Foundations of Financial Economics
Deterministic two-period GE asset pricing

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Functions of finance

- There are several of finance:
  - intertemporal allocation of resources which may or may not be time-dependent (consumption smoothing)
  - inter-state of nature allocation of resources which are uncertain (insurance)
  - financing investment (increase in the resource capacity)
  - matching timing profiles of expenditures and incomes of different agents
  - matching uncertainty profiles of different agents
  - information revelation and pooling
  - distribution of income and wealth
- in this course **we will be mainly concerned with the first two functions**
- in this lecture we deal only with the first in the most simple framework: **two period perfect information models**

Topics covered

- Interest rates, asset pricing and the intertemporal allocation problem under perfect information
- Under several economic environments, defined by
  - Fundamentals: preferences and technology
  - Market structures: Arrow-Debreu securities, financial assets

Syllabus

- Intertemporal consumption preferences
- General equilibrium in a representative agent Arrow-Debreu economy
- General equilibrium in an heterogeneous Arrow-Debreu economy
- General equilibrium in a frictionless finance economy
- General equilibrium in a finance economy with frictions: heterogeneous market participation

- We index variables by time. In the simples case, we have $\mathbb{T} = \{0, 1\}$
- We consider the sequences $\{c_0, c_1\}$ where c_t, for $t = 0, 1$ is consumption in **period** t
- Sequences $\{c_0, c_1\}$ are **ranked** by means of an **intertemporal utility functional** ,

$$U(\{c_0, c_1\}),$$

- The **optimum** is a sequence for which u **is maximum**

Intertemporal choice

Intertemporal utility function

- In this simple model, instead of dealing with **sequences** $\{c_0, c_1\}$ we consider as a **vector of real non-negative numbers** $\mathbf{c} = (c_0, c_1) \in \mathbb{R}^2_+$
- Therefore the **intertemporal utility function** (IUF) can be seen as a mapping $U : \mathbb{R}^2_+ \to \mathbb{R}$,

$$u = U(c_0, c_1)$$

  where u is a number allowing to rank vectors
- Behavioral assumptions can be imposed on the structure of $U(\cdot)$
- General assumption: the intertemporal utility function is **continuous** and **differentiable**

Intertemporal utility

Main properties

- **Positive marginal utility**: increase in consumption in any period increases utility

$$U_0 \equiv \frac{\partial U(c_0, c_1)}{\partial c_0} > 0, \;\; U_1 \equiv \frac{\partial U(c_0, c_1)}{\partial c_1} > 0$$

- **Stationary**: the temporal utility functions are independent of time (but there can be discounting)
- **Impatience**: preference for consumption today rather than in the future
- **Intertemporal properties**: let

$$U(c_0, c_1) = V(u(c_0), v(c_0, c_1)))$$

  V is called the Koopmans aggregator
  - intertemporal independence if $v_{c_0} = 0$
  - intertemporal substitution if $v_{c_0} < 0$
  - intertemporal complementarity (addiction) if $v_{c_0} > 0$
- How can we identify those properties ?

Intertemporal utility

Intertemporal marginal rate of subsrtitution

First, we use the substitution concepts in an intertemporal perspective:

- the **intertemporal marginal rate of substitution** is defined as

$$IMRS_{0,1}(c_0, c_1) = -\frac{dc_1}{dc_0}$$

- **Intuition**: how much we are willing to sacrifice consumption at $t = 1$ (tomorrow) in order to increase une unit of consumption at $t = 0$ (today)
- Taking total differentials to the utility function $U(c_0, c_1)$ such that $dU = 0$

$$U_0(c_0, c_1)dc_0 + U_1(c_0, c_1)dc_1 = 0$$

then

$$\boxed{IMRS_{0,1}(c_0, c_1) = \frac{U_0(c_0, c_1)}{U_1(c_0, c_1)}\bigg|_{U=\text{constant}}}$$

