Foundations of Financial Economics
Household behavior: two period deterministic cases

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Topics for this slide

1. Intertemporal choice
2. Intertemporal household problem subject to an intertemporal budget constraint
3. Intertemporal household problem subject to sequencial budget constraints

Summary

1. We address the structure and economic meaning of the **intertemporal utility function**
$$U(c_0, c_1)$$
   where c_t is consumption in period t
2. Next, we consider the **two simplest benchmark problems** in financial economics
   2.1 The household problem when it can make **real forward contracts** (contract now for delivery in the future)
$$V(q, w) = \max_{c_0, c_1} \{ U(c_0, c_1) : \; c_0 + q\, c_1 = w \}$$
   2.2 The household problem when it can make **spot financial contracts** (buy and sell securities). Example, the "financial approach" is
$$V(q, w) = \max_{c_0, c_1, \theta} \{ U(c_0, c_1) : \text{s.t } c_0 + \theta\, S = y_1, \; c_1 = V\theta + y_2 \}$$
3. Then we take those problems as a building block of a dynamic general equilibrium (DGE) (next lecture)

1. Intertemporal consumption preferences

Consumption sequences

- We **index** variables by time.
- In the simplest case, we have $\mathbb{T} = \{0, 1\}$
- Consider the **sequences** $\{c_0, c_1\}$, where c_t is consumption in **period** $t = 0, 1$
- The **value of sequence** $\{c_0, c_1\}$ is measured by the **intertemporal utility functional**,

$$U = U(\{c_0, c_1\}),$$

- We take a decision at a particular point in time $t = 0$
- The **optimum** is a sequence of consumption in the present and in the future, $\{c_0^*, c_1^*\}$, for which U **is maximum**

Intertemporal choice

- Indexing consumption by time t has two consequences:
  - introduces an **heterogeneity**:
    therefore, we can use **general** concepts and results of choice among **different goods** as in the slide basic utility theory
  - but it also introduces an **order relationship among consumption in different moments in time** c_0 and c_1:
    therefore, we need **particular** concepts and results related to **intertemporal arbitrage**
  - it requires an assumption on information: **perfect information** in this slide
- In this case a intertemporal preference with perfect information

Intertempora l utility

As a generic utility function

- Dealing with **sequences** $\{c_0, c_1\}$ as a **vector of real non-negative numbers** $\mathbf{c} = (c_0, c_1) \in \mathbb{R}^2_+$
- Therefore the **intertemporal utility function** (IUF) can be seen as a mapping $U : \mathbb{R}^2_+ \to \mathbb{R}$,

$$U = U(c_0, c_1)$$

  where U is a number allowing to rank vectors $\mathbf{c} = (c_0, c_1)$
- Behavioral and information assumptions on choice over time, are **implicitly** introduced by mathematical properties of U

Intertemporal utility

Static assumptions

- First assumption: $U(\cdot)$ is **continuous** and **differentiable** in both arguments (c_0, c_1)
- Intuition: we can value sequences of consumption and assess the value of small changes in the quantities of consumption at any point in time (today or tomorrow);
  small changes of the quantities of the consumption generate small changes in value;
  we can assess **how** the intertemporal utility changes for small changes in thedistinguish the value
- Second assumption: **Positive marginal utility**:

$$U_0 \equiv \frac{\partial U(c_0, c_1)}{\partial c_0} > 0, \;\; U_1 \equiv \frac{\partial U(c_0, c_1)}{\partial c_1} > 0$$

  Intuition: consumption in every period is a good, in the sense that it increases intertemporal utility; and there is **no satiation** at every point in time.

