Mathematical Economics
Continuous time: optimal control problem

Paulo Brito

[1] pbrito@iseg.ulisboa.pt
University of Lisbon

December 11, 2018

Continuous time optimal control problem

Find the **state** variable $x^* = (x^*(t))_{0 \leq t \leq T}$ and the **control** variable $u^* = (u^*(t))_{0 \leq t \leq T}$ that solve the problem:

$$\max_u \int_0^T F(t, x(t), u(t))\, dt$$

subject to

$$\dot{x} = G(t, x(t), u(t))$$

- given $x(0) = \phi_0$
- given the horizon T
- and constraints on the terminal value of $x(T)$

$(P1)$ $\quad x(T) = \phi_T$
$(P2)$ $\quad x(T)$ free
$(P3)$ $\quad h(T)x(T) \geq 0.$

Hamiltonian function

- We introduce the Hamiltonian function

$$H(t, x, u, \lambda) \equiv F(t, x, u) + \lambda g(t, x, u)$$

  where $\lambda(t)$ is the **co-state** or **adjoint** variable,

- its derivatives as regards the control variable

$$H_u(t, x, u, \lambda) \equiv \frac{\partial H(t, x, u, \lambda)}{\partial u} = F_u(t, x, u) + \lambda G_u(t, x, u)$$

- and the state variable

$$H_x(t, x, u, \lambda) \equiv \frac{\partial H(t, x, u, \lambda)}{\partial x} = F_x(t, x, u) + \lambda G_x(t, x, u)$$

Pontriyagin's maximum principle

Proposition (Necessary first order conditions)

Let (x^*, u^*) be a solution to the OC problem. Then there is a piecewise continuous function $\lambda(t)$ such that (x^*, u^*, λ) satisfy: Proof

- *the optimality condition*

$$H_u(t, x^*(t), u^*(t), \lambda(t)) = 0, \; 0 \leq t \leq T$$

- *the adjoint equation*

$$\dot{\lambda} = -H_x(t, x^*(t), u^*(t), \lambda(t)), \; 0 < t \leq T$$

- *the admissibility conditions:*

$$\begin{cases} \dot{x}^* = G(t, x^*(t), u^*(t)) & 0 < t \leq T \\ x^*(0) = \phi_0, & t = 0 \end{cases}$$

- *the terminal or transversality condition*

$$\text{(P1) } x(T) = \phi_T, \; \text{(P2) } \lambda(T) = 0, \; \text{(P3) } \lambda(T)x(T) = 0.$$

Optimal control: autonomous discounted infinite horizon problem

Find (x^*, u^*) where $x^* = (x^*(t))_{0 \le t < \infty}$ and $u^* = (u^*(t))_{0 \le t < \infty}$ that solve the OCIH problem:

$$\max_u \int_0^\infty e^{-\rho t} f(x(t), u(t))\, dt$$

subject to

$$\dot{x}(t) = g(x(t), u(t))$$

- and $x(0) = \phi_0$
- alternative terminal conditions

$(P2)$ $\lim_{t \to \infty} x(t)$ free
$(P3)$ $\lim_{t \to \infty} h(t) x(t) \ge 0$.

Current-value Hamiltonian

- We define a **time-independent** current-value Hamiltonian function:

$$\begin{aligned} h(x, u, q) &= f(x, u) + qg(x, u) = \\ &= e^{\rho t} H(t, x, u, \lambda) = \\ &= f(x, u) + qg(x, u) \end{aligned}$$

- as the capitalised value of the discounted Hamiltonian function $H(t, x(t), u(t), \lambda(t))$
- The **current-value co-state variable** is

$$q(t) = e^{\rho t} \lambda(t)$$

Pontriyagin maximum principle

Proposition (Necessary conditions for the OCIHP)

Let $(x^*, u^*) = (x^*(t), u^*(t))_{t\in[0,\infty)}$ *be the solution of the OCIH problem. Then there is a co-state variable* $q(t)$ *such that the solution* (x^*, u^*) *verifies the following conditions:*

- *the optimality condition*

$$h_u(x^*(t), u^*(t), q(t)) = 0,\ 0 \le t < \infty$$

- *the adjoint equation*

$$\dot{q} = \rho q(t) - h_x(x^*(t), u^*(t), q(t)),\ 0 < t < \infty$$

- *the admissibility conditions:*

$$\begin{cases} \dot{x}^* = g(x^*(t), u^*(t)) & 0 < t < \infty \\ x^*(0) = \phi_0, & t = 0 \end{cases}$$

