# The Ramsey growth model

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17.3.2021

# A short history of the model

- ▶ Frank Ramsey (see https://en.wikipedia.org/wiki/Frank_P._Ramsey) made several important contributions in his short life (he died at 26) one of them Ramsey (1928)
- ▶ His contribution was only fully recognized in the early 60's ( Cass (1965), Koopmans (1965)) as presenting a rigorous alternative to the ad-hoc aspects (dynamic inefficiency) of the Solow (1956) model (now we call it **exogenous growth theory**)
- ▶ It was rejoined again in the middle of the 1980's which saw the onset of **endogenous growth theory**
- ▶ It is also the founding rock of the DGE (dynamic general equilibrium theory) of macroeconomics

# The Ramsey model
## The basic idea

- Output is produced by ohysical capital and labor and can be used for investment or for consumption (everything in per capita terms): this introduces an **intratemporal budget constraint**
- **savings** is determined by a **arbitrage between present and future consumption**: it balances two effects:
  - present consumption is a good thing, although its utility decreases with the amount consumed;
  - however, if people sacrifice present consumption to save and increase the capital stock they improve their prospects for more consumption in the future;
- this idea can be formalized by a **intertemporal optimization problem**

# The Ramsey model

- ▶ Production:
  - ▶ closed economy producing a single composite good
  - ▶ production uses two factors: labor and physical capital
  - ▶ production technology: neoclassical (increasing, concave, Inada, CRTS)
- ▶ Reproducible factor:
  - ▶ physical capital (machines)
- ▶ Population:
  - ▶ exogenous (can be constant or increase exponentially)

# The Ramsey model
## Assumptions: continuation

- ▶ Households: optimizing behavior
  - ▶ maximize an intertemporal utility functional with consumption as the control variable
  - ▶ subject to a budget constraint
  - ▶ labor is supplied inelastically
  - ▶ have perfect foresight
- ▶ Equilibrium is Pareto optimal, therefore it is equivalent to a central planer problem

# Ramsey model

▶ In aggregate terms

$$Y(t) = F(A, K(t), L(t)) = AK(t)^{\alpha}L(t)^{1-\alpha}, \ 0 < \alpha < 1$$

where: $A$ TFP productivity, $K$ stock of capital, $L = N$ loabor input = population

▶ In per capita terms:

$$y(t) = Ak(t)^{\alpha}$$

where $y = Y/N$ and $k = K/N$

# Ramsey model
The model: preferences

Preferences: for the representative agent

- ▶ the intertemporal utility functional is

$$V[c] = \int_0^\infty u(c(t)) e^{-\rho t} dt$$

- ▶ $c = C/N$ per capita consumption, $[c] = \big(c(t)\big)_{t\in[0,\infty)}$
- ▶ $\rho > 0$ is the rate of time preference
- ▶ the instantaneous utility function is

$$u(c) = \begin{cases} \dfrac{c^{1-\theta} - 1}{1 - \theta}, & \text{if } \theta \in (0,\infty)/\{1\} \\ \ln(c), & \text{if } \theta = 1 \end{cases}$$

where $1/\theta$ is the elasticity of intertemporal substitution

# Ramsey model
Versions

- We are assuming an **homogeneous agent** (or representative) economy
- There are two versions of the model
  - **centralized** version: maximization of social welfare given the budget constraint
  - **decentralized** (DGE) version: individual maximization of households an firms coordinated by market equilibrium
- As there are no externalities the are **equivalent** (in the sense that generate the same allocations, of consumption and capital through time)

Centralized version:
the Ramsey model:

# Ramsey model
The centralized version

- The central planner is a "benevolent dictator" (acts on behalf of the best interests of the society)
- The central planner solves the problem

$$\max_{(c)_{t \geq 0}} \int_0^\infty \frac{c(t)^{1-\theta} - 1}{1 - \theta} e^{-\rho t} dt$$

$$\text{subject to}$$

$$\dot{k} = Ak(t)^\alpha - c(t) - \delta k(t),$$

$$k(0) = k_0 > 0, \text{ given}$$

$$\lim_{t \to \infty} h(t)k(t) \geq 0$$

physical capital is asymptotically bounded ($h(t)$ is any discount factor)

# Ramsey model

Solving by using the Pontriyagin's max principle

▶ The current-value Hamiltonian is

$$H(c, k, q) = \frac{c^{1-\theta} - 1}{1 - \theta} + q(Ak^{\alpha} - c - \delta k)$$

