The Uzawa-Lucas model
Growth and human capital

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Stylized facts addressed by the model

Since the industrial revolution:

- the population growth rate is smaller than the rate of growth of the economies
- but human capital increase is a major source of long run growth
- there is a permanent increase in the wage rate
- this can only be possible if there is a permanent increase in labor productivity
- education, which became widespread, has been a major source of increase in human capital

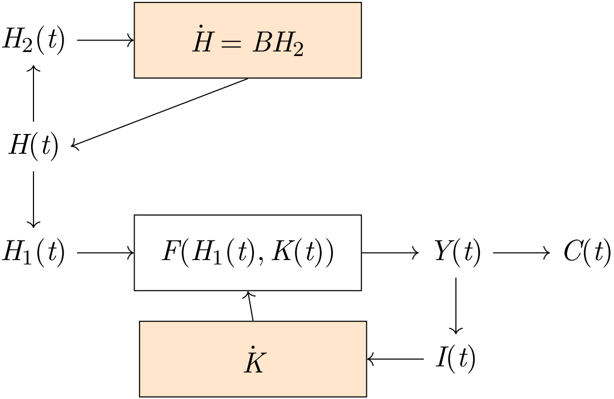
The Uzawa- Lucas model

The economy has the following features:

1. there are **two reproducible** inputs: physical capital and human capital
2. there are **two sectors**: manufacturing and education (training)
   - the manufacturing good is used in consumption and investment
   - the education produces a service which is only used in production
3. consumption/savings are determined by a centralized planner (Ramsey planner)
4. there are versions of the model with or without externalities

The Uzawa- Lucas model

- There are several versions
  - Some extend the AK model: model with no externalities
  - Others extend the Romer model: versions with externalities
- Next we present only the first version (centralized economy with no externalities)


$H_2(t)$
$\dot{H} = BH_2$
$H(t)$
$H_1(t)$
$F(H_1(t), K(t))$
$Y(t)$
$C(t)$
$I(t)$
$\dot{K}$

Assumptions

- the preference structure is analogous to the Ramsey and AK models;
- the education sector uses only human capital as an input and the manufacturing sector uses both factors (physical capital and labor) ;
- both sectors have production functions displaying constant returns to scale;
- there are no externalities;

The model

Variables in levels

- Intertemporal utility

$$\max_{C,K_1,H_1,H_2} \int_0^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

  assumption $\rho + B(\theta - 1) > 0$

- accumulation equations for stocks of physical and human capital

$$\dot{K} = Y_1(t) - C(t)$$
$$\dot{H} = Y_2(t)$$

- allocation constraints of the stocks between the two sectors

$$K(t) = K_1(t)$$
$$H(t) = H_1(t) + H_2(t)$$

- production functions for manufacturing and education

$$Y_1(t) = AK_1(t)^{\alpha} H_1(t)^{1-\alpha}$$
$$Y_2(t) = BH_2(t)$$

Detrending

- We introduce the decomposition

$$K_j(t) = k_j(t)e^{\gamma t},\ H_j(t) = h_j(t)e^{\gamma t},\ j = 1, 2$$

- $\gamma_k = \gamma_h = \gamma$ because a necessary condition for the existence of a balanced growth path is that the rates of growth are equal
- then

$$\frac{\dot{k}_j}{k_j} = \frac{\dot{K}_j}{K_j} - \gamma,\ \frac{\dot{h}_j}{h_j} = \frac{\dot{H}_j}{H_j} - \gamma\ j = 1, 2$$

The model

Detrended variables

- Intertemporal utility

$$\max_{c,k_1,h_1,h_2} \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{-(\rho-\gamma(1-\theta))t} dt$$

- accumulation equations for stocks of physical and human capital

$$\dot{k} = y_1(t) - c(t) - \gamma k(t) \quad (1)$$

$$\dot{h} = y_2(t) - \gamma h(t) \quad (2)$$

- allocation constraints of the stocks between the two sectors

$$k(t) = k_1(t),\ k(0) = k_0 \quad (3)$$

$$h(t) = h_1(t) + h_2(t),\ h(0) = h_0 \quad (4)$$

- production functions for manufacturing and education (because of linear homogeneity)

$$y_1(t) = Ak_1(t)^\alpha h_1(t)^{1-\alpha}$$

$$y_2(t) = Bh_2(t)$$

Solving the model

- Observe that the model is an optimal control problem with:
  - four control variables: c, h_1, h_2, and k_1
  - two state variables: k and h
  - two dynamic constrains (1), (2)
  - two static constrains (3), (4)
- the current-value Hamiltonian is

$$\mathcal{H} = \frac{c(t)^{1-\theta}}{1-\theta} + p_k \left(A k_1^{\alpha} h_1^{1-\alpha} - c - \gamma k \right) + p_h \left(B h_2 - \gamma h \right) + \\ + R(k - k_1) + W(h - h_1 - h_2) \quad (5)$$

p_k, p_h: co-state variables (optimal asset prices)
R, W: Lagrange multipliers (optimal return on capital and wage rates)

