Universidade de Lisboa
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Master in Monetary and Financial Economics
Fundamentals of Financial Economics
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Solutions: only analytical questions

Part 1 1(a) Solving $\mathbf{R}^\top Q^\top = \mathbf{1}$ we find (if $R_2 \neq R_1$) $Q = \big(q_1, q_2\big) = \big(\dfrac{R_2-1}{R_2-R_1}, \dfrac{1-R_1}{R_2-R_1}\big)$. Absence of arbitrage opportunities (AAO) and complete markets (CM) if and only if $R_1 < 1 < R_2$ or $R_2 < 1 < R_1$. From now on we consider the first case.

1(b) Sharpe index ShI $= \dfrac{E[R-1]}{\sigma(R)}$ where $E[R-1] \; = \pi_1(R_1-1)+\pi_2(R_2-1)$ as $\pi_1+\pi_2 = 1$ and $R_1 < 1 < R_2$ has an ambiguous sign and $\sigma(R) = \sqrt{\pi_1\,\pi_2}|R_2 - R_1| > 0$ from AAO and CM positive and finite. ShI would be infinite if $R_1 = R_2$.

1(c) Expected risk premium is $E[R-1]$ it is non-negative if $\pi_1(R_1-1)+\pi_2(R_2-1) \geq 0$. The risk neutral probabilities are

$$\pi_1^q = \frac{q_1}{q_1+q_2} = \frac{R_2-1}{R_2-R_1}$$
$$\pi_2^q = \frac{q_2}{q_1+q_2} = \frac{1-R_1}{R_2-R_1}$$

Then $E[R-1] \geq 0$ if and only if $\dfrac{\pi_1^q}{\pi_2^q} \geq \dfrac{\pi_2}{\pi_1}$

2(a) Solution to the household's problem

$$c_0^* = \frac{h_0}{1+\beta},$$
$$c_{1,s}^* \; = \frac{\beta}{m_s^*}\;\Big(\frac{h_0}{1+\beta}\Big),\; s = 1,2$$

for $m_1^* = \dfrac{R_2-R_1}{\pi_1(R_2-1)}$, $m_2^* = \dfrac{R_2-R_1}{\pi_2(1-R_1)}$ and $h_0 \equiv a_0 + E[M^* Y_1]$, and

$$\theta^* = \frac{1}{R_2-R_1}\Big(-\left(c_{11}^*-y_{11}\right)+\left(c_{12}^*-y_{12}\right)\Big)$$
$$\ell^* = \frac{1}{R_2-R_1}\Big(R_2\left(c_{11}^*-y_{11}\right)+R_1\left(c_{12}^*-y_{12}\right)\Big)$$

2(b) There is full insurance if $c_{11}^* = c_{12}^*$. As

$$c_{11}^* - c_{12}^* = h_0\left(\frac{\beta}{1+\beta}\right)\Big(\frac{1}{m_1^*}-\frac{1}{m_2^*}\Big) = h_0\left(\frac{\beta}{1+\beta}\right) = \ldots = h_0\left(\frac{\beta}{1+\beta}\right)(R_2-R_1)\big(1-E[R]\big)$$

then there is full insurance if and only if $E[R] = 1$ (if we consider the assumption made in 1(a)). Then, if there is full insurance $\theta^* = \dfrac{y_{11}-y_{12}}{R_2-R_1}$. Because $COV(R, Y_1) = \pi_1\,\pi_2\,(R_1-R_2)(y_{11}-y_{12})$ then the agent will take a long (short) position if the return of the risky asset and its endowment at time $t = 1$ are negatively (positively) correlated. One possible intuition: if we want (and can) insure future consumption we should take a debt in a risky asset only if our future income is positively correlated to the associated interest payments. If not we should invest in it.

2(c) Because in a homogenous agent AD economy with $y_{11} \neq y_{12}$ we have at equilibrium $c_{1s}^{eq} = y_{1s}$ there is no full insurance at equilibrium (which is the case of the present pandemics).

