Advanced Mathematical Economics

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Chapter 1

Scalar ODEs

1.1 Introduction

A scalar ordinary differential equation is a functional equation allowing for finding and characterizing a **cross-section distribution** or a **deterministic process** of type $\Big(y(x)\Big)_{\mathrm{X}}$, where $y : \mathrm{X} \to \mathrm{Y} \subseteq \mathbb{R}$, is a mapping $y = y(x)$, where x is a real number belonging to the domain $\mathrm{X} \subseteq \mathbb{R}$, and y is also a real number belonging to the set $\mathrm{Y} \subseteq \mathbb{R}$.

It is specified by using a **variational approach**. This means that the law governing the process is the solution of a functional equation containing the derivative of $y(x)$,

$$y'(x) \equiv \frac{dy(x)}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

Therefore, ODE models phenomena which we describe by specifying a local interaction.

Definition 1. *A* ***scalar ordinary differential equation*** *(ODE) is a functional equation, defined over function $y(x)$, of the form*

$$F(y'(x), y(x), x) = 0, \text{ for } x \in \mathrm{X} \subseteq \mathbb{R} \tag{1.1}$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ is a known continuous and differentiable function.

That is, while function $F(\cdot)$ is known, function $y(\cdot)$ is unknown.

In economics, we call endogenous variable to function $y(\cdot)$, and the ODE commonly takes the form $F(y'(x), y(x), z(x)) = 0$, where $z : \mathrm{X} \to \mathbb{R}$ is called an exogenous variable.

Solving an ODE means finding function $y(x)$ which verifies equation (1.1). The existence, uniqueness, and the properties of the solutions depend on the nature of function $F(\cdot)$.

A well developed theory on the characterization of solutions is provided by the case in which the independent variable is time. In this case, the unknow function is $y : \mathrm{T} \subseteq \mathbb{R} \to \mathbb{R}$ and the derivative is denoted by the Newtonian notation $\dot{y} = \dfrac{dy(t)}{dt}$ and the ODE is

$$F(\dot{y}, y(t), t) = 0$$

or

$$F(\dot{y}, y(t), z(t)) = 0$$

where $z : \mathrm{T} \to \mathbb{R}$ is an exogenous variable.

In most applications, we are interested in characterizing the behavior of the solution of an ODE of type

$$F(y'(x), y(x), x; \varphi) = 0, \text{ if } \ y : \mathrm{X} \to \mathbb{R}$$

or

$$F(\dot{y}, y(t), t; \varphi) = 0, \text{if } \ y : \mathrm{T} \to \mathbb{R}$$

where $\varphi \in \Phi \subseteq \mathbb{R}^m$ is a vector of parameters. When time is the independent variable, the qualitative (geometric) theory of the change in behavior of the solution, that is of the dynamic process $\Big(y(t)\Big)_{t \in \mathrm{T}}$ for different values of the parameter φ is called **bifurcation theory**.

1.2 Classification of scalar ODE

The solutions of an ODE depend crucially on the form of function $F(\cdot)$. Therefore, it is useful to classify ODE's according to its form.

There are several criteria for classification:

First, according to the dependency on the independent variable, we have

- a **non-autonomous** ODE if $F(\cdot)$ depends directly on the dependent variable x, as in equation (1.1);
- a **autonomous** ODE if $F(\cdot)$ does not depend directly on x, i.e., it has form $F(y'(x), y(x)) = 0$.

Second, an ODE is in **explicit** of form if it is written as

$$y'(x) = f(y(x), x), \ x \in \mathrm{X} \subseteq \mathbb{R}. \tag{1.2}$$

otherwise it is called an ODE in **implicit form** as (1.1).

Third, if we consider an ODE in explicit form it is

- **homogeneous** if $f(\cdot)$ is an homogeneous function of y;
- **non-homogeneous** if $f(\cdot)$ is not an homogeneous function of y.

Fourth, scalar ODEs can also be classified according to the linearity properties of $F(\cdot)$

- an ODE is **quasi-linear** if $F(\cdot)$ is a linear function of $y'(x)$. A general form of this equation is
$$A(y(x),x)\,y'(x)+B(y(x),x)=0;$$
where $A:\mathrm{Y}\times\mathrm{X}\rightarrow\mathbb{R}$ and $B:\mathrm{Y}\times\mathrm{X}\rightarrow\mathbb{R}$;

- an ODE is **semi-linear** if $F(\cdot)$ is a linear function of $y'(x)$ and its coefficient is independent of $y(x)$. A general form of this equation is
$$A(x)\,y'(x)+B(y(x),x)=0;$$
where $A:\mathrm{X}\rightarrow\mathbb{R}$ and $B:\mathrm{Y}\times\mathrm{X}\rightarrow\mathbb{R}$;

- an ODE is **linear** if $F(\cdot)$ is a linear function of both $y'(x)$ and $y(x)$. A general form of this equation is
$$A(x)\,y'(x)+B(x)\,y(x)+C(x)=0;$$
where $A:\mathrm{X}\rightarrow\mathbb{R}$, $B:\mathrm{X}\rightarrow\mathbb{R}$, and $C:\mathrm{X}\rightarrow\mathbb{R}$.

- an ODE is **non-linear** if $y'(x)$ enters equation (1.1) is non-linear: example $(y'(x))^2=f(y(x),x)$. In most applied math literature, an ODE in explicit form $y'(x)=f(y,x)$ is called non-linear if function $f(\cdot)$ is a non-linear function.

Fifth, we can distinguish quasi-linear ODE's between **regular** and **singular** equations. If the coefficient function $A(y)\neq 0$ for every $y\in\mathrm{Y}$ then we cay that the ODE is **regular**. In this case, a quasi-linear ODE can be writen in the explicit form $y'(x)=f(y(x))$ for $x\in\mathrm{X}\subseteq\mathbb{R}$. However, if there is at least a value for y, say y^s such that $A(y^s)=0$ then we say that the ODE is **singular**.

Sixth, consider an autonomous ODE in explicit form $y'(x)=f(y(x))$. We have assumed until now that function $f(\cdot)$ is continuous and differentiable. In the case in which $f(y)$ is not continuous or differentiable we say that ODE is **discontinuous**. For instance
$$y'(x)=\begin{cases}f_1(y(x),x) & \text{if } \; h(y(x),x)\leq 0\\ f_2(y(x),x) & \text{if } \; h(y(x),x)>0.\end{cases}$$

1.3 Solving ODEs

In the **scalar ODE in explicit form**
$$y'(x)\equiv\frac{dy(x)}{dx}= \; f(y(x),x),\; x\in\mathbb{R}.$$
a solution exists if there is at least an element in the set of functions $\mathcal{Y}$, that solves the ODE. Solving a scalar ODE means finding a function, say $\phi\mathcal{Y}$, such that $\phi:\mathrm{X}\rightarrow\mathrm{Y}$ such that $\phi^{'}(x)=f(\phi(x),x,\varphi)$.

The question of the existence and uniqueness of solutions is related to number of elements of $\mathcal{Y}$ that satisfy the differential equation.

In general, the known function $f(\cdot)$ constrains the properties of the elements of set $\mathcal{Y}$.

If function$f(\cdot)$ is continuous and differentiable, then the solution of solutions of the differential equation are elements of the space of continuous and differentiable functions.

Most non-linear differential equations do not have **explicit, exact or closed-form** solutions. This is not the case for linear equations of type

$$y'(x) = a(x, \varphi)\, y(x) + b(x, \varphi),$$

which can be solved explicitly (see chapter 2).

The qualitative theory of differential equations essentially addresses the solution of non-linear ODEs by using knowledge about the solution of related linear equations.

1.4 ODE and ODE problems

In general a model (or a problem) involving an ODE takes the form

$$\begin{cases} F(y'(x), y(x), x) = 0 \\ \mathsf{F}[y] \ = \text{constant} \end{cases}$$

where $\mathsf{F}[y]$ is a functional over $y \in \mathcal{Y}$, $\mathsf{F} : \mathcal{F} \to \mathbb{R}$, where $\mathcal{F}$ is the set of functions $y(\cdot)$.

To distinguish the solution to an ODE from the solution to a model (or a problem) involving an ODE, we call **general solution** to the solution of an ODE and **particular solution** to the solution of the latter. Although linear scalar ODE have one unique solution, models (or problems) involving them may not have solutions (if the constraint cannot be satisfied by the solution of the ODE). That is, the fact that a general solution exists and is unique does not imply that the particular solution exists.

The characterization of the solution of a model featuring an ODE has a close relationship to the type of side conditions which are assumed. For instance, in models in which the independent variable is not time the constraint takes the form $\int_{\mathrm{X}} \beta(y(x), x) dx = 0$. In general we have moment conditions, and we are interested in some global characteristics of the solution curve.

In models in which the independent variable is time the constraint sometimes takes to form $\int_{\mathrm{T}} \delta(t-t_0) y(t) dx = y_{t_0}$ where $\delta(\cdot)$ is Dirac's delta generalized function. In this case we fix the value of the function for a particular value of the independent variable time, and want to characterize the evolution of the solution within set Y across time. This leads to the stability and bifurcation analysis of the model: stability regarding some fixed points of Y, existence of invariant sets, dependence of the solution on parameters.

