

# Advanced Mathematical Economics

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## **Part VI**

# **Parabolic partial differential equations**

# Chapter 8

# Parabolic partial differential equations

## 8.1 Introduction

Parabolic partial differential equations involve a known function  $F$  depending on two independent variables  $(t, x)$ , an unknown function of them  $u(t, x)$ , the first partial derivative as regards  $t$  and first and second partial derivatives as regards the "spatial" variable  $x$ :

$$F(t, x, u, u_t, u_x, u_{xx}) = 0$$

where  $u : \mathcal{T} \times \mathcal{X} \rightarrow \mathbb{R}$ , where  $\mathcal{T} \subseteq \mathbb{R}_+$  and  $\mathcal{X} \subseteq \mathbb{R}$ .

In its simplest form,  $F(u_t, u_{xx}) = 0$ , the equation models a distribution featuring dispersion through time, for a cross section variable, generated by spatial contact (think about the time distribution of a pollutant spreading within a lake in which the water is completely still). Equation  $F(u_t, u_x, u_{xx}) = 0$  features both dispersion and advection behaviors (think about the time distribution of a pollutant spreading within a river). Equation  $F(u_t, u, u_x, u_{xx}) = 0$  jointly displays dispersion, advection and growth or decay behaviors (think about a time distribution of a pollutant spreading within a river, in which there is a permanent flow of new pollutants being dumped into the river). The independent terms appear in function  $F(\cdot)$  if there are some time or spatial specific components.

We will also see in the next chapter that there is a close connection between partial differential equations and stochastic differential equations. This implied that continuous-time finance has been using parabolic PDE's since the beginning of the 1970's.

In economics and finance applications it is important to distinguish between **forward** (FPDE) and **backward** (BPDE) parabolic PDE's. While the first are complemented with an initial distribution and generate a flow of distributions forward in time, the latter are complemented with a

terminal distribution and its solution generate a flow of distributions consistent with that terminal constraint. While for FPDE the terminal distribution is unknown, for BPDE the distribution at time  $t = 0$  is unknown. For planar systems, we may have forward, backward or forward-backward (FBPDE) parabolic PDE's. The last case can be seen as a generalization of the saddle-path dynamics for ODE's.

In mathematical finance most applications, such as the Black and Scholes (1973) model, most PDE's are of the backward type. In economics there is recent interest in PDE's related to the topical importance of distribution issues, and, in particular spatial dynamics modelled by BPDE. Optimal control of PDE's and the mean-field games usually lead to FBPDE's.

Again, the body of theory and application of parabolic PDE's is huge. We only present next some very introductory results and applications.

## 8.2 Scalar parabolic PDE's

Let  $u(t, x)$  where  $(t, x) \in \mathcal{T} \times \mathcal{X} \subseteq \mathbb{R} \times \mathbb{R}_+$  is an at least  $C^{2,1}(\mathbb{R}_+, \mathbb{R})$  function<sup>1</sup>. We can define

- linear parabolic PDE

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u + d(t, x)$$

if  $F(\cdot)$  is linear in  $u$  and all its derivatives, and the coefficients are independent from  $u$

- a semi-linear parabolic PDE

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(x, t, u)$$

if  $F(\cdot)$  is linear in the derivatives of  $u$ , and the coefficients are independent from  $u$

- a quasi-linear parabolic PDE

$$u_t = a(x, t, u)u_{xx} + b(x, t, u)u_x + c(x, t, u)$$

if  $F(\cdot)$  is linear in the derivatives of  $u$ , but the coefficients can be functions of  $u$ .

Consider the simplest linear parabolic equation with constant coefficients, sometimes called the **diffusion equation with advection and growth**,

$$u_t = au_{xx} + bu_x + cu + d.$$

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<sup>1</sup>It is, at least, differentiable to the second order as regards  $x$  and to the first order as regards  $t$ .

The time-behavior of  $u$  depends on three terms: a diffusion term,  $au_{xx}$ , a transport term,  $bu_x$ , and a growth term  $cu + d$ . If  $a > 0$  ( $a < 0$ ) the equation is sometimes called a **forward** FPDE (**backward** BPDE) equation, because the diffusion operator works forward (backward) in time. The second term introduces a behavior similar to the first-order PDE: it involves a translation of the solution along the direction  $x$ . The third term generates a time behavior of the whole distribution  $u(x, \cdot)$  in a way similar to a solution of an ordinary differential equation, that is, it involves stability or instability properties.

In the case of a parabolic PDE the stability or instability properties are related to the whole distribution: we have **stability in a distributional sense** if there is a solution  $u(t, x) = \bar{u}(x)$  such that

$$\lim_{t \rightarrow \infty} u(t, x) = \bar{u}(x)$$

where  $\bar{u}(x)$  is a stationary distribution.

An important element regarding the existence and characterization of the solution of PDE's is related to the characteristics of the support of the distribution  $\mathcal{X}$ . We can distinguish between three main cases:

- unbounded or infinite case  $\mathcal{X} = (-\infty, \infty)$
- the semi-bounded or semi-infinite case  $\mathcal{X} = [0, \infty)$  or  $\mathcal{X} = (-\infty, 0]$ , where 0 can be substituted by any finite number
- the bounded case  $\mathcal{X} = (\underline{x}, \bar{x})$  where both limits are finite.

In order to define **problems involving parabolic PDE's** we have to supplement it with a distribution referred to a point in time (an initial distribution for the forward PDE or terminal distribution for a backward PDE), and possibly conditions involving known values for the values of  $u(t, x)$  at the boundaries of  $\mathcal{X}$  (so called boundary conditions), i.e.,  $x \in \partial\mathcal{X}$ .

A problem is said to be **well-posed** if there is a solution to the PDE that verifies jointly the initial (or terminal) and the boundary conditions and it is continuous at those points. In this case we say we have a **classic solution**. If a problem is not well-posed it is **ill-posed**. In this case there are no solutions or classic solutions do not exist (but generalized solutions can exist).

A necessary condition for a problem involving a FPDE to be well posed is that it is supplemented with an initial condition in time, and a necessary condition for a problem BPDE to be well-posed is that it involves a terminal condition in time.

Next we will present the solutions for some simple equations and problems.

### 8.2.1 The heat equation

The simplest linear parabolic PDE is the heat equation, where  $u(t, x)$  and is formalized by the linear forward parabolic PDE

$$u_t - au_{xx} = 0 \quad (8.1)$$

where  $a > 0$ <sup>2</sup>. It describes the dynamics of the temperature  $u(t, x)$  at point  $x$  and time  $t$  when spatial differences in temperature drive the change in distribution of temperature across time. Consider a homogeneous rod with infinite width and let  $u(t, x)$  be the temperature at point  $x$  at time  $t$ . Consider a small segment of the rod between points  $x$  and  $x + \Delta$ . The average change in the temperature in the segment with width  $\Delta x$  is proportional to difference in the temperature gradients with the neighboring segments at both limits of the segment

$$u_t(t, x)\Delta = a(u_x(t, x + \Delta x) - u_x(t, x))$$

where  $u_x(t, y)$  is the temperature gradient at the boundary  $y$ . As the heat flows from hot to colder regions, the temperature in the segment increases if the temperature in the segment is cooler than in the leftward segment  $u_x(t, x) < 0$  then hotter than in the rightward segment  $u_x(t, x + \Delta x) < 0$  and  $|u_x(t, x) - u_x(t, x + \Delta x)| > 0$ .

If we let the width go to zero,  $\Delta x \rightarrow 0$  then we have equation (8.1).

There are several methods to solve this equation depending upon the domain of the variable  $x$  and on the nature of other side conditions related to time,  $t$  or to the spatial dimension  $x$ . We also consider well-posed forward and backward heat equations.

#### The forward heat equation in an infinite domain

The simplest linear PDE is the heat equation for an infinite domain: the dynamics of  $u(., t)$  is explained by the temperature gradient.

$$u_t - au_{xx} = 0, \quad (t, x) \in (-\infty, \infty) \times [0, \infty)$$

where  $a > 0$ . There are several methods to solve this equation. When the domain of the independent variable  $x$  is  $(-\infty, \infty)$ , the most direct method to find a solution is by using Fourier and inverse Fourier transforms (see Appendix 8.A).

The **Fourier transform** of  $u(t, x)$ , taking  $t$  as a parameter, is<sup>3</sup>

$$U(t, \omega) = \mathcal{F}[u(t, x)](\omega) \equiv \int_{-\infty}^{\infty} u(t, x)e^{-2\pi i \omega x} dx \quad (8.2)$$

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<sup>2</sup>See [http://en.wikipedia.org/wiki/Heat\\_equation](http://en.wikipedia.org/wiki/Heat_equation) for the derivation of the equation.

