Economic Growth Theory[1]

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[1]These notes have been prepared for a course in Economic Growth of the Master in Economics. They should be seen as a draft. They have not been fully edited and are not exempt from errors (not only typographical ones). Comments and corrections will be welcome.

Contents

Chapter 1

Introduction

1.1 The object of economic growth theory

The main object of economic growth theory is to study the variation of the per capita aggregate product along time and space.

The main questions it tries to answer are:

- how to describe growth in the very long run or in a shorter run ?
- why rates of growth differ along historical time ?
- why the rates of growth differ among countries ?
- why countries hold inequalities in the GDP per capita while having similar rates of growth ?
- what are the main factors explaining economic growth ?

The main sources of economic growth, as presented by the literature are:

- Physical and biological environments: geography, size, resources, biology;
- Technology: learning by doing, R&D (as an independent activity);
- Population: demography, human capital, social capital;
- Aggregation: externalities, public goods ;
- Economic institutions: inclusive/exclusive, financial institutions, trade openness, patent protection;
- Political Institutions: in a broad sense (inclusive/exclusive, rule of law, enforcement, accountability) or a narrow sense (government intervention, governance)
- Luck (good or bad)

In this course we will study some of those sources of growth that have been introduced into formalised models.

1.2 Some stylized facts

Secular long run perspective:

Historical phases in economic growth:

1. Malthusian trap : (almost) constant rates of growth (-6000 to 1700 CE)
2. Industrial Revolution: transition with modest increases in the rate of growth
3. Modern economic growth: rapid economic growth and Great Divergence: post 1820 and particularly post WW2
4. Nature strikes back: natural limits to growth ?

Ancient growth experience: the Malthusian trap

- the rate of growth was small: between 0% and 0.5%, on average (see figure 1.1)
- increases in production were translated into increases in population and not in the individual welfare Diamond (1997) (see figure 1.2)
- big impact of demographic changes (ex Black-Death (1347-1350)) together with the development of institutions (see Acemoglu and Robinson (2012) on the different responses of Eastern and Western Europe to the Black Death) (see figure 1.3);
- there was a negative correlation between population growth and real wages (see figure 1.4
- non-reproducible resources had an impact on growth because of decreasing returns;
- there were some gains in productivity, although not related to a purposeful activity as R&D;
- there was not a big divergence in terms of GDP per capita (see Table 1.1).

Table 1.1: Ratio richest to poorest region: before the great divergence

1000	1500	1820
1.1:1	2:1	3:1

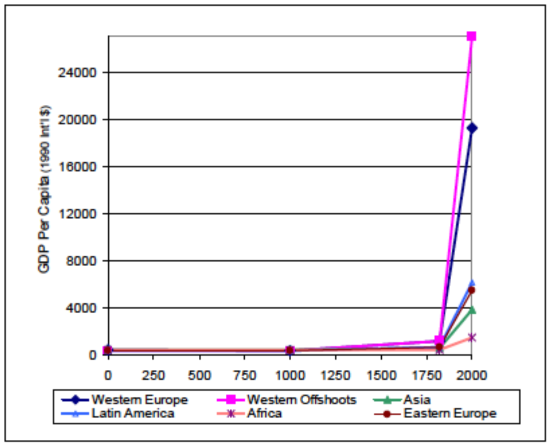

Figure 1: The Evolution of Regional Income Per Capita, 1-2000 CE
(Source: Maddison, 2003)

Figure 1.1: Maddison on the evolution of income per capita

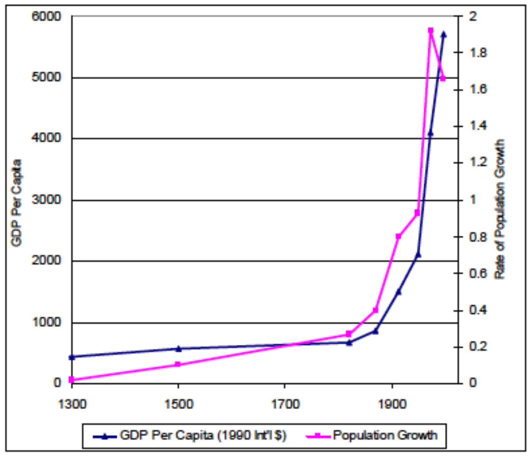

Figure 4: World Population Growth and Income Per Capita
(Source: Maddison, 2001)

Figure 1.2: Maddison on the evolution of population

Modern economic growth:

- modern economic growth may be defined by the existence of permanent positive rates of growth;
- it may have started in the UK around 1800 (see Clark (2007));
- a sequence of other countries then started a modern economic growth process
- this implied that the disparities of the gdp per capita, regarding the leader country, have passed from a ratio of 1 to 1/3 (around 1700) to 1 to 1/16 in 2000 (data from Maddison (2007)). Historians call this process The Great divergence (see table 1.2
- the onset of the modern economic growth process has been contemporaneous with a demographic revolution; except from this it seems to be independent from the rate of growth of the population;
- Therefore, growth is positively correlated with the increase in productivity: wages have had a positive rate of growth and the rate of return of capital looks like following a

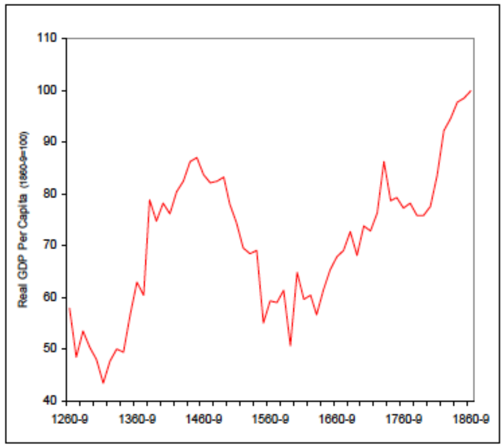

Figure 2: Fluctuations in Real GDP Per Capita in England, 1260-1870 CE
(Source: Clark, 2005)

Figure 1.3: Clark on the UK's real wages

stationary tendency, maybe slightly decreasing if we take secular data (Clark (2007) (see figure 1.5);

- growth is related to the reproducible factors of production: therefore, non-renewable natural resources are not a determinant long run factor of growth;
- technologic progress, and the accumulation of human capital, because there may not exist an upper bound for them (as it is the case for non-reproducible factors), are potentially the best candidates as ultimate sources for long run growth.
- institutions (protection of property rights, contract enforcement, reduction of transactions costs, reduction of uncertainty, etc) provide necessary conditions for growth.

Table 1.2: Ratio richest to poorest region: after the great divergence

1820	1870	1913	1950	2001
3:1	5:1	9:1	15:1	18:1

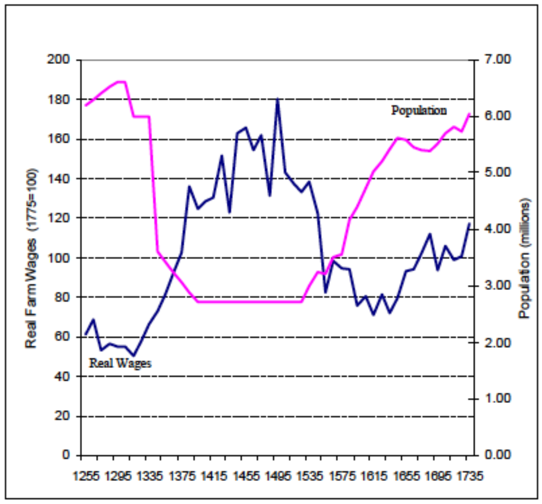

Figure 5: Population and Real Wages in England, 1250-1750 CE
(Source: Clark, 2005)

Figure 1.4: Clark on the UK's real wages and population

Data:

http://www.worldeconomics.com,

http://www.ggdc.net/maddison/maddison-project/home.htm https://pwt.sas.upenn.edu/

1.3 References

Economic history, long run view: Maddison (2007) Galor (2011)

The impact of humans in the planet: the Anthropocene: Lewis and Maslin (2018) https://ourworldindata.org/co2-and-other-greenhouse-gas-emissions

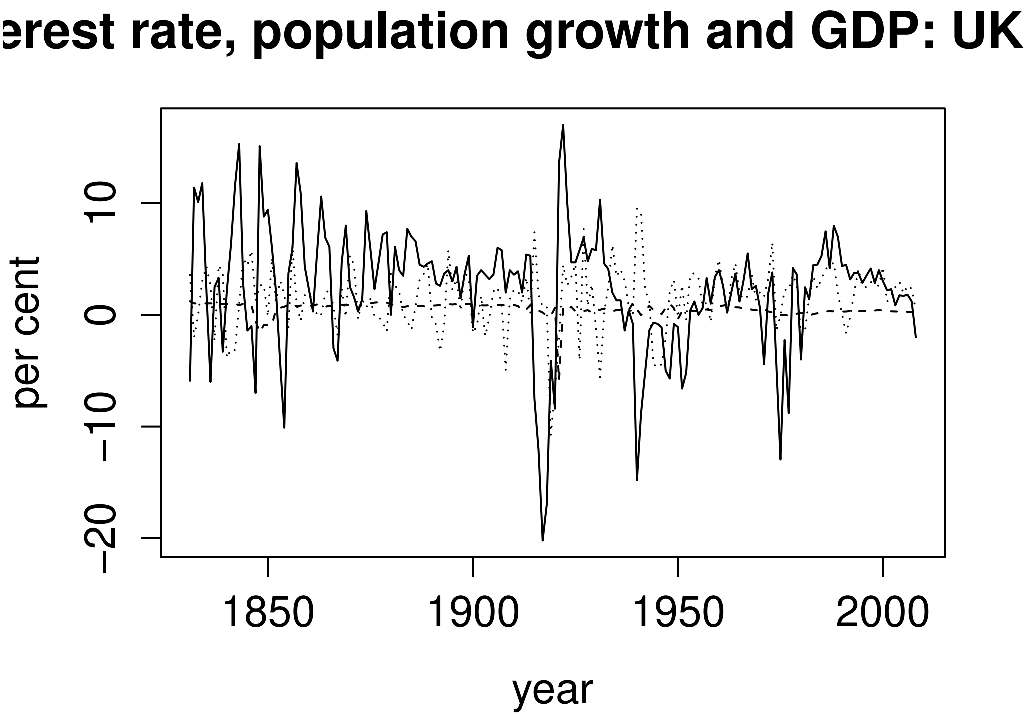

Figure 1.5: UK: post-Malthusian population and GDP growth rates

Chapter 2

The Malthusian model

Malthus (1766-1834) was an English economist and clergyman. In Malthus (1798) he argued that without the practice of "moral restraint" the population tends to increase at a greater rate than its means of subsistence, resulting in the population checks of wars, famines, and epidemics.

We call Malthusian model or trap: increases in productivity do not translate into economic growth but on the increase of population. This model seems to describe well the evidence in figure 1.4.

2.1 A simple version

We consider population, L, grows by the difference between the fertility rate b and the mortality rate m

$$\dot{L} = (b(t) - m)L(t) \tag{2.1}$$

where we assume that the death rate is exogenous and constant and the fertility rate is endogenous and time-varying. Each member of the population has on average b children and a proportion of the population m dies every period.

We assume there is a representative consumer whose problem is to split its income between his consumption and the costs of rearing children. He values both consuming and parenting.

He solves a static problem at every point in time (he does not have an intertemporal problem because he does not save):

$$\max_{c(t),b(t)} u(c(t), b(t))$$

subject to the budget constraint

$$c(t) + \rho b(t) \leq y(t)$$

where c and y are the household consumption and income and ρ is the cost of rearing children. We assume the utility function

$$u(c, b) = c^{1-\gamma} b^{\gamma}$$

where $\gamma > 0$. We can interpret $1/\gamma$ as the Malthusian "moral restraint".

Solving the problem we get the optimal household consumption and birth rate

$$c(t) = (1-\gamma)y(t) \tag{2.2}$$

$$b(t) = \frac{\gamma}{\rho} y(t). \tag{2.3}$$

Therefore equation (2.1) becomes

$$\dot{L} = \left(\frac{\gamma}{\rho} y(t) - m\right) L(t) \tag{2.4}$$

We assume there is a production sector in which the technology is given by the production function, with constant returns to scale but with diminishing marginal returns

$$Y(t) = (AX)^{\alpha} L(t)^{1-\alpha}, \ 0 < \alpha < 1 \tag{2.5}$$

where X is the land input, L is the labour input assumed to be equal to the population and A is the land productivity. Per capita GDP is given by

$$y(t) = \frac{Y(t)}{L(t)} = \left(\frac{AX}{L(t)}\right)^{\alpha}, \tag{2.6}$$

We can solve the Malthusian model for L or for y. We start with the first case, and present (in a compact way) the second case in section 2.1.2

2.1.1 Solving for L

If we consider equation (2.4) and substitute $y(.)$ from equation (2.6) we get the Malthusian dynamic population equation

$$\dot{L} = \left(\frac{\gamma}{\rho}\left(\frac{AX}{L(t)}\right)^{\alpha} - m\right)L(t) \tag{2.7}$$

where

$$n(t) = n(L(t)) \equiv \frac{\gamma}{\rho}\left(\frac{AX}{L(t)}\right)^{\alpha} - m.$$

is the endogenous rate of growth of the population

To characterise the population dynamics, we follow three steps:

1. we determine the existence of steady states (i.e., trajectories of type $L(t) =$ constant for all $t \geq 0$)

2. we study the (global and local) dynamics of population, i.e. the time behaviour of the solution equation (2.7) for any initial value of $L(0)$ at time $t = 0$;

3. perform dynamic comparative analyses, i.e., study how the dynamics of population changes when there are changes in parameters or exogenous variables.

