Advanced Mathematical Economics

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Lecture 3
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Contents

Chapter 5

Scalar non-linear ODE's: the regular case

5.1 Introduction

In this chapter we address non-linear scalar ordinary differential equations, in which time is the independent variable. Non-linear ordinary differential equations provides the most important setup for dynamic systems theory.

Specifically, we look for the existence and characterization of functions $y : \mathrm{T} \subset \mathbb{R}_+ \rightarrow \mathrm{Y} = \mathbb{R}$, that solve the (autonomous) equation

$$F(\dot{y}, y) = 0. \tag{5.1}$$

We say the ODE is **regular** if $F(\dot{y}, y, \cdot)$ is continuous and continuously differentiable in $(\dot{y}, y)$, and furthermore $\dfrac{\partial F(\dot{y}, y)}{\partial \dot{y}} \neq 0$ for every $y \in \mathrm{Y}$. This allows us to write equation (5.1) as $\dot{y} = f(y)$. However, in this chapter we consider ODE's of type

$$\dot{y} = f(y, \varphi),\ f : \mathrm{Y} \times \Phi \rightarrow \mathrm{Y}, \tag{5.2}$$

where $\varphi \in \Phi$ is a set of real-valued parameters. In this case, the solution is the mapping $y : \mathrm{T} \times \Phi \rightarrow \mathrm{Y} \subseteq \mathbb{R}$. We will see that in some cases the value of a parameter (or several parameters) have a fundamental influence on the properties of the solutions for an ODE. We also consider, implicitly, forward ODE's (or initial-value problems).

Most ODEs representing economic theories are not specified explicitly, or when they are explicitly specified are non-linear. Most non-linear equations do not have an explicit (or closed-form solution). Therefore, is there a solution to a non-linear differential equation ? How can we solve, or at least characterize the properties of a given those differential equations ?

First, if $f : \mathrm{Y} \rightarrow \mathrm{Y}$ and $\mathrm{Y} \subseteq \mathbb{R}$ is sufficiently smooth (I.e., it is continuously differentiable up to a higher order) then one unique solution exists, for at least an interval of T[1]. Furthermore, the degree of smoothness of $y(t)$ is the same as that of $f(y)$.

As an application of this theorem we already saw that linear differential equations have one unique explicit solution, this solution exists for the whole T set, and it has the same properties for any value of an initial value $y(0)$. Therefore, we can ay that the solutions we find are **global solutions**. We also saw that they have zero, one or an infinite number of steady states. When a steady state exists and it i unique it was either stable or unstable, which implied that whole space is an attractor set in the first case and a repeller set in the second. Therefore, we have a characterization of the **global dynamic** properties.

Non-linear ODE's display some differences, although we already know that a solution exists and is unique (possibly for a subset of T). First, they may not have explicit solutions[2]. Second, they can have zero, one, or any finite number of steady states. Third, **local and global dynamics** can differ. That is, steady states can can be stable, unstable, or neither stable not stable (called bifurcation points), and, as we can have more than one steady state - but a finite number of them, we can have steady states with different stability properties. This implies that steady states introduce a partition of the state space Y into different subsets with different stability properties. That is, **local dynamics differ from global dynamics**.

In this chapter we consider **regular ODEs**, that is ODEs whose function $f(y, \cdot)$ is continuous and continuously differentiable (in the sense that all the derivatives are finitely-valued), that is $f \in C^n(\mathbb{R})$ for $n \leq 3$. We start with specific normal forms. **normal form** is the simplest ODE, whose exact solution is usually known and represents a whole family of ODEs by topological equivalence.[3]

In the rest of this chapter we present the normal forms (5.2) and in section we present the qualitative theory for 5.3 for non-linear regular scalar ODE's.

5.2 Normal forms

A **normal form** is the simplest ODE, whose exact solution is usually known and represents a whole family of ODEs by topological equivalence.[4]

We present next some important normal forms for scalar ODEs, which have the generic representation by a polynomial vector field

$$\dot{y} = f(y, \varphi) \equiv a_0 + a_1\, y + a_2\, y^2 + a_3\, y^3, \tag{5.3}$$

[1] See Guckenheimer and Holmes (1990).

[2] There are complete lists of differential equations with known explicit solutions in Zwillinger (1998), Canada et al. (2004) or Zaitsev and Polyanin (2003).

[3] In heuristic terms, we say functions $f(y)$ and $g(x)$ are topologically equivalent if there exists a diffeomorphism, i.e., a smooth map h with a smooth inverse h^{-1}, such that if $y = h(x)$ then $h(g(x)) = f(h(x))$. This property may hold globally of locally. The last case is the intuition behind the Grobman-Hartmann theorem.

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where $\varphi \equiv (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$.

Linear scalar ODE's, that have been studied in chapter two, are particular cases in which $a_2 = a_3 = a_4 = 0$. In this chapter we address cases in which at least one of coefficients a_2, a_3 or a_4 is different from zero.

It can be shown that (see (Hale and Koçak, 1991, ch. 2)) the following cases are the most relevant: first, ODE's depending on a single parameter (a), as the Ricatti's equation $\dot{y} = a + y^2$, the quadratic Bernoulli equation, $\dot{y} = ay + y^2$, the cubic equations Bernoulli equation, $\dot{y} = ay - y^3$, the Abel equation $\dot{y} = a + y - y^3$; second, ODE's depending on two parameters (a and b) the Abel's equation, $\dot{y} = a + by - y^3$.

We address the solution of those equations from two approaches: First, an **analytical approach**, seeking to find its explicit solution (when it is known); and, second, a **qualitative (or geometric) approach**, which characterizes qualitatively the possible dynamic behavior of the solution, depending on the value of the parameters. The

5.2.1 The linear equation

In this section we recall linear scalar ODE

$$\dot{y} = b + a\,y,$$

which is a particular case of equation (5.3) with $a_0 = b$, $a_1 = a$, and $a_2 = a_3 = 0$, where a and b are real numbers. Generically, we have $f_{yy}(\cdot) = f_{yyy}(\cdot) = 0$, and $f_y(\cdot) \neq 0$ can have any sign.

It has the a explicit solutions, whose form depends on the parameters

$$y(t) = \begin{cases} y(0) & \text{if } \ a = b = 0 \\ y(0)\,e^{a\,t} & \text{if } \ a \neq 0,\ b = 0 \\ y(0) + b\,t & \text{if } \ a = 0,\ b \neq 0 \\ y(0)\,e^{a\,t} + \bar{y}\ \left(1 - e^{a\,t}\right) & \text{if } \ a \neq 0,\ b \neq 0 \end{cases}$$

where $\bar{y} = -b/a$, and $y(0)$Y is arbitrary.

The qualitative properties are: first, it has one unique steady state, $\bar{y} = -b/a$, if $a \neq 0$, which is asymptotically stable if $a < 0$, or unstable if $a > 0$; second, if $a = 0$ it has an infinite number of steady states, if $b = 0$ or no steady state, if $b \neq 0$.

5.2.2 The Ricatti's equation: saddle-node or fold bifurcation

The quadratic equation

$$\dot{y} = f(y, a) \equiv a + y^2 \tag{5.4}$$

is called **Ricatti** equation. This equation is a particular case of equation (5.3) with $a_0 = a$, $a_2 = 1$, and $a_1 = a_3 = a_4 = 0$. Generically, we have $f_{yy}(\cdot) \neq 0$, $f_a(\cdot) \neq 0$ and $f_y(\cdot) = f_{yyy}(\cdot) = 0$.

It has an explicit solution[5] :

$$y(t) = \begin{cases} \frac{y(0)}{1-y(0)\,t} \ , & \text{if } a = 0 \\ \sqrt{a}\left(\tan\left(\sqrt{a}(t - y(0)^{-1})\right)\right) \ , & \text{if } a > 0 \\ -\sqrt{-a}\left(\tanh\left(\sqrt{-a}(t - y(0)^{-1})\right)\right) \ , & \text{if } a < 0 \end{cases}$$

where $y(0) \in \mathrm{Y}$ is an arbitrary constant belonging to the domain of y.

The behavior solution depends again on the value of the parameter a:

- if $a = 0$, the solution takes an infinite value at a finite time $t = \dfrac{1}{y(0)}$ [6], i.e., $\lim_{t \to -y(0)^{-1}} y(t) = \pm\infty$ and tends asymptotically to a steady state $\bar{y} = 0$, that is $\lim_{t \to \infty} y(t) = 0$ independently of the value of k;

- if $a > 0$ the solution takes infinite values for a periodic sequence of times $t \in \{-y(0)^{-1}, \pi - y(0)^{-1}, 2\pi - y(0)^{-1}, \dots, n\pi - y(0)^{-1}, \dots\}$,

  $$\lim_{t \to n\pi - y(0)^{-1}} y(t) = \pm\infty, \text{ for } n \in \mathbb{N}$$

  and it has no steady state;

- if $a < 0$, the solution converges to

  $$\lim_{t \to \infty} y(t) = \begin{cases} -\sqrt{-a} \ , & \text{if } y(0)^{-1} < \sqrt{-a} \text{ or } -\sqrt{-a} < y(0)^{-1} < \sqrt{-a} \\ +\infty, & \text{if } y(0)^{-1} > \sqrt{-a}. \end{cases}$$

To characterize qualitatively the solution of the differential equation, we have to find the steady state(s), by finding the values of $\bar{y} \ \in \mathrm{Y}$ such that $f(y, a) = 0$, and characterize its local dynamics, by finding the sign of $f_y(\bar{y}, a) = 2\bar{y}$. Therefore, the qualitative dynamic properties of the ODE depend on the value of a:

- existence and number of steady states: the set of steady states is the set $\bar{y} = \{\ y \in \mathrm{Y} : a + y^2 = 0\}$, i.e, the set of stationary solutions to the ODE. We readily find that: (a) if $a > 0$ there are no steady states, (b) if $a = 0$ there is one steady state $\bar{y} = 0$, and (c) if $a < 0$ there are two steady states $\bar{y} \in \{\bar{y}_1, \bar{y}_2\}$ for $\bar{y}_1 = \ -\sqrt{-a} < 0$, and $\bar{y}_2 = \sqrt{-a} > 0$;

- local dynamics at a steady state, can only be determined for $a \leq 0$. Then: (a) if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable, because $f_y(0) = 0$; and (b) if $a < 0$ steady state $\bar{y}_1$ is asymptotically stable and steady state $\bar{y}_2$ is unstable, then , $f_y(\bar{y}_1) = -2\sqrt{-a} < 0$,

[5] See appendix section 5.A.1

[6] This is different to the linear case, v.g., $\dot{y} = y$, whose solution $y(t) = y(0)e^t$, if $y(0) \neq 0$, takes an infinite value only in infinite time.

and $f_y(\bar{y}_2) = 2\sqrt{-a} > 0$. Furthermore, we if $a < 0$ the stable manifold associated to (or the basin of attraction of) steady state $\bar{y}_1$, is

$$\mathcal{W}^s_{\bar{y}_1} = \{\ y \in \mathrm{Y}\ : y < \sqrt{-a}\}.$$

Comparing to the linear case, for the case in which the steady state is asymptotically stable, the stable manifold is a subset of Y not the whole Y.

There is a bifurcation point at $(y, a) = (0, 0) \in \mathrm{Y} \times \Phi$, which is called **saddle-node bifurcation**. This bifurcation point is defined by the subset of points in the state space and the space of the parameters $\mathrm{Y} \times \Phi$, given by $\{\ (y, a) :\ f(y, a) = 0,\ f_y(y, a) = 0\}$, such that a steady state changes its dynamic properties, or its phase diagram: .

The bifurcation point, $(0, 0)$, is determined by solving the following system of equations for (y, a):

$$\begin{cases} f(y, a) = 0 \\ f_y(y, a) = 0 \end{cases} \Leftrightarrow \begin{cases} a + y^2 = 0 \\ 2y = 0. \end{cases}$$

Figure 5.1 shows phase diagrams for the $a < 0$ (panel (a)), for the $a = 0$ (panel (b)), and $a > 0$ (panel (c)). Those diagrams provide a geometrical depiction of the results that we have already obtained: there are no steady states in panel (a); there is a steady state in panel (b) but it is neither stable (an initial value higher than 0 will generate an unstable trajectory) nor unstable (an initial value lower than 0 will generate an asymptotically stable trajectory converging to zero); and in panel (c) there are two steady states one which is stable and one unstable. In the last case, we see that the basin of attraction to steady state $\bar{y}_1$ is limited by steady state $\bar{y}_2$, as shown in $\mathcal{W}^s_{\bar{y}_1}$

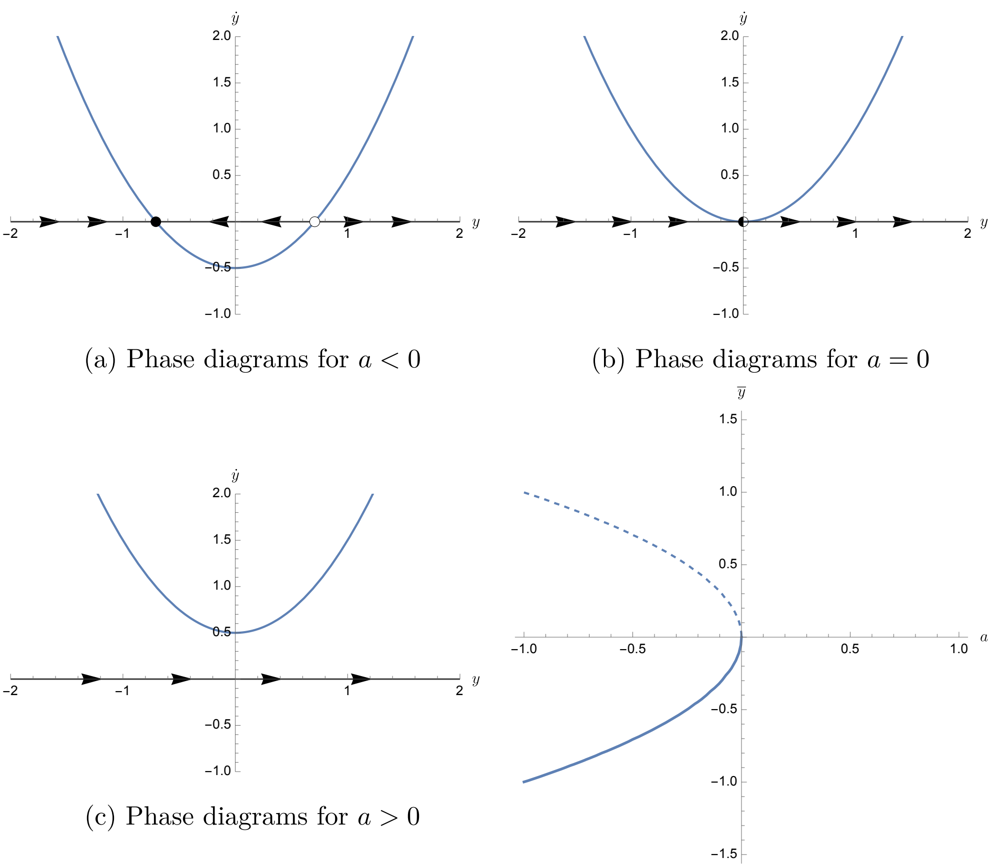
Panel (d) presents the bifurcation diagram for a **saddle-node**. It depict points (a, y) such that $a + y^2 = 0$, say $\bar{y}(a)$, and in solid-line the subset of points such that $f_y(\bar{y}(a)) < 0$ and in dashed-line the subset of points such that $f_y(\bar{y}(a)) > 0$. The first case corresponds to asymptotically stable steady states and the second to unstable steady states. Observe that the curve does not lie in the positive quadrant for a which is the geometrical analogue to the non-existence of steady states. The saddle-node bifurcation point is at the origin $(0, 0)$. This point separates the values for a such that there are two steady states (for $a < 0$) from the points of a such that there is no steady state (for $a > 0$).

5.2.3 Quadratic Bernoulli equation: transcritical bifurcation

The quadratic equation

$$\dot{y} = ay + y^2 \tag{5.5}$$

is called **quadratic Bernoulli** equation. This equation is a particular case of equation (5.3) with $a_1 = a$, $a_2 = 1$, and $a_0 = a_3 = 0$. Generically, we have $f_{yy}(\cdot) \neq 0$ and $f_{ya}(\cdot) \neq 0$. It has the



(a) Phase diagrams for $a < 0$

(b) Phase diagrams for $a = 0$

(c) Phase diagrams for $a > 0$

(d) Bifurcation diagram

Figure 5.1: Phase diagrams and bifurcation diagram for equation (5.4)

explicit solution [7]:

$$y(t) = \begin{cases} \frac{y(0)}{1-y(0)\,t} \ , & \text{if } a = 0 \\ \frac{a}{(1+a/y(0))e^{-at}-1}, & \text{if } a \neq 0 \end{cases}$$

where $y(0)$ is an arbitrary element of Y.

The behavior of the solution is the following (see Figure 5.3):

- if $a < 0$, as we can see in panel (a),

$$\lim_{t\to\infty} y(t) = \begin{cases} 0 \ , & \text{if } y(0) < -a \\ +\infty, & \text{if } y(0) > -a \end{cases}$$

- if $a = 0$, as we can see in panel (a), it behaves as the Ricatti's equation when $a = 0$

$$\lim_{t\to\infty} y(t) = \begin{cases} 0 \ , & \text{if } y(0) < 0 \\ +\infty, & \text{if } y(0) > 0 \end{cases}$$

[7]See appendix section 5.A.2 for the explicit solution for the general Bernoulli ODE.

- if $a > 0$, as we can see in panel (c),

$$\lim_{t\to\infty} y(t) = \begin{cases} -a\ , & \text{if } y(0) < 0 \\ +\infty, & \text{if } y(0) > 0 \end{cases}$$

The dynamic properties depend on the value of a:

- existence and number of steady states: if $a = 0$ there is one steady state $\bar{y} = 0$, if $a \neq 0$ there are two steady states $\bar{y} = \{\ \bar{y}_1, \bar{y}_2\}\ = \{0, -a\}$;

- local dynamics at the steady states: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable, that is, $(y, a) = (0, 0)$ is again a bifurcation point; if $a < 0$ the steady state $\bar{y}_1 = 0$ is asymptotically stable and steady state $\bar{y}_2 = -a$ is unstable; and if $a > 0$ the two steady states change stability with steady state $\bar{y}_1 = 0$ being unstable and steady state $\bar{y}_2 = -a$ being asymptotically stable. Then, a non-empty basin of attraction always exists, because the stable manifolds associated to the asymptotically stable equilibrium points are: if $a < 0$ the stable manifold is

$$\mathcal{W}^s_{\bar{y}_1} = \{\ y \in Y : y < -a\}.$$

  and, if $a > 0$, the stable manifold is

$$\mathcal{W}^s_{\bar{y}_2} = \{\ y \in Y : y < 0\}.$$

The bifurcation point, $(y, a) = (0, 0)$, is determined by solving the following system of equations for (y, a):

$$\begin{cases} f(y, a) = 0, \\ f_y(y, a) = 0, \end{cases} \Leftrightarrow \begin{cases} a\,y + y^2 = 0, \\ a + 2\,y = 0, \end{cases} \Leftrightarrow \begin{cases} 2\,a\,y + 2y^2 = 0, \\ a\,y + 2\,y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} a\,y = 0, \\ a + 2\,y = 0. \end{cases}$$

That bifurcation point is called **transcritical bifurcation**, as Figure 5.3, panel (d), shows that after crossing the bifurcation point the number of steady states is the same (two) but their stability properties change.

