Foundations of Financial Economics *2017/18*
Problem sets 2-5 : Solutions

Paulo Brito

20.4.2018
revised 20.5.2018

Problem set 2

- **Problem**
  Assume the information set has three equiprobable states of nature. A consumer receives the endowment $Y = (y(1+\epsilon), y, y(1-\epsilon))^\top$, where $y > 0$ and $0 < |\epsilon| < 1$. The consumer has the utility functional $\mathbb{E}[\ln(Y)]$.

  1. Find the certainty equivalent for Y.
  2. What would be better: to get Y or a certain amount which would be equal to $\mathbb{E}[Y]$? Justify.
  3. Assume the agent can be in one of two alternative situations: autarky, or in an exchange economy in which the equilibrium price is state-independent $Q = (\bar{q}, \bar{q}, \bar{q})$. Under which situation would the agent be better off ? Justify.

- **Solution**

  1. Let y_c be the certainty equivalent of the endowment Y. Then from $u(y_c) = \mathbb{E}[u(Y)] = \ln(\alpha y)$, where $\alpha = (1-\epsilon^2)^{1/3} \in (0,1)$, we obtain $y_c = \alpha y < y$.
  2. As $\mathbb{E}[Y] = y$ then $u(\mathbb{E}[Y]) = \ln(y) > \ln(\alpha y) = \mathbb{E}[u(Y)]$ then receiving y is better than receiving Y.
  3. Utility of the consumer in autarky $U^A(C) = U^A(Y) = \mathbb{E}[u(Y)] = \ln(\alpha y)$. Utility of the consumer when there is trade $U^T(C) = \ln(y)$. To prove this we solve the consumer problem
  $$\max_C \sum_{s=1}^{3} \frac{1}{3} \ln(c_s) \text{ s.t } \sum_{s=1}^{3} \bar{q} c_s = \sum_{s=1}^{3} \bar{q} y_s$$
  There is only a solution if $\bar{q} = 1/3$. With this assumption we get $C = (c_1, c_2, c_3) = (y, y, y)$. Then $U^T(C) = \mathbb{E}[\ln(y)] = \ln(y)$. Then trade is better.

- **Problem** There are two states of nature with equal probabilities and a lottery with payoffs $Y = \left(\frac{1}{\epsilon}, \frac{1}{1-\epsilon}\right)$, where $0 < \epsilon < 1$ and $\epsilon \neq \frac{1}{2}$. Assume that the utility function is $u(y) = 1 - \frac{1}{y}$.

  (a) Compute the certainty equivalent of the lottery.

  (b) What is better, the lottery or a certain outcome equal to the expected value of the lottery ? Provide an intuition for your result.

  (c) Introduce a proportional transfer (a tax or a subsidy) over the certain outcome with the objective of making the agent indifferent between the two choices in (b). Which value should that transfer take ? Justify.

  (d) Assume that an agent can be in one of the following two environments: (1) autarky, in which case he/she would get the lottery; or (2) in an exchange economy, in which he/she could trade the lottery for a price $Q(\omega_s) = P(\omega_s)$, for $s = 1, 2$. in which environment would he/she be better ? Supply an intuition for your results.

- **Solutions**

  a) Let $y_c = CE[Y]$ be the certainty equivalent. Then we find that $y_c = 2$

  b) Certain outcome $X = \mathbb{E}[Y] = (2\epsilon(1-\epsilon))^{-1} \geq 2$. Three different alternative ways of proving: (1) $X > y_c$; (2) $u(X) - \mathbb{E}[u(Y)] = (1 - 4\epsilon(1-\epsilon))/2 > 0$; (3) by the Jensen inequality $u(X) > \mathbb{E}[u(Y)]$ because $u(y) = 1 - \dfrac{1}{y}$ is concave.

  c) We want to find τ such that $u((1-\tau)\mathbb{E}[Y]) = \mathbb{E}[u(Y)]$. We find $\tau = 1 - 4\epsilon(1-\epsilon) > 0$. It is a tax not a subsidy.

