Advanced Mathematical Economics

Paulo B. Brito
PhD in Economics: 2022-2023
ISEG
Universidade de Lisboa
pbrito@iseg.ulisboa.pt

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Contents

Chapter 3

Planar linear ODE

3.1 Introduction

In this chapter we deal the planar ordinary differential equation (ODE) over function $\mathbf{y} : \mathrm{X} \subseteq \mathbb{R} \rightarrow \mathrm{Y} \subseteq \mathbb{R}^2$ of type

$$\mathbf{F}(\nabla \mathbf{y}(x), \mathbf{y}(x), x) = \mathbf{0}.$$

The equation is planar because the range of $\mathbf{y}$ is two-dimensional, $\mathbf{y} \in \mathrm{Y} \subseteq \mathbb{R}^2$,

$$\mathbf{y}(x) \equiv \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

it is ordinary because the domain of the independent variable has dimension one, $x \in \mathbb{X} \subset \mathbb{R}$, and it is differential because it assumes a variational approach to modelling, that is it is a functional equation containing the gradient

$$\nabla \mathbf{y} \equiv \begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} \frac{dy_1(x)}{dx} \\ \frac{dy_2(x)}{dx} \end{pmatrix}.$$

In this chapter we will consider the following case:

Definition 1. *A* ***planar linear autonomous ordinary differential equation*** *is a functional equation is the following equation: in matrix form*

$$\nabla \mathbf{y}(x) = \mathbf{A}\, \mathbf{y}(x) + \mathbf{B} \tag{3.1}$$

where the coefficient matrices $\mathbf{A} \in \mathbb{R}^{2\times 2}$ *and* $\mathbf{B} \in \mathbb{R}^2$ *have constant elements,*

$$\mathbf{A} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ \mathbf{B} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{3.2}$$

or in expanded form

$$\begin{aligned} y_1'(x) &= a_{11}\, y_1(x) + a_{12}\, y_2(x) + b_1 \\ y_2'(x) &= a_{21}\, y_1(x) + a_{22}\, y_2(x) + b_2. \end{aligned} \tag{3.3}$$

Furthermore, if $\mathbf{B} = \mathbf{0}$ *then the ODE is called* ***homogeneous****, and if* $\mathbf{B} \neq \mathbf{0}$ *it is called* ***non-homogeneous****.*

As with the scalar linear ODE, equation (3.1) (or in form (3.3)) has explicit solutions. However, given its dimension the solutions are more complex. In this chapter we present the general solutions of ODE (3.1) for any independent variable. In the next chapter we consider the case in which the independent variable is time and present the important results on the dynamics that can be generated by a time-dependent ODE.

The content of the chapter is the following: in section 3.2 we review some useful algebra results, in section 3.3 we derive the matrix exponential function. In sections 3.4 and 3.5 we solve the homogeneous and non-homogeneous ODE, respectively.

3.2 Two dimensional matrix algebra results

Matrix $\mathbf{A}$, in equation (3.2) fundamentally determines the solution to differential equation (3.1). It also allows for the characterization of its dynamics as we will see in the next chapter.

It is possible to classify any matrix $\mathbf{A}$ as being:

1. a **canonical matrix** similar to one of the following three matrices, called the **Jordan canonical forms**[1]

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \text{ or } \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{3.4}$$

   belonging to $\mathbb{R}^{2\times 2}$, because λ_-, λ_+ α and β are real numbers. Matrix $\mathbf{\Lambda}_3$ can also be written as

$$\mathbf{\Lambda}_3^c = \begin{pmatrix} \alpha - \beta\, i & 0 \\ 0 & \alpha + \beta\, i \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

   where $i = \sqrt{-1}$ is the imaginary number.

2. or, a **non-canonical matrix** if is of one of the two following forms

$$\mathbf{A}_d \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \text{ or } \mathbf{A}_h \equiv \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{3.5}$$

   where λ, α and β are real numbers.

Two matrices are said to be **similar** if they have the same spectrum. The **spectrum of matrix** $\mathbf{A}$ is a tuple belonging to $\mathbb{C}^2$ (the space of two-dimensional complex numbers)

$$\sigma(\mathbf{A}) = \Big\{\lambda \in \mathbb{C}^2 : \det\left(\mathbf{A} - \lambda\,\mathbf{I}\right) = 0\Big\}.$$

[1] See the appendix 3.A.1 where we gather some useful results from matrix algebra.

where $\mathbf{I}$ is the (2×2) identity matrix.

The elements of $\sigma(\mathbf{A})$ are called the **eigenvalues** of $\mathbf{A}$.

In order to determine the spectrum, we need to find the **characteristic polynomial** associated to matrix $\mathbf{A}$, which is the square polynomial in λ

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - \text{trace}(\mathbf{A})\ \lambda + \det(\mathbf{A}),$$

whose coefficients are the trace and the determinant of $\mathbf{A}$,

$$\text{trace}(\mathbf{A}) \ = a_{11} + a_{22}, \text{ and } \det(\mathbf{A}) \ = a_{11}\, a_{22} - a_{12}\, a_{21}.$$

Equation $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ is called **characteristic equation**. The eigenvalues of matrix $\mathbf{A}$ are the solutions to the characteristic equation:

$$\begin{cases} \lambda_- = \dfrac{\text{trace}(\mathbf{A})}{2} - \sqrt{\Delta(\mathbf{A})}, \\ \lambda_+ = \dfrac{\text{trace}(\mathbf{A})}{2} + \sqrt{\Delta(\mathbf{A})} \end{cases} \tag{3.6}$$

where

$$\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A})$$

is called the discriminant of matrix $\mathbf{A}$.

Eigenvalues of A

Finding the eigenvalues allows us to classify any matrix according to three criteria:

1. the sign of the discriminant allows us to determine if the eigenvalues are real or complex numbers, and to find the Jordan canonical form of matrix $\mathbf{A}$ we can call $\mathbf{\Lambda}$;
2. the sign of the trace and the determinant allows us to sign the eigenvalues if they are real or the sign of their real part if they are complex;
3. their genericity, i.e., the robustness of the classification provided by the previous two criteria to small change in the elements of $\mathbf{A}$

First, the two eigenvalues are real if $\Delta(\mathbf{A}) \geq 0$ and they are complex conjugate if $\Delta(\mathbf{A}) < 0$. In particular, if $\Delta(\mathbf{A}) > 0$ the eigenvalues are real and distinct and satisfy $\lambda_- < \lambda_+$, if $\Delta(\mathbf{A}) = 0$ the eigenvalues are real and multiple and satisfy $\lambda = \lambda_- = \lambda_+ = \dfrac{\text{trace}(\mathbf{A})}{2}$, and if $\Delta(\mathbf{A}) < 0$ they are complex conjugate and satisfy

$$\lambda_\pm = \alpha \pm \beta\, i, \text{ for } i \equiv \sqrt{-1}$$

where $\alpha = \dfrac{\text{trace}(\mathbf{A})}{2}$ and $\beta = \sqrt{|\Delta(\mathbf{A})|}$.

Second, the signs of the real part of both eigenvalues is the same if $\det(\mathbf{A}) \ > 0$ and it is different if $\det(\mathbf{A}) \ < 0$. In the first case they are both positive if $\det(\mathbf{A}) \ > 0$ and $\text{trace}(\mathbf{A}) > 0$ and they are both negative if $\det(\mathbf{A}) \ > 0$ and $\text{trace}(\mathbf{A}) < 0$.

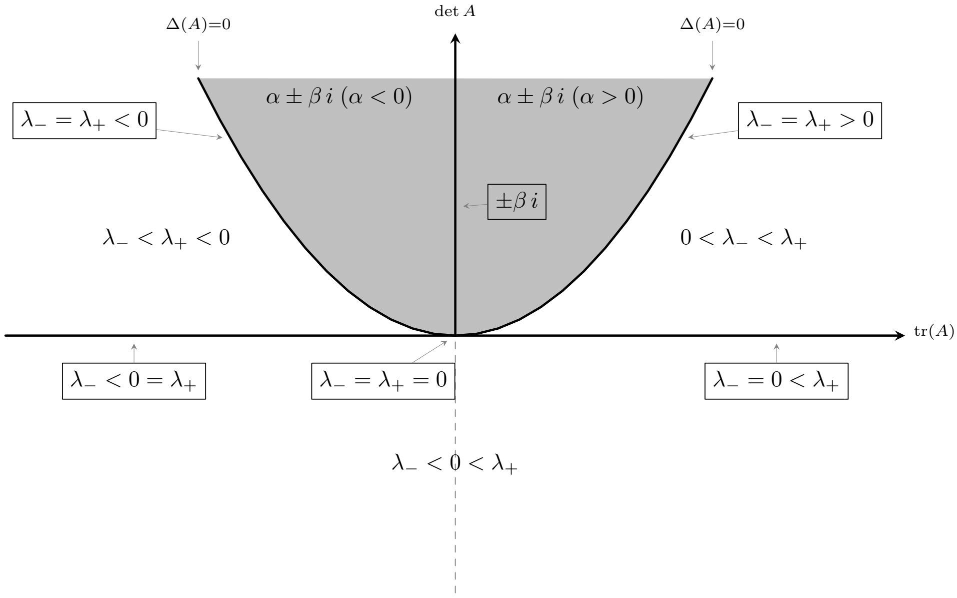

Figure 3.1: Eigenvalues of $\mathbf{A}$ in the $(\text{trace}(\mathbf{A}), \det(\mathbf{A}))$ space. The gray area corresponds to the existence of complex conjugate eigenvalues.

Third, the eigenvalues are generic in the sense that they will not change their type or sign for small changes in the elements of the coefficient matrix $\mathbf{A}$ if $\Delta(\mathbf{A}) \neq 0$, or $\det(\mathbf{A}) \neq 0$, or $\text{trace}(\mathbf{A}) \neq 0$ and $\det(\mathbf{A}) \geq 0$, and they are not generic otherwise, that is if $\Delta(\mathbf{A}) = 0$, or $\det(\mathbf{A}) = 0$, or $\text{trace}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) \geq 0$.

