

Toolkit for economic growth

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Topics

1. Production functions: properties
2. Production functions and input prices
3. Production functions and types of inputs
4. Production functions and technology bias
5. Extensions: continuum of inputs, multi-sector economies
6. Production functions, input accumulation and dynamics
7. Types of growth dynamics and the production function

1. Production function: properties

Production function

- ▶ Production function:

$$y = F(\mathbf{x}) \equiv F(x_1, \dots, x_n)$$

- ▶ $F(\cdot)$ formalizes a **technology**: a transformation of inputs into outputs;
- ▶ y = output of one good (can be used in consumption and/or investment)
- ▶ $\mathbf{x} = (x_1, \dots, x_n)$ bundle of inputs, it is a vector if we assuming a discrete index set
- ▶ x_i input of good i in production (intermediate good or final good)
- ▶ relevant general properties of the technology: marginal changes in inputs, and homogeneity (general change in inputs)
 - ▶ at the input level: necessity, marginal variation, substitutability/complementarity
 - ▶ at the bundle effect: increasing/constant/decreasing returns to scale

Production: general properties

- ▶ Definition: F is **weakly increasing** if $\mathbf{x}^* \geq \mathbf{x} \Rightarrow F(\mathbf{x}^*) \geq F(\mathbf{x})$
- ▶ Meaning: Increase in the quantity of inputs leads to an increase in production (bigger bundle increases output)
- ▶ Derivative:

$$F_i(\mathbf{x}) \equiv \frac{\partial F(\mathbf{x})}{\partial x_i} = \lim_{\epsilon \rightarrow 0} \frac{F(\dots, x_i + \epsilon, \dots) - F(\dots, x_i, \dots)}{\epsilon}$$

- ▶ Gradient: vector of first derivatives

$$DF(\mathbf{x}) = \left(\frac{\partial F(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial F(\mathbf{x})}{\partial x_n} \right)^\top$$

- ▶ if $DF(\mathbf{x}) \geq 0$ then F is weakly increasing

Production: general properties

- Definition: F is **concave** if given any two bundles \mathbf{x} and \mathbf{x}^*

$$F(\mathbf{x}) - F(\mathbf{x}^*) \leq DF(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*)$$

(strictly concave if $<$)

- Meaning: increases in the quantity of inputs increases production less than linearly
- Hessian: matrix of second derivatives

$$D^2 F(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 F(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 F(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 F(\mathbf{x})}{\partial x_n^2} \end{pmatrix}$$

- If $F(\mathbf{x})$ is concave then $D^2 F(\mathbf{x})$ is negative semi-definite.
- $D^2 F(\mathbf{x})$ is negative semi-definite if the principal minors of odd order are negative and the principal minors of even order are positive

Production: general properties

- Definition: F is **homogeneous of degree** η if changing \mathbf{x} by a factor λ , where λ is a positive number changes output by a factor of λ^η

$$\lambda^\eta F(\mathbf{x}) = F(\lambda \mathbf{x}^*)$$

where $\lambda \mathbf{x}^* = (\lambda x_1, \dots, \lambda x_n)^\top$

- η measures the **returns to scale**
- We say the production function displays
 - decreasing returns to scale (DRS) if $\eta < 1$
 - constant returns to scale (CRS) if $\eta = 1$
 - increasing returns to scale (IRS) if $\eta > 1$
- Euler's theorem,

$$\eta F(\mathbf{x}) = DF(\mathbf{x}) \cdot \mathbf{x} = \sum_{i=1}^n \frac{\partial F(\mathbf{x})}{\partial x_i} x_i$$

- A fundamental requirement for the existence of growth is that $F(\cdot)$ is linearly homogeneous if $\eta = 1$

Production function: specific properties

- Necessity: input x_i is **necessary** if output is equal to zero if it is not used

$$x_i = 0 \Rightarrow f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0$$

- Variation in production: measured by the differential

$$dy = DF(\mathbf{x}) \cdot d\mathbf{x} = \sum_{i=1}^n F_i(\mathbf{x}) dx_i$$

- Marginal product (or productivity):

$$MP_i = F_i = \frac{\partial F(\mathbf{x})}{\partial x_i}$$

meaning: variation in production if input i is increased by one unit

$$dy = \sum_{i=1}^n F_i(\mathbf{x}) dx_i = F_i, \text{ if } d\mathbf{x} = (0, \dots, 0, 1, 0, \dots, 0, \dots, 0)$$

