New-Keynesian models of capital accumulation

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1 Introduction

In the benchmark DGE model[1] all the markets are competitive (i.e, all agents are price-takers), and there is only one good which was used both for final consumption and for investment. In this model the equilibrium is Pareto optimal, which makes it equivalent to the Ramsey model of optimal capital allocation.

There are two common features of new-Keynesian models (NK):

- first, equilibrium is not Pareto optimal;
- second, there can be multiple asymptotic equilibria and/or multiple equilibrium paths (indeterminacy), which may justify the need to some type of intervention by economic authorities.

New-Keynesian models are DGE models in which some markets are not competitive. Imperfect competition is usually introduced by separating the final good production from intermediate good production and specifying the existence of a continuum or intermediate goods in which every producer is monopolistic in its own market and there is monopolistic competition among producers. In these models the DGE is not necessarily Pareto optimal.

A fundamental reason for developing those models stems from the need to justify the empirical observation on the existence of the Phillips curve, which apparently renders counterfactual two important characteristics of the classic macroeconomics: the separation of the real and nominal variables, and the neutrality of money. An important strand of New-Keynesian models deals with the existence of nominal rigidities and the need to a micro-founded model to address them. In most of the New-Keynesian models of inflation and unemployment the stock of capital is taken as constant.[2] However, in this lecture we deal with new-Keynesian models in which there is capital accumulation. Section 2 presents a simple new-Keynesian model with monopolistic competition.

Multiple steady states exist if the aggregate rate of return of capital displays non-convexities, stemming from local non-existence of decreasing marginal returns. In section 3 we extend the model of section 2 by considering the transition from a MC regime to a Cournotian monopolistic competition regime (CMC) in which there can be entry of firms and monopolies change to oligopolies in the intermediate goods markets. In this case we will see that (deterministic) multiple steady states can exist. This means that, given some shocks to an economy, the general competitive level can increase or decrease.

However, in some cases non-convexities can be introduced by the existence of externalities. Section 4 presents an abbridged version of the Benhabib and Farmer (1994) model. We see that, if the elasticity of labor supply is sufficiently low relative to the share of labor in the aggregate

[1] See `https://pmbbrito.github.io/cursos/phd/am/am2122_ramsey.pdf`.

[2] See Woodford (2003) or Galí (2008).

production the equilibrium can be indeterminate, in the sense that there are an infinite number of equilibrium paths.

2 A monopolistic competitive model

This section presents a new-Keynesian (NK) model for an economy with a structure similar to the basic RBC model. It features a decentralized with two types of product markets, a final good and a continuum of intermediate goods, and two factor markets (labor and capital). The final good market and the factor markets are competitive (in the sense that all participants are price-takers) but there is monopolistic competition in the intermediate goods' markets.

We will see that the equilibrium representation is similar to the Ramsey model, with the difference that there is a markup over the marginal rate of return of capital.

2.1 The model

In this section we present the problems of the three types of representative agents of the model: households, the final good producer and the generic problem for an intermediate good producer. The last subsection defines the DGE for this economy.

We introduce several simplifying assumptions. In particular, we assume that the final product sector only uses intermediate inputs, all physical capital and labor are used in the intermediate goods' sector, the labor supply is inelastic, there are no adjustment costs for both types of inputs, and there is a classic dichotomy between prices and quantities. This means, in the NK terminology, that there are no real or nominal rigidities.

2.1.1 Household problem

The representative household consumes, offers inelastically labor and invests in a risk-free financial asset. It has an initial level of net financial wealth, and receives a flow of labor and financial income. We assume that its preferences are represented by an intertemporal additive utility functional and a CRRA utility function. Its problem is

$$
\begin{aligned}
\max_{C} V[C] = \int_0^\infty \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t}\, dt \\
& \text{subject to} \\
& \dot{W} = \omega(t)L(t) + r_w(t)W(t) - C(t) \\
& W(0) = W_0, \text{ given} \\
& \lim_{t\to\infty} e^{-R_w(t)} W(t) \geq 0,
\end{aligned}
\tag{1}
$$

where $C(\cdot)$, $L(\cdot)$ and $W(\cdot)$ denote consumption, labor supply, and the net financial wealth, respectively. In addition, $\omega(t)$ and $r_w(t)$ denote the wage rate and the real interest rate and $R(t) = \int_0^t r(s)ds$. Observe that, because the wage rate and the rate of return of capital are endogenous at the general equilibrium level, we set them as variable. All the variables are real, deflated by the final good price $P(t) = 1$.

The first order conditions, are, from Pontryiagin's maximum principle

$$
\dot{W} = \omega(t)L(t) + r_w(t)W(t) - C(t) \tag{2}
$$

$$
\frac{\dot{C}}{C} = \frac{1}{\theta}(r_w(t) - \rho) \tag{3}
$$

together with the initial condition $W(0) = W_0$ and the transversality condition $\lim_{t\to\infty} C(t)^{-\theta}\, W(t)\, e^{-\rho\, t} = 0$.

2.1.2 Final producer problem

Final output uses a continuum of intermediate goods of varieties $j \in [0,1]$ with a constant elasticity of substitution (CES) technology, via a Dixit and Stiglitz (1977) aggregator. Denoting the quantity of intermediate input of variety j used at time t by $x(j,t)$ and by $\mathbf{x} = \big(x(j,t)\big)_{j\in[0,1]}$ the ensemble of all inputs the production function is a functional over $x(\cdot)$,

$$
F\big(\mathbf{x}(t)\big) = \left(\int_0^1 x(j,t)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}},
$$

where the elasticity of substitution between is $\varepsilon > 1$. The profit can be written as a functional over $\mathbf{x}$,

$$
\pi(\mathbf{x}(t)) = P(t)\, Y(t) - C(\mathbf{x}(t)),
$$

where $C(\mathbf{x}) = \int_0^1 p(j,t)\, x(j,t)\, dj$ is total cost (again a functional over $x(\cdot)$).

