```
(*
Solutions of Exercises in the Problem Set
 Calculus of Variations
  Continuous Time
  Paulo Brito : 5.1.2016
*)
(* 2.1.1 *)
F[t_, y_, dy_] = Log[a y + b dy]
F1[t_, y_, dy_] := D[F[t, y, dy], y]
F2[t_, y_, dy_] := D[F[t, y, dy], dy]
EL1 = Factor[Simplify[F1[t, y[t], D[y[t], t]] - D[F2[t, y[t], D[y[t], t]], t]]]
solEL1 := DSolve[{EL1 == 0}, y[t], t]
ys[t_] := Evaluate[y[t] /. eq1]
CC = Assuming[T > 0, Solve[{ys[0] == ϕ, ys[T] == 0}, {C[1], C[2]}]]
sol = Evaluate[ys[t] /. CC]
```

Log[b dy + a y]

$$\frac{a^2\, y[t] + 2\, a\, b\, y'[t] + b^2\, y''[t]}{(a\, y[t] + b\, y'[t])^2}$$

$$\left\{\left\{C[1] \to \phi,\ C[2] \to -\frac{\phi}{T}\right\}\right\}$$

$$\left\{\left\{e^{-\frac{a t}{b}}\, \phi - \frac{e^{-\frac{a t}{b}}\, t\, \phi}{T}\right\}\right\}$$

```
(* 2.1.2 *)

ClearAll[y, ys, EL1, solEL1, LC, CC, sol]
F[t_, y_, dy_] = (dy - a y)^2
F1[t_, y_, dy_] = D[F[t, y, dy], y]
F2[t_, y_, dy_] = D[F[t, y, dy], dy]
EL1 = Factor[Simplify[F1[t, y[t], D[y[t], t]] - D[F2[t, y[t], D[y[t], t]], t]]]
solEL1 = DSolve[{EL1 == 0}, y[t], t]
ys[t_] = Evaluate[y[t] /. solEL1]
LC[t_] = Simplify[F2[t, ys[t], D[ys[t], t]]]
CC = Assuming[T > 0,
  Solve[{ys[0] == y0, Limit[LC[T], t → Infinity] == 0}, {C[1], C[2]}]]
sol = Simplify[Evaluate[ys[t] /. CC]]
```

$(dy - a\, y)^2$

$-2\, a\, (dy - a\, y)$

$2\, (dy - a\, y)$

$2\, \left(a^2\, y[t] - y''[t]\right)$

$\left\{\left\{y[t] \to e^{a t}\, C[1] + e^{-a t}\, C[2]\right\}\right\}$

$\left\{e^{a t}\, C[1] + e^{-a t}\, C[2]\right\}$

$\left\{-4\, a\, e^{-a t}\, C[2]\right\}$

$\{\{C[1] \to y0,\ C[2] \to 0\}\}$

$\left\{\left\{e^{a t}\, y0\right\}\right\}$

```
(* 2.1.3 *)

ClearAll[A, As, EL1, solEL1, LC, CC, sol]
F[t_, A_, dA_] = Exp[-ρ t] Log[r A - dA]
F1[t_, A_, dA_] = D[F[t, A, dA], A]
F2[t_, A_, dA_] = D[F[t, A, dA], dA]
EL1 = Factor[Simplify[F1[t, A[t], D[A[t], t]] - D[F2[t, A[t], D[A[t], t]], t]]]
solEL1 = DSolve[{EL1 == 0}, A[t], t]
As[t_] = Evaluate[A[t] /. solEL1]
CC = Assuming[T > 0, Solve[{As[0] == A0, As[T] == A0}, {C[1], C[2]}]]
sol = Simplify[Evaluate[As[t] /. CC]]
```

$e^{-t\rho}\,\mathrm{Log}[-dA + A\,r]$

$\dfrac{e^{-t\rho}\,r}{-dA + A\,r}$

$-\dfrac{e^{-t\rho}}{-dA + A\,r}$

$\dfrac{e^{-t\rho}\left(r^2 A[t] - r\,\rho\,A[t] - 2\,r\,A'[t] + \rho\,A'[t] + A''[t]\right)}{(r\,A[t] - A'[t])^2}$

