Foundations of Financial Economics
Choice under uncertainty

Paulo Brito

[1]pbrito@iseg.ulisboa.pt
University of Lisbon

March 18, 2022

Topics covered

1. Contingent goods:
   - Definition
   - Comparing contingent goods
2. Probability: revisions
3. Decision under risk:
   - von-Neumann-Morgenstern utility theory
   - Certainty equivalent
   - Attitudes towards risk: risk neutrality and risk aversion
   - Measures of risk
   - The HARA family of utility functions
   - Applications

1. Contingent goods

Contingent goods

informal definition

Contingent goods (or claims or actions): are goods whose **outcomes are state-dependent**, meaning:

- at the moment of decision: the available quantity of the good is uncertain (i.e, *ex-ante* we have **several odds**)
- the actual quantity to be received, the outcome, is revealed afterwards (*ex-post* we have **one realization**)
- **state-dependent**: means that nature chooses which outcome will occur (i.e., the outcome depends on a mechanism out of our control)

Contingent goods

Example: flipping a coin

lottery 1: flipping a coin with **state-dependent outcomes:**

- before flipping a coin the contingent outcome is

odds	head	tail
outcomes	100	0

- after flipping a coin there is only one realization: 0 or 100

lottery 2: flipping a coin with **state-independent outcomes:**

- before flipping a coin the non-contingent outcome is

odds	head	tail
outcomes	50	50

- after flipping a coin we always get: 50

Contingent goods

Example: tossing a dice

lottery 3: dice tossing with state-dependent outcomes:

- before tossing a dice the contingent outcome is

odds	1	2	3	4	5	6
outcomes	100	80	60	40	20	0

- after tossing the dice we will get: 100, or 80 or 60 or 40, or 20, or 0.

Comparing contingent goods

- Question: given two contingent goods (lotteries, investments, actions, contracts) how do we compare them ?
- Answer: we need to reduce it to a **number** which we interpret as its **value**

$$\text{contingent good 1} \rightarrow \text{ Value of contingent good 1} = V_1$$
$$\text{contingent good 2} \rightarrow \text{ Value of contingent good 2} = V_2$$

$$\text{contingent good 1 is better that 2} \quad \Leftrightarrow \quad V_1 > V_2$$

Comparing contingent goods

Example: farmer's problem

Farmer's problem: which crop, vegetables or cereals ?

- before planting: the outcomes and the associated costs (known) are

	income		cost	profit	
weather	rain	drought		rain	drought
vegetables	200	30	50	150	-20
cereals	10	100	20	-10	80

- after planting:
  - vegetables: the profit realization will be: -20 or 150
  - cereals: the profit realization will be: -10 or 80

Comparing contingent goods

Example: investor's problem

Investors's problem: to risk or not to risk ?

- before investing: contingent incomes and the cost are

	income if market is		cost	profit if market is	
market	bull	bear		bull	bear
equity	130	50	100	30	-50
bonds	98	105	100	-2	5

- after investing:
  - in equity: the profit realizations will be: -50 or 30
  - in bonds: profit realizations will be: 5 or -2

Comparing contingent goods

Examples: gambler's problem

Gambler's problem : to flip or not to flip a coin ?

- comparing one non-contingent with another contingent outcome
- Before flipping the coin the alternatives are

	outcomes		cost	profit	
odds	H	T		H	T
flipping	100	0	20	80	- 20
no flipping	50	50	45	5	5

- after flipping:
  - accepts coin flipping: gets 80 or -20
  - rejects coin flipping: gets 5 with certainty

Comparing contingent goods

Examples: insured's problem

Insurance problem: to insure or not to insure ?

- Before insuring, assuming that the coverage is 50%

	outcomes		cost	net income	
damage	no	yes		no	yes
insured	0	- 250	10	-10	- 240
uninsured	0	-500	0	0	-500

- after the contract:
  - insured: net income is : -10 or -240
  - uninsured: net income is : 0 or -500

Comparing contingent goods

Gambler problem: different lottery profiles

- Until this point the states of nature for the alternatives were the same
- But we may want to compare alternatives with different event profiles
- **Example gambler's problem:** which lottery to choose

	income								cost
	coin		dice						
odds	head	tail	1	2	3	4	5	6	
lottery 1	100	0							20
lottery 2			100	80	60	40	20	0	30

Choosing among contingent goods

Characterization of the information environment

Main issues:

