Foundations of Financial Economics
Two period GE for a finance economy

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April 17, 2018

Topics

- The consumer problem in a finance economy
- The consumer problem with complete markets
- The consumer problem with incomplete markets
- GE for a finance economy
- Equilibrium asset prices
- Equivalence with an AD economy

Markets in a finance economy

We assume an endowment finance economy, with the following **markets**:

1. **one real spot market for the good** opening in **every period** $t = 0$ **and** $t = 1$. We set output prices as $P_0 = 1$ and $P_1 = \mathbf{1} = (1, 1, \dots, 1)^\top$;

   Note: differently from the AD economy, now the consumer cannot perform forward contracts in the good but can buy or sell it at time $t = 1$ in the spot market;

2. K **financial spot markets** opening at the **end of period** $t = 0$, where assets paying the (random) payoff V_j (at time $t = 1$) are traded at prices S_j, $j = 1, \dots, K$.

Constraints for the consumer

- The representative consumer receives a (contingent) stream of endowments $\{y_0, Y_1\}$ where $Y_1 = (y_{1,1}, \ldots, y_{1,s}, \ldots, y_{1,N})^\top$
- We say the **portfolio θ finances a (random) consumption sequence** $\{c_0, C_1\}$ if

$$\begin{aligned} c_0 &= y_0 + z_0^\theta \\ C_1 &= Y_1 + Z_1^\theta \end{aligned}$$

  where

$$z_0^\theta = -S\theta,\ Z_1^\theta = V\theta$$

- we can write equivalently

$$\begin{aligned} c_0 + s &= y_0 \\ C_1 &= Y_1 + s\,R \end{aligned}$$

  where $R = V/S$ and $s = S\theta$ are savings

Timing and information for the consumer

- Financial market data

$(S_1, \ldots, S_K)$

0 $\qquad$ $\begin{pmatrix} V_{1,1} & \ldots & V_{K,1} \\ \vdots & \ldots & \vdots \\ V_{1,N} & \ldots & V_{K,N} \end{pmatrix}$ $\qquad$ 1

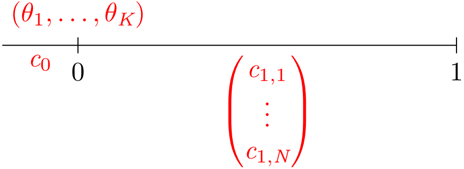
- Endowments

y_0 $\quad$ 0 $\qquad$ $\begin{pmatrix} y_{1,1} \\ \vdots \\ y_{1,N} \end{pmatrix}$ $\qquad$ 1

Timing and information for the consumer

Decisions:

- Portfolio (financial) decisions: θ
- Consumption (real) decisions: c_0, C_1



Budget constraints

Therefore, the consumer/saver has a **random sequence of budget constraints**, conditional on the information he has at $t = 0$:

- for period $t = 0$

$$c_0 = y_0 - \sum_{j=1}^{K} S_j \theta_j$$

- for period $t = 1$

$$\begin{aligned} c_{1,1} &= y_{1,1} + \sum_{j=1}^{K} V_{j,1} \theta_j \\ &\cdots \\ c_{1,s} &= y_{1,s} + \sum_{j=1}^{K} V_{j,s} \theta_j \\ &\cdots \\ c_{1,N} &= y_{1,N} + \sum_{j=1}^{K} V_{j,N} \theta_j \end{aligned}$$

Setting up of the problem

- Assumptions: the agent has a von Neumann Morgenstern utility functional, and operates in a finance economy (S, V)
- Consumer problem:
  find an optimal sequence of consumption, $\{C_t\}_{t=0}^1$**, and portfolio composition,** θ, that maximizes

$$\max_{\{c_0, C_1\}, \theta} \mathbb{E}_0 \left[u(c_0) + \beta u(C_1) \right]$$

subject to the **sequence** of budget constraints

$$\begin{aligned} c_0 &\leq y_0 - S\theta \\ C_1 &\leq Y_1 + V\theta \end{aligned}$$

given (S, V) and $\{Y_t\}_{t=0}^1$.

