EMA 2019-2020:
Problem set 2: non-linear ODEs

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8.2.2020
(revised 11.3.2020)

1 Non-linear differentiable ODE's

1.1 General: scalar

1. Solve the following non-linear ordinary differential equations (explicitly or geometrically)
   (a) $\dot{y} = \frac{1}{2}y^2 - y + 4$ for $y \in \mathbb{R}$;
   (b) $\dot{y} = y^3 - y + 4$ for $y \in \mathbb{R}$;
   (c) $\dot{y} = e^{-\frac{1}{2}y}$ for $y \in \mathbb{R}$;
2. Perform a bifurcation analysis to the following equation: $\dot{y} = ay^2 - b$ for $a \in \mathbb{R}$ and $b > 0$
3. Perform a bifurcation analysis to the following equation: $\dot{y} = y^3 - y^2 + a$ for $(y, a) \in \mathbb{R}^2$.
4. Perform a bifurcation analysis to the following equation: $\dot{y} = ay^b - c$ for $(a, b, c) \in \mathbb{R}^3$.

1.2 Applications: scalar

1. The Verhulst model of population rate assumes that the rate of population growth is deterministic, age-independent and is dependent on the population level,

$$\dot{N} = \gamma(N)N \tag{1}$$

   where $\gamma = \eta(1 - N/K)$, is the growth rate and K is called the carrying capacity. We assume that $N(0) = N_0$ is known. References Banks (1994) `http://en.wikipedia.org/wiki/Logistic_function`
   (a) Solve equation (1)
   (b) Solve the initial value problem

(c) Characterize the dynamics.

2. The Verhulst model of population rate assumes that the rate of population growth is deterministic, age-independent and is dependent on the population level,

$$\dot{N} = \gamma(N)N \tag{2}$$

where $\gamma = \alpha\nu(1 - (N/K)^{\nu})$, is the growth rate and K is called the carrying capacity. We assume that $N(0) = N_0$ is known. References Banks (1994) `http://en.wikipedia.org/wiki/Logistic_function`

   (a) Solve equation (2)
   (b) Solve the initial value problem
   (c) Characterize the dynamics.

3. Solow (1956) is maybe one of the most seminal papers in economics. It features a theory of growth in which the engine of growth baed on capital accumulation. Th main equation is

$$\dot{k} = sAf(k) - nk \tag{3}$$

where k is the capital intensity, $0 < s < 1$ is the savings rate, A is the total factor productivity and n is the rate of growth of the population. The production function in intensity terms $f(k)$ is increasing, concave and has the Inada properties

   (a) Find the equilibrium points of equation (3) and characterise qualitatively the dynamics.
   (b) What is the effect of a productivity shock.
   (c) Is there any bifurcation ?

4. Consider a simplified version of the Skiba model (Skiba (1978)) in which the production function is convex-concave and savings is exogenous, as in the Solow model (Solow (1956)). The capital accumulation equation is

$$\dot{k} = sf(k) - \delta k$$

where k is per-capita capital, s is the savings rate, and δ is the capital depreciation rate. Assume a production function of type

$$f(k) = \frac{1}{2}\left(1 + \mathrm{erf}(\ln(k) - \mu)\right)$$

where $\mathrm{erf}(x) = 2\pi^{-\frac{1}{2}} \int_0^x e^{-s^2} ds$ (see `https://en.wikipedia.org/wiki/Error_function`. Let $k(0) = k_0 > 0$ be given. (Hint: in case of trouble set $(s, \mu, \delta) = (0.15, 0.7, 0.04)$,

   (a) Study the existence and number of steady states depending upon the values of the parameters.

(b) Study the local dynamics at every steady state.

(c) Provide a bifurcation analysis of the model.

5. One of the most popularities utility functions in finance, the hyperbolic absolute risk aversion function (HARA), has several other designations as GLUM (generalized logarithmic utility model Rubinstein (1976)) or generalized logarithm (see (Tsallis, 2009, ch.3)). The fundamentals of the function can be better understood by the fact that it can be derived as a solution to a differential equation

   Let u be a function of the independent variable be $x \in \mathbb{R}_+$ (usually consumption or wealth) . it can be It can be seen as the solution of the differential equation

$$\frac{u^{'}(x)}{u^{''}(x)} = -\left(\eta + \frac{x}{\theta}\right)$$

   in which η and θ are parameters and $u^{'}(x) = \dfrac{du(x)}{dx}$ and $u^{''}(x) = \dfrac{d^2u(x)}{dx^2}$.

   (a) Solve the differential equation.

   (b) Under which conditions the CRRA function $u(x) = \dfrac{x^{1-\theta} - 1}{1-\theta}$ is a solution to the equation ?

   (c) Under which conditions the log function $u(x) = \ln(x)$ is a solution to the equation ? (Hint: note that $f(x) = e^{\ln(f(x))}$)

   (d) Provide an interpretation of function.