Figure 3.1, shows all the possible relevant cases. There are five generic cases (corresponding to two-dimensional subsets), four non-generic cases of co-dimension-one (corresponding to lines) and two co-dimension-two case (the origin). It displays all the following cases:

1. the five generic cases are: (1) if $\text{trace}(\mathbf{A}) > 0$, $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) > 0$ the two eigenvalues are real, different and positive, $\lambda_+ > \lambda_- > 0$; (2) if $\text{trace}(\mathbf{A}) > 0$, $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) < 0$ the two eigenvalues are complex conjugate with positive real parts $\lambda_\pm = \alpha \pm \beta\, i$ with $\alpha > 0$; (3) if $\text{trace}(\mathbf{A}) < 0$, $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) > 0$ the two eigenvalues are real, different, and negative $0 > \lambda_+ > \lambda_-$; (4) if $\text{trace}(\mathbf{A}) < 0$, $\det(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) < 0$ the two eigenvalues are complex conjugate with negative real parts, $\lambda_\pm = \alpha \pm \beta\, i$ with $\alpha < 0$; or (5) if $\det(\mathbf{A}) < 0$ the two eigenvalues are real and with opposite signs $\lambda_+ > 0 > \lambda_-$;

2. the six non-generic cases: (1) if $\text{trace}(\mathbf{A}) > 0$ and $\Delta(\mathbf{A}) = 0$ the two eigenvalues are real, equal and positive $\lambda_+ = \lambda_- > 0$; (2) if $\text{trace}(\mathbf{A}) < 0$ and $\Delta(\mathbf{A}) = 0$ the two eigenvalues are real, equal and negative $\lambda_+ = \lambda_- < 0$; (3) if $\text{trace}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) > 0$ then the two eigenvalues are complex conjugate with zero real part, $\lambda_\pm = \pm\beta\, i$; (4) if $\text{trace}(\mathbf{A}) > 0$ and $\det(\mathbf{A}) = 0$ the two eigenvalues are real one is positive and the other is equal to zero, $\lambda_+ > 0 = \lambda_-$; (5) (4) if $\text{trace}(\mathbf{A}) < 0$ and $\det(\mathbf{A}) = 0$ the two eigenvalues are real one is

negative and the other is equal to zero, $\lambda_+ = 0 < \lambda_-$; or (6) if trace($\mathbf{A}$) $=$ det $(\mathbf{A}) = 0$ both eigenvalues are real and equal to zero, $\lambda_+ = \lambda_- = 0$.

Therefore $\sigma(\mathbf{A}) \in \mathbb{R}^2$ if $\Delta(\mathbf{A}) \geq 0$ and $\sigma(\mathbf{A}) \in \mathbb{C}^2$ if $\Delta(\mathbf{A}) < 0$.

There is a useful result on the relationship between the coefficients of the characteristic equation with elementary operations between the eigenvalues of any matrix $\mathbf{A}$:

$$\lambda_- + \lambda_+ = \text{trace}(\mathbf{A}), \tag{3.7a}$$

$$\lambda_-\, \lambda_+ = \det(\mathbf{A}). \tag{3.7b}$$

Clearly, $\det(\mathbf{A}) < 0$ is a sufficient condition for the existence of two real eigenvalues and is a necessary and sufficient condition for $\lambda_- < 0 < \lambda_+$. This is a very useful result for economic models.

Canonical matrices

There is a close relationship between the discriminant of a matrix $\mathbf{A}$, which is not in a non-canonical form as in equation (3.5), and to its similar Jordan canonical form[2], which we call the **Jordan canonical form of A**.

Lemma 1. ***Jordan canonical form of a matrix* A** *The Jordan canonical form of* $\mathbf{A}$ *is determined by the sign of the discriminant* $\Delta(\mathbf{A})$*: if* $\Delta(\mathbf{A}) > 0$ *then the Jordan canonical form of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_1$*, if* $\Delta(\mathbf{A}) = 0$ *the Jordan canonical of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_2$*, and if* $\Delta(\mathbf{A}) < 0$ *the Jordan canonical form of* $\mathbf{A}$ *is* $\mathbf{\Lambda}_3$*.*

Given any matrix $\mathbf{A}$, and its Jordan canonical form, given in equation (3.4), the fundamental theorem of Algebra states that there is a (non-singular) linear operator $\mathbf{P} \in \mathbb{R}^{2\times 2}$ such that the following relationship holds

$$\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1} \Leftrightarrow \mathbf{\Lambda} = \mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P}. \tag{3.8}$$

Matrix $\mathbf{P}$ is called the **eigenvector matrix** associated to matrix $\mathbf{A}$.

The fact that any matrix $\mathbf{A}$ has a one-to-one relationship with one of the Jordan canonical forms allows us to reduce the determination of the general solution of a planar ODE to the solution of a simpler ODE in which the coefficient matrix is its Jordan canonical form. Next, we can transform back to the original ODE by using $\mathbf{P}$ as an operator.

Non-canonical matrices

For non-canonical matrices, represented in equation (3.5), the spectra are: first, in the case of matrix $\mathbf{A}_d$ there are multiple eigenvalues, $\sigma(\mathbf{A}_d) = \{\lambda, \lambda\}$, although the matrix is not of the form $\mathbf{\Lambda}_2$; and, second, in the case of matrix $\mathbf{A}_h$ the spectrum is $\sigma(\mathbf{A}_h) = \{\alpha + \beta, \alpha - \beta\}$ which are two real and distinct numbers.

[2]See the appendix 3.A.1.

3.3 The two-dimensional matrix exponential function

We saw that the (general) solution of the scalar linear homogeneous equation $y'(x) = a\,y$ is $y(x) = y(x_0)\,e^{ax}$ where $y(x_0)$ is an arbitrary element of $\mathrm{Y} \subseteq \mathbb{R}$ for $x = x_0 \in \mathrm{X}$. Recall that the exponential function has the series representation

$$e^{\lambda x} \equiv \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} = 1 + \lambda x + \frac{1}{2}\,(\lambda x)^2 + \frac{1}{6}\,(\lambda x)^3 + ...$$

For the planar problem we can also define a **matrix exponential** function

$$\mathbf{e}^{\mathbf{Ax}} \;\equiv \sum_{n=0}^{+\infty} \frac{1}{n!} \mathbf{A}^n x^n = \mathbf{I} + \mathbf{A}x + \frac{1}{2}\;\mathbf{A}^2 x^2 + ... \tag{3.9}$$

which is a mapping $\mathbf{e}^{\mathbf{Ax}} : \mathrm{X} \;\rightarrow \mathbb{R}^{2\times 2}$ with the following properties:[3]

Lemma 2 (Properties of matrix exponentials $\mathbf{e}^{\mathbf{Ax}}$)**.** *Matrix exponential function* $\mathbf{e}^{\mathbf{Ax}}$*, defined in equation* (3.9) *has the following properties:*

(i) semigroup property: $\mathbf{e}^{\mathbf{A(x+s)}} = \mathbf{e}^{\mathbf{Ax}}\mathbf{e}^{\mathbf{As}}$

(ii) inverse of the matrix exponencial is the the exponential of the inverse: $\left(\mathbf{e}^{\mathbf{Ax}}\right)^{-1} = \mathbf{e}^{-\mathbf{Ax}}$

(iii) the time derivative commutes: $\frac{d}{dx}\mathbf{e}^{\mathbf{Ax}} = \mathbf{A}\mathbf{e}^{\mathbf{Ax}} = \mathbf{e}^{\mathbf{Ax}}\mathbf{A}$

(iv) if matrices $\mathbf{A}$ *and* $\mathbf{B}$ *commute, (i.e., if* $\mathbf{A}\,\mathbf{B} = \mathbf{B}\,\mathbf{A}$*) then* $\mathbf{e}^{(\mathbf{A}+\mathbf{B})\mathbf{x}} = \mathbf{e}^{\mathbf{Ax}}\mathbf{e}^{\mathbf{Bx}}$

(v) Let $\mathbf{P}$ *be a non-singular and square matrix. Then*

$$\mathbf{e}^{\mathbf{\Lambda x}} = \mathbf{e}^{\mathbf{P^{-1}APx}} = \mathbf{P}^{-1}\mathbf{e}^{\mathbf{Ax}}\mathbf{P}.$$

From Lemma 2 *(v)* as $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{\Lambda}$ then $\mathbf{e}^{\mathbf{\Lambda x}} \; = \mathbf{e}^{\mathbf{P^{-1}APx}} \; = \mathbf{P}^{-1}\mathbf{e}^{\mathbf{Ax}}\mathbf{P}$ or, equivalently

$$\mathbf{e}^{\mathbf{Ax}} = \mathbf{P}\,\mathbf{e}^{\mathbf{\Lambda x}}\,\mathbf{P}^{-1}, \tag{3.10}$$

where $\mathbf{\Lambda}$ is the Jordan canonical form of $\mathbf{A}$.

Therefore, given any matrix $\mathbf{A}$, the exponential matrix $\mathbf{e}^{\mathbf{Ax}}$ is a (2×2) dimensional function of x. It depends on x because , and it depends on x because $\mathbf{e}^{\mathbf{Ax}}$ is a linear transformation of $\mathbf{e}^{\mathbf{\Lambda x}}$ performed by the operator matrix $\mathbf{P}$.

This is an important result which means that the types of solutions, and the associated phase diagrams, can be completely enumerated.

The exponential matrices for the Jordan canonical forms are:

[3]See Hirsch and Smale (1974).

