

The Solow growth model

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Kaldor's stylized facts (1963)

Fact K1 per capita GDP (y) grows along time, and its rate of growth shows no decreasing tendency;

Fact K2 K grows along time;

Fact K3 r (r.o.r of capital) is roughly constant (debatable: it shows a slightly downward tendency for most developing countries);

Fact K4 K/Y is roughly constant;

Fact K5 the shares of capital and labor in the aggregate income are approximately constant;

Fact K6 the growth of Y (p.c.) varies substantially between countries.

Solow (1956) model

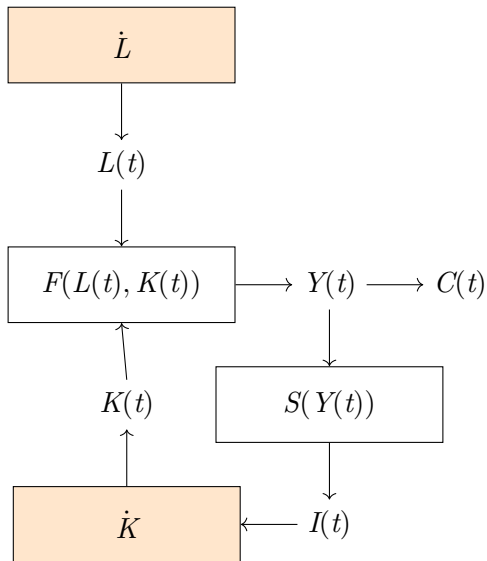
Structure of the economy

- ▶ Environment:
 - ▶ closed economy producing a single composite good
 - ▶ there is only reproducible factor: capital
 - ▶ there are no idle factors (no unemployment)
- ▶ Population:
 - ▶ exogenous
- ▶ Growth engine: capital accumulation

Solow (1956) model

Assumptions

- ▶ Production:
 - ▶ production uses two factors: labor and physical capital
 - ▶ production technology: neoclassical (increasing, concave, Inada, CRTS)
- ▶ Households: add-hoc behaviour
 - ▶ inelastically supply labor
 - ▶ ad-hoc savings proportional to income
 - ▶ static expectations (no anticipations)
- ▶ There is macroeconomic consistency (market clearing), but not necessarily microeconomic consistency (decisions on labor supply, consumption and finance are disconnected)



Solow model

The model: production technology

► Neo-classical production function

$$Y(t) = F(A, K(t), L(t)) = AK(t)^\alpha L(t)^{1-\alpha}, \quad 0 < \alpha < 1$$

where: A productivity, K stock of capital, L labor input

► properties

- constant returns to scale
- increasing in both factors: $\nabla F(K, L) = (F_K, F_L)^\top > \mathbf{0}$
- concave in (K, L)
- Inada

$$\lim_{K \rightarrow 0} F_K(K, L) = \lim_{L \rightarrow 0} F_K(K, L) = +\infty$$

$$\lim_{K \rightarrow \infty} F_K(K, L) = \lim_{L \rightarrow \infty} F_K(K, L) = 0$$

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The model: factor demand and distribution

- ▶ Inverse factor demand functions
 - ▶ the demand K is such that the rate of return of capital equals the marginal productivity of capital

$$r(t) = F_K(K, L) = \alpha \frac{Y(t)}{K(t)}$$

- ▶ the demand L is such that the wage rate equals the marginal productivity of labor

$$w(t) = F_L(K, L) = \alpha \frac{Y(t)}{L(t)}$$

- ▶ from CRTS and Euler's theorem the distribution of income is

$$Y(t) = r(t)K(t) + w(t)L(t)$$

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The model: factor dynamics

► Population growth

$$\dot{N}(t) = nN(t)$$

n is the exogenous rate of growth

► No unemployment (or demand and supply of labor)

$$L(t) = N(t)$$

► Capital accumulation

$$\dot{K} = I(t) - \delta K(t)$$

net investment = gross investment - capital depreciation

$\delta > 0$ rate of depreciation of capital

Solow model: labour market

Consumption and investment

- ▶ **”Keynesian” consumption function**

$$C(t) = (1 - s) Y(t)$$

$0 < s < 1$ is the marginal propensity to consume

- ▶ **investment decisions**

$$S(t) = sY(t)$$

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Macroeconomic equilibrium

- **Equilibrium in the product market**

$$Y(t) = C(t) + I(t)$$

aggregate supply = aggregate demand

- By Walras's law we could "close the model" by the **equilibrium in the capital market**

$$S(t) = I(t)$$

Solow model

GDP per capita

- ▶ The per capita GDP is

$$y(t) \equiv \frac{Y(t)}{N(t)}$$

- ▶ taking log-derivatives w.r.t time we have

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N} \Leftrightarrow g(t) = g_Y(t) - n(t)$$

