular set**:

$$\mathcal{S} \equiv \{(y, \varphi, \epsilon) \in \Omega \times \mathbb{E} :\ \epsilon\, \delta(y, \varphi, \epsilon) = 0\} \tag{6.5}$$

We write it $\mathcal{S}_{\epsilon,\varphi}$ if we consider it as a family of subsets of $\mathbb{Y}$ parameterized by (ϵ, φ), or as $\mathcal{S}_\epsilon$ as a famlily of subsets of Ω.

Therefore, $\mathbb{Y} \times \mathbb{E}/\mathcal{S}$ is the set of regular points.

At a regular point (y, φ, ϵ) we can express equation (6.3) as a differential equation in normal form

$$\dot{y} = \mathcal{F}(y, \varphi, \epsilon) = (\epsilon\delta(y, \varphi, \epsilon))^{-1} A^*(y, \varphi, \epsilon) F(y, \varphi, \epsilon) \tag{6.6}$$

where $A^*(.)$ is the adjoint matrix of $A(.)$. Equivalently

$$\dot{y} = \begin{pmatrix} f_1(y, \varphi, \epsilon) \\ f_2(y, \varphi, \epsilon)/(\epsilon\, \delta(y, \varphi, \epsilon)) \end{pmatrix}.$$

In the previous section we already studied the local of the dynamics of system (6.6)[1]. However, our concern here is related to the consequences of the existence of singularities on the local (and global) dynamics.

An obvious symptom of the consequences of the existence of singularities emerges if we conduct the local dynamic analysis of a model with singularities as if it was regular.

First, a steady state is a family of points $\bar{y}$ such that $f_1(\bar{y}, \varphi, \epsilon) = f_2(\bar{y}, \varphi, \epsilon) = 0$. Let us write the set of steady states as

$$\Gamma_E = \{(y, \varphi, \epsilon) \in \Omega \times \mathbb{E} :\ f_1(y, \varphi, \epsilon) = f_2(y, \varphi, \epsilon) = 0\}.$$

[1]See also Guckenheimer and Holmes (1990) or Kuznetsov (2005).

If set there Γ_E is non-empty, we can compute the Jacobian of $\mathcal{F}$, at every member of that set,

$$D_{\bar{y}} \equiv D_y \mathcal{F}(\bar{y}, \varphi, \epsilon) = \frac{1}{\epsilon \delta(\bar{y}, \varphi, \epsilon)} \begin{pmatrix} f_{1,y_1}(\bar{y}, \varphi, \epsilon) & f_{1,y_2}(\bar{y}, \varphi, \epsilon) \\ f_{2,y_1}(\bar{y}, \varphi, \epsilon) & f_{2,y_2}(\bar{y}, \varphi, \epsilon) \end{pmatrix} \tag{6.7}$$

As the Jacobian has trace and determinant given by

$$\begin{aligned} \mathrm{trace} D_{\bar{y}} &= f_{1,y_1}(\bar{y}, \varphi, \epsilon) + \frac{f_{2,y_2}(\bar{y}, \varphi, \epsilon)}{\epsilon \delta(\bar{y}, \varphi, \epsilon)} \\ \det D_{\bar{y}} &= \frac{f_{1,y_1}(\bar{y}, \varphi, \epsilon) f_{2,y_2}(\bar{y}, \varphi, \epsilon) - f_{1,y_2}(\bar{y}, \varphi, \epsilon) f_{2,y_1}(\bar{y}, \varphi, \epsilon)}{\epsilon \delta(\bar{y}, \varphi, \epsilon)} \end{aligned}$$

then the eigenvalues of the Jacobian $D_{\bar{y}}$ are

$$\lambda_{\pm} = \frac{\mathrm{trace} D_{\bar{y}}}{2} \pm (\Delta D_{\bar{y}})^{1/2}, \text{ for } \Delta D_{\bar{y}} \equiv \left(\frac{\mathrm{trace} D_{\bar{y}}}{2} \right)^2 - \det D_{\bar{y}}.$$

A steady state $(\bar{y}, \varphi, \epsilon)$ is not hyperbolic if at least one eigenvalue $\lambda_{\pm}$ has a zero real real part. If other well known conditions involving higher derivatives of the vector field $\mathcal{F}$ are verified we say we have a *regular bifurcation* at point $(\bar{y}, \varphi, \epsilon)$.

Two consequences from the existence of singularities are already apparent. First, If functions $f_1(.)$ and $f_2(.)$ are sufficiently regular at steady state $\bar{y}$, we readily see that if we evaluate (naïvely) at a singularity it will have infinitely-valued eigenvalues[2]. This is different from regular bifurcation points, because a small change in a parameter changes the sign of at least one eigenvalue not crossing zero but crossing one asymptote. Second, similarly to regular bifurcation points, singularities introduce a bifurcation-like change in the local dynamics. If there is not any further degeneracy, at a singular point the sign of the determinant $\det D_{\bar{y}}$ changes, which means that the dimension of the stable and unstable manifolds changes when a singularity is crossed.

However, the effect of singularities calls for a global analysis of the dynamics.

We introduce the following assumption:

Assumption 1. *Singularities can only be of one of the following two types:* ***singular-perturbation*** *singularities where $\epsilon = 0$ and $\delta(y, \varphi, 0) \neq 0$ for all $(y, \varphi) \in \Omega$, or* ***impasse*** *singularities where $\epsilon \neq 0$ and $\delta(y, \varphi, \epsilon) = 0$ for at least one point $(y, \varphi) \in \Omega$.*

We also present next methods to analyse the local dynamics in the neighbourhood of a singularity. The idea for dealing with the two singularities is similar: because the presence of the singularity generates infinite-valued eigenvalues (or infinitely valued roots for the characteristic equation) we transform locally the system in order to remove the source of the singularity and evaluate the local dynamics in the same way we deal with a regular point).

[2]Even if there is a local bifurcation at a singular steady state the Jacobian evaluated in this way will hide the local dynamic behaviour.

6.2 Fast-slow singularities

We consider some simple cases for scalar and planar equations and introduce next a general method for dealing with fast-slow singularities.

6.2.1 Scalar linear equations

We consider two linear scalar equations in which function $g(y)$ is homogeneous and non-homogeneous.

Homogeneous equation

Consider the linear fast-slow scalar equation, where ϵ takes values in a small interval around 0:

$$\epsilon \dot{y} = \lambda y \tag{6.8}$$

where $\dot{y} = dy/dt$, where $\lambda \neq 0$ is finite and $t \geq 0$. It is easy to see that the rate of growth of y approaches infinity, for all t when ϵ tends to zero: $\lim_{\epsilon \to 0} \dot{y}/y = \lambda/\epsilon = \pm\infty$. This implies, even if $\epsilon \neq 0$, that the rate of growth is still very high, which obfuscates the dynamics.

To uncover the dynamics we remove the source of singularity by redefining time as $\tau = t/\epsilon$, and call t the slow time scale and τ the *fast time scale*. Then we get an associated de-singularized system called associated fast system or layer equation

$$y^{'} = \lambda y \tag{6.9}$$

where $y^{'} = dy/d\tau$.

The pair of equations (6.8) and (6.9) is called the **fast-slow system**. The solutions for both equations are, for the slow equation (6.8)

$$y(t, \epsilon) = y(0)e^{\frac{\lambda}{\epsilon}t} \tag{6.10}$$

and to the fast equation (6.9) is is

$$y(\tau) = y(0)e^{\lambda\tau} \tag{6.11}$$

If $\epsilon \neq 0$ the solution is continuous and smooth at $t = 0$, and the asymptotic behavior of the trajectories, if $y(0) \neq 0$, is the following

$$\lim_{t \to +\infty} y(t, \epsilon) = \begin{cases} +\infty, & \text{if } \lambda/\epsilon > 0 \text{ and } y(0) > 0 \\ -\infty, & \text{if } \lambda/\epsilon > 0 \text{ and } y(0) < 0 \\ 0, & \text{if } \lambda/\epsilon < 0. \end{cases}$$

Therefore, stability depends on the sign of the ratio λ/ϵ. The point $y = 0$ is an attractor if $\lambda/\epsilon < 0$ and is repelling if $\lambda/\epsilon > 0$.

If $\epsilon = 0$, the behavior for $t > 0$ is revealed by the solution of the fast (or layer) equation, equation (6.11): the solution, if $\tau > 0$ ($\tau < 0$) decays monotonically to 0 (increases monotonically to $+\infty$) if $\lambda < 0$ and increases monotonically to $+\infty$ (decays monotonically to 0) if $\lambda > 0$. The trajectories follow the opposite trend.

