Advanced Mathematical Economics

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Part II

Functional calculus and optimal control of ODEs

Chapter 7

Introduction to functional calculus

7.1 Introduction

Functionals are fundamental objects in several fields in applied and theoretical economics. But what is a functional ?

Consider a function of a real variable $f : \mathrm{X} \to \mathbb{R}$, and the set of functions $f(\cdot)$, $\mathcal{F}$ in which several additional properties are imposed. A **functional**, denoted by $\mathsf{F}[f]$ is a mapping of the space of function $\mathcal{F}$, $f \in \mathcal{F}$, into a number. $\mathsf{F} : \mathcal{F} \to \mathbb{R}$.

We can call X an **index-set**. Index sets can be classified according to their internal hierarchy, or according to the continuity of their elements. From the first perspective, there are two types of index sets: they do not have an order structure (they are just a tagging device) or they have an order structure (they are a vector space in which elements are ranked along some dimension, a size, distance, etc). From the second perspective, there are also two types of index sets: discrete, if $\mathrm{X} \subseteq \mathbb{N}$ the index set if finite-dimensional or countable, or continuous, if $\mathrm{X} \subseteq \mathbb{R}$ the index set is infinite-dimensional.

In economics, commonly used continuous functionals over functions $y : \mathrm{X} \to \mathbb{R}$ have the following forms

$$\mathsf{F}[y] \ = \int_{\mathrm{X}} f(x, y(x))\, dx \tag{7.1}$$

or

$$\mathsf{G}[y] \ = g\Big(\int_{\mathrm{X}} f(x, y(x))\, dx \Big) \tag{7.2}$$

where g are both functions with appropriate properties. Finite-dimensional discrete analog functionals over $y : \mathrm{X} \to \mathbb{R}$ are

$$\mathsf{F}[y] \ = \sum_{\mathrm{X}} f(y_x, x) \text{ or } \ \mathsf{G}[y] = g\Big(\sum_{\mathrm{X}} f(y_x, x) \Big).$$

Next we will mostly deal with the continuous case, and with sets $\mathbb{X}$ that can be ordered or not.

As economic problems involve optimizing functionals, such as (7.1) or (7.2), that are constrained or not by algebraic or functional side conditions, we need to use results from generalized calculus, or, more generally from calculus and optimization over normed vector spaces.

Observe that the simplest calculus of variation problem also involves maximizing a functional of type

$$\mathsf{F}[y] \ = \int_{\mathrm{X}} f(x, y(x), y'(x))\, dx,$$

where the integrand function, differently from functionals (7.1) or (7.2) also involves a derivative. We leave calculus of variations problems to another chapter. In this chapter we consider mostly "static" problems.

Next we present the main results from generalized calculus and use them to solve some infinite-dimensional optimization problems. Section 7.2 presents some results on calculus of functionals (or generalized calculus). Section 7.3 presents problems involving maximization of functionals.

7.2 Calculus of functionals

In this section we first present the concept of functional, or Gâteaux, derivative, apply it to linear functionals in subsection 7.2.2. Extensions for functionals involving derivatives and multidimensional functions can be found in subsections in subsections 7.2.4 and 7.2.5.

7.2.1 Functionals and functional derivative

A normed vector space is a vector space (a set of numbers, continuous functions, continuous differentiable functions, bounded functions or distributions) over which a norm is defined. An operator as a mapping between two normed vector spaces.

In this chapter we are interested in operators between spaces of continuous functions and the space of real numbers. We call these operator functionals.

Let $\mathcal{Y}$ be the set of functions y mapping a continuous index set X to the set of real numbers $\mathbb{R}$. In particular, we consider $\mathcal{Y} \ = C^1(\mathrm{X})$, the set of continuous functions defined on X. Recall that a function $y(\cdot)$ is continuous at point $x_0 \in \mathrm{X}$ if $\lim_{x \to x_0} \ y(x) = y(x_0)$ for x for any neighborhood of x_0. In particular, this means that the right and left limits are equal. that is $y(x_0^+) = y(x_0^-) = y(x_0)$ where $y(x_0^-) = \lim_{x \uparrow x_0}$ and $y(x_0^+) = \lim_{x \downarrow x_0}$. Function $y(\dot{)}$ is continuous if it is continuous to all points $x \in \mathrm{X}$.

In this section we introduce **functional** F as a mapping, $\mathsf{F} : \mathcal{Y} \to \mathbb{R}$.

Consider the perturbation of function $y \in \mathcal{Y}$ to $y + dy \in \mathcal{Y}$. A **variation of functional** F is defined as the change in the functional introduced by that perturbation

$$\Delta\mathsf{F}[y] \ = \mathsf{F}[y + dy] - \mathsf{F}[y].$$

As the functional range is a real number this means that $\Delta\mathsf{F}[y]$ is a reasl number as well.

In particular, the **variation of the functional over the direction** $h \in \mathcal{Y}$ is defined as

$$D\,\mathsf{F}[y] \ = \mathsf{F}[y + \epsilon h] - \mathsf{F}[y],$$

where ϵ is a constant.

Definition 1. *The **Gâteaux differential** is defined as the variation of the functional in the direction $\eta \in \mathcal{Y}$ when the constant ε is infinitesimal*

$$D_{\eta(\cdot)}\mathsf{F}[y] \; = \lim_{\varepsilon \to 0} \frac{\mathsf{F}[y+\varepsilon\,\eta] - \mathsf{F}[y]}{\varepsilon}. \tag{7.3}$$

Equivalently, a Taylor expansion yields [1]

$$\mathsf{F}[y+\varepsilon\,\eta] \; = \mathsf{F}[y] \; + D_{\eta(\cdot)}\mathsf{F}[y]\varepsilon + o(\varepsilon^2)$$

In analogy to the case of finite-dimensional analysis[2] it can be shown that the Gâteaux differential is a linear functional of $\eta(x)$. If y is a linear operator in a normed vector space, it is bounded and we can invoke the Riesz-Frechet theorem (Riesz and Sz.-Nagy, 1955, p. 61) which states that if $\mathsf{G}[\eta]$ is a bounded operator in a inner-product space, then there is a ζ such that $\mathsf{G}[\eta]$ can be represented by the inne product $\mathsf{G}[\eta] = \angle\zeta, \eta\angle$ and by the mesure $\|\mathsf{G}\| = \|\zeta\|$. This means that the Gâteux differential of a linear functional admits the representation

$$D_{\eta(\cdot)}\mathsf{F}[y] \; = \int_{\mathrm{X}} d(x)\,\eta(x)\,dx.$$

Let y be a distribution belonging to a space of distributions $\mathcal{Y}$[3] we call **Gâteaux derivative** of the functional $\mathsf{F}[y]$ at a point $x \in \mathrm{X}$, to the Gâteaux differential in which the perturbation is a Dirac-δ generalized function at a singular at the point x,

$$d(x) = \frac{\delta\mathsf{F}[y]}{\delta y(x)} = \int_{\mathrm{X}} d(s)\,\delta(s-x)\,ds.$$

[1]To see this observe that equation (7.3) is equivalent to

$$\lim_{\varepsilon \to 0} \frac{\mathsf{F}[y+\varepsilon\,\eta] - \mathsf{F}[y]}{\varepsilon} - D_{\eta(\cdot)}\mathsf{F}[y] = 0$$

and therefore, to

$$\lim_{\varepsilon \to 0} \frac{\mathsf{F}[y+\epsilon\,\eta] - \mathsf{F}[y] - D_{\eta(\cdot)}\mathsf{F}[y]\,\varepsilon}{\varepsilon} = 0.$$

Writing the numerator as $g(\varepsilon)$, which is possible because a functional has its range in set $\mathbb{R}$, we can use the "little-o" notation, such that $o(\varepsilon) = \lim_{\varepsilon\to 0} \frac{g(\varepsilon)}{\varepsilon} \; = 0$.

[2]This generalizes the concept of directional derivative in elementary calculus. Let $f(\mathbf{y}) = f(y_1, \ldots, y_n)$ the directional derivative in the direction given by the vector $\mathbf{h} \; = (h_1, \ldots, h_n)^\top$ is

$$D_h f(\mathbf{y}) \equiv \lim_{\varepsilon \to 0} \frac{f(\mathbf{y}+\varepsilon\mathbf{h}) - f(\mathbf{y})}{\varepsilon} = \sum_i \frac{\partial f(\mathbf{y})}{\partial y_i} h_i$$

is a linear functional of $\mathbf{h} \; = (h_1, \ldots, h_n)^\top$.

[3]This is the analog to a partial derivative in classical calculus. This is a type of possibly ad-hoc concept which is used in mathematical physics and which is useful for our purposes. In particular it allows to determine elasticities of substitution in a continuum setting. See next...

Therefore, we can represent the Gâteaux diferential for the perturbation by function $\eta \in \mathcal{Y}$ by the linear functional

$$D_{\eta(\cdot)}\mathsf{F}[y] = \int_{\mathrm{X}} \frac{\delta \mathsf{F}[y]}{\delta y(x)}\, \eta(x)\, dx. \tag{7.4}$$

We can extend the previous definitions, to second order variations. The **second order variation of the functional** $\mathsf{F}[y]$, is

$$\Delta^2\mathsf{F}[y] \; = \Delta\, \left(\Delta\mathsf{F}[y+d_1 y] - \Delta\mathsf{F}[y]\right) = \mathsf{F}[y+d_1 y+d_2 y] - \mathsf{F}[y+d_2 y] - \mathsf{F}[y+d_1 y] + \mathsf{F}[y],$$

is the perturbation by $d_2\, y \in \mathcal{Y}$, of the perturbation of the functional by $d_1\, y \in \mathcal{Y}$.

The second-order Gâteaux differential of funcional $\mathsf{F}[y]$ is defined as

$$D_{\eta_1(\cdot),\eta_2(\cdot)}\mathsf{F}[y] = \lim_{\varepsilon_2 \to 0}\; \lim_{\varepsilon_1 \to 0} \frac{\mathsf{F}[y+\varepsilon_1\, \eta_1 + \varepsilon_2\, \eta_2] - \mathsf{F}[y+\varepsilon_2\, \eta_2] - \mathsf{F}[y+\varepsilon_1\, \eta_1] - \mathsf{F}[y]}{\varepsilon_1\, \varepsilon_2},$$

which represents the Gâteaux differential over direction $\eta_2 \in \mathcal{Y}$, of Gâteaux differential over direction $eta_1 \in \mathcal{Y}$.

Using our previous results we can write the second-order "crossed" Gâteaux differential as

$$D_{\eta_1(\cdot),\eta_2(\cdot)}\mathsf{F}[y] = \int_{\mathrm{X}} \int_{\mathrm{X}} \frac{\delta^2 \mathsf{F}[y]}{\delta y(x)\, \delta y(x')}\, \eta_1(x)\, \eta_2(x')\, dx\; dx',$$

and the second-order "crossed" Gâteux derivative,

$$\frac{\delta^2 \mathsf{F}[y]}{\delta y(x)\, \delta y(x')}, \text{ for } x, x' \in \mathrm{X}$$

associated to perturbations $\eta_1(x)$ and $\eta_2(x)$.

We conjecture that there is symmetry

$$D_{\eta_1(\cdot),\eta_2(\cdot)}\mathsf{F}[y] = D_{\eta_2(\cdot),\eta_1(\cdot)}\mathsf{F}[y].$$

The second-order "own" Gâteaux differential as

$$D^2_{\eta(\cdot)}\mathsf{F}[y] = \int_{\mathrm{X}} \int_{\mathrm{X}} \frac{\delta^2 \mathsf{F}[y]}{\delta y(x)^2}\, \eta(x)\, \eta(x')\, dx\, dx',$$

and the second-order "own" Gâteux derivative,

$$\frac{\delta^2 \mathsf{F}[y]}{\delta y(x)^2}, \text{ for } x \in \mathrm{X}$$

associated to perturbations $\eta_1(x)$ and $\eta_2(x)$.

Therefore a generalization of the second order Taylor expansion is

$$\mathsf{F}[y+\varepsilon\, \eta] \; = \mathsf{F}[y] + D_{\eta(\cdot)}\mathsf{F}[y]\, \varepsilon + \frac{1}{2}\, D^2_{\eta(\cdot)}\mathsf{F}[y]\; \varepsilon^2 + o(\varepsilon^3), \tag{7.5}$$

where $D^2_{\eta(\cdot)}\mathsf{F}[y] \equiv D_{\eta(\cdot),\eta(\cdot)}\mathsf{F}[y]$.

There are **general properties of functionals** of one-dimensional functions of one variable, that is $y \in \mathcal{Y}$, $y : \mathrm{X} \to \mathbb{R}$, where $\mathsf{F} : \mathcal{Y} \to \mathbb{R}$.

1. multiplication by a constant: let $a \in \mathbb{R}$ be a number, then $D_{\eta(\cdot)}\{a\,\mathsf{F}[y]\} = aD_{\eta(\cdot)}\mathsf{F}[y]$
2. sum of functionals: let $\mathsf{F}_1[y]$ and $\mathsf{F}_2[y]$ be two functionals, then $D_{\eta(\cdot)}\{\,\mathsf{F}_1[y] + \mathsf{F}_2[y]\} = D_{\eta(\cdot)}\mathsf{F}_1[y] + D_{\eta(\cdot)}\mathsf{F}_2[y]$
3. product rule: let $\mathsf{F}_1[y]$ and $\mathsf{F}_2[y]$ be two functionals, then $D_{\eta(\cdot)}\{\,\mathsf{F}_1[y]\cdot\mathsf{F}_2[y]\} = \{D_{\eta(\cdot)}\mathsf{F}_1[y]\}\cdot \mathsf{F}_2[y] + \{D_{\eta(\cdot)}\mathsf{F}_2[y]\}\cdot\mathsf{F}_1[y]$
4. chain rule: let $f:\mathbb{R}\to\mathbb{R}$ be a monotonic function and $\mathsf{F}_1[y]$ be a functional, then $D_{\eta(\cdot)}\{f(\mathsf{F}_1[y])\Big\} = f'(\mathsf{F}_1[y])\,\{D_{\eta(\cdot)}\mathsf{F}[y]\}$

7.2.2 Linear functionals

In this subsection we consider the following two types of functionals which are common in economics: the linear functional

$$\mathsf{F}[y] = \int_{\mathrm{X}} f(y(x))\,dx, \tag{7.6}$$

where we assume that function $f(\cdot)$ is a smooth function, i.e. $f(\cdot)\in C^2(\mathbb{R})$, and the integral exists, and a functional which is a funtion of a linear functional

$$\mathsf{G}[y] = g(\mathsf{F}[y]) \equiv g\Big(\int_{\mathrm{X}} f(y(x))\,dx\Big) \tag{7.7}$$

where $g(\cdot)$ is also a $C^2(\mathbb{R})$ function.

Using the definitions intruduced in the previou subsection, the generalized differential (in the Gâteaux sense) of functional (7.6) is

$$D_{\eta(\cdot)}\mathsf{F}[y] = \int_{\mathrm{X}} \frac{\delta\mathsf{F}[y]}{\delta y(x)}\,\eta(x)\,dx, \tag{7.8}$$

where the generalized derivative is

$$\frac{\delta\mathsf{F}[y]}{\delta y(x)} = \frac{\partial f(y(x))}{\partial y}, \text{for each } x\in\mathrm{X}.$$

The second-order generalized differential of (7.6) is

$$D^2_{\eta(\cdot)}\mathsf{F}[y] = \int_{\mathrm{X}} \frac{\delta^2\mathsf{F}[y]}{\delta y(x)^2}\,\eta(x)^2\,dx \tag{7.9}$$

where the second-order generalized derivative is

$$\frac{\delta^2\mathsf{F}[y]}{\delta y(x)^2} = \frac{\partial^2 f(y)}{\partial y^2}, \text{for each } x\in\mathrm{X}.$$

Examples: The generalized derivatives for several functionals which are common in economics:

1. for $\mathsf{F}[y] = \int_{\mathrm{X}} a\,dx$, we have $\dfrac{\delta\mathsf{F}[y]}{\delta y(x)} = \dfrac{\delta^2\mathsf{F}[y]}{\delta y(x)^2} = 0$;

2. for $\mathsf{F}[y] = \int_{\mathrm{X}} a\, y(x)\, dx$, we have $\frac{\delta \mathsf{F}[y]}{\delta y(x)} = a$ and $\frac{\delta^2 \mathsf{F}[y]}{\delta y(x)^2} = 0$;

3. for $\mathsf{F}[y] = \int_{\mathrm{X}} y(x)^2\, dx$, we have $\frac{\delta \mathsf{F}[y]}{\delta y(x)} = 2y(x)$ and $\frac{\delta^2 \mathsf{F}[y]}{\delta y(x)^2} = 2$;

4. for $\mathsf{F}[y] = \int_{\mathrm{X}} e^{y(x)}\, dx$, we have $\frac{\delta \mathsf{F}[y]}{\delta y(x)} = \frac{\delta^2 \mathsf{F}[y]}{\delta y(x)^2} = e^{y(x)}$;

5. for $\mathsf{F}[y] = \int_{\mathrm{X}} y(x)^\theta\, dx$, we have $\frac{\delta \mathsf{F}[y]}{\delta y(x)} = \theta\, y(x)^{\theta-1}$ and $\frac{\delta^2 \mathsf{F}[y]}{\delta y(x)^2} = \theta\,(\theta-1)\, y(x)^{\theta-2}$;

6. and for $\mathsf{F}[y] = \int_{\mathrm{X}} e^{g(y(x))}\, dx$, we have $\frac{\delta \mathsf{F}[y]}{\delta y(x)} = g'(y(x))\, e^{g(y(x))}$ and $\frac{\delta^2 \mathsf{F}[y]}{\delta y(x)^2} = g''(y(x))\, e^{g(y(x))} + (g'(y(x)))^2\, e^{g(y(x))}$.

The generalized differential of functional (7.7) is

$$D_{\eta(\cdot)}\mathsf{G}[y] \;=\; \int_{\mathrm{X}} \frac{\delta \mathsf{G}[y]}{\delta y(x)}\, \eta(x)\, dx \tag{7.10}$$

where the generalized derivative is

$$\frac{\delta \mathsf{G}[y]}{\delta y(x)} = g'(\mathsf{F}[y])\, \frac{\partial f(y(x))}{\partial y}, \text{for each } \; x \in \mathrm{X}.$$

The second-order generalized differential of (7.7) is

$$D^2_{\eta(\cdot)}\mathsf{G}[y] \;=\; \int_{\mathrm{X}} \frac{\delta^2 \mathsf{G}[y]}{\delta y(x)^2}\, \eta(x)^2\, dx \tag{7.11}$$

where the second-order generalized derivative is

$$\frac{\delta^2 \mathsf{G}[y]}{\delta y(x)^2} = g''(\mathsf{F}[y]) \left(\frac{\partial f(y(x))}{\partial y}\right)^2 + g'(\mathsf{F}[y])\, \frac{\partial^2 f(y(x))}{\partial y^2}, \text{for each } \; x \in \mathrm{X}., \text{for each } \; x \in \mathrm{X}.$$

Example:

1. for $\mathsf{G}[y] = \left(\int_{\mathrm{X}} y(x)^\theta\, dx\right)^{\frac{1}{\theta}}$, we have $\frac{\delta \mathsf{G}[y]}{\delta y(x)} = \left(\int_{\mathrm{X}} y(x)^\theta\, dx\right)^{\frac{1-\theta}{\theta}} y(x)^{\theta-1} = \left(\frac{\mathsf{G}[y]}{y(x)}\right)^{1-\theta}$.

”Spike” perturbations

In particular, for a ”spike” perturbation at point $x = x_0$, represented by a Dirac’-delta generalized function $\delta(x - x_0)$ [4] we have a functional generalization to the partial derivative[5]

$$D_{\delta(x_0)}\mathsf{F}[y] = \int_{\mathrm{X}} \frac{\delta \mathsf{F}[y]}{\delta y(x)}\, \delta(x - x_0)\, dx = \frac{\delta \mathsf{F}[y]}{\delta y(x_0)} = f'(y(x_0)). \tag{7.12}$$

[4] Dirac-δ is not a function but a distribution. It has the following properties

$$\delta(x - x_0) = \begin{cases} 0 & \text{if } \; x \neq x_0 \\ \infty & \text{if } \; x = x_0 \end{cases},$$

$\int_{-\infty}^{\infty} \delta(x) dx = 1$ and $\int_{-\infty}^{\infty} \delta(x - y_0)\, y(x)\, dx = y(x_0)$.

[5] In mathematical physics this is sometimes called as a Volterra derivative.

7.2.3 Functionals for two-dimensional functions

The previous definitions can be extended to functionals over two-dimensional functions of one variable. In this section we consider functions $\mathbf{y} \in \mathcal{Y}$, where $\mathbf{y} : \mathrm{X} \subseteq \mathbb{R} \to \mathbb{R}^2$, are of type $\mathbf{y}(x) = (y_1(x), y_2(x))^\top$, and functionals $\mathsf{F} : \mathcal{Y} \to \mathbb{R}$.

In particular, we restrain to linear functionals of type

$$\mathsf{F}[\mathbf{y}] \ = \int_{\mathrm{X}} f(x, y_1(x), y_2(x))\, dx. \tag{7.13}$$

In this case, for a perturbation $\boldsymbol{\eta}(x) = (\eta_1(x), \eta_2(x))^\top \in \mathcal{Y}$, the Gâteaux differential is defined

$$D_{\boldsymbol{\eta}(\cdot)}\mathsf{F}[\mathbf{y}] \ = \lim_{\varepsilon \to 0} \frac{\mathsf{F}[\mathbf{y} + \varepsilon\, \boldsymbol{\eta}] - \mathsf{F}[y]}{\varepsilon},$$

which implies that the functional (7.13) has the differential

$$\begin{aligned} D_{\boldsymbol{\eta}(\cdot)}\mathsf{F}[\mathbf{y}] &= \int_{\mathrm{X}} \Big\langle \frac{\delta \mathsf{F}[\mathbf{y}]}{\delta \mathbf{y}(x)}, \boldsymbol{\eta}(x) \Big\rangle\, dx \\ &= \int_{\mathrm{X}} \Big(\frac{\delta \mathsf{F}[\mathbf{y}]}{\delta y_1(x)}\, \eta_1(x) + \frac{\delta \mathsf{F}[\mathbf{y}]}{\delta y_2(x)}\, \eta_2(x) \Big)\, dx, \end{aligned} \tag{7.14}$$

where the generaiized gradient is

$$\frac{\delta \mathsf{F}[\mathbf{y}]}{\delta \mathbf{y}(x)} = \begin{pmatrix} \dfrac{\delta \mathsf{F}[\mathbf{y}]}{\delta y_1(x)} \\ \dfrac{\delta \mathsf{F}[\mathbf{y}]}{\delta y_2(x)} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f(x, y_1(x), y_2(x))}{\partial y_1(x)} \\ \dfrac{\partial f(x, y_1(x), y_2(x))}{\partial y_2(x)} \end{pmatrix}.$$

Example:
The functional $\mathsf{F}[\mathbf{y}] = \int_{\mathrm{X}} y_1(x)^\alpha\, y_2(x)^{1-\alpha}\ dx$, for a number α, then the generalized gradient is

$$\frac{\delta \mathsf{F}[\mathbf{y}]}{\delta \mathbf{y}(x)} = \begin{pmatrix} \alpha y_1(x)^{\alpha-1}\, y_2(x)^{1-\alpha} \\ (1-\alpha) y_1(x)^\alpha\, y_2(x)^{-\alpha} \end{pmatrix}.$$

7.2.4 Functionals involving derivatives

In this subseection it is considered again functionals $\mathsf{F} : \mathcal{Y} \ \to \mathbb{R}$ where $\mathcal{Y}$ is the space of uni-dimensional functions $y : \mathrm{X} \to \mathbb{R}$, where the integrand function $f(.)$ is a continuous and continuously differentiable function of y and of its first and second derivatives. Their functional derivatives are appear in calculus of variations and optimal control problems.

First-order derivatives

The following linear functional is commonly found in economics, in particular in calculus of variations problems,

$$\mathsf{F}[y] \;=\int_{\mathrm{X}} f(x, y(x), y'(x))\, dx, \tag{7.15}$$

where function $f(\cdot) : \mathrm{X}\times\mathrm{Y}\times\mathrm{Y}\to\mathbb{R}$ is assumed to be continuous and continuous differentiable in (y, y').

Lemma 1. *The (first-order) generalized differential is*

$$D_{\eta(\cdot)}\mathsf{F}[y] \;=\int_{\mathrm{X}} \frac{\delta\mathsf{F}[y]}{\delta y(x)}\;\eta(x)\,dx+\int_{\mathrm{X}}\frac{\partial f}{\partial y'}(x)\,\eta(x) \tag{7.16}$$

where the generalized derivative is

$$\frac{\delta\mathsf{F}[y]}{\delta y(x)}=\frac{\partial f(x,y(x),y'(x))}{\partial y}-\frac{d}{dx}\Big(\frac{\partial f(x,y(x),y'(x))}{\partial y'(x)}\Big) \tag{7.17}$$

and

$$\int_{\mathrm{X}}\frac{\partial f}{\partial y'}(x)\,\eta(x)=\frac{\partial f(x_1,y(x_1),y'(x_1))}{\partial y'}\;\eta(x_1)-\frac{\partial f(x_0,y(x_0),y'(x_0))}{\partial y'}\,\eta(x_0),$$

if $\mathrm{X}=[x_0,x_1]$ *is a closed set, or*

$$\int_{\mathrm{X}}\frac{\partial f}{\partial y'}(x)\,\eta(x)=\lim_{x\uparrow x_1}\;\frac{\partial f(x)}{\partial y'}\,\eta(x)-\lim_{x\downarrow x_0}\;\frac{\partial f(x)}{\partial y'}\,\eta(x)$$

if $\mathrm{X}=(x_0,x_1)$ *is an open set.*

Proof. We introduce not only the perturbation $y(x)\to y(x)+\varepsilon\eta(x)$, but alo the perturbation $y'(x)\to y'(x)+\varepsilon\eta'(x)$. From equation (7.14) we have now

$$\begin{aligned}
D_{\eta(\cdot)}\mathsf{F}[y] \;&=\int_{\mathrm{X}}\Big(\frac{\delta\mathsf{F}[y]}{\delta y(x)}\;\eta(x)+\frac{\delta\mathsf{F}[y]}{\delta y'(x)}\;\eta'(x)\Big)\,dx\\
&=\int_{\mathrm{X}}\frac{\delta\mathsf{F}[y]}{\delta y(x)}\;\eta(x)\,dx+\int_{\mathrm{X}}\frac{\delta\mathsf{F}[y]}{\delta y'(x)}\;\eta'(x)\,dx\\
&=\int_{\mathrm{X}}\frac{\partial f(x,y(x),y'(x))}{\partial y}\;\eta(x)\,dx+\int_{\mathrm{X}}\frac{\partial f(x,y(x),y'(x))}{\partial y'}\;\eta'(x)\,dx
\end{aligned}$$

Integrating the second integral by parts yields

$$\int_{\mathrm{X}}\frac{\partial f(x,y(x),y'(x))}{\partial y'}\;\eta'(x)\,dx=\int_{\mathrm{X}}\frac{\partial f(x,y(x),y'(x))}{\partial y'}\,\eta(x)-\int_{\mathrm{X}}\frac{d}{dx}\Big(\frac{\partial f(x,y(x),y'(x))}{\partial y'}\Big)\,\eta(x)\,dx,$$

that, upon substitution, yields the differential (7.16). □

Second-order derivatives

Now consider the integral where function f depends on the first and second derivatives of y,

$$\mathsf{F}[y] \;=\int_{\mathrm{X}} f(x,y(x),y'(x),y''(x))\,dx. \tag{7.18}$$

where we assume function $f(\cdot)$ is continuous and continuous differentiable at least up to the second order in (y, y', y'').

