Foundations of Financial Economics
Deterministic GE asset pricing: two-period case

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Syllabus

- 1. General equilibrium in a representative agent Arrow-Debreu economy
- 2. General equilibrium in an heterogeneous agent Arrow-Debreu economy
- 3. General equilibrium in a frictionless finance economy
- 4. General equilibrium in a finance economy with frictions: heterogeneous market participation

Two-period general equilibrium models

- Next we will address the (macro) **determination of the interest rate** in two-period general equilibrium models under perfect information (i.e., certainty)
- We consider two (equivalent) approaches and models
  - a micro-economic approach: Arrow-Debreu simultaneous equilibrium economy
  - a finance (or macro-finance) approach:a finance sequencial equilibrium economy

1. General equilibrium in a representative agent Arrow-Debreu economy

Arrow Debreu model

Assumptions

- Now, we go from a microeconomic to a macroeconomic perspective

H1 Assume there is perfect information: **deterministic general equilibrium**

H2 Assume all agents are homogeneous (in behavior and in endowments): **representative agent economy**

H3 Assume that there is an **exogenous** sequence of resources sustaining consumption: **endowment economy**

H4 Assume that all trade is done at time $t = 0$: an **Arrow-Debreu economy**

- We want to determine (endogenously) the price q: the **Arrow-Debreu** price

Arrow Debreu model

The setup

- Assume that the resource of the economy takes the form of a **flow** of non-durable goods, that can be collected both at time $t = 0$ and $t = 1$, $\{y_0, y_1\}$.
- Again, assume that **trade can only take place at time** $t = 0$: this means that the price for contracts for delivery at time $t = 1$ has to be set at time $t = 0$. We call q the **Arrow-Debreu price**
- From this, wealth at time $t = 0$ is equal to the **present value of the flow of endowments**

$$w = y_0 + q\, y_1$$

Arrow Debreu model

The setup: continuation

- All the participants have **perfect information** (over prices and endowments referring to period $t = 1$) and solve their micro-economic problems;
- At every period, total consumption must be equal to total endowment (market equilibrium);
- **Representative agent economy**: we assume that all consumers solve the same problem (same utility function and same endowments);
- What is the equilibrium forward price q ?

Arrow Debreu model

General equilibrium for a representative agent economy

Definition 1

General equilibrium *in this economy is defined by the sequence of consumption and by the AD price,* $(\{c_0^{eq}, c_1^{eq}\}, q^{eq})$, *given* $\{y_0, y_1\}$, *such that*

- *the consumer solves the problem, given* q,

$$\max_{c_0, c_1}\{U(c_0, c_1) : c_0 + q\, c_1 \leq y_0 + q y_1\}$$

- *markets clear (demand = supply) in every period*

$$c_0 = y_0,$$
$$c_1 = y_1$$

Arrow Debreu model

General equilibrium for a representative agent economy

- General equilibrium conditions: (c_0, c_1, q) is determined from

$$\begin{cases} qU_0(c_0, c_1) = U_1(c_0, c_1) & \text{(micro: intertemporal optimality condition)} \\ c_0 + qc_1 = y_0 + qy_1 & \text{(micro: budget constraint)} \\ c_0 = y_0 & \text{(aggregate: market clearing for } t = 0\text{)} \\ c_1 = y_1 & \text{(aggregate: market clearing for } t = 1\text{)} \end{cases}$$

- There are only three independent conditions (Walras's law)

$$\begin{cases} qU_0(c_0, c_1) = U_1(c_0, c_1) \\ c_0^{eq} = y_0 \\ c_1^{eq} = y_1 \end{cases}$$

- In a representative agent economy there is **no trade** (consumption is equal to the endowment)

Arrow Debreu model

Equilibrium AD price

- Then **the equilibrium AD price** should be equal to the inverse of the $IMRS_{0,1}$

$$\boxed{q^{eq} = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)}}$$

- **We need more structure on preferences to get explicit results**

Arrow Debreu model

Example 1: additive utility

- For an intertemporally additive utility function

$$U(c_0, c_1) = u(c_0) + \beta\, u(c_1)$$

- General equilibrium determination

$$\begin{cases} qu'(c_0) = \beta\, u'(c_1) & \text{(micro: intertemporal optimality condition)} \\ c_0 + qc_1 = y_0 + qy_1 & \text{(micro: budget constraint)} \\ c_0 = y_0 & \text{(aggregate: market clearing for } t = 0\text{)} \\ c_1 = y_1 & \text{(aggregate: market clearing for } t = 1\text{)} \end{cases}$$

