EMA 2019-2020:
Problem set 1: linear ODE's

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1 Linear scalar ODE's

1.1 Autonomous ODE's

1.1.1 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following scalar ordinary differential equations and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -\frac{1}{2}y$;

(b) $\dot{y} = \frac{1}{2}y$;

(c) $\dot{y} = 2y$;

(d) $\dot{y} = -2y$;

(e) $\dot{y} = 0$;

(f) $\dot{y} = 2$;

(g) $\dot{y} = -2$;

1.1.2 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following scalar ordinary differential equations and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -\frac{1}{2}y + 1$;

(b) $\dot{y} = \frac{1}{2}y - 1$;

(c) $\dot{y} = 2y - 2$;

(d) $\dot{y} = -2y + 2$;

(e) $\dot{y} = ay - 2$ for $a \in (-2, 2)$

(f) $\dot{y} = y + b$ for $b \in (-1, 1)$

1.1.3 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following initial value problems and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -0.5y + 1$, for $t \geq 0$ and $y(0) = 1$ for $t = 0$;

(b) $\dot{y} = 0.5y - 1$, for $t \geq 0$ and $y(0) = 1$ for $t = 0$.

1.1.4 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following terminal value problems and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -0.5y + 1$, for $t \geq 0$ and $\lim_{t\to\infty} y(t) = \overline{y}$, where $\overline{y}$ is the steady state;

(b) $\dot{y} = 0.5y - 1$, for $t \geq 0$ and $\lim_{t\to\infty} e^{-0.5t} y(t) = 0$.

1.2 Non-autonomous ODE's

1.2.1 Consider the scalar ODE

$$\dot{y} = ay + b(t),\ y : [0, \infty) \to \mathbb{R}$$

where

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^*, \\ b_1 & \text{if } t^* \leq t < \infty. \end{cases}$$

(a) Assume that $a \neq 0$ and $y(0) = y_0$ is given. Solve the initial value problem.

(b) Assume that $a > 0$ and $\lim_{t\to\infty} y(t)e^{-at} = 0$. Solve the terminal-value problem.

1.2.2 Consider the scalar ODE

$$\dot{y} = ay + b(t),\ y : [0, \infty) \to \mathbb{R}$$

where

$$b(t) = \begin{cases} b & \text{if } 0 \leq t < t^*, \\ b + \Delta b & \text{if } t^* \leq t < t^* + \Delta t, \\ b & \text{if } t^* + \Delta t \leq t < \infty, \end{cases}$$

where $\Delta t > 0$.

(a) Assume that $a \neq 0$ and $y(0) = y_0$ is given. Solve the initial value problem.

(b) Assume that $a > 0$ and $\lim_{t\to\infty} y(t)e^{-at} = 0$. Solve the terminal-value problem.

1.3 Applications

1.3.1 The simplest model of population dynamics assumes that the rate of population growth is deterministic, age-independent, and constant:

$$\dot{N} = \nu N.\ N : \mathbb{R}_+ \to \mathbb{R}_+, \tag{1}$$

where $N(t)$ is the population at time t and $\nu \equiv \beta - \mu$ is the net rate of growth, β is the fertility rate and μ is the mortality rate. We assume that $N(0) = N_0 \geq 0$ is given. (References Banks (1994) see also `http://en.wikipedia.org/wiki/Exponential_growth`)

(a) Solve equation (1).

(b) Solve the initial value problem.

(c) Characterize the dynamics.

1.3.2 The stock-flow dynamics is generically represented by an equation of type,

$$\dot{A} = \pi + rA,\ A : \mathbb{R}_+ \to \mathbb{R} \tag{2}$$

where A is the stock of an asset at time t, π is net income and r is the rate of return. Assume that $r > 0$

(a) Solve equation (2) and characterise qualitatively the dynamics.

(b) Assuming we know $A(0) = A_0$, solve the initial value problem.

(c) Assuming we introduce a solvability requirement $\lim_{t\to\infty} A(t)e^{-rt} = 0$, determine the initial level of $A(0)$.

1.3.3 Sargent and Wallace (1973) is one of the first papers to deal with perfect foresight dynamics. The main equation of the paper was

$$\dot{p} = \beta(p - m(t)),\ p : \mathbb{R}_+ \to \mathbb{R} \tag{3}$$

where p and m are the logs of the price index and nominal money supply and $\beta > 0$

(a) Solve equation (3).

(b) Setting $p(0) = p_0$, where p_0 is known, solve the initial value problem. Does the solution to this problem makes economic sense (hint: recall the expected relationship between increases in the money supply and the price evolution) ?

(c) Let m is constant. Assuming there are no speculative bubbles, i.e, $\lim_{t\to\infty} p(t)e^{-\beta t} = 0$, determine $p(0)$.

(d) Modify the previous results assuming that there is an anticipated (to time $\tau > 0$ and finite) monetary shock.

2 Planar ODE's

2.1 General

2.1.1 Let $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{A}$ can take one of the following values

$$\text{a)} \begin{pmatrix} -3 & 1 \\ -1 & -5 \end{pmatrix},\ \text{b)} \begin{pmatrix} -3 & 2 \\ -1 & -6 \end{pmatrix},\ \text{c)} \begin{pmatrix} -4 & 4 \\ -2 & -4 \end{pmatrix},$$

(a) Solve the planar ODE's and characterise analytically and geometrically the solutions for each case.