First order conditions for an interior solution

- optimal consumption

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \Leftrightarrow c^{-\theta} = p_k \tag{6}$$

- optimal allocation of human and physical capital between the two sectors, $r \equiv R/p_k$ and $w \equiv W/p_h$,

$$\frac{\partial \mathcal{H}}{\partial k_1} = 0 \quad \Leftrightarrow \quad \alpha y_1 = r k_1 \tag{7}$$

$$\frac{\partial \mathcal{H}}{\partial h_1} = 0 \quad \Leftrightarrow \quad (1-\alpha) p_k y_1 = w p_h h_1 \tag{8}$$

$$\frac{\partial \mathcal{H}}{\partial h_2} = 0 \quad \Leftrightarrow \quad w = B \tag{9}$$

- conditions for the Lagrange multipliers R and W

$$\frac{\partial \mathcal{H}}{\partial R} = 0 \quad \Leftrightarrow \quad k = k_1 \tag{10}$$

$$\frac{\partial \mathcal{H}}{\partial W} = 0 \quad \Leftrightarrow \quad h = h_1 + h_2 \tag{11}$$

First order conditions for an interior solution
(continuation)

- Euler equations

$$\dot{p}_k = p_k(\rho + \gamma(\theta - 1)) - \frac{\partial \mathcal{H}}{\partial k} = p_k(\rho + \gamma\theta - r) \quad (12)$$

$$\dot{p}_h = p_h(\rho + \gamma(\theta - 1)) - \frac{\partial \mathcal{H}}{\partial h} = p_h(\rho + \gamma\theta - B) \quad (13)$$

- transversality conditions

$$\lim_{t\to\infty} e^{-(\rho+\gamma(\theta-1))t} (p_k(t)k(t) + p_h(t)h(t)) = 0 \quad (14)$$

- admissibility conditions

$$\dot{k} = y_1(t) - c(t) - \gamma k(t) \quad (15)$$

$$\dot{h} = y_2(t) - \gamma h(t) \quad (16)$$

Solution for returns and allocations

- Solving equations (7)-(11) for k_1, h_1, h_2, r and w, we get

$$
\begin{aligned}
r &= r(\pi) \equiv \left(\alpha_0 A(\pi/B)^{1-\alpha}\right)^{\frac{1}{\alpha}}, \text{for } \alpha_0 \equiv \alpha^\alpha (1-\alpha)^{(1-\alpha)} \\
w &= B \\
k_1 &= k \\
h_1 &= \left(\frac{r(\pi)}{\alpha A}\right)^{\frac{1}{1-\alpha}} k \\
h_2 &= h - h_1
\end{aligned}
$$

- we define the relative prices as

$$\pi \equiv \frac{p_k}{p_h}$$

Solution for sectoral outputs

- If we substitute in the production function of both sectors we get a linear system

$$
\begin{aligned}
y_1 &= a_1(\pi)k, \text{ where } a_1 = \frac{r(\pi)}{\alpha} > 0 \\
y_2 &= a_2(\pi)k + Bh, \text{ where } a_2 = -B\left(\frac{r(\pi)}{\alpha A}\right)^{\frac{1}{1-\alpha}} < 0
\end{aligned}
$$

- then an increase in the relative price $\pi = p_k/p_h$ increases the output of manufactures and reduces the output of the educational sector

Long run growth rate and factor returns

- From equation (13), setting $\dot{p}_h = 0$ we get the long-run growth rate
$$\bar{\gamma} = \frac{B-\rho}{\theta}$$
increases with the productivity of the educational sector
- From equation (13) and (12) we have a long-run arbitrage condition
$$\bar{r} = \bar{w} = B$$
- then the long-run relative price is
$$\bar{\pi} = \frac{\bar{p}_k}{\bar{p}_h} = \left(\frac{\alpha_0 A}{B}\right)^{\frac{1}{1-\alpha}}$$

Other long run relationships

- ratio between the state variables

$$\frac{\bar{k}}{\bar{h}} = \eta \equiv -\frac{B - \bar{\gamma}}{\bar{a}_2} = \bar{\pi}\left(\frac{\bar{\gamma} - B}{B}\right)\left(\frac{\alpha}{1-\alpha}\right)$$

  because

$$\bar{a}_2 = -B\left(\frac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}} = -\frac{B}{\bar{\pi}}\left(\frac{1-\alpha}{\alpha}\right) < 0$$

