The Malthusian growth model

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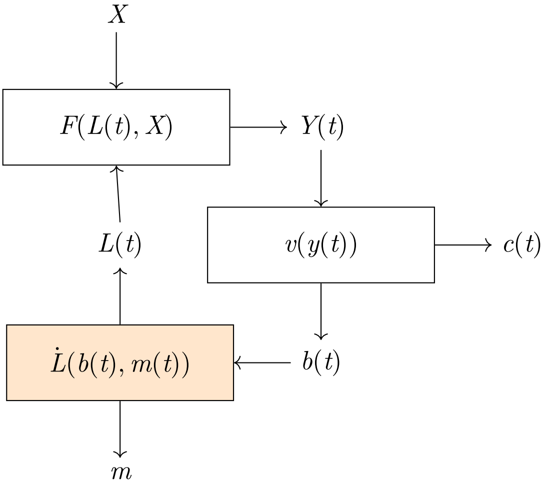
Assumptions

- Production:
  - production uses two factors: labor and land
  - the production function has constant returns to scale
  - the only reproducible factor is labor, and it faces decreasing marginal returns
- Population:
  - fertility is endogenous and mortality is exogenous
- Farmers:
  - households are land-owners
  - they choose among consumption and child-rearing
  - there are no savings



where $v(y) = \max\{u(c, b) : c + \rho b \leq y\}$ and $y = \frac{Y}{L}$

The Malthus model as a growth model

- Defining the GDP per capita as

$$y(t) = \frac{Y(t)}{L(t)}$$

- we want to know what this theory implies for
  - the rate of growth of GDP $g(t) = \frac{\dot{y}(t)}{y(t)}$
  - the steady state level of GDP $\bar{y}$
  - and the dynamics: i.e. separating $g(t)$ into transition and long-run dynamics

The model

Production

- **Production function**

$$Y(t) = (AX)^{\alpha} L(t)^{1-\alpha},\ 0 < \alpha < 1$$

  where: A productivity, X stock of land, L labor input

- displays constant returns to scale

$$(\lambda AX)^{\alpha}(\lambda L)^{1-\alpha} = \lambda Y$$

- implication: the Euler theorem holds

$$Y = \frac{\partial Y}{\partial L} L + \frac{\partial Y}{\partial X} X$$

The model
Production

- it has positive marginal returns

$$\frac{\partial Y}{\partial L} = (1-\alpha)\frac{Y}{L} > 0, \frac{\partial Y}{\partial X} = \alpha\frac{Y}{X} > 0$$

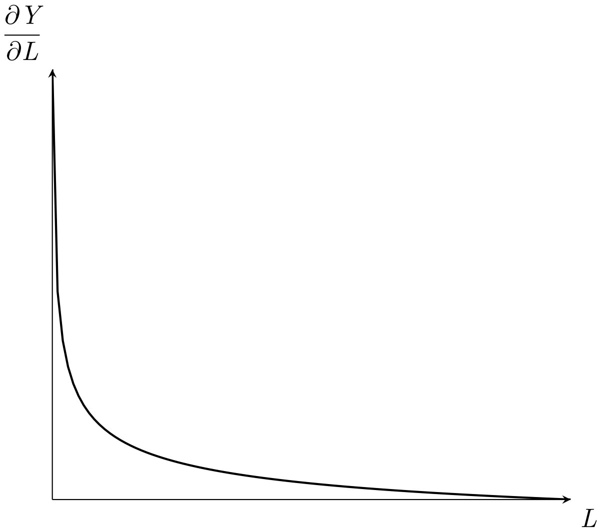
- but decreasing marginal returns

$$\frac{\partial^2 Y}{\partial L^2} = -\alpha(1-\alpha)\frac{Y}{L} < 0, \frac{\partial^2 Y}{\partial X^2} = -\alpha(1-\alpha)\frac{Y}{X} < 0$$