1.5 Backward and forward ODE

Consider a particular point $x_0 \in \mathrm{X} \subseteq \mathbb{R}$, because $\mathbb{R}$ has an order structure. We can classify further the ODE in the integral from as a **forward** ODE if

$$y(x) = y(x_0) + \int_{x_0}^{x} f(y(s), s)\, ds$$

or as a **backward** ODE if

$$y(x^0) = y(x) + \int_{x}^{x^0} f(y(s), s)\, ds.$$

In economics, when the independent variable is time, this distinction is very important. It distinguishes the dynamics generated from past events, usually related to stocks or quantities variables, from anticipated events, usually related to prices or return variables.

Chapter 2

Scalar linear ODE

A **linear scalar ODE** in explicit form is an ODE in which $f(\cdot)$ is a linear function of y. The most general form is

$$y'(x) = f(y(x), x) \equiv a(x)\, y + b(x),\ y : \mathrm{X} \subseteq \mathbb{R} \to \mathrm{Y} \subseteq \mathbb{R} \tag{2.1}$$

where $a(\cdot)$ and $b(\cdot)$ are known functions over X. Linear ODE's have explicit solutions. Its existence allow for an analytic derivation of the solution. In addition, we address the qualitative (or geometrical) properties of the solutions, and to the associated problems. We also present the particular definitions and characterizations which have been used in economics.

In section 2.1 we deal with autonomous equations and in section 2.2 equations.

2.1 Autonomous equations

In subsection 2.1.1 we present analytical solution to scalar linear equations over an arbitrary real domain X. In subssection 2.1.2 we deal with problems associated to that equation. In subsection 2.1.3 we address the particular characterization of autonomous scalar ODEs in which the independent variable is time. This provides a first approach to dynamic systems in subsection 2.1.4. Subsection presents problems for time-dependent ODEs 2.1.5, and subsection 2.1.6.

2.1.1 Analytical solutions for an arbitrary independent variable

In this section we assume that the independent variable is an arbitrary number $x \in \mathrm{X} \subseteq \mathbb{R}$.

We start with the scalar linear **autonomous** ODE

$$y'(x) = a\, y(x) + b,\ x \in \mathbb{R}.$$

If $b = 0$ the equation is **homogeneous** and if $b \neq 0$ the equation is **non-homogeneous**. The reason for this is simple: while $f(y) = a\,y$ is an homogeneous function, $f(y) = a\,y + b$ is non-homogenous.[1]

We start with the simplest ODE, the homogeneous equation,

$$y'(x) = a\,y(x),\ x \in \mathbb{R} \tag{2.2}$$

with an arbitrary real coefficient, $a \in \mathbb{R}$.

There are several methods for solving this equation[2]. We present two methods that will be useful in subsequent chapters: the methods of separation of variables and recursive integration method.

Proposition 1. *The unique solution of ODE* (2.2) *is a function* y

$$y(x) = y(x_0)\,e^{a\,(x-x_0)} \text{ if } a \neq 0 \tag{2.3a}$$

$$y(x) = y(x_0) \text{ if } a = 0 \tag{2.3b}$$

where $y(x_0)$ *is the arbitrary element of* Y *for an arbitrary* $x_0 \in$ X.

Proof. Using separation of variables approach. It involves four steps: first, as $y'(x) \equiv dy(x)/dx$ we can write equation (2.2) in an equivalent way, by separating y from x

$$\frac{dy}{y} = a\,dx.$$

Second, we integrate both sides of the equation by quadrature, assuming that $x \geq x_0$,

$$\int_{y(x_0)}^{y(x)} \frac{dy}{y} = \int_{x_0}^{x} a\,ds.$$

Third, we simplify both sides of the equation by computing the elementary integrals

$$\int_{y(x_0)}^{y(x)} d\ln(y) = a \int_{x_0}^{x} ds \iff \ln(y(x)) - \ln(y(x_0)) = a\,(x - x_0).$$

Taking exponentials of the two sides, we find equation (2.3a) if $a \neq 0$ and equation (2.3b) In the special case in which $a = 0$ □

Proof. It is instructive[3] to use another method of proof, by observing that the equation (2.2) can be written as

$$y(x) = y(x_0) + \int_{x_0}^{x} a\,y(s)ds,$$

[1]Recall that function $f(x)$ is homogeneous of degree n if multiplying the independent variable by an arbitrary real number λ then the value of the function multiplied by λ^n, that is $f(\lambda\,x) = \lambda^n\,f(x)$.

[2]There are several methods we can employ to find the proof (separation of variables, Laplace transforms, Fourier transforms, transforming into an integral equation, using the concept of generating function, just to name a few). See Zwillinger (1998)

[3]When we deal with planar ODE's or stochastic differential equations.

substituting $y(s)$ inside the integral yields

$$\begin{aligned} y(x) &= y(x_0) + \int_{x_0}^{x} a \left(y(x_0) + \int_{x_0}^{s} a\, y(s')\, ds' \right) ds \\ &= y(x_0) + a\, y(x_0) \int_{x_0}^{x} ds + \int_{x_0}^{x} \int_{x_0}^{s} a^2\, y(s')\, ds' \Big) ds \\ &= y(x_0) + a\, y(x_0)\, (x - x_0) + \int_{x_0}^{x} \int_{x_0}^{s} a^2\, y(s')\, ds'\, ds \end{aligned}$$

factoring and substituting again the solution $y(s')$ inside the integral yields

$$\begin{aligned} y(x) &= y(x_0)\,(1 + a\,(x - x_0)) + \int_{x_0}^{x} \int_{x_0}^{s} a^2 \left(y(x_0) + \int_{x_0}^{s'} a\, y(s'')\, ds'' \right) ds'\, ds \\ &= y(x_0) \left(1 + a\,(x - x_0) + a^2 \int_{x_0}^{x} \int_{x_0}^{s} ds'\, ds \right) + \int_{x_0}^{x} \int_{x_0}^{s} \int_{x_0}^{s'} a^3\, y(s'')\, ds''\, ds'\, ds \\ &= y(x_0) \left(1 + a\,(x - x_0) + \frac{a^2\,(x - x_0)^2}{2} \right) + \int_{x_0}^{x} \int_{x_0}^{s} \int_{x_0}^{s'} a^3\, y(s'')\, ds''\, ds'\, ds. \end{aligned}$$

If we continue we find

$$y(x) = y(x_0) \sum_{n=0}^{\infty} \frac{a^n\,(x - x_0)^n}{n!}$$

which is the series representation of the exponential in solution (2.3a). □

Equation (**??**) is called a **general solution**. As it can be seen it depends on an arbitrary point $(x_0, y(x_0))$ in the space $\mathrm{X} \times \mathrm{Y}$, that is on an arbitrary value for the independent variable and the associated value for the dependent variable. This should be intuitive given the fact that we are using variational approach for uncovering the economic phenomenon that the we want to study.

The ODE formalism describes the change in the dependent variable from a marginal change in the independent variable. We will see next how the complete specification of the behavior or y is provided by a side-condition.

Now consider a scalar linear **autonomous non-homogeneous** ODE

$$y'(x) = a\, y(x) + b,\ x \in \mathbb{R} \tag{2.4}$$

with an arbitrary real coefficient, $a \in \mathbb{R}$ and $b \neq 0$.

Proposition 2. *Consider the ODE* (2.4) *where b is a non-zero real number, The unique solution of that ODE is function y*

$$y(x) = \bar{y} + (y(x_0) - \bar{y})\, e^{a\,(x - x_0)} \text{ if } a \neq 0 \tag{2.5a}$$

$$y(x) = y(x_0) + b\,(x - x_0) \text{ if } a = 0 \tag{2.5b}$$

where

$$\bar{y} = -\frac{b}{a}$$

where $y(x_0)$ is the arbitrary element of Y *for an arbitrary $x_0 \in \mathrm{X}$.*

Proof. First assume that $a \neq 0$. Then, there is a number $\bar{y} \;=-b/a$ such that if $y(x)=\bar{y}$ then $d\bar{y}/dx=0$. Introduce a change in variables $z(x)=y(x)-\bar{y}$. Then $z'(x)=y'(x)=a\,y(x)+b=a\,(z(x)+\bar{y})+b=a\,z(x)$ form the definition of $\bar{y}$. We already know that the solution of $z'(x)=a\,z(x)$ is $z(x)=z(x_0)\,e^{a\,(x-x_0)}$. Mapping back to y we have

$$y(x)-\bar{y} \;=\; z(x_0)\,e^{a\,(x-x_0)}-\bar{y} \;=\big(y(x_0)-\bar{y}\big)\,e^{a\,(x-x_0)}-\bar{y}.$$

Then $y(x)=\bar{y}+\big(y(x_0)-\bar{y}\big)\,e^{a\,(x-x_0)}$ if $a\neq 0$, as in equation (2.5a).

Now assume that $a=0$. Using l'Hôpital's rule.

$$\begin{aligned} y(x) &= \lim_{a\to 0}\left(\; y(x_0)e^{a\,(x-x_0)}-\frac{b\,(1-e^{a\,(x-x_0)})}{a}\;\right)\\ &= y(x_0)-\lim_{a\to 0}\frac{\dfrac{d}{da}\;b\left(1-e^{a\,(x-x_0)}\right)}{\dfrac{d}{da}\;a}=\\ &= y(x_0)-\frac{-b(x-x_0)}{1}, \end{aligned}$$

yields equation (2.5b) □

2.1.2 Problems involving autonomous ODEs

The choice of pair $(x_0, y(x_0))$, in any of the equations (2.19) and (2.21), and therefore the exact determination of their solution, depends on the side conditions we impose. We can generically say that they take the form of a functional.

