

Foundations of Financial Economics
Deterministic GE asset pricing: two-period
case

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Functions of finance

- ▶ Finance encompasses several functions, v.g:
 - ▶ intertemporal allocation of resources, which may or may not be time-dependent (consumption smoothing)
 - ▶ inter-state of nature allocation of resources which are uncertain (insurance)
 - ▶ financing investment (increase in the resource capacity)
 - ▶ matching timing profiles of expenditures and incomes of different agents
 - ▶ matching uncertainty profiles of different agents
 - ▶ information revelation and pooling
 - ▶ distribution of income and wealth
- ▶ in this course **we will be mainly concerned with the first two functions**
- ▶ in this lecture we deal only with the **first function** in the most simple framework: **two period perfect information models**

Topics covered

- ▶ Interest rates, asset prices and the intertemporal allocation problem under perfect information
- ▶ Under several economic environments, defined by
 - ▶ Fundamentals: preferences and technology
 - ▶ Market structures: Arrow-Debreu securities, financial assets

Syllabus

- ▶ 1. Intertemporal consumption preferences
- ▶ 2. General equilibrium in a representative agent Arrow-Debreu economy
- ▶ 3. General equilibrium in an heterogeneous agent Arrow-Debreu economy
- ▶ 4. General equilibrium in a frictionless finance economy
- ▶ 5. General equilibrium in a finance economy with frictions: heterogeneous market participation

1. Intertemporal consumption preferences

Intertemporal utility

- ▶ We **index** variables by time.
- ▶ In the simplest case, we have $\mathbb{T} = \{0, 1\}$
- ▶ Consider the sequences $\{c_0, c_1\}$, where c_t is consumption in **period** $t = 0, 1$
- ▶ The **value** of a sequence $\{c_0, c_1\}$ is measured by the **intertemporal utility functional** ,

$$U = U(\{c_0, c_1\}),$$

- ▶ The **optimum** is a sequence for which U is **maximum**

Intertemporal choice

- ▶ Behavioral assumptions are implicitly introduced through the mathematical properties of U
- ▶ In this case a **intertemporal preference** structure
- ▶ **Indexing consumption by time** t has two consequences:
 - ▶ introduces an **heterogeneity**:
therefore, we can use **general** concepts and results of choice among **different goods** as in the slide **basic utility theory**
 - ▶ but it also introduces an **order relationship among consumption in different moments in time** c_0 and c_1 :
therefore, we need **particular** concepts and results related to **intertemporal arbitrage**

Intertemporal utility

As a generic utility function

- ▶ Dealing with **sequences** $\{c_0, c_1\}$ as a **vector of real non-negative numbers** $\mathbf{c} = (c_0, c_1) \in \mathbb{R}_+^2$
- ▶ Therefore the **intertemporal utility function** (IUF) can be seen as a mapping $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$,

$$U = U(c_0, c_1)$$

where U is a number allowing to rank vectors $\mathbf{c} = (c_0, c_1)$

Intertemporal utility

As a generic utility function

- ▶ First assumption: $U(\cdot)$ is **continuous** and **differentiable** in both arguments
- ▶ Second assumption: **Positive marginal utility:**

$$U_0 \equiv \frac{\partial U(c_0, c_1)}{\partial c_0} > 0, \quad U_1 \equiv \frac{\partial U(c_0, c_1)}{\partial c_1} > 0$$

Intuition: increase in consumption in any period increases utility and there is no satiation

- ▶ Third assumption: the Allen-Uzawa elasticities exist

$$\varepsilon_{tt'}(c_0, c_1) = -\frac{U_{tt'}(c_0, c_1) c_{t'}}{U_t(c_0, c_1)} \text{ for } t, t' = 0, 1$$

types of **intertemporal dependence**

$$\varepsilon_{t,t'} \begin{cases} > 0, & \text{intertemporal substitutability} \\ = 0, & \text{intertemporal independence} \\ < 0 & \text{intertemporal complementarity} \end{cases}$$

these are gross or Edgeworth relationships

Intertemporal utility

As a generic utility function: cont.

- ▶ The **intertemporal marginal rate of substitution** is defined as

$$IMRS_{0,1}(c_0, c_1) = -\frac{dc_1}{dc_0}$$

- ▶ **Intuition:** how much we are willing to sacrifice consumption at $t = 1$ (tomorrow) in order to increase one unit of consumption at $t = 0$ (today)
- ▶ For a compensated change in c_0 and c_1 such that $dU = 0$, we have

$$U_0(c_0, c_1)dc_0 + U_1(c_0, c_1)dc_1 = 0$$

then the IMRS is equal to the ratio of the marginal utilities

$$IMRS_{0,1}(c_0, c_1) = \frac{U_0(c_0, c_1)}{U_1(c_0, c_1)} \Big|_{U=\text{constant}}$$

Intertemporal utility

As a generic utility function: cont.