- and is Inada

The model

Inada property



The model

Production efficiency

- production efficiency:

$$\max_{L,X}\{Y(L,X) - wL - RX\}$$

  where w is the wage rate and R are is land rent

- and competitive markets lead to

$$w(L,X) = \frac{\partial Y}{\partial L} = (1-\alpha)\frac{Y}{L} > 0$$
$$R(L,X) = \frac{\partial Y}{\partial X} = \alpha\frac{Y}{X} > 0$$

- with the properties

$$w_L < 0, w_X > 0;\ R_X < 0, R_L > 0$$

Farmers' problem

Endogenous rate of population growth

- There are L farmers; who get percapita income from farming and decide which part to consume and which part allocate to raising offspring, by deciding the number of offspring (Beckerian model)
- **Household's (farmer's) problem**

$$\max_{c(t),b(t)} \{c(t)^{1-\gamma} b(t)^{\gamma} : \ c(t) + \rho b(t) = y(t)\}$$

  $\rho =$ relative cost of raising children

- solution

$$c(t) = (1-\gamma) y(t)$$
$$b(t) = \frac{\gamma}{\rho} y(t)$$

The model

Population dynamics

- **Population growth**

$$\dot{L} \equiv \frac{dL(t)}{dt} = (b(t) - m)L(t)$$

- where the fertility rate $b(t)$
- the mortality rate is exogenous m
- the initial level of population is assumed to be given by number L_0

$$L(t)|_{t=0} = L(0) = L_0$$

The Malthusian model

Endogenous rate of population growth

- Then

$$\dot{L} = \left(\frac{\gamma}{\rho} y(t) - m \right) L(t)$$

- where the per capita GDP is

$$y(t) \equiv \frac{Y(t)}{L(t)} = \left(\frac{AX}{L(t)} \right)^{\alpha}$$

- There are two approaches to **solving the model**
  - Approach 1: solve the differential equation for L and substitute in y to get the dynamics of growth
  - Approach 2: get a differential equation in y and solve it

Detour

Per-capita rate of growth arithmetics

- taking log-derivatives w.r.t time we have
$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L}$$
- that we denote by
$$g(t) = g_Y(t) - n(t)$$
- as the per capita GDP is
$$y(t) \equiv \frac{Y(t)}{L(t)} = \left(\frac{AX}{L(t)}\right)^{\alpha}$$
- Then
$$g(t) = \frac{\dot{y}}{y} = -\alpha\frac{\dot{L}}{L}$$

Solving the Malthusian model

Approach 1: solving for L

- If we substitute y in the dynamic equation for L we have the initial value problem

$$\begin{cases} \dot{L} = \left(\frac{\gamma}{\rho}\left(\frac{AX}{L(t)}\right)^{\alpha} - m\right) L(t), & t \geq 0 \\ L(0) = L_0 \text{given} & t = 0 \end{cases}$$

- we can solve it
- then differentiate it and substitute in

$$g(t) = -\alpha \frac{\dot{L}}{L}$$

to get model's explanation for the per-capita growth of the economy

Solving the Malthusian model

Approach 2: solving for y directly

- from

$$\frac{\dot{y}}{y} = -\alpha \frac{\dot{L}}{L}$$

- we get the dynamic equation for the GDP per capita

$$\dot{y} = -\alpha \left(\frac{\gamma}{\rho} y(t) - m \right) y(t) \tag{1}$$

  together with the initial value

$$y(0) = y_0 = (AX)^{\alpha} L_0^{1-\alpha}$$

Solving the Malthusian model

Explicit solution for y

- Equation (1) has two steady states $y^* = \{0, \bar{y}\}$ where

$$\bar{y} = \frac{m\rho}{\gamma}$$

- we can re-write the growth equation as

$$\dot{y} = \alpha \frac{\gamma}{\rho} \left(\bar{y} - y(t)\right) y(t)$$

Explicit solution for y

- This is a Bernoulli differential equation with has an explicit solution appendix

$$y(t) = \left[\frac{1}{\bar{y}} + \left(\frac{1}{y(0)} - \frac{1}{\bar{y}}\right) e^{-\alpha m t}\right]^{-1}, \text{for } 0 \leq t < \infty$$

