# The Ramsey growth model

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# A short history of the model

- Frank Ramsey (see https://en.wikipedia.org/wiki/Frank_P._Ramsey) made several important contributions in his short life (he died at 26) one of them Ramsey (1928)
- His contribution was only fully recognized in the early 1960's ( Cass (1965), Koopmans (1965)) as presenting a rigorous alternative to the ad-hoc aspects (dynamic inefficiency) of the Solow (1956) model (now we call it **exogenous growth theory**)
- It was rejoined again in the middle of the 1980's which saw the onset of **endogenous growth theory**
- It is also the founding rock of the DGE (dynamic general equilibrium theory) of macroeconomics

# The Ramsey model

## The basic idea

- Output is produced by physical capital and labor and can be used for investment or for consumption (everything in per capita terms): this introduces an **intratemporal budget constraint**
- **savings** is determined by a **arbitrage between present and future consumption**: it balances two effects:
  - present consumption is a good thing, although its utility decreases with the amount consumed;
  - however, if people sacrifice present consumption, by saving and increasing the capital stock, they improve their prospects for more consumption in the future;
- this idea can be formalized as an **intertemporal optimization problem**

# The Ramsey model

## Assumptions

- Production:
  - closed economy producing a single composite good
  - production uses two factors: labor and physical capital
  - production technology: neoclassical (increasing, concave, Inada, CRTS)
- Reproducible factor:
  - physical capital (machines)
- Population:
  - exogenous (can be constant or increase exponentially)

# The Ramsey model

## Assumptions: continuation

- Households: optimizing behavior
  - maximize an intertemporal utility functional with consumption as the control variable
  - subject to a budget constraint
  - labor is supplied inelastically
  - have perfect foresight
- Equilibrium is Pareto optimal, therefore it is equivalent to a social welfare problem

# Ramsey model

## The model: production technology

- In aggregate terms

$$Y(t) = F(A, K(t), L(t)) = AK(t)^{\alpha} L(t)^{1-\alpha}, \; 0 < \alpha < 1$$

  where: $A$ TFP productivity, $K$ stock of capital, $L = N$ loabor input = population

- In per capita terms:

$$y(t) = Ak(t)^{\alpha}$$

  where $y = Y/N$ and $k = K/N$

# Ramsey model

## The model: preferences

Preferences: of the representative agent

- the intertemporal utility functional is

$$V[c] = \int_0^{\infty} u(c(t))\, e^{-\rho t} dt$$

- $c = C/N$ per capita consumption, $[c] = \big(c(t)\big)_{t \in [0,\infty)}$
- $\rho > 0$ is the rate of time preference
- the instantaneous utility function is

$$u(c) = \begin{cases} \dfrac{c^{1-\theta} - 1}{1-\theta}, & \text{if } \theta \in (0,\infty)/\{1\} \\ \ln(c), & \text{if } \theta = 1 \end{cases}$$

where $1/\theta$ is the elasticity of intertemporal substitution

# Ramsey model

## Versions

- We are assuming an **homogeneous agent** (or representative) economy
- There are two versions of the model
  - **centralized** version: maximization of social welfare given the budget constraint
  - **decentralized** (DGE) version: individual maximization of households an firms coordinated by market equilibrium
- As there are no externalities they are **equivalent** (in the sense that generate the **same allocations**, of consumption and capital, over time)

Centralized version:
the Ramsey model:

# Ramsey model

## The centralized version

- The central planner is a "benevolent dictator" (acts on behalf of the best interests of the society)
- The central planner solves the problem

$$\max_{(c)_{t\geq 0}} \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

$$\begin{aligned}
&\text{subject to}\\
&\dot{k} = G(k) \equiv Ak(t)^\alpha - c(t) - \delta k(t),\\
&k(0) = k_0 > 0, \text{ given}\\
&\lim_{t\to\infty} h(t)k(t) \geq 0
\end{aligned}$$

physical capital is asymptotically bounded ($h(t)$ is any discount factor)

# Ramsey model

## Solving by using the Pontriyagin's max principle

- The current-value Hamiltonian is

$$H(c,k,q) = \frac{c^{1-\theta}-1}{1-\theta} + q\left(Ak^{\alpha} - c - \delta k\right)$$