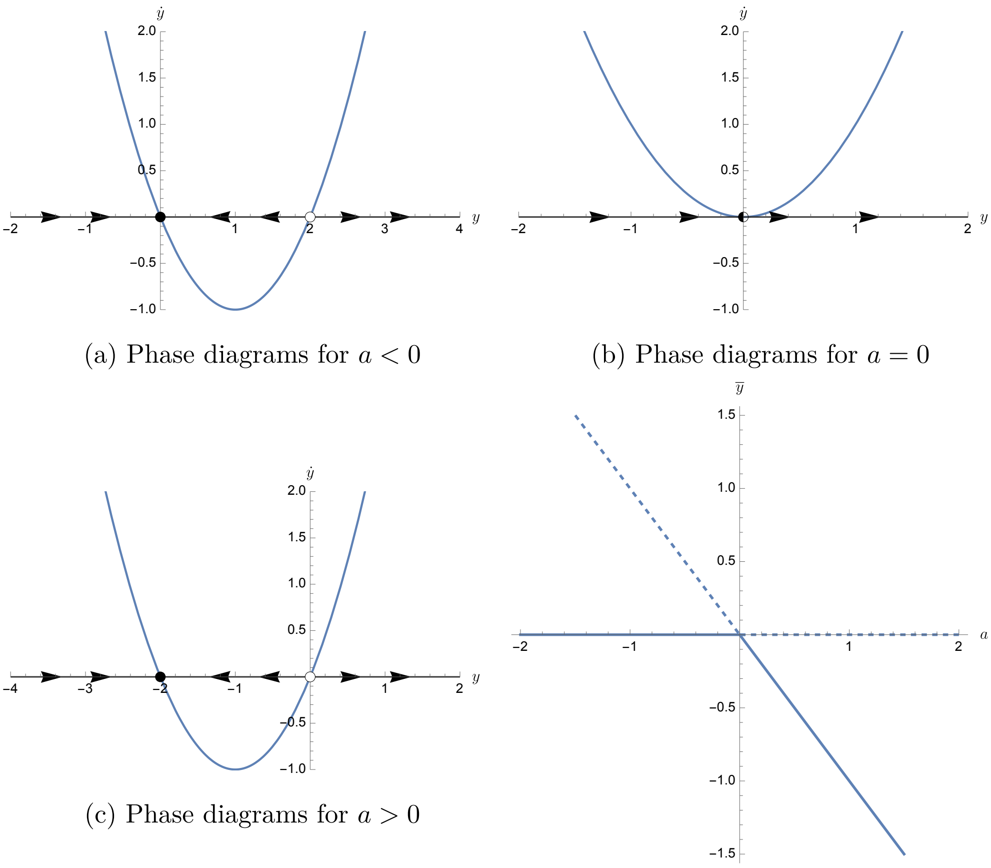
Exercise Show that $\dot{y} = a\,y - y^2$ also has a transcritical bifurcation.

5.2.4 Bernoulli's cubic equation: subcritical pitchfork

The equation

$$\dot{y} = ay - y^3 \tag{5.6}$$

is a **cubic Bernoulli** equation. This equation is a particular case of equation (5.3) with $a_1 = a$, $a_3 = -1$, and $a_0 = a_2 = 0$. Generically, we have $f_{yyy}(\cdot) \neq 0$ and $f_{ya}(\cdot) \neq 0$.



(a) Phase diagrams for $a < 0$

(b) Phase diagrams for $a = 0$

(c) Phase diagrams for $a > 0$

(d) Bifurcation diagram

Figure 5.2: Phase diagrams and bifurcation diagram for equation (5.5)

Being a Bernoulli equation, it also has an explicit solution:

$$y(t) = \begin{cases} \left(y(0)^{-2} \ + 2\,t\right)^{-\frac{1}{2}} & \text{if } \ a = 0 \\ \pm\sqrt{a}\ \left[1 - \left(1 - \frac{a}{y(0)^{-2}}\right) e^{-2at}\ \right]^{-1/2} & \text{if } \ a \neq 0 \end{cases}$$

where $y(0)$ is an arbitrary element of Y. The solution trajectories have the following properties for different values of the parameter a (see Figure 5.3):

- if $a \leq 0$, panels (a) and (b) show that, for any $y(0) \in \mathrm{Y}$, we have $\lim_{t\to\infty} y(t) = 0$;
- if $a > 0$, panel (c) shows that

$$\lim_{t\to\infty} y(t) = \begin{cases} -\sqrt{a}\ , & \text{if } y(0) < 0 \\ \sqrt{a}\ , & \text{if } 0 < y(0) < \sqrt{a} \\ +\infty, & \text{if } y(0) > \sqrt{a} \end{cases}$$

The dynamic properties depend on the value of a:

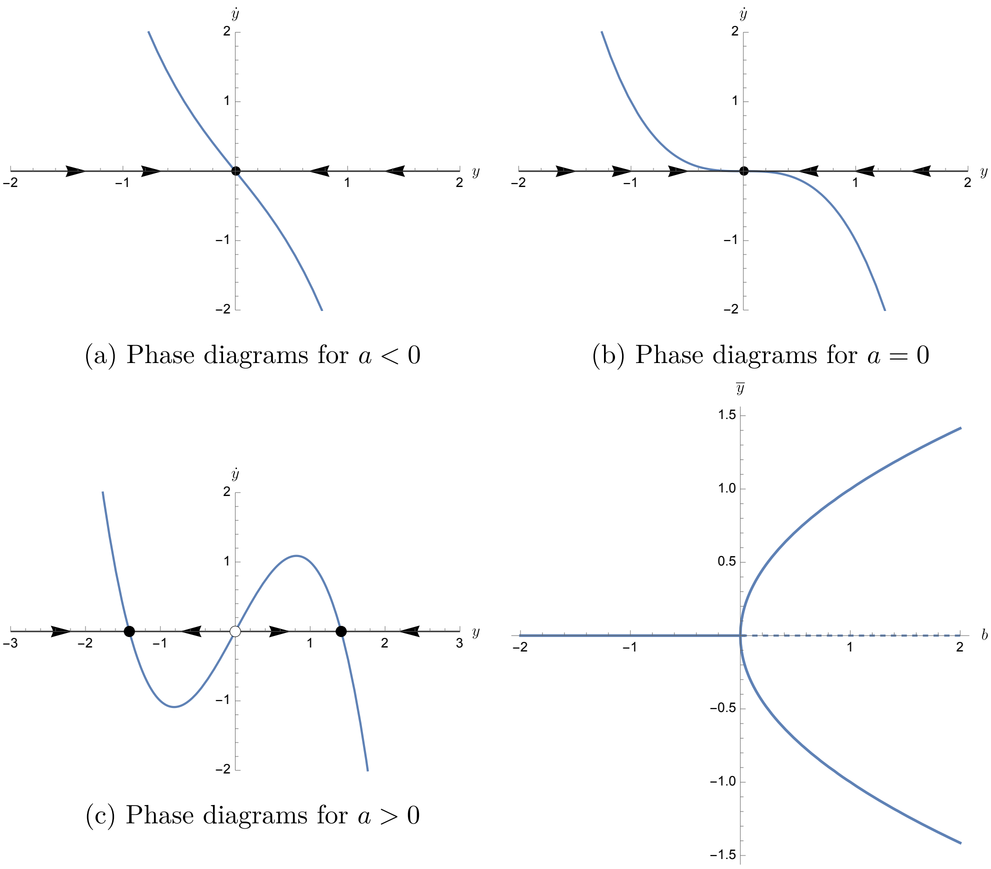
- existence and number of steady states: there is **one** steady state $\bar{y} = 0$ and if $a \leq 0$ and there are **three** steady states $\bar{y} = \{0, -\sqrt{a}, \sqrt{a}\}$ if $a > 0$;
- local dynamics at the steady states: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable; if $a < 0$ steady state $\bar{y} = 0$ is asymptotically stable; and if $a > 0$ steady state $\bar{y} = 0$ is unstable and the other two steady states $\bar{y} = -\sqrt{a}$ and $\bar{y} = \sqrt{a}$ are asymptotically stable.

To find the bifurcation point, we solve jointly $f(y, a) = 0$ and $f_y(y, a) = 0$ for (y, a), yielding

$$\begin{cases} a\,y - y^3 = 0 \\ a - 3\,y^2 = 0 \end{cases} \iff \begin{cases} 3\,a\,y - 3\,y^3 = 0 \\ a\,y - 3\,y^3 = 0 \end{cases} \iff \begin{cases} 2\,a\,y = 0 \\ a - 3\,y^2 = 0 \end{cases}$$

then the there is a **subcritical pitchfork** at $(y, a) = (0, 0)$.

Figure **??** shows two phase diagrams and the bifurcation diagram.



(a) Phase diagrams for $a < 0$

(b) Phase diagrams for $a = 0$

(c) Phase diagrams for $a > 0$

(d) Bifurcation diagram

Figure 5.3: Phase diagrams and bifurcation diagram for equation (5.6).

Exercise: Study the solution for equation $\dot{y} = ay + y^3$. Show that point $(y, a) = (0, 0)$ is also a bifurcation point called **supercritical pitchfork**.

5.2.5 Abel's equation: hysteresis

The following ODE

$$\dot{y} = a + y - y^3 \tag{5.7}$$

is called an Abel equation of the first kind, in which the following properties hold: $f_{yyy}(\cdot) \neq 0$, $f_y(\cdot) \neq 0$ and $f_a\ (\cdot) \neq 0$.

Although closed form solutions have been found recently [8] they are too cumbersome to report. If $a = 0$ the Abel's equation reduces to a particular Bernoulli's equation (5.6) $\dot{y} = y - y^3$.

Equation (5.7) can have one, two or three equilibrium points, which are the real roots of the polynomial equation $f(y, a) \equiv a + y - y^3 = 0$.

We can determine bifurcation points in the space $\mathrm{Y} \times \Phi$ by solving for (y, a)

$$\begin{cases} f(y, a) = 0, \\ f_y(y, a) = 0. \end{cases}$$

Because

$$\begin{cases} a + y - y^3 = 0, \\ 1 - 3y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3(a + y) - 3y^3 = 0, \\ y - 3y^3 = 0, \end{cases} \Leftrightarrow \begin{cases} 3a + 2y = 0 \\ y = \pm\sqrt{1/3}, \end{cases}$$

we readily find that the ODE (5.7) has two critical points, called **hysteresis** points:

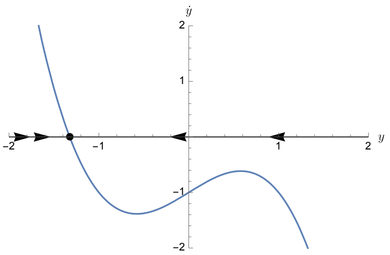
$$(y, a) = \left\{ \left(-\sqrt{\frac{1}{3}}, \frac{2}{3}\sqrt{\frac{1}{3}}\right), \left(\sqrt{\frac{1}{3}}, -\frac{2}{3}\sqrt{\frac{1}{3}}\right) \right\}.$$

Figure 5.4 shows that:

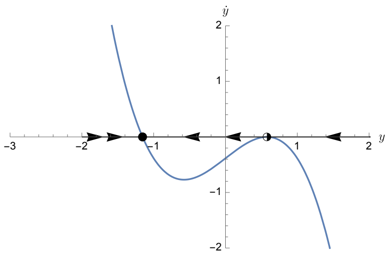
- for $a > \frac{2}{3}\sqrt{\frac{1}{3}}$ or for $a < -\frac{2}{3}\sqrt{\frac{1}{3}}$ there is one asymptotically stable steady state, see panels (a) and (e), respectively;
- for $a = \frac{2}{3}\sqrt{\frac{1}{3}}$ there are two steady states: one asymptotically stable equilibrium and a bifurcation point for $\bar{y} = \sqrt{\frac{1}{3}}$, see panel (b);
- for $a = -\frac{2}{3}\sqrt{\frac{1}{3}}$ there are two steady states: one asymptotically stable equilibrium and a bifurcation point for $\bar{y} = -\sqrt{\frac{1}{3}}$, see panel (d);
- for $-\frac{2}{3}\sqrt{\frac{1}{3}} < a < \frac{2}{3}\sqrt{\frac{1}{3}}$ there are three steady states, two asymptotically stable (the extreme ones) and one unstable, see panel (c). (the middle one)

Panel (f) in Figure 5.4, which illustrates the bifurcation diagram, displays the hysteresis curve. If we start will small values of a there will be a unique asymptotically steady state. If a a bifurcation is reached such that the number of steady states will increase to three, with an emergence of a

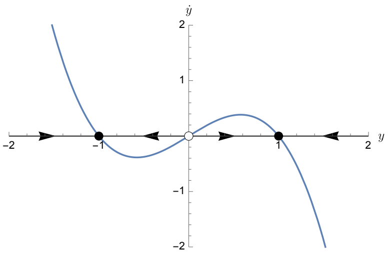
[8] For known closed form solutions of ODEs see, Zaitsev and Polyanin (2003) or Zwillinger (1998).



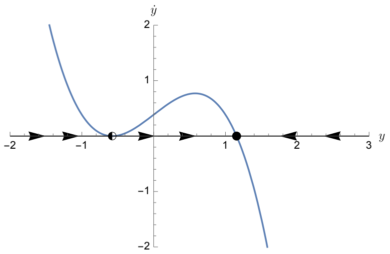
(a) Phase diagrams for $a < -\frac{2}{3}\sqrt{\frac{1}{3}}$



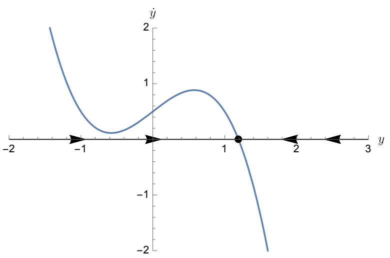
(b) Phase diagrams for $a = -\frac{2}{3}\sqrt{\frac{1}{3}}$



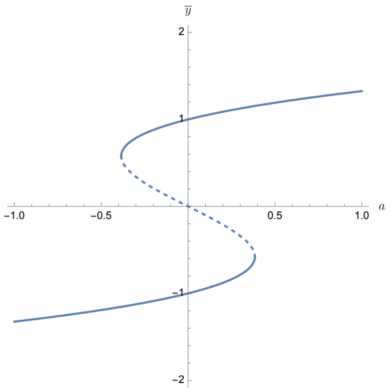
(c) Phase diagrams for $-\frac{2}{3}\sqrt{\frac{1}{3}} < a < \frac{2}{3}\sqrt{\frac{1}{3}}$



(d) Phase diagrams for $a = \frac{2}{3}\sqrt{\frac{1}{3}}$



(e) Phase diagrams for $a > \frac{2}{3}\sqrt{\frac{1}{3}}$



(f) Bifurcation diagram

Figure 5.4: Phase diagrams and bifurcation diagram for equation (5.7).

new intermediate steady state and a another asymptotically steady state, corresponding to greater values for y. This scenario will be maintained for greater values of a until a new bifurcation is reached. Further increases of a will imply that we will have again one unique asymptotically stable steady state greater than in the start of the exercise.

For intermediate values of a, specifically for $-\frac{2}{3}\sqrt{\frac{1}{3}} < a < \frac{2}{3}\sqrt{\frac{1}{3}}$, we have three steady states such that $\bar{y}_1 < \bar{y}_2 = 0 < \bar{y}_3$, where $\bar{y}_1$ and $\bar{y}_3$ are asymptotically stable and $\bar{y}_2$ is unstable. This type of phase diagram will not occur in linear models. In particular, the state space Y is partitioned as $\mathrm{Y} = \mathcal{W}^s_{\bar{y}_1} \cup \{\bar{y}_2\} \cup \mathcal{W}\ s_{\bar{y}_3}$ where $\mathcal{W}^s_{\bar{y}_1}$ and $\mathcal{W}^s_{\bar{y}_3}$ are the stable manifolds associated to steady states $\bar{y}_1$ and $\bar{y}_3$, respectively, which are

$$\mathcal{W}^s_{\bar{y}_1} = \Big\{ y \in \mathrm{Y} : y < \bar{y}_2 \Big\} \text{ and } \mathcal{W}^s_{\bar{y}_3} = \Big\{ y \in \mathrm{Y} : y > \bar{y}_2 \Big\}.$$

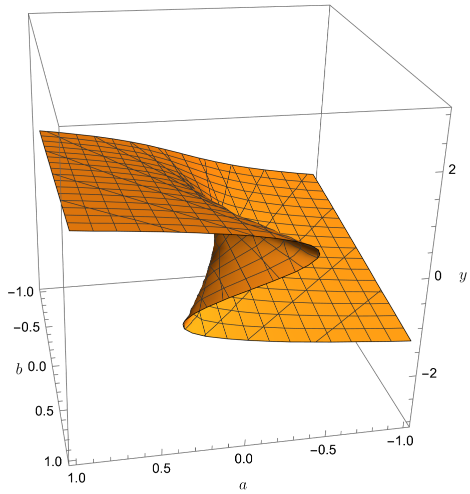


Figure 5.5: Bifurcation diagram for equation $\dot{y} = a + by - y^3$

5.2.6 Cubic equation: cusp

The next ODE

$$\dot{y} = f(y, a, b) \equiv a + by - y^3 \tag{5.8}$$

is also an Abel equation of the first kind. It has two parameters, a and b, and has the following properties: $f_{yyy}(\cdot) \neq 0$ and $f_a\ (\cdot) \neq 0$ and $f_y(\cdot)$ can have any sign depending on the parameter b. This last property allow for critical changes of its solution.

This ODE can have one, two or three equilibrium points, depending on the values of the parameters a and b. We can determine them by solving the cubic polynomial equation $a+b\,y-y^3 = 0$ (see appendix section 5.A.3). The number of real roots of this polynomial depends on the value of

$$\Delta \equiv \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3; \tag{5.9}$$

which is called the discriminant. It is known that: if $\Delta < 0$ there are three real roots, if $\Delta = 0$ there are two real roots (one is multiple), and if $\Delta > 0$ there is one real root and a pair of complex conjugate roots.

Therefore, regarding our ODE: if $\Delta < 0$ there are three steady states, if $\Delta = 0$ there are two steady states, and if $\Delta > 0$ there is one steady state.

We already know we can determine critical points by solving the system:

$$\begin{cases} f(y, a, b) = 0 \\ f_y(y, a, b) = 0. \end{cases} \tag{5.10}$$

Applying to equation (5.8) we have

$$\begin{cases} a + b\,y - y^3 = 0, \\ b - 3\,y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3\,a + 2\,b\,y = 0, \\ b - 3\,y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3\,a + 2\,b\,y = 0 \\ 2\,b^2 + 9\,a\,y = 0, \end{cases} \Leftrightarrow \begin{cases} 27\,a^2 + 18\,a\,b\,y = 0 \\ 4\,b^3 + 18\,a\,b\,y = 0. \end{cases}$$

The solutions to the system must verify

$$18\,a\,b\,y = -12\,a^2 = -4\,b^3 \Leftrightarrow \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3 = 0$$

that is $\Delta = 0$. Function $f(y, a, b) = 0$ traces out a surface in the three-dimensional space for (a, b, y) called **cusp** which is depicted in Figure 5.5 [9]. Because we have two parameters, the bifurcation loci, obtained from system (5.10) defines a line in the three-dimensional space (a, b, y). We can see how it changes by imagining horizontal slices in Figure 5.5 and project them in the (a, b)-plane. This would convince us that if $a = 0$ we would get the bifurcation diagram for the pitchfork, for equation (5.6), and if $a \neq 0$ and $b = 1$ we obtain the hysteresis diagram, for equation (5.7). This result would be natural because those two equations are a particular case of the cusp equation.

Those points refer to just one level of degeneracy, and are given by a particular relationship between two parameters, and not by specific values for them. This is the reason that we call them **co-dimension one bifurcation points**. As we have two parameters, we can find a higher level of degeneracy by finding a particular values of (y, a, b) that solves the system

$$\begin{cases} f(y, a, b) = 0 \\ f_y(y, a, b) = 0 \\ f_b(y, a, b) = 0, \end{cases}$$

and call them **co-dimension two bifurcation points**. We find that this point is unique $(y, a, b) = (0, 0, 0)$ and is called a **cusp point**. Looking at Figure 5.5 we observe that this point separates two types of bifurcation diagrams we have already seen: first, if we project the cusp surface into a vertical plane passing through this point we obtain a bifurcation diagram for the pitchfork, and, second, if we project the cusp surface into a horizontal plane passing through this point we obtain a bifurcation diagram for the hysteresis.

