EMA 2022-2023:
Problem set 1: linear ODE's

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1 Linear scalar ODE's

1.1 Autonomous ODE's

1.1.1 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following scalar ordinary differential equations and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -\frac{1}{2}y$;
(b) $\dot{y} = \frac{1}{2}y$;
(c) $\dot{y} = 2y$;
(d) $\dot{y} = -2y$;
(e) $\dot{y} = 0$;
(f) $\dot{y} = 2$;
(g) $\dot{y} = -2$;

1.1.2 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following scalar ordinary differential equations and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -\frac{1}{2}y + 1$;
(b) $\dot{y} = \frac{1}{2}y - 1$;
(c) $\dot{y} = 2y - 2$;
(d) $\dot{y} = -2y + 2$;
(e) $\dot{y} = ay - 2$ for $a \in (-2, 2)$
(f) $\dot{y} = y + b$ for $b \in (-1, 1)$

1.1.3 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following initial value problems and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -0.5y + 1$, for $t \geq 0$ and $y(0) = 1$ for $t = 0$;

(b) $\dot{y} = 0.5y - 1$, for $t \geq 0$ and $y(0) = 1$ for $t = 0$.

1.1.4 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following terminal value problems and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -0.5y + 1$, for $t \geq 0$ and $\lim_{t\to\infty} y(t) = \overline{y}$, where $\overline{y}$ is the steady state;

(b) $\dot{y} = 0.5y - 1$, for $t \geq 0$ and $\lim_{t\to\infty} e^{-0.5t} y(t) = 0$.

1.1.5 Perform a bifurcation analysis to the following equation $\dot{y} = ay + b$ for $a \in [-2, 2]$ and $b \in (-1, 1)$.

1.1.6 Let $y = y(t)$ is a function, $y : \mathbb{R}_+ \to \mathbb{R}$. Consider the terminal value problem

$$\begin{cases} \dot{y} = gy + b & t \geq 0 \\ \lim_{t\to\infty} y(t) = \overline{y} \end{cases}$$

where $\overline{y}$ is the steady state, and g and $b \neq 0$ are constants.

(a) Assume that $g < 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

(b) Assume that $g > 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

1.1.7 Consider the following problem

$$\begin{cases} \dot{y} = \lambda\big(\bar{y} - y\big) & \text{for } t \in \mathbb{R}_+ \\ \int_0^\infty y(t)\,\phi(t)\,dt = \bar{y} \end{cases}$$

where $\lambda > 0$ and $\phi(t) = \lambda\, e^{-\lambda t}$. Observe that $\phi(t)$ is a distribution.

(a) Solve the problem.

(b) Provide an intuition for the problem and its solution.

1.2 Non-autonomous ODE's

1.2.1 Consider the scalar ODE

$$\dot{y} = ay + b(t),\ y : [0, \infty) \to \mathbb{R}$$

where

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^*, \\ b_1 & \text{if } t^* \leq t < \infty. \end{cases}$$

(a) Assume that $a \neq 0$ and $y(0) = y_0$ is given. Solve the initial value problem.

(b) Assume that $a > 0$ and $\lim_{t\to\infty} y(t)e^{-at} = 0$. Solve the terminal-value problem.

1.2.2 Consider the scalar ODE

$$\dot{y} = ay + b(t),\ y : [0, \infty) \to \mathbb{R}$$

where

$$b(t) = \begin{cases} b & \text{if } 0 \le t < t^*, \\ b + \Delta b & \text{if } t^* \le t < t^* + \Delta t, \\ b & \text{if } t^* + \Delta t \le t < \infty, \end{cases}$$

where $\Delta t > 0$.

(a) Assume that $a \neq 0$ and $y(0) = y_0$ is given. Solve the initial value problem.

(b) Assume that $a > 0$ and $\lim_{t\to\infty} y(t)e^{-at} = 0$. Solve the terminal-value problem.

1.2.3 A utility function, $u(x)$, is said to display constant relative risk aversion if is satisfies

$$\frac{u''(x)}{u'(x)} = -\alpha,\ x \in \mathbb{R}_+$$

where $\alpha > 0$ is called the coefficient of absolute risk aversion.

(a) Find the general solution to the ODE.

(b) The popular form of the CARA utility function in the literature is $u(x) = -\dfrac{e^{-\alpha x}}{\alpha}$. Assuming that the constraint $\alpha \int_0^\infty u'(x)dx = 1$ is satisfied, find the condition which is implicitly assumed in the previous popular form.