- what is the **source** of uncertainty:
  - objective (equal for all agents): risk
  - subjective (different among agents): uncertainty
- **knowledge**:
  - common: risk
  - asymmetric: information (moral hazard, adverse selection)
- **characterization** of the odds:
  - precise: distribution over exact odds
  - imprecise: ambiguity (distribution over a distribution of the odds)
- distribution of contingent **outcomes**:
  - known model: specific relationship between odds and outcomes
  - model uncertainty: uncertain relationship between odds and outcomes

2. Probability: revisions

- **Information** is given by the probability space: $(\Omega, \mathcal{F}, \mathbb{P})$ (sets of events and a probability over them):
- Space of **pure events** (or states of nature):

$$\Omega = \{\omega_1 \dots \omega_N\}$$

  Examples:
  coin $\Omega = \{head, tail\}$,
  dice $\Omega = \{1, \dots, 6\}$,
  weather: $\Omega = \{rain, sunshine\}$
- Set of **all** events: $\mathcal{F}$:
  Example: coin $\mathcal{F} = \{head, tail, (head\, and\, tail)\}$

Probability spaces

Probabilities

- $\mathbb{P}$ probability:
  - is a **mapping** (a function) that assigns to an event a number between 0 and 1

$$\omega_s \mapsto P(\omega_s) \in [0, 1]$$

  - its sum for all events is equal to one

$$\sum_{s=1}^{N} P(\omega_s) = 1$$

- We write $\pi_s = P(\omega_s) \in [0, 1]$: then

$$\boxed{0 \leq \pi_s \leq 1, \text{ and } \sum_{s=1}^{N} \pi_s = 1}$$

Probability spaces

Probabilities

- **Any mapping with those properties can be formally seen as a probability mapping**
- Classification of events:
  certain event or almost sure: it is an event $\omega = \omega^c \in \Omega$ such that $P(\omega_s) = 1$
  negligible event: it is an event $\omega = \omega^n \in \Omega$ such that $P(\omega^n) = 0$
- Meaning: this classification depends on the way we build function $P(\cdot)$

Random variables

- To quantity of our contingent goods is a random variable
- A **random variable** X is a mapping between events and a real number

$$X : \mathcal{F} \to \mathbb{R}$$

- In the following we write $X = X(\omega)$, that is

$$X = \begin{pmatrix} X(\omega_1) \\ \dots \\ X(\omega_s) \\ \dots \\ X(\omega_N) \end{pmatrix} = \begin{pmatrix} x_1 \\ \dots \\ x_s \\ \dots \\ x_N \end{pmatrix}$$

- where x_s **is the outcome** if the event ω_s is **realized** (ex: draw head after flipping a coin)
- Next we concentrate in the outcomes which are realized and let the events be implicit

Random variables realizations and probabilities

- The information we usually assume regards the states of nature, their probabilities and their outcomes

	states	1	...	s	...	N
P	probabilities	π_1	...	π_s	...	π_N
X	outcomes	x_1	...	x_s	...	x_N

Statistics for a random variable

- Most common statistics
  - Mean (arithmetic) is a measure of position:

$$\mathbb{E}[X] = \sum_{s=1}^{N} \pi_s \, x_s$$

  - Variance and standard deviation is a measure of dispersion:

$$\mathbb{V}[X] = \sum_{s=1}^{N} \pi_s \left(x_s - \mathbb{E}[X]\right)^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2, \; \sigma[X] = \sqrt{\mathbb{V}[X]}$$

  - $\mathbb{V}[X]$ is always non-negative (and it is zero for a deterministic variable

Random variables and statistics

	1	...	s	...	N	statistics
P	π_1	...	π_s	...	π_N	$\mathbb{E}[X]$, $\mathbb{V}[X]$
X	x_1	...	x_s	...	x_N	

- The mean and the variance combine information on **both** the probabilities and the outcomes

Statistics for two random variables

- Sometimes we have two random variables

states	1	$\dots$	s	$\dots$	N
P	π_1	$\dots$	π_s	$\dots$	π_N
X	x_1	$\dots$	x_s	$\dots$	x_N
Y	y_1	$\dots$	y_s	$\dots$	y_N

- Means:

$$\mathbb{E}[X] = \sum_{s=1}^{N} \pi_s\, x_s,\ \mathbb{E}[Y] = \sum_{s=1}^{N} \pi_s\, x_s$$

- Variances:

$$\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2,\ \mathbb{V}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

- Covariance

$$\mathrm{Cov}[X, Y] = \mathbb{E}[X\, Y] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

$$\text{Correlation coefficient: } \rho_{X,Y} = \frac{\mathrm{Cov}[X, Y]}{\sigma[X]\,\sigma[Y]}$$