Intertemporal utility

Intertemporal elasticity of subsrtitution

- **Intertemporal elasticity of substitution**

$$EIS_{0,1}(c_0, c_1) = \frac{d \ln(c_1/c_0)}{d \ln IMRS_{0,1}(c_0, c_1)}$$
$$= \frac{c_0\, U_0 + c_1\, U_1}{c_1\, U_1 \varepsilon_{00} - 2 c_0\, U_0 \varepsilon_{01} + c_0\, U_0 \varepsilon_{11}}$$

where $\varepsilon_{ij} = -\dfrac{U_{ij} c_j}{U_i}$ for $i = 0, 1$ and $U_{ij} = \dfrac{\partial^2 U}{\partial c_i c_j}$

- **Intuition**: how much does the rate of growth of the ratio c_1/c_0 changes for a one percent increase in the $IMRS$. This provides a scale-free measure of the preferences regarding the behavioral assumptions concerning the intertemporal allocation of consumption

Intertemporal utility

Impatience and intertemporal complementarity

- Second, assume we start from a stationary consumption process, i.e, $c_0 = c_1 = c$ a constant
- We measure **impatience** by the relative change in consumption at period 1 relative to period 0. We say the **IUF displays impatience** if

$$IMRS_{0,1}(c) = \frac{U_0(c,c)}{U_1(c,c)} > 1$$

  This means that the reduction in consumption in period $t = 1$ should be bigger than the increase in consumption in period $t = 0$, $-(c_1 - c) > c_0 - c$, to keep utility constant. This means that **consumption at $t = 0$ has more value than consumption at $t = 1$**
- **Intertemporal dependence** can be determined by the Allen-Uzawa elasticity ε_{01}.

$$\varepsilon_{0,1}(c) = -\frac{U_{01}(c)\,c}{U_0(c)} \begin{cases} > 0, & \text{intertemporal substitutability} \\ = 0, & \text{intertemporal independence} \\ < 0 & \text{intertemporal complementarity} \end{cases}$$

Intertemporal utility

Example: additive IUF

- **Assumption 1**: the IUF is Intertemporally additive

$$U(c_0, c_1) = u(c_0) + \beta u(c_1), \text{ where } \beta \equiv \frac{1}{1+\rho}$$

  where $\beta \in (0,1)$ is the psychological discount factor and ρ is the rate of time preference
- **Assumption 2**: u is increasing and concave $u^{''}(c_t) < 0 < u^{'}(c_t)$, $t = 0, 1$
- Period marginal utilities depend only on the consumption of the same period: intertemporally independence

Intertemporal utility

Case 1: additive IUF, cont

- Marginal utilities for c_t, $t = 0, 1$ are
$$U_0 = u^{'}(c_0), \;\; U_1 = \beta u^{'}(c_1)$$
- Derivatives of marginal utilities for c_t, $t = 0, 1$ are
$$U_{00} = u^{''}(c_0), \;\; U_{01} = 0, U_{11} = \beta u^{''}(c_1)$$
- The *IMRS* is
$$IMRS_{0,1} = \frac{U_0}{U_1} = \frac{u^{'}(c_0)}{\beta u^{'}(c_1)}$$
Therefore: marginal utility for period $t = 0$ is proportional to the discounted marginal utility for period $t = 1$ (from the perspective of period $t = 0$)
$$u^{'}(c_0) = \beta u^{'}(c_1)\, IMRS_{0,1}$$
we will see an analogous equation again and again translating the idea of intertemporal arbitrage.

Intertemporal utility

Case 1: additive IUF, cont

- The Allen-Uzawa elasticities are

$$\varepsilon_{00}(c_0) = -\frac{u''(c_0)c_0}{u'(c_0)},\ \varepsilon_{01} = 0,\ \varepsilon_{11}(c_1) = -\frac{u''(c_1)c_1}{u'(c_1)}$$

- The elasticity of intertemporal substitution between period 0 and 1 is

$$EIS_{0,1}(c_0, c_1) = \frac{c_0 u'(c_0) + \beta c_1 u'(c_1)}{\beta c_1 u'(c_1)\varepsilon_{00}(c_0) + c_0 u'(c_0)\varepsilon_{11}(c_1)}$$