5.3 Qualitative theory

The qualitative (or geometrical) theory of differential equations characterizes the local dynamics of a non-linear differential equation by characterizing the dynamics for a linearized equation closed to a steady state.

Assuming that $f \in C^3(\mathbb{R})$, and given a value for a parameter $\varphi = \varphi_0$, and if there is a steady state $\bar{y} \in \{\, y : f(y, \varphi_0) = 0\}$, a Taylor expansion of the ODE in in a neiighborhood of $\bar{y}$ yields[10]

$$\dot{y} \;=\; f_y(\bar{y}, \varphi_0)\,(y - \bar{y}_0) + \frac{1}{2!}\, f_{yy}\;(\bar{y}, \varphi_0)\,(y - \bar{y}_0)^2 + \frac{1}{3!}\, f_{yyy}\;(\bar{y}, \varphi_0)\,(y - \bar{y}_0)^3 + o((y - \bar{y}_0)^4)$$

[9] This was one of the famous cases of catastrophe theory very popular in the 1980's see `https://en.wikipedia.org/wiki/Catastrophe_theory`.

[10] See Appendix 5.B for the little-o notation.

If for any point $(\bar{y}, \varphi_0)$ we have $f_y\ (\bar{y}_0, \varphi_0) \neq 0$ and $f_{yy}\ (\bar{y}_0, \varphi_0) = f_{yyy}\ (\bar{y}_0, \varphi_0) = 0$, then a non-linear ODE can be approximated locally by a linear ODE

$$\dot{y}\ = \lambda\,(y - \bar{y}),\ \lambda \equiv f_y(\bar{y}_0, \varphi_0)$$

If the steady state is unique, or there are no values for the parameter φ such that $f_y(\bar{y}, \varphi_0) = 0$ then the ODE is topologically equivalent to a linear ODE. This means that if we introduce an arbitrary change in parameter $\varphi \in \Phi$ the phase diagram will not change its properties (i.e., the dynamic properties are **global**).

In this case when we can perform comparative dynamics exercises (globally). If there is a value for the parameter such that the previous condition holds, then we have a bifurcation point and global dynamics differ from local dynamics. In this we can conduct a bifurcation analysis, which consists in finding equivalence to the normal forms which were presented in the previous section.

Example Consider the scalar ODE

$$\dot{y} = f(y) \equiv y^\alpha - a \tag{5.11}$$

where a and α are two constants, with $a > 0$, and $y \in \mathbb{R}_+$. Then there is an unique steady state $\bar{y} = a^{\frac{1}{\alpha}}$. As

$$f_y(y) = \alpha y^{\alpha-1}$$

then

$$f_y(\bar{y}) = \alpha a^{\frac{\alpha-1}{\alpha}}.$$

Set $\lambda \equiv f_y(\bar{y})$. Therefore the steady state is hyperbolic if $\alpha \neq 0$ and it is non-hyperbolic if $\alpha = 0$. In addition, if $\alpha < 0$ the hyperbolic steady state $\bar{y}$ is asymptotically stable and if $\alpha > 0$ it is unstable.

If $\alpha \neq 0$ we can perform a first-order Taylor expansion of the ODE (5.11) in the neighborhood of the steady state

$$\dot{y} = \lambda(y - \bar{y}) + o((y - \bar{y}))$$

which means that the solution to (5.11) can be locally approximated by

$$y(t) = \bar{y} + (k - \bar{y})e^{\lambda t}$$

for any $k \in \mathbb{R}_+$. In particular, if we fix $y(0) = y_0$ then $k = y_0$.

5.3.1 Comparative dynamics exercises

Assume there is only one parameter, say $\varphi \in \Phi \subseteq \mathbb{R}$. A steady state is a mapping $\bar{y} : \Phi \to \mathrm{Y}$. We call long-run multiplier to the derivative $\bar{y}'(\varphi) \equiv \dfrac{d\bar{y}(\varphi)}{d\phi}$. For a variation of the parameter $d\varphi = \varphi_1 - \varphi_0$ we obtain the change in the steady state $d\bar{y}\ = \bar{y}'(\varphi)\,d\varphi$, where $d\bar{y} = \bar{y}_1 - \bar{y}_0$.

A comparative dynamics exercise consists in studying the path of the system when, starting from a steady state, there is a change in parameter φ. This entails approximating the ODE by the linear ODE

$$\dot{y}\ = \lambda\,dy(t) + b\,d\varphi,\ \text{for}\ \ \lambda \equiv\ f_y(\bar{y}, \varphi_0),\ b \equiv\ f_\varphi\ (\bar{y}, \varphi_0),$$

where $dy(t) = y(t) - \bar{y}_0$. The solution to this linear (variational) ODE is

$$dy(t) = d\bar{y}\ \left(1 - e^{\lambda\, t}\right), \text{for}\ \ t \in [0, \infty)$$

where $\ d\bar{y} = -\dfrac{b}{\lambda}\, d\varphi$. Observe that, because we assume we are at time $t = 0$ at a steady state we have $dy(0) = y(0) - \bar{y}_0 = \bar{y}_0 - \bar{y}_0 = 0$. If the steady state is asymptotically stable, i.e., if $\lambda < 0$ then

$$\lim_{t\to\infty} \frac{dy(t)}{d\varphi} = \frac{d\bar{y}}{d\varphi} = \bar{y}'(\varphi) = -\frac{b}{\lambda}$$

is the long-run multiplier. The solution to this ODE yields the **short-run multipliers**.

We obtain, therefore, a small perturbation of the phase diagram, which no changes in the qualitative dynamics, only a quantitative change in the steady state after we have perturb it by a change in a parameter (or parameters).

5.3.2 Bifurcations for scalar ODE's

Bifurcations, as we saw, occur when a small change in a parameter change the phase diagram, or the qualitative properties of the dynamics.

For the regular scalar ODE we have found the following types of bifurcations.

Fold bifurcation (see (Kuznetsov, 2005, ch. 3.3)): Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yy}(0, 0) \neq 0,\ f_{\varphi}(0, 0) \neq 0.$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi \pm y^2,$$

that is to the Ricatti's model (5.4).

Transcritical bifurcation: Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yy}(0, 0) \neq 0,\ f_{\varphi y}(0, 0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^2$$

that is to the Bernoulli model (5.5).

Pitchfork bifurcation: Let $f \in C^2(\mathbb{R})$ and consider $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yyy}(0, 0) \neq 0,\ f_{\varphi y}(0, 0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^3$$

that is to the Bernoulli model (5.6).

5.4 References

- (Hale and Koçak, 1991, Part I , III): very good introduction.
- Brock and Malliaris (1989), (Grass et al., 2008, ch.2) has a compact presentation of all the important results with some examples in economics and management science.

5.A Appendix: solutions to some ODEs

5.A.1 Solution of Ricatti's equation (5.4)

Start with the case: $a = 0$. Separating variables, we have

$$\frac{dy}{y^2} = dt$$

integrating both sides

$$\int_{y(0)}^{y(t)} \frac{dy}{y^2} = \int_0^t ds \Leftrightarrow -\frac{1}{y(t)} = t - \frac{1}{y(0)}.$$

Then the solution is

$$y(t) = \frac{y(0)}{1 - y(0)\, t}.$$

Now let $a \neq 0$. By using the same method we have

$$\frac{dy}{a + y^2} = dt. \tag{5.12}$$

At this point it is convenient to note that

$$\frac{d \tan^{-1}(x)}{dx} = \frac{1}{1+x^2}\ , \ \frac{d \tanh^{-1}(x)}{dx} = \frac{1}{1-x^2},$$

where

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}, \quad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Then we should deal separately with the cases $a > 0$ and $a < 0$. If $a > 0$ integrating equation (5.12)

$$\int \frac{dy}{a+y^2} = dt \Leftrightarrow \frac{1}{\sqrt{a}} \int \frac{1}{1+x^2} dx = t + k \Leftrightarrow \frac{1}{\sqrt{a}} \tan^{-1}(x) = t + k$$

where we defined $x = y/\sqrt{a}$. Solving the last equation for x and mapping back to y we get

$$y(t) = \sqrt{a} \left(\tan\left(\sqrt{a}(t+k) \right) \right).$$

If $a < 0$ we integrate equation (5.12) by using a similar transformation, but instead with $x = y/\sqrt{-a}$ to get

$$\int \frac{dy}{a+y^2} = dt \Leftrightarrow -\frac{1}{\sqrt{-a}} \int \frac{1}{1-x^2} dx = t + k \Leftrightarrow -\frac{1}{\sqrt{-a}} \tanh^{-1}(x) = t + k.$$

Then

$$y(t) = -\sqrt{-a} \left(\tanh\left(\sqrt{-a}(t+k) \right) \right).$$

5.A.2 Solution for a general Bernoulli equation

Consider the Bernoulli equation

$$\dot{y} = ay + by^{\eta},\ a \neq 0,\ b \neq 0 \tag{5.13}$$

where $y : T \to \mathbb{R}$. We intoduce a first transformation $z(t) = y(t)^{1-\eta}$, which leads to a linear ODE

$$\dot{z} = (1-\eta)(az + b) \tag{5.14}$$

beause

$$\begin{aligned} \dot{z} &= (1-\eta)y^{-\eta}\dot{y} = \\ &= (1-\eta)(ay^{1-\eta} + b) = \\ &= (1-\eta)(az + b). \end{aligned}$$

To solve equation (5.14) we introduce a second transformation $w(t) = z(t) + \frac{b}{a}$. Observing that $\dot{w} = \dot{z}$ we obtain a homogeneous ODE $\dot{w} = a(1-\eta)w$ which has solution

$$w(t) = w(0)\, e^{a(1-\eta)t}.$$

Then the solution to equation (5.14) is

$$z(t) = -\frac{b}{a} + \Big(w(0) + \frac{b}{a}\Big)e^{a(1-\eta)t}$$

because $w(0) = y(0) + \frac{b}{a}$.

We finally get the solution for the Bernoulli equation (5.13)

$$y(t) = \Big[-\frac{b}{a} + \Big(y(0)^{1-\eta} + \frac{b}{a}\Big)\, e^{a(1-\eta)t}\Big]^{\frac{1}{1-\eta}} \tag{5.15}$$

because $z(0) = y(0)^{1-\eta}$.

If $a = 0$ the solution is

$$y(t) = \Big(y(0)^{1-\eta}\ + b\,(1-\eta)\, t\Big)^{\frac{1}{1-\eta}}.$$

Æxercise: Prove this.

5.A.3 Solution to the cubic polynomial equation

In this section we present the solutions to the cubic polynomial equation [11]

$$x^3 + \alpha_2\, x^2 + \alpha_1\, x + \alpha_0 = 0.$$

By performing a Tschirnhausen transformation (see King (1996)) we can transform into a simpler, irreducible, equation. Defining $y = x - \dfrac{\alpha_2}{3}$ we obtain the polynomial

$$y^3 + \Big(\alpha_1 - \frac{\alpha_2^2}{3}\Big)\, y + \alpha_0 - \frac{\alpha_1\,\alpha_2}{3} + \frac{2\,\alpha_2^3}{27}\ = 0$$

[11] See `cubicpolynomialhistory` for the history of this equation.

This equation is a (monic) cubic polynomial equation, of type

$$y^3 - by - a = 0. \tag{5.16}$$

If $a = 0$ we have $y\,(y^2 - b\,y) = 0$ and the solutions are $y = 0$ and the solutions of the quadratic equation are $y = \pm\sqrt{b}$.

If $a \neq 0$ we prove that the solutions of the monic cubic equation are

$$y_j = w^{j-1}\theta^{\frac{1}{3}} + \frac{b}{3}\left(\omega^{j-1}\theta^{\frac{1}{3}}\right)^{-1},\ j = 1, 2, 3. \tag{5.17}$$

where ω and θ are presented next.

Write $y = u + v$. Then we get the equivalent representation

$$u^3 + v^3 + 3\left(uv - \frac{b}{3}\right)(u + v) - a = 0.$$

As $y = u + v \neq 0$, u and v solve simultaneously

$$\begin{cases} u^3 + v^3 = a \\ uv = \dfrac{b}{3} \end{cases} \Leftrightarrow \begin{cases} u^3u^3 + u^3v^3 - u^3a = 0 \\ u^3v^3 = \left(\dfrac{b}{3}\right)^3 \end{cases} \Leftrightarrow \begin{cases} u^6 - au^3 + \left(\dfrac{b}{3}\right)^3 = 0 \\ uv = \frac{b}{3}. \end{cases}$$

The first equation is a quadratic polynomial in u^3 which has roots

$$u^3 = \frac{a}{2} \pm \sqrt{\Delta},\ \text{where } \Delta \equiv \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3$$

where Δ is the discriminant in equation (5.9). We can take any solution of the previous equation and set $\theta \equiv \frac{a}{2} + \sqrt{\Delta}$.

At this stage it is useful to observe that the solutions of equation $x^3 = 1$ are

$$x_1 = 1,\ x_2 = \omega,\ x_3 = \omega^2.$$

where $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2}(1 - \sqrt{3}i)$ and $\omega^2 = e^{\frac{4\pi i}{3}} = -\frac{1}{2}(1 + \sqrt{3}i)$. Therefore $u^3 = \theta$ has also three solutions

$$u_1 = \theta^{\frac{1}{3}},\ u_2 = \omega\theta^{\frac{1}{3}},\ u_3 = \omega^2\theta^{\frac{1}{3}}$$

and because $v = \frac{b}{3u}$, we finally obtain equation (5.17), which are the solutions to equation (5.16).

5.B Appendix: approximating functions

Assume we have a function $f(x)$ for $x \in \mathrm{X} \subseteq \mathbb{R}$. If function $f(.)$ is non-linear, or non explicitly specified, we need sometimes to compare it with another function whose behaviour can be simpler, or we need to know only about how it grows or decays, or its concavity or symmetry, without worrying about the details. The notations $O(.)$ and $o(.)$ (called big-O and littlele-o notations) provide a usefull approach.

5.B.1 Big O notation

A function is of **constant order** if there is a non-zero constant c such that we can write, equivalently

$$\lim_{x\to\infty} \frac{f(x)}{c} = 1 \,\text{or}\, \lim_{x\to\infty} f(x) = c$$

We say in this case that

$$f(x) \in O(1)$$

More generally:

Definition 1. *A function $f(x)$ **is big-O of function** $g(x)$, written as $f(x) \in O(g(x))$ (sometimes written $f(x) = O(g(x))$ if $f(x)$ is of the same order than $g(x)$ (they grow or decay at the same rate)*

$$f(x) \in O(g(x)) \; or \; \lim_{x\to\infty} \frac{f(x)}{g(x)} = c$$

or $\frac{f(x)}{g(x)} \in O(1)$.

Examples:

- $ax + b \in O(x)$ if $a \neq 0$
- $ax^2 + bx + c \in O(x^2)$ if $a \neq 0$
- $ax^{-2} + bx^{-1} \in O(x^{-1})$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \in O(x^2)$

5.B.2 Little o notation

There is an associate function, the *little-o* notation that means that a functions is asymptotically smaller than another function.

Function $f(x)$ is asymptotically smaller than a non-zero constant c if

$$\lim_{x\to\infty} \frac{f(x)}{c} = 0, \,\text{or} f(x) \in o(1)$$

In general, function $f(x)$ *is little-o of function* $g(x)$, written as of $f(x) \in o(g(x))$ if for $n \to \infty$ there is a number N such that

$$|f(x)| < \epsilon |x| \,\text{for}\, n > N$$

Definition 2. *If $g(x) \neq 0$ in all its domain them $f(x)$ is a small-o of function $g(x)$, denoted $f(x) \in o(g(x))$ if and only if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

Examples:

Notation	Limit definition (a)	Limit definition (b)
$f \in o(g)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$\limsup \frac{f(x)}{g(x)} = 0$
$f \in O(g)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$\limsup \frac{f(x)}{g(x)} < \infty$

- $2x = o(x^2)$
- $e^x = 1 + x + \frac{1}{2}x^2 + o(x^3)$,
- $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^4)$
- $\ln(x) \in o(x)$

For a frequent function used in economics $f(x) = x^\alpha$ where $\alpha > 0$; we have

- if $0 \le \alpha < 1$ then $f(x)$ if $o(x)$
- if $1 \le \alpha < 2$ then $f(x)$ if $o(x^2)$
- if $2 \le \alpha < 3$ then $f(x)$ if $o(x^3)$
- if $n - 1 \le \alpha < n$, for n an integer, then $f(x)$ if $o(x^n)$.

If $f(x) \in o(g(x))$ then $f(x) \in O(g(x))$ but the converse may not be true.
There are several **properties** for $O(.)$ and $o(.)$ functions, this is a small sample:

- If $f(x) \in O(g(x))$ then $c\,f(x) \in O(g(x))$ for any constant c,
- If $f_1(x)$ and $f_2(x)$ are both $O(g(x))$ then $f_1(x) + f_2(x) \in O(g(x))$,
- If $f_1(x) \in O(g(x))$ and $f_2(x) \in o(g(x))$ then $f_1(x) + f_2(x) \in O(g(x))$,
- If $f_1(x) \in O(f_2(x))$ and $f_2(x) \in o(g(x))$ then $f_1(x) \in o(g(x))$,
- $x^n o(x^m) = o\left(x^{n+m}\right)$
- $o\left(x^n\right) o\left(x^m\right) = o\left(x^{n+m}\right)$
- $x^m = o\left(x^n\right)$ if $n < m$

5.B.3 Calculus with big-O and little-o

The function f has a strong derivative at x if

$$f(x + \epsilon) = f(x) + f^{'}(x)\epsilon + O(\epsilon)$$

for ϵ sufficiently small.

If f is n-times differentiable at x the Taylor's theorem states

$$f(x+\epsilon) = f(x) + f'(x)\,\epsilon + \frac{f''(x)}{2}\ \epsilon^2 + ... + \frac{f^{(n)}}{n!}\epsilon^n + O(\epsilon^n).$$

Weak derivative

$$f(x+\epsilon) = f(x) + f^{'}(x)\epsilon + o(\epsilon^2)$$

The Lagrange formula can be written as

$$f(x+\epsilon) = f(x) + f^{'}(x)\epsilon + \frac{f^{''}(x)}{2!}\ \epsilon^2 + ... + \frac{f^{(n)}(x)}{n!}\ \epsilon^n + o(\epsilon^{n+1}).$$

Chapter 6

Planar non-linear ODE's: the regular case

In this chapter we assume again that time, $t \in \mathrm{T} \subseteq \mathbb{R}_+$, is the independent variable, the state space, $\mathrm{Y} \subset \mathbb{R}^2$, has dimension two, and there is a vector of parameters, $\varphi \in \Phi \subseteq \mathbb{R}^n$, for $n \geq 1$. We try to find or characterize functions $\mathbf{y} : \mathrm{T} \times \Phi \ \to \mathrm{Y} = \mathbb{R}^2$, that solve the (autonomous) equation planar differential equation

$$\mathbf{F}(\dot{\mathbf{y}}, \mathbf{y}, \varphi) = \mathbf{0} \tag{6.1}$$

We say the ODE is **regular** if $\mathbf{F}(\dot{\mathbf{y}}, \mathbf{y}, \cdot)$ is continuous and continuously differentiable function, and furthermore det $\dfrac{D\,\mathbf{F}(\dot{\mathbf{y}}, \mathbf{y}, \varphi)}{d\dot{y}} \neq 0$ for every $\mathbf{y} \in \mathrm{Y}$.

This allows us to write equation (5.1) as

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi). \tag{6.2}$$

Expanding, we have,

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \varphi) \\ \dot{y}_2 &= f_2(y_1, y_2, \varphi). \end{aligned} \tag{6.3}$$

In the previous chapter we studied the following normal form of scalar ODEs: first, ODE's depending on a single parameter (a), as the Ricatti's equation $\dot{y} = a + y^2$, the quadratic Bernoulli equation, $\dot{y} = ay + y^2$, the cubic equations Bernoulli equation, $\dot{y} = ay - y^3$, the Abel equation $\dot{y} = a + y - y^3$; second, ODE's depending on two parameters (a and b) the Abel's equation, $\dot{y} = a + by - y^3$.