Functions of random variables

- Consider a function of a random variable: $f(X)$ and let $f_s = f(x_s)$

states	1	...	s	...	N
P	π_1	...	π_s	...	π_N
X	x_1	...	x_s	...	x_N
$f(X)$	f_1	...	f_s	...	f_N

- Statistics
  - Mean and variance

$$\mathbb{E}[f(X)] = \sum_{s=1}^{N} \pi_s f(x_s), \; \mathbb{V}[f(X)] = \mathbb{E}[f(X)^2] - \mathbb{E}[f(X)]^2$$

  - A useful result: **Jensen inequality**:

$$\boxed{\text{if } f(\cdot) \text{ is concave} \Rightarrow f\big(\mathbb{E}[X]\big) \geq \mathbb{E}[f(X)]}$$

$$\boxed{\text{if } f(\cdot) \text{ is linear} \Rightarrow f\big(\mathbb{E}[X]\big) = \mathbb{E}[f(X)]}$$

Jensen's inequality for a concave function

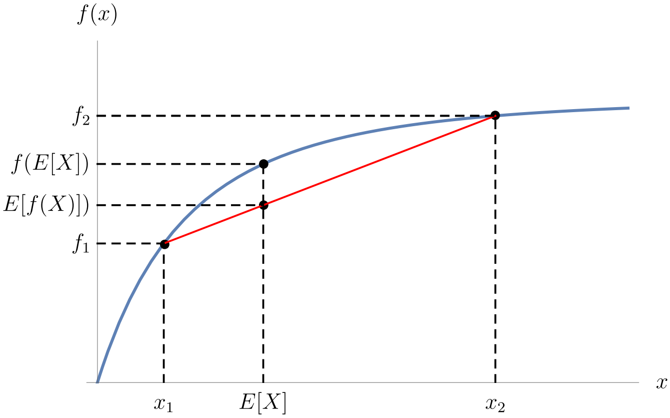


Figure: Jensen inequality for a concave function

Useful results

- Assume there are only **two states of nature**

states	1	2	
P	π_1	π_2	$\pi_1 + \pi_2 = 1$
X	x_1	x_2	
Y	y_1	y_2	

- Mean: $\mathbb{E}[X] = \pi_1\, x_1 + \pi_2\, x_2$
- Variance $\mathbb{V}[X] = \pi_1\, \pi_2\, (x_1 - x_2)^2$
- Standard deviation $\sigma[X] = \sqrt{\pi_1\, \pi_2}\, |x_1 - x_2|$
- Covariance: $\text{Cov}[X, Y] = \pi_1\, \pi_2\, (x_1 - x_2)\, (y_1 - y_2)$
- Correlation: $\rho[X, Y] = \dfrac{(x_1 - x_2)\, (y_1 - y_2)}{|x_1 - x_2|\, |y_1 - y_2|}$
- Exercise: Prove this

Useful results

- Consider the data

states	1	2	
P	π_1	π_2	$\pi_1 + \pi_2 = 1$
X	x_1	x_2	
$f(X)$	f_1	f_2	

- Jensen inequality: if $f(\cdot)$ is concave

$$f\big(\pi_1 x_1 + \pi_2\, x_2\big) \geq \pi_1\, f(x_1) + \pi_2\, f(x_2)$$

- An example: if $f(x) = \ln(x)$ prove that

$$\ln\left(\mathbb{E}[X]\right) > \mathbb{E}[\ln X] \iff \mathbb{E}[X] > e^{\mathbb{E}[\ln X]} = \mathbb{GE}[X]$$

where $\mathbb{GE}[X] = x_1^{\pi_1}\, x_2^{\pi_2}$ is the geometric mean

3. Decision under risk

3.1 Von-Neuman Morgenstern utility theory

Decision under risk

Notation:

- Ω space of states of nature

$$\Omega = \{\omega_1, \dots, \omega_N\}$$

- $\mathbb{P}$ is an **objective** probability distribution over states of nature

$$\mathbb{P} = (\pi_1, \dots, \pi_N)$$

  where $0 \leq \pi_s \leq 1$ and $\sum_{s=1}^{N} \pi_s = 1$
- X a **contingent good** with possible outcomes

$$X = (x_1, \dots, x_s, \dots x_N)$$

Decision under risk

Information environment

- **Information**:
  - we **know**:
    the probability space $(\Omega, \mathbb{P})$
    the outcomes for a contingent good X are common knowledge and are unique;
  - we **do not know**:
    which state of nature will materialize, and therefore, the realization $X = x$ of X
- Question: what is the value of X ?