▶ the optimality conditions are

$$\frac{\partial H}{\partial c} = 0 \iff c^{-\theta}(t) = q(t), \ t \in [0, \infty)$$

$$\dot{q} = \rho q - \frac{\partial H}{\partial k} \iff \dot{q} = q(t)\left(\rho + \delta - \alpha Ak(t)^{\alpha - 1}\right), \ t \in [0, \infty)$$

$$\lim_{t \to \infty} q(t)k(t)e^{-\rho t} = 0$$

▶ the admissibility conditions

$$\dot{k} = Ak(t)^{\alpha} - c(t) - \delta k(t), \ t \in [0, \infty)$$

$$k(0) = k_0 > 0, \ t = 0$$

# Ramsey model

► An optimum path $(c^*(t), k^*(t))_{t \in [0, +\infty)}$ is the solution of the (MHDS)

$$
\begin{aligned}
\dot{c} &= \frac{c}{\theta}\left(r(k(t)) - \rho - \delta)\right) \\
\dot{k} &= Ak(t)^\alpha - c(t) - \delta k(t) \\
0 &= \lim_{t \to \infty} c(t)^{-\theta} k(t) e^{-\rho t} \\
k(0) &= k_0 \text{ given}
\end{aligned}
$$

► with the (gross) rate of return for capital

$$
r(k) = \alpha A k^{\alpha - 1}
$$

# Solving the Ramsey model

- In general the Ramsey **does not have an explicit solution** (also called exact or closed form)
- Analytical methods for finding the solution:
    - **linear approximation** of the solution to converging to the steady state, which satisfies the transversality constraint;
    - **exact solution**: **transforming** the MHDS into a known differential equation (only for that very special case)
- We can only find an **exact solution** for the case $\theta = \alpha$ (which is counterfactual)
- In all cases, **it is always a good idea to build the phase diagram**

# Steady states

▶ Steady states are fixed points of the system

$$\frac{c^*}{\theta}\left(r(k^*) - \rho)\right) = 0,$$
$$c^* = A(k^*)^\alpha - \delta k^*.$$

▶ there are three steady states

$$(c^*, k^*) = \{(0,0), (0, (A/\delta)^{1/(1-\alpha)}), (\bar{c}, \bar{k})\}$$

for

$$\bar{k} = \left(\frac{\alpha A}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}, \ \bar{c} = \frac{\rho + \delta(1-\alpha)}{\alpha}\bar{k}$$

▶ the last one verifies the transversality condition (the second does not: check)

## Steady states: and long-run level of GDP

▶ Then the unique steady state which satisfies the initial and the transversality conditions is

$$\bar{k} = \left( \frac{\alpha A}{\delta + \rho} \right)^{\frac{1}{1-\alpha}} , \ \bar{c} = \frac{\rho + \delta(1 - \alpha)}{\alpha} \bar{k}$$

▶ the associated long-run GDP is

$$\bar{y} = A\bar{k}^{\alpha} = \left[ A \left( \frac{\alpha}{\delta + \rho} \right)^{\alpha} \right]^{\frac{1}{1-\alpha}} . \tag{1}$$

# Approximate solution

- there is no explicit solution
- we study dynamics of the approximate system in a neighbourhood of $(\bar{c}, \bar{k})$
- the linearised MHDS is

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{\bar{c}r^{'}(\bar{k})}{\theta} \\ -1 & \rho \end{pmatrix} \begin{pmatrix} c(t) - \bar{c} \\ k(t) - \bar{k} \end{pmatrix}$$

- where $r^{'} = (\alpha - 1)\alpha A k^{\alpha-2}|_{k=\bar{k}} = -\dfrac{(1-\alpha)\rho}{\bar{k}} < 0$
- and $\dfrac{\bar{c}r^{'}(\bar{k})}{\theta} = -d \equiv -\dfrac{(1-\alpha)\,\rho\,(\rho + \delta(1-\alpha))}{\alpha\theta} < 0$

# Approximate solution

- the system is of type $\dot{x} = Jx$
- where the Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} 0 & -d \\ -1 & \rho \end{pmatrix}$$

- the solution is of type

$$x(t) = h_s \mathbf{V}^s e^{\lambda_s t} + h_u \mathbf{V}^u e^{\lambda_u t}$$

- where $\lambda_j$ are the eigenvalues and $\mathbf{V}^j$ are the associated eigenvectors of $J$ and $h_s$ are arbitrary constants

# Approximate solution

▶ the eigenvalues of $\mathbf{J}$ are

$$
\begin{aligned}
\lambda_u &= \frac{\rho}{2} + \left[\left(\frac{\rho}{2}\right)^2 + d\right]^{1/2} > \rho > 0 \\
\lambda_s &= \frac{\rho}{2} - \left[\left(\frac{\rho}{2}\right)^2 + d\right]^{1/2} < 0
\end{aligned}
$$