- **Problem** Let there be uncertainty characterized by two states of nature with equal probabilities. A lottery has payoffs $Y = (y_1, y_2) = (e^{\epsilon}, e^{-\epsilon})$, where $\epsilon > 0$, and the behavior of an agent is charaterized by a von-Neumann Morgenstern utility functional with a logarithmic Bernoulli utility function.

  (a) Find the certainty equivalent of lottery Y.

  (b) Which is better, the lottery or a certain payoff equal to $\mathbb{E}[Y]$? Describe and give an intuition on the possible approaches to come up with an answer.

  (c) Assume you introduce an flat tax over the certain payoff $\mathbb{E}[Y]$. What would be the level of the tax such that the agent would be indifferent between the penalized certain outcome or the lottery. Provide an intuition.

- **Solution**

  (a) Let $y_c = CE(Y)$ be the certainty equivalent. Then we find $y_c = 1$

  (b) The expected value of lottery Y is $\mathbb{E}(Y) = \dfrac{e^{\epsilon} + e^{-\epsilon}}{2}$. Observe that

  $$\frac{\partial \mathbb{E}(Y)}{\partial \epsilon} = \frac{e^{\epsilon} - e^{-\epsilon}}{2}, \ \frac{\partial^2 \mathbb{E}(Y)}{\partial \epsilon^2} = \frac{e^{\epsilon} + e^{-\epsilon}}{2} > 0$$

  this means that $\mathbb{E}(Y)$ is a convex function of ϵ and reaches a minimum at $\epsilon = 0$. And as $\mathbb{E}(Y|\epsilon = 0) = 2$ then $\mathbb{E}(Y|\epsilon > 0) > 1$. Three ways to compare: (1) $y_c = 1 < \mathbb{E}(Y)$; (2) $\ln(y_c) = 0 < \ln\left((e^{\epsilon} + e^{-\epsilon})/2\right)$; (3) as $u(y_s) = \ln(y_s)$ is concave the Jensen inequality implies $u(\mathbb{E}[Y]) > \mathbb{E}[u(Y)]$

  (c) Penalized certain outcome $\mathbb{E}[Y] - T$. Then $T = \frac{e^{\epsilon}+e^{-\epsilon}}{2} - 1 > 0$ for $\epsilon > 0$.

Problem set 3

- **Problem** Consider Arrow-Debreu economies with the data that follows: (1) the information is given by a binomial tree with two periods, $\mathbb{T} = \{0, 1\}$ and N states of nature for period 1; (2) the endowment distribution for the period $t = 1$ is $y_{1,s} = y_0 \cdot (1+\gamma_s)$, for state $s = 1, \ldots, N$; (3) agents are homogenous; (4) the representative agent has a discounted time-additive, von-Neumann-Morgenstern utility functional with a CARA Bernoulli utility function,

$$u(C) = -\frac{e^{-\lambda C}}{\lambda},\ \lambda > 0$$

  (a) Define the equilibrium, and provide an intuition.

  (b) Determine the solution for the consumer problem, and provide an intuition.

  (c) Determine the equilibrium stochastic discount factor. Assuming that $\mathbb{E}[\Gamma] = \gamma > 0$ find a bound to the expected value of the stochastic discount factor by using Jensen's inequality. Provide an intuition for your results.

- **Solution**

  (b)

$$\begin{aligned} c_0^* &= \frac{1}{1+\mathbb{E}[m]}\left(h_0 + \frac{1}{\lambda}\mathbb{E}[m \ln{(m/\beta)}]\right) \\ c_{1,s}^* &= c_0^* - \frac{1}{\lambda}\ln\left(\frac{m_s}{\beta}\right) \end{aligned}$$

  (c) We find that the SDF is $m_s = \beta e^{-\lambda y_0 \gamma_s}$. This is a convex function of γ_s. Therefore using the Jensen's inequality for convex functions we have

$$\mathbb{E}[M] = \beta\mathbb{E}\left[e^{-\lambda y_0 \gamma}\right] > \beta e^{-\lambda y_0 \mathbb{E}[\Gamma]} = \beta e^{-\lambda y_0 \gamma}$$

- **Problem** Consider endowment economy in which the information be given by a two-period binomial tree, the endowment process, $\{y_0, Y_1\}$, verifies $y_0 = 1$ and $Y_1 = (1-\gamma, 1+\gamma)$ for $0 < \gamma < 1$, the intertemporal utility functional is time additive, discounted and von-Neumann-Morgenstern, with a linear Bernoulli utility function $u(c) = a\,c$, for $a > 0$ constant.