Production function: specific properties

- ▶ We say input i is **productive** if $F_i(\mathbf{x}) > 0$ for a particular value of \mathbf{x}
- ▶ If $F(\mathbf{x})$ is strictly increasing then all inputs are productive.
- ▶ **Inada property** if $MP_i \in (0, \infty)$

$$\lim_{x_i \rightarrow 0} F_i(\mathbf{x}) = +\infty, \text{ and } \lim_{x_i \rightarrow \infty} F_i(\mathbf{x}) = 0$$

- ▶ A technology is non-Inada if there are bounds (superior or inferior) in the MP of any input (or $MP = \text{constant}$)
- ▶ Productiveness can be a global (as in Inada case) or a local property: in the second case, it is possible to have bundles \mathbf{x}' or \mathbf{x}'' such that $F_i(\mathbf{x}') = 0$ or $F_i(\mathbf{x}'') < 0$ (saturation and congestion)

Production function: technology

- ▶ Change in the marginal product:

$$MP_{ij} = F_{ij} = \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j}, \text{ for any pair, } i, j = 1, \dots, n$$

- ▶ The marginal product for input i , MP_i is
 - ▶ decreasing if $MP_{ii} < 0$
 - ▶ constant if $MP_{ii} = 0$
 - ▶ increasing if $MP_{ii} > 0$
- ▶ If the technology is concave (among all other conditions) we have:

$$F_{ii}(\mathbf{x}) \leq 0 \text{ for all } i = 1, \dots, n \text{ (decreasing MP)}$$

$$F_{ii}(\mathbf{x}) F_{jj}(\mathbf{x}) - F_{ij}(\mathbf{x})^2 \geq 0, \text{ for all pairs } i \neq j = 1, \dots, n$$

- ▶ If a technology is concave then the MP_i , for all i , are non-increasing.

Production function: technology

- ▶ Allen-Uzawa elasticities

$$(\text{own elasticities}) \epsilon_{ii} = -\frac{F_{ii}(\mathbf{x})x_i}{F_i(\mathbf{x})}, \quad i = 1, \dots, n$$

$$(\text{crossed elasticities}) \epsilon_{ij} = -\frac{F_{ij}(\mathbf{x})x_j}{F_i(\mathbf{x})}, \quad i, j = 1, \dots, n$$

- ▶ Then regarding "own" elasticities
 - ▶ If MP_i is constant then the "own" elasticity is equal to zero
 - ▶ If MP_i is decreasing then the "own" elasticity is positive
- ▶ For "crossed" elasticities, we say there is **gross or Edgeworth**:
 - ▶ substitutability if $\epsilon_{ij} > 0$ ($F_{ij} < 0$)
 - ▶ independence if $\epsilon_{ij} = 0$ ($F_{ij} = 0$)
 - ▶ complementarity if $\epsilon_{ij} < 0$ ($F_{ij} > 0$)
- ▶ do not confuse with Hicksian substitutability which is evaluated from the change of demand as regards input prices

Production function: technology

- Compensated changes in two inputs: variations in inputs i and j such that the output is constant

$$dy = F_i(\mathbf{x})dx_i + F_j(\mathbf{x})dx_j = 0$$

- **Marginal rate of substitution** between inputs i and j : compensated change in j for a unit change in i

$$MRS_{ij} \equiv -\frac{dx_j}{dx_i} \Big|_{dy=0}$$

- Then: it is equal to the ratio of marginal products

$$MRS_{ij} = \frac{F_i(\mathbf{x})}{F_j(\mathbf{x})}, \quad i \neq j = 1, \dots, n$$

Production function: technology

- Elasticity of substitution between inputs i and j

$$ES_{ij} \equiv \frac{d \ln (x_j / x_i)}{d \ln MRS_{ij}(\mathbf{x})} \Big|_{dy=0}$$

- then

$$ES_{ij}(\mathbf{x}) = \frac{x_i F_i(\mathbf{x}) + x_j F_j(\mathbf{x})}{x_j F_j(\mathbf{x}) \epsilon_{ii}(\mathbf{x}) - 2x_i F_i(\mathbf{x}) \epsilon_{ij}(\mathbf{x}) + x_i F_i(\mathbf{x}) \epsilon_{jj}(\mathbf{x})}$$