Expected utility theory

vNM utility

Definition 1

*Expected or **von-Neumann Morgenstern** utility functional*

$$U(X) = \mathbb{E}[u(X)] \equiv \sum_{s=1}^{N} \pi_s u(x_s)$$

*where $u(x)$ is **Bernoulli** utility function*

- Do not confuse: $U(X)$ value of one lottery with $u(x_s)$ value of one outcome
  - $U(X)$ is a value measure of the contingent good
  - $u(x_s)$ measures **the value of outcome** x_s

- **Assumption**: a contingent good X is valued by the Expected or **von-Neumann Morgenstern** utility

- **Properties of the expected utility function**
  - **state-independent** valuation of the outcomes $u(x_s)$ and not $u_s(x_s)$:
    $u(x_s)$ **only** depends on the outcome x_s and **not** on the state of nature s (symmetric evaluation of good and bad states)
  - **linear in probabilities**:
    the utility of the contingent good $U(X)$ is a linear function of the probabilities π_s
  - **information context**:
    $U(X)$ refers to choices in a context of risk because the odds are known and $\mathbb{P}$ are objective probabilities
  - **attitude towards risk**:
    is implicit in the shape of $u(.)$ (in particular in its concavity).

Expected utility theory

Comparing contingent goods

- Consider two contingent goods with outcomes

$$X = (x_1, \ldots, x_N)\,, \;\; Y = (y_1, \ldots, y_N)$$

- we can rank them using the relationship

$$\boxed{X \text{ is prefered to } \; Y \Leftrightarrow \mathbb{E}[u(X)] > \mathbb{E}[u(Y)]}$$

that is $U(X) > U(Y) \Leftrightarrow \mathbb{E}[u(X)] > \mathbb{E}[u(Y)]$

$$\mathbb{E}[u(X)] > \mathbb{E}[u(Y)] \Leftrightarrow \sum_{s=1}^{N} \pi_s u(x_s) > \sum_{s=1}^{N} \pi_s u(y_s)$$

- There is **indifference** between X and Y if

$$\boxed{U(X) = U(Y) \Leftrightarrow \mathbb{E}[u(X)] = \mathbb{E}[u(Y)]}$$

Examples: coin flipping

- Odds: $\Omega = \{head, tail\}$
- Probabilities: $\mathbb{P} = \Big(P(\{head\}, P(\{tail\}\Big) = \Big(\frac{1}{2}, \frac{1}{2}\Big)$
- Outcomes: $X = (X(\{head\}, X(\{tail\}) = (60, 10)$
- Value of flipping a coin

$$U(X) = \frac{1}{2}u(60) + \frac{1}{2}u(10)$$

Examples: dice tossing

- Odds: $\Omega = \{1, \dots, 6\}$
- Probabilities: $\mathbb{P} = \Big(P(\{1\}, \dots, P(\{6\}\Big) = \Big(\frac{1}{6}, \dots, \frac{1}{6}\Big)$
- Outcomes: $Y = \big(Y(\{1\}, \dots, Y(\{6\}\big) = \big(10, 20, 30, 40, 50, 60\big)$
- Value of tossing a dice is

$$U(Y) = \frac{1}{6}u(10) + \frac{1}{6}u(20) + \ldots + \frac{1}{6}u(60)$$

- whether $U(X) \gtreqless U(Y)$ depends on the utility function

Expected utility theory

Comparing one contingent good with a non-contingent good

- given one contingent good $X = (x_1, \dots, x_N)$ and one non-contingent good z,
- we can rank them using the relationship

$$X \text{ is prefered to } z \Leftrightarrow U(X) \geq u(z)$$

- Obs: a non-contingent good is a particular contingent good such that $Z = (z, \dots, z)$. In this case

$$U(X) = U(Z) \Leftrightarrow \mathbb{E}[u(X)] = \mathbb{E}[U(Z)] = \sum_{s=1}^{N} \pi_s u(z) = u(z)$$

  because $\sum_{s=1}^{N} \pi_s = 1$.
- There is indifference between X and z if

$$\boxed{\mathbb{E}[u(X)] = u(z)}$$

3.2 Certainty equivalent

Expected utility theory

Certainty equivalent

Definition 2

Certainty equivalent *(CE) is the certain outcome, x^c, which has the same utility as a contingent good X*

$$x^c = u^{-1}\left(\mathbb{E}[u(X)]\right) = u^{-1}\left(\mathbb{E}\left[\sum_{s=1}^{N} \pi_s u(x_s)\right]\right)$$

