Foundations of Financial Economics
Two period SGE: production economy

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Topics

- The AD model for a production economy
- Asset pricing consequences

Arrow-Debreu economy with production

Endowment and production economies

- Continue to consider a two-period Arrow-Debreu economy with uncertainty for period 1;
- In an endowment economy the sequence of of the endowments of the good, $\{y_0, Y_1\}$ is independent of the agent decision;
- In a production economy the agent's endowment of goods in period 1 is dependent upon the savings decisions in period 0 and on a state-contingent productivity shock
- How do Arrow-Debreu prices (or the stochastic discount factor) change when comparing the two economies ?

Assumptions

- Assume a two-period Arrow-Debreu economy with uncertainty for period 1: $s \in \{1, \ldots, N\}$;
- With production, the supply of goods is given by the sequence $\{y_0, Y_1\}$, where

$$\begin{aligned} y_0 &= a_0 k_0, \; t = 0 \\ y_{1,s} &= a_{1,s} k_1, \; t = 1, \; s = 1, \ldots, N \end{aligned}$$

  where k_t is the capital stock and A_t is a time-dependent productivity parameter.
- Therefore $Y_1 = A_1 k_1$, that is

$$Y_1 = \begin{pmatrix} y_{1,1} \\ \ldots \\ y_{1,s} \\ \ldots \\ y_{1,N} \end{pmatrix} = \begin{pmatrix} a_{1,1} \\ \ldots \\ a_{1,s} \\ \ldots \\ a_{1,N} \end{pmatrix} k_1$$

Assumptions

- The capital stock at the beginning of period 1 is determined by savings in period 0

$$k_1 - k_0 = s_0 = y_0 - c_0$$

- Then the availability of the good at period 1 depends on the consumption at period 0

$$Y_1 = A_1(k_0 + s_0) = A_1((1 + a_0)k_0 - c_0)$$

- The **distribution** of income available at period 1 is

$$y_{1,s} = a_{1,s}\left((1 + a_0)k_0 - c_0\right),\ s = 1, \ldots, N$$

Assumptions

- The growth rate of the economy is endogenous

$$1 + g_s = \frac{y_{1,s}}{y_0} = (1 + \gamma_s)\left(1 + a_0 - \frac{c_0}{k_0}\right),$$

  where productivity grows at the rate γ_s

$$1 + \gamma_s = \frac{a_{1,s}}{a_0}$$

- If there is no stochastic productivity growth then the increase in income will be deterministic

$$1 + g = (1 + \gamma)\left(1 + a_0 - \frac{c_0}{k_0}\right),$$

- Even if $\gamma = 0$ we can have growth if $1 + a_0 > \dfrac{c_0}{k_0}$.

The stochastic discount factor with production

Questions

- How does the existence of a new motive for allocation of resources (investment in productive capital) changes the stochastic discount factor ?
- What are the consequences for asset prices (or asset rates of return ?)
- To answer those questions we need a general equilibrium model. We consider next the Arrow-Debreu economy (or the equivalent finance economy with complete markets)

Arrow-Debreu economy with production

The agent's problem

- von Neumann-Morgenstern utility functional

$$\max_{c_0, C_1} u(c_0) + \beta \sum_{s=1}^{N} \pi_s u(c_{1,s})$$

- intertemporal constraint (where $Q = (q_s)_{s=1}^{N}$ are the Arrow-Debreu prices)

$$c_0 - y_0 + \sum_{s=1}^{N} q_s(c_{1,s} - y_{1,s}) \leq 0$$

- the constraint is equivalent to

$$c_0 - a_0 k_0 + \sum_{s=1}^{N} \pi m_s \left[c_{1,s} - a_{1,s}\left((1 + a_0)k_0 - c_0\right)\right] \leq 0$$

Arrow-Debreu economy with production

first order conditions of optimality

Assuming no satiation: i.e. $u^{'}(c) > 0$ for all $c > 0$,

- intertemporal arbitrage condition

$$q_s u^{'}(c_0^*) = \beta \pi_s u^{'}(c_{1,s}^*) \left(1 + \sum_{s=1}^{N} q_s a_{1,s}\right), \; s = 1, \ldots, N$$

- intertemporal constraint

$$c_0^* - a_0 k_0 + \sum_{s=1}^{N} q_s \left[c_{1,s}^* - a_{1,s}\left((1 + a_0)k_0 - c_0^*\right)\right] = 0$$

Arrow-Debreu economy with production

first order conditions of optimality

Using $M = (m_s)_{s=1}^N$ is the stochastic discount factor where $q_s = \pi_s m_s$, the representative consumer optimal path $\{c_0^*, C_1^*\}$ is obtained from

$$m_s u^{'}(c_0^*) = \beta u^{'}(c_{1,s}^*)\,(1 + \mathbb{E}[MA_1])\,,\ s = 1, \ldots, N$$

$$c_0^*(1 + \mathbb{E}[MA_1]) + \sum_{s=1}^N \pi_s m_s c_{1,s}^* = (a_0(1 + \mathbb{E}[MA_1]) + \mathbb{E}[MA_1])\, k_0$$

where

$$\mathbb{E}[MA_1] = \sum_{s=1}^N \pi m_s a_{1,s}$$

is the expected discount value of the future productivity.