Intertemporal utility

Case 1: additive IUF, cont

For a stationary consumption path $\{c, c\}$ we find:

- The IMRS is independent from c and

$$IMRS_{0,1}(c) = \frac{1}{\beta} = 1 + \rho > 1$$

  this means that the IUF displays **impatience**, and this effect is captured by time discounting

- It displays **intertemporal dependence** because

$$\varepsilon_{0,1}(c) = 0$$

- The IES is

$$IES_{0,1}(c) = -\frac{u'(c)}{u''(c)c} > 0$$

Intertemporal utility

Case 1: example

- Utility function (generalized logarithm)

$$u(c) = \frac{c^{1-\zeta} - 1}{1-\zeta}$$

- if $\zeta = 1$ we have $u(c) = \ln(c)$ (Prove this)
- Derivatives

$$U_0 = c_0^{-\zeta},\ U_1 = \beta\, c_1^{-\zeta},\ U_{00} = -\zeta\, c_0^{-\zeta-1},\ U_{01} = 0,\ U_{11} = -\zeta\, c_1^{-\zeta-1}$$

- The IMRS is

$$IMRS_{0,1} = \frac{1}{\beta}\left(\frac{c_1}{c_0}\right)^{\zeta}$$

- The UA elasticities are constant $\varepsilon_{00} = \varepsilon_{11} = \zeta$
- The IES is also constant

$$EIS_{0,1} = \frac{1}{\zeta}$$

This is why it is usually to call ζ **the inverse of the elasticity of intertemporal substitution** .

Intertemporal utility

Non additive IUF

- Case 2: The **Uzawa and Epstein-Hynes** case

$$U(c_0, c_1) = u(c_0) + b(c_0)v(c_1)$$

the discount factor is endogenous i.e. $\beta = b(c)$ with $b^{'}(.) < 0$ (rich people are more patient)

The crossed AU elasticity is for a stationary sequence is

$$\varepsilon_{0,1}(c) = -\frac{b^{'}(c)v^{'}(c)c}{u^{'}(c) + b^{'}(c)v(c)}$$

displays intertemporal dependence

- Case 3: **Habit formation**

$$U(c_0, c_1) = u(c_0) + \beta v(c_0, c_1).$$

where $v_{c_0}(c_0, c_1) < 0$.

The crossed AU elasticity is for a stationary sequence is

$$\varepsilon_{0,1}(c) = -\frac{\beta v_{c_0 c_1}(c) c}{u'(c) + \beta v_{c_0}(c)} c$$

can display intertemporal substitutability, independence or complementarity depending on the relationship between time discounting and the relative importance of habits, i.e., the magnitude of $v_{c_0}(c)$

Intertemporal utility

Case 3: habit formation example

- IUF
$$U(c_0, c_1) = \ln(c_0) + \beta \ln\left(\frac{c_1}{c_0}\right)^{\zeta},\ \zeta > 0$$

- Derivatives
$$U_0 = \frac{1-\beta\zeta}{c_0},\ U_1 = \frac{\beta\zeta}{c_1},\ U_{00} = -\frac{1-\beta}{c_0^2},\ U_{01} = 0,\ U_{11} = -\frac{\beta\zeta}{c_1^2}$$

- The IMRS is
$$IMRS_{0,1}(c_0, c_1) = \left(\frac{1-\beta\zeta}{\beta\zeta}\right)\frac{c_1}{c_0}$$

- The UA elasticities are constant
$$\varepsilon_{00} = \varepsilon_{11} = 1,\ \varepsilon_{01} = 0$$

- The IES is also constant
$$EIS_{0,1}(c_0, c_1) = 1$$

  for any (c_0, c_1)

Intertemporal utility

Case 2: habit formation example, cont

For a stationary sequence $c_0 = c_1 = c$

- The IMRS

$$IMRS_{0,1}(c) = \frac{1-\beta\zeta}{\beta\zeta}$$

  the utility displays impatience if $\zeta < \frac{1}{2\beta} = \frac{1+\rho}{2}$. Intuition: there is impatience (according to the above definition) if the weight of past consumption is not too strong

- As $\varepsilon_{01} = 0 = 0$ the model displays intertemporal independence (but this is special to this example).