Equivalently: given u and $\mathbb{P}$, CE is the certain outcome such that the consumer is indifferent between X and x^c

$$u(x^c) = \mathbb{E}[u(X)] \Leftrightarrow u(z) = \sum_{s=1}^{N} \pi_s u(x_s)$$

- **Example:** the certainty equivalent of flipping a coin is the outcome z such that

$$x^c = u^{-1}\left(\frac{1}{2}u(60) + \frac{1}{2}u(10)\right)$$

3.3 Attitudes towards risk

Expected utility theory

Risk neutrality

Definition 3

For any contingent good, X, we say there is ***risk neutrality*** *if the utility function $u(.)$ has the property*

$$\mathbb{E}[u(X)] = u(\mathbb{E}[X])$$

- Risk neutrality: the expected utility of the contingent good is equal to the utility of the average outcome
- Risk neutrality: the certainty equivalent of X is equal to the average of X (see next)

Expected utility theory

Risk neutrality

Proposition 1

*There is **risk neutrality** if and only if the Bernoulli utility function $u(.)$ **is linear***

$$\sum_s \pi_s u(x_s) = u(\sum_s p_s x_s)$$

Expected utility theory

Risk aversion

Definition 4

For any contingent good, X, we say there is ***risk aversion*** *if*

$$\mathbb{E}[u(X)] < u(\mathbb{E}[X])$$

- Risk aversion: the expected utility of the contingent good is smaller than the utility of the average outcome
- Risk aversion: the certainty equivalent of X is smaller to the average of X (see next)

Proposition 2

*There is **risk aversion** if and only if the utility function $u(.)$ **is concave***

Proof: the Jensen inequality states that if $u(.)$ is strictly concave then

$$\mathbb{E}[u(X)] < u[E(X)] \Leftrightarrow \sum_{s=1}^{N} \pi_s u(x_s) < u\left(\sum_{j=1}^{N} x_s \pi_s\right).$$

Jensen's inequality and risk aversion $u(x)$

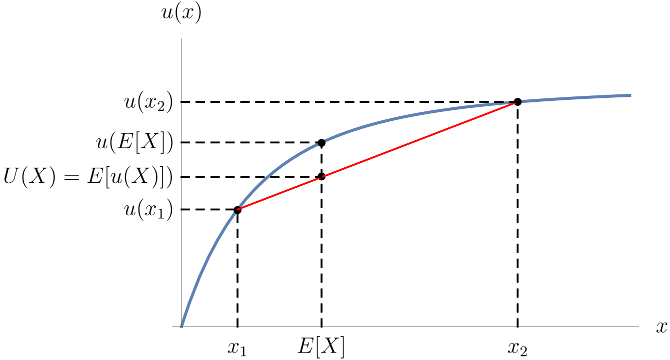


Figure: Jensen's inequality $\mathbb{E}[u(X)] < u[E(X)]$

Expected utility theory

Risk neutrality, risk aversion and the certainty equivalent

- Using the certainty equivalent definition $u(x^c) = \mathbb{E}[u(X)]$ and if $\mathbb{E}[u(X)] \leq u\big(\mathbb{E}[X]\big)$ then (look at the Jensen inequality figure)

$$\mathbb{E}[X] = u^{-1}\left(u(\mathbb{E}[X])\right) \geq u^{-1}\left(\mathbb{E}[u(X)]\right)$$

  then

- There is **risk neutrality** if and only if

$$x^c = \mathbb{E}[X]$$

  the certainty equivalent is equal to the expected value of the outcome

- there is **risk aversion** if and only if

$$x^c < \mathbb{E}[X]$$

  certainty equivalent is smaller than the expected value of the outcome

Certainty equivalent for a concave $u(x)$

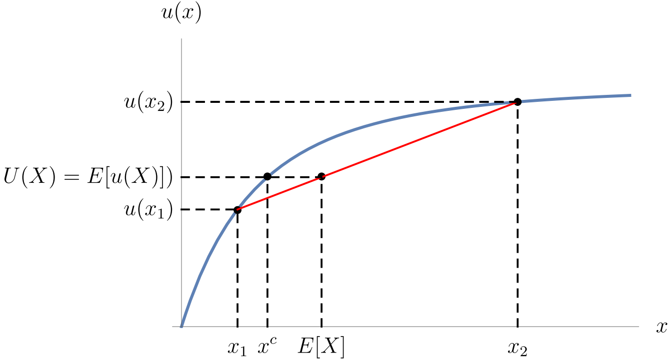


Figure: Certainty equivalent and mean outcome: $x^c < \mathbb{E}[X]$

Expected utility theory

Risk premium

Definition 5

Risk premium *is defined by the difference between the expected value and the certainty equivalent*