Intertemporal utility

Static assumptions

- Third assumption: non-increasing marginal utility for every period

$$U_{00} \equiv \frac{\partial^2 U(c_0, c_1)}{\partial c_0^2} \leq 0, \text{ and } U_{11} \equiv \frac{\partial^2 U(c_0, c_1)}{\partial c_1^2} \leq 0$$

Intuition: remember last slide

Intertemporal utility

Dynamic assumptions

- **Definition**: the **intertemporal marginal rate of substitution** is

$$IMRS_{0,1}(c_0, c_1) = -\frac{dc_1}{dc_0}$$

- **Intuition**: how much are we willing to sacrifice consumption at $t = 1$ (tomorrow) in order to increase une unit of consumption at $t = 0$ (today) ?
- For a compensated change in c_0 and c_1 such that $dU = 0$, we have

$$U_0(c_0, c_1)dc_0 + U_1(c_0, c_1)dc_1 = 0$$

then the IMRS is equal to the ratio of the marginal utilities

$$\boxed{IMRS_{0,1}(c_0, c_1) = \frac{U_0(c_0, c_1)}{U_1(c_0, c_1)}\Big|_{U=\text{constant}}}$$

Intertemporal utility

Dynamic assumptions

- Fourth assumption: we can have the following types of **intertemporal dependence**. Using the gross or Edgeworth elasticity

$$\begin{cases} \text{intertemporal substitutability} & (\mathrm{U}_{0,1} < 0 \text{ that is } \uparrow c_1 \implies \downarrow U_0(c_0, c_1 \\ \text{intertemporal independence} & (\mathrm{U}_{0,1} = 0 \text{ that is } \uparrow c_1 \implies = U_0(c_0, c \\ \text{intertemporal complementarity} & (\mathrm{U}_{0,1} > 0 \text{ that is } \uparrow c_1 \implies \uparrow U_0(c_0, c_1 \end{cases}$$

Intertemporal utility

Dynamic assumptions

- **Intertemporal elasticity of substitution**

$$\boxed{EIS_{0,1}(c_0, c_1) = \frac{d\ln(c_1/c_0)}{d\ln IMRS_{0,1}(c_0, c_1)} = \frac{c_0\, U_0 + c_1\, U_1}{c_1\, U_1\varepsilon_{00} - 2c_0\, U_0\varepsilon_{01} + c_0\, U_0\varepsilon_{11}}}$$

where $\varepsilon_{ij} = -\frac{U_{ij}c_j}{U_i}$ for $i = 0, 1$ and $U_{ij} = \frac{\partial^2 U}{\partial c_i c_j}$

- **Intuition**: how much does the rate of growth of the ratio c_1/c_0 changes for a one percent increase in the $IMRS$.
  This provides a measure of relative **intertemporal substitution/complementarity** of consumption
- In particular: $EIS_{0,1} > 0$ if there is intertemporal substitution, $EIS_{0,1} = 0$ intertemporal independence and $EIS_{0,1} < 0$ intertemporal complementarity (again in the Edgeworth sense)

Intertemporal choice

Introducing the time arrow

- Time can be introduced parametrically or via temporal utility functions
- **Discounting**: time is introduced via a time-weight: usually a **discount factor**

  $$\{\beta^t\}_{t=0}^T = \{1, \beta, \beta^2, \dots, \beta^t \dots\}$$

  where

  $$\beta \equiv \frac{1}{1+\rho}$$

  where ρ is the **rate of time preference**: as $\rho > 0 \implies 0 < \beta < 1$
  Example:

  $$U[\{c\}] = u(c_0) + \beta\, u(c_1)$$

- **Temporal utility functions** dependent on time: example the "temporal" preferences are different for different time periods

  $$U[\{c\}] = U(u_0(c_0), u_1(c_0, c_1))$$

Intertemporal utility

Main assumptions regarding intertemporal preferences

Definition 1

Stationary preferences*: if the temporal utility functions are independent of time (but there can be discounting)*

Definition 2

Impatience*: if there is a preference for consumption today, at* $t = 0$ *rather than in the future* $t = 1, 2, \dots$

Intertemporal utility

Main assumptions regarding intertemporal preferences

Definition 3

There is

- ***intertemporal independence*** *if* $\varepsilon_{0,1} = 0$
- ***intertemporal substitution*** *if* $\varepsilon_{0,1} > 0$
- ***intertemporal complementarity*** *(addiction) if* $\varepsilon_{0,1} > 0$

How can we identify those properties in particular utility functions ?