► Intertemporal elasticity of substitution

$$EIS_{0,1}(c_0, c_1) = \frac{d \ln(c_1/c_0)}{d \ln IMRS_{0,1}(c_0, c_1)} = \frac{c_0 U_0 + c_1 U_1}{c_1 U_1 \varepsilon_{00} - 2c_0 U_0 \varepsilon_{01} + c_0 U_0 \varepsilon_{11}}$$

$$\text{where } \varepsilon_{ij} = -\frac{U_{ij}c_j}{U_i} \text{ for } i = 0, 1 \text{ and } U_{ij} = \frac{\partial^2 U}{\partial c_i \partial c_j}$$

- **Intuition:** how much does the rate of growth of the ratio c_1/c_0 changes for a one percent increase in the *IMRS*.
This provides a measure of relative **intertemporal substitution/complementarity** of consumption
- In particular: $EIS_{0,1} > 0$ if there is intertemporal substitution, $EIS_{0,1} = 0$ intertemporal independence and $EIS_{0,1} < 0$ intertemporal complementarity (again in the Edgeworth sense)

Intertemporal choice

Introducing the time arrow

- ▶ Time can be introduced parametrically

$$U[\{c\}] = U[\{c_0, c_1\}; \mathbb{T}]$$

- ▶ **Discounting:** time is introduced via a time-weight: usually a **discount factor**

$$\{\beta_t\}_{t=0}^T = \{1, \beta, \beta^2, \dots, \beta^t \dots\}$$

where

$$\beta \equiv \frac{1}{1 + \rho}$$

where ρ is the **rate of time preference**

- ▶ **Temporal utility functions** dependent on time: example the "temporal" preferences are different from different time periods

$$U[\{c\}] = U(u_0(c_0), u_1(c_0, x_1))$$

Intertemporal utility

Main assumptions regarding intertemporal preferences

We define:

- ▶ **Stationary preferences** if the temporal utility functions are independent of time (but there can be discounting)
- ▶ **Impatience**: if there is a preference for consumption today, at $t = 0$ rather than in the future $t = 1, 2, \dots$
- ▶ Intertemporal allocation: let

$$U(c_0, c_1) = V(u(c_0), v(c_0, c_1))$$

where V is called the Koopmans aggregator. There is

- ▶ **intertemporal independence** if $v_{c_0} = 0$
- ▶ **intertemporal substitution** if $v_{c_0} < 0$
- ▶ **intertemporal complementarity** (addiction) if $v_{c_0} > 0$
- ▶ How can we identify those properties in particular utility functions ?

Intertemporal utility

Impatience and intertemporal complementarity

- ▶ Consider a stationary consumption process, i.e, $c_0 = c_1 = \bar{c}$ a constant
- ▶ **Impatience**: can be determined by using the *IMRS*. We say the **IUF displays impatience** if

$$IMRS_{0,1}(\bar{c}) = \frac{U_0(\bar{c})}{U_1(\bar{c})} > 1$$

Intuition: to keep intertemporal utility constant, if we increase c_0 by one unit the reduction in consumption in period $t = 1$ be bigger then one unit. This means that **consumption at $t = 0$ has more value than consumption at $t = 1$.**

- ▶ **Intertemporal dependence** can be determined by the Allen-Uzawa elasticity ε_{01} .

$$\varepsilon_{0,1}(\bar{c}) = -\frac{U_{01}(\bar{c}) \bar{c}}{U_0(\bar{c})} \begin{cases} > 0, & \text{intertemporal substitutability} \\ = 0, & \text{intertemporal independence} \\ < 0 & \text{intertemporal complementarity} \end{cases}$$

Intertemporal utility

Example 1: additive IUF

This is the simplest intertemporal utility function:

- **Assumption 1:** the IUF is intertemporally **additive**

$$U(c_0, c_1) = u(c_0) + \beta u(c_1), \text{ where } \beta \equiv \frac{1}{1 + \rho}$$

where $\beta \in (0, 1)$ is the psychological discount factor and ρ is the rate of time preference

- **Assumption 2:** u is increasing and concave $u''(c_t) < 0 < u'(c_t)$, $t = 0, 1$

Intertemporal utility

Additive IUF

- ▶ Marginal utilities for c_t , $t = 0, 1$ are

$$U_0 = u'(c_0), \quad U_1 = \beta u'(c_1)$$

- ▶ Derivatives of marginal utilities for c_t , $t = 0, 1$ are

$$U_{00} = u''(c_0), \quad U_{01} = 0, \quad U_{11} = \beta u''(c_1)$$

- ▶ The *IMRS* is

$$IMRS_{0,1} = \frac{U_0}{U_1} = \frac{u'(c_0)}{\beta u'(c_1)}$$

Therefore: marginal utility for period $t = 0$ is proportional to the discounted marginal utility for period $t = 1$ (from the perspective of period $t = 0$)

$$u'(c_0) = \beta u'(c_1) IMRS_{0,1}$$

we will see an analogous equation again and again translating the idea of intertemporal arbitrage.