Setting up of the problem

- Expanding the expressions, we have an equivalent representation :

$$\max_{\{c_0,(c_{1,1},\dots,c_{1,N})\},(\theta_1,\dots,\theta_K)} u(c_0) + \beta \sum_{s=1}^{N} \pi_s u(c_{1,s})$$

- subject to the $N+1$ restrictions

$$\begin{aligned} c_0 &\leq y_0 - \sum_{j=1}^{K} S_j \theta_j \\ &\dots \\ c_{1,s} &\leq y_{1,s} + \sum_{j=1}^{K} V_{j,s} \theta_j, \; s = 1, \dots, N \\ &\dots \end{aligned}$$

- Differences to the problem in an AD economy:
(1) in the AD economy the consumer has a **single intertemporal** constraint;
(2) in a finance economy he/she has a **sequence of period (intratemporal)** constraints

Solving the consumer's problem

Lagrangean

- Next we assume there is no satiation, i.e., $u^{'}(c) > 0$ for all $c > 0$
- The Lagrangean for this problem is

$$\mathcal{L} = u(c_0) + \beta \sum_{s=1}^{N} \pi_s u(c_{1,s}) + \lambda_0 \left(y_0 - \sum_{j=1}^{K} S_j \theta_j - c_0 \right) + \\ + \sum_{s=1}^{N} \lambda_{1,s} \left(y_{1,s} + \sum_{j=1}^{K} V_{j,s} \theta_j - c_{1,s} \right)$$

- we have to maximize it for: $K + 2 \times (1 + N)$ variables:

  $1 + N$ consumption variables: c_0, $(c_{1,1}, \dots, c_{1,N})$,
  K portfolio components: $(\theta_1, \dots, \theta_K)$
  $1 + N$ Lagrange multipliers: λ_0, $(\lambda_{1,1}, \dots, \lambda_{1,N})$

Solving the consumer's problem

First order conditions

- optimality conditions for consumption: $1 + N$ equations

$$\begin{aligned} u^{'}(\hat{c}_0) &= \lambda_0, \ (t = 0) \\ \beta \pi_s u^{'}(\hat{c}_{1s}) &= \hat{\lambda}_{1s}, \ \ (t = 1, s = 1, \ldots, N) \end{aligned}$$

- the optimality conditions for portfolio expenditure: K equations

$$\lambda_0 S_j = \sum_{s=1}^{N} \hat{\lambda}_{1s} V_{js}, \ j = 1, \ldots, K, \ (t = 0 \text{ vs } t = 1)$$

- and the period budget constraints, evaluated at the optimum: $1 + N$ equations

$$\begin{aligned} \hat{c}_0 + \sum_{j=1}^{K} S_j \theta_j &= y_0, \ (t = 0) \\ \hat{c}_{1s} &= y_{1s} + \sum_{j=1}^{K} V_{js} \theta_j, \ (t = 1, s = 1, \ldots, N) \end{aligned}$$

Solving the consumer's problem

- If we define the **shadow price of the state of nature** s

$$\hat{q}_s \equiv \frac{\lambda_{1s}}{\lambda_0},\ s = 1, \ldots, N$$

then we get an arbitrage condition for consumption

$$\beta \pi_s u^{'}(\hat{c}_{1s}) = \hat{q}_s u^{'}(\hat{c}_0),\ s = 1, \ldots, N \tag{1}$$

the optimal portfolio conditions become

$$S_j = \sum_{s=1}^{N} \hat{q}_s V_{js},\ j = 1, \ldots, K \tag{2}$$

Solving the consumer's problem

and the budget constraints

$$\hat{c}_0 = y_0 - \sum_{j=1}^{K} s_j \theta_j, \tag{3}$$

$$\hat{c}_{1s} = y_{1s} + \sum_{j=1}^{K} V_{js} \theta_j, \ \ s = 1, \ldots, N \tag{4}$$

Solving the consumer's problem

In complete or incomplete financial markets

- Complete financial markets:
  - in this case $\det(V) \neq 0$
  - then **real and financial decision separate** ($\hat{Q}$ can be uniquely determined from (S, V))
  - allowing for explicit solutions
- Incomplete financial markets:
  - in this case $\det(V) = 0$ (or $K < N$)
  - then **real and financial decision do not separate** (Q cannot be uniquely determined from (S, V))
  - rarely allowing for explicit solutions