If we assume that $\mathrm{X}=[\underline{x},\bar{x}]$ the side conditions take generically the form of a functional

$$\int_{\underline{x}}^{\bar{x}} C(y(x),x)\,dx=\text{constant} \tag{2.6}$$

where $C(\cdot)$ is a known function. A **problem** involving the (2.4) ODE can be defined as

$$\begin{cases} y'(x)=a\,y+b\\ \int_{\underline{x}}^{\bar{x}}\;C(y(x),x)\,dx=\text{constant}\end{cases} \tag{2.7}$$

Although we know that the differential equation has an unique solution, the solution to the problem may not exist.

If it exists, we call it a **particular solution** , which is which is an exact solution for $y(\cdot)$.

The solution to poblem (2.7), if it exists, is of the form

$$y(x)=v(\underline{x},\bar{x})\,e^{a\,(x-\underline{x})}$$

where $v(\underline{x},\bar{x})$ is a function of the two limit of X.

Example: exponential distribution Let $\mathrm{X} = [0, \infty)$ and consider the problem

$$\begin{cases} y'(x) = a\, y & x \in [0, \infty) \\ \int_0^\infty \; y(x)\, dx = 1 \end{cases}$$

We say that the general solution of the ODE is $y(x) = y(0)\, e^{a\, x}$, taking $x_0 = 0$, where $y(0)$ is an arbitrary value for the solution associated to $x = 0$. Substituting in the side condition we have, formally,

$$\int_0^\infty \; y(0)\, e^{a\, x} \; dx = 1$$

which we will need to solve for $y(0)$. However: if $a \geq 0$ there is no real number that solves that equation, and if $a < 0$ we can find a real number that solves it. Therefore, the solution to the problem will only exist for $a < 0$. In this case, we find $y(0) = -a$. Therefore, the particular solution is

$$y(x) = -a\, e^{a\, x}, \; x \in [0, \infty) \text{ for } \; a < 0.$$

$\square$

The functional (2.6) is sufficiently general to encompass cases in which we fix a value for y, as $y(x_0) = y_0$, for a particular point x_0 in the domain (or its closure) X. For instance, for the side condition

$$\int_{\underline{x}}^{\bar{x}} \; \delta(x - x_0)\, y(x) \; dx = y_0$$

where y_0 is a know number. If the solution exists, we can find $y(x_0)$ by solving

$$\int_{\underline{x}}^{\bar{x}} \; \delta(x - x_0)\, y(x_0)\, e^{a\,(x - x_0)}\, dx = y_0.$$

Example: exponential distribution Let $\mathrm{X} = [0, \infty)$ and consider the problem

$$\begin{cases} y'(x) = a\, y & x \in [0, \infty) \\ \int_0^\infty \; \delta(x) y(x)\, dx = 1 \end{cases}$$

The solution is $y(x) = e^{a\, x}$ for any real number a. $\square$

2.1.3 Scalar autonomous ODEs with time as the independent variable

Most applications of differential equations have time, t, as the independent variable. In this case, the convention is to use Newton's notation for the derivative, i.e., $\dot{y} \equiv \dfrac{dy(t)}{dt}$ and write the general scalar linear ODE as

$$\dot{y} = a(t)\, y + b(t), \; t \in \mathrm{T} \subseteq \mathbb{R}_+ \tag{2.8}$$

where $y : \mathrm{T} \to \mathbb{R}$. This is a non-autonomous and non-homogeneous equation.[4]

[4]If we redefine the independent variable as $t = \tau$ we can transform the non-autonomous scalar linear ODE into a planar non-linear equation: $\dot{y_1} = \lambda(y_2) y_1 + \beta(y_2)$ $\dot{y}_2 = 1$ where $y_2(t) = t$. This means that the behavior of the solution when the coefficients are functions of time can be quite different.

The existence and uniqueness of solutions for linear ODE has already been established in the previous section (becaue we are able to find explicit solutions) to the ODE and to ODE problems. However, in the case of time dependent ODEs there is a rich geometrical theory for the characterization of the solutions. That is, we can describe the process $\left(y(t)\right)_{t\in \mathrm{T}}$ without the nedd to explicitly solving the ODE. As most non-linear ODEs do not have explicit solutions, the characterization of their solutions may be possible by comparing them, at least locally, to the linear ODEs. In particular, the qualitative theory for ODE is based upon the local approximation of non-linear ODE by linear ODE and by verifying conditions under which a non-linear ODE is (topologically) equivalent to a linear ODE (at least locally).

Next we present solutions to autonomous equations, present the qualitative theory of the solutions of linear present solutions to non-autonomous equations, and describe their main applications to economics.

Analytical solution

A scalar ODE is **autonomous** if the coefficients are constant, i.e, they are independent of the exogenous variable t,

$$\dot{y} = a\, y + b \tag{2.9}$$

where $(a, b) \in \Phi \subseteq \mathbb{R}^2$ are known constants. From now on we let initial value of the independent variable be equal to zero, that is $t_0 = 0$.

Using the results of the previous section, we can state (without proof) that

Proposition 3 (Solutions to equation (2.8))**.** *For a given pair of parameters $(a, b) \in \Phi$ the solution is a unique mapping $\phi : \mathrm{T} \times \mathrm{Y} \times \Phi \to \mathrm{Y}$. In particular,*

$$y(t) = \phi(t, y(0); a, b) = \begin{cases} y(0)\, e^{a\,t} - \dfrac{b}{a}\left(1 - e^{a\,t}\right) & \text{if } a \neq 0, \text{ and } b \neq 0 \\ y(0)\, e^{a\,t} & \text{if } a \neq 0, \text{ and } b = 0 \\ y(0) + b\,t & \text{if } a = 0, \text{ and } b \neq 0 \\ y(0) & \text{if } a = b = 0 \end{cases} \quad \text{for any } t \in \mathrm{T}, \tag{2.10}$$

where $y(t\,|\,t = 0) = y(0)$ is an arbitrary element of Y.

The solution of a scalar linear a autonomous ODE is a function $y(t) = \phi(t, y(0); a, b)$ of time and on an arbitrary element of Y, whose behavior depends on the parameters a and b. **Characterizing the dynamics** generated by the ODE means tracking the behavior of the path $\Big(y(t)\Big)_{t\in\mathrm{T}}$ traveled within Y when time independent variable changes from $t = 0$ to $t = \infty$.

Geometrical representation

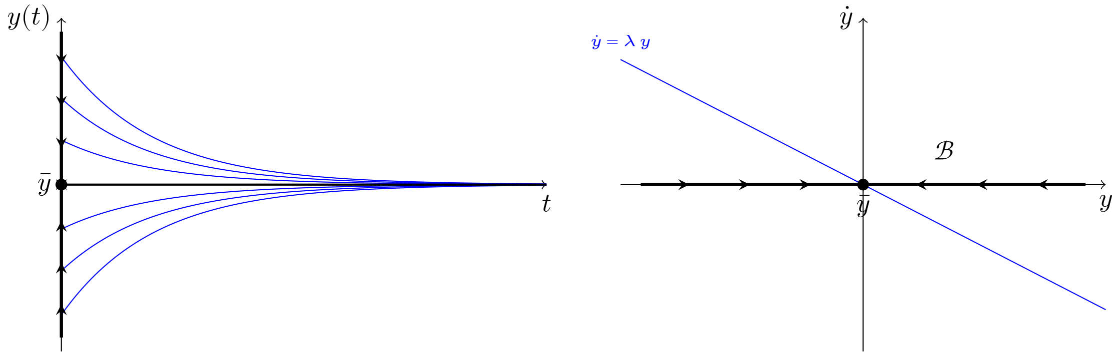

Figure 2.1: Trajectories (left) and phase diagram (right) of ODE $\dot{y} = ay$ for $a < 0$

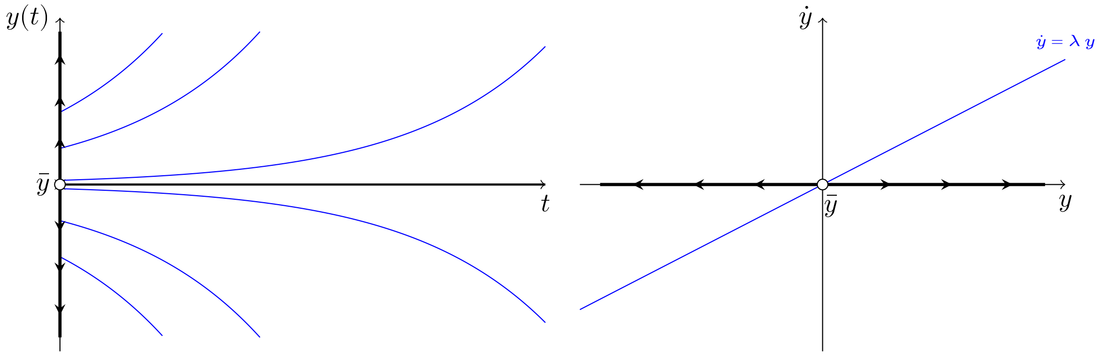

Figure 2.2: Trajectories (left) and phase diagram (right) of ODE $\dot{y} = ay$ for $a > 0$

The trajectories for the autonomous and homogeneous ODE $\dot{y} = a\, y$ for different initial values $y(0)$ are represented in the left panels of figures 2.1, for the case in which $a < 0$, and in 2.2, for the case in which $a > 0$. We observe that when $a < 0$, independently of $y(0)$, all the trajectories converge to $y = 0$, asymptotically, and that when $a > 0$, all the trajectories diverge to $+\infty$ if $y(0) > 0$ and to $-\infty$ if $y(0) < 0$. However, in both cases if $y(0) = 0$ the trajectories are stationary, that is $y(t) = 0$ for any $t \in [0, \infty)$.

