Mathematical Economics
Continuous time: calculus of variations

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Continuous time calculus of variations problem

- Find the path $x^* = (x^*(t) :\ 0 \le t \le T)$ that solves the problem

$$\max_{(x(t))_{t \in [0,T]}} \int_0^T F(t, x(t), \dot{x}(t))\, dt$$

- given $x(0)$
- given the horizon T
- and possibly other constraints on the value of $x(T)$

Calculus of variations: simplest problem

- Find $x^* = (x^*(t))_{0 \leq t \leq T}$ that solves the problem:

$$\max_{(x(t))_{t \in [0,T]}} \int_0^T F(t, x(t), \dot{x}(t))\, dt$$

given $x(0) = \phi_0$ and $x(T) = \phi_T$.

CV simplest problem: solution

Proposition (Necessary first order conditions)

Let $x^* = (x^*(t))_{0 \le t \le T}$ *be the solution to the simplest CV problem. Then* x^* *verifies the following conditions*

$$\begin{cases} F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt}\left(F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\right) = 0 & \textit{Euler equation} \\ x^*(0) = \phi_0 & \textit{initial condition} \\ x^*(T) = \phi_T & \textit{terminal condition} \end{cases}$$

Proof

The Euler-Lagrange equation

- The Euler equation is a 2nd order ODE (ordinary differential equation)

$$F_x^* = F_{\dot{x}t}^* + F_{\dot{x}x}^* \dot{x}^* + F_{\dot{x}\dot{x}}^* \ddot{x}^*, \; 0 \leq t \leq T.$$

Cake eating problem (terminal state given)

- Find $C^* = (C^*(t))_{0 \leq t \leq T}$ that

$$\max_{(C(t))_{t \in [0,T]}} \int_0^T \ln(C(t))\, e^{-\rho t} dt$$

- subject to

$$\dot{W}(t) = -C(t), \text{ for } t \in [0, T]$$

given $W(0) = \phi$ and $W(T) = 0$

- Formulated as a CV problem: find $W^* = (W^*(t))_{0 \leq t \leq T}$ that

$$\max_{(W(t))_{t \in [0,T]}} \int_0^T \ln(-\dot{W}(t))\, e^{-\rho t} dt$$

given $W(0) = \phi$ and $W(T) = 0$

Cake eating problem (terminal state given)

- Euler equation

$$\frac{d}{dt}\left(e^{-\rho t}\frac{1}{\dot{W}}\right) = 0 \Leftrightarrow \ddot{W} + \rho\dot{W} = 0$$

- Solution:
  - We know that $C(t) = -\dot{W}(t)$, therefore, the Euler equation becomes $\dot{C} = -\rho C(t)$ with solution

$$C(t) = C(0)e^{-\rho t}$$

    where $C(0)$ is unknwon.
  - But $\dot{W}(t) = \frac{dW(t)}{dt} = -C(t)$ can be separated as $dW = -C(t)dt$. Integrating

$$\int_{W(0)}^{W(t)} dW = -\int_0^t C(s)ds \Leftrightarrow W(t) - W(0) = -C(0)\int_0^t e^{-\rho s}ds$$

  - But $W(0) = \phi$, and

$$\int_0^t e^{-\rho s}ds = \frac{1}{\rho}\left(1 - e^{-\rho t}\right)$$

Cake eating problem (fixed terminal state)

- Then
$$W(t) = \phi - \frac{C(0)}{\rho}\left(1 - e^{-\rho t}\right)$$
where $C(0)$ is still unknown.
- To find $C(0)$ we use the terminal condition $W(T) = 0$. Therefore
$$\phi - \frac{C(0)}{\rho}\left(1 - e^{-\rho T}\right) = 0 \Rightarrow \frac{C(0)}{\rho} = \phi\left(1 - e^{-\rho T}\right)^{-1}$$
- The solution to the cake-eating problem is
$$W^*(t) = \frac{e^{-\rho t} - e^{-\rho T}}{1 - e^{-\rho T}}\phi, \text{ for } t \in [0, T]$$
- compare with the discrete time analog.

Calculus of variations: free end-point problem

- Find $x = (x(t))_{0 \leq t \leq T}$ that

$$\max_{(x(t))_{t \in [0,T]}} \int_0^T F(t, x(t), \dot{x}(t))\, dt$$

given $x(0) = \phi_0$, and $x(T)$ is free.