$\left\{\left\{A[t] \to e^{t\,(r-\rho)}\,C[1] + e^{r\,t}\,C[2]\right\}\right\}$

$\left\{e^{t\,(r-\rho)}\,C[1] + e^{r\,t}\,C[2]\right\}$

$\left\{\left\{C[1] \to \dfrac{A0\left(-1 + e^{r\,T}\right)}{e^{r\,T} - e^{T\,(r-\rho)}},\ C[2] \to -\dfrac{A0\left(-1 + e^{T\,(r-\rho)}\right)}{e^{r\,T} - e^{T\,(r-\rho)}}\right\}\right\}$

$\left\{\left\{\dfrac{A0\,e^{r\,t - r\,T - t\,\rho}\left(-e^{T\rho} + e^{(t+T)\,\rho} + e^{T\,(r+\rho)} - e^{r\,T + t\,\rho}\right)}{-1 + e^{T\rho}}\right\}\right\}$

```
(* 2.1.4 *)
```

```
ClearAll[A, As, EL1, solEL1, LC, CC, sol]
F[t_, A_, dA_] = Exp[-ρ t] Log[r A - dA]
F1[t_, A_, dA_] = D[F[t, A, dA], A]
F2[t_, A_, dA_] = D[F[t, A, dA], dA]
EL1 = Factor[Simplify[F1[t, A[t], D[A[t], t]] - D[F2[t, A[t], D[A[t], t]], t]]]
solEL1 = DSolve[{EL1 == 0}, A[t], t]
As[t_] = Evaluate[A[t] /. solEL1]
LC[t_] = Simplify[F2[t, As[t], D[As[t], t]]]
CC = Assuming[T > 0, Solve[{As[0] == A0, LC[T] As[T] == 0}, {C[1], C[2]}]]
sol = Simplify[Evaluate[As[t] /. CC]]
```

$e^{-t\rho}\,\mathrm{Log}[-dA + A\,r]$

$\dfrac{e^{-t\rho}\,r}{-dA + A\,r}$

$-\dfrac{e^{-t\rho}}{-dA + A\,r}$

$\dfrac{e^{-t\rho}\left(r^2\,A[t] - r\,\rho\,A[t] - 2\,r\,A'[t] + \rho\,A'[t] + A''[t]\right)}{(r\,A[t] - A'[t])^2}$

$\left\{\left\{A[t] \to e^{t\,(r-\rho)}\,C[1] + e^{r\,t}\,C[2]\right\}\right\}$

$\left\{e^{t\,(r-\rho)}\,C[1] + e^{r\,t}\,C[2]\right\}$

$\left\{-\dfrac{e^{-r\,t}}{\rho\,C[1]}\right\}$

$\left\{\left\{C[1] \to \dfrac{A0\,e^{r\,T}}{e^{r\,T} - e^{T\,(r-\rho)}},\ C[2] \to -\dfrac{A0\,e^{T\,(r-\rho)}}{e^{r\,T} - e^{T\,(r-\rho)}}\right\}\right\}$

$\left\{\left\{\dfrac{A0\,e^{r\,t}\left(-1 + e^{(-t+T)\,\rho}\right)}{-1 + e^{T\,\rho}}\right\}\right\}$

```
(* 2.1.5 *)
```

```
ClearAll[A, As, EL1, solEL1, LC, CC, sol]
F[t_, A_, dA_] = Exp[-ρ t] Log[r A - dA]
F1[t_, A_, dA_] = D[F[t, A, dA], A]
F2[t_, A_, dA_] = D[F[t, A, dA], dA]
EL1 = Factor[Simplify[F1[t, A[t], D[A[t], t]] - D[F2[t, A[t], D[A[t], t]], t]]]
solEL1 = DSolve[{EL1 == 0}, A[t], t]
As[t_] = Evaluate[A[t] /. solEL1]
LC[t_] = Simplify[F2[t, As[t], D[As[t], t]]]
CC = Assuming[T > 0,
  Solve[{As[0] == A0, Limit[LC[T] As[T], T → Infinity] == 0}, {C[1], C[2]}]]
sol = Simplify[Evaluate[As[t] /. CC]]
```

$e^{-t\rho}\,\mathrm{Log}[-dA + A\,r]$

$\dfrac{e^{-t\rho}\,r}{-dA + A\,r}$

$-\dfrac{e^{-t\rho}}{-dA + A\,r}$

$\dfrac{e^{-t\rho}\left(r^2\,A[t] - r\,\rho\,A[t] - 2\,r\,A'[t] + \rho\,A'[t] + A''[t]\right)}{(r\,A[t] - A'[t])^2}$

