Growth economics

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1 Introduction

Next we present the analytical part of the solution for some problems. The justitication and the interpretation is left to the students.

2 Problem set 1: Malthus

- **Problem 4** Assume that the representative consumer solves the problem: $\max_{c,b}\{u(c,b) : c+\rho b \leq y\}$ where c is consumption, b is the birth rate, ρ is the cost of raising children and y is per capita income. Assume that the utility function is

  $$u(c,b) = \ln(c) + \phi \ln(b),\ \phi > 0$$

  and the aggregate production function is Cobb-Douglas $Y = (AX)^{\alpha} L^{1-\alpha}$, with $0 < \alpha < 1$, where X is the stock of land, A is land-specific productivity and L is population. Population growth is $\dot{L}/L = b - m$, where the mortality rate, m, is constant and exogenous, and $L(0) = L_0 > 0$ is given. Land productivity grows at a rate $\gamma > 0$.

  1. Defining $\ell \equiv L/A$, obtain a differential equation for ℓ.
  2. Study the qualitative dynamics of the model. Provide an intuition for your results.
  3. Derive the growth facts (long run growth rate, long run per capita output and transition dynamics). What are the effects of an increase in γ ?

- **Solution**

  1. $\dot{\ell} = \ell\left(\psi(X/\ell)^{\alpha} - (m+\gamma)\right)$ for $\psi \equiv \frac{\phi}{\rho(1+\phi)}$;
  2. Steady states $\ell^* = \{\ 0, \bar{\ell}\}$ with $\bar{\ell} = \left(\frac{\psi}{m+\gamma}\right)^{\frac{1}{\alpha}} X$, local dynamics $\frac{\partial \dot{\ell}}{\partial \ell}(\bar{\ell}) = -\alpha(m+\gamma)$. Solving explicitly we have

     $$\ell(t) = \left(\bar{\ell}^{\alpha} + (\ell(0)^{\alpha} - \bar{\ell}^{\alpha})e^{-\alpha(m+\gamma)t}\right)^{\frac{1}{\alpha}}$$

     The conclusions are the same: if $\ell(0) > 0$ then ℓ converges to the steady state $\bar{\ell}$.
  3. Growth facts: as

     $$y(t) = \frac{X^{\alpha}}{\bar{\ell}^{\alpha} + (\ell(0)^{\alpha} - \bar{\ell}^{\alpha})e^{-\alpha(m+\gamma)t}}$$

     then $\lim_{t\to\infty} y(t) = \dfrac{m+\gamma}{\psi}$: no long run growth, there is transitional dynamics and the steady state level of output per capita dependes on m, γ and parameters associated to consumer problem (ψ)

3 Problem set 2

- **Problem: Solow 8** Assume that the Solow model is a good representation of the capital accumulation dynamics for two countries, labelled by 1 and 2, respectively. Let the economies have the same preferences and the same demographic data, but differ as regards the initial capital intensity, $k_i(0)$ and the TFP. The Solow accumulation equation would be

$$\dot{k}_i = sA_i k_i(t)^\alpha - nk_i(t),\ i = 1, 2.$$

  Assume that: $k_1(0) > k_2(0)$, $A_1 < A_2$, $0 < s < 1$, $0 < \alpha < 1$ and $n \geq 0$.

  1. Characterize the differences in the growth dynamics between the two countries.
  2. Will there be convergence ? If affirmative, which kind of convergence ?
  3. Assuming there is some form of catch up, provide a measure of its timing ?

- **Solution**

  1. $\gamma_1 = \gamma_2 = 0$, $\bar{y}_1 < \bar{y}_2$ and $\lambda_1 = \lambda_2$ where $\bar{y}_i = \left(A_i \left(\frac{s}{n}\right)^\alpha\right)^{\frac{1}{1-\alpha}}$, $\lambda_i = -(1-\alpha)n$
  2. $t \approx \dfrac{1}{(\alpha-1)n} \ln\left(\dfrac{\bar{k}_2 - \bar{k}_1}{\bar{k}_2 - \bar{k}_1 + k_1(0) - k_2(0)}\right)$

- **Problem: Solow 3** Consider a version of the Solow model, in which: (1) the savings function is $S(t) = sY(t)$, with $0 < s < 1$; (2) the population, L grows at a constant rate $n > 0$, $\dot{L} = nL(t)$, (3) there is no depreciation of capital, and (4) the technology is CES (constant elasticity of substitution)

$$Y(t) = F(K(t), L(t)) = \left(\alpha K(t)^{-\eta} + (1-\alpha)L(t)^{-\eta}\right)^{-1/\eta},\ 0 < \alpha < 1,\ \eta > -1,\ \eta \neq 0$$

  1. Derive the accumulation equation for the detrended capital stock $k(t) \equiv K(t)/L(t)$.
  2. Determine analytically the long run level for k, its stability properties, and discuss its economic meaning.
  3. Study the effect of a permanent increase in n on the long run growth, transition, and the level of the product.