  1. Define, explicitly, the Arrow-Debreu equilibrium for this economy.
  2. Write the equilibrium conditions. Under which conditions an equilibrium exists ? Is it unique ? Justify.
  3. Find the stochastic discount factor and provide an economic intuition for its value.

- **Solution**

  b) Equilibrium conditions

$$\begin{aligned} a &= \lambda \\ \beta a &= m_s \lambda, \ s = 1, 2 \\ c_0 + \mathbb{E}[MC_1] &= y_0 + \mathbb{E}[MY_1] \\ c_0 &= y_0 \\ C_1 &= Y_1 \end{aligned}$$

  existence conditions $m_1 = m_2 = \beta$; the equilibrium is unique.

  c) $m_1 = m_2 = \beta$: with neutral preferences and homogeneous agents the stochastic discount factor is state-independent (even though there is aggregate uncertainty)

Problem set 4

Arbitrage pricing theory

- **Problem**
Consider a finance economy in which there are three assets with the vector of prices and payoff matrix given by

$$S = \left(p, 1, \frac{1}{1+r}\right), \quad V = \begin{pmatrix} r-\epsilon & r+\epsilon & 1 \\ r+\epsilon & r-\epsilon & 1 \end{pmatrix}$$

where we assume that $r > 0$, $p > 0$ and $\epsilon \geq 0$.

  1. Under which conditions there are no arbitrage opportunities ?
  2. Under the conditions that you imposed in the answer to the previous point, what would be the meaning of market completeness in this economy ? Determine the conditions for the existence of market completeness.
  3. Compute the price for the first asset, p, by building a replicating transactions' strategy. Explain.

- **Solution**

  1. There are no arbitrage opportunities if the following conditions hold: $\epsilon > 1$ and $\max\{0, (r-\epsilon)/(1+r)\} < p < (r+\epsilon)/(1+r)$. This is the condition that guarantees $q_s^{1,2} > 0$, $q_s^{1,3} > 0$ and $q_s^{2,3} > 0$, for $s = 1, 2$ where $q_s^{i,j}$ is the price of state of nature s for the sub-market composed by assets i and j.
  2. Markets are complete if $r > 1$ and $p = (r-1)/(1+r)$. Markets are complete if and only if the prices of the states of nature are unique: $q_1 = q_1^{1,2} = q_1^{1,3} = q_1^{2,3}$ and $q_2 = q_2^{1,2} = q_2^{1,3} = q_2^{2,3}$.
  3. Let θ_k be the number of asset k in a portfolio. The replicating portfolio is $\theta_2 = -1$ and $\theta_3 = 2r$ and the price of asset 1 is, again, $p = (r-1)/(1+r)$.

- **Problem** Consider two (alternative) financial markets, A and B, characterized by the return matrices $(N \times K)$

$$R = \begin{pmatrix} 1 & 1+a & b \\ 1 & 1-a & b \end{pmatrix}, \text{ for financial market } A, \text{ and } R = \begin{pmatrix} 1 & 1 \\ 1+a & 1-a \\ 1-b & 1+b \end{pmatrix}, \text{ for financial market } B$$

where a and b can take any real value.

1. Under which conditions there are no arbitrage opportunities and there is market completeness in the financial market A
2. Under which conditions there are no arbitrage opportunities and there is completeness in the financial market B

- **Solution**

a) There is only a market (in the sense that there is a vector of prices of the state of nature if $b = 1$. In this case there are no arbitrage opportunities and there is market completeness and $q = (1/2, 1/2)$

b) The vector of prices of the states of nature is $q = (q_1, q_2, q_3) = (x, b(1-x)/(a+b), a(1-x)/(a+b)$ for an arbitrary x. There is always market incompleteness. There are no arbitrage opportunities if $x \in (0, 1)$ and $\text{sign}(a) = \text{sign}(b)$

Equilibrium asset prices

- **Problem** Consider a finance economy in which there are two assets with the vector of prices $S = (1, 1)$ and the payoff matrix

$$V = \begin{pmatrix} 1+r-\epsilon & 1+r+\epsilon \\ 1+r+\epsilon & 1+r-\epsilon \end{pmatrix}$$

where we assume that $r > 0$ and ϵ can take any sign. The two states of nature have equal probabilities.