CV free endpoint problem: solution

- Necessary conditions: the solution to the CV problem $x^* = (x^*(t))_{0 \leq t \leq T}$ verifies:

$$\begin{cases} F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt}\left(F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\right) = 0 & \text{Euler equation} \\ x^*(0) = \phi_0 & \text{initial condition} \\ F_{\dot{x}}(T, x^*(T), \dot{x}^*(T)) = 0 & \text{transversality condition} \end{cases}$$

Cake eating problem (free terminal state)

- As in the previous case, the Euler equation and the initial condition yield

$$W(t) = \phi + \frac{C(0)}{\rho}\left(1 - e^{-\rho t}\right)$$

  where $C(0)$ is unknwon.

- To find $C(0)$ we use now the transversality condition $F_{\dot{W}}(T) = 0$

$$F_{\dot{W}}(T) = \frac{e^{-\rho T}}{\dot{W}(T)} = -\frac{e^{-\rho T}}{C(T)} = -\frac{e^{-\rho T}}{C(0)\,e^{-\rho T}} = -\frac{1}{C(0)}$$

- Then $F_{\dot{W}}(T) = 0$ if and only if $C(0) = \infty$ which means that our problem is misspecified.

Calculus of variations: constrained terminal state

- Find $x = (x(t))_{0 \leq t \leq T}$ that

$$\max_{(x(t))_{t \in [0,\,T]}} \int_0^T F(t, x(t), \dot{x}(t))\, dt$$

given $x(0) = \phi_0$, and $x(T) \geq \phi_T$ where ϕ_0 and ϕ_T are given.

CV constrained terminal state problem: solution

- Necessary conditions: the solution to the CV problem $x^* = (x^*(t))_{0 \leq t \leq T}$ verifies:

$$\begin{cases} F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt}\left(F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\right) = 0 & \text{Euler equation} \\ x^*(0) = \phi_0 & \text{initial condition} \\ F_{\dot{x}}(T, x^*(T), \dot{x}^*(T))\left(x^*(T) - \phi_T\right) = 0 & \text{transversality condition} \end{cases}$$

Cake eating problem with $\phi_T = 0$

- As in the previous case, the Euler equation and the initial condition yield

$$W(t) = \phi - \frac{C(0)}{\rho}\left(1 - e^{-\rho t}\right)$$

  where $C(0)$ is an arbitrary constant.

- To find $C(0)$ we use now the transversality condition $F_{\dot{W}}(T)\,W(T) = 0$

$$F_{\dot{W}}(T)\,W(T) = -\frac{\phi - \frac{C(0)}{\rho}\left(1 - e^{-\rho T}\right)}{C(0)} = 0$$

  to find again

$$\frac{C(0)}{\rho} = \phi\left(1 - e^{-\rho T}\right)^{-1}$$

- Then the solution is **formally (but not conceptually)** the same as in the fixed terminal state problem.

Calculus of variations: discounted infinite horizon

- Find $x = (x(t))_{t \in \mathbb{R}_+}$ that

$$\max_{(x(t))_{t \in [0,T]}} \int_0^\infty f(x(t), \dot{x}(t)) e^{-\rho t} dt, \ \rho t$$

  given $x(0) = \phi_0$
- Euler equation

$$e^{-\rho t} f_x(x^*, \dot{x}^*) - \frac{d}{dt}\left(e^{-\rho t} f_{\dot{x}}(x^*, \dot{x}^*)\right) = 0$$

- is equivalent to the 2nd order ODE

$$f_x(x^*, \dot{x}^*) + \rho f_{\dot{x}}(x^*, \dot{x}^*) - f_{\dot{x}x}(x^*, \dot{x}^*)\dot{x} - f_{\dot{x}\dot{x}}(x^*, \dot{x}^*)\ddot{x} = 0$$

Calculus of variations: discounted infinite horizon constrained terminal value

- Find $x = (x(t))_{t\in\mathbb{R}_+}$ that

$$\max_{(x(t))_{t\in[0,T]}} \int_0^\infty f(x(t), \dot{x}(t))e^{-\rho t}dt,\ \rho t$$

given $x(0) = \phi_0$, $\lim_{t\to\infty} x(t) \geq 0$

- Necessary conditions:

$$\begin{cases} f_x(x^*, \dot{x}^*) + \rho f_{\dot{x}}(x^*, \dot{x}^*) - f_{\dot{x}x}(x^*, \dot{x}^*)\dot{x} - f_{\dot{x}\dot{x}}(x^*, \dot{x}^*)\ddot{x} = 0 \\ x^*(0) = \phi \\ \lim_{t\to\infty} e^{-\rho t} f_{\dot{x}}(x^*(t), \dot{x}^*(t))x^*(t) = 0 \end{cases}$$

