The benchmark DGE model

Paulo B. Brito
Advanced Macroeconomics (PhD in Economics): 2020-2021
ISEG
Universidade de Lisboa
pbrito@iseg.ulisboa.pt

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Contents

1 Introduction

In this note we present the simplest version of the dominant current view in macroeconomics, the dynamic general equilibrium framework.

We can distinguish between two views on macroeconomics, assuming that its birth as a separate field in economics started after the great depression of the late 1920's.

There is a dilemma in macroeconomic modelling between the consistency, at the micro level, of the actions of agents in all the markets they participate and the aggregation and coordination of agents, at the macro level. This dilemma has been addressed in two different ways along the histoty of macroeconomic modelling [1]

The IS-LM model stressed the aggregative aspect by sacrificing micro consistency. The model central core is based on market equilibrium relationships between the main macroeconomic aggregates, in which the behavioral functions are introduced separately. For instance, the consumption function is dependent on income, the investment function is dependent on the interest rate and the demand for money is dependent on income and the interest rate, but it does not consider that all those decisions are taken simultaneously and are tied down by financial constraints.

An alternative approach emerged, which we now call the DGE approach, which core aspect is built around getting the micro consistency right. The actions of agents in different markets are modelled by specific (dynamic) microeconomic models, which are made consistent at the aggregate level by market equilibrium conditions. That is consistency was achieved at the cost of some unrealism by casting away complexities generated by aggregating heterogenous agents. However, this approach is evolving to introduce heterogeneous agents, at the cost of models becoming very large and mathematically complex.

In this note we provide a simple introduction to the simplest DGE models.

Section 3 presents the centralized verision of the model, and the decentralized version is presented in section 4. In both of the previous section the labor supply is exogenous. Section 5 presents a endogenous labor version of the model.

2 The basic DGE model and the stylised facts

Some specialization emerged between macroeconomics and growth economics, which is not clear in some macroeconomic textbooks (as Romer, 2019 or Alogoskoufis, 2019), but it is clear in growth textbooks (see Acemoglu, 2009).

For the purpose of this course, we would like to make this distinction clear. While growth economics deals with the **trend** of the economy, in particular with average rates of growth for

[1] This dilemma is not unique to economics.

long periods of time, macroeconomic deals with the deviations from that trend, the **cycle**. This means that while a growth model has to generate a theory on the **rate of growth** of the economy, a macromodel has to generate a theory on the **deviations of the economy relative to a stationary state**, i.e., the business cycle.

This implies that growth economics and macroeconomics look at different economic dimensions. Growth economics cares about technological change, the dynamics of human capital, natural capital, and innovations, for instance. Macroeconomics deals with the short run adjustments in consumption in investment, between its expenditure, financial and capacity building aspects, in the labor market, pricing adjustments, for instance.

Let us remember the basic instantaneous product market equilibrium

$$y(t) = c(t) + i(t) + g(t) + x(t) - m(t)$$

where y is output (supply), and c, i and x represent consumption, investment expenditures, and exports, and m imports, which holds for every point in time t. The trade balance is $tb(t) = x(t) - m(t)$

Uribe and Schmitt-Grohé, 2017, ch 1 present several stylized facts for business cycles. In this note we are interested in the following dimensions

1. all the expenditure components are pro-cyclical (i.e., positively correlated with output) , with the exception of the trade balance which is counter-cyclical
2. consumption and government expenditures are less volatile than output while investment and trade balance are more volatile
3. consumption is less volatile in developed than in less developed countries.
4. consumption is more persistent than output and investment less persistent.

All those variables have a type of mean-reverting behavior, i.e, they fluctuate around their trend.

There are several elements a model has to explain, from those facts. In particular we might want to understand what explains

1. the stability mechanism that forces variables to adjust to their stationary value ?
2. the deviations from their trend ?
3. the persistent behavior of consumption ?
4. the volatile behavior of investment ?

5. the higher persistency of consumption as regards output ?

The simplest DGE model provides some explanations for those questions (that we might check quantitatively):

1. the stability mechanism is provided by the decreasing marginal return of capital

2. the deviations from a stationary state is brought about by supply shocks (increase in productivity) or demand shocks (preference to anticipate consumption)

3. the persistent behavior of consumption is produced by intertemporal consumption substitution, that is, by a preference for smooth consumption trajectories, and by the possibility to use asset markets for intertemporal transactions

4. the volatile behavior of investment is generated by the volatile behavior of savings, which finances intertemporal substitution in consumption

5. the higher persistency of consumption as regards output is again explained by the existence of mechanism for intertemoral allocation of goods together with a high elasticity of intertemporal substitution in consumption.

We can have other explanations for the different persistencies of consumption and output: life-cycle behavior in consumption, habit persistence, precautionary motives, just to name a few.

Next we address a model in which the dominant feature is the existence of intertemporal substitution in consumption.

3 The Ramsey model

The Ramsey, 1928 model (in the version stablished by Cass, 1965 and Koopmans, 1965) has become the founding stone of the current dominant model in macroeconomics, that has been christened the DGE (dynamic general equilibrium) approach to macroeconomics.

One of the main strong points, and also one of the main weaknesses, is the debatable nature of the model, as a normative (how the economy should work ?) or a positive (how the economy actually works ?) approach to the economy. Although the initial view of Ramsey was normative, some researchers use it as an acceptable positive description of the economy[2].

[2]Not wanting to delve into the philosophy of economics, maybe several economists (aware or unaware) follow Friedman, 1953's view that if a model, even if it looks very abstract, generates relationship between variables statistically similar to actual time series, it is a good representation of the economy. For a recent debate on models and the economy see Sargent, 2008, and for background in the philosophy of economics see https://plato.stanford.edu/archives/win2013/entries/economics/

Consider an economy populated by N homogeneous households or agents, which we assume that it is constant. Consider the problem in which the economy produces and consumes a durable good throughout time, given an initial stock of the durable good.

Assuming that the stock of the durable good can be increased by production, which depends on the existing stock of the durable good, and can be decreased by consumption, what should consumption, and therefore the stock of the resource, evolve across time ? In order to answer that question we should find a way to value the state of the economy.

Therefore, the three basic elements of the model are:

1. A possibility of shifting, and a constraint on, the use of the resource through time;

2. A technology allowing the transformation of the existing stock of the resource into a different stock in the future;

3. A preference ordering over different future paths of consumption.

In the case of the Ramsey model the resource is considered to be physical capital, denoted by K, the output of the economy is denoted by Y and aggregate consumption is denoted by C. All those quantities are functions of time t which is assumed to be a non-negative real variable belonging to set $\mathrm{T} = \mathbb{R}_+$ (or $\mathrm{T} = \mathbb{R}_+ \cup \infty$. We take the present time as $t = 0$.

Now and in the rest of the lecture it is important to distinguish: mappings, values and paths. Taking the example of consumption, we can have consumption as a function $C(\cdot) : \mathrm{T} \to \mathbb{R}_+$, the level of consumption at a particular point in time $C(t)$, for $t \in \mathrm{T}$ and the path of consumption $\mathbf{C} = \big(C(t)\big)_{t\in\mathrm{T}}$. Off course, a path $\mathbf{C}$ can be traced out if we know function $C(\cdot)$ and evaluate at every point in time $t \in \mathrm{T}$.

Per capita variables are denoted by small cap symbols. Therefore $k \equiv K/N$, $y \equiv Y/N$ and $c \equiv C/N$ are, respectively, capital intensity, per-capita output and per-capita consumption. We assume all those variables take non-negative real values. We can denote accordingly per-capita consumption as a function, $c(\cdot)$, as a value at time t, $c(t)$ and as a path $\mathbf{c} = \big(c(t)\big)_{t\in\mathrm{T}}$ and analogously for the other variables.

3.1 Constraints

The are three constraints on the allocation of the good in the economy: first, the initial capital stock is given $k(0) = k_0 > 0$ is known and is positive, second, the present-value of the terminal capital stock is non-negative $\lim_{t\to\infty} k(t)e^{-R(t)}$, where $R(t)$ is a rate of return; and third, there is an **instantaneous budget constraint** $\dot{k}(t) = y(t) - c(t)$.

Let us assume we can write

$$\dot{k} = r(t)k(t) + z(t), \quad t \in [0, \infty) \tag{1a}$$

$$k(0) = k_0, \ t = 0 \tag{1b}$$

$$\lim_{t \to \infty} k(t)e^{-R(t)} \geq 0 \tag{1c}$$

where $r(t)$ is the instantaneous rate of return on capital, and $R(t) = \int_0^t r(s)ds$, and

From those three constraints, we can derive a **intertemporal budget constraint**. Solving (2b) we find

$$k(t) = e^{R(t)}\Big(k(0) + \int_0^t e^{-R(s)}z(s)ds\Big).$$

Multiplying by $e^{-R(t)}$ and considering conditions (2c) and (2d) we have the intertemporal budget constraint

$$k_0 \geq -\int_0^\infty e^{-R(s)}z(s)ds$$

which means that those conditions are equivalent to requiring that the initial capital stock is larger (or equal) to the present value of the net uses in consumption of the good in the future. As k_0 is bounded then the path of the consumption $\mathbf{c}$ should also be bounded in present value terms.

There are two alternative equivalent ways of seeing the instantaneous budget constraint. First, defining savings at time t as non-consumed output, $s(t) = y(t) - c(t)$, the instantaneous budget constraint is equivalent to investment equal savings: $\dot{k} = s(t)$. Second, as we assume that there is no capital depreciation, gross investment is equal to investment expenditures and is equal to net investment (or capital formation), $i(t) = \dot{k}$. Therefore, the budget constraint can be seen as a balance between the origin of output and its uses $y(t) = c(t) + i(t)$.

3.2 Assumptions

The technology of this economy is described by the production function,

$$Y = F(K, L)$$

where aggregate output, Y, is produced with two inputs the stocks of physical capital, K and labor L. We assume there is no unemployment, then $L = N$. The production function is assumed to display increasing returns, $F_K(\cdot) > 0$, $F_L(\cdot) > 0$, to be concave and to have constant returns to scale. Mathematically the last property is translated by the requirement the $F(K, L)$ is an homogeneous of degree one function. Thus $Y = F(K, N)$.

The three properties, display different roles in the model. Constant returns to scale (CRS), allows to recast the production in per capita terms, increasing returns allows for a positive effect of the increase in resources on the possibility for having consumption in the future without completely depleting the stock of capital and concavity provides a stable dynamics.

From the CRS assumption, we have

$$y = \frac{Y}{N} = \frac{F(K,N)}{N} = F\big(\frac{K}{N},1\big) = F(k,1) = f(k).$$

Preferences over aggregate consumption are ranked according to a (cardinal) social welfare functional $\mathsf{U}[\mathbf{C}] = N\mathsf{U}[\mathbf{c}]$, where

$$\mathsf{U}[\mathbf{c}] \equiv \int_0^\infty u(c(t))\, e^{-\rho t} dt.$$

where is the intertemporal utility functional for a single agent (or household). Here we assume an institutional mechanism that derives social utility by summing up utility of every household (or dynasty) assumed to be infinitely living.