$$\mathcal{R}(X) = \mathbb{E}[X] - x^c$$

- Intuition: given the utility function, this is the value the agent is **willing to pay for not bearing risk**
- Therefore:
  - If there is risk neutrality then $\mathcal{R}(X) = 0$, the agent is not willing to pay nor to receive in order to bear risk
  - If there is risk aversion then $\mathcal{R}(X) > 0$, the agent is willing to pay to avoid bearing risk

Risk premium for a concave $u(x)$

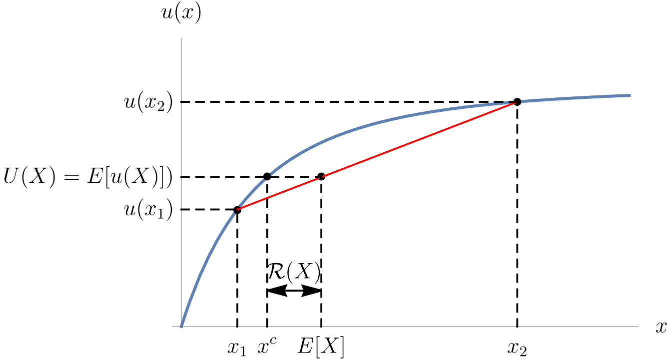


Figure: Risk premium $\mathcal{R}(X) = \mathbb{E}[X] - x^c$

3.4 Measure of risk

Measures of risk

- **Risk and the shape of u:**
  if u is linear it represents risk neutrality
  if $u(.)$ is concave then it represents risk aversion
- **Arrow-Pratt measures of risk aversion**:
  1. coefficient of **absolute** risk aversion:
  $$\varrho_a \equiv -\frac{u''(x)}{u'(x)}$$
  2. coefficient of **relative** risk aversion
  $$\varrho_r \equiv -\frac{x\, u''(x)}{u'(x)}$$
  3. coefficient of **prudence**
  $$\varrho_p \equiv -\frac{x\, u'''(x)}{u''(x)}$$

3.5 The HARA family of utility functions

HARA family of utility functions

- Meaning: hyperbolic absolute risk aversion

$$\boxed{u(x) = \frac{\gamma - 1}{\gamma}\left(\frac{\alpha x}{\gamma - 1} + \beta\right)^{\gamma}} \tag{1}$$

- Cases: (prove this)
  1. linear: if $\beta = 0$ and $\gamma = 1$

     $$u(x) = ax$$

     properties: risk neutrality
  2. quadratic : if $\gamma = 2$

     $$u(x) = ax - \frac{b}{2}x^2, \text{ for } x < \frac{2a}{b}$$

     properties: risk aversion, has a satiation point $x = \frac{2a}{b}$

HARA family of utility functions

1. CARA: if $\gamma \to \infty$, (note that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$)

$$u(x) = -\frac{e^{-\lambda x}}{\lambda}$$

   properties: constant absolute risk aversion (CARA), variable relative risk aversion, scale-dependent

2. CRRA: if $\gamma = 1-\theta$ and $\beta = 0$

$$u(x) = \begin{cases} \ln(x) & \text{if } \theta = 1 \\ \dfrac{x^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \end{cases}$$

   (if $\theta = 1$ note that $\lim_{n\to 0}\frac{x^n-1}{n} = \ln(x)$)
   properties: constant relative risk aversion (CRRA); scale-independent

3.6 Applications

Example 1: comparing contingent goods

Coin flipping vs dice tossing

- Take our previous case:

$$U(X) = \frac{1}{2}u(60) + \frac{1}{2}u(10)$$

or

$$U(Y) = \frac{1}{6}u(10)+\frac{1}{6}u(20)+\frac{1}{6}u(30)+\frac{1}{6}u(40)+\frac{1}{6}u(50)+\frac{1}{6}u(60)$$