▶ satisfying $\lambda_s + \lambda_u = \rho > 0$, $\lambda_s \lambda_u = -d$
▶ then $(\bar{c}, \bar{k})$ is a **saddle-point**

## Approximate solution

- the eigenvectors are determined as follows
- $\mathbf{V}^s$ solves the homogeneous system

$$(\mathbf{J} - \lambda_s \mathbf{I}_2) \mathbf{V}^s = \mathbf{0}$$

- that is

$$\begin{pmatrix} -\lambda_s & -d \\ -1 & \rho - \lambda_s \end{pmatrix} \begin{pmatrix} \mathbf{V}_1^s \\ \mathbf{V}_2^s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- the members of vector $\mathbf{V}^s$ should satisfy

$$\frac{\mathbf{V}_1^s}{\mathbf{V}_2^s} = -\frac{d}{\lambda_s} = \lambda_u \Rightarrow \mathbf{V}^s = \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix}$$

(because $\rho - \lambda_s = \lambda_u$)

- for $\mathbf{V}^u$ we find (prove this)

$$\frac{\mathbf{V}_1^u}{\mathbf{V}_2^u} = -\frac{d}{\lambda_u} = \lambda_s \Rightarrow \mathbf{V}^u = \begin{pmatrix} \lambda_s \\ 1 \end{pmatrix}$$

## Approximate solution

▶ Then the general solution is

$$\begin{pmatrix} c(t) - \bar{c} \\ k(t) - \bar{k} \end{pmatrix} = h_s \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix} e^{\lambda_s t} + h_u \begin{pmatrix} \lambda_s \\ 1 \end{pmatrix} e^{\lambda_u t}$$

▶ We determine $h_s$ and $h_u$ by forcing the general solution to satisfy the two remaining conditions

$$\lim_{t \to \infty} \frac{k(t)}{c(t)^\theta} e^{-\rho t} = 0, \text{ and } k(0) = k_0$$

  ▶ the first condition holds if
    $\lim_{t \to \infty}(c(t) - \bar{c}) = \lim_{t \to \infty}(k(t) - \bar{k}) = 0$, i.e., they converge
    to the steady state, which is obtained by eliminating the
    effect of $e^{\lambda_u t}$ (wich converges to $\infty$) by setting $h_u = 0$
  ▶ the second condition holds if

$$k(0) = \bar{k} + h_s - \bar{k} = k_0 \rightarrow h_s = k_0$$

# Approximate solution

▶ the approximate solution is, for $t \in [0, \infty)$

$$
\begin{aligned}
c(t) &= \bar{c} + \lambda_u (k_0 - \bar{k}) e^{\lambda_s t}, \\
k(t) &= \bar{k} + (k_0 - \bar{k}) e^{\lambda_s t}.
\end{aligned}
$$

# Approximate solution

▶ at $t = 0$ we have

$$\begin{pmatrix} c(0) \\ k(0) \end{pmatrix} = \begin{pmatrix} \bar{c} + \lambda_u(k_0 - \bar{k}) \\ k_0 \end{pmatrix}$$

observe that $\lambda_u$ gives the variation of consumption as $c(0) - \bar{c} = \lambda_u(k_0 - \bar{k})$ and the initial consumption is determined from **future data** ($\bar{c}$ and $\bar{k}$)

▶ asymptotically (i.e., in the long run)

$$\lim_{t \to \infty} \begin{pmatrix} c(t) \\ k(t) \end{pmatrix} = \begin{pmatrix} \bar{c} \\ \bar{k} \end{pmatrix} = \begin{pmatrix} \frac{\rho + \delta(1-\alpha)}{\alpha} \\ 1 \end{pmatrix} \bar{k}$$

the solution converges to the steady state (this means that the transversality condition is satisfied)

▶ **the saddle path dynamics implies that the solution is unique**

# Ramsey model

Case $\theta \neq \alpha$: phase diagrams for $\theta < \alpha$
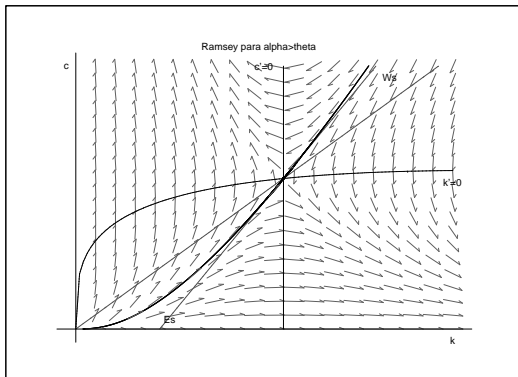


