Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2017-2018

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Exam: **Época Normal**
10.1.2018 (18.00h-20.30h)

Closed book exam.

1. `[2 points]` Consider the terminal value problem $\dot{y} = gy + b$, for $t \geq 0$ and $\lim_{t\to\infty} y(t) = \overline{y}$, where $\overline{y}$ is the steady state. Let $b \neq 0$:
   (a) assume that $g < 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically;
   (b) assume that $g > 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

2. `[3 points]` Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

   (a) Solve the ODE.
   (b) Draw the phase diagram and characterize it.
   (c) Let $y_1(0) = 0$ and $y_2(0) = 1$. Solve the initial value problem.

3. `[3 points]` Consider the following ODE:

$$\dot{y}_1 = y_1(1 - y_2^2),$$
$$\dot{y}_2 = y_2(y_1 - 1)$$

   (a) Find the equilibrium points and characterize them.
   (b) Prove that the equation has the first integral: $V(y_1, y_2) = \log\left(y_1 y_2^2\right) - \left(y_1 + y_2^2\right)$.
   (c) Draw the phase diagram and provide an intuition for it.

4. `[3 points]` Sidrausky [?] wrote an influential article on money and economic growth. Assume a representative agent economy in which the stock of financial wealth in real terms, a, is the sum of the stock of real money and equity, k, $a = k + m$. The budget constraint is $\dot{a} = y + (\mu - \pi)m - c$, where c is real consumption, y is the income from labor and dividends, μm are nominal money transfers (subsidies minus taxes), and π is the inflation rate. Investment in equity produces a flow of income $y = f(k)$. The intertemporal utility functional is

$$J(c, m) = \int_0^{+\infty} u(c, m) e^{-\delta t} dt,$$

if we assume that the utility of cash holdings is related to the reduction of time devoted to transactions of goods. The decision variables for maximizing J are the consumption flow and the stock of money.

Assume that the production function, $f(k)$ is increasing, concave and Inada, and the utility function, $u(c,m)$ is increasing, concave, Inada and separable (i.e, $u_{cm}(c,m) = u_{mc}(c,m) = 0$). Assume that that $a(0) = a_0 > 0$ and that $\lim_{t\to\infty} e^{-(\mu-\pi)t} a(t) \geq 0$.

(a) Write the problem as an optimal control problem and the first order conditions according to the Pontriyagin's principle (Hint: use a as the stat evariable). Are those conditions necessary and sufficient ?

(b) Determine the steady state(s) and the local dynamic properties

(c) Depict the phase diagram.

(d) Find the short-run and the long run multipliers for a permament increase in the rate of growth of money. Provide a geometric representation

5. `[3 points]` Consider the first-order partial differential equation $y_t(t,x) + ax\, y_x(t,x) = 0$, where $y = y(t,x)$ and $(t,x) \in (0,\infty) \times (-\infty,\infty)$.

   (a) Find the solution (hint use the method of characteristics).

   (b) Let $y(0,x) = \delta(x - x_0)$, where $\delta(.)$ is Dirac's delta function and $x_0 > 0$. Find the solution to the initial-value problem.

6. `[3 points]` Consider the parabolic partial differential equation $u_t - u_{xx} = 0$, where $u = u(x,t)$ and $(x,t) \in (-\infty,\infty) \times (0,\infty)$.

   (a) Find the solution to the PDE.

   (b) If $u(x,0) = \delta(x - x_0)$, where $\delta(.)$ is Dirac's delta "function"and $x_0 > 0$ find the solution to the initial-value problem.

7. `[3 points]` The diffusion equation is

$$dX(t) = \gamma X(t)dt + \sigma dW(t)$$

   for $X(0) = x_0$.

   (a) Find the solution of the initial value problem

   (b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

   (c) Derive the forward Kolmogorov equation for the density associated to $X(t) = x > 0$, assuming that $X(0) = 0$.