- From Walras's law: the independent equations are

$$\begin{cases} qu'(c_0) = \beta\, u'(c_1) \\ c_0 = y_0 \\ c_1 = y_1 \end{cases}$$

Arrow Debreu model

Example 1: additive utility

Proposition 1

Then, for an intertemporally additive utility the ***equilibrium AD price*** *is*

$$\boxed{q^{eq}(\beta, y_0, y_1) = \beta \frac{u^{'}(y_1)}{u^{'}(y_0)}}$$

- concavity of $u(.)$, i.e., $u^{''}(c) < 0$, implies

$$\frac{\partial q^{eq}(y_0, y_1)}{\partial y_0} = -\beta \frac{u^{'}(y_1) u^{''}(y_0)}{(u^{'}(y_0))^2} > 0, \; \frac{\partial q^{eq}(y_0, y_1)}{\partial y_1} = \beta \frac{u^{''}(y_1)}{u^{'}(y_0)} < 0$$

- Then q it increases (decreases) with an excess supply of present (future) **relative** to future (present) supply

Arrow Debreu model

Example 1: additive utility

- Particular case: if $u(c) = \ln(c)$ then

$$\boxed{q^{eq}(y_0, y_1) = \beta \frac{y_0}{y_1} = \frac{\beta}{1+\gamma}}$$

- or, if we set $y_1 = (1+\gamma)\, y_0$ where γ is the **anticipated rate of growth**, and recall $\beta = (1+\rho)^{-1}$ then

$$\boxed{q^{eq}(y_0, y_1) = \frac{1}{(1+\gamma)\,(1+\rho)}}$$

the AD price decreases with the rate of time preference and the anticipated rate of growth (more resources in the future lower prices for buying them)

Arrow Debreu model

Example 2: habit formation

- For the habit formation intertemporal utility

$$U(c_0, c_1) = u(c_0) + \beta\, v(c_0, c_1)$$

- General equilibrium determination

$$\begin{cases} q\big(u'(c_0) + \beta\, v_{c_0}(c_0, c_1)\big) = \beta\, v_{c_1}(c_0, c_1)\big) \\ c_0 + qc_1 = y_0 + qy_1 \\ c_0 = y_0 \\ c_1 = y_1 \end{cases}$$

- From Walras's law: the independent equations are

$$\begin{cases} q\big(u'(c_0) + \beta\, v_{c_0}(c_0, c_1)\big) = \beta\, v_{c_1}(c_0, c_1)\big) \\ c_0 = y_0 \\ c_1 = y_1 \end{cases}$$

Arrow Debreu model

Example 2: habit formation

- For the habit formation utility function

$$q^{eq}(y_0, y_1) = \beta \frac{v_{c_1}(y_0, y_1)}{u'(y_0) + \beta v_{c_0}(y_0, y_1)}$$

- May have the same qualitative properties than with the additive model: recall

$$q^{eq}_{y_0} > 0, \text{and } q^{eq}_{y_1} < 0$$

Arrow Debreu model

Example 2: habit formation

- Example: setting $U(c_0, c_1) = \ln(c_0) + \beta \ln\left[\left(\frac{c_1}{c_0}\right)^{\zeta}\right]$ displaying intertemporal substitution

$$\boxed{q^{eq} = \frac{\beta\zeta}{y_1}\left(\frac{1}{y_0} - \beta\zeta\frac{1}{y_0}\right)^{-1} = \frac{\beta\zeta y_0}{(1-\beta\zeta)y_1} = \frac{\beta\zeta}{(1-\beta\zeta)(1+\gamma)}}$$

- again $q^{eq}_{y_0} > 0$
- however:
  $q^{eq}_{y_1} < 0$ if $\zeta < \frac{1}{\beta} = 1 + \rho$ (low weight of habits)
  $q^{eq}_{y_1} > 0$ if $\zeta > \frac{1}{\beta} = 1 + \rho$ (high weight of habits)