Going back to the slow-time scale, we see that if $\epsilon = 0$ the solution is **discontinuous at** $t = 0$:

$$y(t, 0) = \begin{cases} y(0), & \text{if } t = 0 \\ 0, & \text{if } \lambda < 0, \text{ and } \forall\, t > 0 \\ \infty, & \text{if } \lambda > 0, \text{ and } \forall\, t > 0 \end{cases}$$

Then, for a fixed λ, crossing $\epsilon = 0$ means that there is a change in the stability properties of the flow. For small ϵ the solution decays monotonically (or increases monotonically), it becomes discontinuous for $\epsilon = 0$ and changes stability properties upon crossing that value. The type of discontinuity at $\epsilon = 0$ is determined by the sign of λ (and is uncovered by the dynamics of the layer system: stable or unstable).

Non-homogeneous equation

It may not be clear in the former example, but the convergence to a finite value is referred to convergence to a steady state.

Consider the scalar linear fast-slow system

$$\epsilon \dot{y} = \lambda y + b \tag{6.12}$$

$$y^{'} = \lambda y + b \tag{6.13}$$

where $\lambda \neq 0$ and $b \neq 0$. Then there is an unique steady state $\bar{y} = -b/\lambda$.

The solutions for the slow time and the fast time are

$$y(t, \epsilon) = \bar{y} + (y(0) - \bar{y})e^{\frac{\lambda}{\epsilon}t}$$

and

$$y(\tau) = \bar{y} + (y(0) - \bar{y})e^{\lambda\tau}$$

If $\epsilon \neq 0$ the slow solution is continuous and smooth in t, and if $y(0) \neq \bar{y}$

$$\lim_{t \to +\infty} y(t, \epsilon) = \begin{cases} +\infty, & \text{if } \lambda/\epsilon > 0 \text{ and } y(0) > \bar{y} \\ -\infty, & \text{if } \lambda/\epsilon > 0 \text{ and } y(0) < \bar{y} \\ \bar{y}, & \text{if } \lambda/\epsilon < 0 \text{ for any } y(0). \end{cases}$$

If $\epsilon = 0$ we get again a discontinuous solution of t and (using the information extracted form the solution of the layer equation)

$$y(t,0) = \begin{cases} y(0), & \text{if } t = 0 \\ \bar{y}, & \text{if } \lambda < 0 \text{ and } \forall t > 0 \\ \infty, & \text{if } \lambda > 0 \text{ and } \forall t > 0. \end{cases}$$

The layer equation uncovers the dynamics close to $\epsilon = 0$. It has two components: a constant *outer* solution $\bar{y}$ and a initial layer correction $(y(0) - \bar{y})e^{\lambda\tau}$ which decays monotonically to zero if $\lambda < 0$ and $\tau \to \infty$, or increases monotonically if $\lambda > 0$.

Then the direction of the discontinuous "jump" for the slow dynamics is given by the monotonic dynamics of the associated fast equation.

6.2.2 Linear planar equations

The linear planar system can be represented by the equation

$$A_\epsilon \dot{y} = A^s y + B^s$$

where is, for instance,

$$A_\epsilon = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}.$$

We already know from Chapter 3 that $A^s = P\Lambda P^{-1}$ where Λ is the Jordan normal form of A^s. Next we deal with two cases in which matrix A^s is already in a Jordan normal form.

Uncoupled system

Consider the fast-slow system over $\mathbb{Y} = \mathbb{R}^2$ with two uncoupled equations

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1, & y_1^{'} &= \epsilon\lambda_1 y_1, \\ \epsilon\dot{y}_2 &= \lambda_2 y_2, & y_2^{'} &= \lambda_2 y_2, \end{aligned} \tag{6.14}$$

where $\varphi = (\lambda_1, \lambda_2) \neq (0, 0)$ is a real-valued vector.

The solutions to equation (6.14), for a generic ϵ, in the two time scales are

$$y(t,\epsilon) = \begin{pmatrix} y_1(t,\epsilon) \\ y_2(t,\epsilon) \end{pmatrix} = \begin{pmatrix} y_1(0)e^{\lambda_1 t} \\ y_2(0)e^{\frac{\lambda_2}{\epsilon}t} \end{pmatrix}$$

$$y(\tau) = \begin{pmatrix} y_1(\tau) \\ y_2(\tau) \end{pmatrix} = \begin{pmatrix} y_1(0)e^{\epsilon\lambda_1\tau} \\ y_2(0)e^{\lambda_2\tau} \end{pmatrix}$$

for $\epsilon\tau = t$. Throughout we assume that $t \geq 0$.

The equilibrium point is $\bar{y} = (\bar{y}_1, \bar{y}_2) = (0,0)$. It is a: (1) a stable node if $\lambda_1 < 0$ and $\lambda_2/\epsilon < 0$, (2) an unstable node if $\lambda_1 > 0$ and $\lambda_2/\epsilon > 0$; and a saddle point if $\lambda_1 < 0$ and $\lambda_2/\epsilon > 0$ or $\lambda_1 > 0$ and $\lambda_2/\epsilon < 0$.

If $\epsilon = 0$ the fast-slow system becomes

$$\begin{aligned} \dot{y}_1 &= \lambda_1 \, y_1, \qquad & y_1^{'} &= 0, \\ 0 &= \lambda_2 \, y_2, \qquad & y_2^{'} &= \lambda_2 \, y_2, \end{aligned} \tag{6.15}$$

For $y_2(0) \neq 0$ we have

$$y(t,0) = (y_1(t,0), y_2(t,0))^\top = \begin{cases} (y_1(0), y_2(0))^\top, & \text{if } t = 0 \\ \left(y_1(0)e^{\lambda_1 t}, 0\right)^\top, & \text{if } \lambda_2 < 0,\ t > 0 \\ \left(y_1(0)e^{\lambda_1 t}, \infty\right)^\top, & \text{if } \lambda_2 > 0, t > 0 \end{cases}.$$

$$y(\tau) = (y_1(\tau), y_2(\tau))^\top = \left(y_1(0), y_2(0)\, e^{\lambda_2 \tau}\right)$$

We observe that while the slow solution is discontinuous at $t = 0$ the fast solution is everywhere continuous for τ. In addition while the value for slow-y_2 jumps to its asymptotic value for a small increase in t after $t = 0$, the fast-y_2 converges monotonically to its steady state $\bar{y}_2 = 0$ if $\lambda_2\epsilon < 0$ and it increases unboundedly if $\lambda_2\epsilon > 0$ if $t > 0$.

The set of critical points, that we denote by $\mathcal{S}_0^p$, coincides with the y_1 axis: $\mathcal{S}_0^p = \{(y, \varphi) \in \Omega : y_2 = 0\}$. It is a set of attracting points if $\lambda_2\epsilon < 0$ and of repelling points if $\lambda_2\epsilon > 0$

Figures 6.1 and 6.2 depict two cases where $\lambda_1 < 0$ and $\lambda_2 < 0$ and ϵ converges to zero from positive values (figure 6.1 left) or from negative values (figure 6.1 right).

Two observations are important: First, in those graphs, the initial point $(y_1, y_2) = (1,1)$ belongs to the stable subspace in the case of figure 6.1 and to the unstable subspace in the case of 6.2, and therefore for a given value of λ_2 (negative in the case of the figure) the local dynamics of the solution, and the dimension of the stable and unstable subspaces is not changed when ϵ converges to zero. This illustrates Fenichel (1979) theorem. Second, when ϵ changes sign (keeping λ_2 constant) changes the dimension of the stable subspaces, in an equivalent way as would changing the sign of λ_2 from negative to positive while keeping ϵ constant and positive.

Therefore we can see $\epsilon = 0$ as a bifurcation point.

Focus system

In the previous example if ϵ changes by crossing zero we saw that it acts as a bifurcation point by changing the dimension of the local stable and unstable subspaces, in addition to generating a

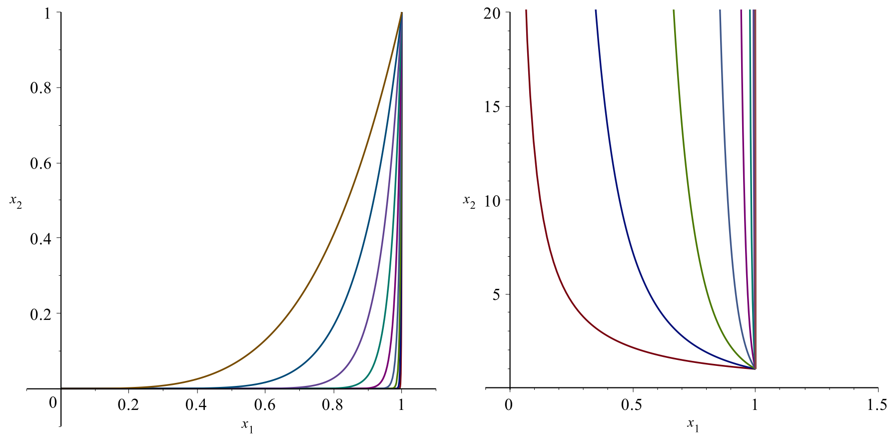


Figure 6.1: Left figure: phase diagram for $\lambda_1 < 0$ and $\lambda_2 \epsilon < 0$. We depict paths for a sequence of ϵ converging to zero . It is clear that when $\epsilon = 0$ the solution becomes discontinuous at $t = 0$ where $(x_1(0), x_2(0)) = (1, 1)$ at $t = 0$ and $(x_1(0^+), x_2(0^+)) = (1, 0)$. Right figure: Phase diagrams for $\lambda_1 < 0$ and $\lambda_2\epsilon > 0$. We depict paths for a sequence of positive ϵ converging to zero. It is clear that when $\epsilon = 0$ the solution becomes discontinuous at $t = 0$ where $(x_1(0), x_2(0)) = (1, 1)$ at $t = 0$ and $(x_1(0^+), x_2(0^+)) = (1, +\infty)$.

discontinuous behaviour at $\epsilon = 0$. However, the next example also shows that it can generate for non-zero values a change in the monotonicity of the solution.

