Economic Growth Theory:
Problem set 1: Malthusian and Solow models

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Malthusian growth theory

1. Assume that the representative consumer solves the problem: $\max_{c,b}\{u(c,b) : c + \rho b \leq y\}$ where c is consumption, b is the birth rate, ρ is the cost of raising children and y is income. Assume that the utility function is

$$u(c,b) = \frac{(cb^{\phi})^{1-\theta}}{1-\theta},\ \theta > 0,\ \phi > 0$$

and the aggregate production function is Cobb-Douglas $Y = X^{\alpha}L^{1-\alpha}$, with $0 < \alpha < 1$, where X is the stock of land. Population growth is $\dot{L}/L = b - m$, where the mortality rate, m, is constant and exogenous, and $L(0)$ is given.

   (a) Prove that the fertility rate is $b = (\phi/(1+\phi))(y/\rho)$, and determine the dynamic equation for population growth.

   (b) Determine the steady state population level and study their dynamic behaviour.

   (c) What is the effect of an increase in m and ρ ?

   (d) Supply an economic intuition for the the previous results.

2. Assume that the representative consumer solves the problem: $\max_{c,b}\{u(c,b) : c + \rho b \leq y\}$ where c is consumption, b is the birth rate, ρ is the cost of raising children and y is income. Assume that the utility function is

$$u(c,b) = \frac{(cb^{\phi})^{1-\theta}}{1-\theta},\ \theta > 0,\ \phi > 0$$

and the aggregate production function is Cobb-Douglas $Y = AX^{\alpha}L^{1-\alpha}$, with $0 < \alpha < 1$, where A is the total factor productivity, X is the stock of land. Population growth is $\dot{L}/L = b - m$, where the mortality rate, m, is constant and exogenous, and $L(0)$ is given. Assume that the total factor productivity grows with a constant rate such that $\alpha m > \gamma > 0$

   (a) The per capita GDP in efficiency units is $y = Y/(LA)$. Write a differential equation for y.

   (b) Study, qualitatively, the solution for that differential equation.

   (c) Does long run growth exists for this economy ? Supply an economic intuition.

3. Let the dynamics of population be described by the differential equation $\dot{L} = (b - m)L$, where L is total population. Assume that both the birth and the mortality rates are functions of per capita GDP, y, where $b = \beta y$ and $m = \mu/y$, for $\beta > 0$ and $\mu > 0$. The technology for this economy is represented by the production function $Y = X^{\alpha}L^{1-\alpha}$ where $0 < \alpha < 1$, and X denoted the land endowment. The initial population is given, $L(0) = L_0$.

   (a) Write an equation for the population dynamics.

   (b) Determine the steady state level of population and study their dynamic behaviour.

   (c) What are the effects of a reduction in the mortality rate parameter μ, on population and the per capita GDP ?

   (d) Does long run growth exists for this economy ? Provide an economic intuition.

4. Assume that the representative consumer solves the problem: $\max_{c,b}\{u(c,b) : c + \rho b \leq y\}$ where c is consumption, b is the birth rate, ρ is the cost of raising children and y is per capita income. Assume that the utility function is

$$u(c,b) = \ln(c) + \phi \ln(b),\ \phi > 0$$

and the aggregate production function is Cobb-Douglas $Y = (AX)^{\alpha} L^{1-\alpha}$, with $0 < \alpha < 1$, where X is the stock of land, A is land-specific productivity and L is population. Population growth is $\dot{L}/L = b - m$, where the mortality rate, m, is constant and exogenous, and $L(0) = L_0 > 0$ is given. Land productivity grows at a rate $\gamma > 0$.

(a) Defining $\ell \equiv L/A$, obtain a differential equation for ℓ.

(b) Study the qualitative dynamics of the model. Provide an intuition for your results.

(c) Derive the growth facts (long run growth rate, long run per capita output and transition dynamics). What are the effects of an increase in γ ?

5. Assume that the representative consumer solves the problem: $\max_{c,b}\{\ln(cb^{\phi}) : c + \rho b \leq y\}$ where c is consumption, b is the birth rate, $\rho > 0$ is the cost of raising children, $\phi > 0$ is the love-for-children parameter, and y is per capita income. The aggregate production function is CES

$$Y = \Big(\alpha(AX)^{\eta} + (1-\alpha)L^{\eta}\Big)^{\frac{1}{\eta}},$$

with $0 < \alpha < 1$ and $\eta > 0$, where X is the stock of land, A is land-specific productivity and L is population. Population growth is $\dot{L}/L = b - m$, where the mortality rate, m, is constant and exogenous, and $L(0) = L_0 > 0$ is given. Land productivity grows at a rate $\gamma > 0$.

(a) Prove that the differential equation for the per capita product, $y \equiv Y/L$ is

$$\dot{y} = \Big(1 - \alpha - y^{\eta}\Big)\Big(\beta y - m\Big)\, y^{1-\eta},$$

where $\beta \equiv \frac{\phi}{\rho(1+\phi)} > 0$.