Two-period general equilibrium models

- Next we will address the determination of the interest rate in two-period general equilibrium models under perfect information (i.e., certainty)
- We consider two (equivalent) approaches and models
  - a micro-economic approach: Arrow-Debreu simultaneous equilibrium economy
  - a finance (or macro-finance) approach:a finance sequencial equilibrium economy
- For each model we proceed in two steps:
  - present and solve the consumer problem in each economy
  - we define and determine the general equilibrium

Arrow Debreu model

The consumer problem: set-up

- A consumer has an asset (resource, endowment) in positive amount ($w > 0$) which allows for a sequence of consumption in two periods, $\{c_0, c_1\}$, today c_0 and in the future c_1.
- There is a market for **forward contracts** allowing for contracting today for delivery in the future, at a price set today, $q > 0$. We take the price paid today as a *numéraire* and all the variables are denominated at todays' price
- The value of the consumption sequence is assessed by an intertemporal utility function: $U(c_0, c_1)$;
- The budget constraint, referring to payments made today, is

$$c_0 + qc_1 \leq w$$

Arrow Debreu model

The consumer problem: ptimality conditions

- Formally, the intertemporal problem for the consumer is

$$v(w) = \max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + q\, c_1 \leq w \}$$

- The (interior) optimum (c_0^*, c_1^*) satisfies the conditions

$$\begin{cases} qU_0(c_0^*, c_1^*) = U_1(c_0^*, c_1^*) & \text{(optimality condition)} \\ c_0^* + pc_1^* = w & \text{(budget constraint)} \end{cases}$$

Arrow Debreu model

Optimality: interpretation

- At the optimum: the IMRS is equal to the relative price (internal = external valuation)

$$IMRS^*_{0,1} = IMRS_{0,1}(c^*_0, c^*_1) = \frac{U_0(c^*_0, c^*_1)}{U_1(c^*_0, c^*_1)} = \frac{1}{q}$$

- Intuition: at the optimum increasing one euro of consumption tomorrow should be matched by a reduction in $1/q$ euro of consumption today, ie $dc^*_0 = -qdc^*_1$
- Therefore q is an **intertemporal relative price**: i.e., is an opportunity cost for changing the sequence of consumption through time.

Arrow Debreu model

Assumptions

- Now, we go from a microeconomic to a macroeconomic perspective

H1 Assume there is perfect information: **deterministic general equilibrium**

H2 Assume all agents are equal: **representative agent economy**

H3 Assume that there is an exogenous sequence of resources sustaining consumption: **endowment economy**

H4 Assume that all trade is done at time $t = 0$: an **Arrow-Debreu economy**

- We want to determine (endogenously) the price q: the **Arrow-Debreu** price

Arrow Debreu model

The setup

- Assume that the resource of the economy takes the form of a **flow** of non-durable goods, that can be collected both at time $t = 0$ and $t = 1$, $\{y_0, y_1\}$.
- Again, assume that trade can only take place at time $t = 0$, this means that the price for contracts for delivery at time $t = 1$ has to be set at time $t = 0$. We call q the **Arrow-Debreu price**
- From this, wealth at time $t = 0$ is equal to the **present value of the flow of endowments**

$$w = y_0 + qy_1$$

Arrow Debreu model

The setup: continuation

- All the participants have **perfect information** on prices and endowments referring to period $t = 1$ and solve a problem similar to our previous micro-economic problem;
- At every period, total consumption must be equal to total endowment;
- **Representative agent economy**: we assume that all consumers solve the same problem (same utility function and same endowments);
- What is the equilibrium forward price q ?