Solving the consumer's problem

In a complete financial market

- The shadow price of the states of nature is (using the optimal portfolio conditions) equation (2)

$$S^\top = V^\top \hat{Q}^\top \Rightarrow \hat{Q}^\top = (V^\top)^{-1} S^\top$$

  where $\hat{Q} = (\hat{q}_1, \ldots, \hat{q}_N)$

- If $K = N$ and $\det(V) \neq 0$ then

$$(V^\top)^{-1} S^\top = \hat{Q}^\top$$

  is equivalent to

$$\boxed{\hat{Q} = SV^{-1}}$$

  depends only on market (exogenous) data

Solving the consumer's problem

In a complete financial market

- Equation (4) is

$$\sum_{j=1}^{K} \theta_j V_{js} = c_{1s} - y_{1s}, \; s = 1, \ldots, N$$

- or

$$\begin{cases} \theta_1 V_{11} + \ldots + \theta_K V_{K1} = c_{11} - y_{11} \\ \ldots \\ \theta_1 V_{1N} + \ldots + \theta_K V_{KN} = c_{1N} - y_{1N} \end{cases}$$

  or

$$V\theta = C_1 - Y_1$$

- Again, if $K = N$ and $\det(V) \neq 0$ then the optimal portfolio composition as a linear function of $C_1 - Y_1$

$$\boxed{\hat{\theta} = (V)^{-1}(\hat{C}_1 - Y_1)}$$

Solving the consumer's problem

In a complete financial market

- Substituting in equation (3)

$$\hat{c}_0 = y_0 - S\hat{\theta} = S(V)^{-1}(\hat{C}_1 - Y_1) = \hat{Q}(\hat{C}_1 - Y_1)$$

(is identical to the constraint for the consumer problem in an AD economy)

Solving the consumer's problem

In a complete financial market

- Therefore, we can obtain the optimal consumption path from equation (1) and this equation

$$\beta \pi_s u^{'}(c_{1,s}) = \hat{q}_s u^{'}(c_0), \; s = 1, \dots, N$$
$$\hat{c}_0 + \sum_{s=1}^{N} \hat{q}_s \hat{c}_{1s} = H_0 = y_0 + \sum_{s=1}^{N} \hat{q}_s y_{1s}$$

  where H_0 is total wealth (present value of endowments)

- using the definition of stochastic discount factor $\hat{m}_s = \hat{q}_s / \pi_s$ we have equivalently

$$\boxed{\begin{array}{r} \beta u^{'}(c_{1,s}) = \hat{m}_s u^{'}(c_0), \; s = 1, \dots, N \\ \hat{c}_0 + \mathbb{E}[M \hat{C}_1] = H_0 = y_0 + \mathbb{E}[M\, Y_1] \end{array}}$$

Solving the consumer's problem

In a complete financial market

- **Example**: for a log utility function, $u(c) = \ln(c)$ we have

$$\frac{\beta}{c_{1,s}} = \frac{\hat{m}_s}{c_0}, \;\; s = 1, \dots, N$$

- then $c_0 + \mathbb{E}[M\,C_1] = c_0 + \sum_{s=1}^{N} m_s \dfrac{\beta}{m_s} c_0 = (1+\beta)c_0 = H_0$
- Therefore

$$\hat{c}_0 = \frac{1}{1+\beta} H_0$$

$$\hat{c}_{1,s} = \frac{\beta}{(1+\beta)m_s} H_0$$

- Conclusion: consumption is a function of wealth defined as the present value of endowments (using a shadow discount factor, which in a complete financial market is equal to the market stochastic discount factor)

Solving the consumer's problem

In an incomplete financial market

- if $\det(V) = 0$ (or $K < N$) then the financial market is incomplete
- and we have to solve **jointly**, for c_0, C_1, Q, θ the equations

$$\begin{cases} \beta\pi_s u'(c_{1,s}) = \hat{q}_s u'(c_0), & s = 1, \ldots, N \\ \sum_{s=1}^{N} \hat{q}_s V_{js} = S_j, & j = 1, \ldots, K \\ \hat{c}_0 = y_0 - \sum_{j=1}^{K} S_j \theta_j & \\ \hat{c}_{1s} = y_{1s} + \sum_{j=1}^{K} V_{js}\theta_j, & s = 1, \ldots, N \end{cases}$$