Then we derive the growth implications, because per-capita growth is endogenous as regards population dynamics, see equation (2.6)

Steady state

Steady states are values of the population L such that, when they are reached the population remains constant.

We take equation (2.7) and write it as

$$\dot{L} = n(L; A, X, \gamma, \rho, m, \alpha)L \tag{2.8}$$

or, shortly, $\dot{L} = n(L, .)L$. Steady states are defined by

$$L^{ss} \equiv \{L \geq 0 : \; n(L, .)L = 0\}.$$

Applying this definition to equation (2.7) we see that it has two steady states $L^{ss} = 0$ and $L^{ss} = \bar{L}$ where

$$\bar{L} = AX\left(\frac{\gamma}{m\rho}\right)^{1/\alpha} \tag{2.9}$$

Therefore:

- there are two steady state levels for the population: zero or a positive stationary level $\bar{L}$;
- $\bar{L}$ is a positive function of the land productivity, A, and of the dimension of the land, X, and is a negative function of the lack of "moral restraint" , $1/\gamma$, of the mortality rate, m and of the cost of rearing children ρ. We can derive this results analytically by computing the long run multipliers [1]

$$\frac{\partial \bar{L}}{\partial X} = \frac{\bar{L}}{X} > 0, \; \frac{\partial \bar{L}}{\partial A} = \frac{\bar{L}}{A} > 0, \; \frac{\partial \bar{L}}{\partial m} = -\frac{\bar{L}}{\alpha m} < 0$$

[1] The long run multipliers are determined by computing the derivatives of function $\bar{L} = AX(\gamma/(m\rho))^{1/alpha}$ and evaluate it at the value of $\bar{L}$.

- as the steady state population density is

$$d^* = \frac{\bar{L}}{X} = A\left(\frac{\gamma}{m\rho}\right)^{1/\alpha}$$

  we see that except for X all the other effects change total population because they change population density.

Dynamics

The characterisation of he dynamics of population as given by this theory can be done by solving equation (2.7), which embeds that theory. There are basically three methods for solving that equation: explicit methods (i,e, getting an explicit solution), approximated methods (i.e, studying the solution in the neighbourhood of the steady states), or numerical methods ("solving" it by using numerical approximations).

Only in rare situations we can solve a differential equation explicitly. Fortunately this is the case of equation (2.7). It is indeed a Bernoulli equation.

Next we solve it explicitly and using a qualitative approximation.

Global dynamics The following explicit solution of equation (2.7) is determined in the appendix:

$$L(t) = \bar{L}\left[1 + \left(\left(\frac{L(0)}{\bar{L}}\right)^{\alpha} - 1\right)e^{-\alpha m t}\right]^{1/\alpha},\ t \in [0, \infty) \tag{2.10}$$

It is a global solution in the sense that it gives the exact solution for any initial given value of population $L(0) = L_0 \neq 0$.

We can see that:

- if $L_0 = \bar{L}$ then $L(t) = \bar{L}$ for all $t \geq 0$: if we start from a steady state the solution is constant through time

- if $L_0 \neq \bar{L}$ then $L(0) = L_0$ at time $t = 0$ and $L(\infty) = \bar{L}$: the solution tends asymptotically to the steady state $\bar{L}$

- for given values of the parameters, the population grows along time only if $L_0 < \bar{L}$ and it diminishes along time if $L_0 > \bar{L}$. We only have transient dynamics.

Local dynamics In most cases we can only study local dynamics, i.e., the behaviour close to a steady state. If we define

$$\lambda = \frac{\partial \dot{L}}{\partial L}(L^{ss})$$

we approximate the non-linear equation (2.7) by the linear equation

$$\dot{L} = \lambda(L - L^{ss}). \tag{2.11}$$

We evaluate the local dynamics in the neighbourhood of every steady state L^{ss} by determining

$$\frac{\partial \dot{L}}{\partial L}(L^{ss}) = -m + \frac{(1-\alpha)\gamma}{\rho}\,(AX)^{\alpha}\,(L^{ss})^{-\alpha}$$

Then:

- $L^{ss} = 0$ is unstable because
$$\frac{\partial \dot{L}}{\partial L}(0) = +\infty$$

- $L^{ss} = \bar{L}$ is stable because
$$\frac{\partial \dot{L}}{\partial L}(\bar{L}) = -\alpha m < 0$$

Linearised solution The solution of equation (2.11) approximated at $\bar{L}$ is

$$L(t) \approx \bar{L} + (L(0) - \bar{L})e^{-\alpha m t} \tag{2.12}$$

for $L(0)$ in a small neighbourhood of $\bar{L}$

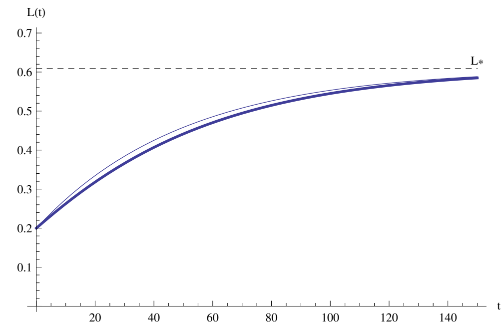

Figure 2.1: Explicit solution in boldface and linearised solution for equation (2.7)

Figure 2.1 presents the explicit solution and the linearised solution for the following parameter values: $\alpha = 2/3$, $\gamma = 0.01$, $m = 0.03$, $\rho = 10$, $A = 1$, $X = 100$, which implies $\bar{L} = 0.609$, for the initial population $L(0) = 0.2$:

As we see the linear approximation is accurate. Also, as $L(0) < \bar{L}$ population grows through time and tends asymptotically to $\bar{L}$, that is $\lim_{t\to\infty} L(t) = \bar{L}$.

Comparative dynamics

In order to study comparative dynamics we consider again equation (2.8), and for studying, for example productivity shocks we write it as

$$\dot{L} = G(L, A)$$

Now, consider a permanent productivity increase increases the steady state population level. If the economy is in a steady state $\bar{L}(A_0)$, a productivity increase from $A = A_0$ to $A = A_1 > A_0$. Increases population from $\bar{L}(A_0)$ to $\bar{L}(A_1) > \bar{L}(A_0)$:

$$\Delta\bar{L} = \bar{L}(A_1) - \bar{L}(A_0) \approx \frac{\partial \bar{L}}{\partial A}(A_0)\Delta A$$

where $\Delta A \equiv A_1 - A_0$.

We could take the global solution (2.10). However, the common method is to use a linear approximation:

$$\dot{L} = \frac{\partial G}{\partial L}(\bar{L}_0)(L - \bar{L}_0) + \frac{\partial G}{\partial A}(\bar{L}_0)\Delta A$$

where

$$\frac{\partial G}{\partial L}(\bar{L}_0) = -\alpha m,$$

and

$$\frac{\partial G}{\partial A}(\bar{L}_0) = \frac{\gamma}{\rho}\bar{L}_0\frac{\partial y}{\partial A} = \alpha\frac{\gamma}{\rho}\bar{y}(A_0)\frac{\bar{L}_0}{A_0} = \alpha m\frac{\bar{L}_0}{A_0}$$

Note that

$$\bar{L}_1 - \bar{L}_0 = -\frac{\partial G/\partial A}{\partial G/\partial L}(\bar{L}_0)\Delta A = \frac{\partial \bar{L}}{\partial A}\Delta A = \frac{\bar{L}_0}{A_0}\Delta A$$

Then (prove this)

$$L(t) - \bar{L}_0 = \left(\bar{L}_1 - \bar{L}_0\right)\left(1 - e^{-\alpha m t}\right),\ t \in [0, \infty).$$

Observe that $L(0) = \bar{L}_0$ and $L(\infty) = \bar{L}_1$.

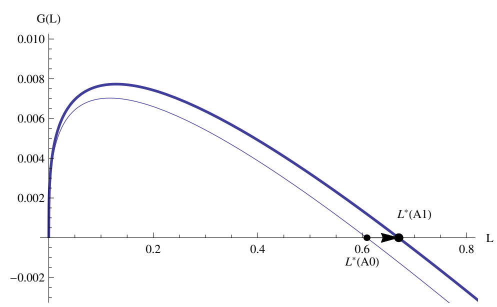

Figure 2.2: Effect of a permanent productivity increase

Geometrically, see figure 2.2, the economy is initially at point $\bar{L}(A_0)$. If there is a permanent productivity shock the steady state becomes $\bar{L}(A_1)$. Then, at time $t = 0$ the value $\bar{L}(A_0)$ is not a steady state equilibrium. This generates a transitional increase in population along time towards $\bar{L}(A_1)$.

2.1.2 Solving for y

As

$$y(t) = \left(\frac{AX}{L(t)}\right)^{\alpha}$$

we can derive the growth implications from the solutions for L which were derived in the previous section.

One alternative way to solve the Malthusian model is to derive a differential equation on y and deal directly with it

First observe that the rate of growth of GDP per capita is monotonously related to the rate of growth of population

$$g_y(t) \equiv \frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = (1-\alpha)\frac{\dot{L}}{L} - \frac{\dot{L}}{L} = -\alpha\frac{\dot{L}}{L}$$

Then

$$\dot{y}(t) = -\alpha\left(\frac{\gamma}{\rho}y(t) - m\right)y(t) \tag{2.13}$$

This is a differential equation that has also two steady states, $y^{ss} = \{0, \bar{y}\}$ where

$$\bar{y} = \frac{m\rho}{\gamma}.$$

Then, we can write equation (2.13) as (2.14)

$$\dot{y}(t) = \frac{\alpha\gamma}{\rho}\left(\bar{y} - y(t)\right)y(t) \tag{2.14}$$

Assume that we know $y(0) = y_0$. Then the explicit solution for this equation is

$$y(t) = \left[\frac{1}{\bar{y}} + \left(\frac{1}{y_0} - \frac{1}{\bar{y}}\right)e^{-\alpha m t}\right]^{-1} \tag{2.15}$$

then the rate of growth of the GDP is

$$g_y(t) = \frac{\alpha\gamma}{\rho}\left(\bar{y} - y(t)\right) = \frac{\alpha\gamma}{\rho}\left(\bar{y} - \frac{1}{\frac{1}{\bar{y}} + \left(\frac{1}{y_0} - \frac{1}{\bar{y}}\right)e^{-\alpha m t}}\right)$$

Exercise: prove this

The solution for the linearised equation,

$$\dot{y}(t) = \alpha m \left(\bar{y} - y(t)\right)$$

is

$$y(t) = \bar{y} + (y_0 - \bar{y})e^{-\alpha m t}$$

In this case the approximated growth rate is

$$g_y(t) = \alpha m \frac{\bar{y} - y_0}{y_0 + (1 - e^{\alpha m t})\bar{y}}$$

Exercise: prove this

In both types of solutions the dynamics of the gdp per capita is depicted in Figure 2.3 (with the same parameters as before)

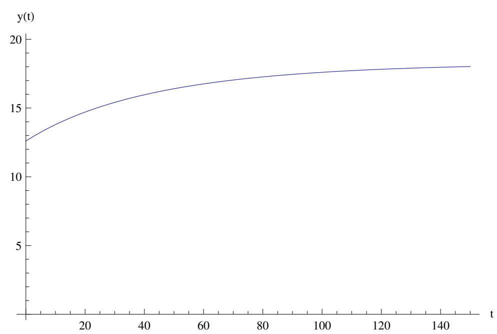

Figure 2.3: Per capita GDP dynamics

Therefore:

- there is no long run growth because

$$\lim_{t\to\infty} g_y(t) = 0$$

- the model only explains the level of the long run per capita GDP: it depends positively on the rate of mortality, the cost of rearing children and on the "moral restraint" $(1/\gamma)$.

