Mathematical Economics
Introduction

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This part of the Math Econ course

- Deals with **dynamic optimization**
- Which is the **main** building block for modern macroeconomics, financial economics and growth economics.
- We will set up and solve problems in both discrete and continuous time
- And using three different methods

Options for the course

- We address the simplest problems, all having with **explicit solutions** (most dynamic optimization problems don't have closed form solutions)
- Some heuristic proofs are provided.
- Pre-requisites: elementary calculus, difference and differential equations
  I assume you will be able to solve scalar and planar linear difference and differential equations

Syllabus

1. Discrete time problems
   1. Optimal control problem: Pontriyagin maximum principle
   2. Optimal control problem with infinite horizon: dynamic programming
2. Continuous time
   1. Optimal control problem: Pontriyagin maximum principle
   2. Optimal control problem with infinite horizon: dynamic programming

Course material

- Slides, classnotes, and problem sets will be posted at `https://pmbbrito.github.io/cursos/master/em/em_m_2021.html`
- Corrections may be introduced until 18th December: check the date of the document.

References

- Discrete time:

  de la Fuente, A (2000), *Mathematical Methods and Models for Economists*, Cambridge
- Continuous time:

  Kamien, M. I. and N. L. Schwartz (1991), *Dynamic Optimization*, 2nd ed, Elsevier

  Chiang, A. (1992), *Elements of Dynamic Optimization*, McGraw-Hill
- Other references in my classnotes.

Discrete time dynamic optimization

Optimal control

- Consider:
  - the set of independent variables, v.g., time $\mathcal{T} \subseteq \mathbb{N}$
  $$\mathcal{T} = \{0, \ldots, T\}$$
  - the control variable $u : \mathcal{T} \to \mathbb{R}$, i.e, sequences
  $$u \equiv \{u_0, \ldots, u_t, \ldots, u_{T-1}\}$$
  - that controls the state variable $x : \mathcal{T} \to \mathbb{R}$, i.e, the sequences
  $$x \equiv \{x_0, \ldots, x_t, \ldots, x_T\}$$
- The problem is to **find the best sequences**
  $$u^* \equiv \{u_0^*, \ldots, u_t^*, \ldots, u_{T-1}^*\}$$
  and
  $$x^* \equiv \{x_0^*, \ldots, x_t^*, \ldots, x_T^*\}$$
  satisfying some **admissibility** and an **optimality** criterium.

Optimal control problem

- Given:
  - T: the terminal time
  - $\mathcal{X}$ the set of all **admissible** sequences $(x, u) = \{(x_t, u_t)\}_{t \in \mathcal{T}}$:
    - satisfying the **difference equation**

$$x_{t+1} = G(x_t, u_t, t),$$

    - plus **some initial or terminal conditions** (over x_0 and/or x_T)
  - and the **value functional**

$$J(u, x) \equiv \sum_{t=0}^{T-1} F(t, u_t, x_t)$$

where $F_t = F(t, u_t, x_t, t)$ is called the **objective function**

Optimal control problem

- **OC problem**: find the optimal sequences $u^* \equiv \{u_0^*, \ldots, u_T^*\}$ and $x^* \equiv \{x_0^*, \ldots, x_T^*\}$ that maximize $J(u, x)$.
- The **value** of the optimal sequence (u^*, x^*) is a number:

$$J^* \equiv J(x^*) = \max_u \{J(u, x) : \ (u, x) \in \mathcal{X}\}$$

- Dynamic versus static optimization:
  what makes the model dynamic is the fact that the control changes the **variation** of the state variable
- Intuition:
  there is an **intertemporal trade-off** between the value (cost) of the control (in $F(\cdot, u_t, \cdot)$) and the cost (benefit) of changing x (in $x_{t+1} = G(x_t, u_t, t)$)

Application: resource depletion

- Let:
  W_t = the level of the resource at time t (i.e, at the beginning of period t)
  C_t = consumption in period t
  T = horizon (terminal time)
- The problem:
  Find sequences $W \equiv \{W_t\}_{t=0}^{T}$ and $C \equiv \{C_t\}_{t=0}^{T-1}$ that solve the problem:

$$\max_{C} \sum_{t=0}^{T-1} \beta^t u(C_t)$$

subject to

$$\begin{cases} W_{t+1} = W_t - C_t, \quad t = 0, 1, \ldots, T-1 \\ W_0 \text{ given} \\ \text{other conditions} \end{cases}$$