Intertemporal utility

Additive IU

- ▶ The Allen-Uzawa elasticities are

$$\varepsilon_{00}(c_0) = -\frac{u''(c_0)c_0}{u'(c_0)}, \varepsilon_{01} = 0, \varepsilon_{11}(c_1) = -\frac{u''(c_1)c_1}{u'(c_1)}$$

- ▶ The elasticity of intertemporal substitution between period 0 and 1 is

$$EIS_{0,1}(c_0, c_1) = \frac{c_0 u'(c_0) + \beta c_1 u'(c_1)}{\beta c_1 u'(c_1) \varepsilon_{00}(c_0) + c_0 u'(c_0) \varepsilon_{11}(c_1)}$$

Intertemporal utility

Additive IUF

For a stationary consumption path $\{\bar{c}, \bar{c}\}$ we find:

- ▶ The IMRS is independent from \bar{c} and

$$IMRS_{0,1}(\bar{c}) = \frac{1}{\beta} = 1 + \rho > 1$$

this means that the IUF displays **impatience**, and this effect is captured by time discounting

- ▶ It displays **intertemporal independence** because

$$\varepsilon_{0,1}(\bar{c}) = 0$$

Intuition: the marginal valuation of consumption at time $t = 1$ is independent of the history of consumption

- ▶ The IES is

$$IES_{0,1}(\bar{c}) = -\frac{u'(\bar{c})}{u''(\bar{c})\bar{c}} > 0$$

Intuition: this is interpreted as a measure of the preference for **consumption smoothing** through time

Intertemporal utility

Additive IUF

Particular case:

- Utility function (generalized logarithm)

$$u(c) = \frac{c^{1-\zeta} - 1}{1 - \zeta}$$

- if $\zeta = 1$ we have $u(c) = \ln(c)$ (Prove this)
- Derivatives

$$U_0 = c_0^{-\zeta}, \quad U_1 = \beta c_1^{-\zeta}, \quad U_{00} = -\zeta c_0^{-\zeta-1}, \quad U_{01} = 0, \quad U_{11} = -\zeta c_1^{-\zeta-1}$$

- The IMRS is

$$IMRS_{0,1} = \frac{1}{\beta} \left(\frac{c_1}{c_0} \right)^\zeta$$

- The UA elasticities are constant $\varepsilon_{00} = \varepsilon_{11} = \zeta$
- The IES is also constant

$$EIS_{0,1} = \frac{1}{\zeta}$$

this is why ζ is called the **inverse of the elasticity of intertemporal substitution**

Intertemporal utility

Non-additive IUF

- Case 1: **Uzawa and Epstein-Hynes** utility

$$U(c_0, c_1) = u(c_0) + b(c_0)v(c_1)$$

the discount factor is endogenous i.e. $\beta = b(c)$ with $b'(\cdot) < 0$
(rich people are more patient)

The crossed AU elasticity is for a stationary sequence is

$$\varepsilon_{0,1}(c) = -\frac{b'(c)v'(c)c}{u'(c) + b'(c)v(c)}$$

displays intertemporal dependence

Intertemporal utility

Non additive IUF

► Case 2: **Habit formation**

$$U(c_0, c_1) = u(c_0) + \beta v(c_0, c_1).$$

where $v_{c_0}(c_0, c_1) < 0$.

The crossed AU elasticity is for a stationary sequence is

$$\varepsilon_{0,1}(c) = -\frac{\beta v_{c_0 c_1}(c)c}{u'(c) + \beta v_{c_0}(c)}c$$

can display intertemporal substitutability, independence or complementarity depending on the relationship between time discounting and the relative importance of habits, i.e., the magnitude of $v_{c_0}(c)$

Intertemporal utility

Case 3: habit formation example

- IUF

$$U(c_0, c_1) = \ln(c_0) + \beta \ln\left(\frac{c_1}{c_0}\right)^\zeta, \quad \zeta > 0$$

- Derivatives

$$U_0 = \frac{1 - \beta\zeta}{c_0}, \quad U_1 = \frac{\beta\zeta}{c_1}, \quad U_{00} = -\frac{1 - \beta}{c_0^2}, \quad U_{01} = 0, \quad U_{11} = -\frac{\beta\zeta}{c_1^2}$$

- The IMRS is

$$IMRS_{0,1}(c_0, c_1) = \left(\frac{1 - \beta\zeta}{\beta\zeta}\right) \frac{c_1}{c_0}$$

- The AU elasticities are constant

$$\varepsilon_{00} = \varepsilon_{11} = 1, \quad \varepsilon_{01} = 0$$

- The IES is also constant

$$EIS_{0,1}(c_0, c_1) = 1$$

for any (c_0, c_1)

Intertemporal utility

Case 2: habit formation example, cont

For a stationary sequence $c_0 = c_1 = c$

- The IMRS

$$IMRS_{0,1}(c) = \frac{1 - \beta\zeta}{\beta\zeta}$$

the utility displays impatience if $\zeta < \frac{1}{2\beta} = \frac{1 + \rho}{2}$.