Implications for wages

If we consider the problem of a farm

$$\max_{X,L(t)}\{Y(t) - w(t)L(t) - r(t)X\}$$

where w and r are the real wage rate and rent rate, the first order condition yields

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)y(t)$$

so the wage rate is monotonously related to the GDP per capita, where $1-\alpha$ is the share of labour over GDP (which is usually calibrated as 2/3 in those economies) (see Figure 2.4)

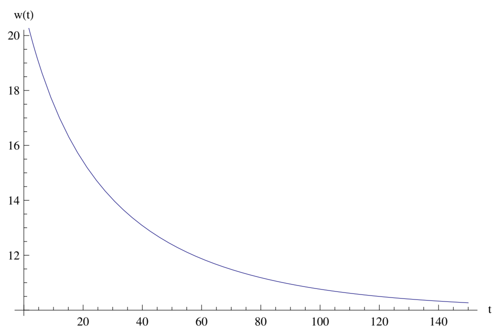

Figure 2.4: Wage rate dynamics

2.1.3 Summing up

Therefore, the model seems to match the relationship between population and wage rate in England (and in Western Europe as whole) before the onset of industrial revolution. But it does not explain why the rate of growth may be positive, even though by very small numbers (around 0.5%).

Does increases in productivity provide an explanation ?

2.2 Trend increase in land productivity

Now assume there is a trend growth in the productivity of the use of land

$$\dot{A} = g_A A,\ g_A > 0$$

Now the rate of growth of per capita GDP is

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \alpha\frac{\dot{A}}{A} + (1-\alpha)\frac{\dot{L}}{L} - \frac{\dot{L}}{L} = \alpha\left(g_A + m - \frac{\gamma}{\rho}y\right).$$

The dynamics for y is given by an analogous equation

$$\dot{y} = \frac{\alpha\gamma}{\rho}\left(\bar{y} - y(t)\right)y(t)$$

where

$$\bar{y} = \frac{(g_A + m)\rho}{\gamma}$$

and the solutions are formally similar.

Again:

- the economy tends to the long run level, if $y(0) > 0$, $\bar{y}$ where which are a positive function of the growth in productivity;
- there is no long run growth
$$\lim_{t\to\infty} g_y(t) = 0$$
- why ? increases in productivity increase, asymptotically, aggregate GDP, Y and L by the same rate, and, therefore the per-capita GDP does not display permanent growth.

Therefore, in this economy a trend increase in the productivity of land does not generate long run growth but increases the level of the GDP.

2.3 Trend increase in labor productivity

Let us consider the production function

$$Y = X^{\alpha} (hL)^{1-\alpha}$$

Now assume there is a trend growth in the productivity of labour

$$\dot{h} = g_h h, \; g_h > 0$$

Using the same method as in the previous section we arrive at

$$\dot{y} = \frac{\alpha\gamma}{\rho} (\bar{y} - y) \, y$$

where

$$\bar{y} = \frac{\rho \left((1-\alpha) g_h + \alpha m \right)}{\alpha\gamma},$$

which is essentially similar to the previous case: (i) there is no long run growth; (ii) tendential increases in productivity only increase the long run levels of population and of the GDP per capita (which could be handy for nations involved in military conflicts).

2.4 Learning-by-doing

In some specific cases long-term growth can be generated in a Malthusian economy. Consider the following case in which there is learning-by-doing, meaning that there can be an endogenous productivity enhancement by experience of past production.

Let us assume a production function of type

$$Y(t) = A(t) X^{\alpha} L(t)^{1-\alpha}$$

where A is the total factor productivity (TFP).

We assume that there is learning-by-doing which takes the particular form [2] formalised by

$$A(t) = \beta \int_{-\infty}^{t} e^{-\mu(t-s)} A(s) y(s) ds \tag{2.16}$$

This means that the experience, measured by the past production influences TFP in a linear way, with a (small) coefficient $\beta > 0$, but there is some forgetting, modelled by the discount factor $e^{-\mu(t-s)}$, where $\mu > 0$ is the rate in which some knowledge is lost. We assume that past experience influences current productivity through the interaction of the TFP with GDP per capital.

Time-differentiating equation (2.16), we get [3]

$$\dot{A} = (\beta y(t) - \mu) A(t)$$

Then

$$g_y(t) = \frac{\dot{y}(t)}{y(t)} = \frac{\dot{A}(t)}{A(t)} - \alpha \frac{\dot{L}(t)}{L(t)} = \left(\beta - \alpha \frac{\gamma}{\rho}\right) y(t) + \alpha m - \mu$$

Existence of long-run growth In the particular case in which $\beta = \alpha\gamma/\rho$ and $\alpha > \mu/m$ then

$$\dot{y} = (\alpha m - \mu) y$$

and

$$y(t) = y_0 e^{\alpha m - \mu) t}.$$

[2]There is some evidence that in some activities, which lasted for sufficiently long periods without having been hampered by wars or other political or natural disruptions) this process may have been working: ex, international shipping.

[3]We use Leibnitz's formula for a derivative of an integral. Let $F(t) = \int_{a(t)}^{b(t)} f(t, x(s)) ds$. Then

$$\frac{dF}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(t, x(s))}{\partial t} ds + b^{'}(t) f(t, x(b(t))) - a^{'}(t) f(t, x(b(t))).$$

then there is long run growth, that is

$$g_y(t) = \alpha m - \mu$$

and

$$\lim_{t\to\infty} g_y(t) > 0$$

The constraints can be equivalently written as

$$\frac{\mu}{m} < \alpha = \frac{\beta\rho}{\gamma}$$

mean that there is long run growth if the share of land in the GDP is bounded by the decay in the learning process compounded by the rate of mortality and by the effect of learning weighted by the cost of rearing children times the "moral restraint".

2.5 Bibliography

References: Galor (2005) Ashraf and Galor (2011) and Galor (2011):ch 3.

Appendix

2.A Solution of equation (2.7)

Equation (2.7) is a Bernoulli differential equation. In order to solve it we simplify it in two steps: First observe that it can be equivalently written as

$$\dot{L} = mL(t)\left(\left(\frac{\bar{L}}{L(t)}\right)^{\alpha} - 1\right)$$

and, second, considerthe change in variables $z(t) = L(t)^{\alpha}$. If we time differentiate z, we get

$$\begin{aligned} \dot{z} &= \alpha L(t)^{\alpha-1}\dot{L} \\ &= \alpha L(t)^{\alpha-1} m L(t) \left(\left(\frac{\bar{L}}{L(t)} \right)^{\alpha} - 1 \right) \\ &= \alpha m \left((\bar{L})^{\alpha} - L(t)^{\alpha} \right) \\ &= \alpha m \left(\bar{z} - z(t) \right) \end{aligned}$$

This is a linear equation which has the solution

$$z(t) = \bar{z} + (z(0) - \bar{z}) e^{-\alpha m t}$$

where $\bar{z} = (\bar{L})^{\alpha}$. Then the solution of the differential equation for the population dynamics is

$$L(t) = \bar{L} \left[1 + \left(\left(\frac{L(0)}{\bar{L}} \right)^{\alpha} - 1 \right) e^{-\alpha m t} \right]^{1/\alpha}, \; t \in [0, \infty)$$

2.B Solution of equation (2.14)

Equation (2.14) can be written as

$$\dot{y}(t) = \left(\alpha m - \frac{\alpha \gamma}{\rho} y(t) \right) y(t)$$

This is a Bernoulli equation (see) with parameter $\alpha = 2$. To solve it, we set $x(t) = 1/y(t)$. Then we get a linear ODE

$$\dot{x} = \frac{\alpha \gamma}{\rho} - \alpha m x(t)$$

which has steady state

$$\bar{x} = \frac{\gamma}{\rho m} = \frac{1}{\bar{y}}$$

In order to solve this equation we define

$$z(t) = x(t) - \bar{x}$$

and get the ODE

$$\dot{z} = -\alpha m z(t)$$

which has the solution

$$z(t) = z(0)e^{-\alpha m t}$$

then

$$x(t) = \bar{x} + (x(0) - \bar{x})e^{-\alpha m t}$$

because $x(t) = \bar{x} - z(t)$ and $z(0) = x(0) - \bar{x}$. Finally, because $y(t) = 1/x(t)$, we get the solution for the GDP per capita (2.15).

Chapter 3

Modern growth accounting and modern growth facts

In this chapter we will study in a general framework some fundamental ideas in growth theory:

1. level and rate of growth of per capita GDP

2. transition and long run growth

3. convergence: β-convergence and σ-convergence

We say:

- there is economic growth if there is a permanent increase in the GDP per capita, i.e, $g_y(t) > 0$ for all $t \in [0, \infty)$

- there is an increase in the level but no growth if the level of GDP ($\lim_{t\to\infty} y(t) = \bar{y} > y(0)$) increases but $\lim_{t\to\infty} g_y(t) = 0$

- there is only transitional dynamics if, given an initial level $y(0)$ and $\lim_{t\to\infty} y(t) = \bar{y} = y(0)$).

In a nutshell we can conclude the following about the mechanics of growth generating economic growth

- to have economic growth we should have at least one reproducible factor (or mechanism) and non-decreasing marginal returns at the aggregate level;
- if there are non-reproducible factors but there are permanent positive shifts in the levels of factors of production or there are decreasing marginal returns, we may have an increase in the level of output ($\bar{y}$), but no economic growth (i.e., long run growth of GDP per capita rate equal to zero);
- if there are non-reproducible and temporary shifts in factors, we will not have long run increase in the level of output ($\bar{y}$), and no economic growth, but only transition dynamics.

3.1 A general framework for growth theory

The Malthusian model did not generate a positive long-run growth rate of per-capita GDP, except in the case in which we had learning-by-doing.

No-growth is related to one or both of the following conditions: (1) there are no reproducible factors; or (2) they are produced with technologies featuring decreasing returns.

Economic growth theories present a view on the behaviour of per capita GDP at time $t \geq 0$, $Y(t)$.

Most economic growth models have two basic types of equations:

1. one production function: in which the aggregate GDP is a function of several factors of production

$$Y(t) = F(\mathbf{Z}(t), \varphi), \quad \mathbf{Z}(t) = (K(t), H(t), A(t), Q(t), X(t)) \tag{3.1}$$

where $t \geq 0$ represents time, and

$K(t)$ = stock of physical capital p.c.;

$H(t)$ = stock of human capital p.c.;

$A(t)$ = total factor productivity;

$Q(t)$ = variables measuring technical progress, ideas, R& D ;

$X(t)$ = other variables (v.g., population, industrial structure, institutions, environment, etc);

φ = vector of parameters related with: technology, preferences, demographics, institutions.

2. the factors of production are reproducible and can be accumulated, and their behaviour is described by one differential equation or a system of differential equations (if there is more than one factor)
$$\dot{\mathbf{Z}} = G(\mathbf{Z}(t)). \tag{3.2}$$

We can decompose the aggregate per capita GDP (exactly or approximately) as

$$Y(t) = \left((y(0) - \bar{y})e^{\lambda t} + \bar{y}\right) e^{\gamma t}. \tag{3.3}$$

In this equation three components are present:

1. the long run growth rate: γ

2. the long run growth level: $\bar{y}$

3. the transition component: $(y(0) - \bar{y})e^{\lambda t}$, and in particular the growth rate along the transition path λ

Note that in general we observe $Y(t)$ and $y(0)$ but we do not observe γ, $\bar{y}$ and λ.

Sometimes we call balanced growth path (BGP) to the (non-observed) aggregate per capita long run per capita GDP level.

$$\bar{Y}(t) = \bar{y}e^{\gamma t}$$

For the case of Portugal see Figure 3.1

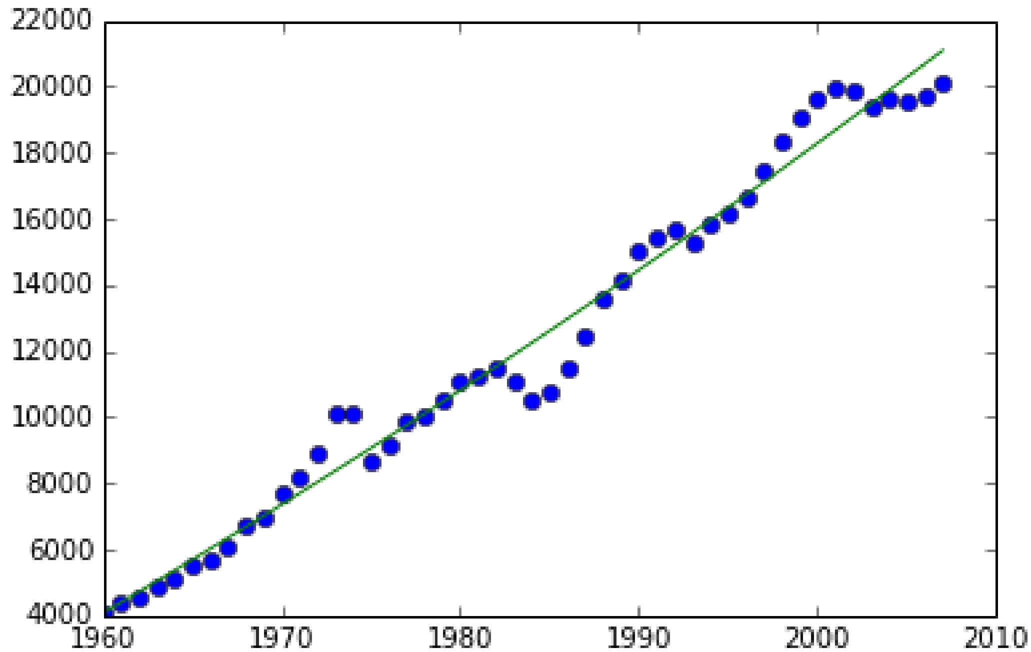

Figure 3.1: Portugal gdp per capita: observed and BGP. See python code in the appendx

We want to obtain those components γ, $\bar{y}$ and λ from the behaviour of the factors of production or other dynamic driving forces which solve equations (3.2).