There are three main features of this utility functional t$\mathsf{U}[\mathbf{c}]$ hat worth mentioning: first, it displays impatience, second, it displays additive separability and intertemporal substitution in comsumption can be measured by the relative concavity of $u(\cdot)$: $\sigma(c) \equiv -\frac{u''(c)\,c}{u'(c)}$.

3.3 Optimal allocation

Definition 1 (Optimal allocation)**.** *An optimal allocation* $(\mathbf{c}^*, \mathbf{k}^*) = \big(c^*(t), k^*(t)\big)_{t\in T}$ *is an allocation that solves the problem*

$$\max_{c(\cdot)} \int_0^\infty u(c(t))\, e^{-\rho t} dt \tag{2a}$$

subject to

$$\dot{k} = f(k) - c, \quad t \in [0,\infty) \tag{2b}$$

$$k(0) = k_0, \; t = 0 \tag{2c}$$

$$\lim_{t\to\infty} k(t) e^{-R(t)} \geq 0 \tag{2d}$$

There are two approaches to solving this problem that became the most used in the literature [3]: the Pontryiagin's maximum principle (PMP) and the dynamic programming approach (DPP).

Observe that the problem is fundamentally to find the best way to use an initial resource, k_0, from now to the infinite future knowing that, if we don't use it all at every point in time there is a

[3]There is also the calculus of variations approach.

technology that allows it to instantaneaously grow. Therefore the problem is valuing the resource by the use in consumption that it allows in the future.

Therefore, we are deriving a value function [4] for the initial resource

$$v(k_0) = \mathsf{U}[\mathbf{c}^*] = \int_0^\infty u(c^*(t))\, e^{-\rho t} dt.$$

This idea is used by the DPP.

3.4 Finding the solution through the PMP

We write the current-value Hamiltonian function

$$H(k, q, c) = u(c) + q\,(f(k) - c)$$

An optimal allocation $(\mathbf{c}^*, \mathbf{k}^*)$ satisfies the following optimality, adjoint and transversality conditions

$$u^{'}(c^*(t)) = q(t) \tag{3a}$$

$$\dot{q} = q\big(\rho - f'(k)\big) \tag{3b}$$

$$\lim_{t\to\infty} q(t)\, k(t)\, e^{-\rho t} = 0 \tag{3c}$$

together with the admissibility constraints (2b)-(2c), evaluated at $k^*(t)$. Off course $k^*(0) = k_0$ should hold. If the utility function is strictly concave, if if $u''(c) < 0$ on all its domain, we can apply the implicit function theorem to get $q = Q(c)$ and obtain the so-called Euler equation

$$u''(c)\, \dot{c} = u'(c)\big(\rho - f'(k)\big).$$

Using the previously defined elasticity of the utility function $\sigma(c)$ the modified Hamiltonian dynamic system (MHDS) which provides the optimality conditions

$$\dot{k}^* = f(k^*) - c^* \tag{4a}$$

$$\dot{c}^* = \frac{c^*}{\sigma(c^*)}\big(f'(k^*) - \rho\big) \tag{4b}$$

$$k^*(0) = k_0 \tag{4c}$$

$$\lim_{t\to\infty} u'(c^*(t))\, k^*(t)\, e^{-\rho t} = 0. \tag{4d}$$

[4] Mathematically it is a function, not a functional, because if we were able to find the optimal solution for consumption it would be of type $c^* = C(t, k_0)$. If we plugged it into the functional $\mathsf{U}[\mathbf{c}]$, and integrate on time, we would obtain a function of k_0. The problem only in very special cases we can obtain explicitly function $C(t, k_0)$.

The last condition forces the solution to converge asymptotically (i.e., when $T \to \infty$) to a constant or to a function of time that should grow at a non-positive rate[5]

The PMP operates in the following way.

Let us assume that there is one unique solution to the problem. If this is the case, given an initial value for $(k(0), c(0))$ the system (4a)-(4b) traces out one single path. However, we only know $k(0) = k_0$ and for most values of $c(0)$ the solution will not satisfy the transversality condition (4d). On the other hand, uniqueness of solutions to the problem is equivalent to requiring that there will be only one value for $c(0)$ such that the solution which satisfies (4d). If this condition only holds for for constant values of k and c, this means that the unique solution converges to a steady state of the saddle-point type.

From all this we learn two things: first, the crucial step in obtaining the solution to the problem is to find the initial value for consumption $c^*(0)$, and, second, this value will be determined backwards from the terminal conditions which is usually a steady state of saddle-point type. This is only possible if there is some type of foresight, as is clear from the fact that the consumer values the present value of all the consumption path $(c(t))_{t\in[0,\infty)}$.

Therefore, an optimal solution will contain both forward and backward mechanisms. The first, represented by the instantaneous budget constraint, propagates the capital stock into the future through savings. The second, backward (anticipating) mechanism is related to the incentives for consumption, guided by both the return on production relative to impatience and the attitude of the consumer as regards intertemporal consumption substitution, which determines savings at every point in time.

Next we will show how to find the solution for a particular case.

3.5 Finding the solution through the DPP approach

The DPP approach operates in a different way. It tries to find a rule that, given the observed capital stock in a point in time, would enable to find the optimal consumption.

We call value function to

$$v(k(t)) = \mathsf{U}[\mathbf{c}^*; t] \equiv \max_{c(\cdot)} \int_t^\infty u(c^*(s))e^{-\rho(s-t)}\, ds$$

which provides a valuation of the existing level of the capital stock as a resource allowing for consumption in the future. This notation highlights the fact that the value of the stock of the durable good at time t, $k(t)$, if used efficiently allows us to attain the intertemporal utility level $\mathsf{U}[\mathbf{c}^*; t]$, at the optimum.

[5]This is the case of endogenous growth models for which $f(k)$ is a linear function.

Applying the principle of dynamic programming optimal allocation $(\mathbf{c}^*, \mathbf{k}^*)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\rho\, v(k) = \max_c \big\{u(c) + v'(k)\big(f(k) - c\big)\big\}. \tag{5}$$

The optimal consumption satisfies

$$u'(c^*) = v'(k) \tag{6}$$

If again the utility function has no singularities, we can find implictly the **optimal policy function**

$$c^* = C(k) = (u')^{-1}(v'(k)) \tag{7}$$

which, substituting on the HJB equation, yields

$$\rho\, v(k) = u(C(k)) + v'(k)\big(f(k) - C(k)\big) \tag{8}$$

which is an implicit ordinary differential equation. If we are able to find the optimal policy function, we can determine the solution by solving

$$\begin{aligned} \dot{k} &= f(k) - C(k), \ \text{for}\, t \in \mathrm{T} \\ k(0) &= k_0. \end{aligned} \tag{9}$$

Comparing the optimality conditions from the PMP, equation (3a), and from the DPP, equation (6), we observe that, in the infinite horizon case, the adjoint variable q of the first is formally equal to the marginal value of capital, $v'(k)$ of the second. This allows us to interpret q as a shadow-value of capital.

Remembering our discussing that the solution for c is forward looking, when using the PMP, with the policy function in equation (7) we observe that the last equation implicitly incorporates the transversality condition (4d). That is, it translates the backward solution of the adjoint equation together with the transversality condition, equations (4b) and (4d) into a single backward rule (7). This is why solving the problem by the DPP is said to be applying recursive methods[6]

There are advantages and disadvantages in applying the PMP or the DPP to solving the Ramsey-like types of models. The main difficulty in applying the PMP is related to finding the initial value of consumption by transforming the MHDS into a initial-terminal value problem. However, it allows for using the richness of results from dynamic systems theory for, at least, characterizing analytically the optimal solution. The main difficulty in applying the DPP is solving the implicit ordinary differential equation (8). However, the recursive structure of the solution provides a simpler interpretation of the characteristics of the solution.

[6]Ljungqvist and Sargent, 2018 apply systematically this approach for discrete time versions of macromodels.

3.6 A benchmark application: the isoelastic-Cobb-Douglas case

The Ramsey model in the general form just presented does not have an explicit solution because we have left the utility function, $u\cdot)$, and the production function, $f(\dot{)}$, unspecified.

Whatever the method we use, in general we cannot find explicitly the optimal allocation $(\mathbf{c}^*, \mathbf{k}^*)$, even if we consider explicit utility and production functions. Even if we specified those functions in most cases we cannot obtain explicit (i.e„ exact) solution and we have to use approximate or numerical methods to find them. This is unfortunate, because a complete understanding of the nature of the solution can only be obtained by knowing the explicit solution.[7]

A benchmark particular case in which the utility function is a generalized logarithm[8]

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta}, \text{ for } \theta > 0$$

where $\sigma(c) = \theta$, and the production function is Cobb-Douglas $Y = AK^\alpha L^{1-\alpha}$, yielding

$$y = f(k) = A\,k^\alpha,\ A > 0,\, 0 < \alpha < 1.$$

If we want to determine or characterize the optimal allocation $(\mathbf{c}^*, \mathbf{k}^*)$, we have two alternative methods: First, use the PMP and solve the particular version of the MHDS system together with its initial and transversality conditions, (4a)-(4d),

$$\dot{k}^* = A\,(k^*)^\alpha - c^* \tag{10a}$$

$$\dot{c}^* = \frac{c^*}{\theta}\big(\alpha\,A\,(k^*)^{\alpha-1} - \rho\big) \tag{10b}$$

$$k^*(0) = k_0 \tag{10c}$$

$$\lim_{t\to\infty} (c^*(t))^{-\theta}\,k^*(t)\,e^{-\rho t} = 0 \tag{10d}$$

Because the initial stock of capital is given, from equation (23c) and, as we will see, consumption at time $t = 0$ is implicitly determined from the transversality condition (**??**), we call k a **predetermined** variable and c a **non-predetermined** variable. If we are able to find $c(0)$ uniquely, we say that **the optimal path is determined**.

Second, use the DPP, solve the particular version of the HJB equation (5)

$$\rho\,v(k) = \max_c \left\{\frac{c^{1-\theta}-1}{1-\theta} + v'(k)\big(A\,k^\alpha - c\big)\right\} \tag{11}$$

[7] It is not uncommon to find in the literature inaccurate presentations of the solutions to the Ramsey problem.

