Mathematical Economics

Deterministic dynamic optimization

Continuous time

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Contents

Chapter 1

Introduction

This note deals with dynamic optimisation in continuous time, for one-dimensional problems.

The state of a system is represented by function $x : \mathbb{T} \to \mathbb{R}$, where the domain of the independent variable is a subset of $\mathbb{R}_+$, $\mathbb{T} \subseteq \mathbb{R}_+$, and $x(t) \in \mathbb{R}$, that is, we deal with scalar, or one-dimensional, problems. We represent the **path** or orbit of x by $x = (x(t))_{t\in\mathbb{T}}$ to distinguish it from the value taken at time t, $x(t)$.

Again, we want to find optimal paths, $x^* = (x^*(t))_{t\in\mathbb{T}}$ which solve problems of dynamic optimization problems. Two ypes of problems are considered

1. calculus of variations problems

2. optimal control problems

Optimal control problems can be solved by one of the following methods

1. by using the Pontriyagin's principle

2. by using the dynamic programming.

Some optimal control problems can be converted into calculus of variations problems, although the reverse is not always possible. This means that calculus of variations can provide another method for solving optimal control problems.

The **calculus of variations problem** is : find a flow $x^* \equiv (x^*(t))_{t\in\mathbb{T}}$, where $\mathbb{T} = [0, T]$, or $\mathbb{T} = [0, T)$, that maximises the value functional

$$V[x] = \int_0^T F(\dot{x}(t), x(t), t)dt + S(x(T), T)$$

given the initial value $x(0) = x_0$ and possibly some additional information on the terminal state $x(T)$ and/or T. [1].

The **optimal control problem** is: find the flows $x^* \equiv (x^*(t))_{t\in[0,T]}$, $u^* \equiv (u^*(t))_{t\in[0,T]}$ such that u^* maximizes the value functional

$$V[x, u] = \int_0^T F(u(t), x(t), t)dt + S(x(T), T)$$

subject to the differential equation

$$\dot{x} = g(t, u(t), x/t))$$

given the initial value $x(0) = x_0$ and possibly some additional information on the terminal state $x(T)$ and/or T.

The two most important examples in economics are:

- the consumption-asset accumulation problem

  $$\max_C \int_0^T u(C)t))e^{-\rho t}dt$$

  subject to $\dot{W} = Y - C(t) + rW$ where $W(0) = W_0$ is the initial level of financial wealth and some other conditions can be imposed

- the production investment problem for a firm

  $$\max_I \int_0^T PY(K(t)) - D(I(t), K(t))e^{-rt}dt + S(K(T), T)$$

  subject to $\dot{K} = I(t) - \delta K$, where $K(0) = K_0$ is the initial level of the capital stock.

[1]We use the common notation for time-derivatives $\dot{x}(t) \equiv \frac{dx(t)}{dt}$.

Chapter 2

Calculus of variations

The Calculus of Variations (CV) one important difference regarding static optimization. While statis optimization consists in finding an extremum **number** the calculus of variations consists in finding a best **function**. This function has its domain in a specified space, and the solution traces out a path in that space. Therefore, we can also say that the CV consists in finding a best path (in time, in some, or along any other domain).

However, CV has an important analogy with static optimization.

In static optimization we want to maximize a **function** (satisfying given constraints): it generally takes the form

$$x^* = \arg\ \max f(x).$$

The objective function, $f(x)$ is a mapping

$$f : \text{number} \to \text{number}$$

We look for, in the range of the function $f(.)$, the maximum, $f(x^*)$, and find x^* which is the solution to our problem (if it is unique).

In CV we want to maximize a **functional** (satisfying given constraints): it takes the form

$$(x^*(t))_{t \in \mathbb{T}} = \arg\ \max V[x] = \int_0^T F(t, x(t), \dot{x}(t))dt.$$

The objective functional, $V[x]$ is a mapping

$$V : \text{function} \rightarrow \text{number}$$

Again we look for, in the range of the functional $V[x]$, the maximum $V[x^*]$, and find $(x^*(t))_{t\in\mathbb{T}}$ as the solution to our problem (if it is unique).

Next we look at the CV problem from one of the two equivalent perspectives:

- as a problem of finding the element $x^* \in \mathscr{X}$ on the space of functions, $\mathscr{X}$, whose elements are mapping $x : \mathbb{T} \rightarrow \mathcal{X} \subseteq \mathbb{R}$ such that they maximize the functional $V[x]$, i.e.,

$$V[x^*] = \max_x V[x]$$

- as a problem of finding the path $(x^*(t))_{t\in\mathbb{T}}$ that maximizes the value functional

$$V[x^*] = \max_{(x(t))_{t\in\mathbb{T}}} \int_0^T F(t, x(t), \dot{x}(t))dt.$$

Next we consider the following general calculus of variations problem with **value functional**

$$V[x] = \int_0^T F(t, x(t), \dot{x}(t))dt + S(x(T), T) \tag{2.1}$$

together with **boundary** and **terminal time** conditions. From now on we call $x : \mathbb{T} \rightarrow \mathbb{R}$ the **state variable**, because it measures the system we are trying to optimize.

Calculus of variations problems have the following components:

1. the **value functional** as in equation (2.1) has two components: the **intertemporal objective functional** (in the independent variable is time)

$$J[x] = \int_0^T F(t, x(t), \dot{x}(t))dt$$

   and the **scrap function** $S(x(T), T)$. While $J[x]$ evaluates a path $S(x(T), T)$ evaluates the terminal value of the state. We assume throughout that both $F(.)$ and $S(.)$ are continuous, differentiable and smooth ($F \in C^2(\mathbb{R}^3)$ and $S \in C^2(\mathbb{R}^2)$) in all its arguments and that $F(.)$ concave in $(x, \dot{x})$;

2. **boundary conditions** always involve a fixed initial state, $x(0) = x_0$, and one of the following conditions regarding the **terminal state and time**:

   (a) fixed terminal state and time $x(T) = x_T$ and T fixed

   (b) free terminal state and T fixed

   (c) strongly constrained terminal state $\psi(x(T), T) = 0$ and T fixed

   (d) weakly constrained terminal state $\psi(x(T), T) \geq 0$ and T fixed

   (e) strongly constrained terminal state $\psi(x(T), T) = 0$ and T free

   (f) weakly constrained terminal state $\psi(x(T), T) \geq 0$ and T free

We can also consider other side constraints over the state of the system:

1. iso-perimetric constraints
$$\int_0^T G(t, x(t))dt \leq 0$$

2. instantaneous constraints
$$h(t, x(t), \dot{x}(t)) \leq 0, \text{ for } t \in (0, T)$$

We will consider the following problems:

- the simplest problem (see section 2.1.1): find $\max_x V[x]$ such that T is finite and fixed and $x(0) = x_0$ and $x(T) = x_T$ are given;

- the simplest free terminal state problem: find $\max_x V[x]$ such that T is finite and given and $x(0) = x_0$ is given and $x(T)$ is free;

- infinite horizon discounted problem: $F(x(t), \dot{x}(t), t) = e^{-\rho t} f(x(t), \dot{x}(t))$, where $\rho > 0$ and $T = \infty$

2.1 The simplest problems

2.1.1 Fixed end state and time

The problem: find a path $x^* = (x^*(t))_{t\in\mathbb{T}}$ that solves the problem

$$V[x^*] = \max_{(x(t))_{t\in[0,T]}} \int_0^T F(t, x(t), \dot{x}(t))dt$$
$$\text{subject to} \begin{cases} x(0) = x_0 \text{ given} & t = 0 \\ x(T) = x_T \text{ given} & t = T \\ T \text{ given} \end{cases} \tag{CV1}$$

We say that x **is admissible** if it verifies the restrictions of the problem, i.e if $x \in \mathscr{X}$ where is the set of functions satisfying the initial and terminal conditions $\mathscr{X} = \{x(t) \in \mathbb{R} \ : \ x(0) = x_0, \text{ and } x(T) = x_T]$. We say that a path x^* *is optimal* if it is admissible and it maximizes the functional (2.1); therefore it is a solution of the problem (CV1).

Proposition 1 (**First-order necessary conditions for problem** (CV1))**.** *Assume that x^* is optimal. Then it verifies the Euler equation together with the boundary conditions*

$$\begin{cases} F^*_x(t) = \frac{d}{dt}\left(F^*_{\dot{x}}(t)\right), & 0 \le t \le T \text{ (Euler-Lagrange equation)} \\ x^*(0) = x_0, & t = 0 \text{ (boundary condition: initial)} \\ x^*(T) = x_T, & t = T \text{ (boundary condition: terminal)} \end{cases} \tag{2.2}$$

We use the notations $F^*(t) = F(t, x^*(t), \dot{x}^*(t))$, $F^*_x(t) = F_x(t, x^*(t), \dot{x}^*(t))$ and $F^*_{\dot{x}}(t) = F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))$ where

$$F_x(t, x^*(t), \dot{x}^*(t)) \equiv \left.\frac{\partial F(t, x(t), \dot{x}(t))}{\partial x}\right|_{x=x^*}$$

and

$$F_{\dot{x}}(t, x(t), \dot{x}(t)) \equiv \left.\frac{\partial F(t, x(t), \dot{x}(t))}{\partial \dot{x}}\right|_{x=x^*}.$$

Denoting the second order partial derivatives, evaluated at the optimal path x^*, by F^*_{xx}, $F^*_{\dot{x}\dot{x}}$ and $F^*_{x\dot{x}}$ we observe that

$$\frac{d}{dt}F^*_{\dot{x}} = F^*_{\dot{x}t} + F^*_{\dot{x}x}\dot{x}^* + F^*_{\dot{x}\dot{x}}\ddot{x}^*.$$

This implies that the Euler equation can be written as a second order ordinary differential equation, if $F^*_{\dot{x}\dot{x}}\ddot{x}^* \neq 0$

$$F^*_x = F^*_{\dot{x}t} + F^*_{\dot{x}x}\dot{x}^* + F^*_{\dot{x}\dot{x}}\ddot{x}^*,\ 0 \leq t \leq T.$$

Example 1: the cake eating problem Let a cake have size W_0 at the initial time $t = 0$ and assume we want to eat the cake completely until time $T > 0$. The problem is: which is the best eating strategy, or, equivalently, what should be the size of the cake during the interval between $t = 0$ and $t = T$. We are characterized by the fact that we are impatient, and value instantaneoulsly the cake by a logarithmic function.

Formally, the problem is to find the optimal flows of cake munching $C^* = (C^*(t))_{t\in[0,T]}$ and of the size of the cake $W^* = (W^*(t))_{t\in[0,T]}$ such that

$$\max_C \int_0^T \ln(C(t))e^{-\rho t}dt,\ \text{subject to } \dot{W} = -C,\ t \in (0, T),\ W(0) = W_0,\ W(T) = 0 \tag{2.3}$$

where $W_0 > 0$ is given. In that form, the problem is formulated as an optimal control problem. To represent it as a calculus of variations problem, we write it directly over the state variable W:

$$\max_W \int_0^T \ln(-\dot{W}(t))e^{-\rho t}dt,$$

subject to

$$\begin{cases} W(0) = W_0 & \text{for } t = 0 \\ W(T) = 0 & \text{for } t = T. \end{cases} \tag{2.4}$$

The value function is $F(t, W, \dot{W}) = \ln(-\dot{W})e^{-\rho t}$, is only defined for $\dot{W} < 0$, and has derivatives

$$F_W(t, W, \dot{W}) = 0,\ F_{\dot{W}}(t, W, \dot{W}) = -\frac{e^{-\rho t}}{\dot{W}(t)}$$

Then

$$\frac{d}{dt}F_{\dot{W}}(t, W, \dot{W}) = -\frac{\ddot{W}}{\dot{W}}e^{-\rho t}$$

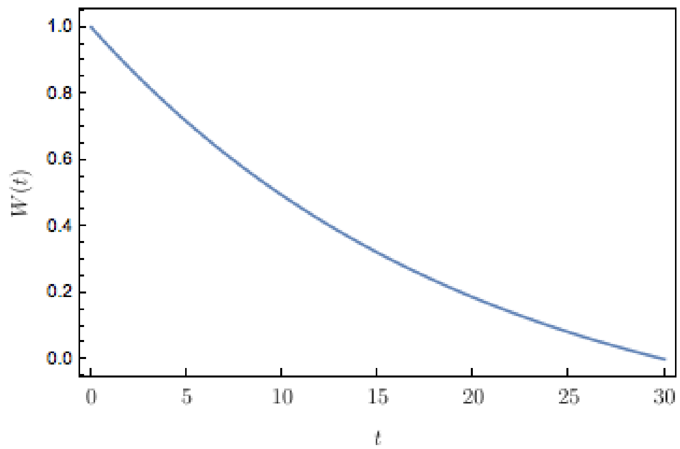

Figure 2.1: Optimal path for the cake problem

and the Euler-Lagrange equation becomes

$$\ddot{W} + \rho \dot{W} = 0. \tag{2.5}$$

This is a second order linear ordinary differential equation. The solution to the problem is function $W^*(t)$ that solves the system (2.5) together with the constraints (2.4):

$$W^*(t) = W_0 \frac{e^{-\rho t} - e^{-\rho T}}{1 - e^{-\rho T}},\ t \in [0, T], \tag{2.6}$$

and for consumption, as $C^*(t) = -dW^*(t)/dt$,

$$C^*(t) = \rho W_0 \frac{e^{-\rho t}}{1 - e^{-\rho T}},\ t \in [0, T]. \tag{2.7}$$

The solution for the cake size if depicted in Figure 2.1

Solving the model

Steps: (1) solving the EL equation together with the initial condition; (2) using the boundary terminal condition

There are several methods to solve the equation:

We can use two alternative methods to find the optimal size of the cake $(W^*)_{t\in[0,T]}$

Method 1: recursive solution If we defining $C(t) = -\dot{W}(t)$, we can transform equation (2.5) into an equivalent system of first order ODE

$$\begin{cases} \dot{W} = -C(t) \\ \dot{C} = -\rho C \end{cases}$$

where $W(0) = W_0$. The (general) solutions are

As it is a recursive system, we solve the two equations independently. Solving the second equation we get

$$C(t) = C(0)e^{-\rho t}$$

where $C(0)$ is unknown. Then, plugging this expression into the solution of the first equation we have

$$\begin{aligned} W(t) &= W_0 - \int_0^t C(s)ds = \\ &= W_0 - \int_0^t C(0)e^{-\rho s}ds. \end{aligned}$$

Then the solution of the EL equation together with the initial condition is

$$W(t) = W_0 - \frac{C(0)}{\rho}\left(1 - e^{-\rho t}\right). \tag{2.8}$$

We determine $C(0)$ by considering the terminal oundary condition $W(T) = 0$ to get the optimal initial consumption

$$C^*(0) = -\frac{\rho}{1 - e^{-\rho T}} W_0.$$

By substituting in equation (2.8) we get the optimal size of the cake (2.6).

Method 2: solving equation (2.5) **as a first order planar ordinary differential equation** We can transform the second order ODE (2.5) into the first order planar linear ODE, $\dot{y} = Ay$, where $y = (y_1, y_2)^\top$ by making $y_1 = \dot{W}^*$ and $y_2 = \ddot{W}^* = \dot{y}_1$, and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\rho \end{pmatrix}$$

Matrix A has the Jordan form

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & -\rho \end{pmatrix}$$

and eigenvector matrix

$$P = \begin{pmatrix} 1 & 1 \\ 0 & -\rho \end{pmatrix}$$

Therefore, the solution for equation $\dot{y} = Ay$ is

$$\begin{aligned} y(t) &= Pe^{\mathbf{\Lambda} t}P^{-1}k = \\ &= \frac{1}{\rho}\left(\begin{array}{cc} 1 & 1 \\ 0 & -\rho \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & e^{-\rho t} \end{array}\right)\left(\begin{array}{cc} \rho & 1 \\ 0 & -1 \end{array}\right)\left(\begin{array}{c} k_1 \\ k_2 \end{array}\right). \end{aligned}$$

As $W^*(t) = y_1(t)$ we get the general solution for W_t^*,

$$W^*(t) = k_1 + \frac{k_2}{\rho}\left(1 - e^{-\rho t}\right)$$

We use the initial and the terminal conditions $W(0) = W_0$ and $W(T) = 0$ to determine

$$k_1 = W_0,\ k_2 = -\frac{\rho}{1 - e^{-\rho T}}W_0.$$

And substituting back to $W^*(t)$, we get the optimal path for the cake size as in equation (2.6).

Example 2: the firm problem Consider a firm with a given initial capital $K(0) = K_0$. The firm produces output $Y = K$ with capital and increases its capital by investment $I(t) = \dot{K}(t)$. However the firm faces adjustment costs in investment, which imply that investment expenditures are $D(I) = \frac{1}{2}I^2$. The firm's cash flow is $\pi = PY - D(I)$, where $PY = PK$ are the firm's sales.

The value of the firm is quantified by the value functional which is the present value of cash flows, discounted at the market interest rate

$$V[K] = \int_0^T \pi(t)e^{-r\,t}\,dt,$$

where we assume that T is finite.