- the ratio $\frac{\bar{k}}{\bar{h}}$ is positive because of the transversality condition holds if and only if

$$\rho + \bar{\gamma}(\theta - 1) = \frac{\rho + B(\theta - 1)}{\theta} = B - \bar{\gamma} > 0$$

- the long run level of consumption is

$$\bar{c} = c(p_k) = \beta\bar{k},\ \beta \equiv \frac{B}{\alpha} - \bar{\gamma} > 0$$

The MHDS

- substituting $\gamma = \bar{\gamma}$ the MHDS becomes

$$\begin{aligned} \dot{p}_k &= p_k(B - r(p_k/p_h)) && (17) \\ \dot{p}_h &= 0 && (18) \\ \dot{k} &= (a_1 r(p_k/p_h)) - \bar{\gamma})\, k - c(p_k) && (19) \\ \dot{h} &= a_2(r(p_k/p_h))k - (B - \bar{\gamma})h && (20) \end{aligned}$$

Initial conditions and the BGP

- If the initial conditions verifies

$$k_0 = \eta h_0$$

- then the economy will evolve along the BGP such that

$$\bar{K}(t) = \eta h_0 e^{\bar{\gamma} t}, \bar{H}(t) = h_0 e^{\bar{\gamma} t}$$

- If the initial conditions verifies

$$k_0 \neq \eta h_0$$

- then there will be transitional dynamics

Transitional dynamics

- The system (17)-(20) is non-linear. A linear approximation in the neighborhood of the BGP is

$$\begin{pmatrix} \dot{p}_k \\ \dot{p}_h \\ \dot{k} \\ \dot{h} \end{pmatrix} = J \begin{pmatrix} p_k - \bar{p}_k \\ p_h - \bar{p}_h \\ k - \bar{k} \\ h - \bar{h} \end{pmatrix}$$

where

$$\bar{\mathbf{J}} = \begin{pmatrix} \mu - \beta & \bar{\pi}(\beta - \mu) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta(\theta-\alpha)-\alpha\mu)\bar{k}}{\alpha\theta\bar{p}_k} & -\frac{(\beta-\mu)\bar{k}}{\alpha\bar{p}_h} & \beta & 0 \\ -\dfrac{\mu\bar{h}}{\alpha\bar{p}_k} & \dfrac{\mu\bar{h}}{\alpha\bar{p}_h} & -\dfrac{\mu}{\eta} & \mu \end{pmatrix}$$

Local dynamics in the neighborhood of the BGP

- The characteristic polynomial of the Jacobian J is

$$C(\mathbf{J},\lambda) = \lambda\left(\lambda - (\mu - \beta)\right)(\lambda - \beta)(\lambda - \mu),$$

- Then the eigenvalues are

$$\lambda_1 = \mu - \beta < 0,\ \lambda_2 = 0,\ \lambda_3 = \beta > 0,\ \lambda_4 = B - \bar{\gamma} > 0$$

- there is transitional dynamics: because $\lambda_1 < 0$ which is

$$\lambda_2 = \left.\frac{\partial \dot{p}_k}{p_k}\right|_{BGP} = \mu - \beta < 0$$

Local dynamics in the neighborhood of the BGP

- solving the system (see my revised notes chapter 7)

$$
\begin{aligned}
h(t) &= h_\infty + (h_0 - h_\infty)e^{(\mu-\beta)t} \\
k(t) &= \eta h_\infty + (k_0 - \eta h_\infty)e^{(\mu-\beta)t} \\
p_k(t) &= \bar{p}_k \left[1 + \frac{\theta\alpha(2\beta-\mu)}{\mu(\theta-\alpha)} \left(\frac{h_0 - h_\infty}{h_\infty}\right) e^{(\mu-\beta)t}\right] \\
p_h(t) &= \bar{p}_h
\end{aligned}
$$

where

$$
h_\infty = \frac{k_0\mu(\theta-\alpha) + \eta h_0(\beta(\alpha+\theta) - \mu\theta)}{\eta(\beta(\alpha+\theta) - \alpha\mu)}
$$

if we take $\bar{h} = h_\infty$. This implies $\bar{p}_k = (\beta\eta h_\infty)^{-\theta}$ and $\bar{p}_h = \bar{\pi}\bar{p}_k$.