Lemma 2. *The Gâteaux differential of functional* (7.18) *is*

$$D_{\eta(\cdot)}\mathsf{F}[y] \;=\int_{\mathrm{X}} \frac{\delta \mathsf{F}[y]}{\delta y(x)}\;\eta(x)\,dx+\int_{\mathrm{X}} \frac{\partial f(x)}{\partial y'}\,\eta(x)+\int_{\mathrm{X}} \frac{\partial f(x)}{\partial y''}\,\eta'(x)-\int_{\mathrm{X}} \frac{d}{dx}\Big(\frac{\partial f(x)}{\partial y'}\Big)\,\eta(x) \tag{7.19}$$

where we denote $F(x)=f(x,y(x),y'(x),y''(x))$*, the Gâteaux derivative is*

$$\frac{\delta \mathsf{F}[y]}{\delta y(x)}=\frac{\partial F(x)}{\partial y}-\frac{d}{dx}\,\Big(\frac{\partial F(x)}{\partial y'}\Big)+\frac{d^2}{dx^2}\,\Big(\frac{\partial F(x)}{\partial y''}\Big), \tag{7.20}$$

and, in the case in which set $\mathrm{X}=[x_0,x_1]$ *is closed*

$$\begin{aligned}
&\int_{\mathrm{X}} \frac{\partial F(x)}{\partial y'}\,\eta(x)=\frac{\partial F(x_1)}{\partial y'}\,\eta(x_1)-\frac{\partial F(x_0)}{\partial y'}\,\eta(x_0)\\
&\int_{\mathrm{X}} \frac{\partial F(x)}{\partial y''}\,\eta'(x)=\frac{\partial F(x_1)}{\partial y''}\;\;\eta'(x_1)-\frac{\partial F(x_0)}{\partial y''}\;\;\eta'(x_0)\\
&\int_{\mathrm{X}} \frac{d}{dx}\Big(\frac{\partial F(x)}{\partial y''}\Big)\,\eta(x)=\frac{d}{dx}\Big(\frac{\partial F(x_1)}{\partial y''}\Big)\,\eta(x_1)-\frac{d}{dx}\Big(\frac{\partial F(x_0)}{\partial y''}\Big)\,\eta(x_0).
\end{aligned}$$

Proof. In this case we have

$$\begin{aligned}
D_{\eta(\cdot)}\mathsf{F}[y] \;&=\int_{\mathrm{X}} \Big(\frac{\delta \mathsf{F}[y]}{\delta y(x)}\;\;\eta(x)+\frac{\delta \mathsf{F}[y]}{\delta y'(x)}\;\;\eta'(x)+\frac{\delta \mathsf{F}[y]}{\delta y''(x)}\;\;\eta''(x)\Big)\,dx\\
&=\int_{\mathrm{X}} \frac{\delta \mathsf{F}[y]}{\delta y(x)}\;\;\eta(x)\,dx+\int_{\mathrm{X}} \frac{\delta \mathsf{F}[y]}{\delta y'(x)}\;\;\eta'(x)\,dx+\int_{\mathrm{X}} \frac{\delta \mathsf{F}[y]}{\delta y''(x)}\;\;\eta''(x)\,dx\\
&=\int_{\mathrm{X}} \frac{\partial F(x)}{\partial y}\;\;\eta(x)\,dx+\int_{\mathrm{X}} \frac{\partial F(x)}{\partial y'}\;\;\eta'(x)\,dx+\int_{\mathrm{X}} \frac{\partial F(x)}{\partial y''}\;\;\eta''(x)\,dx.
\end{aligned}$$

Integrating by parts the second integral, we have

$$\int_{\mathrm{X}} \frac{\partial F(x)}{\partial y'}\;\;\eta'(x)\,dx=\int_{\mathrm{X}} \frac{\partial F(x)}{\partial y'}\,\eta(x)-\int_{\mathrm{X}} \frac{d}{dx}\,\Big(\frac{\partial F(x)}{\partial y'}\Big)\,\eta(x)\,dx.$$

Integrating by parts the third integral, [6] we have

$$\begin{aligned}
\int_{\mathrm{X}} \frac{\partial f(x)}{\partial y''}\;\;\eta''(x)\,dx=&\int_{\mathrm{X}} \frac{\partial f(x)}{\partial y''}\,\eta'(x)-\int_{\mathrm{X}} \frac{d}{dx}\;\frac{\partial f(x)}{\partial y''}\,\eta'(x)\,dx\\
&+\int_{\mathrm{X}} \frac{d^2}{dx^2}\;\frac{\partial f(x)}{\partial y''}\,\eta(x)\,dx.
\end{aligned}$$

□

[6]This is because

$$\int uv''\,dx=uv'\;-\int u'v'\,dx=uv'\;-\Big(u'v-\int u''\,v dx\Big).$$

7.2.5 Functionals over higher-dimensional independent variables

We can extend those definitions for functionals $\mathsf{F}[y]$ over functions of higher dimensional independent variables $\mathbf{x} \in \mathrm{X} \subseteq R^n$. This is the case of higher-dimensional calculus of variations problems. In this section we deal with functionals $\mathsf{F} : \mathcal{Y} \to \mathbb{R}$ where $y \in \mathcal{Y}$ is the space of functions $y : \mathrm{X} \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$, that is $y(\mathbf{x}) = y(x_1, x_2)$.

The simplest functional is

$$\mathsf{F}[y] = \int_{\mathrm{X}} f(\mathbf{x}, y(\mathbf{x}))\, d\mathbf{x} = \int_{\mathrm{X}_1} \int_{\mathrm{X}_2} f(x_1, x_2, y(x_1, x_2))\, dx_2\, dx_1 \tag{7.21}$$

where $f(\cdot, y)$ is continuously diffferentiable in y and $\mathrm{X} = \mathrm{X}_1 \times \mathrm{X}_2 \subseteq \mathbb{R}^2$.

Introducing the perturbation $y(\mathbf{x}) \to y(\mathbf{x}) + \varepsilon\, \eta(\mathbf{x})$, the Gâteaux differential is

$$D_{\eta(\cdot)}\ \mathsf{F}[y] = \int_{\mathrm{X}} \frac{\delta \mathsf{F}[y]}{\delta y(\mathbf{x})}\ \eta(\mathbf{x})\, d\mathbf{x} = \int_{\mathrm{X}_1} \int_{\mathrm{X}_2} \frac{\delta \mathsf{F}[y]}{\delta y(x_1, x_2)}\, \eta(x_1, x_2)\, dx_2\, dx_1 \tag{7.22}$$

where the Gâteaux derivative is

$$\frac{\delta \mathsf{F}[y]}{\delta y(\mathbf{x})} = \frac{\partial f(\mathbf{x}, y(\mathbf{x}))}{\partial y} \tag{7.23}$$

For the functional with first derivatives,

$$\begin{aligned} \mathsf{F}[y] &= \int_{\mathrm{X}} f(\mathbf{x}, y(\mathbf{x}), \nabla y(\mathbf{x}))\, d\mathbf{x} = \\ &\int_{\mathrm{X}_1} \int_{\mathrm{X}_2} f(x_1, x_2, y(x_1, x_2), y_{x_1}(x_1, x_2), y_{x_2}(x_1, x_2))\, dx_2\, dx_1 \end{aligned} \tag{7.24}$$

where $f(\cdot, y)$ is continuously diffferentiable in $(y, \nabla y)$, $\mathrm{X} = \mathrm{X}_1 \times \mathrm{X}_2 \subseteq \mathbb{R}^2$ and $y_{x_i}(\mathbf{x})$ denote the partial derivatives for $i = 1, 2$. Introducing, the perturbations $y(\mathbf{x}) \to y(\mathbf{x}) + \varepsilon\, \eta(\mathbf{x})$ and $\nabla y(\mathbf{x}) \to \nabla y(\mathbf{x}) + \varepsilon\, \nabla \eta(\mathbf{x})$

The Gâteaux differential is

$$D_{\eta(\cdot)}\ \mathsf{F}[y] = \int_{\mathrm{X}} \frac{\delta \mathsf{F}[y]}{\delta y(\mathbf{x})}\ \eta(\mathbf{x})\, d\mathbf{x} = \int_{\mathrm{X}_1} \int_{\mathrm{X}_2} \frac{\delta \mathsf{F}[y]}{\delta y(x_1, x_2)}\, \eta(x_1, x_2)\, dx_2\, dx_1 \tag{7.25}$$

where the Gâteaux derivative is

$$\frac{\delta \mathsf{F}[y]}{\delta y(x_1, x_2)} = \frac{\partial f(\mathbf{x}, y(\mathbf{x})}{\partial y} \tag{7.26}$$

7.2.6 Applications

Next we present some infinite-dimensional functionals which are relatively common in economic models.

7.2.7 Generalized means

The generalized mean is very common functional which is used in economics.

Let there be different varieties indexed over a continuous index set I, which can have an order structure or not. It will have an order structure if the index refers to different locations and does not have an order structure if it refers to qualitative differences amons goods. The distribution $\Big(x(i)\Big)_{i\in \mathrm{I}}$ denotes the quantities of goods for all varieties, where $x : \mathrm{I} \to \mathbb{R}$ is our variable of interest.

Definition 2. *The **generalized mean** is defined as the functional* [7]

$$\mathsf{M}_\rho[x] = \left(\int_{\mathrm{I}} w(i)\, x(i)^\rho \, di \right)^{\frac{1}{\rho}}, \; \textit{for } \rho \in [-\infty, \infty] \tag{7.27}$$

where $w : \mathrm{I} \to (0,1)$ *is a weighting function such that*

$$\int_{\mathrm{I}} w(i)\, di = 1.$$

This functional can be seen as an infinite-dimensional generalization of the CES aggregator.

Proposition 1. *The generalized mean expression in equation* (7.27) *encompasses several several special cases for different values of* ρ*:*

1. *if* $\rho = 0$ *then* $\mathsf{M}_0[x] = \exp\Big(\int_{\mathrm{I}} \ln\left(x(i)^{w(i)}\right) di \Big) = \int_{\mathrm{I}} x(i)^{w(i)} \, di$*, is a geometric mean, and, in particular a generalized Cobb-Douglas function for a continuum;*
2. *if* $\rho = 1$ *then* $\mathsf{M}_1[x] = \int_{\mathrm{I}} w(i)\, x(i)\, di$*, is a generalized arithmetic mean;*
3. *if* $\rho = -1$ *then* $\mathsf{M}_{-1}[x] = \dfrac{1}{\int_{\mathrm{I}} \dfrac{w(i)}{x(i)}\, di}$ *it is a harmonic mean;*
4. *if* $\rho = -\infty$ *then* $\mathsf{M}_{-\infty}[x] = \min\Big\{ \big(x(i)\big)_{i\in \mathrm{I}} \Big\}$ *(in particular a generalized Leontieff production function)*
5. *if* $\rho = \infty$ *then* $\mathsf{M}_{\infty}[x] = \max\Big\{ \big(x(i)\big)_{i\in \mathrm{I}} \Big\}$

In general $\min_x[x] \leq \mathsf{M}_\rho[x] \leq \max_x[x]$

Proof. We can write equation (7.27) equivalently as

$$M_\rho[x] = \exp\left\{ \ln\left[\left(\int_{\mathrm{I}} w(i)\, x(i)^\rho \, di \right)^{\frac{1}{\rho}} \right] \right\} = \exp\left\{ \frac{1}{\rho} \ln\Big(\int_{\mathrm{I}} w(i)\, x(i)^\rho \, di \Big) \right\}.$$

[7] See Bullen (2003).

For the case $\rho = 0$ we use l'Hôpital's rule

$$\lim_{\rho\to 0} \frac{\ln\left(\int_{\mathrm{I}} w(i)\, x(i)^{\rho}\, di\right)}{\rho} = \lim_{\rho\to 0} \frac{\int_{\mathrm{I}} w(i)\, \ln\left(x(i)\right] x(i)^{\rho}\, di}{1}$$
$$= \int_{\mathrm{I}} w(i)\, \ln\left(x(i)\right) di = \int_{\mathrm{I}} \ln\left(x(i)^{w(i)}\right) di$$

Then $M_0[x] = \exp\left(\int_{\mathrm{I}} \ln\left(x(i)^{w(i)}\right)\right) di$. The generalized mean for $\rho = 1$ and $\rho = -1$ are obtained by direct substitution.

If function $x : \mathbb{R} \to$ X and $x \in L_{\infty}(\mathrm{X})$ then there is a maximum $x^* = ||x||_{\infty}$ and we can write

$$M_{\infty}[x] = \lim_{\rho\to\infty} x^* \left(\int_{\mathrm{I}} w(i) \left(\frac{x(i)}{x^*}\right)^{\rho} di\right)^{\frac{1}{\rho}} = \lim_{\psi\to 0} x^* \left(\int_{\mathrm{I}} w(i) \left(\frac{x(i)}{x^*}\right)^{\frac{1}{\psi}} di\right)^{\psi} = x^*$$

where $\psi = 1/\rho$. For any ρ we have $M_{\rho}[x] = M_{-\rho}\left[\frac{1}{x}\right]$. If $x \in L_{\infty}(\mathrm{X})$ as $\max\left[\frac{1}{x}\right] = \min[x]$ then

$$M_{-\infty}[x] = \lim_{\rho\to\infty} \frac{1}{M_{\rho}\left[\frac{1}{x}\right]} = \min[x]$$

□

We can calculate generalized derivatives of first and second order, elasticities and elasticities of substitution.

The generalized first-order derivative of equation (7.27), for a finite ρ, is

$$\frac{\delta \mathsf{M}_{\rho}[x]}{\delta x(i)} = w(i) \left(\frac{\mathsf{M}_{\rho}[x]}{x(i)}\right)^{1-\rho}, \text{ for any } \; i \in \mathrm{I},$$

and the generalized second-order derivative is

$$\frac{\delta^2 \mathsf{M}_{\rho}[x]}{\delta x(i) \delta x(j)} = (1-\rho)\, \frac{w(i)\, w(j)}{x(i)} \left(\frac{\mathsf{M}_{\rho}[x]}{x(ij}\right)^{1-\rho} \left(\frac{\mathsf{M}_{\rho}[x]}{x(i)}\right)^{-\rho}, \text{ for any } \; i, j \in \mathrm{I}.$$

7.2.8 Varieties in consumption theory

Let $c(i)$ be the quantity of good of variety $i \in \mathrm{I}$ and $c = \left(c(i)\right)_{i\in\mathrm{I}}$ the distribution of consumption among varieties. Index set I can be ordered or not, although in many application it is a non-ordered continuum of varieties. Sometimes the index set is specified as I $= [0, N]$, where N measures the span of varieties, where an increae in N can account for horizontal innovation. We denote by $c(\iota)$ the quantity of a specific variety $i = \iota \in \mathrm{I}$. We can see $c(\iota)$ as a product of the distribution c with a ι dentered Dirac' delta generalized function, $c(\iota) = \int_{\mathrm{I}} \delta(i-\iota)\, c(i)\, di$. The set of variety distributions c is the function space $\mathcal{C}$.

A utility functional $\mathsf{U}[c]$ maps a variety distribution into value, which is a number, i.e. $\mathsf{U} : \mathcal{C} \to \mathbb{R}$.

There are two main types of utility functionals in the literature $\mathsf{U}[c] = \int_{\mathrm{I}} u(c(i))\, di$, where $u(c)$ i increasing and strictly concave (assuming no satiation), $u''(c) < 0 < u'(c)$ for $c \in (0, \infty)$. [8]

Another common utility functional in the literature is the Dixit and Stiglitz (1977)[9] utility functional

$$\mathsf{U}[c] = \left(\int_{\mathrm{I}} c(i)^{1-\gamma} di \right)^{\frac{1}{1-\gamma}}, \tag{7.28}$$

which is analogous to functional (7.7) This utility functional is called a CES (constant elasticity of substitution) utility functional for reason that will be clear bellow. Observe that although it i similar it is not a generalized mean.

Using our previous definitions and results, in particular equation (7.10), a change in the variety distribution, $c \to c + \varepsilon\, \zeta \in \mathcal{C}$ will lead to a variation in utility given by Gâteaux differential

$$D_{\zeta(\cdot)}\ \mathsf{U}[c] \ = \int_{\mathrm{I}} \frac{\delta \mathsf{U}[c]}{\delta c(i)} \zeta(i)\, di, \text{ for any } \ i \in \mathrm{I}$$

where the marginal utility of variety i is given by the Gâteaux derivative

$$\frac{\delta \mathsf{U}[c]}{\delta c(i)} = \frac{1}{1-\gamma} \left(\int_{\mathrm{I}} c(i)^{1-\gamma} di \right)^{\frac{\gamma}{1-\gamma}} (1-\gamma) c(i)^{-\gamma} = \mathsf{U}[c]^{\gamma} c(i)^{-\gamma}$$

Therefore, the **marginal utility** for a specific variety $\iota \in \mathrm{I}$ is represented by the Gâteaux differential for a perturbation $\zeta(i) = \delta(i - \iota)$, that is

$$\frac{\delta \mathsf{U}[c]}{\delta c(\iota)} = \left(\frac{\mathsf{U}[c]}{c(\iota)} \right)^{\gamma}, \text{ for } \ i = \iota \in \mathrm{I}.$$

This allow to determine the marginal rate of substitution between varieties ι and ι'

$$MRS_{\iota,\iota'} = \frac{\frac{\delta \mathsf{U}[c]}{\delta c(\iota)}}{\frac{\delta \mathsf{U}[c]}{\delta c(\iota')}} = \left(\frac{c(\iota)}{c(\iota')} \right)^{-\gamma}, \text{for any } \ \iota, \iota' \in \mathrm{I}.$$

We can also use the second-order Gâteaux derivatives to derive the implicit properties of the utility functional (7.28) as regards the change in marginal utility and the substitution between varieties. Generalized versions of the Allen-Uzawa elasticities are,

$$\varepsilon_{\iota,\iota} \ = -\frac{\frac{\delta^2 \mathsf{U}[c]}{\delta c(\iota)^2} c(\iota)}{\frac{\delta \mathsf{U}[c]}{\delta c(\iota)}} = \gamma \left(1 - \left(\frac{\mathsf{U}[c]}{c(\iota)} \right)^{\gamma - 1} \right), \text{for any } \ \iota \in \mathrm{I}$$

[8]See, for recent contribution, Evgeny Zhelobodko and Thisse (2012) or Dhingra and Morrow (2019).

[9]See also Krugman (1980) or Melitz (2003).

which means there is decreasing marginal utility for variety ι if $\gamma > 1$ and $\mathsf{U}[c] < c(\iota)$, and

$$\varepsilon_{\iota,\iota'} = -\frac{\dfrac{\delta^2 \mathsf{U}[c]}{\delta c(\iota)\,\delta c(\iota')}\, c(\iota')}{\dfrac{\delta \mathsf{U}[c]}{\delta c(\iota)}} = -\gamma \left(\frac{\mathsf{U}[c]}{c(\iota')}\right)^{\gamma-1}, \text{for any } \iota \in \mathrm{I}$$

which that varieties ι and ι' are Edgeworth complements. We can alo determine a generalized elasticity of vaiety substitution

$$EVS_{\iota,\iota'} = \frac{c(\iota)\,\dfrac{\delta \mathsf{U}[c]}{\delta c(\iota)} + c(\iota')\,\dfrac{\delta \mathsf{U}[c]}{\delta c(\iota')}}{c(\iota)\,\dfrac{\delta \mathsf{U}[c]}{\delta c(\iota)}\,\varepsilon_{\iota',\iota'} - 2\,c(\iota)\,\dfrac{\delta \mathsf{U}[c]}{\delta c(\iota)}\,\varepsilon_{\iota,\iota'} + c(\iota')\,\dfrac{\delta \mathsf{U}[c]}{\delta c(\iota')}\,\varepsilon_{\iota,\iota}}$$

Substituting the previous formulas and simplifying we find

$$EVS_{\iota,\iota'} = \gamma^{-1},$$

which justifies the usual classification of utility functional (7.28) as CES (constant elasticity of substitution) between varieties.

7.2.9 Infinite-dimensional production functions

In this subsection we extend the concept of functional and generalized derivatives for production with a continuum of inputs, which are usually intermediate goods or heterogeneous labor or capital inputs. In the first case the index set is usually not ordered, but in the two last cases it can be ordered is, for instance, workers are ranked by skill level.

Let $x(i)$ be the quantity of input of varieties $i \in \mathrm{I}$. We denote x_j the quantity of variety $j \in \mathrm{I}$. We can see $x(j) = \int_{\mathrm{I}} \delta(i-j) q(i) di$. We denote x a variety distribution $x = (x(i))_{\mathrm{I}}$. The set of variety distributions belong to the function space $\mathcal{X}$.

A production function is a functional $\mathsf{F}[x]$ which maps the variety distribution into real production, a non-negative number, i.e. $\mathsf{F} : \mathcal{X} \to \mathbb{R}_+$. We can write it as

$$y = \mathsf{F}[x].$$

There are several production functions used in the literature (see, for instance, Parenti et al. (2017) and Bucci and Ushchev (2016)):

- the generalized AK production function

$$\mathsf{F}[x] = \int_{\mathrm{I}} A(i)\, x(i)\, di;$$

  in which $A(i)$ is the specific marginal productivity of input $i \in \mathrm{I}$,

- the generalized constant elasticity of substitution production function

$$\mathsf{F}[x] \;= \Big(\int_{\mathrm{I}} A(i)\, x(i)^{\frac{\epsilon-1}{\epsilon}}\, di \Big)^{\frac{\epsilon}{\epsilon-1}} \tag{7.29}$$

which is the infinite-dimensional analog of $F(\{x\}\ ) = \Big(\sum_{i\in \mathrm{I}} A_i\, x_i^{\frac{\epsilon-1}{\epsilon}} \Big)^{\frac{\epsilon}{\epsilon-1}}$, for $\mathrm{I} = \{i_1, \dots, i_n\}$;

- the O-ring production function (see Kremer (1993))

$$\mathsf{F}[x] \;= \exp\Big(\int_{\mathrm{I}} \gamma(i) \log\left(x(i)\right) di \Big)$$

where $\int_{\mathrm{I}} \gamma(i) = 1$. This function is the infinite-dimensional of the finite-dimensional analog of the discrete geometrical average $F(\{x\}\ ) = \prod_{i\in \mathrm{I}} x_i^{\gamma_i}$ for $\sum_{\mathrm{I}} \gamma_i = 1$, which makes the O-ring production function an infinite-dimensional analog to the Cobb-Douglas production function;

- the translog production function

$$F[x] \;= \exp\left(\int_{\mathrm{I}} \gamma(i) \log\left(x(i)\right) + \gamma(i) \log\left(x(i)^2\right) di \right).$$

Using a similar approach to the utility functional, for the CES production function (7.29), we have the marginal productivity for a particular input $i \in \mathrm{I}$,

$$\frac{\delta \mathrm{F}[x]}{\partial x(i)} = A(i) \Big(\frac{\mathrm{F}[x]}{x(i)} \Big)^{\frac{1}{\varepsilon}} = A(i) \Big(\frac{y}{x(i)} \Big)^{\frac{1}{\varepsilon}},$$

which implies that the marginal rate of transformation between inputs ι and ι' is

$$MRT_{\iota,\iota'} \;= \frac{\dfrac{\delta \mathrm{F}[x]}{\partial x(\iota)}}{\dfrac{\delta \mathrm{F}[x]}{\partial x(\iota')}} = \frac{A(\iota)}{A(\iota')} \Big(\frac{x(\iota')}{x(\iota)} \Big)^{\frac{1}{\varepsilon}}, \text{ for any } \iota, \iota' \in \mathrm{I}.$$

The (generalized) Allen-Uzawa elasticities are, for variety ι

$$\varepsilon_{\iota,\iota} \;= \frac{1}{\varepsilon} \left(1 - A(i) \Big(\frac{y}{x(\iota)} \Big)^{\frac{1-\varepsilon}{\varepsilon}} \right), \text{for any } \iota \in \mathrm{I}$$

and for varieties ι and ι'

$$\varepsilon_{\iota,\iota'} \;= -\frac{A(\iota')}{\varepsilon} \Big(\frac{y}{x(\iota')} \Big)^{\frac{1-\varepsilon}{\varepsilon}} \text{ for any } \iota \neq \iota' \in \mathrm{I}$$

which, again, tells us that varieties ι and ι' are (generalized) Edgeworth complements.