Lemma 3 (Matrix exponential functions for Jordan canonical forms)**.** *Let* $\mathbf{\Lambda}$ *be a matrix in an arbitrary Jordan canonical form, as in equation* (3.4)*, and let* λ_-*,* λ_+*,* λ*,* α *and* β *be real numbers. Then,*

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ *then*

$$\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{x}} = \mathbf{e}^{\boldsymbol{\Lambda_1}\mathbf{x}} = \begin{pmatrix} e^{\lambda_- x} & 0 \\ 0 & e^{\lambda_+ x} \end{pmatrix}. \tag{3.11}$$

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ *then*

$$\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{x}} = \mathbf{e}^{\boldsymbol{\Lambda_2}\mathbf{x}} = e^{\lambda x} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{3.12}$$

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ *then*

$$\mathbf{e}^{\boldsymbol{\Lambda}\mathbf{x}} = \mathbf{e}^{\boldsymbol{\Lambda_3}\mathbf{x}} = e^{\alpha x} \begin{pmatrix} \cos \beta x & \sin \beta x \\ -\sin \beta x & \cos \beta x \end{pmatrix} \; or \; \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix} \tag{3.13}$$

Proof. Consider the definition of matrix exponential, equation (3.9) and the Jordan canonical form matrices in equation (3.4). In the first case, we have

$$\mathbf{e}^{\boldsymbol{\Lambda_1}\mathbf{x}} = \mathbf{I}_2 + \mathbf{\Lambda}_1 x + \frac{1}{2} (\mathbf{\Lambda}_1)^2 x^2 + ... = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_- x & 0 \\ 0 & \lambda_+ x \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \lambda_-^2 x^2 & 0 \\ 0 & \lambda_+^2 x^2 \end{pmatrix} + ...$$

then, performing the matrix additions,

$$\mathbf{e}^{\boldsymbol{\Lambda_1}\mathbf{x}} = \begin{pmatrix} 1 + \lambda_- x + \frac{1}{2}\lambda_-^2 x^2 + ... & 0 \\ 0 & 1 + \lambda_+ x + \frac{1}{2}\lambda_+^2 x^2 + ... \end{pmatrix} = \begin{pmatrix} e^{\lambda_- x} & 0 \\ 0 & e^{\lambda_+ x} \end{pmatrix}$$

because $e^y = \sum_{n=0}^{+\infty} \frac{y^n}{n!}$. That result is straightforward to obtain because the Jordan matrix is diagonal. This is not the case for Jordan matrix $\mathbf{\Lambda}_2$, though. But if we decompose $\mathbf{\Lambda}_2$ as

$$\mathbf{\Lambda}_2 = \mathbf{\Lambda}_{2,1} + \mathbf{\Lambda}_{2,2} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and because the two matrices commute, i.e. $\mathbf{\Lambda}_{2,1}\mathbf{\Lambda}_{2,2} = \mathbf{\Lambda}_{2,2}\mathbf{\Lambda}_{2,1}$, then applying property (iv) of Lemma 2 we obtain

$$\mathbf{e}^{\boldsymbol{\Lambda_2}\mathbf{x}} = \mathbf{e}^{(\boldsymbol{\Lambda_{2,1}} + \boldsymbol{\Lambda_{2,2}})\mathbf{x}} = \mathbf{e}^{\boldsymbol{\Lambda_{2,1}}\mathbf{x}}\, \mathbf{e}^{\boldsymbol{\Lambda_{2,2}}\mathbf{x}}$$

where

$$\mathbf{e}^{\boldsymbol{\Lambda_{2,1}}\mathbf{x}} = \begin{pmatrix} e^{\lambda x} & 0 \\ 0 & e^{\lambda x} \end{pmatrix} = e^{\lambda x}\mathbf{I}_2.$$

Using again formula (3.9) for matrix $\mathbf{\Lambda}_{2,2}$ we get

$$\mathbf{e}^{\boldsymbol{\Lambda_{2,2}}\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \frac{x^2}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + ... = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

therefore multiplying by matrix $\mathbf{e}^{\boldsymbol{\Lambda}_{2,1}\mathbf{x}}$ yields (3.12).

In the third case, $\mathbf{\Lambda}_3$ is again non-diagonal, but it can also be decomposed into the sum of two matrices, $\mathbf{\Lambda}_{3,1}$ and $\mathbf{\Lambda}_{3,2}$, that commute

$$\mathbf{\Lambda}_3 = \mathbf{\Lambda}_{3,1} + \mathbf{\Lambda}_{3,2} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Applying again property (iv) of Lemma 2 we get

$$\mathbf{e}^{\boldsymbol{\Lambda}_3\mathbf{x}} = \mathbf{e}^{\boldsymbol{\Lambda}_{3,1}\mathbf{x}}\,\mathbf{e}^{\boldsymbol{\Lambda}_{3,2}\mathbf{x}},$$

where

$$\mathbf{e}^{\boldsymbol{\Lambda}_{3,1}\mathbf{x}} = e^{\alpha x}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using again formula (3.9) for matrix $\mathbf{\Lambda}_{3,2}$ we get

$$\mathbf{e}^{\boldsymbol{\Lambda}_{3,2}\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \beta\,x \\ -\beta\,x & 0 \end{pmatrix} + \frac{x^2}{2}\begin{pmatrix} \beta^2\,x^2 & 0 \\ 0 & -\beta^2\,x^2 \end{pmatrix} + \ldots = \begin{pmatrix} \cos\beta\,x & \sin\beta\,x \\ -\sin\beta\,x & \cos\beta\,x \end{pmatrix},$$

because $\sin y = \sum_{n=0}^{+\infty}\dfrac{y^{2n+1}}{(2n+1)}$ and $\cos y = \sum_{n=0}^{+\infty}\dfrac{y^{2n}}{(2n)}$, we obtain (3.13). □

For non-canonical matrices we have to specifically determine their exponential matrix:

Lemma 4 (Matrix exponential functions for non-canonical matrices)**.** *Let matrix* $\mathbf{A}$ *be in one of the two non-canonical forms, as in equation* (3.5)*. Then their matrices exponential functions are:*

1. *If* $\mathbf{A} = \mathbf{A}_d$*, then*
$$\mathbf{e}^{\mathbf{A_d x}} = e^{\lambda x}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.14}$$

2. *if* $\mathbf{A} = \mathbf{A}_h$*, then* [4]
$$\mathbf{e}^{\mathbf{A}_h t} = e^{\alpha\,x}\begin{pmatrix} \cosh(\beta\,x) & \sinh(\beta\,x) \\ \sinh(\beta\,x) & \cosh(\beta\,x) \end{pmatrix} \tag{3.15}$$

Proof. We know that $\mathbf{A} = \mathbf{P\Lambda P}^{-1}$, where $\mathbf{\Lambda}$ is the Jordan form of $\mathbf{A}$. Then $\mathbf{e}^{\mathbf{Ax}} = \mathbf{e}^{\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}\mathbf{x}} = \mathbf{Pe}^{\mathbf{x}}\mathbf{P}^{-1}$ by property (v) of Lemma 2. Matrix $\mathbf{A} = \mathbf{A}_d$ has two equal real eigenvalues equal to λ and, because it is diagonal it satisfies $\mathbf{A}_d\,\mathbf{P}_d = \mathbf{P}_d\,\mathbf{A}_d$. Therefore $\mathbf{P}_d = \mathbf{I}$ and

$$\mathbf{e}^{\mathbf{A}_d x} = \mathbf{P}\,e^{\lambda x}\,\mathbf{I}\,\mathbf{P}^{-1} = e^{\lambda x}\,\mathbf{I}.$$

Matrix $\mathbf{A} = \mathbf{A}_h$ has the real spectrum $\sigma = \{\,\alpha+\beta, \alpha-\beta\}$ and has eigenvector matrix

$$\mathbf{P}_h = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

[4] Recall $\cosh(\beta\,x) = \frac{1}{2}(e^{\beta x} + e^{-\beta x})$ and $\sinh(\beta\,x) = \frac{1}{2}(e^{\beta x} - e^{-\beta x})$

Therefore, the exponential matrix is

$$\mathbf{e}^{\mathbf{A}_h x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(\alpha+\beta)x} & 0 \\ 0 & e^{(\alpha+\beta)x} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

which, expanding the matrix multiplication, yields matrix (3.15). □

Summing up, the matrix exponential function can be reduced to two formal cases:

1. if matrix $\mathbf{A}$ is canonical, the matrix exponential is given by equation (3.10), which depends on the matrix exponential of its Jordan canonical form, which take one of following three forms (3.11), (3.12), or (3.13), depending on the spectrum of $\mathbf{A}$;

2. if matrix $\mathbf{A}$ is non-canonical, as in equation (3.5), its matrix exponential function is either given by equation (3.14) or (3.15).

3.4 The homogeneous planar ODE

In this section we present the general solution to the homogeneous linear planar ODE, that is to equation

$$\nabla \mathbf{y} = \mathbf{A}\, \mathbf{y},\ y : \mathrm{X} \subseteq \mathbb{R} \to \mathrm{Y} \subseteq \mathbb{R}^2. \tag{3.16}$$

Proposition 1 (General solution to the homogenous ODE (3.16))**.** *Consider the ODE* (3.16)*, for any real matrix* $\mathbf{A} \in \mathbb{R}^{2\times 2}$*. The unique solution is the mapping* $\mathbf{\Phi} : \mathrm{X} \ \times \mathrm{Y} \to \mathrm{Y} \subseteq \mathbb{R}^2$*,*

$$\mathbf{y}(x) = \mathbf{\Phi}(x, x_0, \mathbf{y}(x_0)) \equiv \mathbf{e}^{\mathbf{A}\,(x-x_0)}\, \mathbf{y}(x_0) \text{ for } x \geq x_0 \in \mathrm{X} \tag{3.17}$$

where $\mathbf{y}(x_0) \in \mathrm{Y}$ *is arbitrary.*

Proof. We can verify that the solution to equation (3.16) is (3.17). The derivative of (3.17) satisfies, from Lemma 2 (iii),

$$\frac{d}{dx}\mathbf{y}(x) = \frac{d}{dx}\mathbf{e}^{\mathbf{A}\,(x-x_0)}\mathbf{y}(x_0) = \mathbf{A}\, \mathbf{e}^{\mathbf{A}\,(x-x_0)}\mathbf{y}(x_0) = \mathbf{A}\, \mathbf{y}(x),$$

for any real matrix $\mathbf{A}$. □

We see that the solution is of the form $\mathbf{y}(x) = \Psi(x, x_0)\, \mathbf{y}(x_0)$ where

$$\Psi(x, x_0) = \mathbf{e}^{\mathbf{A}\,(x-x_0)}$$

is the **matrix exponential function** which encodes the dependence of the general solution of the ODE to the independent variable x.