Consider the fast-slow system

$$\begin{aligned} \dot{y}_1 &= \alpha y_1 + \beta y_2 \qquad & y_1^{'} &= \epsilon(\alpha y_1 + \beta y_2) \\ \epsilon \dot{y}_2 &= -\beta y_1 + \alpha y_2 \qquad & y_2^{'} &= -\beta y_1 + \alpha y_2 \end{aligned} \tag{6.16}$$

where we assume that $\alpha \neq 0$ and $\beta \neq 0$. This system has an unique equilibrium point $\bar{y} = (0, 0)$ and $\mathcal{S}_0^p$ is given by the curve $y_2 = -(\beta/\alpha)y_1$.

At first look, that equilibrium point seems to be a stable focus (if $\alpha < 0$), an unstable focus (if $\alpha > 0$) or a centre (if $\alpha = 0$) (see chapter 3). However, if $\epsilon \neq 0$ we can write the slow equation as $\dot{y} = Ay$ where

$$A = \begin{pmatrix} \alpha & \beta \\ -\frac{\beta}{\epsilon} & \frac{\alpha}{\epsilon} \end{pmatrix}$$

has

$$\text{trace}(A) = \frac{\alpha(1 + \epsilon)}{\epsilon}, \ \det(A) = \frac{\alpha^2 + \beta^2}{\epsilon}$$

which has eigenvalues

$$\lambda_{\pm} = \frac{\alpha(1 + \epsilon)}{2\epsilon} \pm \left(\frac{\alpha^2(\epsilon - 1)^2 - 4\epsilon\beta^2}{(2\epsilon)^2} \right)^{1/2}.$$

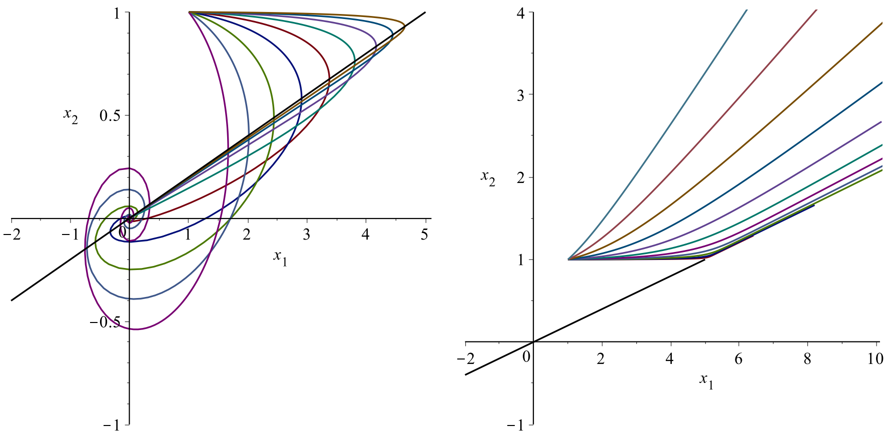


Figure 6.2: Phase diagrams for equation (6.16) for $\alpha < 0$, $\beta > 0$ and ϵ converging to zero from positive values (left figure) and negative values (right figure). When ϵ is closer to zero the trajectories become monotonous

If $-1 < \epsilon < 0$ we see that the equilibrium point is a saddle point, and if $\epsilon > 0$ it is a stable (unstable) node or focus if $\alpha < 0$ ($\alpha > 0$) or a centre if $\alpha = 0$.

If we define

$$\epsilon^*_- \equiv \frac{\alpha^2 + 2\beta^2 - 2\sqrt{\alpha^2\beta^2 + \beta^4}}{\alpha^2}, \text{ and } \epsilon^*_+ \equiv \frac{\alpha^2 + 2\beta^2 + 2\sqrt{\alpha^2\beta^2 + \beta^4}}{\alpha^2},$$

then the equilibrium point is a focus if $\epsilon \in (\epsilon^*_-, \epsilon^*_+)$ or it is a node if $\epsilon < \epsilon^*_-$ or $\epsilon > \epsilon^*_+$.

At point $\epsilon = 0$, the solution of the fast system is $y_1 = k$ and $y_2(\tau) = (\alpha/\beta)k + (y_2(0) - (\alpha/\beta)k)e^{\alpha\tau}$ where k is any value for y_1. If $\tau > 0$ the solution converges to the surface $y_2 = (\alpha/\beta)y_1$ if $\alpha < 0$. If $y_1 = 0$ is at the steady state value, the solution for y_2 converges to the equilibrium value, $y_2 = 0$, as well.

We get the family of phase diagrams for $\alpha < 0$, $\beta > 0$ and ϵ converging to zero from positive values (figure 6.2 left panel) and from negative values (figure 6.2 right panel). Observe in the left panel that if $\epsilon > 0$ and large the equilibrium point $(0, 0)$ is a stable focus, and the transition is non-monotonic. If ϵ approaches zero the trajectory becomes monotonic and when $\epsilon = 0^+$ it converges along the surface to the equilibrium value. If $\epsilon < 0$ it does just the opposite. As in this case the initial point is a saddle point and the initial point which is depicted, $(y_1(0), y_2(0) = (1, 1)$ belongs to the unstable sub space (E^u) then the solution becomes unbounded when T grows to infinity. However, if $\epsilon \to 0^-$ the trajectory also tends to the surface $\mathcal{S}^p_0$ but it is divergent.

In sum: in both cases we observe that the solution converges to the set $\mathcal{S}^p_0$ which is associated to a stable steady state in the first case and an unstable steady state in the second.

6.2.3 Planar non-linear equation

In this subsection we deal with the dynamics of the non-linear system

$$\begin{aligned} \dot{y}_1 &= f_1(y, \varphi, \epsilon), \\ \epsilon\, \dot{y}_2 &= f_2(y, \varphi, \epsilon) \end{aligned} \tag{6.17}$$

when $\epsilon \to 0$ and with bifurcation phenomena associated with crossing that value. In matrix form

$$A_\epsilon \dot{y} = \mathcal{F}^s(y, \varphi, \epsilon)$$

where

$$A_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix},\ \mathcal{F}^s(y, \varphi, \epsilon) = \begin{pmatrix} f_1(y, \varphi, \epsilon) \\ f_2(y, \varphi, \epsilon) \end{pmatrix}.$$

Equations of type (6.17), when the parameter ϵ is small and can take the zero value, are called singular perturbed or fast-slow systems (see Kuehn (2015) for a recent textbook presentation). This is because those systems contain two different time scales for each variable: a slow time scale for y_1 and a fast time scale for y_2. Time t is called slow time and time $\tau = t/\epsilon$ is called fast time. As the time derivatives in equation (6.17) refer to the slow time scale we call it slow system. On the fast time scale equation (6.17) becomes

$$\begin{aligned} y^{'} &= \frac{dy}{d\tau} = \epsilon\, f_1(y, \varphi, \epsilon), \\ y^{'} &= \frac{dy}{d\tau} = f_2(y, \varphi, \epsilon) \end{aligned} \tag{6.18}$$

that is written matricially as

$$y^{'} = \mathcal{F}^f(y, \varphi, \epsilon)$$

The fast-slow paths are described by equations (6.17) and (6.18).

If $\epsilon \neq 0$ the dynamics of equation (6.17) is regular and is well know. The main aspects which interest us here are the following. First, given any initial value $y(0) \in \mathbb{Y}$ a solution exists and is unique for all $t \in [0, \infty)$, and it is (at least) continuous in t. Second, at a regular equilibrium point $\bar{y} \in \Gamma_E$ the local dynamics is characterized by the eigenvalues of the Jacobian (6.7) for the slow-system evaluated at an equilibrium point (if Γ_E is non-empty). Clearly, there may exist values for ϵ associated to regular bifurcation points or to a change in the phase diagram (for example changing monotonous adjustment paths to oscillatory).