In principle, we could consider combining all the previous scalar normal forms to have an idea of the number of possible cases, and extend the previous method to study the dynamics. That method consisted in finding critical points, corresponding to steady states and values of the parameters such that the derivatives of the steady variables would be equal to zero. However, for planar equation, to fully characterise the dynamics, we may have to study local dynamics in invariant orbits other than steady states. In general there are, at least, three types of **invariant orbits** that do not exist in planar linear models: homoclinic, and heteroclinic orbits and limit cycles.

We addressed the solution of most scalar ODE by using both analytical and qualitative or geometrical approaches. For non-linear planar there are not, in general, known analytical sioluttion. Therefore, we use a qualitative approach to finding and characterizing the most well known types of dynamics.

Next, in section(6.1) we present some most well known normal forms for planar ODEs, and in section 6.2 we present the qualitative theory for non-linear regular planar ODE's.

6.1 Normal forms

A **normal form** is the simplest ODE, whose exact solution is usually known and represents a whole family of ODEs by topological equivalence.[1]

We present next some important normal forms for scalar ODEs, which have the generic representation by a polynomial vector field

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi) \equiv \mathbf{A}_0 + \mathbf{A}_1 \mathbf{y} + \mathbf{A}_2 \mathbf{y}^{\otimes 2} + \mathbf{A}_3 \mathbf{y}^{\otimes 3}, \tag{6.4}$$

where $\mathbf{y}^{\otimes 2} = \left(y_1^2, y_1 y_2, y_2^2\right)^\top$, $\mathbf{y}^{\otimes 3} = \left(y_1^3, y_1^2 y_2, y_1 y_2^2, y_2^3\right)^\top$, $\mathbf{A}_0 \in \mathbb{R}^2$, $\mathbf{A}_1 \in \mathbb{R}^{2\times 2}$, $\mathbf{A}_2 \in \mathbb{R}^{2\times 3}$, and $\mathbf{A}_3 \in \mathbb{R}^{2\times 4}$, and $\boldsymbol{\varphi} \equiv (\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) \in \mathbb{R}^4$.

Given the very large number of cases we consider just a few associated to new types of dynamic phenomena.

Equation (6.4) covers the case of linear scalar ODE's, as a particular case in which $\mathbf{A}_2 = \mathbf{0}_{(2\times 3)}$ and $\mathbf{A}_3 = \mathbf{0}_{(2\times 4)}$. Therefore, a linear planar ODE can be written as

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi) \equiv \mathbf{A}_0 + \mathbf{A}_1 \mathbf{y}.$$

Linear ODE have explicit solutions. We considered all cases in chapter three. Again we found that all solutions correspond to global solutions and the phase diagrams display global dynamics (global asymptotic stability, global instability, or a global center manifold).

As for the scalar ODEs this is not the case for non-linear ODE's. However, given the very large number of cases we deal only with a selected number of cases displaying some well studied bifurcations (see Kuznetsov (2005).)

6.1.1 Heteroclinic orbits

We say there is an **heteroclinic orbit** if, in a planar ODE in which there are at least two steady states, say $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$, there are solutions $\mathbf{y}(t)$ that entirely lie in a curve joining $\bar{\mathbf{y}}^1$ to $\bar{\mathbf{y}}^2$

[1] In heuristic terms, we say functions $f(y)$ and $g(x)$ are topologically equivalent if there exists a diffeomorphism, i.e., a smooth map h with a smooth inverse h^{-1}, such that if $y = h(x)$ then $h(g(x)) = f(h(x))$. This property may hold globally of locally. The last case is the intuition behind the Grobman-Hartmann theorem.

say Het($\mathbf{y}$). Therefore, if $\mathbf{y}(0) \in \text{Het}(\mathbf{y})$ then $\mathbf{y}(t) \in \text{Het}(\mathbf{y})$ for $t > 0$ and either $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^1$ and $\lim_{t\to-\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^2$ or $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^2$ and $\lim_{t\to-\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^1$. Heteroclinics can exist if the stability type of the steady states are different or they are equal, with the exception of the case in which they are stable nodes of foci. For instance, they can connect one stable or one unstable node and a saddle point, or they can connect a stable and one unstable node, or they can connect two saddle points. In the last case, we say there is a **saddle connection**.

Heteroclinic networks can also exist when there are more than two steady states which are connected.

However, heteroclinic orbits can be generic, if they exist for a non-degenerate subset of parameter values, or they can be non-generic, if they only exist for a degenerate subset of parameters. In the last case we have a heteroclinic bifurcation.

Generic heteroclinic orbits

Although there are several normal forms generating generic heteroclinic orbits, we focus next in the following case:[2]

$$\begin{aligned} \dot{y}_1 &= a y_1 y_2 \\ \dot{y}_2 &= 1 + y_1^2 - y_2^2 \end{aligned} \tag{6.5}$$

where $a \neq 0$. [3] This equation has two steady states: $\bar{\mathbf{y}}^1 = (0, -1)$ and $\bar{\mathbf{y}}^2 = (0, 1)$. Calling,

$$\mathbf{f}(\mathbf{y}) = \begin{pmatrix} a y_1 y_2 \\ 1 + y_1^2 - y_2^2 \end{pmatrix}$$

we have the Jacobian, evaluated at any point $\mathbf{y} \in Y$,

$$J(\mathbf{y}) \equiv D_{\mathbf{y}} \mathbf{f}(\mathbf{y}) = \begin{pmatrix} a y_2 & a y_1 \\ 2 y_1 & -2 y_2 \end{pmatrix},$$

which has trace and determinant depending on the parameter a

$$\begin{aligned} \text{trace}\,(J(\mathbf{y})) &= (a-2) y_2 \\ \det\,(J(\mathbf{y})) &= -2a(y_1^2 + y_2^2). \end{aligned}$$

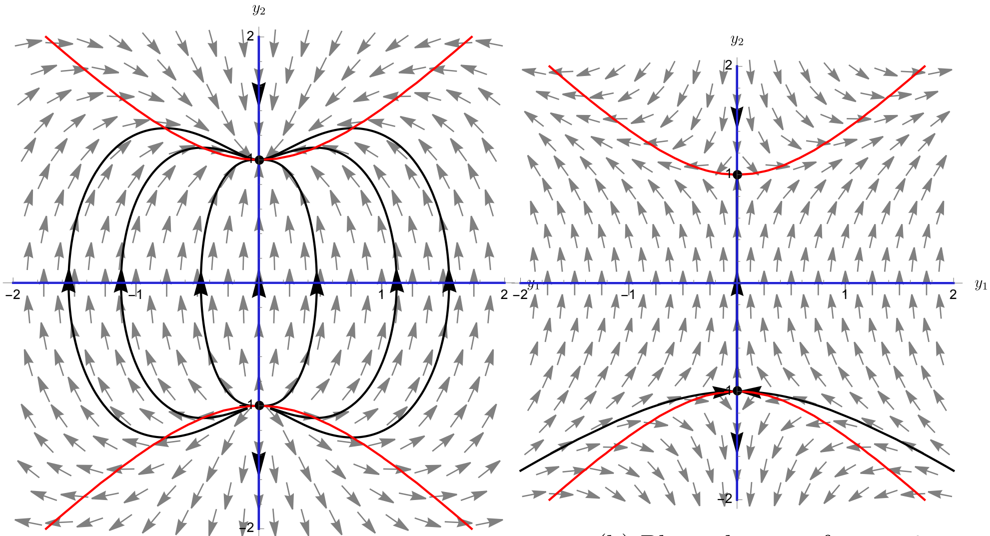
Then, remembering again that we assumed $a \neq 0$ and because, for any steady state $y_1^2 + y_2^2 > 0$ then $\det\big(J(\mathbf{y})\big) > 0$ if $a < 0$ and $\det\big(J(\mathbf{y})\big) < 0$ if $a > 0$.

Therefore, if $a < 0$, steady state $\bar{\mathbf{y}}^1$ is an unstable node, because $\text{trace}\big(J(\bar{\mathbf{y}}^1)\big) > 0$ and steady state $\bar{\mathbf{y}}^2$ is a stable node, because $\text{trace}(J(\bar{\mathbf{y}}^2)) < 0$. Then for any trajectory starting from any element of $\mathbf{y} \neq \bar{\mathbf{y}}^2$ there is convergence to steady state $\bar{\mathbf{y}}^2$ (see the left subfigure in figure 6.1).

[2] This is a particular case of equation (6.4) with

$$\mathbf{A}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{A}_1 = \mathbf{0}_{(2\times 2)}, \ \mathbf{A}_2 = \begin{pmatrix} 0 & a & 0 \\ 1 & 0 & -1 \end{pmatrix}, \text{ and } \mathbf{A}_3 = \mathbf{0}_{(2\times 4)}.$$

[3] If $a = 0$ this equation reduces to a scalar Ricatti equation.



(a) Phase diagram for $a < 0$

(b) Phase diagram for $a > 0$

Figure 6.1: Phase diagrams for equation (6.5)

If we denote $\text{Het}(\mathbf{y})$ the set points connecting $\bar{\mathbf{y}}^2$ to $\bar{\mathbf{y}}^1$ we readily see that $\text{Het}(\mathbf{y}) = \mathrm{Y}/\{\bar{\mathbf{y}}_1\}$, which means there is an infinite number of heteroclinic orbits, and that this set is coincident with the stable manifold $\mathcal{W}^s_{\bar{\mathbf{y}}^2}$ (see subfigure (a) in Figure 6.1).

However, if $a > 0$ both steady states, $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$, are saddle points, because $\det\left(J(\bar{\mathbf{y}}^1)\right) = \det\left(J(\bar{\mathbf{y}}^2)\right) < 0$. In this case, there is one heteroclinic surface

$$\text{Het}(\mathbf{y}) = \{\ (y_1, y_2) : y_1 = 0,\ -1 \leq y_2 \leq 1\}$$

which is the locus of points connecting $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$ such that for any initial value $\mathbf{y}(0) \in \text{Het}(\mathbf{y})$ the solution will converge to $\bar{\mathbf{y}}^2$ (see subfigure (b) in Figure 6.1). In this case $\text{Het}(\mathbf{y})$ is the set of points belonging to the intersection of the unstable manifold of $\bar{\mathbf{y}}^1$ and to the stable manifold of $\bar{\mathbf{y}}^2$: $\text{Het}(\mathbf{y}) = \mathcal{W}^u_{\bar{\mathbf{y}}^1} \cap \mathcal{W}^s_{\bar{\mathbf{y}}^2}$.

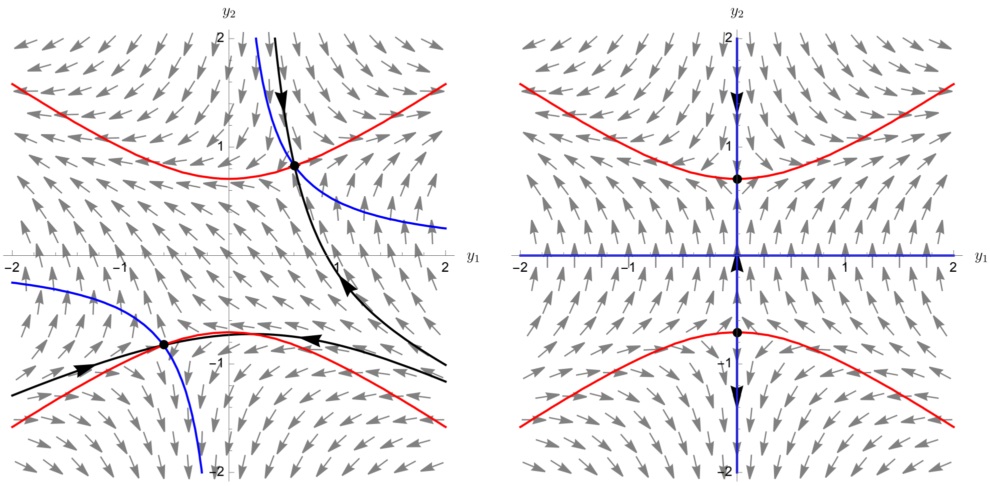
At last, we should notice that in both cases the heteroclinic orbits are **generic**, in the sense that they persist for a wide range of values for parameter a. This is not the case for the next example.

Non-generic case: heteroclinic saddle connection bifurcation

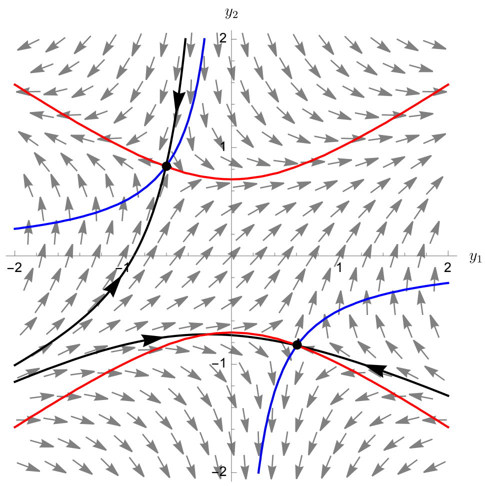
Assuming a related but slightly different normal form generates an heteroclinic bifurcation meaning we may have a bifurcation parameter that when it crosses a specific value heteroclinic orbits cease to exist. The following model is studied, for instance, in (Hale and Koçak, 1991, p.210).

$$\begin{aligned} \dot{y}_1 &= \lambda + 2y_1y_2 \\ \dot{y}_2 &= 1 + y_1^2 - y_2^2 \end{aligned} \tag{6.6}$$

As Figure 6.2. shows, there are always two steady states and they are both saddle points. If $\lambda \neq 0$ there are no trajectories connecting the two steady states. However, if $\lambda = 0$ there is an heteroclinic trajectory connecting the two saddle points (indeed this is the same case as subfigure (b) in Figure 6.1. In this case we say there is a **heteroclinic saddle connection bifurcation**, for $\lambda = 0$.



(a) Phase diagrams for $\lambda < 0$ (b) Phase diagrams for $\lambda = 0$



(c) Phase diagrams for $\lambda > 0$

Figure 6.2: Phase diagrams for equation (6.6).

6.1.2 Homoclinic orbits

We say there is an **homoclinic orbit** if, in a planar ODE, there is a subset of points $\text{Hom}(\mathbf{y})$ connecting a steady state with itself. This is only possible if the steady state $\bar{\mathbf{y}}$ is a saddle point in which the stable manifold contains a closed curve, that we call homoclinic curve. Because of this fact, homoclinic orbits exist jointly with periodic trajectories.

Again, homoclinic orbits can be generic or non-generic. Next we illustrate both cases.

Generic homoclinic orbits

Consider the non-linear planar ODE depending on one parameter, a, of form[4]

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 - a y_1^2. \end{aligned} \tag{6.7}$$

If $a \neq 0$ there are two steady states $\bar{\mathbf{y}}^1 = (0,0)$ and $\bar{\mathbf{y}}^2 = (1/a, 0)$. The Jacobian, evaluated at any point $\mathbf{y} \in \mathrm{Y}$,

$$J(\mathbf{y}) \equiv D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ 1 - 2ay_1 & 0 \end{pmatrix}$$

has following the trace and the determinant

$$\begin{aligned} \mathrm{trace}\left(J(\mathbf{y})\right) &= 0, \\ \det\left(J(\mathbf{y})\right) &= 2ay_1 - 1. \end{aligned}$$

If $a = 0$ this ODE reduces to a linear planar ODE, there is only one equilibrium point $\bar{\mathbf{y}} = \mathbf{0}$, which is a saddle point.

If $a \neq 0$, it is easy to see that steady state $\bar{\mathbf{y}}^1$ is always a saddle point, because $\det\left(J(\bar{\mathbf{y}}^1)\right) = -1 < 0$, and the steady state $\bar{\mathbf{y}}^2$ is always locally a center, because $\det\left(J(\bar{\mathbf{y}}^2)\right) = 1 > 0$ and $\mathrm{trace}\left(J(\bar{\mathbf{y}}^2)\right) = 0$, for any value of a.

Furthermore, we can prove that there is an invariant curve, such that solutions follow a potential or first integral curve which is constant.

In order to see this we introduce a **Lyapunov function** which is a differentiable function $H(\mathbf{y})$ such that the time derivative is $\dot{H} = D_{\mathbf{y}}H \cdot \dot{\mathbf{y}}$, that is $\dot{H} = H_{y_1}\dot{y}_1 + H_{y_2}\dot{y}_2$. A **first integral** is a set of points $\mathbf{y} \in \mathrm{Y}$, different from $\bar{\mathbf{y}}$, such that $\dot{H} = 0$. In this case the orbits passing through those points allow for a conservation of energy in some sense and $H(\mathbf{y}(t)) = \text{constant}$. For values such that $H(\mathbf{y}(t)) = 0$ that curve passes through a steady state.

For this case consider the function

$$H(y_1, y_2) = -\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{a}{3}y_1^3.$$

If we time-differentiate this Lyapunov function and substitute equations (6.7) we have

$$\begin{aligned} \dot{H} &= (ay_1 - 1)y_1\dot{y}_1 + y_2\dot{y}_2 = \\ &= (ay_1 - 1)y_1y_2 + y_2y_1(1 - ay_1) = \\ &= 0. \end{aligned}$$

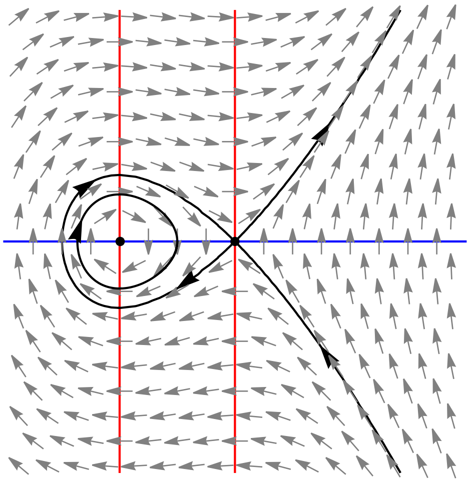
[4]This is a particular case of equation (6.4) with

$$\mathbf{A}_0 = \mathbf{0}_{(2\times 1)},\ \mathbf{A}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & 0 \end{pmatrix},\ \text{and}\ \mathbf{A}_3 = \mathbf{0}_{(2\times 4)}.$$

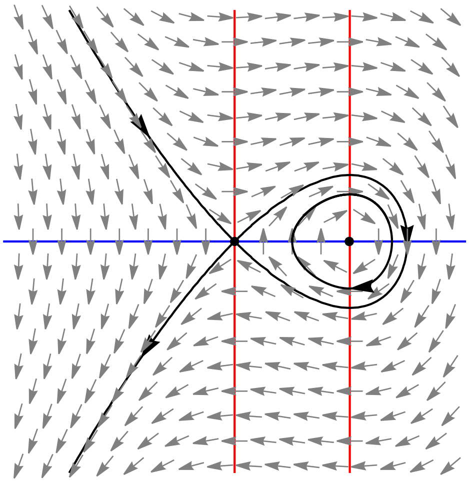
Then $\dot{H} = 0$, for any values of $\mathbf{y}$ and a. We call homoclinic surface to the set of points such that there are homoclinic orbits. In our case, homoclinic orbits converge both for $t \to \infty$ and $t \to -\infty$ to point $\bar{\mathbf{y}}^1$. Therefore the homoclinic surface is the set of points

$$\mathrm{Hom}(\bar{\mathbf{y}}^1) = \{\ (y_1, y_2) : H(y_1, y_2) = 0,\ \mathrm{sign}(\bar{y}_1) = \mathrm{sign}(a)\}$$

Figure 6.3 depicts phase diagrams for the case in which $a < 0$ (subfigure (a)) and $a > 0$ (sub-figure (b)).