The elasticity of variety substitution is constant

$$EVS_{\iota,\iota'} = \varepsilon.$$

7.2.10 In information theory and econometrics

In, according to some authors, one of the most influential paper in the XX century Shannon (1948) introduces a measure of information by entropy. Entropy is a the functional

$$\mathrm{H}[f] \ = -\int_{\mathrm{X}} f(x)\ln(f(x))dx,$$

which is used in an information theoretic approach to econometrics (see Judge and Mittelhammer (2012)). The functional derivative is

$$\frac{\delta \mathrm{H}[f]}{\delta f(x)} = -\big(\ln\left(f(x)\right) \ +1\big).$$

7.3 Problems involving functionals

Several (static) problems in economics, and in other sciences, are cast a maximization of functionals with pr without constraints. The first order conditions of those problems involve solving functional equations.

In the next subsection 7.3.1 we define extremes of functionals, and, in particular maxima of functionals. In subsections 7.3.2 and 7.3.3 optimum conditions for maximum problems without or with constraints, respectively, are obtained. In the next section 7.4 we present applications in microeconomics and statistics.

7.3.1 Extremes of functionals

The function $y^* \in \mathcal{Y}$ is an **extremum** of functional $\mathsf{F}[y]$ if the value of functional, for that function y^*, has the value $\mathsf{F}[y^*]$, and any small arbitrary perturbation $\eta \in \mathcal{Y}$, such that $y = y^* + \varepsilon + \eta \in \mathcal{Y}$ will deviate the functional from that value. That is $\mathrm{F}[y] \ \neq \mathrm{F}[y^*]$.

That is locally we have

$$D_{\eta(\cdot)} \ \mathsf{F}[y^*] \ = 0, \text{ for } \ y^* \ \in \mathcal{Y}.$$

An extreme $y^* \in \mathcal{Y}$ is a **maximum**, only if, for an arbitrary perturbation η, such that $y = y^* \ + \varepsilon\, \eta$ we have $\mathsf{F}[y^*] \geq \mathsf{F}[y]$, and it is a minimum only if $\mathsf{F}[y^*] \leq \mathsf{F}[y]$.

From the generalized Taylor expansion in equation (7.5) a necessary condition of second order for a maximum is

$$D^2_{\eta(\cdot)} \ \mathsf{F}[y^*] \ \leq 0.$$

7.3.2 Maximum of functionals: unconstrained problem

Consider the problem: find $y \in \mathcal{Y}$ that solves the problem

$$\max_{y(\cdot)} \; \mathsf{F}[y] \tag{P1}$$

Lemma 3. *A necessary condition for a maximum is that the Gâteaux derivative of $F[y]$ satisfies*

$$\frac{\delta \mathsf{F}[y^*]}{\delta y(x)} = \frac{\delta \mathsf{F}[y]}{\delta y(x)}\Big|_{y^*(x)} = 0. \tag{7.30}$$

Proof. Assume we know the optimum $y^*(x)$. The objective functional evaluated at the optimum is $\mathsf{F}[y^*]$, which is a number. Introducing an arbitrary admissible perturbation $\eta(x) \in \mathcal{Y}$ at the optimum, we obtain $y(x) = y^*(x) + \varepsilon\,\eta(x)$ which has the value $\mathsf{F}[y] = \mathsf{F}[y^* + \varepsilon\,\eta]$. If $y^*(x)$ is an optimum then we should have $\mathsf{F}[y^*] \geq \mathsf{F}[y]$. Expanding $\mathsf{F}[y]$ in a neighborhood of the optimum we have

$$\mathsf{F}[y] \; = \mathsf{F}[y^*] + D_{\eta(\cdot)}\mathsf{F}[y^*]\,\varepsilon + \frac{1}{2}\; D_{\eta(\cdot),\eta(\cdot)}\mathsf{F}[y^*]\,\varepsilon^2 + o(\varepsilon^3)$$

If y^* is a maximum it satisfies $D^2_{\eta(\cdot)}\mathsf{F}[y^*] \leq 0$. Therefore we can have $\mathsf{F}[y^*] \geq \mathsf{F}[y]$ only if

$$D_{\eta(\cdot)}\mathsf{F}[y^*] = \lim_{\varepsilon \to 0} \; \frac{\mathrm{F}[y^* + \varepsilon\,\eta] - \mathrm{F}[y^*]}{\varepsilon} = 0.$$

□

The Taylor expansion also allows us to find a sufficient condition. Assume that: first, the functional is concave in the sense that it satisfies the condition:

$$D^2_{\eta(\cdot)}\;\mathsf{F}[y] \;\leq 0, \text{ for any } \; y \in \mathcal{Y}$$

and, second, there is an element of $\mathcal{Y}$, $y^*(x)$ satisfying the condition $D_{\eta(\cdot)}\mathsf{F}[y^*] = 0$. Then $y^*(x) \in \mathcal{Y}$ is a maximum.

Example For functional (7.6) we have, assuming that we know $y^*(x)$, and for any admissible perturbation $y^*(x) \to y(x) = y^*(x) + \varepsilon\eta(x)$, we have

$$\mathrm{F}[y] \; - \mathrm{F}[y^*] = \int_{\mathrm{X}} \; f(y^*(x) + \varepsilon\eta(x)) - f(y^*(x))\,dx$$

Therefore, the Gâteaux differential is

$$D_{\eta(\cdot)}\mathsf{F}[y^*] = \int_{\mathrm{X}} \; f'(y^*(x))\,\eta(x)\,dx$$

which implies that, for any admissible perturbation, the first-order conditions for a maximum is

$$f'(y^*(x)) = 0 \text{ for each } \; x \in \mathrm{X}.$$

Exercise Prove that the maximum for functional (7.7) is

$$g'(\mathrm{F}[y^*]) \;\; f'(y^*(x)) = 0 \text{ for each } \; x \in \mathrm{X}.$$

7.3.3 Constrained maximum of functionals

In economics we are generally interested in problems defined by the maximization of a functional depending on constraints. We next consider two types of problems which differ as regards the nature of their constraints: problems with functional constraints and problems with point-wise constraints. The first type of constraints involve all the distribution and the second type of constraint are more stringent because they should hold at every point of the distribution.

We restrain to problems involving linear functionals.

Problems with functional constraints

Consider the two linear functionals over function $y : \mathrm{X} \to \mathbb{R}$ belonging to the set $y \in \mathcal{Y}$ of bounded functions:

$$\mathsf{F}[y] = \int_{\mathrm{X}} f(y(x), x) dx$$

and

$$\mathsf{G}[y] = \int_{\mathrm{X}} g(y(x), x) dx$$

The first problem is: find function $y \in \mathcal{Y}$ that solves

$$\begin{aligned} &\max_{y(\cdot)} \int_{\mathrm{X}} f(y(x), x)\, dx \\ &\text{subject to} \\ &\int_{\mathrm{X}} g(y(x), x) dx = 0 \end{aligned} \tag{P2}$$

Lemma 4. *The function $y^*(x)$ for $x \in \mathrm{X}$ and th enumber λ are necessary conditions for an optimum of problem* (P2) *if and only if they satisfy*

$$\frac{\partial f(y^*(x), x)}{\partial y} + \lambda \frac{\partial g(y^*(x), x)}{\partial y} = 0, \text{ for each } x \in \mathrm{X} \tag{7.31a}$$

$$\int_{\mathrm{X}} g(y^*(x), x) dx = 0, \tag{7.31b}$$

where the first condition involves a Gâteaux derivative for every point in $x \in \mathrm{X}$, the second is an equation for the whole distribution.

Proof. We define a generalized Lagrangean functional We define the Lagrangean functional can be written as a parameterized (by λ) functional

$$\mathsf{L}[y] = \mathsf{F}[y] + \lambda\, \mathsf{G}[y] = \int_{\mathrm{X}} L(y(x), x, \lambda)\, dx.$$

where the Lagrangean (function) is

$$L(y(x), x, \lambda) \equiv f(y(x), x) + \lambda\, g(y(x), x),$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. At the optimum $(y^*(x))_{x\in \mathrm{X}}$ we have $\mathrm{L}[y^*] = \mathrm{F}[y^*]$. Introducing an admissible perturbation $y^*(x) \to y(x) = y^*(x) + \varepsilon\, \eta(x) \in \mathcal{Y}$ and taking the limit we have the Gâteux differential

$$D_{\eta(\cdot)}\ \mathrm{L}[y^*]\ = \int_{\mathrm{X}} \Big(\frac{\partial f(y^*(x))}{\partial y} + \lambda\, \frac{\partial g(y^*(x))}{\partial y}\Big)\, \eta(x)\, dx.$$

The maximum of the functional is attained only if $D_{\eta(\cdot)}\ \mathrm{L}[y^*]\ = 0$, for any admissible perturbation. Therefore the necessary condition for an optimum, which must satisfy the constraint of the problem, equation (7.31b), is in equation(7.31a). □

Problems with point-wise constraints

The previous problem involved a constraint on all the distribution. Now we consider problems in which there are point-wise constraints for every component of the distribution

$$\begin{aligned} &\max_{y(\cdot)} \int_{\mathrm{X}} f(y(x), x)\, dx \\ &\quad \text{subject to} \\ &\quad g(y(x), x) = 0,\ \text{for all } x \in \mathrm{X}. \end{aligned} \tag{P3}$$

Lemma 5. *The pair of functions $y^*(x)$ and $\lambda(x)$ for $x \in \mathrm{X}$ are necessary conditions for an optimum of problem* (P3) *if and only if they satisfy*

$$\frac{\partial f(y^*(x), x)}{\partial y} + \lambda(x)\, \frac{\partial g(y^*(x), x)}{\partial y} = 0,\ \textit{for each}\ \ x \in \mathrm{X} \tag{7.32a}$$

$$g(y^*(x), x) = 0,\ \textit{for each}\ \ x \in \mathrm{X} \tag{7.32b}$$

where both equations the first condition involves a Gâteaux derivative and the second a classical derivative.

Proof. While in problem P2 we had one constraint, in this case we have an infinity of constraints. Therefore, we have to introduce a Lagrangean function $\lambda : \mathrm{X} \to \mathbb{R}$ (instead of a Lagrange multiplier $\lambda \in \mathbb{R}$ as in problem P2). The Lagrangean functional is now

$$\mathsf{L}[y, \lambda] = \int_{\mathrm{X}} f(y(x), x) + \lambda(x)\, g(y(x), x) dx.$$

Therefore, the Gâteux differential for an admissible perturbation, $\eta(\cdot)$, in a neighborhood of the optimum $y^*(\cdot)$ is now

$$D_{\eta(\cdot)}\ \mathrm{L}[y^*]\ = \int_{\mathrm{X}} \Big(\frac{\partial f(y^*(x))}{\partial y} + \lambda(x)\, \frac{\partial g(y^*(x))}{\partial y}\Big)\, \eta(x)\, dx.$$

Setting it to zero yields condition (7.32a). □

7.4 Applications

In this subsection we present applications of functionals to problems in microeconomics, to the consumer problems 7.4.1 to firms' poblems 7.4.2

7.4.1 Applications to the consumer problem

The optimal choice of varieties

Assume we have a continuum of varieties I and a basket of consumption containing different varieties is $c = \big(c(i)\big)_{i \in \mathrm{I}}$ where $c(i) \geq 0$ is the quantity of variety $i \in \mathrm{I}$ in basket c. The composition of the basket can be seen as a function mapping between the space of varieties and a real number, the quantity consumed: $c : \mathrm{I} \rightarrow \mathbb{R}_{+}$, belonging to a space of positively-valued bounded functions, $c \in \mathcal{C}$.

The value of the basket is measured by the utility functional $\mathsf{U} : \mathcal{C} \rightarrow \mathbb{R}$:

$$\mathsf{U}[c] = \left(\int_{\mathrm{I}} c(i)^{1-\gamma} di \right)^{\frac{1}{1-\gamma}}.$$

The consumer has income $y > 0$ which can be spent on the purchase of baskets. The total expenditure is the functional

$$\mathsf{E}[c] = \int_{\mathrm{I}} p(i)\, c(i)\, di$$

where $p(.)$ is the relative price of variety relative to the income deflator. Assuming that there are no savings, and denoting nominal income by Y, the budget constraint is $\mathsf{E}[p, c] = Y$.

The consumer problem is to find the basket $\Big(c(i)\Big)_{i \in \mathrm{I}}$ that solves the problem

$$\begin{aligned} &\max_{c(\cdot)} \mathrm{U}[c] = \left(\int_{\mathrm{I}} c(i)^{1-\gamma} di \right)^{\frac{1}{1-\gamma}} \\ &\text{subject to} \\ &\mathsf{E}[c] = \int_{\mathrm{I}} p(i)\, c(i)\, di = Y. \end{aligned} \tag{PC}$$

The solution is the basket $c^* = \big(c^*(i)\big)_{i \in \mathrm{I}}$ where

$$c^*(i) = \frac{Y}{P} \left(\frac{P}{p(i)} \right)^{\frac{1}{\gamma}}, \text{ for each } i \in \mathrm{I}. \tag{7.33}$$

where $P = \left(\int_{\mathrm{I}} p(i)^{\frac{\gamma-1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}}$ is the true cost of living index.

The Lagrangean functional can be interpreted as the indirect utility functional

$$\mathsf{L}[c; \lambda] = \mathsf{U}[c] - \lambda \mathsf{E}[c] = \left(\int_{\mathrm{I}} c(i)^{1-\gamma} di \right)^{\frac{1}{1-\gamma}} - \lambda \int_{\mathrm{I}} p(i)\, c(i)\, di.$$

The first order conditions for problem (PC) are

$$\frac{\delta \mathsf{L}[c^*]}{\delta c(i)} = \frac{\delta \mathsf{U}[c^*]}{\delta c(i)} - \lambda \frac{\delta \mathsf{E}[c^*]}{\delta c(i)} = 0, \text{ for each } i \in \mathrm{I}, \tag{7.34a}$$

$$\mathsf{E}[c^*] = y. \tag{7.34b}$$

The optimality condition (7.34a) is equivalent to

$$\left(\frac{c(i)}{\mathsf{U}[c]}\right)^{-\gamma} = \lambda\, p(i) \iff c^*(i) = \mathsf{U}[c^*]\Big(\lambda\, p(i)\Big)^{-\frac{1}{\gamma}}.$$

Substituting in the utility functional yields

$$\mathsf{U}[c^*] = \Big(\int_{\mathrm{I}} \Big(\mathsf{U}[c^*]\Big(\lambda\, p(i)\Big)^{-\frac{1}{\gamma}}\Big)^{1-\gamma}\, di\Big)^{\frac{1}{1-\gamma}}$$

allows us to find λ as

$$\frac{1}{\lambda} = P \equiv \Big(\int_{\mathrm{I}} p(i)^{\frac{\gamma-1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}},$$

that we interpret as the consumer price deflactor (or the true index of cost of living) using as numéraire the income deflactor. Substuting in the constraint (7.34b), and simplifying, yields $P\,\mathsf{U}[c^*] = Y$. Substituting again in the expression for $c(i)$ yields the solution to the consumer problem in equation (7.33).

Several comparative statics results can be obtained from equation (7.33): First, consumption of variety i is a linear function of real income $y = Y/P$, where nominal income is deflated by the true cost of living index P,

Second, observing that the P is a functional of the distribution of prices,

$$P = \mathsf{P}[p] = \Big(\int_{\mathrm{I}} p(i)^{\frac{\gamma-1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}},$$

then $c^*(i) = \mathsf{C}^*(i)[p]$ can be seen as a functional over the price distribution. As

$$\frac{\delta \mathsf{P}[p]}{\delta p(i)} = \left(\frac{P}{p(i)}\right)^{\frac{1}{\gamma}}, \text{ for any } i \in \mathrm{I},$$

using the previous results on functional differentiation, we find the response of the demand for variety i to increases in its own price

$$\frac{\delta \mathsf{C}^*(i)[p]}{\delta p(i)} = \frac{c^*(i)}{\gamma\, p(i)}\Big(1 + (\gamma-1)\left(\frac{P}{p(i)}\right)^{\frac{1-\gamma}{\gamma}}\Big),$$

and

$$\frac{\delta \mathsf{C}^*(i)[p]}{\delta p(j)} = c^*(i)\,\frac{(1-\gamma)}{\gamma}\, p(j)^{-\frac{1}{\gamma}}, \text{ for any } j \neq i \in \mathrm{I}$$

for the change of the price of any other variety.

We see that the responses to both own and other price changes depend on the elasticity of variety substitution $EVS_{i,j} = 1/\gamma$ (see subsection 7.2.8). If $p(i) \approx P$ and $EVS_{i,j} > 1$ then an increase in the own price, $p(i)$, decreases the consumption of variety i and an increase in the price of any other variety, $p(j)$ for $j \neq i$ increases its consumption. This means that although different varieties are Edgeworth complements, they are Hicks substitutable.

7.4.2 Application to production theory

Single-product firm

This problem is a component of most new-Keynesian models, as the problem for a final producer using a continuum of inputs $x = (x(i))_{i\in \mathrm{I}}$. When bundles of inputs are Infinite-dimensional, as x, models usually consider they refer to intermediary goods not factors of production.

Total sales in a competitive sales market are $S(p, y) = py$, where p is the final good price and is given to the firm, and y is the output. Assume that the firm has a generalized constant elasticity of substitution (CES) production function

$$y = \mathsf{F}[x] \equiv \Big(\int_{\mathrm{I}} A(i)\, x(i)^{\frac{\epsilon-1}{\epsilon}}\, di\Big)^{\frac{\epsilon}{\epsilon-1}}$$

with specific factor productivity $A(i)$. Therefore, nominal sales are a functional

$$\mathsf{S}[x] \;= p\,\mathsf{F}[x],$$

where p is the price of the output. The total cost is also a functional

$$\mathsf{C}[x] \;= \int_{\mathrm{I}} w(i)x(i)di,$$

where $w(i)$ denotes the price of input $i \in \mathrm{I}$. We can denote by $A = \big(A(i)\big)_{i\in\mathrm{I}}$ and $w = \big(w(i)\big)_{i\in\mathrm{I}}$ the profiles of productivity and costs associated to the input profile $x = \big(x(i)\big)_{i\in\mathrm{I}}$.

Therefore, firms profits are also a functional over the x

$$\Pi[x] = p\,\mathsf{F}[x] - \mathsf{C}[x].$$

Assuming a price-taker firm in all the markets, both the in the product market and the I markets for inputs, the firm's problem is to find the optimal input bundle $X^* = \big(x^*\big)_{i\in\mathrm{I}}$ that solves the problem

$$\max_{x(\cdot)} \Pi[x] = p\Big(\int_{\mathrm{I}} A(i)\, x(i)^{\frac{\epsilon-1}{\epsilon}}\, di\Big)^{\frac{\epsilon}{\epsilon-1}} - \int_{\mathrm{I}} w(i)x(i)di$$

subject to

$$\Big(\int_{\mathrm{I}} A(i)\, x(i)^{\frac{\epsilon-1}{\epsilon}}\, di\Big)^{\frac{\epsilon}{\epsilon-1}} = d$$

where d is the market demand. In equilibrium $y = \mathsf{F}[x] = d$.

Exercise: Prove that the solution is

$$x^*(i) = \Big(\frac{W\,A(i)}{w(i)}\Big)^{\epsilon}\; d, \text{for each } \; i \in \mathrm{I}$$

where $W = \mathsf{W}[A, w] \equiv \Big(\int_{\mathrm{I}} A(i)^{\epsilon}\, w(i)^{1-\epsilon}\, di\Big)^{\frac{1}{1-\epsilon}}$ is a measure of the aggregate marginal costs.

7.4.3 Optimal taxation

Let $\theta \in \Theta \subseteq \mathbb{R}_+$ denote the skill level of the population and let $f(\theta)$ be their density. Therefore $\int_\Theta f(\theta)\, d\theta = 1$. Let $\ell(\theta)$, $y(\theta) = \theta\, \ell(\theta)$, and $c(\theta)$ denote the work effort, income and consumption, respectively, by people of skill level θ. The utility for a household with skill level θ is $u(\theta) = U(c(\theta), \ell(\theta))$.

Assume the government has an exogenous expenditure level G and wants to find an optimal tax schedule $T(\theta) = \tau(\theta)\, y(\theta)$ which implements a social optimum. Assume that the social optimal criterium is the average social welfare

$$\mathsf{W}[u] = \int_\Theta W\big(u(\theta)\big)\, f(\theta)\, d\theta.$$

Assume that the central planner not only observes the distribution of income $y(\theta) = \theta\, \ell(\theta)$ but has also complete information on the work effort, that is it is able to separate the productivity θ from the work effort $\ell(\theta)$.

The problem is simpler if we use the implicit function theorem to solve $u(\theta) = U(c(\theta), \ell(\theta))$ for $c(\theta) = C\big(u(\theta), \ell(\theta)\big)$.

We can find the optimal allocation of utility and work effort by solving the functional problem

$$\begin{aligned} \max_{u(\cdot), \ell(\cdot)} \; & \int_\Theta W\big(u(\theta)\big)\, f(\theta)\, d\theta \\ & \text{subject to} \\ & \int_\Theta \Big(\theta\, \ell(\theta) - C\big(u(\theta), \ell(\theta)\big)\Big)\, f(\theta)\, d\theta = G \end{aligned}$$

If we can find c^* and ℓ^* then the optimal optimal tax-transfer policy that implements the social optimum, and finances the government expenditure G, has the following property

$$T^*\ (\theta) = y^*\ - c^*.$$

7.4.4 Statistics: optimal derivation of statistical distributions

Maximum entropy with a simple constraint

Problem 1 find the continuous maximum entropy for function $f : \mathrm{X} \to (0, 1)$ with support $\mathrm{X} = [a, b]$ where $b > a$, such that it is a distribution. Formally, we seek to find function $f(\cdot)$ that solves the problem

$$\begin{aligned}\max_{f(\cdot)}\ \mathsf{H}[f] &= -\int_a^b f(x)\ln(f(x))\,dx \text{ (entropy functional)}\\ &\text{subject to}\\ \mathsf{G}[f] &= \int_a^b f(x)dx = 1 \text{ (functional constraint)}\end{aligned}$$

The solution is: function f follows a uniform distribution with support $\mathrm{X} = [a, b]$

$$f^*(x) = \frac{1}{b-a} \text{ for any } \ x \in \mathrm{X} = [a, b].$$

To prove this result we write the Lagrangean functional

$$\mathsf{L}[f] = \int_a^b L(f(x), \lambda)dx = \int_a^b -f(x)\ln(f(x)) - \lambda f(x)\,dx$$

where λ is a Lagrange multiplier. The previously obtained the first-order conditions yield

$$\frac{\delta \mathsf{L}[f^*]}{\delta f(x)} = \frac{\partial L(f(x))}{\partial f} = -\big(\ln(f(x)) + 1\big) - \lambda = 0$$

if and only if $f(x) = e^{-(1+\lambda)}$. Substituting in the constraint

$$\mathsf{G}[f] = \int_a^b e^{-(1+\lambda)}\,dx = 1,$$

we find $e^{-(1+\lambda)} = \frac{1}{b-a}$ and therefore $1 + \lambda = \ln(b - a)$. Therefore $f^*(x) = 1/(b - a)$.

Maximum entropy with constraints in the first and second moments

Problem 2:[10] find the continuous maximum entropy distribution with support $(-\infty, \infty)$ such that the average satisfies $\mathbb{E}[x] = \mu$ and the variance satisfies $\mathbb{E}[(x - \mu)^2] = \sigma^2$, where μ and σ are real numbers. Recall that

$$\begin{aligned}\mathbb{E}[x] &= \int_{-\infty}^{\infty} x f(x)dx,\\ \mathbb{V}[x] &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx.\end{aligned}$$

[10]See (Shannon, 1948, p.36), or (Cover and Thomas, 2006, ch. 12)

Formally, the problem is: find $f : \mathrm{X} \to (0,1)$, where $\mathrm{X} = (-\infty, \infty)$ that solves the problem

$$\begin{aligned}
\max_{f(\cdot)} \mathsf{H}[f] \;&= -\int_a^b f(x)\ln(f(x))\,dx,\\
&\text{subject to}\\
&\int_{-\infty}^{\infty} f(x)dx = 1 \text{ (f is a density function)}\\
&\int_{-\infty}^{\infty} xf(x)dx = \mu \text{ (with average equal to μ)}\\
&\int_{-\infty}^{\infty} x^2 f(x)dx = \sigma^2 + \mu^2 \text{ (with variance equal to σ^2).}
\end{aligned}$$

The last restriction is equivalent to $\mathbb{V}[x] = \sigma^2$, because

$$\mathbb{V}[x] \;= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - 2\int_{-\infty}^{\infty} \mu x f(x)dx + \int_{-\infty}^{\infty} \mu^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2.$$

Observation: the problem is to assume we have an unknown distribution which is required to have an average equal to μ and a variance equal to σ^2, two known numbers, what is this distribution if we require the principle of maximum entropy to apply ? The answer is: the normal distribution.

The Lagrangean functional is

$$\begin{aligned}
\mathsf{L}\,[f] \;&= \int_a^b L(x, f(x))\,dx =\\
&= \int_a^b -f(x)\ln(f(x)) - \lambda_0 f(x) - \lambda_1 x f(x) - \lambda_2 x^2 f(x) dx
\end{aligned}$$

where λ_0, λ_1 and λ_2 are Lagrange multipliers (they are all numbers).