The goal of the firm is to maximize its value, $V^* = V[K^*]$ which is attained by choosing an optimal capital accumulation path.

If we assume that we know the terminal time T and the terminal level of capital K_T the firm's problem, formulated as a calculus of variations problem is to find

$$\max_K V[K] = \int_0^T \left(PK(t) - \frac{1}{2}\dot{K}(t)^2 \right) e^{-rt} dt \tag{2.9}$$

subject to

$$\begin{cases} K(0) = K_0 \text{ given} & \text{for } t = 0 \\ K(T) = K_T \text{ given} & \text{for } t = T. \end{cases} \tag{2.10}$$

The objective function

$$F(t, K(t), \dot{K}(t)) = \left(PK(t) - \frac{1}{2}(\dot{K}(t))^2 \right) e^{-r\,t}$$

has derivatives

$$F_K(t, K, \dot{K}) = Pe^{-r\,t},\ F_{\dot{K}}(t, K, \dot{K}) = -\dot{K}(t)e^{-r\,t}.$$

Because

$$\frac{d}{dt}\left(F_{\dot{K}}(t, K, \dot{K}) \right) = -\left(\ddot{K} - r\dot{K} \right) e^{-r\,t},$$

the Euler-Lagrange equation is

$$\ddot{K} - r\dot{K} + P = 0.$$

Therefore, the first order conditions are

$$\begin{cases} \ddot{K} - r\dot{K} + P = 0 & 0 \le t \le T \\ K(0) = K_0 \text{ given} & t = 0 \\ K(T) = K_T \text{ given} & t = T. \end{cases} \tag{2.11}$$

To solve the system, we proceed in two steps: first, we solve the Euler-Lagrange (EL) equation together with the initial condition and, second we constrain the solution to verify the terminal boundary condition.

To solve the EL equation let us define $I(t) = \dot{K}(t)$. Then we can transform the second order ordinary differential equation (ODE) into a recursive system of first ODE's

$$\begin{cases} \dot{K} = I(t),\ K(0) = K_0 \\ \dot{I} = rI(t) - P \end{cases}$$

The (general) solution of the first equation is

$$K(t) = K_0 + \int_0^t I(s)ds.$$

Investment $I(.)$ is the obtained from teh second equation. Solving the second ODE we find

$$I(t) = \bar{I} + (I(0) - \bar{I})e^{rt},\ 0 \le t \le T$$

where $\bar{I} = \dfrac{P}{r}$ is known and $I(0)$ is unknown.

Then substituting back in the solution for K we find

$$K(t) = K_0 + \int_0^t \bar{I}ds + \int_0^t (I(0) - \bar{I})e^{rs}ds.$$

If we perform the integrations we get the solution of the EL equation

$$K(t) = K_0 + \bar{I}t + \frac{(I(0) - \bar{I})}{r}(e^{rt} - 1) \tag{2.12}$$

which still depends on the unknown initial investment $I(0)$ which is unknown up to this point.

Its determination is the second step in solving the firm's problem we pointed out previously.

From the boundary-terminal condition we get the equation

$$K(t)|_{t=T} = K_0 + \bar{I}T + \frac{(I(0) - \bar{I})}{r}(e^{rT} - 1) = K_T$$

that we solve for $I(0)$ to get

$$I^*(0) = \bar{I} + \frac{r\left(K_T - K_0 - \bar{I}T\right)}{e^{rT} - 1} \text{ for } t = 0. \tag{2.13}$$

Observe that while $I(0)$ is a generic representation of the initial investment $I^*(0)$ is its optimal value, which is a specific function of the data of the problem: the interest rate, the price (recall that we defined $\bar{I} = P/r$), the initial and terminal levels of capital and the horizon of the firm, T.

Substituting $I(0)$ by $I^*(0)$ in equation (2.12) we find the solution for the problem:

$$K^* = K_0 + \bar{I}t + \left(K_T - K_0 - \bar{I}T\right)\left(\frac{e^{rt} - 1}{e^{rT} - 1}\right), \text{for } 0 \le t \le T. \tag{2.14}$$

The solution is depicted in Figure 2.2. Observe that $K^*(0) = K_0$ and $K^*(T) = K_T$ as required by the data of the problem.

The solution for the cake size if depicted in Figure 2.2

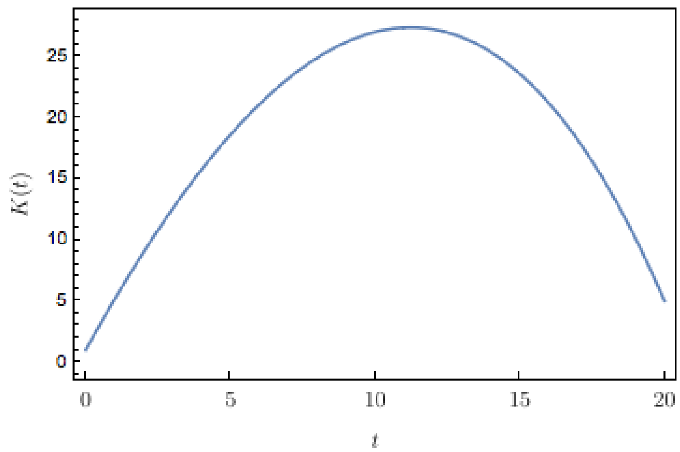

Figure 2.2: Optimal path for the firm's problem for $r = 0.05$, $P = 10$, $K_0 = 1$, $K_T = 5$ and $T = 20$.

2.1.2 The simplest free terminal state problem

Consider the problem: find a path $x^* = (x^*(t))_{t\in\mathbb{T}}$ that solves the problem

$$V[x^*] = \max_{(x(t))_{t\in[0,T]}} \int_0^T F(t, x(t), \dot{x}(t))dt$$
$$\text{subject to} \begin{cases} x(0) = x_0 \text{ given} & t = 0 \\ x(T) \text{ free} & t = T \\ T \text{ given} \end{cases} \tag{CV2}$$

Proposition 2 (First-order necessary conditions for problem (CV2)**).** *Assume that x^* is optimal. Then it verifies the Euler-Lagrange equation the boundary and the transversality conditions,*

$$\begin{cases} F^*_x(t) = \frac{d}{dt}\left(F^*_{\dot{x}}(t)\right), & 0 \le t \le T \text{ (Euler-Lagrange equation)} \\ x^*(0) = x_0, & t = 0 \text{ (boundary condition: initial} \\ F^*_{\dot{x}}(T) = 0, & t = T \text{ (transversality condition)} \end{cases} \tag{2.15}$$

Comparing optimality conditions (2.15) with (2.2) we observe that the boundary condition $x^*(T) = x_T$ is substituted by the transversality condition

$$F^*_{\dot{x}}(T) = F_{\dot{x}}(T, x^*(T), \dot{x}^*(T)) = 0.$$

The intuition is: not having a specified terminal value for the state variable, we determine its optimal level by the value for which the terminal gain is zero. If $F(T, x(T), \dot{x}(T)) > 0$ we could increase the value $V[x]$ by increasing $x(T)$ while if $F_{\dot{x}}(T, x(T), \dot{x}(T)) < 0$ terminal for x would be decreasing the value. Therefore the optimal terminal value is natural: $F_{\dot{x}}(T, x(T), \dot{x}(T)) = 0$.

Example 1: the cake-eating problem The only difference regarding the optimality conditions consists in the substitution of the terminal condition $W(T) = 0$ by $F_{\dot{W}}(T) = 0$. But

$$F_{\dot{W}}(T, W(T), \dot{W}(T)) = -\frac{e^{-\rho T}}{\dot{W}(T)} = -\frac{1}{C(0)}$$

which means that the problem has no solution because $\frac{1}{C(0)} = 0$ has no solution in the space of the real numbers ($\mathbb{R}$).

This result is a consequence of an incompleteness in the formulation of the problem: there should be some constraints in the path of consumption which is related to the fact that the initial level of the cake, W_0, is finite, and that it cannot be negative in any point in time $W(t) \geq 0$ for $0 \leq t \leq T$.

Example 2: the firm problem Considering again the firm's problem, the objective functional is the same (see equation (2.9) but the constraints are now

$$\begin{cases} K(0) = K_0 \text{ given} & \text{for } t = 0 \\ K(T) = K_T \text{ given} & \text{for } t = T. \end{cases} \tag{2.16}$$

instead of (2.10). Then as both the EL equation and the initial condition are formally the same, equation (2.12) is still valid. Comparing with the determination of the solution for the fixed terminal capital stock, in this case we have to determine the unknown initial investment $I(0)$ by using the transversality condition in (2.15) instead of the terminal boundary condition as in (2.2).

As we had $I(t) = \dot{K}$ then

$$\begin{aligned} F_{\dot{K}}(T, K(T), \dot{K}) &= -I(T)e^{-rT} = \\ &= -\left(\bar{I} + (I(0) - \bar{I})e^{rT}\right)e^{-rT} = \\ &= \bar{I}(1 - e^{-rT}) - I(0). \end{aligned}$$

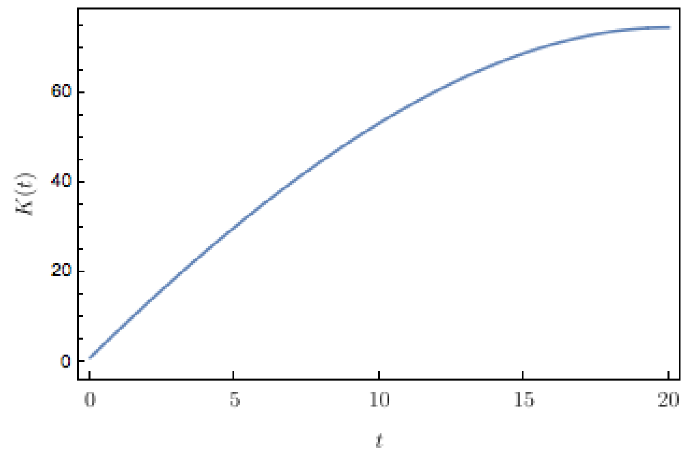

Figure 2.3: Optimal path for the firm's problem for $r = 0.05$, $P = 10$, $K_0 = 1$, and $T = 20$.

Setting $F_{\dot{K}}(T, K(T), \dot{K}) = 0$ then we find the optimal initial investment in this case

$$I^*(0) = \bar{I}(1 - e^{-rT}),\ t = 0 \tag{2.17}$$

which is different from (2.13). Then, we obtain the optimal capital stock solution Then

$$K^*(t) = K_0 + \bar{I}\left[t + \frac{1}{r}(1 - e^{-rT})\right],\ 0 \leq t \leq T \tag{2.18}$$

The solution for the cake size if depicted in Figure 2.3. Comparing with 2.2, that was build with the same data, with the exception of the terminal capital stock K_T we observe that if that constraint was not in place the firm had an incentive to invest much more, in particular closer to the terminal time.

2.1.3 The free terminal time problem

Consider the problem: find a path $x^* = (x^*(t))_{t\in\mathbb{T}}$ that solves the problem

$$V[x^*] = \max_{(x(t))_{t\in[0,T]}} \int_0^T F(t, x(t), \dot{x}(t))dt$$
$$\text{subject to} \begin{cases} x(0) = x_0 \text{ given} & t = 0 \\ x(T) = x_T \text{ given} & t = T \\ T \text{ free} \end{cases} \tag{CV3i}$$

Proposition 3 (First-order necessary conditions for problem (CV2)). *Assume that x^* is optimal. Then it verifies the Euler-Lagrange equation the boundary and the transversality conditions,*

$$\begin{cases} F_x^*(t) = \frac{d}{dt}\left(F_{\dot{x}}^*(t)\right), & 0 \le t \le T^* \text{ (Euler-Lagrange equation)} \\ x^*(0) = x_0, & t = 0 \text{ (boundary condition: initial)} \\ x^*(^*) = x_T, & t = T^* \text{ (boundary condition: terminal)} \\ F^*(T^*) - \dot{x}^*(T^*)F_{\dot{x}}^*(T^*) = 0, & t = T^* \text{ (transversality condition)} \end{cases} \tag{2.19}$$

Comparing optimality condition (2.19) with (**??**) and (**??**) we observe that the transversality condition allows for the determination of the **optimal stopping time** T^*

$$F^*(T^*) - \dot{x}^*(T^*)F_{\dot{x}}^*(T^*) == F(T^*, x_T, \dot{x}^*(T^*)) - \dot{x}^*(T^*)F_{\dot{x}}^*(T^*, x_T, \dot{x}^*(T^*)) = 0$$

where we know x_T but have to determine $\dot{x}^*(T^*)$ and T^*.

If we have the problem where the constraints are

$$\begin{cases} x(0) = x_0 \text{ given} & t = 0 \\ x(T) \text{ free} & t = T \\ T \text{ free} \end{cases} \tag{CV3ii}$$

the transversality conditions would be

$$\begin{cases} F_{\dot{x}}^*(T^*) = 0 \\ F^*(T^*) = 0 \end{cases}$$

The next table summarizes the previous results

Table 2.1: Boundary conditions for the finite time Calculus of Variations problem

Problem	T	$x(T)$	conditions	conditions
(CV1)	fixed	fixed	T	x_T
(CV2)	fixed	free	T	$F^*_{\dot{x}}(T) = 0$
(CV3i)	free	fixed	$F^*(T^*) - \dot{x}^*(T^*)F^*_{\dot{x}}(T^*) = 0$	x_T
(CV3ii)	free	free	$F^*(T^*) = 0$	$F^*_{\dot{x}}(T) = 0$

Example 2: Firm's problem Consider the firms problem where T is free but the terminal level of the capital stock is given $K(T) = K_T$. Let T^* be the free terminal time, where

$$T^* = \{T : \ \tau(T) = 0\}$$

where

$$\tau(T) \equiv F(T) - \dot{K}F_{\dot{K}}(T) = e^{-rT}\left(PK_T + \frac{I(T)^2}{2}\right)$$

where $I(T) = I(t)|_{t=T}$. Then

$$I(T) = \bar{I} + (K_T - K_0 - \bar{I}T)\frac{1}{1-e^{-rT}}$$

Therefore

$$\tau(T) = e^{-rT}\left(PK_T + \frac{((K_T - K_0 - \bar{I}T))^2}{2}\left(\frac{1}{1-e^{-rT}}\right)^2\right).$$

For data used in 2.2 we plot function $\tau(T)$ in Figure 2.4: in this case the optimal stopping time it $T^* \to \infty$, i.e, the firm has no incentive to exit.

2.2 Terminal constraints and scrap values

Regarding the problem studied in the previous section we introduce two changes:

1. we introduce a terminal equality constraint: $R(x(T), T) \leq 0$
2. we introduce a scrap value function: $S(x(T), T)$

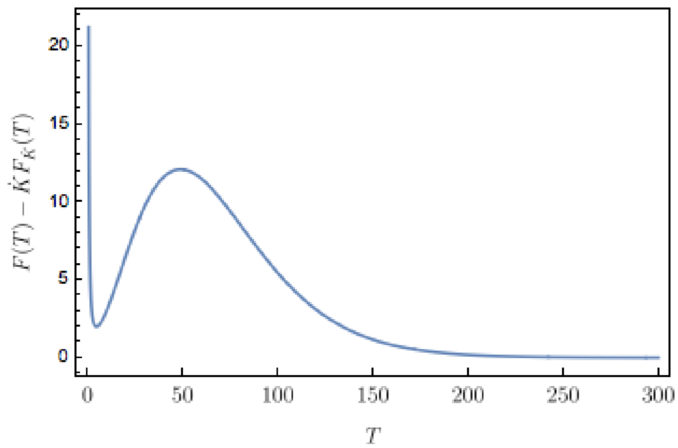

Figure 2.4: Transversality condition for the firm's problem $r = 0.05$, $P = 10$, $K_0 = 1$ and $T = 20$.

The first extension allows for having a constraint on the pair of terminal time and terminal value level, conditioning the value of them both in a more general level that their terminal value. We can think of three possibilities: (1) T could be fixed and $X(T)$ can be determined by the constraint; (2) $X(T)$ can be fixed and T can be determined by the constraint, or (3) both T and $X(T)$ are free but their value is constrained.

The first extension means that the terminal value of the state variable $x(T)$ has a value which is not captured by the value functional. For example, in the case of the firm's problem, if T is the terminal time of a factory the value of the firm should include the present value of selling the capital at the end of its planned lifetime.

Next we define and solve problems with equality and inequality constraints, having a non-zero scrap value, but with a fixed terminal time.