(a) Determine the state prices and characterize the finance market regarding the existence of arbitrage opportunities and completeness.

(b) Let a risk-free bond be introduced with face value equal to one. Use arbitrage pricing theory to determine the risk free interest rate (hint: compute the replicating portfolio and use the cost of that replicating portfolio to determine the risk free interest rate). Provide an intuition to your results.

(c) Compute the Sharpe indices for the two risky assets. Explain

(d) Assume a general equilibrium finance economy in which the representantive consumer has a von-Neumann-Morgenstern utility functional, in which the states of nature have equal probabilities, and a logarithmic Bernoulli utility function. Write the equilibrium conditions for the risk-premia of both risky assets. Under which conditions can an equilibrium stochastic discount factor exist in this economy ? Explain.

- **Solution**

(a) State prices are $q_1 = q_2 = \frac{1}{2(1+r)}$. There are no arbitrage opportunities and markets are complete (if $\epsilon \neq 0$)

(b) Replicating portfolio $\theta_1 = \theta_2 = \frac{1}{2(1+r)}$. Bond price $\frac{1}{1+r}$

(c) Sharpe indices for assets 1 and 2 are equal to zero

(d) The equilibrium equity premia for assets 1 and 2 satisfy, denoting by $M = (m_1, m_2)$ the stocastic discount factor,

$$\begin{aligned} -\epsilon(m_1 - m_2) &= 0 \\ \epsilon(m_1 - m_2) &= 0. \end{aligned}$$

As $m_s = \frac{\beta}{1+\gamma_s}$, where $1 + \gamma_s = \frac{y_{1,s}}{y_0}$, then the equilibrium only exists if there is no aggregate uncertainty, i.e, $y_{1,1} = y_{1,2}$.

- **Problem** Let information be given by a two-period binomial tree, with probabilities $(\pi_1, \pi_2) = (\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon)$, for $\varepsilon \in (0, \frac{1}{2})$, and assume an endowment finance economy in which there are no arbitrage opportunities and agents are homogeneous. Further, assume that agents have the intertemporal utility function $\mathbb{E}_0\left[\sum_{t=0}^{1} \beta^t \ln(C_t)\right]$, for $0 < \beta < 1$, and assume that endowments are $Y_0 = 1$ and $Y_1 = (1-\gamma, 1+\gamma)$, for $0 < \gamma < 1$. The financial market is characterized by the existence two assets with the price vector and the payoff matrix

$$S = \left(\tfrac{1}{1+r}, 1\right), \text{ and } V = \begin{pmatrix} 1, 1+r+\nu \\ 1, 1+r-\nu \end{pmatrix}$$

for $r > 0$.

1. Find the stochastic discount factor. Justify your answer.
2. Find the Sharpe index. Justify your answer.
3. For which values of γ the Hansen-Jagannathan bound is verified ? Provide an intuition for your result.

- **Solution**

(a) Because the asset markets are complete the equilibrium stochastic discount factor is $M = \left(\frac{\beta}{1+\gamma}, \frac{\beta}{1-\gamma}\right)$.

(b) The equity premium is $R - R^f = (\nu, -\nu)$ then it has expected value $\mathbb{E}[R - R^f] = \pi\nu - (1-\pi)\nu = 2\varepsilon\nu$ and standard deviation $\sigma[R - R^f] = 2\nu\sqrt{\frac{1}{4} - \varepsilon^2}$. Then the Sharpe index is

$$\frac{\mathbb{E}[R - R^f]}{\sigma[R - R^f]} = \frac{\varepsilon}{\sqrt{\frac{1}{4} - \varepsilon^2}}$$

(c) Then Hansen-Jagannathan bound for the Sharpe index is

$$\frac{\mathbb{E}[R - R^f]}{\sigma[R - R^f]} \le \frac{\sigma(M)}{\mathbb{E}(M)}.$$