The final producer is a price-taker in both markets, inputs and output, seeks to maximize the total costs for producing the output quantity $Y(t)$, where given the ensemble of input prices $\mathbf{p} = \big(p(j,t)\big)_{j\in[0,1]}$

$$\max_{x(\cdot,t)} \pi(\mathbf{x}(t)) \text{ s.t. } F\big(\mathbf{x}(t)\big) = Y(t) \tag{4}$$

The demand for input j is[3]

$$x(j,t) = \left(\frac{p(j,t)}{P(t)}\right)^{-\varepsilon} Y(t), \text{ for every } j \in [0,1], \text{ and } t \in [0,\infty) \tag{5}$$

where because of perfect competition the market price is equal to the marginal cost of producing one unit of output

$$P(t) = P^*(t) = \left(\int_0^1 p(j,t)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \quad t \in [0,\infty).$$

Because we have no monetary variables we will set $P(t) = 1$, for all t. We see that the demand of variety j is negatively related with its own price and positively related with the output product price and the level of activity Y.

2.1.3 Intermediate producer of variety j

We assume that the number of industries, which produce intermediate inputs, is fixed and is normalized to one.

Next we present and solve the problem for the firm in any of the industry $j \in [0,1]$, assuming that this producer is a monopolist. This is a case of monopolistic competition (MC) because, although the firm in any industry is a price-setter on the market for its output, $p(j,\cdot)$, it has to compete with all other industries in the supply of intermediate products to the producer of the final good, whose demand function is given in equation (5). An increase in the price of product j will generate a reduction in its demand.

The technology for product j is benchmark: It uses labour and capital with a Cobb-Douglas technology

$$x(j,t) = A\, k(j,t)^{\alpha} \ell(j,t)^{1-\alpha} \tag{6}$$

where $k(j,t)$ and $\ell(j,t)$ are the capital and labour inputs for producing $x(j,t)$.

In order to simplify the model we assume a homogeneous technology across industries. In particular, it is assumed that the TFP is not product specific, that is $A(j) = A$ for every $j \in [0,1]$.

The instantaneous profit for producer j is, in real terms evaluated at the price of the final good

$$\pi(j,t) = \frac{x(j,t)p(j,t)}{P(t)} - w(t)\ell(j,t).$$

[3]See Appendix A.

where $x(j,t)$ is equal to the demand for intermediate product j by the final producer. Substituting the demand function (5) yields

$$
\begin{aligned}
\pi(j,t) &= \frac{x(j,t)p(j,t)}{P(t)} - w(t)\ell(j,t) = \\
&= \left(\frac{x(j,t)}{Y(t)}\right)^{-\frac{1}{\varepsilon}} x(j,t) - \omega(t)\ell(j,t) = \\
&= x(j,t)^{1-\mu}Y(t)^{\mu} - \omega(t)\ell(j,t)
\end{aligned}
$$

where $\mu \equiv 1/\varepsilon \in (0,1)$ is the Lerner index. If we substitute the production function (6), the profit of the intermediate producer j becomes

$$
\pi(j,t) = \left(Ak(j,t)^{\alpha}\ell(j,t)^{1-\alpha}\right)^{1-\mu} Y(t)^{\mu} - \omega(t)\ell(j,t). \tag{7}
$$

As we assume that capital is used in the production of intermediate goods, the problem of each intermediate producer is dynamic. We assume the classic Jorgenson (1967) model for the producer in which there are no adjustment costs in investment.

The firm's objective is to maximize the present-value of the cash-flow discounted by the market interest rate, subject to the accumulation equation for capital:

$$
\begin{aligned}
&\max_{i(j,t),\ell(j,t)} \int_0^{\infty} \left(\pi(j,t) - i(j,t)\right) e^{-R(s)}\, dt \\
&\text{subject to} \\
&\frac{dk(j,t)}{dt} = i(j,t) - \delta k(j,t), \text{ for each } t \in [0,\infty) \\
&k(j,0) = k_0, \text{ given}
\end{aligned} \tag{8}
$$

for $R(t) = \int_0^t r(s)\, ds$, where $r(t)$ is the market rate of return of capital, and we assume that $k(j,t)$ is asymptotically non-negative. The optimum demand for labor and capital are symmetric, in the sense that they are the same for producer of all intermediate goods,[4]

$$
\begin{aligned}
\ell^*(t) &= (1-\mu)\left(\frac{1-\alpha}{w(t)}\right)\pi^*(t), \text{ for each } j \in [0,1] \\
k^*(t) &= (1-\mu)\left(\frac{\alpha}{r(t)+\delta}\right)\pi^*(t), \text{ for each } j \in [0,1]
\end{aligned} \tag{9}
$$

where

$$
\pi^*(t) = \left[(1-\mu)\, A \left(\frac{\alpha}{r(t)+\delta}\right)^{\alpha}\left(\frac{1-\alpha}{w(t)}\right)^{1-\alpha}\right]^{\frac{1-\mu}{\mu}} Y(t), \text{ for each } j \in [0,1]
$$

[4]See Appendix B.

is the optimal level of profit, which is a function of the cost of labor and capital inputs and of the output of the final good. Therefore, the optimal supply of intermediate inputs is also symmetric across industries, that is $x^*(j,t) = x^*(t)$ for all $j \in [0,1]$ where

$$x^*(t) = Y(t)^{-\frac{\mu}{1-\mu}}\, \pi^*(t)^{\frac{1}{1-\mu}},\ \text{for each } j \in [0,1],$$

is also a function of the final output and of the profit of the intermediate producer. Differently from the final producer case, the profit is different from zero because there is monopolistic competition (MC): the intermediate producer of variety j is a monopolist in its own market but competes with the producer of all the other varieties that enter in the production of the final output $Y(t)$. It has a Lerner-markup given by $\mu = \frac{\varepsilon-1}{\varepsilon}$ which decreases with the elasticity of substitution ε.