If $\epsilon = 0$ we define the critical set which is a subset of the singularity set defined in (6.7):

$$\mathcal{S}^p \equiv \{(y, \varphi) \in \mathbb{Y} :\ \epsilon = 0\}. \tag{6.19}$$

In addition, the fast-slow system becomes

$$\begin{aligned} \dot{y}_1 &= f_1(y,\varphi,0), \\ 0 &= f_2(y,\varphi,0) \end{aligned} \tag{6.20}$$

and the fast system is

$$\begin{aligned} y_1^{'} &= 0, \\ y_2^{'} &= f_2(y,\varphi,0). \end{aligned} \tag{6.21}$$

In (6.20) equation $f_2^s(y,\varphi,0)=0$ defines a the family of manifolds. We call it *singular-perturbed critical subset*

$$\mathcal{S}_0^p = \{(y,\varphi) \in \mathcal{S}^p :\ f_2(y,\varphi,0)=0\}\,.$$

If function $f_2(y_1,y_2,.)$ is a sufficiently smooth as a function of y_2 the implicit function theorem allows us to write $y_2 = h(y_1)$. Upon substitution in equation (6.20) we find that within the critical subset $\mathcal{S}_0^p$ the dynamics if driven by the scalar differential equation for the slow variable y_1

$$\dot{y}_1 = f_1(y_1, h(y_1,\varphi),\varphi,0).$$

Therefore, along $\mathcal{S}_0^p$ y_1 is a continuous function of the slow time scale t, while y_2 is discontinuous. In order to uncover the dynamics out of the singular-perturbed critical subset, $\mathbb{Y}/\mathcal{S}_0^p$, and the type of discontinuity of y_2, we use the fast system (6.21). For any constant $y_1=y_1^*$ the dynamics of y_2 is given by the scalar differential equation

$$y_2^{'} = f_2(y_1^*, y_2, \varphi, 0).$$

Therefore, at the fast time scale, τ, y_2 is a continuous function while keeping y_1 constant.

In order to characterise the dynamics over $\mathcal{S}_0^p$ we introduce a stratification over it.

We say point $y_p \in \mathcal{S}_0^p$ is *fast-slow regular* if $f_{2,y_2}^s(y_p,\varphi,0) \neq 0$. A point $y_s \in \mathcal{S}_0^p$ is called *fast-slow singular* if $f_{2,y_2}^s(y_s,\varphi,0)=0$.

The set of *fast-slow singular points* is defined as

$$\mathcal{S}_1^p = \left\{(y,\varphi) \in \mathcal{S}^p :\ f_{2,y_2}^s(y,\varphi,0)=0\right\} \tag{6.22}$$

In addition, observe that the set of fast-slow regular points is the equivalent to the set of hyperbolic equilibrium points and the set of singular points is the set of non-hyperbolic points of the fast system.

The set $\mathcal{S}_1^p$ introduces a partition over the set $\mathcal{S}_0^p$ into two subsets: the set of *fast-slow regular attracting* points if $f^s_{2,y_2}(y,\varphi,0) < 0$ and *fast-slow regular repelling* points if $f^s_{2,y_2}(y,\varphi,0) > 0$.

Therefore, we have the following partition: $\mathcal{S}_0^p = \mathcal{S}_0^{p,a} \cup \mathcal{S}_0^{p,r} \cup \mathcal{S}_1^p$, where $\mathcal{S}_0^{p,a}$ and $\mathcal{S}_0^{p,r}$ are, respectively, the sets of fast-slow regular attracting and repelling points, respectively.

We can partition $\mathcal{S}_1^p$ further. It is important to observe that, while the set of fast-slow critical points $\mathcal{S}_0^p$ describes a one-dimensional manifold (with attracting and/or repelling segments), over the phase space $\mathbb{Y}$, the set of fast-slow singular points, $\mathcal{S}_1^p$, describe zero-dimansional manifold (i.e. a point) over $\mathbb{Y}$. This means that a further stratification of $\mathcal{S}_1^p$ involves, in general, a stratification on the parameter set $\mathbb{Y} \times \mathbb{E}$.

There are already some results on fast-slow singular bifurcation analysis. In our case we can have fold-like, transcritical and hysteresis fast-slow bifurcations (see (Kuehn, 2015, ch 3, ch 8)). This would induce a partition over set $\mathcal{S}_1^p$, among several sets which could be denoted as $\mathcal{S}_2^p$.

For instance there is a fast-slow fold bifurcation points exists if the following conditions hold

$$f_2(y,\varphi,0) = 0,\ f_{2,y_2}(y,\varphi,0) = 0,\ f_{2,y_2y_2}(y,\varphi,0) \neq 0,\ f_{2,y_1}(y,\varphi,0) \neq 0,\ f_1(y,\varphi,0) = 0$$

One of the most important results on the fast-slow system, the Fenichel (1979) theorem proves that in a neighbourhood of $\mathcal{S}_0^p$, that is for $\epsilon = 0$, the dimension of the stable manifold is not changed by a small variation of the parameter ϵ

Also, when $\epsilon \to 0$ and the equilibrium point is stable, the trajectories will approach a monotonic behaviour independently from the eigenvalues of the Jacobian (6.7) seen as a function of ϵ, $D^s_{\bar{y}}(\epsilon)$, having complex roots for values of ϵ sufficiently higher than ϵ. This means there that will be a critical level for ϵ such that oscillatory dynamics will change to monotonous.

The example in the previous subsection illustrate this important theorem.

6.3 Impasse singularities

In this subsection we study the dynamics of system (6.4) (or (6.3) in matrix form) under the assumption that ϵ has a fixed value different from zero and that function $\delta(.)$ can be equal to zero. We use the constrained-systems approach (see Zhitomirskii (1993), Llibre et al. (2002) and Riaza (2008)).

6.3.1 Scalar examples

Consider the equation, where $y : \mathbb{R}_+ \to \mathbb{Y} = \mathbb{R}$:

$$(1-y)\dot{y} = 1 \tag{6.23}$$

Let $y(0)=y_0$ and define $\tau \equiv (y_0-1)^2 > 0$. The general solution for equation is not unique

$$y(t) = \begin{cases} 1+(2(\tau-t))^{1/2} \\ 1-(2(\tau-t))^{1/2}. \end{cases}$$

To prove this we use the separation of variables method:

$$\int_{y(0)}^{y(t)} (1-y)dy = \int_0^t ds \Leftrightarrow y - \frac{1}{2}y^2\Big|_{y=y(0)}^{y=y(t)} = t \Leftrightarrow (y(t)-1)^2 = 2(\tau(y(0))-t) \Leftrightarrow y(t)-1 = \pm\sqrt{2(\tau-t)}.$$

The solution exists for the interval $t \in [0,\tau]$ and does not exist for the interval $t \in (\tau,+\infty)$, because $(2(\tau-t))^{1/2}$ is not a real number for t in this interval.

For the initial value problem we start to note that at $t=0$ we have

$$y(0) = \begin{cases} 1+|y_0-1| = y_0 & \text{if } y_0 > 1 \\ 1-|y_0-1| = y_0 & \text{if } y_0 < 1 \end{cases}$$

then the solution for the initial-value problem is

$$y(t) = \begin{cases} 1+(2(\tau(y_0)-t))^{1/2} & \text{if } y_0 \geq 1, \text{ and } t \in [0,\tau(y_0)] \\ 1-(2(\tau(y_0)-t))^{1/2}. & \text{if } y_0 \leq 1 \text{ and } t \in [0,\tau(y_0)] \end{cases}$$

If $y_0 < 1$ ($y_0 > 1$) the solution increases (decreases) until time $t=\tau$ when it reaches $y(\tau)=1$, and there is no continuation for $t > \tau$.

The solution has the following properties: (1) there is no steady state; (2) for any initial value $y(0)=y_0 \neq 1$ the solution is only defined for a finite time period $[0,\tau(y_0)]$ dependent on the initial value; (3) the solution branch depends upon the initial value be smaller of higher that $y=1$.

Figure 6.3 depicts the phase diagram and some trajectories: it can be seen that the solution cannot be continued (in finite time) when they reach the surface $\mathcal{S}_0^i$ which is in this case an impasse surface of the attractor type. Also note that the time of flight until impact, τ is dependent on the initial level for $y(0)$. We call *impasse point* to point $y=y^i=1$.

