

# Mathematical Economics

## *Problem set 2020/21*

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## 1 Deterministic dynamic optimisation: discrete time

### 1.1 Optimal control and the Pontryagin's principle

**1.1.1** Consider the optimal control problem:  $\max_u - \sum_{t=0}^3 (2 - u_t)^2$  subject to  $y_{t+1} = 1/2y_t + u_t$  such that  $y_0 = y_4 = 1$ .

- (a) Determine first order conditions from the Pontryagin's maximum principle.
- (b) Determine the solution of the problem.

**1.1.2** Consider the optimal control problem:  $\max_u - \sum_{t=0}^3 (2 - u_t)^2$  subject to  $y_{t+1} = 1/2y_t + u_t$  such that  $y_0 = 1$  and  $y_4$  is free.

- (a) Determine first order conditions from the Pontryagin's maximum principle.
- (b) Determine the solution of the problem.

**1.1.3** Assume that there is a cake whose size at time  $t \in \{0, 1, \dots, T\}$ , where  $T$  is finite, is  $W_t$ . A consumer wants to eat it in  $T$  periods. The initial size of the cake is  $W_0 = \phi > 0$ . The consumer has a psychological discount factor  $0 < \beta < 1$  and the period utility function is isoelastic  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$  with  $\sigma > 0$ .

- (a) Determine first order conditions from the Pontryagin's maximum principle.
- (b) Determine the solution of the problem.

**1.1.4** Consider the optimal control problem:  $\max_{\{u\}} \sum_{t=0}^3 y_t - (2 - u_t)^2$  subject to  $y_{t+1} = 1/2(y_t - u_t)$  e  $y_0 = 0$  e  $y_4 = 45/2$ .

- (a) Write the first order conditions according to Pontryagin's principle.

(b) Solve the problem, that is determine the optimal sequences  $\{y_t^*\}_{t=0}^4$  and  $\{u_t^*\}_{t=0}^4$

**1.1.5** Assume that there is a cake whose size at time  $t \in \{0, 1, \dots, T\}$ , for a finite  $T$ , is  $W_t$ . A consumer wants to eat it in  $T$  periods; that is  $W_T = 0$ . The initial size of the cake is  $W_0 = \phi > 0$ . The consumer has a psychological discount factor  $0 < \beta < 1$  and the period utility function is isoelastic  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$  with  $\sigma > 0$ .

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

**1.1.6** Assume that there is a cake whose size at time  $t \in \{0, 1, \dots, \infty\}$  is  $W_t$ . A consumer wants to eat it with the terminal constraint  $\lim_{t \rightarrow \infty} W_t \geq 0$ . The initial size of the cake is  $W_0 = \phi > 0$ . The consumer has a psychological discount factor  $0 < \beta < 1$  and the period utility function is isoelastic  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$  with  $\sigma > 0$ .

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

**1.1.7** Assume that that a consumer has an endowment denoted by  $W_t$  at time  $t \in \{0, 1, \dots, T\}$ . The horizon  $T$  is finite. The endowment evolves over time as  $W_{t+1} = (1+r)W_t - C_t$ , where  $C_t$  is the amount of the endowment consumed at time  $t$  and  $r > 0$  is a parameter. Assume that  $W_0 = \phi > 0$  and that the consumer wants to have  $W_T = \phi$ . The consumer has a psychological discount factor  $0 < \beta < 1$  and the period utility function is logarithmic.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

**1.1.8** Assume that that a consumer has an endowment denoted by  $W_t$  at time  $t \in \{0, 1, \dots, \infty\}$ . The endowment evolves over time as  $W_{t+1} = (1+r)W_t - C_t$ , where  $C_t$  is the amount of the endowment consumed at time  $t$  and  $r > 0$  is a parameter. Assume that  $W_0 = \phi > 0$  and that the consumer wants to satisfy  $\lim_{t \rightarrow \infty} e^{-rt}W_t \geq 0$ . The consumer has a psychological discount factor  $0 < \beta < 1$  and the period utility function is logarithmic.

(a) Write the first order conditions from the Pontryagin's maximum principle.

(b) Solve the problem.