- We will rank them assuming
  1. a linear utility function $u(x) = x$
  2. a logarithmic utility function $u(x) = \ln(x)$
- Observe that the two contingent goods have the same expected value

$$\mathbb{E}[X] = 35 \; \mathbb{E}[Y] = 35$$

Example 1: comparing contingent goods

Coin flipping vs dice tossing: linear utility

- If $u(x) = x$
  - $U(X) = \mathbb{E}[u(x)] = \frac{1}{2}60 + \frac{1}{2}10 = 35$
  - $U(Y) = \mathbb{E}[u(y)] = \frac{1}{6}10 + \ldots + \frac{1}{6}60 = 35$
- Then there is risk neutrality

$$\mathbb{E}[u(x)] = \mathbb{E}[X] = 35,\ \mathbb{E}[u(y)] = \mathbb{E}[Y] = 35$$

- and we are indifferent between the two lotteries because $\mathbb{E}[X] = \mathbb{E}[Y]$

Example 1: comparing contingent goods

Coin flipping vs dice tossing: log utility

- If $u(x) = \ln(x)$
  - $U(X) = \dfrac{1}{2}\ln(60) + \dfrac{1}{2}\ln(10) \approx 3.20$ and
    $u(\mathbb{E}[X]) = \ln(\mathbb{E}[X]) = \ln(35) \approx 3.56$,
    $x_X^c \approx 24.5$ (certainty equivalent)
  - $U(Y) = \dfrac{1}{6}\ln(10) + \ldots + \dfrac{1}{6}\ln(60) \approx 3.40$ and
    $u(\mathbb{E}[Y]) = \ln(\mathbb{E}[Y]) \approx 3.56$
    $x_Y^c \approx 29.9$ (certainty equivalent)
- there is risk aversion: $x_X^c < \mathbb{E}[X]$ and $x_Y^c < \mathbb{E}[Y]$ and the certainty equivalents are smaller than the
- as $U(X) < U(Y)$ (or $x_X^c < x_Y^c$) we see that Y is better than X

Example 2: comparing contingent and non-contingent goods with log-utility

The problem

Assumptions

- **contingent good**: has the possible outcomes $Y = (y_1, \dots, y_N)$ with probabilities $\pi = (\pi_1, \dots, \pi_N)$
- **non-contingent good**: has the payoff $\bar{y}$ where $\bar{y} = \mathbb{E}[Y] = \sum_{s=1}^{N} \pi_s y_s$ with probability 1
- **utility**: the agent has a vNM utility functional with a logarithmic Bernoulli utility function.

Would it be better if he received the certain amount or the contingent good ?

Example 2: comparing contingent and non-contingent goods with log-utility

The solution

1. the value for the non-contingent payoff $\bar{y}$ is

$$\ln(\bar{y}) = \ln(\mathbb{E}[Y]) = \ln\left(\sum_{s=1}^{N} \pi_s y_s\right)$$

   has the certainty equivalent

$$e^{\ln(\mathbb{E}[Y])} = \mathbb{E}[Y]$$

2. the value for the contingent payoff y is

$$U(Y) = \sum_{s=1}^{N} \pi_s \ln(y_s) = \mathbb{E}[\ln Y] = \ln(G\mathbb{E}[Y])$$

   where $G\mathbb{E}[Y] = \prod_{s=1}^{N} y_s^{\pi_s}$ is the geometric mean of Y
3. the certainty equivalent is

$$e^{\ln(G\mathbb{E}[Y])} = G\mathbb{E}[Y])$$

Example 2: comparing contingent and non-contingent goods with log-utility

The solution: cont

- Because the arithmetic average is larger than the geometric

$$\mathbb{E}[Y] > G\mathbb{E}[Y]$$

then he would be better off if he received the average endowment rather than the certainty equivalent

- The risk premium is

$$\mathcal{R}(Y) = \mathbb{E}[Y] - G\mathbb{E}[Y] > 0$$

Example 3: the value of insurance

The problem

- Let there be two states of nature $\Omega = \{L, H\}$ with probabilities $\mathbb{P} = (p, 1-p)$ $0 \leq p \leq 1$
- consider the outcomes
  - without insurance

    $$X = (x_L, x_H) = (x - L, x)$$

    where $L > 0$ is a potential damage and there is full coverage
  - with full insurance : $y_L = y_H = y$

    $$Y = (y, y) = (x - L + L - qL, x - qL) = (x - qL, x - qL)$$

    where q is the cost of the insurance
- Given L under which conditions we would prefer to be insured ?