1.2.4 Consider the scalar ODE

$$\frac{y'(x)\,x}{y(x)} = \mu,\ x \in \mathbb{R}$$

where μ is a constant.

(a) Prove that the general solution follows a power law.

(b) Impose conditions on the parameter and an initial value such that the solution satisfies

$$\int_{x_0}^{\infty} y(x)dx = 1$$

1.2.5 Consider the scalar ODE problem

$$\begin{cases} \dfrac{y'(x)}{y(x)} = -x, & x \in \mathbb{R} \\ \int_{-\infty}^{\infty} y(x)dx = 1 \end{cases}$$

(a) Prove that the solution is the standard Gaussian probability density function $y(x) \sim N(0, 1)$

1.2.6 Consider the scalar problem

$$\begin{cases} \dfrac{y'(x)}{y(x)} = -\dfrac{1+\ln(x)}{x}, & x \in (0,\infty) \\ \int_0^\infty y(x)dx = 1 \end{cases}$$

(a) Prove that the solution is the standard lognormal density function $y \sim LN(0,1)$.

1.2.7 Consider the scalar ODE

$$y'(x) = -\mu\, y(x) + \beta,\ x \in \mathbb{R}_+. \tag{1}$$

where $\mu > 0$.

(a) Solve the ODE.

(b) Find the particular solution of the ODE satisfying $\int_0^\infty \big(y(x) - \bar{y}\big)\, dx = 1$ where $\bar{y}$ is the stationary solution of (1).

1.2.8 Consider the initial value problem (IVP)

$$\begin{cases} \dot{y} + \lambda\, y = f(t), & t \in \mathbb{R}_+ \\ y(0) = 0 & t = 0 \end{cases}$$

where λ is a constant, and $f(\cdot)$ is an arbitrary function, not necessarily continuous, which is sometimes called a "driving force".

(a) Prove that the solution to the IVP is a convolution, $y(t) = f(t) * g(t)$, where f is a driving force and g is a function sometimes called unit impulse response function (IRF).[1] Provide an intuition for this fact.

(b) Assume that $\lambda > 0$. If we consider y represents the variation of a macroeconomic variable subject to a temporary shock

$$f(t) = \begin{cases} \alpha, & 0 \le t \le \tau \\ 0 & t > \tau, \end{cases}$$

where $\alpha > 0$ is a constant. Find the solution to the problem. Draw the solution path.

1.3 Applications

1.3.1 The simplest model of population dynamics assumes that the rate of population growth is deterministic, age-independent, and constant:

$$\dot{N} = \nu N.\ N : \mathbb{R}_+ \to \mathbb{R}_+, \tag{2}$$

[1] A convolution of two functions $f(x)$ and $g(x)$, for $x \in \mathbb{R}$, is a function $h = f * g$ such that $h(x) = \int_0^x f(t)\, g(x-t)\, dt$.

where $N(t)$ is the population at time t and $\nu \equiv \beta - \mu$ is the net rate of growth, β is the fertility rate and μ is the mortality rate. We assume that $N(0) = N_0 \geq 0$ is given. (References **?** see also `http://en.wikipedia.org/wiki/Exponential_growth`)

(a) Solve equation (2).

(b) Solve the initial value problem.

(c) Characterize the dynamics.

1.3.2 The stock-flow dynamics is generically represented by an equation of type,

$$\dot{A} = \pi + rA, \;\; A : \mathbb{R}_+ \to \mathbb{R} \tag{3}$$

where A is the stock of an asset at time t, π is net income and r is the rate of return. Assume that $r > 0$

(a) Solve equation (3) and characterise qualitatively the dynamics.

(b) Assuming we know $A(0) = A_0$, solve the initial value problem.

(c) Assuming we introduce a solvability requirement $\lim_{t\to\infty} A(t)e^{-rt} = 0$, determine the initial level of $A(0)$.

1.3.3 **?** is one of the first papers to deal with perfect foresight dynamics. The main equation of the paper was

$$\dot{p} = \beta(p - m(t)), \;\; p : \mathbb{R}_+ \to \mathbb{R} \tag{4}$$

where p and m are the logs of the price index and nominal money supply and $\beta > 0$

(a) Solve equation (4).