2.2.1 Equality terminal constraints

First we consider the problem with equality constraints:

find a path $x^* = (x^*(t))_{t\in\mathbb{T}}$ such that

$$
\begin{aligned}
& V[x^*] = \max_{(x(t))_{t\in[0,T]}} \int_0^T F(t, x(t), \dot{x}(t))dt + S(x(T), T) \\
& \text{subject to} \begin{cases} x(0) = x_0 \text{ given} & t = 0 \\ R(x(T), T) = 0 & t = T \\ T \text{ given} \end{cases}
\end{aligned}
\tag{CV4}
$$

Proposition 4 (First-order necessary conditions for problem (CV4)). *Assume that x^* is optimal. Then it verifies the Euler-Lagrange the initial boundary condition*

$$
\begin{cases}
F_x^*(t) = \frac{d}{dt}\left(F_{\dot{x}}^*(t)\right), & 0 \leq t \leq T \textit{ (Euler-Lagrange equation)} \\
x^*(0) = x_0, & t = 0 \textit{ (boundary condition: initial)} \\
R^*(T) = 0 & t = T \textit{ (boundary condition: terminal)} \\
F_{\dot{x}}^*(T) + S_x^*(T) + \psi R_x^*(T) = 0, & t = T, \textit{ (transversality condition)}
\end{cases}
\tag{2.20}
$$

where ψ is the Lagrange multiplier associated to the constraint $R(x(T), T) = 0$;

We denoted $S^*(T) = S(x^*(T), T)$, $S_x^*(T) = S_x(x^*(T), T)$ $R_x^*(T) = R(x^*(T), T)$ and $R_x^*(T) = R(x^*(T), T)$.

There are two observations we should make concerning this case:

1. Now we have to determine $x(T)$ and the Lagrange multiplier ψ. We determine $x(T)$ by solving the constraint $R(x(T), T) = 0$ to $x(T)$ and substitute into the transversality condition to get ψ

2. The scrap function will only influence the value of ψ, i.e., the terminal shadow price of $x(T)$ and not the dynamics of the solution.

2.2.2 Inequality terminal constraint

Consider the problem: find a path $x^* = (x^*(t))_{t\in\mathbb{T}}$ that solves the problem

$$V[x^*] = \max_{(x(t))_{t\in[0,T]}} \int_0^T F(t, x(t), \dot{x}(t))dt + S(x(T), T)$$

$$\text{subject to} \begin{cases} x(0) = x_0 \text{ given} & t = 0 \\ R(x(T), T) \geq 0 & t = T \\ T \text{ given} \end{cases} \tag{CV5}$$

Proposition 5 (First-order necessary conditions for problem (CV5)**).** *Assume that x^* is optimal. Then it verifies the Euler-Lagrange the initial boundary condition*

$$\begin{cases} F^*_x(t) = \frac{d}{dt}\left(F^*_{\dot{x}}(t)\right), & 0 \leq t \leq T \text{ (Euler-Lagrange equation)} \\ x^*(0) = x_0, & t = 0 \text{ (boundary condition: initial)} \end{cases}$$

together with the Kuhn-Tucker conditions for the terminal state

$$\begin{cases} \phi R^*(T) = 0,\ R^*(T) \geq 0,\ \psi^* \geq 0 & t = T \\ F^*_{\dot{x}}(T) + S^*_x(T) + \psi^* R^*_x(T) = 0, & t = T \end{cases} \tag{2.21}$$

where ψ is the Lagrange multiplier associated to the constraint $R(x(T), T) \geq 0$;

The next table gathers the results for the cases in which T is fixed:

Table 2.2: Boundary and transversality conditions for T fixed

Problem	T	$x(T)$	conditions	conditions
(CV1)	fixed	fixed	T	x_T
(CV2)	fixed	free	T	$F^*_{\dot{x}}(T) + S^*_x(T) = 0$
(CV4)	fixed	$R^*(T) = 0$	T	$F^*_{\dot{x}}(T) + S^*_x(T) + \psi R^*_x(T) = 0$
(CV5)	fixed	$R^*(T) \geq 0, \psi \geq 0, \psi R^*(T) = 0$	T	$F^*_{\dot{x}}(T) + S^*_x(T) + \psi R^*_x(T) = 0$

Observe that when $\psi^* > 0$ then the case (CV5) is equivalent to (CV2).

Now we consider one particular case in which the terminal time is free and the terminal state is constrained

$$V[x^*] = \max_{(x(t))_{t\in[0,T]}} \int_0^T F(t, x(t), \dot{x}(t))dt$$
$$\text{subject to} \begin{cases} x(0) = x_0 \text{ given} & t = 0 \\ h(T)x(T) \geq 0 & t = T \\ T \text{ free} \end{cases} \tag{CV6}$$

Proposition 6 (First-order necessary conditions for problem (CV6)). *Assume that x^* is optimal. Then it verifies the Euler-Lagrange the initial boundary condition*

$$\begin{cases} F^*_x(t) = \frac{d}{dt}\left(F^*_{\dot{x}}(t)\right), & 0 \leq t \leq T \text{ (Euler-Lagrange equation)} \\ x^*(0) = x_0, & t = 0 \text{ (boundary condition: initial)} \end{cases}$$

together with the transversality conditions

$$\begin{cases} F^*(T) = 0 & t = T \\ F^*_{\dot{x}}(T)x^*(T) = 0, & t = T \end{cases} \tag{2.22}$$

2.3 Discounted infinite horizon problem

In this case problems in which the terminal time is free and can be determined are not possible[1] and the terminal state is, therefore, always free. It also makes no sense to introduce a scrap value.

Therefore, only two types of problems can exist:

1. problems with free terminal state
2. problems with terminal boundary conditions on the state.

[1]Infinity does not belong to the set of real numbers.

2.3.1 Free asymptotic state

The most common problem in macro-economics involve a infinite horizon discounted utility functional

$$V[x] = \int_0^\infty f(x(t), \dot{x}(t)) e^{-\rho t} dt, \ \rho > 0 \tag{2.23}$$

where $x = (x(t))_{t\in[0,\infty)}$ and we assume $f(.)$ is continuous and $f(t, .)$ is smooth.

The calculus of variations problem is defined as: find $x^* = (x(t))_{t\in[0,\infty)}$ such that

$$\begin{aligned} V[x^*] = \max_{(x(t))_{t\in[0,\infty)}} \int_0^\infty e^{-\rho t} f(x(t), \dot{x}(t)) dt \\ \text{subject to } x(0) = x_0 \text{ given } t = 0 \end{aligned} \tag{2.24}$$

We can take the first order conditions for case (CV3ii) and take $T \to \infty$.

In this case, the Euler equation is a necessary condition and is, evaluated along the optimal trajectory x^*

$$e^{-\rho t} f_x(x^*(t), \dot{x}^*(t)) - \frac{d}{dt}\left(e^{-\rho t} f_{\dot{x}}(x^*(t), \dot{x}^*(t))\right) = 0, \text{ for } t \in [0, \infty)$$

or, if we evaluate the second derivative,

$$f_x(x^*(t), \dot{x}^*(t)) + \rho f_{\dot{x}}(x^*(t), \dot{x}^*(t)) - f_{\dot{x}x}(x, \dot{x})\dot{x}^* - f_{\dot{x}\dot{x}}(x^*(t), \dot{x}^*(t))\ddot{x}^* = 0, \text{ for } t \in [0, \infty) \tag{2.25}$$

For the infinite horizon case, the transversality condition is not necessary in general. In order to solve for the two constants of integration, in addition to the initial condition $x(0) = x_0$, it is assumed that the solution tends to a steady state. In a steady state, $\overline{x}$, as $\dot{x} = \ddot{x} = 0$, we get in equation (2.25)

$$f_x(\overline{x}, 0) + \rho f_{\dot{x}}(\overline{x}, 0) = 0.$$

2.3.2 Calculus of variations: discounted infinite horizon constrained terminal value

The problem is defined as: find x^* that solves the problem

$$
\begin{aligned}
V(x^*) = & \max_{(x(t))_{t\in[0,\infty)}} \int_0^{\infty} e^{-\rho t} f(x(t),\dot{x}(t))dt \\
& \text{subject to } \begin{cases} x(0)=x_0 \text{ given}, & t=0 \\ \lim_{t\to\infty} x(t) \geq 0 \end{cases}
\end{aligned}
\tag{2.26}
$$

The necessary conditions for an optimum are:

$$
\begin{cases}
f_x(x^*,\dot{x}^*) + \rho f_{\dot{x}}(x^*,\dot{x}^*) - f_{\dot{x}x}(x^*,\dot{x}^*)\dot{x} - f_{\dot{x}\dot{x}}(x^*,\dot{x}^*)\ddot{x} = 0 \\
x^*(0) = x_0 \\
\lim_{t\to\infty} e^{-\rho t} f_{\dot{x}}(x^*(t),\dot{x}^*(t))x^*(t) = 0
\end{cases}
$$

We can take the first order conditions from case (CV6) with $T \to \infty$

Example 1: Cake eating problem: infinite horizon

The problem is: find $W^* = (W^*(t))_{t\in\mathbb{R}_+}$ that

$$
\max_W \int_0^{\infty} \ln(-\dot{W}(t))e^{-\rho t}dt
$$

given $W(0) = W_0$ and $\lim_{t\to\infty} W(t) \geq 0$

The first order conditions are:

$$
\begin{cases}
\rho\dot{W}^*(t) + \ddot{W}^*(t) = 0 \\
W^*(0) = W_0 \\
-\lim_{t\to\infty} e^{-\rho t} \frac{W^*(t)}{\dot{W}^*(t)} = 0
\end{cases}
$$

We already found that the general solution from the first equation, after using the initial condition, is

$$
W(t) = W_0 - \frac{k}{\rho}\left(1 - e^{-\rho t}\right).
$$

Then, because $\dot{W}(t) = -ke^{-\rho t}$

$$-\lim_{t\to\infty} e^{-\rho t}\frac{W^*(t)}{\dot{W}^*(t)} = \lim_{t\to\infty}\left(\frac{\rho W_0 - k\left(1-e^{-\rho t}\right)}{ke^{-\rho t}}\right)e^{-\rho t} = \frac{\rho W_0 - k}{k} = 0$$

if and only if $k = \rho W_0$. Then the solution for the cake eating problem is

$$W^*(t) = W_0 e^{-\rho t},\ t \in \mathbb{R}_+.$$

Example 2: firm's problem Let us consider now the problem

$$\max_K \{V[K] : K(0) = K_0, \lim_{t\to\infty} e^{-rt}K(t) \geq 0\}$$

where

$$\max_K V[K] = \int_0^\infty \left(PK(t) - \frac{\dot{K}(t)^2}{2}\right)e^{-rt}dt$$

In this case the transversality condition is

$$\lim_{t\to\infty} F_{\dot{K}}K(t) = -\lim_{t\to\infty} e^{-rt}I(t)K(t) = 0$$

Taking again the capital and investment equations obtained after solving the EL equation we have, remembering that $r > 0$,

$$\begin{aligned}\lim_{t\to\infty} e^{-rt}I(t)K(t) &= e^{-rt}\left(\bar{I} + (I(0)-\bar{I})e^{rt}\right)\left(K_0 + \bar{I}t + \frac{(I(0)-\bar{I})}{r}(e^{rt}-1)\right) = \\ &= \lim_{t\to\infty}(I(0)-\bar{I})\left(K_0 + \bar{I}t + \frac{(I(0)-\bar{I})}{r}(e^{rt}-1)\right)\end{aligned}$$

that can only be zero if and only if $I^*(0) = \bar{I}$. This implies that $I^*(t) = \bar{I}$ for all t and

$$K^* = K_0 + \bar{I}t,\ \text{for } t \in [0,\infty). \tag{2.27}$$

2.4 Canonical representation of CV problems

The Euler equation can be presented under the so-called **canonical representation**, which establishes a relationship between the calculus of variations and the Pontriyagin principle for solving dynamic optimization problems.

If $F_{\dot{x}\dot{x}}\ddot{x} \neq 0$ define the co-state function as

$$p(t) = -F_{\dot{x}}(t, x(t), \dot{x}(t)) \tag{2.28}$$

and we define the Hamiltonian function by

$$H(t, x(t), p(t)) = F(t, x(t), \dot{x}(t)) + p(t)\dot{x}(t).$$

The Hamiltonian has the partial derivatives

$$H_x = \frac{\partial H}{\partial x} = F_x,\ H_p = \frac{\partial H}{\partial p} = \dot{x},\ H_{\dot{x}} = 0$$

because $H_{\dot{x}} = F_{\dot{x}} + p = 0$ from equation (2.28). Then, the canonical representation of the Euler equation is

$$\dot{p} = -H_x(t, x, p),\ \dot{x} = H_p(t, x, p).$$

Observe that the Euler equation is $F_x(.) = d(F_{\dot{x}}(.))/dt = -\dot{p}$.

Therefore the solution for the simplest calculus of variation problem is (if $F(t, x, \dot{x})$ is concave in $(x, \dot{x})$) represented by the paths (x^*, p) such that

$$\begin{cases} \dot{p} = -H_x(t, x^*(t), p(t)) & t \in [0, T] \\ \dot{x} = H_p(t, x^*(t), p(t)) & t \in [0, T] \\ x^*(0) = x_0 & t = 0 \\ x^*(T) = x_0 & t = T \end{cases} \tag{2.29}$$

Bibliographic references Kamien and Schwartz (1991, part I)

Chapter 3

Optimal control

The optimal control problem is a generalization of the calculus of variations problem. In addition to the **state** variable $x : \mathbb{T} \to \mathbb{R}$ we have a **control** variable $u : \mathbb{T} \to \mathbb{R}^m$ which controls the behavior of the state variable. Observe that we may have more than one control variable.

Now the value functional is a function of the

$$V[u, x] = \int_0^T f(u(t), x(t), t)dt$$

and the structure of the economy is represented by the ODE

$$\dot{x} = g(u(t), x(t), t),\ t \in [0, T]$$

given $x(0) = x_0$. The different versions of the problem vary according to differences in the horizon, T, in the terminal state of the economy $x(T)$ and in the existence of other constraints.

In section 3.1 we study the fixed terminal state and time problem, in section **??** we study the fixed terminal time and free state problem and in section **??** the free terminal time problems.

3.1 The simplest problems

3.1.1 The fixed terminal state problem

The simplest problem consists in finding the optimal paths (x^*, u^*), where $x^* \equiv (x^*(t))_{0\leq t\leq T}$ and $u^* \equiv (u^*(t))_{0\leq t\leq T}$, which solve the problem:

$$V[u^*, x^*] = \max_{(u(t))_{t\in[0,T]}} \int_0^T f(t, u(t), x(t))dt$$
$$\text{subject to} \begin{cases} \dot{x}(t) = g(u(t), x(t), t) & 0 \leq t \leq T \\ x(0) = x_0 \text{ given} & t = 0 \\ x(T) = x_T \text{ given} & t = T \\ T \text{ given} \end{cases} \tag{OC1}$$

The functions (x, u) that verify conditions (3.1),

$$\begin{cases} \dot{x}(t) = g(u(t), x(t), t), & \text{for } 0 \leq t \leq T \\ x(0) = x_0 \text{ given} & \text{for } t = 0 \\ x(T) = x_T \text{ given} & \text{for } t = T. \end{cases} \tag{3.1}$$

for every $t \in [0, T]$ are called **admissible solutions**. Optimal solutions (x^*, u^*) are admissible functions which allows us to get the optimal value functional $V[u, x]$. Generally, while $x(t)$ is a continuous function for every $t \in [0, T]$, $u(t)$ should be piecewise continuous in $t \in [0, T]$, but continuous it $t \in (0, T]$, that is we only require that they are continuous at t converges to zero by positive values.

We call **Hamiltonian** to the function

$$H(t, x, u, \lambda) \equiv f(t, x, u) + \lambda g(t, u, x) \tag{3.2}$$

where function $\lambda : \mathbb{T} \to \mathbb{R}$ is called the **co-state or adjoint variable**.

The **maximized Hamiltonian** is a function of the state and co-state variables

$$H^*(t, x, \lambda) = \max_u H(t, x, u, \lambda),$$

then

$$H^*(t,x,\lambda) = f(t,x^*,u^*) + \lambda g(t,u^*,x^*).$$

Proposition 7 (**Necessary conditions by the Pontriyagin's maximum principle**). *Let (x^*,u^*) be a solution to the problem (**??**)-(**??**). Then there is a piecewise continuous function $\lambda(.)$ such that $(x^*(.),u^*(.),\lambda(.))$ simultaneously satisfy:*

1. *the admissibility conditions: state dynamic constraint and boundary-initial condition*

$$\begin{cases} \dot{x}^* = \frac{\partial H^*(t)}{\partial \lambda} = g(t,x^*(t),u^*(t)), & 0 < t \le T \\ x(0) = x_0, & t = 0 \\ x(T) = x_T, & t = T \end{cases} \tag{3.3}$$

2. *the multiplier equation*

$$\dot{\lambda} = -\frac{\partial H^*(t)}{\partial x} = -f_x(t,x^*(t),u^*(t)) - \lambda(t) g_x(t,x^*(t),u^*(t)),\ 0 < t \le T \tag{3.4}$$

3. *the optimality condition:*

$$\frac{\partial H^*(t)}{\partial u} = f_u(t,x^*(t),u^*(t)) + \lambda(t) g_u(t,x^*(t),u^*(t)) = 0, \ \ 0 \le t \le T. \tag{3.5}$$

Proposition 8 (**Interpretation of the co-state variable**). *The co-state variable can be interpreted as the marginal value of the state variable,*

$$\delta_{x(t)} V[u^*,x^*] = \lambda(t)$$

where $\delta_{x(t)} V[u^*,x^*]$ is a "spike" perturbation of the state variable at time t.