Intertemporal utility

Impatience and intertemporal complementarity

- We consider a stationary consumption process, i.e,

$$c_0 = c_1 = \bar{c} \text{ constant}$$

- Determining impatience: we use $IMRS_{0,1}$. We say the **IUF displays impatience** if

$$IMRS_{0,1}(\bar{c}) = \frac{U_0(\bar{c})}{U_1(\bar{c})} > 1$$

  Intuition: to keep intertemporal utility constant, if we increase c_0 by one unit the reduction in consumption in period $t = 1$ be bigger then one unit. This means that **consumption at** $t = 0$ **has more value than consumption at** $t = 1$.

- **Intertemporal dependence** can be determined by the Allen-Uzawa elasticity ε_{01}.

$$\varepsilon_{0,1}(\bar{c}) = -\frac{U_{01}(\bar{c})\,\bar{c}}{U_0(\bar{c})} \begin{cases} > 0, & \text{intertemporal substitutability} \\ = 0, & \text{intertemporal independence} \\ < 0 & \text{intertemporal complementarity} \end{cases}$$

Intertemporal utility

Example 1: additive IUF

This is the simplest intertemporal utility function:

- **Assumption 1**: the IUF is intertemporally **additive**

$$U(c_0, c_1) = u(c_0) + \beta u(c_1), \text{ where } \beta \equiv \frac{1}{1+\rho}$$

  where $\beta \in (0,1)$ is the psychological discount factor and ρ is the rate of time preference, and $u(\cdot)$ is called the Bernoulli utility function

- **Assumption 2**: u is increasing and concave $u^{''}(c_t) < 0 < u^{'}(c_t)$, $t = 0, 1$

Intertemporal utility

Example 1: additive IUF

- Marginal utilities for c_t, $t = 0, 1$ are positive

$$U_0 = u^{'}(c_0) > 0, \;\; U_1 = \beta u^{'}(c_1) > 0$$

- Utility function is concave

$$U_{00} = u^{''}(c_0), \;\; U_{01} = 0, U_{11} = \beta u^{''}(c_1)$$

- The *IMRS* is

$$IMRS_{0,1} = \frac{U_0}{U_1} = \frac{u^{'}(c_0)}{\beta u^{'}(c_1)}$$

Therefore: marginal utility for period $t = 0$ is proportional to the discounted marginal utility for period $t = 1$ (from the perspective of period $t = 0$)

$$\boxed{u^{'}(c_0) = \beta u^{'}(c_1)\, IMRS_{0,1}}$$

we will see an analogous equation again and again translating the idea of intertemporal arbitrage.

Intertemporal utility

Example 1: additive IUF

- The Allen-Uzawa elasticities are

$$\varepsilon_{00}(c_0) = -\frac{u^{''}(c_0)c_0}{u^{'}(c_0)},\ \varepsilon_{01} = 0,\ \varepsilon_{11}(c_1) = -\frac{u^{''}(c_1)c_1}{u^{'}(c_1)}$$

  displays intertemporal independence
- The elasticity of intertemporal substitution between period 0 and 1 is

$$EIS_{0,1}(c_0, c_1) = \frac{c_0 u^{'}(c_0) + \beta c_1 u^{'}(c_1)}{\beta c_1 u^{'}(c_1)\varepsilon_{00}(c_0) + c_0 u^{'}(c_0)\varepsilon_{11}(c_1)}$$

Intertemporal utility

Example 1: additive IUF

For a stationary consumption path $\{\bar{c}, \bar{c}\}$ we find:

- The IMRS is independent from $\bar{c}$ and

$$IMRS_{0,1}(\bar{c}) = \frac{1}{\beta} = 1 + \rho > 1$$

  this means that the IUF displays **impatience**, and this effect is captured by time discounting

- It displays **intertemporal independence** because

$$\varepsilon_{0,1}(\bar{c}) = 0$$

  Intuition: the marginal valuation of consumption at time $t = 1$ is independent of the history of consumption