Figure: Exact (dark) and approximate (light) solutions

# Ramsey model

Case $\theta \neq \alpha$: phase diagrams for $\theta > \alpha$



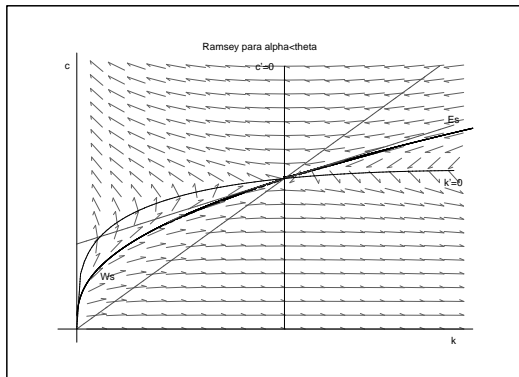Figure: Exact (dark) and approximate (light) solutions

# Ramsey model

▶ there is an explicit solution:

$$
\begin{aligned}
c(t) &= \frac{\delta + \rho(1-\alpha)}{\alpha} k(t), \\
r(t) &= \frac{r(0)(\delta + \rho)}{r(0) + (\delta + \rho - r(0))e^{-[(1-\alpha)(\delta+\rho)/\alpha]t}},
\end{aligned}
$$

with $k(t) = (\alpha A / r(t))^{1/(1-\alpha)}$

▶ given $k(0)$ we get explicitly

$$
c(0) = \frac{\delta + \rho(1-\alpha)}{\alpha} k(0)
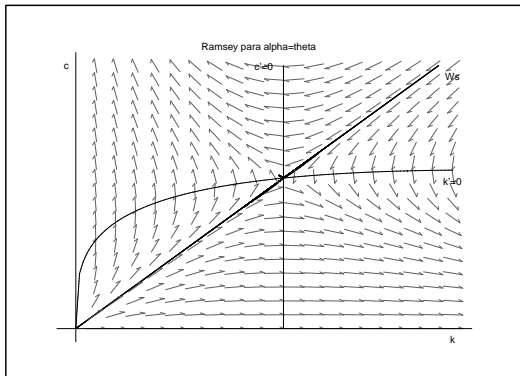$$

▶ convergences asymptotically to the steady state,

$$
\begin{aligned}
\lim_{t\to\infty} c(t) &= \bar{c} \\
\lim_{t\to\infty} r(t) &= \bar{r} = \delta + \rho \\
\lim_{t\to\infty} k(t) &= \bar{k}
\end{aligned}
$$

# Ramsey model

Case $\theta = \alpha$: phase diagram

# Ramsey model
## Properties of the solution paths

1. if $k(0) \neq \bar{k}$ then $\lim_{t \to \infty} k(t) = \bar{k}$,

2. given any initial value for $k$, $k(0)$, there is only a value for $c$, $c(0)$ which is determined endogenously such that $\lim_{t \to \infty} c(t) = \bar{c}$;

3. **the solution is determinate, i.e, unique**: this is the only solution for the ode system such that the transversality condition holds;

4. the saddle path is asymptotically tangent to the straight line

$$c(t) - \bar{c} = \lambda_u (k(t) - \bar{k})$$

# Ramsey model
## Per-capita GDP dynamics

- the **approximate** per-capita output path is generated by

$$y(t) = \left[ \bar{y}^{1/\alpha} + (y(0)^{1/\alpha} - \bar{y}^{1/\alpha})e^{\lambda_s t} \right]^{\alpha} \qquad (2)$$

the model only displays transitional dynamics as $\lambda_s < 0$.

- if $\theta = \alpha$: GDP exact dynamics is generated by

$$y(t) = A \left[ \frac{\alpha A k(0)^{\alpha-1}(\delta + \rho)}{\alpha A k(0)^{\alpha-1} + (\delta + \rho - \alpha A k(0)^{\alpha-1})e^{-[(1-\alpha)(\delta+\rho)/\alpha]t}} \right]^{\alpha},$$

- in any case the solution converges asymptotically to the steady state

$$\lim_{t \to \infty} y(t) = \bar{y} = \left[ A \left( \frac{\alpha}{\delta + \rho} \right)^{\alpha} \right]^{1/(1-\alpha)}$$

# The Ramsey model
Growth implications

▶ there is **no long-run growth** $\bar{g} = 0$

▶ the **long-run level** $\bar{y}$ depends on $(A, \delta, \rho, \alpha)$: productivity, the rate of depreciation, the rate of time preference (impatience) and on the income shares (see equation (1));

▶ there is **only transitional dynamics**: the **speed** and the pattern of convergence depends on the relationship between the capital share, $\alpha$, in income and the intertemporal elasticity of substitution $\theta$ (see equation (2)). This is because

$$\lambda_s = \frac{\rho}{2} - \left[ \left(\frac{\rho}{2}\right)^2 + \frac{(1-\alpha)\,\rho\,(\rho + \delta(1-\alpha))}{\alpha\theta} \right]^{\frac{1}{2}} < 0$$

the higher $|\lambda_s|$ is the faster the transition speed is.