Application: consumption-investment problem

- Let:
  $C_t =$ consumption in period t
  $u_t = u(C_t) =$ value of consumption in period t
  $A_t =$ net financial wealth at time t
  $Y_t =$ non-financial flow of income in period t
  $r =$ interest rate
- The problem:
  Find sequences $A \equiv \{A_t\}_{t=0}^{T}$ and $C \equiv \{C_t\}_{t=0}^{T-1}$ that solve the problem

$$\max_C \sum_{t=0}^{T-1} \beta^t u(C_t)$$

subject to

$$\begin{cases} A_{t+1} = Y_t + (1+r)A_t - C_t, \quad t = 0, 1, \dots, T-1 \\ A_0 \text{ given} \\ \text{other conditions} \end{cases}$$

Applications: firm's investment

- Let:
  $\pi_t = \pi(K_t, I_t)$ = firm's cash-flow in period t
  K_t = stock of capital at time t
  I_t = gross investment in period t
  r = interest rate (assumed to be constant)
  $R(T, X_T)$ = scrap value
- The problem:
  Find sequences $K \equiv \{K_t\}_{t=0}^{T}$ and $I \equiv \{I_t\}_{t=0}^{T-1}$ that solve the problem

$$\max_{I} \sum_{t=0}^{T-1} \left(\frac{1}{1+r}\right)^t \pi(K_t, I_t) + R(T, K_T)$$

subject to

$$\begin{cases} K_{t+1} = I_t - (1+\delta)K_t, \quad t = 0, 1, \dots, T-1 \\ K_0 \text{ given} \\ \text{other conditions} \end{cases}$$

Applications: economic growth models

- **Endogenous growth model**: Find sequences $K \equiv \{K_t\}_{t=0}^{\infty}$ and $C \equiv \{C_t\}_{t=0}^{\infty}$ that solve

$$\max_C \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t),\ K_{t+1} = (1+A)K_t - C_t,\ \text{other conditions} \right\}$$

$K_t =$ stock of physical capital, $C_t =$ consumption

- **Ramsey model**: find sequences $K \equiv \{K_t\}_{t=0}^{\infty}$ and $C \equiv \{C_t\}_{t=0}^{\infty}$ that solve

$$\max_C \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t),\ K_{t+1} = K_t + F(K_t) - C_t,\ \text{other conditions} \right\}$$

$K_t =$ stock of physical capital, $C_t =$ consumption

Continuous time dynamic optimization

Optimal control

- Consider:
  - the set $\mathcal{T} = [0, T]$ or $[0, \infty)$
  - a possibly discontinuous and differentiable function $u : \mathcal{T} \to \mathbb{R}$ that controls a continuous and differentiable function $x : \mathcal{T} \to \mathbb{R}^m$, $m \geq 1$
- The problem is to find **the best flows**

$$u^* = \Big(u^*(t) \Big)_{t \in \mathcal{T}}$$

and

$$x^* = \Big(x^*(t) \Big)_{t \in \mathcal{T}}$$

satisfying some **admissibility** and an **optimality** criterium

Optimal control

- Consider:
  - the set of independent variables, v.g., time $\mathcal{T} \subseteq \mathbb{R}$, in particular

$$\mathcal{T} = \{0, \dots, T\}$$

    $\mathcal{T} = [0, T]$ or $[0, \infty)$
  - the state variable which is a continuous and differentiable function $x : \mathcal{T} \to \mathbb{R}$ or, equivalently the **flow**

$$x = \Big(x(t)\Big)_{t \in \mathcal{T}}$$

  - the function $x : \mathcal{T} \to \mathbb{R}$ is continuous and differentiable, or, equivalently the flow

$$x = \Big(x(t)\Big)_{t \in \mathcal{T}}$$

- The problem is to **find the best flow**

$$x^* = \Big(x^*(t)\Big)_{t \in \mathcal{T}}$$

satisfying some **admissibility** and an **optimality** criterium

Optimal control problem

- Given:
  - T the terminal time if $\mathcal{T}$ is finite
  - the set $\mathcal{X}$ of trajectories $(x, u) = ((x(t), u(t))_{t \in \mathcal{T}}$
    - satisfying a ordinary differential equation