From (a) we have $\mathbb{E}[M] = \beta\left(\frac{1 - 2\varepsilon\gamma}{(1+\gamma)(1-\gamma)}\right)$ and $\sigma[M] = \frac{2\beta\gamma}{(1+\gamma)(1-\gamma)}\sqrt{\frac{1}{4} - \varepsilon^2}$. Then the HJ bound holds if and only if

$$\frac{\varepsilon}{\sqrt{\frac{1}{4} - \varepsilon^2}} < \frac{2\gamma\sqrt{\frac{1}{4} - \varepsilon^2}}{1 - 2\gamma\varepsilon},$$

that is if $\gamma \ge 2\varepsilon$.

- **Problem** Let information be given by a two-period binomial tree, with equal probabilities for the two states of nature, and assume an endowment finance economy in which there are no arbitrage opportunities and agents are homogeneous. Further, assume that agents have a standard intertemporal utility functional (i.e, time additive and von-Neumann-Morgenstern) with logarithmic Bernoulli utility function and assume that the endowment at time 1 is $Y_1 = ((1-\gamma)\, y_0, (1+\gamma)\, y_0)$ where $0 < \gamma < 1$. The financial market is characterized by the existence of a bond with unit face value (to be paid at time $t = 1$) and a risky asset with price S and payoff $(2\, v\, S, v\, S)$.

  (a) Find the stochastic discount factor.

  (b) Find the Sharpe index. Justify your reasoning.

  (c) Find the Hansen-Jagannathan bound. Explain its meaning.

  (d) For which values of γ the equity premium puzzle holds ? Provide an intuition for your result.

- **Solution**

  (a) $M = (\beta(1-\gamma)^{-1}, \beta(1+\gamma)^{-1})$

  (b) Returns: risk-free asset $R^f = (1+i, 1+i)$ risky asset $R = (1+r_1, 1+r_2) = (2v, v)$ risk premium $= R - R^f = (r_1 - i, r_2 - i)$. Expected risk premium $= \mathbb{E}[R - R^f] = (3/2)v - 1 - i$ Standard deviation: $\sigma[R - R^f] = v/2$. Sharpe index $(3v - 2(1+i))/v$ where i is the rate of return for the riskless asset.

  (c) $\mathbb{E}[M] = \beta \left(\frac{(1-\gamma}{2} + \frac{(1+\gamma}{2} \right) = \frac{\beta}{(1-\gamma^2)}$, $\sigma(M) = \frac{\beta\gamma}{(1+\gamma)(1-\gamma)}$ then the HJ bound $\sigma[M]/\mathbb{E}[M] = \gamma$

  d) The equity premium puzzle is verified if $\gamma \geq (3v - 2(1+i))/v$.

- **Problem** Consider a finance economy in which there is a risk-free asset with return $1+i$, for $i>0$, and a risky asset with return $R=(1+i+\varepsilon, 1+i-\varepsilon)$ for $0<\varepsilon<1$. Assume that the representative consumer has a logarithmic Bernoulli utility function and the endowment process is $Y=\{y_0, Y_1\}$ where $Y_1=((1+\gamma)y_0,(1-\gamma)y_0)^\top$ for $0<\gamma<1$.

  1. Find the probabilities for the two states of nature assuming that there are no arbitrage opportunities (Hint: use the condition $\mathbb{E}[M(R-R^f]]=0$).
  2. Find the Sharpe index for the risk premium. Justify.
  3. Find the Hansen-Jaganathan bound. Justify.
  4. Does the equity premium puzzle holds in this case ? Provide an intuition.

- **Solution**

  (a) The stochastic discount factor is $M=\left(\frac{\beta}{1+\gamma},\frac{\beta}{1-\gamma}\right)$ and the equity premium is $EP=(\varepsilon,-\varepsilon)$. Solving
  $$\mathbb{E}[MEP]=0 \Leftrightarrow \pi(1-\gamma)+(1-\pi)(1+\gamma)=0$$
  then $\pi=\frac{1+\gamma}{2}$ and $1-\pi=\frac{1-\gamma}{2}$.