**1.1.9** Assume that that a consumer has an endowment denoted by  $W_t$  at time  $t \in \{0, 1, \dots, T\}$ . The horizon  $T$  is finite. The endowment evolves over time as  $W_{t+1} = (1+r)W_t - C_t$ , where  $C_t$  is the amount of the endowment consumed at time  $t$  and  $r > 0$  is a parameter. Assume that  $W_0 = \phi > 0$  and that the consumer wants to have  $W_T = \phi$ . The consumer has a psychological discount factor  $0 < \beta < 1$  and the period utility function is isoelastic  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$  with  $\sigma > 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**1.1.10** Assume that a consumer has an endowment denoted by  $W_t$  at time  $t \in \{0, 1, \dots, \infty\}$ . The endowment evolves over time as  $W_{t+1} = (1+r)W_t - C_t$ , where  $C_t$  is the amount of the endowment consumed at time  $t$  and  $r > 0$  is a parameter. Assume that  $W_0 = \phi > 0$  and that the consumer has the terminal constraint  $\lim_{t \rightarrow \infty} W_t \geq 0$ . The consumer has a psychological discount factor  $0 < \beta < 1$  and the period utility function is isoelastic  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$  with  $\sigma > 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**1.1.11** Find the optimal investment sequence for a firm,  $\{I_t\}_{t=0}^{\infty}$ , that maximizes the value functional  $\sum_{t=0}^{\infty} (1+r)^{-t} \pi_t$ , where  $r > 0$  is the market interest rate. The cash flow in period  $t$  is  $\pi_t = AK_t - I_t(1 + \xi I_t)$ , where  $K_t$  is the capital stock, and  $A > 0$  and  $\xi > 0$  are productivity, and investment cost parameters, respectively. The restrictions of the problem are: the accumulation equation  $K_{t+1} = I_t + (1 - \delta)K_t$ , where  $\delta \in [0, 1)$  is the rate of depreciation of capital, and the initial capital stock is given,  $K_0 = \phi > 0$ . Assume that  $A > r + \delta$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**1.1.12** A representative consumer has the utility function  $u = \ln(c_t)$  and has a constant intertemporal discount factor  $\beta^t$ , with  $0 < \beta < 1$ , and a finite lifetime  $T$ , she/he has the budget constraint  $a_{t+1} = y - c_t + (1+r)a_t$ , and has to bequeath the same wealth received at birth  $a_0 = a_T = A$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**1.1.13** A representative consumer has the utility function

$$u = B - \zeta^{-1} e^{-\zeta C_t},$$

where  $B > 0$  and  $\zeta > 0$ , and has a constant intertemporal discount factor  $\beta^t$ , with  $0 < \beta < 1$ , and a finite lifetime  $T$ , she/he has the budget constraint  $A_{t+1} = A_t - C_t$ , and  $A_0 = \phi$  and  $A_T = 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**1.1.14** A representative consumer has the utility function

$$u = B - \zeta^{-1} e^{-\zeta C_t},$$

where  $B > 0$  and  $\zeta > 0$ , and has a constant intertemporal discount factor  $\beta^t$ , with  $0 < \beta < 1$ , and a finite lifetime  $T$ , she/he has the budget constraint  $A_{t+1} = Y - C_t + (1+r)A_t$ , and has to bequeath the same wealth received at birth  $A_0 = A_T = A$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**1.1.15** Find the optimal investment sequence,  $\{I_t\}_{t=0}^T$ , that maximizes the value functional

$$\sum_{t=0}^T \left( \frac{1}{1+r} \right)^t (pK_t - I_t(1 + (1/2)I_t))$$

where  $K_t$  is the capital stock,  $r > 0$  is the market interest rate, and  $p > 0$  is a productivity parameter. The restrictions of the problem are: the accumulation equation is  $K_{t+1} = I_t + (1 - \delta)K_t$ , where  $\delta$  is the rate of depreciation of capital, and the initial and terminal capital stock is given by  $K_0 = K_T = \phi > 0$ . Assume that  $p > r + \delta$  and  $\delta \in [0, 1)$ .

- (a) Write the problem as an optimal problem and determine the optimality conditions from the Pontryagin's maximum principle.
- (b) Find an explicit solution for  $K_t$ . Justify and give an intuition for your results.

**1.1.16** A representative consumer wants to maximize the intertemporal utility functional  $\sum_{t=0}^{\infty} \beta^t \ln(C_t Z_t^{1-\alpha})$ , where  $0 < \alpha < 1$  and  $0 < \beta < 1$ , by using consumption  $C_t$  as a control variable. The variable  $Z_t$  denotes habits and is governed by the difference equation  $Z_{t+1} = \delta(Z_t - C_t)$ , where  $\delta > 0$ . The following initial and terminal conditions hold:  $Z(0) = Z_0 > 0$ , and  $\lim_{t \rightarrow \infty} \beta^t Z(t) \geq 0$ .

- (a) Write the first order optimality conditions from the Pontryagin's maximum principle.
- (b) Solve the problem, and provide an intuition to your results.

