Foundations of Financial Economics
Choice under uncertainty

Paulo Brito

[1] pbrito@iseg.ulisboa.pt
University of Lisbon

March 20, 2020

Topics covered

- Contingent goods:
  - Definition
  - Comparing contingent goods
- Decision under risk:
  - von-Neumann-Morgenstern utility theory
  - Certainty equivalent
  - Attitudes towards risk: risk neutrality and risk aversion
  - Measures of risk
  - The HARA family of utility functions

Contingent goods

informal definition

Contingent goods (or claims or actions): are goods whose **outcomes are state-dependent**, meaning:

- the quantity of the good to be available is uncertain at the moment of decision (i.e, *ex-ante* we have **several odds**)
- the actual quantity to be received, the outcome, is revealed afterwards (*ex-post* we have **one realization**)
- **state-dependent**: means that nature chooses which outcome will occur (i.e., the outcome depends on a mechanism out of our control)

Contingent goods

Example: flipping a coin

lottery 1: flipping a coin with **state-dependent outcomes:**

- before flipping a coin the contingent outcome is

odds	head	tail
outcomes	100	0

- after flipping a coin there is only one realization: 0 or 100

lottery 2: flipping a coin with **state-independent outcomes:**

- before flipping a coin the non-contingent outcome is

odds	head	tail
outcomes	50	50

- after flipping a coin we always get: 50

Contingent goods

Example: tossing a dice

lottery 3: dice tossing with state-dependent outcomes:

- before tossing a dice the contingent outcome is

odds	1	2	3	4	5	6
outcomes	100	80	60	40	20	0

- after tossing the dice we will get: 100, or 80 or 60 or 40, or 20, or 0.

Comparing contingent goods

- Question: given two contingent goods (lotteries, investments, actions, contracts) how do we compare them ?
- Answer: we need to reduce to a **number** which we interpret as its **value**

$$\text{contingent good 1} \rightarrow \text{ Value of contingent good 1} = V_1$$
$$\text{contingent good 2} \rightarrow \text{ Value of contingent good 2} = V_2$$

$$\text{contingent good 1 is better that 2} \quad \Leftrightarrow \quad V_1 > V_2$$

Comparing contingent goods

Example: farmer's problem

Farmer's problem: which crop, vegetables or cereals ?

- before planting: the outcomes and the associated costs (known)nare

	income		cost	profit	
weather	rain	drought		rain	drought
vegetables	200	30	50	150	-20
cereals	10	100	20	-10	80

- after planting:
  - vegetables: the profit realization will be: -20 or 150
  - cereals: the profit realization will be: -10 or 80

Comparing contingent goods

Example: investor's problem

Investors's problem: to risk or not to risk ?

- before investing: contingent incomes and the cost are

	income if market is		cost	profit if market is	
market	bull	bear		bull	bear
equity	130	50	100	30	-50
bonds	98	105	100	-2	5

- after investing:
  - in equity: the profit realizations will be: -50 or 30
  - in bonds: profit realizations will be: 5 or -2

Comparing contingent goods

Examples: gambler's problem

Gambler's problem : to flip or not to flip a coin ?

- comparing one non-contingent with another contingent outcome
- Before flipping the coin the alternatives are

	outcomes		cost	profit	
odds	H	T		H	T
flipping	100	0	20	80	- 20
no flipping	50	50	45	5	5

- after flipping:
  - accepts coin flipping: gets 80 or -20
  - rejects coin flipping: gets 5 with certainty

Comparing contingent goods

Examples: insured's problem

Insurance problem: to insure or not to insure ?

- Before insuring, assuming that the coverage is 50%

	outcomes		cost	net income	
damage	no	yes		no	yes
insured	0	- 250	10	-10	- 240
uninsured	0	-500	0	0	-500

- after damage:
  - insured: net income is : -10 or -240
  - uninsured: net income is : 0 or -500

Comparing contingent goods

Examples: tax evasion

Tax dodger problem: to report or or not to report the true income ?