<sup>3</sup>There are different definitions of Fourier transforms, we use the definition by, v.g., Kammler (2000).

where  $i^2 = -1$  and the inverse Fourier transform is

$$u(t, x) = \mathcal{F}^{-1}[U(t, \omega)](x) \equiv \int_{-\infty}^{\infty} U(t, \omega) e^{2\pi i \omega x} d\omega. \quad (8.3)$$

We will use the following representations of the derivatives of function  $u(t, x)$ : the time-derivative is

$$u_t(t, x) = \int_{-\infty}^{\infty} U_t(t, \omega) e^{2\pi i \omega x} d\omega,$$

and the first and second derivatives as regards the independent variable are

$$u_x(t, x) = \int_{-\infty}^{\infty} 2\pi \omega i U(t, \omega) e^{2\pi i \omega x} d\omega,$$

and

$$u_{xx}(t, x) = \int_{-\infty}^{\infty} (2\pi \omega i)^2 U(t, \omega) e^{2\pi i \omega x} d\omega = - \int_{-\infty}^{\infty} (2\pi \omega)^2 U(t, \omega) e^{2\pi i \omega x} d\omega.$$

Let  $u(t, x)$  be a convolution, i.e.

$$u(t, x) = v(t, x) * y(t, x) \equiv \int_{-\infty}^{\infty} v(t, \xi) y(t, x - \xi) d\xi,$$

then

$$U(t, \omega) = \mathcal{F}[u(t, x)](\omega) = V(t, \omega) Y(t, \omega)$$

where  $V(t, \omega) = \mathcal{F}[v(t, x)](\omega)$  and  $Y(t, \omega) = \mathcal{F}[y(t, x)](\omega)$ , and

$$u(t, x) = \mathcal{F}^{-1}[U(t, \omega)](x) = \mathcal{F}^{-1}[V(t, \omega) Y(t, \omega)](x) = v(t, x) * y(t, x).$$

Using those properties of the Fourier Transforms, then the PDE (8.1) can be written as

$$0 = u_t - au_{xx} = \int_{-\infty}^{\infty} (U_t(t, \omega) + a(2\pi \omega)^2 U(t, \omega)) e^{2\pi i \omega x} d\omega,$$

which is verified if and only if the linear ODE over  $U(t, \omega)$  holds

$$U_t(t, \omega) = -(2\pi \omega)^2 a U(t, \omega).$$

The solution for this ODE is

$$U(t, \omega) = K(\omega) G(t, \omega)$$

where  $G(\cdot)$  is called the **Gaussian kernel**

$$G(\omega, t) \equiv \begin{cases} 1, & t = 0 \\ e^{-(2\pi \omega)^2 at}, & t > 0 \end{cases}$$

and the function  $K(\omega)$  is arbitrary. In order to obtain the solution in terms of the original function,  $u(t, x)$ , we perform an inverse Fourier transform again

$$u(t, x) = \mathcal{F}^{-1}(U(t, \omega)) = \mathcal{F}^{-1}(K(\omega)G(t, \omega)) = k(x) * g(t, x)$$

where  $k(x) * g(t, x)$  is a convolution, that is

$$k(x) * g(t, x) = \int_{-\infty}^{\infty} k(\xi)g(x - \xi, t)d\xi.$$

Using the tables in the Appendix, the Gaussian kernel in the initial variable is

$$g(t, x) = \begin{cases} \delta(x), & t = 0 \\ (4\pi at)^{-1/2} e^{-\frac{x^2}{4at}}, & t > 0 \end{cases}$$

where  $\delta(\cdot)$  is the Dirac's delta function.

Therefore

$$u(t, x) = \begin{cases} k(x), & t = 0 \\ \frac{1}{2\sqrt{\pi at}} \int_{-\infty}^{\infty} k(\xi)e^{-\frac{(x-\xi)^2}{4at}} d\xi, & t > 0 \end{cases} \quad (8.4)$$

where  $k(x)$  is an arbitrary but bounded function, i.e. verifying  $\int_{-\infty}^{\infty} |k(x)|dx < \infty$ . We applied a property of the Dirac's delta function, namely  $\int_{-\infty}^{\infty} k(\xi)\delta(x - \xi)d\xi = k(x)$ .

### The forward heat equation in a semi-infinite domain

Now consider the equation

$$u_t - au_{xx} = 0, \quad (t, x) \in [0, \infty) \times [0, \infty) \quad (8.5)$$

where  $a > 0$ . We solve this equation by using the **method of images**. It consists in introducing the following extension to the arbitrary function  $k(x)$

$$\tilde{k}(x) = \begin{cases} k(x), & \text{if } k \geq 0 \\ -k(-x) & \text{if } k < 0 \end{cases}$$

where  $k(\cdot)$  is an odd function verifying  $k(-x) = -k(x)$ . Using the solution (8.4) for  $t > 0$  we have

$$\begin{aligned} u(t, x) &= \frac{1}{2\sqrt{\pi at}} \int_{-\infty}^{\infty} \tilde{k}(\xi)e^{-\frac{(x-\xi)^2}{4at}} d\xi = \\ &= \frac{1}{2\sqrt{\pi at}} \left( \int_{-\infty}^0 \tilde{k}(\xi)e^{-\frac{(x-\xi)^2}{4at}} d\xi + \int_0^{\infty} \tilde{k}(\xi)e^{-\frac{(x-\xi)^2}{4at}} d\xi \right) = \\ &= \frac{1}{2\sqrt{\pi at}} \left( - \int_0^{\infty} k(\xi)e^{-\frac{(x+\xi)^2}{4at}} d\xi + \int_0^{\infty} k(\xi)e^{-\frac{(x-\xi)^2}{4at}} d\xi \right) \end{aligned}$$

where the last step involves integration by substitution: i.e., if we define  $u = -x$  for  $x \in [0, \infty)$  then  $\int_{-\infty}^0 f(u)du = -\int_0^\infty f(-x)dx = \int_0^\infty f(-x)dx$ . Then the solution of equation (8.5) is

$$u(t, x) = \frac{1}{2\sqrt{\pi at}} \int_0^\infty k(\xi) \left( e^{-\frac{(x-\xi)^2}{4at}} - e^{-\frac{(x+\xi)^2}{4at}} \right) d\xi, \quad t > 0.$$

**Initial value problem** Now we consider a well-posed linear FPDE. Assume we know the distribution at time  $t = 0$ , then we have an **initial value problem**

$$\begin{cases} u_t = au_{xx}, & (t, x) \in (-\infty, \infty) \times (0, \infty) \\ u(0, x) = \phi(x) & (t, x) \in (-\infty, \infty) \times \{t = 0\} \end{cases} \quad (8.6)$$

where  $a > 0$ . Applying (8.4), the solution is

$$u(t, x) = \begin{cases} \phi(x), & t = 0 \\ \int_{-\infty}^\infty \phi(\xi) (4\pi at)^{-1/2} e^{-\frac{(x-\xi)^2}{4at}} d\xi, & t > 0 \end{cases}$$

because  $\int_{-\infty}^\infty \phi(\xi) \delta(x - \xi) d\xi = \phi(x)$ . Figure 8.1 illustrates the behavior of the solution for  $a = 1$  and  $\phi(x) = e^{-x^2}$ , which is simplified to

$$u(t, x) = \frac{1}{\sqrt{4a(t+1)}} e^{-\frac{x^2}{4a(t+1)}}.$$

As can be seen, the solution decays through time and converges to a homogeneous distribution

$$\lim_{t \rightarrow \infty} u(t, x) = 0, \quad \forall x \in (-\infty, \infty)$$

**Piecewise-constant initial condition** We consider the heat equation with the initial condition

$$\phi(x) = \begin{cases} \phi_0, & \text{if } x \in [\underline{x}, \bar{x}] \\ 0 & \text{if } x \notin [\underline{x}, \bar{x}] \end{cases}$$

where  $\underline{x} < \bar{x}$  are both finite. In this case, the solution to the problem is

$$u(t, x) = \phi_0 \left[ \Phi \left( \frac{x - \underline{x}}{\sqrt{2at}} \right) - \Phi \left( \frac{x - \bar{x}}{\sqrt{2at}} \right) \right] \quad (8.7)$$

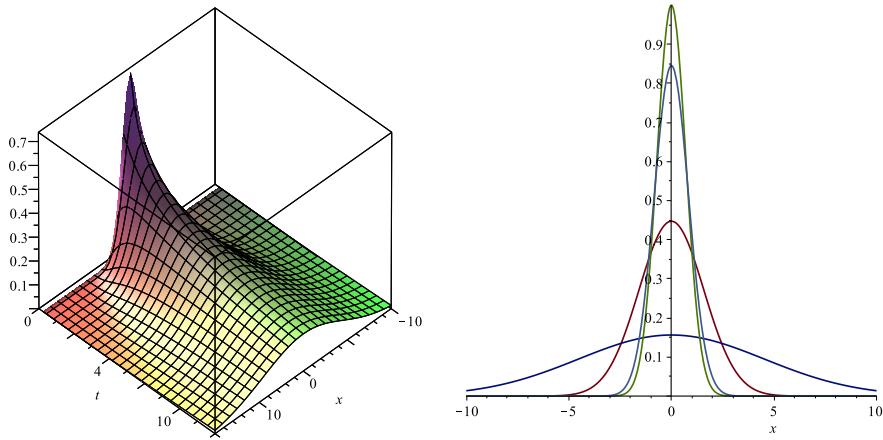


Figure 8.1: Solution for the initial value problem for the heat equation with  $a = 1$  and  $\phi(x) = e^{-x^2}$ .

where  $\Phi(z)$  is the standard normal distribution function

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{z^2}{2}} dz \in (0, 1).$$

Observe that  $\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = (2\pi)^{\frac{1}{2}}$ .