As we see next, there are some requirements on functions $F(.)$ and $G(.)$ if we want a model which is capable of generating long run growth: $F(.)$ has to be linearly homogeneous and $G(.)$ should be linear or be approximated by a linear function.

3.2 GDP growth and factor accumulation dynamics

We start with the one-dimensional case.

3.2.1 The simplest case

Let us start with the case in which there is only one reproducible factor, i.e, with the one-dimensional case.

Assume that the production function is linear,

$$Y = AZ \tag{3.4}$$

where A is the factor's Z productivity and that the factor accumulation is given by the linear differential equation

$$\dot{Z} = gZ, \text{ where } \dot{Z} \equiv \frac{dZ(t)}{dt}, \ Z(0) = z_0 > 0 \text{ given} \tag{3.5}$$

Then, solving the differential equation and using the initial value, we get

$$Z(t) = z_0 e^{gt}$$

which implies that the GDP becomes

$$Y(t) = Az_0 e^{gt}.$$

Conclusion: the model which is composed by equations (3.4) and (3.5)

1. only generates long run growth and does not display transitional dynamics
2. the long run growth rate is $\gamma = g$ and the level along the BGP as $\bar{y} = Az_0$

3.2.2 Linear input accumulation functions

Now assume there are n reproducible factors that is

$$Y(t) = F(Z_1(t), \dots, Z_n(t))$$

where $F(.)$ is linearly homogeneous [1] and that

$$\dot{Z}_i(t) = g_i Z_i,\ Z_i(0) = z_{i0} > 0 \text{ given.}$$

Again we can solve the initial value problem for each Z_i we get

$$Z_i(t) = z_{i,0} e^{g_i t}$$

If we substitute in the production function we get

$$Y(t) = F\left(z_{1,0} e^{g_1 t}, \dots, z_{n,0} e^{g_n t}\right)$$

There are several possible cases cases.

Equal growth rates If $g_1 = \dots = g_n = g$ and because the production function is linearly homogeneous, we can write

$$Y(t) = F\left(z_{1,0}, \dots, z_{n,0}\right) e^{gt}$$

which has the form

$$Y(t) = y_0\, e^{gt}$$

then Y is the level along the BGP with $\bar{y} = y_0 = F\left(z_{1,0}, \dots, z_{n,0}\right)$ and there is no transition.

[1] Let $f(x_1, \dots, x_n)$. We say that $f(.)$ is homogeneous function of degree ν if, given a constant λ we have $f(\lambda x_1, \dots, \lambda x_n) = \lambda^\nu f(x_1, \dots, x_N)$. The function is linearly homogeneous if the former property holds with $\lambda = 1$.

Different growth rates Let there be at least two different g_i. As we want to derive a BGP, assume there is a rate of growth g. If we use the homogeneity of $F(.)$ then

$$Y(t) = F\left(z_{1,0}e^{(g_1-g)t}, \dots, z_{n,0}e^{(g_n-g)t}\right) e^{gt} = y(t)e^{gt}$$

The GDP will converge to a BGP if and only if

$$\lim_{t\to\infty} y(t) = \bar{y}$$

where $\bar{y}$ is positive and bounded.

This can only occur under certain conditions. For instance

1. if F has a geometrical average structure

   $$Y = F(\prod_{i=1}^{n} Z_i^{\alpha_i})$$

   where $\sum_{i=1}^{n} \alpha_i = 1$ then the rate of growth is the arithmetic mean of the rates of growth of the components

   $$g = \sum_{i=1}^{n} \alpha_i g_i$$

   and there is usually no transition

2. if $F(.)$ has a separable structure, that it has the following structure,

   $$F(Z_1, \dots, Z_n) = F_0(f_1(Z_1) + \dots + f_n(Z_n))F_1(Z_m)$$

   and is linear homogeneous and $g = g_m = \max\{g_1, \dots, g_n\}$ then there is a BGP and there is transitional dynamics.

Conclusion: when there are several factors, a necessary condition for the existence of a BGP is that their rates of growth should be monotonously related.

Examples: see Problem set 1

3.2.3 General non-linear factors' accumulation dynamics

The one-dimensional case

Assume that Z represents the reproducible factors and is of dimension one. Generally, the accumulation of reproducible factors is modelled by a dynamic system

$$\dot{Z} = G(Z), \text{ where } \dot{Z} \equiv \frac{dZ(t)}{dt}, \; Z(0) = z_0 \text{ given} \tag{3.6}$$

where $G(Z)$ is linearly homogenous.

Let us write Z as

$$Z(t) = z(t)e^{\gamma t}, \tag{3.7}$$

where $z(t)$ is the detrended value for $Z(t)$ and γ is its long run rate of growth.

Taking time log-derivatives, $d\ln(Z(t))/dln(t)$, in equation (3.7), we get

$$\frac{\dot{Z}}{Z} = \frac{\dot{z}}{z} + \gamma \tag{3.8}$$

or

$$\dot{z} = \dot{Z}e^{-\gamma t} - \gamma z$$

From the linear homogeneity of $G(.)$ and the decomposition (3.7)

$$G(Z) = G(ze^{\gamma t}) = e^{\gamma t}G(z),$$

or

$$G(z) = e^{-\gamma t}G(Z).$$

Then, the former representation of Z is valid, as is the following representation in the trendless variables

$$\dot{z} = h(z) = G(z) - \gamma z \tag{3.9}$$

where $h(.)$ is usually linear.

Depending on the particular model under study, we may determine γ:

- if we have an exogenous growth model, γ is determined from the exogenous variable which grows exponentially ;

- if we have an endogenous growth model, γ is determined from an intertemporal arbitrage conditions (or its equivalent as an dynamic equation for consumption).

Let $\bar{z}$ be a stationary equilibrium point of equation (11.10), i.e.

$$\bar{z} \in \{z : \; g(z) - \gamma z = 0\}$$

Assume that there is at least one value $\bar{z} > 0$.

Then, we say that

$$\bar{Z}(t) = \bar{z}e^{\gamma t},$$

is the value of Z along the balanced growth path.

Linearizing equation (11.10) in a neighborhood of $\bar{z}$ we get the linear system

$$\dot{z} = \lambda(z - \bar{z}), \; \lambda \equiv \left.\frac{dh(z)}{dz}\right|_{z=\bar{z}}$$

that has solution

$$z(t) - \bar{z} = ke^{\lambda t} \tag{3.10}$$

where k is an arbitrary constant to be determined from initial (at $t = 0$) or terminal ($t = \infty$) conditions.

Then:

- if $\lambda = 0$ then we say that there is no transitional dynamics. This means that $z(t) = \bar{z}$ and $Z(t) = \bar{z}e^{\gamma t}$ for all $t \geq 0$

- if $\lambda < 0$ then we say that there is transitional dynamics. This means that $\lim_{t\to\infty} z(t) = \bar{z}$, i.e. the variable tends asymptotically to its BGP. If in equation (3.10) we set $k = z(0) - \bar{z}$ and substitute in equation (3.7) then

$$Z(t) = \left(\bar{z} + (z(0) - \bar{z})e^{\lambda t}\right) e^{\gamma t}$$

Several reproducible factors

The same reasoning applies when we have more than one reproducible factor. In this case

$$\dot{Z} = G(Z),\ \text{where } \dot{Z} \equiv \frac{dZ(t)}{dt} \tag{3.11}$$

and $G(Z) = (g_1(Z), \dots, g_n(Z))^\top$ and all $g_i(Z)$ are linearly homogeneous as regards $Z \in \mathbb{R}^n$.

We assume every component of Z has the representation

$$Z_i(t) = z_i(t)e^{\gamma_i t},\ i = 1, \dots, n, \tag{3.12}$$

where $z_i(t)$ is the detrended value and γ_i is the long run growth rate of variable $Z_i(t)$.

Again, from the linear homogeneity of $G(.)$ and the decomposition (3.12), we have

$$G(Z) = G(ze^{\gamma t}) = e^{\gamma t}G(z),$$

in which component i is

$$g_i\left(z_1e^{\gamma_1 t}, \dots, z_ne^{\gamma_n t}\right) = e^{\gamma_i t}g_i(z_1, \dots, z_n),\ i = 1, \dots, n$$

which means that the rates of growth are related and tend to have a common factor.

Taking time derivatives from equation (3.12), we get

$$\frac{\dot{Z}_i}{Z_i} = \frac{\dot{z}_i}{z_i} + \gamma_i,\ i = 1, \dots, n, \tag{3.13}$$

Then, we have the representation in the trendless variables

$$\dot{z}_i = h_i(z_1(t), \dots, z_n(t)) = g_i(z_1(t), \dots, z_n(t)) - \gamma_i z_i(t),\ i = 1, \dots, n, \tag{3.14}$$

where $g(.)$ is homogeneous of degree 1.

Let $\bar{z}_i$ be a stationary equilibrium point of equation (3.12), i.e.

$$\bar{z} \in \{z :\ g(z) - \gamma z = 0\}$$

If $\bar{z}_i$ is a stationary equilibrium value for the detrended variable z_i, we call

$$\bar{Z}_i(t) = \bar{z}_i e^{\gamma_i t}, ,\ i = 1, \dots, n,$$

the value of Z_i along the balanced growth path equation.

Linearizing equation (3.12) in a neighborhood of $\bar{z}_i$ we get the linear system

$$\dot{z}_i = J(z_i(t) - \bar{z}_i) \tag{3.15}$$

where J is the Jacobian matrix, evaluated at the BGP,

$$J = \begin{pmatrix} \left.\frac{\partial h_1}{\partial z_1}\right|_{z=\bar{z}} & \dots & \left.\frac{\partial h_1}{\partial z_n}\right|_{z=\bar{z}} \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial h_n}{\partial z_1}\right|_{z=\bar{z}} & \dots & \left.\frac{\partial h_n}{\partial z_n}\right|_{z=\bar{z}} \end{pmatrix}.$$

The ordinary differential equation (ODE) system (3.15) has solution

$$z(t) = \bar{z} + (z(0) - \bar{z})e^{Jt}$$

If J has at least one negative eigenvalue (or one eigenvalue with negative real part) , then we say that there is transitional dynamics.

If this is the case, we should have $\lim_{t\to\infty} z_i(t) = \bar{z}_i$ and

$$\lim_{t\to\infty} Z(t) = \bar{Z}(t)$$

where

$$Z_i(t) = z_i(t)e^{\gamma_i t}$$

We can obtain the behavior of the GDP per head if we observe that

$$Y(t) = F(Z(t)).$$

3.2.4 Balanced growth paths and transition

The dynamics for GDP, can be obtained because

$$Y = F(Z)$$

We already saw that if a balanced growth path (BGP) exists then the rates of growth of all the components are proportional. Then the output per capita along the BGP can be written as

$$\bar{Y}(t) = \bar{y}e^{\gamma t},\ t \geq 0.$$

Observe that:

- if $\gamma > 0$ we have $\lim_{t\to\infty} \bar{Y}(t) = \infty$, where $\bar{y}$ is a constant;
- If $\gamma = 0$ then $\lim_{t\to\infty} \bar{Y}(t) = \bar{y}$, $Y(t)$ should be in a steady state, and there is no growth.
- If $\gamma < 0$ then $\lim_{t\to\infty} \bar{Y}(t) = 0$ and the economy collapses.