The trajectories for the autonomous and non-homogeneous ODE $\dot{y} = a\, y + b$ for different values initial values $y(0)$ are represented in the left panels of figures 2.3, for the case in which $a < 0$ and $b > 0$, and in 2.4, for the case in which $a > 0$ and $b < 0$. The qualitative behavior of the trajectories, in the sense of being convergent or divergent in time, comparing to the homogeneous ODE, is the same but can be quantitatively different. That is, we observe that when $a < 0$, independently of $y(0)$, all the trajectories converge to a point $y = -b/a$, asymptotically, and that when $a > 0$, all the trajectories again diverge to $+\infty$ if $y(0) > -b/a$ and to $-\infty$ if $y(0) < -b/a$. However, in both cases if $y(0) = -b/a$ the trajectories are stationary, that is $y(t) = -b/a$ for any $t \in [0, \infty)$.

The case $a = b = 0$, where $\dot{y} = 0$, with solution $y(t) = y(0)$ a constant is thus a degenerate case in which the solution is **always** time-invariant, i.e., it is independent from the exogenous variable t and from the initial point $y(0)$. Intuitively we can say that there are no dynamics, or that this corresponds to a boundary case between stability and instability.

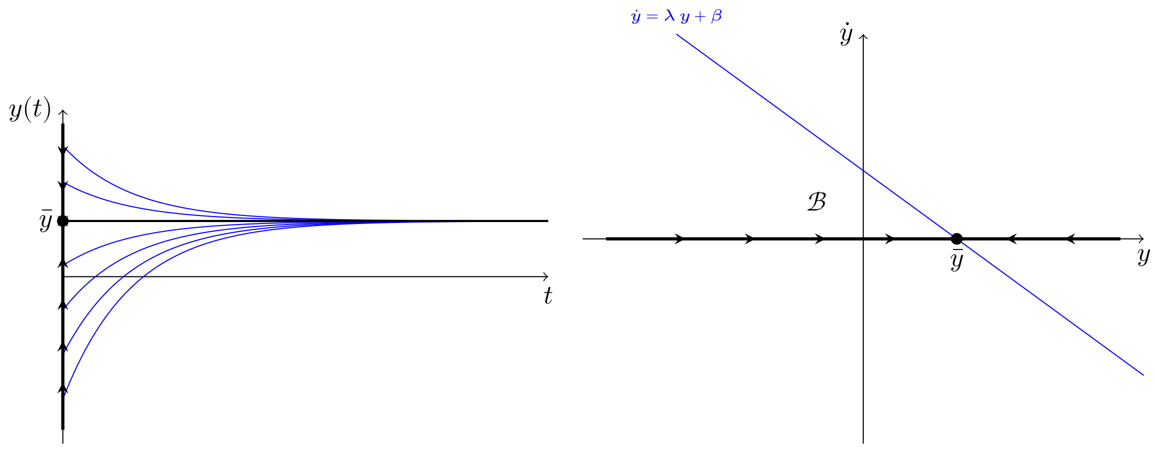

Figure 2.3: Trajectories (left) and phase diagram (right) of ODE $\dot{y} = a\,y + b$ for $a < 0$ and $b > 0$

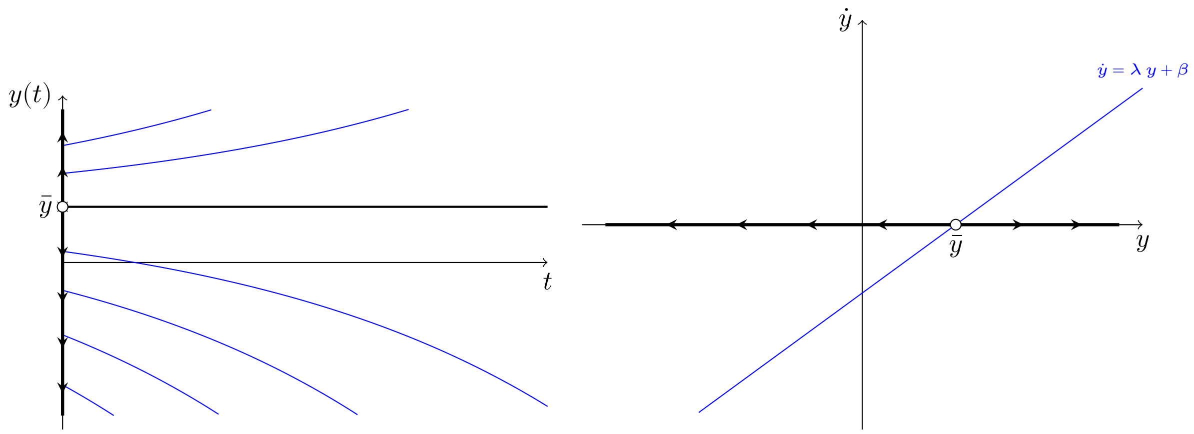

Figure 2.4: Trajectories (left) and phase diagram (right) of ODE $\dot{y} = a\,y + b$ for $a > 0$ and $b < 0$

2.1.4 Qualitative theory and dynamics

This dynamics of the equation is determined by function $f(y) = a\,y + b$.

To introduce qualitative dynamics analysis we introduce some definitions.

Definition 1. *A* ***steady state*** *(or equilibrium point): it is an element in range of y, Y, such that $f(y) = 0$, that is*

$$\bar{y} = \{y \in \mathrm{Y} : f(y) = 0\}.$$

A steady state can be characterized according to its **stability properties**

Definition 2. *A the steady state is* ***asymptotically stable*** *if for any $y = y(x_0) \in \mathrm{Y}$ the flow generated by the ODE $\dot{y} = f(y)$, that we can denote by $(y(t))_{t\in\mathrm{T}}$, has the property*

$$\lim_{t\to\infty} y(t) = y(\infty) = \bar{y}.$$

A steady state is ***unstable*** *if for any $y = y(x_0)$ in a neighborhood of $\bar{y}$, $y(t)$ does not converge to $\bar{y}$.*

This definition allows for introducing a partition over set Y, among **invariant subsets**, which are partitions of set Y containing the whole solution path $(y(t))_{t\in\mathrm{T}}$.

Definition 3. *We call* ***attractor set*** *(or basin of attraction) to the invariant subset of Y such that the solution will converge to the steady state and* ***repelling set*** *to the invariant subset of Y such that the solution will not converge to the steady state.*

Definition 4. *A* ***phase diagram*** *is a graphical representation of the set Y in which we represent the steady states, and the invariant sets. The invariant sets representation includes the representation of the variation of the solution with increasing time.*

We start by applying those definitions to the homogeneous equation $\dot{y} = a\,y$.

The existence and number of **steady states** depend on a

$$\bar{y} = \begin{cases} y(0), & \text{if } a = 0 \\ 0, & \text{if } a \neq 0. \end{cases}$$

In the first case there is an **infinite number** of steady states, consisting in all points in Y, and in the second there is a **single** steady state if $0 \in \mathrm{Y}$, or no steady state if $0 \notin \mathrm{Y}$.

When there is a steady state, that is, when $a \neq 0$ and $0 \in \mathrm{Y}$, we can characterize its **stability properties**:

- if $a < 0$ then $\lim_{t\to\infty} y(t) = 0 = \bar{y}$, for any $y(0)$, then the equilibrium point is asymptotically stable;

- if $a > 0$ then

$$\lim_{t\to\infty} y(t) = \begin{cases} \pm\infty & , \text{if } y(0) \neq 0 \\ 0 & , \text{if } y(0) = 0 \end{cases}$$

  and the equilibrium point $\bar{y}$ is unstable. In this case we say the solution can be non-stationary if $y(0) \neq 0$

Therefore, if $a \neq 0$, and $\bar{y} = 0 \in \mathrm{Y}$, there are only two kinds of possible **invariant sets**:

- if $a < 0$ the basin of attraction for $\bar{y}$ is the whole set Y and Y **is the attraction set**. Then we say $\bar{y}$ is **globally asymptotically stable**;

- if $a > 0$ then $\bar{y}$ is repelling and unstable and Y is the unstable invariant set.

The right-hand panel in Figures 2.1 and 2.2 illustrate the phase diagram for the asymptotically stable and unstable cases, respectively. In the first case we label the basin of attraction of $\bar{y}$ by $\mathcal{B}$.

For the non-homogeneous equation $\dot{y} \ = a\,y + b$ the qualitative dynamics is similar, except for the case in which $a = 0$. For $a \neq 0$, and assuming that $-b/a \in n\mathrm{Y}$, there are only some quantitative differences:

- the steady state is also unique, although it is shifted from $\bar{y} = 0$, if $b = 0$, to $\bar{y} = -b/a$, if $b \neq 0$;

- the stability behavior is qualitatively the same but now relative to the equilibrium point $\bar{y} = -b/a$: it is asymptotically stable if $a < 0$ and it is unstable if $a > 0$.

The right-hand panel in Figures 2.3 and 2.4 illustrate the phase diagram for the asymptotically stable and unstable cases, respectively. In the first case we label the basin of attraction of $\bar{y}$ by $\mathcal{B}$.

The dynamics are qualitatively different when $a = 0$. While in the homogenous case (i.e., if $b = 0$) the solution is stationary and there is an infinite number of steady states (all the elements of Y) in the non-homogeneous case (i.e, if $b \neq 0$) **there are no steady states** and the solution of the ODE is always non-stationary.