Next we presents the several cases for matrix $\Psi(x, x_0)$, starting in subsection 3.4.1 with the cases in which $\mathbf{A}$ is in the canonical Jordan form or it is a non-canonical matrix, and continuing in subsection 3.4.2 with the general cases in which matrix $\mathbf{A}$ is not in the Jordan canonical form, but is similar to a Jordan canonical form.

We will see in the next chapter that the first cases contain the fundamental types of dynamic systems generated by planar linear ODE's.

3.4.1 A in a Jordan canonical form

Consider the ODE (3.16), such that $\mathbf{A} \in \{\ \mathbf{\Lambda}_1,\ \mathbf{\Lambda}_2, \mathbf{\Lambda}_3\}$. From results in section 3.3 the matrix exponentials are

$$\Psi(x) \in \left\{\ \begin{pmatrix} e^{\lambda_- x} & 0 \\ 0 & e^{\lambda_+ x} \end{pmatrix},\ \begin{pmatrix} e^{\lambda x} & x \\ 0 & e^{\lambda x} \end{pmatrix},\ e^{\alpha x} \begin{pmatrix} \cos\beta x & \sin\beta x \\ -\sin\beta x & \cos\beta x \end{pmatrix}\ \right\}. \tag{3.18}$$

If $\mathbf{A} \in \{\ \mathbf{A}_d, \mathbf{A}_h\}$ the matrix exponentials are

$$\Psi(x) \in \left\{ \begin{pmatrix} e^{\lambda x} & 0 \\ 0 & e^{\lambda x} \end{pmatrix},\ e^{\alpha x} \begin{pmatrix} \cosh\beta x & \sinh\beta x \\ \sinh\beta x & \cosh\beta x \end{pmatrix}\ \right\}. \tag{3.19}$$

We can gat more intuition if we expand equation (3.16), we have the following cases:

1. if $\mathbf{A} = \mathbf{\Lambda}_1$, the ODE takes the form

$$\begin{cases} y_1' = \lambda_-\, y_1, \\ y_2' = \lambda_+\, y_2, \end{cases}$$

and has the solution

$$\mathbf{y}(x) = \begin{pmatrix} e^{\lambda_-(x-x_0)} & 0 \\ 0 & e^{\lambda_+(x-x_0)} \end{pmatrix} \mathbf{y}(x_0) = \begin{pmatrix} e^{\lambda_-(x-x_0)}\, y_1(x_0) \\ e^{\lambda_+(x-x_0)} y_2(x_0) \end{pmatrix} \tag{3.20}$$

2. if $\mathbf{A} = \mathbf{\Lambda}_2$, the ODE takes the form

$$\begin{cases} y_1' = \lambda\, y_1 + y_2, \\ y_2' = \lambda\, y_2 \end{cases}$$

and has the solution

$$\mathbf{y}(x) = \begin{pmatrix} e^{\lambda(x-x_0)} & x - x_0 \\ 0 & e^{\lambda(x-x_0)} \end{pmatrix} \mathbf{y}(x_0)\ = \begin{pmatrix} e^{\lambda(x-x_0)}\, y_1(x_0) + y_2(x_0)\,(x - x_0) \\ e^{\lambda(x-x_0)}\, y_2(x_0) \end{pmatrix} \tag{3.21}$$

3. if $\mathbf{A} = \mathbf{\Lambda}_3$, the ODE takes the form

$$\begin{cases} y_1' = \alpha\, y_1 + \beta\, y_2, \\ y_2' = -\beta\, y_1 + \alpha\, y_2; \end{cases}$$

and has the solution

$$\begin{aligned} \mathbf{y}(x) &= e^{\alpha(x-x_0)} \begin{pmatrix} \cos\beta(x-x_0) & \sin\beta(x-x_0) \\ -\sin\beta(x-x_0) & \cos\beta(x-x_0) \end{pmatrix} \mathbf{y}(x_0) \\ &= e^{\alpha(x-x_0)} \begin{pmatrix} y_1(x_0)\cos\beta(x-x_0) + y_2(x_0)\sin\beta(x-x_0) \\ -y_1(x_0)\sin\beta(x-x_0) + y_2(x_0)\cos\beta(x-x_0) \end{pmatrix}. \end{aligned} \tag{3.22}$$

The other two cases, i.e., if $\mathbf{A} = \mathbf{A}_d$ or $\mathbf{A} = \mathbf{A}_h$ have obvious solutions.

Observe that, while solution (3.21) correspond to a non-generic case, at it is relative to the case in which $\Delta(\mathbf{A}) = 0$, the other two cases are relative to both generic and non-generic cases. Therefore, we can have the following non-generic cases:

1. if $\mathbf{A} = \mathbf{\Lambda}_1$ and $\det(\mathbf{A}) = 0$,
$$\mathbf{y}(x) = \begin{pmatrix} y_1(x_0) \\ e^{\lambda_+(x-x_0)}y_2(x_0) \end{pmatrix}, \text{ if } \text{trace}(\mathbf{A}) > 0, \text{ or } \mathbf{y}(x) = \begin{pmatrix} e^{\lambda_-(x-x_0)}y_1(x_0) \\ y_2(x_0) \end{pmatrix}, \text{ if } \text{trace}(\mathbf{A}) < 0 \tag{3.23}$$

2. if $\mathbf{A} = \mathbf{\Lambda}_1$ and $\det(\mathbf{A}) = \text{trace}(\mathbf{A}) = 0$,
$$\mathbf{y}(x) = \begin{pmatrix} y_1(x_0) \\ y_2(x_0) \end{pmatrix} \tag{3.24}$$

3. if $\mathbf{A} = \mathbf{\Lambda}_3$ and $\text{trace}(\mathbf{A}) = 0$
$$\mathbf{y}(x) = \begin{pmatrix} y_1(x_0)\cos\beta(x-x_0) + y_2(x_0)\sin\beta(x-x_0) \\ -y_1(x_0)\sin\beta(x-x_0) + y_2(x_0)\cos\beta(x-x_0) \end{pmatrix}. \tag{3.25}$$

In the first two cases we observe that at least one element of $\mathbf{y}$ is constant, that is, depends only on the arbitrary element $x_0 \in \mathrm{X}$. in the second case the solutions trace out circular curves in Y, passing through a point $\mathbf{y}(x_0)$.

3.4.2 General A matrix

In this section we consider any (canonical) matrix $\mathbf{A}$, with the exception of cases $\mathbf{A}_d$ and $\mathbf{A}_h$, in equation (3.5). Equation (3.17) provides the general solution.

The superposition principle establishes a relationship between the solution of a ODE with a generic coefficient matrix $\mathbf{A}$, and an associated ODE whose coefficient matrix is the Jordan canonical form associated to $\mathbf{A}$, which we denote by $\mathbf{\Lambda}$.

Lemma 5 (Superposition principle). *Consider the coefficient matrix* $\mathbf{A}$ *and let* $\mathbf{P}$ *and* $\mathbf{\Lambda}$ *be its associated eigenvector matrix and Jordan canonical form. Then, then the solution of ODE* (3.35), *with general coefficient matrix* $\mathbf{A}$*, is*
$$\mathbf{y}(x) = \mathbf{P}\,\mathbf{w}(x), \text{ for any } x \in \mathrm{X} \tag{3.26}$$
where $\mathbf{w}$ *is the solution of the ODE* $\mathbf{w}' = \mathbf{\Lambda}\,\mathbf{w}$*, that is* $\mathbf{w}(x) = \Psi(x, x_0)\,\mathbf{w}(x_0)$ *where* $\Psi(x, x_0)$ *is one of the matrices in equation* (3.18) *and* $\mathbf{w}(x_0) = \mathbf{P}^{-1}\ \mathbf{y}(x_0)$.

Proof. Recall the transformation $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where matrix $\mathbf{P}$ is non-singular. Then equation (3.37) yields $\mathbf{w}(x) = \mathbf{P}^{-1}\mathbf{y}(x)$. Taking derivatives for x we find $\mathbf{w}' = \frac{d\mathbf{w}}{dx} = \mathbf{P}^{-1}\,\mathbf{y}' = \mathbf{P}^{-1}\,\mathbf{A}\mathbf{y} = \mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P}\mathbf{y} = \mathbf{\Lambda}\mathbf{w}$. Equation $\mathbf{w}' = \mathbf{\Lambda}\,\mathbf{w}$ has solution $\mathbf{w}(x) = \Psi(x, x_0)\,\mathbf{w}(x_0)$, where $\Psi(x, x_0)$ is the form in (3.18) which is the matrix exponential for the Jordan form which is similar to $\mathbf{A}$. □

We call this transformation the superposition principle because the general solution to an ODE, with a general coefficient matrix, can be written as the sum of two general solutions. In the particular case in which matrix $\mathbf{A}$ has two real distinct eigenvalues, i.e., when $\Delta(\mathbf{A}) > 0$, the solution can be written as

$$\mathbf{y}(x) = \mathbf{P}^1\, w_1(x) + \mathbf{P}^2\, w_2(x)$$

where $\mathbf{P}^1$ and $\mathbf{P}^2$ are the eigenvectors of matrix $\mathbf{A}$.[5] This property is useful for characterizing the dynamics of the solution of an ODE when time is the independent variable.

An alternative form of the solution of a linear homogeneous ODE is

$$\mathbf{y}(x) = \mathbf{P}\, \Psi(x, x_0)\, \mathbf{P}^{-1}\ \ \mathbf{y}(x_0), \text{for any}\ \ x, x_0 \in \mathrm{X}$$

where $\Psi(x, x_0)$ is the matrix exponential associated to $\mathbf{A}$ which is given in equation (3.18).