Now, applying the previous decomposition, the (short run) level the product can be written as

$$Y(t) = y(t)e^{\gamma t}.$$

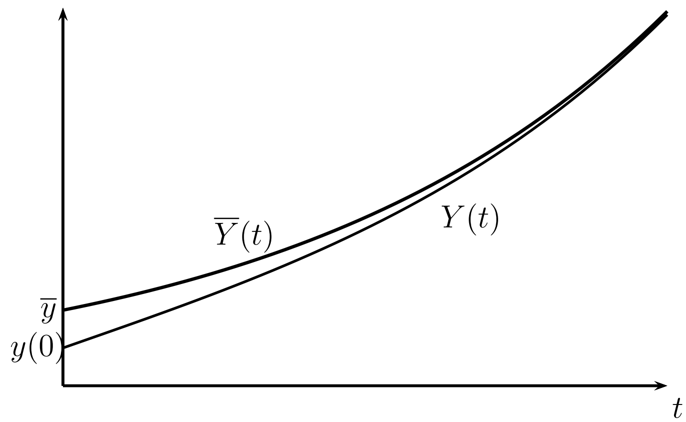

Figure 3.2: The balanced growth path and the short run level. See equation (3.17)

This is equivalent to

$$Y(t) = ([y(t) - \bar{y}] + \bar{y})\, e^{\gamma t}.$$

where $y = F(z)$. If the associated Jacobian J, associated to system (3.15) has at least one non-positive root, $\lambda \leq 0$ then we can perform a first-order taylor approximation

$$y(t) - \bar{y} \approx (y(0) - \bar{y})e^{\lambda t} \tag{3.16}$$

Then the variables in levels can be decomposed as:

$$Y(t) \approx \left((y(0) - \bar{y})e^{\lambda t} + \bar{y}\right) e^{\gamma t} \tag{3.17}$$

Taking log-derivatives, and denoting

$$\gamma_y(t) = \frac{d\ln(Y(t))}{d\ln(t)}$$

then we get [2]

$$\gamma_y(t) = \frac{(y(0) - \bar{y})\lambda e^{\lambda t}}{(y(0) - \bar{y})e^{\lambda t} + \bar{y}} + \gamma \tag{3.18}$$

implying that

$$\lim_{t\to\infty} \gamma_y(t) = \gamma.$$

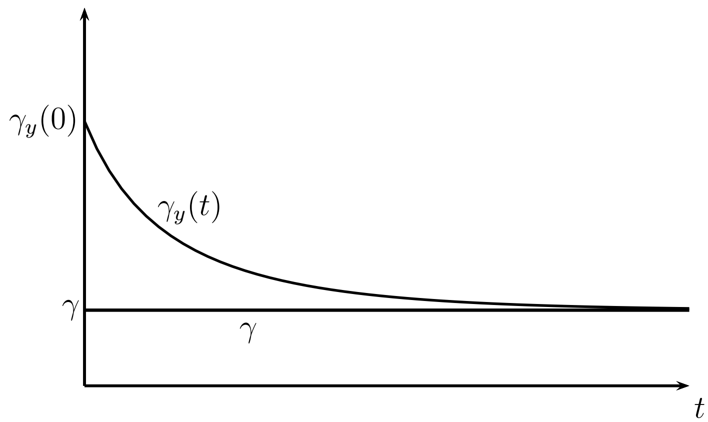

Figure 3.3: Rates of growth: short run and long run. See equation (3.18)

Therefore, the observed variation in $Y(t)$ has three components:

- rate of long run growth, γ;
- level of the variables in the long run, $\bar{y}$;
- transition resulting from the initial level of the product deviating from the long run level, which implies that the short run rate of growth is different from the long run rate of growth.

Particular theories of growth introduce explanations for the three main measures of growth , as dependent on one or more parameters φ:

- $\gamma(\varphi)$ = the rate of long run growth ;
- $\bar{y}(\varphi)$ = the level of the variables in the long run;
- $y(t) - \bar{y}$ the transition (or convergence) towards the BGP.

The are several types of models:

[2] Observe that $\ln(a + b)$ can be approximated by $(a \ln a + b \ln b)/(a + b)$.

- growth rate: (1) models in which the long run rate of growth is zero (the variables tend to a stationary equilibrium) or positive; (2) models in which the rate of growth is exogenous, semi-exogenous or endogenous;

- models with or without transition dynamics.

3.3 Growth accounting

Let us assume that the production function has the Cobb-Douglas form, and we consider growth as generated by n factors of production, which are observable. If we take logs we have

$$\ln Y(t) = \alpha_1 \ln Z_1(t) + \ldots + \alpha_n \ln Z_n(t)$$

where α_i is the share of factor i and we assume $\sum_{i=1}^{n} \alpha_i = 1$.

Taking time derivatives we get a theoretical dependence on the growth rate and the several factors of production

$$\frac{\dot{Y}}{Y} = \alpha_1 \frac{\dot{Z}_1}{Z_1} + \ldots + \alpha_n \frac{\dot{Z}_n}{Z_n}.$$

If we take the model to data we observe a difference between the observed growth rate and the rate of growth accounted by the factors that in theory we assume are the engines of growth:

$$\hat{g}_{\text{Solow}} = \frac{\dot{Y}}{Y} - \alpha_1 \frac{\dot{Z}_1}{Z_1} - \ldots - \alpha_n \frac{\dot{Z}_n}{Z_n}$$

Which is called the Solow residual. We can take $\hat{g}_{Solow}$ as an estimate of the the total factor productivity (TFP) growth rate.

The Solow residual Solow (1957), or the measure of our ignorance, measures the deviation between the observed GDP per capita growth and the factors we think are responsible for it.

In the first applications the factors of production were labor and capital. Barro and Sala-i-Martin (2004):ch 10summarise the results of this approach for the OECD countries:

1. the share of capital was between 34% and 56%

2. the share of labour was between 2% and 40%

3. the TFP was between 16% and 64%

4. and there was a clear productivity slowdown

Aside from the important measurement problems (for example, K is not observed) there can be some endogeneity in $\hat{g}_{Solow}$. The TFP can be endogenous as regards some explanatory variables: externalities and increasing returns (depend on K), or can be related to R&D activities which are to a certain extent endogenous, or can be biased by the the existence of taxes, imperfect competition, etc

3.4 Convergence

If we take logs from equation (3.3) and measure it for several countries or regions $i = 1, \dots, n$, and write $\beta = -\lambda$, we get

$$\ln Y_{i,t} = \gamma_i t + e^{-\beta t} \ln y_{i,0} + (1 - e^{-\beta t}) \ln \bar{y}_i,$$

From this equation we can get the approximation

$$\ln (Y_{i,t}/Y_{i,t-1}) = a_{i,t} + (1 - e^{-\beta}) \ln Y_{i,t-1}$$

1. if $a_{i,t} = a_t$ and β we say there is absolute convergence: poor economies grow at faster rates because they are further away from their own BGP

2. if $a_{i,t}$ varies among economies and β we say there is relative convergence: BGP growth rates vary among economies and every economy converges to its own BGP.

The evidence tends to be supportive to the existence of absolute convergence between regions of major countries or between countries with a similar state of development, but tends to reject the existence of absolute convergence between countries in different stages of development.

3.5 References

References: Barro and Sala-i-Martin (2004):ch. 10, 11, 12, Acemoglu (2009):ch. 1, 3.

Appendix

3.A Python code for Figure 3.1

Chapter 4

Exogenous growth theory: capital accumulation dynamics

Next, we present the two founding models of exogenous growth theory:

1. the Solow (1956) model
2. the Ramsey (1928)-Cass (1965)-Koopmans (1965) model

The models are distinct by the fact that the Solow model has an ad-hoc consumption function and the Ramsey-Cass-Koopmans model consumption is derived from intertemporal utility maximixation.

Founding contributions: Solow (1956)-Swan (1956) model of capital accumulation

General assumptions :

- the economy is closed: there is no international trade in goods or capital;
- a single composite good is produced and used for consumption and investment;
- the good is produced, using capital and labor, by a neoclassical technology;

- investment allows for capital accumulation, therefore physical capital is a reproducible input;
- population grows at an exogenous rate, n, and there is full participation in production;
- all markets, factor, product and financial markets are perfectly competitive.

The model is ad-hoc in the sense that the behavior of agents is translated into structural forms for the consumption, production and investment, independently from a consistent modelling of their microeconomic fundamentals. Although the firms' problem can be directly derived from static firm optimisation, the joint problem of production and investment would not produce the same solutions (check !).

The basic questions that the model addresses are: can capital accumulation explain growth ? how does the capital accumulation behaves along time and what are the explanatory variables ?

Main results:

1. GDP dynamics is generated by capital accumulation and population growth;
2. the long run per capita growth rate is zero;
3. the long run GDP level depends on population growth rate, savings rate and technological parameters;
4. the model displays transition dynamics, driven by decreasing marginal returns. This means that the short run rate of growth is driven by the difference between the initial capital intensity relative to the long run level.

4.1 The Solow-Swan model

Consumers

Consumers' receive income and consume and save. Their behavior is characterized by the Keynesian savings function. Consumers receive income, Y, from labour supplied to firms and from ownership of firms. They save a constant proportion, s of income

$$S(t) = sY(t),\ 0 < s < 1,$$

and, therefore, they also consume a constant propostion of income

$$C(t) = (1 - s)Y(t).$$

Firms

Firms have two types of decisions: (1) static - how to allocate capital and labour inputs to production; (2) dynamic - how to expand activity. In this model, the decisions are not taken jointly

The technology is described by the production function, $F(K, L)$, which is neoclassical:

- positive marginal returns: $F_K = \partial F/\partial K > 0$ and $F_L = \partial F/\partial L > 0$. This means that both factors are necessary;
- decreasing marginal returns and concavity in (K, L): $F_{KK} < 0$, $F_{LL} < 0$ and $F_{KK}F_{LL} - F_{KL}^2 > 0$. This means that there are decreasing marginal returns and that the two factors are substitutable;
- it is homogeneous of degree one, meaning that there are constant returns to scale, for a constant λ it verifies
$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

- it is of the Inada type: that is, it verifies

$$\lim_{j\to 0} F_j(K,L) = \infty,\ \lim_{j\to\infty} F_j(K,L) = 0,\ j = K, L.$$

  This means that the marginal productivity is globally decreasing (i.e, for every $K > 0$ and $L > 0$), for both factors.

From the linear homogeneity property , we can write the model in intensity terms

$$F(K,L) = LF(K/L,1) = Lf(k),\ \text{where} k \equiv K/L,$$

Firms determine the optimal activity level by minimizing costs of production: i.e., they solve the problem, for every moment in time, t

$$\min_{K(t),L(t)} \{r(t)K(t) + w(t)L(t) :\ AF(K(t), L(t) \geq Y(t)\}$$

where the rate of return on capital, r and the wage rate, w, are assumed to be given to the representative firm, K and L denote the capital and labour inputs, and A is the total factor productivity (we assume that technical progress is not biased).

The first order conditions for the firm maximization are

$$\begin{aligned} r(t) &= AF_K(K(t), L(t)) = Af^{'}(k(t)), \\ w(t) &= AF_L(K(t), L(t)) = A[f(k(t)) - k(t)f^{'}(k(t))] \end{aligned}$$

that is, if the factor markets are competitive, then the rates of return are equalized to the marginal productivities.

From Euler's theorem (which is a generic property of the homogeneous functions)

$$Y = F_K K + F_L L$$

and from the first order conditions we have an income distribution equation

$$Y = rK + wL..$$

With perfect capital markets, they take the price of capital as given and demand capital inelasticaly in order to meet the accounting relationship

$$I(t) = \dot{K} + \delta K(t)$$

gross investment, I, is equal to net investment which represents the increase in the capital stock, $\dot{K}$, plus depreciation, where δ is the rate of depreciation of capital.

Population growth

Population grows according to

$$\dot{N} = nN, \tag{4.1}$$

where n is the rate of population growth, and $N(0) = N_0$, the initial population, is given.

The solution of equation (4.1) is

$$N(t) = N(0)e^{nt}$$

In this model it is assumed that the labour input is supplied inelasticaly, there is not unemployment and there is full participation. This implies that the aggregate labour input is equal to total population

$$L(t) = N(t), t \geq 0.$$

From now on, we assume that the technology is represented by a Cobb-Douglas production function

$$Y(t) = F(K(t), L(t)) = AK(t)^{\alpha}L(t)^{1-\alpha}, 0 < \alpha < 1$$

where A represents the total factor productivity and α the share of capital in aggregate income

Equilibrium

Given the flow $[N(t)]_{t=0}^{\infty}$, an equilibrium for this economy is defined by the collection of flows $[C(t), I(t), Y(t), K(t), L(t)]_{t=0}^{\infty}$ such that each agent behaves according to the behavior rules that we have just derived (firm's optimization and ad-hoc consumption behavior) and the product market equilibrium condition

$$Y(t) = C(t) + I(t)$$

holds.

4.2 Equilibrium capital accumulation dynamics

We can see that ultimately C, I and L are dependent on K and N. Therefore the general equilibrium can be determined from the solution of the following system of ordinary differential equations (ODE) in the level variables (K, L) (we do not use N from now on):

$$\dot{K} = sAK^{\alpha}N^{1-\alpha} - \delta K \tag{4.2}$$

$$\dot{N} = nN \tag{4.3}$$

Exercise: prove this.