[8] We can write it as $u(c) = \log_\theta(c)$. A particular case, when $\theta = 1$ is the logarithmic utility function $u(c) = \log_1(c) = \log(c)$. Prove this by nothing that $u(c) = \log(e^{u(c)})$ and using the l'Hôpital rule.

obtain optimal consumption from the optimal policy function

$$c^* = C(k) = \big(v'(k)\big)^{-\theta},$$

and solve the ODE problem

$$\dot{k} = A\,k^{\alpha} - C(k)$$

together with the initial condition $k(0) = k_0$ to obtain $\mathbf{k}^* = \big(k^*(t)\big)_{t \in [0,\infty)}$.

At last we can answer two fundamental macroeconomic questions: First, how does consumption and output respond to demand and supply shocks, represented by shifts in the parameters A and ρ ? Second, does the type of response matches the stylized facts ?

The best way to answer those questions would involve obtaining exact (also called closed form) solutions. Even in this isoelastic-Cobb-Douglas case, for generic values of the parameters, there is no known explicit solution, which explains why researchers resort to approximate analytical of numerical methods to solve it.

3.6.1 The case $\theta = \alpha$

However, if $\theta = \alpha$ we can obtain an explicit solution can be obtained. Although this relationship is counterfactual [9] because most empirical research, using macro and micro data, finds a realistic value for θ to be in an interval centered at $\theta = 2$, this case is important because it allows for an understanding of our previous discussion.

If we have any hope of getting an explicit solution, the hard problem is solving the backward looking part, i.e., obtaining consumption as a function of the capital stock: $c = C(k)$. We prove in the appendix[10] that both methods yield **one unique solution** such that

$$c = C(k) \equiv \frac{\rho}{\alpha}\,k.$$

Then the capital equation is obtained by solving [11]

$$\begin{cases} \dot{k}^* = A\,(k^*)^{\alpha} - \dfrac{\rho}{\alpha}\,k^* & \text{for}\, t \in (0,\infty) \\ k^*(0) = k_0 \end{cases} \tag{12}$$

which is

$$k^*(t) = \left[\bar{k}^{1-\alpha} + \left(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\right) e^{-\rho\left(\frac{1-\alpha}{\alpha}\right)t}\right]^{\frac{1}{1-\alpha}},\ \text{for}\, t \in [0,\infty), \tag{13}$$

[9] References benchmark parameter values: Basu and Fernald, 1997, Hall, 1988.

[10] See appendix A.

[11] Observe, in the initial condition, that while $k^*(0)$ refers to the optimal solution $k^*(t)$ evaluated at $t = 0$, which is not known at this point, k_0 refers to an observed level of the percapita capital stock which we can obtain from published statistics.

where

$$\bar{k} = \left(\frac{\alpha\, A}{\rho}\right)^{\frac{1}{1-\alpha}} \tag{14}$$

is the steady state level of the capital stock.

From this result we can get explicit solutions for consumption $c^*(t) = \frac{\rho}{\alpha} k^*(t)$, output $y^*(t) = A\left(k^*(t)\right)^{\alpha}$, savings $s(t) = y(t) - c(t)$ and the interest rate $r(t) = \alpha\, A\left(k^*(t)\right)^{\alpha-1}$. For instance,

$$y^*(t) = \left[\bar{y}^{\frac{1-\alpha}{\alpha}} + \left(y_0^{\frac{1-\alpha}{\alpha}} - \bar{y}^{\frac{1-\alpha}{\alpha}}\right) e^{-\rho\left(\frac{1-\alpha}{\alpha}\right)t}\right]^{\frac{\alpha}{1-\alpha}}, \text{ for } t \in [0, \infty) \tag{15a}$$

$$r^*(t) = \left[\frac{1}{\rho} + \left(\frac{1}{r(0)} - \frac{1}{\rho}\right) e^{-\rho\left(\frac{1-\alpha}{\alpha}\right)t}\right]^{-1}, \text{ for } t \in [0, \infty). \tag{15b}$$

where the steady state output

$$y(\infty) = \bar{y} = \bar{c} = \left(A\left(\frac{\alpha}{\rho}\right)^{\alpha}\right)^{\frac{1}{1-\alpha}} = \frac{\rho}{\alpha}\,\bar{k}. \tag{16}$$

We only have the following type of dynamic adjustments (see 1):

- If $k_0 = \bar{k}$ stationary solution: consumption and the capital stock, and therefore output will stay the same, if there is no anticipated shock in any parameter. In this case the rate of return on capital, and therefore, the real interest rate is equal to the rate of time preference $r(\bar{k}) = \rho$.

- If $k_0 > \bar{k}$ increasing solution: if the level of consumption is too low given the level of production, there is positive savings, which allows through capital accumulation to increase output and therefore consumption in the future. There is substitution of present consumption by future consumption. This adjustment is slowed down by the fact that there are decreasing marginal returns: increasing the production capacity does not increase output commensurably. Stability is brought about by the fact that the increase in the stock of capital reduces the rate of return until it converges to ρ from above ($\lim_{k \to \bar{k}^-} r(k) = \rho$).

- If $k_0 < \bar{k}$ decreasing solution: if the level of consumption is higher than output, part of the durable good is deccumulate, which decreases the capital stock and therefore output. This forces consumption to reduce as well. In this case, because there is too much capital the rate of return is below the rate of time preference. Stability is brought about by the fact that the decrease in the stock of capital reduces the rate of return until it converges to ρ from below ($\lim_{k \to \bar{k}^+} r(k) = \rho$).

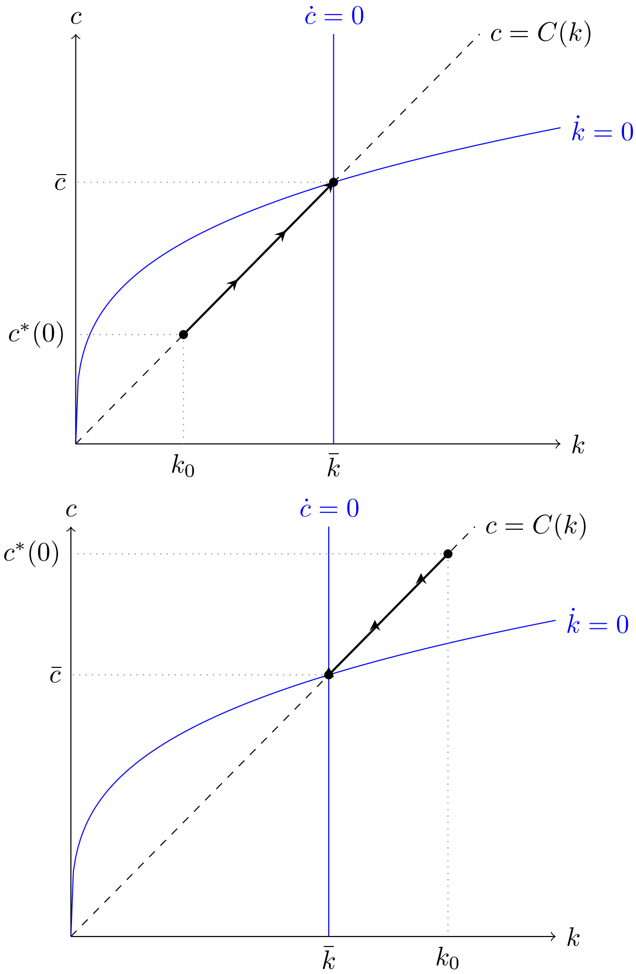

Figure 1: Phase diagrams for the exact optimal adjustments for c and k for the $\theta = \alpha$ case. The adjustment depends on the relationship between the initial level of capital, k_0 and the steady state level $\bar{k}$: if $k_0 < \bar{k}$, as in the upper panel, there will be both an increase in consumption and the stock of capital; and if $k_0 > \bar{k}$, as in the lower panel, both will decrease.

3.6.2 The general case with $\theta \neq \alpha$

Using any of the two approaches, the PMP and DPP, we cannot find an exact solution to the optimal allocation in this case. If we want to have an analytical derivation of the solution, we can only resort to a linear approximation. In this case, the PMP allows for a simpler derivation of an approximative solution[12]. We proceed as follows: first, we obtain a non-zero steady state of the model (which in this case is unique); second, we perform a first-order Taylor approximation of the MHDS (23a)-(23b) in the neighborhood of the steady state , obtaining a variational MHDS; third, we solve the variational MHDS by requiring that the it converges to the steady state and being consistent with the known initial value of the predetermined variable k. This means that the solution satisfies the transversality condition (**??**).

This steady state is given by the positive pair (k, c) such that $r(k) = \rho$ and $c = f(k)$, that is

$$(\bar{k}, \bar{c}) = \Big\{(k, c) \in \mathbb{R}^2_{++} : \alpha\, A\, k^{\alpha-1} = \rho,\ c = A\, k^{\alpha}\Big\}.$$

The steady state is the same as in the equal to the previous case, where $\bar{k}$ is given in equation (14) and consumption, $\bar{c}$, is given in equation (16). Again, in the steady state $\bar{c} = \bar{y}$. This means that the parameter θ does not influence the steady state, only the adjustment towards it.

The variational MHDS, in the neighborhood of the steady state $(\bar{k}, \bar{c})$, is represented by the linear-ODE

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \mathbf{J} \begin{pmatrix} k(t) - \bar{k} \\ c(t) - \bar{c} \end{pmatrix}, \text{where } \mathbf{J} \equiv \begin{pmatrix} \rho & -1 \\ \dfrac{\bar{c} r^{'}(\bar{k})}{\theta} & 0 \end{pmatrix} \tag{17}$$

As we prove in the Appendix B the solution is

$$\begin{aligned} k(t) &= \bar{k} + (k(0) - \bar{k}) e^{\lambda_s t} \\ c(t) &= \bar{c} + \lambda_u (k(0) - \bar{k}) e^{\lambda_s t} \end{aligned} \tag{18}$$

where $\lambda_s < 0 < \lambda_u$ are the eigenvalues of the Jacobian matrix $\mathbf{J}$, satisfying the properties $\lambda_s + \lambda_u = \rho > 0$ and $\lambda_s \lambda_u = \det(J) < 0$, where the the determinant of the Jacobian $\mathbf{J}$ is dependent on the parameters of the model

$$\det(\mathbf{J}) = -\Big(\frac{1-\alpha}{\alpha}\Big)\frac{\rho^2}{\theta} < 0.$$

Therefore

$$\lambda_s = \frac{\rho}{2}\left(1 - \Big(1 + \Big(\frac{1-\alpha}{\alpha}\Big)\frac{4}{\theta}\Big)^{\frac{1}{2}}\right),\ \lambda_u = \frac{\rho}{2}\left(1 + \Big(1 + \Big(\frac{1-\alpha}{\alpha}\Big)\frac{4}{\theta}\Big)^{\frac{1}{2}}\right)$$

[12]However, some people find the DPP as simpler to get a numerical solution.

both λ_s and λ_u depend upon the parameters ρ, α and θ, that is on the substitution between labor and capital in production, on the rate of time preference (which measures impatience) and on the intertemporal substitution in consumption. If $\theta \to \infty$, meaning that there is no intertemporal substitution in consumption we have $\lim_{\theta\to\infty} \lambda_s = \lim_{\theta\to\infty} \lambda_u = \rho$ which means that there will be no savings and consumption will be equal to output. Therefore the smaller is θ the higher (positive or negative) savings will be and consumption will be less instantanepusly correlated with income.