2.1.4 Aggregation

Because different inputs are measured in real terms, they can be measured in different physical units. Therefore, in order to build the aggregate capital stock and labor input we need to choose an appropriate aggregator.

If we use the Dixit and Stiglitz (1977) aggregator we have the aggregate demand for capital and labor

$$K(t) = \left(\int_0^1 k^*(j,t)^{\frac{\varepsilon-1}{\varepsilon}}\, dj\right)^{\frac{\varepsilon}{\varepsilon-1}} = k^*(t)$$

and

$$L(t) = \left(\int_0^1 \ell^*(j,t)^{\frac{\varepsilon-1}{\varepsilon}}\, dj\right)^{\frac{\varepsilon}{\varepsilon-1}} = \ell^*(t).$$

where $k^*(t)$ and $\ell^*(t)$ are given in equation (9). Total investment is also obtained from

$$I(t) = \left(\int_0^1 i^*(j,t)^{\frac{\varepsilon-1}{\varepsilon}}\, dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

although it is indeterminate, as is well known in a Jorgenson (1967) model, but can be determined at the equilibrium level.

2.1.5 General equilibrium

The **general equilibrium** is defined by the aggregate flows of consumption, $\big(C(t)\big)_{t\in[0,\infty)}$, output $\big(Y(t)\big)_{t\in[0,\infty)}$, and by the allocation flows of production, capital input, labor input and prices of intermediate-goods, $\Big(\big(x(j,t), k(j,t), \ell(j,t), p(j,t)\big)_{j\in[0,1]}\Big)_{t\in[0,\infty)}$, the rate of return of capital $\big(r(t)\big)_{t\in[0,\infty)}$, and the wage rate $\big(\omega(t)\big)_{t\in[0,\infty)}$ such that

1. households solve their problem, in equation (1), given the rate of return and the wage rate;

2. the final producer solve its problem, in equation (4), given the prices of the intermediate goods;

3. every intermediate producer, $j \in [0,1]$, solve its problem, in equation (8), given the rate of return and the wage rate;

4. the balance sheet and consistency conditions hold: $W(t) = K(t)$ (households own firms);

5. the market clearing conditions, for the final good market

$$Y(t) = C(t) + I(t), \text{ for each } t \in [0, \infty)$$

and for the factor markets

$$L(t) = 1, \text{ and } r_w(t) = r(t) - \delta, \text{ for each } t \in [0, \infty).$$

2.2 General equilibrium representation and dynamics

There are several consequences of the previous definition (we skip next the optimum notation). First, the supply of the final good, given the demand at equilibrium of the intermediate inputs is

$$Y(t) = \left(\int_0^1 x(t)^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}} = x(t) = Y(t)^{-\frac{\mu}{1-\mu}} \, \pi^*(t)^{\frac{1}{1-\mu}}$$

is equal to the profit of intermediate goods' firms $Y(t) = \pi^*(t)$. Second, as $\pi(t) = \Big(A\, k(t)^{\alpha}\, \ell(t)^{1-\alpha} \Big)^{1-\mu} Y(t)^{\mu}$ then the aggregate output is a Cobb-Dougkes function of the aggregate capital and labor stocks demanded by firms: $Y(t) = A\, k(t)^{\alpha}\, \ell(t)^{1-\alpha}$.

Third, from the market clearing conditions (and aggregation) then the aggregate output

$$Y(t) = A\, k(t)^{\alpha}.$$

Fourth, the total income distributed to households is $\omega(t)\, \ell(t) + r_w(t)\, W(t) = \omega(t) + r(t)\, k(t)$. But, at the aggregate, $\omega(t) + (r(t) + \delta)\, k(t) = (1-\mu)\, \pi(t)$. Therefore,

$$\omega(t)\, \ell(t) + r_w(t)\, W(t) = (1-\mu) Y(t) - \delta\, k(t).$$

Then, the budget constraint of the household $\dot{W} = \omega(t)\, \ell(t) + r_w(t)\, W(t) - C(t)$ is equivalent at the equilibrium to

$$\dot{k} = (1-\mu) Y(t) - \delta\, k(t) - C(t).$$

We could obtain the same condition from the final good's market equilibrium $Y(t) = C(t)+I(t) = C(t) + \dot{k} - \delta\, k(t)$

At last, the equilibrium condition in the capital market becomes

$$r_w(t) = r(t) - \delta = \alpha\,(1-\mu)\frac{\pi^*(t)}{k(t)} - \delta = \alpha\,(1-\mu)A\,k(t)^{\alpha-1} - \delta,$$

which allows us to write the Euler equation as $\dot{C} = \frac{C}{\theta}\big(r(t) - \rho - \delta\big)$.

2.3 Characterizing the equilibrium

Therefore the equilibrium is the solution to the system

$$\begin{aligned}\dot{k} &= y(t) - C(t) - \delta k(t)\\ \dot{C} &= \frac{C(t)}{\theta}\big(r(t) - \rho - \delta\big)\end{aligned}$$

where

$$r(t) = (1-\mu)\alpha k(t)^{\alpha-1},\ \text{and}\ y(t) = (1-\mu)k(t)^{\alpha}$$

which looks like the Ramsey model with a distortion introduced by $m \equiv 1-\mu$ which is an (exogenous) markup.

