

Mathematical Economics

Problem set 2020/21

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1 Deterministic dynamic optimisation: discrete time

1.1 Optimal control and the Pontryagin's principle

1.2.1 Consider the optimal control problem: $\max_u - \sum_{t=0}^3 (2 - u_t)^2$ subject to $y_{t+1} = 1/2y_t + u_t$ such that $y_0 = y_4 = 1$.

- (a) Determine first order conditions from the Pontryagin's maximum principle.
- (b) Determine the solution of the problem.

1.2.2 Consider the optimal control problem: $\max_u - \sum_{t=0}^3 (2 - u_t)^2$ subject to $y_{t+1} = 1/2y_t + u_t$ such that $y_0 = 1$ and y_4 is free.

- (a) Determine first order conditions from the Pontryagin's maximum principle.
- (b) Determine the solution of the problem.

1.2.3 Assume that there is a cake whose size at time $t \in \{0, 1, \dots, T\}$, where T is finite, is W_t . A consumer wants to eat it in T periods. The initial size of the cake is $W_0 = \phi > 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

- (a) Determine first order conditions from the Pontryagin's maximum principle.
- (b) Determine the solution of the problem.

1.2.4 Consider the optimal control problem: $\max_{\{u\}} \sum_{t=0}^3 y_t - (2 - u_t)^2$ subject to $y_{t+1} = 1/2(y_t - u_t)$ e $y_0 = 0$ e $y_4 = 45/2$.

- (a) Write the first order conditions according to Pontryagin's principle.

(b) Solve the problem, that is determine the optimal sequences $\{y_t^*\}_{t=0}^4$ and $\{u_t^*\}_{t=0}^4$

1.2.5 Assume that there is a cake whose size at time $t \in \{0, 1, \dots, T\}$, for a finite T , is W_t . A consumer wants to eat it in T periods; that is $W_T = 0$. The initial size of the cake is $W_0 = \phi > 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

1.2.6 Assume that there is a cake whose size at time $t \in \{0, 1, \dots, \infty\}$ is W_t . A consumer wants to eat it with the terminal constraint $\lim_{t \rightarrow \infty} W_t \geq 0$. The initial size of the cake is $W_0 = \phi > 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

1.2.7 Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \dots, T\}$. The horizon T is finite. The endowment evolves over time as $W_{t+1} = (1+r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer wants to have $W_T = \phi$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is logarithmic.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

1.2.8 Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \dots, \infty\}$. The endowment evolves over time as $W_{t+1} = (1+r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer wants to satisfy $\lim_{t \rightarrow \infty} e^{-rt}W_t \geq 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is logarithmic.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

1.2.9 Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \dots, T\}$. The horizon T is finite. The endowment evolves over time as $W_{t+1} = (1+r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer wants to have $W_T = \phi$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

1.2.10 Assume that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \dots, \infty\}$. The endowment evolves over time as $W_{t+1} = (1+r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer has the terminal constraint $\lim_{t \rightarrow \infty} W_t \geq 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

1.2.11 Find the optimal investment sequence for a firm, $\{I_t\}_{t=0}^{\infty}$, that maximizes the value functional $\sum_{t=0}^{\infty} (1+r)^{-t} \pi_t$, where $r > 0$ is the market interest rate. The cash flow in period t is $\pi_t = AK_t - I_t(1 + \xi I_t)$, where K_t is the capital stock, and $A > 0$ and $\xi > 0$ are productivity, and investment cost parameters, respectively. The restrictions of the problem are: the accumulation equation $K_{t+1} = I_t + (1 - \delta)K_t$, where $\delta \in [0, 1)$ is the rate of depreciation of capital, and the initial capital stock is given, $K_0 = \phi > 0$. Assume that $A > r + \delta$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

1.2.12 A representative consumer has the utility function $u = \ln(c_t)$ and has a constant intertemporal discount factor β^t , with $0 < \beta < 1$, and a finite lifetime T , she/he has the budget constraint $a_{t+1} = y - c_t + (1+r)a_t$, and has to bequeath the same wealth received at birth $a_0 = a_T = A$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