(a) Phase diagrams for $a < 0$



(b) Phase diagrams for $a > 0$

Figure 6.3: Phase diagrams for equation (6.7).

We see that the homoclinic trajectories are generic, i.e, they exist for different subsets of values of the parameters. This is not always the case as we show next.[5]

Non-generic case: Homoclinic or saddle-loop bifurcation

This model is studied, for instance, in (Hale and Koçak, 1991, p.210) and (Kuznetsov, 2005, ch. 6.2). It is a non-linear ODE depending on one parameter, a, of type

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 + a\, y_2 - y_1^2. \end{aligned} \tag{6.8}$$

In this case, we have

$$\mathbf{f}(\mathbf{y}, a) = \begin{pmatrix} y_2 \\ y_1 + a\, y_2 - y_1^2. \end{pmatrix}.$$

The Jacobian for any point $\mathbf{y} = (y_1, y_2)$,

$$D_{\mathbf{y}}\mathbf{f}(\mathbf{y}, a) = \begin{pmatrix} 0 & 1 \\ 1 - 2y_1 & a \end{pmatrix}.$$

[5]n that Figure we also see that inside the homoclinic loop there is an infinite number of periodic orbits. Applying the Bendixson-Dulac criterium we see that $f_{1,y_1} + f_{2,y_2} = 0$ for any (y_1, y_2).

The eigenvalues of this Jacobian are functions of the variables and of the parameter a,

$$\lambda_{\pm}(\mathbf{y}, a) = \frac{a}{2} \pm \left[\ \left(\frac{a}{2}\right)^2 + 1 - 2y_1 \ \right]^{\frac{1}{2}}.$$

The set of steady states $\bar{\mathbf{y}} = \{ \ \mathbf{y} : \mathbf{f}(\mathbf{y}, a) = \mathbf{0}\}$. For equation (6.8) we have two steady states,

$$\bar{\mathbf{y}}^1 = \begin{pmatrix} \bar{y}_1^1 \\ \bar{y}_2^1 \end{pmatrix} = \mathbf{0}, \ \bar{\mathbf{y}}^2 = \begin{pmatrix} \bar{y}_1^2 \\ \bar{y}_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If we evaluate the eigenvalues at the steady state $\bar{\mathbf{y}}^1 = (0, 0)$, we find it is a saddle point, because the eigenvalues of the Jacobian $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^1)$ are

$$\lambda_{\pm}^1 \equiv \lambda_{\pm}(\bar{\mathbf{y}}^1, a) = \frac{a}{2} \pm \left[\ \left(\frac{a}{2}\right)^2 + 1 \ \right]^{\frac{1}{2}}$$

yielding $\lambda_-^1 < 0 < \lambda_+^1$. At the steady state $\bar{\mathbf{y}}^2 = (1, 0)$ the eigenvalues of the Jacobian $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^2)$ are

$$\lambda_{\pm}^2 = \lambda_{\pm}(\bar{\mathbf{y}}^2, a) = \frac{a}{2} \pm \left[\ \left(\frac{a}{2}\right)^2 - 1 \ \right]^{\frac{1}{2}}$$

yielding $\text{sign}(\text{Re}(\lambda_{\pm}(\bar{\mathbf{y}}^2, a)) = \text{sign}(a)$.

Therefore steady state $\bar{\mathbf{y}}^1$ is always a saddle point, and steady state $\bar{\mathbf{y}}^2$ is a stable node or a stable focus if $a < 0$, it is an unstable node or an unstable focus if $a > 0$, or it is a centre if $a = 0$.

When $a = 0$ another type of dynamics occurs. We introduce the following Lyapunov function

$$H(y_1, y_2) = -\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{3}y_1^3.$$

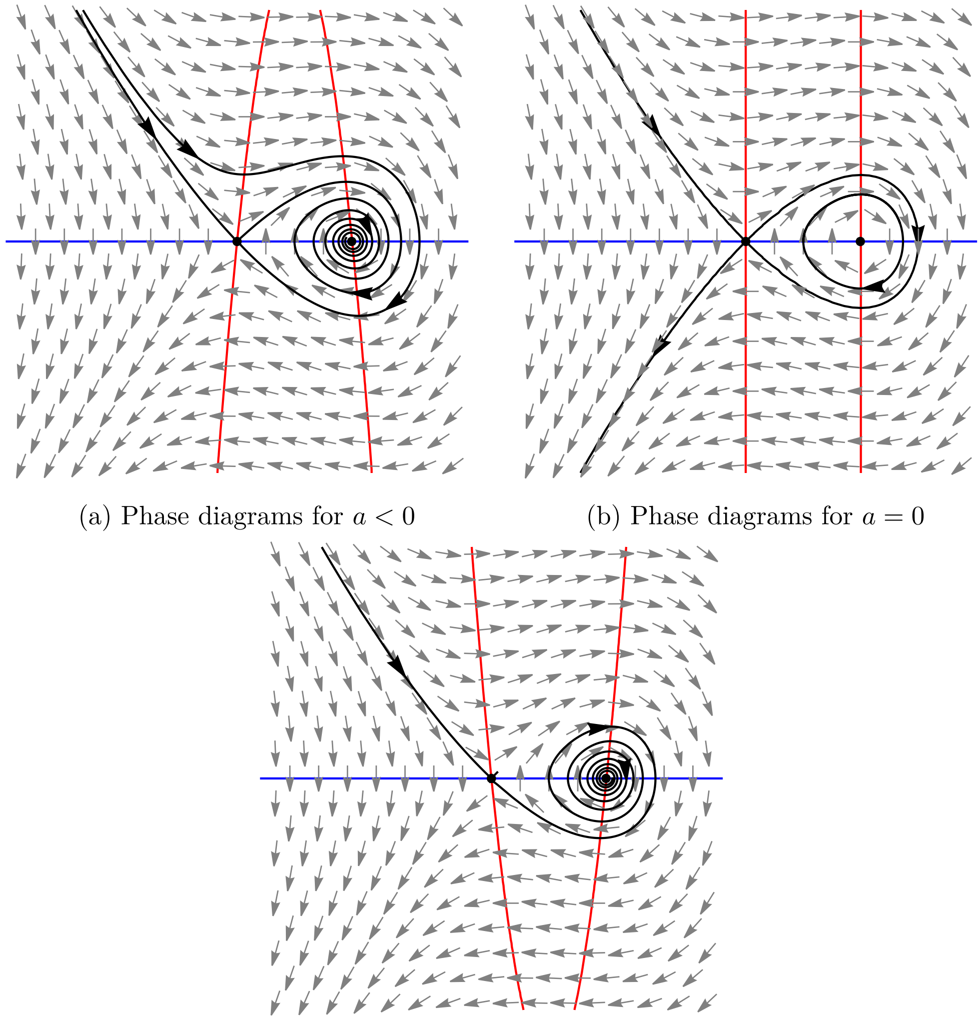
and prove that it can only be a first integral if $a = 0$. To show this, if we time-differentiate this Lyapunov function and substitute equations (6.8) we have

$$\begin{aligned} \dot{H} &= (y_1 - 1)y_1\dot{y}_1 + y_2\dot{y}_2 = \\ &= (y_1 - 1)y_1y_2 + y_2y_1(1 - y_1) + ay_2^2 = \\ &= ay_2^2. \end{aligned}$$

Then $\dot{H} = 0$, for any values of $\mathbf{y}$, if and only if and only if $a = 0$.

In our case this generates an **homoclinic orbit** which is a trajectory that exits a steady state and returns to the same steady state. In this case, a homoclinic orbit exists if $a = 0$ and it does not exist if $a \neq 0$.

Figure 6.4 shows the phase diagrams for the cases $a < 0$, $a = 0$ and $a > 0$: first, if $a < 0$ (see subfigure (a)) there is a saddle point and a stable focus, and a heteroclinic orbit connecting the two steady states, second, if $a = 0$ (see subfigure (b)) there is a saddle point, and an infinite number of centres surrounded by an homoclinic orbit; at last If $a > 0$ (see subfigure (c)) there is a saddle point and an unstable focus, and an heteroclinic orbit connecting the two steady states.



(a) Phase diagrams for $a < 0$

(b) Phase diagrams for $a = 0$

(c) Phase diagrams for $a > 0$

Figure 6.4: Phase diagrams for equation (6.8)

6.1.3 Limit cycles

This model is studied, for instance, in (Hale and Koçak, 1991, p.212): [6]

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2) \equiv y_2 + y_1(\lambda - y_1^2 - y_2^2) \\ \dot{y}_2 &= f_2(y_1, y_2) \equiv -y_1 + y_2(\lambda - y_1^2 - y_2^2) \end{aligned} \tag{6.9}$$

It has a single steady state $\bar{\mathbf{y}} = (0, 0)$. However, it has another invariant curve. In order to see this, we compute the Jacobian

$$J(\mathbf{y}) = \begin{pmatrix} \lambda - 3y_1^2 - y_2^2 & 1 - 2y_1y_2 \\ -1 - 2y_1y_2 & \lambda - y_1^2 - 3y_2^2 \end{pmatrix}$$

which has eigenvalues

$$\lambda_{\pm} = \lambda - 2(y_1^2 + y_2^2) \pm \sqrt{(y_1^2 + y_2^2 + 1)\,(y_1^2 + y_2^2 - 1)}$$

In figure 6.5 we see the following: if $\lambda < 0$ there will be only one steady state which is a stable node with multiplicity, although the speed of convergence to the steady state increases very much when λ converges to zero, if $\lambda > 0$ a **limit circle** appears and the steady state becomes a unstable focus. According to the **Bendixson-Dulac criterium** (see Theorem 3) as

$$\frac{\partial f_1(y_1, y_2)}{\partial y_1} + \frac{\partial f_2(y_1, y_2)}{\partial y_2} = 4\left(\frac{\lambda}{2} - (y_1^2 + y_2^2)\right)$$

changes sign for $\lambda > 0$, in a subset of $\mathbf{y}$, then a closed curve can exist. This closed curve is a limit cycle which is a curve such that $y^1 + y^2 = \lambda/2$. To prove this, we transform the system in polar coordinates (see Appendix to chapter 1) and get [7]

$$\begin{aligned} \dot{r} &= r(\lambda - r^2) \\ \dot{\theta} &= -1 \end{aligned}$$

there is thus a periodic orbit with radius $\bar{r} = \sqrt{\lambda}$.

[6] This is a particular case of equation (6.4) with

$$\mathbf{A}_0 = \mathbf{0}_{(2\times 1)},\ \mathbf{A}_1 = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix},\ \mathbf{A}_2 = \mathbf{0}_{(2\times 3)},\ \text{and}\ \mathbf{A}_3 = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}.$$

[7] We define $r^2 = y_1^2 + y_2^2$ and $\theta = \arctan\frac{y_2}{y_1}$, and take time derivatives, obtaining

$$\begin{aligned} \dot{r} &= \frac{y_1\dot{y}_1 + y_2\dot{y}_2}{r} \\ \dot{\theta} &= \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{r^2} \end{aligned}$$

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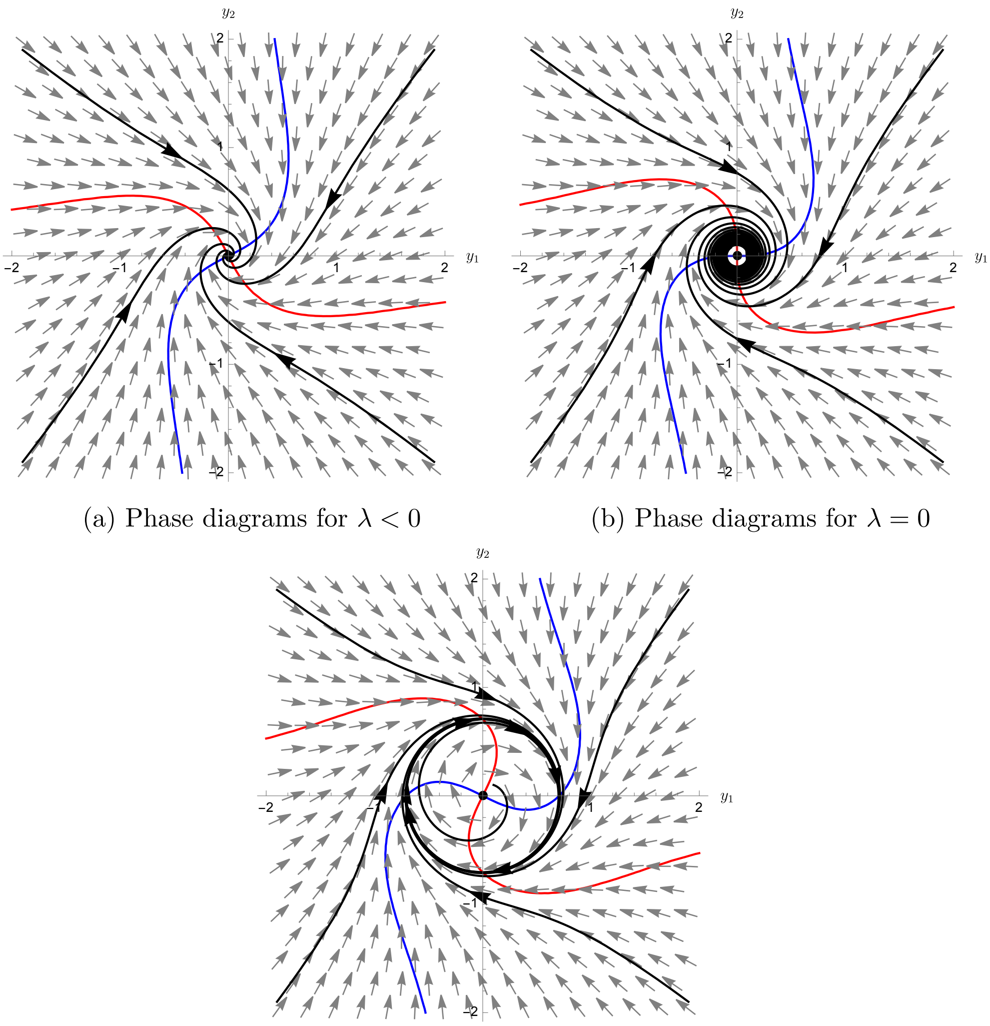


(a) Phase diagrams for $\lambda < 0$

(b) Phase diagrams for $\lambda = 0$

(c) Phase diagrams for $\lambda > 0$

Figure 6.5: Phase diagrams for equation (6.9)

6.1.4 Co-dimension two bifurcations

Until this subsection we have only considered non-generic cases unfolded by one parameter. Next we see one common case of a co-dimensio two bifurcation

Bogdanov-Takens bifurcation

The normal form of the Bogdanov-Takens bifurcation (see (Kuznetsov, 2005, p. 311)) is [8]

$$\begin{aligned}\dot{y}_1 &= f_1(y_1, y_2) \equiv y_2\\ \dot{y}_2 &= f_2(y_1, y_2) \equiv a + b\,y_1 + y_1^2 + c\,y_1\,y_2,\end{aligned} \tag{6.10}$$

where $c = \pm 1$. It displays several bifurcations: first, co-dimension one local bifurcations (saddle-node and Andronov-Hopf) and a global bifurcation: homoclinic bifurcation.

We see that: if $a > \left(\frac{b}{2}\right)^2 > 0$ there are no steady states (see subfigure (a) in Figure 6.6), if $a = \left(\frac{b}{2}\right)^2 > 0$ there is one steady state $\bar{\mathbf{y}} = (-\frac{b}{/}2, 0)$ (see subfigures (e) and (f) in Figure 6.6) and if $a < \left(\frac{b}{2}\right)^2$ there are two steady states

$$\bar{\mathbf{y}} = \left\{ \left(-\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - a}, 0\right), \left(-\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - a}, 0\right)\right\}$$

(see subfigure (b) in Figure 6.6). We can show that there are the following bifurcation curves: We have a transcritical bifurcation for $(a, b) \in T$ where

$$T = \left\{(a, b) : a - \left(\frac{b}{2}\right)^2 = 0\right\}.$$

We have an Andronov-Hopf biburcation $(a, b) \in H$

$$H = \left\{(a, b) : a = 0,\ b < 0\right\},$$

and we have a co-dimension-two if $(a, b) \in P$

$$P = \left\{(a, b) : a + \frac{6}{25}\,b^2,\ b < 0\right\}.$$

There a generic homoclinic orbit, given by the Lyapunov function

$$H(y_1, y_2) = -\frac{y_1^2}{2} + \frac{y_2^2}{2} + b\,\frac{y_1^3}{3},\ \text{for}\ \ b < 0.$$

Figure 6.7 shows the bifurcation diagram when $c = -1$, and Figure 6.6 illustrates representative phase diagrams

[8] This is a particular case of equation (6.4) with

$$\mathbf{A}_0 = \begin{pmatrix} 0 \\ a \end{pmatrix},\ \mathbf{A}_1 = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix},\ \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \pm 1 & 0 \end{pmatrix},\ \text{and}\ \ \mathbf{A}_3 = \mathbf{0}_{(2\times 4)}.$$

6.2 Qualitative theory of ODE

Next we present a short introduction to the qualitative (or geometrical) theory of ODE's.

We consider a generic ODE

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}),\ \mathbf{f} : \mathrm{Y} \to \mathrm{Y},\ \mathbf{y} : \mathrm{T} \to \mathrm{Y} \tag{6.11}$$

where $\mathbf{f} \in C^1(\mathrm{Y})$, i.e., $f(.)$ is continuously differentiable up to the first order.

The qualitative theory of ODEs consists in finding a (topological) equivalence between a non-linear (or even incompletely defined) function $\mathbf{f}(.)$ and a linear or a normal form ODE. This allows us to characterize the dynamics in the neighborhood of a steady state or of a periodic orbit or other invariant sets (homoclinic and heteroclinic orbits or limit cycles). If there are more than one invariant orbit or steady state we distinguish between local dynamics (in the neighborhood of a steady state or invariant orbit) from global dynamics (in all set y). If there is only one invariant set then local dynamics is qualitatively equivalent to global dynamics.

One important component of qualitative theory is **bifurcation analysis**, which consists in describing the change in the dynamics (that is, in the phase diagram) when one or more parameters take different values within its domain.

6.2.1 Local analysis

We study local dynamics of equation (6.11) by performing a local analysis close to an equilibrium point or a periodic orbit. There are three important results that form the basis of the local analysis: the Grobman-Hartmann, the manifold and the Poincaré-Bendixson theorems. The first two are related to using the knowledge on the solutions of an equivalent linearized ODE to study the local properties close to the a steady-state for a non-linear ODE and the third introduces a criterium for finding periodic orbits.

Equivalence with linear ODE's

Assume there is (at least) one equilibrium point $\bar{\mathbf{y}} \in \{\ \mathbf{y} \in Y \subseteq \mathbb{R}^n : \mathbf{f}(\mathbf{y}) = \mathbf{0}\}$, for $n \geq 1$, and consider the Jacobian of $\mathbf{f}(.)$ evaluated at that equilibrium point

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \frac{\partial f_1(\bar{y})}{\partial y_1} & \dots & \frac{\partial f_1(\bar{y})}{\partial y_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n(\bar{y})}{\partial y_1} & \dots & \frac{\partial f_n(\bar{y})}{\partial y_n} \end{pmatrix}.$$

In this section we consider the case $n = 2,$ therefore

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \frac{\partial f_1(\bar{y})}{\partial y_1} & \frac{\partial f_1(\bar{y})}{\partial y_2} \\ \frac{\partial f_2(\bar{y})}{\partial y_1} & \frac{\partial f_2(\bar{y})}{\partial y_2} \end{pmatrix}.$$

An equilibrium point is **hyperbolic** if the Jacobian J has no eigenvalues with zero real parts. An equilibrium point is **non-hyperbolic** if the Jacobian has at least one eigenvalue with zero real part.