Several other observations can be made. First, if $k(0) \neq \bar{k}$ then the solution converges asymptotically to the steady state $\lim_{t\to\infty} k(t) = \bar{k}$, because $\lim_{t\to\infty} e^{-\lambda_s t} = 0$. The speed of convergence can be measured by the half-life of the adjustment [13] which is, in this case $\tau = -\dfrac{\log(1/2)}{\lambda_s} > 0$. This means the higher in absolute value λ_s is the quicker will be the adjustment. Again, a small θ will speed uf convergence to the steady state.

Second, again the solution displays determinacy. Given any initial value for k, $k(0)$, there is only a value for c, $c(0)$ which is determined endogenously such that $\lim_{t\to\infty} c(t) = \bar{c}$. The exact solution for consumption throughout time is tangent to a linear approximation

$$c(t) - \bar{c} = \lambda_u(k(t) - \bar{k}).$$

3.6.3 Approximations to the optimal path

The difference between the exact solution and the linear approximation can be seen in Figure 2. The linear approximation can capture the qualitative dynamics of the optimal trajectory, although it can quantitatively be slightly different if the initial value of k_0 is far away from the steady state (or the economy is perturbed by an unanticipated shock in any parameter.

There are, in the literature, some misunderstandings regarding the nature of the solution. Sometimes people use the fact that the steady state is a saddle-point, in which the stable manifold passing through the steady state is unique, by using this fact to derive the solution. This can be done as a device to characterize the solution, but introduces some misunderstanding on the mathematical meaning of the uniqueness of the solution to an optimal control problem. If we look at the derivation of the exact solution for the $\theta = \alpha$ case makes this clear: the uniqueness of the policy function is associated to the existence of a stable-manifold of dimension one means that the solution of the problem is unique (it will never wander around the phase diagram).

From a dynamic systems perspective, the solution traces out an heteroclinic trajectory [14] linking $(0,0)$ to $(\bar{k},\bar{c})$ and following a smooth trajectory for (k,c) higher than $(\bar{k},\bar{c})$. That is, the policy function is geometrically equivalent to tha theteroclinic trajectory. For the case $\theta = \alpha$ the line

[13] The half life is defined by $\tau = \left\{t : k(t) - k_0 = \dfrac{\bar{k} - k_0}{2}\right\}$, i.e, by the time needed such to travel half of the distance between the initial, k_0, and the steady state, $\bar{k}$, level for the stock of capital.

[14] An heteroclinic trajectory is a path linking, in a dynamic system, two steady states.

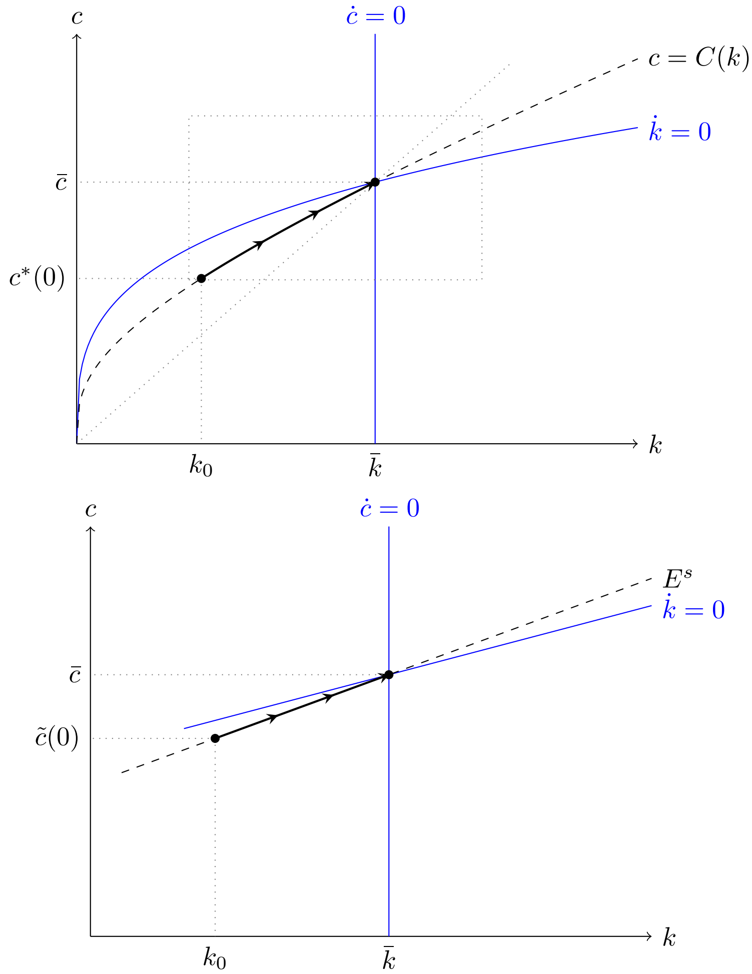

Figure 2: Phase diagrams for the exact optimal adjustments for c and k: the upper panel is the exact solution and the lower panel is the linear approximation

$c = \frac{\rho}{\alpha}k$ is that heteroclinic trajectory (which we saw is the only locus where the solution lies). If $\theta > \alpha$ the policy function will be "trapped" between the same line $c = \frac{\rho}{\alpha}k$ and the isocline for k, $c = f(k) = A\,k^{\alpha}$. Therefore, a better approximation to the policy function (or the stable manifold in the language of the dynamic systems) would be given by an average between the two curves, as

$$c = S(k) = \omega \frac{\rho}{\alpha}\, k + (1-\omega)\, A\, k^{\alpha}$$

where ω is a weighting factor. We could determine ω such that the slope of curve $S(k)$ in the neighborhood of the steady state would be the same as the slope of the stable eigenvector which we saw that was given by λ_u. Therefore

$$\omega = \{w : S'(\bar{k}) = \lambda_u\}.$$

Performing the calculation we find

$$\omega = \frac{\alpha\,(\lambda_u - \rho)}{\rho(1-\alpha)}.$$

Therefore a "heteroclinic" approximation of the policy function is

$$c = S(k) = \frac{\lambda_u - \rho}{1-\alpha}\, k + \frac{\rho - \alpha\,\lambda_u}{\rho(1-\alpha)}\, A\, k^{\alpha}$$

3.6.4 Consequences of supply and demand shocks

Assuming we depart from a steady state Figure 3 display the dynamic adjustments following a non-anticipated, permanent and constant supply shock (i.e., an increase in productivity from A to $A' > A$) and a non-anticipated, permanent and constant demand shock (i.e., an increase in productivity from ρ to $\rho' > \rho$).

In the case of a positive supply shock, starting from point A (in the upper panel of Figure 3) consumption, responding to the new anticipated path for the rate of return increases continuously, to point B. As there is also an increase in output, which is larger than the increase in consumption, there is positive savings which through investment sets in motion a process of further capital accumulation, increase in consumption, and positive savings which drives a transition dynamics process towards the new steady state, point C, in which both consumption, output and the capital stock will be higher.

In the case of a positive demand shock, there is also an immediate increase in consumption, because the implicit relationship between present and future consumption becomes unbalanced at point A (in the lower panel): consumers prefer consuming now than in the future. Because output does not change, this generates negative savings (i.e, the durable good is used more for consumption rather than investment) which reduces the stock of capital and output, forcing a reduction in consumption. The process is stabilized by the facts that further reduction in the stock of capital raises the rate of return of capital until we have $(\bar{k}') = \rho'$.

This type of dynamics justifies labelling capital as predetermined and consumption as non-pretedermined: capital moves continuously when there is savings, while consumption responds to news regarding the future evolution of the economy.

The most common approach to obtain an analytical derivation of the whole process is through a **comparative dynamics exercise**. As $\det \mathbf{J} \neq 0$ we could approximate the response function for the model for a change in parameter φ by

$$\partial_\varphi k(t) = \partial_\varphi \bar{k}\,(1 - e^{\lambda_s\, t})\ t \in [0, \infty)$$

where $\partial_\varphi \bar{k}$ is the long run multiplier and $\partial_\varphi k(t)$ is the short run multiplier for the stock of capital for a permanent, non-anticipated shock in the parameter φ. For consumption one would get

$$\partial_\varphi c(t) = \partial_\varphi \bar{c}\, - \lambda_u\, \partial_\varphi \bar{k}(e^{\lambda_s\, t}\ t \in [0, \infty).$$

We call impact multiplier to the short run multiplier evaluated at time $t = 0$. Performing the calculation we have

$$\partial_\varphi k(0) = 0,\ \partial_\varphi c(0) = \partial_\varphi \bar{c}\, - \lambda_u\, \partial_\varphi \bar{k}$$

where we have an analytical confirmation to the fact that k only changes continuously and that the initial change in c responds to the long-run anticipated behavior of the economy.

In order to calculate the long-run multipliers we use the implicit function theorem. Looking to the MHDS system, in particular to the dynamic equations (23a)-(23b) we see that at the steady state

$$\begin{cases} y(A, k) - c = 0 \\ \dfrac{c}{\theta}\big(r(A, k) - \rho\big) = 0 \end{cases}$$

Computing differentials for both the endogenous variables and the parameters, evaluating the derivatives at the steady state, we have, for a change in A [15]

$$\begin{pmatrix} \partial_k y(\bar{k}) & -1 \\ \dfrac{\bar{c}\,\partial_k r(\bar{k})}{\theta} & \dfrac{r(\bar{k}) - \rho}{\theta} \end{pmatrix} \begin{pmatrix} dk \\ dc \end{pmatrix} + \begin{pmatrix} \partial_A\, y(\bar{k}) \\ \dfrac{\bar{c}\partial_A\, r(\bar{k})}{\theta} \end{pmatrix} dA$$

where, at the steady state $r(\bar{k}) - \rho$. As the first Jacobian matrix is matrix $\mathbf{J}$ and we already found that it has a non-zero determinant, then we can invert it to get the expressions for the long-run

[15] We use the following notation. Let a function be $y = f(x, \varphi)$ we write $\partial_x y = \dfrac{\partial f(x, \varphi)}{\partial x}$.