Let us denote the per capita variables in small caps. Then, the capital intensity is

$$k(t) \equiv K(t)/N(t)$$

Therefore

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{N}}{N}$$

From the linear homogeneity of the production function, it is easy to see that we get the fundamental equation for capital accumulation of the Solow-Swan model:

$$\dot{k} = sAk^{\alpha} - (n + \delta)k. \tag{4.4}$$

The equilibrium capital intensity path $[k(t)]_{t=0}^{\infty}$ is the solution of equation (4.4) given $k(0) = K(0)/N(0)$. That is equilibrium for this economy is a function of the path for capital intensity, which is a function of the initial level of capital intensity

$$[k(t, k_0)]_{t=0}^{\infty}$$

where $k(t) = k(t, k_0)$ and $k(0, k_0) = k_0$. As we will see, we will be able to get function $k(t)$ explicitly.

If we determine the solution for the capital intensity $k(t, k_0)$ then we can determine the equilibrium GDP behavior from

$$Y(t) = A(k(t, k_0)N(t))^{\alpha}(N(t))^{1-\alpha} = A(k(t, k_0))^{\alpha} N_0 e^{nt}$$

Next we find the solutions of the Solow equation (4.4). First, observe that there are two types of equilibrium solutions: (1) stationary solutions; (2) time-varying solutions

1. Stationary equilibrium The stationary equilibrium for k, is a particular equilibrium (in the sense of the previous definition), such that the capital intensity is constant, $[k^*]_{t=0}^{\infty}$

It can be determined from the steady states of differential equation (4.4) : $k^{ss} = \{k \geq 0 \ : \ \dot{k} = 0\}$.

Then, it is determined from

$$k^{ss} = \{k \geq 0 \ : \ k(sAk^{1-\alpha} - (n + \delta)) = 0\}.$$

We can see that there are two fixed points

$$k^{ss} = \{0, \bar{k}\}$$

where

$$\bar{k} = \left(\frac{sA}{n + \delta}\right)^{1/(1-\alpha)}$$

From now on, we will only consider the interior point: $k^{ss} = \bar{k}$. The steady-state per capita GDP becomes

$$\bar{y} = A\left(\frac{sA}{n+\delta}\right)^{\alpha/(1-\alpha)},$$

for the interior solution.

We can also determine the long run gross rate of return

$$\overline{r} = f^{'}(\overline{k}) = \frac{\alpha}{s}(n+\delta)$$

If $k(0) = \overline{k}$ then the equilibrium path for k is $[\overline{k}]_{t=0}^{\infty}$: that is, the economy is in a BGP

$$Y(t) = \bar{Y}(t) = \overline{y}N_0 e^{nt}$$

and grows at a positive rate if $n > 0$ or is in a steady state if $n = 0$, which is equal to the rate of population growth. Then the per capita growth rate γ is zero.

2. Non-stationary equilibria In order to determine and characterize the other possible, non-stationary, equilibria we should solve the differential equation for capital accumulation (4.4), given $k(0) = k_0 \neq \overline{k}$.

We can do it in two alternative ways: [1]:

- we can solve it explicitly
- or perform a qualitative dynamics study through linearization in the neighborhood of the BGP

[1]The first method can be used because we can find a closed form solution for the capital accumulation equation. If this is not the case we could only use the second method.

First approach: determining a closed form solution of the Solow equation Observe that the solution for equation (4.4) is a Bernoulli ordinary differential equation. Then the solution is [2]

$$k(t)=\left[\frac{sA}{n+\delta}+e^{-(1-\alpha)(n+\delta)t}\left(k(0)^{1-\alpha}-\frac{sA}{n+\delta}\right)\right]^{\dfrac{1}{1-\alpha}},t\in[0,\infty).$$

Then the level for the aggregate GDP is

$$Y(t)=A\left[\frac{sA}{n+\delta}+e^{-(1-\alpha)(n+\delta)t}\left(k(0)^{1-\alpha}-\frac{sA}{n+\delta}\right)\right]^{\dfrac{\alpha}{1-\alpha}}N(0)e^{nt},t\in[0,\infty),$$

which has the form $Y(t)=y(t)N(0)e^{\gamma t}$ where

$$\begin{aligned}
y(t) &= A\left[\frac{sA}{n+\delta}+e^{-(1-\alpha)(n+\delta)t}\left(k(0)^{1-\alpha}-\frac{sA}{n+\delta}\right)\right]^{\dfrac{\alpha}{1-\alpha}}\\
\gamma &= n.
\end{aligned}$$

Observe that

$$y(t)=\left[\bar{y}^{(1-\alpha)/\alpha}+e^{-(1-\alpha)(n+\delta)t}\left(y(0)^{(1-\alpha)/\alpha}-\bar{y}^{(1-\alpha)/\alpha}\right)\right]^{\dfrac{\alpha}{1-\alpha}}$$

represents the transition component, because

$$\lim_{t\to\infty}y(t)=\bar{y}.$$

Second approach: approximate solution by linearization Most ordinary differential equations do not have closed form solutions, and therefore we can only determine an approximation. The next method is the most common:

Observe again that equation (4.4) has an unique interior equilibrium such that $k=\bar{k}>0$,

$$\bar{k}=\left(\frac{sA}{n+\delta}\right)^{1/(1-\alpha)}$$

[2]See the appendix

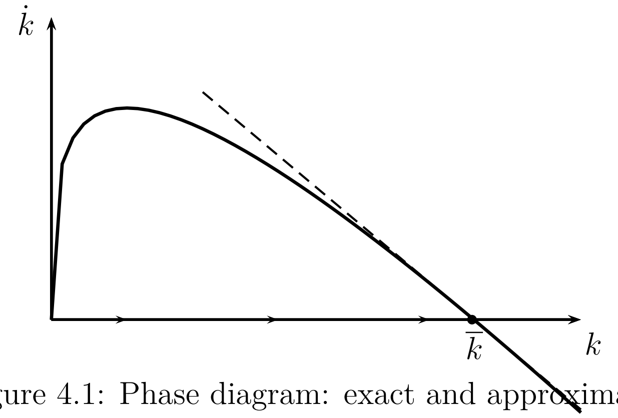

Figure 4.1: Phase diagram: exact and approximated

Performing a first-order Taylor approximation in the neighborhood of $\bar{k}$ we get

$$\dot{k} = \lambda(k(t) - \bar{k}) + \text{h.o.t}$$

where

$$\begin{aligned} \lambda &= \left.\frac{\partial \dot{k}}{\partial k}\right|_{k=\bar{k}} = \\ &= s\alpha A\bar{k}^{\alpha-1} - (n+\delta) = \\ &= -(n+\delta)(1-\alpha) < 0. \end{aligned}$$

In terms of the real rate of return we have

$$\lambda = -s\bar{r}\frac{1-\alpha}{\alpha} < 0$$

because $\bar{r} = \alpha(n+\delta)/s$, then the steady state $\bar{k}$ is asymptotically stable (see 4.2).

Then, if $k(0) \neq \bar{k}$ the level of the GDP has the form

$$Y(t) = y(t)N(0)e^{nt} = Ak(t)^{\alpha}e^{nt} \approx A\left(\bar{k} + (k(0) - \bar{k})e^{\lambda t}\right)^{\alpha} N(0)e^{nt}.$$

Again , $\lim_{t\to\infty} Y(t) = \bar{Y}(t)$.

Results:

1. the long run growth rate is equal to the rate of growth of the population. As we take γ as the growth rate of per-capita GDP then

$$\gamma = 0$$

2. the long run level increases with productivity and the rate of savings and decreases and decreases with the rate of depreciation and the rate of growth of population:

$$\bar{y} = \bar{y}(A, s, n, \delta) = A^{1/(1-\alpha)} \left(\frac{s}{n+\delta}\right)^{\alpha/(1-\alpha)}$$

where the first two have positive effects and the last two have negative effects,

$$\frac{\partial \bar{y}}{\partial A} = \frac{\bar{y}}{(1-\alpha)A} > 0, \frac{\partial \bar{y}}{\partial s} = \frac{\alpha \bar{y}}{(1-\alpha)s} > 0,$$

$$\frac{\partial \bar{y}}{\partial \delta} = -\frac{\alpha \bar{y}}{(1-\alpha)n} < 0, \frac{\partial \bar{y}}{\partial n} = -\frac{\alpha \bar{y}}{(1-\alpha)\delta} < 0,$$

3. the short run rate of convergence may be measured by λ.

To see the last point, we may measure the speed of adjustment by the half-life of the adjustment: given an initial stock of capital $k(0)$ it measures the time taken for half of the adjustment to the steady state to be produced. Formally

$$\tau = \left\{ t \ : \ k(t) - k(0) = \frac{\bar{k} - k(0)}{2} \right\}.$$

Form the previous version of the mode, we

$$\tau = \frac{\ln(1/2)}{\lambda} > 0$$

it is smaller the larger the absolute value of λ is (off course $\ln(1/2) < 0$.

Conclusions:

- without technical progress, the long run growth rate is equal to the rate of growth of the population, which means the per capita growth rate is zero: there is no long-run growth;
- the other parameters (savings applied in the accumulation of physical capital and technological parameters) have only level (on $\bar{y}$) and transitional effects (on $y(t)-\bar{y}$));
- as $\lim_{t\to\infty} -\dfrac{\partial \ln \dot{K}}{\partial \ln K} = \lim_{t\to\infty} \beta(t) = (1-\alpha)(n+\delta)$ there is, what has been called afterwards, $\beta-$convergence;
- the mechanism which produces convergence is the following:
  savings allow for net capital accumulation, but the presence of decreasing marginal returns imply that the this effect decreases with increases in the level of capital.

References : First paper: Solow (1956). Old presentations: Burmeister and Dobell (1970). New presentations: Barro and Sala-i-Martin (2004), Acemoglu (2009).

4.3 Kaldor stylized facts

The stylized facts in Kaldor (1963) had a deep influence in the ensuing growth literature, both theoretical and empirical:

Fact K1 per capita GDP (y) grows along time, and its rate of growth shows no decreasing tendency;

Fact K2 K grows along time;

Fact K3 r (r.o.r of capital) is roughly constant (debatable: it shows a slightly downward tendency for most developing countries);

Fact K4 K/Y is roughly constant;

Fact K5 the shares of capital and labor in the aggregate income are approximately constant;

Fact K6 the growth of Y (p.c.) varies substantially between countries.

Problems with Solow-Swan model

1. it does not explain fact K1 (trend with positive per capita growth rates);
2. in the 60's it was noted that the steady state $\bar{k}$ and the associated real rate of return

$$r(\bar{k}) = Af^{'}(k)|_{k=\bar{k}} = \alpha A\bar{k}^{\alpha-1} = \alpha(n+\delta)/s.$$

   could not be optimal. If we define the golden rule level of the stock of capital as the level which maximizes consumption, $c(k) = Af(k) - (n+\delta)k$, that is

$$k_g = \text{argmax}\,(Af(k) - (n+\delta)k)$$

   we find that it verifies the condition $r(k_g) = n+\delta$ [3]. As $\bar{r} = \alpha(n+\delta)/s$, the golden rule would be reached if $\alpha/s = 1$. But if $\alpha/s < 1$ then $\bar{k} > k_g$ and there would be dynamic inefficiency because the same consumption level could be obtained with a smaller level of capital. If $\alpha/s > 1$ the steady state level of consumption would be below the golden rule.

Resolution (in the 60's):

1. exogenous technical progress was introduced;
2. the Ramsey (1928)'s model was rediscovered.

[3] If $\dot{k} = 0$ then if we do $c(k) = Af(k) - (n+\delta)k$, Then the maximum of c is reached for $c^{'}(k) = Af^{'}(k) - (n+\delta) = r(k) - (n+\delta) = 0$.

4.4 Exogenous technical progress in the Solow-Swan model

Assumption:

there is labour augmenting technical progress: that is, the labor productivity grows at an exogenous growth rate $\phi > 0$,

$$Y = F(K, A(t)L)$$

where

$$A(t) = A(0)e^{\phi t}, \;\; \phi > 0$$

Intuition: labor becomes more efficient through time.