Theorem 1 (**Grobman-Hartmann theorem**). *Let $\bar{\mathbf{y}}$ be a hyperbolic equilibrium point. Then there is a neighbourhood U of $\bar{\mathbf{y}}$ and a neighborhood U_0 of $\mathbf{y}(0)$ such that the ODE restricted to U is topologically equivalent to the variational equation*

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}}),\ \mathbf{y} - \bar{\mathbf{y}} \in U_0$$

The original paper are Grobman (1959) and Hartman (1964).

Stability properties of $\bar{\mathbf{y}}$ are characterized from the eigenvalues of Jacobian matrix $J(\bar{\mathbf{y}}) = D_y \mathbf{f}(\bar{\mathbf{y}})$.

If all eigenvalues λ of the Jacobian matrix have negative real parts, $\text{Re}(\lambda) < 0$, then $\bar{y}$ is **asymptotically stable**. If there is at least one eigenvalue λ such that $\text{Re}(\lambda) > 0$ then $\bar{y}$ is **unstable**. If the determinant of the Jacobian is negative the steady state is a **saddle point**.

Example Consider the non-linear planar ODE

$$\begin{aligned} \dot{y}_1 &= y_1^\alpha - a,\ 0 < \alpha < 1,\ a \geq 0, \\ \dot{y}_2 &= y_1 - y_2 \end{aligned} \tag{6.12}$$

It has an unique steady state $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2) = (a^{\frac{1}{\alpha}}, a^{\frac{1}{\alpha}})$. The Jacobian evaluated at any point $\mathbf{y}$ is

$$J(\mathbf{y}) = D_{\mathbf{y}} \mathbf{f}(\mathbf{y}) = \begin{pmatrix} \alpha y_1^{\alpha-1} & 0 \\ 1 & -1 \end{pmatrix}.$$

If we approximate the system in a neighborhood of the steady state, $\bar{\mathbf{y}}$, we have the linear planar ODE

$$\dot{\mathbf{y}} \ = J(\bar{\mathbf{y}})\,(\mathbf{y} - \bar{\mathbf{y}})$$

where $J(\bar{\mathbf{y}})$ is the Jacobian evaluated at the steady state,

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}} \mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \alpha a^{\frac{\alpha-1}{\alpha}} & 0 \\ 1 & -1 \end{pmatrix}.$$

We already saw that the solution to this equation is

$$\mathbf{y}(t) = \mathbf{y} + \mathbf{P} \mathbf{e}^{J(\bar{\mathbf{y}})t} \mathbf{h}.$$

Because

$$\begin{aligned} \text{trace}(J(\bar{\mathbf{y}})) &= \alpha a^{\frac{\alpha-1}{\alpha}} - 1 \\ \det\left(J(\bar{\mathbf{y}})\right) &= -\alpha a^{\frac{\alpha-1}{\alpha}} \\ \Delta(J(\bar{\mathbf{y}})) &= \left(\frac{\alpha a^{\frac{\alpha-1}{\alpha}} + 1}{2} \right)^2 \end{aligned}$$

which implies that the eigenvalues of the Jacobian $J(\bar{\mathbf{y}})$ are

$$\lambda_\pm = \frac{\alpha a^{\frac{\alpha-1}{\alpha}} - 1}{2} \pm \frac{\alpha a^{\frac{\alpha-1}{\alpha}} + 1}{2}.$$

that is $\lambda_+ = \alpha a^{\frac{\alpha-1}{\alpha}}$ and $\lambda_- = -1$. Therefore, the steady state is hyperbolic if $\alpha \neq 0$ and non-hyperbolic if $\alpha = 0$.

Furthermore, the steady state is a saddle point if $\alpha > 0$ and it is a stable node if $\alpha < 0$. We can also find the eigenvector matrix of $J(\bar{\mathbf{y}})$,

$$\mathbf{P} = (\mathbf{P}^{+}\mathbf{P}^{-}) = \begin{pmatrix} 1+\lambda_+ & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore, the approximate solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a^{\frac{1}{\alpha}} \\ a^{\frac{1}{\alpha}} \end{pmatrix} + h_+ \begin{pmatrix} 1+\lambda_+ \\ 1 \end{pmatrix} e^{\lambda_+ t} + h_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_- t}.$$

If $\alpha < 0$ the stable eigenspace is $\mathcal{E}^s = \{\ (y_1, y_2) : y_1 = \bar{y}_1\}$, and, if $\alpha > 0$ the stable eigenspace is the whole space, $\mathcal{E}^s = \mathrm{Y}$.

Local manifolds

Consider a neighbourhood $U \subset \mathrm{Y} \subseteq \mathbb{R}^n$ of $\bar{\mathbf{y}}$: the local stable manifold is the set

$$\mathcal{W}^s_{loc}(\bar{\mathbf{y}}) = \{\ \mathbf{k} \in U :\ \lim_{t\to\infty} \mathbf{y}(t,k) = \bar{y},\ y(t,\mathbf{k}) \in U, t \geq 0\}$$

the local unstable manifold is the set

$$\mathcal{W}^u_{loc}(\bar{\mathbf{y}}) = \{\ \mathbf{k} \in U :\ \lim_{t\to\infty} \mathbf{y}(-t,\mathbf{k}) = \bar{\mathbf{y}},\ \mathbf{y}(-t,\mathbf{k}) \in U, t \geq 0\}$$

The center manifold is denoted $\mathcal{W}^c_{loc}(\bar{\mathbf{y}})$. Let n_-, n_+ and n_0 denote the number of eigenvalues of the Jacobian evaluated at steady state $\bar{\mathbf{y}}$ with negative, positive and zero real parts.

Theorem 2 (**Manifold Theorem**). *Suppose there is a steady state $\bar{\mathbf{y}}$ and let $J(\bar{\mathbf{y}})$ be the Jacobian of the ODE* (6.11) *evaluated at that steady state. Then there are local stable, unstable and center manifolds, $\mathcal{W}^s_{loc}(\bar{\mathbf{y}})$, $\mathcal{W}^u_{loc}(\bar{\mathbf{y}})$ and $\mathcal{W}^c_{loc}(\bar{\mathbf{y}})$, of dimensions n_-, n_+ and n_0, respectively, such that $n = n_- + n_+ + n_0$. The local manifolds are tangent to the local eigenspaces $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$ of the (topologically) equivalent linearized ODE*

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}}).$$

The first two, eigenspaces $\mathcal{E}^s$ and $\mathcal{E}^u$, are unique, and $\mathcal{E}^c$need not be unique (see (Grass et al., 2008, ch.2)).

The eigenspaces are spanned by the eigenvectors of the Jacobian matrix $J(\bar{y})$ which are associated to the eigenvalues with negative, positive and zero real parts.

Example 2 Consider example 2 and let $\alpha > 0$ which implies that the steady state $\bar{\mathbf{y}}$ is a saddle point. Because the eigenvector associated to eigenvalue λ_- is $\mathbf{P}^- = (0,1)^\top$, then the stable eigenspace is

$$\mathcal{E}^s = \{(y_1, y_2) \in \mathbb{R}_+ : y_1 = \bar{y}_1 = a^{\frac{1}{\alpha}}\}.$$

The local stable manifold $\mathcal{W}^s_{loc}(\bar{\mathbf{y}})$ is tangent to $\mathcal{E}^s$ in a neighborhood of the steady state.

6.2.2 Periodic orbits

We saw that solution trajectories can converge or diverge not only as regards equilibrium points but also to periodic trajectories (see the Andronov-Hopf model).

The **Poincaré-Bendixson** theorem (see (Hale and Koçak, 1991, p.367)) states that if the limit set is bounded and it is not an equilibrium point it should be a periodic orbit.

In order to determine if there is a periodic orbit in a compact subset of y the Bendixson criterium provides a method for finding it ((Hale and Koçak, 1991, p.373)):

Theorem 3 (**Bendixson-Dulac criterium**). *Let D be a compact region of $y \subseteq \mathbb{R}^2$. If,*

$$div(\mathbf{f})(\mathbf{y}) = f_{1,y_1}(\mathbf{y}) + f_{2,y_2}(\mathbf{y})$$

*has constant sign, for $(y_1, y_2) \in D$, then $\dot{y} = f(y)$ has **not** a constant orbit lying entirely in D.*

6.2.3 Global analysis

While local analysis consists in studying local dynamics in the neighbourhood of steady states or periodic orbits, this may not be enough to characterise the dynamics.

We already saw that there are invariant orbits that are invariant and that cannot be determined by local methods, for instance heteroclinic and homoclinic orbits.

Homoclinic and heteroclinic orbits

There are methods to determine if there are homoclinic or heteroclinic orbits. They essentially consist in building a trapping area for the trajectories and proving there should exist trajectories that do not exit the ”trap”.

Global manifolds

There are global extensions of the local manifolds by continuation in time (in the opposite direction) of the local manifolds: $\mathcal{W}^s(\bar{y})$, $\mathcal{W}^u(\bar{y})$, $\mathcal{W}^c(\bar{y})$.

A trajectory $y(.)$ of the ODE is called a **stable path** of $\bar{y}$ if the orbit $\text{Or}(y_0)$ is contained in the stable manifold $\text{Or}(y_0) \subset \mathcal{W}^s(\bar{y})$ and $\lim_{t\to\infty} y(t, y_0) = \bar{y}$.

A trajectory $y(.)$ of the ODE is called a **unstable path** of $\bar{y}$ if the orbit $\text{Or}(y_0)$ is contained in the stable manifold $\text{Or}(y_0) \subset \mathcal{W}^u(\bar{y})$ and $\lim_{t\to\infty} y(-t, y_0) = \bar{y}$.

6.2.4 Dependence on parameters

We already saw that the solution of linear ODE's, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, may depend on the values for the parameters in the coefficient matrix $\mathbf{A}$.

We can extend this idea to non-linear ODE's of type

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi),\ \varphi \in \Phi \subseteq R^q$$

where φ is a vector of parameters of dimension $q \geq 1$

We can distinguish two types of parameter change:

- **bifurcations** when a parameter change induces a qualitative change in the dynamics, i.e, the phase diagram. By qualitative change we mean change the number or the stability properties of steady states or other invariants. Close to a bifurcation point, a change in a parameter changes the qualitative characteristics of the dynamics;
- **perturbations** when parameter changes do not change the qualitative dynamics, i.e., they do not change the phase diagram. This is typically the case in economics when one performs comparative dynamics exercises.

Bifurcations

If a small variation of the parameter changes the phase diagram we say we have a bifurcation. As you saw, there are local (fixed points) and global bifurcations (heteroclinic connection, etc). Those bifurcations were associated to particular normal forms of both scalar and planar ODEs. This fact allows us to find classes of ODE's which are topologically equivalent to those we have already presented.

Consider the planar ODE

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi),\ \mathbf{y} \in \mathbb{R}^2, \varphi \in \mathbb{R}$$

Andronov-Hopf bifurcation (see (Kuznetsov, 2005, ch. 3.4)): Let $\mathbf{f} \in C^2(\mathbb{R})$ and consider $(\bar{\mathbf{y}}, \varphi_0) = (\mathbf{0}, 0)$ the Jacobian at $(\mathbf{0}, 0)$ has eigenvalues

$$\lambda_{\pm} = \eta(\varphi) \pm i\omega(\varphi)$$

where $\eta(0) = 0$ and $\omega(0) > 0$. If some additional conditions are satisfied then the ODE is locally topologically equivalent to

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

6.2.5 Comparative dynamics in economics

As mentioned, **comparative dynamics** exercises consist in introducing perturbation in a dynamic system: i.e., a small variation of the parameter that does not change the phase diagram. This kind of analysis only makes sense if the steady state is hyperbolic, that is if $\det\left(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)\right) \neq 0$ or $\text{trace}(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)) \neq 0$ if $\det\left(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)\right) > 0$.

In this case let the steady state be for a given value of the parameter $\varphi = \varphi_0$

$$\bar{\mathbf{y}}_0 = \{\ y \in Y : \mathbf{f}(\mathbf{y}, \varphi_0) = \mathbf{0}\}.$$

If $\bar{\mathbf{y}}_0$ is a hyperbolic steady state, then we can expand the ODE into a linear ODE

$$\dot{\mathbf{y}} = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)(\mathbf{y} - \bar{\mathbf{y}}_0) + D_{\varphi}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)(\varphi - \varphi_0). \tag{6.13}$$

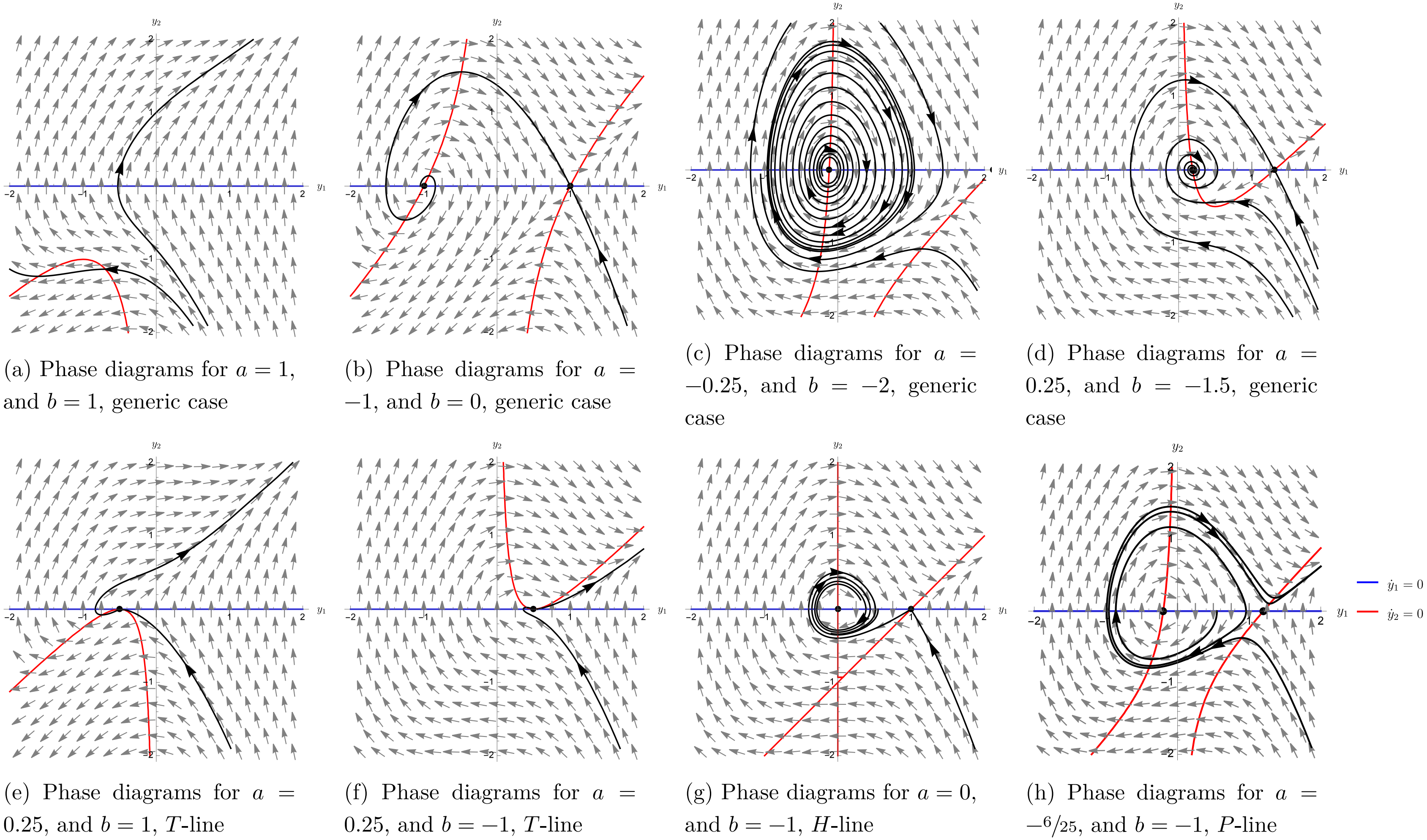
This equation can be solved as a linear ODE. Setting $\varphi = \varphi_0 + \delta_\varphi$ and because $\bar{\mathbf{y}} = \bar{\mathbf{y}}(\varphi)$ and $\bar{\mathbf{y}}_0 = \bar{\mathbf{y}}(\varphi_0)$ we have

$$D_{\varphi}\bar{\mathbf{y}}(\varphi_0) = \lim_{\delta_\varphi \to 0} \frac{\bar{\mathbf{y}}(\varphi_0 + \delta_\varphi) - \bar{\mathbf{y}}(\varphi_0)}{\delta_\varphi} = -D_{\mathbf{y}}^{-1}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)D_{\varphi}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)$$

which are called the **long-run multipliers** associated to a permanent change in φ. Solving the linearized system allows us to have a general solution to the problem of finding the **short-run** or **transition multipliers**, $d\mathbf{y}(t) \equiv \mathbf{y}(t) - \bar{\mathbf{y}}_0$ for a change in the parameter φ.

6.3 References

- (Hale and Koçak, 1991, Part I , III): very good introduction.
- (Guckenheimer and Holmes, 1990, ch. 1, 3, 6) Is a classic reference on the field.
- Kuznetsov (2005) Very complete presentation of bifurcations for planar systems.
- Brock and Malliaris (1989), (Grass et al., 2008, ch.2) has a compact presentation of all the important results with some examples in economics and management science.



(a) Phase diagrams for $a = 1$, and $b = 1$, generic case

(b) Phase diagrams for $a = -1$, and $b = 0$, generic case

(c) Phase diagrams for $a = -0.25$, and $b = -2$, generic case

(d) Phase diagrams for $a = 0.25$, and $b = -1.5$, generic case

(e) Phase diagrams for $a = 0.25$, and $b = 1$, T-line

(f) Phase diagrams for $a = 0.25$, and $b = -1$, T-line

(g) Phase diagrams for $a = 0$, and $b = -1$, H-line

(h) Phase diagrams for $a = -6/25$, and $b = -1$, P-line

Figure 6.6: Phase diagrams for equation (6.10) with $c = -1$. Refer to Figure 6.7.

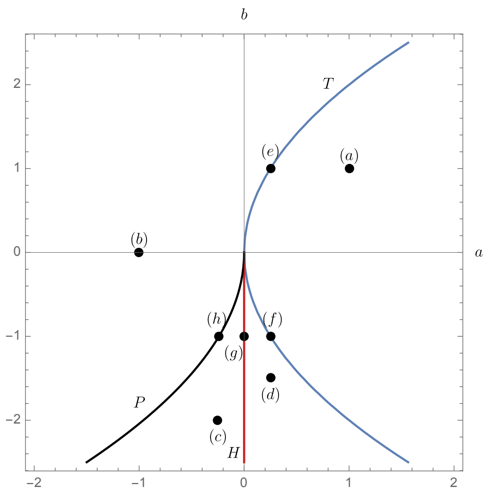


Figure 6.7: Bogdanov-Takens bifurcation diagram. Lines T, P and H refer to transcritical, pitchfork and Hopf bifurcations and points (a) to (b) refer to phase diagrams in Figure 6.6

Chapter 7

Applications to economics

In this chapter we provide some applications of the theory presented in the previous two chapters to economics. Section 7.1 presents scalar models and section 7.2 planar models.

7.1 Scalar ODE

In this section we present two simple models of capital accumulation with a concave production function and with a concave-convex production function.

7.1.1 Capital accumulation with a concave production functions

Solow (1956) is a benchmark contribution to the theory of economic growth. It considers a concave production function

$$y = f(k),\ f(0) = 0, \text{ and } f'(k) > 0.\ f''(k) < 0, \text{ for } \ k > 0$$

where y and k are per capita output and the stock of capital, respectively. As in most of classical economic theory, it is assumed that the law of diminishing returns prevails. Formally, the production function is increasing and concave. The benchmark production function is the Cobb-Dpuglas function

$$y = f(k) = A\,k^{\alpha},\ A > 0,\ 0 < \alpha < 1, \tag{7.1}$$

where A is a productivity parameter and α is the share of capital in national income, y.

