AME 2019-2020:
Problem set 4: PDE's, first-order and parabolic

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1 First-order partial differential equations

1.1 General 1

1. Consider the $y_t(t,x) + y_x(t,x) = 0$. Solve the following cases:

   a) Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and assume that $y(0,x) = y_0(x)$ is an arbitrary continuous function.

   b) Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and assume that $y(0,x) = e^{-x^2}$.

   c) Let $(t,x) \in \mathbb{R}^2_+$ and assume that $y(0,x) = e^{-x^2}$ and $y(t,0) = 0$.

   d) Compare the projected characteristics and provide an intuition.

2. Consider the first-order partial differential equation $y_t(t,x) + y_x(t,x) = 0$, where $y = y(t,x)$ and $(t,x) \in (0,\infty) \times (0,\infty)$.

   a) Find the solution (hint use Laplace transforms).

   b) Let $y(0,x) = e^{-(x-x_0)^2}$. Find the solution to the initial-value problem.

3. Consider the first-order partial differential equation $y_t(t,x) + ax\, y_x(t,x) = 0$, where $y = y(t,x)$ and $(t,x) \in (0,\infty) \times (-\infty,\infty)$.

   (a) Find the solution (hint use the method of characteristics).

   (b) Let $y(0,x) = \delta(x-x_0)$, where $\delta(.)$ is Dirac's delta function and $x_0 > 0$. Find the solution to the initial-value problem.

1.2 General 2

1. Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Prove that the solution for the equation $y_t(t,x) + ay_x(t,x) = b(t,x)$ is $y(t,x) = \int_0^t b(s, x - a(t-s))ds + f(x-at)$ where f is an arbitrary function.

2. Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Prove that the solution for the equation $y_t(t,x) + ay_x(t,x) = cy(t,x)$ is $y(t,x) = f(x-at)e^{ct}$ where f is an arbitrary function.

3. Let $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Prove that the solution for the equation $y_t(t, x) + a y_x(t, x) = c(t, x) y(t, x)$ is $y(t, x) = f(x - at) e^{\int_0^t c(s, x - a(t-s)) ds}$, where f is an arbitrary function.

1.3 Applications

1. Consider a simple version of the McKendrick model where $n(t, a)$ is population with age a at time t. Off course $(t, a) \in \mathbb{R}_+$.

$$\begin{cases} n_t(t, a) + n_a(t, a) = -\mu n(t, a), \\ n(0, a) = e^{0.005 * a(100-a)} \\ n(t, 0) = e^{\beta t} \end{cases}$$

   Let the fertility rates and the mortality rates be $\beta = 0.013$ $\mu = 0.009$.

   a) Determine the dynamic equation for the age distribution of the population $n(t, a)$.

   b) Compute the total population and give an intuition for the solution you have obtained.

2. The forward interest rate, $f(x, t)$, where t is time and x is the time until maturity, verifies a first order PDE

$$f_x - f_t = 0 (x, t) \in (0, \infty) \times (0, \infty) \tag{1}$$

   (a) Find the general solution for equation (1)

   (b) Assume that $f(0, t) = r(t) = e^{-\gamma t}$ where $\gamma > 0$ is the spot interest rate. Find the solution for the boundary value problem.

   (c) Assume that $f(x, 0) = x^\beta$ where $0 < \beta < 1$. Find the solution for the initial value problem.

   (d) Impose the former two conditions and discuss the existence of solutions for the problem.

3. Consider a the age-dependent budget constraint where $w = w(t, a)$ is the net asset position for an agent at age a at time t. Off course $(t, a) \in \mathbb{R}_+$.

$$\begin{cases} w_t(t, a) + w_a(t, a) = s(a) + r w(t, a), \\ w(0, a) = 1 - \left(\dfrac{a}{a_m} \right)^{-\alpha} \\ w(t, 0) = 0 \end{cases}$$

   where r is the rate of return on assets and $\alpha > 0$ (the initial distribution of wealth is a Pareto distribution). Assume that $s(a) = e^{\gamma a(\bar{a} - a)}$ for $\gamma > 0$ and $\bar{a} > a_m$.

   a) Find $w(t, a)$.

   b) Compute the total asset position and give an intuition for the solution you have obtained.

2 Parabolic partial differential equations

2.1 General

1. Consider the following parabolic PDE $u_t(t,x) = au_{xx}(t,x) + b$, where $a > 0$. Solve the following problems:

   a) For $(t,x) \in [0,\infty) \times (-\infty,\infty)$.

   b) For $(t,x) \in [0,\infty) \times [0,\infty)$.

   c) For $(t,x) \in [0,\infty) \times [\underline{x}, \bar{x}]$.