- An agent can evade taxes by reporting truthfully or not, the odds refer to existence of inspection by the taxman.

	income	evasion	tax	penalty		net income	
inspection				no	yes	no	yes
dodge	100	40	10	0	50	90	40
no dodge	100	0	30	0	0	70	70

- after inspection
  - tax dodger: net income will be : 90 or 40
  - tax compliant: net income is : 70 or 70

Comparing contingent goods

Gambler problem: different lottery profiles

- Until this point the states of nature for the alternatives were the same
- But we may want to compare alternatives with different event profiles
- **Example gambler's problem:** which lottery to choose

	income								cost
	coin		dice						
odds	head	tail	1	2	3	4	5	6	
lottery 1	100	0							20
lottery 2			100	80	60	40	20	0	30

Choosing among contingent goods

Characterization of the information environment

Main issues:

- what is the source of uncertainty:
  - objective (equal for all agents): risk
  - subjective (different among agents): uncertainty
- knowledge:
  - common: risk
  - asymmetric: information (moral hazard, adverse selection)
- nature of the odds:
  - precise: distribution over the odds
  - imprecise: ambiguity (distribution over a distribution of the odds)
- distribution of contingent outcomes:
  - known model
  - model uncertainty

Decision under risk

Notation:

- Ω space of states of nature

$$\Omega = \{\omega_1, \ldots, \omega_N\}$$

- $\mathbb{P}$ is an **objective** probability distribution over states of nature

$$\mathbb{P} = (\pi_1, \ldots, \pi_N)$$

  where $0 \leq \pi_s \leq 1$ and $\sum_{s=1}^{N} \pi_s = 1$
- X a **contingent good** with possible outcomes

$$X = (x_1, \ldots, x_s, \ldots x_N)$$

Decision under risk

Information environment

- **Information**:
  - we **know**: the probability space $(\Omega, \mathbb{P})$, and the outcomes for a contingent good X are common knowledge and are unique;
  - we **do not know**: which state of nature will materialize, that is what is the realization $X = x$ of X
- Question: what is the value of X ?

Expected utility theory

Assumptions

- **Assumptions**:
  - the **value of the contingent good** X, is measured by a utility functional
    $$\boxed{U(X) = \mathbb{E}[u(X)]}$$
    called expected utility function or **von-Neumann Morgenstern** utility functional
    (obs: a functional is a mapping vector $\rightarrow$ number)
  - the **Bernoulli** utility function $u(x_s)$ measures **the value of outcome** x_s
  - Expanding
    $$\begin{aligned}\mathbb{E}[u(X)] &= \sum_{s=1}^{N} \pi_s u(x_s)\\ &= \pi_1 u(x_1) + \dots + \pi_s u(x_s) + \ldots + \pi_N u(x_N)\end{aligned}$$
- Do not confuse: $U(X)$ value of one lottery with $u(x_s)$ value of one outcome

Expected utility theory

Properties

- **Properties of the expected utility function**
  - **state-independent** valuation of the outcomes: $u(x_s)$ **only** depends on the outcome x_s and **not** on the state of nature s (no
  - **linear in probabilities**: the utility of the contingent good $U(X)$ is a linear function of the probabilities
  - **information context**: $U(X)$ refers to choices in a context of risk because the odds are known and $\mathbb{P}$ are objective probabilities
  - **attitude towards risk**: is implicit in the shape of $u(.)$ (in particular in its concavity).