Production function: the benchmark case

The benchmark production function is the **generalized mean**

$$y = M_{\sigma}(\mathbf{x}) = \left(\sum_{i=1}^n \alpha_i x_i^{\sigma} \right)^{\frac{1}{\sigma}}, \quad \sigma \in [-\infty, \infty]$$

where α_i is the share of input i satisfying

$$0 \leq \alpha_i \leq 1$$

$$\sum_{i=1}^n \alpha_i = 1$$

- We readily see that it is an homogeneous function, for any value of σ

$$M_{\sigma}(\lambda \mathbf{x}) = \lambda M_{\sigma}(\mathbf{x})$$

thus it represents a **constant returns to scale technology**.

Production function: the benchmark case

Applying our previous concepts we find:

- ▶ Marginal product of input i : all inputs are productive

$$MP_i(\mathbf{x}) = F_i(\mathbf{x}) = \alpha_i \left(\frac{F(\mathbf{x})}{x_i} \right)^{1-\sigma} \geq 0$$

- ▶ Allen-Uzawa elasticities

- ▶ input i has **decreasing** MP_i if $\sigma \leq 1$

$$\epsilon_{ii} = (1 - \sigma) \left(1 - \frac{\alpha_i x_i^\sigma}{F(\mathbf{x})^\sigma} \right) \geq 0 \text{ if } \sigma < 1$$

- ▶ inputs i and j are
 - ▶ gross substitutes if $\sigma > 1$
 - ▶ gross complements if $\sigma < 1$

because

$$\epsilon_{ij} = (\sigma - 1) \frac{\alpha_j x_j^\sigma}{F(\mathbf{x})^\sigma}$$

Production function: the benchmark case

Applying our previous concepts we find:

- ▶ Marginal rate of substitution depends only on the quantities of the two inputs

$$MRS_{ij}(\mathbf{x}) = \frac{\alpha_i}{\alpha_j} \left(\frac{x_j}{x_i} \right)^{1-\sigma}$$

- ▶ Elasticity of substitution is constant, for any pair of inputs

$$ES_{ij}(\mathbf{x}) = \frac{1}{1-\sigma}$$

then i and j are

- ▶ gross substitutes implies $ES_{ij}(\mathbf{x}) > 0$
- ▶ gross complements implies $ES_{ij}(\mathbf{x}) < 0$

Benchmark production function: examples

For different values of σ economists call different names

- ▶ Linear case: if $\sigma = 1$

$$F(\mathbf{x}) = M_1 = \sum_i \alpha_i x_i$$

- ▶ Cobb-Douglas if $\sigma = 0$

$$F(\mathbf{x}) = \prod_i x_i^{\alpha_i} \text{ for } \sum_i \alpha_i = 1$$

Proof:

$$\ln M_0 = \lim_{\sigma \rightarrow 0} \frac{\ln \sum_{i=1}^n \alpha_i x_i^\sigma}{\sigma} = \lim_{\sigma \rightarrow 0} \frac{\sum \alpha_i x_i^\sigma \ln(x_i)}{\sum \alpha_i x_i^\sigma} = \sum \alpha_i \ln(x_i)$$

- ▶ Constant elasticity of substitution (CES)

$$F(\mathbf{x}) = M_\sigma(\mathbf{x}), \text{ if } \sigma < 1 \text{ and is finite}$$

Benchmark production function: examples

- ▶ As $M_\infty = x_{max} = \max\{x_1, \dots, x_n\}$

Proof

$$\lim_{\sigma \rightarrow \infty} M_\sigma = x_{\max} \lim_{\sigma \rightarrow \infty} \sum_i \alpha_i \left(\frac{x_i}{x_{\max}} \right)^\sigma \Big)^{\frac{1}{\sigma}} =$$
$$x_{\max} \lim_{\sigma \rightarrow \infty} \text{const}^{\frac{1}{\sigma}} = x_{\max} \text{ where } \text{const} \in (0, 1)$$