- *the terminal or transversality condition*

$(P2)$ $\lim_{t\to\infty} e^{-\rho t} q(t) = 0$

$(P3)$ $\lim_{t\to\infty} e^{-\rho t} q(t) x(t) = 0.$

Example: Resource depletion problem

- The problem (N is non-renewable resource endowment)

$$\max_C \int^{\infty} e^{-\rho t} \ln\left(C(t)\right) dt, \ \rho > 0$$

subject to

$$\begin{cases} \dot{N}(t) = -C(t), \ t \in [0, \infty) \\ N(0) = N_0, \ \text{given} \\ \lim_{t\to\infty} N(t) \geq 0 \end{cases}$$

- Current-value Hamiltonian

$$h = \ln\left(C\right) + q(-C)$$

- First order conditions:

$$\begin{aligned} C(t) &= 1/q(t) \\ \dot{q} &= \rho q(t) \\ \lim_{t\to\infty} e^{-\rho t} q(t) N(t) &= 0 \\ \dot{N} &= -C(t) \\ N(0) &= N_0 \end{aligned}$$

Example: Resource depletion problem

- The maximized Hamiltonian dynamic system (MHDS)

$$\begin{aligned} \dot{C} &= -\rho C(t) \\ \dot{N} &= -C(t) \\ N(0) &= N_0 \\ \lim_{t\to\infty} e^{-\rho t}\frac{N(t)}{C(t)} &= 0 \end{aligned}$$

- To solve the MHDS:
  - 1st step: let $z(t) = N(t)/C(t)$ and solve

$$\begin{cases} \dot{z} = -1 + \rho z \\ \lim_{t\to\infty} e^{-\rho t} z(t) = 0 \end{cases} \Rightarrow z(t) = \frac{1}{\rho},\ t \in [0,\infty)$$

  - 2nd step: substitute in the dynamic constraint

$$\begin{cases} \dot{N} = -C(t) = -N(t)/z(t) = -\rho N(t) \\ N(0) = N_0 \end{cases} \Rightarrow N(t) = N_0 e^{-\rho t},\ t \in [0,\infty)$$

Example: Resource depletion problem

- Characterization of the solution

$$N^*(t) = N_0 e^{-\rho t} \text{ for } t \in [0, \infty)$$

- there is asymptotic extinction

$$\lim_{t \to \infty} N^*(t) = 0$$

- speed of the adjustment: half-life of the process

$$\tau \equiv \left\{ t : \; N^*(t) = \frac{N(0) - N^*(\infty)}{2} \right\} = -\frac{\ln(1/2)}{\rho}$$

if $\rho = 0.02$ then $\tau \approx 34.6574$ years

The MHDS

- In sufficiently smooth cases we have qualitative results on the optimal path for (x, q) (or for (u, x))
- If $\partial^2 h/\partial u^2 \neq 0$, we can solve the optimality condition for u, $u^* = u(x^*, q)$
- then we get the modified Hamiltonian dynamic system:

$$\begin{cases} \dot{q} = \dot{q}(q, x) = \rho q - h_x(x, u(x, q)) \\ \dot{x} = \dot{k}(q, x) = g(x, u(x, q)) \end{cases}$$

- Assume the MHDS has a fixed point $(\bar{q}, \bar{x})$.
- In the neighbourhood of $(\bar{x}, \bar{q})$ we can approximate the MHDS by the linear system

$$\begin{pmatrix} \dot{q}(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial x} \\ \frac{\partial \dot{x}}{\partial q} & \frac{\partial \dot{x}}{\partial x} \end{pmatrix} \begin{pmatrix} q(t) - \bar{q} \\ x(t) - \bar{x} \end{pmatrix} = J \begin{pmatrix} q(t) - \bar{q} \\ x(t) - \bar{x} \end{pmatrix}$$

The MHDS

- If we write $h^*(x, q) = h(x, u(x, q), q)$ then The Jacobian has the structure

$$J = \begin{pmatrix} \rho - a & b \\ c & a \end{pmatrix} = \begin{pmatrix} \rho - h^*_{xq}(\bar{x}, \bar{q}) & -h^*_{xx}(\bar{x}, \bar{q}) \\ h^*_{qq}(\bar{x}, \bar{q}) & h^*_{qx}(\bar{x}, \bar{q}) \end{pmatrix}$$

- has trace and determinant (for most cases):

$$\mathrm{tr}(J) = \rho > 0, \; \det(J) = a(\rho - a) - bc < 0$$

- Interpretation:
  - the equilibrium point $(\bar{x}, \bar{q})$ is a saddle point. The stable manifold associated with $(\bar{x}, \bar{q})$ is the solution set of the OC problem.
  - this means that the solution to the OC problem is unique.