Expected utility theory

Comparing contingent goods

- Consider two contingent goods with outcomes

$$X = (x_1, \ldots, x_N), \;\; Y = (y_1, \ldots, y_N)$$

- we can rank them using the relationship

$$\boxed{X \text{ is prefered to } \; Y \Leftrightarrow \mathbb{E}[u(X)] > \mathbb{E}[u(Y)]}$$

that is $U(X) > U(Y) \Leftrightarrow \mathbb{E}[u(X)] > \mathbb{E}[u(Y)]$

$$\mathbb{E}[u(X)] > \mathbb{E}[u(Y)] \Leftrightarrow \sum_{s=1}^{N} \pi_s u(x_s) > \sum_{s=1}^{N} \pi_s u(y_s)$$

- There is **indifference** between X and Y if

$$\boxed{U(X) = U(Y) \Leftrightarrow \mathbb{E}[u(X)] = \mathbb{E}[u(Y)]}$$

Examples: coin flipping

- Odds: $\Omega = \{head, tail\}$
- Probabilities: $\mathbb{P} = \Big(P(\{head\}, P(\{tail\}\Big) = \left(\frac{1}{2}, \frac{1}{2}\right)$
- Outcomes: $X = (X(\{head\}, X(\{tail\}) = (60, 10)$
- Value of flipping a coin

$$U(X) = \frac{1}{2}u(60) + \frac{1}{2}u(10)$$

Expected utility theory

Comparing contingent goods

Examples: dice tossing

- Odds: $\Omega = \{1, \ldots, 6\}$
- Probabilities: $\mathbb{P} = \Big(P(\{1\}, \ldots, P(\{6\}\Big) = \Big(\frac{1}{6}, \ldots, \frac{1}{6}\Big)$
- Outcomes: $Y = \big(Y(\{1\}, \ldots, Y(\{6\}\big) = \big(10, 20, 30, 40, 50, 60\big)$
- Value of tossing a dice is

$$U(Y) = \frac{1}{6}u(10) + \frac{1}{6}u(20) + \ldots + \frac{1}{6}u(60)$$

- whether $U(X) \gtreqless U(Y)$ depends on the utility function

Expected utility theory

Comparing one contingent good with a non-contingent good

- given one contingent good $X = (x_1, \ldots, x_N)$ and one non-contingent good z,
- we can rank them using the relationship

$$X \text{ is prefered to } Z \Leftrightarrow U(X) \geq u(z)$$

- Obs: a non-contingent good is a particular contingent good such that $Z = (z, \ldots, z)$. In this case

$$U(X) = U(Z) \Leftrightarrow \mathbb{E}[u(X)] = \mathbb{E}[U(Z)] = \sum_{s=1}^{N} \pi_s u(z) = u(z)$$

  because $\sum_{s=1}^{N} \pi_s = 1$.
- There is indifference between X and z if

$$\boxed{\mathbb{E}[u(X)] = u(z)}$$

Expected utility theory

Certainty equivalent

Definition: certainty equivalent is the certain outcome, x^c, which has the same utility as a contingent good X

$$\boxed{x^c = u^{-1}\left(\mathbb{E}[u(X)]\right) = u^{-1}\left(\mathbb{E}\left[\sum_{s=1}^{N} \pi_s u(x_s)\right]\right)}$$

- Equivalently: given u and $\mathbb{P}$, CE is the certain outcome such that the consumer is indifferent between X and x^c

$$u(x^c) = \mathbb{E}[u(X)] \Leftrightarrow u(z) = \sum_{s=1}^{N} \pi_s u(x_s)$$

- **Example:** the certainty equivalent of flipping a coin is the outcome z such that

$$x^c = u^{-1}\left(\frac{1}{2}u(60) + \frac{1}{2}u(10)\right)$$

- **Definition**: for any contingent good, X, we say there is risk neutrality if the utility function $u(.)$ has the property

$$\boxed{\mathbb{E}[u(X)] = u(\mathbb{E}[X])}$$

Proposition: there is **risk neutrality** if and only if the utility function $u(.)$ **is linear**

$$\sum_s \pi_s u(x_s) = u(\sum_s p_s x_s)$$

Expected utility theory

Risk aversion

- **Definition**: for any contingent good, X, we say there is risk aversion if the utility function $u(.)$ has the property