Intuition: there is impatience (according to the above definition) if the weight of past consumption is not too strong

- As $\varepsilon_{01} = 0 = 0$ the model displays intertemporal independence (but this is special to this example).

2. General equilibrium in a representative agent Arrow-Debreu economy

Two-period general equilibrium models

- ▶ Next we will address the (macro) **determination of the interest rate** in two-period general equilibrium models under perfect information (i.e., certainty)
- ▶ We consider two (equivalent) approaches and models
 - ▶ a micro-economic approach: Arrow-Debreu simultaneous equilibrium economy
 - ▶ a finance (or macro-finance) approach: a finance sequential equilibrium economy
- ▶ For each model we proceed in two steps:
 - ▶ present and solve the consumer problem in each economy
 - ▶ we define and determine the general equilibrium

Arrow Debreu model

The consumer problem: set-up

- ▶ A consumer has an asset (resource, endowment) in positive amount ($w > 0$) which allows for a sequence of consumption in two periods, $\{c_0, c_1\}$, today c_0 and in the future c_1 .
- ▶ There is a market for **forward contracts** allowing for contracting today for delivery in the future, at a price set today, $q > 0$. We take the price paid today as a *numéraire* and all the variables are denominated at today's price
- ▶ The value of the consumption sequence is assessed by an intertemporal utility function: $U(c_0, c_1)$;
- ▶ The **budget constraint**, referring to payments made today, is

$$c_0 + q c_1 \leq w$$

Arrow Debreu model

The consumer problem

- Formally, the intertemporal problem for the consumer is

$$v(w) = \max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + q c_1 \leq w \}$$

- The (interior) optimum (c_0^*, c_1^*) satisfies the conditions

$$\begin{cases} qU_0(c_0^*, c_1^*) = U_1(c_0^*, c_1^*) & \text{(optimality condition)} \\ c_0^* + pc_1^* = w & \text{(budget constraint)} \end{cases}$$

Arrow Debreu model

Optimality: interpretation

- ▶ At the optimum: the IMRS is equal to the relative price (internal = external valuation)

$$IMRS_{0,1}^* = IMRS_{0,1}(c_0^*, c_1^*) = \frac{U_0(c_0^*, c_1^*)}{U_1(c_0^*, c_1^*)} = \frac{1}{q}$$

- ▶ Intuition: at the optimum **increasing one euro of consumption tomorrow should be matched by a reduction in $1/q$ euro of consumption today, ie $dc_0^* = -qdc_1^*$**
- ▶ Therefore q is an **intertemporal relative price**: i.e., an opportunity cost for changing the consumption sequence across time.

Arrow Debreu model

Assumptions

- ▶ Now, we go from a microeconomic to a macroeconomic perspective
- H1 Assume there is perfect information: **deterministic general equilibrium**
- H2 Assume all agents are equal: **representative agent economy**
- H3 Assume that there is an exogenous sequence of resources sustaining consumption: **endowment economy**
- H4 Assume that all trade is done at time $t = 0$: an **Arrow-Debreu economy**
 - ▶ We want to determine (endogenously) the price q : the **Arrow-Debreu price**

Arrow Debreu model

The setup

- ▶ Assume that the resource of the economy takes the form of a **flow** of non-durable goods, that can be collected both at time $t = 0$ and $t = 1$, $\{y_0, y_1\}$.
- ▶ Again, assume that **trade can only take place at time $t = 0$** : this means that the price for contracts for delivery at time $t = 1$ has to be set at time $t = 0$. We call q the **Arrow-Debreu price**
- ▶ From this, wealth at time $t = 0$ is equal to the **present value of the flow of endowments**

$$w = y_0 + q y_1$$

Arrow Debreu model

The setup: continuation

- ▶ All the participants have **perfect information** (over prices and endowments referring to period $t = 1$) and solve their micro-economic problems;
- ▶ At every period, total consumption must be equal to total endowment (market equilibrium);
- ▶ **Representative agent economy**: we assume that all consumers solve the same problem (same utility function and same endowments);
- ▶ What is the equilibrium forward price q ?