or solve $\hat{Q}$ (dimension N) and $\hat{\theta}$ (dimension K) from the $N + K$ equations

$$\begin{cases} \beta\pi_s u'\left(y_{1s} + \sum_{j=1}^{K} V_{js}\theta_j\right) = \hat{q}_s u'\left(y_0 - \sum_{j=1}^{K} S_j\theta_j\right), & s = 1, \ldots, N \\ \sum_{s=1}^{N} \hat{q}_s V_{js} = S_j, & j = 1, \ldots, K \end{cases}$$

and substitute in the budget constraint expressions to get $\hat{c}_0$ and $\hat{C}_1$;

Solving the consumer's problem

In an incomplete financial market (cont.)

- Only for the case of quadratic preferences, because $u^{'}(x)$ is linear in x, we can obtain **explicit solutions** for θ and Q (in all other case usually we cannot get explicit solutions, or existence and uniqueness may not be guaranteed)
- Why ? The origin of the problem is related to the fact that we cannot get $\hat{q}_s$ directly from S and V, i.e, we cannot have $\hat{Q} = Q = SV^{-1}$, that is, **we cannot equate market and shadow prices for the states of nature**

Solving the consumer's problem

In an incomplete financial market (cont.)

Example:

- Let $u(c) = ac - \frac{b}{2}c^2$, assume that $N = 2$ and $K = 1$ where the price is S and the payoff is $V = (v_{1,1}, v_{1,2})^\top$
- the first-order conditions are, if we substitute $\hat{q}$,

$$\begin{aligned} u^{'}(\hat{c}_0) &= \beta\left[\pi u^{'}(\hat{c}_{1,1})R_1 + (1-\pi)u^{'}(\hat{c}_{1,2})R_2\right] && (5)\\ \hat{c}_{1,1} &= y_{1,1} + (y_0 - \hat{c}_0)R_1 && (6)\\ \hat{c}_{1,2} &= y_{1,2} + (y_0 - \hat{c}_0)R_2 && (7)\\ \hat{\theta} &= \frac{y_0 - \hat{c}_0}{S} && (8) \end{aligned}$$

- as $u^{'}(c) = a - bc$ we can solve equation (5) for $\hat{c}_0$ to get

$$\hat{c}_0 = \frac{a(1 - \mathbb{E}(R)) - b\left[\mathbb{E}[RY_1] + y_0\mathbb{E}[R^2]\right]}{b\left(1 + \beta\mathbb{E}[R^2]\right)}$$

Characterizing consumer's behavior

Intertemporal arbitrage condition

Proposition 1

The ***intertemporal arbitrage condition*** *relating consumption and portfolio behavior at the individual level*

$$S_j u^{'}(\hat{c}_0) = \beta \mathbb{E}[u^{'}(\hat{C}_1) V_j], \;\; j = 1, \ldots, K$$

holds irrespective of the completeness of markets, for any asset j. We have an equivalent expression

$$u^{'}(\hat{c}_0) = \beta \mathbb{E}[u^{'}(\hat{C}_1) R_j], \;\; j = 1, \ldots, K$$

using asset return R_j.

Characterizing consumer's behavior

Intertemporal arbitrage condition

- Proof: **Independently of the financial market structure**, we have two optimality conditions

$$\begin{aligned} \beta \pi_s u^{'}(\hat{c}_{1,s}) &= \hat{q}_s u^{'}(\hat{c}_0),\ s = 1, \ldots, N \\ S_j &= \sum_{s=1}^{N} q_s V_{js},\ j = 1, \ldots, K \end{aligned}$$

- multplying the second equation by $u^{'}(\hat{c}_0)$, we have

$$u^{'}(\hat{c}_0) S_j = \sum_{s=1}^{N} q_s u^{'}(\hat{c}_0) V_{js}$$

then

$$S_j u^{'}(\hat{c}_0) = \beta \sum_{s=1}^{N} \pi_s u^{'}(\hat{c}_{1s}) V_{js} = \beta \mathbb{E}[u^{'}(\hat{c}_1) V_j],\ \ j = 1, \ldots, K$$

- and use the definition $R_j = V_j / S_j$ we get the intertemporal arbitrage condition for the consumer.