2. General equilibrium in an heterogeneous agent Arrow-Debreu economy

Arrow Debreu model

Assumptions

- The previous model is more general than it looks

H1 idem

H2 Assume heterogeneity in endowments

H3 idem

H4 idem

- What are the consequences for the equilibrium q

Arrow-Debreu economy

Beyond the representative agent case

- Assumptions: there are two groups of agents
  - **with the same preferences**
  - but **endowments are different** (y_t^i is the endowment of agent i at time t)

$$y^1 = \{y_0^1, y_1^1\}, y^2 = \{y_0^2, y_1^2\}$$

  and we assume $y^1 \neq y^2$

- The aggregate **flow of total endowments** are

$$y_0 = y_0^1 + y_0^2$$
$$y_1 = y_1^1 + y_1^2$$

- The general equilibrium is now

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

Definition 2

General equilibrium *in this economy is defined by the allocations* $(\{c_0^{1,eq}, c_1^{1,eq}\}, \{c_0^{2,eq}, c_1^{2,eq})\}$ *and the price* q^{eq} *such that*

- *consumer* $i \in \{1, 2\}$ *solves the problem*

$$\max_{c_0^i, c_1^i} \{ U(c_0^i, c_1^i) : c_0^i + q\, c_1^i \leq y_0^i + q\, y_1^i \}, \; for\ i = 1, 2$$

- *aggregation conditions*

$$c_t = c_t^1 + c_t^2, for\ t = 0, 1$$
$$y_t = y_t^1 + y_t^2, for\ t = 0, 1$$

- *market clearing for* $t = 0, 1,$

$$c_t = y_t, \; for\ t = 0, 1$$

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- General equilibrium conditions (considering that the Walras' law holds)

$$\begin{cases} qU_0(c_0^1, c_1^1) = U_1(c_0^1, c_1^1) & \text{(optimality condition for agent 1)} \\ qU_0(c_0^2, c_1^2) = U_1(c_0^2, c_1^2) & \text{(optimality condition for agent 2)} \\ c_t = c_t^1 + c_t^2 & \text{(aggregation of consumption for } t = 0, 1) \\ y_t = y_t^1 + y_t^2 & \text{(aggregation of endowment for } t = 0, 1) \\ c_t = y_t & \text{(market clearing for period } t = 0, 1) \end{cases}$$

- In this case we **can have trade**, because $c_t^1 - y_t^1 = y_t^2 - c_t^2$ can be different from zero, but the budget constraint should hold for every agent. (check this !)

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- Because we assumed homogeneity in preferences, $U(.,.)$ is the same for both consumers.
- Therefore, it also holds for the aggregate consumption

$$qU_0(c_0, c_1) = U_1(c_0, c_1)$$

that is

$$qU_0\big(c_0^1 + c_0^2, c_1^1 + c_2^1\big) = U_1(c_0, c_1)$$

Arrow Debreu model

General equilibrium for a heterogeneous agent economy

- Using the market clearing conditions we have again

$$\boxed{q^{eq} = \frac{U_1(y_0, y_1)}{U_0(y_0, y_1)} = \frac{U_1(y_0^1 + y_0^2, y_1^1 + y_1^2)}{U_0(y_0^1 + y_0^2, y_1^1 + y_1^2)}}$$

- Conclusion: if agents have **homogeneous preferences** but **heterogeneous endowments** the distribution of income between agents **has no influence** the AD price. It is only determined by the aggregate endowment
- if agents have **heterogenous preferences** , this result **will not hold** in general (distribution matters)

3. General equilibrium in a frictionless finance economy

Finance economy model

Assumptions

- Now we change the market structure

H1 idem

H2 Assume a representative agent economy

H3 idem

H4 Assume a **sequence of spot asset markets** (for the good opening at $t = 0$ and $t = 1$, for an asset opening at $t = 0$)

- What is the equilibrium asset price

Finance economy model

The economy

- Two periods and full information
- We still assume that the agent receives a flow of endowments $y = \{y_0, y_1\}$ The agent can consume the totality of the income, or not, at the end of period 1
- Agents can reallocate the endowments through a **spot financial** contract
- There is an asset (that can be seen as a durable good) that agents can lend and borrow at period $t = 0$ paying or receiving an interest income at period $t = 1$. The asset is in non-negative net supply at the beginning to period $t = 0$ and there is a market for the asset at time $t = 0$.
- Every agent has now a **sequence of budget constraints** (because trade in the good market can take place at period 1)