3.1.2 The free terminal state problem

The simplest problem consists in finding the optimal paths (x^*, u^*), where $x^* \equiv (x^*(t))_{0\leq t\leq T}$ and $u^* \equiv (u^*(t))_{0\leq t\leq T}$, that solve the problem:

$$V[u^*, x^*] = \max_{(u(t))_{t\in[0,T]}} \int_0^T f(t, u(t), x(t))dt$$
$$\text{subject to} \begin{cases} \dot{x}(t) = g(u(t), x(t), t) & 0 \leq t \leq T \\ x(0) = x_0 \text{ given} & t = 0 \\ T \text{ given} \end{cases} \tag{OC2}$$

and $x(T)$ is free.

Proposition 9 (**Necessary conditions by the Pontriyagin's maximum principle for problem** (OC2))**.** *Let* (x^*, u^*) *be a solution to the problem* (OC2)*. Then there is a piecewise continuous function* $\lambda(.)$ *such that* $(x^*(.), u^*(.), \lambda(.))$ *simultaneously satisfy:*

1. *the admissibility conditions: state dynamic constraint and boundary-initial condition*
$$\begin{cases} \dot{x}^* = \frac{\partial H^*(t)}{\partial \lambda} = g(t, x^*(t), u^*(t)), & 0 < t \leq T \\ x(0) = x_0, & t = 0 \end{cases} \tag{3.6}$$

2. *the multiplier equation and the transversality condition:*
$$\begin{cases} \dot{\lambda} = -\frac{\partial H^*(t)}{\partial x} = -f_x(t, x^*(t), u^*(t)) - \lambda(t) g_x(t, x^*(t), u^*(t)), & 0 < t \leq T \\ \lambda(T) = 0, & t = T \end{cases} \tag{3.7}$$

3. *the optimality condition* (3.5)

Using the interpretation of the co-state variable in Proposition 8 we can have an intuition on the meaning of the transversality condition. Setting for $t = T$, we get

$$\lambda_T = \delta_{x(T)} V[u^*, x^*] = 0,$$

that is, if the terminal value of the state variable is free it is optimally determined such that there is no change in the optimal value functional.

3.1.3 Free terminal time

As for the Calculus of Variations problems we consider the case in which the terminal time is free and should be optimally chosen:

$$V[u^*, x^*] = \max_{(u(t))_{t\in[0,T]}} \int_0^T f(t, u(t), x(t))dt$$
$$\text{subject to} \begin{cases} \dot{x}(t) = g(u(t), x(t), t) & 0 \le t \le T \\ x(0) = x_0 \text{ given} & t = 0 \\ T \text{ free} \end{cases} \tag{OC3}$$

where we can consider two alternatives: (1) the case for which $x(T) = x_T$ is fixed and (2) the case for which $x(T)$ is free.

Proposition 10 (**Necessary conditions by the Pontriyagin's maximum principle for problem** (OC3)**).** *Let* (x^*, u^*) *be a solution to the problem* (OC2) *and* T^* *the optimal terminal time. Then there is a piecewise continuous function* $\lambda(.)$ *such that* $(x^*(.), u^*(.), \lambda(.))$ *and* T^* *satisfy:*

1. *the admissibility conditions:*

$$\begin{cases} \dot{x}^* = \frac{\partial H^*(t)}{\partial \lambda} = g(t, x^*(t), u^*(t)), & 0 < t \le T^* \\ x(0) = x_0, & t = 0x(T^*) = x_T \ (\textit{if} \ \ x(T) = x_T \textit{ is given}) \end{cases} \tag{3.8}$$

2. *the multiplier equation and the transversality condition:*

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^*(t)}{\partial x} = -f_x(t, x^*(t), u^*(t)) - \lambda(t) g_x(t, x^*(t), u^*(t)), & 0 < t \le T^* \\ \lambda(T) = 0, & t = T^* \ (\textit{if} \ \ x(T) \textit{ is free}) \end{cases} \tag{3.9}$$

3. *the optimality condition for the control variable* (3.5) *and the optimality condition for the terminal time*

$$H^*(T^*) = H(T^*, u^*(T^*), x^*(T^*), \lambda(T^*)) + + \frac{d(\lambda x)}{dt}(T^*) = 0 \tag{3.10}$$

In equation (3.10) we denote

$$\frac{d(\lambda x)}{dt}(T^*) = \frac{d}{dt}\left(\lambda(t)x(t)\right)\Big|_{t=T^*}$$

3.1.4 Examples

The cake eating problem

Consider, again, problem (2.3). The current value hamiltonian is

$$H(t) = \ln\left(C(t)\right)e^{-\rho t} - \lambda(t)C(t)$$

where λ is the co-state variable. The first order conditions are

$$\begin{aligned}
\frac{\partial H^*(t)}{\partial C(t)} &= \frac{1}{e^{\rho t}C^*(t)} - \lambda(t) = 0, \\
\dot{\lambda} &= -\frac{\partial H^*}{\partial W} = 0 \\
\dot{W}^* &= -C^*(t) \\
W^*(0) &= W_0 \\
W^*(T) &= 0.
\end{aligned}$$

Taking time derivatives of the first equation we find

$$\dot{\lambda} = -\frac{\dot{C}e^{\rho t} + \rho C e^{\rho t}}{(Ce^{\rho t})^2}.$$

Because $\dot{\lambda} = 0$ from multiplier equation then $\dot{C} + \rho C = 0$. Therefore, the optimality conditions become

$$\begin{aligned}
\dot{C}^* &= -\rho C^*(t) \\
\dot{W}^* &= -C^*(t) \\
W^*(0) &= W_0 \\
W^*(T) &= 0.
\end{aligned}$$

The two first equations form a linear planar ODE of type $\dot{y} = Ay$, where

$$A = \begin{pmatrix} -\rho & 0 \\ -1 & 0 \end{pmatrix}.$$

The solution is

$$\begin{aligned} \begin{pmatrix} C^*(t) \\ W^*(t) \end{pmatrix} &= \frac{1}{\rho}\begin{pmatrix} 0 & \rho \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-\rho t} \end{pmatrix}\begin{pmatrix} -1 & \rho \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \\ &= \begin{pmatrix} -k_1 e^{-\rho t} \\ k_2 - \frac{1}{\rho}\left(1 - e^{-\rho t}\right) \end{pmatrix}. \end{aligned}$$

Using the initial and the terminal conditions, $W(0) = W_0$ and $W(T) = 0$, we get the same solutions as for the calculus of variation version, equations (2.7)-(2.6).

The firm's problem

We consider again the same firm's problem, state it as an optimal control problem and solve it using the Pontriyagin's principle.

The firm problem is to find the optimal investment path leading to capital accumulation allowing the maximization of the value of a firm. The value of the firm is defined by the present value of the firm's cash flows

$$V[I, K] = \int_0^T \left(PK(t) - \frac{I(t)^2}{2}\right) e^{-rt} dt$$

where the price P and the interest rate r are both positive, subject to the constraints

$$\begin{cases} \dot{K} = I, & \text{for } 0 \leq t \leq T \\ K(0) = K_0, & \text{for } t = 0, \\ T > 0 \text{ fixed} \end{cases}$$

Investment, I, is the control variable, and the stock of capital, K, is the state variable.

We consider two cases: (1) the terminal capital stock is fixed $K(T) = K_T > 0$ is known; (2) the terminal capital is free.

To apply the Pontriyagin's principle we need to find the Hamiltonian:

$$H(t, I, K, \lambda) = \left(PK - \frac{I^2}{2}\right) e^{-rt} + \lambda I.$$

The first order conditions are: the optimality condition for the first problem are

$$\begin{aligned}
&H_I(t, I, K, \lambda) = 0 \;\Leftrightarrow\; -Ie^{-rt} + \lambda = 0 \\
&\dot{\lambda} = -H_K(t, I, K, \lambda) \;\Leftrightarrow\; \dot{\lambda} = -Pe^{-rt} \\
&\dot{K} = I \\
&K(0) = K_0 \\
&K(T) = K_T
\end{aligned}$$

and for the second problem they are the same with the exception of the terminal condition which is substituted by

$$\lambda(T) = 0.$$

From the static optimality condition, and introducing the time dependence, we have

$$\lambda(t) = I(t)e^{-rt}.$$

If we take the time derivative and equate to the multiplier equation we find

$$\dot{\lambda} = \dot{I}e^{-rt} - rIe^{-rt} = -Pe^{-rt}$$

we get $\dot{I} = rI - P$. Defining

$$\bar{I} = \frac{P}{r}$$

we have the MHDS (maximized Hamiltonian dynamic system)

$$\begin{aligned}
\dot{K} &= I && (3.11) \\
\dot{I} &= r(I - \bar{I}) && (3.12) \\
K(0) &= K_0 && (3.13)
\end{aligned}$$

that should be solved with the constraint

$$K(T) = K_T \tag{3.14}$$

for the fixed terminal state problem, or with the constraint

$$e^{-rT}I(T) = 0 \tag{3.15}$$

for the free terminal state problem.

As the system (3.11)-(3.12) is recursive we can solve it in the following way: first, solve equation (3.12) to get

$$I(t) = \bar{I} + (I(0) - \bar{I})e^{rt} \tag{3.16}$$

where $\bar{I}$ is known but $I(0)$ is unknown; second solve equation (3.11) and to get

$$K(t) = K(0) + \int_0^t I(s)ds.$$

But we know $K(0)$, by equation (3.13) and we already found $I(s)$. Then

$$K(t) = K_0 + \int_0^t \left(\bar{I} + (I(0) - \bar{I})e^{rs}\right) ds.$$

Performing the integration we have

$$K(t) = K_0 + \bar{I}t + (I(0) - \bar{I})\frac{e^{rt} - 1}{r}, \text{ for } 0 \leq t \leq T \tag{3.17}$$

where, again, $I(0)$ is unknown.

In order to determine the optimal initial investment $I^*(0)$, we consider the terminal condition (for the fixed terminal state problem) or the transversality condition (for the free terminal state problem).

For the fixed terminal state problem we have $I^*(0) = \{I(0) : K(T) = K_T\}$ where $K(T)$ is the general solution for the capital stock in equation (3.17) evaluated at $t = T$. Therefore

$$I^*(0) = \bar{I} + \frac{K_T - K_0 - \bar{I}T}{e^{rT} - 1}$$

which implies that the solution to the fixed terminal state problem is

$$\begin{aligned} I^*(t) &= \bar{I} + \left(\frac{K_T - K_0 - \bar{I}T}{e^{rT} - 1}\right) e^{rt}, \text{ for } 0 \leq t \leq T && (3.18) \\ K^*(t) &= K_0 + \bar{I}t + \left(K_T - K_0 - \bar{I}T\right) \left(\frac{e^{rt} - 1}{e^{rT} - 1}\right), \text{ for } 0 \leq t \leq T. && (3.19) \end{aligned}$$

For the free terminal state problem we have $I^*(0) = \{I(0) : \lambda(T) = 0\}$ where $\lambda(T) = e^{-rT}I(T) = 0$ where $I(T)$ is the general solution for the capital stock in equation (3.16) evaluated at $t = T$. Therefore

$$I^*(0) = \bar{I}\left(1 - e^{-rT}\right)$$

which implies that the solution to the free terminal state problem is

$$\begin{aligned} I^*(t) &= \bar{I}\left(1 - e^{-r(T-t)}\right), \text{ for } 0 \leq t \leq T && (3.20) \\ K^*(t) &= K_0 + \bar{I}\left(t + e^{-rT}\left(1 - e^{rt}\right)\right), \text{ for } 0 \leq t \leq T. && (3.21) \end{aligned}$$

3.2 Terminal constraints

Now consider the problem

$$V[u^*, x^*] = \max_{(u(t))_{t\in[0,T]}} \int_0^T f(t, u(t), x(t))dt$$
$$\text{subject to} \begin{cases} \dot{x}(t) = g(u(t), x(t), t) & 0 \leq t \leq T \\ x(0) = x_0 \text{ given} & t = 0 \\ h(T)x(T) \geq 0 \text{ given} & t = T \\ T \text{ given} \end{cases} \tag{OC4}$$

that is, we assume that the weighted terminal state is bounded below by zero.

Proposition 11 (**Necessary conditions by the Pontriyagin's maximum principle for problem** (OC4))**.** *Let (x^*, u^*) be a solution to the problem* (OC4)*. Then there is a piecewise continuous function $\lambda(.)$ such that $(x^*(.), u^*(.), \lambda(.))$ simultaneously satisfy:*

1. *the admissibility conditions: state dynamic constraint and boundary-initial condition as in equation* (3.6)

2. *the multiplier equation and the transversality condition:*

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^*(t)}{\partial x} = -f_x(t, x^*(t), u^*(t)) - \lambda(t)g_x(t, x^*(t), u^*(t)), & 0 < t \leq T \\ \lambda(T)x(T) = 0, & t = T \end{cases} \tag{3.22}$$

3. *the optimality condition* (3.5)

The necessary first order conditions from the Pontryiagin's maximum principle are formally identical to the cases (OC1) and (OC2) with the exception that the transversality condition is now

$$\lambda(T)x(T) = 0.$$

The next table gather the the terminal boundary and/or transversality conditions for all cases studied previously:

Table 3.1: Boundary conditions for finite time Optimal Control problems

Problem	T	$x(T)$	conditions	conditions
(OC1)	fixed	fixed	T	x_T
(OC2)	fixed	free	T	$\lambda(T) = 0$
(OC3)	free	fixed	$H^*(T^*) + \frac{d(\lambda x)}{dt}(T^*) = 0$	x_T
(OC3)	free	free	$H^*(T^*) + \frac{d(\lambda x)}{dt}(T^*) = 0$	$\lambda(T^*) = 0$
(OC4)	fixed	constrained	T	$\lambda(T)x(T) = 0$

3.2.1 Example

The firm's problem

Consider the firm problem with the terminal constraint

$$e^{-rT}K(T) \geq 0.$$

In this case the first order conditions are as in equations (3.11), (3.12) and (3.13) but we have now the transversality condition $\lambda(T)K(T) = 0$ which is equivalent to

$$e^{-rT}I(T)K(T) = 0 \tag{3.23}$$

In this case we have two potential solutions: (1) the solution we have found for the free terminal state problem (equations (3.20)-(3.21)); or (2) a solution similar to the fixed terminal capital (equations (3.18)-(3.19)) but with $K_T = 0$.

3.3 Discounted infinite horizon problems

For the infinite horizon discounted problem, which is the most common case in macroeconomics and growth theory, the value functional is

$$v[u,x]\int_0^{\infty} f(u(t),x(t))e^{-\rho t}dt, \tag{3.24}$$

where the objective function, $f(u,x)$ is autonomous (does not depend directly on time) but is multiplied by a discount factor $e^{-\rho t}$, where $\rho \geq 0$. If the he discount rate, ρ, is positive $e^{-\rho t}$ is a time weight which has the maximum at time $t=0$, $e^{-\rho t}|_{t=0}=1$, is decreasing in time and converges to zero, $\lim_{t\to\infty} e^{-\rho t}=0$.

The infinity horizon is a case in which time is not specified, and therefore the case in which we have a fixed terminal value does not make sense, but is not free either, in the sense that we cannot chose terminal time as a decision variable. Therefore we can consider two cases which have a common value function

$$V[u^*,x^*]=\max_{(u(t))_{t\in[0,\infty)}} V[u,x]$$

the same dynamic constraint and initial condition

$$\begin{cases} \dot{x}(t)=g(u(t),x(t)) & t\in[0,\infty) \\ x(0)=x_0 \text{ given} & t=0 \end{cases}$$

but different terminal conditions:

- the free terminal state problem

$$\lim_{t\to\infty} x(t) \text{ free} \tag{OC5}$$

- and the constrained terminal constraint problem

$$\lim_{t\to\infty} h(t)x(t)\geq 0 \tag{OC6}$$

Observe we considered the case in which the vector field $g(u,x)$ is also an autonomous function, i.e, they are time-independent.

The hamiltonian function is now called the **discounted** hamiltonian and is defined by

$$H(t, x(t), u(t), \lambda(t)) \equiv f(x(t), u(t))e^{-\rho t} + \lambda(t)g(u(t), x(t)),\ t \in \mathbb{R}_+$$

where $\lambda(t)$ is the discounted co-state variable.

It is convenient to use the **current value** hamiltonian h instead of the present value Hamiltonian H,

$$h(x(t), u(t), q(t)) = f(x(t), u(t)) + q(t)g(x(t), u(t)) = e^{\rho t}H(t, x(t), u(t), \lambda(t))$$

where the current value co-state variable is defined as

$$q(t) \equiv e^{\rho t}\lambda(t).$$

The current-value maximized hamiltonian is

$$h^*(x(t), q(t)) = \max_{u(t)} h(x(t), q(t), u(t))\ t \in \mathbb{R}_+.$$

Passing the horizon to infinity comes with the cost of introducing some difficulties regarding the transversality condition, as compared with the finite horizon case. It ceases to be a necessary condition and becomes a sufficient condition. There are some ways to overcome this. One consists in imposing that the solution should converge to a steady state. Another, that we will consider next is to assume that functions $f(u, x)$ and $g(u, x)$ are concave (not necessarily strictly concave in the case of $g(.)$) in (u, v). In this case, The Pontriyagin's principle gives necessary and sufficient conditions for optimality.