General equilibrium asset pricing

Homogeneous agent economy

Definition: **General equilibrium with rational expectations for an homogeneous economy** : it is the (random) sequence of consumption and of optimal portfolio and asset prices $(\{c_0^*, C_1^*\}, \theta^*, S^*)$ where $C_1^* = (c_{11}^*, \ldots, c_{1N}^*)$, $\theta^* = (\theta_1^*, \ldots, \theta_K^*)$ and $S^* = (S_1^*, \ldots, S_K^*)$ such that, given $\{y_0, Y_1\}$ and V:

- (1) (c^*, θ^*) is the solution of the consumers' problem where consumers have common knowledge and rational expectations over y and V ;
- (2) asset markets clear (when assets are in zero net supply)

$$\theta^* = 0.$$

- (3) the product market is in equilibrium,

$$\begin{aligned} c_0^* &= y_0, \\ c_{1s}^* &= y_{1s}, \quad s = 1, \ldots, N. \end{aligned}$$

Equilibrium asset price

Homogeneous agent economy

Proposition 2

Consider the DGSE just defined and assume the utility function with no satiation. Then there is an equilibrium stochastic discount factor $M^* = (m_1^*, \ldots, m_N^*)$, *where*

$$m_s^* = \beta \frac{u^{'}(y_{1,s})}{u^{'}(y_0)}, \; s = 1, \ldots, N,$$

such that the equilibrium price for asset j *is*

$$S_j^* = \mathbb{E}[M^* V_j], \; j =, 1, \ldots, K.$$

Equilibrium asset price

Homogeneous agent economy

- Proof: in an **homogeneous agent** economy, the equilibrium conditions (i,e $(c_0^*, C_1^*, \theta^*, S^*)$) are

$$\begin{cases} S_j^* u^{'}(c_0^*) = \beta \mathbb{E}[u^{'}(c_1^*) V_j], & j = 1, \ldots, K \\ \theta_j = 0, & j = 1, \ldots, K \\ c_0^* = y_0, & \\ c_{1,s}^* = y_{1,s}^*, & s = 1, \ldots, N \end{cases}$$

  we just need to substitute the equilibrium values of consumption into the consumer's arbitrage condition
- Therefore, the **equilibrium exists and is unique**.
- Observation: in **heterogenous** agents' finance economies the existence and uniqueness is not guaranteed: in particular if the financial markets are incomplete (remember that in these economies we do not have in general $\hat{q} = Q$ and Q is not unique).

Equilibrium asset price

Homogeneous agent economy

- Proof cont.: If we define the **equilibrium** stochastic discount factor as the random variable $M^* = (m_1^*, \ldots, m_N^*)$ such that

$$m_s^* = \beta \frac{u'(y_{1,s})}{u'(y_0)}, \; s = 1, \ldots, N$$

- Then equilibrium asset prices can be obtained explicitly from the **equilibrium arbitrage condition for asset market** j

$$S_j^* = \sum_{s=1}^{N} \pi_s^* m_s^* V_{j,s}, \; j =, 1, \ldots, K$$

Equilibrium

- **Example**: if the utility function is CRRA
$$u(C) = \frac{C^{1-\eta}}{1-\eta}$$
and the endowment follows the process
$$Y_1 = (1+\Gamma)y_0$$
where $\Gamma = (\gamma_1, \dots, \gamma_N)$
- The equilibrium stochastic discount factor is a distribution
$$m_s^* = \beta(1+\gamma_s)^{-\eta},\ s = 1, \dots, N$$
or in vector notation
$$M^* = \beta(1+\Gamma)^{-\eta}$$
- And the equilibrium asset price is, for any asset j
$$S_j^* = \beta\mathbb{E}[(1+\Gamma)^{-\eta}\, V_j].$$

Equilibrium asset return

Homogeneous agent economy

- Equivalently, for the equilibrium return for asset j is

$$\boxed{\mathbb{E}[M^* R_j^*] = 1,\ j = 1, \ldots, K.}$$

- This implies, that at equilibrium the following arbitrage condition holds

$$\mathbb{E}[M^* R_1^*] = \mathbb{E}[M^* R_2^*] \ldots = \mathbb{E}[M^* R_K^*] = 1$$