The Ramsey model

- **The problem**: find the optimal allocation of savings through time in order to maximize the time aggregate of the discounted **value** of consumption (in utility terms), when there is a technology of production displaying decreasing marginal returns:

$$\max_{C} \int_0^{\infty} e^{-\rho t} U(C(t))\, dt, \ \rho > 0,$$

subject to

$$\dot{K}(t) = F(K(t)) - C(t), \ t \in [0, \infty)$$

- $K(0) = K_0$ given and $\lim_{t \to \infty} e^{-\rho t} K(t) \geq 0$
- Utility and production functions, $u(C)$ and $F(K)$; are increasing, concave and Inada :

$$U^{'}(.) > 0, \ U^{''}(.) \leq 0, F^{'}(.) > 0, \ F^{''}(.) \leq 0$$

$$U^{'}(0) = \infty, \ U^{'}(\infty) = 0, \ F^{'}(0) = \infty, \ F^{'}(\infty) = 0$$

The Ramsey model: optimality conditions

- The current-value Hamiltonian

$$h(C, K, Q) = U(C) + Q(F(K) - C)$$

- The Pontriyagin's f.o.c

$$\begin{aligned}
U^{'}(C(t)) &= Q(t) \\
\dot{Q} &= Q(t)\left(\rho - F^{'}(K(t))\right) \\
\lim_{t \to \infty} e^{-\rho t} Q(t) K(t) &= 0 \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0
\end{aligned}$$

The Ramsey model: the non-linear MHDS

- The MHDS

$$
\begin{aligned}
\dot{C} &= \frac{C(t)}{\sigma(C(t))}\left(r(K(t)) - \rho\right) \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0 > 0 \\
0 &= \lim_{t\to\infty} e^{-\rho t} U^{'}(C(t))K(t)
\end{aligned}
$$

  where $r(K) \equiv F^{'}(K)$ and $\sigma(C) \equiv -\frac{U^{''}(C)C}{U^{'}(C)}$

- The MHDS has no explicit solution: we can only use qualitative methods:
  - determine the steady state(s)
  - linearize the system around the candidate steady states
  - solve the linearized MHDS

The Ramsey model: the linearized MHDS

- The steady state (if $K > 0$ and $C > 0$)

$$\begin{aligned} F^{'}(\bar{K}) &= \rho \Rightarrow \bar{K} = (F^{'})^{-1}(\rho) \\ \bar{C} &= F(\bar{K}) \end{aligned}$$

- Linearized system

$$\begin{pmatrix} \dot{C} \\ \dot{K} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\bar{C}}{\sigma(\bar{C})} F^{''}(\bar{K}) \\ -1 & \rho \end{pmatrix} \begin{pmatrix} C(t) - \bar{C} \\ K(t) - \bar{K} \end{pmatrix}$$

- The jacobian J has trace and determinant:

$$\mathrm{tr}(J) = \rho, \ \det(J) = \frac{\bar{C}}{\sigma(\bar{C})} F^{''}(\bar{K}) < 0$$

then $(\bar{C}, \bar{K})$ is a saddle point

The Ramsey model: the linearized MHDS

- There is one unique interior steady state (satisfying $K > 0$ and $C > 0$)

$$
\begin{aligned}
F^{'}(\bar{K}) &= \rho \Rightarrow \bar{K} = (F^{'})^{-1}(\rho) \\
\bar{C} &= F(\bar{K})
\end{aligned}
$$

  It exists and is unique because of the Inada properties of the production function

- Define the variations as regards the steadys state: $c(t) = C(t) - \bar{C}$, $k(t) = K(t) - \bar{K}$
- The linearized or variational MHDS is

$$
\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\bar{C}}{\sigma(\bar{C})} F^{''}(\bar{K}) \\ -1 & \rho \end{pmatrix} \begin{pmatrix} c(t) \\ k(t) \end{pmatrix}
$$