The output can be consumed or used to increase the capital stock, through investment. The macroeconomic equilibrium, in a closed economy, requires gross per capita investment, consisting on net investment plus depreciation, $\dot{k} + \delta\,k$, to be equal to savings. In Solow (1956)'s model it is assumed that savings has a somewhat mechanic explanation as the part of the income which is not consumed. Therefore, savings is given by $s\,y$ where $0 < s < 1$. I assume a constant population.

The Solow (1956) capital accumulation equation is

$$\dot{k} = s\,f(k) - \delta\,k,\ k \in \mathbb{R}_+. \tag{7.2}$$

We can solve this ODE analytically, provided we specify a suitable production function, or qualitatively.

If we assume a Cobb-Douglas production function (7.1) then equation (7.2) is a Bernoulli ODE, and therefore it has a closed form (that is global) solution .

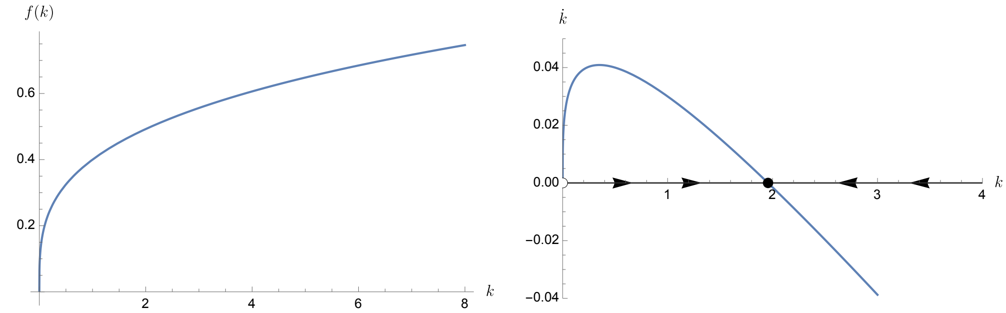
Exercise Find it.

To find a qualitative, or approximated, solution to equation (7.2) we start by finding its steady states. It is easy to see that the ODE (7.2) has two steady states $k = 0$ and $\bar{k} = \{k : s\,f(k) = \delta\,k\}$. We can also observe that the positive steady state can be linearly approximated by a linear ODE

$$\dot{k} = \lambda\,(k - \bar{k}),\ \text{where}\ \ \lambda \equiv (s\,f'(\bar{k}) - \delta) < 0$$

because $f'(\bar{k}) < \frac{f(\bar{k})}{\bar{k}} = \frac{\delta}{s}$ from the global concavity of the production function.

Therefore, the Solow (1956) model can be locally approximated at the steady state $\bar{k}$ by a linear ODE displaying asymptotic stability.



(a) Production function

(b) Phase diagram for a high B

Figure 7.1: Concave production function and the accumulation of capital

7.1.2 Capital accumulation with concave-convex production functions

The Solow model displays convergence to a steady state. However, it does not capture well the process of structural transformation that has been found to take place when some economies pass through a process of development. In particular, there seems to be some form of stability for poor and rich economies and the process of industrialization displaying some form of instability.

This process can be captured by assuming instead a concave-convex production function, see v.g. Skiba (1978). For instance, assume the following production function

$$y = f(k, B) = k^{\alpha}\left(A + \frac{k^{3-\alpha}}{2\,(k^3 + B^3)}\right),\ A > 0,\ 0 < \alpha < 1,\ B > 0 \tag{7.3}$$

where B can be seen as a parameter representing generic inefficiencies in transforming the capital stock into output. This production function is depicted in Figure 7.2 panel (a).

The capital accumulation equation, assuming a Solow economy, in which the savings function is ad-hoc is

$$\dot{k} \ = s\, f(k,B) - \delta\, k,\ k : \mathrm{T} \to \mathbb{R}_+$$

where $k(0) = k_0 > 0$ is given.

We can have an extension to the Solow model for the dynamics of capital accumulation by plugging this function into equation (7.2). We readily see that as the depreciation is a linear function $\delta\, k$ we may have two generic cases, with one or three steady states, and possibly two non-generic cases.

Figure 7.2 displays three different generic phase diagrams for different values for B: first, if B is high (see panel (b)) there are high ineficiencies and there is only a asymptotically stable steady state for a low level of k - and therefore of output y; second, for a relatively lower level of B (see panel (c)) there are three steady states, two extreme asymptotically stable steady states separated by an unstable steady state; and, at last, if B is low (see panel (d)) there will be again an asymptotically stable steady state with a high level for k (and y).

This seems to capture several apparent facts: (a) there is a poverty trap which seems hard to escape from; (b) several institutional changes are required in the process of development, which is looks like a locally unstable process featuring high growth rates, but there is no guarantee that the economy will not fall back to poverty; and, at last mature economies also seem to be in a relatively stable state.

Looking at our previous analysis and to the bifurcation diagram in panel (e) of Figure 7.2 we have a phase diagram which is qualitatively similar to the hysteresis case (see Figure 5.4)

Choosing B as the bifurcation parameter, and denoting

$$F(k,B) = s\, k^{\alpha} \left(A + \frac{k^{3-\alpha}}{2\,(k^3 + B^3)} \right) - \delta\, k$$

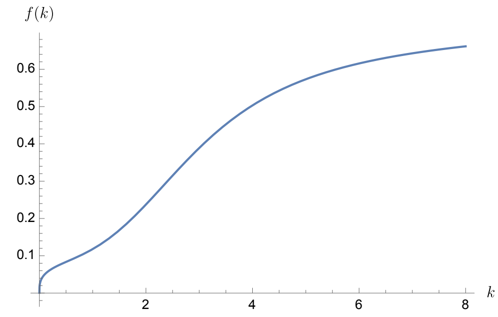
the histeresis points are found by solving

$$\begin{cases} F(k,B) = 0 & \iff s\, f(k,B) = \delta\, k \iff \dfrac{f(k,B)}{k} = \dfrac{\delta}{s} \\ F_k(k,B) = 0 & \iff s\, f_k(k,B) = \delta \iff f_k(k,B) = \dfrac{\delta}{s}. \end{cases}$$

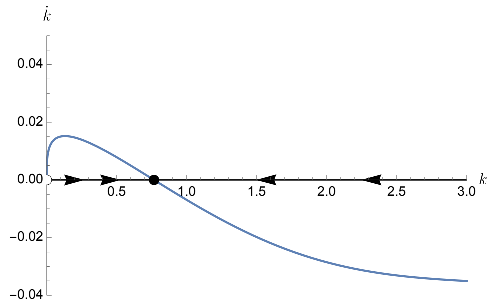
Therefore, at a bifurcation point we have $f(k,B) = f_k(k,B)\, k$ meaning that the production function is locally concave. This allows for, although we cannot determine explicitly the bifurcation points explicitly, to conclude that: first, a steady state $\bar{k}$ is locally asymptotically stable, that is $F_k(\bar{k},B) < 0$, if the production function is locally concave; second, a steady state $\bar{k}$ is locally unstable, that is $F_k(\bar{k},B) > 0$, if the production function is locally convex. To see this prove that,

$$F(k,B) = 0,\ \text{and}\ \ F_k(k,B) \lesseqgtr 0 \iff f(k,B) \gtreqless f_k(k,B)\, k.$$

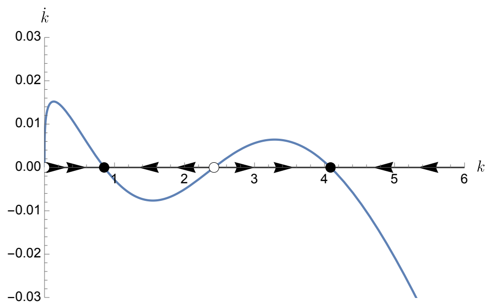
As we see in panel (c) in 7.2 a locally unstable steady state only occurs for intermediate values of B and the production function is locally convex, i.e., displaying increasing marginal returns. This steady state, say k^c, is indeed a critical point that separates the basins of attraction to the lower steady state from the higher steady state. This means that if $k(0) < k^c$ the economy will converge to the lower steady state and if $k(0) > k^c$ it will converge to the higher steady state. A small change in the other parameters, s, δ, or A will change k^c which means that we can only ascertain the effects of changes in those parameters (and in B) by means of a bifurcation analysis, not a comparative dynamics exercise.



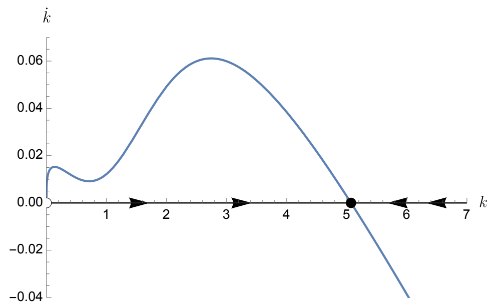
(a) Production function



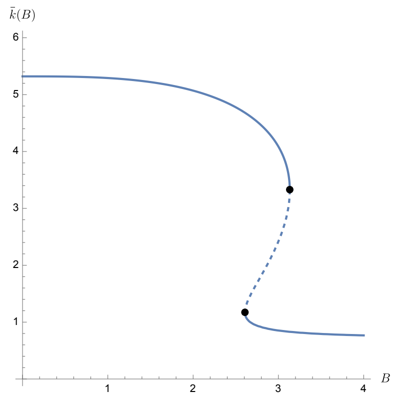
(b) Phase diagram for a high B



(c) Phase diagram for a medium B



(d) Phase diagram for a low B



(e) Bifurcation diagram

Figure 7.2: Concave-convex production function and the accumulation of capital

7.2 Planar ODE

In this section we consider optimal growth models in which the central planner has the same production functions as in the previous section, see subsections 7.2.1 and **??**. We also present a decentralized model with externalities in which the can perform a bifurcation analysis over a single steady state in subsection **??**

7.2.1 Ramsey model

The Ramsey (1928) model (see also Cass (1965) and Koopmans (1965)) is the workhorse of modern macroeconomics and growth theory. It is a normative model (but can also be seen as a positive model if its behavior fits the data) on the optimal choice of consumption and where savings leads to the accumulation of capital, and therefore to future consumption. Therefore, the optimal trade-off between present and future consumption guides the accumulation of capital.

We will derive the optimality conditions when we study optimal control. In this section we assume that there are two primitives for the model related with technology and preferences: (1) the production function, $f(k)$ and (2) the elasticity of intertemporal substitution $\eta(c)$ and the rate of time preference ρ.

The first order conditions for an optimum take the form of two non-linear differential equations. Let k and c denote per-capita physical capital and consumption, respectively, and let the two variables be non-negative. That is $(k, c) \in \mathbb{R}^2_+$. The Ramsey model is the planar ODE

$$\dot{k} = f(k) - c - \delta\, k \tag{7.4}$$
$$\dot{c} = \eta(c)\, c\, \left(f^{'}(k) - \rho - \delta\right), \tag{7.5}$$

supplemented with an initial value for k, $k(0) = k_0$ and the transversality condition $\lim_{t\to\infty} u^{'}(c)k(t)e^{-\rho t} = 0$, where $\eta(c) = -\frac{u'(c)}{u''(c)\, c}$. For this section we will be concerned with trajectories that are bounded asymptotically, that is converging to a steady state.

The ODE system (also called modified Hamiltonian dynamic system MHDS) is non-linear when the two primitive functions are not completely specified, as is the case with system (7.4)-(7.5). Next we assume a smooth case and the following assumptions

1. preferences are specified by a constant elasticity of intertemporal substitution, $\eta(c) = \eta > 0$ is constant;

2. the rate of time preference is positive $\rho > 0$;

3. the production function is of the Inada type: it is positive for positive levels of capital, it is monotonously increasing and globally concave. Formally: $f(0) = 0$, $f(k) > 0$ for $k > 0$, $f^{'}(k) > 0$, $\lim_{k\to 0} f^{'}(k) = +\infty$, $\lim_{k\to+\infty} f^{'}(k) = 0$, and $f^{''}(k) < 0$ for all $k \in \mathbb{R}_+$. [1]

[1]Observe that $f(k)$ is locally but not globally Lipschitz, i.e, a small change in k close to zero induces a large change in $f(k)$.

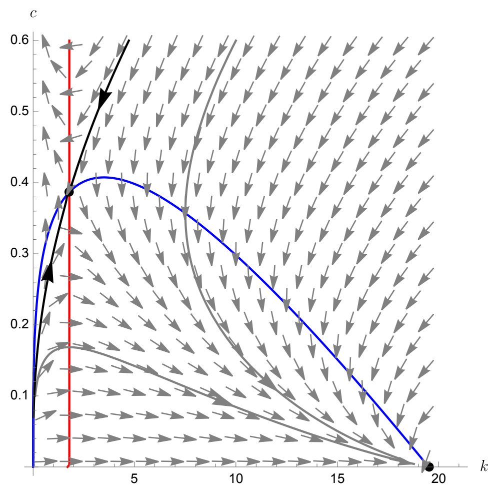


Figure 7.3: Ramsey model

Given the smoothness of the vector field, i.e, of functions $f_1(k,c) \equiv f(k) - c - \delta\, k$ and $f_2(k,c) \equiv \eta c(f^{'}(k) - \rho)$, we know that a solution exists and it is unique. Therefore, in order to characterize the dynamics we can use the qualitative theory of ODE's presented previously in this section.

In particular we will

1. determine the existence and number of steady states
2. characterize them regarding hyperbolicity and local dynamics, performing, if necessary, a local bifurcation analysis
3. try to find other invariant trajectories of a global nature
4. conduct comparative dynamics analysis in the neigborhood of relevant hyperbolic steady states.

Figure **??** depicts the phase diagram for the case in which $\delta > 0$. As we can see there are three steady states: two steady states given by $c = 0$, that is $(k, c) = \{\ (0,0),\ \ (k', 0)\}$ where $k' = \{k\,:\, f(k) = \delta\, k\}$, and a steady state where $\bar{k} = (f')^{-1}(\rho + \delta)$. It is clear that none of the steady states for which $c = 0$ satisfy the transversality condition, which means that they cannot be solutions to the Ramsey model.

From now on we deal formally with the case $\delta = 0$ and study it in more detail.

Steady states Any steady-state, $(\bar{k}, \bar{c})$, belongs to the set

$$(\bar{k}, \bar{c}) = \{\ (k,c) \in \mathbb{R}^2_+ :\ \dot{k} = \dot{c} = 0\}\ = \{\ (0,0), (k^*, c^*)\}$$

where $k^* = g(\rho)$, where $g(.) = (f^{'})^{-1}(.)$ and $c^* = f(k^*) = f(g(\rho))$.

To prove the existence and uniqueness of a positive steady state level for k we use the Inada and global concavity properties of the production function: first, $\dot{c} = 0$ if there is a value k that solves the equation $f^{'}(k) = \rho$; second, because $\rho > 0$ is finite and $f^{'}(k) \in (0, \infty)$ then there is at least one value for k that solves that equation; at last, because the function $f(.)$ is globally strictly concave then $f^{'}(k)$ is monotonously decreasing which implies that the solution is unique.

Characterizing the steady states In order to characterize the steady states, we find the Jacobian of system (7.4)-(7.5), is

$$D_{(k,c)}\mathbf{F}(k,c) = \begin{pmatrix} f^{'}(k) & -1 \\ \eta c f^{''}(k) & \eta\left(f^{'}(k) - \rho\right). \end{pmatrix} \tag{7.6}$$

The eigenvalues of $D_{(k,c)}\mathbf{F}(k,c)$ evaluated at steady state $(\bar{k}, \bar{c}) = (0,0)$ are

$$\lambda_s^0 = \eta(f^{'}(0) - \rho) = +\infty,$$
$$\lambda_u^0 = f^{'}(0) = +\infty,$$

which means that this steady state is singular (see chapter 8). This is a consequence of the fact that $f(k)$ is not locally Lipschitz close to $k = 0$.

For steady state $(\bar{k}, \bar{c}) = (k^*, c^*)$, the trace and the determinant of the Jacobian are

$$\text{trace}(D_{(k,c)}\mathbf{F}(k^*, c^*)) = \rho > 0, \ \det(D_{(k,c)}F(k^*, c^*)) = \eta c^* f^{''}(k^*) < 0$$

and the eigenvalues are

$$\lambda_s^* = \frac{\rho}{2} - \left(\left(\frac{\rho}{2}\right)^2 - \ \eta c^* f^{''}(k^*)\right)^{\frac{1}{2}} < 0, \ \lambda_u^* = \frac{\rho}{2} + \left(\left(\frac{\rho}{2}\right)^2 - \ \eta c^* f^{''}(k^*)\right)^{\frac{1}{2}} > 0$$

satisfy the relationships

$$\lambda_s^* + \lambda_u^* = \rho, \ \lambda_s^* \lambda_u^* = \eta c^* f^{''}(k^*) < 0.$$

The steady state (k^*, c^*) is also hyperbolic and it is a saddle-point. The intuition behind this property is transparent when we look at the expression for the determinant: the mechanism generating stability is related to the existence of decreasing marginal returns in producion. Because capital accumulation is equal to savings, and savings sustains future increases in consumption by increasing production, the existence of decreasing marginal returns implies that the marginal increase in production will tend to zero thus stopping the incentives for future capital accumulation.

As the Jacobians of system (7.4)-(7.5), evaluated at every steady state, does not have eigenvalues with zero real parts both steady states are hyperbolic and there are no local bifurcation points.

In addition, from the Grobman-Hartmann theorem the system (7.4)-(7.5) can be approximated by a (topologically equivalent) linear system in the neighborhood of every steady state.

Let us consider the steady state (k^*, c^*). As the Jacobian in this case is

$$D_{(k,c)}\mathbf{F}(k^*, c^*) = \begin{pmatrix} \rho & -1 \\ \eta c^* f^{''}(k^*) & 0 \end{pmatrix}$$

we can consider the **variational system**

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \eta c^* f^{''}(k^*) & 0 \end{pmatrix} \begin{pmatrix} k - k^* \\ c - c^* \end{pmatrix}$$

as giving the approximated dynamics in the neighborhood of the steady state (k^*, c^*).

Because

$$D_{(k,c)}\mathbf{F}(k^*, c^*) - \lambda_s^* \mathbf{I}_2 = \begin{pmatrix} \rho - \lambda_s^* & -1 \\ \eta c^* f^{''}(k^*) & -\lambda_s^* \end{pmatrix} = \begin{pmatrix} \lambda_u^* & -1 \\ \lambda_s^* \lambda_u^* & -\lambda_s^* \end{pmatrix}$$

we get the eigenvector associated to λ_s^*

$$\mathbf{P}_s^* = (1, \lambda_u^*)^\top.$$

This implies that the stable eigenspace of the linearized ODE,

$$\mathcal{E}^s = \{\ (k, c) \in N^* : c = \lambda_u^* k\}$$

gives the locus of points in the domain, wich are tangent to the local stable manifold for the original ODE (7.4)-(7.5)

$$\mathcal{W}_{loc}^s = \{\ (k, c) \in N^* : \lim_{t\to\infty} (k(t), c(t)) = (k^*, c^*)\}$$

where $N^* = \{\ (k, c) \in \mathbb{R}_+^2 : ||(k, c) - (k^*, c^*)|| < \delta\}$ for a small δ.

Global invariants We can prove that there is an heteroclinic orbit connecting steady states $(0, 0)$ and (k^*, c^*). Furthermore, the points in that orbit belong to the stable manifold

$$\mathcal{W}^s = \{\ (k, c) \in \mathbb{R}_+^2 : \lim_{t\to\infty} (k(t), c(t)) = (k^*, c^*)\},$$

and take the form $c = h(k)$. Although we cannot determine explicitly the function $h(.)$ we can prove that it exists (see Figure **??**).