The solution of equation (8.7) is illustrated in Figure 8.4.

In order to prove this result, applying the general solution in equation (8.4) yields the solution of the initial-value problem

$$u(t, x) = \frac{\phi_0}{2\sqrt{\pi at}} \int_{\underline{x}}^{\bar{x}} e^{-\frac{(x-\xi)^2}{4at}} d\xi.$$

To simplify the expression, we make the transformation of variables  $z \equiv (x - \xi)/\sqrt{2at}$ , and denote  $\bar{z} \equiv (\bar{x} - \xi)/\sqrt{2at}$  and  $\underline{z} \equiv (\underline{x} - \xi)/\sqrt{2at}$ . Then, the solution simplifies <sup>4</sup>

$$\begin{aligned} \frac{1}{\sqrt{4\pi at}} \int_{\underline{x}}^{\bar{x}} e^{-(x-\xi)^2/4at} d\xi &= -\frac{\sqrt{2at}}{\sqrt{4\pi at}} \int_{(\underline{x}-\xi)/\sqrt{2at}}^{(\bar{x}-\xi)/\sqrt{2at}} e^{-z^2/2} dz \\ &= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{(\bar{x}-\underline{x})/\sqrt{2at}} e^{-z^2/2} dz - \int_{-\infty}^{(\underline{x}-\bar{x})/\sqrt{2at}} e^{-z^2/2} dz \right) = \\ &= \phi_0 \Phi\left(\frac{x-\underline{x}}{\sqrt{2at}}\right) - \Phi\left(\frac{x-\bar{x}}{\sqrt{2at}}\right). \end{aligned}$$

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<sup>4</sup>Recall that, if we set  $z = \varphi(\xi)$  and  $\xi \in (a, b)$  then

$$\int_{\varphi(a)}^{\varphi(b)} f(z) dz = \int_a^b f(\varphi(\xi)) d\xi.$$

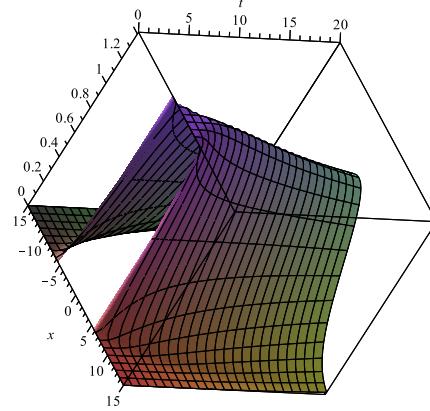


Figure 8.2: Solution for the initial value problem for the heat equation with  $a = 1$  and piecewise initial condition.

### 8.2.2 The simplest backward parabolic PDE

Now consider the backward equation

$$u_t + au_{xx} = 0, \quad (t, x) \in (-\infty, \infty) \times [0, T]$$

where  $a > 0$  defined for an infinite domain for  $x$  and a finite domain for  $t$ .

In order to solve it we introduce a change in variables  $\tau = T - t$  and consider a change in the variable  $v(\tau, x) = u(t(\tau), x)$  where  $t(\tau) = T - \tau$ . As

$$v_\tau(\tau, x) = -u_t(t(\tau), x), \text{ and } v_{xx}(\tau, x) = u_{xx}(t(\tau), x)$$

Then  $u_t(t, x) = -au_{xx}(t, x)$  is equivalent to

$$v_\tau(\tau, x) = av_{xx}(\tau, x).$$

Using the solution already found in equation (8.4) we get

$$v(\tau, x) = \begin{cases} k(x), & \tau = 0 \\ \int_{-\infty}^{\infty} k(\xi) (4\pi a \tau)^{-1/2} e^{-\frac{(x-\xi)^2}{4a\tau}} d\xi, & \tau \in (0, T). \end{cases}$$

Therefore, transforming back to  $u(t, x)$  we have

$$u(t, x) = \begin{cases} k(x), & t = T \\ \frac{1}{\sqrt{4\pi a(T-t)}} \int_{-\infty}^{\infty} k(\xi) e^{-\frac{(x-\xi)^2}{4a(T-t)}} d\xi, & t \in (0, T) \end{cases}$$

A problem involving a backward PDE is only well posed if together with the PDE we have a terminal condition, for example  $u(T, x) = \phi_T(x)$ . In this case the value of the variable at time  $t = 0$  becomes endogenous.

Consider the **terminal-value problem**

$$\begin{cases} u_t = -au_{xx}, & (t, x) \in (-\infty, \infty) \times (0, T] \\ u(T, x) = \phi_T(x) & (t, x) \in (-\infty, \infty) \times \{t = T\}. \end{cases}$$

The solution is

$$u(t, x) = \begin{cases} \phi_T(x), & t = T \\ \frac{1}{\sqrt{4\pi a(T-t)}} \int_{-\infty}^{\infty} \phi_T(\xi) e^{-\frac{(x-\xi)^2}{4a(T-t)}} d\xi, & t \in (0, T) \end{cases}$$

The initial distribution can be obtained by setting  $t = 0$

$$u(0, T) = \frac{1}{\sqrt{4\pi a(T-t)}} \int_{-\infty}^{\infty} \phi_T(\xi) e^{-\frac{(x-\xi)^2}{4aT}} d\xi.$$

### 8.2.3 The homogeneous diffusion equation

Now we consider the (forward) diffusion equation

$$u_t = au_{xx} + bu, \quad (t, x) \in (-\infty, \infty) \times (0, \infty)$$

where  $a > 0$  and  $b \neq 0$ . In order to solve the equation, we can follow one of two alternative methods:

1. apply the Fourier transform method to transform the PDE into a parameterized ODE, solve it, and apply inverse Fourier transforms;
2. , or, transform the equation into a heat equation, solve the heat equation and transform back to the initial variables.

We follow next the second method.

Define  $v(t, x) = e^{-bt}u(t, x)$ . Then  $v_t = -be^{-bt}u + e^{-bt}u_t$  and  $v_{xx} = e^{-bt}u_{xx}$ . Therefore we get the heat equation in the transform function

$$v_t = av_{xx}.$$

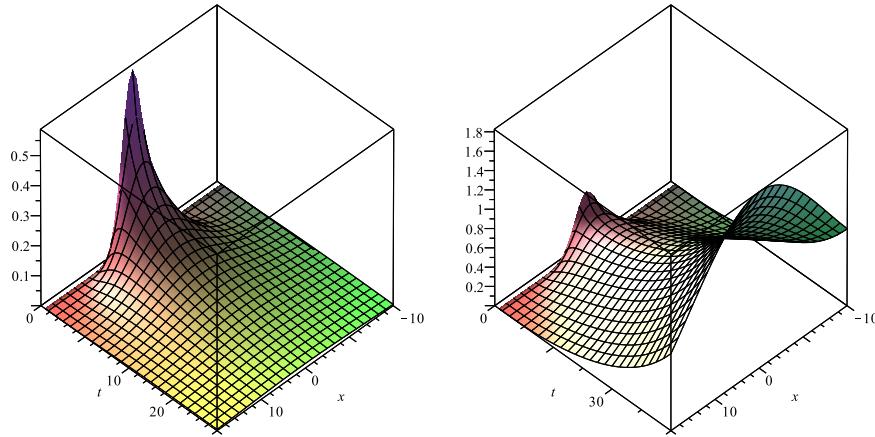


Figure 8.3: Solution for the initial value problem for the heat equation with  $a = 1$  and  $\phi(x) = e^{-x^2}$  and  $b = -0.1$  and  $b = 0.1$ .

The solution for the initial value problem, where  $u(0, x) = \phi(x)$ , is

$$u(t, x) = \begin{cases} \int_{-\infty}^{\infty} \phi(\xi) \delta(x - \xi) d\xi = \phi(x), & t = 0 \\ \frac{e^{bt}}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} \phi(\xi) e^{-\frac{(x-\xi)^2}{4at}} d\xi, & t > 0. \end{cases}$$

The dynamics of the solution depends crucially on the sign of  $b$ :

$$\lim_{t \rightarrow \infty} u(t, x) = \begin{cases} 0 & \text{if } b < 0 \\ \infty & \text{if } b > 0 \end{cases}$$

Figure 8.3 illustrates the cases in which  $b < 0$  and  $b > 0$ . In both cases we see that the long-time behavior of the solution is commanded by  $e^{bt}$ : if  $b > 0$  then  $\lim_{t \rightarrow \infty} u(t, x) = 0$  and if  $b > 0$  then  $\lim_{t \rightarrow \infty} u(t, x) \propto \lim_{t \rightarrow \infty} e^{bt} = \infty$ .