- **Solution**

  1. Accumulation equation: $\dot{k} = s\left(\alpha k^{-\eta} + 1 - \alpha\right)^{-\frac{1}{\eta}} - nk$
  2. Steady state: $\bar{k} = \left(\dfrac{(s/n)^\eta - \alpha}{1-\alpha}\right)^{\frac{1}{\eta}}$. Stability properties: the steady state is asymptotically stable because

$$\lambda = \left.\frac{\partial \dot{k}}{\partial k}\right|_{k=\bar{k}} = -n\left(1 - \alpha(n/s)^\eta\right) < 0$$

  3. Effect of a shock in n: (1) no effect on the long run growth rate: $\gamma = 0$; (2) negative effect on the long run level of GDP $\bar{y} = f(\bar{k}) = \left(\alpha\bar{k}^{-\eta} + 1 - \alpha\right)^{-\frac{1}{\eta}}$ because $f'(k) > 0$ and

$$\frac{\partial \bar{k}}{\partial n} = -(s/n)^{1+\eta}\left(\frac{(s/n)^\eta - \alpha}{1-\alpha}\right)^{\frac{1-\eta}{\eta}} < 0$$

     (3) transition effect $\frac{\partial \lambda}{\partial n} < 0$

- **Ramsey 8** Consider a version of the Ramsey model with constant population where the objective utility functional for the central planner is:

$$\max_{c} \int_0^{\infty} ln(c(t) - \bar{c})e^{-\rho t} dt,$$

where $\rho > 0$ and $\bar{c} > \frac{\rho}{\alpha}$ is a minimum level of consumption, subject to

$$\dot{k} = Ak(t)^{\alpha} - c(t), \; 0 < \alpha < 1$$

where c and k are the per capita consumption and capital stock. We also assume that $k(0) = k_0$ is given and that the stock of capital is bounded.

1. Apply the Pontryiagin's principle and determine the optimality conditions as a dynamic system in (c, k).
2. Draw the phase diagram.
3. Determine the steady states and study their local stability properties.
4. Find an approximate solution to the problem in the neighborhood of the steady state associated with a maximum consumption.
5. Determine the effects of a permanent increase in productivity, A.

- **Solution**

1. The MHDS

$$\begin{aligned}
\dot{c} &= (c - \bar{c})\,(r(k) - \rho) \\
\dot{k} &= Ak^{\alpha} - c \\
0 &= \lim_{t \to \infty} \frac{k(t)}{c(t) - \bar{c}} e^{-\rho t} \\
K(0) &= K_0
\end{aligned}$$

3 Steady states: corner steady state $\left(\bar{c}, \left(\frac{\bar{c}}{A}\right)^{\frac{1}{\alpha}}\right)$ interior steady state

$$(c^*, k^*) = \left(A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}}, \left(\frac{\alpha A}{\rho}\right)^{\frac{1}{1-\alpha}}\right)$$

satisfying $c^* > \bar{c}$. Local dynamics for the interior steady state: eigenvalues of the Jacobian

$$\lambda_{s,u} = \frac{\rho}{2} \pm \left(\left(\frac{\rho}{2}\right)^2 - D\right)^{\frac{1}{2}}$$

where $D = -(\bar{c} \; - \bar{c})^2(1-\alpha)\rho(k^*)^{-1} < 0$. It is a saddle point.

4 Approximate solution in the neighborhood of (c^*, k^*):

$$c(t) = c^* + \lambda_u(k_0 - k^*)e^{\lambda_s t}$$
$$k(t) = k^* + (k_0 - k^*)e^{\lambda_s t}$$

5 Effects of a permanent unit increase in A. Asymptotically consumption and capital increase by

$$\begin{aligned}
\frac{\partial c^*}{\partial A} &= \frac{\rho}{\alpha(1-\alpha)} \frac{k^*}{A} > 0 \\
\frac{\partial k^*}{\partial A} &= \frac{1}{1-\alpha} \frac{k^*}{A} > 0
\end{aligned}$$

At time $t = 0$ consumption jumps by

$$\frac{\partial c(0)}{\partial A} = \frac{\partial c^*}{\partial A} - \lambda_u \frac{\partial k^*}{\partial A} = \left(\frac{(1-\alpha)\rho + \alpha\lambda_s}{\alpha(1-\alpha)} \right) \frac{k^*}{A}$$

which is ambiguous.