The dynamics of the model is similar to the Ramsey model with the difference that it converges to the steady state (K^*_M, C^*_M)

$$\begin{aligned}K^*_M &= \Big(\frac{(1-\mu)\,\alpha\,A}{\rho+\delta}\Big)^{\frac{1}{1-\alpha}}\\ C^*_M &= \frac{\rho+\delta\,(1-\alpha)}{\alpha}\,K^*_M.\end{aligned} \tag{10}$$

Clearly the steady state stock of capital in this NK model is smaller than in the competitive case: $K^*_M < K^*_C = \Big(\frac{\alpha\,A}{\rho+\delta}\Big)^{\frac{1}{1-\alpha}}$. This implies that the steady state consumption level is smaller as well $C^*_M < C^*_C = \frac{\rho+\delta\,(1-\alpha)}{\alpha}\,K^*_C$. The first case refers to the monopolistic case and the second to the competitive case (se Figure 1)

3 A Cournotian monopolistic competitive model

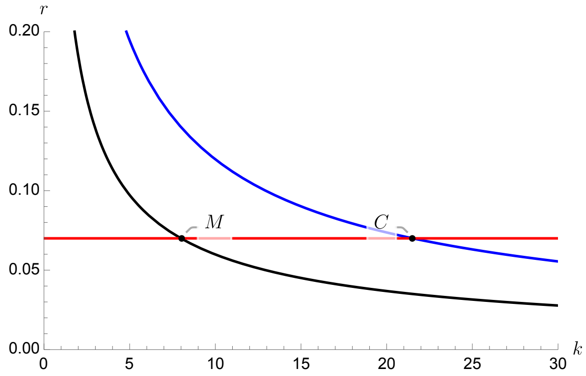

Figure 1: Competitive monopolistic and competitive equilibria

Until now, we have dealt with two limiting situations: either there is full competition (the DGE-Ramsey case) or monopoly in the intermediate product sector (the previous case). Furthermore, in the first case the markup is equal to zero and in the second markups are positive but constant.

A more realistic situation is the one in which there is an intertemediate level of competition and markups are endogenous and can vary between sectors. In this case we should allow for entry in every sector.

We assume that here are $z(t) \in (0,1]$ industries j, that is $j \in [0, z(t)]$. We also assume that the number of firms in any industry, denoted by $n(j,t)$ can be larger than one. If $n(j,t) = 1$ this means that there is only one producer, a monopolist, in industry j at time t.

Entry can take two forms: if $z(t) < 1$ then an entrant can start a new industry, thus becoming a monopolist; but if $z(t) = 1$ then entry can only be done by entering an existing industry, thus increasing the level of competition in industry j. Of course exit can have the inverse type of effect.

In this section we present an abridged version of Brito et al. (2013).

3.1 The model

Next we present the problems for the representative household, for the producer of the final good, for a firm in an intermediate good industry, equilibrium at the industry level and the general equilibrium.

Household's problem

$$
\begin{aligned}
\max_{c(\cdot),\ell(\cdot)} & \int_0^\infty \ln\left(c(t)\right) e^{-\rho\, t}\, dt \\
& \text{subject to} \\
& \dot{k} = w(t) + r(t)\, k(t) + \pi(t) - c - \delta\ k \\
& k(0) = k_0
\end{aligned}
$$

Optimal consumption satisfies the Keynes-Ramsey rule

$$
\dot{c} = c\left(r(t) - \rho - \delta\right)
$$

Final production sector

The mass of internediate goods at time t is $0 < z(t) \leq 1$. The final good uses a continuum of intermediate goods $j \in [0, z(t)]$ with technology specified by the CES production function

$$
Y(t) = z(t)^{\frac{1}{1-\varepsilon}} \left(\int_0^{z(t)} x(j,t)^{\frac{\varepsilon-1}{\varepsilon}}\, dj \right)^{\frac{\varepsilon}{1-\varepsilon}}.
$$

Assuming again that the price of the final good is $P(t) = 1$, for every $t \in [0, \infty)$, we find the demand for variety j

$$
x^d(j,t) = p(j,t)^{-\varepsilon}\, \frac{Y(t)}{z(t)} \tag{11}
$$

where

$$
1 = \frac{1}{z(t)} \int_0^{z(t)} p(j,t)^{1-\varepsilon} dj.
$$

Intermediate production at the industry level

Industry $j \in [0, z(t)]$, for $0 < z(t) \leq 1$ is characterized by the price $p(j,t)$ and by total output

$$
x(j,t) = \int_1^{n(j,t)} x(i,j,t)\, di
$$

where $n(j,t) \geq 1$ is the number of firms in industry j and $x(i,j,t)$ is the output of firm i producing variety j. We already know that the demand for variety j is given by equation (11).

Problem for an intermediate production firm

Now we consider the problem for a firm i that has entered any industry j.

The production technology for the representative firm i in industry j is

$$x(i,j,t) = A\,k(i,j,t)^{\alpha}\,\ell(i,j,t)^{1-\alpha} - \phi$$

where ϕ are fixed costs.

The problem for a firm i in industry j is

$$\max_{x(\cdot),\ell(\cdot),k(\cdot)} \pi(i,j,t) = p(j,t)\,x(i,j,t) - w(t)\ell(i,j,t) - r(t)\,k(i,j,t)$$
$$\text{subject to}$$
$$x(i,j,t) = A\,k(i,j,t)^{\alpha}\,\ell(i,j,t)^{1-\alpha} - \phi$$

with the inverse demand function

$$p(j,t) = \left(x(j,t)\frac{z(t)}{Y(t)}\right)^{-\frac{1}{\varepsilon}}$$

where $x(j,t) = \int_1^{n(j,t)} x(i,j,t)\,di$ is the total demand for the product of industry j. Therefore

$$\frac{\delta p(j,t)}{\delta x(i,j,t)} = -\frac{1}{\varepsilon}\frac{p(j,t)}{x(i,j,t)}$$

The first order conditions are

$$\begin{aligned}(1-\mu(i,j,t))\,(1-\alpha)\,A\left(\frac{k(i,j,t)}{\ell(i,j,t)}\right)^{\alpha} &= \frac{w(t)}{p(j,t)}\\ (1-\mu(i,j,t))\,\alpha\,A\left(\frac{k(i,j,t)}{\ell(i,j,t)}\right)^{\alpha-1} &= \frac{r(t)}{p(j,t)}\end{aligned}$$

where $\mu(i,j,t) = \dfrac{x(i,j,t)}{\varepsilon\,x(j,t)} \in (0,1)$ is the Lerner index for firm i in industry j.