- satisfies $\lim_{t\to\infty} y(t) = \bar{y}$

Explicit solution for g

- the GDP growth rate is

$$g(t) = \alpha m \left[1 - \left(1 + \left(\frac{\bar{y}}{y(0)} - 1 \right) e^{-\alpha m t} \right)^{-1} \right], \text{for } 0 \leq t < \infty$$

Malthusian model

Properties

- there is **no long run growth**, because $\lim_{t\to\infty} g(t) = 0$
- the **long run level** of GDP per capita

$$\bar{y} = \frac{m\rho}{\gamma}$$

increases with the mortality rate, the cost or rearing children and the "moral restraint" (no productivity effects)

- there is **only transitional dynamics**
  - if the initial GDP $y(0)$ is small then there is an increase in time of the GDP $g(t) > 0$
  - if the initial GDP $y(0)$ is large then there is an decrease in time of the GDP $g(t) > 0$

Malthusian model

Mechanics of the model

- if $y(0)$ is large so is the wage rate $w(0) = (1 - \alpha)y(0)$
- this implies that the initial fertility rate is higher, $b(0) = \frac{\gamma}{\rho}y(0)$
- population increases, which increases output,
- but decreases the rate of growth of GDP

$$g(t) = -\alpha n(t)$$

because there are decreasing marginal returns due to the fact that X is fixed.

Malthusian model

Trajectories: y, L and w

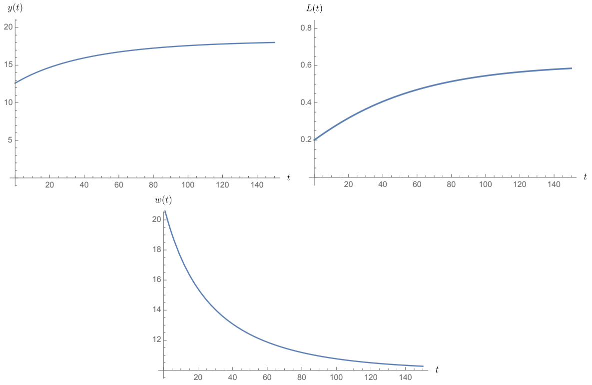


Figure: Parameter values: $\alpha = 2/3$, $m = 0.03$, $\gamma = 0.01$, $\rho = 10$, $A = 1$, $X = 100$, $y(0) < \bar{y}$

Malthusian model

Exponential increase in land productivity

Can increases in land-productivity generate long-run growth ?

- now $Y(t) = (A(t)X)^{\alpha} L(t)^{1-\alpha}$
- where $\dot{A} = g_A A, \; g_A > 0$
- this implies

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{A}}{A} + (1-\alpha)\frac{\dot{L}}{L} - \frac{\dot{L}}{L} = \alpha \left(m + g_A - \frac{\gamma}{\rho} y(t) \right)$$

- there is **no increase in the long run growth rate**

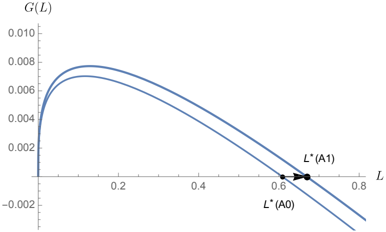
$$\lim_{t \to \infty} g(t) = 0$$

- there is **an increase in GDP level**

$$\bar{y} = \frac{(g_A + m)\rho}{\gamma}$$

Malthusian model and land productivity

Phase diagram for an increase in A



Can an increase in the productivity of labor generate long-run growth ?