The Ramsey model: solving the linearized MHDS

The linearized MHDS and the solution to the Ramsey problem

- The linearized MHDS and the solution to the Ramsey problem is

$$\dot{\mathbf{x}}(t) = \mathbf{J}\,\mathbf{x}(t)$$

  where $\mathbf{J}$ is the Jacobian of the MHDS evaluated at the steady state

- To find **approximate solutions to Ramsey model** that **satisfy the transversality condition** in the neighborhood of the steady state,
  - we solve the linearized system (general solution)
  - find the particular solution such that: (1) there is convergence to the steady state; (2) it starts from $K(0) = k_0$ implying $k(0) = k_0 - \bar{K}$.
- it is useful to draw the phase diagram

The Ramsey model: solving the linearized MHDS

The general solution of the linearized MHDS

- The general solution is

$$\begin{pmatrix} c(t) \\ k(t) \end{pmatrix} = \mathbf{x}(t) = h_- \mathbf{P}_- e^{\lambda_- t} + h_+ \mathbf{P}_+ e^{\lambda_+ t}$$

  where $\lambda_\pm$ are the eigenvalues and $\mathbf{P}_\mp$ are the associated eigenvectors of matrix $\mathbf{J}$

- **Finding the eigenvalues**
  - The jacobian $\mathbf{J}$ has trace and determinant:

$$\mathrm{tr}(\mathbf{J}) = \rho, \ \det(\mathbf{J}) = \frac{\bar{C}}{\sigma(\bar{C})} F''(\bar{K}) < 0$$

  - then the eigenvalues of J (solutions of $\det(J - \lambda I_2) = 0$,

$$\lambda_- = \frac{\rho}{2} - \sqrt{\Delta} < 0, \ \lambda_+ = \frac{\rho}{2} + \sqrt{\Delta} > \rho$$

    where the discriminant of J is

$$\Delta = \left(\frac{\rho}{2}\right)^2 - \frac{\bar{C}}{\sigma(\bar{C})} F''(\bar{K}) > \left(\frac{\rho}{2}\right)^2$$

- Then $(\bar{C}, \bar{K})$ is a saddle point

The Ramsey model: solving the linearized MHDS

The general solution of the linearized MHDS

- Eigenvector matrix of $\mathbf{J}$

$$\mathbf{P} = (\mathbf{P}_-|P_+) = \begin{pmatrix} \lambda_+ & \lambda_- \\ 1 & 1 \end{pmatrix}$$

where $\mathbf{P}_j$, $i = -, +$, is calculated the solution to

$$(\mathbf{J} - \lambda^i \mathbf{I}_2)\mathbf{P}_i = \mathbf{0}, \ i = -, +$$

- Then

$$\begin{pmatrix} c(t) \\ k(t) \end{pmatrix} = h_- \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix} e^{\lambda_- t} + h_+ \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix} e^{\lambda_+ t}$$

The Ramsey model: solving the linearized MHDS

The particular solution of the linearized MHDS

- We have to determine the constants: h_- and h_+ such that the the solution converges to the steady state and the initial value for $K(0) = K_0$ is satisfied:
  - convergence to the steady state

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} k(t) = 0 \Leftrightarrow h_+ = 0$$

  - starting from $k(0) = K_0 - \bar{K}$

$$k(t)|_{t=0} = K_0 - \bar{K} \Leftrightarrow h_- = K_0 - \bar{K}$$

- Therefore, the linearized solution is

$$\begin{pmatrix} c^*(t) \\ k^*(t) \end{pmatrix} = \left(K_0 - \bar{K}\right) \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix} e^{\lambda_- t}$$

The Ramsey model: the approximate solution

- The **approximate** solution to the Ramsey model is, therefore,

$$\begin{pmatrix} C^*(t) \\ K^*(t) \end{pmatrix} = \begin{pmatrix} \bar{C} \\ \bar{K} \end{pmatrix} + (K_0 - \bar{K}) \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix} e^{\lambda_- t}, \ t \in [0, \infty)$$

- Solution at $t = 0$

$$\begin{pmatrix} C^*(0) \\ K^*(0) \end{pmatrix} = \begin{pmatrix} \bar{C} + \lambda_+ (K_0 - \bar{K}) \\ K_0 \end{pmatrix}$$