Table 2.1, which we can call a **bifurcation table**, summarizes the main types of dynamics for the scalar linear autonomous ODE:

Table 2.1: Types of dynamics for the linear scalar ODE

	$a < 0$	$a = 0$	$a > 0$
$b = 0$	one steady state	infinite number of steady states	one steady state
$b \neq 0$	asymptotically stable	no steady states	unstable

2.1.5 Problems for time-dependent scalar ODEs

We can also draw a distinction between an ODE and a **problem** involving an ODE. Again, to get an intuition of the difference, observe that equations (2.10) involve a dependence on an arbitrary point $y(0) \in \mathrm{Y}$, which is the reason why called **general solutions**.

In the case of time-dependent ODE's the side constraint is usually defined by a constraint on the solution for a specific point in time[5]

The evolution described by the ODE can be done forward in time (if we know the initial point) or backward in time (if we know a terminal point). With this additional information we can sometimes uniquely determine a forward or a backward path. An **ODE problem** involving side-conditions and the solution it is called **particular solution**[6] In the first case, we call the ODE a **forward ODE** because the solution will be obtained from future instants (assuming that the present time is $t = 0$) and in the second case we call the ODE a **backward ODE**.

We can have several types of side-conditions but in economics the two most common conditions are initial conditions, if the point $(t, y(t)) = (0, y_0)$ is know, or terminal conditions, if the point $(t, y(t)) = (T, y_T)$ for finite-time problems where $\mathrm{T} = [0, T]$, or if $\lim_{t\to\infty} \; y(t) = y_\infty$ for infinite-time problems where $\mathrm{T} = [0, \infty)$,

In the first case, we have an **initial-value problem**

$$\begin{cases} \dot{y} \; = a\, y + b \text{ for } \; t \in \mathrm{T} & (2.11a) \\ \quad y(0) = y_0 \text{ for } \; t = 0, \; y_0 \in \mathrm{Y} & (2.11b) \end{cases}$$

While $y(0)$ represents function $y(t)$ evaluated at $t = 0$, y_0 is a number belonging to the range of y. If $a \neq 0$ the solution is unique. If furthermore, $b \neq 0$ the solution to the previous problem is (prove this)

$$y(t) = \bar{y} \; + (y_0 - \bar{y})\, e^{a\, t}, \; t \in \mathrm{T}$$

for $\bar{y} \; = -b/a$.

A common **terminal-value problem** is the following

$$\begin{cases} \dot{y} \; = a\, y + b \text{ for } \; t \in \mathrm{T} & (2.12a) \\ \; y(T) = y_T \text{ for } \; t = T, \; y_T \in \mathrm{Y} & (2.12b) \end{cases}$$

While $y(T)$ represents function $y(t)$ evaluated at $t = T$, y_T is a number belonging to the range of y. If $a \neq 0$ and $b \neq 0$ the solution is unique

$$y(t) = \bar{y} \; + (y_T - \bar{y})\, e^{-a\,(T-t)}, \; t \in \mathrm{T} = [0, T].$$

In this case, we observe that $y(0) = \bar{y} \; + (y_T - \bar{y})\, e^{-a\,T}$ becomes endogenous.

[5]Which is equivalent to having a side constraint which is a functional over a Dirac-delta.

[6]Again, although the solution to a linear ODE always exists, the solution to an ODE problem may not exist if the side conditions are incompatible with the general solution of the ODE.

A common **infinite-horizon terminal-value problem** in economics is the following

$$\begin{cases} \dot{y} = a\,y + b \text{ for } t \in \mathrm{T} = [0,\infty) & (2.13a) \\ \lim_{t\to\infty} e^{-\mu\,t}\; y(t) = 0 \text{ where } \mu > 0 & (2.13b) \end{cases}$$

We can prove that solutions always exist, but are not necessarily unique. Specifically: (1) if $a < \mu$ there is an infinite number of solutions

$$y(t) = \bar{y} + (y(0) - \bar{y})\, e^{a\,t},\ t \in [0,\infty)$$

where $y(0)$ is an arbitrary value for y. In this case we say that the solution to the problem is **indeterminate**; (2) if $a \geq \mu$ then the solution is unique and stationary

$$y(t) = \bar{y} \text{ for all } t \in [0,\infty).$$

In this case we say that the solution to the problem is **determinate**.

To prove this: first, take the appropriate solution to the ODE (2.13a) from equation (2.10),

$$y(t) = \bar{y} + (y(0) - \bar{y})\, e^{a\,t},\ t \in \mathrm{T} = [0,\infty)$$

where $y(0)$ is an arbitrary number from Y; second, write

$$\begin{aligned} \lim_{t\to\infty} e^{-\mu\,t}\; y(t) &= \lim_{t\to\infty} e^{-\mu\,t}\Big(\bar{y} + (y(0) - \bar{y})\, e^{a\,t}\Big) \\ &= 0 + \lim_{t\to\infty} (y(0) - \bar{y})\, e^{(a-\mu)\,t}, \end{aligned}$$

at last, from equation (2.13b) we should have

$$\lim_{t\to\infty} (y(0) - \bar{y})\, e^{(a-\mu)\,t} = 0,$$

which is verified for any $y(0) - \bar{y}$ if $a - \mu < 0$ and only for $y(0) - \bar{y} = 0$ if $a - \mu \geq 0$.

Therefore, uniqueness (and existence) of the solution of an ODE is not the same as uniqueness of a problem involving an ODE. And this distinction has important conceptual differences in economic applications

2.1.6 Economic applications

In macro-economic and growth models we use the following classification of variables and economic equilibrium

- **pre-determined** and **non-pre-determined** variables: the first are observed and the second are anticipated, that is, we have information for $t = 0$ for the first type of variables and we have asymptotic beliefs on the second type of variables;
- **stationary** or **non-stationary** variables if they converge to a constant or are unbounded asymptotically (i.e., when $t \to \infty$);

- **determinacy** or **indeterminacy** if an equilibrium or a state of the economy modelled by a differential equation is unique or not

The relationship between them depends on the existence or not of a steady state and on their stability properties, for states within set Y.

For instance

- if a variable is pre-determined the trajectory described by the solution is always determinate, however, it can be stationary (if $a < 0$) or non-stationary (if $a > 0$). The first case is common in models with adaptative expectations, v.g. $\dot{p} = \lambda(\bar{p} - p)$, for $\lambda > 0$ and p is the log of price. The second case is common in endogenous growth models in which the GDP dynamics is given by $\dot{y} = Ay$, where y is GDP per capita;

- if a variable is non-predetermined the trajectory can be determinate if $y(0)$ is determined uniquely and is indeterminate if $y(0)$ can be any value within set Y. For scalar models the solutions are usually stationary if the terminal condition is of the type $\lim_{t\to\infty} y(t)e^{-\mu t} = 0$ for $\mu > 0$.

Table 2.2 summarizes this concepts, used in dynamic general equilibrium models (DGE).

Table 2.2: Classification of equilibrium paths in DGE models

y	$a < 0$	$a = 0$	$a > 0$
pre-determined	determined and stable	determined and stationary	determined and non-stationary
non- pre-determined	indeterminate	ambiguous	determined

Example: budget constraint dynamics

A fundamental differential equation in economics is the budget constraint equation. Let $a(t) \in R$ be the asset position of an economic entity[7] at time t, which is a stock variable which can be read in its balance. If $a > 0$ we say the agent is a net creditor and if $a < 0$ it is a net debtor. Assume that the the asset has an instantaneous return $r(t)$ and that the entity has a flow of non-financial income denoted by $y(t)$ and a flow of expenditures denoted by $e(t)$. One of the iron "laws" of economics is that the change in the asset position, or investment, is equal to savings. Savings, denoted by $s(t)$, is equal to total income minus expenditure. Therefore

$$\dot{a}(t) = s(t) = r(t)\, a(t) + y(t) - e(t), \text{for every } t \in \mathrm{T}. \tag{2.14}$$

Let us assume that all the exogenous variables are constant and parametrically given.

$$\dot{a}(t) = s(t) = r\, a(t) + y - e, \text{for every } t \in \mathrm{T}. \tag{2.15}$$

[7]It can be a household, a the government, or an economy. In the first case, n represents the net asset position, in the second it is usually the government debt, and in the third the net asset position of a country regarding the rest of the world.

We can answer the question: given an initial asset position $a(0) = a_0$ what will be the asset positions in the future ? How will they change for constant, permanent variations in any of the parameters r, y or e or in the initial level a_0 ? We see (2.15) as a forward equation and answer those questions by solving the initial-value problem

$$\begin{cases} \dot{a}(t) = r\,a(t) + y - e \text{ for } t \in \mathrm{T} & (2.16a) \\ a(0) = a_0 \text{ for } t = 0 & (2.16b) \end{cases}$$

The solution is

$$a(t) = \bar{a} + (a_0 - \bar{a})e^{r\,t},\ \bar{a} = -\frac{y-e}{r}\ t \in [0, \infty).$$

If $r > 0$ we can see that the process $\left(a(t)\right)_{t\in[0,\infty)}$ is unstable: if $a_0 > \bar{a}$, then $\lim_{t\to\infty} a(t) = +\infty$ and the agent will become a very large (indeed unboundedly large) creditor; if $a_0 < \bar{a}$, then $\lim_{t\to\infty} a(t) = -\infty$ and the agent will become a very large (indeed unboundly large) debtor; or if $a_0 = \bar{a}$ its asset position will be stationary $a(t) = \bar{a}$ for any t.