Arrow Debreu model

General equilibrium for a representative agent economy

General equilibrium in this economy is defined by (c_0^*, c_1^*, q^*) such that

- the consumer solves the problem

$$\max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + q\, c_1 \leq y_0 + q y_1 \}$$

- markets clear

$$c_0 = y_0,$$
$$c_1 = y_1$$

Arrow Debreu model

General equilibrium for a representative agent economy

- General equilibrium conditions: (c_0, c_1, q) is determined from

$$\begin{cases} qU_0(c_0, c_1) = U_1(c_0, c_1) & \text{(micro:optimality condition)} \\ c_0 + qc_1 = y_0 + qy_1 & \text{(micro:budget constraint)} \\ c_0 = y_0 & \text{(aggregate: market clearing for } t = 0\text{)} \\ c_1 = y_1 & \text{(aggregate: market clearing for } t = 1\text{)} \end{cases}$$

- There are only three independent conditions (Walras's law)

$$\begin{cases} qU_0(c_0, c_1) = U_1(c_0, c_1) \\ c_0^* = y_0 \\ c_1^* = y_1 \end{cases}$$

- In a representative agent economy there is **no trade** (consumption is equal to the endowment)

Arrow Debreu model

Equilibrium AD price

- Then **the equilibrium AD price** is

$$\boxed{q^* = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)}}$$

- We call $m = IMRS_{0,1}$ the **discount factor**
- Equivalently: the **(deterministic) equilibrium discount factor** is

$$\boxed{m^*(y_0, y_1) = \frac{1}{q^*} = \frac{U_0(y_0, y_1)}{U_1(y_0, y_1)}}$$

- The AD price (discounf factor) depends on the present and future endowments
- **We need more structure on preferences to get explicit results**

Arrow Debreu model

AD price and utility functions

- For an intertemporally additive utility function

$$\boxed{q^*(y_0, y_1) = \beta \frac{u'(y_1)}{u'(y_0)}}$$

- concavity of $u(.)$, i.e., $u''(c) < 0$, implies

$$\frac{\partial q^*(y_0, y_1)}{\partial y_0} = -\beta \frac{u'(y_1) u''(y_0)}{(u'(y_0))^2} > 0$$

and

$$\frac{\partial q^*(y_0, y_1)}{\partial y_1} = \beta \frac{u''(y_1)}{u'(y_0)} < 0$$

- The discount factor $m(y_0, y_1)$ decreases (increases) with y_0 (y_1)

Arrow Debreu model

AD price and utility functions

- Example: if $u(c) = \ln(c)$ then

$$\boxed{q^*(y_0, y_1) = \beta \frac{y_0}{y_1} = \frac{\beta}{1+\gamma}}$$

or, if we set $y_1 = (1+\gamma)y_0$ where γ is the rate of growth

- For the habit formation utility function

$$q^*(y_0, y_1) = \beta \frac{v_{c_1}(y_0, y_1)}{u'(y_0) + \beta v_{c_0}(y_0, y_1)}$$

- Example: setting $U(c_0, c_1) = \ln(c_0) + \beta \ln \left[\left(\frac{c_1}{c_0} \right)^{\zeta} \right]$ displaying intertemporal substitution then

$$\boxed{q^* = \frac{\beta\zeta}{y_1} \left(\frac{1}{y_0} - \beta\zeta \frac{1}{y_0} \right)^{-1} = \frac{\beta\zeta y_0}{(1-\beta\zeta) y_1} = \frac{\beta\zeta}{(1-\beta\zeta)(1+\gamma)}}$$

has the same properties if $\beta\zeta < 1$

Arrow Debreu model

Assumptions

- The previous model is more general than it looks

H1 idem

H2 Assume heterogeneity in endowments

H3 idem

H4 idem

- What are the consequences for the equilibrium q

Arrow-Debreu economy

Beyond the representative agent case

- Assume there are two agents with the same preferences
- Assume that their endowments are different (y_t^i is the endowment of agent i at time t)

$$y^1 = \{y_0^1, y_1^1\}, y^2 = \{y_0^2, y_1^2\}$$

  and we assume $y^1 \neq y^2$
- The **flow of total endowments** of the economy are

$$y_0 = y_0^1 + y_0^2$$
$$y_1 = y_1^1 + y_1^2$$

- The general equilibrium is now

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

General equilibrium in this economy is defined by the allocations $(c_0^{1*}, c_1^{1*}, c_0^{2*}, c_1^{2*})$ and the price q^* such that