- In words: in equilibrium the mathematical expectation of the discounted returns for all assets is equalized

Equilibrium asset return

When there is a risk-free asset

- Assume there is a risk-free asset with return $R^f = 1 + i$
- At the equilibrium we have

$$\mathbb{E}[MR^f] = 1 \Rightarrow R^f = \frac{1}{\mathbb{E}[M]}$$

- Introduce a probability measure (recall our definition in the last lecture)

$$\pi_s^M = \frac{\pi_s m_s}{\mathbb{E}[M]} = \frac{\pi_s m_s}{\sum_{s=1}^{N} \pi_s m_s}$$

  we get a market probability distribution $\mathbb{P}^M = (\pi_1^M, \dots, \pi_N^M)$

- Then from $\mathbb{E}[M^* R_j^*] = 1$ we have

$$\boxed{\mathbb{E}^M[R_1^*] = \mathbb{E}^M[R_2^*] \dots = \mathbb{E}^M[R_K^*] = R^f}$$

  the expected return for all assets are equal and equal to the risk free rate.

Equity premium (in equilibrium)

- Assume there are two assets:
  - an equity or any risky asset with return $R = (1 + r) = (1 + r_1, \dots 1 + r_N)^\top$
  - a risk-free asset with return $R^f = 1 + i$
- The **equity premium** is the differences in the rates of return: $R - R^f = r - i$ where

$$r - i = (r_1 - i, \dots, r_N - i)^\top$$

- and the **Sharpe index** is

$$\frac{\mathbb{E}[r - i]}{\sigma[r]}$$

where $\mathbb{E}[r - i] = \sum_{s=1}^{N} \pi_s (r_s - i)$ and $\sigma[r] = \sum_{s=1}^{N} \pi_s r_s^2$ are the expected value and the standard deviation for the risk premium

Equilibrium equity premium

- From the previous model the equilibrium equity premium is

$$\mathbb{E}[M(r-i)] = 0$$

- Assuming a CRRA utility function and a growing endowment economy we have

$$\mathbb{E}[(1+\Gamma)^{-\eta}(r-i)] = 0$$

- It can be proved that the Sharpe index should verify, if this theory is correct

$$\boxed{\frac{|\mathbb{E}[r-i]|}{\sigma[r]} \leq \sigma[(1+\Gamma)^{-\eta}]}$$

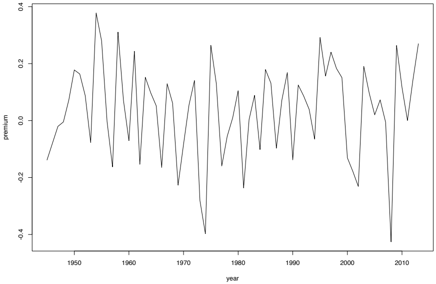
(prove this)

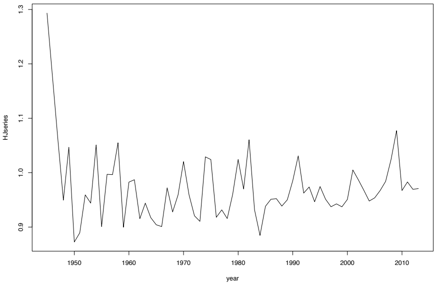
- The term, $\sigma[(1+\Gamma)^{-\eta}]$ is called the **Hansen-Jagannathan bound**.

Equity premium puzzle

- Equity premium puzzle: Mehra and Prescott (1985) provide a simple test to the theory
- In our framework this can be seen simply by comparing data with the theory:
  - **Mehra and Prescott (US)** the Sharpe ratio is roughly around 0.5
  $$\frac{\mathbb{E}[r-i]}{\sigma[r]} \approx \frac{0.07}{0.16} \approx 0.45$$
  - **Theory**: for a log utility ($\eta = 1$) the standard deviation of the stochastic discount factor (taking real per capita growth) is
  $$\sigma[(1+\Gamma)^{-1}] \approx 0.01$$
  - but the Hansen-Jaganathan bounds will only hold for $\eta > 20$, which is considered irrealistic