#### 8.2.4 The general homogeneous diffusion equation

Now we consider the (forward) diffusion equation

$$\begin{cases} u_t(t, x) = au_{xx}(t, x) + bu_x + cu(t, x), & (t, x) \in (0, \infty) \times (-\infty, \infty) \\ u(0, x) = \phi(x), & (t, x) \in \{t = 0\} \times (-\infty, \infty), \end{cases} \quad (8.8)$$

where  $a > 0$ ,  $b \neq 0$  and  $c \neq 0$ .

We will solve this problem using the Fourier transform representation of equation  $u_t - (au_{xx} + bu_x + cu) = 0$ . Using inverse Fourier transforms yields

$$u_t(t, x) - au_{xx}(t, x) - bu_x(t, x) - cu(t, x) = \int_{-\infty}^{\infty} e^{2\pi i \omega x} [U_t(t, \omega) + \lambda(\omega)U(t, \omega)] d\omega = 0.$$

where the coefficient is a complex-valued function of  $\omega$ <sup>5</sup>

$$\lambda(\omega) \equiv a(2\pi\omega)^2 - 2\pi ib\omega - c.$$

Therefore, the PDE (8.8) is equivalent to the linear ODE parameterized by  $\omega$

$$U_t(t, \omega) = -\lambda(\omega)U(t, \omega), \quad (t, \omega) \in \mathbb{R}_+ \times \mathbb{R},$$

which has the explicit solution

$$U(t, \omega) = \Phi(\omega)G(t, \omega), \quad \text{for } t \in [0, \infty)$$

where  $\Phi(\omega) = \mathcal{F}[\phi(x)](\omega)$  is the Fourier transform of the initial distribution, and  $G(t, \omega)$  is the Gaussian kernel

$$G(t, \omega) = e^{-\lambda(\omega)t}, \quad \text{for } t > 0.$$

We obtain the solution of problem (8.8) by applying the inverse Fourier transform

$$\begin{aligned} u(t, x) &= \mathcal{F}^{-1}[U(t, \omega)](x) = \\ &= \mathcal{F}^{-1}[\Phi(\omega)G(t, \omega)](x) = \\ &= \int_{-\infty}^{\infty} \phi(s)g(t, x - s)ds \end{aligned}$$

where (see the Appendix 8.A )

$$g(t, y) = \mathcal{F}^{-1}\left[e^{-\lambda(\omega)t}\right] = \frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{y^2 + 2bty + (b^2 - 4ac)t^2}{4at}\right), \quad (8.9)$$

because  $at > 0$ .

### 8.2.5 Non-autonomous linear equation

Next we consider two non-autonomous equations in which there is one term depending on the independent variables  $(t, x)$

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<sup>5</sup>The advection term, involving the first derivative has a complex-valued the Fourier transform representation

$$u_x(t, x) = \frac{\partial}{\partial x} \left( \int_{-\infty}^{\infty} U(t, \omega) e^{2\pi i x \omega} d\omega \right) = \int_{-\infty}^{\infty} 2\pi \omega i U(t, \omega) e^{2\pi i x \omega} d\omega.$$

**Non-homogeneous heat equation** The non-homogeneous (forward) heat equation

$$u_t - au_{xx} - b(t, x) = 0, \quad (t, x) \in (-\infty, \infty) \times (0, \infty) \quad (8.10)$$

this equation has a component which is not affected by  $u$ , although it changes with  $(t, x)$ .

In order to solve it, we again use inverse Fourier transforms to get an equivalent ODE in transformed variables  $U(t, \omega)$ ,

$$U(t, \omega) = -\lambda(\omega)U(t, \omega) + B(t, \omega)$$

where  $B(t, \omega) = \mathcal{F}[b(t, x)](\omega)$ . The solution to equation (8.10) is

$$U(t, \omega) = K(\omega)G(t, \omega) + \int_0^t B(s, \omega) G(t-s, \omega) ds.$$

Applying inverse Fourier transforms yields

$$u(t, x) = k(x) * g(t, x) + \int_0^t b(s, x) * g(t-s, x) ds.$$

Therefore, the solution to the parabolic PDE (8.10) is, for  $t > 0$ ,

$$u(t, x) = \frac{1}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} k(\xi) e^{-\frac{(x-\xi)^2}{4at}} d\xi + \int_0^t \frac{1}{\sqrt{4\pi a(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4a(t-s)}} b(s, \xi) d\xi ds.$$

The solution can converge to a spatially non-homogenous distribution.

### Non-autonomous diffusion equation

Consider the equation

$$u_t = au_{xx} + bu + d(x), \quad (t, x) \in (-\infty, \infty) \times (0, \infty)$$

where  $a > 0$ . It can be proved (see Exercise 1) that the solution for  $t > 0$  is

$$u(t, x) = \frac{e^{bt}}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} \phi(\xi) e^{-\frac{(x-\xi)^2}{4at}} d\xi + \frac{1}{\sqrt{4\pi a(t-\tau)}} \int_0^t e^{b(t-\tau)} \int_{-\infty}^{\infty} d(\xi) e^{-\frac{(x-\xi)^2}{4a(t-\tau)}} d\xi d\tau$$

## 8.3 Economic applications

### 8.3.1 The distributional Solow model

In Brito (2004) we prove that in an economy in which the capital stock is distributed in an heterogeneous way between regions,  $K(t, x)$ , if there is an infinite support, and there are free capital flows between regions, the budget constraint for the location  $x$  can be represented by the parabolic PDE.

Consider the accounting balance between savings and internal and external investment for a region  $x$  at time  $t$

$$I(t, x) + T(t, x) = S(t, x)$$

where  $I(t, x)$  and  $S(t, x)$  is investment and domestic savings of location  $x$  at time  $t$  and  $T(t, x)$  is the savings flowing to other regions.

Assume that the capital flow for a region of length  $\Delta x$  is symmetric to the capital distribution gradient to neighboring regions:

$$T(t, x)\Delta x = -(K_x(x + \Delta x, t) - K_x(t, x))$$

that is capital flows proportionaly and in a reverse direction to the "spatial gradient" of the capital distribution: regions with high capital intensity will tend to "leak" capital to other regions. If  $\Delta x \rightarrow 0$  leads to  $T(t, x) = -K_{xx}(t, x)$ .

If there is no depreciation then  $I(t, x) = K_t(t, x)$ . If the technology is  $AK$  and the savings rate is determined as in the Solow model then  $S(t, x) = sAK(t, x)$  where  $0 < s < 1$  and  $A$  is assume to be spatially homogeneous.

Therefore we obtain a distributional Solow equation for an economy composed by heterogenous regions

$$K_t = K_{xx} + sAK, (t, x) \in (-\infty, \infty) \times (0, \infty)$$

We can define a spatially-homogenous balanced growth path (BGP) as

$$\bar{K}(t) = \bar{K}e^{\gamma t}$$

where  $\gamma = sA$ .

Then, if we define the deviations as regards the BGP as  $k(t, x) = K(t, x)e^{-\gamma t}$ , we observe that the transitional dynamics is given by the solution of the equation

$$k_t = k_{xx}$$

which is the heat equation. Therefore, given the initial distribution of the capital stock  $K(x, 0) = k_0(x)$  the solution for this spatial  $AK$  model is

$$K(t, x) = \begin{cases} k_0(x), & t = 0 \\ e^{\gamma t} \int_{-\infty}^{\infty} k_0(\xi) (4\pi t)^{-1/2} e^{-\frac{(x-\xi)^2}{4t}} d\xi, & t > 0 \end{cases}$$

and the solution is similar to the case depicted in Figure 8.4 when  $b > 0$ .

The main conclusion is that: (1) there is long run growth; (2), if there are homogenous technologies and preferences the asymptotic distribution will become spatially homogeneous. That is: the so-called  $\beta$ - and  $\sigma$ - convergences can be made consistent !

### 8.3.2 The option pricing model

The Black and Scholes (1973) model is a case in which a research paper had an immense impact on the operation of the economy. It is related to the onset of derivative markets and basically gave birth to stochastic finance<sup>6</sup>.

It provides a formula (the so called Black-Scholes formula) for the value of an European call option when there are two assets, a riskless asset with interest rate  $r$  and a underlying asset whose price,  $S$  which follows a diffusion process (in a stochastic sense):  $dS = \mu S dt + \sigma S dB$  where  $dB$  is the standard Brownian motion (see next chapter). An European call option offers the right to buy the underlying asset at time  $T$  for a price  $K$  fixed at time  $t = 0$ , which is conventioned to be the moment of the contract.