(b) Let $\mathbf{B} = (1, 1)$. Solve the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, for the three cases, and characterise analytically and geometrically the solutions.

(c) Consider the ODE's of the last question. Let $\mathbf{y}(0) = (0, 0)$. Solve the initial value problems. Characterise analytically and geometrically the solutions of the initial value problem.

2.1.2 Let $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{A}$ can take one of the following values

$$\text{a)} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \text{ b)} \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, \text{ c)} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \text{d)} \begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}, \text{e)} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$

(a) Solve the planar ode and characterise analytically and geometrically the solutions

(b) Let $\mathbf{B} = (1, 1)$. Solve the planar ode $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ and characterise analytically and geometrically the solutions

(c) Consider the ODE's of the last question. Let $\mathbf{y}(0) = (0, 0)$. Solve the initial value problems. Characterise analytically and geometrically the solutions of the initial value problem.

2.1.3 Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ a & -2 \end{pmatrix},$$

for $a \in \mathbb{R}$. Let a take any value on its domain. Determine the different solutions and characterise them.

2.1.4 Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} a & 1 \\ 1 & -3a \end{pmatrix},$$

for $a \in \mathbb{R}$. Let a take any value on its domain. Determine the different solutions and characterise them.

2.2 Applications

2.2.1 Consider a continuous time version of a two-state Markov process $\dot{y} = My$, where the transition matrix is

$$M = \begin{pmatrix} p-1 & 1-p \\ 1-q & q-1 \end{pmatrix}$$

for $0 < p < 1$ and $0 < q < 1$

(a) solve the differential equation;

(b) let $y(0) = (1, 2)$. Solve the initial value problem;

(c) draw the phase diagram.

2.2.2 Consider a continuous time version of a two-state Markov process $\dot{y} = My$, where the transition matrix is

$$M = \begin{pmatrix} -\pi_1 & \pi_1 \\ \pi_2 & -\pi_2 \end{pmatrix}$$

for $0 < \pi_1 < 1$ and $0 < \pi_2 < 1$

(a) solve the differential equation;

(b) let $y(0) = (0, 1)$ and solve the initial value problem;

(c) draw the phase diagram associated to the initial value problem.

2.2.3 Consider the wage-price dynamics for an economy in which there is perfect foresight in the product market and in which wages do not fully adjust to imbalances in the labour market. The economy is represented by a planar ODE

$$\begin{aligned} \dot{p} &= \lambda(p - m) \\ \dot{w} &= \gamma(p - w - N) \end{aligned}$$

where λ and γ are positive parameters and m and N are exogenous variables (money and population respectively). Assume that $w(0) = w_0$ is given and that prices verify $\lim_{t\to\infty} p(t) = \overline{p}$:

(a) determine the fixed point;

(b) solve the ODE;

(c) solve the mixed initial-terminal value problem;

(d) draw the phase diagram;

(e) which consequences will arise from an increase in the money supply ?

2.2.4 This is inspired in the Calvo (1983) model. Assume an imperfectly competitive economy in which the firms have the following rule for setting prices: $x(t) = \delta \int_t^{+\infty} (p^*(s) + \beta y^*(s))e^{-\delta(t-s)} ds$, for $\beta > 0$ and $\delta > 0$, where x is the price set by each firm, p is the aggregate price index, y is the aggregate level of activity, and δ denotes the (constant) probability for price revisions. All the variables are logarithms and the star represents expectations. Differentiating, we have equivalently $\dot{x} = \delta(x - p^* - \beta y^*)$. We assume that the aggregate price level is a weighted average of the prices set by all firms and it is given by $p(t) = \delta \int_{-\infty}^{t} x(s)e^{-\delta(t-s)} ds$, or equivalently $\dot{p} = \delta(x - p)$.

Equilibria in the goods and monetary markets implies that the following reduced form equation holds $y(t) = a(m(t) - p(t)) + b\pi^*(t)$, where m is the (log) of the money stock and $\pi \equiv \dot{p}$ is the inflation rate. The nominal money stock m is constant and exogenous. At last, assume that in this economy agents have perfect foresight (i.e, $p^* = p$). All the parameters are positive.

a) Obtain a planar ODE over (p, x), representing this economy

b) Perform a qualitative analysis of the local dynamics. Assume that $\beta b < 1 < \beta(a+b)$.

c) Assume there is a non-anticipated and permanent shock in m. Study the comparative dynamics assuming that x is non-predetermined and p is pre-determined.

d) Discuss the goodness of the choice of x as a non-predetermined variable, versus the alternative in which p is non-predetermined. Does it makes sense to assume that both variables are non-predetermined ? What would be the comparative dynamics in this case ?

References

Banks, R. B. (1994). *Growth and Diffusion Phenomena*. Springer-Verlag.

Calvo, G. A. (1983). Staggered contracts and exchange rate policy. In Frenkel, J. A., editor, *Exchange Rates and International Macroeconomics*, pages 235 – 258. University of Chicago Press.

Sargent, T. J. and Wallace, N. (1973). The stability of models of money and growth with perfect foresight. *Econometrica*, 41:1043–8.