Part 2 |

1(a) Applying the formulas [1], we find $IMRS_{0,1} = \left(\frac{1-\mu}{\mu}\right)\left(\frac{c_1}{c_0}\right)^{1-\eta}$, $\varepsilon_{0,0} = (1-\eta)\left(1-(1-\mu)\left(\frac{U}{c_0}\right)^{-\eta}\right)$, $\varepsilon_{0,1} = -(1-\eta)\,\mu\left(\frac{U}{c_1}\right)^{-\eta}$, $\varepsilon_{1,1} = (1-\eta)\left(1-\mu\left(\frac{U}{c_1}\right)^{-\eta}\right)$ and $IES_{0,1} = \frac{1}{1-\eta}$. Then, for a constant sequence $\{c, c\}$ we find

$$IMRS_{0,1}(c) = \frac{1-\mu}{\mu},\ \varepsilon_{0,1}(c) = -\mu(1-\eta),\ IES_{0,1}(c) = \frac{1}{1-\eta}$$

therefore: there is impatience if $0 < \mu < 1/2$, intertemporal substitutability (IS) if $\eta > 1$, intertemporal independence (II) if $\eta = 1$ and intertemporal complementarity (IC) if $\eta < 1$.

1(b) Solution to the representative household problem in an AD economy: $c_0^* = \frac{h_0}{1+q\,\phi(q)}$ and $c_1^* = \frac{\phi(q)\,h_0}{1+q\,\phi(q)}$, where $\phi(q) = \left(\left(\frac{1-\mu}{\mu}\right)q\right)^{\frac{1}{\eta-1}}$ and $h_0 = y_0 + qy_1$. There are two effects on c_0 of a change in q: first, a wealth effect $h'(q) = y_1 > 0$ and a substitution/complementarity effect

$$\frac{\partial}{\partial q}\left(\frac{1}{1+q\,\phi(q)}\right) = \frac{\eta}{(1-\eta)(1+q\phi(q))^2} \gtrless 0 \text{ if } \eta \lessgtr 1$$

i.e. the substitution effect is positive (negative) if there is IC (IS)

1(c) $q^{eq} = \frac{\mu}{1-\mu}(1+\gamma)^{\eta-1}$ the related interest rate is $1+r^{eq} = \frac{1-\mu}{\mu}(1+\gamma)^{1-\eta}$. Properties (see 1(a)): impatience implies $r^{eq} > 0$, and the interest rate response to growth of the endowment depends on η

$$\frac{\partial(1+r^{eq})}{\partial(1+\gamma)} = (1-\eta)(1+r^{eq})/(1+\gamma) \gtrless 0 \text{ if } \eta \lessgtr 1.$$

2(a) $CE(C_1) = e^{E[\ln(C_1)]}$ and $E[C_1] = e^{\ln(E[C_1])}$ then $E[C_1] = CE(C_1)$ if C_1 is state independent and, by Jensen's inequality, $E[C_1] > CE(C_1)$ if it is state-dependent. Therefore, the utility function displays risk-aversion.

[1] On the elementary calculus toolkit for economists: derivatives of a generalized mean, which is a function pervasive in economics (for production functions, price indexes, etc)

$$F = F(x_1, \dots, x_i, \dots, x_n) = \Big(\sum_{i=1}^{n} w_i x_i^{\eta}\Big)^{\frac{1}{\eta}}$$

If $n = 2$ and $\eta = 2$ and $w_i = 1$ this is the Pythagorean equation already known at least 3000 years ago.
First derivatives:

$$\begin{aligned} F_i &= \frac{\partial F}{\partial x_i} = \frac{1}{\eta}\Big(\sum_{i=1}^{n} w_i x_i^{\eta}\Big)^{\frac{1}{\eta}-1} w_i \eta x_i^{\eta-1} = \left(\Big(\sum_{i=1}^{n} w_i x_i^{\eta}\Big)^{\frac{1}{\eta}}\right)^{1-\eta} w_i x_i^{\eta-1} \\ &= F^{1-\eta} w_i x_i^{\eta-1} = w_i\Big(\frac{F}{x_i}\Big)^{1-\eta} \end{aligned}$$