For the Cobb-Douglas technology we get

$$Y = K^{\alpha}(A(t)L)^{1-\alpha}$$

If we define the per capita variables in efficiency units, we get

$$y(t) \equiv \frac{Y(t)}{A(t)L(t)}, \; k(t) \equiv \frac{K(t)}{A(t)L(t)}.$$

Or

$$K(t) = k(t)e^{(\phi+n)t}$$

Now, we have, with full participation, again, $L(t) = N(t)$

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{N}}{N} - \frac{\dot{A}}{A} = \frac{\dot{K}}{K} - \phi - n$$

The equation for capital accumulation becomes

$$\dot{k} = sk^{\alpha} - (\phi + n + \delta)k, \tag{4.5}$$

and

$$r(k) = \alpha k^{\alpha-1}$$

For $k > 0$ we still have an unique equilibrium point

$$\bar{k} = \left(\frac{s}{n+\phi+\delta}\right)^{1/(1-\alpha)}$$

which is asymptotically stable

$$\begin{aligned}
\lambda(\phi) &= \left.\frac{\partial \dot{k}}{\partial k}\right|_{k=\bar{k}} = \\
&= sr(\bar{k}) - (n+\phi+\delta) = \\
&= -(n+\phi+\delta)(1-\alpha) < 0
\end{aligned}$$

Along the BGP

$$\bar{K}(t) = \bar{k}e^{(\phi+n)t} \quad \bar{Y}(t) = \bar{k}^{\alpha}e^{(\phi+n)t}$$

and representing by per capita variables we get

$$\frac{\bar{K}(t)}{N(t)} = \bar{k}e^{\phi t} \quad \frac{\bar{Y}(t)}{N(t)} = \bar{k}^{\alpha}e^{\phi t}$$

Conclusions:

- we have now not only transitional dynamics but also long term dynamics, if we consider the variables in per capita terms $y(t) = Y(t)/L(t)$ and $K(t)/L(t)$;

- the long run growth rate is positive, it is equal to ϕ, but it is exogenous

$$\gamma = \phi$$

;

- the level of the product in the long run depends: positively on s, and negatively on n, ϕ, and δ;
- the conclusions regarding convergence are the same qualitatively, however, the rate of convergence is higher because $\lambda(\phi) > \lambda$.

Chapter 5

Exogenous growth theory: the Ramsey model

After the publication of a key work on the mathematical theory of optimal control Pontryagin et al. (1962), some economists, in particular, Cass (1965) and Koopmans (1965), readdressed the Ramsey (1928) model. This model has become, after a interregnum in the 1970's, the work horse of modern macroeconomics and growth theories, particularly after the middle 1980's .

References: Arrow and Kurz (1971) present the state-of-the-art in the beginning of the 1970's, and Barro and Sala-i-Martin (2004):ch. 2, and Acemoglu (2009):ch. 8 are recent textbooks.

There are two versions of the model:
- for a centralized economy;
- for a decentralized economy: dynamic general equilibrium model (DGE).

As the two versions are equivalent we say that the DGE is Pareto optimal.

5.1 The Cass (1965)- Koopmans (1965) model

The general structure of the model is characterised by:

- the economy has the same productive structure as in the Solow's model,
- but consumption is determined as the solution of an intertemporal optimizing model: they evaluate consumption, and therefore savings in order to maximize lifetime utility;
- there is a central planner who decides on the optimal path of consumption and capital accumulation of the economy.
- we assume that population is constant, that is $n = 0$.

Technology

- the structure of the economy, as far as the characterization of the markets and the technology are concerned, is essentially the same as in the Solow's model, and is represented by the per capita capital accumulation equation

$$\dot{k} = Af(k) - c - \delta k$$

with $k(0) = K(0)/N$ given. We assume from now on a Cobb-Douglas technology

$$f(k) = k^{\alpha},\ 0 < \alpha < 1;$$

- the population is constant N;

Preferences

Agents are homogeneous, and evaluate the paths of consumption, $[c(t)]_{t\geq 0}$, through the intertemporal utility functional

$$V([c]_{t\geq 0}) = \int_0^{\infty} u(c(t))e^{-\rho t}dt$$

where:

- the intertemporal utility function is time-additive and is time-consistent (preferences are stationary), displays impatience, and dynastic;

- $u(c)$ is increasing and Inada: there is no satiation and the marginal utility is decreasing, the marginal utility for $c = 0$ is infinite and the marginal utility for $c = \infty$ is equal to zero. This property implies that c is always a good (as opposed to bad);

- the most common utility function is

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}$$

  where θ is the inverse of the elasticity of intertemporal substitution: if $0 < \theta < 1$ it is elastic, meaning that there rate of intertemporalsubstitution is large, and if $\theta > 1$ it has low elasticity, meaning that agents do not shift consumption intertemporally very much [1];

- $\rho > 0$ is the rate of time preference;

- $e^{-\rho t}$ is the intertemporal discount factor, from the $t = 0$ perspective, and translates the existence of impatience;

[1] For time-additive utility functionals the elasticity of intertemporal substitution is $\text{EIS} = -u^{'}(c)/(u^{''}(c)c$. To see this consider the problem $\max_{c(.)}\{\int_0^T e^{-\rho t}u(c(t))dt\ :\ \ \dot{A} = rA - c\}$. The Euler equation for this problem is $ru^{'}(c(t)) = \rho u^{'}(c(t)) - u^{''}(c(t))\dot{c}$ that is $\dot{c}/c = -u^{'}(c(t))(r-\rho)/(u^{''}(c(t))c(t)$ We define the

$$\text{EIS} \equiv \frac{\partial(\dot{c}/c)}{\partial(r-\rho)}$$

Which produces the previous expression. For an isoelastic function $\text{EIS} = 1/\theta$.

Central planner

There is a benevolent dictator who evaluates the flows of consumption at the aggregate level with the same preference structure as the individual consumers.

Formally, we have the following optimal control model: the central planner wants to determine the optimal paths $[c^*(t), k^*(t)]_{t=0}^{\infty}$ such that

$$\max_{[c]_{t\geq 0}} V([c]_{t\geq 0}), \quad \text{where } V([c]_{t\geq 0}) = \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{k} = Ak(t)^\alpha - c(t) - \delta k(t),$$

given $k(0) = k_0$. It is also assumed that the physical capital is asymptotically bounded by $\lim_{t\to\infty} h(t)k(t) \geq 0$.

The modified Hamiltonian dynamic system

In order to determine the solution for the centralized problem, we write the Hamiltonian

$$H(c, k, q) = \frac{c^{1-\theta}-1}{1-\theta} + q(Ak^\alpha - c - \delta k)$$

The first order conditions, from the Pontriyagin's maximum principle are

$$\begin{aligned} \frac{\partial H}{\partial c} &= 0 \\ \frac{\partial H}{\partial k} &= \rho q - \dot{q} \\ \lim_{t\to\infty} q(t)k(t)e^{-\rho t} &= 0 \end{aligned}$$

for an admissible trajectory verifying the restrictions of the problem.

The first order conditions are

$$\begin{aligned} c^{-\theta} &= q \\ \dot{q} &= q\left(\rho + \delta - \alpha A k^{\alpha-1}\right) \\ \lim_{t\to\infty} q(t)k(t)e^{-\rho t} &= 0 \end{aligned}$$

As

$$\frac{\dot{q}}{q} = -\theta \frac{\dot{c}}{c}$$

we can solve the first order conditions for (c, k).

The optimal flow $[c^*(t), k^*(t)]_{t \in [0,+\infty)}$, verifies the following two ordinary differential equations, which are called the modified Hamiltonian dynamic system (MHDS)

$$\dot{c} = \frac{c}{\theta}\left(r(k(t)) - \rho)\right) \tag{5.1}$$

$$\dot{k} = Ak(t)^\alpha - c(t) - \delta k(t) \tag{5.2}$$

together with the terninal (or transversality) and the initial condition

$$0 = \lim_{t \to \infty} c(t)^{-\theta} k(t) e^{-\rho t} \tag{5.3}$$

$$k(0) = k_0 \text{ given} \tag{5.4}$$

where $r = \alpha A k^{\alpha-1} - \delta$ is the net rate of return for capital

Solving the MHDS

To solve the problem composed by equations (5.1)-(5.4) we have to resort to qualitative methods, because, for almost all parameter values, it has not an explicit solution.

Stationary solution Again we determine the stationary solution

$$(c^*, k^*) = \{(c, k) : \ \dot{c} = \dot{k} = 0\}$$

The fixed points are solution of the equations:

$$\begin{aligned} \frac{c^*}{\theta}\left(r(k^*) - \rho)\right) &= 0, \\ c^* = A(k^*)^\alpha - \delta k^*. \end{aligned}$$

This non-linear ordinary differential equation (ODE) system (5.1)-(5.2) has three fixed points:

$$(c^*, k^*) = \{(0,0), (0, (A/\delta)^{1/(1-\alpha)}), (\bar{c}, \bar{k})\}$$

where

$$\begin{aligned}
\bar{r} &= \rho \\
\bar{k} &= \left(\frac{\alpha A}{\delta+\rho}\right)^{1/(1-\alpha)} \\
\bar{c} &= \frac{\rho+\delta(1-\alpha)}{\alpha}\bar{k}
\end{aligned}$$

Then

$$\bar{y} = A\bar{k}^{\alpha} = \left[A\left(\frac{\alpha}{\delta+\rho}\right)^{\alpha}\right]^{1/(1-\alpha)}. \tag{5.5}$$

The first solution, $c^* = 0$ and $k^* = 0$, has no economic meaning (and raises technical problems), the second, $c^* = 0$ and $k^* = (A/\delta)^{1/(1-\alpha)}$, does not verify the transversality condition, equation (5.3), which implies that it does not verify one admissibility condition. The unique admissible fixed point with economic significance is $c^* = \bar{c}$, and $k^* = \bar{k}$.

Other non-stationary solution Next, we will study the approximate solution in the neighborhood of the stationary equilibrium point, $c(t) - \bar{c}$ and $k(t) - \bar{k}$. If we perform a first order Taylor expansion in a neighborhood of $(\bar{c}, \bar{k})$, we get the ode linear system

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} 0 & \bar{c}r'(\bar{k})/\theta \\ -1 & \rho \end{pmatrix} \begin{pmatrix} c(t) - \bar{c} \\ k(t) - \bar{k} \end{pmatrix}$$

The jacobian matrix, J, has the following trace

$$\mathrm{tr}(J) = \rho > 0$$

and determinant

$$\begin{aligned}\det(J) &= \bar{c}r^{'}(\bar{k})/\theta = \\ &= \bar{c}\alpha(\alpha-1)A\bar{k}^{\alpha-2}/\theta = \\ &= -(1-\alpha)\frac{(\rho+\delta(1-\alpha))(\rho+\delta)}{\alpha\theta} < 0.\end{aligned}$$

Then, the eigenvalues for matrix J are

$$\lambda_s = \frac{\rho}{2} - \left[\left(\frac{\rho}{2}\right)^2 - \det(J)\right]^{1/2} < 0,\ \lambda_u = \frac{\rho}{2} + \left[\left(\frac{\rho}{2}\right)^2 - \det(J)\right]^{1/2} > 0.$$

There is a useful relationship between the trace and the determinant of J and the eigenvalues

$$\lambda_s + \lambda_u = \mathrm{tr}(J) = \rho$$

and

$$\lambda_s\lambda_u = \det(J).$$

Then $\lambda_u + \lambda_s = \rho > 0$ and $\lambda_s\lambda_u < 0$. Therefore the steady state $(\bar{c}, \bar{k})$ is a saddle point.

The eigenvector matrix associated to J [2] is

$$\begin{pmatrix} \lambda_u & \lambda_s \\ 1 & 1 \end{pmatrix}$$

and therefore, the general solution is

$$\begin{pmatrix} c(t)-\bar{c} \\ k(t)-\bar{k} \end{pmatrix} = h_1 \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix} e^{\lambda_s t} + h_2 \begin{pmatrix} \lambda_s \\ 1 \end{pmatrix} e^{\lambda_u t}$$

We eliminate explosive trajectories if we set $h_2 = 0$. Then

$$\begin{pmatrix} c(t)-\bar{c} \\ k(t)-\bar{k} \end{pmatrix} = h_1 \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix} e^{\lambda_s t}$$

[2] We determine the column P^j by solving the homogeneous system $(J-\lambda_j I)P^j = 0$, where I is the (2×2) identity matrix, for non-zero solutions.