3.5 Non-homogeneous equation

In this section we present solutions to the autonomous non-homogenous planar linear ODE

$$\nabla \mathbf{y} = \mathbf{A}\, \mathbf{y} + \mathbf{B}, \tag{3.27}$$

where $\mathbf{B}$ can be any real vector. In subsection 3.5.1 we assume that matrix $\mathbf{A}$ is in a Jordan canonical form, that is $\mathbf{A} = \mathbf{\Lambda}$ where $\mathbf{\Lambda} \in \{\, \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3\}$, and in subsection 3.5.2 we consider an arbitrary coefficient matrix $\mathbf{A}$.

3.5.1 A in a Jordan canonical form

In this subsection we present the unique solutions of the planar linear ODE

$$\nabla \mathbf{y} = \mathbf{\Lambda} \mathbf{y} + \mathbf{B}. \tag{3.28}$$

It can take only one of the three expanded forms

$$\begin{cases} y_1' = \lambda_-\, y_1 + b_1, \\ y_2' = \lambda_+\, y_2 + b_2, \end{cases} \quad \begin{cases} y_1' = \lambda\, y_1 + y_2 + b_1, \\ y_2' = \lambda\, y_2 + b_2, \end{cases} \quad \text{and} \begin{cases} y_1' = \alpha\, y_1 + \beta\, y_2 + b_1, \\ y_2' = -\beta\, y_1 + \alpha\, y_2 + b_2 \end{cases} .$$

To study this equation it is useful to consider its **set of invariant solutions**, i.e., solutions in Y which are independent from $x \in n\mathrm{X}$,

$$\bar{\mathbf{y}}\ = \Big\{ \mathbf{y} \in \mathrm{Y} :\ \mathbf{\Lambda} \mathbf{y} + \mathbf{B} = \mathbf{0} \Big\}.$$

We show next that this set is non-empty, meaning invariant. solutions always exist, but it can contain several elements, meaning that invariant solutions may not be unique.

[5] Recall the the eigenvector matrix is the concatenation of the those two eigenvectors, $\mathbf{P} = \mathbf{P}^1 | \mathbf{P}^2$.

Lemma 6. *An invariant state always exists, and has the form*

$$\bar{\mathbf{y}} = -\mathbf{\Lambda}^{+}\,\mathbf{B} + \left(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\right)\mathbf{y} \tag{3.29}$$

where $\mathbf{\Lambda}^{+}$ *is the Moore-Penrose inverse of* $\mathbf{\Lambda}$ *and* $\mathbf{y}$ *is an arbitrary element of* Y*. If* $\det(\mathbf{\Lambda}) \neq 0$ *the invariant state is unique, and if* $\det(\mathbf{\Lambda}) = 0$ *there is an infinite number of invariant states.*

Proof. See (Magnus and Neudecker, 1988, p36). □

The following cases are possible.

Non-degenerate case If $\det(\mathbf{\Lambda}) \neq 0$ then the Moore-Penrose inverse is the classical inverse, that is

$$\mathbf{\Lambda}^{+} = \mathbf{\Lambda}^{-1} = \frac{\mathrm{adj}^{\top}(\mathbf{\Lambda})}{\det(\mathbf{\Lambda})},$$

where $\mathrm{adj}^{\top}(\mathbf{\Lambda})$ is the transposed of the adjoint matrix $\mathbf{\Lambda}$. The classic inverse satisfies the property $\mathbf{\Lambda}^{-1}\,\mathbf{\Lambda} = \mathbf{I}$. Then, the invariant state is unique, and from equation (3.29), it is

$$\bar{\mathbf{y}} = -\mathbf{\Lambda}^{-1}\,\mathbf{B}.$$

If $\mathbf{B} = \mathbf{0}$ then the invariant state is $\bar{\mathbf{y}} = \mathbf{0}$. In both cases, the invariant state is a **single point** in the set Y.

Degenerate cases If $\det(\mathbf{\Lambda}) = 0$ then $\Delta(\mathbf{\Lambda}) > 0$. Then all the eigenvalues are real, which means that the Jordan matrix $\mathbf{\Lambda}$ is diagonal, and it has at least one eigenvalue which is equal to zero. There is one zero eigenvalue if $\mathrm{trace}(\mathbf{A}) \neq 0$ and two zero eigenvalues if $\mathrm{trace}(\mathbf{A}) = 0$. This means that the Jordan matrix can only be one of the following three cases

$$\mathbf{\Lambda} \in \left\{ \begin{pmatrix} \lambda_- & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \lambda_+ \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{3.30}$$

The associated Moore-Penrose inverses are

$$\mathbf{\Lambda}^{+} \in \left\{ \begin{pmatrix} \frac{1}{\lambda_-} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\lambda_+} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{3.31}$$

Therefore, substituting those matrices in equation (3.29) we find

$$\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda} \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and there is always an infinite number of invariant states depending on the arbitrary element $\mathbf{y} \in \mathrm{Y}$. It is useful to consider further two possibilities: first, if $\mathrm{trace}(\mathbf{A}) \neq 0$ from equation (3.29), we find the invariant states are

$$\bar{\mathbf{y}} = \begin{pmatrix} -\frac{b_1}{\lambda_-} \\ y_2 \end{pmatrix}, \text{ or } \bar{\mathbf{y}} = \begin{pmatrix} y_1 \\ -\frac{b_2}{\lambda_+} \end{pmatrix}. \tag{3.32}$$

In both cases the invariant states set defines a **one-dimensional linear manifold** (i.e, a line) in set Y: in the first case it it is a line passing through $y_1 = -\dfrac{b_1}{\lambda_-}$ (a vertical line in a Cartesian diagram) and in the second it is a line passing through $y_2 = -\dfrac{b_2}{\lambda_+}$ (a horizontal line in a Cartesian diagram); and, second, if trace$(\mathbf{\Lambda}) = 0$ there is also an infinite number of invariant states

$$\bar{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathrm{Y}, \tag{3.33}$$

which means that the set of invariant states is coincident with set Y, i.e., $\bar{y} = \mathrm{Y}$, which we can see as a **two-dimensional manifold** (a surface).

Corollary 1. *An invariant state always exists, it is unique if* $\det(\mathbf{\Lambda}) \neq 0$ *and there is in infinite number if* $\det(\mathbf{\Lambda}) = 0$.

Next, we obtain a general form for the solution of ODE (3.28), for any matrices $\mathbf{\Lambda}$ and $\mathbf{B}$.

Proposition 2 (General solution to the non-homogenous ODE (3.28))**.** *Consider the ODE* (3.28) *for an arbitrary real vector* $\mathbf{B} \in \mathbb{R}^2$. *The solution to the ODE always exists and is uniquely given by*

$$\mathbf{y}(x) = -\mathbf{\Lambda}^+ \mathbf{B} + \mathbf{e}^{\mathbf{\Lambda}(x-x_0)} (\mathbf{y}(x_0) - \mathbf{\Lambda}^+ \mathbf{B}) + (\mathbf{I} - \mathbf{\Lambda}^+ \mathbf{\Lambda}) \mathbf{B} (x - x_0), \text{ for } x, x_0 \in \mathrm{X} \tag{3.34}$$

where $\mathbf{e}^{\mathbf{\Lambda}(x-x_0)}$ *is the appropriate matrix exponential given in equation* (3.18), $\mathbf{y}(x_0)$ *is an arbitrary element of* Y, *associated to an arbitrary* $x_0 \in \mathrm{X}$.

Proof. We start with the case in which $\det(\mathbf{\Lambda}) \neq 0$. Then again, matrix $\mathbf{\Lambda}$ has a unique classical inverse, $\mathbf{\Lambda}^+ = \mathbf{\Lambda}^{-1}$, which implies that $\bar{\mathbf{y}} = -\mathbf{\Lambda}^{-1}\mathbf{B}$ and $\mathbf{I} - \mathbf{\Lambda}^+\mathbf{\Lambda} = \mathbf{0}$. Define $\mathbf{z}(x) = \mathbf{y}(x) - \bar{\mathbf{y}}$ where $\mathbf{y}$ is given in equation (3.29). Then $\nabla\mathbf{z} = \nabla\mathbf{y} = \mathbf{\Lambda}\,\mathbf{y} + \mathbf{B} = \mathbf{\Lambda}(\mathbf{y} - \bar{\mathbf{y}}) = \mathbf{\Lambda}\,\mathbf{z}$, yields a homogenous ODE $\nabla\mathbf{z} = \mathbf{\Lambda}\,\mathbf{z}$, whose solution is, from equation (3.17), $\mathbf{z}(x) = e^{\mathbf{\Lambda}(x-x_0)}\mathbf{z}(x_0)$. Going back to the original variables we have

$$\mathbf{y}(x) = -\mathbf{\Lambda}^{-1}\mathbf{B} + e^{\mathbf{\Lambda}(x-x_0)} (\mathbf{y}(x_0) + \mathbf{\Lambda}^{-1}\mathbf{B})$$

If $\det(\mathbf{A}) = 0$ the coefficient matrix itakes one of the forms in equation (3.30). Therefore, the ODE's can take one of the following forms

$$\begin{cases} y_1' = \lambda_-\, y_1 + b_1, \\ y_2' = b_2, \end{cases} \quad \begin{cases} y_1' = b_1, \\ y_2' = \lambda_+\, y_2 + b_2, \end{cases} \quad \text{or} \quad \begin{cases} y_1' = b_1, \\ y_2' = b_2. \end{cases}$$

Using the results for the scalar ODE, the solutions are

$$\begin{cases} y_1(x) = -\dfrac{b_1}{\lambda_-} + e^{\lambda_-(x-x_0)} \left(y_1(x_0) + \dfrac{b_1}{\lambda_-}\right) \\ y_2(x) = y_2(x_0) + b_2\,(x - x_0) \end{cases} \quad \begin{cases} y_1(x) = y_1(x_0) + b_1\,(x - x_0) \\ y_2(x) = -\dfrac{b_2}{\lambda_+} + e^{\lambda_+(x-x_0)} \left(y_2(x_0) + \dfrac{b_2}{\lambda_+}\right) \end{cases} \quad \text{or}$$

$$\begin{cases} y_1(x) = y_1(x_0) + b_1\,(x - x_0) \\ y_2(x) = y_2(x_0) + b_2(x - x_0). \end{cases}$$

If we consider that: first, the invariant states in the first and second cases are the same we obtained in equation (3.32) , for the first two cases, and (3.33) for the third case; second, the exponential matrices are, respectively

$$\begin{pmatrix} e^{\lambda_- x} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda_+ x} \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

third, their Jordan matrices in equation (3.30); and, at last, their Moore-Penrose inverses iin equation (3.31), then we see that equation (3.34) is the matrix form encompassing all cases. □

If $\det(\mathbf{A}) \neq 0$ then the solution can be written as

$$\mathbf{y}(x) = \bar{\mathbf{y}} + \Psi(x, x_0)\left(\mathbf{y}(x_0) - \bar{\mathbf{y}}\right),\ x, x_0 \in \mathrm{X}$$

where $\bar{\mathbf{y}} = -\mathbf{\Lambda}^{-1}\ \mathbf{B}$ is the unique invariant state, and $\Psi(x, x_0) = e^{\mathbf{\Lambda}(x-x_0)}$ is the matrix exponential.