Some times, a represents a ratio of debt over another indexing variable (as population, prices, GDP, etc). In this case, r represent the interest rate net of rate of growth of the indexing, which makes possible that $r \leq 0$. In this case, particularly when $r < 0$ the dynamics change radically: the process for $\left(a(t)\right)_{t\in[0,\infty)}$ becomes asymptotically stable and converges to $\bar{a}$, for any initial asset position a_0.

We can see how the solution of the equation changes with variations in the parameters. For instance, we call finding $\frac{\partial a(t)}{\partial r}$ an exercise of comparative dynamics. This should not be confused with finding $\frac{\partial \bar{a}}{\partial r}$ which is a comparative statics exercise.

A different question is: what is the sustainable level of the asset position $a(0)$? The question posed like this is close to meaningless, before we translate mathematically "sustainability" by some criterium. One commonly used is: we say that the asset position is sustainable if the asymptotic present value of a is equal to zero, that is

$$\begin{cases} \dot{a}(t) = r\,a(t) + y - e \text{ for } t \in \mathrm{T} = [0, \infty) & (2.17a) \\ \lim_{t\to\infty} a(t)e^{-\rho\,t} = 0,\ \rho > 0 & (2.17b) \end{cases}$$

Using our previous example, the answer depends on the relationship between r and ρ: if $r < \rho$ then the solution is

$$a(t) = \bar{a} + (a(0) - \bar{a})e^{r\,t},\ \bar{a} = -\frac{y-e}{r}\ t \in [0, \infty).$$

any initial asset position, $a(0)$ is sustainable; however if $r \geq \rho$ then the solution is

$$a(t) = \bar{a}, \text{ for all } t \in [0, \infty),$$

which means that $a(0) = \bar{a}$. If the entity is an initial debtor, say $a_0 < 0$ then this level of debt is sustainable only if it satisfies, for every point in time, $y - e = -ra_0 > 0$, i.e, its income is permanently higher than its expenditure.

2.2 Non-autonomous equations

In subsection 2.2.1 we present analytical solutions for non-autonomous equations over an arbitrary real domain X. In subsection 2.1.3 we address the particular characterization of autonomous scalar ODEs in which the independent variable is time. In all subsections, we also study related problems and provide some examples.

2.2.1 Analytical solutions for an arbitrary independent variable

Let us start with the **non-autonomous homogeneous** equation,

$$y'(x) = a(x)\,y(x),\ y : \mathrm{X} \to \ \mathrm{Y} \tag{2.18}$$

where both X are Y are subsets of $\mathbb{R}$. The coefficient function $a(x)$ is an arbitrary function $a : \mathrm{X} \to \mathbb{R}$. It can be a constant, piecewise constant, or be an arbitrary function of x.

Proposition 4. *The unique solution of ODE* (2.18) *is a function* y

$$y(x) = y(x_0)\,e^{\int_{x_0}^{x}\ a(s)\,ds},\ \textit{for any}\ \ x, x_0 \in \mathrm{X} \tag{2.19}$$

where x_0 *is an element of* X *and* $y(x_0)$ *is the arbitrary element of* Y *associated to it.*

Proof. We use the method of separation of variables to determine the solution for ODE (2.18). Recalling that we denoted the derivative as $\frac{dy}{dx} = y'(x)$ we can write the ODE (2.18) as $\frac{dy}{dx} = a(x)\,y$. Using integration by quadratures we find

$$\int_{y(x_0)}^{y(x)} \frac{dy}{y} = \int_{x_0}^{x} a(s)\,ds$$

as the anti-derivative of $\frac{1}{y}$ is $\ln y$ then $\ln\left(\frac{y(x)}{y(x_0)}\right) = \int_{x_0}^{x}\ a(s)\,ds$ Taking exponentials for both sides yields the general solution (2.19). $\square$

The **non-autonomous and non-homogeneous** scalar linear ODE is

$$y'(x) = a(x)y + b(x),\ y : \mathrm{X} \to \ \mathrm{Y} \tag{2.20}$$

where, again, both X are Y are subsets of $\mathbb{R}$.

Proposition 5. *The unique solution of ODE* (2.20) *is a function* y

$$y(x) = y(x_0)\,e^{\int_{x_0}^{x}\ a(s)\,ds} + \int_{x_0}^{x} e^{\int_{s}^{x} a(z)\,dz}\,b(s)ds \tag{2.21}$$

where x_0 *is an element of* X *and* $y(x_0)$ *is the arbitrary element of* Y *associated to it.*

Proof. We apply the variation of constant method[8]. First, we consider the solution for the homogeneous equation, such that $b(x) = 0$ for all $x \in \mathrm{X}$. From equation (2.19) its solution for the fixed interval (x_0, x), such that $x_0 < x$ is

$$y_h(x, y_0) = y_0\, e^{\int_{x_0}^{x} a(s)ds}.$$

But we expect the solution to equation (2.20) to be, for an arbitrary $x > x_0$,

$$y(x) = y_h(x, y_0(x)) = y_0(x)\, e^{\int_{x_0}^{x} a(s)ds}. \tag{2.22}$$

Taking derivatives for x and using the Leibniz's rule [9] we obtain

$$y'(x) = y_0'(x)\, e^{\int_{x_0}^{x} a(s)ds} + y_0(x)\, a(x)\, e^{\int_{x_0}^{x} a(s)ds} = y_0'(x)\, e^{\int_{x_0}^{x} a(s)ds} + a(x)\, y(x)$$

which should be equal to equation (2.20). By equating the right-hand sides of both equations we get the ODE

$$y_0'(x) = b(x) e^{-\int_{x_0}^{x} a(s)ds}.$$

As function $y_0(.)$ is continuous, from the fundamental theorem of calculus $\int_{x_0}^{x} y_0'(s)\, ds = y_0(x) - y_0(x_0)$, and

$$y_0(x) = y_0(x_0) + \int_{x_0}^{x} b(s) e^{-\int_{x_0}^{s} a(z)dz} ds.$$

Substituting in equation (2.22) and because $y_0(x_0) = y(x_0)$ we finally get solution (2.21) □

Problems involving non-autonomous ODEs

The choice of pair $(x_0, y(x_0))$, in any of the equations (2.19) and (2.21), and therefore the exact determination of their solution, depends on the side conditions we impose. We can generically see that they take the form of a functional. Assuming that $\underline{x}$ and $\bar{x}$ denote the infimum and the supremum of X, the side conditions take generically the form

$$\int_{\underline{x}}^{\bar{x}} C(y(x), x)\, dx = \text{constant}$$

where $C(\cdot)$ is a knwon function. In this case we consider (2.19) in the form

$$y(x) = v(\underline{x}, \bar{x})\, e^{\int_{\underline{x}}^{x} a(s)\, ds}$$

where $v(\underline{x}, \bar{x})$ is a number.

[8]Due to Lagrange (1811).

[9]Let $h(x) = \int_{b(x)}^{b(x)} f(x, s) ds$. The Leibniz rule states that $\dfrac{dh(x)}{dx} = b'(x) f(x, b(x)) - a'(x) f(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, s)\, ds$.

Analogously for equation (2.21), and solve

$$\int_{\underline{x}}^{\bar{x}} C(v(\underline{x}, \bar{x})\, e^{\int_{\underline{x}}^{x} a(s)\, ds}, x)\, dx = \text{constant}$$

for $v(\underline{x}, \bar{x})$. This would allows us to obtain a **particular solution** which is an exact solution for $y(\cdot)$.

This approach is sufficiently general to encompass cases in which we fix a value for y, as $y(x_0) = y_0$, for a particular point x_0 in the domain (or its closure) X. For instance, for the side condition

$$\int_{\underline{x}}^{\bar{x}} \delta(x - x_0)\, y(x)\ dx = y_0$$

where y_0 is a know number. As

$$\int_{\underline{x}}^{\bar{x}} \delta(x - x_0)\, y(x)\, dx = y(x_0).$$

Then, the particular solution is

$$y(x) = y_0\, e^{\int_{\underline{x}}^{x} a(s)\, ds}.$$

Next we see some applications of models involving non-autonomous ODE's, which include particular side constraints.

Some applications

Example: Utility theory

The generalized logarithmic function[10]

$$u(x) = \ln_\sigma(x) \equiv \frac{x^{1-\sigma} - 1}{1 - \sigma} \text{ for } \sigma > 0$$

has many uses, not only in economics. In economics, in deterministic models it is called the iso-elastic utility function, or, in stochastic models, it is called constant relative risk aversion (CRRA) Bernoulli utility function CRRA.

Using the analysis in Pratt (1964) it can be showed that it is a solution of the problem

$$\begin{cases} -\dfrac{u''(x)\, x}{u'(x)} = \sigma,\ x \in \mathrm{X} = (0, \infty) & (2.23a) \\ \sigma \displaystyle\int_1^\infty \frac{u'(x)}{x}\ dx = 1 & (2.23b) \\ u(1) = 0 & (2.23c) \end{cases}$$

The first equation is a definition of the relative risk aversion, as the symmetric of the elasticity of $u(\cdot)$ being a constant equal to σ. The first constraint conditions the relative slope of $u(\cdot)$ on all

[10] If $\sigma = 1$ it can be shown that it is $u(x) = \ln(x)$.

its domain and the last constraint fixes the value of utility of consumption at one (this condition makes transparent that a logarithm is hidden behind the utility function).