Arrow Debreu model

General equilibrium for a representative agent economy

General equilibrium in this economy is defined by (c_0^*, c_1^*, q^*) such that

- ▶ the consumer solves the problem

$$\max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + q c_1 \leq y_0 + q y_1 \}$$

- ▶ markets clear

$$c_0 = y_0,$$

$$c_1 = y_1$$

Arrow Debreu model

General equilibrium for a representative agent economy

- ▶ General equilibrium conditions: (c_0, c_1, q) is determined from

$$\begin{cases} qU_0(c_0, c_1) = U_1(c_0, c_1) & (\text{micro: optimality condition}) \\ c_0 + qc_1 = y_0 + qy_1 & (\text{micro: budget constraint}) \\ c_0 = y_0 & (\text{aggregate: market clearing for } t=0) \\ c_1 = y_1 & (\text{aggregate: market clearing for } t=1) \end{cases}$$

- ▶ There are only three independent conditions (Walras's law)

$$\begin{cases} qU_0(c_0, c_1) = U_1(c_0, c_1) \\ c_0^* = y_0 \\ c_1^* = y_1 \end{cases}$$

- ▶ In a representative agent economy there is **no trade**
(consumption is equal to the endowment)

Arrow Debreu model

Equilibrium AD price

- ▶ Then **the equilibrium AD price** is

$$q^* = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)}$$

- ▶ **We need more structure on preferences to get explicit results**

Arrow Debreu model

Equilibrium discount factor

- ▶ We call $m = IMRS_{0,1}$ the **discount factor** where

$$m = 1 + r$$

where r is the **risk-free market interest rate**

- ▶ Equivalently: the **(deterministic) equilibrium discount factor** is

$$m^*(y_0, y_1) = \frac{1}{q^*} = \frac{U_0(y_0, y_1)}{U_1(y_0, y_1)}$$

- ▶ In an AD economy the discount factor, and therefore the interest rate, depends on the present and future endowments $r^* = r(y_0, y_1)$.

Arrow Debreu model

AD price and utility functions

- ▶ For an intertemporally additive utility function

$$q^*(y_0, y_1) = \beta \frac{u'(y_1)}{u'(y_0)}$$

- ▶ concavity of $u(\cdot)$, i.e., $u''(c) < 0$, implies

$$\frac{\partial q^*(y_0, y_1)}{\partial y_0} = -\beta \frac{u'(y_1)u''(y_0)}{(u'(y_0))^2} > 0$$

and

$$\frac{\partial q^*(y_0, y_1)}{\partial y_1} = \beta \frac{u''(y_1)}{u'(y_0)} < 0$$

- ▶ Then it increases (decreases) with an excess supply of present (future) **relative** to future (present) supply

Arrow Debreu model

AD discount factor and utility functions

- ▶ For an intertemporally additive utility function

$$m^*(y_0, y_1) = \beta \frac{u'(y_0)}{u'(y_1)}$$

- ▶ concavity of $u(\cdot)$, i.e., $u''(c) < 0$, implies

$$\frac{\partial m^*(y_0, y_1)}{\partial y_0} = \beta \frac{u''(y_0)}{u'(y_1)} < 0, \quad \frac{\partial m^*(y_0, y_1)}{\partial y_1} = -\beta \frac{u'(y_0)u''(y_1)}{(u'(y_1))^2} > 0$$

- ▶ As $m = 1 + r$ then

$$\frac{\partial r^*(y_0, y_1)}{\partial y_0} < 0, \quad \frac{\partial r^*(y_0, y_1)}{\partial y_1} > 0$$

- ▶ The interest rate decreases (increases) with an excess supply of present (future) **relative** to future (present) supply

Arrow Debreu model

AD price and utility functions

- ▶ Example: if $u(c) = \ln(c)$ then

$$q^*(y_0, y_1) = \beta \frac{y_0}{y_1} = \frac{\beta}{1 + \gamma}$$

or, if we set $y_1 = (1 + \gamma)y_0$ where γ is the rate of growth

- ▶ or equivalently

$$m^*(y_0, y_1) = \frac{1}{\beta} \frac{y_1}{y_0} = \frac{1 + \gamma}{\beta} = (1 + \gamma)(1 + \rho)$$

- ▶ or

$$1 + r^* = (1 + \gamma)(1 + \rho)$$

interest rate grows with the rate of time preference
(impatience) and the anticipated rate of growth

Arrow Debreu model

AD price and utility functions

- For the habit formation utility function

$$q^*(y_0, y_1) = \beta \frac{v_{c_1}(y_0, y_1)}{u'(y_0) + \beta v_{c_0}(y_0, y_1)}$$