The first order condition is

$$\frac{\delta \mathsf{L}[f^*]}{\delta f(x)} = -\ln(f(x)) - 1 - \lambda_0 - \lambda_1 x - \lambda_2 x^2 = 0$$

Therefore

$$f(x) = e^{-(1+\lambda_0+\lambda_1 x+\lambda_2 x^2)}$$

There are two methods for finding the solution:

1. First approach: conjecture that the solution is a Gaussian integral

$$g(x) = ae^{-\frac{(x-b)^2}{2c^2}}$$

   where a, b, and c are undetermined coefficients, and should be c a real and positive number. If the conjecture on the general form of the solution is correct, then we can obtain the three parameters by substituting this function in the three constraints. Fortunately, this is the case because we obtain a system of three equations in the three unknowns a, b, and c:

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)dx &= a\sqrt{2\pi c}\\ \int_{-\infty}^{\infty} xg(x)dx &= ba\sqrt{2\pi c}\\ \int_{-\infty}^{\infty} x^2 g(x)dx &= a\sqrt{2\pi c}\,(b^2+c)\end{aligned}$$

Solving the system yields $a = \dfrac{1}{2\pi\sigma^2}$, $b = \mu$ and $c = \sigma^2$, implying function $g(x)$ becomes

$$g(x) = e^{\log(a) - \frac{1}{2c}(x^2 - 2bx + b^2)}.$$

Matching the exponent with $f(x)$, we find the Lagrange multipliers

$$\lambda_0 = -1 - \ln(2\pi\sigma^2)^{-\frac{1}{2}} + \frac{\mu^2}{2\sigma^2},\ \lambda_1 = -\frac{\mu}{\sigma^2},\ \lambda_2 = \frac{1}{2\sigma^2}.$$

2. Second approach: alternatively, we can substitute our candidate solution in the constraints and try to determine the Lagrange multipliers. Assuming that $\mathrm{Re}(\lambda_2) > 0$, we find

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \sqrt{\frac{\pi}{\lambda_2}}e^{-1-\lambda_0+\frac{\lambda_1^2}{4\lambda_2}} = 1\\ \int_{-\infty}^{\infty} xf(x)dx &= -\frac{\lambda_1}{2\lambda_2}\int_{-\infty}^{\infty} f(x)dx = \mu\\ \int_{-\infty}^{\infty} x^2 f(x)dx &= \left(\frac{\lambda_1^2 + 2\lambda_2}{(2\lambda_2)^2}\right)\int_{-\infty}^{\infty} f(x)dx = \sigma^2 + \mu^2.\end{aligned}$$

We obtain a system of three equations for the Lagrange multipliers

$$\begin{aligned}&\sqrt{\frac{\pi}{\lambda_2}}e^{-1-\lambda_0+\frac{\lambda_1^2}{4\lambda_2}} = 1\\ &-\frac{\lambda_1}{2\lambda_2} = \mu\\ &\left(\frac{\lambda_1^2 + 2\lambda_2}{(2\lambda_2)^2}\right) = \sigma^2 + \mu^2.\end{aligned}$$

If we solve it we obtain the same result.

The optimal density function is the Normal distribution:

$$f^*(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

7.5 References

Generalized calculus in economics: Ok (2007), which observes that in most textbooks in functional analysis there is not much applications to differential and integral calculus and applications to optimizations.

Generalized calculus in applied mathematics: (Siddiqi, 2018, ch 5), (Drabek and Milota, 2013, ch 3)

Application of functional analysis to optimization, however with an emphasis in non-differentiable functions Clarke (2013)

Chapter 8

Introduction to optimal control: the maximum principle approach

8.1 Introduction

The Pontriagin's maximum principle (PMP), see Pontryagin et al. (1962), provides first order necessary conditions for the optimal control problem, from a slightly different approach than the Calculus of Variations approach. Although the two approaches lead to the same solution, when applied to the same simple problems, the PMP can be more flexible when dealing with some type of problems (singular problems, for instance). As we will see that the first-order conditions are a system of two ordinary differential equations (and not an implicit second order ODE as in the calculus of variations approach) it allows a more direct use of results from the theory of ODE's, and also to use geometrical methods when the constitutive functions of the problem are not completely specified or the first-order conditions do not have explicit solutions. In this chapter we will consider optimal control problems on a general state space X, and in the next chapter we consider problems in which the independent variable is time.

We denote again the independent variable by x and assume it has the domain $\mathrm{X} \subseteq \mathbb{R}$. We can write $\mathrm{X} = [x_0, x_1]$ if it is a closed set, or $\mathrm{X} = (x_0, x_1)$ if it is open, where $x_0 < x_1$ are fixed or determined optimally, and x_1 can be bounded or unbounded.

The optimal control problem has two variables we need to find: the **state variable**, denoted by $y(x)$ and the **control variable**, denoted by $u(x)$. As we consider only problems in which the state variable is of dimension one, the state variable is a mapping $y : \mathrm{X} \to \mathrm{Y} \subseteq R$ and the control variable is a mapping $u : \mathrm{X} \to \mathrm{U} \subseteq R^m$. That is, we may have more $m \geq 1$ control variables. Again it is important to distinguish between the **point-wise level** of variables, i.e $y(x')$ and $u(x')$, for a specific $x = x'$, from the **curves or profiles traced out in the range of variables** $(y(x))_{x \in \mathrm{X}}$ and $(u(x))_{x \in \mathrm{X}}$.

A solution to an optimal control problem allows for finding optimal curves (optimal trajectories), traced out in the specified or optimized domain, (or time interval) according to one criterium (a functional) and given some constraints. The constraints can be be specified point-wise for all the domain of the independent variable, can be specified in particular points (usually boundary points in X), or for all the domain.

If we can find an optimality criterium for the pointwise behavior of the state and control variables, say functions $y^*(x)$ and $u^*(x)$ for every $x \in \mathrm{X}$ then we can trace-out the optimal curves $y^* = (y^*(x))_{x\in\mathrm{X}}$ and $u^* = (u^*(x))_{x\in\mathrm{X}}$.

An **optimal control problem** consists in finding functions $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, where $\mathcal{Y} \in C^1(\mathrm{X})$, the set of continuous and continuously differentiable functions $y : \mathrm{X} \to \mathrm{Y} \subseteq \mathbb{R}$, and $\mathcal{U} \in PC^1(\mathrm{X})$, the set of piecewise continuous functions $u : \mathrm{X} \to U \subseteq \mathbb{R}^m$, such that

$$y'(x) = G(y(x), u(x), x), \text{ for } x \in [x_0, x_1] \tag{8.1}$$

that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_{x_0}^{x_1} F(x, y(x), u(x))\, dx \tag{8.2}$$

with side conditions, for the aggregate curve or point-wise on the boundaries of sets X and Y. The additional data is related to the information concerning the boundary values of the independent variable x_0 and x_1 and/or the boundary values for the state variable $y(x_0)$ and $y(x_1)$.

The necessary conditions for an optimum according to the **Pontriyagin's maximum principle** are set by using the **Hamiltonian** function, defined as

$$H(x, y, u, \lambda) = F(x, y, u) + \lambda\, G(x, y, u).$$

where λ, called the co-state variable, is a piecewise continuous mapping $\lambda : X \to \mathbb{R}$.

Next we present the optimality conditions for a bounded domain, in section 8.2 , and problems with integral constraints in section 8.3.

8.2 Constraints on boundaries

In this subsection we assume that the data of the problem includes the boundary values for the independent variable: i.e., x_0 and x_1 are known. The optimal control problem is to find an optimal control curve $(u^*(x))_{x\in[x_0,x_1]}$ that maximizes the functional (8.2) subject to ODE constraint (8.1) and, possibly additional information for the state variables at the boundary values for the independent variable.

In other words: the bounds of the domain X are known and the limits of the curves $y \in \mathrm{Y}$, traced out by $y(x)$ for $x \in \mathrm{X}$, may be known or may be chosen optimally.

Formally, the problem is

$$\max_{u(\cdot)} \int_{x_0}^{x_1} F(x, y(x), u(x))\, dx$$
$$\begin{aligned} &\text{subject to} \\ &y' = G(y(x), u(x), x), \text{ for } x \in [x_0, x_1] \\ &x_0 \text{ and } x_1 \text{ given} \\ &\text{conditions on } y(x_0) \text{ and } y(x_1) \end{aligned} \tag{P1}$$

We can consider the following cases:

(a) both boundary values are known: $y(x_0) = y_0$ and $y(x_1) = y_1$ fixed; (P1a)

(b) the lower boundary value is known: $y(x_0) = y_0$ fixed and $y(x_1)$ free; (P1b)

(c) the upper boundary value is known: $y(x_0)$ free and $y(x_1) = y_1$ fixed; (P1c)

(d) both boundary values are free: $y(x_0)$ and $y(x_1)$ free;. (P1d)

Proposition 1. ***[First order necessary conditions for fixed boundary values of the independent variable]*** *Let (y^*, u^*) be a solution (curve) to the OC problem* (P1) *in which one of the conditions* (P1a), *or* (P1b), *or* (P1c) *or* (P1d) *is introduced. Then there is a piecewise continuous function $\lambda : [x_0, x_1] \to \mathbb{R}$, called co-state variable, such that the curves (y^*, u^*, λ) satisfy the following conditions:*

- *the optimality condition* [1]*:*

$$H_u(x, y^*(x), u^*(x), \lambda(x)) = 0, \text{ for each } x \in [x_0, x_1] \tag{8.4}$$

- *the multiplier equation*

$$\lambda' = -H_y(x, y^*(x), u^*(x), \lambda(x)), \text{ for each } x \in (x_0, x_1) \tag{8.5}$$

- *the constraint of the problem:*

$$y^{*'}(x) = G(x, y^*(x), u^*(x)), \text{ for each } x \in (x_0, x_1) \tag{8.6}$$

- *and the adjoint conditions associated to the boundary conditions* (P1a) *to* (P1d)
  - *for problem* (P1a)

$$y^*(x_0) = y_0 \text{ for } x = x_0, \text{ and } y^*(x_1) = y_1 \text{ for } x = x_1, \tag{8.7}$$

[1] We use the notation $H_u(x, y(x), u(x), \lambda(x)) \equiv \frac{\partial H(x,y(x),u(x),\lambda(x))}{\partial u}$ is the derivative evaluated at point $x \in \mathrm{X}$ for any curves (y, u, λ) and $H_u(x, y^*(x), u^*(x), \lambda(x))$ is the derivative evaluated for the optimal curves (y^*, u^*). The derivatives for y are denoted in analogous way.

– *for problem* (P1b)

$$y^*(x_0) = y_0 \textit{ for } x = x_0, \textit{ and } \lambda(x_1) = 0 \textit{ for } x = x_1, \tag{8.8}$$

– *for problem* (P1c)

$$\lambda(x_0) = 0 \textit{ for } x = x_0, \textit{ and } y^*(x_1) = y_1 \textit{ for } x = x_1, \tag{8.9}$$

– *for problem* (P1d)

$$\lambda(x_0) = 0 \textit{ for } x = x_0, \textit{ and } \lambda(x_1) = 0 \textit{ for } x = x_1. \tag{8.10}$$

Proof. (Heuristic) Let $u^* = (u^*(x))_{x\in \mathrm{X}}$ be an optimal control curve and let $y^* = (y^*(x))_{x\in \mathrm{X}}$ be the associated state curve. The value of the problem is

$$\mathsf{J}[y^*, u^*] = \int_{x_0}^{x_1} F(x, y^*(x), u^*(x))\, dx.$$

The pair (y^*, u^*) is an optimiser if $\mathsf{J}[y^*, u^*] \geq \mathsf{J}[y, u]$ is satisfied for any other admissible pair of functions $(u(x), y(x))$.

It is convenient to write

$$\begin{aligned}
\mathsf{J}[y^*, u^*] &= \int_{x_0}^{x_1} F(x, y^*(x), u^*(x))dx = \\
&= \int_{x_0}^{x_1} \left[F(x, y^*(x), u^*(x)) + \lambda(x)(G(x, y^*(x), u^*(x)) - y^{*'}(x))\right] dx = \\
&= \int_{x_0}^{x_1} \left[H\Big(x, y^*(x), u^*(x), \lambda(x)\Big) - y^{*'}(x)\lambda(x)\right] dx
\end{aligned}$$

We introduce a perturbation on the optimal state-control pair $(y, u) = (y^*, u^*) + \varepsilon\, \boldsymbol{\eta}$, where ε is a constant and $\boldsymbol{\eta} = (\eta_y, \eta_u)$. The admissible perturbations differ for the different versions of the problem: for (P1a) we have $\eta_y(x_0) = \eta_y(x_1) = 0$, for (P1b) we have $\eta_y(x_0) = 0$ and $\eta_y(x_1) \neq 0$, for (P1c) we have $\eta_y(x_0) \neq 0$ and $\eta_y(x_1) = 0$, and for (P1d) we have $\eta_y(x_0) \neq 0$ and $\eta_y(x_1) \neq 0$.

The first-order Taylor approximation of the functional at (y^*, u^*) is

$$\mathsf{J}[y, u] = \mathsf{J}[y^*, u^*] + D_{\boldsymbol{\eta}(\cdot)}\ \mathsf{J}[y^*, u^*]\, \epsilon + o(\epsilon^2)$$

where the Gâteaux differential is

$$\begin{aligned}
D_{\boldsymbol{\eta}(\cdot)}\ \mathsf{J}[y^*, u^*] &= \int_{x_0}^{x_1} \Big\{\ H_u(x, y^*(x), u^*(x), \lambda(x))\, \eta_u(x) + H_y(x, y^*(x), u^*(x), \lambda(x))\, \eta_y(x) - \lambda(x)\eta_y'(x) \Big\}dx = \\
&= \int_{x_0}^{x_1} \Big\{H_u(x, y^*(x), u^*(x), \lambda(x))\, \eta_u(x) + (H_y(x, y^*(x), u^*(x), \lambda(x)) + \lambda'(x))\, \eta_y(x)\Big\}\ dx + \\
&+ \lambda(x_0)\eta_y(x_0) - \lambda(x_1)\eta_y(x_1).
\end{aligned}$$

Then $\mathsf{J}[y, u] \leq \mathsf{J}[y^*, u^*]$ only if $D_{\boldsymbol{\eta}(\cdot)}\ \mathsf{J}[y^*, u^*] = 0$, which, using similar arguments as in the case of the calculus of variations problem, is equivalent to the Pontriyagin's conditions: $H_u(.) = \lambda' - H_y(.) = 0$. The adjoint constraints should verify $\lambda(x_0)\eta_y(x_0) = \lambda(x_1)\eta_y(x_1) = 0$. From this and the admissibility values for $\eta_y(x_0)$ and $\eta_y(x_1)$ then the adjoint constraints are as in equations (8.7) to (8.10) $\square$

8.2.1 Constraints on the boundary values of the independent variable

In this subsection we consider the case in which one or both bounds in the domain of independent variables can be optimally chosen, i.e $x \in \mathrm{X}^* = [x_0^*, x_1^*]$, where one or both x_j^*, for $j = 0, 1$ are free, but the boundary values for the state variable are fixed: i.e. $y(x_0^*) = y_0$ and/or $y(x_1^*) = y_1$ are fixed. The optimal control problem is to find the optimal cut-off values for the independent variable, x_0^* and/or x_1^* and an optimal control $\big(u^*(x)\big)_{x \in [x_0^*, x_1^*]}$ that maximizes the functional (8.2) subject to ODE constraint (8.1).

In other words: the bounds of the domain X^* can be known or can be chosen optimally while the limits of the curves $y \in \mathrm{Y}$, traced out by $y(x)$ for $x \in \mathrm{X}^*$ are known.

Formally, the problem is

$$\begin{aligned}
\max_{u(\cdot)} \int_{x_0}^{x_1} & F(x, y(x), u(x))\, dx \\
& \text{subject to} \\
& y' = G(y(x), u(x), x), \text{ for } x \in [x_0, x_1] \\
& y(x_0) = y_0 \text{ and } y(x_1) = x_1 \text{ given} \\
& \text{conditions on } x_0 \text{ and } x_1
\end{aligned} \tag{P2}$$

We can consider the following cases:

$$\text{(a) both cut-offs are known: } x_0 \text{ and } \ x_1 \text{ fixed;} \tag{P2a}$$

$$\text{(b) the lower cut-off is known: } x_0 \text{ fixed and } \ x_1 \text{ free;} \tag{P2b}$$

$$\text{(c) the upper cut-off is known: } x_1 \text{ free and } \ x_1 \text{ fixed;} \tag{P2c}$$

$$\text{(d) both cut-offs are free: } x_0 \text{ and } \ x_1 \text{ free;.} \tag{P2d}$$

Proposition 2 (First order necessary conditions for free domain and fixed boundary state variable optimal control problems). *Let (y^*, u^*) be a solution curve to the OC problem* (P2) *where $y(x_0) = y_0$ and $y(x_1) = y_1$ are fixed. Then there is an optimal domain for the independent variable $x^* = [x_0^*, x_1^*] \subset \mathbb{R}$, a piecewise continuous function $\lambda : x^* \to \mathbb{R}$, called co-state variable, such that (y^*, u^*, λ) satisfy the optimality condition* (8.4)*, the multiplier equation* (8.5) *and the ODE constraint of the problem* (8.6)*, all for $x \in Int(\mathrm{X}^*)$ and the adjoint conditions associated to the boundary conditions* (P2a) *to* (P2d)

- *for problem* (P2a) $y^*(x_0) = y_0$ *and* $y^*(x_1^*) = y_1$ *and* x_0 *and* x_1 *are fixed;*
- *for problem* (P2b) $y^*(x_0) = y_0$ *and* $y^*(x_1^*) = y_1$ *and*

$$x_0^* = x_0 \textit{ and } \ H(x_1^*, y_1, u^*(x_1^*)) - {y^*}'(x_1^*)\lambda(x_1^*) = 0; \tag{8.12}$$

- *for problem* (P2c) $y^*(x_0^*) = y_0$ *and* $y^*(x_1) = y_1$ *and*

$$H(x_0^*, y_0, u^*(x_0^*)) - y^{*'}(x_0^*)\lambda(x_0^*) = 0 \text{ and } x_1^* = x_1; \tag{8.13}$$

- *for problem* (P2d) $y^*(x_0^*) = y_0$ *and* $y^*(x_1^*) = y_1$ *and*

$$H(x_0^*, y_0, u^*(x_0^*)) - y^{*'}(x_0^*)\lambda(x_0^*) = 0 \text{ and } H(x_1^*, y_1, u^*(x_1^*)) - y^{*'}(x_1^*)\lambda(x_1^*) = 0. \tag{8.14}$$

Proof. In this case in addition to the optimal pair (y^*, u^*) we have to find optimal boundary values for the independent variables, x_0^* and x_1^*. Using the same method as in the previous proof, but introducing additional perturbations for the terminal values of the independent variables, $x_j = x_j^* + \varepsilon\,\chi_j$ for $j \in \{0, 1\}$, [2] the Gâteaux differential of the value functional becomes

$$\begin{aligned} D_{(\eta(\cdot),\chi)}\mathsf{J}[y^*, u^*; x^*] = &\int_{x_0^*}^{x_1^*} \big(H_u(x, y^*(x), u^*(x), \lambda(x))\eta_u(x) + H_y(x, y^*(x), u^*(x), \lambda(x))\eta_y(x) - \lambda(x)\eta_y'(x)\big)\, dx + \\ &+ H(x, y^*(x), u^*(x), \lambda(x))\big|_{x=x_1^*}\, \chi_1 - H(x, y^*(x), u^*(x), \lambda(x))\big|_{x=x_0^*}\, \chi_0. \end{aligned}$$

Denoting $H^*(x) = H(x, y^*(x), u^*(x), \lambda(x))$, and integrating by parts, yields

$$\begin{aligned} D_{(\eta(\cdot),\chi)}\mathsf{J}[y^*, u^*; x^*] \ = &\int_{x_0^*}^{x_1^*} \big(H_u^*(x)\eta_u(x) + \big(H_y^*(x) + \lambda'(x)\big)\,\eta_y(x)\big)\, dt + \lambda(x_0^*)\eta_y(x_0^*) - \lambda(x_1^*)\eta_y(x_1^*) + \\ &+ H^*(x_1^*)\chi_1 - H^*(x_0^*)\chi_0. \end{aligned}$$

Using the approximation $\eta(x_j) \approx \eta(x_j^*) + y'(x_j^*)\,\chi_j$, for $j = 0, 1$, [3] yields

$$\begin{aligned} D_{(\eta(\cdot),\chi)}\mathsf{J}[y^*, u^*; x^*] = &\int_{x_0^*}^{x_1^*} \big(H_u^*(x)\eta_u(x) + \big(H_y^*(x) + \lambda'(x)\big)\,\eta_y(x)\big)\, dt + \lambda(x_0^*)\eta_0 - \lambda(x_1^*)\eta_1 + \\ &+ \Big(H^*(x_1^*) - y^{*'}(x_1^*)\lambda(x_1^*)\Big)\,\chi_1 - \Big(H^*(x_0^*) - y^{*'}(x_0^*)\lambda(x_0^*)\Big)\,\chi_0 \end{aligned} \tag{8.15}$$

The adjoint necessary conditions for the optimum, because $\eta_1 = \eta_0 = 0$, are presented, for the different versions of the problem, in equations (8.12) to (8.14). □

8.2.2 A taxonomy for optimal control problems

This is a general case that encompasses combinations of all the previous cases: we assume both the domains of the independent variables and the boundary values of the state variables are free. That is x_0 and/or x_1 can be fixed or free and $y(x_0)$ and/or $y(x_1)$ can be fixed or free.

When there is a free boundary condition, for the independent variable x or for the state variable $y(x)$, it should be optimized. In the first case, the optimal control problem is to find the optimal cut-off values for the indepdendent variable, x_0^* and/or x_1^* and an optimal control $(u^*(x))_{t\in[x_0^*, x_1^*]}$ that maximizes the functional (8.2) subject to ODE constraint (**??**) and having fixed or free boundary

[2] For more details see the proofs of propositions **??** and 1, in chapter **??**.

[3] As in the proof of Proposition **??** in chapter **??**, and analogous to equation (**??**).

values for the state variable. In the second case, the optimal control problem is to find the optimal boundary values for the state variable , $y^*(x_0)$ and/or $y^*(x_1)$ and an optimal control $(u^*(x))_{t\in[x_0,x_1]}$ that maximizes the functional (8.2) subject to ODE constraint (**??**) and having fixed or free cut-off values for the independent variable.

The necessary conditions include the optimality condition (8.4), the multiplier equation (8.5) and the ODE constraint of the problem (8.6), all for $x \in \text{int}(x^*)$. To get the adjoint condition associated to the terminal values of the state variable, when they need to be optimized, are obtained by setting in equation (8.15), $\eta_0 \neq 0$ and $\eta_1 \neq 0$. Therefore, the adjoint condition associated to $y^*(x_j^*)$ and $\lambda(x_j^*) = 0$, implying that the adjoint condition associated to the optimal boundary value of the independent variable, x_j^* is $H^*(x_j^*) = 0$, for $j = 0, 1$.

The adjoint conditions presented in Table 8.1 cover the 16 possible cases and it is the analogue to Table **??** for the calculus of variations problem.

Table 8.1: Adjoint conditions for bounded domain OC problems

data		optimum	
x_j	$y(x_j)$	x_j^*	$y^*(x_j^*)$
fixed	fixed	x_j	y_j
fixed	free	x_j	$\lambda(x_j) = 0$
free	fixed	$H(x_j^*, y_j, u^*(x_j^*)) - y^{*'}(x_j^*)\lambda(x_j^*) = 0$	y_j
free	free	$H(x_j^*, y^*(x_j^*), u^*(x_j^*)) = 0$	$\lambda(x_j^*) = 0$

The index j refers to the lower boundary when $j = 0$ and to the upper boundary when $j = 1$, y_j and x_j refer to the cases when the values are fixed, and y_j^* and x_j^* when they are optimally determined.

8.2.3 Example

8.3 Extension: integral constraints

In this section we present an optimal control problem in which there is a integral constraint of the iso-perimetric type, that is a constraint involving the integration of a known function of the state, and or control variables for all the domain of the independent variable. This case should not be confused to other types of optimal control problems in which there are integral constraints only for a sub-domain of the independent variable. While in the case we deal here, the constraint is of dimension zero (it is a scalar) in the second case the constraint is infinite-dimensional. This means that while in the case we address here we associate an adjoint variable, in the second case we to introduce an adjoint function.

Let the the independent variable be $x \in \mathrm{X} \subseteq \mathbb{R}$, where $\mathrm{X} = [x_0, x_1]$, the state variable be $y : \mathrm{X} \to\to \mathbb{R}$, and the control variable be $u : \mathrm{X} \to \mathbb{R}$.

We consider the following constraints

$$\int_{x_0}^{x_1} G_0\left(x, y(x), u(x)\right) dx \leq \bar{G} \tag{8.16a}$$

$$\frac{dy(x)}{dx} = G_1\left(x, y(x), u(x)\right)\; x \in \mathrm{X} \tag{8.16b}$$

$$x_0, x_1, y(x_0), y(x_1) \text{ free} \tag{8.16c}$$

where $\bar{G}$ is a constant.

The problem is

$$\begin{aligned}\max_{x_0, x_1, u(\cdot)} & \int_{x_0}^{x_1} F(x, y(x), y^{'}(x))dx \\ & \text{subject to} \\ & \int_{x_0}^{x_1} G_0(x, y(x), u(x))dx \leq \bar{G} \\ & \frac{dy(x)}{dx} = G_1(x, y(x), u(x)),\; x \in \mathrm{X} \\ & x_0, x_1, y(x_0), y(x_1) \text{ free}\end{aligned} \tag{ICP}$$

This problem optimal control problem has one functional constraint of the iso-perimetric type, (8.16a), one ordinary differential equation constraint, (8.16b), and has free initial and terminal indices and free initial and terminal values for the state variable .

There are several versions for this problem. For instance: (1) the simplest problem is the one in which x_0, x_1, $y(x_0)$ and $y(x_1)$ are fixed; (2) the free terminal problem which is common in optimal control problems in which the index variable is time in which x_0 and $y(x_0)$ are known and x_1 and $y(x_1)$ are free; (3) a problem in which the limit values of the indices, x_0 and x_1, are fixed and the state values, $y(x_0)$ and $y(x_1)$, are free; or (4) a problem in which the limit values of the indices, x_0 and x_1, are free and the state values, $y(x_0)$ and $y(x_1)$, are fixed.