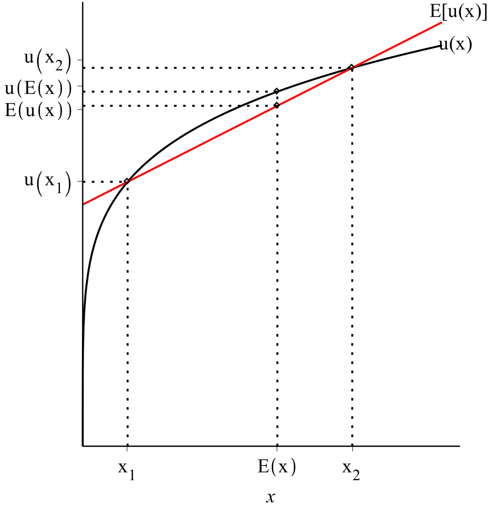
$$\mathbb{E}[u(X)] < u(\mathbb{E}[X])$$

> **Proposition**: there is **risk aversion** if and only if the utility function $u(.)$ **is concave**.

Proof: the Jensen inequality states that if $u(.)$ is strictly concave then

$$\mathbb{E}[u(X)] < u[E(X)] \Leftrightarrow \sum_{s=1}^{N} \pi_s u(x_s) < u\left(\sum_{j=1}^{N} x_s \pi_s\right).$$

Jensen's inequality and risk aversion $u(x)$



Expected utility theory

Risk neutrality, risk aversion and the certainty equivalent

- Using the certainty equivalent definition $u(x^c) = \mathbb{E}[u(X)]$ and if $\mathbb{E}[u(X)] \leq u\big(\mathbb{E}[X]\big)$ then (look at the Jensen inequality figure)

$$\mathbb{E}[X] = u^{-1}\left(u(\mathbb{E}[X])\right) \geq u^{-1}\left(\mathbb{E}[u(X)]\right)$$

  then
- There is **risk neutrality** if and only if

$$x^c = \mathbb{E}[X]$$

  the certainty equivalent is equal to the expected value of the outcome
- here is **risk neutrality** if and only if

$$x^c < \mathbb{E}[X]$$

  certainty equivalent is smaller than the expected value of the outcome

Expected utility theory
Risk premium

- **Risk premium** is defined by the difference between the expected value and the certainty equivalent

$$\mathcal{R}(X) = \mathbb{E}[X] - x^c$$

- Intuition: given the utility function, this is the value the agent is **willing to pay for not bearing risk**
- Therefore:
  - If there is risk neutrality then $\mathcal{R}(X) = 0$, the agent is not willing to pay nor to receive in order to bear risk
  - If there is risk aversion then $\mathcal{R}(X) > 0$, the agent is willing to pay to avoid bearing risk

Measures of risk

- **Risk and the shape of u:**
  if u is linear it represents risk neutrality
  if $u(.)$ is concave then it represents risk aversion
- **Arrow-Pratt measures of risk aversion**:
  1. coefficient of **absolute** risk aversion:
  $$\varrho_a \equiv -\frac{u^{''}(x)}{u^{'}(x)}$$
  2. coefficient of **relative** risk aversion
  $$\varrho_r \equiv -\frac{xu^{''}(x)}{u^{'}(x)}$$
  3. coefficient of **prudence**
  $$\varrho_p \equiv -\frac{xu^{'''}(x)}{u^{''}(x)}$$

HARA family of utility functions

- Meaning: hyperbolic absolute risk aversion

$$\boxed{u(x) = \frac{\gamma - 1}{\gamma}\left(\frac{\alpha x}{\gamma - 1} + \beta\right)^{\gamma}} \tag{1}$$