- the optimality conditions are

$$\frac{\partial H}{\partial c} = 0 \text{ (optimality condition)}$$

$$\dot{q} = \rho q - \frac{\partial H}{\partial k} \text{ (Keynes-Ramsey rule)}$$

$$\lim_{t\to\infty} q(t)k(t)e^{-\rho t} = 0 \text{ (transversality condition)}$$

- the admissibility conditions

$$\dot{k} = G(k) \text{ (aggregate constraint)}$$

$$k(0) = k_0 > 0 \text{ (initial condition)}$$

# Ramsey model

## Solving by using the Pontriyagin's max principle

- The current-value Hamiltonian is

$$H(c,k,q) = \frac{c^{1-\theta}-1}{1-\theta} + q\left(Ak^{\alpha} - c - \delta k\right)$$

- the optimality conditions are

$$c^{-\theta}(t) = q(t), \text{ for } t \in [0,\infty)$$

$$\dot{q} = q(t)\left(\rho + \delta - \alpha A k(t)^{\alpha-1}\right), \text{ for } t \in [0,\infty)$$

$$\lim_{t\to\infty} q(t)k(t)e^{-\rho t} = 0, \text{ for } t = \infty$$

- the admissibility conditions

$$\dot{k} = Ak(t)^{\alpha} - c(t) - \delta k(t), \text{ for } t \in [0,\infty)$$

$$k(0) = k_0 > 0, \text{ for } t = 0$$

# Ramsey model

## The modified Hamiltonian dynamic system

- An optimum path $(c^*(t), k^*(t))_{t\in[0,+\infty)}$ is the solution of the (MHDS)

$$
\begin{aligned}
\dot{k} &= Ak(t)^\alpha - c(t) - \delta k(t) && (1)\\
\dot{c} &= \frac{c}{\theta}\left(r(k(t)) - \rho - \delta)\right) && (2)\\
0 &= \lim_{t\to\infty} c(t)^{-\theta} k(t) e^{-\rho t} && (3)\\
k(0) &= k_0 \text{ given} && (4)
\end{aligned}
$$

- where the (gross) rate of return of capital

$$r(k) = \alpha A k^{\alpha-1}$$

# Solving the Ramsey model

- In general this system **does not have an explicit solution** (also called exact or closed form)
- We can only find an **exact solution** for the case $\theta = \alpha$ (which is counterfactual)
- Analytical methods for finding the solution (unique way to solve it if $\theta \neq \alpha$): **linear approximation** of the solution converging to the steady state, which satisfies the transversality constraint
- In all cases, **it is always a good idea to build the phase diagram**

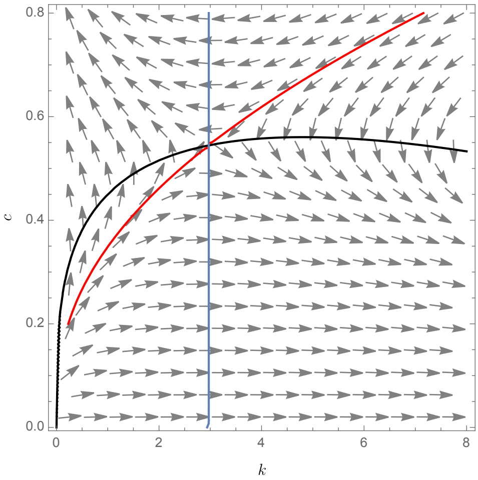
# Solving the Ramsey model

## By linear approximation

- step 0: try to draw the phase diagram of system (1)-(2)
- step1: find the steady state (consistent with the transversality and initial conditions)
- step 2: find the linear approximation of system (1)-(2) in the neighborhood of the steady state
- step 3: find the general solution of the linearized system
- step 4: find the particular solution by using the transversality and the initial conditions
- step 5: sit back and try to understand the meaning of the solution

# Solving the Ramsey model

## step 0: phase diagram



# Solving the Ramsey model

## step 1: Steady states

- Steady states are fixed points of the system

$$
\begin{aligned}
c^{ss} &= A(k^{ss})^{\alpha} - \delta k^{ss}, \\
\frac{c^{ss}}{\theta}\left(r(k^{ss}) - (\rho+\delta)\right) &= 0.
\end{aligned}
$$

- there are three steady states

$$(k^{ss}, c^{ss}) = \{(0,0), ((A/\delta)^{1/(1-\alpha)}, 0), (\bar{k}, \bar{c})\}$$

  for

$$\bar{k} = \left(\frac{\alpha A}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}, \ \bar{c} = \beta\, \bar{k}$$

  where $\beta \equiv \frac{\rho+\delta(1-\alpha)}{\alpha}$
- $(\bar{k}, \bar{c})$ satisfies the transversality condition but $((A/\delta)^{1/(1-\alpha)}, 0)$ does not