Equilbrium in industry j

In a monopoly then $n(j) = 1$ then $x(i,j,t) = x(j,t)$ and the markup is $\mu(i,j,t) = \mu(j,t) = 1/\varepsilon$. In an oligopoly then $n(j,t) > 1$ and then $0 < \mu(i,j,t) < 1/\varepsilon$.

In a symmetric equilibrium $x(i,j,t) = x(j,t)/n(j,t)$ then the Lerner index in industry j is

$$\mu(j,t) = \frac{1}{\varepsilon\,n(j,t)}$$

A symmetric inter-industry equilibrium yields

$$n(j,t) = n(t),\ \text{and}\ \mu(j,t) = \frac{1}{\varepsilon n(t)}\text{for every } j \in [0, z(t)]$$

The profit for a firm in any industry is

$$\pi = \mu\, A\, K^{\alpha} - z\, n\, \phi.$$

Entry is still undefined. There are two regimes:

1. there is MC when $n = 1$ and $z < 1$. An entrant has to start a new industry, implying that the zero-profit condition determines $z \in (0, 1)$. IN this case the markup is as in the previous section
$$\mu = \frac{1}{\varepsilon}$$

2. there is CMC when $z = 1$ and $n > 1$. Entry increases competition within industries, and therefore determines n. As $n\, \varepsilon\, \mu = 1$ the free entry condition ($\Pi = 0$). In this case $\pi = \mu\, A\, K^{\alpha} - z\, n\, \phi$ with $\mu = \frac{1}{\varepsilon n}$ which yields an endogenous markup
$$\mu = m(K) = \sqrt{\frac{\phi}{\varepsilon A K^{\alpha}}}$$

3. at the boundary we have $z = n = 1$

Therefore, the markup is
$$\mu = \mu(K) = \min\left\{\frac{1}{\varepsilon}, m(K)\right\}$$

General equilibrium

Aggregating over industries, as in the previous section, we find the final output

$$Y = AK^{\alpha} - z\, n\, \phi$$

and the rate of return of capital
$$r(k) = (1-\mu)\alpha A K^{\alpha-1}$$

Market clearing condition for the final good's market

$$Y(t) = C(t) + I(t) = C(t) + \dot{K} + \delta\, K(t)$$

Therefore,
$$\dot{K} = (1-\mu) A K^{\alpha} - C - \delta\, K$$

3.2 Characterizing general equilibrium

We can represent the general equilibrium by a dynamic system with two regimes:

$$\dot{K} = Y(K) - \delta K - C$$
$$\dot{C} = C\left(r(K) - (\rho + \delta)\right)$$

where the aggregate output is

$$Y(K) = (1 - \mu(K))\, A\, K^{\alpha}$$

and the rate of return of capital is

$$r(K) = (1 - \mu(K))\, \alpha\, A\, K^{\alpha-1}.$$

In Brito et al. (2013) it is proved that there exists a critical value for the inverse of the Frisch elasticity:

$$\bar{\varepsilon} \equiv \frac{2-\alpha}{2\,(1-\alpha)} > 1$$

and a two critical value for the fixe cost

$$\tilde{\phi} \equiv \frac{1}{\varepsilon}\Big(A\Big(\frac{\alpha\,(\varepsilon-1)}{(\rho+\delta)\,\varepsilon}\Big)^{\alpha}\Big)^{\frac{1}{1-\alpha}}$$

and

$$\bar{\phi} \equiv \frac{\varepsilon}{\bar{\varepsilon}^2}\Big(A\Big(\frac{\alpha\,(\bar{\varepsilon}-1)}{(\rho+\delta)\,\bar{\varepsilon}}\Big)^{\alpha}\Big)^{\frac{1}{1-\alpha}}$$

such that (see Figure 2):

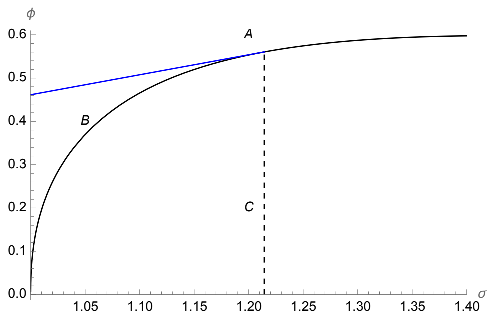

Figure 2: Bifurcation diagram

A if $\varepsilon > 1$ and $\phi > \max\{\bar{\varepsilon}, \bar{\tilde{}}\}$ then there is a single steady state in which there is MC (this is the case in the previous section). This steady state is also a saddle point (see Figure 3)

B if $1 < \varepsilon < \bar{\varepsilon}$ and $\tilde{\varepsilon} < \phi < \bar{\varepsilon}$ there are three steady states: a MC steady state and two CMC steady states, one for a low level of entry and the second for a high level of entry. The first steady state is close to a monopoly steady state. One of the CMC steady states corresponds to a low level of entry, a high markup, and to a low level of capital stock, and is unstable. The second CMC steady state has a high level of entry, a low level of markups. and the steady state level of capital capital stock is also high, and it is a saddle point. Therefore, it is (locally) close to a Ramsey case (see Figure 3);

C if $\varepsilon > 1$ and $0 < \phi < \max\{\bar{\varepsilon}, \tilde{\varepsilon}\}$ then there is a single steady state in which there is CMC. This case is closer to the Ramsey model. This steady state is also a saddle point. (see Figure 3)