- ▶ Leontieff production function if $\sigma = -\infty$

$$F(\mathbf{x}) = M_{-\infty}(\mathbf{x}) = x_{min} = \min\{x_1, \dots, x_n\}$$

Proof:

$$M_{-\infty}(\mathbf{x}) = \frac{1}{M_\infty(1/\mathbf{x})}$$

2. Production function and input prices

Problem for a competitive firm

- ▶ **Assumption:** competitive product and input markets
- ▶ Total cost:

$$C(\mathbf{x}, \mathbf{w}) = \mathbf{w} \cdot \mathbf{x} = \sum_{i=1}^n w_i x_i$$

- ▶ Marginal cost of input i

$$MC_i = \frac{\partial C(\mathbf{x}, \mathbf{w})}{\partial x_i}$$

- ▶ Return, assuming an unit cost for the product :

$$RT = 1 \times y = F(\mathbf{x})$$

- ▶ Profit: equal to total return minus total cost

$$\pi(\mathbf{x}, \mathbf{w}) = F(\mathbf{x}) - C(\mathbf{x}, \mathbf{w}) = F(x_1, \dots, x_n) - \sum_{i=1}^n w_i x_i$$

Input demand functions

- ▶ The firm's primal problem (simpler problem) is to maximize the profit by choosing a vector of inputs

$$\pi^*(\mathbf{w}) = \max_{\mathbf{x}} \pi(\mathbf{x}, \mathbf{w}) \text{ s.t. } F(\mathbf{x}) \leq y$$

where y is output

- ▶ We write the Lagrangean

$$L(\mathbf{x}, \lambda) = F(\mathbf{x}) - C(\mathbf{x}, \mathbf{w}) + \lambda (y - F(\mathbf{x}))$$

where λ is the Lagrange multiplier

- ▶ Optimum conditions, for an interior solution are

$$\begin{cases} (1 - \lambda) F_i(\mathbf{x}) = w_i, & \text{for } i = 1, \dots, n \\ F(\mathbf{x}) = y \end{cases}$$

- ▶ If there is an interior solution we will find the (Hicksian) demand functions for all inputs

$$x^* = X_i(\mathbf{w}, y)$$

as functions of input prices and output.

Input demand functions

For the benchmark case, the Inada conditions guarantee existence and uniqueness of solutions to the producer problem:

- ▶ solving the optimality condition we find

$$x_i = \left(\frac{\alpha_i (1 - \lambda)}{w_i} \right)^{\frac{1}{1-\sigma}} F(\mathbf{x})$$

- ▶ substituting in the constraint yields

$$(1 - \lambda)^{\frac{1}{1-\sigma}} = \frac{1}{P(\mathbf{w})}$$

where $P(\mathbf{w})$ is a producer price index

$$P(\mathbf{w}) \equiv \left(\sum_{i=1}^n \alpha_i \left(\frac{w_i}{\alpha_i} \right)^{\frac{\sigma}{\sigma-1}} \right)^{\frac{1}{\sigma}}$$

Input demand functions

- ▶ The optimal demand (Hicksian) functions are

$$x_i^* = X_i(\mathbf{w}, y) = \left(\frac{w_i}{\alpha_i}\right)^{\frac{1}{\sigma-1}} \frac{y}{P(\mathbf{w})}$$

proportional to the real demand, where the deflator is the producer price index

- ▶ Comparative statics for prices: elasticities

- ▶ for "own price" changes

$$\frac{\partial X_i}{\partial w_i} \frac{w_i}{X_i} = \frac{1}{\sigma - 1} \left(1 - \frac{\alpha_i \left(\frac{w_i}{\alpha_i}\right)^{\frac{\sigma}{\sigma-1}}}{P(\mathbf{w})^\sigma} \right), \text{ for } i = 1, \dots, n$$

- ▶ for "crossed price" changes

$$\frac{\partial X_i}{\partial w_j} \frac{w_i}{X_i} = -\frac{1}{\sigma - 1} \frac{\alpha_i \left(\frac{w_i}{\alpha_i}\right)^{\frac{\sigma}{\sigma-1}}}{P(\mathbf{w})^\sigma}, \text{ for } i \neq j = 1, \dots, n$$