$$\dot{x} = G(t, u(t), x(t))$$

    - and one initial condition $x(0) = x_0$ and possibly one terminal condition

  the **value functional**

$$J(x, u) \equiv \int_0^T F(t, x(t), u(t))\, dt$$

Optimal control problem

- **CT OC problem**: find $u^* \equiv (u^*(t))_{t\in\mathcal{T}}$ and $x^* \equiv (x^*(t))_{t\in\mathcal{T}}$, belonging to $\mathcal{X}$, that maximize the functional $J(x, u)$
- The optimal value for the program is:

$$J^* \equiv J(x^*) = \max_u \{J(x, u) : \ (x, u) \in \mathcal{X}\}$$

- Dynamic optimization:
  what makes the model dynamic is the fact that the control u changes the **variation** of the state variable x
- Intuition:
  there is an **intertemporal trade-off** between the value (cost) of the control (In $F(\cdot, u(t))$) and the cost (benefit) on the instantaneous change of the level of x, ($\dot{x} = G(x(t), u(t), t)$.

Application: resource depletion

- Let:
  $C(t)$ = consumption at time t
  $u(t) = u(C(t))$ = value of consumption at time t
  $W(t)$ resource level at time t,
  $\dot{W}(t) = \frac{dW(t)}{dt}$ instantaneous change in $W(t)$
- The problem:
  Find the **flows** $W \equiv (W(t))_{t=0}^{T}$ and $C = \equiv (C(t))_{t=0}^{T}$ that solve the problem

$$\max_{C} \int_{t=0}^{T} u(C(t))\, e^{-\rho t} dt$$

subject to

$$\begin{cases} \dot{W}(t) = -C(t), & t \in [0, T] \\ W(0) \text{ given} \\ \text{other conditions} \end{cases}$$

Application: consumption-investment problem

- Let:
  $C(t)$ = consumption at time t
  $u(t) = u(C(t))$ = value of consumption at time t
  $A(t)$ = net financial wealth at time t $Y(t)$ = non-financial flow of income at time t
  r = interest rate
- The problem:
  Find the **flows** $A \equiv (A(t))_{t=0}^{T}$ and $C \equiv (C_t)_{t=0}^{T}$ that solve the problem

$$\max_{C} \int_{t=0}^{T} u(C(t))e^{-\rho t}dt$$

subject to

$$\begin{cases} \dot{A} = Y(t) + rA(t) - C(t), & t \in [0, T] \\ A(0) \text{ given} \\ \text{other conditions} \end{cases}$$

Application: firm's investment problem

- Let:
  $\pi(t) = \pi(K(t), I(t))$ = firm's cash-flow at time t
  $K(t)$ = stock of capital at time t
  $I(t)$ = gross investment at time t
  r = interest rate (assumed to be constant)
  $R(T, K(T))$ = scrap value
- The problem:
  Find the **flows** $K \equiv (K(t))_{t=0}^{T}$ and $I \equiv (I(t))_{t=0}^{T}$ that solve the problem

$$\max_{I} \int_{0}^{T} \pi(K(t), I(t)) e^{-rt} dt + R(T, K(T))$$

subject to

$$\begin{cases} \dot{K}(t) = I(t) - \delta K(t), & t \in [0, T] \\ \text{other conditions} \end{cases}$$

Applications: economic growth models

- **Simple endogenous growth model**: find flows $K \equiv (K(t))_{t=0}^{T}$ and $C \equiv (C_t)_{t=0}^{T}$ that solve

$$\max_{C} \left\{ \int_{t=0}^{\infty} u(C(t))\, e^{-\rho t} dt : \ \dot{K} = AK(t) - C(t) - \delta K(t), \text{ other conditions} \right\}$$

  $K(t)$ stock of physical capital, $C(t)$ consumption flow at time t

- **Ramsey model**: find flows $K \equiv (K(t))_{t=0}^{T}$ and $C \equiv (C_t)_{t=0}^{T}$ that solve

$$\max_{C} \left\{ \int_{t=0}^{\infty} u(C(t))\, e^{-\rho t} dt : \ \dot{K} = F(K(t)) - C(t) - \delta K(t), \right.$$
$$\left. \text{plus other conditions} \right\}$$