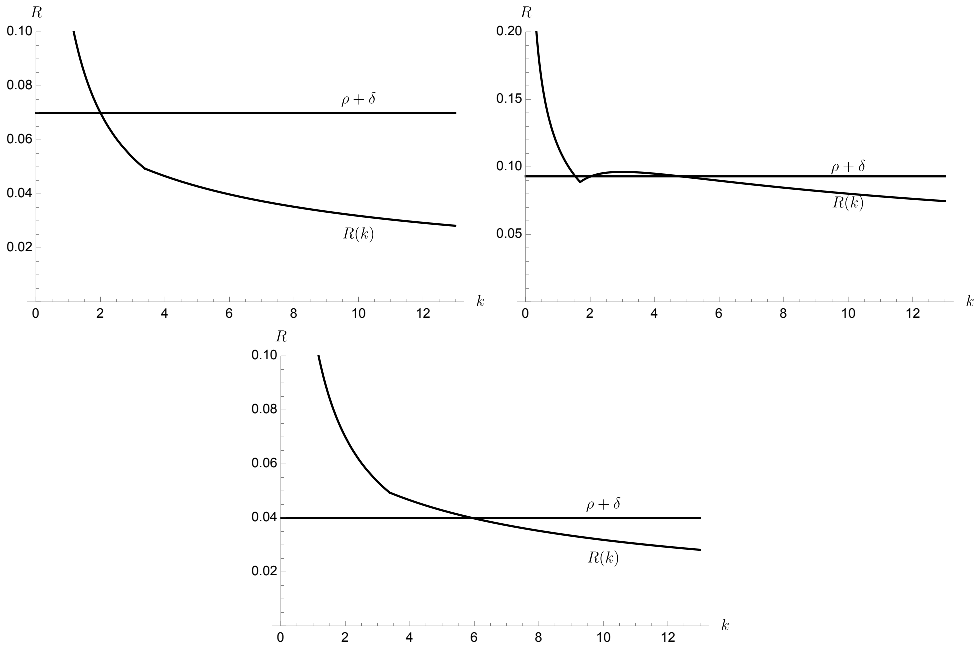

Figure 3: Steady state for the three cases

In the case B there are multiple steady states.

If the initial point $K(0) = K_0$ is lower than the middle equilibrium point (K_M) the economy will converge to the lower steady state in which there is a smaller number of sectors $z(\infty) < 1$ and there is a monopolist in every sector. Observe that, even if the K_0 corresponds to a case in which $z(0) = 1$ and $n(0) > 1$ there will be a reduction in the number of firms until it reaches a point in which $z(t) = n(t) = 1$ and, from that point on, some sectors disapear.

If the initial point $K(0) = K_0$ is higher then the middle equilibrium point (K_M) then the number of industries will be kept constant $z(t) = 1$ and there will be entry in all industries until the aggregate stock of capital satisfies

$$r(K_L) = (1 - \mu(K_l))\, \alpha\, A\, K_L^{\alpha-1} = \rho + \delta$$

which implies a steady state number of firms per industry equal to $n(K_l) = \sqrt{\frac{A\, K_L^{\alpha}}{\phi\, \varepsilon}}$.

A reduction in productivity can have the effect of changing the transition dynamics from the second case to the first case.

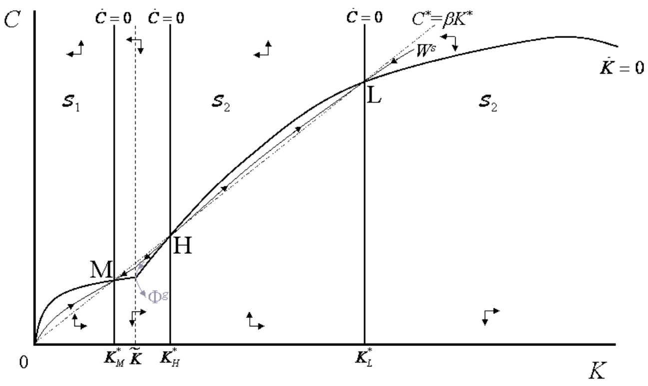

Figure 4: Phase diagram for the case B

4 A new-Keynesian model with externalities

In all those cases the steady state is locally determinate: given an initial level for the stock of capital there is a unique trajectory converging to the steady state, even though, with a different initial level for the stock of capital the economy can converge to a **different** steady state.

Next we present a case in which the is an unique steady state but there are multiple trajectories converging to it: that is, the equilibrium is indeterminate.

Benhabib and Farmer (1994) is an important paper featuring a new-Keynesian model in which there are increasing returns to scale and externalities similar to Romer (1990) model in growth theory. In particular, it presents conditions under which the general equilibrium can be indeterminate even though agents have perfect foresight.

4.1 The model

The **household production function** is

$$y = Ak^a\, \ell^b, \text{for } a > 0,\ b > 0,$$

where productivity is a function of the aggregate capial and labor inputs

$$A = K^{\alpha-a}\, L^{\beta-b},$$

That is, the aggregate capital and labor inputs generate a positive externality over the individual household. This can account for several different factors: infrastructures, network externalities, etc.

The **aggregate production function** is

$$Y = K^{\alpha}\, L^{\beta},\ \text{for } \alpha > 0,\ \beta > 0.$$

W assume that $\alpha+\beta > 1$, meaning that there are increasing returns to scale at the aggregate level, and $\alpha > a$ and $\beta > b$.

We can simplify the problem by assuming that the representative household consumes and does home-production and decides over investment

The **general equilibrium** for this economy is represented by the paths of capital $\big(k(t)\big)_{t\geq 0}$, labor effort $\big(\ell(t)\big)_{t\geq 0}$, consumption $\big(c(t)\big)_{t\geq 0}$ such that:

1. the representative household solves the problem

$$\begin{aligned} \max_{c(\cdot),\ell(\cdot)} \int_0^{\infty} & \ln\left(c(t)\right) - \frac{\ell(t)^{1+\xi}}{1+\xi}\, e^{-\rho\, t}\, dt \\ & \text{subject to} \\ & \dot{k} = y - Ak^{\alpha}\, \ell^{\beta} - c - \delta\ k \\ & k(0) = k_0 \end{aligned}$$

   given the aggregate paths of capital $\big(K(t)\big)_{t\geq 0}$ and labor input $\big(L(t)\big)_{t\geq 0}$;

2. the micro-macro consistency conditions

$$C(t) = c(t),\ K(t) = k(t),\ \text{and } L(t) = \ell(t),\ \text{for every } t \in [0,\infty)$$

   hold.