We already know that the steady state $(0, 0)$ is an unstable node, which means that any small deviation will set a diverging path, and, because steady state (k^*, c^*) is a saddle point there is one unique path converging to it. There is an heteroclinic orbit if this path starts from from $(0, 0)$. In order to prove this is the case we can consider a "trapping area" $T = \{\ (k, c) : c \le f(k),\ 0 \le k \le k^*\}$, where the isoclines $\dot{k} = 0$ and $\dot{c} = 0$ define the boundaries $S_1 = \{\ (k, c) : c = f(k), 0 \le k \le k^*\}$ and $S_2 = \{(k, c) : 0 \le c \le c^*,\ k = k^*\}$. We can see that all the trajectories coming from inside will exit T: first, the trajectories that cross S_1 will exit T because $\dot{k}|_{S_1} = 0$ and $\dot{c}|_{S_1} = \eta c(f^{'}(k) - \rho) = \eta f(k)(f^{'}(k) - f^{'}(k^*)) > 0$ because $f^{'}(k) > f^{'}(k^*)$ for $k < k^*$, second all trajectories that cross S_2 will exit T because $\dot{k}|_{S_2} = f(k^*) - c = c^* - c > 0$ and $\dot{c}|_{S_2} = 0$.

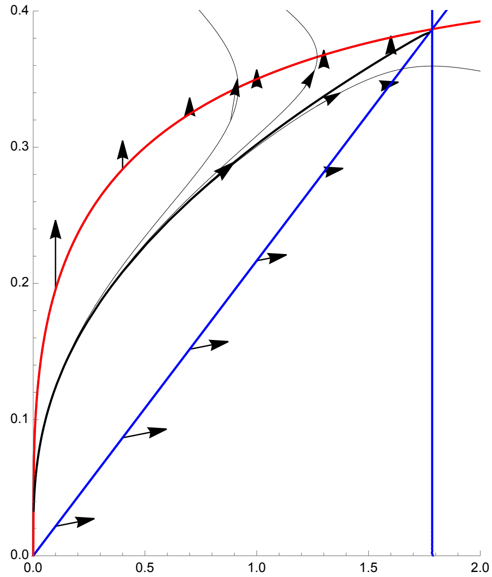


Figure 7.4: Heteroclinic orbit in the Ramsey mode

Comparative dynamics Let us consider the steady state (k^*, c^*). As we saw that it is an hyperbolic point, small perturbations by a parameter will not change the local dynamic properties of the steady state, only its quantitative level. Therefore, we can perform a comparative dynamics exercise in its neighborhood.

Assume we start at a steady state and introduce a small change in ρ. As the steady state is a function of ϕ, this means that, after the change, the steady state will move and the initial point is not a steady state. That is we can see it as an arbitrary initial point out of the (new) steady state. From hyperbolicity, the new steady state is still a saddle point, which means that the small perturbation will generate unbounded orbits unless there is a "jump" to the new stable manifold associated to the new steady state. This is the intuition behind the comparative dynamics exercise in most perfect foresight macro models (see **?** and **?**) that we illustrate next. We basically assume that variable k is continuous in time (it is pre-determined) and that c is piecewise continuous in time (it is non-predetermined).

Formally, as we also saw that it is a function of the rate of time preference, let us introduce a permanent change in its value from ρ to $\rho + d\rho$. This will introduce a time-dependent change in the two variables, from (k^*, c^*) to $(k(t), c(t))$ where $k(t) = k^* + dk(t)$ and $c(t) = c^* + dc(t)$. In order to find $(dk(t), dc(t))$ we make a first-order Taylor expansion on (k, c) generated by $d\rho$ to get

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = D_{(k,c)}\mathbf{F}(k^*, c^*) \begin{pmatrix} dk(t) \\ dc(t) \end{pmatrix} + D_\rho \mathbf{F}(k^*, c^*) d\rho \tag{7.7}$$

where $D_\rho \mathbf{F}(k^*, c^*) = (0, -\eta c^*)^\top$. This is a linear planar non-homogeneous ODE.

From $\dot{k} = \dot{c} = 0$ we can find the long-run multipliers

$$\begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} = \begin{pmatrix} \frac{dk^*}{d\rho} \\ \frac{dc^*}{d\rho} \end{pmatrix} = -\Big(D_{(k,c)} \mathbf{F}(k^*, c^*) \Big)^{-1} D_\rho \mathbf{F}(k^*, c^*),$$

that is

$$\begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \eta c^* f''(k^*) & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \eta c^* \end{pmatrix}.$$

Then $\partial_\rho k^* = \dfrac{1}{f''(k^*)} < 0$ and $\partial_\rho c^* = \rho \partial_\rho k^* < 0$. A permanent unanticipated change in ρ will reduce the long run capital stock and consumption level.

We are only interested in the trajectories that converge to the new steady state after a perturbation, $k^* + \partial_\rho k^* d\rho$ and $c^* + \partial_\rho c^* d\rho$, that is a saddle point. In order to make sure this is the case, we solve the variational system for the saddle path to get

$$\begin{pmatrix} \partial_\rho k(t) \\ \partial_\rho c(t) \end{pmatrix} = \begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} + x \begin{pmatrix} 1 \\ \lambda_u^* \end{pmatrix} e^{\lambda_s^* t},$$

where x is a positive arbitrary element. If we assume that the variable k is pre-determined, that is it can only be changed in a continuous way from the initial steady state value k^*, we set $\partial_\rho k(0) = 0$. Then, from

$$\partial_\rho k(0) = \partial_\rho k^* + x = 0 \Rightarrow x = -\partial_\rho k^*$$

At last we obtain the **short-run multipliers**

$$\begin{aligned} \partial_\rho k(t) &= \frac{1}{f''(k^*)} (1 - e^{\lambda_s^* t}) \\ \partial_\rho c(t) &= \frac{1}{f''(k^*)} (\rho - \lambda_u \, e^{\lambda_s^* t}) \end{aligned}$$

for $t \in [0, \infty)$. In particular we get the impact multipliers, for $t = 0$

$$\begin{aligned} \partial_\rho k(0) &= 0 \\ \partial_\rho c(0) &= \partial_\rho k^* \lambda_s > 0 \end{aligned}$$

which quantify the "jump" to the new stable eigenspace, and the long-run multipliers

$$\begin{aligned} \lim_{t\to\infty} \partial_\rho k(t) &= \partial_\rho k^* < 0 \\ \lim_{t\to\infty} \partial_\rho c(t) &= \rho \partial_\rho k^* = \partial_\rho c^* < 0. \end{aligned}$$

Therefore, on impact consumption increases, which reduces capital accumulation, which reduces again consumption through time. The process stops because the reduction in the per-capita stock will increase marginal productivity which reduces the incentives for further reduction in consumption.

Observe also that we should have a "jump" to the stable manifold to have convergence towards the new steady state. As we have determined convergence to the steady state within the stable eigenspace of the variational system, the trajectory we have determined is qualitatively but not quantitatively exact.

7.2.2 The Skiba (1978) model

In this section we present a explicitly specified version of the Skiba (1978) model. The model has the same structure of the Ramsey model in which the technology is represented nit by a concave production function but by a concave-convex production function from subsection 7.1.2.

The MHDS is

$$\begin{aligned}\dot{k} &= f(k,B) - c - \delta\, k\\ \dot{c} &= \frac{c}{\sigma}\ \left(f_k(k,B) - \rho - \delta\, k\right)\end{aligned}$$

where the production function is

$$f(k,B) = A\, k^{\alpha}\ + \frac{k^3}{2\,(k^3+B^3)},$$

and the marginal return to capital is

$$f_k(k,B) = \alpha\, A\, k^{\alpha-1}\ \frac{3\, B^3\, k^2}{2\,(k^3+B^3)^4}\Big).$$

As in the Ramsey model, the non-zero steady state levels for the capital stock is

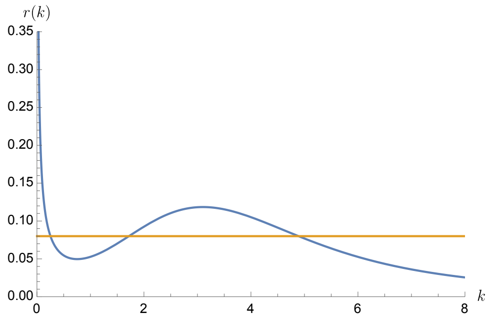
$$\bar{k}\ \in \Big\{\ k>0:\ f_k(k,B) = \rho+\delta\Big\}$$

For intermediate values of B we can have three steady states (see panel (a) of Figure **??**). If we compare to the non-optimizing version of the model from subsection (7.1.2) we can see that the steady state stock of capital is determined from an arbitrage condition and not from the equilibrium between savings and investment for a stationaary capital. This means that the steady state capital stock is determined in efficiency terms. The macroeconomic balance condition determines the steady state consumption level, instead.

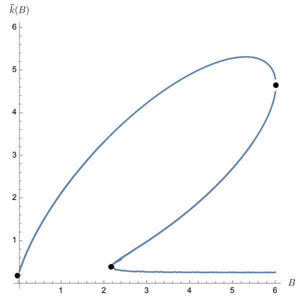
Panel (b) of Figure **??** displays the bifurcation diagram and panel (c) the phase diagram for intermediate values for B.

As for the "ad-hoc" model in section 7.1.2 there are again three steady states. However the two extreme steady states are now Ramsey-like saddle points and the intermediate case is an unstable focus.

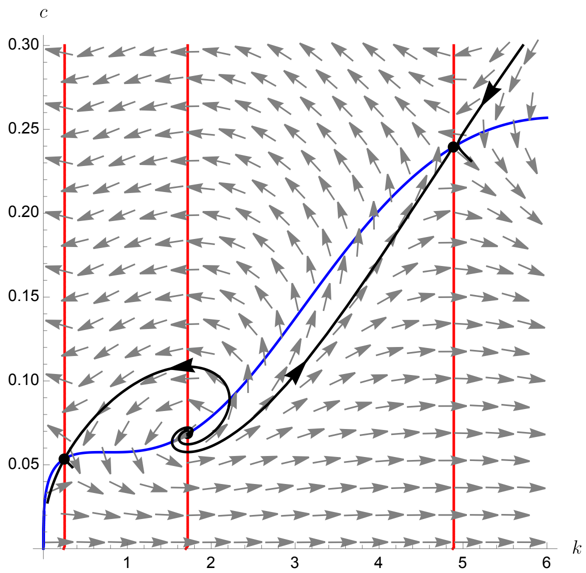
However, differently from the "ad-hoc" model the separation between the convergence to the lower and upper steady states is done through the stable manifolds passing through those steady states. Those two curves can be seen as heteroclinic orbits connecting the intermediate steady states to one of those two. This implies that given an initial level for the capital stock coincident with the intermediate steady states, there are three possible paths, depending on the initial level of consumption: first, if c is at the steady state level the capital stock will remain stationary, second, if it is lower there will be convergence to the lower steady state, and, third, if it is higher then there will be convergence to the higher steady state. There are two vertical curves, called DNS-curve (named after **?** and Skiba (1978)), which define an interval in which if the initial capital is within those boundaries it there are multiple optimal trajectories, but if it is outside that interval, then there will be convergence to only one steady state.



(a) Steady state condition $f_k(k, B) = \rho + \delta$



(b) Bifurcation diagram



(c) Phase diagram for intermediate B

Figure 7.5: Phase diagram for the Skiba (1978) model

7.2.3 The ?: bifurcations

Next we consider an extension of **?**. This is a non-linear endogenous growth model with externalities. In this case we will see that, when a steady state exists, it can have different types of local stability properties depending on the parameters.

We introduce two externalities a (flow) externality on preferences and a (stock) production externality.

Preference externalities are introduced by assuming that the utility function is $u = u(c, C)$, where c household's cosumption and C represents aggregate consumption. This means that the value of consuming c, at the household level, depends on the. aggregate consumption. This encompasses two cases: (i) the case in which C decreases utility, $u_{cC}(c, C) = \dfrac{\partial^2 u}{\partial c \partial C} < 0$, which is typical the case of function *going along with the Joneses* in which the utility depends on the difference between the household consumption and the aggregate; (ii) the case in which C decreases utility, $u_{cC}(c, C) = \dfrac{\partial^2 u}{\partial c \partial C} > 0$ where aggregate consumption is a public good C, in particular if C is directly or indirectly dependent of natural resources or the level of some infrastructures which can be accessed by households.

Technological externalities are introduced by assuming that the production function is $y = f(k, K)$ where k is the capital stock at the firm's level and K is the aggregate capital stock. We can assume that both the level of production and the marginal productivity of capital depend on the size of the economy which is measured by K. This also encompasses two cases: (i) the case in which K increases marginal productivity, $f_{kK}(k, K) = \dfrac{\partial^2 f}{\partial k \partial K} > 0$, in which agglomeration generates external economies to the firm with capital k, which is the case studied in **?**; (ii) the case in which K decreases marginal productivity, $f_{kK}(k, K) = \dfrac{\partial^2 f}{\partial k \partial K} < 0$ where the aggregate size of K generates congestion.

Furthermore we assume that households are homogeneous and have a mass equal to one.

Behavior of the representative household

The problem for the representative household is

$$\max_{c} \int_0^{+\infty} \frac{\left(cC^b\right)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\begin{aligned} \dot{k} &= A\, k^{\alpha}\, K^{\beta} - c \\ k(0) &= k_0, \\ \lim_{t \to \infty} k(t) e^{-\rho t} &\geq 0, \end{aligned}$$

where $\theta > 0$, $0 < \alpha < 1$ and b and β can be any real number.

Observe that both the utility function $u(\cdot)$ and the production function $f(\cdot)$ are increasing and concave as regards the households decision variables c and k. Furthermore, because

$$u_{cC} = b(1-\theta) c^{-\theta}\, C^{b(1-\theta)-1}, \text{and } f_{kK} = \alpha\, \beta\, k^{\alpha-1}\, K^{\beta-1},$$

if $b(1-\theta) > 0$ ($b(1-\theta) < 0$) there is a positive (negative) consumption externality, and if $\beta > 0$ ($\beta < 0$) there is a positive (negative) production externality.

The Hamiltonian, at the household level, is

$$H = \frac{c^{1-a} C^b}{1-a} + q\left(Ak^{\alpha} K^{\beta} - c\right),$$

where q is the co-state variable, and the necessary (and sufficient) first order conditions are

$$\begin{aligned} c^{-a} C^b &= q \\ \dot{q} &= q\left(\rho - \alpha A k^{\alpha-1} K^{\beta}\right) \\ \lim_{t \to +\infty} q(t)\, k(t)\, e^{-\rho t} &= 0 \end{aligned}$$

together with the constraints.

We can represent the solution by the system of ODE

$$\dot{k} = A\,k^{\alpha}\,K^{\beta} - c \tag{7.8a}$$

$$\dot{c} = \frac{c}{\theta}\Big(\alpha A k^{\alpha-1} K^{\beta} + b(1-\theta)\frac{\dot{C}}{C} - \rho\Big) \tag{7.8b}$$

General equilibrium

The general equilibrium is defined by the path $\big(C(t), K(t)\big)_{t\in\mathbb{R}_+}$ such that: (1) the representative consumer solves his problem, (2) there are micro-macro consistency conditions and (3) market clearing.

In this economy the micro-macro consistency conditions are $C = c$ e $K = k$. This means that, although agents do not decide on the macro variables, the level of the macro variables should be consistent with the aggregation of individual agents. As we assume agents are homogeneous then the macro variables should be equal to the household level variables if we assume that the mass of agents is equal to one.

Therefore the DGE is defined by the trajectories $\{c(t), k(t)\}_{t=0}^{+\infty}$ which solve the system

$$\dot{k} = A\,k^{\alpha+\beta} - c \tag{7.9a}$$

$$\dot{c} = \frac{c}{\theta + b\,(\theta-1)}\Big(\alpha A k^{\alpha+\beta-1} - \rho\Big) \tag{7.9b}$$

Steady state and local dynamics

Depending on the value of the parameters, we have two cases:

1. if $\alpha + \beta = 1$ there is not a steady state. In this case the solution is a balanced growth path

2. if $\alpha + \beta \neq 1$ there is only a steady state such that both variables are positive.

If $\alpha + \beta \neq 1$ then there is a unique steady state in $\mathbf{R}^2_{++}$

$$\begin{aligned} \bar{k} &= \Big(\frac{\alpha A}{\rho}\Big)^{\frac{1}{1-(\alpha+\beta)}} \\ \bar{c} &= A\bar{k}^{\alpha+\beta}. \end{aligned}$$

The Jacobian of system (**??**)-(**??**), evaluated at the steady state (again we are assuming that $\alpha + \beta \neq 1$) is

$$J := \begin{pmatrix} \dfrac{\rho\,(\alpha+\beta)}{\alpha} & -1 \\ \dfrac{\rho^2\,(\alpha+\beta-1)}{\alpha\,(\theta+b(\theta-1)} & 0 \end{pmatrix}$$

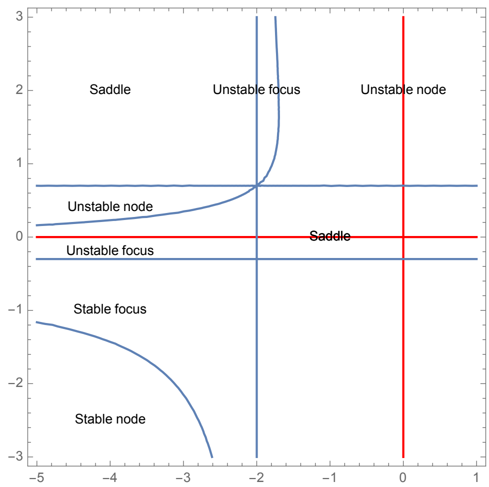


Figure 7.6: Bifurcation diagram in (b, β) for and $0 < \alpha < 1$, $\theta > 1$. The red lines separate negative from positive externalities at the consumption level (the vertical line) and at the production level (horizontal line).

which has trace, determinant

$$\text{trace}(J) = \frac{\rho(\alpha+\beta)}{\alpha}$$
$$\det(J) = \frac{\rho^2(\alpha+\beta-1)}{\alpha(\theta+b(\theta-1)}$$

Comparing to a Ramsey model we see that the trace is different from ρ and both the trace and the determinant can have any sign:

1. the trace can have any sign and is different, if $\beta \neq 0$ from the trace of the Jacobian in a Ramsey-like model: $\text{trace}(J) > 0$ if $\beta > -\alpha$ and $\text{trace}(J) \leq 0$ if $\beta > -\alpha$

2. the determninant can have any sign: in particular $\det(J) > 0$ if $\text{sign}(\alpha+\beta-1) = \text{sign}(\theta + b(\theta-1)$, $\det(J) < 0$ if $\text{sign}(\alpha+\beta-1) \neq \text{sign}(\theta+b(\theta-1)$ and $\det(J) = 0$ if $\alpha+\beta = 1$, which is a case in which there is a BGP.

Characterization of equilibrium dynamics

The following cases are possible (see Figure **??**):

Determinate equilibrium: this is associated to a steady state which is a saddle point, because there is a unique trajectory converging to the steady state. This case occurs if $b(\theta-1) > -\theta$ and $\beta < 1-\alpha$ or $b(\theta-1) < -\theta$ and $\beta > 1-\alpha$. This means that determinacy can exist with several types of combinations of externalities: (1) congestion in production, mild negative externalities in consumption or positive externalities in consumption; (2) positive externalities in production and negative externalities in consumption;

Indeterminate equilibrium: there is an infinite number of trajectories converging to the steady state if the steady state is a stable node or a stable focus. In this case expectations are self-fulfilling, in which, even when expectations are not coordinated the economy converges to the steady state. This case occurs only if $\alpha + \beta < 0$ and $b(\theta - 1) < -\theta < 0$, which means there should be congestion in production and negative externalities in consumption.

Balanced growth path: if $\alpha + \beta = 1$, this case can occur with or without externalities in consumption

Overdeterminate equilibrium: if the steady state is an unstable node or focus the equilibrium will be reduced to the steady state. Any deviation from it will violate the transversality condition and, therefore, cannot be an equilibrium (in the economic sense). This case would occur, for instance, with high positive externalities in both consumption and production.

The Ramsey model will be a particular case in which the two red lines meet.

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