Under the assumption that there are no arbitrage opportunities Black and Scholes (1973) proved that the price of the option  $V = V(t, S)$  is a function of time,  $t \in (0, T)$  and the price of an underlying asset  $S \in (0, \infty)$  follows the backward parabolic PDE and has a terminal constraint

$$\begin{cases} V_t(t, S) = -\frac{\sigma^2 S^2}{2} V_{SS}(t, S) - rSV_S(t, S) + rV(t, S), & (t, S) \in [0, T] \times (0, \infty) \\ V(T, S) = \max\{S - K, 0\}, & (t, S) \in \{t = T\} \times (0, \infty). \end{cases} \quad (8.11)$$

The first equation is valid for any financial option having the same underlying asset dynamics, and the terminal constraint is characteristic of the European call option: because the writer sells the right, but not the obligation, to purchase the underlying asset at the price  $K$  at time  $t = T$ , the buyer is only interested in that purchase if he can sell it at the prevailing market price  $S = S(T)$  when that price is higher than the exercise price  $K$ . In this case the payoff will be  $S(T) - K$ . Otherwise he will not execute the option and the terminal payoff will be zero.

The two boundary constraints

$$\begin{aligned} V(t, 0) &= 0, & (t, S) \in [0, T] \times \{S = 0\} \\ \lim_{S \rightarrow \infty} V(t, S) &= S, & (t, S) \in [0, T] \times \{S \rightarrow \infty\}, \end{aligned}$$

are sometimes referred to, but they are redundant.

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<sup>6</sup>Myron Scholes was awarded the Nobel prize in 1997, together with Robert Merton another important contributer to stochastic finance, precisely for this formula. Fisher Black was deceased at the time.

The same structure occurs in the Merton's model (see Merton (1974)) which is a seminal paper on the pricing of default bonds. It was the first model on the so-called structural approach to modelling credit risk which is on the foundation of the credit risk models used by rating agencies (see Duffie and Singleton (2003)). In essence, this model assumes that the value of the firm follows a linear diffusion process and it considers the issuance of a bond with an expiring date  $T$  whose indenture gives it absolute priority on the value of the firm at the expiry date. This means that either if the value of the firm is smaller than the face value of the bond the creditor takes possession of the firm and in the opposite case it recovers the face value. In this case, we can interpret the position of the equity owner as holding an European call option over the value of the firm with strike price equal to the face value of the debt and the creditor as having an European put option security.

The price of the European call option <sup>7</sup>, given the former assumptions is given by

$$V(t, S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \quad t \in [0, T] \quad (8.12)$$

where  $\Phi(\cdot)$  is cumulative Gaussian density function such that  $\Phi(d) = \mathbb{P}(x \leq d)$  where

$$d_1 = \frac{\ln(S/K) + (T-t)\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \quad (8.13)$$

$$d_2 = \frac{\ln(S/K) + (T-t)\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \quad (8.14)$$

*Proof.* In order to solve the B-S PDE, which is a non-linear backward parabolic PDE, we transform it to a quasi-linear parabolic forward PDE, by applying the transformations:  $t(\tau) = T - \tau$  and  $S = Ke^x$  and setting  $u(\tau, x) = V(t(\tau), S(x))$ . We can transform the option-pricing problem to the equivalent initial-value problem PDE equivalent to (8.11)

$$\begin{cases} u_\tau = \frac{\sigma^2}{2}u_{xx} + \left(r - \frac{\sigma^2}{2}\right)u_x - ru, & (\tau, x) \in [0, T] \times (-\infty, \infty) \\ u(0, x) = u_0(x) \end{cases} \quad (8.15)$$

where

$$u_0(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ K(e^x - 1), & \text{if } x > 0 \end{cases}$$

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<sup>7</sup>For the credit risk model  $S$  would be the value of assets of a firm,  $K$  would be the face value of loan, and  $T$  the term of the loan.

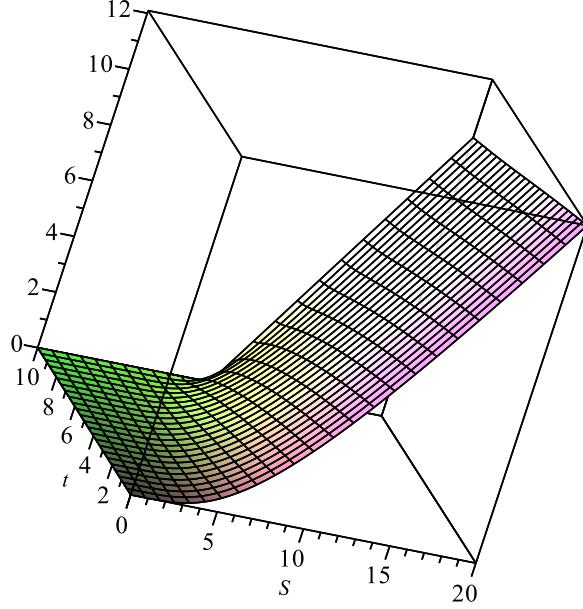


Figure 8.4: Solution for the Black and Scholes model, for  $r = 0.02$ ,  $T = 20$ ,  $\sigma = 0.2$ , and  $K = 10$ .

The PDE is a particular example of equation (8.8), which implies that the solution is

$$\begin{aligned} u(\tau, x) &= \int_{-\infty}^0 0 g(\tau, x - s) ds + K \int_0^\infty (e^s - 1) g(\tau, x - s) ds \\ &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \int_0^\infty (e^s - 1) e^{h(\tau,x-s)} ds \end{aligned}$$

where (from equation (8.9))

$$h(\tau, y) \equiv -\frac{y^2 + 2\tau \left(r - \frac{\sigma^2}{2}\right)y + \left(r + \frac{\sigma^2}{2}\right)^2 \tau^2}{2\tau\sigma^2}.$$

Then

$$\begin{aligned} u(\tau, x) &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \left( \int_0^\infty e^{s+h(\tau,x-s)} ds - \int_0^\infty e^{h(\tau,x-s)} ds \right) \\ &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} (I_1 - I_2). \end{aligned}$$

In order to simplify the integrals it is useful to remember the forms of the error function,  $\text{erf}(x)$ , and of the Gaussian cumulative distribution  $\Phi(x)$ ,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^x e^{-z^2} dz, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz$$

which are related as

$$\Phi(x) = \frac{1}{2} \left[ 1 + \operatorname{erf} \left( \frac{x}{\sqrt{2}} \right) \right].$$

After some algebra we obtain

$$\begin{aligned} s + h(\tau, x - s) &= x - \frac{1}{2}(\delta_1(s))^2 \\ h(\tau, x - s) &= -r\tau - \frac{1}{2}(\delta_2(s))^2 \end{aligned}$$

where

$$\delta_1(s) \equiv \frac{x - s + \left( r + \frac{\sigma^2}{2} \right)}{\sigma\sqrt{\tau}}, \text{ and } \delta_2(s) \equiv \frac{x - s + \left( r - \frac{\sigma^2}{2} \right)}{\sigma\sqrt{\tau}}.$$

Then <sup>8</sup>

$$\begin{aligned} I_1 &= e^x \int_0^\infty e^{-\frac{1}{2}(\delta_1(s))^2} ds = \\ &= -\sigma\sqrt{\tau}e^x \int_{d_1}^{-\infty} e^{-\frac{1}{2}\delta_1^2} d\delta_1 = \\ &= \sqrt{\sigma^2\tau}e^x \int_{-\infty}^{d_1} e^{-\frac{1}{2}\delta_1^2} d\delta_1 = \\ &= \sqrt{2\pi\sigma^2\tau}e^x \Phi(d_1) \end{aligned}$$

where  $d_1 = \delta_1(0)$  as in equation (8.13) for  $\tau = T - t$ , and also, writing that  $d_2 = \delta_2(0)$ , as in equation (8.14) for  $\tau = T - t$ ,

$$\begin{aligned} I_2 &= e^{-r\tau} \int_0^\infty e^{-\frac{1}{2}(\delta_2(s))^2} ds = \\ &= -\sigma\sqrt{\tau}e^{-r\tau} \int_{d_2}^{-\infty} e^{-\frac{1}{2}\delta_2^2} d\delta_2 = \\ &= \sqrt{\sigma^2\tau}e^{-r\tau} \int_{-\infty}^{d_2} e^{-\frac{1}{2}\delta_2^2} d\delta_2 = \\ &= \sqrt{2\pi\sigma^2\tau}e^{-r\tau} \Phi(d_2) \end{aligned}$$

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<sup>8</sup>We use integration by transformation of variables: if we define  $z = \varphi(s)$  where  $\varphi : [a, b] \rightarrow \mathcal{I}$  and  $f : \mathcal{I} \rightarrow \mathbb{R}$  we have that

$$\int_{\varphi(a)}^{\varphi(b)} f(z) dz = \int_a^b f(\varphi(s)) \varphi'(s) ds.$$

Thus

$$u(\tau, x) = K (e^x \Phi(d_1) - e^{-r\tau} \Phi(d_2))$$

and transforming back  $V(t, S) = u(T - t, \ln(S/K))$  we get equation (8.12).  $\square$

Observe this is a backward parabolic PDE, which implies that the terminal condition determines the particular solution.