First-order conditions for the household

$$\begin{aligned} \dot{c} &= c\left(a\,\frac{\hat{y}}{k} - \rho - \delta\right) \\ \hat{\ell}^{1+\xi} &= \frac{b\,\hat{y}}{c} \end{aligned}$$

where $\hat{y} = A(K, L)\, k^a\, \ell^b$. Substituting the micro-macro consistency conditions, defining

$$\epsilon \equiv \beta - (1 + \xi)$$

and representing the dynamic system in (K, L) we have (see Brito et al. (2017))

$$\dot{K} = K^{\alpha} L^{\beta}(1 - bL^{\epsilon-\beta}) \ ,$$
$$\dot{L} = \frac{L}{\epsilon} \left[K^{\alpha-1} L^{\beta} \left(a - \alpha(1 - bL^{\epsilon-\beta})\right) - \rho\right],$$

where ϵ can take any sign.

There is a unique positive steady-state

$$\bar{K} = (a/\rho)^{\frac{1}{1-\alpha}}\ \bar{L}^{\frac{\beta}{1-\alpha}}, \text{ and } \bar{L} = b^{\frac{1}{\beta-\epsilon}}.$$

If we determine the Jacobian, evaluated at the steady state we obtain the trace and determinant

$$\text{tr}DF(\bar{K}, \bar{L}) = \frac{\rho^2(1-\alpha)(\beta-\epsilon)}{a\epsilon}, \ \det DF(\bar{K}, \bar{L}) = \frac{\rho\,(\beta(a-\alpha) + \epsilon\alpha)}{a\epsilon}.$$

We can easily see that both these quantities can take infinite values for $\epsilon = 0$, which is a case we exclude from now on. With the previous assumptions on the parameters we find that the dimension of the stable manifold depends on ϵ: for $\epsilon < 0$ the steady-state is either a stable focus or node and for $\epsilon > 0$ it is a saddle, as can be seen in Figure 5 which represents the phase diagram in the space (K, L):

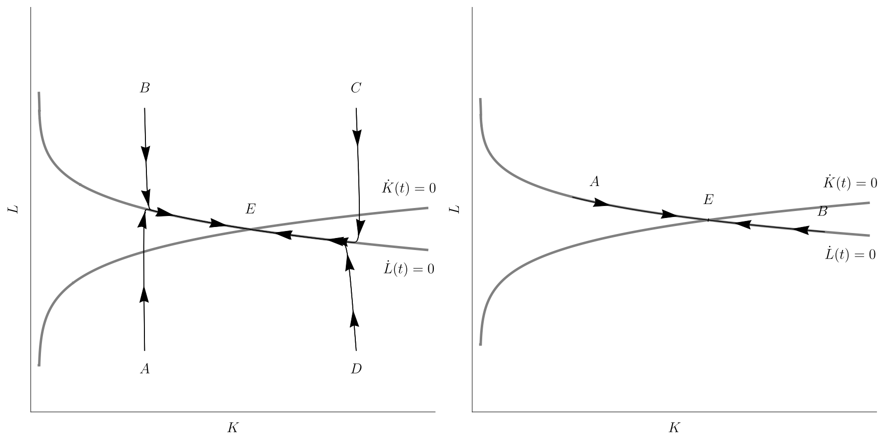

Figure 5: Phase diagrams for the Benhabib and Farmer (1994) model: LHS case $\epsilon > 0$, RHS case $\epsilon < 0$

On the left-hand-side (LHS) panel, we represent the phase diagram for $\epsilon > 0$. We can observe that there is a unique steady-state equilibrium represented by point E. Since there are two negative eigenvalues associated with this stationary point, all DGE paths converge asymptotically to E. However, the steady-state is locally and globally indeterminate, as there is an infinite number of initial values for L, for a given K_0, leading to the long-run equilibrium. We can also see that L adjusts very fast so that the trajectory quickly approaches the isocline $\dot{L} = 0$ and then K starts adjusting more slowly until the steady-state is reached.

On the right-hand-side (RHS) panel, we represent the phase diagram for $\epsilon < 0$, also small. Now, the unique steady-state is locally and globally determinate, as there is one positive and one negative eigenvalue associated with it. For each initial level for the capital stock, K_0, there is only one value of L, such that convergence to the steady-state is asymptotically verified . Notice that the stable manifold associated to the steady-state, $\mathcal{W}^s$, stays very close to the $\dot{L} = 0$ isocline, meaning that labor adjusts faster than capital, as in the case $\epsilon > 0$.[5]

The first (second) case occurs for a high (low) share of labor in the aggregate product or for a high (low) Frisch elasticity (which is again equal to $1/\xi$). Therefore, indeterminacy is more probable if the adjustment of labor supply is very high. For benchmark values of the parameters $\beta = 0.7$ and $\xi = 2$ the second case can be taken as a benchmark.

5 Conclusion

We have presented in this note three main aspects of new-Keynesian models:

1. existence of distortions generating an inefficient general equilibrium
2. possible existence of multiple steady states: dependence of the equilibrium path on the initial level of pre-determined variables and of the parameters of the model
3. possible existence of indeterminacy: multiple paths converging to a steady state.

We can also distinguish between local indeterminacy and global indeterminacy. There is glocal indeterminacy when we combine the existence of multiple steady states with the existence of at least one steady state which is locally determinate (i.e, in a modele there are steady states which are locally determinate and others which are locally indeterminate).[6]

[5]For $\epsilon = 0$, there is a degenerate case where the adjustment of L is automatic, so that the stable manifold coincides with the $\dot{L} = 0$.

[6]For an example in growth theory see Brito and Venditti (2010).