Solow model

The model: the rate of growth

- ▶ The per capita GDP is

$$y(t) \equiv \frac{Y(t)}{N(t)} = A \left(\frac{K(t)}{N(t)} \right)^\alpha = Ak(t)^\alpha$$

defining the capital intensity by

$$k \equiv \frac{K}{L} = \frac{K}{N}$$

- ▶ Then

$$g(t) = \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} = \alpha g_k(t)$$

- ▶ the rate of growth is a linear function of the rate of growth of the capital intensity
- ▶ but the ratio between the two is less than one

$$\frac{g(y)}{g_k(t)} = \alpha \in (0, 1)$$

Solow model

The capital accumulation equation

- ▶ the dynamic equations of the model are

$$\begin{cases} \dot{K} &= sAK^\alpha N^{1-\alpha} - \delta K \\ \dot{N} &= nN \end{cases}$$

- ▶ using the de definition of the capital intensity k we obtain

$$\begin{cases} \dot{k} = sAk^\alpha - (n + \delta)k & t \geq 0 \\ k(0) = k_0, & t = 0 \end{cases}$$

- ▶ Then the dynamics for per capita GDP is

$$\begin{cases} \dot{y} = \alpha \left(sA^{\frac{1}{\alpha}} y^{1-\frac{1}{\alpha}}(t) - (n + \delta) \right) y(t) & t > 0 \\ y(0) = y_0 = Ak_0^\alpha, & t = 0 \end{cases}$$

- ▶ We can solve the equation for k or for y

Solow model

Explicit solution for k

- ▶ There are two steady states levels $k^* = \{0, \bar{k}\}$ where

$$\bar{k} = \left(\frac{sA}{n + \delta} \right)^{\frac{1}{1-\alpha}}$$

- ▶ has the solution proof

$$k(t) = \left[\bar{k}^{1-\alpha} + (k_0^{1-\alpha} - \bar{k}^{1-\alpha}) e^{\lambda t} \right]^{\frac{1}{1-\alpha}}, \quad t \in [0, \infty)$$

where

$$\lambda \equiv -(1 - \alpha)(n + \delta) < 0$$

- ▶ The growth rate of the capital intensity is

$$g_k(t) = -(n + \delta) \left(\frac{(k_0^{1-\alpha} - \bar{k}^{1-\alpha}) e^{\lambda t}}{\bar{k}^{1-\alpha} + (k_0^{1-\alpha} - \bar{k}^{1-\alpha}) e^{\lambda t}} \right)$$

Solow model

Properties of the solution

- ▶ The solution is continuous on k_0

$$k(0) = k(t|t=0) = k_0$$

- ▶ If $k_0 > 0$, k converges asymptotically to \bar{k}

$$\lim_{t \rightarrow \infty} k(t) = \bar{k}$$

independently of the initial value k_0 .

- ▶ Equivalently

$$\lim_{t \rightarrow \infty} g_k(t) = 0 \text{ because } \lim_{t \rightarrow \infty} e^{\lambda t} = 0$$

Solow model

Mechanics of the model

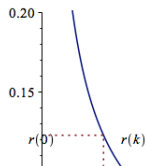
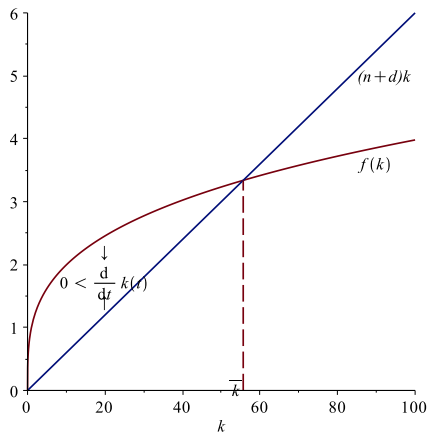
- ▶ we can write Solow's equation as

$$g_k(t) = \frac{\dot{k}}{k} = \frac{s}{\alpha} r(k(t)) - (n + \delta)$$

- ▶ low $k(0)$ means $r(0)$ is high relative to $n + \delta$
- ▶ this implies high incentive for saving and for accumulating capital
- ▶ but capital accumulation decreases the marginal productivity of capital because $r_k(k) = \frac{\partial r(k)}{\partial k} < 0$, which reduce progressively the incentives to accumulate capital
- ▶ which stops asymptotically the incentives to accumulate capital
- ▶ notice that in the long run capital increases just to cover $(n + \delta)$