Decentralized version:
dynamic general equilibrium (DGE) model

# The Neoclassical DGE model
Assumption

- ▶ Representative household: has initial financial wealth $b(0)$, receives has wage income $w$ and financial income $(rb)$, and decides on consumption $(c)$ and savings $(\dot{b})$ ;

- ▶ Households own firms with physical capital $(k)$ which is only financed by bonds: thus $b = k$. Firms transform capital and labor into output $(y)$

- ▶ There are accounting restrictions.

- ▶ All markets are competitive

- ▶ Other assumptions: infinite-lived households with isoelastic utility and Cobb-Douglas production, function and no frictions.

# Household problem

▶ Household's problem: maximize discounted intertemporal utility subject to a financial constraint

$$\max_{c(.)} \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

$$\text{subject to}$$

$$\dot{b} = r(t)b(t) + w(t) - c(t), \ t \geq 0$$

$$b(0) = b_0$$

$$\lim_{t \to \infty} e^{-\int_t^\infty r(s)ds} \geq 0$$

where $b = $bonds, $w = $wage

# Household problem

- ▶ Optimality conditions

$$\dot{c} = \frac{c(t)}{\theta} \left( r(t) - \rho \right)$$

$$\lim_{t \to \infty} e^{-\rho t} c(t)^{-\theta} b(t) = 0$$

- ▶ Admissibility conditions

$$\dot{b} = r(t)b(t) + w(t) - c(t), \ t \geq 0$$

$$b(0) = b_0$$

# The firm's problem

▶ Firm's problem (price taker in all the markets): maximizes present value of profits

$$\max_i \int_0^\infty \left(Ak(t)^\alpha - w(t) - i(t)\right) e^{-R(t)} dt$$
$$\text{subject to}$$
$$\dot{k} = i - \delta k$$
$$k(0) = k_0$$

▶ observations

   ▶ the discount factor is the (endogeneous) market interest rate $R(t) = \int_t^\infty r(s)\, ds$
   ▶ the control variable: investment expenditure
   ▶ no adjustment cost: investment expendiure is equal to gross investment
   ▶ constraint: net investment = gross investment minus ddepreciation

# The firm's problem

- Hamiltonian

$$H(i, k, q) = Ak^{\alpha} - w - i + q(1 - \delta k)$$

- Optimality conditions:
$$\frac{\partial H(i, k, q)}{\partial i} = 0 \iff q(t) = 1, \text{ for all } t \geq 0$$

- Canonical equation

$$\dot{q} = q\left(r(t) + \delta - \alpha\, A\, k^{\alpha-1}\right)$$

- Then
$$r(t) = \alpha A k(t)^{\alpha-1} - \delta, \text{ for all } t \geq 0$$

# The general equilibrium determination

▶ Micro-macro constraints and equilibrium conditions:
  ▶ Accounting identity $b(t) = k(t)$,
  ▶ Then $\dot{b}(t) = \dot{k}(t)$,
  ▶ Market equilibrium condition

$$y = c + i$$

▶ From

$$\dot{b}(t) = \dot{k}(t) \iff rb + w - c = i - \delta k \iff rk + w + \delta k = y$$
$$\iff y - \delta k = Ak^\alpha - \delta k = rk + w$$

▶ Then we get

$$\dot{k} = A k^\alpha - c - \delta k$$

# The general equilibrium

▶ We obtain the same dynamic system as in the Ramsey model

$$\dot{c} = \frac{c(t)}{\theta} \left( r(t) - \rho \right)$$
$$\dot{k} = Ak(t)^{\alpha} - c(t) - \delta k(t)$$

▶ Then the allocations of $c$ and $k$ are equal: we say that the **equilibrium is Pareto efficient**)

# References

- ► Ramsey (1928), Cass (1965) Koopmans (1965)
- ► (Acemoglu, 2009, ch. 8) , (Aghion and Howitt, 2009, ch. 1), (Aghion and Howitt, 2009, ch. 1), (Barro and Sala-i-Martin, 2004, ch. 2)

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