- Example: setting $U(c_0, c_1) = \ln(c_0) + \beta \ln \left[\left(\frac{c_1}{c_0} \right)^\zeta \right]$ displaying intertemporal substitution then

$$q^* = \frac{\beta\zeta}{y_1} \left(\frac{1}{y_0} - \beta\zeta \frac{1}{y_0} \right)^{-1} = \frac{\beta\zeta y_0}{(1 - \beta\zeta)y_1} = \frac{\beta\zeta}{(1 - \beta\zeta)(1 + \gamma)}$$

has the same properties if $\beta\zeta < 1$

3. General equilibrium in an heterogeneous agent Arrow-Debreu economy

Arrow Debreu model

Assumptions

- ▶ The previous model is more general than it looks

H1 idem

H2 Assume heterogeneity in endowments

H3 idem

H4 idem

- ▶ What are the consequences for the equilibrium q

Arrow-Debreu economy

Beyond the representative agent case

- ▶ Assumptions: there are two groups of agents
 - ▶ **with the same preferences**
 - ▶ **but endowments are different** (y_t^i is the endowment of agent i at time t)

$$y^1 = \{y_0^1, y_1^1\}, y^2 = \{y_0^2, y_1^2\}$$

and we assume $y^1 \neq y^2$

- ▶ The aggregate **flow of total endowments** are

$$y_0 = y_0^1 + y_0^2$$

$$y_1 = y_1^1 + y_1^2$$

- ▶ The general equilibrium is now

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

General equilibrium in this economy is defined by the allocations $(c_0^{1*}, c_1^{1*}, c_0^{2*}, c_1^{2*})$ and the price q^* such that

- ▶ consumer $i \in \{1, 2\}$ solves the problem

$$\max_{c_0^i, c_1^i} \{ U(c_0^i, c_1^i) : c_0^i + q c_1^i \leq y_0^i + q y_1^i \}, \text{ for } i = 1, 2$$

- ▶ market clearing hold for $t = 0, 1$,

$$c_0 = y_0, \quad c_1 = y_1$$

where $c_t = c_t^1 + c_t^2$ for $t = 1, 2$ and $y_t = y_t^1 + y_t^2$

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- General equilibrium conditions (considering that the Walras' law holds)

$$\left\{ \begin{array}{ll} qU_0(c_0^1, c_1^1) = U_1(c_0^1, c_1^1) & \text{(optimality condition for agent 1)} \\ qU_0(c_0^2, c_1^2) = U_1(c_0^2, c_1^2) & \text{(optimality condition for agent 2)} \\ c_t = y_t & \text{(market clearing for period } t = 1, 2) \\ c_t = c_t^1 + c_t^2 & \text{(aggregation of consumption for } t) \\ y_t = y_t^1 + y_t^2 & \text{(aggregation of endowment for } t) \end{array} \right.$$

- In this case there can be trade, because $c_t^1 - y_t^1 = y_t^2 - c_t^2$ can be different from zero, but the budget constraint should hold for every agent. (check this !)

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- ▶ Because we assumed homogeneity in preferences $U(.,.)$ is the same for both consumers.
- ▶ Therefore, it also holds for the aggregate consumption

$$qU_0(c_0, c_1) = U_1(c_0, c_1)$$

that is

$$qU_0(c_0^1 + c_0^2, c_1^1 + c_1^2) = U_1(c_0, c_1)$$

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- ▶ Using the market clearing conditions we have again

$$q^* = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)} = \frac{U_1(y_0^1 + y_0^2, y_1^1 + y_1^2)}{U_0(y_0^1 + y_0^2, y_1^1 + y_1^2)}$$

- ▶ Conclusion: if agents are **homogeneous as regards preferences** but are **heterogeneous as regards endowments** the **distribution of income between agents has no influence** the AD price. It is only determined by the aggregate endowment
- ▶ If there is heterogeneity in preferences, this result **will not hold** in general.

4. General equilibrium in a frictionless finance economy

Finance economy model

Assumptions

► Now we change the market structure

H1 idem

H2 Assume a representative agent economy

H3 idem

H4 Assume a sequence of asset markets

► What is the equilibrium asset price

Finance economy model

The economy

- ▶ Assume there is a spot market for the good opening at every period $t = 0$ and $t = 1$;
- ▶ There is an asset (that can be seen as a durable good) that agents can lend and borrow at period $t = 0$ paying or receiving an interest income at period $t = 1$. The asset is in non-negative net supply at the beginning to period $t = 0$ and there is a market for the asset at time $t = 0$.
- ▶ We still assume that the agent receives a flow of endowments $y = \{y_0, y_1\}$ The agent can consume the totality of the income, or not, at the end of period 1
- ▶ Every agent has now a **sequence of budget constraints** (because trade in the good market can take place at period 1)

Finance economy model

Micro-economic problem in the finance economy

- ▶ The problem (assuming an additive intertemporal utility)

$$\max_{c_0, c_1, a_1, a_2} U(c_0, c_1) = u(c_0) + \beta u(c_1) :$$

- ▶ subject to

$$\begin{cases} c_0 + a_1 = y_0 + a_0 \\ c_1 = y_1 + (1 + r)a_1 - a_2 \end{cases}$$

where a_0 is the level of the asset at beginning of period 0 and a_1 and a_2 are the levels at the end of period 0 and 1, and r is the real interest rate.