## 8.4 Bibiography

- Mathematics of PDE's: introductory Olver (2014), Salsa (2016) and (Pinsky, 2003, ch 5). Advanced (Evans, 2010, ch 3).
- Applications to economics (with more advanced material) : Achdou et al. (2014)
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- Applications to finance: asset pricing Björk (2004) and Cvitanić and Zapatero (2004) , credit risk Bielecki and Rutkowski (2004). Advanced Pham (2009).

## 8.A Appendix: Fourier transforms

Consider a function  $u(x)$  such that  $x \in \mathbb{R}$  and  $\int_{-\infty}^{\infty} |u(x)|dx < \infty$ . We can define a pair of generalized functions, the Fourier transform of  $u(x)$ ,  $U(\omega) = F[u(x)](\omega)$  and the inverse Fourier transform  $F^{-1}[U(\omega)](x) = u(x)$  (using the definition of Kammler (2000) ), where

$$U(\omega) = \int_{-\infty}^{\infty} u(x)e^{-2\pi i \omega x} dx$$

where  $i^2 = -1$  and

$$u(x) = \int_{-\infty}^{\infty} U(\omega)e^{2\pi i \omega x} d\omega.$$

There are some useful properties of the Fourier transform:

- it preserves linearity,

$$F[u(x) + cv(x)] = U(\omega) + cV(\omega)$$

and

$$F^{-1}[U(\omega) + cV(\omega)] = u(x) + cv(x)$$

- if the function  $u(.)$  has a parameter, say  $t$ , then  $u = u(x, t)$  and

$$u_t(x, t) = \int_{-\infty}^{\infty} U_t(\omega)e^{2\pi i \omega x} d\omega$$

- the first and second derivatives can be expressed as the inverse transforms

$$u_x(x) = \int_{-\infty}^{\infty} U(\omega)2\pi i \omega e^{2\pi i \omega x} d\omega$$

and

$$u_{xx}(x) = \int_{-\infty}^{\infty} U(\omega)(2\pi i \omega)^2 e^{2\pi i \omega x} d\omega$$

- a convolution between two functions  $u(x)$  and  $v(x)$  is defined as

$$u(x) * v(x) = \int_{-\infty}^{\infty} u(y)v(x-y)dy.$$

The inverse Fourier transform of a product of two Fourier transforms is a convolution,

$$u(x) * v(x) = \int_{-\infty}^{\infty} U(\omega)V(\omega)e^{2\pi i \omega x} d\omega$$

Table 8.1: Fourier and inverse Fourier transforms

$f(x)$ for $-\infty < x < \infty$	$F(\omega)$ for $-\infty < \omega < \infty$	obs
$k\delta(x)$	$k$	$k$ constant
$k$	$k\delta(\omega)$	$k$ constant
$\frac{1}{\sqrt{4\pi a}}e^{-\frac{x^2}{4a}}$	$e^{-a(2\pi\omega)^2}$	$a > 0$
$\frac{1}{\sqrt{4\pi a}}e^{-\frac{(x+b)^2}{4a}}$	$e^{-a(2\pi\omega)^2+b(2\pi i\omega)}$	$a > 0, b \in \mathbb{R}$
$\frac{1}{\sqrt{4\pi a}}e^{c-\frac{x^2}{4a}}$	$e^{-a(2\pi\omega)^2+c}$	$a > 0, c \in \mathbb{R}$
$\frac{1}{\sqrt{4\pi a}}e^{c-\frac{(x+b)^2}{4a}}$	$e^{-a(2\pi\omega)^2+b(2\pi i\omega)+c}$	$a > 0, (b, c) \in \mathbb{R}^2$
$u(x) * v(x)$	$U(\omega)V(\omega)$	

# Chapter 10

## Optimal control of parabolic partial differential equations

### 10.1 Introduction

The optimal control of parabolic PDE is sometimes called optimal control of distributions.

Similarly to the optimal control of ODE's, the first order conditions include a system of forward-backward parabolic PDE's together with boundary conditions. The requirements for the existence of solutions are clearly very strong, because the general solutions of the PDE system may not allow for the boundary conditions to be satisfied. Ill-posedness is, therefore, an important issue here.

### 10.2 A simple optimal control problem

Next we consider a simple optimal control problem for a system governed by a parabolic PDE.

Because the purpose of this section is introductory, we assume two simple boundary conditions: a initial time condition and two "spatial" (Neumann) conditions such that

$$\max_u = \int_0^\infty \int_{-\infty}^\infty f(t, x, u(t, x), y(t, x)) dx dt \quad (10.1)$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + g(t, x, u(t, x), y(t, x)), & \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ y(0, x) = y_0(x), & \text{for } (t, x) \in \{t = 0\} \times \mathbb{R} \\ \lim_{x \rightarrow} \frac{\partial y(t, x)}{\partial x} = 0 & \text{for } (t, x) \in \mathbb{R}_+ \times \{(x = -\infty), (x = \infty)\} \end{cases} \quad (10.2)$$

We assume that function  $f(\cdot)$  is measurable, in the sense <sup>1</sup>

$$\int_{\mathbb{R}_+ \times \mathbb{R}} |f(t, x)| d(x, t) < \infty$$

We define the Hamiltonian

$$H(t, x, u, y, \lambda) = f(t, x, u, y) + \lambda(t, x)g(t, x, u, y)$$

and call  $\lambda = \lambda(t, x)$  the co-state variable.

**Proposition 1 (Necessary first-order conditions ).** *Let  $(u^*, y^*)$  be a solution to problem (10.1)-(10.2). Then there is a co-state variable  $\lambda$  such that the following necessary conditions hold*

$$\frac{\partial H^*(t, x)}{\partial u} = 0 \quad (10.3)$$

$$\frac{\partial \lambda(t, x)}{\partial t} = -\frac{\partial^2 \lambda(t, x)}{\partial x^2} - \frac{\partial H^*(t, x)}{\partial y} \quad (10.4)$$

$$\lim_{t \rightarrow \infty} \lambda(t, x) = 0 \quad (10.5)$$

$$\lim_{x \rightarrow \pm\infty} \frac{\partial \lambda(t, x)}{\partial x} = 0. \quad (10.6)$$

together with equations (10.2)

See the proof in the appendix. Equation (10.3) is a static optimality condition, that if function  $H(u, \cdot)$  is sufficiently smooth, allows for the determination of the optimal control  $u^*$  as a function of the co-state variable, and the state variable. Equation (10.4) is a Euler-equation. In this case it is a backward parabolic PDE which encodes the incentives for changing the control variable. Equations (10.5) and (10.6) are transversality conditions, which are dual to the boundary conditions in (10.2) related to the asymptotic properties

### 10.3 Optimal average control problem

We consider a optimal control problem for a system governed by an autonomous semi-linear parabolic PDE where the planner wants to maximize the spatial-average of the present-value utility.

We denote the state and control variables by, respectively,  $y(x, t)$  and  $u(x, t)$  and assume  $(x, t) \in \mathbb{R} \times \mathbb{R}_+$  and the constraints are

$$y_t = \sigma^2 y_{xx} + g(y, u)$$

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<sup>1</sup>This condition requires that the function is bounded for every value of  $u(\cdot)$  and  $y(\cdot)$  and allow for the use of Fubini's theorem, i.e, for the interchange of the integration for  $t$  and  $x$ . Intuitively, we should consider functions such that the order of integration does not matter.

where we the initial condition is given and a terminal condition holds

$$y(x, 0) = \phi(x), \quad x \in \mathbb{R} \quad (10.7)$$

$$\lim_{t \rightarrow \infty} R(t)y(x, t) \geq 0, \quad x \in \mathbb{R} \quad (10.8)$$

where  $R(t) \leq R_0 e^{-\rho t}$ , where  $R_0$  is a constant, and a technical condition regarding the spatial distribution is also introduced

$$\lim_{x \rightarrow \pm\infty} \frac{y(x, t)}{x} = 0, \quad t \in \mathbb{R}_+. \quad (10.9)$$

The objective functional is consists in an uniform distribution for a time-discounted utility function

$$J(y, u) = \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x \int_0^\infty f(y(\xi, t), u(\xi, t)) e^{-\rho t} dt d\xi \quad (10.10)$$

where  $\rho > 0$  and  $f(\cdot)$  is continuous and differentiable.

The optimal control problem consists in finding the optimal processes for  $u$  that maximizes the functional (10.10) subject to the constraints (10.7)-(10.9).

The Hamiltonian function

$$H(y, u, q) \equiv f(y, u) + qg(y, u)$$

where  $q(x, t)$  is the current value co-state variable.

The necessary first order conditions, according to the Pontryagin's maximum principle are the following.