## 1.2 Dynamic programming

**1.2.1** Consider the infinite-horizon benchmark consumption-investment problem

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = (1+r)W_t - C_t, W_0 = \phi$$

where  $W_t$  is the stock of financial wealth in the beginning of period  $t$ , and  $C_t$  is consumption in period  $t$ ,  $\beta$  the discount factor ( $\beta \in (0, 1)$ ),  $r > 0$  is the market interest rate, and  $\phi > 0$  is the initial stock of wealth.

- (a) Write the HJB equation.
- (b) Solve the HJB equation (hint: consider the trial function  $V(W) = a + b \ln(W)$  where  $a$  and  $b$  are undetermined coefficients)
- (c) Find the solution to the problem.

**1.2.2** Consider the infinite-horizon cake-eating problem

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\theta} - 1}{1-\theta}, \text{ subject to } W_{t+1} = W_t - C_t, W_0 = \phi$$

where  $W_t$  is the stock of financial wealth in the beginning of period  $t$ , and  $C_t$  is consumption in period  $t$ ,  $\beta$  the discount factor ( $\beta \in (0, 1)$ ),  $\theta > 0$  is the inverse of the elasticity of intertemporal substitution, and  $\phi > 0$  is the initial stock of wealth.

- (a) Write the HJB equation.
- (b) Solve the HJB equation (hint: consider the trial function  $V(W) = a + b W^{1-\theta}$  where  $a$  and  $b$  are undetermined coefficients)
- (c) Find the solution to the problem.

**1.2.3** Consider the infinite-horizon benchmark consumption-investment problem

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = (1+r)W_t - C_t, W_0 = \phi$$

where  $W_t$  is the stock of financial wealth in the beginning of period  $t$ , and  $C_t$  is consumption in period  $t$ ,  $\beta$  the discount factor ( $\beta \in (0, 1)$ ) and  $r > 0$  is the market interest rate, and  $\phi > 0$  is the initial stock of wealth.

- (a) Write the HJB equation.
- (b) Solve the HJB equation (hint: consider the trial function  $V(W) = a + b W^{1-\theta}$  where  $a$  and  $b$  are undetermined coefficients)

(c) Find the solution to the problem.

**1.2.4** Consider the infinite-horizon problem for a firm:

$$\begin{aligned} \max_{\{I_t\}} \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t A K_t - I_t, \\ \text{subject to} \\ K_{t+1} = (1-\delta) K_t + \sqrt{I_t}, \text{ for } t \in \{0, \dots, \infty\} \\ K_0 = \phi, \text{ for } t = 0 \end{aligned}$$

where  $K_t$  is capital stock of the firm,  $Y_t = A K_t$  is the output,  $A > 0$  is a productivity parameter,  $I_t$  is the investment expenditure,  $0 < \delta < 1$  is the rate of depreciation of capital and  $r > 0$  is the market interest rate.

(a) Write the HJB equation.

(b) Solve the HJB equation (hint: consider the trial function  $V(K) = a + b K$  where  $a$  and  $b$  are undetermined coefficients)

(c) Find the solution to the problem.