We define the Hamiltonian function

$$H\left(x, y, u, \lambda_0, \lambda_1\right) = F\left(x, y, u\right) - \lambda_0 G_0\left(x, y, u\right) + \lambda_1 G_1\left(x, y, u\right)$$

where λ_0 is a constant, and λ_1 is a mapping $\lambda_1 : \mathrm{X} \to \mathbb{R}$. The maximized Hamiltonian, is

$$H^*(x) = H(x, y^*(x), y^*(x), \lambda_0, \lambda_1(x)) = \max_{u(\cdot)}\; H\left(x, y(x), u(x), \lambda_0, \lambda_1(x)\right), \text{ for each }\; x \in \mathrm{X}.$$

Proposition 3. ***[First order necessary conditions for the integral-constrained optimal control problem]*** *Let* (y^*, u^*) *be a solution to the OC problem* (ICP)*. Then there is a variable* λ_0 *and a piecewise continuous function* $\lambda : \mathrm{X} \to \mathbb{R}$ *, such that* $(y^*, u^*, \lambda_0, \lambda_1)$ *satisfy the following conditions:*

- *the optimality condition:*
$$H_u^*(x) = 0,\ \textit{for}\ x \in [x_0^*, x_1^*] \tag{8.17}$$

- *the multiplier equation*
$$\lambda_1^{'}(x) + H_y^*(x) = 0,\ \textit{for}\ x \in [x_0^*, x_1^*] \tag{8.18}$$

- *initial and terminal conditions associated to the independent and state variables*
$$\lambda_1(x_j^*) = 0,\ \textit{for}\ j = 0, 1 \tag{8.19a}$$
$$H^*(x_j^*) - \lambda_j(x_j^*)\, y^{*'}(x_j^*) = 0,\ \textit{for}\ j = 0, 1 \tag{8.19b}$$

- *for admissible solutions, i.e., satisfying*
$$\int_{x_0^*}^{x_1^*} G_0\left(x, y^*(x), u^*(x)\right) dx = \bar{G} \tag{8.20a}$$
$$y^{*'}(x) = G_1\left(x, y^*(x), u^*(x)\right)\ x \in (x_0^*, x_1^*) \tag{8.20b}$$

Proof. The value functional at the optimum is
$$\mathsf{J}^* \ = \int_{x_0^*}^{x_1^*} F\left(x, y^*(x), u^*(x)\right) dx. \tag{8.21}$$

Equivalently, substituting the definition of the Hamiltonian and using integration by parts
$$\begin{aligned}\mathsf{J}^* &= \int_{x_0^*}^{x_1^*} \Big(H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)\right) - y^{*'}(x)\Big)\, dx + \lambda_0 \bar{G} \\ &= \int_{x_0^*}^{x_1^*} \Big(H(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)) + \lambda_1^{'}(x) y^*(x)\Big)\, dx + \lambda_1(x_1^*)\, y^*\left(x_1^*\right) - \lambda_1\left(x_0^*\right) y^*\left(x_0^*\right) + \lambda_0\, \bar{G}\end{aligned}$$

Introduce the arbitrary (functional) perturbations $y^*(x) \to y(x) = y^*(x) + \varepsilon\eta_y(x)$, $u^*(x) \to u(x) = u^*(x) + \varepsilon\eta_u(x)$, and the (point) perturbations $x_t^* \to x_t = x_t^* + \varepsilon\chi_t$, for $t = 0, 1$ and $y_t^* \to y_t = y_t^* + \varepsilon\iota_t$, for $t = 0, 1$, such that
$$\iota(x_t^*) = \iota_t - y^{'}(x_t^*)\, \chi_t,\ t = 0, 1 \tag{8.22}$$

At the optimum $\delta\mathsf{J}[y^*, u^*] = 0$ where the variational derivative is
$$\delta\mathsf{J}[y^*, u^*] = \lim_{\epsilon \to 0} \frac{\Delta\mathsf{J}}{\epsilon}$$

where $\Delta\mathsf{J} = \mathsf{J}[y^* + \varepsilon\iota, u^* + \varepsilon\eta_u] - \mathsf{J}[y^*, u^*]$. Using derivations from the previous problems we find
$$\begin{aligned}\Delta\mathsf{J}[y, u] = \int_{x_0^*}^{x_1^*} &\Big[H\left(x, y^*(x) + \varepsilon\eta_y(x), u^*(x) + \varepsilon\eta_u(x), \lambda_0, \lambda_1(x)\right) - H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)\right) + \\ &+ \lambda_1^{'}(x)\left(y^*(x) + \varepsilon\eta_y(x) - y^*(x)\right)\Big]\ dx + \\ &+ \lambda_1(x_1^*)\left(y^*(x_1^*) + \varepsilon\eta_y(x_1^*)\right) - \lambda_1(x_0^*)\left(y^*(x_0^*) + \varepsilon\eta_y(x_0^*)\right) - \lambda_1(x_1^*) y^*(x_1^*) + \lambda_1(x_0^*) y^*(x_0^*) + \\ &+ \Big(H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)\right)\big|_{x = x_1^*}\Big)\, \chi_1 - \Big(H\left(x, y^*(x), u^*(x), \lambda_0, \lambda_1(x)\right)\big|_{x = x_0^*}\Big)\, \chi_0\end{aligned}$$

Using a first-order Taylor approximation and equation (8.22), collecting terms, factoring out and simplifying the notation we have,

$$\begin{aligned}
\delta \mathsf{J}[y,u] \; &= \int_{x_0^*}^{x_1^*} \left[\, H_u^*(x)\eta_u(x) + \left(H_y^*(x) + \lambda_1^{'}(x)\right)\eta_y(x)\right]\; dx + \lambda_1(x_1^*)\eta_y(x_1^*) - \lambda_1(x_0^*)\eta_y(x_0^*) + \\
&+ H^*(x_1^*)\chi_1 - H^*(x_0^*)\chi_0 \; = \\
&= \int_{x_0^*}^{x_1^*} \left[\, H_u^*(x)\eta_u(x) + \left(H_y^*(x) + \lambda_1^{'}(x)\right)\eta_y(x)\right]\; dx + \\
&+ \lambda_1(x_1^*)\iota_1 - \lambda_1(x_0^*)\iota_0 + \left(H^*(x_1^*) - \lambda_1(x_1^*)(y^*)^{'}(x_1^*)\right)\chi_1 - \left(H^*(x_0^*) - \lambda_1(x_0^*)(y^*)^{'}(x_0^*)\right)\chi_0,
\end{aligned}$$

at the optimum $\delta\mathsf{J}\;[y^*,u^*] = 0$ from which we derive equations (8.17)-(8.19b).

$\square$

Chapter 9

Introduction to optimal control: the maximum principle approach in the time domain

9.1 Introduction

If we can find an optimality criterium for the pointwise behavior of the state and control variables, say functions $y^*(x)$ and $u^*(x)$ for every $t \in \mathrm{T}$ (or $y^*(t)$ and $u^*(t)$ for every $t \in \mathrm{T}$) then we can trace-out the optimal curves $y^* = (y^*(x))_{t\in\mathrm{T}}$ and $u^* = (u^*(x))_{t\in\mathrm{T}}$.

In most applications in macroeconomics and growth theory the independent variable is time. In this case there is a rich characterisation of the solutions by seen them through the lens of dynamic systems theory. In particular, there is a closed connection between the existence and uniqueness of solutions and the fact they have a saddle-point structure.

In problems in which time is the independent variable we use instead t as the independent variable and $\mathrm{T} \subseteq \mathbb{R}_+$ as its domain, and we usually set $t_0 = 0$, and $t_1 = T$ finite or $t_1 = \infty$. Again, the optimal control problem features two variables we need to find: the **state variable**, $y(t)$, and the **control variable**, $u(t)$. We consider again only problems in which the state variable is of dimension one, the state variable is a mapping $y : \mathrm{T} \to \mathrm{Y} \subseteq R$ and the control variable is a mapping $u : \mathrm{T} \to \mathrm{U} \subseteq R^m$, where $m \geq 1$, and it is important to distinguish between the **point-wise level** of variables, i.e $y(t')$ and $u(')$, for a specific $t = t'$, from the **paths or trajectories** if the time-domain $y = (y(t))_{t\in\mathrm{T}}$ and $u = (u(t))_{t\in\mathrm{T}}$.

A solution to an optimal control problem allows for finding optimal trajectories, y^* and u^*, traced out in a interval according to one criterium (a functional) and given some constraints. The constraints can be be specified point-wise for all the domain of the independent variable, can be specified in particular points (usually boundary points in T), or for all the domain (we will see this case in the next chapter).

Formally, the optimal control problem consists in finding functions $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, where $\mathcal{Y} \in C^1(\mathbb{R})$, the set of continous and continuously differentiable functions $y : \mathrm{T} \to \mathrm{Y} \subseteq \mathbb{R}$, and $\mathcal{U} \in PC^1(\mathbb{R})$, the set of piecewise continuous functions $u : \mathrm{T} \to U \subseteq \mathbb{R}^m$ such that

$$\dot{y} = G(y(t), u(t), t), \text{ for } t \in [t_0, t_1] \tag{9.1}$$

that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_{t_0}^{t_1} F(t, y(t), u(t))dt \tag{9.2}$$

with additional data, for the aggregate curve or point-wise on the boundaries of sets T and Y. The additional data is related to the information concerning the boundary values of the independent variable t_0 and t_1 and the boundary values for the state variable $y(t_0)$ and $y(t_1)$.

The necessary conditions for an optimum according to the **Pontriyagin's maximum principle** are set by using the **Hamiltonian** function, defined as

$$H(t, y, u, \lambda) = F(t, y, u) + \lambda\, G(t, y, u).$$

where λ is a piecewise continuous function $\lambda : \mathrm{T} \to \mathbb{R}$.

Next we present the optimality conditions for a bounded domain, not necessarily time, in section **??** and an example in section **??**. Then we move to the time domain particular problems in section 9.3 and present several economic applications.

9.2 Constraints on boundaries states and times

Just for completeness, in this section we deal with the cases analogous to section 8.2 of chapter 8. We skip proofs as they were already provides in that chapter.

9.2.1 Constraints on the boundary values of the state variables

In this subsection we assume that the data of the problem includes the boundary values for the independent variable: i.e., t_0 and t_1 are known. The optimal control problem is to find an optimal control curve $(u^*(t))_{t \in [t_0, t_1]}$ that maximizes the functional (9.2) subject to ODE constraint (9.1) and, possibly additional information for the state variables at the boundary values for the independent variable.

In other words: the bounds of the domain T are known and the limits of the curves $y \in \mathrm{Y}$, traced out by $y(t)$ for $t \in \mathrm{T}$, may be known or may be chosen optimally.

Formally, the problem is

$$\max_{u(\cdot)} \int_{t_0}^{t_1} F(t, y(t), u(t))\, dt \tag{P1}$$
$$\begin{aligned}
&\text{subject to} \\
&\dot{y} = G(y(t), u(t), t), \text{ for } t \in [t_0, t_1] \\
&t_0 \text{ and } t_1 \text{ given} \\
&\text{conditions on } y(t_0) \text{ and } y(t_1)
\end{aligned}$$

We can consider the following cases:

(a) both boundary values are known: $y(t_0) = y_0$ and $y(t_1) = y_1$ fixed; (P1a)

(b) the lower boundary value is known: $y(t_0) = y_0$ fixed and $y(t_1)$ free; (P1b)

(c) the upper boundary value is known: $y(t_0)$ free and $y(t_1) = y_1$ fixed; (P1c)

(d) both boundary values are free: $y(t_0)$ and $y(t_1)$ free; . (P1d)

Proposition 1. *[**First order necessary conditions for fixed boundary values of the independent variable**] Let (y^*, u^*) be a solution (curve) to the OC problem* (P2) *in which one of the conditions* (P2a)*, or* (P2b)*, or* (P2c) *or* (P2d) *is introduced. Then there is a piecewise continuous function $\lambda : [t_0, t_1] \to \mathbb{R}$, called co-state variable, such that the curves (y^*, u^*, λ) satisfy the following conditions:*

- *the optimality condition* [1]*:*

$$H_u(t, y^*(t), u^*(t), \lambda(t)) = 0, \text{ for each } t \in [t_0, t_1] \tag{9.4}$$

- *the multiplier equation*

$$\dot{\lambda} = -H_y(t, y^*(t), u^*(t), \lambda(t)), \text{ for each } t \in (t_0, t_1) \tag{9.5}$$

- *the constraint of the problem:*

$$\dot{y}^* = G(t, y^*(t), u^*(t)), \text{ for each } t \in (t_0, t_1) \tag{9.6}$$

- *and the adjoint conditions associated to the boundary conditions* (P2a) *to* (P2d)
  - *for problem* (P2a)

$$y^*(t_0) = y_0 \text{ for } t = t_0, \text{ and } y^*(t_1) = y_1 \text{ for } t = t_1, \tag{9.7}$$

[1]We use the notation $H_u(x, y(t), u(t), \lambda(x)) \equiv \frac{\partial H(x,y(t),u(t),\lambda(x))}{\partial u}$ is the derivative evaluated at point $t \in \mathrm{T}$ for any curves (y, u, λ) and $H_u(x, y^*(x), u^*(x), \lambda(x))$ is the derivative evaluated for the optimal curves (y^*, u^*). The derivatives for y are denoted in analogous way.

– *for problem* (P2b)

$$y^*(t_0) = y_0 \; for\, t = t_0, \;\; and \;\; \lambda(t_1) = 0 \; for\, t = t_1, \tag{9.8}$$

– *for problem* (P2c)

$$\lambda(t_0) = 0 \; for\, t = t_0, \;\; and \;\; y^*(t_1) = y_1 \; for\, t = t_1, \tag{9.9}$$

– *for problem* (P2d)

$$\lambda(t_0) = 0 \; for\, t = t_0, \;\; and \;\; \lambda(t_1) = 0 \; for\, t = t_1. \tag{9.10}$$

9.2.2 Constraints on the boundary values of the independent variable

In this subsection we consider the case in which one or both bounds in the domain of independent variables can be optimally chosen, i.e $t \in \mathrm{T}^* = t_0^*, t_1^*]$, where one or both t_j^*, for $j = 0, 1$ are free, but the boundary values for the state variable are fixed: i.e. $y(t_0^*) = y_0$ and/or $y(t_1^*) = y_1$ are fixed. The optimal control problem is to find the optimal limit values for the indepdendent variable, t_0^* and/or t_1^* and an optimal control $\left(u^*(t)\right)_{t \in [t_0^*, t_1^*]}$ that maximizes the functional (9.2) subject to ODE constraint (9.1).

In other words: the bounds of the domain T^* can be known or can be chosen optimally while the limits of the curves $y \in \mathrm{Y}$, traced out by $y(t)$ for $t \in \mathrm{T}^*$ are known.

Formally, the problem is

$$\begin{aligned} \max_{u(\cdot)} \int_{t_0}^{t_1} & F(t, y(t), u(t))\, dt \\ & \text{subject to} \\ & y' = G(y(t), u(t), t), \text{ for } t \in [t_0, t_1] \\ & y(t_0) = y_0 \text{ and } y(t_1) = t_1 \text{ given} \\ & \text{conditions on}\, t_0 \text{ and } t_1 \end{aligned} \tag{P2}$$

We can consider the following cases:

$$\begin{aligned} &\text{(a) both limits are known: } t_0 \text{ and } \; t_1 \text{ fixed;} && \text{(P2a)} \\ &\text{(b) the lower limit is known: } t_0 \text{ fixed } \text{ and } \; t_1 \text{ free;} && \text{(P2b)} \\ &\text{(c) the upper limit is known: } t_1 \text{ free } \text{ and } \; t_1 \text{ fixed;} && \text{(P2c)} \\ &\text{(d) both limits are free: } t_0 \text{ and } \; t_1 \text{ free;}\,. && \text{(P2d)} \end{aligned}$$

Proposition 2 (First order necessary conditions for free domain and fixed boundary state variable optimal control problems). *Let* (y^*, u^*) *be a solution curve to the OC problem* (P2) *where* $y(t_0) = y_0$ *and* $y(t_1) = y_1$ *are fixed. Then there is an optimal domain for the independent variable* $t^* = [t_0^*, t_1^*] \subset \mathbb{R}$*, a piecewise continuous function* $\lambda : t^* \to \mathbb{R}$ *, called co-state variable, such that* (y^*, u^*, λ) *satisfy the optimality condition* (9.4)*, the multiplier equation* (9.5) *and the ODE constraint of the problem* (9.6)*, all for* $t \in Int(\mathrm{T}^*)$ *and the adjoint conditions associated to the boundary conditions* (P2a) *to* (P2d)

- *for problem* (P2a) $y^*(t_0) = y_0$ *and* $y^*(t_1^*) = y_1$ *and* t_0 *and* t_1 *are fixed;*
- *for problem* (P2b) $y^*(t_0) = y_0$ *and* $y^*(t_1^*) = y_1$ *and*
$$t_0^* = t_0 \ \textit{and} \ \ H(t_1^*, y_1, u^*(t_1^*)) - \dot{y}^*(t_1^*)\lambda(t_1^*) = 0; \tag{9.12}$$
- *for problem* (P2c) $y^*(t_0^*) = y_0$ *and* $y^*(t_1) = y_1$ *and*
$$H(t_0^*, y_0, u^*(t_0^*)) - \dot{y}^*(t_0^*)\lambda(t_0^*) = 0 \ \textit{and} \ \ t_1^* = t_1; \tag{9.13}$$
- *for problem* (P2d) $y^*(t_0^*) = y_0$ *and* $y^*(t_1^*) = y_1$ *and*
$$H(t_0^*, y_0, u^*(t_0^*)) - \dot{y}^*(t_0^*)\lambda(t_0^*) = 0 \ \textit{and} \ \ H(t_1^*, y_1, u^*(t_1^*)) - \dot{y}^*(t_1^*)\lambda(t_1^*) = 0. \tag{9.14}$$

9.2.3 Summing up

This is a general case that encompasses combinations of all the previous cases: we assume both the domains of the independent variables and the boundary values of the state variables are free. That is t_0 and/or t_1 can be fixed or free and $y(t_0)$ and/or $y(t_1)$ can be fixed or free.

When there is a free boundary condition, for the independent variable t or for the state variable $y(t)$, it should be optimized. In the first case, the optimal control problem is to find the optimal limit values for the indepdendent variable, t_0^* and/or t_1^* and an optimal control $(u^*(t))_{t\in[t_0^*,t_1^*]}$ that maximizes the functional (9.2) subject to ODE constraint (9.1) and having fixed or free boundary values for the state variable. In the second case, the optimal control problem is to find the optimal boundary values for the state variable , $y^*(t_0)$ and/or $y^*(t_1)$ and an optimal control $(u^*(t))_{t\in[t_0,t_1]}$ that maximizes the functional (9.2) subject to ODE constraint (9.1) and having fixed or free limit values for the time interval.

The necessary conditions include the optimality condition (9.4), the multiplier equation (9.5) and the ODE constraint of the problem (9.6), all for $t \in \mathrm{int}(t^*)$. To get the adjoint condition associated to the terminal values of the state variable, when they need to be optimized, are obtained by setting in equation (8.15), $\eta_0 \neq 0$ and $\eta_1 \neq 0$. Therefore, the adjoint condition associated to $y^*(t_j^*)$ and $\lambda(t_j^*) = 0$, implying that the adjoint condition associated to the optimal boundary value of the independent variable, t_j^* is $H^*(t_j^*) = 0$, for $j = 0, 1$.

The adjoint conditions presented in Table 9.1 cover the 16 possible cases and it is the analogue to Table 8.1 for optimal control problems in any other domain.

Table 9.1: Adjoint conditions for bounded domain OC problems

data		optimum	
x_j	$y(x_j)$	t_j^*	$y^*(t_j^*)$
fixed	fixed	x_j	y_j
fixed	free	x_j	$\lambda(x_j) = 0$
free	fixed	$H(t_j^*, y_j, u^*(t_j^*)) - \dot{y}^*(t_j^*)\lambda(t_j^*) = 0$	y_j
free	free	$H(t_j^*, y^*(t_j^*), u^*(t_j^*)) = 0$	$\lambda(t_j^*) = 0$

The index refers to the lower boundary when $j = 0$ and to the upper boundary when $j = 1$

9.3 Specific time domain problems

Next we present two problems which are common when the independent variable is time: the constrained terminal state problem and the discounted infinite horizon problem. While the second is typical from time-domain problems, the first can also occur in general x-domain problems. If this is the case we can simply adapt the results from the previous section.

In both problem we take already presented objective functional and dynamic constraint, in equations (9.2) and (9.1), respectively.

9.3.1 Constrained terminal state problem

A common problem in macroeconomics is the following: the set of independent variables is known such as $t_0 = 0$ and $t_1 = \bar{t}$, the initial value of the state value is fixed, $y(0) = y_0$, the structure of the economy given by the ODE (**??**), the value functional is , and we assume that the terminal value for the state variable is constrained by $R(\bar{t}, y(\bar{t})) \geq 0$ where $y(\bar{t})$ is free. Our goal is to determine the optimal trajectories for the state variable y^* $\left(y^*(t)\right)_{t \in \mathrm{T}}$ and the u^* $\left(u^*(t)\right)_{t \in \mathrm{T}}$.

Formally the problem is

$$
\begin{aligned}
\max_{u(\cdot)} \int_0^{\bar{t}} & F(t, u(t), y(t))\, dt \\
& \text{subject to} \\
& \dot{y} = G(t, u(t), y), \text{ for } \ t \in \mathrm{T} \\
& \bar{t} \text{ given} \\
& y(0) = y_0 \text{ fixed} \\
& R(\bar{t}, y(\bar{t})) \geq 0,
\end{aligned}
\tag{Pt1}
$$

where $\mathrm{T} = [0, \bar{t}]$ and functions $R(\cdot)$, $F(\cdot)$, and $G(\cdot)$ are known.

The Hamiltonian function is

$$
H(t) = H(t, u, y, \lambda) \equiv F(t, u, y) + \lambda\, G(t, u, y), \text{ for } \ t \in \mathrm{T}
$$

where λ is called co-state (or adjoint) and is a piecewise continuous function $\lambda : \mathrm{T} \to \mathbb{R}$. We assume H to be continuous, and continuously differentiable, and except otherwise mentioned that $H_{uu}(t) \neq 0$ for every $t \in \mathrm{T}$.

Proposition 3. ***1st order necessary conditions for the constrained terminal value problem*** *Let (y^*, u^*) be the solution trajectories for problem Pt1. Then it satisfies*

- *the optimality condition*

$$H_u(t, u^*, (t) y^*(t), \lambda(t)) = 0, \text{ for each } t \in \mathrm{T}; \tag{9.15}$$

- *the multiplier equation*

$$\dot{\lambda} = \; - H_y(t, u^*(t), y^*(t), \lambda(t)) = 0, \text{ for each } t \in \mathrm{T}; \tag{9.16}$$

- *the transversality condition*

$$\lambda(\bar{t}) R(\bar{t}, y^*(\bar{t})) = 0, \tag{9.17}$$

and the admissibility conditions:

$$\dot{y}^* \; = \; G(t, u^*(t), y^*(t)) = 0, \text{ for each } t \in \mathrm{T}; \tag{9.18}$$

and

$$y^*(0) = y_0, \text{ for } t = 0 \tag{9.19}$$

Proof. In this case the value at the optimum is

$$\mathsf{J}[y^*, u^*] = \int_0^{\bar{t}} \left(H(t, u^*(t), y^*(t), \lambda(t)) - \dot{y}^*(t)\lambda(t) \right) \, dt + \psi R(\bar{t}, y(\bar{t}))$$

where ψ is a Lagrange multiplier. The functional derivative, for an arbitrary perturbation $(\delta y, \delta u) = \varepsilon(\eta_y, \eta_u)$ around (y^*, u^*), is now

$$\begin{aligned} \delta_{(\eta_y(\cdot), \eta_u(\cdot))} \mathsf{J}[y^*, u^*] = & \int_0^{\bar{t}} \Big[H_u(t, y^*(t), u^*(t), \lambda(t)) \eta_u(t) + (H_y(t, y^*(t), u^*(t), \lambda(t)) + \dot{\lambda}(t)) \eta_y(t) \Big] \, dt + \\ & + \lambda(0) \eta_y(0) + \left(\psi R_y(\bar{t}, y^*(\bar{t})) - \lambda(\bar{t}) \right) \eta_y(\bar{t}), \end{aligned}$$

where admissible perturbations satisfy $\eta_y(0) = 0$ and $\eta_y(\bar{t}) \neq 0$. Given the inequality constraint, the KKT conditions

$$R(\bar{t}, y^*(\bar{t})) \geq 0, \; \psi \geq 0, \; \psi R(\bar{t}, y^*(\bar{t})) = 0,$$

are also necessary for an optimum. Setting $H_u^*(t) = \dot{\lambda}\ (t) - H_y^*(t) = \eta_y(0) = 0$, as presented in conditions (9.15)-(9.16). At last, because $\eta_y(\bar{t}) \neq 0$, the remaining necessary condition for an optimum is $\psi R_y(\bar{t}, y^*(\bar{t})) - \lambda(\bar{t}) = 0$. Multiplying both terms by $R(\bar{t}, y^*(\bar{t}))$ and using the KKT condition yields condition (9.17). $\square$

9.3.2 Infinite horizon problems

Necessary conditions for an optimum

The benchmark problem in macroeconomics and growth theory is the (autonomous) **discounted infinite horizon problem** is

$$\begin{aligned} &\max_{u(\cdot)} \int_0^{\infty} e^{-\rho t} f(y(t), u(t)) dt \\ &\text{subject to} \\ &\dot{y} = g(y, u), \ \text{for} \ \ t \in \mathrm{T} \\ &y(0) = y_0 \text{ fixed} \\ &\text{boundary conditions at infinity.} \end{aligned} \tag{Pt2}$$

Two versions, related to different boundary conditions, are usually considered

$$\lim_{t \to \infty} y(t) \text{ is free} \tag{Pt2a}$$

$$\lim_{t \to \infty} R(t, y(t)) \geq 0 \tag{Pt2b}$$

where $\rho > 0$, and function $R(t, y)$ is known and takes the form of a solvability or sustainability condition. Observe that the utility function is $F(t, y(t), u(t)) \equiv e^{-\rho t} f(y(t), u(t))$ is directly dependent on time by a discount factor, which is a bounded function of time, and we consider a version of the problem in which the constraint ODE os autonomous.

For discounted optimal control problems define the **current-value Hamiltonian** function

$$\begin{aligned} h(y(t), u(t), q(t)) &= f(y(t), u(t)) + q(t)\, g(y(t), u(t)) = \\ &= e^{-\rho t}\, H(t, y(t), u(t), \lambda(t)). \end{aligned}$$

where $q(t) = e^{\rho t} \lambda(t)$ is the current-value co-state variable. Consistently with the previous definitions we call **discounted Hamiltonian** and **discounted co-state variable** to $H(t, y, u, \lambda)$ and λ, respectively. Again q (or λ) are piecewise continuous functions $q : \mathrm{T} \to \mathbb{R}$ (or $\lambda : \mathrm{T} \to \mathbb{R}$).