# Solving the Ramsey model

## step 2: linearized MHDS

- The linearised MHDS in the neighbourhood of $(\bar{c}, \bar{k})$ is

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \dfrac{\bar{c}r^{'}(\bar{k})}{\theta} & 0 \end{pmatrix} \begin{pmatrix} k(t) - \bar{k} \\ c(t) - \bar{c} \end{pmatrix}$$

- where $r^{'}(\bar{k}) = (\alpha - 1)\alpha A k^{\alpha-2}\Big|_{k=\bar{k}} = -\dfrac{(1-\alpha)(\rho+\delta)}{\bar{k}} < 0$
- and $\dfrac{\bar{c}r^{'}(\bar{k})}{\theta} = -d \equiv -\dfrac{(1-\alpha)\,\beta\,(\rho+\delta)}{\theta} < 0$

# Solving the Ramsey model

## step 3: finding the general solution of the linearized MHDS

- the system is of type $\dot{x} = Jx$
- where the Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} \rho & -1 \\ -d & 0 \end{pmatrix}$$

- the solution is of type

$$x(t) = h_s \mathbf{V}^s e^{\lambda_s t} + h_u \mathbf{V}^u e^{\lambda_u t}$$

- where $\lambda_j$ are the eigenvalues, and $\mathbf{V}^j$ are the associated eigenvectors of $J$, and $(h_s, h_u)$ are arbitrary constants

# Solving the Ramsey model

## step 3: finding the general solution of the linearized MHDS

- the eigenvalues of $\mathbf{J}$ are

$$
\begin{aligned}
\lambda_u &= \frac{\rho}{2} + \left[\left(\frac{\rho}{2}\right)^2 + d\right]^{1/2} > \rho > 0 \\
\lambda_s &= \frac{\rho}{2} - \left[\left(\frac{\rho}{2}\right)^2 + d\right]^{1/2} < 0
\end{aligned}
$$

- satisfying $\lambda_s + \lambda_u = \rho > 0$, $\lambda_s \lambda_u = -d < 0$
- then $(\bar{k}, \bar{c})$ is a **saddle-point**

# Solving the Ramsey model

## step 3: finding the general solution of the linearized MHDS

- the eigenvectors are determined as follows
- $\mathbf{V}^s$ solves the homogeneous system

$$\left(\mathbf{J} - \lambda_s \mathbf{I}_2\right) \mathbf{V}^s = \mathbf{0}$$

- that is

$$\begin{pmatrix} \rho - \lambda_s & -1 \\ -d & -\lambda_s \end{pmatrix} \begin{pmatrix} \mathbf{V}_1^s \\ \mathbf{V}_2^s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- the members of vector $\mathbf{V}^s$ should satisfy (because $\rho - \lambda_s = \lambda_u$ and $\lambda_s \lambda_u = -d$)

$$\lambda_u \mathbf{V}_1^s - \mathbf{V}_2^s = 0 \implies \mathbf{V}^s = \begin{pmatrix} 1 \\ \lambda_u \end{pmatrix}$$

- for $\mathbf{V}^u$ we find (prove this)

$$\lambda_s \mathbf{V}_1^u - \mathbf{V}_2^u = 0 \implies \mathbf{V}^u = \begin{pmatrix} 1 \\ \lambda_s \end{pmatrix}$$

# Solving the Ramsey model

## step 4: finding the particular solution of the linearized MHDS

- Then the general solution is

$$\begin{pmatrix} k(t)-\bar{k} \\ c(t)-\bar{c} \end{pmatrix} = h_s \begin{pmatrix} 1 \\ \lambda_u \end{pmatrix} e^{\lambda_s t} + h_u \begin{pmatrix} 1 \\ \lambda_s \end{pmatrix} e^{\lambda_u t}$$

- We determine $h_s$ and $h_u$ by forcing the general solution to satisfy the two remaining conditions

$$\lim_{t\to\infty} \frac{k(t)}{c(t)^{\theta}} e^{-\rho t} = 0, \text{ and } k(0) = k_0$$