Let $x = (x(t))_{t\in\mathbb{R}_+}$ and $u = (u(t))_{t\in\mathbb{R}_+}$ be admissible paths for the state and the control variables.

Proposition 12 (**The maximum principle of Pontryiagin for infinite horizon discounted problems**).
Consider the optimal control problem, with one of the terminal conditions (OC5) *or* (OC6)*, and assume that functions $f(u, x)$ and $g(u, x)$ are concave. Then (x^*, u^*) is a solution of the optimal control problem if and only if there is piecewise continuous function $q(t)$ such that (x^*, u^*, q) simultaneously satisfy:*

- *the admissibility condition:*

$$\begin{cases} \dot{x}^* = \frac{\partial h^*}{\partial q} = g(x^*(t), u^*(t)), & t \in (0, +\infty) \\ x(0) = x_0, & t = 0, \end{cases} \tag{3.25}$$

- *the multiplier equation*

$$\dot{q} = \rho q - \frac{\partial h^*}{\partial x} = \rho q(t) - f_x(x^*(t), u^*(t)) - q(t) g_x(x^*(t), u^*(t)), t \in (0, +\infty), \tag{3.26}$$

- *the transversality condition for problem* (OC5)

$$\lim_{t \to \infty} e^{-\rho t} q(t) = 0, \tag{3.27}$$

- *or the transversality condition for problem* (OC6)

$$\lim_{t \to \infty} e^{-\rho t} q(t) x(t) = 0, \tag{3.28}$$

- *and the optimality condition:*

$$\frac{\partial h^*}{\partial u} = f_u(x^*(t), u^*(t)) + q(t) g_u(x^*(t), u^*(t)) = 0, \quad t \in [0, +\infty]. \tag{3.29}$$

To make the make the boundary and transversality conditions for the infinite horizon clear we present them again in a table

Table 3.2: Boundary conditions for infinite horizon Optimal Control problems

Problem	boundary condition	transversality condition
(OC5)	$\lim_{t\to\infty} x(t)$ free	$\lim_{t\to\infty} q(t)e^{-\rho t} = 0$
(OC6)	$\lim_{t\to\infty} h(t)x(t) \geq 0$	$\lim_{t\to\infty} q(t)x(t)e^{-\rho t} = 0$

3.3.1 Examples

Resource depletion problem

Assume there is a non-renewable resource where $N(t)$ denotes its endowment at time t and that the resource is depleted by consumption. What would be the optimal rate of depletion if in the economy there is a representative agent who maximizes the discounted intertemporal path of consumption and we assume that the asymptotic value of the resource cannot be negative.

The problem is

$$\max_{C} \int_0^{\infty} e^{-\rho t} \ln\left(C(t)\right)dt,\ \rho > 0$$

subject to

$$\begin{cases} \dot{N}(t) = -C(t),\ t \in [0, \infty) \\ N(0) = N_0,\ \text{given} \\ \lim_{t\to\infty} N(t) \geq 0 \end{cases}$$

The current-vale hamiltonian is

$$h(t) = ln(C(t)) - q(t)C(t)$$

and the first order conditions are:

$$\begin{aligned} C(t) &= 1/q(t) \\ \dot{q} &= \rho q(t) \\ \lim_{t\to\infty} e^{-\rho t} q(t)N(t) &= 0 \\ \dot{N} &= -C(t) \\ N(0) &= N_0 \end{aligned}$$

If we differentiate the optimality condition and substitute the Euler equation we get the MHDS

$$\begin{aligned} \dot{C} &= -\rho C(t) \\ \dot{N} &= -C(t) \\ N(0) &= N_0 \\ \lim_{t\to\infty} e^{-\rho t} \frac{N(t)}{C(t)} &= 0 \end{aligned}$$

We can solve the MHDS in two steps:

- 1st step: we define $z(t) \equiv N(t)/C(t)$ and consider the transversality condition

$$\begin{cases} \dot{z} = -1 + \rho z \\ \lim_{t\to\infty} e^{-\rho t} z(t) = 0 \end{cases}$$

  the solution is constant

$$z(t) = \frac{1}{\rho},\ t \in [0, \infty)$$

- 2nd step: we substitute the solution for z in

$$\begin{cases} \dot{N} = -C(t) = -N(t)/z(t) \\ N(0) = N_0 \end{cases}$$

and solve for $N(t)$ to get the solution for the resource endowment

$$N^*(t) = N_0 e^{-\rho t},\ t \in [0, \infty).$$

Characterization of the solution:

- there is asymptotic extinction

$$\lim_{t\to\infty} N^*(t) = 0$$

- the speed of adjustment can be assessed by computing the half-life of the process

$$\tau \equiv \left\{ t :\ N^*(t) = \frac{N(0) - N^*(\infty)}{2} \right\} = -\frac{\ln(1/2)}{\rho}$$

  if $\rho = 0.02$ then $\tau \approx 34.6574$ years

3.4 Linear MHDS

Assume that $H(u, \cdot)$ is sufficiently smooth. Using the static condition to find $q = -\dfrac{F_u(u,x,t)}{G_u(u,x,t)}$, take the time derivative and use the Euler equation to obtain a system of differential equations

$$\dot{x} = M_1(u,k)$$
$$\dot{u} = M_2(u,k)$$

We call maximized Hamiltonian dynamic system (MHDS) to this system. It yields a general solution to the optimal control problem, asa function of two arbitrary constants. The solution of the optimal control problem is the particular solution of this system which is obtained substituting in the initial and transversality (or terminal) conditions.

When the maximized Hamiltonian dynamic system is linear we can obtain, in most cases, explicit solutions to the optimal control problem, by using one of the following methods:

1. **method 1**: if the system is recursive, we can use each equation independently. We obtain a two-dimensional general solution depending on two arbitrary constants. At last, we use the initial and transversality conditions fo obtain the explicit (or particular) solution in the end;

2. **method 2**: we introduce a transformation of variables reducing the system to a backward scalar ODE whose solution depends on one constant. We use the transversality condition to find a particular value of that constant. We use the inverse transformation to obtain the value of one of the control variable as a function of the state variable. We substitute in the dynamic constraint of the problem and use the initial condition to obtain the solution;

3. **method 3**: solve the coupled ODE equations jointly to obtain a general solution and use the initial and transversality conditions (this is the only method available when the system is not recursive)

Next we apply the three methods to find the solution to the consumption-investment problem

The consumption-investment problem

The problem is to find the optimal consumption and asset paths, $\big(c^*(t), a^*(t)\big)_{t=0}^{\infty}$ which maximize an intertemporal utility functional subject to an instantaneous budget constraint and an asymptotic solvability condition, given an initial level of the asset $a(0) = a_0$:

$$\max_c \int_0^\infty \ln\left(c(t)\right) e^{-\rho t}\, dt,\ \rho > 0$$

$$\text{subject to}$$

$$\dot{a} = r\,a - c,\ t \in [0, \infty)$$

$$a(0) = a_0,\ \text{given},\ \{t = 0\}$$

$$\lim_{t\to\infty} e^{-r\,t}\, a(t) \geq 0,\ \{t = \infty\}$$

The control variable is c, the state variable is a and the current-value Hamiltonian is

$$h = \ln\left(c\right) + q\left(r\,a - c\right).$$

The first order conditions are

$$c(t) = 1/q(t)$$

$$\dot{q} = \big(\rho - r\big)\, q,\ \lim_{t\to\infty} e^{-\rho t} q(t) a(t) = 0$$

$$\dot{a} = ra - c,\ a(0) = a_0$$

The optimality condition $c(t) = 1/q(t)$, allows us to determine $\frac{\dot{c}}{c} = -\frac{\dot{q}}{q} = (r - \rho)$ and write the transversality condition as $\lim_{t\to\infty} e^{-\rho t}\frac{a(t)}{c(t)} = 0$. Therefore the maximized Hamiltonian dynamic system, together with the initial and the transversality condition is

$$\dot{c} = \big(\rho - r\big)\, c,$$

$$\dot{a} = ra - c,$$

$$a(0) = a_0,$$

$$\lim_{t\to\infty} e^{-\rho t}\frac{a(t)}{c(t)} = 0.$$

Method 1: solving the problem recursively

First step: solve the Euler equation $\dot{c} = (r-\rho)c$, to obtain

$$c(t) = c(0)\, e^{(r-\rho)\,t},$$

where $c(0)$ is unknown.

Second step: substitute in the budget constraint $\dot{a} = r\,a - c(0)\,e^{(r-\rho)\,t}$, and solve, knowing that $a(0) = a_0$

$$\begin{aligned} a(t) &= e^{r\,t}\left(a_0 - \int_0^t e^{-r\,s}\,c(s)\,ds\right) \\ &= e^{r\,t}\left(a_0 - c(0)\int_0^t e^{-\rho\,s}\,ds\right) \\ &= e^{r\,t}\left(a_0 + \frac{c(0)}{\rho}\left(e^{-\rho\,t} - 1\right)\right) \end{aligned}$$

Third step: substitute in the transversality condition to find $c(0)$,

$$\begin{aligned} \lim_{t\to\infty} e^{-\rho t}\frac{a(t)}{c(t)} &= \lim_{t\to\infty} e^{-\rho t}\frac{e^{(r-\rho)t}}{e^{(r-\rho)t}\,c(0)}\left(a_0 + \frac{c(0)}{\rho}\left(e^{-\rho\,t} - 1\right)\right) \\ &= \lim_{t\to\infty}\left(\frac{a_0}{c(0)} - \frac{1}{\rho} + \frac{e^{-\rho t}}{\rho}\right) \\ &= \frac{a_0}{c(0)} - \frac{1}{\rho} \\ &= 0 \;\Rightarrow c^*(0) = \rho\,a_0 \end{aligned}$$

Fourth step: substitute in the general solutions to the budget constraint and to the Euler equation, to obtain the particular solutions

$$\begin{aligned} a^*(t) &= a_0\,e^{(r-\rho)t},\ t\in[0,\infty) \\ c^*(t) &= \rho\,a_0\,e^{(r-\rho)t},\ t\in[0,\infty). \end{aligned} \tag{3.30}$$

Method 2: backward solution

First step: come up with a trial function $z(t) = \frac{a(t)}{c(t)}$. Then

$$\frac{\dot{z}}{z} = \frac{\dot{a}}{a} - \frac{\dot{c}}{c}$$

Second step: substitute from the ODE's in the MHDS and obtain a backward problem

$$\begin{cases} \dot{z} = \rho z - 1 \\ \lim_{t\to\infty} e^{-\rho t} z(t) = 0 \end{cases}$$

Third step: solve the backward problem. The general solution of the ODE is

$$z(t) = \bar{z} + (z(0) - \bar{z})\, e^{\rho t},$$

where $z(0)$ is unknown and $\bar{z} = \frac{1}{\rho}$. To get the particular solution substitute in the transversality condition

$$\begin{aligned} \lim_{t\to\infty} e^{-\rho t} z(t) &= \lim_{t\to\infty} e^{-\rho t} \left(\bar{z} + (z(0) - \bar{z})\, e^{\rho t}\right) \\ &= \lim_{t\to\infty} e^{-\rho t}\, \bar{z} + (z(0) - \bar{z}) \\ &= z(0) - \bar{z} = 0 \Rightarrow z(0) = \bar{z} = \rho^{-1} \end{aligned}$$

Then $c(t) = \frac{a(t)}{z(t)} = \rho\, a(t)$

Fourth step: substitute in the budget constraint and solve the initial-value problem

$$\begin{cases} \dot{a} = (r - \rho)\, a, & t \in [0, \infty) \\ a(0) = a_0, & t = 0 \end{cases}$$

We obtain the same solution (3.30).

Method 3: general method for linear MHDS

If the MHDS is linear we can use general formulas for the solution of linear ODE's (see Appendix B). First step: observe that the MHDS is a linear ODE system. Defining

$$\mathbf{X}(t) = \begin{pmatrix} a(t) \\ c(t) \end{pmatrix}$$

it can be written in matrix form as

$$\dot{\mathbf{X}} = \mathbf{A}\,\mathbf{X},\ \text{where}\ \mathbf{A} = \begin{pmatrix} r & -1 \\ 0 & r-\rho \end{pmatrix}.$$

The steady state of this system is

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Second step: find the general solution of this system we know it is

$$\mathbf{X}(t) = \mathbf{e}^{\mathbf{A}t}\,\mathbf{X}(0),\ \text{for}\, t \in [0, \infty)$$

where the vector $\mathbf{X}(0) = \begin{pmatrix} a(0) \\ c(0) \end{pmatrix} = \begin{pmatrix} a_0 \\ c(0) \end{pmatrix}$ where $c(0)$ is unknwon.

Third step: the hard part is to find $\mathbf{e}^{\mathbf{A}t}$. As the eigenvalues of $\mathbf{A}$ are

$$\lambda_- = r - \rho < \lambda_+ = r$$

and the eigenvector matrix is

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ \rho & 0 \end{pmatrix},$$

the exponential matrix is

$$\mathbf{e}^{\mathbf{A}t} = \begin{pmatrix} 1 & 1 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} e^{(r-\rho)\,t} & 0 \\ 0 & e^{r\,t} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\rho} \\ 1 & -\frac{1}{\rho} \end{pmatrix} = \begin{pmatrix} e^{r\,t} & \frac{1}{\rho}\left(e^{(r-\rho)\,t} - e^{r\,t}\right) \\ 0 & e^{(r-\rho)\,t} \end{pmatrix}.$$

Therefore the general solution to the MHDS is

$$\begin{pmatrix} a(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} a(0)\,e^{r\,t} + \frac{c(0)}{\rho}\left(e^{(r-\rho)\,t} - e^{r\,t}\right) \\ c(0)\,e^{(r-\rho)\,t} \end{pmatrix}$$

Fourth step: the initial condition is satisfied if $a(t)|_{t=0} = a_0$, which implies $a(0) = a_0$. The transversality condition is satisfied if

$$\begin{aligned} \lim_{t\to\infty} e^{-\rho t}\frac{a(t)}{c(t)} &= \lim_{t\to\infty} \left(\frac{a_0}{c(0)}\,e^{\rho t} + \frac{1}{\rho}\left(1 - e^{\rho t}\right)\right) \\ &= \frac{a_0}{c(0)} - \frac{1}{\rho} \\ &= 0. \end{aligned}$$

It is satisfied if $c^*(0) = \rho\, a_0$, again.

Fifth step: we substitute again $c(0) = \rho\, a_0$ in the general solution to get the same (particular) solution to our problem (3.30).

3.5 The Hamiltonian dynamic system

Most optimal control problems don't have explicit solutions. However, in infinite-horizon optimal control problems we can obtain an approximate solution towards a steady state, as a result of the particular type of dynamics displayed by the MHDS.

We have been assuming that functions $f(.)$ and $g(.)$ are continuous and differentiable as regards (u, x). If in addition assume that $\partial^2 h(u, x)/\partial u^2 \neq 0$ (or $\det(h_{uu}) \neq 0$ if there is more than one control variable). Then, from the implicit function theorem, we can determine locally, from the optimality condition (**??**)

$$u^*(t) = u^*(x(t), q(t)),\ t \in \mathbb{R}_+$$

where

$$\frac{\partial u^*}{\partial q} = -h_{uu}^{-1} g_u, \qquad \frac{\partial u^*}{\partial x} = -h_{uu}^{-1} h_{ux},$$

where

$$h_{uu} = f_{uu} + q g_{uu},\ h_{uq} = g_u,\ h_{ux} = f_{ux} + q g_{ux}.$$

Then

$$h_x^*(x, q) = f_x(x^*, u^*(x^*, q)) - q g_x(x^*, u^*(x^*, q))$$

and

$$h_q^*(x, q) = f(x^*, u^*(x^*, q)).$$

If we substitute the control variables into the differential equations in (**??**) -(**??**) we get a mixed initial-terminal value problem for the planar ordinary differential equation which is called the **modified hamiltonian dynamic system**

$$\dot{q} = \rho q(t) - h_x(x^*(t), q(t)) \tag{3.31}$$

$$\dot{x}^* = h_q(x^*(t), q(t)) \tag{3.32}$$

together with the initial condition $x^*(0) = x_0$ and the transversality condition $\lim_{t\to+\infty} e^{-\rho t}q(t) = 0$ or $\lim_{t\to+\infty} e^{-\rho t}q(t)x^*(t) = 0$.

Proposition 13 (Local dynamics for the MHDS).
Let functions $f(.)$ and $g(.)$ be continuous and differentiable and let the MHDS have a fixed point. Then the local stable manifold of the MHDS has dimension 1 if and only if the determinant of its Jacobian evaluated in the neighborhood of its fixed point is negative. The steady state can never be a stable node or focus.