$\{\{A[t] \to e^{t\,(r-\rho)}\,C[1] + e^{r\,t}\,C[2]\}\}$

$\{e^{t\,(r-\rho)}\,C[1] + e^{r\,t}\,C[2]\}$

$\{-\dfrac{e^{-r\,t}}{\rho\,C[1]}\}$

$\{\{C[1] \to \dfrac{A0\,e^{\rho\,\infty}}{-1 + e^{\rho\,\infty}},\ C[2] \to -\dfrac{A0}{-1 + e^{\rho\,\infty}}\}\}$

$\{\{A0\,e^{r\,t}\}\}$

```
(* 2.1.6 *)
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```
ClearAll[A, As, EL1, solEL1, LC, CC, sol]
F[t_, A_, dA_] = Exp[-ρ t] (r A - dA) ^ (1 - σ) (1 - σ) ^ (-1)
F1[t_, A_, dA_] = D[F[t, A, dA], A]
F2[t_, A_, dA_] = D[F[t, A, dA], dA]
EL1 = Factor[Simplify[F1[t, A[t], D[A[t], t]] - D[F2[t, A[t], D[A[t], t]], t]]]
solEL1 = DSolve[{EL1 == 0}, A[t], t]
As[t_] = Evaluate[A[t] /. solEL1]
CC = Assuming[T > 0, Solve[{As[0] == A0, As[T] == A0}, {C[1], C[2]}]]
sol = Simplify[Evaluate[As[t] /. CC]]
```

$$\frac{e^{-t\rho}\,(-dA + A\,r)^{1-\sigma}}{1-\sigma}$$

$$e^{-t\rho}\,r\,(-dA + A\,r)^{-\sigma}$$

$$-e^{-t\rho}\,(-dA + A\,r)^{-\sigma}$$

$$e^{-t\rho}\,(r\,A[t] - A'[t])^{-1-\sigma}\left(r^2\,A[t] - r\,\rho\,A[t] - r\,A'[t] + \rho\,A'[t] - r\,\sigma\,A'[t] + \sigma\,A''[t]\right)$$

$$\left\{\left\{A[t] \to e^{\frac{t\,(r-\rho)}{\sigma}}\,C[1] + e^{r\,t}\,C[2]\right\}\right\}$$

$$\left\{e^{\frac{t\,(r-\rho)}{\sigma}}\,C[1] + e^{r\,t}\,C[2]\right\}$$

$$\left\{\left\{C[1] \to \frac{A0\left(-1 + e^{r\,T}\right)}{e^{r\,T} - e^{\frac{T\,(r-\rho)}{\sigma}}},\ C[2] \to -\frac{A0\left(-1 + e^{\frac{T\,(r-\rho)}{\sigma}}\right)}{e^{r\,T} - e^{\frac{T\,(r-\rho)}{\sigma}}}\right\}\right\}$$

$$\left\{\left\{\frac{A0\left(e^{r\,t} + e^{r\,T + \frac{t\,(r-\rho)}{\sigma}} - e^{r\,t + \frac{T\,(r-\rho)}{\sigma}} - e^{\frac{t\,(r-\rho)}{\sigma}}\right)}{e^{r\,T} - e^{\frac{T\,(r-\rho)}{\sigma}}}\right\}\right\}$$

```
(* 2.1.10 *)
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```
ClearAll[p, ps, p0, TC, EL1, solEL1, CC, sol, px]
F[t_, p_, dp_] = -((dp + pn)^2) Exp[-ρ t]
F1[t_, p_, dp_] = D[F[t, p, dp], p]
F2[t_, p_, dp_] = D[F[t, p, dp], dp]
EL1 = Factor[Simplify[F1[t, p[t], D[p[t], t]] - D[F2[t, p[t], D[p[t], t]], t]]]
solEL1 = DSolve[{EL1 == 0}, p[t], t]
ps[t_] = Evaluate[p[t] /. solEL1]
TC[t_] = Simplify[D[ps[t], t]]
CC = Assuming[T > 0, Solve[{ps[0] == p0, TC[T] == 0}, {C[1], C[2]}]]
sol = Factor[Simplify[Evaluate[ps[t] /. CC]]]
px[ρ_, pn_, p0_, T_, t_] = Factor[Simplify[Evaluate[ps[t] /. CC]]]
Plot[px[0.02, 0.02, -0.01, 200, t], {t, 0, 200}]
```