**Proposition 2** (Necessary conditions for the optimal average problem). *Assume there are optimal processes for the state and the control variable,  $y^* = (y^*(x, t))_{(x,t) \in \mathbb{R} \times \mathbb{R}_+}$  and  $u^* = (u^*(x, t))_{(x,t) \in \mathbb{R} \times \mathbb{R}_+}$  then there is a (current-value) co-state variable  $q(x, t)$  such that the following conditions hold:*

- the optimality condition

$$H_u(y^*(x, t), u^*(x, t), q(x, t)) = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_{++}$$

- the distributional Euler equation

$$q_t = -\sigma^2 q_{xx} + q(\rho - H_x(y^*(x, t), u^*(x, t), q(x, t))), \quad (x, t) \in \mathbb{R} \times \mathbb{R}_{++}$$

- the boundary condition, dual to equation (10.9)

$$\lim_{x \rightarrow \pm\infty} e^{-\rho t} \frac{q(x, t)}{x} = 0$$

- the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lim_{x \rightarrow \infty} \int_{-x}^x q(\xi, t) y(\xi, t) d\xi = 0$$

## 10.4 Application: the distributional AK model

As application consider a simple model in which there is a central planner in a dynamic economy who wants to maximize the average (un-weighted) utility of an economy composed with heterogeneous agents. Assume that the heterogeneity is only given by their initial asset position. Each agent produces a different quantity of a good, depending only on their endowment of capital and the central planner assigns consumption which varies between consumers and can be different from their production (given the capital endowment). Therefore there is a distribution of savings in the economy allowing some agents to use more (less) capital than they have at the beginning of every (infinitesimal) period.

In Brito (2011) we present this model with more detail.

The problem is to maximize the average intertemporal discounted utility of consumption,  $C$ ,

$$J(C, K) \equiv \max_{[C]} \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x \int_0^\infty \frac{C(\xi, t)^{1-\theta}}{1-\theta} e^{-\rho t} dt d\xi. \quad (10.11)$$

subject to the distributional capital accumulation equation

$$\frac{\partial K(x, t)}{\partial t} = \tau^2 \frac{\partial^2 K(x, t)}{\partial x^2} + AK(x, t) - C(x, t) \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}_+ \quad (10.12)$$

and the terminal, the boundary and the initial conditions

$$\lim_{t \rightarrow \infty} e^{-\int_0^t r(x, s) ds} K(x, t) \geq 0, \quad \forall x \in \mathbb{R} \quad (10.13)$$

$$\lim_{x \rightarrow \mp\infty} \frac{K(x, t)}{x} = 0, \quad \forall t \in \mathbb{R}_+ \quad (10.14)$$

$$K(x, 0) = \phi(x), \quad \forall x \in \mathbb{R} \text{ given.} \quad (10.15)$$

$$\frac{\partial C}{\partial t} = -\tau^2 \left[ \frac{\partial^2 C}{\partial x^2} - \frac{1+\theta}{C} \left( \frac{\partial C}{\partial x} \right)^2 \right] + \gamma C, \quad x \in \mathbb{R}, \quad t > 0 \quad (10.16)$$

$$\frac{\partial K}{\partial t} = \tau^2 \frac{\partial^2 K}{\partial x^2} + AK - C, \quad x \in \mathbb{R}, \quad t > 0 \quad (10.17)$$

(10.18)

where the endogenous growth rate is

$$\gamma \equiv \frac{A - \rho}{\theta}, \quad (10.19)$$

the transversality condition

$$\lim_{t \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x e^{-\rho t} K(\xi, t) C(\xi, t)^{-\theta} d\xi = 0, \quad (10.20)$$

and the dual boundary conditions

$$\lim_{x \rightarrow \pm\infty} \left( e^{\rho t} C(x, t)^\theta x \right)^{-1} = 0, \quad t \geq 0. \quad (10.21)$$

In system (10.16)-(10.17),  $r$  is the net total factor productivity and  $\gamma$  is equal to the endogenous growth rate in the benchmark homogeneous  $AK$  model. The initial condition  $K(x, 0) = \phi(x)$  and the boundary condition (10.14) should also hold.

The coupled system (10.16)-(10.17) has the closed form solution

$$K(x, t) = e^{\gamma t} k(x, t), \quad C(x, t) = e^{\gamma t} c(x, t), \quad t \geq 0, \quad x \in \mathbb{R} \quad (10.22)$$

where

$$k(x, t) = \frac{1}{2\tau\sqrt{\pi\theta t}} \int_{-\infty}^{\infty} \phi(\xi) e^{-\left(\frac{x-\xi}{2\tau}\right)^2 \frac{1}{\theta t}} d\xi, \quad t > 0, \quad x \in \mathbb{R}$$

and

$$c(x, t) = \frac{1}{2\tau\sqrt{\pi\theta t}} \int_{-\infty}^{\infty} \phi(\xi) \left[ r - \gamma + (\theta - 1) \left( \frac{1}{2\theta t} - \left( \frac{x - \xi}{2\tau\theta t} \right)^2 \right) \right] e^{-\left(\frac{x-\xi}{2\tau}\right)^2 \frac{1}{\theta t}} d\xi, \quad t > 0, \quad x \in \mathbb{R}.$$

A necessary condition for the existence of a solution of the centralized problem is that  $A > \gamma$ . The model displays convergence to a time-unbounded balanced growth path similar to the homogeneous-agent  $AK$  model. Figure ?? presents a graphic depiction of the detrended-solution for  $\phi(x) = e^{|x|}$ . We observe that the initial heterogeneity is eliminated along the transition

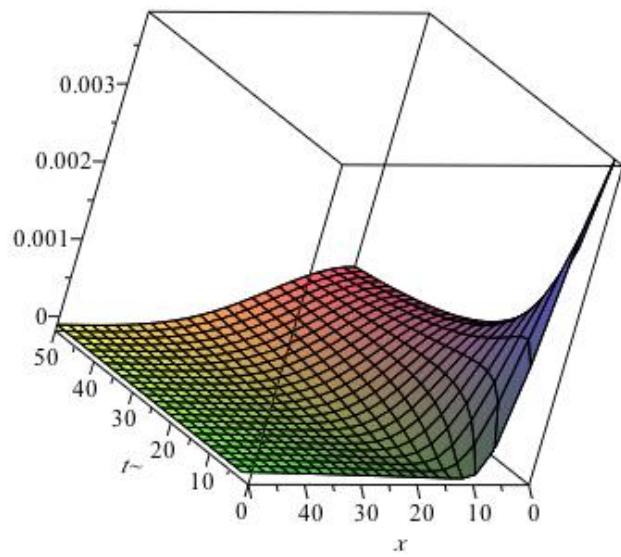
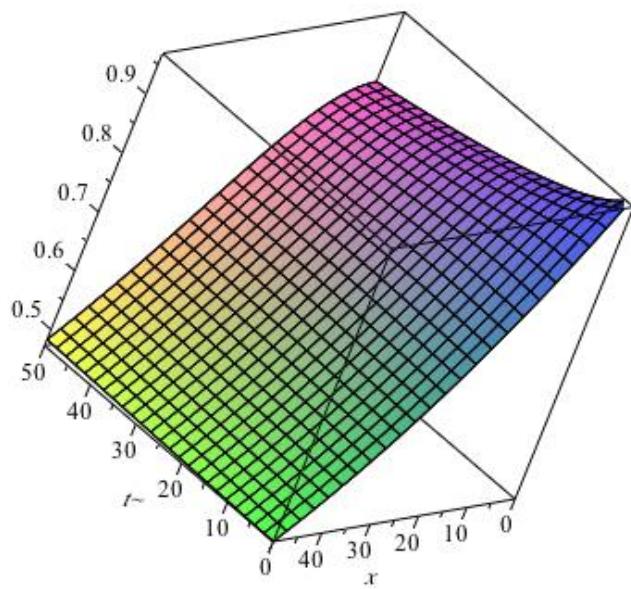


Figure 10.1: Convergence to a homogeneous stationary state: local dynamics for  $k(x, t)$  and  $c(x, t)$  for the case  $AK$ .

## 10.5 References

- Optimal control problem of partial differential equations or an optimal distributed control problem Butkovskiy (1969), Lions (1971), Derzko et al. (1984) or Neittaanmaki and Tiba (1994) present optimality results with varying generality. We draw mainly upon the last two references. See also the textbooks: Fattorini (1999) and Tröltzsch (2010).
- Applications in economics Carlson et al. (1996, chap.9)
- This section draws upon Brito (2004) and Brito (2011).

### .1 Proofs

*Proof of Proposition 1.* The value functional is

$$V[u, y] = \int_0^\infty \int_{-\infty}^\infty f(t, x, u(t, x), y(t, x)) dx dt,$$

considering the constraint we have

$$\begin{aligned} V[u, y] &= \int_0^\infty \int_{-\infty}^\infty \left[ f(t, x, u(t, x), y(t, x)) + \lambda(t, x) \left( g(t, x, u(t, x), y(t, x)) - \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial x^2} \right) \right] dx dt = \\ &= \int_0^\infty \int_{-\infty}^\infty \left[ H(t, x, u(t, x), y(t, x), \lambda(t, x)) + \left( \frac{\partial \lambda(t, x)}{\partial t} + \frac{\partial^2 \lambda(t, x)}{\partial x^2}(t, x) \right) y(t, x) \right] dx dt + \\ &\quad - \int_{-\infty}^\infty (\lambda(t, x)y(t, x)|_{t=0}^\infty) dx + \int_0^\infty \left( \lambda(t, x) \frac{\partial y(t, x)}{\partial x} - \frac{\partial \lambda(t, x)}{\partial x} y(t, x) \Big|_{x=-\infty}^\infty \right) dt \end{aligned}$$

for any control and state variables.