1.2.13 A representative consumer has the utility function

$$u = B - \zeta^{-1} e^{-\zeta C_t},$$

where $B > 0$ and $\zeta > 0$, and has a constant intertemporal discount factor β^t , with $0 < \beta < 1$, and a finite lifetime T , she/he has the budget constraint $A_{t+1} = A_t - C_t$, and $A_0 = \phi$ and $A_T = 0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

1.2.14 A representative consumer has the utility function

$$u = B - \zeta^{-1} e^{-\zeta C_t},$$

where $B > 0$ and $\zeta > 0$, and has a constant intertemporal discount factor β^t , with $0 < \beta < 1$, and a finite lifetime T , she/he has the budget constraint $A_{t+1} = Y - C_t + (1+r)A_t$, and has to bequeath the same wealth received at birth $A_0 = A_T = A$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

1.2.15 Find the optimal investment sequence, $\{I_t\}_{t=0}^T$, that maximizes the value functional

$$\sum_{t=0}^T \left(\frac{1}{1+r} \right)^t (pK_t - I_t(1 + (1/2)I_t))$$

where K_t is the capital stock, $r > 0$ is the market interest rate, and $p > 0$ is a productivity parameter. The restrictions of the problem are: the accumulation equation is $K_{t+1} = I_t + (1 - \delta)K_t$, where δ is the rate of depreciation of capital, and the initial and terminal capital stock is given by $K_0 = K_T = \phi > 0$. Assume that $p > r + \delta$ and $\delta \in [0, 1)$.

- (a) Write the problem as an optimal problem and determine the optimality conditions from the Pontryagin's maximum principle.
- (b) Find an explicit solution for K_t . Justify and give an intuition for your results.

1.2.16 A representative consumer wants to maximize the intertemporal utility functional $\sum_{t=0}^{\infty} \beta^t \ln(C_t^\alpha Z_t^{1-\alpha})$, where $0 < \alpha < 1$ and $0 < \beta < 1$, by using consumption C_t as a control variable. The variable Z_t denotes habits and is governed by the difference equation $Z_{t+1} = \delta(Z_t - C_t)$, where $\delta > 0$. The following initial and terminal conditions hold: $Z(0) = Z_0 > 0$, and $\lim_{t \rightarrow \infty} \beta^t Z(t) \geq 0$.

- (a) Write the first order optimality conditions from the Pontryagin's maximum principle.
- (b) Solve the problem, and provide an intuition to your results.

Deterministic dynamic optimisation: continuous time

Optimal control: Pontryagin's principle

2.2.1 A representative consumer has the utility functional $\int_0^T e^{-\rho t} \ln(C(t)) dt$, where $\rho > 0$, and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial and terminal wealth $A(0) = A(T) = A_0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

2.2.2 A representative consumer has the utility functional $\int_0^T e^{-\rho t} \ln(C(t)) dt$, where $\rho > 0$, and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial wealth $A(0) = A_0$ and $A(T) \geq 0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

2.2.3 A representative consumer has the utility functional $\int_0^\infty e^{-\rho t} \ln(C(t)) dt$, where $\rho > 0$, and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial wealth $A(0) = A_0$ and $\lim_{t \rightarrow \infty} A(t) \geq 0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

2.2.4 A representative consumer has the utility functional $\int_0^T e^{-\rho t} (1-\sigma)^{-1} (C(t))^{1-\sigma} dt$, where $\rho > 0$ and $\sigma > 0$ (but $\sigma \neq 1$), and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial and terminal wealth $A(0) = A(T) = A_0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

2.2.5 A representative consumer has the utility functional $\int_0^T e^{-\rho t} (1-\sigma)^{-1} (C(t))^{1-\sigma} dt$, where $\rho > 0$, and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial wealth $A(0) = A_0$ and $A(T) \geq 0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

2.2.6 A representative consumer has the utility functional $\int_0^\infty e^{-\rho t} (1-\sigma)^{-1} (C(t))^{1-\sigma} dt$, where $\rho > 0$. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial wealth $A(0) = A_0$ and $\lim_{t \rightarrow \infty} A(t) \geq 0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

2.2.7 Solve $\max_C \int_0^\infty e^{-\rho t} (B - \zeta \exp(-(C(t)/\zeta)) dt$, $\zeta > 0$ subject to $\dot{A} = rA - C$ where $A(0) = A_0$ given and $\lim_{t \rightarrow \infty} e^{-rt} A(t) \geq 0$.