  - the first condition holds if $\lim_{t\to\infty}(c(t)-\bar{c}) = \lim_{t\to\infty}(k(t)-\bar{k}) = 0$, i.e., they converge to the steady state, which is obtained by eliminating the effect of $e^{\lambda_u t}$ (because $\lim_{t\to\infty} e^{\lambda_u t} = \infty$) by setting $h_u = 0$
  - the second condition holds if

$$k(0)-\bar{k} = h_s = k_0 - \bar{k} \implies h_s = k_0 - \bar{k}$$

# Solving the Ramsey model

## step 4: finding the particular solution of the linearized MHDS

- the approximate solution is, for $t \in [0, \infty)$

$$\begin{aligned} k(t) &= \bar{k} + (k_0 - \bar{k})e^{\lambda_s t} \\ c(t) &= \bar{c} + \lambda_u (k_0 - \bar{k})e^{\lambda_s t}. \end{aligned} \tag{5}$$

- where

$$\bar{k} = \left( \frac{\alpha A}{\delta + \rho} \right)^{\frac{1}{1-\alpha}}, \; \bar{c} = \beta \, \bar{k}$$

  and $\lambda_u > \rho > 0 > \lambda_s$.

# Solving the Ramsey model

## step 5: understanding the solution

- at $t = 0$ we have

$$\begin{pmatrix} k(0) \\ c(0) \end{pmatrix} = \begin{pmatrix} k_0 \\ \bar{c} + \lambda_u (k_0 - \bar{k}) \end{pmatrix}$$

  observe that $\lambda_u$ gives the variation of consumption as $c(0) - \bar{c} = \lambda_u (k_0 - \bar{k})$ and the initial consumption is determined from **future data** ($\bar{c}$ and $\bar{k}$)

- asymptotically (i.e., in the long run)

$$\lim_{t \to \infty} \begin{pmatrix} k(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \end{pmatrix} \bar{k}$$

  the solution converges to the steady state (this means that the transversality condition is satisfied)

- **the saddle path dynamics implies that the solution is unique**. We say it is **determinate**

# Ramsey model

## Case $\theta \neq \alpha$ benchmark case: phase diagrams for $\theta > \alpha$

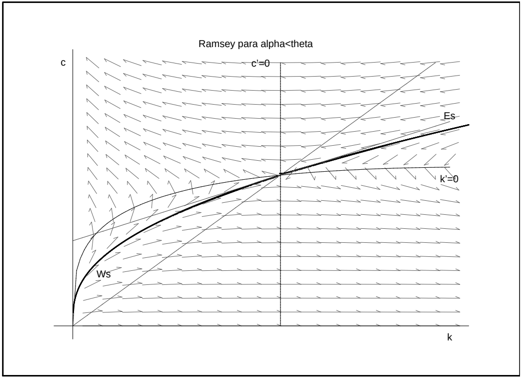



Figure: Exact (dark) and approximate (light) solutions

# Ramsey model

## Case $\theta \neq \alpha$: phase diagrams for $\theta < \alpha$

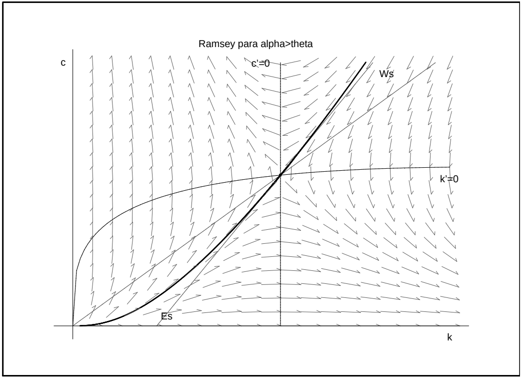
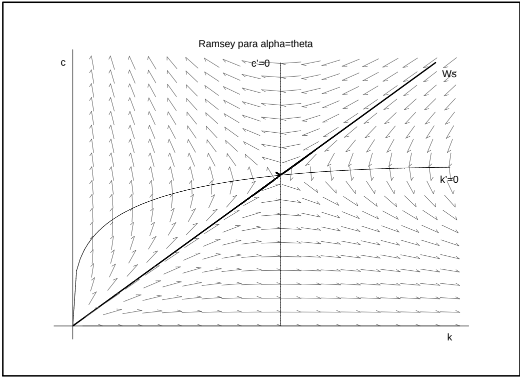