Finance economy without frictions

General equilibrium for a representative agent finance economy

Definition 3

General equilibrium *in this economy is defined by the sequence of consumption* $\{c_0^{eq}, c_1^{eq}\}$ *and the return* R^{eq}*, given* $\{y_0, y_1\}$*, such that*

- *the consumer solves the problem, given* R*,*

$$\max_{c_0, c_1, a_1} U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

$$\text{subject to}$$

$$c_0 + a_1 = y_0 + a_0$$

$$c_1 = y_1 + R\, a_1$$

- *and the market for the good clears*

$$\begin{cases} c_0 = y_0 \\ c_1 = y_1 \end{cases}$$

Finance economy without frictions

General equilibrium for a representative agent finance economy

- The equilibrium equations are (from Walras's law)

$$
\begin{aligned}
m\, u^{'}(c_0) &= \beta\, u^{'}(c_1) \\
c_0 &= a_0 + y_0 \\
c_1 &= y_1
\end{aligned}
$$

- The **equilibrium discount factor** is

$$
m^{eq} = m(a_0, y_0, y_1) = \beta \frac{u^{'}(y_1)}{u^{'}(a_0 + y_0)}
$$

- Because $R = \dfrac{1}{m}$ and $\beta = \dfrac{1}{1+\rho}$ where ρ is the psychological discount factor

Finance economy without frictions

General equilibrium for a representative agent finance economy

- The **equilibrium discount factor** is

$$\boxed{m^{eq} = m^{eq}(\beta, y_0, y_1, a_0) = \frac{\beta\, u^{'}(y_1)}{u^{'}(y_0 + a_0)}}$$

- concavity of $u(.)$, i.e., $u^{''}(c) < 0$, implies

$$\frac{\partial m^{eq}}{\partial y_0} = \frac{\partial m^{eq}}{\partial a_0} = -\beta \frac{u^{'}(y_1) u^{''}(y_0 + a_0)}{(u^{'}(y_0 + a_0))^2} > 0$$

$$\frac{\partial m^{eq}}{\partial y_1} = \beta \frac{u^{''}(y_0)}{u^{'}(y_1)} < 0$$

- There are two main effects:
  - a direct effect: high y_0 or a_0 **increase** the DF
  - an anticipation effect: high y_1 **decreases** the DF

Finance economy without frictions

Asset return in a frictionless economy

- The **equilibrium asset return** (recall)

$$\boxed{R^{eq} = 1 + r^{eq} = (1+\rho)\frac{u^{'}(a_0 + y_0)}{u^{'}(y_1)}}$$

- But $R^{eq} = R(a_0, y_0, y_1)$, with partial derivatives

$$\frac{\partial R}{\partial a_0} = \frac{\partial R}{\partial y_0} = (1+\rho)\frac{u^{''}(a_0 + y_0)}{u^{'}(y_1)} < 0$$

$$\frac{\partial R}{\partial y_1} = -(1+\rho)\frac{u^{''}(y_1)u^{'}(a_0 + y_0)}{(u^{'}(y_1))^2} > 0$$

- There are two main effects:
  - a direct effect: high y_0 or a_0 **reduce** the interest rate
  - an anticipation effect: high y_1 **increases** the interest rate

Finance economy without frictions

Savings

- Savings is equal to zero at equilibrium

$$s^{eq}$$

4. General equilibrium in a finance economy with frictions: heterogeneous market participation

A simple finance economy with frictions

Assumptions

- Now we introduce heterogeneity

H1 idem

H2 Assume agents face financing constraints

H3 idem

H4 idem

- What is the equilibrium asset price

Finance economy with heterogeneity

Heterogenous participation

- Assume there are two agents in the economy: agent b is a borrower and agent l is a lender, the only one that has positive assets at time 0 ($a_0^l > 0$, $a_0^b = 0$)
- To simplify, assume agent b is the only one that receives the flow of endowments $\{y_0, y_1\}$ and agent b can only earn interest income
- Assume there are no constraints in the credit market
- Assume that agents have homogeneous preferences

Finance economy with heterogeneity

- The **lender**'s problem is

$$\max_{c_0^l, c_1^l} \{u(c_0^l) + \beta u(c_1^l) : \; c_0^l + l^l = a_0, \; c_1^l = (1+r) l^l\}$$