It is natural that the optimal trajectory should be a saddle path: given the initial value of the state variable, $x(0)$, the optimal path that verifies the first order conditions is unique for u and x.

Application: The Ramsey (1928)-Cass (1965) model

The Ramsey problem is:

$$\max_C \int_0^\infty e^{-\rho t}U(C(t))dt,\ \rho > 0,$$

subject to

$$\dot{K}(t) = F(K(t)) - C(t),\ t \in [0, \infty)$$

and $K(0) = K_0$ given and $\lim_{t\to\infty} e^{-\rho t}K(t) \geq 0$.

We assume that $u(C)$ and $F(K)$ are Increasing, concave and Inada:

$$U^{'}(.) > 0,\ U^{''}(.) < 0, F^{'}(.) > 0,\ F^{''}(.) < 0$$

$$U^{'}(0) = \infty,\ U^{'}(\infty) = 0,\ F^{'}(0) = \infty,\ F^{'}(\infty) = 0$$

The current-value Hamiltonian is

$$h(C(t), K(t), Q(t)) = U(C(t)) + Q(t)(F(K(t)) - C(t)).$$

The first order conditions according to the Pontriyagin's principle are:

$$
\begin{aligned}
U^{'}(C(t)) &= Q(t),\ t \in \mathbb{R}_{+} \\
\dot{Q} &= Q(t)\left(\rho - F^{'}(K(t))\right),\ t \in \mathbb{R}_{+} \\
\lim_{t\to\infty} e^{-\rho t} Q(t) K(t) &= 0 \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0
\end{aligned}
$$

The MHDS is

$$
\begin{aligned}
\dot{C} &= \frac{C(t)}{\sigma(C(t))}\left(F^{'}(K(t)) - \rho\right) \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0 > 0 \\
0 &= \lim_{t\to\infty} e^{-\rho t} U^{'}(C(t)) K(t)
\end{aligned}
$$

where

$$
\sigma(C) \equiv -\frac{U^{''}(C)C}{U^{'}(C)}
$$

is the elasticity of intertemporal substitution.

As we do not specified functions $U(.)$ and $F(.)$, the MHDS has no explicit solution. But we can use a qualitative approach in order to characterise the solution.

In order to do it, we first determine the steady state(s) $(\bar{C}, \bar{K})$, linearize the MDHS in the neighbourhood of the steady states, check if the transversality condition holds, and then characterise the linearised dynamics in the neighbourhood of an admissible steady state.

The steady state (if $K > 0$)

$$
\begin{aligned}
F^{'}(\bar{K}) &= \rho \Rightarrow \bar{K} = (F^{'})^{-1}(\rho) \\
\bar{C} &= F(\bar{K})
\end{aligned}
$$

Linearized system

$$\begin{pmatrix} \dot{C} \\ \dot{K} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\bar{C}}{\sigma(\bar{C})}F''(\bar{K}) \\ -1 & \rho \end{pmatrix} \begin{pmatrix} C(t)-\bar{C} \\ K(t)-\bar{K} \end{pmatrix}$$

The jacobian matrix

$$\mathbf{J} = \begin{pmatrix} 0 & \frac{\bar{C}}{\sigma(\bar{C})}F''(\bar{K}) \\ -1 & \rho \end{pmatrix}$$

has trace and determinant:

$$\mathrm{tr}(\mathbf{J}) = \rho > 0,\ \det(\mathbf{J}) = \frac{\bar{C}}{\sigma(\bar{C})}F''(\bar{K}) < 0$$

. Then $(\bar{C}, \bar{K})$ is a saddle point. In order to see this we compute the eigenvalues of $\mathbf{J}$

$$\lambda^s = \frac{\rho}{2} - \sqrt{\Delta} < 0,\ \lambda^u = \frac{\rho}{2} + \sqrt{\Delta} > \rho$$

where the discriminant is

$$\Delta = \left(\frac{\rho}{2}\right)^2 - \frac{\bar{C}}{\sigma(\bar{C})}F''(\bar{K}) > \left(\frac{\rho}{2}\right)^2 > 0.$$

The Jordan canonical form of matrix J is

$$\Lambda = \begin{pmatrix} \lambda^s & 0 \\ 0 & \lambda^u \end{pmatrix}$$

The eigenvector matrix is

$$\mathbf{V} = (V^s V^u) = \begin{pmatrix} \lambda^u & \lambda^s \\ 1 & 1 \end{pmatrix}.$$

To calculate it using the general formula for eigenvector V^i for $i = s, u$, $(J - \lambda^i I_2)V^i = 0$. Then in this model,

$$\begin{pmatrix} -\lambda^s & \frac{\bar{C}}{\sigma(\bar{C})}F''(\bar{K}) \\ -1 & \rho-\lambda^s \end{pmatrix} \begin{pmatrix} V_1^s \\ V_2^s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \frac{V_1^s}{V_2^s} = \frac{\rho-\lambda^s}{1} = \frac{\frac{\bar{C}}{\sigma(\bar{C})}F''(\bar{K})}{\lambda^s}$$

and

$$\begin{pmatrix} -\lambda^u & \frac{\bar{C}}{\sigma(\bar{C})}F''(\bar{K}) \\ -1 & \rho-\lambda^u \end{pmatrix} \begin{pmatrix} V_1^u \\ V_2^u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \frac{V_1^u}{V_2^u} = \frac{\rho-\lambda^u}{1} = \frac{\frac{\bar{C}}{\sigma(\bar{C})}F''(\bar{K})}{\lambda^u}$$

But from the general relationship between the eigenvalues and the trace and the determinant of matrix $\mathbf{J}$

$$\begin{cases} \lambda^s + \lambda^u = \mathrm{tr}(\mathbf{J}) = \rho \\ \lambda^s \lambda^u = \det(\mathbf{J}) = \frac{\bar{C}}{\sigma(\bar{C})} F''(\bar{K}) \end{cases}$$

we get

$$\frac{V_1^s}{V_2^s} = \lambda^u, \; \frac{V_1^u}{V_2^u} = \lambda^s$$

The general solution to the linearised system is

$$\begin{pmatrix} C(t) - \bar{C} \\ K(t) - \bar{K} \end{pmatrix} = h_s V^s e^{\lambda^s t} + h_u V^u e^{\lambda^u t}$$

where h_s and h_u are arbitrary constants. However $\lim_{t\to\infty} e^{\lambda^u t} = \infty$ which violates the transversality constraint. Therefore we set $h_u = 0$.

In order to evaluate h_s we use the information available on the initial point. Taking the solution for $K(t)$ and evaluating at $t = 0$ we have

$$\left.(K(t) - \bar{K})\right|_{t=0} = K(0) - \bar{K} = h_s$$

Then $h_s = K_0 - \bar{K}$ using the data on the problem $K(0) = K_0$

The optimal solution (C^*, K^*) is tangent, asymptotically, to the local stable manifold:

$$\begin{pmatrix} C^*(t) \\ K^*(t) \end{pmatrix} = \begin{pmatrix} \bar{C} \\ \bar{K} \end{pmatrix} + (K_0 - \bar{K}) \begin{pmatrix} \lambda^u \\ 1 \end{pmatrix} e^{\lambda^s t}, \; t \in [0, \infty).$$

This means that the local stable manifold has slope higher than the isocline $\dot{K}(C, K) = 0$

$$\left.\frac{dC}{dK}\right|_{W^s} (\overline{C}, \overline{K}) = \lambda^u > \left.\frac{dC}{dK}\right|_{\dot{K}} (\overline{C}, \overline{K}) = F'(\bar{K}) = \rho$$

3.6 Bibliographic references

- Introductory: Chiang (1992)

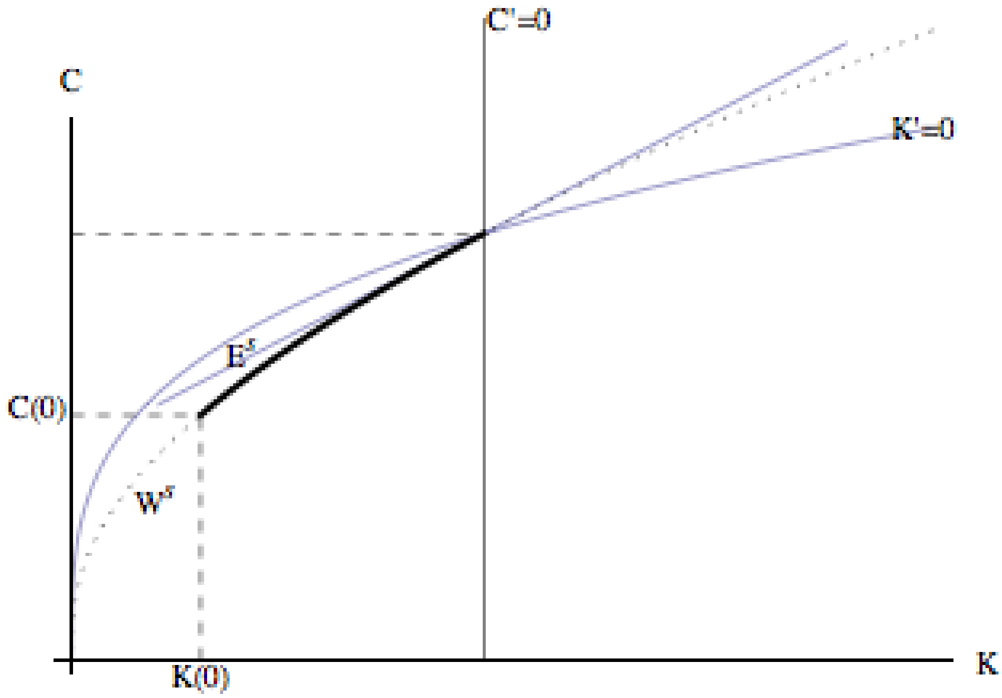

Figure 3.1: Ramsey model: phase diagram

- A little more advanced: Kamien and Schwartz (1991, part II)
- The original presentation is in Pontryagin et al. (1962)

Chapter 4

Dynamic programming

4.1 Finite horizon

We consider again the simplest optimal control problem with a fixed terminal time and free terminal state. The problem is to find the path $(u, x) = (u(t), x(t))_{t \in [t_0, t_1]}$ for $t_0 \leq t \leq t_1$, where $0 \leq t_0 < t_1 < \infty$, i.e, find functions (u^*, x^*) that:

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t)) dt$$

subject to

$$\begin{cases} \dot{x} = g(t, x(t), u(t)), & \text{for } t_0 \leq t \leq t_1 \\ x(t_0) = x_0 \text{ given} & \text{for } t = t_0 \\ x(t_1) \text{ free} & \text{for } t = t_1 \\ t_1 \text{ given} & \end{cases}$$

The value functional, evaluated at the optimum, is a function of the data of the problem, and in particular of the initial time and initial value of the state variable

$$\mathcal{V}(t_0, x_0) = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t)) dt.$$

We can extend this definition to the the optimal remaining value for starting at any time $t \in [t_0, t_1]$

when the state variable has level $x(t) = x_t$, assumed to be known: then

$$\mathcal{V}(t, x_t) = \int_t^{t_1} f(t, x^*(t), u^*(t))dt.$$

The remaining value for the terminal time is $\mathcal{V}(t_1, x(t_1)) = 0$, if there is no scrap value.

The **principle of dynamic programming** states that the solution to the optimal control problem should verify the following property: the control variable is determined in such a way that it maximizes the instantaneous variation of the value functional, assuming that the same strategy is followed for all the other times.

By using the principle of the dynamic programming Bellman (1957) obtained the following first-order necessary condition for optimality

Proposition 14 (**First order necessary conditions for optimality from the Dynamic Programming principle**).
*Let $\mathcal{V} \in C^2(\mathbb{T}, \mathbb{R})$. Then the value function which is associated to the optimal path $((x^*(t), u^*(t))_{t_0 \le t \le t_1}$ the **Hamilton-Jacobi-Bellman equation***

$$-\mathcal{V}_t(t, x) = \max_u [f(t, x, u) + \mathcal{V}_x(t, x)g(t, x, u)]. \tag{4.1}$$

Observations:

- In the DP theory the function $u^* = h(t, x)$ is called the **policy function** . Then the HJB equation may be written as

$$-\mathcal{V}_t(t, x) = f(t, x, h(t, x)) + \mathcal{V}_x(t, x)g(t, x, h(t, x))].$$

- Though the differentiability of $\mathcal{V}$ is assured for the functions f and g which are common in the economics literature, we can get explicit solutions, for $V(.)$ and for $h(.)$, only in very rare cases. Proving that $\mathcal{V}$ is differentiable, even in the case in which we cannot determine it explicitly is hard and requires proficiency in Functional Analysis.

Relationship with the Pontriyagin's principle:

(1) If we apply the transformation $\lambda(t) = \mathcal{V}_x(t, x(t))$ we get the following relationship with the Hamiltonian function which is used by the Pontriyagin's principle: $-\mathcal{V}_t(t, x) = H^*(t, x, \lambda)$;

(2) If $\mathcal{V}$ is sufficienty differentiable, we can use the principle of DP to get necessary conditions for optimality similar to the Pontriyagin principle.

The maximum condition is

$$f_u + \mathcal{V}_x g_u = f_u + \lambda g_u = 0$$

and the canonical equations are: as $\dot{\lambda} = \frac{\partial \mathcal{V}_x}{\partial t} = \mathcal{V}_{xt} + \mathcal{V}_{xx} g$ and differenting the HJB as regards x, implies $-\mathcal{V}_{tx} = f_x + \mathcal{V}_{xx} g + \mathcal{V}_x g_x$, therefore the canonical equation results

$$-\dot{\lambda} = f_x + \lambda g_x.$$

(3) Differently from the Pontryiagin's principle which defines a dynamic system of the form $\left(\mathbb{T}, \mathbb{R}^2, \varphi_t = (q(t), x(t))\right)$, the principle of dynamic programming defines a dynamic system as $((\mathbb{T}, \mathbb{R}), \mathbb{R}, v_{t,x} = \mathcal{V}(t, x)))$. That is, if defines a recursive mechanism in all or in a subset of the state space.

4.2 Infinite horizon discounted problem

Now we consider the discounted infinite-horizon problem

$$\max_u \int_0^\infty f(x(t), u(t)) e^{-\rho t} dt$$

subject to constraints

$$\begin{cases} \dot{x} = g(x, u), & \text{for } t \in [0, \infty) \\ x(0) = x_0 \text{ given} \end{cases}$$

We can additionally introduce constraints on the asymptotic value of the state variable x.

Proposition 15 (**First order necessary conditions for optimality from the Dynamic Programming principle**). *Let $\mathcal{V}$ be continuously differentiable in (t, x). Then the value function associated to the optimal path $((x^*(t), u^*(t))_{t_0 \le t < +\infty}$ verifies the fundamental non-linear ODE called the* ***Hamilton-Jacobi-Bellman equation***

$$\rho V(x) = \max_u \left\{ f(x, u) + V^{'}(x) g(x, u) \right\}, \tag{4.2}$$

where $V^{'}(x) \equiv \dfrac{dV(x)}{dx}$.

Differently from the HJB equation for the finite horizon, equation (4.1), the HJB equation for the infinite horizon, equation (4.2) is an ordinary differential equation. Unfortunately they are implicit functions and only in very rare cases they have closed form solutions.

If we determine the policy function $u^* = h(x)$ and substitute in the HJB equation get the implicit ODE

$$\rho V(x) = f(x, h(x)) + V'(x) g(x, h(x)),$$

where the independent variable is x and the unknown function is $V(x)$. This means that, while the Calculus of Variations and the Pontriyagin principle allow for the determination of the complete time path, the dynamic programming approach determines a **recursion** mechanism allowing fot solving the optimal control problem. This is a reason that the optimal policy function is interpreted as a feedback mechanism. Once it is determined, i.e., once we know $u^* = h(x)$ we can obtain the path of the state equation by solving the initial-value problem

$$\begin{cases} \dot{x}^* = g(x^*, h(x^*)), & \text{for } t \in [0, \infty) \\ x^*(0) = x_0 & \text{for } t = 0. \end{cases}$$

The difficulty here is: to determine $h(x)$ we have to obtain first $V(x)$ because $u = h(x)$ solves

$$f_u(x, u) + V^{'}(x) g_u(x, u) = 0.$$

4.2.1 Solving the HJB equation explicitly

In some cases we can solve the HJB equation (4.2) in an explicit way.

For example

1. the function $f(x,u) = f(u)$ is homogeneous;

2. the function $g(x,u) = ax + bu + c$ is linear.

In this case we usually can solve the HJB equation applying the **method of the undetermined coefficients**.

In this case we assume a trial function

$$v(W) = A + Bf(ax + c)$$

where A and B are undetermined, $f(.)$ is the objective function and a and c are the initial parameters. After substituting the trial function in the HJB equation we should be able to determine the coefficients A and B.

The next examples shows how the method works.