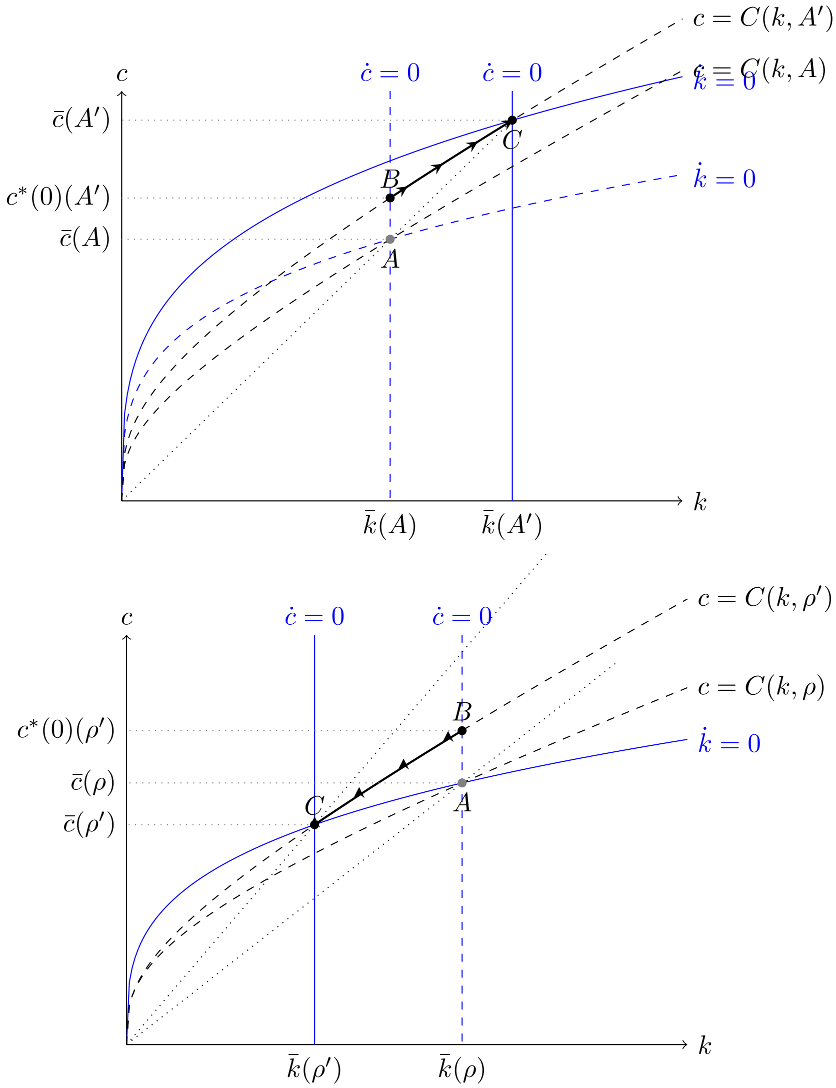

Figure 3: Dynamic adjustments for non-anticipated, permanent and constant supply (upper panel) and a demand (lower panel) shocks. In both cases I assume that the economy is in the steady state (point A). Immediately after the shock, becomes off-balance and there is an immediate jump in consumption (point B) to the new curve $C(k)$ (or the stable eigenspace in the linearized version). Across time it converges asymptotically to the new steady state (point C). We see that while a supply shock generates a pro-cyclical adjustment, the demand shock has a transient countercyclical adjustment.

multipliers

$$\begin{pmatrix} \partial_A \bar{k} \\ \partial_A \bar{c} \end{pmatrix} = \begin{pmatrix} \frac{dk}{dA}\big|_{k=\bar{k}} \\ \frac{dc}{dA}\big|_{c=\bar{c}} \end{pmatrix} = - \begin{pmatrix} \partial_k\, y(\bar{k}) & -1 \\ \frac{\bar{c}\partial_k\, r(\bar{k})}{\theta} & \frac{r(\bar{k}) - \rho}{\theta} \end{pmatrix}^{-1} \begin{pmatrix} \partial_A\, y(\bar{k}) \\ \frac{\bar{c}\,\partial_A\, r(\bar{k})}{\theta} \end{pmatrix}.$$

For the case of a shock in A we find comparative dynamics exercise for the cases depicted in Figure 3, performing simplifications and evaluating the derivatives at the steady state, we find the long run multipliers

$$\begin{pmatrix} \partial_A \bar{k} \\ \partial_A \bar{c} \end{pmatrix} = - \begin{pmatrix} \frac{\bar{c}}{A} \\ \frac{\rho\,\bar{c}}{\theta\, A} \end{pmatrix} = \begin{pmatrix} \frac{\bar{k}}{(1-\alpha)\, A} \\ \frac{\bar{c}}{(1-\alpha)\, A} \end{pmatrix},$$

which are both positive. Therefore, the impact multipliers for consumption is

$$\partial_A c(0) = \Big(\frac{\rho - \alpha\,\lambda_u}{\alpha}\Big)\frac{\bar{k}}{(1-\alpha)\, A}$$

which is positive if and only if $\rho - \alpha\,\lambda_u = \lambda_s + (1-\alpha)\lambda_u > 0$.

3.7 Empirical implementation

If we would like to to take this model to data we can fix parameters $\alpha = 0.3$, $\theta = 2$ and $\rho = 0.02$. Taking statistics for the GDP per capita (in Portugal in 2019)[16] as $y = 23.7$ and the capital output ratio as $k/y = 3.43$ if we calibrate $A = 6.5$.

4 The simplest dynamic general equilibrium (DGE) model

4.1 The model

Instead of the existence of a central planner as a coordinating device, in a market economy the coordination is make through market transactions. We assume production is done in firms which are owned by households. This allows us to distinguish between financial capital, a and physical capital k. Firms are a device to transform financial capital into physical capital. They distribute not only capital income but also wage income to households as a result of the production process. Firms are also on the supply side of the output market where demand comes from consumption and investment expenditures.

[16]See https://www.ine.pt/xportal/xmain?xpid=INE&xpgid=ine_destaques&DESTAQUESdest_boui=306571350&DESTAQUESmodo=2.

Households As owners of firms, household also have to finance investment expenditures through savings. We assume that the utility functional is time additive

$$\mathsf{U}[c] = \int_0^\infty u(c(t))\, e^{-\rho\, t}\, dt$$

The household problem is

$$v(a_0) = \max_c \mathsf{U}[c] \tag{19a}$$

subject to

$$\dot{a} = r(t)\, a + w(t)\ell(t) - c(t),\ \text{for}\, t \in (0, \infty) \tag{19b}$$

$$a(0) = a_0 \tag{19c}$$

$$\lim_{t\to\infty} a(t) e^{-\int_t^\infty r(s)ds} \geq 0 \tag{19d}$$

$$\tag{19e}$$

The budget constraint (27b) has the following meaning: income to consumers include capital income ra and labor income w, where it is assumed that consumers supply inelastically one unit of labor $\ell(t) = 1$; the difference between total income and consumption are equal to savings $s(t) = r(t)\, a + w(t)\ell(t) - c(t)$ which, if positive, represent an increase in the ownership financial assets, which in this economy we assume it is deterministic. Differently from k in the Ramsey model $a(t)$ can be positive, if the household is a net creditor, or negative, if the household is a net debtor, at time t.

The last condition is called in the literature the non-Ponzi game condition (NPG). It essentially means that households can not expect to be a net debtor asymptotically. It can only be a net debtor in the short run. [17] Taken together, the constraints to the household mean that it cannot expect to use more than its initial asset position a_0, if it is a net creditor (i.e., if $a_0 > 0$) or has to repay its initial level of indebtedness, if it is an initial net debtor (i.e, if $a_0 < 0$).

In order to see this, integrate equation (27b) together with the initial condition (27c) to get

$$a(t) = e^{\int_0^t r(s)ds}\Big(a_0 + \int_0^t e^{-\int_0^s r(z)dz}\, w(s) - c(s)\, ds\Big]$$

multiply both terms by the discount factor $e^{-\int_0^t r(s)ds}$

$$e^{-\int_0^t r(s)ds}\, a(t) = a_0 + \int_0^t e^{-\int_0^s r(z)dz}\, w(s) - c(s)\, ds$$

pass to the limit as $t \to \infty$

$$\lim_{t\to\infty} e^{-\int_0^t r(s)ds}\, a(t) = a_0 + \int_0^\infty e^{-\int_0^s r(z)dz}\, w(s) - c(s)\, ds.$$

[17] Sometimes people call this condition the transversality condition, which is a misnomer.

Introduce the NPG condition,

$$\int_0^\infty e^{-\int_0^t r(s)ds}\, c(t)\, dt \leq a_0 + h^\infty(0)$$

where

$$h^\infty(0) \equiv \int_0^\infty e^{-\int_0^t r(s)ds}\, w(t)\, dt$$

is the lifetime human capital, i.e, the present value (at the time of planning) of the future wages discounted by the market interest rate.

Therefore the three constraints are equivalent to requiring that the present value of consumption does not exceed the initial financial and human capital, measured by the present value of future wage earnings.

Firms This capital represents participation in firms. Therefore, the firms balance sheet takes the form of $K(t) = N\, a(t)$, where we assume $K(t)$ to be aggregate capital and N population (or the number of households) assumed to be homogeneous and constant.

The firms problem is solved for every moment in time

$$\max_{K(t),L(t)} \Pi(K(t), L(t)) = F(K(t), L(t)) - r(t)K(t) - w(t)L(t)$$

where we assume that $F(\cdot)$ is homogeneous of degree one. We assume firms are price takers in all the markets and that there is no unemployment, that is $L(t) = N$.

Because of linear homogeneity, the firms problem may be simplified to

$$\max_{k(t)} \pi(k(t)) = f(k(t)) - r(t)k(t) - w(t). \tag{20}$$

Definition 2 (DGE)**.** *A dynamic general equilibrium is an allocation* $\left(c^{eq}(t), k^{eq}(t), y^{eq}(t), w^{eq}(t), r^{eq}(t)\right)_{t\in T}$ *such that*

- *every household solves problem* (19a)-(27d) *taking the interest rate as given* r *and* w*, but having perfect foresight on their determination;*
- *firms solve problem* (20)*;*
- *The compatibility condition holds:* $a(t) = k(t)$*.*
- *Labor, capital and goods markets clear. In particular, the product market equilibrium condition holds:*

$$y(t) = c(t) + \dot{k}(t), \textit{for every}\, t \in T.$$

Solution to the household's problem, using the PMP yield the first order conditions

$$\dot{c} = \frac{c}{\sigma(c)}\big(r(t) - \rho\big) \tag{21a}$$

$$\lim_{t\to\infty} u'(c(t))\, a(t) e^{-\rho t} = 0 \tag{21b}$$

together with the constraints (19a)-(27c). Equation (21a) provides an arbitrage condition between consumption and increasing the investment in financial assets and equation (21b) is the transversality condition, which is a dual condition associated to the NPG condition (27d) by requiring that the asset position has no value in present-value terms.