$-e^{-t\rho}(dp+pn)^2$

0

$-2e^{-t\rho}(dp+pn)$

$-2e^{-t\rho}(pn\,\rho+\rho\,p'[t]-p''[t])$

$\left\{\left\{p[t]\to -pn\,t+\frac{e^{t\rho}C[1]}{\rho}+C[2]\right\}\right\}$

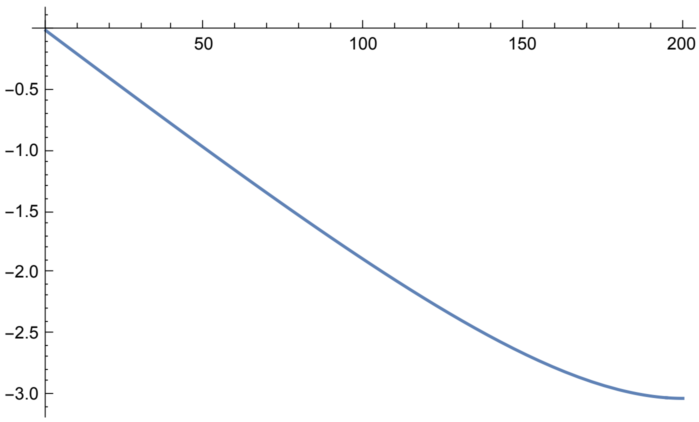
$\left\{-pn\,t+\frac{e^{t\rho}C[1]}{\rho}+C[2]\right\}$

$\left\{-pn+e^{t\rho}C[1]\right\}$

$\left\{\left\{C[1]\to e^{-T\rho}pn,\ C[2]\to \frac{e^{-T\rho}\left(-pn+e^{T\rho}p0\,\rho\right)}{\rho}\right\}\right\}$

$\left\{\left\{\frac{e^{-T\rho}\left(-pn+e^{t\rho}pn+e^{T\rho}p0\,\rho-e^{T\rho}pn\,t\,\rho\right)}{\rho}\right\}\right\}$

$\left\{\left\{\frac{e^{-T\rho}\left(-pn+e^{t\rho}pn+e^{T\rho}p0\,\rho-e^{T\rho}pn\,t\,\rho\right)}{\rho}\right\}\right\}$

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(* 2.1.13 *)
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```
ClearAll[y, ys, EL1, solEL1, LC, CC, sol]
F[t_, y_, dy_] = - (dy - y) ^2
F1[t_, y_, dy_] = D[F[t, y, dy], y]
F2[t_, y_, dy_] = D[F[t, y, dy], dy]
EL1 = Factor[Simplify[F1[t, y[t], D[y[t], t]] - D[F2[t, y[t], D[y[t], t]], t]]]
solEL1 = DSolve[{EL1 == 0}, y[t], t]
ys[t_] = Evaluate[y[t] /. solEL1]
LC[t_] = Simplify[F2[t, ys[t], D[ys[t], t]]]
CC = Assuming[T > 0, Solve[{ys[0] == 1, LC[T] == 0}, {C[1], C[2]}]]
sol = Simplify[Evaluate[ys[t] /. CC]]
```

$-(dy - y)^2$

$2\,(dy - y)$

$-2\,(dy - y)$

$-2\,(y[t] - y''[t])$

$\{\{y[t] \to e^{t}\,C[1] + e^{-t}\,C[2]\}\}$

$\{e^{t}\,C[1] + e^{-t}\,C[2]\}$

$\{4\,e^{-t}\,C[2]\}$

$\{\{C[1] \to 1,\ C[2] \to 0\}\}$

$\{\{e^{t}\}\}$

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(* 2.1.14 *)
(* corrected 12.1.2016 *)
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```
ClearAll[p, ps, EL1, solEL1, LC, CC, sol]
F[t_, p_, dp_] = -((un - dp)^2 + p^2) Exp[-ρ t]
F1[t_, p_, dp_] = D[F[t, p, dp], p]
F2[t_, p_, dp_] = D[F[t, p, dp], dp]
EL1 = Factor[Simplify[F1[t, p[t], D[p[t], t]] - D[F2[t, p[t], D[p[t], t]], t]]]
solEL1 = DSolve[{EL1 == 0}, p[t], t]
ps[t_] = Evaluate[p[t] /. solEL1]
TC[t_] = Simplify[D[ps[t], t]]
CC = Assuming[T > 0, Solve[{ys[0] == p0, TC[T] == 0}, {C[1], C[2]}]]
sol = Factor[Simplify[Evaluate[ps[t] /. CC]]]
```