- Because l^l is free it can be simplified to

$$\max_{l^l} \{u(a_0 - l^l) + \beta u((1+r) l^l)\}$$

- The optimality condition is

$$u^{'}(a_0 - l^l) = \beta(1+r) u^{'}((1+r) l^l)$$

or equivalently

$$u^{'}(c_0^l) = \beta(1+r) u^{'}(c_1^l)$$

Finance economy with heterogeneity

Agents' problems

- The **borrower** problem is

$$\max_{c_0^b, c_1^b} \{u(c_0^b) + \beta u(c_1^b) : \; c_0^b = y_0 + l^b, \; c_1^l + (1+r)l^b = y_1\}$$

- Because l^b is free it can be simplified to

$$\max_{l^b} \{u(y_0 + l^b) + \beta u(y_1 - (1+r)l^b)\}$$

- The optimality condition is

$$u^{'}(y_0 + l^b) = \beta(1+r)u^{'}(y_1 - (1+r)l^b)$$

or equivalently

$$u^{'}(c_0^b) = \beta(1+r)u^{'}(c_1^b)$$

Finance economy with heterogeneity

Equilibrium equations

- The equilibrium equations are

$$\begin{aligned} u^{'}(c_0^l) &= \beta(1+r)u^{'}(c_1^l) \\ u^{'}(c_0^b) &= \beta(1+r)u^{'}(c_1^b) \\ c_0^l + c_0^b &= y_0 + a_0 \\ c_1^l + c_1^b &= y_1 \end{aligned}$$

- Because preferences are homogeneous we can use the same argument as before, to get

$$u^{'}(y_0 + a_0) = \beta(1+r)u^{'}(y_1)$$

Finance economy with heterogeneity

Equilibrium interest rate

- The **equilibrium return** is again

$$\boxed{R^{eq} = 1 + r^{eq} = (1+\rho)\frac{u^{'}(y_0 + a_0)}{u^{'}(y_1)}}$$

- Is formally similar to the representative agent economy case;
- Again, we have
  - negative liquidity effect $R^{eq}_{a_0} < 0$;
  - a negative income effect, $R^{eq}_{y_0} < 0$
  - a positive anticipation effect, $R^{eq}_{y_1} > 0$
- Conclusion: without other sources of heterogeneity, limited participation has no effect on the market interest rate.

Taking the model to data

- In the long run we have (see introduction):
  - an upward trend of the growth rates
  - a downward trend of the interest rates
- then: the simple model has the **wrong** correlation (why ?)
- In the shorter run we have a different scenario
  https://www.jornaldenegocios.pt/mercados/detalhe/
  juros-nos-eua-e-pib-nao-divergiam-tanto-desde-1966-e-dao-margem-para

Taking the model to data

- data from http://www.nber.org/papers/w24112.pdf
  $R_{safe} = 1.0188$ (average safe return) $R_{wealth} = 1.0678$ (average wealth return) $\gamma = 0.0287$ (average rate of growth)
- calibrated parameters: $\rho = 0.02$
- Utility functions
  - isoelastic utility function

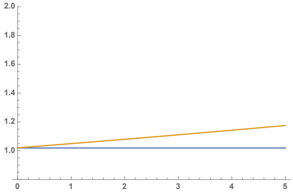
$$U(c_0, c_1) = \frac{c_0^{1-\zeta} - 1}{1-\zeta} + \beta \frac{c_1^{1-\zeta} - 1}{1-\zeta}$$

  - habit formation:

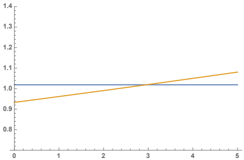
$$U(c_0, c_1) = \ln c_0 + \beta \left[\frac{\left(\frac{c_1}{c_0}\right)^{\phi(1-\zeta)} - 1}{1-\zeta} \right]$$

Taking the model to data

- Additive utility: interest rate puzzle (the model over predicts the observed risk-free interest rate, for any value of the *EIS*)



- Habit formation: it is possible to find values for the parameter ϕ, in the case $\phi \approx 0.5$ such that the model matches the observed R for "acceptable" values fo ζ



Questions

- The previous results hold for cases in which there is
  - full information (deterministic general equilibrium)
  - agents have homogeneous preferences (with or without homogeneous resources)
  - frictionless economy (for the case of a finance economy)
- Do those results hold under:
  - imperfect information (uncertainty) ?
  - heterogeneity in agents' preferences ?
  - frictions in a finance economy (ex: credit constraints) ?