- consumer $i \in \{1, 2\}$ solves the problem

$$\max_{c_0^i, c_1^i} \{ U(c_0^i, c_1^i) : c_0^i + q\, c_1^i \leq y_0^i + q y_1^i \}, \text{ for } i = 1, 2$$

- market clearing hold for $t = 0, 1$,

$$c_0 = y_0, \ c_1 = y_1$$

where $c_t = c_t^1 + c_t^2$ for $t = 1, 2$ and $y_t = y_t^1 + y_t^2$

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- General equilibrium conditions (considering that the Walras' law holds)

$$\begin{cases} qU_0(c_0^1, c_1^1) = U_1(c_0^1, c_1^1) & \text{(optimality condition for agent 1)} \\ qU_0(c_0^2, c_1^2) = U_1(c_0^2, c_1^2) & \text{(optimality condition for agent 2)} \\ c_t = y_t & \text{(market clearing for period } t = 1, 2\text{)} \\ c_t = c_t^1 + c_t^2 & \text{(aggregation of consumption for } t\text{)} \\ y_t = y_t^1 + y_t^2 & \text{(aggregation of endowment for } t\text{)} \end{cases}$$

- In this case there can be trade, because $c_t^1 - y_t^1 = y_t^2 - c_t^2$ can be different from zero, but the budget constraint should hold for every agent. (check this !)

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- Because we assumed homogeneity in preferences $U(.,.)$ is the same for both consumers.
- Therefore, it also holds for the aggregate consumption

$$qU_0(c_0, c_1) = U_1(c_0, c_1)$$

that is

$$qU_0\big(c_0^1 + c_0^2, c_1^1 + c_2^1\big) = U_1(c_0, c_1)$$

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- Using the market clearing conditions we have again

$$\boxed{q^* = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)} = \frac{U_1(y_0^1 + y_0^2, y_1^1 + y_1^2)}{U_0(y_0^1 + y_0^2, y_1^1 + y_1^2)}}$$

- Conclusion: if agents are **homogeneous as regards preferences** but are **heterogeneous as regards endowments** the distribution of income between agents **has no influence** the AD price. It is only determined by the aggregate endowment
- If there is heterogeneity in preferences, this result **will not hold** in general.

Finance economy model

Assumptions

- Now we change the market structure

H1 idem

H2 Assume a representative agent economy

H3 idem

H4 Assume a sequence of asset markets

- What is the equilibrium asset price

Finance economy model

The economy

- Assume there is a spot market for the good opening at every period $t = 0$ and $t = 1$;
- There is an asset (that can be seen as a durable good) that agents can lend and borrow at period $t = 0$ paying or receiving an interest income at period $t = 1$. The asset is in non-negative net supply at the beginning to period $t = 0$ and there is a market for the asset at time $t = 0$.
- We still assume that the agent receives a flow of endowments $y = \{y_0, y_1\}$ The agent can consume the totality if the income, or not, at the end of period 1
- Every agent has now a **sequence of budget constraints** (because trade in the good market can take place at period 1)

Finance economy model

Micro-economic problem in the finance economy

- The problem

$$\max_{c_0, c_1, a_1, a_2} U(c_0, c_1) = u(c_0) + \beta u(c_1) :$$

- subject to

$$\begin{cases} c_0 + a_1 = y_0 + a_0 \\ c_1 = y_1 + (1+r)a_1 - a_2 \end{cases}$$

  where a_0 is the level of the asset at beginning of period 0 and a_1 and a_2 are the levels at the end of period 0 and 1, and r is the real interest rate.