1.3 General: planar

1. Consider the following non-linear ODE

$$(1)\ \begin{aligned}\dot{y}_1 &= -y_2,\\ \dot{y}_2 &= -y_1 - y_2^2,\end{aligned} \quad (2)\ \begin{aligned}\dot{y}_1 &= y_2,\\ \dot{y}_2 &= -y_1 - y_2^3,\end{aligned} \quad (3)\ \begin{aligned}\dot{y}_1 &= y_2,\\ \dot{y}_2 &= -y_1 + y_1^3,\end{aligned} \quad (4)\ \begin{aligned}\dot{y}_1 &= y_1(y_1 + y_2 - 1),\\ \dot{y}_2 &= y_2(y_1 - y_2 + 1),\end{aligned}$$

   (a) Find the steady states.

   (b) Study the local dynamics.

   (c) Draw the phase diagrams.

2. Consider the following ODE's, depending on a parameter $\lambda \in (-\infty, \infty)$:

$$(1)\ \begin{aligned}\dot{y}_1 &= \lambda y_2,\\ \dot{y}_2 &= -y_1 - y_2^2,\end{aligned} \quad (2)\ \begin{aligned}\dot{y}_1 &= y_2,\\ \dot{y}_2 &= \lambda y_1 - y_2^2,\end{aligned} \quad (3)\ \begin{aligned}\dot{y}_1 &= y_2 + \lambda y_1^2,\\ \dot{y}_2 &= y_1 - y_2^2,\end{aligned}$$

   (a) Find the equilibrium points.

   (b) Find possible bifurcation points and characterize them.

   (c) Draw phase diagrams for the generic cases.

1.4 Applications: planar 1

1. The Lotka-Volterra model (see Kot (2001) or `http://en.wikipedia.org/wiki/Lotka-Volterra_equation`) is an important model in mathematical biology. It models the dynamics of the interaction between predators and preys. In its simplest formulation it is

$$\begin{aligned} \dot{N}_1 &= N_1(\alpha - \beta N_2) \\ \dot{N}_2 &= -N_2(\gamma - \delta N_1) \end{aligned}$$

   where N_1 is the number of prey (for example, rabbits) and N_2 is the number of some predator (for example, foxes) and the rates of growth of preys and predators are $\gamma_1 = \alpha - \beta N_2$ and $\gamma_2 = \delta N_1 - \gamma$.

   (a) find the equilibrium points and study the local dynamics
   (b) draw the phase diagram
   (c) prove that the model solution verifies the first integral $V = -\delta\, N_1 + \gamma\, \log(N_1) - \beta\, N_2 + \alpha\, \log(N_2)$ such that $\dot{V} = 0$. Discuss the dynamic properties when this condition holds

2. Consider the dynamics of a two-sector growth model

$$\begin{aligned} \dot{k}_1 &= A_1 k_1^{\alpha} k_2^{1-\alpha} - \delta_1 k_1,\ 0 < \alpha < 1 \\ \dot{k}_2 &= A_2 k_1^{1-\beta} k_2^{\beta} - \delta_2 k_2,\ 0 < \beta < 1 \end{aligned} \tag{4}$$

   where k_i is the capital stock of sector i, $A_i > 0$ and $\delta_i > 0$ are the total factor productivity and the rate of depreciation for section $i \in \{1, 2\}$. Assume that $k_1 > 0$ and $k_2 > 0$.

   (a) Under which conditions equilibrium points of equation (4) exist ? Characterize their dynamic properties.
   (b) If the previous condition(s) don't hold which type of dynamics is displayed by the solution of equation (4) ? Provide an intuition for your results.

3. The Ramsey Ramsey (1928) model addresses the dynamics of capital accumulation (or savings) as a way to maximize intertemporal consumption. The optimality conditions features a non-linear planar ODE together with an initial and a terminal condition. However, it is instructive to look at the dynamics of the ODE's which provide candidates for solutions. We have the following ODE over capital stock and consumption, $(k, c) \in \mathbb{R}_+$, assuming we have a Cobb-Douglas production function and an isoelastic utility function,

$$\begin{aligned} \dot{k} &= k^{\alpha} - c - \delta k \\ \dot{c} &= \frac{c}{\sigma}\left(\rho + \delta - \alpha k^{\alpha-1}\right) \end{aligned}$$

   where $0 < \alpha < 1$ is the income share for capital, $\delta > 0$ is the capital depreciation rate, ρ is the rate of time preference, and $\sigma > 0$ is the inverse of the elasticity of intertemporal substitution for consumption.

(a) Find the steady states.

(b) Determine their local dynamics.

(c) Draw the phase diagram.

(d) Prove that there are two heteroclinic trajectories connecting all the steady states.