Observe the current-value Hamiltonian is time-independent. If the constraint is time-dependen, i.e, if $g(t, y, u)$ then Hamiltonian is also explicitly time dependent

$$h(t, y(t), u(t), q(t)) = f(y(t), u(t)) + q(t)\, g(t, y(t), u(t)).$$

Proposition 4 (**First order necessary conditions: Pontriyagin maximum principle**). *Let (y^*, u^*) be the optimal state and control trajectory pair. Then there is a PC^1 continuous co-state variable q such that the following conditions hold:*

- *the optimality condition*

$$h_u(y^*(t), u^*(t), q(t)) = 0, \ \textit{for each} \ \ t \in [0, \infty) \tag{9.21}$$

- *the multipliers equation for the current co-state variable (also called adjoint equation)*

$$\dot{q} = \rho q - h_y(y^*(t), u^*(t), q(t)), \ \textit{for each} \ \ t \in [0, \infty) \tag{9.22}$$

- *the transversality condition or* (Pt2b)*:*
  - *associated to* (Pt2a)

$$\lim_{t \to \infty} e^{-\rho t} q(t) = 0, \tag{9.23}$$

  - *associated to* (Pt2b)

$$\lim_{t \to \infty} e^{-\rho t} q(t) y^*(t) = 0; \tag{9.24}$$

- *and the admissibility conditions*

$$\dot{y}^* = g(y^*(t), u^*(t)), \ \textit{for each} \ \ t \in [0, \infty) \tag{9.25a}$$

$$y^*(0) = y_0, \ \textit{for} \ \ t = 0. \tag{9.25b}$$

Proof. We can see this proposition as a particular case of Propositions 1 and Proposition 3. □

Remark: if the constraint ODE is non-autonomous, i.e, if $\dot{y} = g(t, y(t), u(t))$ equations (9.22) and (9.25a) will become non-autonomous ODE's, because the current value Hamiltonian becomes (directly) time-dependent, i.e. $h(t) = h(t, y(t), u(t), q(t)) = f(y(t), u(t)) + q(t)\, g(t, y(t), u(t))$.

Sufficient conditions for an optimum

The nature of the transversality condition (9.24) is a difficult technical issue associated with the solution to infinite-horizon optimal control problems (see Michel (1982) and Kamihigashi (2001)).

From now on, we assume the **Arrow sufficiency condition**: $h^*_{uu} = h(y^*, u^*, q) \leq 0$. This condition guarantees that the first-order necessary conditions for an extremum, presented in Proposition 4, are also necessary.

9.4 The dynamics of optimal control problems

The infinite-horizon discounted problem is the central structure to macroeconomics and growth theory since the 1960's.

In several applications the constitutive functions $f(y, u)$ and $g(y, u)$ are specified explicitly, or, if they have a non-linear structure the resulting system of ODE's usually cannot be solved explicitly. However, we can have a geometric interpretation for the solution of an optimal control problem in regular cases.

Writing

$$h(y, u, q) = f(y, u) + q\, g(y, u),$$

the necessary (and possibly sufficient as well) conditions for the infinite-horizon discounted optimal control problem, presented in Proposition 4, can be compactly presented as a differential-algebraic system:

$$\begin{aligned} \dot{y} &= g(y,u) \\ \dot{q} &= \rho q - h_y(y,u,q). \\ 0 &= h_u(y,u,q) \end{aligned} \tag{9.26}$$

Observe that $h_q(y,u,q) = g(y,u)$, $h_{qy}(y,u,q) = h_{yq}(y,u,q) = g_y(y,u)$, and $h_{qu}(y,u,q) = h_{uq}(y,u,q) = g_u(y,u)$.

The (local) existence and uniqueness of a steady state can be assessed from the Jacobian of system (9.26) evaluated at $\dot{y} = \dot{q} = 0$, that if

$$\mathbf{F}(y,u,q) = \begin{pmatrix} g_y(y,u) & g_u(y,u) & 0 \\ -h_{yy}(y,u,q) & -h_{uy}(y,u,q) & \rho - g_y(y,u) \\ h_{yu}(y,u,q) & h_{uu}(y,u,q) & g_u(y,u) \end{pmatrix}.$$

The determinant of $\mathbf{F}$ is

$$\det(\mathbf{F})(y,u,q) = (g_y - \rho)\,(g_y\, h_{uu} - g_u\, h_{uy}) + g_u\,(g_u\, h_{yy} - g_y\, h_{yu})$$

where all the partial derivatives are evaluated at an arbitrary point (y,u,q). As $h_{yy}(y,u,q) = f_{yy}(y,u) + q\, g_{yy}(y,u)$, and if the functions are continuous $h_{yu}(y,u,q) = h_{yu}(y,u,q) = f_{yu}(y,u) + q\, g_{yu}(y,u) = f_{uy}(y,u) + q\, g_{uy}(y,u)$ and $h_{uu}(y,u,q) = f_{uu}(y,u) + q\, g_{uu}(y,u)$, we readily see that:

1. if $f(y,u)$ and $g(y,u)$ are linear functions both in the state and the control variable, that is in (y,u), all the second derivatives of $h_{(}\cdot,q)$ are equal to zero, then $\det(\mathbf{F})(y,u,q) = 0$, which implies that either a steady state does not exist, or there is an infinite number of steady states (and possibly a solution to the problem does not exist);

2. if h is linear in the control variable, that is if $h_{uu} = h_{yu} = h_{uy} = 0$ then $\det(\mathbf{F}) = -(g_u)^2\, h_{yy}$, where g_u is a constant, and a steady state can exist, if $h_{yy} \neq 0$;

3. in several models applied to macroeconomics, the objective function is independent of the state variable, that is $f = f(u)$ wthich implies $h_{uy} = q\, g_{uy}$ and $h_{yy} = 0$ In this case, we have

$$\det(\mathbf{F}) = (g_y - \rho)\,(g_y\, h_{uu} - g_u\, q\, g_{uy}) - g_u\, g_y\, q\, g_{yu}$$

   which does not rule out the existence of a steady state. However, if the constraint function is linear in u we cannot rule out that $\det(\mathbf{F}) = 0$. Indeed, this is the case in simple endogenous growth models and simple models for the representative household.

4. if functions $f(yu)$ and $g(y,u)$ are nonlinear, having locally $\det(\mathbf{F})(y,u,q) = 0$ provides a necessary condition for the existence of local bifurcations and of potentially complex dynamics in the solution of the optimal control problem. The number of cases is potentially enormous

and complex. In abstract we can say that any bounded trajectory converging to a steady state or a limit cycle is a candidate for optimality, although we cannot rule out the possibility of existence of multiple solutions.

If functions $f(.)$ and $g(.)$ are sufficiently smooth we may qualitative characterize the optimal dynamics of (y, q) (or for (y, u)).

The algebraic equation in system (9.26) allows us to determine uniquely the control variable. If $\partial^2 h/\partial u^2 \neq 0$, the implicit function theorem allows for obtaining from the optimality condition for u, $h_u(u, y, q) = 0$, an implicit representation of the control as a function of the state and co-state variables $u = U(y, q)$.

Assume that $h_{uu}(y, u, q) \neq 0$ for any $(y, u, q) \in \mathrm{Y} \times \mathrm{U} \times \mathrm{Q}$. Then, from the implicit function theorem, we can find uniquely the control as a function of the state and the co-state variables

$$u = U(y, q)$$

where, from the implicit function theorem

$$U_y = -\frac{h_{uy}}{h_{uu}}, \text{ and } U_q = -\frac{g_u}{h_{uu}}.$$

If we substitute this control representation in the differential equations of (9.26) we obtain the **modified Hamiltonian dynamic system** (MHDS) as a non-linear planar ODE,

$$\begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \mathbf{M}\ (y, q) \equiv \begin{pmatrix} g(y, U(y, q)) \\ \rho\, q - h_y(y, U(y, q), q) \end{pmatrix}. \tag{9.27}$$

The Jacobian of the MHDS is

$$\begin{aligned} D\mathbf{M}\ (y, q) &= \begin{pmatrix} \dfrac{\partial \dot{y}(y, q)}{\partial y} & \dfrac{\partial \dot{y}(y, q)}{\partial q} \\ \dfrac{\partial \dot{q}(y, q)}{\partial y} & \dfrac{\partial \dot{q}(y, q)}{\partial q} \end{pmatrix} \\ &= \begin{pmatrix} g_y(y, q) + g_u(y, q)\, U_y(y, q) & g_u(y, q)\, U_q(y, q) \\ -h_{yy}(y, q) - h_{yu}(y, q)\, U_y(y, q) & \rho - h_{yu}(y, q)\, U_q(y, q) - h_{yq}(y, q) \end{pmatrix}. \end{aligned}$$

Lemma 6. *Let $\rho > 0$ and assume that $h_{uu}(y, u, q) \neq 0$ for any $(y, u, q) \in \mathrm{Y} \times \mathrm{U} \times \mathrm{Q}$. Then the Jacobian $D\mathbf{M}\ (y, q)$ has trace$D\ \mathbf{M}\ (y, q) = \rho > 0$ for any point $(y, q) \in \mathrm{Y} \times \mathrm{Q}$.*

Proof. Has $g_u\, U_y = -g_u \dfrac{h_{uy}}{h_{uu}} = h_{uy}\ U_q = -h_{uy}\ \dfrac{g_u}{h_{uu}}$, we readily see that the Jacobian matrix has the following

$$D\mathbf{M}\ (y, q) = \begin{pmatrix} M_{11}(y, q) & M_{12}\ (y, q) \\ M_{21}(y, q) & \rho - M_{11}(y, q) \end{pmatrix}$$

Therefore trace $D\mathbf{M}\ (y, q) = M_{11}\ + \rho - M_{11} = \rho > 0$ $\square$

Observe that this result has a global nature. The only requirement is that $h_{uu} \neq 0$ globally. This assures that there is steady state and that there are no singularities [2]

Assume the MHDS has, at leat, one steady state, $(\bar{y}, \bar{q}) = \{(y,q) : \dot{y} = \dot{q} = 0\}$. In the neighbourhood of $(\bar{y}, \bar{q})$ we can approximate the non-linear MHDS (9.27) by the linear system

$$\begin{pmatrix} \dot{y}(t) \\ \dot{q}(t) \end{pmatrix} = D_{(y,q)}\mathbf{M}(\bar{y}, \bar{q}) \begin{pmatrix} y(t) - \bar{y} \\ q(t) - \bar{q} \end{pmatrix}$$

where the Jacobian, evaluated at the steady state $(\bar{y}, \bar{q})$ is the matrix of constants

$$D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q}) = \begin{pmatrix} \dfrac{\partial \dot{y}(\bar{y}, \bar{q})}{\partial y} & \dfrac{\partial \dot{y}(\bar{y}, \bar{q})}{\partial q} \\ \dfrac{\partial \dot{q}(\bar{y}, \bar{q})}{\partial y} & \dfrac{\partial \dot{q}(\bar{y}, \bar{q})}{\partial q} \end{pmatrix}.$$

If functions $f(.)$ and $g(.)$ have no singularities we can obtain a generic characterization of the dynamics of the MHDS, and, therefore, of the solution to the optimal control problem.

Proposition 5. *Let there be a steady state $(\bar{y}, \bar{q})$ for the MHDS system. This steady state can never be locally a stable node or focus. There is transitional dynamics converging to the steady state only if it is a saddle point, that is if* $\det(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})) < 0$.

Proof. Evaluating the Jacobian at a steady state $(\bar{y}, \bar{q})$, with $\bar{u} = u(\bar{y}, \bar{q})$, we find that the Jacobian matrix becomes a matrix of constants

$$D_{(y,q)}\mathbf{M}\ (\bar{y}, \bar{q}) = \begin{pmatrix} \bar{g}_y - \dfrac{\bar{g}_u \bar{h}_{uy}}{\bar{h}_{uu}} & -\dfrac{(\bar{g}_u)^2}{\bar{h}_{uu}} \\ -\bar{h}_{yy} + \dfrac{(\bar{h}_{uy})^2}{\bar{h}_{uu}} & \rho - \bar{g}_y + \dfrac{\bar{g}_u \bar{h}_{uy}}{\bar{h}_{uu}} \end{pmatrix}$$

where $\bar{g}_y = g(u(\bar{y}, \bar{q}), \bar{y})$, etc[3]. Observe that the Jacobian matrix has a particular structure

$$D_{(y,q)}\mathbf{M}(\bar{y}, \bar{q}) = \begin{pmatrix} a & b \\ c & \rho - a \end{pmatrix}. \tag{9.28}$$

impiying that the trace is equal to the rate of time preference,

$$\text{trace}\ (D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})) = \rho > 0 \tag{9.29}$$

and is always positive and

$$\det(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})) = a(\rho - a) - bc. \tag{9.30}$$

This implies that, if there is a steady state, it can never a stable node or focus. Therefore, it can be an unstable node or focus if $\det(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})) > 0$, a saddle-point if $\det(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})) < 0$ or a degenerate saddle node if $\det(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})) = 0$. There can only be transitional dynamics if it is a saddle-point. □

[2] In Brito et al. (2017) we deal with the case in which we can have locally $h_{uu} = 0$. This tends to make the solution to be non-unique locally for a subset of the space $\mathrm{Y} \times \mathrm{Q}$.

[3] because if $h(.)$ is continuous then $h_{uy}(.) = h_{yu}(.)$.

Then we can conclude the following:

1. in generic cases the equilibrium point $(\bar{y}, \bar{q})$ is a saddle point if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) < 0$, and:
   (a) the stable manifold associated with $(\bar{y}, \bar{q})$
   $$W^s = \{\ (y, q) \in \mathrm{Y} \times \mathrm{Q}\ \subseteq \mathbb{R}^2 : \lim_{t\to}(y(t), q(t)) = (\bar{y}, \bar{q})\}$$
   passing through point $y(0) = y_0$ **is the solution set of the OC problem**;
   (b) the solution to the OC problem is (at least locally) unique;
   (c) the optimal trajectory is asymptotically tangent to the stable eigenspace E^s associated to Jacobian $D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})$

2. if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) > 0$ the equilibrium point is unstable. In this case the candidate solutions tend to diverge away from the steady state. The existence of a solution depends on the satisfaction of the transversality condition (equation (9.25b)). If this is the case, the solution to the optimal control problem is **non-stationary**, that is, it will be asymptotically unbounded;

3. if $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) = 0$, there will be an infinite number of candidate steady states. The system is "anchored" however by the initial condition $y(0) = y_0$, which means that the solution to the optimal control problem exists, it is unique, but it is **stationary**;

A relatively common problem in economics, involve functions $f(y, u)$ and $g(y, u)$, in which g is a linear function and f is independent from the state variable, that is $f = f(u)$ with $f_{uu}\ < 0$ and $g(y, u) = a\, y + bu$. In this case, which is common in economics, the last two cases tend to occur:

1. the MHDS has the structure of matrix (9.28) with possibly the coefficients a, b and c linearly dependent upon y and/or q. In this case a steady state (unless at $y = u = 0$) and the MHDS displays unbounded growth. This means that evaluating the MHDS at the steady state the determinant (9.30) will be positive. The existence of a solution depends on the satisfaction of the transversality condition (equation (9.25b)). If this is the case, the solution to the optimal control problem is **non-stationary**, that is, it will be asymptotically unbounded;

2. the MHDS has the structure of matrix (9.28) with $a = \rho$ and $b = 0$. In this case, the $\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) = 0$ and there will be an infinite number of candidate steady states. The system is "anchored" however by the initial condition $y(0) = y_0$, which means that the solution to the optimal control problem exists, it is unique, but it is **stationary**;

Conclusion

Therefore, if there is a unique solution to the infinite-horizon discounted optimal control problem, and if the constitutive functions are monotonic, three types of solutions can occur:

1. a unique time varying solution converging to a steady state;
2. a stationary solution;
3. an time-varying unbounded solution which grows in time such that is satisfies the transversality condition.

A last message: be careful if your optimal control problem has linear constitutive functions.

9.5 Economic applications

We consider the same problems as in the calculus of variations section.

9.5.1 Two simple problems

Example 1: Resource depletion problem

The (non-renewable) resource depletion problem can now be solved by using the Pontriyagin's principle. Recall that the problem is

$$\max_{c(\cdot)} \int_0^{\infty} e^{-\rho t} \ln{(c(t))} dt,\ \rho > 0$$

subject to

$$\begin{cases} \dot{w}(t) = -c(t),\ t \in [0, \infty) \\ w(0) = w_0,\ \text{given} \\ \lim_{t\to\infty} w(t) \geq 0. \end{cases}$$

In this problem, the control variable is consumption, C, and the state variable is the remaining level of the resource, W. What is the best path for consumption-depletion ?

For applying the Pontryiagin maximum principle we write the current-value Hamiltonian

$$h = \ln{(c)} - q\, c.$$

The first order conditions are

$$\begin{aligned} c(t) &= 1/q(t) \\ \dot{q} &= \rho q(t) \\ \lim_{t\to\infty} e^{-\rho t} q(t) w(t) &= 0 \\ \dot{w} &= -c(t) \\ w(0) &= w_0 : \end{aligned}$$

and can be written as a planar differential equation in (w, c), together with the initial and the transversality condition is

$$\begin{aligned} \dot{w} &= -c(t) \\ \dot{c} &= -\rho c(t) \\ w(0) &= w_0 \\ \lim_{t\to\infty} e^{-\rho t}\frac{w(t)}{c(t)} &= 0 \end{aligned}$$

If we want to find the solution we must solve the system, together with the conditions on time.

There are several ways to solve it. Here is a simple one. First, define $z(t) \equiv w(t)/c(t)$. Time-differenting and substituting, we get the scalar terminal-value problem

$$\begin{cases} \dot{z} = -1 + \rho z \\ \lim_{t\to\infty} e^{-\rho t} z(t) = 0 \end{cases}$$

which has a constant solution $z(t) = \frac{1}{\rho}$ for every $t \in [0, \infty)$. Second, substitute $c(t) = w(t)/z(t) = \rho w(t)$. therefore,

$$\begin{cases} \dot{w} = -c(t) = -\rho w(t) \\ w(0) = w_0 \end{cases}$$

Then $w^*(t) = w_0 e^{-\rho t}$ for $t \in [0, \infty)$ and $c^*(t) = \rho w^*(t)$.

Characterization of the solution: there is asymptotic extinction

$$\lim_{t\to\infty} w^*(t) = 0,$$

at a speed given by the half-life of the process

$$\tau \equiv \left\{ t: \ w^*(t) = \frac{w(0) - w^*(\infty)}{2} \right\} = -\frac{\ln(1/2)}{\rho}$$

if $\rho = 0.02$ then $\tau \approx 34.6574$ years.

Example 2: the consumption-savings problem

Problem: find the functions $(a(t), c(t))$ pair that maximizes the functional

$$\max_{c(\cdot)} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\theta} - 1}{1-\theta} dt, \ \rho > 0$$

subject to

$$\begin{cases} \dot{a}(t) = y - c(t) + r\,a, \ t \in [0, \infty) \\ a(0) = a_0, \text{ given} \\ \lim_{t\to\infty} a(t)^{-r\,t} \geq 0. \end{cases}$$

In this problem, the control variable is consumption, c, and the state variable is the level of net wealth, a. The current value Hamiltonian is

$$h(a,c,q) = \frac{c^{1-\theta}-1}{1-\theta} + q\,(y - c + r\,a)$$

and the first order conditions according to the Pontiyagin's principle are

$$\begin{cases} c(t)^{-\theta} = q(t) \\ \dot{q} = q\,(\rho - r) \\ \dot{a} = y - c + r\,a \\ a(0) = a_0 \\ \lim_{t\to\infty} q(t)\,a(t)e^{-\rho t} = 0 \end{cases}$$

As

$$\frac{\dot{q}}{q} = -\theta\,\frac{\dot{c}}{c}$$

we can obtain the solution by solving the mixed initial-terminal value problem for ODE's

$$\begin{cases} \dot{a} = y - c + r\,a \\ \dot{c} = \gamma\,c \\ a(0) = a_0 \\ \lim_{t\to\infty} c(t)^{-\theta} a(t)\,e^{-\rho t} = 0 \end{cases}$$

where again $\gamma \equiv \dfrac{r-\rho}{\theta}$. We present and discuss next the solution to this problem.

9.5.2 Qualitatively specified problems

Next we present a general Ramsey (1928) model in which the behavioral functions are qualitatively specified. This allows us to study the qualitative solution to the optimal control problem.

The Ramsey problem is:

$$\begin{aligned} \max_{c(\cdot)} \int_0^\infty & e^{-\rho t} U(c(t))dt,\ \rho > 0, \\ & \text{subject to} \\ & \dot{k}(t) = F(k(t)) - c(t),\ t \in [0,\infty) \\ & k(0) = k_0 \text{ fixed} \\ & \lim_{t\to\infty} e^{-\rho t} k(t) \geq 0. \end{aligned}$$

We also assume that $(k,c) : \mathbb{R}_+ \to \mathbb{R}^2_+$. In this problem the control variable is c and the state variable is the stock of capital k.

The utility and the production functions, $u(c)$ and $F(k)$, are usually assumed to have the following properties: Increasing, concave and Inada :

$$U'(.) > 0,\ U''(.) < 0, F'(.) > 0,\ F''(.) < 0$$

$$U'(0) = \infty,\ U'(\infty) = 0,\ F'(0) = \infty,\ F'(\infty) = 0.$$

Although we do not have explicit utility and production functions we can still characterize the optimal consumption-accumulation process (we are using the Grobman-Hartmann theorem).

The current-value Hamiltonian is

$$h(c,k,q) = U(c) + q\,(F(k) - c)$$

The necessary (and sufficient) conditions according to Pontriyagin's maximum principle are

$$\begin{aligned}
U^{'}(c(t)) &= q(t) \\
\dot{q} &= q(t)\left(\rho - F^{'}(k(t))\right) \\
\lim_{t\to\infty} e^{-\rho t}\, q(t)\, k(t) &= 0 \\
\dot{k} &= F(k(t)) - c(t) \\
k(0) &= k_0
\end{aligned}$$

The MHDS and the initial and transversality conditions become

$$\begin{aligned}
\dot{k} &= F(k(t)) - c(t) \\
\dot{c} &= \frac{c(t)}{\theta(c(t))}\left(F^{'}(k(t)) - \rho\right) \\
k(0) &= k_0 > 0 \\
0 &= \lim_{t\to\infty} e^{-\rho t} U^{'}(c(t))k(t)
\end{aligned}$$

where $\theta(c) = -\frac{U^{''}(c)\,c}{U^{'}(c)} > 0$ is the inverse of the elasticity of intertemporal substitution.

The MHDS has no explicit solution (it is not even explicitly defined) : we can only use **qualitative methods**. They consist in:

- determining the steady state(s): $(\bar{c}, \bar{k})$
- characterizing the linearised dynamics (it is useful to build a phase diagram).

The steady state (if $k > 0$) is

$$\begin{aligned}
F^{'}(\bar{k}) &= \rho \Rightarrow \bar{k} = (F^{'})^{-1}(\rho) \\
\bar{c} &= F(\bar{k})
\end{aligned}$$

The linearized MHDS is

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \frac{\bar{c}}{\theta(\bar{c})} F''(\bar{k}) & 0 \end{pmatrix} \begin{pmatrix} k(t) - \bar{k} \\ c(t) - \bar{c} \end{pmatrix}$$

where we denote $D\mathbf{M}$ the Jacobian matrix. The jacobian J has trace and determinant:

$$\mathrm{tr}(D\mathbf{M}) = \rho, \ \det(D\mathbf{M}) = \frac{\bar{c}}{\theta(\bar{c})} F''(\bar{k}) < 0$$

the steady state $(\bar{c}, \bar{k})$ is a saddle point. The eigenvalues of $D\mathbf{M}$ are

$$\lambda_s = \frac{\rho}{2} - \sqrt{\Delta} \ < 0, \ \lambda_u = \frac{\rho}{2} + \sqrt{\delta} \ > \rho > 0$$

where the discriminant is

$$\Delta = \ \left(\frac{\rho}{2}\right)^2 - \frac{\bar{c}}{\theta(\bar{c})} F''(\bar{k}) > \left(\frac{\rho}{2}\right)^2.$$

and the eigenvector matrix of $D\mathbf{M}$ is

$$\mathbf{P} \ = (\mathbf{P}^s \mathbf{P}^u) = \begin{pmatrix} 1 & 1 \\ \lambda_u & \lambda_s \end{pmatrix}$$

Then the approximate solution for the Ramsey problem, in the neighbourhood of the steady state, is

$$\begin{pmatrix} k^*(t) \\ c^*(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \end{pmatrix} + k_0 \begin{pmatrix} 1 \\ \lambda_u \end{pmatrix} e^{\lambda_s t}, \ t \in [0, \infty) \tag{9.31}$$

Then the local stable manifold has slope higher than the isocline $\dot{k}(C, K) = 0$

$$\left.\frac{dc}{dk}\right|_{W^s} = \lambda_u > \left.\frac{dc}{dk}\right|_{\dot{k}} = F'(\bar{k}) = \rho$$

Geometrically (see figure 9.1) the **approximate** solution (9.31) belongs to the stable sub space E^s

$$E^s = \{ \ (k, c) : (c - \bar{c}) = \lambda_u (k - \bar{k})\}$$

while the **exact** solution belongs to the stable manifold W^s (which cannot be determined explicitly). Observe that while the slope of the isocline in the neighborhood of the steady is flatter then the slope of the stable manifold

$$\left.\frac{dc}{dk}\right|_{\dot{k}=0} = F'(\bar{k}) = \rho < \left.\frac{dc}{dk}\right|_{W^s} = \lambda_u$$

meaning that the solution approaches the steady state by accumulating (reducing) capital is the initial capital level is smaller (bigger) than the steady state level.