- Cases: (prove this)
  1. linear: if $\beta = 0$ and $\gamma = 1$

$$u(x) = ax$$

     properties: risk neutrality
  2. quadratic : if $\gamma = 2$

$$u(x) = ax - \frac{b}{2}x^2, \text{ for } x < \frac{2a}{b}$$

     properties: risk aversion, has a satiation point $x = \frac{2a}{b}$

HARA family of utility functions

1. CARA: if $\gamma \to \infty$, (note that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$)

$$u(x) = -\frac{e^{-\lambda x}}{\lambda}$$

   properties: constant absolute risk aversion (CARA), variable relative risk aversion, scale-dependent

2. CRRA: if $\gamma = 1-\theta$ and $\beta = 0$

$$u(x) = \begin{cases} \ln(x) & \text{if } \theta = 1 \\ \dfrac{x^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \end{cases}$$

   (if $\theta = 1$ note that $\lim_{n\to 0}\frac{x^n-1}{n} = \ln(x)$)
   properties: constant relative risk aversion (CRRA); scale-independent

Comparing contingent goods

Coin flipping vs dice tossing

- Take our previous case:

$$U(X) = \frac{1}{2}u(60) + \frac{1}{2}u(10)$$

  or

$$U(Y) = \frac{1}{6}u(10) + \frac{1}{6}u(20) + \frac{1}{6}u(30) + \frac{1}{6}u(40) + \frac{1}{6}u(50) + \frac{1}{6}u(60)$$

- We will rank them assuming
  1. a linear utility function $u(x) = x$
  2. a logarithmic utility function $u(x) = \ln(x)$
- Observe that the two contingent goods have the same expected value

$$\mathbb{E}[X] = 35 \ \mathbb{E}[Y] = 35$$

Comparing contingent goods

Coin flipping vs dice tossing: linear utility

- If $u(x) = x$
  - $U(X) = \mathbb{E}[u(x)] = \frac{1}{2}60 + \frac{1}{2}10 = 35$
  - $U(Y) = \mathbb{E}[u(y)] = \frac{1}{6}10 + \ldots + \frac{1}{6}60 = 35$
- Then there is risk neutrality

$$\mathbb{E}[u(x)] = \mathbb{E}[X] = 35, \ \mathbb{E}[u(y)] = \mathbb{E}[Y] = 35$$

- and we are indifferent between the two lotteries because $\mathbb{E}[X] = \mathbb{E}[Y]$

Comparing contingent goods

Coin flipping vs dice tossing: log utility

- If $u(x) = \ln(x)$
  - $U(X) = \frac{1}{2}\ln(60) + \frac{1}{2}\ln(10) \approx 3.20$ and
    $u(\mathbb{E}[X]) = \ln(\mathbb{E}[X]) = \ln(35) \approx 3.56$,
    $x_X^c \approx 24.5$ (certainty equivalent)
  - $U(Y) = \frac{1}{6}\ln(10) + \ldots + \frac{1}{6}\ln(60) \approx 3.40$ and
    $u(\mathbb{E}[Y]) = \ln(\mathbb{E}[Y]) \approx 3.56$
    $x_Y^c \approx 29.9$ (certainty equivalent)
- there is risk aversion: $x_X^c < \mathbb{E}[X]$ and $x_Y^c < \mathbb{E}[Y]$ and the certainty equivalents are smaller than the
- as $U(X) < U(Y)$ (or $x_X^c < x_Y^c$) we see that Y is better than X

Choosing among contingent and non-contingent goods with log-utility

The problem

Assumptions

- **contingent good**: has the possible outcomes $Y = (y_1, \ldots, y_N)$ with probabilities $\pi = (\pi_1, \ldots, \pi_N)$
- **non-contingent good**: has the payoff $\bar{y}$ where $\bar{y} = \mathbb{E}[Y] = \sum_{s=1}^{N} \pi_s y_s$ with probability 1
- **utility**: the agent has a vNM utility functional with a logarithmic Bernoulli utility function.

Would it be better if he received the certain amount or the contingent good ?