Figure: Exact (dark) and approximate (light) solutions

# Ramsey model

## Case $\theta = \alpha$: phase diagram



# Ramsey model

## Growth implications

- The unique steady state which satisfies the initial and the transversality conditions is

$$\bar{k} = \left(\frac{\alpha A}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}, \ \bar{c} = \beta \, \bar{k}$$

for $\beta \equiv \frac{\rho+\delta(1-\alpha)}{\alpha}$

- the associated long-run GDP is

$$\boxed{\bar{y} = A\bar{k}^{\alpha} = \left[A\left(\frac{\alpha}{\delta + \rho}\right)^{\alpha}\right]^{\frac{1}{1-\alpha}}.} \tag{6}$$

# Ramsey model

## Per-capita GDP dynamics

- the **approximate** per-capita output path is generated by

$$y(t) = \left[\bar{y}^{1/\alpha} + (y(0)^{1/\alpha} - \bar{y}^{1/\alpha})e^{\lambda_s t}\right]^{\alpha} \tag{7}$$

the model only displays transitional dynamics as $\lambda_s < 0$.

- the solution converges asymptotically to the steady state

$$\lim_{t\to\infty} y(t) = \bar{y} = \left[A\left(\frac{\alpha}{\delta+\rho}\right)^{\alpha}\right]^{1/(1-\alpha)}$$

# The Ramsey model

## Growth implications

1. there is **no long-run growth** $\bar{g} = 0$
2. the **long-run level** $\bar{y}$ depends on $(A, \delta, \rho, \alpha)$: productivity, the rate of depreciation, the rate of time preference (impatience) and on the income shares (see equation (6));
3. there is **only transitional dynamics**: the **speed** and the pattern of convergence depends on the relationship between the capital share, $\alpha$, in income and the intertemporal elasticity of substitution $\theta$ (see equation (7)). This is because

$$\lambda_s = \frac{\rho}{2} - \left[\left(\frac{\rho}{2}\right)^2 + \frac{(1-\alpha)\,\beta\,(\rho+\delta)}{\theta}\right]^{\frac{1}{2}} < 0$$

the higher $|\lambda_s|$ is the faster the transition speed is.

Decentralized version:
dynamic general equilibrium (DGE) model

# The Neoclassical DGE model

## Assumption

- Representative household: has initial financial wealth $b(0)$, receives has wage income $w$ and financial income $(rb)$, and decides on consumption $(c)$ and savings $(\dot{b})$ ;
- Households own firms with physical capital $(k)$ which is only financed by bonds: thus $b = k$. Firms transform capital and labor into output $(y)$
- There are accounting restrictions.
- All markets are competitive
- Other assumptions: infinite-lived households with isoelastic utility and Cobb-Douglas production, function and no frictions.

# Household problem

- Household's problem: maximize discounted intertemporal utility subject to a financial constraint

$$\max_{c(.)} \int_0^{\infty} \frac{c(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{b} = r(t)b(t) + w(t) - c(t),\ t \geq 0$$

$$b(0) = b_0$$

$$\lim_{t \to \infty} e^{-\int_t^{\infty} r(s)\, ds} \geq 0$$

where $b =$bonds, $w =$wage

# Household problem

- Optimality conditions

$$\dot{c} = \frac{c(t)}{\theta}\left(r(t) - \rho\right)$$
$$\lim_{t \to \infty} e^{-\rho t} c(t)^{-\theta} b(t) = 0$$

- Admissibility conditions

$$\dot{b} = r(t)b(t) + w(t) - c(t), \ t \geq 0$$
$$b(0) = b_0$$

# The firm's problem

- Firm's problem (price taker in all the markets): maximizes present value of profits

$$\max_i \int_0^\infty \left(Ak(t)^\alpha - w(t) - i(t)\right) e^{-R(t)}\,dt$$

subject to

$$\dot{k} = i - \delta k$$

$$k(0) = k_0$$

- observations
  - the discount factor is the (endogeneous) market interest rate $R(t) = \int_t^\infty r(s)\,ds$
  - the control variable: investment expenditure
  - no adjustment cost: investment expendiure is equal to gross investment
  - constraint: net investment = gross investment minus depreciation