(b) Setting $p(0) = p_0$, where p_0 is known, solve the initial value problem. Does the solution to this problem makes economic sense (hint: recall the expected relationship between increases in the money supply and the price evolution) ?

(c) Let m is constant. Assuming there are no speculative bubbles, i.e, $\lim_{t\to\infty} p(t)e^{-\beta t} = 0$, determine $p(0)$.

(d) Modify the previous results assuming that there is an anticipated (to time $\tau > 0$ and finite) monetary shock.

1.3.4 The government budget constraint, in nominal variables, is

$$\dot{B} = D + iB,$$

where $B(t)$ is the stock of government bonds at time t, (where $B : \mathbb{R}_+ \to \mathbb{R}$), D is the primary deficit, and i is the interest rate on government bonds. Assume that the GDP, Y, follows the process $\dot{Y} = gY$. All variables are in nominal terms.

(a) Let $b \equiv B/Y$ and $d \equiv D/Y$. Which types of dynamic behavior for b one should expect ?

(b) Assuming we know $b(0) = b_0$, solve the initial value problem.

(c) If we introduce a solvability requirement such that $\lim_{t\to\infty} b(t)e^{-rt} = 0$, determine the initial level of $b(0)$, assuming that $r \equiv i - g > 0$.

1.3.5 Let the government budget constraint be $\dot{b} = -\tau(t) + rb(t)$ where $b(t)$ is the government debt and $\tau(t)$ is the time-varying primary surplus, at time $t \geq 0$, and $r > 0$ is the interest rate on the government debt. Assume that the government adopts a fiscal rule taking the form $\dot{\tau} = \gamma b(t) - \xi\tau(t)$ where $\gamma > 0$. Assume that the initial level of the debt is given $b(0) = b_0$.

(a) If we assume that $r > \xi$, under which conditions on the parameters of the fiscal rule can the government reach the following goal: $\lim_{t\to\infty} b(t) = 0$?

(b) Assuming the previous condition determine the paths for the government debt and primary surplus.

(c) What should be the initial surplus $\tau(0)$? Provide an intuition for this result.

1.3.6 Let x be the log of the nominal exchange rate for a country with a flexible exchange rate regime. The Fisher open equation for the behavior of the rate of depreciation is $\dot{x} = i(t) - i^*(t)$, where i and i^* are the domestic and international nominal interest rates, respectively. Assume that the domestic interest rate is a linear function of the nominal exchange rate $i(t) = \lambda\, x(t)$ where λ is a positive constant. Assume that there are no speculative bubbles. Therefore, the problem is

$$\begin{cases} \dot{x} = \lambda x - i^*(t), & \text{for } t \in (0, \infty) \\ \lim_{t\to\infty} e^{-\lambda t} x(t) = 0. \end{cases}$$

(a) Solve the problem.

(b) Assume there is an anticipated, but temporary change in the international interest rate, such that

$$\Delta i^*(t) = \begin{cases} 0 & \text{for } 0 \leq t < T_0 \\ d & \text{for } T_0 \leq t < T_0 + \Delta T \\ 0 & \text{for } T_0 + \Delta T \leq t < \infty \end{cases}$$

for $T_0 > 0$, $\Delta T > 0$ and a constant $d \neq 0$. Find the response of the nominal exchange rate for $t \in [0, \infty)$.

1.3.7 Consider a household having the budget constraint $\dot{a} = r\, a + y(t) - c$, where a is the time-varying net asset position, c is consumption (exogenous and constant), and r is the rate of return on assets. Assume the household expects income, y, to have two stages

$$y(t) = \begin{cases} y_0 & 0 \leq t \leq t_s \\ y_1 & t_s < t < \infty \end{cases}$$

where $y_0 = y_1 + \Delta y$, for $\Delta y > 0$, and the switching time satisfies $t_s \in (0, \infty)$. Assume that $r > 0$ and that $c > 0$.

(a) Assume that $a(0) = a_0$ is known. Solve the initial value problem (hint: find the solutions for the two stages). Provide a geometrical intuition.

(b) Assume instead that there is a terminal constraint $\lim_{t\to\infty} a(t)e^{-r\,t} = 0$. Solve the terminal value problem (hint: in this case $a(0)$ should be determined).

(c) Compare and discuss the difference between the two solutions. Provide an economic intuition assuming that the two stages for an individual are employed/unemployed or active/retired.