Let us assume we know the optimal control and state variables  $u^*(t, x)$  and  $y^*(t, x)$  then

$$V[u^*, y^*] = \int_0^\infty \int_{-\infty}^\infty f(t, x, u^*(t, x), y^*(t, x)) dx dt,$$

and let  $u(t, x)$  and  $y(t, x)$  be admissible perturbations over the optimal levels

$$\begin{aligned} u(t, x) &= u^*(t, x) + \varepsilon h_u(t, x) \\ y(t, x) &= y^*(t, x) + \varepsilon h_y(t, x) \end{aligned}$$

where  $\varepsilon$  is a constant, and  $h_y(0, x) = 0$ , for every  $x \in \mathbb{R}$ , and  $\lim_{x \pm \infty} \frac{\partial h_y(t, x)}{\partial x} = 0$ , for every  $t \in \mathbb{R}_+$ .

The integral derivative evaluated at  $\varepsilon = 0$  is

$$\delta V[u^*, y^*] = \int_0^\infty \int_{-\infty}^\infty \left[ \frac{\partial H^*(t, x)}{\partial u} h_u(t, x) + \left( \frac{\partial H^*(t, x)}{\partial y} + \frac{\partial \lambda(t, x)}{\partial t} + \frac{\partial^2 \lambda(t, x)}{\partial x^2} \right) h_y(t, x) \right] dx dt - \\ - \int_{-\infty}^\infty (\lambda(t, x) h_y(t, x)|_{t=0}^\infty) dx + \int_0^\infty \left( \lambda(t, x) \frac{\partial h_y(t, x)}{\partial x} - \frac{\partial \lambda(t, x)}{\partial x} h_y(t, x) \Big|_{x=-\infty}^\infty \right) dt =$$

where  $H^*(t, x) = H(t, xu^*(t, x), y^*(t, x), \lambda(t, x))$ . From admissibility conditions, we have

$$\delta V[u^*, y^*] = \int_0^\infty \int_{-\infty}^\infty \left[ \frac{\partial H^*(t, x)}{\partial u} h_u(t, x) + \left( \frac{\partial H^*(t, x)}{\partial y} + \frac{\partial \lambda(t, x)}{\partial t} + \frac{\partial^2 \lambda(t, x)}{\partial x^2} \right) h_y(t, x) \right] dx dt - \\ - \lim_{t \rightarrow \infty} \int_{-\infty}^\infty \lambda(t, x) h_y(t, x) dx - \int_0^\infty \left( \frac{\partial \lambda(t, x)}{\partial x} h_y(t, x) \Big|_{x=-\infty}^\infty \right) dt$$

Optimality requires that  $V[u^*, y^*] \geq V[u, y]$  which holds only if  $\delta V[u^*, y^*] = 0$ . Then optimality conditions are as in equation (10.2).  $\square$

*Proof of Proposition 2.* Let us assume that there is a solution  $(u^*, y^*)$ , for the problem, and define the value function as

$$V[u^*, y^*] = \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x \int_0^\infty f(y^*(\xi, t), u^*(\xi, t)) e^{-\rho t} dt d\xi.$$

Consider a small continuous perturbation  $(u(\epsilon), y(\epsilon)) = \{(u(x, t), y(x, t)) : (x, t) \in \mathbb{R} \times \mathbb{R}_+\}$ , where  $\epsilon$  is any positive constant, such that  $u(x, t) = u^*(x, t) + \epsilon h_u(x, t)$  and  $y(x, t) = y^*(x, t) + \epsilon h_y(x, t)$ , for  $t > 0$ , and  $h_u(x, 0) = h_y(x, 0) = 0$ , for every  $x \in \mathbb{R}$ . The value of this strategy is

$$V(\epsilon) = \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x \int_0^\infty f(u(\xi, t), y(\xi, t)) e^{-\rho t} dt d\xi.$$

But,

$$V(\epsilon) := \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x \int_0^\infty f(u(\xi, t), y(\xi, t)) e^{-\rho t} dt d\xi - \\ - \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x \int_0^\infty \lambda(\xi, t) \left[ \frac{\partial y(\xi, t)}{\partial t} - \frac{\partial^2 y(\xi, t)}{\partial \xi^2} - g(u(\xi, t), y(\xi, t)) \right] dt d\xi + \\ + \lim_{t \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x e^{-r(\xi, t)} \mu(\xi, t) y(\xi, t) d\xi \quad (23)$$

where  $\lambda(\cdot)$  is the co-state variable and  $\mu(\cdot)$  is a Lagrange multiplier associated with the solvability condition. In the optimum, the Kuhn-Tucker condition should hold

$$\lim_{t \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x e^{-r(\xi, t)} \mu(\xi, t) y(\xi, t) d\xi = 0.$$

By using integration by parts we find that

$$\int_0^\infty \lambda(x, t) \frac{\partial y(x, t)}{\partial t} dt = \lambda(x, t)y(x, t)|_{t=0}^\infty - \int_0^\infty \frac{\partial \lambda(x, t)}{\partial t} y(x, t) dt$$

and that

$$\begin{aligned} & \int_{-x}^x \int_0^\infty \lambda(\xi, t) \frac{\partial^2 y(\xi, t)}{\partial \xi^2} dt d\xi = \\ &= \int_0^\infty \lambda(\xi, t) \frac{\partial y(\xi, t)}{\partial \xi} \Big|_{\xi=-x}^x - y(\xi, t) \frac{\partial \lambda(\xi, t)}{\partial \xi} \Big|_{\xi=-x}^x dt + \int_{-x}^x \int_0^\infty \frac{\partial^2 \lambda(\xi, t)}{\partial \xi^2} y(\xi, t) dt d\xi, \end{aligned} \quad (24)$$

where the second term is canceled by the boundary conditions (10.14). Then

$$\begin{aligned} V(\epsilon) &= \lim_{x \rightarrow \infty} \frac{1}{2x} \int_{-x}^x \int_0^\infty (f(u(\xi, t), y(\xi, t)) e^{-\rho t} + \\ &+ \frac{\partial \lambda(\xi, t)}{\partial t} y(\xi, t) + \frac{\partial^2 \lambda(\xi, t)}{\partial \xi^2} y(\xi, t) + \lambda(\xi, t) g(u(\xi, t), y(\xi, t))) dt d\xi - \\ &- \lim_{x \rightarrow \infty} \frac{1}{2x} \left( \int_{-x}^x \lambda(\xi, t) y(\xi, t)|_{t=0}^\infty d\xi + \int_0^\infty \lambda(\xi, t) \frac{\partial y(\xi, t)}{\partial \xi} \Big|_{\xi=-x}^x dt \right) \end{aligned}$$

If an optimal solution exists, then we may characterize it by applying the variational principle,

$$\frac{\partial J(u^*, y^*)}{\partial \epsilon} = \lim_{\epsilon \rightarrow 0} \frac{J(u(\epsilon), y(\epsilon)) - J(u^*, y^*)}{\epsilon} = 0.$$

But, defining the Hamiltonian function as  $H(u, y, \lambda) = f(u, y) + \lambda g(u, y)$ , then

$$\begin{aligned} \frac{\partial J}{\partial \epsilon} &= \lim_{x \rightarrow \infty} \frac{1}{2x} \left\{ \int_{-x}^x \int_0^\infty [H_u(u^*(\xi, t), y^*(\xi, t), \lambda(\xi, t)) h_u(\xi, t) + \right. \\ &+ \left( \frac{\partial \lambda(\xi, t)}{\partial t} + \frac{\partial^2 \lambda(\xi, t)}{\partial \xi^2} + \lambda(\xi, t) H_x(u^*(\xi, t), y^*(\xi, t), \lambda(\xi, t)) \right) h_y(\xi, t)] dt d\xi \\ &- \int_{-x}^x \lambda(\xi, t) h_y(\xi, t)|_{t=0}^\infty d\xi + \int_0^\infty \lambda(\xi, t) \frac{\partial h_y(\xi, t)}{\partial \xi} \Big|_{\xi=-x}^x dt - \\ &\left. - \lim_{t \rightarrow \infty} \int_{-x}^x \mu(\xi, t) e^{-r(\xi, t)} h_y(\xi, t) d\xi \right\}. \end{aligned} \quad (25)$$

The last and the third to last expressions are canceled if  $\lim_{t \rightarrow \infty} [\mu(x, t) e^{-r(x, t)} - \lambda(t, x)] = 0$ , and by the fact that  $h_k(x, 0) = 0$ , for any  $x$ . Then, substituting in the Kuhn-Tucker condition we get a generalized transversality condition. We get the first order conditions by equating to zero all the remaining components of  $\frac{\partial J}{\partial \epsilon}$ . Equations (??)-(??) are obtained by simply making  $q(x, t) = e^{\rho t} \lambda(x, t)$ .  $\square$

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