The AK model

Paulo Brito

pbrito@iseg.ulisboa.pt

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The AK model

- This is the simplest endogenous growth model
- The economy has the following features:
  1. population is constant and normalised to one $N = 1$
  2. there only one reproducible input: physical capital
  3. the economy produces one good with a CRS technology (using only capital)
  4. the good is used in consumption and investment (it is a closed economy)
  5. the consumer solves an intertemporal optimization problem

The AK model

Two versions of the model

- Decentralized version: there is no state and the allocation of capital through time is determined by market equilibrium
- Centralized version: there is a central planner ("benevolent dictator") that determines the optimal allocation of capital by maximizing the intertemporal social welfare
- As there are no externalities or other distortions, the two versions are equivalent: in this case we say that the **equilibrium allocations are Pareto optimal**
- When there are externalities (see the Romer model) the two economies lead to different allocations: then **equilibrium allocations are not Pareto optimal**

Assumptions

- Technology: production function:

$$Y = AK$$

- Economy's constraint: capital accumulation equation

$$\dot{K} = Y - \delta K - C$$

- Preferences: utility functional

$$\int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

The model: centralized version

- The central planner determines the optimal paths $(C(t), K(t))_{t\in[0,\infty)}$ by solving the problem

$$\max_{[C(t)]_{t\geq 0}} \int_0^\infty \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\begin{array}{rcl} \dot{K} & = & AK(t) - C(t) - \delta K, \\ K(0) & = & k_0, \text{ given, } \text{ t} = 0 \\ \lim_{t\to+\infty} e^{-At} K(t) & \geq & 0. \end{array}$$

- assumption: $A > \delta$

The MHDS

We determine the growth facts on $Y(t) = AK(t)$ as a solution of the MHDS (maximised hamiltonian dynamic system) solving

- two dynamic equations

$$\begin{aligned} \dot{C} &= C(A - \rho - \delta)/\theta && (1) \\ \dot{K} &= AK - C - \delta K, && (2) \\ & && (3) \end{aligned}$$

- initial and the transversality conditions

$$\begin{aligned} 0 &= \lim_{t \to \infty} C(t)^{-\theta} K(t) e^{-\rho t} && (4) \\ K(0) &= K_0, \text{ given} && (5) \end{aligned}$$

Growth in the AK model: conclusions

- The capital stock solution is

$$K(t) = \bar{K}(t) = k_0 e^{\gamma t},\ t \in [0, \infty)$$

- which implies that the output is

$$Y(t) = \bar{Y}(t) = A k_0 e^{\gamma t},\ t \in [0, \infty)$$

- Conclusion (growth facts):
  - the (endogenous) long run rate of growth is

$$\boxed{\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0}$$

  - the long run level is $\bar{y} = A k_0$
  - there is no transitional dynamics $\lambda = 0$ (this is counterfactual)

Solution method

for endogenous growth models

How do we obtain the previous results ?

1. Write variables as: $X(t) = x(t)e^{\gamma_x t}$
   (level = detrended $\times$ trend)
2. Rewrite the MHDS for the detrended variables by introducing assumptions on the rates of growth (call it **detrended MHDS**) such that it is an autonomous ODE
3. Determine the long run growth rate ($\bar{\gamma}$) from the steady state of the detrended MHDS
4. Introduce the long run growth rate in the detrended MHDS and solve for the detrended variables, k and $y = Ak$
5. Get the final solution for K and, therefore, for $Y = AK$

Step 1 : detrending variables

- Separation of transition, (k, c), and long-run trend $(e^{\gamma_k t}, e^{\gamma_c t})$

$$K(t) = k(t)e^{\gamma_k t}, \quad C(t) = c(t)e^{\gamma_c t},$$

- Then (because $c(t) = C(t)e^{\gamma_c t}$ and $k(t) = K(t)e^{\gamma_k t}$)

$$\frac{\dot{c}}{c} = \frac{\dot{C}}{C} - \gamma_c$$
$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \gamma_k$$

Step 2 : building the detrended MHDS

- Substituting $\dot{C}/C$ and $\dot{K}/K$ we get

$$\frac{\dot{c}}{c} = \frac{A-\rho-\delta}{\theta} - \gamma_c$$
$$\frac{\dot{k}}{k} = A - \delta - \frac{c}{k} e^{(\gamma_c - \gamma_k)t} \gamma_k$$