3.5.2 Generic A matrix

In this section we solve the general planar ODE

$$\nabla \mathbf{y} = \mathbf{A}\mathbf{y} + \mathbf{B} \tag{3.35}$$

where matrix $\mathbf{A}$ is not necessarily in a Jordan canonical form and $\mathbf{B}$ can be any real vector. This covers both the homogenous case in which $\mathbf{B} = \mathbf{0}$ and the non-homogeneous case in which $\mathbf{B} \neq \mathbf{0}$.

Proposition 3 (Invariant state for the non-homogenous ODE (3.35))**.** *invariant states for equation* (3.35) *exist and are given by*

$$\bar{\mathbf{y}} = -\mathbf{A}^{+}\,\mathbf{B} + \left(\mathbf{I} - \mathbf{A}^{+}\,\mathbf{A}\right)\mathbf{y}, \tag{3.36}$$

where $\mathbf{A}^{+} = \mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{P}^{-1}$ *is the Moore-Penrose inverse of* $\mathbf{A}$*, and* $\mathbf{y}$ *is an arbitrary element of* Y*.*

Proof. Multiplying equation (3.37) by $\mathbf{P}$ we get

$$\begin{aligned}
\bar{\mathbf{y}} &= \mathbf{P}\,\bar{\mathbf{w}} \\
&= -\mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{P}^{-1}\,\mathbf{B} + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{w}(0) \\
&= -A^{+}\,\mathbf{B} + \mathbf{P}(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda})\,\mathbf{P}^{-1}\ y(0) \\
&= -A^{+}\mathbf{B} + (\mathbf{P}\,\mathbf{P}^{-1} - \mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}\ )\,\mathbf{y}(0) \\
&= -A^{+}\mathbf{B} + (\mathbf{I} - \mathbf{A}^{+}\,\mathbf{P}\,\mathbf{P}^{-1}\ \ \mathbf{A})\,\mathbf{y}(0) \\
&= -A^{+}\mathbf{B} + (\mathbf{I} - \mathbf{A}^{+}\ \ \mathbf{A})\,\mathbf{y}(0)
\end{aligned}$$

□

In order to find the solution of the ODE (3.35), we start by presenting two useful results:

Lemma 7. *Consider the coefficient matrix* $\mathbf{A}$ *and let* $\mathbf{P}$ *and* $\mathbf{\Lambda}$ *be its associated eigenvector matrix and Jordan canonical form. Then, the ODE* (3.35) *with general coefficient matrix* $\mathbf{A}$ *can be transformed into an ODE with coefficient matrix* $\mathbf{\Lambda}$

$$\mathbf{y}(x) = \mathbf{P}\,\mathbf{w}(x) \tag{3.37}$$

where $\mathbf{P}$ *is the eigenvector matrix associated to* $\mathbf{A}$ *and* $\mathbf{w}(x)$ *is the solution of the ODE*

$$\nabla \mathbf{w} = \mathbf{\Lambda}\,\mathbf{w} + \mathbf{P}^{-1}\,\mathbf{B} \tag{3.38}$$

Proof. Recall that any matrix satisfies $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where matrix $\mathbf{P}$ is non-singular. Then we can introduce a unique linear transformation $\mathbf{w}(x) = \mathbf{P}^{-1}\mathbf{y}(x)$. Then

$$\nabla \mathbf{w} = \mathbf{P}^{-1}\,\nabla \mathbf{y} = \mathbf{P}^{-1}\left(\mathbf{A}\mathbf{y} + \mathbf{B}\right) = \mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} + \mathbf{P}^{-1}\mathbf{B} = \mathbf{\Lambda}\mathbf{w} + \mathbf{P}^{-1}\mathbf{B}.$$

□

Lemma 8. *The solution to the ODE transformed coordinates* $\mathbf{w}$*, equation* (3.38) *is*

$$\mathbf{w}(x) = \bar{\mathbf{w}} \; + \mathbf{e}^{\mathbf{\Lambda}(x-x_0)}(\mathbf{w}(x_0) - \bar{\mathbf{w}}) + \left(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\right)\mathbf{P}^{-1}\,\mathbf{B}\,(x - x_0) \tag{3.39}$$

where

$$\bar{\mathbf{w}} \; = -\mathbf{\Lambda}^{+}\,\mathbf{P}^{-1}\,\mathbf{B} + \left(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\right)\mathbf{w}(0)$$

and $\mathbf{w}(x_0) = \mathbf{P}^{-1}\,\mathbf{y}(x_0)$ *for an arbitrary* $\mathbf{y}(x_0)$*.*

Proof. ODE (3.38) is a non-homogeneous ODE in which the coefficient matrix is in the Jordan canonical form. Comparing with equation (3.28) we find that instead of $\mathbf{B}$ we now have $\mathbf{P}^{-1}\mathbf{B}$. By performing this substitution in the solution to the last ODE, in equation (3.34) we find the solution of the transformed ODE in equation (3.39). □

The general solution to equation (3.35) exists and is uniquely given in the next proposition:

Proposition 4 (Solution for the non-homogenous ODE (3.35))**.** *Consider the ODE* (3.35) *for any matrix* $\mathbf{A} \in \mathbb{R}^{2\times 2}$ *and vector* $\mathbf{B} \in \mathbb{R}^{2}$*. The solution to the ODE always exist and is uniquely given by*

$$\mathbf{y}(x) = \bar{\mathbf{y}} + \; \mathbf{e}^{\mathbf{A}\,(x-x_0)}\left(\mathbf{y}(x_0) - \bar{\mathbf{y}}\right) + \left(\mathbf{I} - \mathbf{A}^{+}\,\mathbf{A}\right)\mathbf{B}\,(x - x_0), \; \textit{for}\, x, x_0 \in \mathrm{X}, \tag{3.40}$$

where the invariant state $\bar{\mathbf{y}}$ *is given in equation* (3.36)*, and* $\mathbf{y}(x_0)$ *is an arbitrary element of* $\mathbf{y}$ *for* $x = x_0$*.*

Proof. Multiplying equation (3.37) by $\mathbf{P}$ we get the inverse transformation $\mathbf{y}(x) = \mathbf{P}\,\mathbf{w}(x)$. Using the solution for the transformed variables in equation (3.39) yields

$$\begin{aligned}
y(x) &= \mathbf{P}\,\bar{\mathbf{w}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}(x-x_0)}(\mathbf{w}(0) - \bar{\mathbf{w}}) + \mathbf{P}\big(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\big)\,\mathbf{P}^{-1}\,\mathbf{B}\,(x - x_0)\\
&= \bar{\mathbf{y}} \; + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}(x-x_0)}\mathbf{P}^{-1}(\mathbf{y}(x_0) - \bar{\mathbf{y}}) + \mathbf{P}\big(\mathbf{I} - \mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\big)\,\mathbf{P}^{-1}\,\mathbf{B}\,(x - x_0)\\
&= \bar{\mathbf{y}} + e^{\mathbf{A}(x-x_0)}(\mathbf{y}(0) - \bar{\mathbf{y}}) + \left(\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^{+}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}\;\right)\mathbf{B}\,(x - x_0)
\end{aligned}$$

which gives equation (3.40). □

Next we present the specific forms for the ODE (3.35).

Solutions for $\det(\mathbf{A}) \neq 0$ **cases**

If $\det(\mathbf{A}) \neq 0$ then $\mathbf{A}^{+} = \mathbf{A}^{-1}$ then there is a unique invariant state

$$\bar{y} = -\mathbf{A}^{-1}\,\mathbf{B}.$$

Expanding the previous formula, we have

$$\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = -\frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22}\,b_1 - a_{12}\,b_2 \\ -a_{21}\,b_1 - a_{11}\,b_2 \end{pmatrix}.$$

The solution (3.40) takes the particular form

$$\mathbf{y}(x) = \bar{\mathbf{y}} + \mathbf{e}^{\mathbf{A}(x-x_0)}\,(\mathbf{y}(x_0) - \bar{\mathbf{y}}), \tag{3.41}$$

where $\mathbf{e}^{\mathbf{A}t} = \mathbf{P}\,\mathbf{e}^{\mathbf{\Lambda}t}\,\mathbf{P}^{-1}$, where $\mathbf{e}^{\mathbf{\Lambda}t}$ is the matrix exponential of the Jordan canonical form which is similar to $\mathbf{A}$. It is useful in applications to write the solution (3.41) as

$$\mathbf{y}(x) = \bar{\mathbf{y}} + \mathbf{P}\,\mathbf{e}^{\mathbf{\Lambda}x}\;\mathbf{k}(x_0)$$

where $\mathbf{k}(x_0) = \mathbf{P}^{-1}(\mathbf{y}(x_0) - \bar{\mathbf{y}})$. Writing the eigenvector matrix $\mathbf{P}$ as [6]

$$\mathbf{P} = \mathbf{P}^{-}|\mathbf{P}^{+} = \begin{pmatrix} P_1^{-} & P_1^{+} \\ P_2^{-} & P_2^{+} \end{pmatrix},$$

then

$$\mathbf{k}(x_0) = \begin{pmatrix} k_1(x_0) \\ k_2(x_0) \end{pmatrix} = \frac{1}{\det(\mathbf{P})} \begin{pmatrix} P_2^{+} & -P_1^{+} \\ -P_2^{-} & P_1^{-} \end{pmatrix} \begin{pmatrix} y_1(x_0) - \bar{y}_1 \\ y_2(x_0) - \bar{y}_2 \end{pmatrix}.$$

in which $\mathbf{y}(x_0)$ is an arbitrary element of $\mathbf{y}$ for $x = x_0$.