Equity premium puzzle: data





Equity premium puzzle

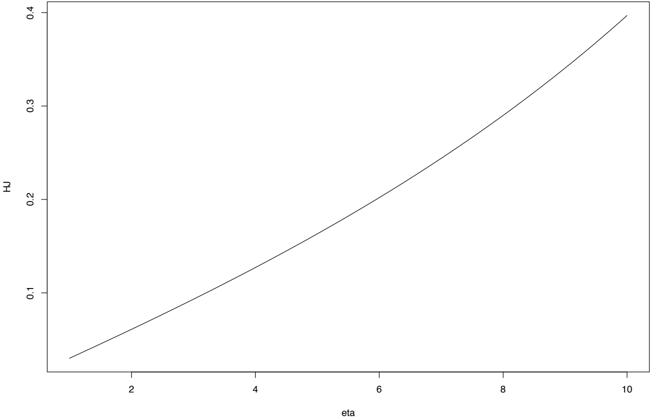


Figure: HJ bounds for different values of η

Equity premium puzzle

- The Puzzle: the risk premium is much larger in the data as compared to what is implied by the model.
- Intuition: the response of consumption to (future) income shocks is much stronger in the model than in the data
- Solution: hundreds of papers, books, Phd theses with different solutions: how to account for the smoother dynamics of consumption (or less responding of asset holdings regarding changes in the rates of return) in the data in comparison with the model.

Equivalence between Radner and Arrow-Debreu equilibria

Proposition 3

Consider a finance economy in which there is absence of arbitrage opportunities and the markets are complete. Then the general equilibrium in this finance economy and in a Arrow-Debreu economy with the same fundamentals (preferences and endowments) are equivalent

Equivalence between Radner and Arrow-Debreu equilibria

Proof

The consumers' problems' in the two economies are

- the intertemporal utility functional is the same

$$\mathbb{E}_0 \left[u(c_0) + \beta u(C_1) \right]$$

- in the **AD** economy it has a single constraint

$$c_0 - y_0 + q(C_1 - Y_1) = 0$$

- but in the **finance** economy there is a sequence of period budget constraints

$$c_0 - y_0 + S\theta = 0$$
$$C_1 - Y_1 - V\theta = 0$$

Equivalence between Radner and Arrow-Debreu equilibria

Proof (cont.)

If **markets are complete** :

- then we can **transform the sequence of period budget constraints** for a finance economy in an **unique intertemporal budget constraint**:
  1. because $\det(V) \neq 0$ then from the $t = 1$ constraints we get uniquely
  $$\theta = V^{-1}(C_1 - Y_1)$$
  2. then substituting at the $t = 0$ constraint
  $$c_0 - y_0 + SV^{-1}(C_1 - Y_1) = 0$$
  where $Q = SV^{-1} >> 0$ is the vector of AD prices **if there are no arbitrage opportunities**.

Equivalence between Radner and Arrow-Debreu equilibria

- When financial markets are **complete** : we determine the prices of AD contracts from asset prices and payoffs as $Q^\top = (R^\top)^{-1}\mathbf{1}$.

  The general equilibria in the two economies are equivalent because:

  - market equilibrium conditions are the same $c_0^* = y_0$ and $C_1^* = Y_1$
  - the state prices of the states of nature are equal $q_s = \beta\pi_s(u^{'}(y_{1,s})/u^{'}(y_0))$

- In a **homogeneous agent economy** the equivalence always exists irrespective of the finance market structure

- In **heterogeneous agent economies** the equivalence always exists if the financial markets are complete, but equivalence may not hold if financial markets are incomplete.

Take away

- When markets are complete the own state price of the consumer is equal to the market state price and the household problem is formally similar to the problem for an agent in an Arrow-Debreu economy
- When markets are incomplete, although we may solve the household problems, it may not have closed form solutions
- In a representative agent economy the arbitrage condition holds irrespective of the market being complete or incomplete

$$u^{'}(c_0) = \beta E[u^{'}(C_1)R_j] \text{ for every asset j}$$

- In a simple representative agent economy the equilibrium stochastic discount factor is negatively correlated to the growth rate of the endowments, if agents are risk averse
- The simple version of the model needs to be changed in order to match empirical data (equity premium puzzle)
- Associated problem set: Problem set 4