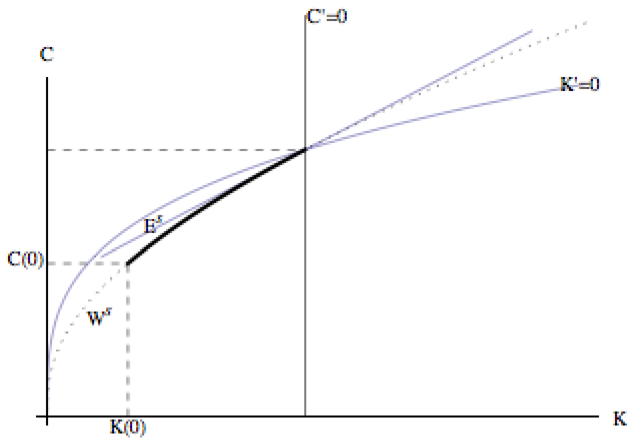

Figure 9.1: The phase diagram for the Ramsey model: it depicts the isoclines $\dot{c} = 0$ and $\dot{k} = 0$, the stable manifold W^s and the stable eigenspace, E^s, which is tangent asymptotically to the stable manifold. The exact solution follows along the stable manifold, but we have determined just the approximation along the stable eigenspace.

9.5.3 Unbounded solutions

In the previous section we saw that if the solution converges to a steady state we can have a qualitative characterization of the solution appealing to the Grobman-Hartman theorem. However, in some cases, in particular in endogenous growth theory models, solutions may not converge to a steady state, or the solution which interests us can be unbounded in time.

In particular, the consumer-saver problem may have an unbounded solution:

$$\max_{c(\cdot)} \int_0^\infty e^{-\rho t} U(C(t))dt, \ \rho > 0,$$
$$\text{subject to}$$
$$\dot{A}(t) = Y - C + r\,A, \ t \in [0, \infty)$$
$$A(0) = A_0 \text{ fixed}$$
$$\lim_{t\to\infty} e^{-\rho t} A(t) \geq 0.$$

If we write the MHDS in the (A, Q) space, we have

$$\begin{cases} \dot{A} = Y - Q^{-\frac{1}{\theta}} + rA \\ \dot{Q} = Q(\rho - r) \end{cases}$$

the solution of the optimal control problem are the solutions of that MHDS together with the inicial and transversality conditions

$$A(0) = a_0, \ \lim_{t\to\infty} Q(t)A(t)e^{-\rho t} = 0.$$

There are two interesting cases. First, if $r = \rho$ then there is an infinity of stationary solutions satisfying $Q^{-\frac{1}{\theta}} = Y + rA$. Second, if $r \neq \rho$ it has no steady state in $\mathbb{R}$. To see this note that, $\dot{Q} = 0$ if and only if $Q = 0$ but then $\dot{A} = 0$ can only be reached asymptotically when $A \to \infty$.

We can have a clearer characterization if we recast the problem in the (A, C) spac. Recall that in this case we have the MHDS

$$\begin{cases} \dot{A} = Y - C + rA \\ \dot{C} = \gamma C, \end{cases}$$

where

$$\gamma \equiv \frac{r - \rho}{\theta},$$

which, for the moment, we assume has an ambiguous sign.

The solution of the optimal control problem are the solutions of that MHDS together with the inicial and transversality conditions

$$\begin{cases} A(0) = a_0, \\ \lim_{t\to\infty} c(t)^{-\frac{1}{\theta}} A(t) e^{-\rho t} = 0. \end{cases}$$

The MHDS is linear planar ODE with coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} r & -1 \\ 0 & \gamma \end{pmatrix}$$

that has eigenvalues

$$\lambda_- = \gamma, \ \lambda_+ = r > 0.$$

and has eigenvector matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ r - \gamma & 0 \end{pmatrix}$$

The solution to the MHDS is, for $\gamma \neq 0$

$$\begin{pmatrix} A(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} -\frac{Y}{r} \\ 0 \end{pmatrix} + h_- \begin{pmatrix} 1 \\ r - \gamma \end{pmatrix} e^{\gamma t} + h_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{rt}.$$

For later use, observe that the trajectories starting from $A(0) = a_0$ and travelling along the eigenspace associated to eigenvalue λ_- are

$$\begin{pmatrix} A(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} -\frac{Y}{r} \\ 0 \end{pmatrix} + (A_0 + \frac{Y}{r}) \begin{pmatrix} 1 \\ r - \gamma \end{pmatrix} e^{\gamma t}.$$

that is

$$\mathbb{E}^- = \left\{ (A, C) \in \mathbb{R} \times \mathbb{R}_+ : \ C = (r - \gamma) \left(A + \frac{Y}{r} \right) \right\}.$$

We saw that the only requirement for the transversality condition to be me, and therefore for the optimal control problem to have a solution was $r > \gamma$. Even if we keep this assumption, three cases are possible

1. if $r < \rho$ then $\lambda_{-} = \gamma < 0$ and and the steady state $(\bar{A}, \bar{C}) = (-Y/r, 0)$ is a saddle-point. The solution of the optimal control problem, which lies along the stable manifold converges to $c^*(\infty) = 0$ and $A^*(\infty) = -Y/r < 0$. The steady state is a saddle point. The intuition is: the consumer is more impatient than the market and therefore will be asymptotically a debtor to a point in which it can collateralize the debt by its human capital $A(\infty) + H(0) = 0$;

2. if $\gamma < r = \rho$ then $\lambda_{-} = 0$ and the solution is constant $c^*(t) = Y + rA_0$ and $A^*(t) = A_0$ for all $t \in [0, \infty)$. This was the case corresponding to the existence of an infinite number of equilibria when the characterization is conducted in the (A, Q) space;

3. if $r > \rho$ then $\lambda_{-} = \gamma > 0$ and the steady state $(\bar{A}, \bar{C}) = (-Y/r, 0)$ is an unstable node. In this case, there are admissible solutions only if $A_0 \geq -Y/r$, otherwise consumption would be negative. However, if $A_0 > -Y/r$ there is an admissible solution to the optimal control problem but it is unbounded.

The question the last case poses is the following. First, if we look at the MHDS as a dynamical system we would say that it is unstable but most of the qualitative theory of ODE characterizes the dynamics close to a steady state. But we already found that this case is indeed a solution to the optimal control problem. How can we reconcile the two points ?

A way to deal with the last type of behavior is to consider convergence of the solution to a kind of invariant structure and to consider convergence to that structure. An approach which is used in the economic growth literature (see Acemoglu (2009)) is to consider convergence to an exponential solution, called **balanced growth path**, such that the initial and the transversality conditions hold.

The method proceeds along five steps.

First, define the variables using multiplicative deviations along an exponential trends with proportional growth rates. In our case we try the case in which the rates of growth are equal

$$A(t) = a(t)e^{gt},\ c(t) = c(t)e^{gt}$$

Second, obtain the dynamic system for the detrended variables (a, c). If we observe that

$$\frac{\dot{a}}{a} = \frac{\dot{A}}{A} - g,\ \frac{\dot{c}}{c} = \frac{\dot{C}}{C} - g,$$

we get

$$\begin{cases} \dot{a} = Ye^{-gt} - c + (r - g)a \\ \dot{c} = (\gamma - g)c \end{cases}$$

Third, obtain g from a stationary solution to the system in detrended variables. In our case setting $g = \gamma$ transforms the previous system to

$$\begin{cases} \dot{a} = Ye^{-\gamma t} - c + (r - \gamma)a \\ \dot{c} = 0 \end{cases}$$

which implies that $c(t) = \bar{c}$ which is an unknown constant. Setting $a(0) = A_0$ and $c(t) = \bar{c}$ we can solve the equation for the detrended asset holdings

$$a(t) = \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1 - e^{-rt}\right) \right) e^{(r-\gamma)t} + \frac{\bar{c}}{r-\gamma}.$$

Fourth, we can determine $\bar{c}$ from the transversality condition

$$\begin{aligned}
\lim_{t\to\infty} (c(t))^{-\theta} A(t) e^{-\rho t} &= \lim_{t\to\infty} \bar{c}^{-\theta} e^{(\gamma(1-\theta)-\rho)t} a(t) = \\
&= \lim_{t\to\infty} \bar{c}^{-\theta} e^{(\gamma(\theta-1)-\rho+r-\gamma)t} \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1 - e^{-rt}\right) + \frac{\bar{c}}{r-\gamma} e^{-(r-\gamma)t} \right) = \\
&= \lim_{t\to\infty} \bar{c}^{-\theta} \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1 - e^{-rt}\right) + \frac{\bar{c}}{r-\gamma} e^{-(r-\gamma)t} \right) = \\
&= \bar{c}^{-\theta} \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r} \right) = 0
\end{aligned}$$

if and only if $\bar{c} = c^* = (r-\gamma)\left(A_0 + \dfrac{Y}{r} \right)$.

At last we get the solution

$$c^*(t) = c^* e^{\gamma t},\ A^*(t) = a^*(t) e^{\gamma t}$$

where

$$c^* = (r-\gamma)\left(a_0 + \frac{Y}{r} \right),\ a^*(t) = A_0 + \frac{Y}{r}\left(1 - e^{-\gamma t}\right).$$

We see that

$$c^*(t) = (r-\gamma)\left(A^*(t) + \frac{Y}{r} \right), \text{for } t \in [0, \infty)$$

which means that the solution to the optimal control problem evolves along the eigenspace associated to the eigenvalue $\lambda_{\_}$ (see figure 9.2).

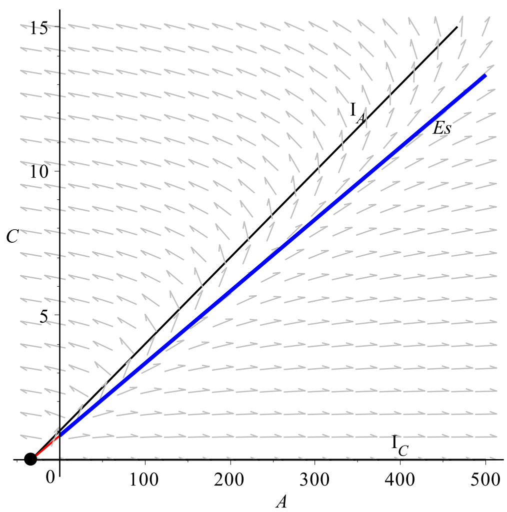

Figure 9.2: Phase diagram for the benchmark consumer problem for the case $r > \gamma$.

If $r < \rho$, and therefore $\gamma < 0$, the solution evolves along the eigenspace associated to $\lambda_{\_}$ butit converges to the steady state in which $A(\infty) = -Y/r$. In this case $\mathbb{E}^- = \mathbb{E}^s$ that is this is the stable eigenspace (which as the model is linear is the stable manifold).

From this we have a geometrical interpretation of the solution to the optimal control problem: if $r \neq \rho$ the solution will belong to the eigenspace $\mathbb{E}^-$, and it converges to the steady state if $r < \rho$ and diverges from it if $r > \rho$.

This illustrates, and reinforces, the fact that interpreting phase diagrams for MHDS of optimal control problems should be done with care: if the optimal control problem has a single solution, the geometrical analog of it is also unique.

9.6 Bibliography

- Introductory textbooks: Kamien and Schwartz (1991), Chiang (1992);
- A little less introductory: Weber (2011) ;
- Complete references, particularly for optimal control applied to management and economics Grass et al. (2008), Sethi (2019);
- The seminal contributions: optimal control Pontryagin et al. (1962);
- Other important contributions: Fleming and Rishel (1975);
- Historical contributions in economics: Arrow and Kurz (1970), Seierstad and Sydsæter (1987), Intriligator (2002),
- Applications: Kamien and Schwartz (1991) (management) , Grass et al. (2008) (economics, terror and drugs), Tu (1984)
- Applications to economic growth: Acemoglu (2009).
- Optimal control and non-linear dynamics: Brock and Malliaris (1989)
- Brief histories of optimal control: Sargeant (2000) or Peschl and Plail (2009)

Chapter 10

Introduction to the dynamic programming approach

The dynamic programming principle provides another approach to solve optimal control problems. It is based upon the principle of dynamic programming that states basically that finding an optimal solution to the problem may be achieved by finding the best rule for the immediate (infinitesimal in continuous time) step. In the literature it is associated to a recursive approach to solving the problem or to finding a feedback rule.

The first order conditions according to the principle of dynamic programming are represented by the Hamilton-Jacobi-Bellman (HJB) equation which is a partial differential equation (for the finite horizon problem) or a implicit ordinary differential equation (for the infinite horizon problem).

It has some advantages and some disadvantages when compared to the maximum principle approach:

- on the plus side: it provides a clearer intuition to dynamic optimization, it allows for an easier extension to stochastic optimal control problems, and, it is preferred by researchers who prefer to solve problems numerically;

- on the minus side: giving place to a differential equation, the terminal condition to solving that equation is not always clear (in fact it is a way to circumvent the need to a terminal condition; studying the differentiability properties of the HJB equation requires advanced functional analysis; and does not lead immediately to the use of the qualitative theory of ODE's to qualitative analysing of the solution. However, w will see that this shortcoming can be eliminated by the use of the envelope theorem.

10.1 The finite horizon case

Consider again the free terminal state optimal control problem: among functions $y \in \mathrm{Y}$ and $u \in U$ satisfying

$$\dot{y} = G(y(t), u(t), t), \text{ for } t \in [0, \bar{t}] \tag{10.1}$$

and $y(0) = y_0$ find the pair (y^*, u^*) that maximize the functional

$$\mathsf{J}[y, u] \equiv \int_0^{\bar{t}} F(t, y(t), u(t))dt \tag{10.2}$$

where $\bar{t}$ is given and $y^*(\bar{t})$ is free.

Proposition 1 (Necessary conditions according to the principle of dynamic programming). *Consider the optimal state and control functions $y^* \in \mathrm{Y}$ and $u^* \in U$ for the optimal control problem with free terminal state. Then the **Hamilton-Jacobi-Bellman** equation must hold*

$$-V_t(t, y) = \max_{u \in \mathcal{U}} \left\{ F(t, y, u) + V_y(t, y)G(t, y, u) \right\} \tag{10.3}$$

for all $t \in [0, \bar{t})$ and all $y \in Y \subseteq \mathbb{R}$.

Proof. (heuristic) We define the functional over the state and control functions continuing from an arbitrary time $t \geq 0$: $(y, u) : [t, \bar{t}] \to Y \times U \subseteq \mathbb{R}^2$

$$\mathsf{J}[y, u](t) = \int_t^{\bar{t}} F(s, y(s), u(s))ds.$$

and call **value function** to

$$V(t, y(t)) \equiv \max_{(u(s)|s \in [t, \bar{t}])} \mathsf{J}[y, u; t]$$

for $y(t) \in Y$.

The **Principle of dynamic programming optimality** states the following: for every $(t, y) \in [0, \bar{t}] \times Y$ and every $\Delta t \in (0, \bar{t} - t]$ the value function satisfies

$$V(t, y(t)) = \max_{(u(s)|s \in [t, t+\Delta t])} \left\{ \int_t^{t+\Delta t} F(s, y(s), u(s))ds + V(t + \Delta t, y(t + \Delta t)) \right\}$$

where

$$y(t + \Delta t) = y(t) + G(t, y(t), u(t))\Delta t + o(\Delta t).$$

Performing a first-order Taylor expansion we get

$$V(t + \Delta t, y(t + \Delta t)) = V(t, y(t)) + V_t(t, y(t))\Delta t + V_y(t, y(t))G(t, y(t), u(t))\Delta t + o(\Delta t)$$

(this requires that V is C^1). If the interval Δt is sufficiently small we can use the mean-value theorem

$$\int_t^{t+\Delta t} F(s, y(s), u(s))ds = F(t, y(t), u(t))\Delta t$$

Then

$$V(t,y(t)) = \max_{(u(s)|s\in[t,t+\Delta t])} \left\{ F(t,y(t),u(t))\Delta t + V(t,y(t)) + V_t(t,y(t))\Delta t + V_y(t,y)g(t,y(t),u(t))\Delta t + o(\Delta t)\right\}.$$

Cancelling out $V(t,y(t))$, dividing by Δt, taking $\Delta t \to 0$ and observing that the pair $(t,y(t))$ is an arbitrary element of $T \times Y$ we get the HJB equation (10.3). □

For solving the optimal control problem, while the Pontriagyin's principle provides necessary conditions in a form of a initial-terminal value problem for a planar ODE, the principle of the dynamic programming provides a formula for evaluating the value of our resource in a recursive way and independent of time.

The HJB equation (10.3) is a PDE (partial differential equation).

10.2 Infinite horizon discounted optimal control problem

The infinite horizon discounted optimal control problem is, again, to find functions $u^* \in \mathrm{U}$ and $u^* \in \mathrm{Y}$ satisfying

$$\begin{cases} \dot{y} = g(y(t),u(t)), & t \in [0,\infty) \\ y(0) = y_0, \\ \lim_{t\to\infty} h(t)y(t) \geq 0 \end{cases}$$

that maximize the objective functional

$$\mathsf{J}[y,u] \equiv \int_0^\infty e^{-\rho t} f(y(t),u(t))dt$$

Proposition 2 (Necessary conditions according to the principle of dynamic programming for the infinite horizon problem). *Let (y^*,u^*) be the solution to the discounted infinite horizon problem. Then it satisfies the HJB equation*

$$\rho v(y) = \max_u \{\, f(y,u) + v'(y)g(y,u)\} \tag{10.4}$$

Proof. For $y(t) = y$ the value function is

$$V(t,y) \equiv \int_t^\infty e^{-\rho s} f(y^*(s),u^*(s))ds$$

Multiplying by a inverse of the discount factor, the value function becomes independent of the initial time,

$$e^{\rho t}V(t,y) = \int_t^\infty e^{-\rho(s-t)} f(y^*(s),u^*(s))ds = v(y).$$

If we take derivatives of $V(t,y) = e^{-\rho t}v(y)$, we have $V_t(t,y) = -\rho\, e^{-\rho t}v(y)$ and $V_y(t,y) = e^{-\rho t}v'(y)$, which after substituting in equation (10.3) yields equation (10.4). □

In the case of the discounted infinite horizon the HJB equation is not a PDE but an ODE in implicit form. In order to see this we need to determine another important element of the DP approach: the policy function.

If we define the function $h(u, y) \equiv f(y, u) + v^{'}(y)g(y, u)$ the HJB equation (10.4) can be written as

$$\rho v(y) = \max_{u} h(u, v).$$

We can obtain the optimal control from the first-order condition

$$\frac{\partial h(u, y)}{\partial u} = 0.$$

If function $h(u, v)$ is monotonic as regards u, by appealing to the implicit function theorem, we can obtain the optimal control as a function of the state variable, $u^* = \pi(y)$. Function $\pi(.)$ in the DP literature is called **policy function**. It gives the optimal control as a function of the state variable. This is why it is called a **feedback control** problem.

The reason for this is the following. If we substitute the policy function in equation (10.4) we finally obtain the HJB equation as an ODE in implicit form

$$\rho v(y) = f(\pi(y), y) + v^{'}(y)g(\pi(y), y)$$

where the state variable y is the independent variable and the value function, $v(y)$, is the unknown function.

If we are able to determine a solution to this equation, we can usually specify the utility function, which means that we are able to obtain the optimal control as a function of the state variable. We can obtain the solution to the optimal control problem by substituting in the ODE constraint to get

$$\dot{y} \ = g(y, \pi(y)), \ t \in [0, \infty)$$

which, together with the initial condition $y(0) = y_0$, would, hopefully, allow for the determination of the solution for the state variable.

If we can find the policy function, then obtaining the optimal dynamics for y reduces to solving an initial-value problem instead of a mixed initial-terminal value problem (or two-point boundary value problem) as is the case when we use the calculus of variations of the Pontriyagin's principle approaches.

However, only in a very small number of cases we can obtain closed form solutions to the HJB equation. Next we show some cases in which this is possible.

10.3 Applications

10.3.1 Example 1: The resource depletion problem

We solve again resource-depletion problem for an infinite horizon

$$\max_{C} \int_0^{\infty} e^{-\rho t} \ln{(C(t))} dt, \text{ s.t } \dot{W} = -C,\ W(0) = W_0$$

by using the DP principle.

The HJB equation is

$$\rho v(W) = \max_{C} \left[\ \ln{(C)} + v^{'}(W)(-C)\right]$$

Policy function

$$\frac{1}{C^*} - v^{'}(W) = 0 \Leftrightarrow C^* = \left(v^{'}(W)\right)^{-1}$$

Then the HJB becomes

$$\rho v(W) = -\ \ln{(v^{'}(W))} - 1$$

The textbook method for solving the HJB equation through is by using the **method of undetermined coefficients** after we make a conjecture over the form of the value function (no constructive way here).

Assume the trial function

$$v(W) = a + b \ln{(W)}$$

As $v^{'}(W) = b/W$ and substituting and collecting terms we get

$$\rho a + 1 + \ln{(b)} = \ln{(W)}\,(1 - \rho b)$$

then $b = 1/\rho$ and $a = (\ln \rho - 1)/\rho$.

Then:

$$v(W) = \frac{\ln \rho - 1 + \ln{(W)}}{\rho},\ C^* = (v^{'}(W))^{-1} = \rho W$$

A second method: the HJB equation is an ODE, where W is the independent variable, so we can try to solve it (this is a constructive method).

The HJB is equivalent to

$$v^{'}(W) = e^{-(1+\rho v(W))}$$

ODE $y^{'}(x) = e^{(a+by(x))}$ has the closed form solution

$$y(x) = \frac{1}{b}\ \left(-a + \ln\left(-\frac{1}{b(k+x)}\ \right)\right)$$

where k is an arbitrary constant. Then we determine

$$v(W) = -\frac{1}{\rho}\left(1 + \ln\left(\frac{1}{\rho(W+k)}\ \right)\right)$$

and

$$C^* = \left(V^{'}(W)\right)^{-1} = \rho(W+k)$$

Substituting in the constraint $\dot{W} = -C = -\rho(W+k)$, we get the solution

$$W(t) = -k + (W(0)+k)e^{-\rho t}.$$

The problem is somewhat incompletely specified, which reveals a potential problem when using the DP approach.

In our case, as it is natural to assume that $\lim_{t\to\infty} W(t) = 0$ we would obtain $k = 0$ and therefore we would get the same solution as from using the CV and Pontriyagin's approaches:

$$C^*(t) = \rho W_0 e^{-\rho t},\ W^*(t) = W_0 e^{-\rho t},\ \text{for } t \in [0,\infty).$$

10.3.2 Example 2: The benchmark consumption-savings problem

Applying the HJB equation (10.4) to our problem we have

$$\rho v(A) = \max_C \left\{ \frac{C^{1-\theta}-1}{1-\theta} + v^{'}(A)(Y - C + rA) \right\}. \tag{10.5}$$

Define a indirect utility function by

$$\tilde{u}(v'(A)) = \max_C \left\{ \frac{C^{1-\theta}-1}{1-\theta} - v^{'}(A)\, C \right\}$$

and total wealth, summing up human and financial wealth, by $W(A) \equiv \dfrac{Y}{r} + A$, then the HJB equation (10.5) at the optimum is a implicit ODE

$$\rho v(A) = \tilde{u}(v'(A)) + r\, v^{'}(A)\, W(A). \tag{10.6}$$

Solving the static utility problem we get the optimum policy for consumption

$$C^* = \pi(A) \equiv \left(v^{'}(A)\right)^{-\frac{1}{\theta}}.$$

as a function of the (unknown) marginal value function, and upon substitution yields

$$\tilde{u}(v'(A)) = \frac{1}{1-\theta}\Big(\left(v^{'}(A)\right)^{\frac{\theta-1}{\theta}} - 1 \Big).$$

Therefore, equation (10.6) becomes

$$\rho\, v(A) = \frac{\theta}{1-\theta} \left(v^{'}(A)\right)^{\frac{\theta-1}{\theta}} - \frac{1}{1-\theta} + r\, v^{'}(A)\, W(A) \tag{10.7}$$

To solve this (implicit ODE) equation, we use again the method of undetermined coefficients. Conjecturing the trial function

$$v(A) = a + b\, W(A)^{1-\theta},$$

with arbitrary parameters a and b. Then

$$v^{'}(A) = b\,(1-\theta)\, W(A)^{-\theta}$$

and after substitution in equation (10.7) we get

$$a\rho + \frac{1}{1-\theta} = W(A)^{1-\theta}\, b\,\theta \Big[\Big(b\,(1-\theta)\Big)^{-1/\theta} - (r-\gamma)\Big]$$

where we have again $\gamma \equiv (r-\rho)/\sigma$. Setting both sides to zero, yields

$$a = \frac{1}{\rho(\theta-1)} \text{ and } b = \frac{(r-\gamma)^{-\theta}}{1-\theta}$$

Then, the value function is

$$v(A) = \frac{1}{1-\theta}\Big[\,(r-\gamma)^{-\theta}\left(\frac{Y}{r}+A\right)^{1-\theta} - \frac{1}{\rho}\Big].$$

Taking the derivative as regards A and substituting in the policy function for C, we find the optimal consumption in feedback form

$$C^*(A) = (r-\gamma)\left(\frac{Y}{r}+A\right)$$

which only makes sense if $r > \gamma$.

We can get the optimal asset path by substituting optimal consumption in the budget constraint

$$\dot{A}^* = Y + rA - C^*(A) = \gamma\left(\frac{Y}{r}+A\right).$$

Solving this equation with $A(0) = A_0$ we get the optimal paths for asset holdings

$$A^*(t) = -\frac{Y}{r} + \left(\frac{Y}{r}+A_0\right)e^{\gamma t}, \text{ for } t \in [0,\infty),$$

and consumption

$$C^*(t) = (r-\gamma)\left(\frac{Y}{r}+A_0\right)e^{\gamma t}, \text{ for } t \in [0,\infty).$$

Exercise Prove, by setting $\theta = 1$, that the value function for $u(C) = \ln(C)$ is

$$V(A) = \frac{1}{\rho}\Big[\,\frac{r-\rho}{\rho} + \ln(\rho\, W(A))\,\Big].$$

Hint: use the property $f(x) = \exp(\ln f(x))$ and use the l'Hôpital theorem.