- The IES is

$$IES_{0,1}(\bar{c}) = -\frac{u^{'}(\bar{c})}{u^{''}(\bar{c})\bar{c}} > 0$$

  Intuition: this is interpreted as a measure of the preference for **consumption smoothing** through time

Intertemporal utility

Example 1: additive IUF

Particular case:

- Isoelastic utility function

$$u(c) = \begin{cases} \dfrac{c^{1-\zeta} - 1}{1-\zeta} & \text{if } \zeta \neq 1 \\ \ln(c) & \text{if } \zeta = 1 \end{cases}$$

- Derivatives

$$U_0 = c_0^{-\zeta},\ U_1 = \beta\, c_1^{-\zeta},\ U_{00} = -\zeta\, c_0^{-\zeta-1},\ U_{01} = 0,\ U_{11} = -\zeta\, c_1^{-\zeta-1}$$

- The IMRS is

$$IMRS_{0,1} = \frac{1}{\beta}\left(\frac{c_1}{c_0}\right)^{\zeta}$$

Intertemporal utility

Example 1: additive IUF

For a stationary consumption path $c_0 = c_1 = \bar{c}$:

- The IMRS is

$$IMRS_{0,1} = \frac{1}{\beta} > 1$$

  It displays **impatience**
- The AU elasticities are constant $\varepsilon_{00} = \varepsilon_{11} = \zeta$
- The IES is also constant

$$EIS_{0,1} = \frac{1}{\zeta}$$

  this is why ζ is called the **inverse of the elasticity of intertemporal substitution** .

Intertemporal utility

Example 2: endogenous discount factor

- **Uzawa and Epstein-Hynes** utility

$$U(c_0, c_1) = u(c_0) + b(c_0)v(c_1)$$

the discount factor is endogenous i.e. $\beta = b(c)$ with $b^{'}(.) < 0$ (rich people are more patient) $v^{'}(.) > 0$

The crossed AU elasticity is for a stationary sequence is

$$\varepsilon_{0,1}(c) = -\frac{b^{'}(c)v^{'}(c)c}{u^{'}(c) + b^{'}(c)v(c)} \neq 0$$

displays intertemporal dependence

Intertemporal utility

Example 3: habit formation

- **Habit formation**

$$U(c_0, c_1) = u(c_0) + \beta v(c_0, c_1).$$

for $v_{c_0}(c_0, c_1) < 0$ where c_0 refer to consumption habits
The crossed AU elasticity is for a stationary sequence is

$$\varepsilon_{0,1}(c) = -\frac{\beta v_{c_0 c_1}(c) c}{u'(c) + \beta v_{c_0}(c)} c$$

can display intertemporal substitutability, independence or complementarity depending on the relationship between time discounting and the relative importance of habits, i.e., the magnitude of $v_{c_0}(c)$

Intertemporal utility

Example 3: habit formation

- Consider the intertemporal utility function

$$U(c_0, c_1) = \ln(c_0) + \beta \ln\left(\left(\frac{c_1}{c_0}\right)^{\eta}\right),\ \eta > 0$$

- Derivatives

$$U_0 = \frac{1-\beta\eta}{c_0},\ U_1 = \frac{\beta\eta}{c_1},\ U_{00} = -\frac{1-\beta\,\eta}{c_0^2},\ U_{01} = 0,\ U_{11} = -\frac{\beta\eta}{c_1^2}$$

- The IMRS is

$$IMRS_{0,1}(c_0, c_1) = \left(\frac{1-\beta\eta}{\beta\eta}\right)\frac{c_1}{c_0}$$

- The AU elasticities are constant

$$\varepsilon_{00} = \varepsilon_{11} = 1,\ \varepsilon_{01} = 0$$

- The IES is also constant

$$EIS_{0,1}(c_0, c_1) = 1$$

  for any (c_0, c_1)

Intertemporal utility

Case 2: habit formation example, cont

For a stationary sequence $c_0 = c_1 = c$

- The IMRS
$$IMRS_{0,1}(c) = \frac{1-\beta\zeta}{\beta\zeta}$$
the utility displays impatience if $\zeta < \dfrac{1}{2\beta} = \dfrac{1+\rho}{2}$.
Intuition: there is impatience (according to the above definition) if the weight of past consumption is not too strong
- As $\varepsilon_{01} = 0 = 0$ the model displays intertemporal independence (but this is special to this example).