## Deterministic dynamic optimisation: continuous time

### Optimal control: Pontryagin's principle

**2.1.1** A representative consumer has the utility functional  $\int_0^T e^{-\rho t} \ln(C(t))dt$ , where  $\rho > 0$ , and  $T$  is finite. She/he has the budget constraint  $\dot{A}(t) = -C(t) + rA(t)$ , and has initial and terminal wealth  $A(0) = A(T) = A_0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.2** A representative consumer has the utility functional  $\int_0^T e^{-\rho t} \ln(C(t))dt$ , where  $\rho > 0$ , and  $T$  is finite. She/he has the budget constraint  $\dot{A}(t) = -C(t) + rA(t)$ , and has initial wealth  $A(0) = A_0$  and  $A(T) \geq 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.3** A representative consumer has the utility functional  $\int_0^\infty e^{-\rho t} \ln(C(t))dt$ , where  $\rho > 0$ , and  $T$  is finite. She/he has the budget constraint  $\dot{A}(t) = -C(t) + rA(t)$ , and has initial wealth  $A(0) = A_0$  and  $\lim_{t \rightarrow \infty} A(t) \geq 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.4** A representative consumer has the utility functional  $\int_0^T e^{-\rho t} (1-\sigma)^{-1} (C(t))^{1-\sigma} dt$ , where  $\rho > 0$  and  $\sigma > 0$  (but  $\sigma \neq 1$ ), and  $T$  is finite. She/he has the budget constraint  $\dot{A}(t) = -C(t) + rA(t)$ , and has initial and terminal wealth  $A(0) = A(T) = A_0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.5** A representative consumer has the utility functional  $\int_0^T e^{-\rho t} (1-\sigma)^{-1} (C(t))^{1-\sigma} dt$ , where  $\rho > 0$ , and  $T$  is finite. She/he has the budget constraint  $\dot{A}(t) = -C(t) + rA(t)$ , and has initial wealth  $A(0) = A_0$  and  $A(T) \geq 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.6** A representative consumer has the utility functional  $\int_0^\infty e^{-\rho t} (1-\sigma)^{-1} (C(t))^{1-\sigma} dt$ , where  $\rho > 0$ . She/he has the budget constraint  $\dot{A}(t) = -C(t) + rA(t)$ , and has initial wealth  $A(0) = A_0$  and  $\lim_{t \rightarrow \infty} A(t) \geq 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.7** Solve  $\max_C \int_0^\infty e^{-\rho t} (B - \zeta \exp -(C(t)/\zeta) ) dt$ ,  $\zeta > 0$  subject to  $\dot{A} = rA - C$  where  $A(0) = A_0$  given and  $\lim_{t \rightarrow \infty} e^{-rt} A(t) \geq 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.8** Solve  $\max_\pi \int_0^T -(\pi(t))^2 e^{-\rho t} dt$  subject to  $\dot{\pi} = \pi - \bar{\pi}$ , where  $\rho > 0$ , subject to  $\pi(0) = 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.9** Solve  $\max_I \int_0^\infty e^{-rt} (pK(t) - qI(t)^2) dt$  subject to  $\dot{K} = I(t) - \delta K(t)$   $K(0) = k_0 > 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.10** Solve  $\max_I \int_0^\infty e^{-rt} (pK(t) - qI(t)(1 + \xi I(t)/K(t))) dt$  subject to  $\dot{K} = I(t) - \delta K(t)$   $K(0) = k_0 > 0$ .

- (a) Write the first order conditions from the Pontryagin's maximum principle.
- (b) Solve the problem.

**2.1.11** Consider the following endogenous growth model:

$$\max_C \int_0^\infty \frac{1}{1-\sigma} C(t)^{1-\sigma} e^{-\rho t} dt, \text{ subject to } \dot{K} = Y(t) - C(t)$$

together with  $K(0) = K_0$  given and  $\lim_{t \rightarrow \infty} e^{-At} K(t) \geq 0$ . The production function is linear  $Y(t) = AK(t)$  and the parameters verify:  $\rho > 0$ ,  $\sigma > 1$  and  $A > 0$ .

- (a) Write the first order conditions according to the maximum principle of Pontryagin.
- (b) Solve the problem. Under which conditions the solution displays unbounded growth ?

**2.1.12** A representative consumer wants to maximize the intertemporal utility functional  $\int_0^\infty e^{-\rho t} \ln(C(t)) dt$ , where  $\rho > 0$ , by using consumption  $C(\cdot)$  as a control variable. She/he has initial wealth  $A(0) = A_0$ , and the instantaneous budget constraint is  $\dot{A}(t) = (1 - \tau)(Y + rA(t)) - C(t)$ , where income  $Y$  is constant and positive, and the income tax rate verifies  $0 < \tau < 1$ . The non-Ponzi game condition  $\lim_{t \rightarrow \infty} e^{-rt} A(t) \geq 0$  holds.



- (a) Write the first order optimality conditions from the Pontryagin's maximum principle.
- (b) Solve the problem, and supply an intuition for your results.

**2.1.13** Assuming that  $x(\cdot)$  is a state variable and  $u(\cdot)$  is a control variable, consider the optimal control problem

$$\max_{(u(t))_{t=0}^{\infty}} \int_0^{\infty} (x(t)^2 + u(t)^2) e^{-\rho t} dt$$

subject to  $\dot{x} = \alpha(x-u)$  and  $x(0) = \phi$  and  $\lim_{t \rightarrow \infty} x(t)e^{-\rho t} = 0$ . Assume that  $0 < \rho < 2\alpha$  and that  $\phi > 0$

- (a) Determine the optimality conditions from the Pontryagin's maximum principle.
- (b) Find an explicit solution for the optimal state variable  $x(\cdot)$ . Justify.

## Dynamic programming

**2.2.1** Consider the optimal control problem  $\max_u - \int_0^4 (2-u)^2 dt$  subject to  $\dot{y} = 1/2y(t) + u(t)$  for  $t \in [0, 4]$  and  $y(0) = 1$ .

- (a) Write the HJB equation.
- (b) Determine the optimal policy function.