The solution to the firm's problem yield the equations

$$f'(k(t)) = r(t) \tag{22a}$$

$$f(k(t) - k(t)\, f'(k(t)) = w(t). \tag{22b}$$

The first equation means that the optimal the return on capital (obtained from production) is equal, at the optimum, to the interest rate, and the second means that the marginal return from employing labor is equal to the wage rate.

Because labor supplied inelastically to firms and households supply capital to firms, equations (22a) and (22b) are identically market clearing conditions for capital and labor market. As $y(t) = f(k(t)) = f'(k(t))k(t) + w(t) = r(t)k(t) + w(t)$ then we get the equilibrium total income to households.

From the compatibility condition we have $a(t) = k(t)$, we see that the household's budget constraint is formally identical to the market equilibrium condition $\dot{k} = y(t) - c(t)$.

Therefore, we can find the DGE $\big(c^{eq}(t), k^{eq}(t), y^{eq}(t), w^{eq}(t), r^{eq}(t)\big)_{t\in\mathrm{T}}$ by solving a dynamic system which is formally identical to the first optimum conditions for an optimal allocation $\big(c^*(t), k^*(t)\big)_{t\in\mathrm{T}}$ as in the Ramsey model (4a)-(4d).

In the case of the isoelastic-Cobb-Douglas case the DGE satisfies the following dynamic system

$$\dot{k}^{eq} = A\,(k^{eq})^{\alpha} - c^{eq} \tag{23a}$$

$$\dot{c}^{eq} = \frac{c^{eq}}{\theta}\big(\alpha\, A\,(k^{eq})^{\alpha-1} - \rho\big) \tag{23b}$$

$$k^{eq}(0) = k_0 \tag{23c}$$

$$\lim_{t\to\infty} (c^{eq}(t))^{-\theta}\, k^{eq}(t)\, e^{-\rho t} = 0 \tag{23d}$$

This **means that the DGE is Pareto optimal**, and all our previous analysis can be reinterpreted as a result of a decentralized market allocation.

4.2 Interpreting the DGE

We have used the PMP to obtain the first order conditions for an optimum. There is a potential pitfall in using a recursive approach by using the DPP which allows for a discussion of one important feature of the DGE.

Assume that the utility function is $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$ and consider the consumer problem (**??**)-(**??**) taking the rate of return and the wage rate as constants $r(t) = r$ and $w(t) = w$. The HJB equation is

$$\rho\, v(a) = \max_c \left\{ \frac{c^{1-\theta}-1}{1-\theta} + v'(a)\big(r\,a + w - c\big) \right\}.$$

The policy function is $c = C(a) = (v'(a))^{-\frac{1}{\theta}}$. Substituting in the HJB equation yeilds the implicit ODE

$$\rho\, v(a) = \frac{1}{1-\theta}\Big(\theta\,(v'(a))^{-\frac{\theta-1}{\theta}} - 1\Big) + v'(a)\,\big(r\,a + w\big)$$

Using a similar method as the one used in Appendix B we explicitly find the value function

$$v(a) = \frac{1}{1-\theta}\left[\Big(\frac{\rho + r\,(\theta-1)}{\theta}\Big)^{-\theta}\Big(\frac{r\,a+w}{r}\Big)^{1-\theta} - \frac{1}{\rho}\right]$$

which implies that the optimal consumption is given by the policy function

$$c = C(a) = \Big(\frac{\rho + r\,(\theta-1)}{\theta}\Big)\Big(\frac{r\,a+w}{r}\Big). \tag{24}$$

If we consider the optimality conditions for the firm, the consistency condition $a = k$ and the market equilibrium condition we see that the DGE becomes a recursive system, where

$$\dot{k} = A\,k^{\alpha} - C(k)$$

and

$$C(k) = \frac{\rho}{\alpha\,\theta}\,k + \frac{\theta-1}{\theta}\,A\,k^{\alpha}, \tag{25}$$

is market policy function. If we solve the differential equation we obtain

$$k(t) = \Big(\bar{k}^{1-\alpha} + \big(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\big)\,e^{-\tilde{\lambda}\,t}\Big)^{\frac{1}{1-\alpha}}$$

where

$$\tilde{\lambda} = \frac{\rho}{\theta}\Big(\frac{1-\alpha}{\alpha}\Big).$$

If we compare this solution to the Ramsey case dealt in section 3.6.2, we observe: first, now we obtain a closed form solution while for the Ramsey model we could not obtain a closed form solution; second, we obtain an explicit solution to solution path (25) while this was not possible for the centralized model; third, the asimptoticaly both consumption and the capital stock converge to the same steady state; and fourth, the main difference is related to the speed of adjustment, which is now exactly $\tilde{\lambda}$ and it was approximated by λ_s in the Ramsey case.

Looking to equation (25) we observe that, as in our discussion to the ”heteroclinic” approach to approximating the optimal path for the centralized problem, it is an average of the same two schedules $c = \dfrac{\rho}{\alpha}k$ and $c = A\,k^{\alpha}$, but now we are able to find the explicit weight $\omega = \frac{1}{\theta}$. This means that a linear solution along curve $c = \dfrac{\rho}{\alpha}k$ will occur for $\theta = 1$ and not $\theta = \alpha$ as in the Ramsey model. This means that compared to the model in 3.6.2 the equilibrium trajectory determined in this way is shifted down, which means that **there is over-saving**.

Next we show that the reason for this is that solving the model this way implicitly assumes that the consumer is myopic and does not take into consideration that saving now depresses the future interest rate because of the existence of decreasing marginal returns.

We next prove that the optimal consumption satisfies (see the proof in Appendix C)

$$c^{eq}(t) = \frac{a(t) + h^{\infty}(t)}{\int_t^{\infty} e^{-\int_t^s \gamma(z)dz} ds}, \quad \text{for } t \in (0, \infty) \tag{26}$$

where

$$\gamma(t) \equiv \frac{(\theta - 1)\,r(t) + \rho}{\theta}$$

and

$$h^{\infty}(t) \equiv \int_t^{\infty} e^{-\int_t^s r(s)ds}\, w(t)\, dt$$

is the human capital at time t. In this case the rate of consumption growth is endogenous and time varying, because the interest rate is a function of the capital stock $r(t) = \alpha\, A\, k(t)^{1-\alpha}$ which depends on consumption. If the interest rate were constant, i.e, $r(t) = r$ for time t onward, we would obtain the policy function in equation (24).

This clarifies why the equilibrium in this case is different from the one derived from using the PMP together with the market equilibrium condition for the rate of return as in the previous subsection: if the consumer takes both the interest rate and the wage rates as constants it does not incorporate the effect of the increase in savings on capital accumulation which will increase wages and decrease the rate of return on capital. It is as an externality which is not internalized.

Therefore, if we assume perfect foresight, the PMP approach allows for the right determination of the DGE path. The DPP approach seems simpler to use if there technology of production is linear (as in endogenous growth models) or in models in which there is not perfect foresight.

5 Endogenous labor supply

In the simple version of the model, we assumed that every household supplied elastically one unit of labor supply, i.e. $\ell = 1$, where ℓ represents work effort (hours worked plus effort per hour).

The utility functional of households depends now on the paths of both consumption and work effort $\big(c(t), \ell(t)\big)_{t\in\mathbb{R}_+}$,

$$\mathsf{U}[c,\ell] = \int_0^\infty u(c(t), \ell(t))\, e^{-\rho\, t}\, dt$$

The household's problem now is

$$v(a_0) = \max_{c,\ell} \mathsf{U}[c,\ell] \tag{27a}$$

subject to

$$\dot{a} = r(t)\, a + w(t)\ell - c(t),\ \text{for}\, t \in (0,\infty) \tag{27b}$$

$$a(0) = a_0 \tag{27c}$$

$$\lim_{t\to\infty} a(t)e^{-\int_t^\infty r(s)ds} \geq 0 \tag{27d}$$

$$\tag{27e}$$

This firms problem is the same as in the previous DGE model.

In a perfect foresight enviroment the DGE is Pareto optimal and is equivalent to the following centralized problem

$$v(k_0) = \max_{c,\ell} \int_0^\infty u(c(t), \ell(t))\, e^{-\rho\, t}\, dt \tag{28a}$$

subject to

$$\dot{k} = f\big(k(t), \ell(t)\big)c(t),\ \text{for}\, t \in (0,\infty) \tag{28b}$$

$$k(0) = k_0 \tag{28c}$$

$$\lim_{t\to\infty} k(t)e^{-\int_t^\infty r(s)ds} \geq 0 \tag{28d}$$

$$\tag{28e}$$

The model is said to be neo-classical if the utility function, $u(c,\ell)$ is increasing and concave in c and decreasing and convex in ℓ and the production function is increasing in both inputs, displays decreasing marginal returns and is concave, and both functions are sufficiently smooth such that there are no-singularities allowing for the implicit function theorem to be invoked to determine uniquely the marginal utilities and marginal productivities.

There are two main versions of the model, depending on the separability of the utility function:

1. the additive utility case, in which

$$u(c,\ell) = u(c) - v(\ell)$$

   in which $u''(c) < 0 < u'(c)$ and $v'(\ell) < 0 < v''(\ell)$

2. non-additively separable cases comprise several possibilities:

   (a) the Cobb-Douglas case

$$u(c,\ell) = u(v(c,\ell)) = \frac{\left(c^{\gamma}(1-\ell)^{1-\gamma}\right)^{1-\theta} - 1}{1-\theta}$$

   (b) the Greenwood, Hercowitz, and Huffman, 1988, or GHH case

$$u(c,\ell) = u(c - v(\ell))$$

   (c) the King, Plosser, and Rebelo, 1988 or KPR case

$$u(c,\ell) = u(c - v(c,\ell))$$

Defining the current-value Hamiltonian function

$$H(k,q,c,\ell) = u(c,\ell) + q\big(f(k,\ell) - c\big)$$

the necessary conditions for an optimum $\big(c^*(t), \ell^*(t), k^*(t)\big)_{t\in\mathbb{R}_+}$ are [18]

$$u_c(c,\ell) = q \tag{29a}$$

$$u_\ell(c,\ell) = q\, f_\ell(k,\ell) \tag{29b}$$

$$\dot{q} = q\big(\rho - f_k(k,\ell)\big) \tag{29c}$$

$$\lim_{t\to\infty} q(t)\, k(t)\, e^{-\rho t} = 0 \tag{29d}$$

together with the admissibility conditions (28b) and (28c), where the partial derivatives of the utility function are denoted by $u_j(c,\ell) \equiv \frac{\partial u(c,\ell)}{\partial j}$, for $j = c, \ell$ and the marginal productivities are denoted by $f_j(k,\ell) \equiv \frac{\partial f(k,\ell)}{\partial j}$, for $j = k, \ell$.

The strategy to solving the model is different depending on the the assumption regarding the separability of the utility function.

[18] From now, although we are refering to optimal allocations, we do not introduce the notation * in cases in which there is no ambiguity.