2. Intertemporal household problem with an intertemporal budget constraint

Contract environment

- A household has a resource (endowment) in positive amount ($w > 0$)
- Wants to consume over two periods, $\{c_0, c_1\}$, today c_0 and in the future c_1.
- There is a market for **forward contracts** allowing for contracting today for delivery in the future, at a price set today, $q > 0$. We take the price paid today as a *numéraire* and all the variables are denominated at todays' price
- The value of the consumption sequence is assessed by an intertemporal utility function: $U(c_0, c_1)$;
- The **budget constraint**, referring to payments made today, is

$$c_0 + q\, c_1 \leq w$$

The household problem

- The intertemporal problem for the consumer is

$$v(w) = \max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + q\, c_1 \leq w \}$$

- The (interior) optimum (c_0^*, c_1^*) satisfies the conditions

$$\begin{cases} qU_0(c_0^*, c_1^*) = U_1(c_0^*, c_1^*) & \text{(optimality condition)} \\ c_0^* + pc_1^* = w & \text{(budget constraint)} \end{cases}$$

The household problem: interpretation

- At the optimum: the IMRS is equal to the relative price (internal = external valuation)

$$IMRS^*_{0,1} = IMRS_{0,1}(c_0^*, c_1^*) = \frac{U_0(c_0^*, c_1^*)}{U_1(c_0^*, c_1^*)} = \frac{1}{q}$$

- Intuition: at the optimum increasing one euro of consumption tomorrow should be matched by a reduction in $1/q$ euro of consumption today, ie $dc_0^* = -q dc_1^*$
- Therefore q is an **intertemporal relative price**: i.e., an opportunity cost for changing the consumption sequence across time.

The household problem

Example

- The simplest problem

$$v(w) = \max_{c_0, c_1} \{ \ln(c_0) + \beta \ln(c_1) : c_0 + q\, c_1 = w \}$$

- The Lagrangean is

$$\mathcal{L}(c_0, c_1, \lambda) = \ln(c_0) + \beta \ln(c_1) + \lambda (w - c_0 - q\, c_1)$$

- The first order conditions

$$\begin{cases} \dfrac{\partial L}{\partial c_0} = \dfrac{1}{c_0} - \lambda = 0 & \Longrightarrow \; c_0 = \dfrac{1}{\lambda} \\ \dfrac{\partial L}{\partial c_1} = \dfrac{\beta}{c_1} - q\lambda = 0 & \Longrightarrow \; c_1 = \dfrac{\beta}{\lambda\, q} \\ \dfrac{\partial L}{\partial \lambda} = c_0^* + q c_1^* = w & \Longrightarrow \; \dfrac{1}{\lambda} = \dfrac{w}{1+\beta} \end{cases}$$

The household problem

Example

- The solution is

$$\begin{cases} c_0^* = \dfrac{w}{1+\beta} \\ c_1^* = \dfrac{\beta\, w}{q\,(1+\beta)} \end{cases}$$

- Interpretation:
  1. although there are only trading at $t = 0$ consumption is positive in every period
  2. consumption in every period depends on total wealth
  3. consumption is increasing, stationary or decreasing in time depending on the ratio $\frac{\beta}{q}$: i.e., on the relative internal valuation of time relative to the market price of forward contracts
  4. if $\frac{\beta}{q} = 1$ there is complete **consumption smoothing over time**