- other constraints:

$$c_0 \geq 0, c_1 \geq 0, a_1 \text{free}, a_2 \geq 0$$

- Next, we prove that, it will never be optimal to have $a_2 > 0$

Finance economy model

Optimality of $a_2 = 0$

- Substitute c_0 and c_1 in the utility function, assume that $\beta > 0$ and r is finite, and consider the constraint for a_2

$$\max_{a_1, a_2} \{u(y_0 + a_0 - a_1) + \beta u(y_1 + (1+r)a_1 - a_2) : a_2 \geq 0\}$$

- The first order conditions are

$$\begin{aligned} u^{'}(c_0) &= \beta(1+r)u^{'}(c_1) \\ \beta u^{'}(c_1) &= \lambda \\ \lambda a_2 &= 0,\ \lambda \geq 0,\ a_2 \geq 0 \end{aligned}$$

- We have $a_2 > 0$ if and only if $\lambda = 0$, but in this case either there is satiation or $c_1 \to \infty$ and $c_0 \to \infty$. But this is only possible if $a_0 - \to \infty$. Therefore we should have $a_2 = 0$ and $\lambda > 0$.

Finance economy model

The consumer problem in a frictionless case

- Taking $a_2 = 0$ and assuming a_1 is free (i.e., the consumer can borrow or lend freely) we can eliminate a_1 in the sequence of budget constraints, to get

$$c_0 + mc_1 = a_0 + y_0 + my_1$$

where q^m is the **market discount factor**

$$m \equiv \frac{1}{1+r} \equiv \frac{1}{R}$$

- This implies that the **sequence of budget constraints** is equivalent to an **intertemporal budget constraint** formally similar to the constraint in the Arrow-Debreu economy.

$$c_0 + mc_1 = y_0 + mc_1$$

Finance economy without frictions

General equilibrium for a representative agent finance economy

General equilibrium in this economy is defined by $(c_0^*, c_1^*, m*)$ such that

- the consumer solves the problem

$$\max_{c_0, c_1} \{ U(c_0, c_1) : c_0 + mc_1 = a_0 + y_0 + mc_1 \}$$

- market clearing hold

$$c_0 = y_0, \; c_1 = y_1$$

Finance economy without frictions

General equilibrium for a representative agent finance economy

- The equilibrium equations are (from Walras's law)

$$
\begin{aligned}
mu'(c_0) &= \beta u'(c_1) \\
c_0 &= a_0 + y_0 \\
c_1 &= y_1
\end{aligned}
$$

- The **equilibrium discount factor** is

$$
m^* = m(a_0, y_0, y_1) = \beta \frac{u'(y_1)}{u'(a_0 + y_0)}
$$

- Because $R = \dfrac{1}{m}$ and $\beta = \dfrac{1}{1+\rho}$ where ρ is the psychological discount factor

Finance economy without frictions

Asset return in a frictionless economy

- The **equilibrium asset return** (recall)

$$\boxed{R^* = 1 + r^* = (1+\rho)\frac{u^{'}(y_1)}{u^{'}(a_0 + y_0)}}$$

- But $R^* = R(a_0, y_0, y_1)$, with partial derivatives

$$\frac{\partial R}{\partial a_0} = \frac{\partial R}{\partial y_0} = (1+\rho)\frac{u^{''}(a_0 + y_0)}{u^{'}(y_1)} < 0$$

$$\frac{\partial R}{\partial y_1} = -(1+\rho)\frac{u^{''}(y_1)u^{'}(a_0 + y_0)}{(u^{'}(y_1))^2} > 0$$

- There are two main effects:
  - a direct effect: high y_0 or a_0 **reduce** the interest rate
  - an anticipation effect: high y_1 **increases** the interest rate

A simple finance economy

Assumptions

- Now we introduce heterogeneity

H1 idem

H2 Assume agents face financing constraints

H3 idem

H4 idem

- What is the equilibrium asset price

Finance economy with heterogeneity

Heterogenous participation

- Assume there are two agents in the economy: agent b is a borrower and agent l is a lender, the only one that has positive assets at time 0 ($a_0^l > 0$, $a_0^b = 0$)
- To simplify, assume agent b is the only one that receives the flow of endowments $\{y_0, y_1\}$ and agent b can only earn interest income
- Assume there are no constraints in the credit market
- Assume that agents have homogeneous preferences