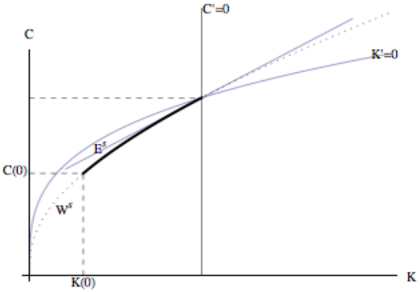
  the initial value of consumption is determined endogenously

- Asymptotic solution

$$\lim_{t \to \infty} \begin{pmatrix} C^*(t) \\ K^*(t) \end{pmatrix} = \begin{pmatrix} \bar{C} \\ \bar{K} \end{pmatrix}$$

  the solution tends to the fixed point of the MHDS.

Ramsey model: phase diagram



Exact solution (stable manifold - W_-), linearized solution (stable eigenspace - E_-).
Close to the steady state W_- has slope equal to to the slope of E_-, and they are higher than the slope of the isocline $\dot{K}(C, K) = 0$

$$\left.\frac{dC}{dK}\right|_{W_-} = \left.\frac{dC}{dK}\right|_{E_-} = \lambda_+ > \left.\frac{dC}{dK}\right|_{\dot{K}} = F^{'}(\bar{K}) = \rho$$

Principle of the Dynamic Programming

- Consider again the infinite horizon discounted optimal control problem

$$\max_u \int_0^\infty e^{-\rho t} f(x(t), u(t))\, dt$$

subject to

$$\begin{cases} \dot{x} = g(x, u), & 0 \leq t < \infty \\ x(0) = \phi, & t = 0 \\ \lim_{t \to \infty} h(t)x(t) \geq 0 & \end{cases}$$

Principle of the Dynamic Programming

Proposition

Let (x^*, u^*) be a solution of the discounted infinite horizon optimal control problem. Then for every $t \in [0, \infty)$, $x^*(t) = x$, and $u^*(t) = u$ satisfy the Hamilton-Jacobi-Bellman equation

$$\rho V(x) = \max_u \left[f(x, u) + V^{'}(x) g(x, u) \right]$$

We call $u^* = h(x) = \arg\max_u \{f(x, u) + V^{'}(x) g(x, u)\}$ the policy function.

Application: the consumption-investment problem

A representative household (or economy) wants to find the trajectories for consumption and asset holdings, $(C(t))_{t=0}^{\infty}$, and $(W(t))_{t=0}^{\infty}$ that solve the problem:

$$\max_{C} \int_{0}^{\infty} e^{-\rho t} \ln\left(C(t)\right) dt,$$

subject to

$$\begin{cases} \dot{W} = rW - C & \text{(budget constraint)} \\ W(0) = W_0 & \text{(initial wealth} \\ \lim_{t\to\infty} e^{-r\,t} W(t) \geq 0 & \text{(NPG constraint)} \end{cases}$$

Application: (cont.)

- **First step:** Write the HJB equation

$$\rho V(W) = \max_{C} \left[\ln(C) + V'(W)(rW - C) \right]$$

- **Second step:** find the policy function for consumption

$$\frac{1}{C^*} - V'(W) = 0 \Leftrightarrow C^* = \left(V'(W) \right)^{-1}$$

- **Third step:** substitute it into the HJB equation

$$\rho V(W) = -\ln(V'(W)) + V'(W) rW - 1$$

Application: (cont.)

- **Fourth step:** solving the HJB equation by using the **method of undermined coefficients** (i.e., a and b)
  - put forward a **trial function** v.g.

$$V(W) = a + b\ln(W) \Rightarrow V^{'}(W) = \frac{b}{W}$$

  - substitute into the HJB equation

$$\rho\left(a + b\ln(W)\right) = -\ln(b) + \ln(W) + rb - 1$$

  - separate terms with $\ln(W)$

$$(\rho b - 1)\ln(W) = -\ln(b) + rb - 1 - \rho a$$

  - determine the coefficients such that both sides are equal to zero: then

$$b = \frac{1}{\rho} \text{ and } a = \frac{1}{\rho}\left(\ln\rho + \frac{r}{\rho} - 1\right)$$

  - Then our conjecture was correct and

$$V(W) = \frac{1}{\rho}\left(\ln\rho + \frac{r-\rho}{\rho} + \ln(W)\right)$$

Application: (cont.)