This form simplifies the computation of second derivatives, and therefore, of elasticities

$$\begin{aligned} F_{ii} = \frac{\partial^2 F}{\partial x_i^2} &= (1-\eta)w_i\Big(\frac{F}{x_i}\Big)^{-\eta}\Big(F_i - \frac{F}{x_i}\Big)\frac{1}{x_i} = (1-\eta)w_i\Big(\frac{F}{x_i}\Big)^{-\eta}\Big(w_i\Big(\frac{F}{x_i}\Big)^{1-\eta} - \frac{F}{x_i}\Big)\frac{1}{x_i} \\ &= -(1-\eta)w_i\Big(\frac{F}{x_i}\Big)^{1-\eta}\Big(1 - w_i\Big(\frac{F}{x_i}\Big)^{-\eta}\Big)\frac{1}{x_i} \\ F_{ij} = \frac{\partial^2 F}{\partial x_i\, x_j} &= w_i(1-\eta)\Big(\frac{F}{x_i}\Big)^{-\eta}\Big(\frac{F_j}{x_i}\Big) = w_i(1-\eta)\Big(\frac{F}{x_i}\Big)^{-\eta}\Big(w_j\Big(\frac{F}{x_j}\Big)^{1-\eta}\Big)\frac{1}{x_i} \\ &= w_i w_j(1-\eta)\Big(\frac{F}{x_i}\Big)^{1-\eta}\Big(\frac{F}{x_j}\Big)^{1-\eta}\frac{1}{F} \end{aligned}$$

2(b) The solution to the household problem is

$$
\begin{aligned}
c_0^* &= \frac{h}{1+\Psi(M)} \text{ for } \Psi(M) \equiv \left((\frac{1-\mu}{\mu}) \, E[\ln{(M)}]^{\eta} \right)^{\frac{1}{\eta-1}} \\
c_{1s}^* &= \frac{\Psi(M)}{m_s} \frac{h}{1+\Psi(M)}, \; s = 1, 2
\end{aligned}
$$

where $m_s = q_s/\pi_s$ is the stochastic discount factor and $h = y_0 + E[MY_1]$. There are still two effects of a change in q_s on c_0 wealth and substitution/complementarity effects as in the deterministic case (although the second is slighly different)

2(c) The equilibrium stochastic discount factor (SDF) M^* is a distribution such that

$$
m_s^* = \frac{\mu}{(1-\mu)} e^{\eta E[\ln{(1+\Gamma)}]} \frac{1}{1+\gamma_s}, \; s = 1, 2
$$

The covariance of the SDF with the growth factor $1+\Gamma$ is COV $(M, 1+\Gamma) = E[M\,(1+\Gamma)] - E[M]\,E[1+\Gamma]$ but as $E[M\,(1+\Gamma)] = \frac{\mu}{(1-\mu)} e^{\eta E[\ln{(1+\Gamma)}]}$ and $E[M] = \frac{\mu}{(1-\mu)} e^{\eta E[\ln{(1+\Gamma)}]} E\left[\frac{1}{1+\Gamma}\right] = E[M\,(1+\Gamma)]\; E\left[\frac{1}{1+\Gamma}\right]$, then $\text{COV}(M, 1+\Gamma) = E[M\,(1+\Gamma)] \left(1 - E\left[\frac{1}{1+\Gamma}\right] E[1+\Gamma]\right) < E[M\,(1+\Gamma)] \left(1 - \frac{E[1+\Gamma]}{E[1+\Gamma]}\right) = 0$ because $E\left[\frac{1}{1+\Gamma}\right] > E[1+\Gamma]$ by Jensen's inequality. Conclusion: independently from η there is a negative (positive) correlation between M $(R = 1/M)$ and the growth factor.