5.1 Additively separable utility

If the utility function is additively separable, then $u_c(c,\ell) = u'(c)$ and $u_\ell(c,\ell) = v'(\ell)$. Then we can use equation (29a) to eliminate q, yielding the arbitrage equation for labor, equation (29b),

$$v'(\ell) = u'(c)\, w(k,\ell)$$

where the wage rate is equal to the marginal productivity of labor: $w(k,\ell) = f_\ell(k,\ell)$. If the utility and the production functions are sufficiently smooth, we can apply the implicit function theorem to obtain the optimal work effort as a function of consumption and the stock of capital

$$\ell = L(c,k).$$

Because

$$\begin{aligned} L_c(c,k) &= -\frac{u''(c)\, f_\ell(\ell,k)}{v''(\ell) + u'(c)\, f_{\ell\ell}(k,\ell)} < 0 \\ L_k(c,k) &= -\frac{u'(c)\, f_{\ell k}(\ell,k)}{v''(\ell) + u'(c)\, f_{\ell\ell}(k,\ell)} > 0 \end{aligned}$$

the optimal work effort decreases with consumption (because consumption and work effort and Edgeworth substitutable) and increases with the stock of capital, because it increases the wage rate $w_k(k,\ell) = f_{\ell k}(\ell,k) > 0$.

As we are able to isolate q, and we are able to obtain the optimal labor effort uniquely (if there are no singularities) then the MHDS becomes

$$\dot{c} = \frac{c}{\sigma(c)}\big(R(k,c) - \rho\big) \tag{30a}$$

$$\dot{k} = F(k,c) - c \tag{30b}$$

$$k(0) = k_0 \tag{30c}$$

$$\lim_{t\to\infty} u'(c(t))\, k(t)\, e^{-\rho t} = 0 \tag{30d}$$

where the rate of return of capital is $R(k,c) = r\big(k, L(c,k)\big) = f_k\big(k, L(c,k)\big)$ and output $Y(k,c) = y\big(k, L(c,k)\big) = f\big(k, L(c,k)\big)$ are both functions of the stock of capital and consumption, because the work effort is endogenous and depends on the leisure-consumption choices of the household.

Clearly, the reduced form rate of return decreasing with both consumption and the capital stock $R_c(c,k) = f_{k\ell}(k,\ell)\, L_c(c,k) < 0$ and

$$\begin{aligned} R_k(c,k) &= f_{kk}(k,\ell) + f_{k\ell}(k,\ell)\, L_k(c,k) \\ &= \frac{u'(c)\,\big(f_{kk}(\ell,k) f_{\ell\ell}(\ell,k) - (f_{k\ell}(\ell,k)^2\big) + v''(\ell) f_{kk}(k,\ell)}{v''(\ell) + u'(c)\, f_{\ell\ell}(k,\ell)} < 0 \end{aligned}$$

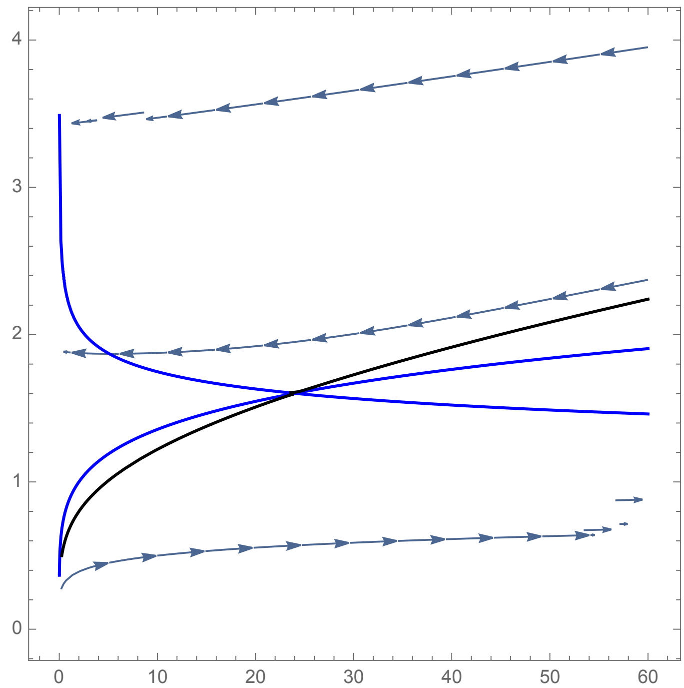

Figure 4: Endogenous labor

because of the concavity of $f(\cdot)$. Furthermore, we find that $Y_c(c,k) = f_\ell(k,\ell)L_c(c,k) < 0$ and $Y_k(c,k) = f_k(k,\ell) + f_\ell(k,\ell)L_k(c,k) > 0$.

This implies that the isocline $\dot{c} = 0$ is negatively sloped in the diagram $(k,c) = 0$ (and not vertical as in the previous case) and the isocline $\dot{k} = 0$ is positively sloped (see Figure 4.

We also observe in the phase diagram 4 that the stable manifold is positively sloped, which means that along the optimal paths consumption and the capital stock are positively related. Given the properties of function $L(c,k)$ this implies that the adjustment of the labor effort is ambiguous, because of its responds positively to the increase in wages but negatively to the increase in consumption.

5.2 Non-additively separable utility

If the utility function is non-separable, we cannot separate the effect of consumption over q. This means that the marginal utility of consumption responds to both changes in consumption and labor effort. In this case, we can use the implicit function theorem in equation (29a) and (29b) to solve both consumption and labor effort as a function of the adjoint variable q, $c = C(q,k)$ and $\ell = L(q,k)$. In appendix D we prove that all the partial derivatives in functions $C(\cdot)$ and $L(\cdot)$ have ambiguous signs. However, it we expect that $L_k > 0$ and $C_q < 0$ The MHDS becomes

$$\dot{q} = q\big(\rho - f_k\big(k, L(q,k)\big)\big) \tag{31a}$$

$$\dot{k} = y\big(k, L(q,k)\big) - C(q,k) \tag{31b}$$

$$\lim_{t\to\infty} q(t)\,k(t)\,e^{-\rho t} = 0 \tag{31c}$$

We provide in the problem set 1 several particular cases of this model.

6 References

The original Ramsey model is presented in Ramsey, 1928 and was rediscovered by Cass, 1965 and Koopmans, 1965.

There are several textbook presentations of this model: in continuous time, recent presentations can be found in Heijdra, 2009, sec. 14.5 Acemoglu, 2009, ch.8, Romer, 2019, ch 2 and Alogoskoufis, 2019, ch. 4.

The model with endogenous labor can be found in Wickens, 2008, sec 4.6 (discrete time version) and Heijdra, 2009, ch. 7.

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A Solving the Ramsey problem for $\theta = \alpha$

A.1 Using the PMP

Looking at the Euler equation (23b) we can write it $\dot{c} = \gamma_c(t)c(t)$, where γ_c is the (variable growth rate of consumption. It has a solution of type

$$c(t) = c(0)e^{\int_0^t \gamma(s)\,ds}, \text{ where } \gamma(t) \equiv Ak^{\alpha-1}(t) - \frac{\rho}{\alpha}$$

where we do no know $c(0)$. The idea is to obtain it from the transversality condition (**??**). There is another difficulty related to the fact that γ is a function if k and the transversality equation also depends on k. In order to circumvent both difficulties define

$$z(t) \equiv \frac{k(t)}{c(t)}$$

Therefore, taking log derivatives of time and substituting equations (23a) and (23b) we find ´

$$\frac{\dot{z}}{z} = \frac{\dot{k}}{k} - \frac{\dot{c}}{c} = A\,k^{\alpha-1} - \frac{c}{k} - \Big(Ak^{\alpha-1}(t) - \frac{\rho}{\alpha}\Big) = -\frac{1}{z} + \frac{\rho}{\alpha}$$

we find a linear ODE

$$\dot{z} = \frac{\rho}{\alpha}\,z - 1$$

which has solution

$$z(t) = \frac{\alpha}{\rho} + \Big(z(0) - \frac{\alpha}{\rho}\Big)e^{\frac{\rho}{\alpha}\,t}$$

where we do not know $z(0) = k(0)/c(0)$ because $c(0)$ is unknown. Therefore, the transversality condition becomes

$$\begin{aligned}
\lim_{t\to\infty} c(t)^{-\alpha}\,k(t)\,e^{-\rho\,t} &= \lim_{t\to\infty} c(t)^{1-\alpha}\,z(t)\,e^{-\rho\,t} = \\
&= \lim_{t\to\infty} c(t)^{1-\alpha}\Big[\frac{\alpha}{\rho}\,e^{-\rho\,t} + \Big(\frac{k(0)}{c(0)} - \frac{\alpha}{\rho}\Big)e^{\rho\left(\frac{1-\alpha}{\alpha}\right)\,t}\Big] = \\
&\text{(because } z(0) = k(0)/c(0)\text{)} \\
&= 0 + \lim_{t\to\infty} c(t)^{1-\alpha}\Big[\Big(\frac{k(0)}{c(0)} - \frac{\alpha}{\rho}\Big)e^{\rho\left(\frac{1-\alpha}{\alpha}\right)\,t}\Big] = \\
&\text{(because } \lim_{t\to\infty} e^{-\rho\,t} = 0\text{)} \\
&= 0
\end{aligned}$$

if and only if $c(0) = \frac{\rho}{\alpha}k(0)$ because $c(t) > 0$ and $\lim_{t\to\infty} e^{\rho\left(\frac{1-\alpha}{\alpha}\right)\,t} = \infty$. Then, $z(t) = \bar{z} = \frac{\alpha}{\rho}$ is a constant, which means that $c(t) = \frac{\rho}{\alpha}k(t)$ for all $t \in [0, \infty)$.

Then, if we choose the optimal consumption rule, we can obtain the optimal path for capital by solving problem (12). This is a Bernoulli ODE than can be transformed into a linear ODE. Define $x(t) \equiv k(t)^{1-\alpha}$ as

$$\frac{\dot{x}}{x} = (1-\alpha)\,\frac{\dot{k}}{k} = (1-\alpha)\Big(Ak^{\alpha-1} - \frac{\rho}{\alpha}\Big) = (1-\alpha)\Big(\frac{A}{x} - \frac{\rho}{\alpha}\Big) \Rightarrow \dot{x} = (1-\alpha)\Big(A - \frac{\rho}{\alpha}\,x\Big)$$

This equation has the solution

$$x(t) = \frac{\alpha\,A}{\rho} + \Big(x(0) - \frac{\alpha\,A}{\rho}\Big)\,e^{-\rho\left(\frac{1-\alpha}{\alpha}\right)t}.$$

By transforming back to k we find the optimal capital accumulation as a function of time (13).