- (a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

2.2.8 Solve $\max_{\pi} \int_0^T -(\pi(t))^2 e^{-\rho t} dt$ subject to $\dot{\pi} = \pi - \bar{\pi}$, where $\rho > 0$, subject to $\pi(0) = 0$.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

2.2.9 Solve $\max_I \int_0^{\infty} e^{-rt} (pK(t) - qI(t)^2) dt$ subject to $\dot{K} = I(t) - \delta K(t)$ $K(0) = k_0 > 0$.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

2.2.10 Solve $\max_I \int_0^{\infty} e^{-rt} (pK(t) - qI(t)(1 + \xi I(t)/K(t))) dt$ subject to $\dot{K} = I(t) - \delta K(t)$ $K(0) = k_0 > 0$.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

2.2.11 Consider the following endogenous growth model:

$$\max_C \int_0^{\infty} \frac{1}{1-\sigma} C(t)^{1-\sigma} e^{-\rho t} dt, \text{ subject to } \dot{K} = Y(t) - C(t)$$

together with $K(0) = K_0$ given and $\lim_{t \rightarrow \infty} e^{-At} K(t) \geq 0$. The production function is linear $Y(t) = AK(t)$ and the parameters verify: $\rho > 0$, $\sigma > 1$ and $A > 0$.

(a) Write the first order conditions according to the maximum principle of Pontryagin.

(b) Solve the problem. Under which conditions the solution displays unbounded growth?

2.2.12 A representative consumer wants to maximize the intertemporal utility functional $\int_0^{\infty} e^{-\rho t} \ln(C(t)) dt$, where $\rho > 0$, by using consumption $C(\cdot)$ as a control variable. She/he has initial wealth $A(0) = A_0$, and the instantaneous budget constraint is $\dot{A}(t) = (1 - \tau)(Y + rA(t)) - C(t)$, where income Y is constant and positive, and the income tax rate verifies $0 < \tau < 1$. The non-Ponzi game condition $\lim_{t \rightarrow \infty} e^{-rt} A(t) \geq 0$ holds.

(a) Write the first order optimality conditions from the Pontryagin's maximum principle.

(b) Solve the problem, and supply an intuition for your results.

2.2.13 Assuming that $x(\cdot)$ is a state variable and $u(\cdot)$ is a control variable, consider the optimal control problem

$$\max_{(u(t))_{t=0}^{\infty}} \int_0^{\infty} (x(t)^2 + u(t)^2) e^{-\rho t} dt$$

subject to $\dot{x} = \alpha(x-u)$ and $x(0) = \phi$ and $\lim_{t \rightarrow \infty} x(t)e^{-\rho t} = 0$. Assume that $0 < \rho < 2\alpha$ and that $\phi > 0$

- (a) Determine the optimality conditions from the Pontryagin's maximum principle.
- (b) Find an explicit solution for the optimal state variable $x(\cdot)$. Justify.

Dynamic programming

2.3.1 Consider the optimal control problem $\max_u - \int_0^4 (2-u)^2 dt$ subject to $\dot{y} = 1/2y(t) + u(t)$ for $t \in [0, 4]$ and $y(0) = 1$.

- (a) Write the HJB equation.
- (b) Determine the optimal policy function.

2.3.2 Assume that there is an endowment whose size at time $t \in [0, T]$, where T is finite, is $W(t)$. A consumer wants to consume until time T . That is $W(T) = 0$. The initial size of the cake is $W(0) = \phi > 0$. The consumer has a psychological rate of time preference $\rho > 0$ and a static logarithmic utility function. Determine the optimal consumption strategy using the principle of dynamic programming.

- (a) Write the HJB equation.
- (b) Determine the optimal policy function.