(b) Prove that if $y(0) < \min\left\{\frac{m}{\beta}, (1-\alpha)^{\frac{1}{\eta}}\right\}$ the economy will collapse, i.e. $\lim_{t\to\infty} y(t) = 0$ and if $y(0) > \min\left\{\frac{m}{\beta}, (1-\alpha)^{\frac{1}{\eta}}\right\}$ the economy will converge to $\max\left\{\frac{m}{\beta}, (1-\alpha)^{\frac{1}{\eta}}\right\}$ (Hint: draw the phase diagram)

(c) Discuss the economic intuition of those results.

6. Assume two economies $i = E, P$ which are equal in every respect, except that economy E obtained an increase in its land endowment (for instance by becoming an empire). Thus $X_E > X_P$. In the two economies there is a representative farmer who solves the problem: $\max_{c,b}\{u(c,b) : \; c_i + \rho b_i \leq y_i\}$ where c_i is consumption, b_i is the birth rate, and y_i is per-capita income, in country $i = E.P$, and ρ is the cost of raising children. Assume that the utility function is

$$u(c,b) = \ln\left(cb^{\phi}\right), \; \phi > 0$$

and the aggregate production function for country i is Cobb-Douglas $Y_i = AX_i^{\alpha}L_i^{1-\alpha}$, with $0 < \alpha < 1$, where A is the total factor productivity, X is the stock of land. Population growth is $\dot{L}/L = b - m$, where the mortality rate, m, is constant and exogenous, and $L(0)$ is given.

   (a) Write a differential equation for y_i.

   (b) What are the growth consequences to become an empire for country E ?

   (c) Is that realistic ? How would you change the model in order to obtain growth effects from increasing X_E ?

Solow growth theory

1. Consider a version of the Solow model, in which: (1) the savings function is $S(t) = sY(t)$, with $0 < s < 1$; (2) the population, L grows at a constant rate $n > 0$, $\dot{L} = nL(t)$, (3) there is no depreciation of capital, and (4) the technology is linear

$$Y(t) = AK(t)$$

   (a) Derive the accumulation equation for the detrended capital stock $k(t) \equiv K(t)/L(t)$.

   (b) Determine analyticaly the long run level for k, and discuss its economic meaning.

   (c) Will there be transitional dynamics in this model ?

   (d) Interpret the results for the properties of the model regarding the existence of a balanced growth path, the existence of transition dynamics, the existence of endogenous growth, and the effects of n over long run growth, transition, and the level effects.

2. Consider a version of the Solow model, in which: (1) the savings function is $S(t) = sY(t)$, with $0 < s < 1$; (2) the population, L grows at a constant rate $n > 0$, $\dot{L} = nL(t)$, (3) there is no depreciation of capital, and (4) the production technology is given by a Cobb-Douglas function

   (a) Derive the dynamic equation for the detrended output $y(t) \equiv Y(t)/L(t)$.

   (b) Solve the equation in (a) explicitly.

   (c) Solve the equation in (a) by approximation methods.

   (d) Supply an intuition for the results you obtained.

3. Consider a version of the Solow model, in which: (1) the savings function is $S(t) = sY(t)$, with $0 < s < 1$; (2) the population, L grows at a constant rate $n > 0$, $\dot{L} = nL(t)$, (3) there is no depreciation of capital, and (4) the technology is CES (constant elasticity of substitution)

$$Y(t) = F(K(t), L(t)) = \left(\alpha K(t)^{-\eta} + (1-\alpha)L(t)^{-\eta}\right)^{-1/\eta},\ 0 < \alpha < 1,\ \eta > -1,\ \eta \neq 0$$

   (a) Derive the accumulation equation for the detrended capital stock $k(t) \equiv K(t)/L(t)$.

(b) Determine analytically the long run level for k, its stability properties, and discuss its economic meaning.

(c) Study the effect of a permanent increase in n on the long run growth, transition, and the level of the product.

4. Consider a version of the Solow model, in which there are two types of labor: skilled L_s, and unskilled labor L_u. The proportion of population with each skill is constant, such that $\ell = L_u/L$ and $1-\ell = L_s/L$, where $0 < \ell < 1$. The total population, L, grows at a constant rate $n > 0$. The technology of production involves a complementarity between capital and unskilled labor and a substitution between them and skilled labor. It is represented by the production function

$$Y(t) = (K(t) + L_u(t))^{\alpha}(AL_s(t))^{1-\alpha}$$

where $0 < \alpha < 1$ and $A > 1$ measures the specific productivity of skilled labor. The savings function is $S(t) = sY(t)$, with $0 < s < 1$, and there is no depreciation of capital.

(a) Derive the accumulation equation for the detrended capital stock $k(t) \equiv K(t)/L(t)$.

(b) Prove there is a unique long run level for k. Is uniqueness related to the Inada properties , for k , of the production function ?