A.2 Using the DPP

The HJB equation, if $\theta = \alpha$, is

$$\rho\,v(k) = \max_c \Big\{\frac{c^{1-\alpha}-1}{1-\alpha} + v'(k)\big(A\,k^\alpha - c\big)\Big\}$$

The optimality condition for consumption is

$$c^{-\alpha} = v'(k) \Rightarrow C(k) = (v'(k))^{-\frac{1}{\alpha}}$$

After substitutiong $C(k)$ we get the HJB at the optimum as an implicit differencial equation $v(k)$ on k,

$$\rho\,v(k) = \frac{\alpha}{1-\alpha}(v'(k))^{\frac{\alpha-1}{\alpha}} + v'(k)\,A\,k^\alpha - \frac{1}{1-\alpha}$$

Although this is a highly non-linear equation we can obtain an explicit solution by using the method of undetermined coefficients. Unfortunately, it does not provide a constructive way to obtain the solution: we conjecture a functional form of the equation, depending on unknown parameters; if the functional form is right, by substituting in the HJB equation, we would obtain the values of those parameters. Off course that depends on assuming that the solution has an explicit functional form and in our ability to find it.

Let us conjecture that the solution is of type

$$v(k) = \beta_0 + \beta_1\,k^{1-\alpha}$$

which has derivative

$$v'(k) = \beta_1\,(1-\alpha)\,k^{-\alpha}$$

Substituting in the HJB equation, we have

$$\rho\big(\beta_0 + \beta_1\,k^{1-\alpha}\big) = \frac{\alpha}{1-\alpha}\Big(\beta_1\,(1-\alpha)\Big)^{\frac{\alpha-1}{\alpha}}\,k^{1-\alpha} + A\,\beta_1\,.(1-\alpha) - \frac{1}{1-\alpha}$$

We can find β_0 and β_1 by matching the term that depends on $k^{1-\alpha}$ as the remaining term by solving the following system for β_0 and β_1,

$$\begin{cases} \rho\,\beta_0 = A\,\beta_1\,.(1-\alpha) - \dfrac{1}{1-\alpha} \\ \rho\,\beta_1 = \dfrac{\alpha}{1-\alpha}\Big(\beta_1\,(1-\alpha)\Big)^{\frac{\alpha-1}{\alpha}} \end{cases}$$

Therefore the solution to the HJB equation is

$$v(k) = \frac{1}{\rho}\Big(A\left(\frac{\alpha}{\rho}\right)^{\alpha} - \frac{1}{1-\alpha}\Big) + \frac{1}{1-\alpha}\left(\frac{\alpha}{\rho}\right)^{\alpha} k^{1-\alpha}$$

and the optimal policy function is ´

$$c^* = C(k) = v'(k)^{-\frac{1}{k}} = \frac{\rho}{\alpha}\,k$$

which is the same as the one we obtained by using the PMP. We follow the same approach to find the optimal capital accumulation function $k^*(t)$ and $c^*(t)$.

B Solving the Ramsey problem for the case $\theta \neq \alpha$

Consider the variational MHDS (17). The jacobian matrix, J, has a positive trace, $\mathrm{tr}(J) = \rho > 0$ and a negative determinant, because

$$\det(J) = \frac{\bar{c}r^{'}(\bar{k})}{\theta} = \frac{\bar{c}\alpha(\alpha-1)A\bar{k}^{\alpha-2}}{\theta} < 0$$

Then, the eigenvalues are both real and

$$\lambda_s = \frac{\rho}{2} - \left[\left(\frac{\rho}{2}\right)^2 - \det(J)\right]^{1/2} < 0,\ \lambda_u = \frac{\rho}{2} + \left[\left(\frac{\rho}{2}\right)^2 - \det(J)\right]^{1/2} > 0.$$

Therefore the steady state $(\bar{c}, \bar{k})$ is a saddle point.

The eigenvector matrix associated to J [19] is

$$\begin{pmatrix} 1 & 1 \\ \lambda_u & \lambda_s \end{pmatrix}$$

and therefore, the general solution is

$$\begin{pmatrix} k(t) - \bar{k} \\ c(t) - \bar{c} \end{pmatrix} = h_1 \begin{pmatrix} 1 \\ \lambda_u \end{pmatrix} n e^{\lambda_s t} + h_2 \begin{pmatrix} 1 \\ \lambda_s \end{pmatrix} e^{\lambda_u t},$$

where h_1 and h_2 are arbitrary constants. We determine them by requiring that the solution will converge asymptotically to a steady state and the predetermined variable k satisfies $k(0) = k_0$. As the explosive dynamics is generated by $e^{\lambda_u\,t}$ we set $h_2 = 0$ and by setting $h_1 = k(0) - \bar{k}$ the initial condition is satisfied. Then we obtain the approximate solution (18).

[19]We determine the column P^j by solving the homogeneous system $(J - \lambda_j I)P^j = 0$, where I is the (2×2) identity matrix, for non-zero solutions.

C Proof of equation (26)

Solving the household's Euler equation (21a) for the isoelastic case we get

$$c(t) = c(0)e^{\int_0^t \gamma_c(s)ds}, \text{ for } \gamma_c(t) = \frac{r(t)-\rho}{\gamma}$$

on the other hand integrating the budget constraint we get and substituting consumption yields

$$a(t) = e^{\int_0^t r(s)ds} a_0 + \int_0^t e^{\int_s^t r(z)dz} w(s)ds - c(0) \int_0^t e^{\int_s^t r(z)dz} e^{\int_0^s \gamma_c(z).dz} ds$$

Therefore, after some algebra we have

$$c(t)^{-\theta}\, a(t)\, e^{-\rho t} = c(0)^{-\theta} \left[a_0 + h^t(0) - c(0) \int_0^t e^{-\int_0^s \gamma(z)dz} ds \right]$$

where $\gamma(t) = \dfrac{(\theta-1)r(t)+\rho}{\theta}$ and $h^t(0) = \int_0^t e^{-\int_0^s r(z)\,dz}\, w(s)\, ds$. The transversality equation, $\lim_{t\to\infty} c(t)^{-\theta}\, a(t)\, e^{-\rho t} = 0$, holds if and only if

$$c^*(0) = \frac{a_0 + h^\infty(0)}{\int_0^\infty e^{-\int_t^s \gamma(z)dz} ds}$$

which is the optimal initial level of consumption from the point of view of the household equation.

Therefore, households consumption for any time $t \in (0,\infty)$ becomes

$$c^*(t) = \frac{\big(a_0 + h^\infty(0)\big)\, e^{\int_0^t \gamma_c(s)ds}}{\int_0^\infty e^{-\int_t^s \gamma(z)dz} ds}$$

and the optimal level of net wealth at time t is

$$\begin{aligned}
a^*(t) = e^{\int_0^t r(s)ds} \Big(a_0 + h^t(0) - c^*(0) \int_0^t e^{-\int_0^s (r(z)-\gamma_c(z))dz} ds \Big) = \\
&= e^{\int_0^t r(s)ds} \Big(a_0 + h^\infty(0) - h^\infty(t) e^{-\int_0^t r(s)ds} - c^*(0) \int_0^t e^{-\int_0^s \gamma(z))dz} ds \Big) = \\
&= -h^\infty(t) + e^{\int_0^t r(s)ds} \Big(c^*(0) \int_0^\infty e^{-\int_0^t \gamma(s)ds} dt - c^*(0) \int_0^t e^{-\int_0^s \gamma(z))dz} ds \Big) = \\
&= -h^\infty(t) + c^*(0)\, e^{\int_0^t r(s)ds} \Big(\int_0^\infty e^{-\int_0^t \gamma(s)ds} dt - \int_0^t e^{-\int_0^s \gamma(z))dz} ds \Big) = \\
&= -h^\infty(t) + c^*(t)\, e^{\int_0^t \gamma(s)ds} \Big(\int_0^t e^{-\int_0^s \gamma(s)ds} dt + e^{-\int_0^t \gamma(s)ds} \int_t^\infty e^{-\int_t^s \gamma(z)dz} ds - \int_0^t e^{-\int_0^s \gamma(z))dz} ds \Big) = \\
&= -h^\infty(t) + c^*(t) \int_t^\infty e^{-\int_t^s \gamma(z)dz} ds
\end{aligned}$$

which solving for $c^*(t)$ is equation (26). We have used the fact that $h^\infty(0) = h^t(0) + e^{-\int_0^t r(s)ds} h^\infty(t)$.

D Behavioral functions for the non-additive separable model

Differencing the optimality conditions, equation (29a) and (29b), we have

$$\begin{pmatrix} u_{cc}(c,\ell) & u_{c\ell}(c,\ell) \\ u_{c\ell}(c,\ell) & u_{\ell\ell} - q\, f_{\ell\ell}(k,\ell) \end{pmatrix} \begin{pmatrix} dc \\ d\ell \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f_{\ell}(k,\ell) & q f_{\ell k}(k,\ell) \end{pmatrix} \begin{pmatrix} dq \\ dk \end{pmatrix}.$$

Writing

$$\begin{pmatrix} \frac{dc}{dq} & \frac{dc}{dk} \\ \frac{d\ell}{dq} & \frac{d\ell}{dk} \end{pmatrix} = \begin{pmatrix} C_q(q,k) & C_k(q,k) \\ L_q(q,k) & L_k(q,k) \end{pmatrix}$$

we have

$$\begin{aligned}
\begin{pmatrix} C_q & C_k \\ L_q & L_k \end{pmatrix} &= \begin{pmatrix} u_{cc}(c,\ell) & u_{c\ell}(c,\ell) \\ u_{c\ell}(c,\ell) & u_{\ell\ell} - q\, f_{\ell\ell}(k,\ell) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ f_{\ell}(k,\ell) & q f_{\ell k}(k,\ell) \end{pmatrix} \\
&= \frac{1}{D} \begin{pmatrix} u_{\ell\ell} - q\, f_{\ell\ell}(k,\ell) & -u_{c\ell}(c,\ell) \\ -u_{c\ell}(c,\ell) & u_{cc}(c,\ell) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f_{\ell}(k,\ell) & q f_{\ell k}(k,\ell) \end{pmatrix} \\
&= \frac{1}{D} \begin{pmatrix} u_{\ell\ell} - q\, f_{\ell\ell}(k,\ell) - u_{c\ell}(c,\ell) f_{\ell}(k,\ell) & -u_{c\ell}(c,\ell)\, q\, f_{\ell k}(k,\ell) \\ -u_{c\ell}(c,\ell) + u_{cc}(c,\ell) f_{\ell}(k,\ell) & u_{cc}(c,\ell)\, q\, f_{\ell k}(k,\ell) \end{pmatrix}
\end{aligned}$$

where $D \equiv u_{cc}u_{\ell\ell} - u_{c\ell}^2 - q u_{cc} f_{\ell\ell}$ has an ambiguous sign.