Advanced macroeconomics 2021-2022
Problem set: consumption, savings, and asset accumulation

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1 Intertemporal utility

[1]

1 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \log\big(c(t)\big)\, e^{-\rho t}\, dt,\ \rho > 0$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

2 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T -\frac{1}{\zeta}\, e^{-\zeta\, c(t)}\, e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0.$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

3 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \frac{c(t)^{1-\theta} - 1}{1-\theta}\, e^{-\rho t}\, dt,\ \rho > 0,\ \theta > 0.$$

[1]Questions are marked with asterisks depending on their degree of difficulty.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

4 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \left(c(t) - \frac{\beta}{2}c(t)^2\right) e^{-\rho t}\, dt,\ \rho > 0,\ \beta > 0.$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

5 Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T c(t)\, e^{-\rho t}\, dt,\ \rho > 0,\ \theta > 0.$$

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(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

8* Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \log\big(c(t)\big)\, e^{\int_0^t \log\big(c(s)\big)\,ds}\, dt,$$

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

9* Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \log\big(c(t) - \zeta h(t)\big)\, e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0,$$

where

$$\dot{h} = \eta\,(c - h)$$

and $h(0) = h_0$, where h_0 is given.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

10* Consider the following utility functional over a consumption path $c = \big(c(t)\big)_{t\in[0,T]}$,

$$\mathsf{U}[c] = \int_0^T \log\left(\frac{c(t)}{h(t)^{\zeta}}\right) e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0,$$

where

$$\dot{h} = \eta\,(c - h)$$

and $h(0) = h_0$, where h_0 is given.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

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(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

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$$\mathsf{U}[c] = \int_0^T \log\left(\frac{c(t)}{h(t)^{\zeta}}\right) e^{-\rho t}\, dt,\ \rho > 0,\ \zeta > 0,$$

where

$$\dot{h} = \eta\, c - \delta\, h,\ \text{for } 0 < \delta. < 1$$

and $h(0) = h_0$, where h_0 is given.

(a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.

(b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

2 Consumer problems

1 Assume that a consumer has utility functional

$$\mathsf{U}[c] = \int_0^T \left(c(t) - \frac{\beta}{2}c(t)^2\right) e^{-\rho t}\, dt,\ \rho > 0,\ \beta > 0.$$

and no constraints on consumer.

(a) Find the optimal consumption function

(b) Would the solution change if consumer had an initial wealth $w(0) = w_0$ and no further constraints on wealth ?

(c) Would the solution change if consumer had an initial wealth $w(0) = w_0 > 0$ and had a constraint on wealth such that $w(t) \geq 0$?

(d) Discuss the previous results.

2 Consider the problem

$$\max_c \int_0^T \log(c(t))\, e^{-\rho t} dt$$

subject to

$$\dot a(t) = r\, a - c(t), \text{ for } t \in \mathrm{T}$$

$$a(t) \in [\underline{\mathrm{a}}, \infty), \text{ for every } t \in [0, T]$$

$$a(0) = a_0 > \max\{0, \underline{\mathrm{a}}\} \text{ given}$$

(a) Find the optimality conditions.

(b) Find the solution to the problem. Under which conditions it is optimum to saturate the borrowing constraint at the terminal time T ?

(c) Provide an intuition for your results.

3** Consider the problem

$$\max_c \int_0^\infty \log(c(t))\, e^{-\rho t} dt$$

subject to

$$\dot a(t) = \begin{cases} r_1\, a + y - c(t), & \text{for } 0 \le t < t_s \\ r_2\, a + y - c(t), & \text{for } t_s \le t < \infty \end{cases}$$

$$a(0) = a_0 > \text{ given}$$

$$\lim_{t \to \infty} e^{-r_2\, t} a(t) \ge 0$$

where $y > 0$ and $r_1 > \rho > r_2$.

(a) Find the optimality conditions.

(b) Find the solution to the problem.

(c) Draw the phase diagram.

(d) Provide an intuition for your results.

3 Comparative dynamics

1 Consider the problem

$$\max_{c} \int_{0}^{\infty} \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt$$
$$\text{subject to}$$
$$\dot{a}(t) = (1-\tau)\big(r\, a + w\big) - c(t), \text{ for } t \in \mathbb{R}_{+}$$
$$a(0) = a_0 \text{ given}$$
$$\lim_{t\to\infty} a(t)\, e^{-r\,t} \geq 0$$

where $0 < \tau < 1$ is the income tax rate.

(a) Find the optimality conditions.

(b) Study the comparative dynamics effects for an anticipated, permanent and constant increase in the income tax rate τ.

(c) Provide an intuition for your results.

2 Consider the problem

$$\max_{c} \int_{0}^{\infty} -\frac{e^{-\xi c(t)}}{\xi}\, e^{-\rho t} dt, \; \xi > 0$$
$$\text{subject to}$$
$$\dot{a}(t) = (1-\tau)\big(r\, a + w\big) - c, \text{ for } t \in \mathbb{R}_{+}$$
$$a(0) = a_0 \text{ given}$$
$$\lim_{t\to\infty} a(t)\, e^{-r\,t} \geq 0,$$

where $0 < \tau < 1$ is the income tax rate.

(a) Find the optimality conditions.

(b) Study the comparative dynamics effects for an anticipated, permanent and constant increase in the income tax rate τ.

(c) Provide an intuition for your results.

4 Habit formation

1 Consider the problem

$$\max_c \int_0^\infty \log\big(c(t) - \zeta\, h(t)\big)\, e^{-\rho t} dt$$

subject to

$$\dot{a}(t) = r\, a + w - c(t), \text{ for } t \in \mathbb{R}_+$$

$$\dot{h}(t) = \eta\,\big(c - h\big) \text{ for } t \in \mathbb{R}_+$$

$$a(0) = a_0 \text{ given}$$

$$h(0) = h_0 \text{ given}$$

$$\lim_{t\to\infty} a(t)\, e^{-r\, t} \geq 0$$

for $\rho > 0$, $0 < \zeta < 1$ and $\eta > 0$.

(a) Find the first order conditions

(b) Under which conditions there will be transitional dynamics

(c) . Discuss the dynamics of consumption and income for a non-anticipated, permanent and constant increase in non-financial income w.

(d) Is consumption response perfectly correlated with income ? Why ?

2 Consider the problem

$$\max_c \int_0^\infty \log\big(c(t) h(t)^{-\zeta}\big)\, e^{-\rho t} dt$$

subject to

$$\dot{a}(t) = r\, a + w - c(t), \text{ for } t \in \mathbb{R}_+$$

$$\dot{h}(t) = \eta\,\big(c - h\big) \text{ for } t \in \mathbb{R}_+$$

$$a(0) = a_0 \text{ given}$$

$$h(0) = h_0 \text{ given}$$

$$\lim_{t\to\infty} a(t)\, e^{-r\, t} \geq 0$$

for $\rho > 0$, $0 < \zeta < 1$ and $\eta > 0$.

(a) Find the first order conditions

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(d) Is consumption response perfectly correlated with income ? Why ?