Equation (2.23a) is a second order ODE. We can transform it into a first order ODE by defining $z(x) = \ln u'(x) \iff u'(x) = e^{z(x)}$. Then we obtain an the linear ODE

$$z'(x) = b(x) \equiv -\frac{\sigma}{x}.$$

This equation has solution $z(x) = z(x_0) + \int_{x_0}^{x} b(s)ds$, which we can prove simplifies to $z(x) = z(x_0) - \sigma \ln\left(\frac{x}{x_0}\right)$, for an arbitrary $x_0 > 0$. Therefore

$$u'(x) = e^{z(x)} = u'(x_0)\left(\frac{x}{x_0}\right)^{-\sigma}$$

which is again a linear differential equation. Solving it, and observing that $u'(x_0)$ is an arbitrary constant, yields

$$\begin{aligned} u(x) &= u(x_0) + u'(x_0)\int_{x_0}^{x}\left(\frac{s}{x_0}\right)^{-\sigma} ds \\ &= u(x_0) + u'(x_0)x_0^{\sigma}\left(\frac{x^{1-\sigma}}{1-\sigma} - \frac{x_0^{1-\sigma}}{1-\sigma}\right). \end{aligned} \tag{2.24}$$

The two side conditions (2.23b) and (2.23c) allow us, in principle, to determine the arbitrary constants $u(x_0)$ and $u'(x_0)$. First, using the expression obtained for $u'(x)$ we have

$$\int_{x_0}^{\infty} \frac{u'(x)}{x} dx = u'(x_0)\, x_0^{\sigma} \int_{x_0}^{\infty} x^{-\sigma-1}\ dx = \frac{u'(x_0)}{\sigma},$$

which, considering constraint (2.23b) for $x_0 = 1$, we have $\frac{u'(1)}{\sigma} = \frac{1}{\sigma}$ that is, $u'(1) = 1$. Setting again $x_0 = 1$ in equation (2.24) we have

$$u(x) = u(1) + \frac{x^{1-\sigma} - 1}{1-\sigma}$$

which, upon introducing side-condition (2.23c) yields the generalized logarithm.

Example: the Gaussian distribution

We can derive the standard Gaussian probability density function from the ODE problem,

$$\begin{cases} y'(x) = -x\, y(x), \text{ for } x \in \mathrm{X} = (-\infty, \infty) & (2.25a) \\ \int_{-\infty}^{\infty} y(x)\ dx = 1 & (2.25b) \end{cases}$$

While equation (2.25a) means that the rate of decay between two adjacent points in X is equal to the value of x in which we measure it, equation (2.25b) constraints $(y(x))_{\mathrm{X}}$ to be a distribution.

The solution to the problem is

$$y(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \tag{2.26}$$

To prove this, we already know that the solution to equation (2.25a) is

$$y(x) = y(x_0)\, e^{\int_{x_0}^{x} s\, ds} = y(x_0)\, e^{-\frac{x^2}{2}+\frac{x_0^2}{2}} = y(x_0)\, e^{\frac{x_0^2}{2}} \; e^{-\frac{x^2}{2}}$$

for an arbitrary point $(x_0, y(x_0))$. As[11]

$$\int_{-\infty}^{\infty} y(x)\; dx = y(x_0)\, e^{\frac{x_0^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\; dx = y(x_0)\, e^{\frac{x_0^2}{2}} \sqrt{2\pi}$$

if we substitute this general solution in the constraint (2.25b) we have $y(x_0)\, e^{\frac{x_0^2}{2}} \sqrt{2\pi} = 1$, which yields the standard Gaussian probability density function (2.26).

2.2.2 Scalar non-autonomous ODEs with time as the independent variable

The scalar linear non-autonomous having time as an independent variable, has been already written in equation (2.8).

From Proposition 5 its general solution is, adapting equation (2.21)

$$y(t) = y(0)\, e^{\int_0^t a(s)\, ds}, \text{ if } b(t) = 0 \tag{2.27a}$$

$$y(t) = y(0)\, e^{\int_0^t a(s)\, ds} + \int_0^t e^{\int_s^t a(z)\, dz}\, b(s) ds, \text{ if } b(t) \neq 0 \tag{2.27b}$$

where $y(0)$ is an arbitrary element of Y associated to it.

A common **initial value problem** is

$$\begin{cases} \dot{y} = a(t)\, y + b(t) \text{ for } t \in \mathrm{T} & (2.28a) \\ y(0) = y_0 \text{ for } t = 0,\; y_0 \in \mathrm{Y} & (2.28b) \end{cases}$$

has the solution

$$y(t) = y_0\, e^{\int_0^t a(s)\, ds} + \int_0^t e^{\int_s^t a(z)\, dz}\, b(s) ds,\; t \in \mathrm{T}$$

Exercise: prove this.

A common terminal value problem is

$$\begin{cases} \dot{y} = a(t)\, y + b(t) \text{ for } t \in \mathrm{T} & (2.29a) \\ \lim_{t\to\infty} e^{-\int_0^t a(s) ds}\; y(t) = 0 \text{ for } t = 0,\; y_0 \in \mathrm{Y}\,. & (2.29b) \end{cases}$$

Has the solution

$$y(t) = -\int_t^{\infty} e^{-\int_t^s a(z) dz} b(s) ds$$

[11] The Gaussian integral is $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\; dx = \sqrt{2\pi}$.

Example: Budget constraint We can now consider the budget constraint (2.14) with time varying interest rate and primary surplus. A more realistic presentation, if we consider the dynamics of the government debt, is

$$b(t) = r(t)\, b + d(t), \text{ for } \; t \in \mathrm{T} \tag{2.30}$$

where b is the government debt, r is the interest rate and $d = e - y$ is the primary deficit, the difference between government expenditures (excluding the service of debt) and income (from taxes and other sources). We assume all the variables are in nominal terms. If they are in real terms r represents the real interest rate, and if they are deflated by the GDP, r is the difference between the nominal interest rate and the growth rate of the GDP.

We can include the ODE (2.30) in two different types of problems: initial or terminal value problems. In the initial value problem the stock of debt, $b(\cdot)$, is defined as a pre-determined variable and in the terminal value problem is defined as a non-predetermined variable.

In an initial value problem,

$$\begin{cases} b(t) = r(t)\, b + d(t), & \text{for } \; t \in \mathrm{T} \\ b(0) = b_0 \text{ given} \end{cases}$$

we ask the question: given the initial level of debt $b(0) = b_0$ and our forecast over the future paths of the interest rate and the primary deficit, $\left(r(t), d(t)\right)_{t \in [0,\infty)}$, what will be the future behavior of the government debt ?

The answer is

$$b(t) = b_0\, e^{\int_0^t \; r(s)\, ds} + \int_0^t \; e^{\int_s^t \; r(z)\, dz}\, d(s) ds, \; t \in \mathrm{T}$$

A terminal value problem could be

$$\begin{cases} b(t) = r(t)\, b + d(t), & \text{for } \; t \in \mathrm{T} \\ \lim_{t \to \infty} \; e^{-\int_0^t \; r(s)\, ds}\, b(t) = 0 \end{cases}$$

we ask the question: given our forecast over the future paths of the interest rate and the primary deficit, $\left(r(t), d(t)\right)_{t \in [0,\infty)}$, what should be the initial level of debt such that the dynamics of debt is sustainable (or solvent) ?

The answer is

$$b(0) = -\int_0^\infty \; e^{\int_s^t \; r(z)\, dz}\, d(s) ds,$$

where the right-hand side term is the symmetric to the present value of the future primary deficits. That is, if the present value of future primary deficits is negative (positive) the government should be in a positive (negative) initial asset position -i.e., having a sovereign wealth or investment fund.

Another common application in economics is related to studying the effects of anticipated shocks in exogenous variables. Again the perspectives, and questions, related to initial and terminal value problems are different.

We can distinguish between:

- non-anticipated and anticipated shocks
- temporary and permanent shocks

Example: anticipated shocks for a pre-determined variable Let us assume that y is a pre-determined variable, in which we know the value at time $t = 0$, and assume there will be an additive shock in a prescribed future date, t^*, in an exogenous variable that affects the dynamics of y.

We can address this case through the initial value problem

$$\begin{cases} \dot{y} = ay + b(t), & \text{for } t \in [0, \infty) \\ y(0) = y_0 \text{ given}, & \text{for } t = 0 \end{cases} \tag{2.31}$$

where $a \neq 0$ by assumption and

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^* \\ b_1 & \text{if } t^* \leq t < \infty. \end{cases}$$

The solution to problem (2.31) is

$$y(t) = \begin{cases} y_0 e^{at} + \dfrac{b_0}{a}\left(e^{at} - 1\right) & \text{if } 0 \leq t < t^* \\ y_0 e^{at} + \dfrac{b_0}{a} e^{at} + \left(\dfrac{b_1 - b_0}{a}\right) e^{a(t-t^*)} - \dfrac{b_1}{a} & \text{if } t^* \leq t < \infty \end{cases}$$

and for the case in which $b_1 > b_0$ the solution is depicted in Figure 2.5.

Observe that the solution, at any point in time, is capitalizing on the past changes of the variable $b(t)$. It only responds to the shock **after** it is observed, at time $t = t^*$ ($t = 3$ in the example in the figure).