Equation 2

Now consider the equation

$$(1-y)\dot{y} = -1 \tag{6.24}$$

Using the same method, we get the general solution

$$y(t) = \begin{cases} 1+(2(\tau+t))^{1/2} \\ 1-(2(\tau+t))^{1/2}. \end{cases}$$

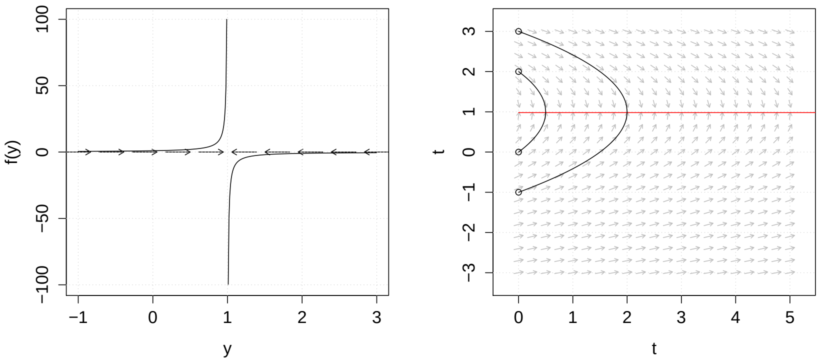


Figure 6.3: Phase diagram and trajectories for equation (6.23)

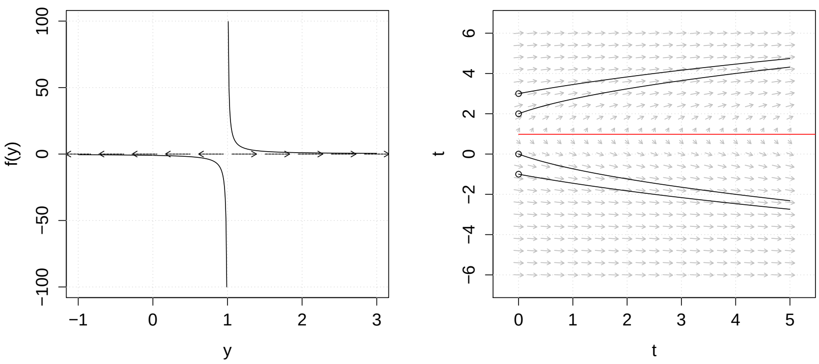


Figure 6.4: Phase diagram and trajectories for equation (6.23)

then we solution for the initial value problem is

$$y(t) = \begin{cases} 1 + (2(\tau + t))^{1/2} & \text{if}\, y_0 > 1 \\ 1 - (2(\tau + t))^{1/2} & \text{if}\, y_0 < 1 \end{cases}$$

and the solution can now be evaluated for $t \in [0, \infty)$. In this case the singular value $y^s = 1$ is repelling because $\lim_{t\to\infty} = \pm\infty$ (see figure 6.4). As with equation (6.23) there are no steady states, but now the impasse point $y^i = 1$ is a repeller point.

Example 3

Now consider the equation

$$(1 - y)\dot{y} = y \tag{6.25}$$

and consider the initial value $y(0) = y_0$.

Let $\tau \equiv y_0 - 1 - \ln(y_0)$. The initial value problem has the explicit solution

$$y(t) = \begin{cases} -W_0\left(-y_0 e^{t-y_0}\right) & \text{if}\, y_0 \leq 0, \text{ and}\, t \in [0, +\infty) \\ -W_0\left(-y_0 e^{t-y_0}\right) & \text{if}\, 0 < y_0 < 1, \text{ and}\, t \in [0, \tau] \\ -W_{-1}\left(-y_0 e^{t-y_0}\right) & \text{if}\, y_0 > 1, \text{ and}\, t \in [0, \tau] \end{cases}$$

where $W_0(.)$ and $W_{-1}(.)$ are the principal and secondary branches of the Lambert-W function (see Brito et al. (2008)).

To prove this, we employ the same method of separation of variables:

$$\int_{y(0)}^{y(t)} \left(\frac{1}{y} - 1\right) dy = t \Leftrightarrow \ln y(t) - y(t) = \ln y(0) - y(0) + t \Leftrightarrow y(t)e^{-y(t)} = y(0)e^{t-y(0)} \Leftrightarrow$$
$$\Leftrightarrow y(t) = -W\left(-y(0)e^{t-y(0)}\right)$$

This is because the Lambert-W function is defined as a solution of equation $xe^x = a$, $x = W(a)$. The Lambert function has the following properties that interest us here: (a) $W(x)$ is defined for the domain $x \in [-e^{-1}, +\infty)$, (b) it is single-valued and increasing for $x > 0$, but it has two values for $x \in [-e^{-1}, 0)$, $W_0(x) \in [-1, 0)$ and $W_{-1}(x) \in [-1, -\infty)$; (c) at point $x = -e^{-1}$ we have $W_0(-e^{-1}) = W_{-1}(-e^{-1}) = -1$.

Therefore, letting $\eta(t) = -y(0)e^{t-y(0)}$ we observe that $\eta(t) = -e^{-1}$ if and only if $t = \eta$ and also, because of the properties of the W-function we conclude: (a) if $y_0 < 0$ then $\eta(t) > 0$ and the solution is $W(\eta(t)) = W_0(\eta(t))$ is unique; (b) if $y_0 > 0$ then the solution is only defined for $\eta(t) \leq -e^{-1}$ that is for $t \leq \tau$ and the solution has two branches; (c) at the point $t = \tau$ then $W(\eta(\tau)) = -1$ where the solution is $y(\tau) = 1$; (d) by continuity at point $y_0 = 0$ we see that for $0 < y_0 < 1$ the solution is given by branch $W_0(\eta(t))$ and for $y_0 > 1$ by branch $W_1(\eta(t))$.

Figure 6.5 presents the phase diagram and some trajectories. Now we see that there is a steady state at point $\bar{y} = 0$ and an impasse point is an attractor point.

We see that the steady state is a regular point. We can study its local dynamics as usual. Write equation (6.25) as

$$\dot{y} = f(y) \equiv \frac{y}{1-y}.$$

Function $f(y)$ has the derivative

$$Df(y) = f^{'}(y) = \frac{1}{(1-y)^2}$$

Then, at the steady state as $f^{'}(0) = 1 > 0$ we see that it is unstable.

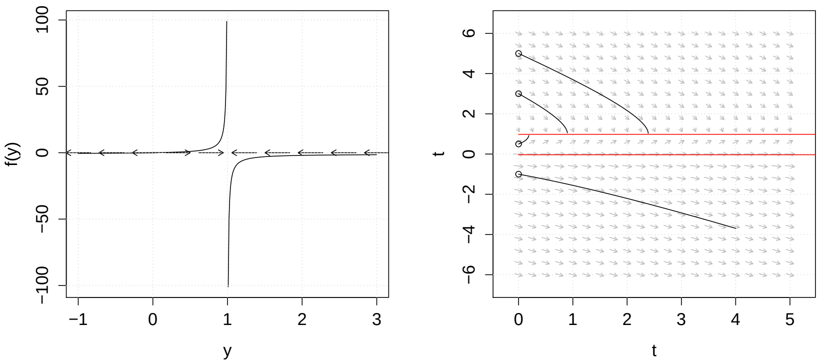


Figure 6.5: Phase diagram and trajectories for equation (6.25)

6.3.2 Planar examples

We use the foliowing definitions and results. First, we write the planar system in the form $A(y)\dot{y} = F(y)$, where

$$\begin{pmatrix} a_{11}(y_1, y_2) & a_{12}(y_1, y_2) \\ a_{21}(y_1, y_2) & a_{22}(y_1, y_2) \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2). \end{pmatrix}$$

We define

$$\delta(y_1, y_2) \equiv \det A(y_1, y_2).$$

We call **impasse point** to a point (y_1^s, y_2^s) such that $\delta(y_1^s, y_2^s) = 0$ and **regular point** to a point (y_1, y_2) such that $\delta(y_1, y_2) \neq 0$. The **impasse set** or impasse surface is defined as

$$\mathcal{S}_0^i = \{(y_1, y_2) \in \mathcal{Y} : \ \delta(y_1, y_2) = 0\}.$$

An impasse point is called **impasse-regular** if

$$\frac{\partial \delta(y_1^s, y_2^s)}{\partial y_1} f_1(y_1^s, y_2^s) + \frac{\partial \delta(y_1^s, y_2^s)}{\partial y_2} f_2(y_1^s, y_2^s) \neq 0$$

if this expression is positive a point is called **impasse-repeller** and if it is negative it is an **impasse-attractor** point. We call **impasse singular** to a point such that

$$\frac{\partial \delta(y_1, y_2)}{\partial y_1} f_1(y_1, y_2) + \frac{\partial \delta(y_1, y_2)}{\partial y_2} f_2(y_1, y_2) = 0$$

This expression can be equal to zero because the gradient of $\delta(y_1, y_2)$ is zero or if $F(y_1, y_2) = 0$. In the first case the trajectory is tangent to $\mathcal{S}_0^i$ and in the second case it is transversal. At second type of points we can study the local dynamics in a similar way of an equilibrium point, but by means of the so-called **regularized system** $\dot{y} = \mathrm{adj}(A)F(y)$, that is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} a_{22}(y_1, y_2) & -a_{12}(y_1, y_2) \\ -a_{21}(y_1, y_2) & a_{11}(y_1, y_2) \end{pmatrix} \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

the eigenvalues of the Jacobian of this system, evaluated at a point such that $\delta(y_1, y_2) = 0$ and $f_1(y_1, y_2) = 0$, allows us to classify the impasse-singular point as a transversal saddle,node or node. We will see next examples of those points.