- ▶ other constraints:

$$c_0 \geq 0, c_1 \geq 0, a_1 \text{ free}, a_2 \geq 0$$

- ▶ Next, we prove that, it will never be optimal to have $a_2 > 0$

Finance economy model

Optimality of $a_2 = 0$

- ▶ Substitute c_0 and c_1 in the utility function, assume that $\beta > 0$ and r is finite, and consider the constraint for a_2

$$\max_{a_1, a_2} \{u(y_0 + a_0 - a_1) + \beta u(y_1 + (1 + r)a_1 - a_2) : a_2 \geq 0\}$$

- ▶ The first order conditions are

$$\begin{aligned}u'(c_0) &= \beta(1 + r)u'(c_1) \\ \beta u'(c_1) &= \lambda \\ \lambda a_2 &= 0, \lambda \geq 0, a_2 \geq 0\end{aligned}$$

- ▶ We have $a_2 > 0$ if and only if $\lambda = 0$, but in this case either there is satiation or $c_1 \rightarrow \infty$ and $c_0 \rightarrow \infty$. But this is only possible if $a_0 \rightarrow \infty$. Therefore we should have $a_2 = 0$ and $\lambda > 0$.

Finance economy model

The consumer problem in a frictionless case

- ▶ Taking $a_2 = 0$ and assuming a_1 is free (i.e., the consumer can borrow or lend freely) we can eliminate a_1 in the sequence of budget constraints, to get

$$c_0 + mc_1 = a_0 + y_0 + my_1$$

where m is the **market discount factor**

$$m \equiv \frac{1}{1+r} \equiv \frac{1}{R}$$

- ▶ This implies that the **sequence of budget constraints** is equivalent to an **intertemporal budget constraint** formally similar to the constraint in the Arrow-Debreu economy.

$$c_0 + mc_1 = y_0 + mc_1$$

Finance economy without frictions

General equilibrium for a representative agent finance economy

General equilibrium in this economy is defined by (c_0^*, c_1^*, m^*) such that

- ▶ the consumer solves the problem

$$\max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + mc_1 = a_0 + y_0 + mc_1 \}$$

- ▶ market clearing hold

$$c_0 = y_0, \quad c_1 = y_1$$

Finance economy without frictions

General equilibrium for a representative agent finance economy

- ▶ The equilibrium equations are (from Walras's law)

$$mu'(c_0) = \beta u'(c_1)$$

$$c_0 = a_0 + y_0$$

$$c_1 = y_1$$

- ▶ The **equilibrium discount factor** is

$$m^* = m(a_0, y_0, y_1) = \beta \frac{u'(y_1)}{u'(a_0 + y_0)}$$

- ▶ Because $R = \frac{1}{m}$ and $\beta = \frac{1}{1 + \rho}$ where ρ is the psychological discount factor

Finance economy without frictions

Asset return in a frictionless economy

- ▶ The **equilibrium asset return** (recall)

$$R^* = 1 + r^* = (1 + \rho) \frac{u'(a_0 + y_0)}{u'(y_1)}$$

- ▶ But $R^* = R(a_0, y_0, y_1)$, with partial derivatives

$$\frac{\partial R}{\partial a_0} = \frac{\partial R}{\partial y_0} = (1 + \rho) \frac{u''(a_0 + y_0)}{u'(y_1)} < 0$$

$$\frac{\partial R}{\partial y_1} = -(1 + \rho) \frac{u''(y_1)u'(a_0 + y_0)}{(u'(y_1))^2} > 0$$

- ▶ There are two main effects:
 - ▶ a direct effect: high y_0 or a_0 **reduce** the interest rate
 - ▶ an anticipation effect: high y_1 **increases** the interest rate

5. General equilibrium in a finance economy with frictions: heterogeneous market participation

A simple finance economy with frictions

Assumptions

► Now we introduce heterogeneity

H1 idem

H2 Assume agents face financing constraints

H3 idem

H4 idem

► What is the equilibrium asset price

Finance economy with heterogeneity

Heterogenous participation

- ▶ Assume there are two agents in the economy: agent b is a borrower and agent l is a lender, the only one that has positive assets at time 0 ($a_0^l > 0$, $a_0^b = 0$)
- ▶ To simplify, assume agent b is the only one that receives the flow of endowments $\{y_0, y_1\}$ and agent b can only earn interest income
- ▶ Assume there are no constraints in the credit market
- ▶ Assume that agents have homogeneous preferences