4.2.2 Examples

Application: the cake eating problem

Consider again problem (2.3). Now, we want to solve it by using the principle of the dynamic programming. In order to do it, we have to determine the value function $V = V(W)$ which solves the HJB equation

$$\rho V(W) = \max_{C} \left\{ \ln(C) + V^{'}(W)(-C) \right\}$$

The optimal policy for consumption is determined from

$$C^{*} = \left(V^{'}(W) \right)^{-1}$$

If we substitute back into the HJB equation we get the partial differential equation

$$\begin{aligned} \rho V(W) &= \ln(C^{*}) - V^{'}(W)C^{*} = \\ &= \ln\left[\left(V^{'}(W) \right)^{-1} \right] - \frac{V^{'}(W)}{V^{'}(W)} \end{aligned}$$

Then the HJB becomes an ordinary differential equation

$$\rho V(W) = -\ln V^{'}(W) - 1.$$

To solve it, i.e, in order to determine $V(W)$ we can use the method of determined coefficients by introduction a trial function

$$V(W) = a + b \ln W$$

where a and b are constants to be determined, if our conjecture is right. With this function, the HJB equation comes

$$\rho(a + b \ln W) = \ln(W) - \ln b - 1$$

if we set $b = 1/\rho$ we eliminate the term in $\ln W$ and get

$$a = -(1 - \ln(\rho))/\rho.$$

Therefore, solution for the HJB equation is

$$V(W) = \frac{-1 + \ln(\rho) + \ln W}{\rho}$$

and the optimal policy for consumption is $C = \rho W$. Then

$$C^{*}(t) = \rho W(t),\ t \in [0, \infty)$$

and

$$\dot{W}^{*} = -\rho W^{*}(t),\ t \in [0, \infty)$$

and the solution for the problem is has the solution

$$W^{*}(t) = W(0)e^{-\rho t},\ C^{*}(t) = \rho W(0)e^{-\rho t}\ \ t \in [0, \infty).$$

Application: the consumption-investment problem

We consider now the consumption-investment problem with isoelastic utility

$$\max_{c(\cdot)} n \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t}\, dt,\ \theta>0,\ \rho>0$$

subject to

$$\dot{a} = r\,a - c$$

$$a(0)=a_0$$

$$\lim_{t\to\infty} e^{-rt}\, a(t) \geq 0$$

where we assume that the agent has a positive initial net wealth position $a_0 > 0$.

The Hamilton-Jacobi-Bellman equation is

$$\rho\, v(a) = \max_c \left\{ \frac{c^{1-\theta}-1}{1-\theta} + v'(a)\left(r\,a - c\right)\right\}.$$

Therefore the maximum consumption satisfies,

$$c^{-\theta} = v'(a) \Leftrightarrow c = C(v'(a)) \equiv (v'(a))^{-\frac{1}{\theta}}.$$

Substitution c by $C(v'(a))$ in the HJB equation yields the optimum HJB equation

$$\rho\, v(a) = \frac{1}{1-\theta}\left(\theta\, (v'(a))^{\frac{\theta-1}{\theta}} - 1\right) + r\, a\, v'(a).$$

Because the utility function is homogeneous and the constraint of the problem is linear this equation has a closed form solution.

We try

$$v(a) = \alpha + \beta\, a^{1-\theta}$$

where α and β are two undetermined coefficients. As $v'(a) = (1-\theta)\,\beta\, a^{-\theta}$ then the optimum HJB equation is

$$\rho\,(\alpha + \beta\, a^{1-\theta}) = \theta\, \beta^{\frac{\theta-1}{\theta}}\,(1-\theta)^{-\frac{1}{\theta}}\, a^{1-\theta} - \frac{1}{1-\theta} + r\,\beta\,(1-\theta)\, a^{1-\theta}.$$

Separating the terms in $a^{1-\theta}$ this is equivalent to

$$\alpha\,\rho + \frac{1}{1-\theta} = \beta\, a^{1-\theta}\left(\theta\Big(\beta(1-\theta)\Big)^{-\frac{1}{\theta}} + r\,(1-\theta) - \rho\right)$$

setting both side to zero we find

$$\alpha = -\frac{1}{\rho\,(1-\theta)}, \text{and } \beta = \frac{1}{1-\theta}\Big(\frac{\rho + r\,(\theta-1)}{\theta}\Big)^{-\theta}$$

Therefore the optimal value function is

$$v(a) = \frac{1}{1-\theta}\Big(\Big(\frac{\rho + r\,(\theta-1)}{\theta}\Big)^{-\theta} a^{1-\theta} - \frac{1}{\rho}\Big).$$

Therefore, the optimal policy function is linear in a

$$c^* = C(a) = v'(a)^{-\frac{1}{\theta}} = \Big(\frac{\rho + r\,(\theta-1)}{\theta}\Big)\, a.$$

Substituting in the budget constraint yields,

$$\dot{a} = r\,a - C(a) = \gamma_a\,a, \text{ where } \gamma_a \equiv \frac{r-\rho}{\theta}.$$

Solving, together with the initial condition, yields the solution to the asset holdings of the agent

$$a^*(t) = a_0\, e^{\gamma_a\, t},\ t \in [0, \infty)$$

and the optimal consumption

$$c^*(t) = \big(r - \gamma_a\big)\, a_0\, e^{\gamma_a\, t},\ t \in [0, \infty).$$

We can check that this satisfies the solvability condition is satisfied

$$\lim_{t\to\infty} e^{-rt} a(t) = \lim_{t\to\infty} e^{\gamma_a - rt} a_0 = 0$$

because $r > \gamma_a$.

4.3 References

Some textbooks are:

- Intermediate: Kamien and Schwartz (1991, part II, section 21)
- Advanced: Grass et al. (2008)

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Appendices

Appendix A

Proofs

A.1 Proofs of chapter 2

Proof of Proposition 1. Suppose that we know the optimum x^*. Then the value of the path is

$$V[x^*] = \int_0^T F(t, x^*(t), \dot{x}^*(t))dt.$$

Consider an admissible perturbed path, x, such that $x(t) = x^*(t) + \varepsilon h(t)$ for all $t \in \mathbb{T}$. If it is admissible then the increment should verify $h(0) = h(T) = 0$. The value functional for this path is

$$V[x] = \int_0^T F(t, x^*(t) + \varepsilon h(t), \dot{x}^*(t) + \varepsilon \dot{h}(t))dt.$$

The variation of the functional V, after introducing the perturbation by $\varepsilon h(.)$ can be seen as a function of ε

$$\begin{aligned} \nu(\varepsilon) &= V[x^* + \varepsilon h] - V(x^*) = \\ &= \int_0^T \Big(F(t, x^*(t) + \varepsilon h(t), \dot{x}^*(t) + \varepsilon \dot{h}(t)) - F(t, x^*(t), \dot{x}^*(t)) \Big) \, dt \end{aligned}$$

A functional derivative (in the sense of Gâteaux) , evaluated at the optimal path, is

$$\delta V(x^*) = \lim_{\varepsilon \to 0} \frac{V(x^* + \varepsilon h) - V(x^*)}{\varepsilon} = v^{'}(0)$$

If x^* is optimum then $\delta V(x^*) = \nu^{'}(0) = 0$. We can compute $\nu^{'}(0)$ by using the Taylor theorem,

$$\begin{aligned} V[x] - V(x^*) &= \int_0^T \Big(F_x(t, x^*(t), \dot{x}^*(t))\varepsilon h(t) + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\varepsilon \dot{h}(t) + \dots \Big)\, dt \approx \\ &\approx \varepsilon \left(\int_0^T \Big(F_x(t, x^*(t), \dot{x}^*(t)) h(t) + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) \dot{h}(t) \Big)\, dt \right). \end{aligned}$$

Then $\delta V(x^*) = \upsilon^{'}(0)$

$$\begin{aligned} \nu^{'}(0) &= \int_0^T \Big(F_x(t, x^*(t), \dot{x}^*(t)) h(t) + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) \dot{h}(t) \Big)\, dt = \\ &= \int_0^T F_x(t, x^*(t), \dot{x}^*(t)) h(t) dt + \\ &\quad + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) h(t)|_{t=0}^T - \int_0^T \frac{d}{dt} F_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) h(t) dt = \\ &= \int_0^T \left(F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} F_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) \right) h(t) dt + \\ &\quad F_{\dot{x}}(T, x^*(T), \dot{x}^*(T)) h(T) - F_{\dot{x}}(T, x^*(T), \dot{x}^*(T)) h(T) \end{aligned}$$

using integration by parts. We can write compactly the necessary condition as

$$\nu^{'}(0) = \int_0^T \left(F_x^*(t) - \frac{d}{dt} F_{\dot{x}}^*(t) \right) h(t) dt + F_{\dot{x}}^*(T) h(T) - F_{\dot{x}}^*(0) h(0) = 0. \tag{A.1}$$

The admissibility conditions for the perturbation state $h(0) = h(T) = 0$. Then condition (A.1) becomes

$$\nu^{'}(0) = \int_0^T \left(F_x^*(t) - \frac{d}{dt} F_{\dot{x}}^*(t) \right) h(t) dt = 0.$$

Using the fact that given two continuous functions $g(t)$ and $h(t)$ and if $\int_0^T g(t)h(t)dt = 0$ for every function $h(t)$ such that $h(0) = h(T) = 0$ then $g(t) = 0$ for every $t \in [0, T]$. Therefore $\delta V[x^*] = \nu^{'}(0) = 0$ if and only if $F_x^*(t) = \frac{d}{dt} F_{\dot{x}}^*(t)$, which should hold for every $t \in [0, T]$. □

Proof of Proposition 2. We use, adapting, the same method of proof as in proposition 1. Again, we assume we know the solution to the problem x^*, that is a mapping $x^* : [0, T] \to \mathbb{R}$ such that $x^*(0) = x_0$. We introduce the perturbation $x(t) = x^*(t) + \varepsilon h(t)$. To be admissible $h(t)$ only has to satisfy $h(0) = 0$ and $h(T) \neq 0$ because $x^*(T)$ is free and should be determined.

In this case equation (A.1), after setting $h(0) = 0$ becomes

$$\nu^{'}(0) = \int_0^T \left(F_x^*(t) - \frac{d}{dt} F_{\dot{x}}^*(t) \right) h(t) dt + F_{\dot{x}}^*(T) h(T) = 0.$$

Then, x^* is optimum if and only if the Euler-Lagrange equation holds and if $F^*(T) = 0$ for $t = T$. $\square$

Proof of Proposition 3. Assuming we know the optimum x^*, the value is

$$V[x^*] = \int_0^{T^*} F(t, x^*(t), \dot{x}^*(t))dt$$

where the terminal time is $T = T^*$ is optimum. Let us assume that $x^*(T) = x_T$ and $T = T^*$ are both free. In this case we perturb both the terminal value of the state variable and the terminal time $x^*(T^*)$ and T^* by $\delta x(t) = \eta h(t)$ and $\delta T = \varepsilon\tau(T)$. Now the value functional becomes

$$V[x] = \int_0^{\mathbb{T}^*+\varepsilon\tau(T)} F(t, x^*(t) + \varepsilon h(t), \dot{x}^*(t) + \varepsilon\dot{h}(t))dt$$

To calculate $\delta V[x^*]$ we need to consider jointly the variation of the terminal state and the terminal time. To do that we let $T = T^* + \varepsilon\tau(t))$ and determine

$$\begin{aligned}\delta x(T) = \varepsilon h(T) &= x(T^* + \varepsilon\tau(T)) - x^*(T^*) = \\ &= x(T^*) - x^*(T^*) + \dot{x}^*(T^*)\delta T = \\ &= \varepsilon\left(h(T^*) + \dot{x}^*(T^*)\tau(T)\right)\end{aligned}$$

Therefore at the optimum

$$\begin{aligned}\nu^{'}(0) &= F^*(T^*)\tau(T) + \int_0^{T^*} (F^*_x(t) - \frac{d}{dt}(F^*_{\dot{x}}(t)))dt + F^*_{\dot{x}}(T^*)(h(T) - \dot{x}^*(T^*)\tau(T)) = \\ &= \int_0^{T^*} (F^*_x(t) - \frac{d}{dt}(F^*_{\dot{x}}(t)))dt + F^*_{\dot{x}}(T^*)h(T) + (F^*(T^*) - \dot{x}^*(T^*)F^*_{\dot{x}}(T^*))\tau(T) = 0\end{aligned}$$

For the case (CV3ii) as $h(T) = 0$ the transversality condition is $F^*(T^*) - \dot{x}^*(T^*)F^*_{\dot{x}}(T^*) = 0$ and, for case (CV3ii) we have two we have the same condition together with $F^*_{\dot{x}}(T^*) = 0$ $\square$

Proof of Proposition 4. Assuming we know the optimum x^*, the value is

$$V[x^*] = \int_0^{T} F(t, x^*(t), \dot{x}^*(t))dt + S(x^*(T), T).$$

The present of the boundary constraint makes it useful to define the Lagrangean We can define a Lagrangean, for any function $x \in \mathscr{X}$ and the Lagrange multiplier ψ

$$L[x] = V[x] + \psi R(x(T), T).$$

First, let us consider first the case (CV3i), that is, let us assume that T is fixed and $x(T)$ is free but constrained. Again we introduce the perturbation $x = x^* + \varepsilon h$ where $x(t) = x^*(t) + \varepsilon h(t)$ where $h(0) = 0$ and $h(t) \neq 0$ for every $t \in (0, T]$. Then

$$L[x^* + \varepsilon h] = V[x^* + \varepsilon h] + \psi R(x^*(T) + \varepsilon h(T), T)$$

Then

$$\delta L[x^*] = \int_0^T \left(F_x^*(t) - \frac{d}{dt} F_{\dot{x}}^*(t) \right) h(t) dt + \left(F_{\dot{x}}^*(T) + S_x^*(T) + \psi R_x^*(T) \right) h(T) = 0.$$

where $R^*(T) = R(x^*(T), T)$ and $R_x^*(T) = R_x(x^*(T), T)$. At the optimum, in addition to the Euler-Lagrange condition, the transversality condition is now $F_{\dot{x}}^*(T) + S_x^*(T) + \psi R_x^*(T) = 0$. □

Proof of Proposition 6. Take Proposition 6 and observe that $R(x(T), T) = h(T)x(T)$ and $R_x(x(T), T) = h(T)$. Then, for any values of $R(T)$ and ψ we have $F_{\dot{x}}(T) + \psi H(T) = 0$ and $\psi h(T)x(T) = 0$. Then we will always have $F_{\dot{x}}(T)x(T) = -\psi h(T)x(T) = 0$. □

A.2 Proofs of chapter 3

Proof of Proposition 7. Assume the problem has a solution $(u^*, x^*) = ((u(t), x(t))_{t \in [0,T]}$. The value functional takes the following values, at the optimum

$$V[u^*, x^*] = \int_0^T f(u^*(t), x^*(t), t) dt.$$

Next, we introduce perturbations to the state and control variables in the neighborhood of the solution to the problem (OC1): $x(t) = x^*(t) + \varepsilon h_x(t)$ and $u(t) = u^*(t) + \varepsilon h_u(t)$ where ε is a number. Those perturbations should be admissible: that is $h_x(0) = h_x(T) = 0$.

Now, observe that, for any differentiable function $x(t)$ and for any function $u(t)$, the following we have

$$
\begin{aligned}
V[u,x] &= \int_0^T f(u(t),x(t),t)dt = \\
&= \int_0^T f(u(t),x(t),t) + \lambda(t)(g(u(t),x(t),t) - \dot{x}(t))dt \text{ (constraint (3.1)} \\
&= \int_0^T H(u(t),x(t),\lambda(t),t) - \lambda(t)\dot{x}(t)dt \text{ (definition of Hamiltonian)} \\
&= \int_0^T \Big(H(u(t),x(t),\lambda(t),t) + \dot{\lambda}(t)x(t)\Big)\, dt + \lambda(0)x(0) - \lambda(T)x(T) \text{ (integration by parts).}
\end{aligned}
$$

At the optimum we should have $V[u^*,x^*] \geq V[u,x]$ where (u,v) is any other admissible pair of functions. In particular $V[u^*,x^*] \geq V[u^* + \varepsilon h_u, x^* + \varepsilon h_x]$.

Again defining the integral derivative

$$\delta V[u^*,x^*] = \lim_{\varepsilon\to 0} \frac{V[u^* + \varepsilon h_u, x^* + \varepsilon h_x] - V[u^*,x^*]}{\varepsilon} = v^{'}(\varepsilon)\Big|_{\varepsilon=0}$$

where

$$v(\epsilon) = V[u^* + \varepsilon h_u, x^* + \varepsilon h_x] - V[u^*,x^*]$$

a necessary condition for optimality is $v^{'}(0) = 0$.