Input demand functions

- ▶ if $\sigma < 1$ then

$$\frac{\partial X_i}{\partial w_i} \frac{w_i}{X_i} < 0, \text{ and } \frac{\partial X_i}{\partial w_j} \frac{w_j}{X_i} > 0$$

the demand **reduces** with the "own" price and **increases** with any "crossed" price, meaning that inputs i and any other input are **substitutable in the Hicksian sense** (recall they were gross complements in the Edgeworth sense)

- ▶ if $\sigma > 1$ then

$$\frac{\partial X_i}{\partial w_i} \frac{w_i}{X_i} > 0, \text{ and } \frac{\partial X_i}{\partial w_j} \frac{w_j}{X_i} < 0$$

the demand **increases** with the "own" price and **reduces** with any "crossed" price, meaning that inputs i and any other input are **complementary in the Hicksian sense** (recall they were gross substitutable in the Edgeworth sense).

MRS and input prices

- For the benchmark case, setting $F_i(\mathbf{x}) = w_i$ we find

$$x_i^*(\mathbf{w}) = \left(\frac{w_i}{\alpha_i} \right)^{\frac{1}{\sigma-1}} F(\mathbf{x}^*)$$

- Then we have a relationship between factor demands and relative prices,

$$MRS_{ij} = \frac{F_i(\mathbf{x}^*)}{F_j(\mathbf{x}^*)} = \frac{\alpha_i}{\alpha_j} \left(\frac{x_j^*}{x_i^*} \right)^{1-\sigma} = \frac{w_i}{w_j}$$

3. Production functions: types of inputs

Types of inputs

The following distinction is important for economic growth:

- ▶ intermediary goods and factors of production
- ▶ produced inputs and non-produced inputs
- ▶ exogenous inputs and endogenous inputs
- ▶ private inputs and aggregate inputs

Intermediate goods

- ▶ in a given production function: intermediate inputs, \mathbf{z} , enter as flows and factors \mathbf{k} enter as stocks in production functions

$$y = F(\mathbf{z}, \mathbf{k})$$

where

- ▶ intermediate inputs are products of other sectors and use factors of production: example $z_i = f_i(\mathbf{k})$
- ▶ usually for the final use sector they are private goods (i.e., firms pay the full price for their use)
- ▶ they can be produced in a competitive or non-competitive market

Factors of production

- ▶ in a given production function: factors enter as stocks in production functions

$$y = F(\mathbf{k})$$

- ▶ factors of production are usually exogenous to the firm
- ▶ but they can be exogenous or endogenous to the economy
- ▶ when factors of production are produced, their output is a flow which generates a stock-flow dynamics

$$\dot{k}_i = \frac{dk_i(t)}{dt} = G(\mathbf{k})$$

- ▶ durable goods entail necessarily a dynamic mechanism (ex: capital stock)

Factors of production

- ▶ The provision of factor of production can be internal or external to the firm
- ▶ The firm's level production function can be

$$y = f(\mathbf{k}, \mathbf{K})$$

where \mathbf{k} are private factors and \mathbf{K} are external factors

- ▶ their use faces different incentives:
 - ▶ if the factor of production is private the firm has to pay the price w_i for its use
 - ▶ if the factor of production is an externality the firm does not have to pay for its use
- ▶ their existence introduces a distinction between production functions at the firm level $y = f(\mathbf{k}, \mathbf{K})$ and at the aggregate level

$$y = F(\mathbf{K}) = f(\mathbf{k}, \mathbf{K})$$

- ▶ then the properties of the production function, that we just saw, can be different at the firm's level (related to the incentives) and at the aggregate level. For instance, a production function can display DRS at the private level and display CRS at the

4. Production function and technological bias

Production function in input intensity form

- ▶ Production function in efficiency form

$$y = F(\mathbf{A}, \mathbf{x}) \equiv F(A_1 x_1, \dots, A_n x_n)$$

where A_i input augmenting index measuring the specific productivity of input i ,

- ▶ In growth models:
 - ▶ A_i measures specific or aggregate productivity increases
 - ▶ $A_i(t)$, where t is time measures technical progress (or decay),
 - ▶ A_i can be exogenous or endogenous (learning-by-doing, R&D)
- ▶ Here we introduce a first take on the subject

Types of technical progress

- Consider the benchmark production function in intensity form

$$y = F(\mathbf{A}, \mathbf{x}) = \left(\sum_{i=1}^n \alpha_i (A_i x_i)^\sigma \right)^{\frac{1}{\sigma}}$$