Finance economy with heterogeneity

Agents' problems

- ▶ The **lender's** problem is

$$\max_{c_0^l, c_1^l} \{u(c_0^l) + \beta u(c_1^l) : c_0^l + l^l = a_0, c_1^l = (1+r)l^l\}$$

- ▶ Because l^l is free it can be simplified to

$$\max_{l^l} \{u(a_0 - l^l) + \beta u((1+r)l^l)\}$$

- ▶ The optimality condition is

$$u'(a_0 - l^l) = \beta(1+r)u'((1+r)l^l)$$

or equivalently

$$u'(c_0^l) = \beta(1+r)u'(c_1^l)$$

Finance economy with heterogeneity

Agents' problems

- ▶ The **borrower** problem is

$$\max_{c_0^b, c_1^b} \{u(c_0^b) + \beta u(c_1^b) : c_0^b = y_0 + l^b, c_1^l + (1+r)l^b = y_1\}$$

- ▶ Because l^b is free it can be simplified to

$$\max_{l^b} \{u(y_0 + l^b) + \beta u(y_1 - (1+r)l^b)\}$$

- ▶ The optimality condition is

$$u'(y_0 + l^b) = \beta(1+r)u'(y_1 - (1+r)l^b)$$

or equivalently

$$u'(c_0^b) = \beta(1+r)u'(c_1^b)$$

Finance economy with heterogeneity

Equilibrium equations

- The equilibrium equations are

$$u'(c_0^l) = \beta(1+r)u'(c_1^l)$$

$$u'(c_0^b) = \beta(1+r)u'(c_1^b)$$

$$c_0^l + c_0^b = y_0 + a_0$$

$$c_1^l + c_1^b = y_1$$

- Because preferences are homogeneous we can use the same argument as before, to get

$$u'(y_0 + a_0) = \beta(1+r)u'(y_1)$$

Finance economy with heterogeneity

Equilibrium interest rate

- ▶ The **equilibrium return** is again

$$R^* = 1 + r^* = (1 + \rho) \frac{u'(y_0 + a_0)}{u'(y_1)}$$

- ▶ Is formally similar to the representative agent economy case;
- ▶ Again, we have
 - ▶ negative liquidity effect $R_{a_0}^* < 0$;
 - ▶ a negative income effect, $R_{y_0}^* < 0$
 - ▶ a positive anticipation effect, $R_{y_1}^* > 0$
- ▶ Conclusion: without other sources of heterogeneity, limited participation has no effect on the market interest rate.

Taking the model to data

- ▶ In the long run we have (see [introduction](#)):
 - ▶ an upward trend of the growth rates
 - ▶ a downward trend of the interest rates
- ▶ this seems to be at odds with the model
- ▶ why ? What is missing in the model
- ▶ In the shorter run it appears it can do better

Taking the model to data

- ▶ data from <http://www.nber.org/papers/w24112.pdf>
 $R_{safe} = 1.0188$ (average safe return) $R_{wealth} = 1.0678$ (average wealth return) $\gamma = 0.0287$ (average rate of growth)
- ▶ calibrated parameters: $\rho = 0.02$
- ▶ Utility functions
 - ▶ isoelastic utility function

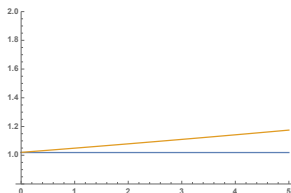
$$U(c_0, c_1) = \frac{c_0^{1-\zeta} - 1}{1-\zeta} + \beta \frac{c_1^{1-\zeta} - 1}{1-\zeta}$$

- ▶ habit formation:

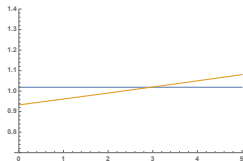
$$U(c_0, c_1) = \ln c_0 + \beta \left[\frac{\left(\frac{c_1}{c_0}\right)^{\phi(1-\zeta)} - 1}{1-\zeta} \right]$$

Taking the model to data

- ▶ Additive utility: interest rate puzzle (the model over predicts the observed risk-free interest rate, for any value of the EIS)



- ▶ Habit formation: it is possible to find values for the parameter ϕ , in the case $\phi \approx 0.5$ such that the model matches the observed R for "acceptable" values for ζ



Questions

- ▶ The previous results hold for cases in which there is
 - ▶ full information (deterministic general equilibrium)
 - ▶ agents have homogeneous preferences (with or without homogeneous resources)
 - ▶ frictionless economy (for the case of a finance economy)
- ▶ Do those results hold under:
 - ▶ imperfect information (uncertainty) ?
 - ▶ heterogeneity in agents' preferences ?
 - ▶ frictions in a finance economy (ex: credit constraints) ?