# The firm's problem

- Hamiltonian

$$H(i, k, q) = Ak^{\alpha} - w - i + q(1 - \delta k)$$

- Optimality conditions:
  $$\frac{\partial H(i, k, q)}{\partial i} = 0 \iff q(t) = 1, \text{ for all } t \geq 0$$
- Canonical equation

$$\dot{q} = q\left(r(t) + \delta - \alpha\, A\, k^{\alpha - 1}\right)$$

- As $q = 1$ then $\dot{q} = 0$ then this is equivalent to

$$r(t) = \alpha A k(t)^{\alpha - 1} - \delta, \text{ for all } t \geq 0$$

# The general equilibrium determination

- Micro-macro constraints and equilibrium conditions:
  - Accounting identity $b(t) = k(t)$,
  - Then $\dot{b}(t) = \dot{k}(t)$,
  - Market equilibrium condition

$$y = c + i$$

- From

$$\dot{b}(t) = \dot{k}(t) \iff rb + w - c = i - \delta k \iff rk + w + \delta k = y$$
$$\iff y - \delta\, k = Ak^{\alpha} - \delta\, k = r\, k + w$$

- Then we get

$$\dot{k} = A\, k^{\alpha} - c - \delta\, k$$

# The general equilibrium

- We obtain the same dynamic system as in the Ramsey model

$$\dot{k} = Ak(t)^{\alpha} - c(t) - \delta k(t)$$

$$\dot{c} = \frac{c(t)}{\theta}\left(r(t) - \rho\right), \text{ where } r(t) = \alpha A k(t)^{\alpha-1} - \delta$$

- Then the allocations of $c$ and $k$ are equal: we say that the **equilibrium is Pareto efficient**)

# Comparative dynamics

- Assume the economy is in a steady state $\bar{c}_0$ and $\bar{k}_0$ for the initial $A_0 = A$. We take $\bar{k}_0$ as an initial point (or previous $k_0$)
- Shock: unanticipated, permanent, increase in TPF $A_1 = A_0 + dA$ for $dA > 0$, or $dA < 0$ as a result of the pandemic or war
- We write $(\bar{k}_1, \bar{c}_1)$ the steady state associated to $A_1$ and take as the new steady state (our previous $(\bar{k}, \bar{c})$)
- Steady state multipliers

$$\frac{\bar{k}_1 - \bar{k}_0}{dA} = d_A \bar{k}(A)\Big|_{\bar{k}=\bar{k}_0, A=A_0} = \frac{\bar{k}_0}{(1-\alpha)\, A_0}$$

$$\frac{\bar{c}_1 - \bar{c}_0}{dA} = d_A \bar{c}(A)\Big|_{\bar{k}=\bar{k}_0, A=A_0} = \beta\, d_A \bar{k}(A)\Big|_{\bar{k}=\bar{k}_0, A=A_0}$$

# Comparative dynamics

- The change in the variables in the transition are

$$d_A k(t) = \frac{k(t) - \bar{k}_0}{dA}, \; d_A c(t) = \frac{c(t) - \bar{c}_0}{dA}$$

- From equation (5) we have

$$d_A k(t) = d_A \bar{k} \left(1 - e^{-\lambda_s t}\right), \text{ for } t \geq 0$$

$$d_A c(t) = d_A \bar{c} - \lambda_u \, d_A \bar{k} \, e^{-\lambda_s t} = d_A \bar{k} \left(\beta - \lambda_u \, e^{-\lambda_s t}\right), \text{ for } t \geq 0$$

- where the long run multipliers are

$$d_A k(\infty) = d_A \bar{k} > 0$$

$$d_A c(\infty) = d_A \bar{c} = \beta \, d_A \bar{k} > 0$$

- the impact multipliers are

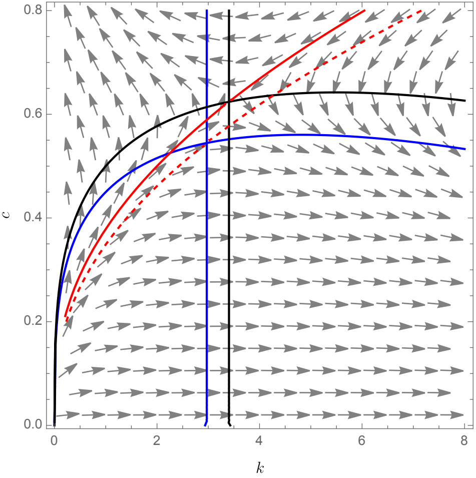
$$d_A k(0) = 0$$

$$d_A c(0) = d_A \bar{k} \left(\beta - \lambda_u\right) \gtreqless 0 \iff \beta \gtreqless \lambda_u$$

(prove this)

# Solving the Ramsey model

## Phase diagram for a productivity shock



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