  (b) The Sharpe index (Si) is
  $$\frac{\mathbb{E}[EP]}{\sigma[EP]}=\frac{\varepsilon\gamma}{\varepsilon\sqrt{(1+\gamma)(1-\gamma)}}=\frac{\gamma}{\sqrt{(1+\gamma)(1-\gamma)}}$$

  (c) The Hansen-Jagannathan (HJb) bound is
  $$\frac{\sigma[M]}{\mathbb{E}[M]}=\frac{\frac{\beta\gamma}{\sqrt{(1+\gamma)(1-\gamma)}}}{\beta}=\frac{\gamma}{\sqrt{(1+\gamma)(1-\gamma)}}$$

  (c) The relationship between Si and HJb for the the non-existence of the equity premium puzzle is
  $$\frac{\mathbb{E}[EP]}{\sigma[EP]}\leq\frac{\sigma[M]}{\mathbb{E}[M]}.$$
  Therefore it holds with equality.

Problem set 5

- **Problem** Consider an Arrow-Debreu (AD) economy with an information tree with two periods and $N > 1$ states of nature for the last period. There are $I > 1$ agents in the economy who are heterogeneous as regards the subjective probabilities associated to the two states of nature, π^i_s, and are homogeneous as regards preferences and endowments. The problem for agent $i \in \{1, \dots, I\}$ is to choose the optimal consumption sequence $\{c^i_0, C^i_1\}$, with $C^i_1 = (c^i_{1,1}, \dots, c^i_{1,N})$, to maximise the intertemporal utility functional

$$U^i(c^i_0, C^i_1) = \ln(c^i_0) + \beta \sum_{s=1}^{N} \pi^i_s \ln(c^i_{1,s})$$

subject to the constraint $c^i_0 + \sum_{s=1}^{N} q_s c^i_{1,s} = y_0 + \sum_{s=1}^{N} q_s y_{1,s}$, where q_s are the AD prices, and $y_{t,s}$ denotes the endowment for agent i at time t for the state of nature s.

(a) Define the AD general equilibrium for this economy.

(b) Solve agent i's problem.

(c) Find the equilibrium AD prices. Provide an intuition for your results (hint: compare with an analogous model in which there is homogeneity in information).

(d) Discuss the consequences of introducing heterogeneity in the endowments of both agents. Focus on both micro and aggregate consequences.

- **Solutions**

b) The Lagrangean for agent i is

$$\mathcal{L}^i = \ln(c^i_0) + \beta \sum_{s=1}^{N} \pi^i_s \ln(c^i_{1,s}) + \lambda^i \left(h - c^i_0 - \sum_{s=1}^{N} q_s c^i_{1,s} \right)$$

where $h = y_0 + \sum_{s=1}^{N} q_s y_{1,s}$ is wealth of agent i, which is equal for all agents. The solution for agent i problem is

$$c^i_0 = \frac{h}{1+\beta}$$
$$c^i_{1,s} = \frac{\beta \pi^i_s}{q_s} \frac{h}{1+\beta}, \; s = 1, \dots, N$$

c) From the market equilibrium conditions

$$\sum_{i=1}^{I} c^i_0 = \sum_{i=1}^{I} y_0 = I y_0$$
$$\sum_{i=1}^{I} c^i_{1,s} = \sum_{i=1}^{I} y_{1,s} = I y_{1,s}, \; s = 1, \dots, N$$

we find

$$\frac{h}{1+\beta} = y_0\, \frac{\beta}{q_s} \frac{h}{1+\beta} \sum_{i=1}^{I} \pi_s^i = I y_{1,s}$$

Defining the average probability of state s among the I agents by

$$\bar{\pi}_s = \frac{\sum_{i=1}^{I} \pi_s^i}{I} = \frac{\pi_s^1 + \ldots \pi_s^i + \ldots + \pi_s^I}{I}$$

we find the equilibrium state price

$$q_s = \beta \frac{y_0}{y_{1,s}} \bar{\pi}_s$$

This is formally similar to the case in which there is homogeneous information $\pi_s^i = \pi_s$ for all $i \in \{1, \ldots, I\}$ where the probability of state s is substituted by the (population) average probability of state s.