(c) Describe the properties of the model regarding the existence of a balanced growth path, of transition dynamics and of endogenous growth.

(d) Assume there is a permanent increase in the proportion of unskilled labour ℓ. Determine the effects over long run growth, the level effects, and the transitional dynamics. (Hint: assume that $\ell < \alpha$ and $s\alpha^{\alpha}(A(1-\alpha))^{1-\alpha} > n$).

5. Consider a version of the Solow (1956) model in which the production function is of the VES (variable elasticity of substitution) type

$$F(K,H) = AK^{\alpha}\left[H + \alpha\beta K\right]^{1-\alpha},\ A > 0,\ 0 < \alpha < 1,\ \beta > -1$$

where K is the stock of physical capital and H is the stock of human capital. Human capital is produced by means of a linear production function $dH(t)/dt = \gamma H(t)$, with

$\gamma > 0$. The accumulation of physical capital is given by $dK(t)/dt = sF(H(t), K(t)) - \delta K(t)$ where $s > 0$ and $\delta > 0$.

(a) Does the production function verifies the necessary conditions for the existence of a balanced growth path ? If your answer is affirmative, derive the expression for the physical capital along the BGP ?

(b) Define $k(t) \equiv K(t)/H(t)$ and determine the accumulation equation for k.

(c) Determine the stationary level k^*, by introducing assumptions over the parameters such that we have $k^* > 0$. With the same assumptions, characterise the local dynamics for $k(t)$.

(d) Supply an intuition for the decomposition of the variation of $K(t)$ between long run growth and transition.

6. Consider a growth model in which technological innovation takes the form of variety expansion. Assume: (1) the final good is produced by a continuum of $N(t)$ perfectly substitutable intermediate goods (i.e., varieties); (2) each variety is produced by a monopolist, although there is free entry in the markets for intermediate goods; (3) the introduction of a new variety, and therefore of a new producer, is done after R&D activities are developed. With those assumptions, we can determine: (1) the production of the final good as a function of the number of varieties

$$Y(t) = \phi N(t), \text{ where } \phi \equiv \left(A\alpha^{2\alpha}\right)^{1/(1-\alpha)}$$

where $0 < \alpha < 1$ is the elasticity of substitution among varieties and $A > 0$ is the exogenous productivity parameter, in the production of the final good; (2) the aggregate income which is generated as a result of imperfect competition in the production of intermediate goods is $X(t) = \alpha^2 \phi N(t)$, and (3) the expenditure in R&D activities is $I(t) = \eta \phi \dot{N}$, where $\eta > 0$ is the average cost of development of a new variety, in units of the final good. Assume, further, that the consumption function for the final good is $C(t) = (1 - s)(Y(t) - X(t))$ where $0 < s < 1$ is the savings rate. The equilibrium condition for the final good market is $Y(t) = C(t) + I(t) + X(t)$.

(a) Derive an equation for the accumulation of varieties.

(b) Determine the solution for $Y(t)$, assuming that $N(0)$ is given. Characterize the growth dynamics for this economy. Give an economic intuition for your results

7. Assuming that the Solow model is a good representation of two economies, A and B. The economies have the same technology of production and the same demographic data, but differ as regards the initial capital intensity k and the savings rate. Let the Solow accumulation equation be

$$\dot{k}_i = s_i A k_i(t)^\alpha - n k_i(t),\ i = A, B.$$

Assume that: $k_A(0) > k_B(0)$, $1 > s_B > s_A > 0$, $A > 0$, $0 < \alpha < 1$ and $n \geq 0$

(a) Characterize the differences in the growth dynamics between the two countries.

(b) Will there be convergence ? If affirmative, which kind of convergence ?

(c) Assuming there is any form of catch up, how can we measured its timing ?

8. Assume that the Solow model is a good representation of the capital accumulation dynamics for two countries, labelled by 1 and 2, respectively. Let the economies have the same preferences and the same demographic data, but differ as regards the initial capital intensity, $k_i(0)$ and the TFP. The Solow accumulation equation would be

$$\dot{k}_i = s A_i k_i(t)^\alpha - n k_i(t),\ i = 1, 2.$$

Assume that: $k_1(0) > k_2(0)$, $A_1 < A_2$, $0 < s < 1$, $0 < \alpha < 1$ and $n \geq 0$.

(a) Characterize the differences in the growth dynamics between the two countries.

(b) Will there be convergence ? If affirmative, which kind of convergence ?

(c) Assuming there is some form of catch up, provide a measure of its timing ?

9. Consider a model in which the production function is Cobb-Douglas, $Y = AK^\alpha L^{1-\alpha}$, population grows as $\dot{L} = nL$ with $n > 0$, there is no depreciation and the consumption function is $C = \nu K$.

(a) Derive the accumulation equation for $k(t) = K(t)/L(t)$.

(b) Determine the interior steady state and study its dynamic properties.

(c) Interpret the growth dimensions (long run rate of growth, long run level and transitional dynamics) which are present in this model. Give an economic interpretation.