Zhitomirskii (1993) presents the following normal forms for planar equations with impasse singularities, also called **constrained ordinary differential equations**:

Impasse-repeller points

$$\begin{aligned} \dot{y}_1 &= 0 \\ y_2\, \dot{y}_2 &= 1 \end{aligned} \tag{6.26}$$

Using the previous methods it is easy to see that the solution of the equation (6.26) is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_1(0) \\ \pm \left(\frac{t}{2} + y_2(0)^2\right)^{1/2} \end{pmatrix}$$

Therefore the solution exists for $t \in [0, \infty)$ and if $y_2(0) > 0$ then $\lim_{t\to\infty} y_2(t) = +\infty$ and if $y_2(0) < 0$ then $\lim_{t\to\infty} y_2(t) = -\infty$. Therefore there is an impasse singularity at point $y_2 = 0$.

Because $\delta(y_1, y_2) = y_2$, we can see that the impasse set $\mathcal{S}_0^i$ is the line $y_2 = 0$, which is geometrically equivalent to the $y_1 = 0$ axis (see figure 6.6). We can also see that

$$\nabla\delta(y_1, y_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ F(y_1, y_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then $\nabla\delta(y_1, y_2) \cdot F(y_1, y_2) = 1 > 0$, for any point, including in the impasse surface. Then we conclude that the singular set is only composed of impasse-regular points which are repeller and there are no impasse singular points.

Impasse-attractor poins

$$\begin{aligned} \dot{y}_1 &= 0 \\ y_2\, \dot{y}_2 &= -1 \end{aligned} \tag{6.27}$$

The solution is now

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_1(0) \\ \pm \left(y_2(0)^2 - \frac{t}{2}\right)^{1/2} \end{pmatrix}.$$

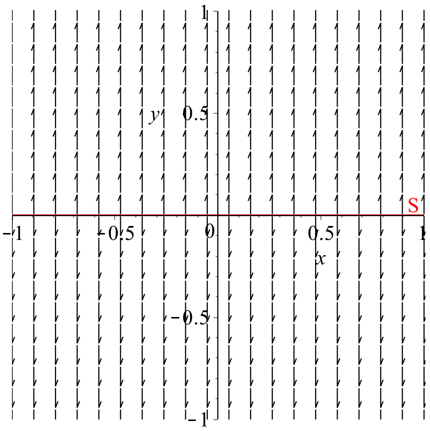


Figure 6.6: Phase diagram and trajectories for equation (6.26)

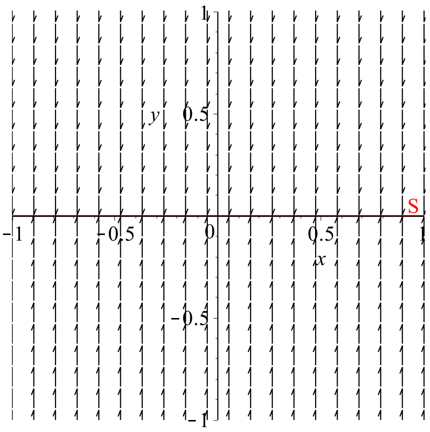


Figure 6.7: Phase diagram and trajectories for equation (**??**)

It is clear that it only exists for $t \in [0, \tau]$, for $\tau \equiv 2(y(0)_2)^2$.

In qualitative terms, we have again $\delta(y_1, y_2) = y_2$ and the singular set is $y_2 = 0$. However we have now $\nabla\delta(y_1, y_2) \cdot F(y_1, y_2) = -1 < 0$. We see now that the impasse surface is repeller (see figure (6.7)).

Impasse-tangent point

$$\begin{aligned} \dot{y}_1 &= 0 \\ (y_1 + y_2^2)\,\dot{y}_2 &= 1 \end{aligned} \tag{6.28}$$

We know again that $y_1(t) = y_1(0)$, an arbitrary initial point, and, if we try to use the method

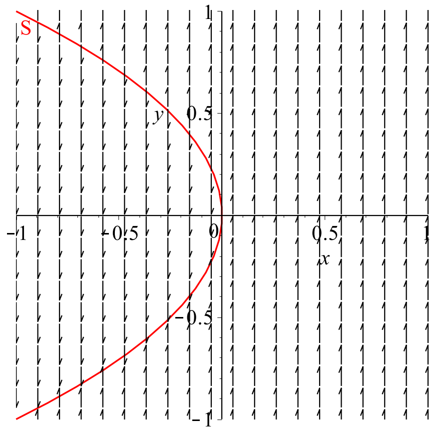


Figure 6.8: Phase diagram and trajectories for equation (6.28)

of separation of variables, $y_2(t)$ is the solution of the cubic equation

$$y_2(t)^3 + 3y_1(0)y_2(t) + 3(t - \tau) = 0$$

where $\tau \equiv -y_2(0)(y_1(0) + (1/3)y_2(0)^2)$. We know that this polynomial equation can have one or three real roots in the generic case or two real roots in the non-generic case.

However, we can apply the qualitative analysis to characterise the solution (see figure 6.8). We have $\delta(y_1, y_2) = y_1 + y_2^2$. This means that that the impasse set is $\mathcal{S}_0^i = \{(y_1, y_2) : y_1 + y_2^2 = 0\}$ a parabola which is only defined for $y_1 \leq 0$ and for all y_2 As $\nabla\delta(y_1, y_2) \cdot F(y_1, y_2) = 2y_2$ then the branch of $\mathcal{S}_0^i$ such that $y_2 > 0$ ($y_2 < 0$) is the locus of impasse-regular repeller (attractor) points.

There is an impasse-singular at $(y_1, y_2) = (0, 0)$ because at this point both $\nabla\delta(y_1, y_2) \cdot F(y_1, y_2) = 0$ and $\nabla\delta(y_1, y_2) = (0, 0)^\top$, which means that a tangent trajectory passes through it.

Transversal node and saddle

The normal form for the existence of an impasse-transversal saddle point or an impasse-transversal node point can be obtained from the system

$$\begin{aligned} \frac{1}{\lambda}\dot{y}_2 - \dot{y}_1 &= 1 \\ (y_2 - y_1)\,\dot{y}_2 &= \lambda y_2 \end{aligned} \tag{6.29}$$

for λ. If $\lambda > 0$ we get a transversal node (see 6.9 left-panel) and if $\lambda > 0$ we get a transversal saddle (see 6.9 right-panel). In both cases the impasse set is

$$\mathcal{S}_0^i = \{(y_1, y_2) : y_1 = y_2\}$$

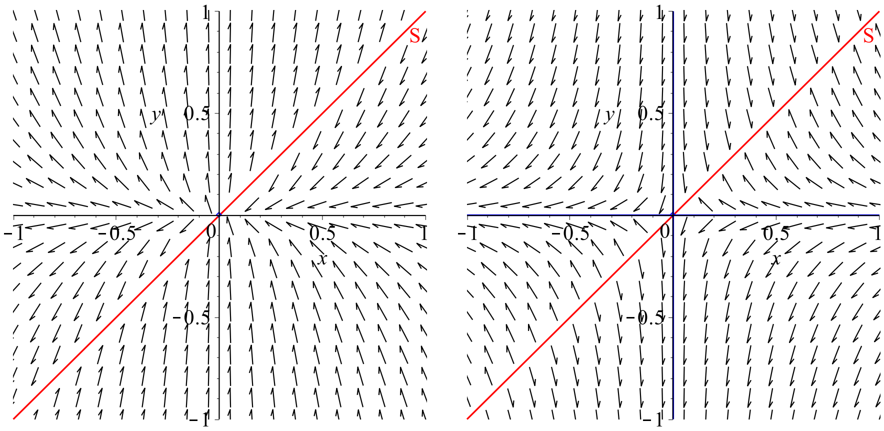


Figure 6.9: Phase diagram and trajectories for equation (6.29) for $\lambda > 0$ and $\lambda < 0$

and there is an impasse-transversal point at the origin $y^i \equiv (y_1^i, y_2^i) = (0, 0)$. Differently from the previous cases there are trajectories that cross the singular surface from the subset $\mathcal{Y}_- = \{(y_1, y_2) : y_2 - y_1 < 0\}$ to the subset $\mathcal{Y}_+ = \{(y_1, y_2) : y_2 - y_1 > 0\}$. However, the local dynamics and the number of crossing trajectories is different and depends on the value of the parameter λ. If $\lambda > 0$ the impasse-transversal point is called an **impasse-transversal node** point and if $\lambda < 0$ it is called an **impasse-transversal saddle** point.