3. Intertemporal household problem with a sequence of budget constraints

Contract environment

- A household receives a sequence of endowments $\{y_0, y_1\}$ in positive amounts
- Wants to consume over two periods, $\{c_0, c_1\}$, today c_0 and in the future c_1.
- There are spot real markets opening at every period with price equal to 1
- There is also a market for **spot financial contracts** opening at every period. In this market the asset (that can be seen as a durable good) agents can lend and borrow at period $t = 0$ paying or receiving an interest income at period $t = 1$.
- Now, every agent has na **sequence of budget constraints** (because trade in the good market can take place at period 1)

Household problem

- The problem (assuming an additive intertemporal utility)

$$\max_{c_0, c_1, a_1, a_2} U(c_0, c_1) = u(c_0) + \beta u(c_1)$$
$$\text{subject to}$$
$$c_0 + a_1 = y_0 + a_0$$
$$c_1 = y_1 + (1+r)a_1 - a_2$$
$$c_0 \geq 0, c_1 \geq 0, a_1 \text{free}, a_2 \geq 0$$

  where a_0 is the level of the asset at beginning of period 0 and a_1 and a_2 are the levels at the end of period 0 and 1, and r is the real interest rate.
- Assumptions: $0 < \beta < 1$, $u''(c) < 0 < u'(c)$ (there is no satiation and $u(\cdot)$ is concave
- The initial resource, a_0, is finite

Household problem

Meaning of the last constraints:

- consumption cannot be negative

$$c_0 \geq 0, c_1 \geq 0$$

- a_1 **free** means the consumer can be in one of the three positions
  - can be a net debtor if $a_1 < 0$
  - can be a net creditor if $a_1 > 0$
  - neither a debtor nor a creditor $a_1 = 0$
- the **non-Ponzi game condition**: cannot be a debtor at the end of the last period

$$a_2 \geq 0$$

- Next, we prove that, it will never be optimal to have $a_2 > 0$

Household problem

Optimality of $a_2 = 0$

- Substitute c_0 and c_1 in the utility function, assume that $\beta > 0$ and r is finite, and consider the constraint for a_2

$$\max_{a_1, a_2} \{u(y_0 + a_0 - a_1) + \beta u(y_1 + (1+r)a_1 - a_2) : a_2 \geq 0\}$$

- The first order conditions are

$$\begin{aligned} u^{'}(c_0) &= \beta(1+r)u^{'}(c_1) \iff IMRS_{0,1}?\frac{\beta}{R} \\ \beta u^{'}(c_1) &= \lambda \\ \lambda a_2 &= 0,\ \lambda \geq 0,\ a_2 \geq 0 \end{aligned}$$

- We have $a_2 > 0$ if and only if $\lambda = 0$, but in this case either there is satiation or $c_1 \to \infty$ and $c_0 \to \infty$. But this is only possible if $a_0 - \to \infty$. Therefore we should have $a_2 = 0$ and $\lambda > 0$.

Household problem

The consumer problem in a frictionless case

- Taking $a_2 = 0$ and assuming a_1 is free (i.e., the consumer can borrow or lend freely) we can eliminate a_1 in the sequence of budget constraints, to get

$$c_0 + mc_1 = a_0 + y_0 + my_1$$

where m is the **market discount factor**

$$m \equiv \frac{1}{1+r} \equiv \frac{1}{R}$$

Household problem

Relationship between the two environments

- This implies that if $w = a_0 + y_0 + m\, y_1$ and the non-Ponzi game condition holds, such that the consumer chooses optimally the last time financial wealth $a_2 = 0$
- if there are no constraints on a_1 then **sequence of budget constraints** is equivalent to an **intertemporal budget constraint** formally similar to the first problem

$$c_0 + mc_1 = a_0 + y_0 + my_1 \iff c_0 + qc_1 = w$$

- if $m = \frac{1}{R} = q$ the stochastic discount factor is the relative price for forward contracts and $w = a_0 + y_0 + my_1$ financial plus "human wealth"

Household problem

Question

- Remember the last slide
- What are the consequences of the existence of a friction taking the form of a constraint in a_1 ?
  - would the problem in the two environments be equivalent ?
  - would the arbitrage condition

$$IMRS_{01} = \frac{\beta}{R}$$

    still hold ?