Solutions for discrete time problems

1.2.4 (a) $y_t^* = (3/2)(-1 + 2^t)$ and $u_t^* = (3/2)(1 - 3(2^t))$ for $t = 0, 1, 2, 3, 4$ or $y_t^* = \{0, 3/2, 9/2, 21/2, 45/2\}$ and $u_t^* = \{-3, -15/2, -33/2, -69/2\}$.

1.2.15 (a) The first order conditions are $\eta_{t+1} = ((1+r)\eta_t - p) \frac{1}{1-\delta}$ and $K_{t+1} = \eta_t - 1 + (1-\delta)K_t$ for $t \in \{0, 1, \dots, T\}$, $K_0 = \phi = K_T$. (b) The problem has the unique solution

$$K_t = k^* + (\phi - k^*) \left(\frac{((1+r^*)^T - 1)(1-\delta)^t + (1 - (1-\delta)^T)(1+r^*)^t}{(1+r^*)^T - (1-\delta)^T} \right)$$

$$\eta_t = \eta^* + \frac{(r^* + \delta)(\phi - k^*)(1 - (1-\delta)^T)}{(1+r^*)^T - (1-\delta)^T} (1+r^*)^t$$

for $t \in \{0, 1, \dots, T\}$, where $1+r^* = \frac{1+r}{1-\delta}$, $k^* = \frac{p-(r+\delta)}{\delta(r+\delta)}$ and $\eta^* = \frac{p}{r+\delta}$.

1.2.16 (a) F.o.c. $C_t^* = (\alpha)/\delta \frac{1}{\eta_t}$, $\eta_{t+1} = \frac{\eta_t}{\beta\delta} - \frac{1-\alpha}{\delta Z_{t+1}^*}$, $Z_{t+1}^* = \delta(Z_t^* - C_t^*)$, $Z_0^* = Z_0$ and $\lim_{t \rightarrow \infty} \beta^t \eta_t Z_t^* = 0$; (b) the solution $Z_t^* = Z_0 \left(\frac{\beta\delta}{\alpha+(1-\alpha)\beta} \right)^t$ and $C_t^* = \frac{\alpha(1-\alpha)}{\alpha+\beta(1-\alpha)} Z_t^*$

Solutions for continuous-time problems

- 2.2.12 (a) The first order conditions include the optimality conditions $1/C(t) = Q(t)$, $\dot{Q}(t) = (\rho - r(1 - \tau))Q(t)$ and $\lim_{t \rightarrow \infty} Q(t)A(t)e^{-\rho t} = 0$, plus the admissibility conditions $\dot{A}(t) = (1 - \tau)(Y + rA(t)) - C(t)$ and $A(0) = A_0$. (b) The problem has the unique solution (where $\gamma \equiv r(1 - \tau) - \rho$), $C(t) = \rho(A_0 + Y/r)e^{\gamma t}$ and $A(t) = -Y/r + (A_0 + Y/r)e^{\gamma t}$ for $t \in [0, \infty)$. Then: if $\gamma < 0$ then $\lim_{t \rightarrow \infty}(C(t), A(t)) = (0, -Y/r)$, if $\gamma = 0$ then $\lim_{t \rightarrow \infty}(C(t), A(t)) = (\rho(A_0 + Y/r), -Y/r)$ or if $\gamma > 0$ then $\lim_{t \rightarrow \infty}(C(t), A(t)) = (+\infty, +\infty)$.
- 2.2.13 (a) $u^*(t) = \frac{\alpha}{2}q(t)$, $\dot{q}(t) = (\alpha - \alpha)q(t) - 2x^*(t)$, $\dot{x}^*(t) = \alpha(x^*(t) - u^*(t))$, $x^*(0) = \phi$ and $\lim_{t \rightarrow \infty} x^*(t)e^{-\rho t} = 0$; (b) $x^*(t) = \phi e^{\lambda_s t}$ where $\lambda_s = \frac{\rho}{2} - \left(\left(\frac{\rho}{2} - \alpha(\rho - 2\alpha)\right)^2\right)^{1/2} < 0$