Choosing among contingent and non-contingent goods with log-utility

The solution

1. the value for the non-contingent payoff z is

$$\ln(\bar{y}) = \ln(\mathbb{E}[Y]) = \ln\left(\sum_{s=1}^{N} \pi_s y_s\right)$$

   has the certainty equivalent

$$e^{\ln(\mathbb{E}[Y])} = \mathbb{E}[Y]$$

2. the value for the contingent payoff y is

$$U(Y) = \sum_{s=1}^{N} \pi_s \ln(y_s) = \mathbb{E}[\ln Y] = \ln(G\mathbb{E}[Y])$$

   where $G\mathbb{E}[Y] = \prod_{s=1}^{N} y_s^{\pi_s}$ is the geometric mean of Y

3. the certainty equivalent is

$$e^{\ln(G\mathbb{E}[Y])} = G\mathbb{E}[Y])$$

Choosing among contingent and non-contingent goods with log-utility

The solution: cont

- Because the arithmetical average is larger than the geometrical

  $$\mathbb{E}[Y] > G\mathbb{E}[Y]$$

  then he would be better off if he received the average endowment rather than the certainty equivalent

- The risk premium will be

  $$\mathcal{R}(Y) = \mathbb{E}[Y] - G\mathbb{E}[Y] > 0$$

Application: the value of insurance

The problem

- Let there be two states of nature $\Omega = \{L, H\}$ with probabilities $\mathbb{P} = (p, 1-p)$ $0 \leq p \leq 1$
- consider the outcomes
  - without insurance

$$X = (x_L, x_H) = (x - L, x)$$

    where $L > 0$ is a potential damage and there is full coverage
  - with full insurance : $y_L = y_H = y$

$$Y = (y, y) = (x - L + L - qL, x - qL) = (x - qL, x - qL)$$

    where q is the cost of the insurance
- Given L under which conditions we would prefer to be insured ?

The value of insurance

The solution

- It is better to be insured if

$$u(y) \geq \mathbb{E}[u(X)]$$

- that is if

$$u(x - qL) \geq pu(x - L) + (1 - p)u(x)$$

The value of insurance

The solution

It is better to be insured

- if $u(.)$ is **linear** then it is better to insure if

$$x - qL \geq p(x-L) + (1-p)x \Leftrightarrow p \geq q$$

  if the cost to insure is lower than the probability of occurring the damage

- if $u(.)$ is **concave** $x - qL$ should be higher than the certainty equivalent of X

$$x - qL \geq v\left(pu(x-L) + (1-p)u(x)\right) \;\; v(.) \equiv u^{-1}(.)$$

  equivalently

$$q \leq \frac{x - v\left(pu(x-L) + (1-p)u(x)\right)}{L}$$

Interpersonal comparison of risk attitudes

- Consider:
  - two agents A and B with different utility functions $u^A(y)$ and $u^B(y)$ and the same infomartion sets
  - and a single contingent income $Y = \big(y_1, \dots y_n\big)$
- Agent A is more risk averse than agent B if
  - her/his utility valuation is lower $U^A(Y) < U^B(Y)$ that is $\mathbb{E}[u^A(Y)] < \mathbb{E}[u^B(Y)]$
  - her/his certainty equivalent is smaller $y^{c,A} < y^{c,B}$
  - her/his risk premium for Y is higher $\mathcal{R}^A(Y) > \mathcal{R}^B(Y)$

References

- (LeRoy and Werner, 2014, Part III), (Lengwiler, 2004, ch. 2), (Altug and Labadie, 2008, ch. 3)

Sumru Altug and Pamela Labadie. *Asset pricing for dynamic economies*. Cambridge University Press, 2008.

Yvan Lengwiler. *Microfoundations of Financial Economics*. Princeton Series in Finance. Princeton University Press, 2004.

Stephen F. LeRoy and Jan Werner. *Principles of Financial Economics*. Cambridge University Press, Cambridge and New York, second edition, 2014.