Cake eating problem: infinite horizon

- Find $W^* = (W^*(t))_{t\in\mathbb{R}_+}$ that

$$\max_{(W(t))_{t\in[0,\infty)}} \int_0^\infty \ln(-\dot{W}(t))\, e^{-\rho t} dt$$

given $W(0) = \phi$ and $\lim_{t\to\infty} W(t) \geq 0$

- The first order conditions are

$$\begin{cases} \rho \dot{W}^*(t) + \ddot{W}^*(t) = 0 \\ W^*(0) = \phi \\ -\lim_{t\to\infty} e^{-\rho t} \frac{W^*(t)}{\dot{W}^*(t)} = 0 \end{cases}$$

Cake eating problem: infinite horizon

- We already found the solution to the Euler equation, given the initial condition to be

$$W(t) = \phi - \frac{C(0)}{\rho}\left(1 - e^{-\rho t}\right)$$

  where $C(0)$ is unknown

- Therefore

$$\dot{W}(t) = -C(0)e^{-\rho t}$$

  then

$$-e^{-\rho t}\frac{W^*(t)}{\dot{W}^*(t)} = \frac{1}{C(0)}\left(\phi - \frac{C(0)}{\rho} + \frac{C(0)}{\rho}e^{-\rho t}\right)$$

- Substituting in the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\frac{W^*(t)}{\dot{W}^*(t)} = \frac{1}{C(0)}\left(\phi - \frac{C(0)}{\rho}\right) = 0$$

  if and only if $C(0) = \rho\phi$

Cake eating problem: infinite horizon

- Therefore, the solution to the problem: find $W^* = (W^*(t))_{t\in\mathbb{R}_+}$ that

$$\max_{(W(t))_{t\in[0,\infty)}} \int_0^\infty \ln(-\dot{W}(t))e^{-\rho t}dt$$

given $W(0) = \phi$ and $\lim_{t\to\infty} W(t) \geq 0$

- is

$$W^*(t) = \phi e^{-\rho t},\ t \in [0,\infty)$$

Proofs

Proof of proposition 1

- Assume we know the solution x^* for the problem.
- The optimal value

$$V(x^*) = \int_0^T F(t, x^*(t), \dot{x}^*(t))\,dt.$$

- For an admissible perturbation $x(t) = x^*(t) + \epsilon h(t)$ the value functional is

$$V(x) = \int_0^T F(t, x^*(t) + \epsilon h(t), \dot{x}^*(t) + \epsilon \dot{h}(t))\,dt$$

- The variation is $V(x) - V(x^*)$ is

$$\begin{aligned}
& \int_0^T \Big(F_x(t, x^*(t), \dot{x}^*(t))\epsilon h(t) + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\epsilon \dot{h}(t) \Big)\,dt \\
= \quad & \epsilon \left(\int_0^T \Big(F_x(t, x^*(t), \dot{x}^*(t)) h(t) + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) \dot{h}(t) \Big)\,dt \right) = \\
= \quad & \epsilon \left(\int_0^T \left(F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} F_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) h(t) \right) dt + \right. \\
& + F_{\dot{x}}(T, x^*(T), \dot{x}^*(T)) h(T) - F_{\dot{x}}(0, x^*(0), \dot{x}^*(0)) h(0) \Big)
\end{aligned}$$

Proof of proposition 1

- A functional (or Gâteux) derivative, evaluated at the optimal path, is

$$\delta V(x^*) = \lim_{\epsilon \to 0} \frac{V(x^* + \epsilon h) - V(x^*)}{\epsilon} = \left| \frac{dV(x^*)}{d\epsilon} \right|_{\epsilon=0}$$

- Then $\delta V(x^*) = 0$ if and only if the following holds:
    - For the simple CV problem:
      $F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} F_{\dot{x}}(t, x^*(t), \dot{x}^*(t) = 0$ and $h(T) = 0$
    - For the free-terminal state CV problem:
      $F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} F_{\dot{x}}(t, x^*(t), \dot{x}^*(t) = 0$ and
      $F_{\dot{x}}(T, x^*(T), \dot{x}^*(T)) = 0$
- Because $h(0) = 0$ is an admissibility condition.

Return.