In the first case there are an infinite number of trajectories crossing the impasse-transversal point y^i from $\mathcal{Y}_-$ to $\mathcal{Y}_+$ and in the second there are two trajectories crossing y^i, one from $\mathcal{Y}_-$ to $\mathcal{Y}_+$ and another in the opposite direction but with a different slope.

To be completed

Transversal focus

The normal form for the existence of an impasse-transversal focus point can be obtained from the system

$$\begin{aligned} \lambda\dot{y}_1 - \dot{y}_2 &= 1 + \lambda^2 \\ y_2\,\dot{y}_2 &= -\lambda y_1 - y_2 \end{aligned} \tag{6.30}$$

for $\lambda > 0$. The phase diagram is in **??** where it displays an **impasse-transversal focus** point. The impasse set is

$$\mathcal{S}_0^i = \{(y_1, y_2) : y_2 = 0\}$$

and there is an impasse-transversal point at the origin $y^i \equiv (y_1^i, y_2^i) = (0, 0)$. This points is locally a focus point. Although there are trajectories that reach this point in finite time from one subset ($\mathcal{Y}_+$

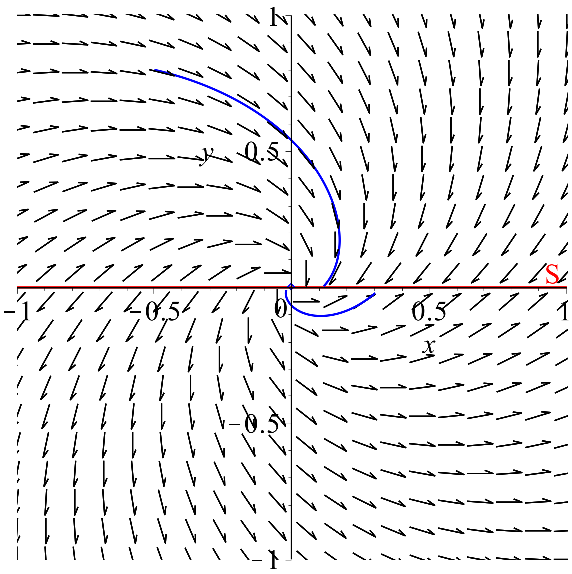


Figure 6.10: Phase diagram and trajectories for equation (6.30)

in figure **??**) and trajectories that leave this point to the other subset ($\mathcal{Y}_-$) it can be proved that there are no crossing trajectories. That is all the trajectories starting in set $\mathcal{Y}_-$ have no continuation after a finite time. It can be proved that the system in polar cooordinates is similar to the attractor case if the initial point is in $\mathcal{Y}_+$.

To be completed

6.4 A general approach

To be corrected

A point y_i in $\mathbb{Y}$ such that $\delta(y_i, \varphi, \epsilon) = 0$ is called an *impasse point*. The *impasse set* is the set of all impasse points

$$\mathcal{S}^i = \{(y, \varphi, \epsilon) \in \mathbb{Y} \times \mathbb{E} : \ \delta(y, \varphi, \epsilon) = 0, \ \epsilon \neq 0\}, \tag{6.31}$$

which is a family of points parameterized by (φ, ϵ) and it is contained in singular set $\mathcal{S}$ defined in equation (6.7).

If the set $\mathcal{S}^i \cup \mathcal{S}^p$ is empty then there are only regular points in set $\mathbb{Y}$ and we can study the dynamics by characterising the solution of equation (6.6) as mentioned earlier.

If the set $\mathcal{S}^i$ is non-empty then we have both regular and impasse points. As equation $\delta(y, \varphi, \epsilon) = 0$ defines, in general, a one-dimensional manifold over the set $\mathbb{Y}$, for a given pair (φ, ϵ), it introduces a partition of $\mathbb{Y}$ into two subspaces of regular points

$$\mathbb{Y}_- = \{y \in \mathbb{Y} :\ \epsilon\,\delta(y, \varphi, \epsilon) < 0\},\ \mathbb{Y}_+ = \{y \in \mathbb{Y} :\ \epsilon\,\delta(y, \varphi, \epsilon) > 0\}.$$

It also introduces a partition over the set of equilibrium points Γ_E into two sets of regular equilibrium points

$$\Gamma_{E_-} = \{y \in \mathbb{Y}_- :\ f_1(y, \varphi, \epsilon) = f_2(y, \varphi, \epsilon) = 0\},\ \Gamma_{E_+} = \{y \in \mathbb{Y} :\ f_1(y, \varphi, \epsilon) = f_2(y, \varphi, \epsilon) = 0\}$$

and a set of impasse-steady state

$$P_E = \{(y, \varphi, \epsilon) \in \mathcal{S}^i :\ f_1(y, \varphi, \epsilon) = f_2(y, \varphi, \epsilon) = 0\}.$$

The dynamics in the neighbourhood of a regular point, belonging to each of the two subsets $\mathbb{Y}_-$ and $\mathbb{Y}_+$ is well known, but the dynamics in the neighbourhood of a singular point is not. A non-empty impasse set introduces some possible local dynamics which do not occur in equations in which it is empty. For instance, if there are two equilibrium points in a regular ODE the dimension of the stable manifolds is different at every equilibrium point. In the impasse surface is non-empty and the there is one steady state in every subset Γ_{E_-} and Γ_{E_+} it is possible that the dimension of the local stable manifolds can be the same (for instance two saddle points). This cannot occur in regular ODE's.

The number of possible dynamics is much richer than in regular ODE's. We will focus on two issues: (1) what kind of dynamics should we expect in the neighbourhood of $\mathcal{S}^i$? (2) Under which conditions there are paths that cross $\mathcal{S}^i$?

For points in $\mathcal{S}^i$ we consider the ODE

$$\dot{y} = \mathcal{F}^r(y, \varphi, \epsilon) = A^*(y, \varphi, \epsilon)\, F(y, \varphi, \epsilon) \tag{6.32}$$

where $\mathcal{F}^r$ is called the *de-singularised vector field.* expanding we have

$$\dot{y} = \begin{pmatrix} \epsilon\,\delta(y, \varphi, \epsilon)\, f_1(y, \varphi, \epsilon) \\ f_2(y, \varphi, \epsilon) \end{pmatrix}.$$

Compared with (6.3) the phase portrait of (6.32) will be the same for points in $\mathbb{Y}_+$ but will invert the sense of the flows in set $\mathbb{Y}_-$. It is introduced because it allows for uncovering the dynamics over set $\mathcal{S}^i$ because it removes the singularities (which originate infinite-valued eigenvalues) and transforms it into an equilibrium point of the de-singularized vector field.

From now on, we assume that an impasse-point is not critical, meaning that for every point $y^i \in \mathcal{S}^i$ the gradient of function $\delta(y^i, .)$ is different from zero. That is

$$D_y\delta(y^i, \varphi, \epsilon) = \left(\delta_{y_1}(y^i, \varphi, \epsilon), \delta_{y_2}(y^i, \varphi, \epsilon)\right)^\top \neq (0, 0)^\top$$

Local dynamics at the impasse set

The approach by Zhitomirskii (1993) and Llibre et al. (2002) is based on the determination of the existence of the kernel and image line fields generated by matrix A and their relationship with $\mathcal{S}^i$. We define two functions [3] $k(y,\varphi,\epsilon) = D_y \det A(y,\varphi,\epsilon)\cdot v$ where $v \in \text{kern}(A(y,\varphi,\epsilon)) - \{(0,0)\}$ and $i(y,\varphi,\epsilon) = \mathcal{F}^r(y,\varphi,\epsilon)$. There are two types of impasse-singular points: a point y^i_k such that $k(y^i_k,\varphi,\epsilon) = 0$ is called *kernel-singular point* and a point y^i_i such that $i(y^i_i,\varphi,\epsilon) = 0$ is called *image-singular point.* A point which is not kernel singular is called *kernel regular* and a point which is not image singular is called *image-regular.*

As $k(y,\varphi,\epsilon) = \epsilon\delta_{y_2}(y,\varphi,\epsilon)$ and $i(y,\varphi,\epsilon) = (0, f_2(y,\varphi,\epsilon))^\top$ the sets of kernel-singular and and image-singular points are

$$\Gamma_K = \{(y,\varphi,\epsilon) \in \mathcal{S}^i :\ \delta_{y_2}(y,\varphi,\epsilon) = 0\}$$

and

$$\Gamma_I = \{(y,\varphi,\epsilon) \in \mathcal{S}^i :\ f_2(y,\varphi,\epsilon) = 0\}$$

which correspond to sets of impasse-singular points. An *impasse-regular* point is a point which is not kernel-singular nor image-singular; that is such that $k(.) \neq 0$ and $i(.) \neq 0$. (Riaza, 2008, p. 164) proves that condition $D_y \det A(y,\varphi,\epsilon)\cdot i(y,\varphi,\epsilon) \neq 0$ is equivalently verified.