Solow model

Mechanics



Solow model

Solution for y

- ▶ Because $y(t) = Ak(t)^\alpha$ and

$$\bar{y} = A\bar{k}^\alpha = A \left(\frac{sA}{n + \delta} \right)^{\frac{\alpha}{1-\alpha}}$$

- ▶ then the GDP per capita varies along time according to

$$y(t) = \left[\bar{y}^{\frac{1-\alpha}{\alpha}} + \left(y_0^{\frac{1-\alpha}{\alpha}} - \bar{y}^{\frac{1-\alpha}{\alpha}} \right) e^{\lambda t} \right]^{\frac{\alpha}{1-\alpha}}, \quad t \in [0, \infty)$$

where

$$\lambda \equiv -(1 - \alpha)(n + \delta) < 0$$

Solow model

Implications for growth

The **implication for growth** are

- ▶ there is **no long run growth**, if $y(0) = y_0 > 0$ then

$$\lim_{t \rightarrow \infty} y(t) = \bar{y} \Rightarrow \lim_{t \rightarrow \infty} g(t) = 0$$

- ▶ the **long run level of GDP per capita** : increases with A , and s and decreases with n and δ
- ▶ only **transitional dynamics** exists, driven by $\lambda = -(1 - \alpha)(n + \delta)$, i.e. it is due to the existence of **decreasing marginal returns to the accumulation factor k**

Solow model

Criticisms

- ▶ a zero long-run rate of growth is counterfactual for industrialised economies since the Industrial revolution
- ▶ capital accumulation displays **dynamic inefficiency**, i.e $\bar{k} > k^{\text{gr}}$ where

$$k^{\text{gr}} = \operatorname{argmax}_k \{c(k) = Ak^\alpha - (n + \delta)k\} = \left(\frac{\alpha A}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

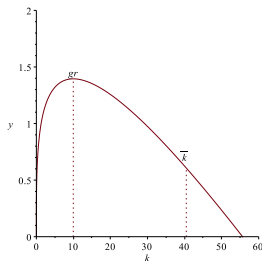


Figure: The golden rule and the steady state \bar{k}

Solow model

Extension: exogenous productivity growth

- ▶ Consider the production function

$$Y(t) = A(t)K(t)^\alpha L(t)^{1-\alpha}, \quad 0 < \alpha < 1$$

- ▶ and that there is exogenous TFP growth

$$A(t) = A_0 e^{g_A t}, \quad g_A > 0$$

- ▶ What are the growth consequences ?

Solow model

Extension: exogenous productivity growth

- ▶ Because

$$y(t) = A(t)k(t)^\alpha$$

- ▶ then

$$g(t) = g_A + \alpha g_k(t)$$

- ▶ as $\lim_{t \rightarrow \infty} g_k(t) = 0$ then

$$\lim_{t \rightarrow \infty} g(t) = g_A > 0$$

- ▶ There is (exogenous) long run growth.

References

- ▶ Solow (1956)
- ▶ (Acemoglu, 2009, ch. 2 and 3) , (Aghion and Howitt, 2009, ch. 1), (Barro and Sala-i-Martin, 2004, ch. 1)

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Princeton University Press, 2009.

Philippe Aghion and Peter Howitt. *The Economics of Growth*.
MIT Press, 2009.

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Appendix

Explicit solution of the Solow model

- ▶ We can re-write the capital accumulation equation as

$$\dot{k} = (n + \delta) \left(\left(\frac{k}{\bar{k}} \right)^{\alpha-1} - 1 \right) k$$

- ▶ use the transformation $z(t) = \left(\frac{k(t)}{\bar{k}} \right)^{1-\alpha}$

- ▶ then

$$\begin{aligned}\dot{z} &= (1 - \alpha) z \frac{\dot{k}}{k} = \\ &= (1 - \alpha)(n + \delta) \left(\frac{1}{z} - 1 \right) z\end{aligned}$$

- ▶ then we get the equivalent ODE

$$\dot{z} = (1 - \alpha)(n + \delta) (1 - z).$$

Appendix

Continuation

- ▶ The ODE

$$\dot{z} = (1 - \alpha)(n + \delta)(1 - z)$$

- ▶ has the solution

$$z(t) = 1 + (z(0) - 1)e^{-(1-\alpha)(n+\delta)t}$$

- ▶ then, transforming back, $k(t) = z(t)^{\frac{1}{1-\alpha}} \bar{k}$, we get

$$k(t) = \bar{k} \left[1 + \left(\left(\frac{k(0)}{\bar{k}} \right)^{1-\alpha} - 1 \right) e^{-(1-\alpha)(n+\delta)t} \right]^{\frac{1}{1-\alpha}}$$