2. Consider the following parabolic PDE $u_t(t,x) = au_{xx}(t,x) + b(t)$, where $a > 0$. Solve the following problems:

   a) For $(t,x) \in [0,\infty) \times (-\infty,\infty)$.

   b) For $(t,x) \in [0,\infty) \times [0,\infty)$.

3. Assume that $(t,x) \in [0,\infty) \times [0,\infty)$ and $u = u(t,x)$ and consider the initial-value problem

$$\begin{cases} u_t = au_{xx} + bu, & (x,t) \in [0,\infty)^2 \\ u(0,x) = \phi(x), & (x,t) \in [0,\infty) \times \{t = 0\}. \end{cases}$$

   a) Find the general solution to the initial-value problem.

   b) Let $\phi(x) = \delta(x)$, where $\delta(.)$ is Dirac's delta "function". Find the solution to the problem. Provide an intuition for the solution.

4. Consider the parabolic partial differential equation $u_t - u_{xx} = 0$, where $u = u(t,x) \in \mathbb{R}$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

   a) Find the solution to the PDE.

   b) If $u(0,x) = \delta(x - x_0)$, where $\delta(.)$ is Dirac's delta "function" and $x_0 > 0$ find the solution to the initial-value problem. Provide an intuitive characterization of the solution.

5. Assume that $(t,x) \in [0,T] \times [0,\infty)$ and $u = u(t,x)$ and consider the initial-value problem

$$\begin{cases} u_t + u_{xx} + bu, & (t,x) \in [0,T] \times [0,\infty) \\ u(T,x) = \phi_T(x), & (t,x) \in \{t = T\} \times [0,\infty). \end{cases}$$

   a) Classify the PDE.

   b) Find the general solution to the PDE.

   c) Find the solution to the problem. Provide an intuition for the solution.

6. Assume that $(t,x) \in [0,\infty) \times [0,\infty)$ and $u = u(t,x)$ and consider the initial-value problem

$$\begin{cases} u_t = u_{xx} + u_x, & (x,t) \in [0,\infty)^2 \\ u(0,x) = \phi(x), & (x,t) \in [0,\infty) \times \{t=0\}. \end{cases}$$

   a) Find the general solution to the PDE.
   b) Find the general solution to the initial-value problem.
   c) Let $\phi(x) = \delta(x)$, where $\delta(.)$ is Dirac's delta "function". Find the solution to the problem. Provide an intuition for the solution.

7. Assume that $(t,x) \in [0,\infty) \times [0,\infty)$ and $u = u(t,x)$ and consider the initial-value problem

$$\begin{cases} u_t = u_{xx} + u_x + b(x), & (x,t) \in [0,\infty)^2 \\ u(0,x) = \phi(x), & (x,t) \in [0,\infty) \times \{t=0\}. \end{cases}$$

   a) Find the general solution to the PDE.
   b) Find the general solution to the initial-value problem.
   c) Let $\phi(x) = \delta(x)$, where $\delta(.)$ is Dirac's delta "function" and $b(x) = x^2$. Find the solution to the problem. Provide an intuition for the solution.

2.2 Applications

1. Let $k(t,x)$ be the capital stock located at time $t \in [0,\infty)$ at location $x \in (-\infty,\infty)$. If capital flows freely between locations, and if it is locally driven by the difference in spatial concentration, it can be proved that the distribution of capital can be modelled by the PDE

$$k_t(t,x) = k_{xx}(t,x) + f(k(t,x)) - c(t,x)$$

   where $f(.)$ is the production function and c is consumption. Let $c = (1-s)f(k)$ where $0 < s < 1$ is the savings rate, and $f(k) = Ak$, where $A > 0$. Assume that the initial distribution of the capital stock is $k(0,x) = e^{-(k-k_0)^2}$ where $k_0 > 0$.

   a) Solve the initial-value problem
   b) Provide an intuition to the solution, and, in particular, to the long-run behavior $\lim_{t\to\infty} k(t,x)$.

2. Consider the parabolic PDE

$$C_t(t,x) = -C_{xx}(t,x) + (r-\rho)C(t,x), \ (t,x) \in [0,\infty) \times [0,\infty)$$

   where r and ρ are both positive, subject to the constraint $\lim_{t\to\infty} e^{-\rho t} C(t,x) = e^{-x^2}$.

   a) Classify the PDE and discuss the well-posedness of the problem.
   b) Solve the equation