**2.2.2** Assume that there is an endowment whose size at time  $t \in [0, T]$ , where  $T$  is finite, is  $W(t)$ . A consumer wants to consume until time  $T$ . That is  $W(T) = 0$ . The initial size of the cake is  $W(0) = \phi > 0$ . The consumer has a psychological rate of time preference  $\rho > 0$  and a static logarithmic utility function. Determine the optimal consumption strategy using the principle of dynamic programming.

- (a) Write the HJB equation.
- (b) Determine the optimal policy function.

## Solutions for discrete time problems

1.1.4 (a)  $y_t^* = (3/2)(-1 + 2^t)$  and  $u_t^* = (3/2)(1 - 3(2^t))$  for  $t = 0, 1, 2, 3, 4$  or  $y_t^* = \{0, 3/2, 9/2, 21/2, 45/2\}$  and  $u_t^* = \{-3, -15/2, -33/2, -69/2\}$ .

1.1.15 (a) The first order conditions are  $\eta_{t+1} = ((1+r)\eta_t - p) \frac{1}{1-\delta}$  and  $K_{t+1} = \eta_t - 1 + (1-\delta)K_t$  for  $t \in \{0, 1, \dots, T\}$ ,  $K_0 = \phi = K_T$ . (b) The problem has the unique solution

$$K_t = k^* + (\phi - k^*) \left( \frac{((1+r^*)^T - 1)(1-\delta)^t + (1 - (1-\delta)^T)(1+r^*)^t}{(1+r^*)^T - (1-\delta)^T} \right)$$

$$\eta_t = \eta^* + \frac{(r^* + \delta)(\phi - k^*)(1 - (1-\delta)^T)}{(1+r^*)^T - (1-\delta)^T} (1+r^*)^t$$

for  $t \in \{0, 1, \dots, T\}$ , where  $1+r^* = \frac{1+r}{1-\delta}$ ,  $k^* = \frac{p-(r+\delta)}{\delta(r+\delta)}$  and  $\eta^* = \frac{p}{r+\delta}$ .

1.1.16 (a) F.o.c.  $C_t^* = (\alpha)/\delta \frac{1}{\eta_t}$ ,  $\eta_{t+1} = \frac{\eta_t}{\beta\delta} - \frac{1-\alpha}{\delta Z_{t+1}^*}$ ,  $Z_{t+1}^* = \delta(Z_t^* - C_t^*)$ ,  $Z_0^* = Z_0$  and  $\lim_{t \rightarrow \infty} \beta^t \eta_t Z_t^* = 0$ ; (b) the solution  $Z_t^* = Z_0 \left( \frac{\beta\delta}{\alpha+(1-\alpha)\beta} \right)^t$  and  $C_t^* = \frac{\alpha(1-\alpha)}{\alpha+\beta(1-\alpha)} Z_t^*$

## Solutions for continuous-time problems

- 2.1.12 (a) The first order conditions include the optimality conditions  $1/C(t) = Q(t)$ ,  $\dot{Q}(t) = (\rho - r(1 - \tau))Q(t)$  and  $\lim_{t \rightarrow \infty} Q(t)A(t)e^{-\rho t} = 0$ , plus the admissibility conditions  $\dot{A}(t) = (1 - \tau)(Y + rA(t)) - C(t)$  and  $A(0) = A_0$ . (b) The problem has the unique solution (where  $\gamma \equiv r(1 - \tau) - \rho$ ),  $C(t) = \rho(A_0 + Y/r)e^{\gamma t}$  and  $A(t) = -Y/r + (A_0 + Y/r)e^{\gamma t}$  for  $t \in [0, \infty)$ . Then: if  $\gamma < 0$  then  $\lim_{t \rightarrow \infty} (C(t), A(t)) = (0, -Y/r)$ , if  $\gamma = 0$  then  $\lim_{t \rightarrow \infty} (C(t), A(t)) = (\rho(A_0 + Y/r), -Y/r)$  or if  $\gamma > 0$  then  $\lim_{t \rightarrow \infty} (C(t), A(t)) = (+\infty, +\infty)$ .
- 2.1.13 (a)  $u^*(t) = \frac{\alpha}{2} q(t)$ ,  $\dot{q}(t) = (\alpha - \alpha)q(t) - 2x^*(t)$ ,  $\dot{x}^*(t) = \alpha(x^*(t) - u^*(t))$ ,  $x^*(0) = \phi$  and  $\lim_{t \rightarrow \infty} x^*(t)e^{-\rho t} = 0$ ; (b)  $x^*(t) = \phi e^{\lambda_s t}$  where  $\lambda_s = \frac{\rho}{2} - \left(\frac{\rho}{2} - \alpha(\rho - 2\alpha)\right)^{1/2} < 0$