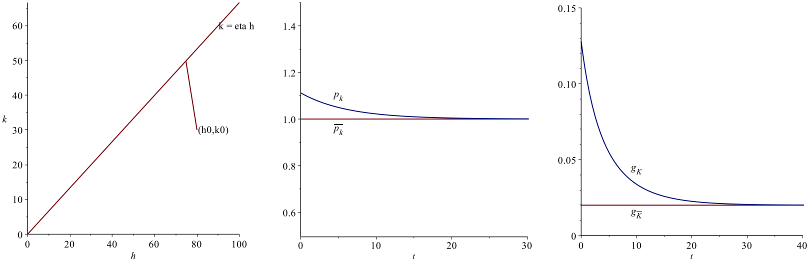


Figure: Uzawa-Lucas model: phase diagram, trajectories for p_k and for the rate of growth of K and $\bar{K}$. Parameter values: $\rho = 0.02$, $\alpha = 0.3$, $\theta = 2$, $A = 0.2$ and $B = 0.06$.

Trajectory for the GDP

- the GDP for the manufacturing sector is

$$Y_1(t) = y_1(t)e^{\bar{\gamma}t}$$

  where

$$y_1(t) = y_{1,\infty}\left(1+\left(\frac{k_0}{\eta h_\infty}-1\right)e^{(\mu-\beta)t}\right)^{\alpha}\left(1+\left(\frac{h_0}{h_\infty}-1\right)e^{(\mu-\beta)t}\right)^{1-\alpha}$$

- Taking $\lim_{t\to\infty} y_1(t) = y_{1,\infty} \equiv A\eta^{\alpha}h_\infty$ we get the BGP

$$\bar{Y}_1(t) \approx y_{1,\infty}e^{\bar{\gamma}t}.$$

Growth implications

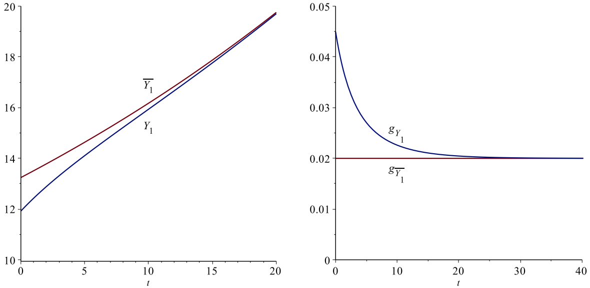


Figure: Uzawa-Lucas model: trajectories for levels and rates of growth of Y_1 and $\bar{Y}_1$

Conclusions

- there is long run growth and the growth rate is a positive function of B
- the long run level of GDP depends on the initial levels of k and h

$$\bar{y}_1 = y_{1,\infty} = A\eta^{\alpha-1}\left(\frac{k_0\mu(\theta-\alpha)+\eta h_0(\beta(\alpha+\theta)-\mu\theta)}{\beta(\alpha+\theta)-\alpha\mu}\right)$$

- if $k_0 = \eta h_0$ the economy will be at a BGP with $\bar{y}_1 = A\eta^{\alpha}h_0$;
- there is transitional dynamics (if $k_0 \neq \eta h_0$) with

$$y_1(t) - \bar{y}_1 \approx e^{(\mu-\beta)t}$$

- the GDP path in levels is

$$Y_1(t) = y_1(t)e^{ga\bar{m}mat}$$

Conclusions

- The driving force for transitional dynamics if $\dot{\pi}/\pi$

$$\dot{\pi}/\pi = B - r(\pi)$$

- if initial capital k_0 is too low relative to ηh_0 then $\pi(0) > \bar{\pi}$ and two effects will occur
  - because $a_1^{'}(\pi) > 0$ and $a_2^{'}(\pi) < 0$ there will be an increase in the ratio $k(t)/h(t)$
  - because $r(\pi) > B$ then $\dot{\pi}/\pi < 0$;
- the adjustment of π will eliminate through time the both the divergences $B - r(\pi)$ and $k(t) - \eta h(t)$ leading to convergence to the BGP.

Conclusions: effect of an increase in B

A positive shock in B (from B_0 to $B_1 > B_0$), will produce the following effects (starting from a BGP)

- an increase in the long-run growth rate $\bar{\gamma}(B_1) > \bar{\gamma}(B_0)$
- an increase in η (because $\frac{\partial \eta}{\partial B} > 0$)
- if before the shock $k_0 = \eta(B_0)h_0$, then after the shock $k_0 < \eta(B_1)h_0$ which means physical capital becomes "too low" relative to human capital
- then the process just described unfolds: $\pi(0) > \bar{\pi}(B_0)$, the interest rate becomes higher then B_1, k accumulates faster than h but π starts to decrease to eliminate the "excess" k.