Arrow-Debreu economy with production

General equilibrium

Is the sequence $\{c_0, C_1\}$ and k_1^* and M such that

- agent's optimality conditions are satisfied

$$m_s u^{'}(c_0) = \beta u^{'}(c_{1,s})\left(1 + \mathbb{E}[MA_1]\right), \; s = 1, \ldots, N \quad (1)$$

$$c_0^*(1 + \mathbb{E}[MA_1]) + \sum_{s=1}^{N} \pi_s m_s c_{1,s}^* = \left(a_0(1 + \mathbb{E}[MA_1]) + \mathbb{E}[MA_1]\right) k_0 \quad (2)$$

- and market equilibrium conditions hold

$$\begin{aligned} c_0 + k_1 - k_0 &= a_0 k_0 && (3) \\ c_{1,s} &= a_{1,s} k_1, \; s = 1, \ldots, N && (4) \end{aligned}$$

Arrow-Debreu economy with production

General equilibrium

- Solving (3) for $k_1 = (1 + a_0)k_0 - c_0$ and substituting in (4) we get

$$c_{1,s} = ((1 + a_0)k_0 - c_0)a_{1,s}$$

- Substituting in (2) we obtain

$$c^* = a_0 k_0$$

- then

$$c^*_{1,s} = a_{1,s}k_0$$

- then

$$k^*_1 = (1 + a_0)k_0 - c^*_0 = k_0$$

- we obtain m^*_s by substituting c_0 and $c_{1,s}$ in equation (1).

Arrow-Debreu economy with production

The equilibrium stochastic discount factor (SDF)

- The equilibrium stochastic discount factor $M = (m_s)_{s=1}^{N}$, where $m_s = \dfrac{q_s}{\pi_s}$ is implictly given by

$$m_s^* = \beta \frac{u^{'}(a_{1,s} k_0)}{u^{'}(a_0 k_0)} \left(1 + \mathbb{E}[M^* A_1]\right)$$

- where

$$\mathbb{E}[MA_1] = \sum_{s=1}^{N} \pi_s m_s^* a_{1,s}$$

- **Observation**: the SDF for a particular state of nature depends on the expected present value of future productivity, which depends on the distribution of M. In order to determine it we can assume a particular utility function.

Arrow-Debreu economy with production

Example: CRRA Bernoulli utility function

- Simplifying assumptions:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \text{ and } a_{1,s} = (1+\gamma_s)a_0$$

- then

$$m_s^* = \beta(1+\gamma_s)^{-\theta}\left(1 + \mathbb{E}[MA_1]\right)$$

- but as $\mathbb{E}[MA_1] = \sum_{s=1}^{N} \pi_s m_s^* a_{1,s}$ we have (because $a_{1,s} = (1+\gamma_s)a_0$)

$$\begin{aligned}\mathbb{E}[MA_1] &= a_0 \sum_{s=1}^{N} \pi_s (1+\gamma_s)\beta(1+\gamma_s)^{-\theta}(1+\mathbb{E}[MA_1]) \\ &= \beta a_0 (1+\mathbb{E}[MA_1])\mathbb{E}[(1+\gamma)^{1-\theta}]\end{aligned}$$

- Solving for $\mathbb{E}[MA_1]$ we obtain

$$\mathbb{E}[MA_1] = \frac{\beta a_0 \mathbb{E}[(1+\gamma)^{1-\theta}]}{1 - \beta a_0 \mathbb{E}[(1+\gamma)^{1-\theta}]}$$

Arrow-Debreu economy with production

- The equilibrium stochastic discount factor for a **production** economy is

$$m_s^{\text{prod}} = \beta(1+\gamma_s)^{-\theta}\Phi,\ s = 1, \dots, N$$

where

$$\Phi \equiv \frac{1}{1-\beta a_0 \mathbb{E}[(1+\gamma)^{1-\theta}]}$$

under the condition $\beta a_0 \mathbb{E}[(1+\gamma)^{1-\theta}] < 1$ then $\Phi > 1$

- We found the equilibrium stochastic discount factor for an **endowment** economy, with the same utility function

$$m_s^{\text{end}} = \beta \frac{u^{'}(y_{1,s})}{u^{'}(y_0)} = \beta(1+\gamma_s)^{-\theta}$$

- Then: for every state of nature the stochastic discount factor is higher in a production economy than in a related endowment economy

$$m_s^{\text{prod}} > m_s^{\text{end}}$$

because $\Phi > 1$.

Arrow-Debreu economy with production

- Φ is a growth factor: the decision on consumption at time 0, in addition to having into consideration the consumption smoothing between time and the states of natures (as in the endowment economy), also takes into account the changes in the level of intertemporal resources because production at time 1 now depends on consumption at time 0.

Finance economy with production

- In a related **finance economy** (with the same fundamentals) we found
$$\mathbb{E}[MR^j] = 1$$
where R^j is the return for any asset j.
- And for a risk free asset
$$R^f = \frac{1}{\mathbb{E}[M]}$$
- Then the risk free interest rate tends to be smaller in a production economy
$$R^{f,\text{prod}} < R^{f,\text{end}}$$