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A Solution of final producer problem (4)

The Lagrangean is, ignoring the time argument,

$$L(\mathbf{x},\lambda) = P\,Y - C(\mathbf{x}) + \lambda\left(F(\mathbf{x}) - Y\right)$$

where λ is the Lagrange multiplier (an unknown constant). The first order conditions are

$$\begin{cases} \dfrac{\delta L(\mathbf{x})}{\delta x(j)} = 0 & \iff \dfrac{\delta C(\mathbf{x})}{\delta x(j)} = \lambda\,\dfrac{\delta F(\mathbf{x})}{\delta x(j)} \iff p(j) = \lambda\left(\dfrac{F(\mathbf{x})}{x(j)}\right)^{\frac{1}{\varepsilon}}, \text{ for every } j \in [0,1] \\ \dfrac{\partial L(\mathbf{x})}{\partial \lambda} = 0 & \iff F(\mathbf{x}) = Y. \end{cases}$$

Then $\lambda^{\varepsilon}\,Y = p(j)^{\varepsilon}\,x(j)$ for any $j \in [0,1]$. Then, the demand for input j is

$$x(j) = \lambda^{\varepsilon} p(j)^{-\varepsilon}\,Y,$$

depends on the Lagrange multiplier. But substituting in the constraint,

$$F(\mathbf{x}) = \left(\int_0^1 \left(\lambda^{\varepsilon} p(j)^{-\varepsilon}\,Y\right)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}} = Y$$

and solving for λ, we find that

$$\lambda^{*} = \left(\int_0^1 p(j)^{1-\varepsilon}\,dj\right)^{\frac{1}{1-\varepsilon}} = P^{*}$$

is the shadow cost of one unit of output.

Substituting in the profit functional yields

$$\pi^{*}(t) = Y(t)\left(P(t) - P^{*}(t)\right).$$

As the firm is price taker then $P^{*}(t) = P(t)$ and there is zero profit for every t. Therefore, the demand function is as in equation (5).

B Solution of the intermediate producer j problem (8)

The current-value Hamiltonian is

$$H(j,t) = \pi(j,t) - i(j,t) - \omega(t)\ell(j,t) + q(j,t)\left(i(j,t) - \delta k(j,t)\right).$$

where $\pi(j,t) = \left(Ak(j,t)^{\alpha}\ell(j,t)^{1-\alpha}\right)^{1-\mu} Y(t)^{\mu}$ The static optimality conditions are

$$\frac{\partial H(j,t)}{\partial \ell(j,t)} = (1-\mu)(1-\alpha)\,\frac{\pi^{*}(j,t)}{\ell(j,t)} - \omega(t) = 0 \tag{12}$$

$$\frac{\partial H(j,t)}{\partial i(j,t)} = -1 + q(j,t) = 0 \tag{13}$$

and the Euler equation and the transversality conditions are

$$\begin{aligned}\frac{dq(j,t)}{dt} &= r(t)q(j,t) - \frac{\partial H(t)}{\partial k(j,t)} \\ &= \left(r(t)+\delta\right) q(j,t) - (1-\mu)\,\alpha\,\frac{\pi^*(j,t)}{k(j,t)}\end{aligned} \tag{14}$$

and $\lim_{t\to\infty} e^{-R(t)}\, q(j,t)\, k(j,t) = 0$ for all j.

Equation (13) yields

$$q(j,t) = q(t) = 1, \text{ for every } j \in [0,1] \text{ and } t \in [0,\infty),$$

that is the shadow value of capital is the same for all sectors $q(t) = q(j,t)$ and it is constant $q(t) = 1$. Therefore, the gross rate of return of capital is obtained from (14)

$$r(t) + \delta = (1-\mu)\,\alpha\,\frac{\pi^*(j,t)}{k^*(j,t)}$$

Equation (12) also involves an arbitrage condition but now in the labor market

$$\omega(t) = (1-\mu)(1-\alpha)\,\frac{\pi^*(j,t)}{\ell^*(j,t)},$$

therefore the total cost is

$$\left(r(t)+\delta\right) k^*(j,t) + \omega(t)\,\ell^*(j,t) = (1-\mu)\,\pi^*(j,t).$$

Substituting in the definition of profit,

$$\pi^*(j,t) = \left[(1-\mu)\,A\left(\frac{\alpha}{r(t)+\delta}\right)^{\alpha}\left(\frac{1-\alpha}{w(t)}\right)^{1-\alpha}\right]^{1-\mu} \pi^*(j,t)^{1-\mu}\,Y(t)^{\mu},$$

yields

$$\pi^*(j,t) = \pi^*(t)\left[(1-\mu)\,A\left(\frac{\alpha}{r(t)+\delta}\right)^{\alpha}\left(\frac{1-\alpha}{w(t)}\right)^{1-\alpha}\right]^{1-\mu} Y(t),$$

which means that profits are symmetric across variety producers. Therefore, factor demand functions are symmetric as well

$$\ell^*(j,t) = \ell^*(t) = (1-\mu)\left(\frac{1-\alpha}{w(t)}\right)\pi^*(t), \text{ and } k^*(j,t) = k^*(t) = (1-\mu)\left(\frac{\alpha}{r(t)+\delta}\right)\pi^*(t).$$

Therefore there is also symmetry in the supply of varieties,

$$\begin{aligned}x^*(j,t) = x^*(t) &= Ak(t)^{\alpha}\ell(t)^{1-\alpha} \\ &= A\left(\frac{\alpha}{r(t)+\delta}\right)^{\alpha}\left(\frac{1-\alpha}{w(t)}\right)^{1-\alpha}\pi^*(t) \\ &= \left(\frac{\pi*(t)}{Y(t)}\right)^{\frac{\mu}{1-\mu}}\pi^*(t) \\ &= Y(t)^{-\frac{\mu}{1-\mu}}\,\pi^*(t)^{\frac{1}{1-\mu}}, \text{ for any } j \in [0,1], \text{ for each } t \in [0,\infty).\end{aligned}$$