4 Problem set 3

- **Problem** *AK* **4**: Consider a centralized economy model in which the central planner's problem is

$$\max_{(C(t))_{t\geq 0}} \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta} L(t)\, e^{-\rho t} dt,$$

subject to the restriction $\dot{\mathbf{K}} = A\mathbf{K}(t) - C(t)L(t) - \delta\mathbf{K}(t)$, given $\mathbf{K}(0) = K_0 > 0$ and $\lim_{t\to\infty} e^{-At}\mathbf{K}(t) \geq 0$, where $\mathbf{K}(t) \equiv K(t)L(t)$ is the aggregate capital stock and $L(t)$ is total population, $C(t)$ is per-capita consumption level, and $K(t)$ is the per-capita capital stock. We assume that population grows exogenously as $L(t) = e^{nt}$, where n is the growth rate of the population. Consider the following assumptions over the parameters: $\theta > 0$ and $0 < \rho < (\theta - 1)(A - \delta) - \theta\, n$ where A is the TFP and δ is the depreciation rate of capital.

  1. Write the central planners's problem in terms of per-capita variables.
  2. Determine the optimality conditions as an initial-terminal value problem in the per-capita variables (C, K).
  3. Specify the model in (per-capita) detrended variables, and determine the long-run (endogenous) growth rate.
  4. Prove that the solution for the detrended variables is $k(t) = K_0$ and $c(t) = \beta K_0$, where $\beta \equiv \dfrac{(\theta-1)(A-\delta) - \theta n + \rho}{\theta}$.
  5. Discuss the growth properties of the model. What are the implications of an increase in the growth rate of population n ?

- **Solution**

  1. $$\max_{C(.)} \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt \text{ s.t. } \dot{K} = (A - \delta - n)K(t) - C(t), \text{ plus terminal conditons}$$

  2. The MHDS in per capita variables

$$\begin{aligned} \dot{C} &= C\left(\frac{A-\delta-\rho}{\theta}\right) \\ \dot{K} &= (A-\delta-n)K - C \\ 0 &= \lim_{t\to\infty} C(t)^{-\theta} K(t) e^{-(\rho-n)t} \\ K(0) &= K_0 \end{aligned}$$

  3. The model in detrended variables $c(t) = C(t)e^{-\gamma t}$ and $k(t) = K(t)e^{-\gamma t}$

$$\begin{aligned} \dot{c} &= c\left(\frac{A-\delta-\rho-\theta\gamma}{\theta}\right) \\ \dot{k} &= (A-\delta-n-\gamma)k - c \\ 0 &= \lim_{t\to\infty} c(t)^{-\theta} k(t) e^{-(\rho-n+\theta(\gamma-1))t} \\ k(0) &= K_0 \end{aligned}$$

Long-run growth rate $\bar{\gamma} = \dfrac{A-\delta-\rho}{\theta}$

4. If $\gamma = \bar{\gamma}$ then the MHDS becomes: $\dot{c} = 0$ and $\dot{k} = \beta k - c$, for $\beta \equiv \dfrac{((\theta-1)(A-\delta)-\theta n+\rho}{\theta}$ and the TVC is $\lim_{t\to\infty} c(t)^{-\theta}k(t)e^{-\beta t} = 0$. The solution for the two differential equations is $c(t) = \bar{c}$ and $k(t) = \dfrac{\bar{c}}{\beta} + \left(k(0) - \dfrac{\bar{c}}{\beta}\right)e^{\beta t}$. Substituting in the transversality condition, we find $\bar{c} = \beta k(0) = \beta K_0$ then $c(t) = \beta K_0$ and $k(t) = K_0$ for all $t \in [0, \infty)$
5. An increase in n neither changes the rate of growth nor the long run level of percapita output nor generates transition dynamics. The only effect is to diminish per capita consumption: $d\beta/dn = -1$

- **Romer 3** Consider an economy in which there are externalities in consumption. The representative consumer has the utility functional

$$\int_0^{+\infty} \frac{1}{1-\theta}\left(C(t)^{1-\beta}(\mathbf{C}(t)^{\beta}\right)^{1-\theta} e^{-\rho t}dt,$$

where $\rho > 0$, $\theta > 0$ and $0 < \beta < 1$, C is the private consumption and $\mathbf{C}$ is the aggregate consumption. The representative agent has the instantaneous budget constraint $\dot{K} = AK - C$, and $K(0)$ is given and K is asymptotically bounded. Assume that $A > \rho$.