- A necessary condition for the MHDS to be autonomous (time-independent) is $e^{(\gamma_c-\gamma_k)t} = 1$, which implies

$$\boxed{\gamma = \gamma_k = \gamma_c}$$

- Therefore, the **detrended MHDS** is

$$\dot{c} = c\left(\frac{A-\rho-\delta}{\theta} - \gamma\right)$$
$$\dot{k} = (A-\delta-\gamma)k - c$$

Step 3 : the long-run growth rates

- Setting $\dot{c} = 0$ we get the long run growth rate

$$\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0$$

- Setting $\dot{k} = 0$ we get the long run ratio

$$\frac{\bar{c}}{\bar{k}} = \beta,$$

where

$$\beta \equiv A - \delta - \bar{\gamma} = \frac{1}{\theta}\left((A - \delta)(\theta - 1) + \rho\right) > 0$$

Step 4: solving the detrended MHDS

- if we substitute the rate of growth $\gamma = \bar{\gamma}$ in the detrended MHDS we have

$$\dot{c} = 0 \tag{6}$$

$$\dot{k} = \beta k - c \tag{7}$$

$$0 = \lim_{t \to +\infty} e^{-\beta t} k(t) c(t)^{-\theta} \tag{8}$$

because

$$\lim_{t \to +\infty} e^{-(\rho + \bar{\gamma}(\theta - 1))t} k(t) c(t)^{-\theta} = \lim_{t \to +\infty} e^{-\beta t} k(t) c(t)^{-\theta}$$

Step 4: solving the detrended MHDS (cont.)

- the solution of equation (6) is an unknown constant

$$c(t) = B$$

  B is an arbitrary constant

- substituting c and solving equation (7) we find

$$k(t) = \left(k_0 - \frac{B}{\beta}\right) e^{\beta t} + \frac{B}{\beta}.$$

- to determine B we substitute in the TVC

$$\begin{aligned}
\lim_{t \to +\infty} e^{-\beta t} k(t) c(t)^{-\theta} &= \lim_{t \to +\infty} e^{-\beta t} \left[\left(k_0 - \frac{B}{\beta}\right) e^{\beta t} + \frac{B}{\beta}\right] B^{-\theta} = \\
&= \lim_{t \to +\infty} \left[k_0 - \frac{B}{\beta}\right] B^{-\theta} = \\
&= 0
\end{aligned}$$

  then we find the unknown constant $B = \beta k_0$

Step 4: solving the detrended MHDS (cont.)

- Therefore the deterended consumption is

$$c(t) = \bar{c} = \beta\, k_0, \text{ for all } t \in [0, \infty)$$

- and the detrended capital stock is

$$k(t) = \bar{k} = \frac{B}{\beta} = k_0 \text{ for all } t \in [0, \infty)$$

- This means that there is no transitional dynamics

Step 5: the solution to the AK model

- The **balanced growth path** BGP is

$$\bar{K}(t) = \bar{k}e^{\gamma t}, \quad \bar{C}(t) = \bar{c}e^{\gamma t}.$$

- where $\gamma = \bar{\gamma}$ is determined from the steady state of the detrended MHDS
- the endogenous rate of growth is

$$\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0$$

- we get additionally the ratio of the levels along the BGP

$$\bar{c} = \beta\bar{k}, \ \bar{k} = k_0$$

- Observe that there is an indeterminacy here: we have two equations ($\dot{c} = 0$ and $\dot{k} = 0$) and three variables ((γ, c, k)). However, the value for k is given at the initial level
- **this is a typical property of the endogenous growth models**.

Discussion

- **Conclusion**: this model provides a **theory for the balanced growth path**.
- Differently from the Ramsey model:
  - it displays long run growth
  - but does not display transition (i.e., business cycle) dynamics
- applying to different countries, it provides a **theory for the long run trend in the growth rates**, provided that growth is only explained by capital accumulation:
  - growth depends positively on total factor **productivity** A and on the elasticity of **intertemporal substitution** in consumption $(1/\theta)$
  - growth depends negatively on the rate of **time preference** ρ and on capital depreciation δ (wear and tear of infrastructures)

References

- (Acemoglu, 2009, ch.11.1)
- Sometimes, researchers call this the Rebelo (1991) model

Daron Acemoglu. *Introduction to Modern Economic Growth.* Princeton University Press, 2009.

Sérgio Rebelo. Long run policy analysis and long run growth. *Journal of Political Economy*, 99(3):500–21, 1991.