- Then the marginal product and the rate of substitution are

$$MP_i = \alpha_i A_i^\sigma \left(\frac{F(\mathbf{A}, \mathbf{x})}{x_i} \right)^{1-\sigma}$$

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j} \left(\frac{A_i}{A_j} \right)^\sigma \left(\frac{x_j}{x_i} \right)^{1-\sigma}$$

- The effect of the technical progress on production depends on:
 - the vector \mathbf{A} , in particular if it is equal for all inputs: Hicks neutral technical progress $A_i = A$ for all i
 - on the substitutability properties of the production function, if \mathbf{A} is heterogeneous

Types of technical progress

There are several concepts of neutrality and bias for technical progress. Here we consider

- ▶ **neutral** technical progress: if the change in any A_i leaves the MRS_{ij} unchanged
- ▶ **biased** technical progress: if the change in any A_i changes the MRS_{ij} .
- ▶ If we consider input prices

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j} \left(\frac{A_i}{A_j} \right)^\sigma \left(\frac{x_j}{x_i} \right)^{1-\sigma} = \frac{w_i}{w_j}$$

- ▶ In the above sense
 - ▶ if $\sigma = 0$ then the technical progress is neutral in this sense
 - ▶ if $\sigma \neq 0$ then the technical progress is biased
- ▶ However, observe that the demand functions $x_i = X_i(\mathbf{A}, \mathbf{w})$ can depend on \mathbf{A} . If $\sigma = 0$ we can interpret this as an income effect.

Types of technical progress

- ▶ We can also write

$$\frac{\alpha_i}{\alpha_j} \left(\frac{A_i x_i}{A_j x_j} \right)^\sigma = \frac{w_i x_i}{w_j x_j}$$

- ▶ Assume that the expenditures in inputs (or factor shares in national income) is constant and $\sigma \neq 0$, then:
 - ▶ if the technical progress is **neutral** the ratio of the inputs remains constant;
 - ▶ if the technical progress is biased an increase in A_i/A_j the technical progress is ***i*-saving**, i.e, there is a reduction in its quantity; the ratio of the two inputs changes.

5.1 Extensions 1: continuum of inputs

Continuum of inputs

- ▶ Dixit-Stiglitz production functions consider a continuum of inputs which makes y a functional

$$y = F[x] = \left(\int_0^N \alpha(i) x(i)^\sigma di \right)^{\frac{1}{\sigma}}$$

- ▶ How to calculate $MP(i)$?
- ▶ we use the functional derivative

$$MP(i) = \frac{\delta F[x]}{\delta x(i)} = \alpha(i) \left(\frac{F[x]}{x(i)} \right)^{1-\sigma}$$

- ▶ all the concepts of production theory can be adapted to this case.
- ▶ in particular

$$MRS(i, j) = \frac{\alpha(i)}{\alpha(j)} \left(\frac{x(j)}{x(i)} \right)^{1-\sigma}$$

and

$$ES(i, j) = \frac{1}{1-\sigma}$$

5.2 Extensions 2: multisector economies

Multisector economies

- If we consider the existence of m production sectors, in this case we have an input-output structure

$$\mathbf{y} = \mathbf{F}(\mathbf{x})$$

where now \mathbf{y} is a $(m \times 1)$ vector and \mathbf{x} is a $(m \times n)$ matrix

$$\mathbf{x} = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ & \dots & \\ x_{j1} & \dots & x_{jn} \\ & \dots & \\ x_{m1} & \dots & x_{mn} \end{pmatrix}$$

and

$$\begin{pmatrix} y_1 \\ \dots \\ y_m \end{pmatrix} = \begin{pmatrix} F_1(x_{11}, \dots, x_{1n}) \\ \dots \\ F_m(x_{m1}, \dots, x_{mn}) \end{pmatrix}$$

- In these economies a vector of prices also exists,
 $\mathbf{p} = (p_1, \dots, p_m)^\top$
- We can extend all the previous concepts component-wise, that is for every sector

Multisector economies

- ▶ We can find the optimal allocation as a solution of the problem

$$\max_{x_{11}, \dots, x_{mn}} \sum_{i=1}^n p_i y_j(\mathbf{x}_j) - \sum_{i=1}^n w_i x_{ji}, \text{ st } \sum_{i=1}^n x_{ji} = x_j$$

- ▶ under some conditions, we can find, at the optimum a relationship as

$$y_j = F^j(\mathbf{x})$$

a supply function for every sector as a function of the aggregate input.