Then the solution for the non-degenerate cases can take the following forms

1. If $\Delta(\mathbf{A}) > 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_1$ and the general solution is

   $$\mathbf{y}(x) = \bar{\mathbf{y}} + k_1(x_0)\,\mathbf{P}^{-}\,e^{\lambda_{-}(x-x_0)} + k_2(x_0)\,\mathbf{P}^{+}\,e^{\lambda_{+}(x-x_0)}$$

   where $\mathbf{P}^{-}$ ($\mathbf{P}^{+}$) is the simple eigenvector associated with λ_{-} (λ_{+}). More specifically

   $$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + k_1(x_0) \begin{pmatrix} P_1^{-} \\ P_2^{-} \end{pmatrix} e^{\lambda_{-}(x-x_0)} + k_2(x_0) \begin{pmatrix} P_1^{+} \\ P_2^{+} \end{pmatrix} e^{\lambda_{+}(x-x_0)}.$$

2. If $\Delta(\mathbf{A}) = 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_2$. The general solution is

   $$\mathbf{y}(x) = \bar{\mathbf{y}} + e^{\lambda(x-x_0)}\left(\mathbf{P}^1(k_1(x_0) + k_2(x_0)\,(x - x_0)) + k_2(x_0)\,\mathbf{P}^2\right)$$

   where $\mathbf{P}^1$ is a simple eigenvector and $\mathbf{P}^2$ is a generalized eigenvector (see the Appendix), or, equivalently

   $$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + e^{\lambda(x-x_0)}\left((k_1(x_0) + k_2(x_0)\,(x - x_0)) \begin{pmatrix} P_1^{-} \\ P_2^{-} \end{pmatrix} + k_2(x_0) \begin{pmatrix} P_1^{+} \\ P_2^{+} \end{pmatrix}\right)$$

[6] Recall that $\mathbf{P}^j$ is the solution of the homogeneous system $(\mathbf{A} - \lambda_j\mathbf{I})\,\mathbf{P}^j = \mathbf{0}$.

3. If $\Delta(\mathbf{A}) < 0$ then the Jordan canonical form of matrix $\mathbf{A}$ is $\mathbf{\Lambda}_3$. The general solution is

$$\mathbf{y}(x) = \overline{\mathbf{y}} + e^{\alpha(x-x_0)}\Big((k_1(x_0)\cos\beta(x-x_0) + k_2(x_0)\,\sin\beta(x-x_0))\mathbf{P}^1 + \\ + (k_2(x_0)\,\cos\beta(x-x_0) - k_1(x_0)\,\sin\beta(x-x_0))\mathbf{P}^2\Big).$$

where $\mathbf{P}$ is a eigenvector (see the Appendix for the determination of the eigenvector matrix in the case in which the eigenvectors are complex) or, equivalently,

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\alpha(x-x_0)}\Bigg\{ \; k_1(x_0)\begin{pmatrix} P_1^1\cos\beta(x-x_0) - P_1^2\,\sin\beta(x-x_0) \\ P_2^1\cos\beta(x-x_0) - P_2^2\sin\beta(x-x_0) \end{pmatrix} + \\ + \; k_2(x_0)\begin{pmatrix} P_1^1\sin\beta(x-x_0) + P_1^2\,\cos\beta(x-x_0) \\ P_2^1\sin\beta(x-x_0) + P_2^2\cos\beta(x-x_0) \end{pmatrix}\Bigg\}.$$

Solutions for $\det(\mathbf{A}) = 0$ **cases**

Degenerate cases occur for $\det(\mathbf{A}) = 0$ implying that $\mathbf{A}^+ \neq \mathbf{A}^{-1}$ and that the Jordan canonical form is diagonal (i.e, of type $\mathbf{\Lambda}_1$ in which one or two of the eigenvalues are equal to zero).

As $\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ then $\mathbf{A}^+ = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}$ and $\mathbf{A}^+\,\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$ where $\mathbf{\Lambda}$ is one of the Jordan forms in equation (3.30) and $\mathbf{\Lambda}^+$ is the associated the Moore-Penrose in equation (3.31), depending on the trace being trace$(\mathbf{A}) \neq 0$ or trace$(\mathbf{A}) = 0$.

First observe that (3.40) can be expanded as

$$\mathbf{y}(x) = -\mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{B} + \; \mathbf{e}^{\mathbf{A}\,(x-x_0)}\,(\mathbf{y}(0) + \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{P}^{-1}\,\mathbf{B}) + (\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1})\,\mathbf{B}\,(x-x_0),$$

where we can see that there are some components which are independent from the particular Jordan form in equation (3.30) and others which depend on the particular Jordan form.

For the first case we have $\tilde{\mathbf{B}} = \mathbf{P}^{-1}\,\mathbf{B}$ and $\mathbf{w}(0) = \mathbf{P}^{-1}\,\mathbf{y}(0)$, and write their expansion as

$$\tilde{\mathbf{B}} = \begin{pmatrix} \tilde{b}_- \\ \tilde{b}_+ \end{pmatrix} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,b_1 + P_1^-\,b_2 \\ P_2^+\,b_1 - P_1^+\,b_2 \end{pmatrix}$$

and

$$\mathbf{w}(x_0) = \begin{pmatrix} w_-(x_0) \\ w_+(x_0) \end{pmatrix} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,y_1(x_0) + P_1^-\,y_2(x_0) \\ P_2^+\,y_1(x_0) - P_1^+\,y_2(x_0) \end{pmatrix}$$

For the second case we have, if $\lambda_- < 0 = \lambda_+$,

$$\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} -P_2^-\,P_1^+ & P_1^-\,P_1^+ \\ -P_2^-\,P_2^+ & P_1^-\,P_2^+ \end{pmatrix}$$

for the case in which $\lambda_- = 0 < \lambda_+$ we have

$$\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \frac{1}{\det(\mathbf{P})}\begin{pmatrix} P_2^+\,P_1^- & -P_1^+\,P_1^- \\ P_2^+\,P_2^- & -P_1^+\,P_2^- \end{pmatrix}$$

and for $\lambda_- = \lambda_+ = 0$ we have $\mathbf{I} - \mathbf{P}\,\mathbf{\Lambda}^+\,\mathbf{\Lambda}\,\mathbf{P}^{-1} = \mathbf{I}$.

Therefore the solutions become

1. if $\lambda_- < 0 = \lambda_+$

$$\mathbf{y}(x) = \mathbf{P}^+ w_+(x_0) - \mathbf{P}^- \frac{\tilde{b}_-}{\lambda_-} + \begin{pmatrix} P_1^- \, e^{\lambda_-(x-x_0)} \\ P_2^- \end{pmatrix} \left(w_-(x_0) + \frac{\tilde{b}_-}{\lambda_-}\right) - \mathbf{P}^+ \tilde{b}_+$$

2. if $\lambda_- = 0 < \lambda_+$

$$\mathbf{y}(x) = \mathbf{P}^- w_-(x_0) - \mathbf{P}^+ \frac{\tilde{b}_+}{\lambda_+} + \begin{pmatrix} P_1^+ \\ P_2^+ \, e^{\lambda_+(x-x_0)} \end{pmatrix} \left(w_+(x_0) + \frac{\tilde{b}_+}{\lambda_+}\right) - \mathbf{P}^- \tilde{b}_-$$

3. for $\lambda_- = \lambda_+ = 0$

$$\mathbf{y}(x) = \mathbf{P} \left(\mathbf{w}(x_0) - \tilde{b}_- \right).$$

3.A Appendix

3.A.1 Review of matrix algebra

Consider matrix $\mathbf{A}$ of order 2 with real entries

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

that is $\mathbf{A} \in \mathbb{R}^{2\times 2}$. The **trace** and the **determinanx** of $\mathbf{A}$ are, respectively,

$$\text{trace}(\mathbf{A}) = a_{11} + a_{22},\ \det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

The kernel (or null space) of matrix $\mathbf{A}$ is a vector $\mathbf{v}$ defined as

$$\text{kern}(\mathbf{A}) = \{\ \mathbf{v} : \mathbf{A}\mathbf{v} = \mathbf{0}\}$$

The dimension of the kernel gives a measure of the linear independence between the rows of $\mathbf{A}$.

The characteristic polynomial of matrix $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda \mathbf{I}_2) = \lambda^2 - \text{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) \tag{3.42}$$

where $\lambda \in \mathbb{C}$ is an eigenvalue, which is complex valued.