The utility function is a generalized logarithm $u(C) = \ln_\theta(C) = \dfrac{C^{1-\theta}-1}{1-\theta}$. Sometimes in the literature people write

$$u(C) = \begin{cases} \dfrac{C^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1 \\ \ln(C) & \textit{if}\, \theta = 1 \end{cases}$$

The problem with this formulation is that if we cannot obtain the value function for the logarithm utility by setting the limit of $\theta = 1$ for the general case $\theta = 1$, which is

$$v(A) = \frac{(r-\gamma)^{-\theta}}{1-\theta} W(A)^{1-\theta}.$$

10.3.3 Example 3: The Ramsey model

The HJB for the Ramsey model is

$$\rho v(k) = \max_{c} \left\{ u(c) + v^{'}(k)\,(F(k) - c) \right\}$$

The optimality condition is

$$u'(c) = v'(k)$$

if u is sufficiently smooth then we obtain the policy function $c = C(k) = (u')^{-1}(v'(k))$. Substituting back in the HJB equation yields the implicit ODE in $v(k)$

$$\rho v(k) = u(C(k)) + v^{'}(k)(F(k) - C(k))$$

which does not have a closed form solution in general.

Exercise: for the case in which $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta}$ and $F(k) = k^{\alpha}$, such that $\theta = \alpha$ prove that a closed form solution can be found.

10.3.4 Example 4: The AK model

The Rebelo (1991) AK model can be seen as a special case of the previous problem in which the HJB function is

$$\rho v(K) = \max_{C} \left\{ \frac{C^{1-\theta}}{1-\theta} + v^{'}(K)\,(AK - C) \right\}$$

Using the same steps as before, we get

$$\rho v(K) = \frac{\theta}{1-\theta} \left(v^{'}(K) \right)^{\frac{\theta-1}{\theta}} + v^{'}(K) AK \tag{10.8}$$

To solve the equation we use again the method of undetermined coefficients and find

$$v(K) = \frac{((A-\gamma)K)^{1-\theta}}{1-\theta}.$$

where

$$\gamma = \frac{A - \rho}{\theta}.$$

The consumption, in the feedback form is,

$$C^{*}(K) = (A - \gamma)K$$

and the budget constraint of the economy is

$$\dot{K}^{*} = AK^{*} - C^{*}(K) = \gamma K^{*}.$$

Considering the given initial level for capital $K(0) = K_0$ we get the optimal paths for capital and output

$$K^{*}(t) = K_0 e^{\gamma t},\; Y^{*}(t) = AK_0 e^{\gamma t},\; \text{for } t \in [0, \infty).$$

10.4 Relationship with the PMP

Consider the HJB equation (10.4).

The optimal policy function $u^* = U(y)$ is obtained from the optimality condition

$$f_u(y, u^*) + v'(y)\, g_u(y, u^*) = 0$$

Write $q = v'(y)$. Next we will show that with this change of variables we will obtain the optimality conditions according to the Pontriyagin's maximum principle.[1]

First, observe that, defining $h(y,u) = f(y,u) + v'(y)\, g(y,u) = f(y,u) + q\, g(y,u)$ at the optimum we have $h_u(y,u^*) = f_u(y,u^*) + q\, g_u(y,u^*) = 0$ is the static optimality condition according to the Pontriyagin's maximum principle.

Taking the derivative of the (10.4), at the optimum, yields

$$\begin{aligned}\rho\, v'(y) &= f_y(y,u^*) + f_u(y,u^*)\, U'(y) + v''(y)\, g(y,u^*) + v'(y)\left(g_y(y,u^*) + g_u(y,u^*\ )\, U'(y)\right)\\ &= f_y(y,u^*) + v''(y)\, g(y,u^*) + v'(y)\, g_y(y,u^*)\end{aligned}$$

using the optimality condition. Therefore, if function $v(\cdot)$ is smooth

$$g(y,u^*) = \frac{\rho\, v'(y) - f_y(y,u^*) - v'(y)\, g_y(y,u^*)}{v''(y)} \ = \frac{\rho\, q - h_y(y,u^*}{v''(y)} \tag{10.9}$$

using the previous definition.The constraint to the problem evaluated at the optimum, $\frac{dy}{dt} = g(y,u^*)$. Taking the time detivative of $q = v'(y)$ implies

$$\frac{dq}{dt} = v''(y)\ \frac{dy}{dt} = v''(y) g(y,u^*) =$$

If we substitute equation (10.9) yields

$$\frac{dq}{dt} = \rho\, q - h_y(y,u^*)$$

which is the multiplier equation from the Pontriyagin's maximum principle [2]

This allows for a qualitative dynamics analiysis of the solution to the optimal control problem obtained via the HJB equation.

10.4.1 Application to the Ramsey model

Consider the Ramsey problem with a CRRA utility function and a Cobb-Douglas production function. The HJB equation is

$$\rho\, v(k) = \max_c \left\{ \frac{c^{1-\theta}-1}{1-\theta} + v'(k)\left(k^\alpha\ - c\right)\right\}$$

[1] Although the DP approach is silent to what to do with the transversality condition. However, Ekeland (2010) shows that the solution to the problem exists if there is a steady state vale for the state variable and at the steady state $\bar{y}$ we should have $\rho\, v(\bar{y}) = v'(\bar{y})$.

[2] See (Beckmann, 1968, p.33).

where $\theta > 0$ and $0 < \alpha < 1$. The policy function is $c^* = C(k) = \left(v'(k)\right)^{-\frac{1}{\theta}}$. Then

$$\frac{dc}{dt} = -\frac{c^*\,v''(k)}{\theta\,v'(k)}\,\frac{dk}{dt}.$$

From the envelop theorem, we obtain

$$\rho\,v'(k) = v''(k)\left(k^{\alpha}\ \ - c^*\right) + \alpha\,k^{\alpha-1}\ \ v'(k)$$

as

$$\frac{dk}{dt} = k^{\alpha}\ \ - c^* = \frac{v'(k)\left(\rho - \alpha\,k^{\alpha-1}\right.}{v''(k)}$$

then

$$\frac{dc}{dt} = c\frac{(\alpha\,k^{\alpha-1} - \rho)}{\theta}$$

which is the Ramsey-Keynes equation associated to the PMP.

10.5 Bibliography

- The seminal contribution: Bellman (1957).
- Other references: Beckmann (1968)
- Recent textbook: Sethi (2019)

Chapter 11

Optimal control of ODE's: extensions

11.1 Introduction

In this chapter we consider some extensions of the simple optimal control problems we dealt in the last chapter

11.2 Singular optimal control

Assume that the Hamiltonian is linear in the control variable. This is equivalent to stating that $H_u\,(y,u,t) = h(y,t)$ and $H_{uu}\,(y,u) = 0$ for all $u \in \mathcal{U}$.

Consider the problem:

$$\begin{aligned}
\max_{u(\cdot)}\ \mathsf{J}[y,u]\ &= \int_0^T\ F(t,y(t),u(t))\,dt \\
&\text{subject to} \\
&\dot{y}\ = G(t,y(t),u(t)) \\
&y(0) = y_0\,\text{given} \\
&R(y(T),T) \geq 0.
\end{aligned} \tag{11.1}$$

In this case define the Hamiltonian function as

$$H(y,u,\lambda_0,\lambda,t) = \lambda_0\,F(y,u,t) + \lambda\,G(y,u,t).$$

An informal version of the Pontriyagin's principle states the following: If (y^*,u^*) is an optimum, then there is a scalar $\lambda_0 \in \{0,1\}$ and a piecewise continuous function $\lambda : \mathrm{T} \to R$ such that:

1. for every $t \in [0,T)$

$$\begin{aligned}
H_u^*(t) &= 0 \\
\dot{\lambda}\ &= -H_y^*(t) \\
\dot{y}\ &= H_\lambda^*\,(t) = G(t,y^*(t),u^*(t))
\end{aligned}$$

2. at the terminal time $t = T$

$$\lambda(T)\, R_y^*(y(T), T) = 0$$

This problem may require enlarging the domain space, also for the state variable y to the set of piecewise-continuous functions $y : \mathrm{T} \to \mathbb{R}$.

Here we are mostly concerned with the household problem with a linear utility function, that is a isoelastic utility function with an infinitely-valued elasticity of substitution. See (Grass et al., 2008, ch. 3.5) for a complete reference.

Example The household problem with a linear utility function

$$\begin{aligned}
\max_{c(\cdot)} \int_0^\infty & c(t)\, e^{-\rho\, t}\, dt \\
& \text{subject to} \\
& \dot{a} \;= r\, a - c \\
& a(0) = a_0, \text{ fixed} \\
& \lim_{t\to\infty} \; e^{-r\, t}\; a(t) \geq 0
\end{aligned}$$

The Hamiltonian function is

$$H(a, c, \lambda_0, \lambda) = \lambda_0 c + \lambda\, (r\, a - c)$$

where λ_0 is a number and λ is a function of time. The first order conditions (observe that the Hamiltonian function is a concave, although not strictly concave function of (c, a)) are

$$\begin{aligned}
\lambda_0 &= \lambda(t), \; t \in \mathrm{T} \\
\dot{\lambda} \; &= \lambda\, (\rho - r), \; t \in \mathrm{T} \\
\lim_{t\to\infty} \; \lambda(t)\, a(t) e^{-\rho t} \; &= 0 \\
\dot{a} \; &= r\, a - c \\
a(0) &= a_0.
\end{aligned}$$

Setting $\lambda_0 = 1$ then $\lambda(t) = 1$ for every $t \in [0, \infty)$, which implies $\dot{\lambda} \; = 0$. A solution only exists if $r = \rho$, which we assume to be the case from now on. Solving the budget constraint and substituting in the transversality condition we should have $a_0 = \int_0^\infty \; e^{-\rho\, s} \; c(s)\, ds$. As there are no more constraints on the functional form for $c(t)$, as in the case with constant elasticity of substitution, the solution for c is indeterminate. That is, there is an infinite number of consumption trajectories that satisfies that constraint. In particular, a constant consumption path $c(t) = c^* = \rho\, a_0$ is a solution.

Setting $\lambda_0 = 0$ then $\lambda(t) = 0$ for every $t \in [0, \infty)$, which implies $\dot{\lambda} \; = 0$. However, this does not require that an existence condition is $r = \rho$. However, even if we assume that $r - \rho$ can have any sign, the transversality condition will be satisfied for any trajectories of a and c. Again, the solution is indeterminate.

From the economic point of view the problem is misspecified. In order to overcome this, either we introduce more curvature in the utility function, or we introduce some adjustment costs in consumption, or bounds in the net asset position of the consumer.

11.3 Two-stage optimal control problems

Assume that the independent variable is time $t \in \mathrm{T} = [t_0, t_2]$ but there is a discontinuity in the objective function andor in the constraint to the problem such that

$$F(t) = F(t, y(t), u(t)) = \begin{cases} F_1(t, y(t), u(t)) & \text{if } \ t_0 \geq t < t_1 \\ F_2(t, y(t), u(t)) & \text{if } \ t_1 \geq t \leq t_2 \end{cases}$$

and/or

$$G(t) = G(t, y(t), u(t)) = \begin{cases} G_1(t, y(t), u(t)) & \text{if } \ t_0 \geq t < t_1 \\ G_2(t, y(t), u(t)) & \text{if } \ t_1 \geq t \leq t_2 \end{cases}$$

where the *switching time* $t_1 \in \mathrm{T} = [t_0, t_2]$, that it, it satisfies $t_0 < t_1 < t_2$, and may be known or may be a decision variable. The optimal control problem in which the switching time is a decision variable is called in the literature a **two-phase optimal control problem**.

The common structure of the problems we address in this section is

$$\begin{aligned} \max_{u(\cdot)} \mathsf{J}[u, y] = & \int_{t_0}^{t_1} F_1(t, y(t), u(t))\, dt + \int_{t_1}^{t_2} F_2(t, y(t), u(t))\, dt \\ & \text{subject to} \\ & \dot{y} \ = \begin{cases} F_1(t, y(t), u(t)) & \text{if } \ t_0 \geq t < t_1 \\ F_2(t, y(t), u(t)) & \text{if } \ t_1 \geq t \leq t_2 \end{cases} \\ & t_0, \text{ and } \ t_2 \text{ fixed} \\ & y(t_0) = y_0 \text{ fixed} \\ & y(t_2) \geq y_2 \text{ constrained} \end{aligned} \tag{PTS}$$

We consider the following versions of the problem depending on the switching conditions:

$$t_1 \text{ fixed and } \ y(t_1) \text{ free} \tag{PTS1}$$

$$t_1 \text{ free but subject to } t_0 \leq t_1 \leq t_2 \text{ and } \ y(t_1) \text{ free} \tag{PTS2}$$

The Hamiltonian function becomes a piecewise continuous, or piecewise differentiable function

$$H(t) = H(t, y(t), u(t), \lambda(t)) = \begin{cases} H_1(t) = F_1(t) + \lambda(t)\, G_1(t) & \text{if } \ t_0 \geq t < t_1 \\ H_2(t) = F_2(t) + \lambda(t)\, G_2(t) & \text{if } \ t_1 \geq t \leq t_2 \end{cases}$$

where the co-state variable is $\lambda : \mathrm{T} \to \mathbb{R}$, and $H_i(t) = H_i(t, y(t), u(t), \lambda(t))$ for $i = 1, 2$.

Proposition 1. *[**First order necessary conditions for the two-stage optimal control problem**] Let (y^*, u^*) be a solution to the OC problem PTS in which one of the conditions (PTS1), or (PTS2) is introduced. Then there is a piecewise continuous function $\lambda : \mathrm{T} \to \mathbb{R}$, called co-state variable, such that (y^*, u^*, λ) satisfy the following conditions:*

- *the optimality condition:*

$$\begin{aligned} H^*_{1,u}(t) &= 0, \ \text{for} \ \ t \in [t_0, t_1^*) \\ H^*_{2,u}(t) &= 0, \ \text{for} \ \ t \in [t_1^*, t_2] \end{aligned} \tag{11.2}$$

  where $H^*_{i,u}(t) = H_{i,u}(t, y^*(t), u^*(t), \lambda(t))$ for $i = 1, 2$;

- *the multiplier equation*

$$\begin{aligned} \dot{\lambda} \ + H^*_{1,y}(t) &= 0, \ \text{for} \ \ t \in [t_0, t_1^*) \\ \dot{\lambda} \ + H^*_{2,y}(t) &= 0, \ \text{for} \ \ t \in [t_1^*, t_2] \end{aligned} \tag{11.3}$$

  where $H^*_{i,y}(t) = H_{i,y}(t, y^*(t), u^*(t), \lambda(t))$ for $i = 1, 2$;

- *the constraint of the problem:*

$$\begin{aligned} \dot{y} \ &= G^*_1(t), \ \text{for} \ \ t \in [t_0, t_1^*) \\ \dot{y} \ &= G^*_2(t), \ \text{for} \ \ t \in [t_1^*, t_2] \end{aligned} \tag{11.4}$$

  where $G^*_i(t) = G_i(t, y^*(t), u^*(t))$ for $i = 1, 2$;

- *the adjoint condition associated to the initial values $(t_0, y(t_0))$: $y^*(t_0) = y_0$;*

- *the adjoint conditions associated to the terminal values $(t_2, y(t_2))$:*

$$\lambda(t_2)\,(y^*(t_2) - y_2) = 0, \ \text{and} \ \lambda(t_2) \geq 0; \tag{11.5}$$

- *and the adjoint conditions associated to switching conditions (PTS1) and (PTS2) are*

  - *for problem (PTS1): assuming that $t_0 < t_1 < t_2$ the switching condition is*

$$\lambda(t_1^-) = \lambda(t_1^+) \tag{11.6}$$

    where $\lambda(t_1^-) = \lim_{t \uparrow t_1} \lambda(t)$ and $\lambda(t_1^+) = \lim_{t \downarrow t_1} \lambda(t)$ are the limits of the co-state variables determined from the first stage (i.e., from $t \in [t_0, t_1]$) and from the second stage (i.e., from $t \in [t_1, t_2]$) respectively;

  - *for problem (PTS2) there are two conditions,*

$$\lambda(t_1^{*-}) = \lambda(t_1^{*+}) \tag{11.7}$$

where $\lambda(t_1^{*-}) = \lim_{t\uparrow t_1^*} \lambda(t)$ *and* $\lambda(t_1^{*+}) = \lim_{t\downarrow t_1^*} \lambda(t)$ *together with one of the one of the following conditions allowing for the determination of the optimal switching time* t_1^**:*

$$\begin{aligned} t_1^* = t_0 < t_2 &\iff H_1^*(t_1^{*-}) < \ H_2^*(t_1^{*+}) \\ t_0 < t_1^* < t_2 &\iff H_1^*(t_1^{*-}) = \ H_2^*(t_1^{*+}) \\ t_0 < t_1^* = t_2 &\iff H_1^*(t_1^{*-}) > \ H_2^*(t_1^{*+}) \end{aligned} \tag{11.8}$$

where $H_1^*(t_1^{*-}) = \lim_{t\uparrow t_1^*} H_1(t)$ *and* $*_2(t_1^{*+}) = \lim_{t\downarrow t_1^*} H_2^*(t)$.

Proof. (Heuristic) Let u^* and y^* be the optimal control and state variable. The value functional, at the optimum, is, for (PTS2) problem,

$$\mathsf{J}^* = \int_{t_0}^{t_1^*} F_1^*(t)\,dt + \int_{t_1^*}^{t_2} F_2^*(t)\,dt$$

and the associated Lagrangean is

$$\mathsf{L}^* = \mathsf{J}^* + \mu_0\,(t_1^* - t_0) + \mu_2\,(t_2 - t_1^*) + \psi\,(y^*\ (t_2) - y_2),$$

where the complementary slackness conditions should hold

$$\begin{aligned} \mu_0\,(t_1^* - t_0) = 0, \mu_0 \geq 0, \ \text{and} \ \ t_1^* \geq t_0 \\ \mu_2\,(t_2 - t_1^*) = 0, \mu_2 \geq 0, \ \text{and} \ \ t_1^* \leq t_2 \\ \psi\,(y^*\ (t_2) - y_2) = 0, \psi \geq 0, \ \text{and} \ \ y^*(t_2) \leq y_2. \end{aligned}$$

Introducing the admissible perturbations $y(t) = y^*(t) + \varepsilon\,\eta_y(t)$ to the state variable, $u(t) = u^*(t) + \varepsilon\,\eta_u(t)$ to the control variable and $t_1 = t^*\ + \varepsilon\,\tau_1$, yields the Gâteaux differential

$$\begin{aligned} \delta_{\eta(\cdot)}\ \mathsf{L}\ [y^*, u^*; t_1^*] \ = & \int_{t_0}^{t_1^*} \Big[\ H_{1,u}^*(t)\,\eta_u(t) + \Big(H_{1,y}^*(t) + \dot{\lambda}\ (t)\Big)\,\eta_y(t)\Big]\,dt + \\ & + \int_{t_1^*}^{t_2} \Big[\ H_{2,u}^*(t)\,\eta_u(t) + \Big(H_{2,y}^*(t) + \dot{\lambda}\ (t)\Big)\,\eta_y(t)\Big]\,dt \\ & + \lambda(t_0)\,\eta(t_0) + (\lambda(t_1^{*+}) - \lambda(t_1^{*-}))\,\eta(t_1^*) + (\psi - \lambda(t_2)\,\eta(t_2) \\ & + \Big(\mu_0 - \mu_2 + H_1^*(t_1^{*-}) - H_2^*(t_1^{*+})\Big)\,\tau_1. \end{aligned}$$

At the optimum $\delta_{\eta(\cdot)}\ \mathsf{L}\ [y^*, u^*; t_1^*] = 0$ should be sarisfied, together with the complementary slackness conditions for admissible perturbations. As the only constraint on the perturbation is $\eta(t_0) = 0$, at the optimum satisfies $H_{1,u}^*(t) = H_{1,y}^*(t) + \dot{\lambda}(t) = \dot{y}\ - G_1^*\ (t) = 0$ for every $t \in [t_0, t_1^{*-})$, $H_{2,u}^*(t) = H_{2,y}^*(t) + \dot{\lambda}(t) = \dot{y}\ - G_2^*\ (t) = 0$ for every $t \in (t_1^{*+}, t?2]$, $\lambda(t_2)(y^*(t_2) - y_2) = 0$, and $\lambda(t_1) = \lambda(t_1)$ if t_1 is fixed, or $\lambda(t_1^{*+}) = \lambda(t_1^{*-})$ together with $\mu_0 - \mu_2 + H_1^*(t_1^{*-}) - H_2^*(t_1^{*+}) = 0$ together with the associated complementary slackness conditions if t_1 should be optimally determined. □

Switching costs: there are version of the problem in which there is a switching cost at time $t = t_1$, depending on the state of the problem, as $\Phi(t_1, y(t_1)$. In this case we have

- for the fixed switching time problem (**??**) instead of condition (11.6) we have

$$\lambda(t_1^-) + \Phi_y\ (t_1, y^*(t_1)) = \lambda(t_1^+)$$

$\Phi_y\ (t, y) = \frac{\partial \Phi(t,y)}{\partial y}$.

- for the free switching time problem (**??**) instead of condition (11.7) we have

$$\lambda(t_1^{*-}) + \Phi_y\ (t_1, y^*(t_1^*) = \lambda(t_1^{*+})$$

and instead of conditions (11.8) we have

$$\begin{aligned} t_1^* = t_0 < t_2 &\iff H_1^*(t_1^{*-}) - \Phi_t\ (t_1, y^*(t_1^*) < \ H_2^*(t_1^{*+}) \\ t_0 < t_1^* < t_2 &\iff H_1^*(t_1^{*-}) - \Phi_t\ (t_1, y^*(t_1^*) = \ H_2^*(t_1^{*+}) \\ t_0 < t_1^* = t_2 &\iff H_1^*(t_1^{*-}) - \Phi_t\ (t_1, y^*(t_1^*) > \ H_2^*(t_1^{*+}) \end{aligned}$$

where $\Phi_t\ (t, y) = \frac{\partial \Phi(t,y)}{\partial t}$.

Observations: we can extend this approach to multple switcing times.

References: Tomiyama (1985), Rossana (1989), Makris (2001). For an application to endogenous growth see Boucekkine et al. (2004).

11.4 Constraints on state and control variables

Assume that the independent variable is time $t \in \mathrm{T} = [0, T]$ and one of the following constraints exist: constraints on state variables such that $Q_2(y(t), t) \leq 0$, or constraints in the state and/or control variables such that $Q_1(ty(t), u(t), t) \leq 0$, for every $t \in \mathrm{T}$.

$$\begin{aligned} \max_{u(\cdot)} \ \mathsf{J}[y, u] \ &= \int_0^T \ F(t, y(t), u(t))\, dt \\ &\text{subject to} \\ &\dot{y} \ = G(t, y(t), u(t)) \\ &y(0) = y_0 \,\text{given} \\ &Q_1(t, y(t), u(t)) \geq 0, \ \text{for any} \ \ t \in \mathrm{T} \\ &Q_2(t, y(t)) \geq 0, \ \text{for any} \ \ t \in \mathrm{T} \\ &R(y(T), T) \geq 0. \end{aligned} \tag{11.9}$$

where $Q_1(\cdot) \geq 0$ is a joint constraint on the state variable and the control variable and $Q_2(\cdot) \geq 0$ is a pure state constraint.

In this problem we have potencial points of discontinuity for the adjoint variable $\lambda(\cdot)$ when the constraints are

In this case an (informal) version of the Pontriyagin maximum principle is presented in Hartl et al. (1995). Define the Hamiltonian function

$$H(y, u, \lambda_0, \lambda, t) = \lambda_0 \, F(y, u, t) + \lambda \, G(y, u, t),$$

and the Lagrangean

$$L(y, u, \lambda_0, \lambda, t) = H(y, u, \lambda_0, \lambda, t) + \nu_1 \, Q_1(y, u, t) + \nu_2 \, Q_1(y, t).$$

An informal version of the Pontriyagin's principle states the following: If (y^*, u^*) is an optimum, then there is a scalar $\lambda_0 \in \mathbb{R}$, a piecewise function $\lambda : \mathrm{T} \to R$, Lagrange multipliers $\nu_1 : \mathrm{T} \to R$, and/or $\nu_2 : \mathrm{T} \to R$, variables $\zeta(t_i$ at the times of discontinuity t_i for $\lambda(\cdot)$, and three constants α and β such that:

1. for every $t \in [0, T)$

$$\begin{aligned} L^*_u(t) &= 0 \\ \dot{\lambda} \ &= -L^*_y(t) \\ \dot{y} \ &= L^*_{\lambda} \ (t) = G(t, y^*(t), u^*(t)) \\ \nu_1(t) \geq 0, \ \text{and} \ \ \nu_1(t) \, Q^*_1(t) &= 0 \\ \nu_2(t) \geq 0, \ \text{and} \ \ \nu_2(t) \, Q^*_2(t) &= 0 \end{aligned}$$

2. at the terminal time $t = T$

$$\begin{aligned} \lambda(T^-) &= \alpha \, Q^*_{2,y} \ (T) + \beta \, R^*_y \ (T) \\ \alpha \geq 0, \ \text{and} \ \ \alpha \, Q^*_2(T) &= 0 \\ \beta \geq 0, \ \text{and} \ \ \beta \, R^*(T) &= 0. \end{aligned}$$

3. for any time t_i at a boundary interval and for any contact time t_i, the co-state variable $\lambda(\cdot)$ may have a discontinuity given by

$$\begin{aligned} \lambda(t_i^-) &= \lambda(t_i^+) + \eta(t_i) \, Q^*_{2,y}(t_i) \\ H^*(t_i^-) &= H^* \ (t_i^+) + \eta(t_i) \, Q^*_{2,t}(t_i) \\ \eta(t_i) \geq 0, \ \text{and} \ \ \eta(t_i) \, Q^*_2(t_i) &= 0 \end{aligned}$$

   where $\lambda(t_i^-) = \lim_{t \uparrow t_i} \ \lambda(t)$ and $\lambda(t_i^+) = \lim_{t \downarrow t_i} \ \lambda(t)$ and an analogous notation for $H^*(t)$.

References: Köhler (1980), Hartl et al. (1995).

11.5 References

The definitive textbook on optimal control is Grass et al. (2008)

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