- **Fifth step:** obtain the optimal consumption rule

$$C^* = (V^{'}(W))^{-1} = \rho W$$

- **Sixth step:** substitute in the budget constraint and define the optimal forward equation (back to the time domain)

$$\begin{cases} \dot{W} = (r-\rho)W \quad W(0) = W_0 \end{cases}$$

- **Seventh step:** solve the initial value problem to obtain the solution

$$W^*(t) = W_0\, e^{(r-\rho)t},\ t \in [0, \infty)$$
$$C^*(t) = \rho\, W_0\, e^{(r-\rho)t},\ t \in [0, \infty)$$

Application: characterizing the solution

- **Eigth step:** (optional) make sure the NPG is satisfied

$$\lim_{t\to\infty} e^{-r\,t} W^*(t) = \lim_{t\to\infty} e^{-\rho\,t} W_0 = 0$$

- **This holds for any values of r and ρ** , although we may have the following solutions
  - if $r < \rho$ then the financial wealth and consumption converge to zero

    $$\lim_{t\to\infty} W^*(t) = \lim_{t\to\infty} C^*(t) = 0$$

  - if $r > \rho$ then the financial wealth and consumption will grow forever

    $$\lim_{t\to\infty} W^*(t) = \lim_{t\to\infty} C^*(t) = \infty$$

  - if $r = \rho$ then the solution is stationary

    $$W^*(t) = W_0,\ C^*(t) = \rho W_0, \text{ for every } t \in [0, \infty)$$

Proofs

Proof of proposition 1

- The value functional is for any paths (x, u)

$$V(x) = \int_0^T f(u(t), x(t), t)\, dt = \text{(definition of } H \text{ function)}$$
$$= \int_0^T H(u(t), x(t), t) - \lambda(t)\dot{x}(t)\, dt = \text{(integration by parts)}$$
$$= \int_0^T \Big(H(u(t), x(t), \lambda(t), t) + \dot{\lambda}(t)x(t) \Big)\, dt + \lambda(0)x(0) - \lambda(T)x(T)$$

- The value at the optimum is

$$V(x^*) = \int_0^T f(u^*(t), x^*(t), t)\, dt =$$
$$= \int_0^T \Big(H(u^*(t), x^*(t), \lambda(t), t) + \dot{\lambda}(t)x^*(t) \Big)\, dt + \lambda(0)x^*(0) - \lambda(T)x^*(T)$$
$$(+\mu\ h(T)x^*(T) \text{ (for case P3))}$$

Proof of proposition 1 (cont.)

- Now we introduce perturbations in the state and co-state variables $x(t) = x^* + \epsilon\, d_x(t)$ and $u(t) = u^* + \epsilon\, d_u(t)$
- The perturbations are admissible if $d_x(0) = 0$ and, for (P1) $d_x(T) = 0$, and $d_x(T)$ is free for (P2) and (P3).
- The optimal should satisfy

$$\delta V(x^*) = \lim_{\epsilon \to 0} \frac{V(x^* + \epsilon h_x) - V(x^*)}{\epsilon} = \frac{dV(x^*)}{d\epsilon} = 0.$$

Proof of proposition 1 (cont.)

- But, writing $H^*(t) = H(u^*(t), x^*(t), \lambda(t), t)$

$$\frac{dV(x^*)}{d\epsilon} = \int_0^T \left[\frac{\partial H^*(t)}{\partial u} h_u(t) + \left(\frac{\partial H^*(t)}{\partial x} + \dot{\lambda}(t) \right) d_x(t) \right] dt + \lambda(0) d_x(0)$$
$$- \lambda(T) d_x(T) (\text{for cases P1 and P2 })$$
$$(+ ((\mu \; h(T) - \lambda(T)) \, h_x(T) \; (\text{for case P3}))$$

  for case P3 the Kuhn-Tucker condition $\mu h(T) x^*(T) = 0$ for $\mu \geq 0$ should also hold.

- Then

$$\delta V(x^*) = 0 \Leftrightarrow \frac{\partial H_t^*}{\partial u} = \frac{\partial H_t^*}{\partial x} + \dot{\lambda} = 0$$

  and $\lambda(T) = 0$ for case P2, and $\lambda(T) x^*(T) = 0$ for case P3.

Return.