The **spectrum** of $\mathbf{A}$ is the **set of eigenvalues**

$$\sigma(\mathbf{A}) \equiv \{\ \ \lambda \in \mathbb{C} : \det(\mathbf{A} - \lambda \mathbf{I}_2)\ \ = 0\}$$

The **eigenvalues** of any 2×2 matrix $\mathbf{A}$ are

$$\lambda_+ = \frac{\text{trace}(\mathbf{A})}{2} + \Delta(\mathbf{A})^{\frac{1}{2}},\ \lambda_- = \frac{\text{trace}(\mathbf{A})}{2} - \Delta(\mathbf{A})^{\frac{1}{2}} \tag{3.43}$$

where the discriminant is

$$\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A}).$$

A useful result on the relationship between the eigenvalues and the trace and the determinant of $\mathbf{A}$:

Lemma 9. *Let λ_+ and λ_- be the eigenvalues of a 2×2 matrix $\mathbf{A}$. Then they are verify:*

$$\begin{aligned} \lambda_+ + \lambda_- &= \text{trace}(\mathbf{A}) \\ \lambda_+\lambda_- &= \det(\mathbf{A}). \end{aligned}$$

Three cases can occur:

1. if $\Delta(\mathbf{A}) > 0$ then λ_+ and λ_- are real and distinct and $\lambda_+ > \lambda_-$
2. if $\Delta(\mathbf{A}) = 0$ then $\lambda_+ = \lambda_- = \lambda = \text{trace}(\mathbf{A})/2$ are real and multiple,

3. if $\Delta(\mathbf{A}) < 0$ then λ_+ and λ_- are complex conjugate $\lambda_+ = \alpha + \beta i$ and $\lambda_- = \alpha - \beta i$ where $\alpha = \frac{\text{tr}(A)}{2}$ and $\beta = \sqrt{|\Delta(\mathbf{A})|}$ and $i = \sqrt{-1}$.

In the last case, we can write the eigenvalues in polar coordinates as

$$\lambda_+ = r(\cos\theta + \sin\theta i),\ \lambda_- = r(\cos\theta - \sin\theta i)$$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\tan\theta = \beta/\alpha$, or

$$\alpha = r\cos\theta,\ \beta = r\sin\theta$$

Jordan canonical forms Two matrices $\mathbf{A}$ and $\mathbf{A}'$ with the equal eigenvalues are called **similar**. This allows for classifying matrices according to their eigenvalues.

The Jordan canonical forms for 2×2 matrices are

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix},\ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},\ \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{3.44}$$

Lemma 10 (Jordan canonical from of matrix $\mathbf{A}$)**.** *Consider any 2×2 matrix with real entries and its discriminant $\Delta(\mathbf{A})$. Then*

1. *If $\Delta(\mathbf{A}) > 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_1$.*

2. *If $\Delta(\mathbf{A}) = 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_2$.*

3. *If $\Delta(\mathbf{A}) < 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_3$.*

The Jordan canonical form $\mathbf{\Lambda}_3$ can also be represented by a diagonal matrix with complex entries

$$\mathbf{\Lambda}_3 = \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix}.$$

In this sense, if $\Delta(\mathbf{A}) \neq 0$ then matrix $\mathbf{A}$ is diagonalizable and it is not diagonalizable if $\Delta(\mathbf{A}) = 0$. Figure 3.1 presents the different cases in a $(\text{trace}(\mathbf{A}), \det(\mathbf{A}))$ diagram.

It has the following information:

- Jordan canonical forms are associated to the following areas: $\mathbf{\Lambda}_1$ is outside the parabola; $\mathbf{\Lambda}_3$ is inside the parabola, and $\mathbf{\Lambda}_2$ is represented by the parabola;

- in the positive orthant the two eigenvalues have positive real parts, in the negative orthant they have negative real parts and bellow the abcissa there are two real eigenvalues with opposite signs;

- the abcissa corresponds to the locus of points in which there is at least one zero-valued eigenvalue, the upper part of the ordinate corresponds to complex eigenvalues with zero real part, and the origin to the case in which there are two eigenvalues equal to zero.

Eigenvectors of A

Lemma 11. *Let* $\mathbf{A}$ *be a* 2×2 *matrix with real entries. Then, there exists a non-singular matrix* $\mathbf{P}$ *such that*

$$\mathbf{A} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$$

where $\boldsymbol{\Lambda}$ *is the Jordan canonical form of* $\mathbf{A}$*, and matrix* $\mathbf{P}$ *is a* 2×2 ***eigenvector*** *matrix associated to* $\mathbf{A}$*.*

There are two types of eigenvectors:

1. **simple eigenvectors** if $\Delta(\mathbf{A}) \neq 0$. In this case the eigenvector is $\mathbf{P} = (\mathbf{P}^-, \mathbf{P}^+)$ concatenating the eigenvectors $\mathbf{P}^-$ and $\mathbf{P}^+$ associated to the eigenvalues λ_+ and λ_-, which are obtained from solving the homogeneous system

$$(\mathbf{A} - \lambda_j \mathbf{I}_2)\mathbf{P}^j = 0, \; j = 1, 2$$

   where $\mathbf{I}_2$ is the identity matrix of order 2. Observe that $\mathbf{P}^j = \text{kern}(\mathbf{A} - \lambda_j \mathbf{I}_2)$, i.e, it is the null space of matrix $(\mathbf{A} - \lambda_j \mathbf{I}_2)$;

2. **generalized eigenvectors** if $\Delta(\mathbf{A}) = 0$, that is, when we have multiple eigenvalues $\lambda_+ = \lambda_- = \lambda$. In this case we determine $\mathbf{P} = (\mathbf{P}^1, \mathbf{P}^2)$ where $\mathbf{P}^1$ is a simple eigenvalue and $\mathbf{P}^2$ is a generalized eigenvalue. They are obtained in the following way: first, $\mathbf{P}^1$ solves $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P}^1 = 0$, where $\mathbf{I} = \mathbf{I}_2$; second, (a) if $(\mathbf{A} - \lambda \mathbf{I})^2 \neq \mathbf{0}$ we determine $\mathbf{P}^2$ from $(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{P}^2 = 0$; however, (b) if $(\mathbf{A} - \lambda \mathbf{I})^2 = \mathbf{0}$ then we determine $\mathbf{P}^2$ from $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P}^2 = \mathbf{P}^1$.

When $\Delta(\mathbf{A}) < 0$ we can use one of the following two approaches:

1. either we write the Jordan matrix as a complex-valued matrix

$$\Lambda_3 = \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix}$$

   and compute $\mathbf{P}^j$ as a complex-valued vector from

$$(\mathbf{A} - \lambda_j \mathbf{I}_2)\mathbf{P}^j = 0,$$

2. or we write the Jordan matrix as a real-valued matrix as in equation (3.44) and compute $\mathbf{P}$ as a real-valued matrix by setting $\mathbf{P} = (\mathbf{u}, \mathbf{v})$ where $\mathbf{Q} = \mathbf{u} + \mathbf{v}i$ is the solution of the homogeneous system

$$(\mathbf{A} - (\alpha + \beta i)\mathbf{I}_2)\mathbf{Q} = 0$$

Conclusion: given a matrix $\mathbf{A}$, we can find matrices $\boldsymbol{\Lambda}$ and $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\Lambda\mathbf{P}^{-1}$ where $\mathbf{P}$ is invertible. Equivalently $\Lambda = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Proposition 5. *The eigenvector matrices associated to the Jordan canonical forms are:*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{3.45}$$

for $\Lambda = \mathbf{\Lambda}_1$, $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ *and* $\mathbf{\Lambda} = \mathbf{\Lambda}_3$, *respectively*

Proof. For $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, because $(\mathbf{\Lambda}_1 - \lambda_+ \mathbf{I})\mathbf{P}^- = 0$ and $(\mathbf{\Lambda}_1 - \lambda\_\mathbf{I})\mathbf{P}^+ = 0$ are

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda_- - \lambda_+ \end{pmatrix} \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda_+ - \lambda_- & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then we get $\mathbf{P} = (\mathbf{P}^- \mathbf{P}^+) = \mathbf{I}$, because $\lambda_+ \neq \lambda_-$. For $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ we determine the simple eigenvector from $(\mathbf{\Lambda}_2 - \lambda \mathbf{I})\mathbf{P}^- = 0$. To determine the second eigenvector as $(\lambda\_ - \lambda \mathbf{I})^2 = \mathbf{0}$, because

$$(\lambda\_ - \lambda \mathbf{I})^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then we use $(\mathbf{\Lambda}_2 - \lambda \mathbf{I})\mathbf{P}^2 = \mathbf{P}^1$,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^+ \\ P_2^+ \end{pmatrix} = \begin{pmatrix} P_1^- \\ P_2^- \end{pmatrix},$$

to get $\mathbf{P}^1 = (1, 0)$ and $\mathbf{P}^2 = (1, 1)$.

For $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ consider eigenvalue $\lambda_+ = \alpha + \beta i$ and assume that there is a complex vector

$$\mathbf{z} = \begin{pmatrix} u_1 + v_1 i \\ u_2 + v_2 i \end{pmatrix}$$

that solves $(\Lambda_3 - (\alpha + \beta i)I)\mathbf{z} = 0$, that is [7]

$$\begin{cases} \beta \left(u_2 + v_1 + (v_2 - u_1)i\right) & = 0 \\ \beta \left((v_2 - u_1) - (u_2 + v_1)i\right) & = 0 \end{cases}$$

then we should have $u_1 = v_2$ and $u_2 = -v_1$. We can arbitrarily set $u_1 = 1$ and $v_1 = 1$, in $\mathbf{P}^1 = (u_1, u_2)^\top$ and $\mathbf{P}_2 = (v_1, v_2)^\top$, to get the third eigenvector matrix. $\square$

Eigenspaces As matrix $\mathbf{P}$ is non singular it forms a basis for vector space $\mathbf{A}$. Then vector space $\mathbf{A}$ can be seen as a direct sum $\mathbf{A} = \mathcal{E}^1 \oplus \mathcal{E}^2$ where

$$\begin{aligned} \mathcal{E}^1 &= \{\text{eigenspace associated with } \lambda_+\} \\ \mathcal{E}^2 &= \{\text{ eigenspace associated with } \lambda\_\}. \end{aligned}$$

[7] We use the rules for sums and multiplications of complex numbers: if $x_1 = a_1 + b_1 i$ and $x_2 = a_2 + b_2 i$, then $x_1 + x_2 = (a_1 + a_2) + (b_1 + b_2)i$ and $x_1 x_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$ because $i^2 = -1$.

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