Then the set of *impasse-regular* points is the open set

$$\mathcal{S}^i_0 = \left\{(y,\varphi,\epsilon) \in \mathcal{S}^i : \delta_{y_2}(y,\varphi,\epsilon) f_2(y,\varphi,\epsilon) \neq 0\right\}$$

and the set of *impasse-singular* points is

$$\mathcal{S}^i_1 = \left\{(y,\varphi,\epsilon) \in \mathcal{S}^i :\ \delta_{y_2}(y,\varphi,\epsilon) = 0, \text{ or } f_2(y,\varphi,\epsilon) = 0\ \right\}$$

The set $\mathcal{S}^i_1$ induces a partition over $\mathcal{S}^i_0$ between two subsets: the set of *attractor impasse points*

$$\mathcal{S}^{i,a}_0 = \left\{y,\varphi,\epsilon) \in \mathcal{S}^i : \epsilon\, \delta_{y_2}(y,\varphi,\epsilon)\, f_2(y,\varphi,\epsilon) < 0\right\}$$

and *repeller impasse points*

$$\mathcal{S}^{i,r}_0 = \left\{y,\varphi,\epsilon) \in \mathcal{S}^i : \epsilon\, \delta_{y_2}(y,\varphi,\epsilon)\, f_2(y,\varphi,\epsilon) > 0\right\}.$$

The geometrical intuition of the several impasse points that were defined is the following: at a repeller impasse point flows depart from $\mathcal{S}^i$ from both sides ($\mathbb{Y}_+$ and $\mathbb{Y}_-$), while at an attractor point they terminate at $\mathcal{S}^i$ in **finite** time, at a kernel-singular point (see figure **??** upper-left panel) a flow is tangent to $\mathcal{S}^i$, also in finite time, and at a image-singular point it is transversal to $\mathcal{S}^i$ and can cross it (in finite or infinite time).

[3] The dot $\cdot$ indicates internal product.

This means that if a trajectory collides with $\mathcal{S}^i$ at an impasse-attractor point it cannot be continued after the time of collision, and cannot be continued until $t \to \infty$, if it collides with a kernel-singular point it will be deflected back to the sub space of $\mathbb{Y}$ ($\mathbb{Y}_-$ or $\mathbb{Y}_+$) from where it came from.

The set $\mathcal{S}_1$ can be stratified further into three types of sets: (1) several types of non-degenerate image-singular and kernel regular points, (2) several types of non-degenerate image-regular and kernel-singular points, and (3) several degenerate image-singular or kernel regular points.

next consider an image-singular and kernel-regular point y_i^i (verifying $\delta(y_i^i, .) = f_2(y_i^i, .) = 0$). It is an equilibrium point of the desingularized vector field (see equation (6.32)). Geometrically it is in the intersection of the isocline $\dot{y}_2 = 0$ and the curve $\delta(yi, .) = 0$ which are both one-dimensional manifolds in $\mathbb{Y}$. The jacobian of the de-singularized vector field, evaluated at an image-singular point is

$$D^r_{y_i^i} \equiv D_y \mathcal{F}^r(yi_i^i, \varphi, \epsilon) = \begin{pmatrix} \epsilon\delta_{y_1}(y_i^i, .) f_1(y_i^i, .) & \epsilon\delta_{y_2}(y_i^i, .) f_1(y_i^i, .) \\ f_{2,y_1}(y_i^i, .) & f_{2,y_2}(y_i^i, .) \end{pmatrix}$$

has trace, determinant and discriminant given by

$$\begin{aligned} \text{trace} D^r_{y_i^i} &= \epsilon\delta_{y_1}(y_i^i, \varphi, \epsilon) f_1(y_i^i, \varphi, \epsilon) + f_{2,y_2}(y_i^i, \varphi, \epsilon) \\ \det D^r_{y_i^i} &= \epsilon f_1(y_i^i, \varphi, \epsilon) \left(\delta_{y_1}(y_i^i, \varphi, \epsilon) f_{2,y_2}(y_i^i, \varphi, \epsilon) - \delta_{y_2}(y_i^i, \varphi, \epsilon) f_{2,y_1}(y_i^i, \varphi, \epsilon)\right) \\ \Delta D^r_{y_i^i} &= \left(\frac{\text{trace} D^r_{y_i^i}}{2}\right)^2 - \det D^r_{y_i^i}. \end{aligned}$$

We say there is a *transversal saddle* if $\det D^r_{y_i^i} < 0$, a transversal node if $\Delta D^r_{y_i^i} > 0$ and a transversal focus if $\Delta D^r_{y_i^i} < 0$. The image-singular point is degenerate if it is non-hyperbolic, that is Jacobian $D^r_{y_i^i}$ has at least one eigenvalue with zero real part. We have stable or unstable nodes and foci with the usual conditions, the eigenvalues have negative or positive real parts. All this results pertain to vector field $\mathcal{F}^r$. They allow to understand the behaviour in the singular vector field $\mathcal{F}$ by removing the singularity: the paths in subspace $\mathbb{Y}_+$ (where $\det A > 0$) and are reversed in subspace $\mathbb{Y}_-$.

Therefore we can have a geometrical interpretation of the local dynamics in the neighbourhood of an image-singular point (see figure **??**). First, if the image-singular point is a transversal saddle (see figure **??** bottom-left panel) there are two one-dimensional stable manifolds, passing through point y_i^i, belonging each to one of the subspaces $\mathbb{Y}_+$ and $\mathbb{Y}_-$, but are not collinear at the image-singular point. Second, if it is a transversal fold (see figure **??** upper-right panel) it is a sink from one of the subspaces partitioned by $\mathcal{S}^i$ and a source from the other subspace: if $\text{trace} D^r_{y_i^i} < 0$ ($\text{trace} D^r_{y_i^i} > 0$) the point yi_i^i has a basin of attraction within subspace $\mathbb{Y}_+$ ($\mathbb{Y}_-$) and has divergent trajectories over subspace $\mathbb{Y}_-$ ($\mathbb{Y}_+$). Third, If it is a transversal focus (see figure **??** bottom-tight panel) the oscillatory trajectories change sign when they cross $\mathcal{S}^i$, they are repelling if they cross at $\mathcal{S}_0^{i,r}$ or are attracted if they cross at $\mathcal{S}_0^{i,a}$. In this case they cannot be continued it time upon contact.

From the previous results three important conclusions emerge. First, a **necessary** condition for a candidate DGE path $y(t)$ to cross $\mathcal{S}^i$ at time t_A is that it is is is an image-singular point: at this point $y(t_A) = y_i^i$ and[4]

$$\mathrm{adj}\,(A(y(t_A), \varphi, \epsilon))\, F(y(t_A), \varphi, \epsilon) = 0,$$

for every given pair (φ, ϵ). Second, a second stronger necessary condition is that the image singular point should be a transversal saddle or a transversal node. Transversal focus cannot generate crossing trajectories. There are other conditions as well, basically associated to the need for the crossing to be in the "right" sense given the location of the initial point and of the steady state (in $\mathbb{Y}_-$ or $\mathbb{Y}_+$). Third, singular steady states, i.e, steady states located in $\mathcal{S}^i$, $\bar{y}_i$, are degenerate image-singular points, because, at a singular steady state we have $f_1(\bar{y}^i, \varphi, \epsilon) = 0$ which implies that $\det D^r_{\bar{y}^i} = 0$. Furthermore, the existence of a steady state is only possible for particular values of parameters (in fact for a one-dimensional manifold in the space $\Phi \times \mathbb{E}$.

We have found some important differences between the previous impasse points: attractive impasse points are reached in finite time and the trajectories reaching them have no continuation in time, transversal saddles have one trajectory that reaches an impasse point in finite time and can be continued afterwards, transversal foci have from one side an infinite number of trajectories that reach an impasse point in finite time and can be continued afterwards, and singular steady states can have one trajectory that reaches the impasse set asymptoticaly, provided that $\mathrm{trace} D^r_{\bar{y}^i} < 0$ (and trajectories came from the "right" side).

Together with singular steady states, degenerate image-singular or kernel-singular points can be assembled into set $\mathcal{S}_2^i$. This last set gathers four types of elements: (1) degenerate image-singular points, (2) degenerate kernel-singular points; (3) kernel-singuar points and (4) steady states belonging to set $\mathcal{S}^i$. Observe that set $\mathcal{S}_2^i$ consists of points in the impasse-singular surface (i.e., manifolds of dimension zero in $\mathbb{Y}$) and manifolds of co-dimension 1 or two in the parameter space $\Phi \times \mathbb{E}$. We cal *impasse bifurcation* points to member of set $\mathcal{S}_2^i$ with the exclusion of degenerate steady states. A degenerate singular steady state point is called a *singularity induced bifurcation point*[5]

[4]See Cardin et al. (2012).

[5]See Llibre et al. (2002) for a complete presentation of local bifurcations for the planar constrained ODE case.

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