6. Production function and input accumulation and dynamics

Investment and savings

- ▶ Assume there is only one factor of production and there are no intermediate goods

$$y = f(x)$$

- ▶ Assume that the good produce is durable and can be used both for consumption and investment, then

$$y = c + \dot{x}$$

- ▶ Let savings be a function of x , $s = s(x) = y - c$
- ▶ Then we have a stock-flow dynamics where

$$\dot{x} = \frac{dx(t)}{dt} = s(x(t))$$

- ▶ The solution to this differential equation, gives

$$x(t) = x_0 + \int_0^t s(x(\tau)) d\tau$$

7. Types of growth dynamics and the production function

Growth dynamics

- ▶ From this solution we can obtain the dynamics of product from

$$y(t) = f(x(t))$$

- ▶ We can also set directly an ODE on the GDP

$$\dot{y} \equiv \frac{dy(t)}{dt} = \mu(y)$$

- ▶ We say **there is long run growth** if the solution to this equation tends asymptotically to an exponential

$$\lim_{t \rightarrow \infty} y(t) \propto e^{\gamma t}, \quad \gamma > 0$$

- ▶ we will see that this requires the **technology to be linear at the aggregate level**. V.g: $y = Ak$

Growth and function $\mu(y)$

- ▶ Long run growth exists for a particular mathematical structure of $\mu(y)$
 - ▶ **logistic law** growth: $\mu(y) = \alpha y(\beta - y)$,
 - ▶ **exponential law** growth: $\mu(y) = \gamma y$,
 - ▶ **power law** growth: $\mu(y) = y^\phi$ for $\phi > 1$,
 - ▶ **linear law** growth: $\mu(y) = \mu$ for $\mu > 0$,
- ▶ **razor edge property of growth models**: although the exponential case is very particular it is the structure underlying (almost) all growth theories

Logistic growth

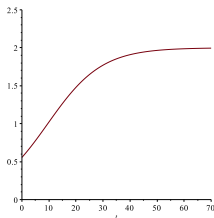


Figure: Logistic growth $\mu(y) = \alpha y(\beta - y)$

- ▶ there is short run (transition) growth
- ▶ but there is no long-run growth

Exponential growth

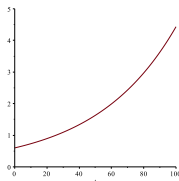


Figure: Exponential growth $\mu(y) = \gamma y$

- ▶ there is no short run (transition) growth
- ▶ but there is long-run growth
- ▶ gdp becomes infinite ($y(t) \rightarrow \infty$) in **infinite** time

Exponential growth

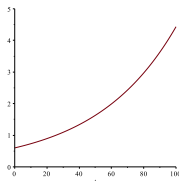


Figure: Exponential growth $\mu(y) = \gamma y$

- ▶ there is no short run (transition) growth
- ▶ but there is long-run growth
- ▶ GDP becomes infinite ($y(t) \rightarrow \infty$) in **infinite** time

Linear growth

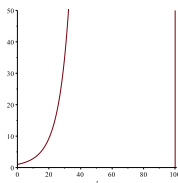


Figure: Linear growth $\mu(y) = \mu$ for $\mu > 0$

- ▶ GDP becomes infinite ($y(t) \rightarrow \infty$) in **infinite** time
- ▶ but the rate of growth is permanently decreasing

Growth and transition dynamics

- In order to have both long run growth and transition dynamics we need to have at least two durable goods

$$\dot{x}_1 = s_1(x_1, x_2)$$

$$\dot{x}_2 = s_2(x_1, x_2)$$

- We have long run growth if we can find a **balanced growth path**, i.e., a solution of type

$$x_1(t) = \phi_1(t) e^{\gamma t},$$

$$x_2(t) = \phi_2(t) e^{\gamma t},$$

where γ is the long run growth rate and $\phi_i(\phi_1(t), \phi_2(t))$ are the transition components;

- This also requires a particular structure of the production functions: they should be CRS (at the aggregate level) for every sector.