Example 3: the value of insurance

The data

states	$s = L$	$s = H$	$\mathbb{E}[\cdot]$
probabilities	p	$1 - p$	
no insurance	$x - L$	x	$p\,(x - L) + (1 - p)\,x$
full insurance	$x - q\,L$	$x - q\,L$	$x - q\,L$

Example 3: the value of insurance

The solution

- It is better to be insured if

$$u(y) \geq \mathbb{E}[u(X)]$$

- that is if

$$u(x - qL) \geq pu(x - L) + (1 - p)u(x)$$

Example 3: the value of insurance

The solution

It is better to be insured

- if $u(.)$ is **linear** then it is better to insure if

$$x - qL \geq p(x - L) + (1 - p)x \Leftrightarrow p \geq q$$

if the **cost to insure is lower than the probability** of occurring the damage

Example 3: the value of insurance

The solution

It is better to be insured

- if $u(.)$ is **concave** $\quad x - qL$ should be higher than the certainty equivalent of X

  $$x - qL \geq v\left(pu(x-L) + (1-p)u(x)\right) \text{ where } v(.) \equiv u^{-1}(.)$$

  equivalently

  $$q \leq \frac{x - v\left(pu(x-L) + (1-p)u(x)\right)}{L}$$

- if $u(x) = \ln(x)$

  $$q \leq \frac{x - (x-L)^p\, x^{1-p}}{L} = \frac{1}{L}\Big(x - \mathbb{GE}[X]\Big)$$

  better to insure if the cost is not too high

Example 4: interpersonal comparison of risk attitudes

The issue

- Consider:
  - two agents A and B with **different** (Bernoulli) utility functions
  $$u^A(y) \text{ and } u^B(y)$$
  - with the **same** information
  $$P = \left(\pi_1, \dots \pi_n\right)$$
  - with **same** contingent income
  $$Y = \left(y_1, \dots y_n\right)$$
- We defined three different, but equivalent, ways of comparing their behavior regarding risk aversion

Example 4: interpersonal comparison of risk attitudes

The issue

- Agent A **is more risk averse than** B if
  - her/his **utility valuation of** Y **is lower**

$$U^A(Y) < U^B(Y) \iff \mathbb{E}[u^A(Y)] < \mathbb{E}[u^B(Y)]$$

  - her/his **certainty equivalent for** Y **is smaller**

$$y^{c,A} < y^{c,B}$$

  - her/his **risk premium for** Y **is higher**

$$\mathcal{R}^A(Y) > \mathcal{R}^B(Y)$$

Example 5: the cost of macroeconomic volatility for Portugal

Period:1970-2014

- The actual rate of growth was stochastic G
- The average growth factor for Portugal is $\mathbb{E}[G] = 1.02039$
- Question: what would be the certainty equivalent growth rate $\mathbb{CE}[G]$?
- How much rate of growth we would be willing to sacrifice to avoid macroeconomic volatility

  (see Problem set 3)

Example 5: the cost of macroeconomic volatility for Portugal

Period:1970-2014

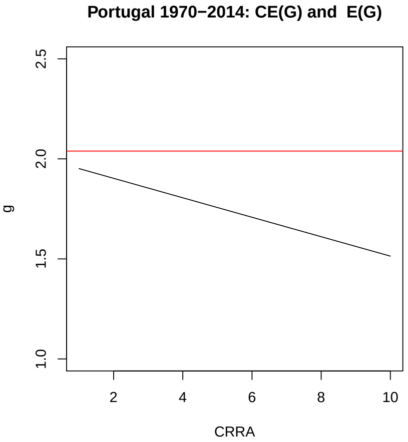
- To answer the question we need
  - from data determine P and G
  - assume an utility function: CRRA for instance

$$u(g) = \frac{g^{1-\zeta} - 1}{1 - \zeta}$$

- Conclusion:
  - the cost is relatively low: less that 0.2% per year (for $\zeta = 2$)
  - increases with the CRRA coefficient

Example 5: the cost of macroeconomic volatility for Portugal

Period:1970-2014



R script

References

- (LeRoy and Werner, 2014, Part III), (Lengwiler, 2004, ch. 2), (Altug and Labadie, 2008, ch. 3)

Sumru Altug and Pamela Labadie. *Asset pricing for dynamic economies*. Cambridge University Press, 2008.

Yvan Lengwiler. *Microfoundations of Financial Economics*. Princeton Series in Finance. Princeton University Press, 2004.

Stephen F. LeRoy and Jan Werner. *Principles of Financial Economics*. Cambridge University Press, Cambridge and New York, second edition, 2014.