But

$$
\begin{aligned}
v(\varepsilon) = \int_0^T & \Big[H(u^*(t) + \varepsilon h_u(t), x^*(t) + \varepsilon h_x(t), \lambda(t), t) - H(u^*(t), x^*(t), \lambda(t), t)) + \dot{\lambda}(t)(x^*(t) + \varepsilon h_x(t) - x^*(t))\Big]\, dt \\
& + \lambda(0)(x^*(0) + \varepsilon h_x(0) - x^*(0)) - \lambda(T)(x^*(T) + \varepsilon h_x(T) - x^*(T)) \\
& = \varepsilon \left\{ \int_0^T \left(\left(\frac{\partial H}{\partial x}(u^*(t), x^*(t), \lambda(t), t) + \dot{\lambda}(t) \right) h_x(t) + \frac{\partial H}{\partial u}(u^*(t), x^*(t), \lambda(t), t) h_u(t) \right) dt + \right. \\
& \left. + \lambda(0)h_x(0) - \lambda(T)h_x(T) \right\}.
\end{aligned}
$$

Therefore

$$v^{'}(0) = \int_0^T \left[\left(\frac{\partial H^*(t)}{\partial x}(t) + \dot{\lambda}(t) \right) h_x(t) + \frac{\partial H^*(t)}{\partial u} h_u(t) \right) dt + \lambda(0)h_x(0) - \lambda(T)h_x(T). \tag{A.2}$$

where we wrote $H^*(t) = H(u^*(t), x^*(t), \lambda(t), t)$. Then $\delta V[u^*,x^*] = 0$ if and only if $v^{'}(0) = 0$ if and only if

$$\frac{\partial H^*(t)}{\partial u} = \frac{\partial H^*(t)}{\partial x} + \dot{\lambda} = 0, \text{ for every } \ 0 < t < T$$

because the admissibility conditions imply $h_x(0) = h_x(T) = 0$. $\square$

Proof of Proposition 8. Consider the solution of the problem $(u^*(t), x^*(t))_{t\in[0,T]}$ and introduce a "spike" perturbation at time $0 < \tau < T$, such that $x(\tau) = x^*(\tau)+a$. $x(t) = x^*(t)+\varepsilon\delta(t-\tau)$ where $\delta(t)$ is Dirac's delta "function". This type a perturbation will generate a change on the solution for time τ onwards.

Now the variation of the value function will be

$$V[u, x] - V[u^*, x^*] = \int_\tau^T (H(t) + \dot{\lambda}(t)x(t) - H^*(t) - \dot{\lambda}(t)x^*(t))dt - \lambda(T)(x(T) - x^*(T)) + \lambda(\tau)\varepsilon,$$

performing a Taylor approximation

$$\begin{aligned} V[u, x] - V[u^*, x^*] \quad &= \quad \int_\tau^T [(H_x^*(t) + \dot{\lambda}(t))(x(t) - x^*(t)) + H_u^*(t)(u(t) - u^*(t))]dt - \\ &\quad -\lambda(T)(x(T) - x^*(T)) + \lambda(\tau)\varepsilon. \end{aligned}$$

Along an optimal path we have, $H_x^*(t) + \dot{\lambda}(t) = H_u^*(t) = 0$ and $x^*(T) = x(T) = x_T$ and therefore defining the partial integral derivative as

$$\delta_{x(t)}V[u^*, x^*] = \lim_{\varepsilon\to 0} \frac{V[u^*, x^* + \varepsilon\delta(t)) - V[u^*, x^*]}{\varepsilon}$$

we have $\delta_{x(t)}V[u^*, x^*] = \lambda(t)$. □

Proof of Proposition 9. The proof follows the same steps as the proof of Proposition 7 but in this case the admissible perturbation only requires that $h_x(0) = 0$. Therefore $h_x(T)$ if free. The implication is that, is that to have $v^{'}(0) = 0$, in equation (A.2) the transversality condition $\lambda(T) = 0$ should hold □

Proof of Proposition 10. In this case the optimal functional is

$$V[u^*, x^*] = \int_0^{T^*} f(t, u^*(t), x^*(t))dt$$

The value function for the perturbed state, control and terminal time is

$$f\left(t, u^*(t) + \varepsilon h_u(t), x^*(t) + \varepsilon h_x(t)\right) dt$$

Using the same reasoning as in the proofs of Propositions 3 and **??** we have the variational derivative and setting $h_x(T) = h_x(T^*) + \dot{x}^*(T^*)h_T(T)$ we get

$$v^{'}(0) = (H^*(T^*) + \dot{\lambda}(T)x^*(T))h_T(T) + \int_0^{T^*} \left[\left(\frac{\partial H^*(t)}{\partial x}(t) + \dot{\lambda}(t) \right) h_x(t) + \frac{\partial H^*(t)}{\partial u} h_u(t) \right) dt -$$

$$- \lambda(T^*)(h_x(T) - \dot{x}^*(T^*)h_T(T)) =$$

$$= \int_0^{T^*} \left[\left(\frac{\partial H^*(t)}{\partial x}(t) + \dot{\lambda}(t) \right) h_x(t) + \frac{\partial H^*(t)}{\partial u} h_u(t) \right] dt - \lambda(T^*)h_x(T) +$$

$$+ \left(H^*(T^*) + \dot{\lambda}(T)x^*(T) + \lambda(T^*)\dot{x}^*(T^*) \right) h_T(T)$$

instead of equation A.2. Note that

$$\dot{\lambda}(T)x^*(T) + \lambda(T^*)\dot{x}^*(T^*) = \frac{d(\lambda x)}{dt}(T^*) = \frac{d}{dt}\left(\lambda(t)x(t)\right)\Big|_{t=T^*}$$

□

Proof of Proposition 11. In order to prove this, observe we introduce the Lagrangian, which becomes after performing the same method as in the proof of Proposition 7

$$L(x) = \int_0^T (H^*(t) + \dot{\lambda}(t)x^*(t))dt - \lambda(T)x^*(T) + \lambda(0)x^*(0) + \nu h(T)x^*(T)$$

where ν is a constant Lagrange multiplier associated to the terminal condition. Then

$$v^{'}(0) = \int_0^T [(H_x^*(t) + \dot{\lambda}(t))h_x(t) + H_u^*(t)h_u(t)]dt + (\nu h(T) - \lambda(T))\, h_x(T)$$

Now, the optimality conditions are

$$H_x^*(t) + \dot{\lambda}(t) = H_u^*(t) = 0,\ t \in [0, T)$$

and

$$\nu h(T) - \lambda(T) = 0.$$

But as terminal state condition is an inequality, from the Kuhn-Tucker theorem the slackness conditions are

$$h(T)x(T) \geq 0,\ \ \nu \geq 0 \quad \text{and} \quad \nu h(T)x(T) = 0.$$

Therefore the transversality condition is $\lambda(T)x(T) = 0$ for any continuous function $h(T)$. □

Proof of Proposition 13. Let $y(t) = (p(t), x(t))$ and assume that the MHDS has a fixed point $\overline{y}$. Then the MHDS may be approximated, under certain conditions, in the neighborhood of $\overline{y}$ by the linear system $\dot{y}(t) = \mathbb{J}(y(t) - \overline{y})$, where $\mathbb{J} = D_y(G(\overline{y}))$. If $\det(h_{uu}) \neq 0$ we see that the jacobian is of type

$$\mathbb{J} = \begin{pmatrix} \dot{q} \\ \dot{x}^* \end{pmatrix} = \begin{pmatrix} \rho - a & b \\ c & a \end{pmatrix}$$

where

$$\begin{aligned} a &:= -\frac{g_u h^*_{ux}}{h^*_{uu}} \\ b &:= -\left(\frac{h^*_{xx}h^*_{uu} - (h^*_{ux})^2}{h^*_{uu}}\right) \\ c &:= -\frac{(g_u)^2}{h^*_{uu}} \end{aligned}$$

if $m = 1$ or

$$\begin{aligned} a &:= -h^*_{xu}u^*_q = h^*_{xu}h^{-1}_{uu}h_{uq} = h^*_{qu}(h^{-1}_{uu})^T h_{ux} \\ b &:= -\left(h^*_{xx} - h^*_{xu}(h^*_{uu})^{-1}h^*_{ux}\right) \\ c &:= -h^*_{qu}(h^*_{uu})^{-1}h^*_{uq} \end{aligned}$$

if $m > 1$, as $h_{xu} = h^T_{ux}$ and $h_{qu} = h^T_{uq}$ (from the continuity of functions $f(.)$ and $g(.)$)are $(1 \times m)$-vectors and h_{xx} is non-singular.

Then we will have

$$\begin{aligned} \mathrm{tr}(\mathbf{J}) &= \rho > 0 \\ \det(\mathbf{J}) &= a(\rho - a) - bc \end{aligned}$$

□

A.3 Proofs of section 4

Proof of Proposition 14. Let $t \in [t_0, t_1)$ and and assume that the value of the state variable at time t is $x^*(t) = x$. The remaining optimal value function

$$\begin{aligned}\mathcal{V}(t,x) &= \int_t^{t_1} f(s, x^*(s), u^*(s))ds \\ &= \max_{(u(s))_{s\in(t,t_1]}} \left(\int_t^{t_1} f(s, x(s), u(s))ds \right) \\ &= \max_{(u(t))_{s\in(t,t_1]}} \left(\int_t^{t+\Delta t} f(s, x(s), u(s))ds + \int_{t+\Delta t}^{t_1} f(s, x(s), u(s))ds \right)\end{aligned}$$

where we assume that $\Delta t > 0$ is small. Using the principle of dynamic programming and the definition of the remaining optimal value function we have

$$\begin{aligned}\mathcal{V}(t,x) &= \max_{(u(s))_{s\in(t,t+\Delta t]}} \left[\int_t^{t+\Delta t} f(s, x(s), u(s))ds + \max_{(u(s))_{s\in(t+\Delta t,t_1]}} \left(\int_{t+\Delta t}^{t_1} f(s, x(s), u(s))ds \right) \right] \\ &= \max_{(u(t))_{s\in(t,t+\Delta t]}} \left[\int_t^{t+\Delta t} f(s, x(s), u(s))ds + \mathcal{V}(t+\Delta t, x+\Delta x) \right]\end{aligned}$$

where the remaining value is evaluated at time $t + \Delta$ with a different value of the state variable resulting from the decision on the optimal control taken in the interval $(t, t + \Delta t)$, that we denote by $x^*(t + \Delta t)$. If we consider that we can approximate the value by $x^*(t + \Delta t) \approx x + \Delta x$, and that the function $\mathcal{V}$ is continuously differentiable, then

$$\mathcal{V}(t+\Delta t, x+\Delta x) = \mathcal{V}(t,x) + \mathcal{V}_t(t,x)\Delta t + \mathcal{V}_x(t,x)\Delta x + \text{h.o.t}$$

where

$$\mathcal{V}_t(t,x) = \frac{\partial \mathcal{V}}{\partial t}, \text{ and } \mathcal{V}_x(t,x) = \frac{\partial \mathcal{V}}{\partial x}.$$

Then

$$\mathcal{V}(t,x) = \max_{u(t)} \Big[f(t, x(t), u(t))\, \Delta t + \mathcal{V}(t,x) + \mathcal{V}_t(t,x)\Delta t + \mathcal{V}_x(t,x)\Delta x \Big].$$

Taking the limit $\Delta t \to 0$,and observing that $\lim_{\Delta t\to 0} \dfrac{\Delta x}{\Delta t} = \dot{x} = g(t,x,u)$, yields , for every $t \in [t_0, t_1]$,

$$-\mathcal{V}_t(t,x) = \max_u [f(t,x,u) + \mathcal{V}_x(t,x) g(t,x,u)].$$

□

Proof of Proposition 15. The value functional at the optimum, is a function of the pair (t_0, x_0),

$$\begin{aligned}\mathcal{V}(t_0, x_0) &= \max_u \left(\int_{t_0}^{+\infty} f(x,u)e^{-\rho t}dt \right) \\ &= e^{-\rho t_0} \max_u \left(\int_{t_0}^{+\infty} f(x,u)e^{-\rho(t-t_0)}dt \right) \\ &= e^{-\rho t_0} V(x_0)\end{aligned}$$

where $V(x_0) = \max_u \left(\int_0^{+\infty} f(x,u)e^{-\rho t}dt \right)$ is independent from t_0 and only depends on x_0, because we also have assumed that $g(.)$ is an autonomous function. As this relation holds initial time, and assuming again that $x^*(t) = x$, we have

$$\mathcal{V}(t,x) = e^{-\rho t} V(x).$$

Therefore

$$\mathcal{V}_t(t,x) = -\rho e^{-\rho t} V(x), \mathcal{V}_x(t,x) = e^{-\rho t} V^{'}(x)$$

and substituting in the HJB equation we get

$$0 = \max_u \left[f(x,u)e^{-\rho t} - \rho e^{-\rho t} V(x) + e^{-\rho t} V^{'}(x) g(x,u) \right]$$

which is equivalent to (4.2). □

Appendix B

Solution of linear ODE's

Consider the linear planar ordinary differential equation (ODE)

$$\dot{\mathbf{X}} = \mathbf{A}\,\mathbf{X} + \mathbf{B}, \text{for}\, t \in [0, T], \tag{B.1}$$

where

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ \dot{\mathbf{X}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix},\ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Assumption 1. $\det(\mathbf{A}) = a_{11}\,a_{22} - a_{12}\,a_{21} \neq 0.$

If assumption 1 holds, then matrix $\mathbf{A}$ has one unique (classical) inverse

$$\mathbf{A}^{-1} = \frac{\mathrm{adj}(\mathbf{A})^\top}{\det(\mathbf{A})} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

and the ODE (B.1) has a unique steady state

$$\bar{\mathbf{X}} = -\mathbf{A}^{-1}\,\mathbf{B}.$$

Any matrix $\mathbf{A}$ satisfies the similarity transformation

$$\mathbf{A} = \mathbf{P}\,\mathbf{\Lambda}\,\mathbf{P}^{-1}$$

where $\mathbf{\Lambda}$ is the Jordan canonical form and $\mathbf{P}$ is the eigenvector matrix. If the system (B.1) is a MHDS the Jordan canonical form is a diagonal matrix

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}$$

where $\lambda_\pm$ are the eigenvalues of $\mathbf{A}$, which are the roots of the characteristic polynomial equation

$$c(\lambda) = \lambda^2 - \mathrm{Trace}(\mathbf{A})\,\lambda + \mathrm{Det}(\mathbf{A}) = 0$$

that is

$$\lambda_\mp = \frac{\mathrm{Trace}(\mathbf{A})}{2} \pm \sqrt{\Delta(\mathbf{A})}$$

where

$$\Delta(\mathbf{A}) \equiv \left(\frac{\mathrm{Trace}(\mathbf{A})}{2}\right)^2 - \mathrm{Det}(\mathbf{A})$$

is the discriminant of $\mathbf{A}$.

From assumption 1, the MHDS has two real eigenvalues satisfying $\lambda_- < \lambda_+$.

The eigenvector matrix is obtained by concatenating the two eigenvectors

$$\mathbf{P} = \mathbf{P}^-|\mathbf{P}^+$$

which are determined To determine the eigenvalues the associated eigenvectors, which are the solutions of the homogeneous equations

$$\left(\mathbf{A} - \lambda_-\mathbf{I}\right)\mathbf{P}^- = \mathbf{0}\ \text{yields}\ \mathbf{P}^-$$

where $\mathbf{I}$ is the (2×2) identity matrix and

$$\left(\mathbf{A} - \lambda_+\mathbf{I}\right)\mathbf{P}^+ = \mathbf{0}\ \text{yields}\ \mathbf{P}^+.$$

The general solution to equation (B.1) is

$$\mathbf{X}(t) = \bar{\mathbf{X}} + \mathbf{e}^{\mathbf{A}t}\,(\mathbf{X}(0) - \bar{\mathbf{X}}),\ \text{for}\ t \in [0, T] \tag{B.2}$$

where

$$\mathbf{e}^{\mathbf{A}t} = \mathbf{P} \begin{pmatrix} e^{\lambda_-\,t} & 0 \\ 0 & e^{\lambda_+\,t} \end{pmatrix} \mathbf{P}^{-1}\ \text{for}\ t \in [0, T].$$

The general solution (B.2) depends on the vector of initial values $\mathbf{X}(0)$ which is arbitrary. To obtain the particular solution to our problem, we use the initial and the transversality (or the terminal) conditions.

Particular case . As matrix exponential $\mathbf{e}^{\mathbf{\Lambda}t}$ is diagonal, it is convenient, particularly in infinite horizon optimal control models, to write the general solution instead as

$$\mathbf{X}(t) = \bar{\mathbf{X}} + h_{-}\mathbf{P}^{-}\, e^{\lambda_{-}t} + h_{+}\mathbf{P}^{+}\, e^{\lambda_{+}t}, \text{ for } t \in [0, T]$$

where h_{-} and h_{+} are two arbitrary constants. In infinite horizon problems if $\lambda_{-} < 0 < \lambda_{+}$ we can observe that $\lim_{t\to\infty} e^{\lambda_{-}t} = 0$ and $\lim_{t\to\infty} e^{\lambda_{+}t} = +\infty$. The transversality generally requires that the solution should be bounded, which lead us, in order to obtain a particular solution, to set $h_{+} = 0$. The other arbitrary constant h_{-} is obtained by forcing the solution to verify the initial condition.