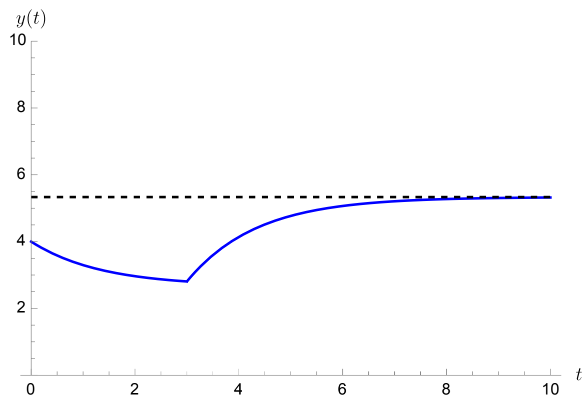

Figure 2.5: Trajectories for problem (2.31)

Example: anticipated shock in an exogenous variable Let us assume that y is a npn-pre-determined variable and assume there will be an additive shock in a prescribed future data in an exogenous variable that affects the dynamics of y. Assume that a rulling-out bubble, a transversality or a sustainability condition should be satisfied.

We can address this case through the terminal value problem

$$\begin{cases} \dot{y} = ay + b(t), & \text{for } t \in [0, \infty) \\ \lim_{t\to\infty} y(t)e^{-at} = 0, & \text{for } t = \infty \end{cases} \tag{2.32}$$

where $a > 0$ and

$$b(t) = \begin{cases} b_0 & \text{if } 0 \le t < t^* \\ b_1 & \text{if } t^* \le t < \infty. \end{cases}$$

The solution to the problem is

$$y(t) = \begin{cases} -\dfrac{b_0}{a} - \left(\dfrac{b_1 - b_0}{a}\right) e^{a(t-t^*)} & \text{if } 0 \le t < t^* \\ -\dfrac{b_1}{a} & \text{if } t^* \le t < \infty \end{cases}$$

Now assume that we have the problem

$$\begin{cases} \dot{y} = ay + b(t), & \text{for } t \in [0, \infty) \\ y(0) = y_0 \text{ given} \\ \lim_{t\to\infty} y(t)e^{-at} = 0, & \text{for } t = \infty \end{cases} \tag{2.33}$$

with the same assumptions for a and $b(t)$. This problem has no solution in the space of continuous functions of time. However, if we allow for solutions in the space of piecewise continuous functions of time we have the following solution

$$y(t) = \begin{cases} y_0 & \text{if } t = 0 \\ -\dfrac{b_0}{a} - \left(\dfrac{b_1 - b_0}{a}\right) e^{a(t-t^*)} & \text{if } 0 < t < t^* \\ -\dfrac{b_1}{a} & \text{if } t^* \le t < \infty. \end{cases}$$

We see that the solution is right continuous, such that

$$y(0) = y_0 \neq y_{0^+} = \lim_{t\downarrow 0} y(t) = -\frac{b_0}{a} - \left(\frac{b_1 - b_0}{a}\right) e^{-at^*}$$

which means there is a discontinuous jump of size $y_{0^+} - y_0$ at time $t = 0$. See Figure 2.6.

Comparing to the initial-value problem we see that the solution has an anticipating feature: for $0 < t < t^*$ the solution depends on the expect value of the variable $b(t)$ **after** its change, b_1, and after the change, for $t \ge t^*$, it is not influenced by the value before the change, b_0.

These two cases illustrate two fundamental types of dynamics in macro-economics. Dynamic general equilibrium models have usualy both dynamics coupled.

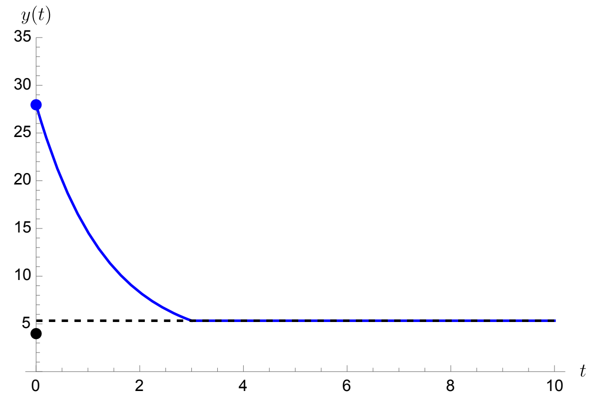

Figure 2.6: Trajectories for problem (2.33). At time $t = 0$ there is a "jump" from the initial value y_0 to

2.2.3 Dynamics for scalar non-autonomous ODEs with time as the independent variable

A scalar non-autonomous ODE (2.8) can or cannot have a steady state.

In the previous section, in problems (2.31) to (2.33), we saw cases in which the solution converged to a steady state.

In this section we consider two equations, which are common in economics, in which the time dependence is additive or multiplicative. We will show that there is convergence to a long-run trajectory defined by function $\bar{y}(t)$. We assume that variable y is pre-determined.

Additive time dependent exogenous variable

Consider the linear scalar ODE

$$\dot{y} = \lambda\, y + b\, z(t), \tag{2.34}$$

where we assume that $\lambda < 0$ and z is a time-dependent exogenous variable. Furthermore, assume that $z(t)$ grows exponentially as $z(t) = e^{\gamma\, t}$, where $\gamma > 0$, and the initial value of the endogenous variable y is $y(0) = y_0$ (given).

The solution to the initial value problem is

$$y(t) = e^{\lambda\, t}\ \left(y(0) - \frac{\beta}{\gamma - \lambda}\right) + \frac{\beta}{\gamma - \lambda} e^{\gamma\, t}. \tag{2.35}$$

If $\lambda < 0$ we see that, for any initial value $y(0)$ then

$$\lim_{t\to\infty}\ y(t) = \bar{y}\ (t) \equiv \frac{\beta}{\gamma - \lambda} e^{\gamma\, t}.$$

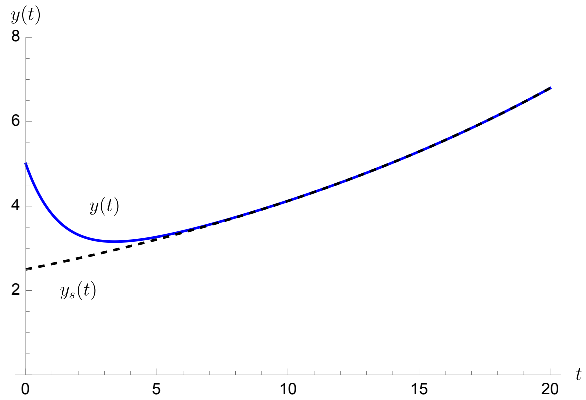

Figure 2.7: Trajectories for $y(t)$ and for $\bar{y}$ from equation (2.34)

Multiplicative time dependent exogenous variable

Next we consider the ODE

$$\dot{y} = z(t)\, y \tag{2.36}$$

in which the exogenous variable, z, enters multiplicatively. We take a common (mean reverting) process for z as given by equation $\dot{z} = \gamma\,(\bar{z} - z)$ where $\gamma > 0$. This equation has solution

$$z(t) = \bar{z} + (z_0 - \bar{z})\, e^{-\gamma\, t}$$

featuring $\lim_{t\to\infty} z(t) = \bar{z}$.

The solution to the initial-value problem is

$$y(t) = y(0)\, \exp\left\{\bar{z}\; t - \frac{z_0 - \bar{z}}{\gamma}\left(e^{-\gamma\, t} - 1\right)\right\} \tag{2.37}$$

We can easily see that

$$\lim_{t\to\infty} y(t) = \bar{t} \equiv y(0)\, e^{\frac{z_0 - \bar{z}}{\gamma}} \; e^{\bar{z}\; t}$$

Figure 2.8 shows both the solution $y(t)$ and the long-run solution $\bar{y}(t)$ in a logarithmic scale.

We see that the solution converges in the long-run to an exponential function with growth rate $\bar{z}$

2.3 References

Mathematics: there is a huge literature on scalar linear ODE, but (Hale and Koçak, 1991, ch 1) is a great modern textbook. See Hubbard (1994) on the history and meaning of differential equations.

Non-automomous equations: (Hale and Koçak, 1991, ch 2), John H. Hubbard (1991)

Applications to economics: Gandolfo (1997).

Problem set on scalar linear ODEs.

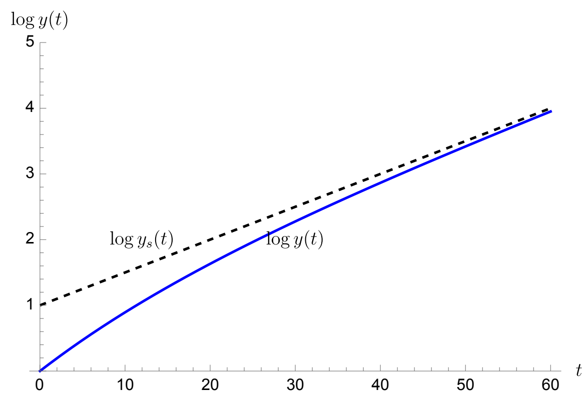

Figure 2.8: Phase diagram and trajectories of equation $\dot{y} = \lambda y + b$ for $\lambda < 0$ and $b > 0$

Bibliography

Gandolfo, G. (1997). *Economic Dynamics*. Springer-Verlag.

Hale, J. and Koçak, H. (1991). *Dynamics and Bifurcations*. Springer-Verlag.

Hubbard, J. H. (1994). What it means to understand a differential equation. *The College Mathematics Journal*, 25(5):372–384.

John H. Hubbard, B. H. W. a. (1991). *Differential Equations: A Dynamical Systems Approach: Ordinary Differential Equations*. Number 5 in Texts in Applied Mathematics. Springer, 1 edition.

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica*, 32:122–136.

Zwillinger (1998). *Handbook of Differential Equations*. Academic Press, third edition.