Finance economy with heterogeneity

Agents' problems

- The **lender** problem is

$$\max_{c_0^l, c_1^l}\{u(c_0^l) + \beta u(c_1^l) : \; c_0^l + l^l = a_0, \; c_1^l = (1+r)l^l\}$$

- Because l^l is free it can be simplified to

$$\max_{l^l}\{u(a_0 - l^l) + \beta u((1+r)l^l)\}$$

- The optimality condition is

$$u^{'}(a_0 - l^l) = \beta(1+r)u^{'}((1+r)l^l)$$

  or equivalently

$$u^{'}(c_0^l) = \beta(1+r)u^{'}(c_1^l)$$

Finance economy with heterogeneity

Agents' problems

- The **borrower** problem is

$$\max_{c_0^b, c_1^b}\{u(c_0^b) + \beta u(c_1^b) : \; c_0^b = y_0 + l^b, \; c_1^l + (1+r)l^b = y_1\}$$

- Because l^b is free it can be simplified to

$$\max_{l^b}\{u(y_0 + l^b) + \beta u(y_1 - (1+r)l^b)\}$$

- The optimality condition is

$$u^{'}(y_0 + l^b) = \beta(1+r)u^{'}(y_1 - (1+r)l^b)$$

or equivalently

$$u^{'}(c_0^b) = \beta(1+r)u^{'}(c_1^b)$$

Finance economy with heterogeneity

Equilibrium equations

- The equilibrium equations are

$$
\begin{aligned}
u^{'}(c_0^l) &= \beta(1+r)u^{'}(c_1^l) \\
u^{'}(c_0^b) &= \beta(1+r)u^{'}(c_1^b) \\
c_0^l + c_0^b &= y_0 + a_0 \\
c_1^l + c_1^b &= y_1
\end{aligned}
$$

- Because preferences are homogeneous we can use the same argument as before, to get

$$u^{'}(y_0 + a_0) = \beta(1+r)u^{'}(y_1)$$

Finance economy with heterogeneity

Equilibrium interest rate

- The **equilibrium return** is again

$$\boxed{R^* = 1 + r^* = (1+\rho)\frac{u^{'}(y_0 + a_0)}{u^{'}(y_1)}}$$

- Is formally similar to the representative agent economy case;
- Again, we have
  - negative liquidity effect $R^*_{a_0} < 0$;
  - a negative income effect, $R^*_{y_0} < 0$
  - a positive anticipation effect, $R^*_{y_1} > 0$

Taking the model to data

- data from `http://www.nber.org/papers/w24112.pdf`
  $R_{safe} = 1.0188$ (average safe return) $R_{wealth} = 1.0678$ (average wealth return) $\gamma = 0.0287$ (average rate of growth)
- calibrated parameters: $\rho = 0.02$
- Utility functions
  - isoelastic utility function

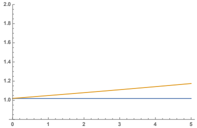
$$U(c_0, c_1) = \frac{c_0^{1-\zeta}}{1-\zeta} + \beta \frac{c_1^{1-\zeta}}{1-\zeta}$$

  - habit formation:

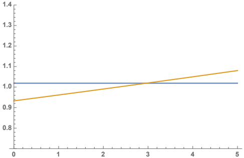
$$U(c_0, c_1) = \ln c_0 + \beta \left[\frac{\left(\frac{c_1}{c_0}\right)^{\phi(1-\zeta)} - 1}{1-\zeta} \right]$$

Taking the model to data

- Additive utility: interest rate puzzle (the model over predicts the observed risk-free interest rate, for any value of the *EIS*)



- Habit formation: it is possible to find values for the parameter ϕ, in the case $\phi \approx 0.5$ such that the model matches the observed R for "acceptable" values fo ζ



Questions

- The previous results hold for cases in which there is
  - full information (deterministic general equilibrium)
  - agents have homogeneous preferences (with or without homogeneous resources)
  - frictionless economy (for the case of a finance economy)
- Do those results hold:
  - Under imperfect information (uncertainty) ?
  - Under heterogeneity in agents' preferences ?
  - Under frictions in a finance economy (ex: credit constraints) ?