1.5 Applications: planar 2

1. Calvo (1983) was one of the first papers to deal with perfect foresight dynamics. The model features a two-dimensional system

$$\begin{aligned} \dot{P} &= \delta(V - P) \\ \dot{V} &= \delta\left(V - P + \beta f(P - \bar{E}, \delta(V - P))\right) \end{aligned} \tag{5}$$

where P and V are the logs of the price and of the price-set contracts, $\bar{E}$ is the inflation expectation, δ and β are positive parameters and function $f(x, y)$ has the following properties: $f(0, 0) = 0$ and $f_x(x, y) > 0$ and $f_y(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$. Let $V(0) = V_0$ be given.

   (a) Find the steady state and characterize the local dynamics for the system (5).

   (b) Find a general linear approximation to the solution of system system (5) in the neighborhood of the steady state.

   (c) Find the solution along the stable eigenspace.

   (d) Draw the phase diagram.

   (e) Assuming that $V_0 > \bar{E}$, which type of adjustment of both P and V can we expect to have.

2. This problem is a version of the Dornbusch (1976) model, which was very influential throughout the 1970's until the end of the century. It features an open economy, both as regards goods and capital international flows, and studies the joint dynamics of the exchange rate and inflation. It was one of the first models to consider rational expectations equilibrium within a Keynesian framework.

   In the model (e, p, p^*, y) denote the logs of the exchange rate, the domestic and the international product deflators and the money stock and i the nominal interest rate. We assume: a) the real supply of money is deflated by the consumer price index $p^c :== \alpha p + (1 - \alpha)(e + p^*)$; b) the domestic demand is a function of the real interest rate $r = i - \dot{p}$ and not by the nominal interest rate (i) as in the original model; c) the log of the international price deflator is always equal to zero $p^* = 0$ and the semi-elasticity of the demand for money as regards the nominal interest rate is equal to one. With these assumptions, we get the planar ODE in (p, e) (where $\alpha \in [0, 1]$ and all the other parameters are positive)

$$\begin{aligned} \dot{p} &= \frac{\pi}{1 - \pi\delta}[g + \delta(e - p) - \sigma i - \bar{y}] \\ \dot{e} &= i - i^* \end{aligned}$$

where the first equation is obtained from the product market equilibrium together with the assumpyions that the prices adjust sluggishly, the second equation is the uncovered interest rate parity condition, and the interest rate is obtained from the equilibrium in the money market

$$i = \phi\bar{y} - m + \alpha p + (1-\alpha)e.$$

We assume that m is an instrument of monetary policy and that $\bar{y}$ is the exogenous supply of goods.

(a) Find the steady states and their local dynamics properties.

(b) Are there bifurcation points ?

(c) Taking the case in which the steady state is a saddle point draw the phase diagram.

(d) Do a comparative dynamics stud, both analytically and geometrically, for one unanticipated and permanent increase in the stock of money.

(e) Assume there is an instantaneous adjustment in the product market, i.e., $\pi \approx \infty$, and that $\sigma = 1$. Assuming we are at $t = 0$ and that there are no speculative bubbles, find the initial jump in the exchange rate for an anticipated increase in money supply at time $t = \tau > 0$. Provide an intuition

2 Dimensions higher than two

1. A seminal paper on the mathematical theory of epidemics is Kermac and McKendrick (1927) (see also Kot (2001), Murray (2003a) and Murray (2003b)). Consider the division of total population $N(t) = S(t) + I(t) + R(t)$ where N is total population, S is the number of subjected, I the number of infected and R the number of removed either by immunisation or death. The dynamic model is

$$\dot{S} = -\beta S I \quad (6)$$
$$\dot{I} = \beta S I - \gamma I \quad (7)$$
$$\dot{R} = \gamma I \quad (8)$$

all the parameters are positive, and have the following meaning: β is the infection rate and γ is the recovery rate.

(a) Study the dynamics of the model and the existence of bifurcations.

(b) Study the effect of an increase in β

2. A benchmark model for infections breakthroughs is an extension of the seminal paper on the mathematical theory of epidemics by Kermac and McKendrick (1927). The total population, at some point in time, $N(t)$ is divided as $N(t) = S(t) + E(t) + I(t) + R(t)$ where S is the number of subjected, E is the number of exposed, I the number

of infected and R the number of removed either by immunisation or death. The SEIR model has the following structure

$$\dot{S} = -\beta \frac{SI}{N} \tag{9}$$

$$\dot{E} = \beta \frac{SI}{N} - \sigma E \tag{10}$$

$$\dot{I} = \sigma E - \gamma I \tag{11}$$

$$\dot{R} = \gamma I. \tag{12}$$

The parameters are positive, and have the following meaning: $\beta = R_0 \gamma$ is the transmission rate, where R_0 is the basic reproduction number, σ is the infection rate and γ is the recovery rate.

(a) Study the dynamics of the model and the existence of bifurcations.

(b) Study the effect of an increase in R_0. You can use the parameters in Wang et al. (2020) for the corona-virus outbreak: they consider the parameters $\sigma = 1/5.2$, $\gamma = 1/18$, and they provide several values for R_0.

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