- now $Y(t) = X^{\alpha}(h(t)\,L(t))^{1-\alpha}$
- where $\dot{h} = g_h h,\ g_h > 0$
- this implies

$$\frac{\dot{y}}{y} = (1-\alpha)\left(\frac{\dot{h}}{h} + \frac{\dot{L}}{L}\right) - \frac{\dot{L}}{L} = \alpha\left[\frac{(1-\alpha)}{\alpha}g_h + m - \frac{\gamma}{\rho}y(t)\right]y(t)$$

- there is **no increase in the long run growth rate**

$$\lim_{t\to\infty} g(t) = 0$$

- there is **an increase in GDP level**

$$\bar{y} = \frac{((1-\alpha)g_h + \alpha m)\rho}{\alpha\gamma}$$

Can learning by doing generate long-run growth ?

- now $Y(t) = (A(t)X)^{\alpha} L(t)^{1-\alpha}$
- from $A(t) = \beta \int_{-\infty}^{t} e^{-\mu(t-s)} A(s) y(s) ds$ we get

$$\dot{A} = (\beta y(t) - \mu)\, A(t)$$

- meaning: past production experience increases productivity
- the dynamic equation for per-capita GDP becomes

$$\frac{\dot{y}}{y} = \left(\beta - \alpha \frac{\gamma}{\rho}\right) y(t) + \alpha m - \mu$$

Can learning by doing generate long-run growth ? Cont.

- If we assume $\beta = \alpha \frac{\gamma}{\rho}$ then

$$\dot{y} = (\alpha m - \mu) y$$

- there is **long run growth** if $\alpha m > \mu$ because

$$g(t) = \alpha m - \mu > 0, \text{for all } t > 0$$

- the GDP level is exogenous

$$y(t) = y_0 e^{(\alpha m - \mu) t}$$

Conclusions

- the existence of decreasing marginal returns to the reproducible factor of production (labor, L) implies that the Malthusian model does not feature long-run growth: there is only transitional dynamics (if initial population is too high, wages will be too low, which generates a fall in fertility and therefore a decrease in population until population is constant)
- **exogenous** permanent increases in productivity will only increase the long-run GDP **level** but will not generate long-run growth
- however, **endogenous** increases in productivity (v.g, generated by learning-by-doing) may generate long run growth (but in this case there is not transition dynamics). Learning-by-doing generates a **reproduction** mechanism.

References

- (Galor, 2011, ch 2, 3)

Oded Galor. *Unified Growth Theory*. Princeton University Press, 2011.

Appendix

Solving a linear ODE's

- The linear ODE
$$\dot{x} = \lambda(x(t) - \bar{x})$$
has an exact solution
$$x(t) = \bar{x} + (k - \bar{x})e^{\lambda t}$$
where k is an arbitrary constant
- The initial value problem
$$\begin{cases} \dot{x} = \lambda(x(t) - \bar{x}) \\ x(0) = x_0 \text{ given} \end{cases}$$
has the exact solution
$$x(t) = \bar{x} + (x_0 - \bar{x})e^{\lambda t}$$

Appendix

The linear and Bernoulli ODE's

- The Bernoulli equation is

$$\dot{x} = \alpha x(t) - \beta x^{\eta}$$

- If we set $z(t) = x(t)^{1-\eta}$ and differentiate

$$\begin{aligned}\dot{z} &= (1-\eta)x(t)^{-\eta}\dot{x} = \\ &= (1-\eta)x(t)^{-\eta}\left(\alpha x(t) - \beta x^{\eta}\right) = \\ &= \lambda\left(z(t) - \bar{z}\right)\end{aligned}$$

- is a linear ODE with solution with $\lambda = (1-\eta)\alpha$ and $\bar{z} = \dfrac{\beta}{\alpha}$

$$z(t) = \bar{z} + \left(k_z - \bar{z}\right) e^{\alpha(1-\eta)t}$$

- transforming back by making $x(t) = z(t)^{\frac{1}{1-\eta}}$

$$x(t) = \left(\frac{\beta}{\alpha} + \left(x(0)^{1-\eta} - \frac{\beta}{\alpha}\right) e^{\alpha(1-\eta)t}\right)^{\frac{1}{1-\eta}}$$