$$e^{-t\rho}\left(-p^2-(-dp+un)^2\right)$$

$$-2\,e^{-t\rho}\,p$$

$$2\,e^{-t\rho}\,(-dp+un)$$

$$2\,e^{-t\rho}\,(un\,\rho-p[t]-\rho\,p'[t]+p''[t])$$

$$\left\{\left\{p[t]\to un\,\rho+e^{\frac{1}{2}t\left(\rho-\sqrt{4+\rho^2}\right)}C[1]+e^{\frac{1}{2}t\left(\rho+\sqrt{4+\rho^2}\right)}C[2]\right\}\right\}$$

$$\left\{un\,\rho+e^{\frac{1}{2}t\left(\rho-\sqrt{4+\rho^2}\right)}C[1]+e^{\frac{1}{2}t\left(\rho+\sqrt{4+\rho^2}\right)}C[2]\right\}$$

$$\left\{\frac{1}{2}\left(e^{\frac{1}{2}t\left(\rho-\sqrt{4+\rho^2}\right)}\left(\rho-\sqrt{4+\rho^2}\right)C[1]+e^{\frac{1}{2}t\left(\rho+\sqrt{4+\rho^2}\right)}\left(\rho+\sqrt{4+\rho^2}\right)C[2]\right)\right\}$$

$$\left\{\left\{C[1]\to-\frac{e^{\frac{1}{2}T\left(\rho+\sqrt{4+\rho^2}\right)}\,p0\left(\rho+\sqrt{4+\rho^2}\right)}{2\left(\frac{1}{2}e^{\frac{1}{2}T\left(\rho-\sqrt{4+\rho^2}\right)}\left(\rho-\sqrt{4+\rho^2}\right)-\frac{1}{2}e^{\frac{1}{2}T\left(\rho+\sqrt{4+\rho^2}\right)}\left(\rho+\sqrt{4+\rho^2}\right)\right)},\right.\right.$$

$$\left.\left.C[2]\to\frac{e^{\frac{1}{2}T\left(\rho-\sqrt{4+\rho^2}\right)}\,p0\left(-\rho+\sqrt{4+\rho^2}\right)}{-e^{\frac{1}{2}T\left(\rho-\sqrt{4+\rho^2}\right)}\rho+e^{\frac{1}{2}T\left(\rho+\sqrt{4+\rho^2}\right)}\rho+e^{\frac{1}{2}T\left(\rho-\sqrt{4+\rho^2}\right)}\sqrt{4+\rho^2}+e^{\frac{1}{2}T\left(\rho+\sqrt{4+\rho^2}\right)}\sqrt{4+\rho^2}}\right\}\right\}$$

$$\left\{\left\{\left(e^{-\frac{1}{2}t\sqrt{4+\rho^2}}\left(-e^{\frac{t\rho}{2}+t\sqrt{4+\rho^2}}\,p0\,\rho+e^{\frac{t\rho}{2}+T\sqrt{4+\rho^2}}\,p0\,\rho-e^{\frac{1}{2}t\sqrt{4+\rho^2}}\,un\,\rho^2+\right.\right.\right.\right.$$
$$e^{\frac{1}{2}(t+2T)\sqrt{4+\rho^2}}\,un\,\rho^2+e^{\frac{t\rho}{2}+t\sqrt{4+\rho^2}}\,p0\sqrt{4+\rho^2}+e^{\frac{t\rho}{2}+T\sqrt{4+\rho^2}}\,p0\sqrt{4+\rho^2}+$$
$$\left.\left.e^{\frac{1}{2}t\sqrt{4+\rho^2}}\,un\,\rho\sqrt{4+\rho^2}+e^{\frac{1}{2}(t+2T)\sqrt{4+\rho^2}}\,un\,\rho\sqrt{4+\rho^2}\right)\right)\Big/$$
$$\left.\left.\left(-\rho+e^{T\sqrt{4+\rho^2}}\rho+\sqrt{4+\rho^2}+e^{T\sqrt{4+\rho^2}}\sqrt{4+\rho^2}\right)\right\}\right\}$$

```
DSolve[D[p[t], t, t] == 0, p[t], t]
```

$$\{\{p[t]\to C[1]+t\,C[2]\}\}$$