1. Write down the optimality conditions for the representative agent. Justify;
2. Introducing the micro-macro consistency condition, determine the general equilibrium of this economy as a dynamic system in (C, K).
3. Solve the dynamic system. Discuss the growth facts we can extract from this model ?

- **Solution**

1.

$$\begin{aligned} Q &= (1-\beta)\left(C^{1-\beta}\mathbf{C}^{\beta}\right)^{1-\theta} C^{-1} \\ \dot{Q} &= Q(\rho - A) \\ \dot{K} &= AK - C \\ 0 &= \lim_{t\to\infty} Q(t)K(t)e^{-\rho t} \\ K(0) &= K_0 \end{aligned}$$

2.

$$\begin{aligned} \dot{C} &= C(A-\rho)/\theta \\ \dot{K} &= AK - C \\ 0 &= \lim_{t\to\infty} (1-\beta)C(t)^{-\theta}K(t)e^{-\rho t} \\ K(0) &= K_0 \end{aligned}$$

3.

$$\begin{aligned} K(t) &= K_0 e^{\gamma t},\ \gamma = (A-\rho)/\theta \\ C(t) &= \beta K(t),\ \beta = (A(\theta-1)+\rho)/\theta \end{aligned}$$

then

$$Y(t) = AK_0 e^{\gamma t}$$

- **Romer 7** Assume an economy with a government in which the public expenditure generates a productive externality. The government finances public expenditures by a tax over total income. Then the government budget constraint is $G(t) = \tau Y(t)$, where $G(t)$ and $Y(t)$ are the levels of government expenditures and aggregate income and τ is the tax rate. Assume that we have a representative household, which determines jointly the consumption, savings and production activities. Therefore, the instantaneous budget constraint for the private agent is $\dot{K}(t) = (1-\tau)Y(t) - C(t) + G(t)$, where total income is equal to total output $Y(t) = AK(t)^{\alpha}G(t)^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$. The intertemporal utility function is

$$\int_0^{+\infty} \ln\left(C(t)\right) e^{-\rho t} dt,$$

where $\rho > 0$. The initial level for the capital stock is given, $K(0) = K_0$, and the asymptotic value of the capital stock is bounded in present-value terms.

1. Determine the optimality conditions for the private agent as a dynamic system in (C, K). Justify;
2. Define and determine a representation for the dynamic general equilibrium (DGE) of this economy, keeping τ as an exogenous parameter. Justify.
3. Under which conditions can we have a BGP ? Write the DGE in detrended variables assuming those conditions from now on. What would be the long run growth rate ?
4. Determine the equilibrium solution for $Y(t)$.
5. Discuss the consequences of an increase in the tax rate, τ, in this economy. Is there any policy which may make the general equilibrium Pareto optimal ?

- **Solution**

1. Optimum conditions for the agent

$$\begin{cases} \dot{C} &= C\left(\alpha(1-\tau)AK^{\alpha-1}G^{1-\alpha} - \rho\right) \\ \dot{K} &= (1-\tau)AK^{\alpha}G^{1-\alpha} - C + G \end{cases}$$

together with $K(0) = K_0$ and $\lim_{t\to\infty} \frac{K(t)}{C(t)}\ e^{-\rho t} = 0$.

2. Equilibrium representation

$$\begin{cases} \dot{C} &= C\left(\alpha(1-\tau)\tilde{A}\ - \rho\right) \\ \dot{K} &= (1-\tau)\tilde{A}K^{\alpha} - C \end{cases}$$

where $\tilde{A} \equiv \left(A\tau^{1-\alpha}\right)^{\frac{1}{\alpha}}$, together with $K(0) = K_0$ and $\lim_{t\to\infty} \frac{K(t)}{C(t)}\ e^{-\rho t} = 0$.

3. The model in detrended variables

$$\begin{cases} \dot{c} &= c\left(\alpha(1-\tau)\tilde{A}\ - \rho - \gamma\right) \\ \dot{k} &= (1-\tau)\tilde{A}k^{\alpha} - c - \gamma k \end{cases}$$

together with $K(0) = K_0$ and $\lim_{t\to\infty} \frac{k(t)}{c(t)}\ e^{-\rho t} = 0$. Long run growth rate: $\bar{\gamma} = \alpha(1-\tau)\tilde{A} - \rho$

4. Equilibrium solution for y: $y(t) = \tilde{A}K_0 e^{\bar{\gamma}t}$
5. A permenent change in τ will change both the long run level and the long-run growth rate

$$\frac{\partial\bar{\gamma}}{\partial\tau} = -\left(1 - \frac{1-\alpha}{\tau}\right)\tilde{A}$$

The GE can be Pareto optimal if $\tau < 1 - \alpha$