The undetermined coefficient h_1 can be determined if we observe that we know $k(0) = k_0$. Evaluating the last expression for $t = 0$, we finally get

$$h_1 = k(0) - \bar{k}$$

then, along the saddle-path, the solution can be approximated by

$$\begin{aligned} c(t) - \bar{c} &= \lambda_u (k(0) - \bar{k}) e^{\lambda_s t}, \\ k(t) - \bar{k} &= (k(0) - \bar{k}) e^{\lambda_s t}. \end{aligned}$$

This means that:

1. if $k(0) \neq \bar{k}$ then $\lim_{t\to\infty} k(t) = \bar{k}$,
2. given any initial value for k, $k(0)$, there is only a value for c, $c(0)$ which is determined endogenously such that $\lim_{t\to\infty} c(t) = \bar{c}$;
3. the solution is determinate, i.e, unique: this is the only solution for the ode system such that the transversality condition holds;
4. the saddle path is asymptotically tangent to the straight line

$$c(t) - \bar{c} = \lambda_u (k(t) - \bar{k})$$

5. we can determine the expression for output along the saddle path

$$\begin{aligned} y(t) &= Ak(t)^\alpha = \\ &= A\left(\bar{k} + (k(0) - \bar{k}) e^{\lambda_s t}\right)^\alpha = \\ &= A\left[(\bar{y}/A)^{\frac{1}{\alpha}} + ((y(0)/A)^{\frac{1}{\alpha}} - (\bar{y}/A)^{\frac{1}{\alpha}}) e^{\lambda_s t}\right]^\alpha = \\ &= \left[\bar{y}^{\frac{1}{\alpha}} + (y(0)^{\frac{1}{\alpha}} - \bar{y}^{\frac{1}{\alpha}}) e^{\lambda_s t}\right]^\alpha \end{aligned}$$

That is

$$y(t) = \left[\bar{y}^{\frac{1}{\alpha}} + (y(0)^{\frac{1}{\alpha}} - \bar{y}^{\frac{1}{\alpha}}) e^{\lambda_s t}\right]^\alpha$$

the model only displays transitional dynamics as $\lambda_s < 0$.

Particular case: explicit solutions In the particular case in which $\theta = \alpha$ then

$$\lambda_u^* = \frac{\rho + \delta(1-\alpha)}{\alpha}$$

and, therefore,

$$c(t) = \lambda_u^* k(t)$$

The general problem has no explicit solution, but if we have the particular case in which $\theta = \alpha$ then the solution is

$$\begin{aligned} c(t) &= \frac{\delta + \rho(1-\alpha)}{\alpha} k(t), \\ r(t) &= \frac{r(0)(\delta+\rho)}{r(0) + (\delta + \rho - r(0))e^{-[(1-\alpha)(\delta+\rho)/\alpha]t}}, \end{aligned}$$

for $t \in [0, \infty)$. As $k(t) = (\alpha A/r(t))^{1/(1-\alpha)}$, then

$$y(t) = A \left[\frac{\alpha A k(0)^{\alpha-1}(\delta+\rho)}{\alpha A k(0)^{\alpha-1} + (\delta + \rho - \alpha A k(0)^{\alpha-1})e^{-[(1-\alpha)(\delta+\rho)/\alpha]t}} \right]^{\alpha},$$

Then, given any initial value for the capital stock, $k(0)$, the economy chooses optimally the initial level of consumption

$$c(0) = \frac{\delta + \rho(1-\alpha)}{\alpha} k(0)$$

and it convergences asymptotically to the steady state,

$$\begin{aligned} \lim_{t\to\infty} c(t) &= \bar{c} = \frac{\rho + \delta(1-\alpha)}{\alpha} \bar{k} \\ \lim_{t\to\infty} r(t) &= \bar{r} = \delta + \rho \\ \lim_{t\to\infty} k(t) &= \bar{k} = \left(\frac{\alpha A}{\delta + \rho} \right)^{1/(1-\alpha)} \end{aligned}$$

and, therefore,

$$\lim_{t\to\infty} y(t) = \bar{y}.$$

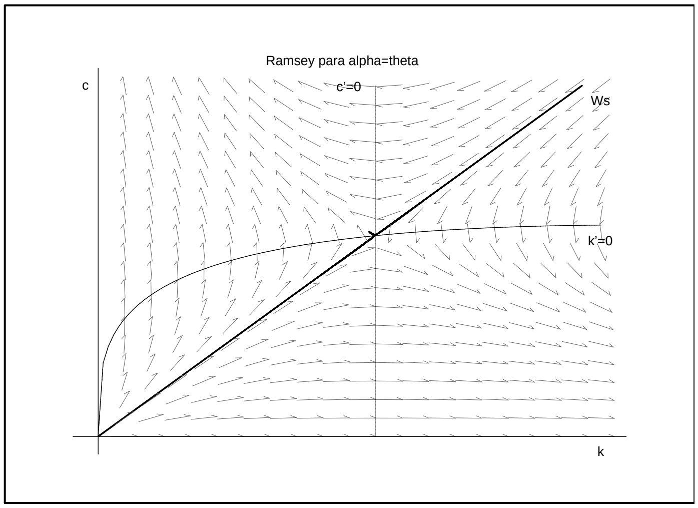

Figure 5.1: Phase diagram for the Ramsey model with $\alpha = \sigma$. It is constructed this way: First, draw the isoclines $\dot{c} = c(r(k) - \rho) = 0$ and $\dot{k} = f(k) - c - \delta k = 0$.Second, observe that the isoclines meet at equilibrium points (or stationary solutions).Third, isoclines divide the state space into four areas, in which the solution is not stationary. Draw the vector field. The vector field indicates the type of variation. Fourth, by considering the directions in the vector field find the stable manifold W^s. In our case the stable manifold converges asymptotically to $c(t) - \bar{c} = \lambda_u(k(t) - \bar{k})$. Fifth, and most importantly, interpret the results. In this case, as $\theta = \alpha$ we know the closed form solution. In the main text, we proved that $c(k) = \beta k(t)$.

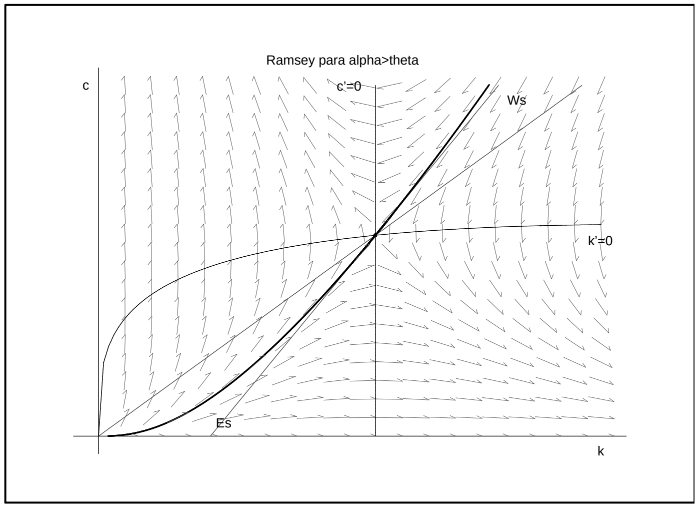

Figure 5.2: Phase diagram for the Ramsey model with $\alpha > \sigma$. In this case, there is no closed form solution, but the solution still converges to $c(t) - \bar{c} = \lambda_u(k(t) - \bar{k})$. As the utility function is not very concave, consumers do not dislike big changes in consumption, and therefore they chose a path in which they save more in the beginning.

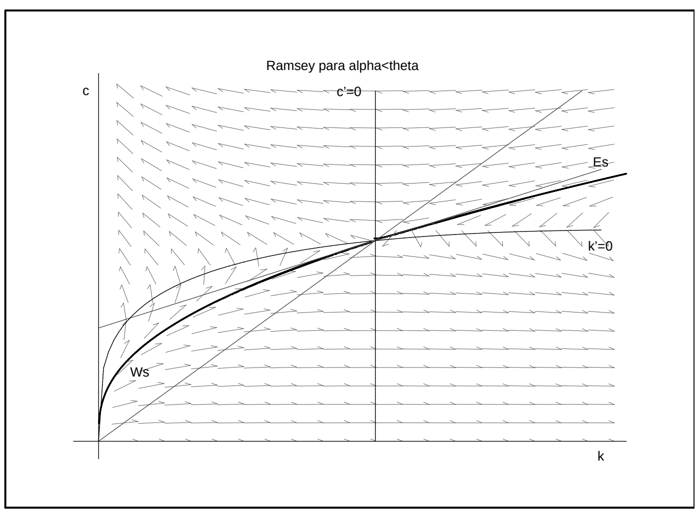

Figure 5.3: Phase diagram for the Ramsey model with $\alpha < \sigma$. Exercise: explain the differences as regards the other parameter configurations.

Conclusions

As the convergence results hold for both the general and the particular cases, we conclude that:

- there is no long term growth, the per capita variables converge to a stationary equilibrium point, $\gamma = 0$ and $\bar{Y}(t) = \bar{y}$;
- but there are level effects: $\bar{y} = \bar{y}(A, \rho, \delta)$ increases with positive variations in A and reduces with ρ and δ;
- the convergence speed is given by the negative eigenvector λ_s,
- the transitional dynamics involves both expectational dynamics, which is essentially unstable, and the effects of the decreasing marginal returns to capital as a force for stability. That is, convergence to a stationary equilibrium point occurs along the saddle path. It has to be followed for the transversality condition to hold;
- the golden rule level of capital is

$$k_g = \text{argmax}\left(Af(k) - \delta k\right)$$

It verifies the condition $r(k_g) = 0$. As we have $r(\bar{k}) = \rho > 0$, because of the concavity of the production function $\bar{k} < k_g$ which means there is dynamic efficiency.

Problems with the Ramsey-Cass-Koopmans model

- it does not present a theory involving positives rates of long term growth : we have $\gamma = 0$,
- we could repair it by introducing exogenous technical progress, in order to generate a long term growth path;
- but by doing that we would merely describe growth, we would not offer an explanation for it ;
- the theory(ies) of endogenous economic growth try to offer explanations for the positive long run growth rate.

Appendix

5.A Solution for the case $\theta = \alpha$

Consider the MHD system (5.1)-(5.2). If we define the variables

$$z(t) \equiv \frac{k(t)}{c(t)}, r(t) = \alpha A k(t)^{\alpha-1}$$

the system is transformed into

$$\dot{z} = (z(t) - \beta) z(t) \tag{5.6}$$

$$\dot{r} = \left((1-\alpha)(\delta + z(t)) - \frac{1-\alpha}{\alpha} r(t) \right) r(t) \tag{5.7}$$

where $\beta = \dfrac{\rho + \delta(1-\alpha)}{\alpha} > 0$ and the transversality condition becomes

$$\lim_{t\to\infty} \alpha A r(t)^{-1} z(t)^{-\alpha} e^{-\rho t} = 0.$$

Solving equation (5.6) (note that it is a Bernoulli ODE) we get

$$z(t) = \frac{\beta z(0)}{z(0) + (\beta - z(0))e^{\beta t}}$$

The transversality condition holds if $r(t)$ or $z(t)^{\alpha} e^{\rho t}$ becomes unbounded (so that the inverse tends to zero. As the solution of $r(t)$ depends on the solution for $z(t)$ and because $z(0) = c(0)/k(0)$ where $k(0) = k_0$ is known, evaluate the second expression.

As

$$z(t)^{-\alpha} e^{-\rho t} = \left(z(0) e^{-\dfrac{\rho}{\alpha} t} + \beta - z(0) \right) e^{\dfrac{\delta(1-\alpha)}{\alpha} t}{}^{\alpha}$$

then

$$\lim_{t \to \infty} z(t)^{-\alpha} e^{-\rho t} = 0$$

if an only if $z(0) = \beta$ which implies that $z(t) = \beta$ is a constant.

Substituting in equation (5.7) we have

$$\dot{r} = \frac{1-\alpha}{\alpha} \left(\bar{r} - r(t) \right) r(t)$$

where $\bar{r} = \rho + \delta > 0$. This equation has solution (it is again a Bernoulli ODE)

$$r(t) = \left(\frac{\bar{r} r(0)}{r(0) + (\bar{r} - r(0)) e^{-\dfrac{(1-\alpha)\bar{r}}{\alpha} t}} \right)^{-1}.$$

It is easy to see that

$$\lim_{t \to r(t)} = \bar{r} = \rho + \delta$$

thus verifying the conjecture we made as the verification of the transversality condition. The initial value for the capital rate of return depends on the initial value for the stock of capital $r(0) = \alpha A k_0^{1-\alpha}$.

Chapter 6

Simplest endogenous growth models

We consider two simple models of endogenous economic growth, that highlight the main mechanism and conceptual approach of the so-called endogenous growth theory.

We present two models in which the growth is generated by capital accumulation:

- the AK model (Rebelo (1991)): the production function depends only on the inputs which are directly used by firms and is linear homogenous. If we have a decentralized economy the equilibrium is Pareto optimal;
- the (k, K) model (Romer (1986)): the production function depends not only on the inputs which are directly used by firms but also on the aggregate inputs (externalities) and is linear homogenous in the total inputs. In this case, if we have a decentralized economy the equilibrium may not be Pareto optimal;

In both cases the aggregate production function displays constant returns to scale as regards both produced inputs (capital and labor).

6.1 The AK model

Assumptions:

- with the exception of the technology the economy is analogous to the economy in the Ramsey (1928) model ;
- the production function (in per capita variables) is linear

$$Y = AK$$

observe that the marginal and the average rates of productivity of capital are equal:

$$r = \frac{\partial Y}{\partial K} = \frac{Y}{K} = A$$

- let us assume that $0 \leq A - \delta - \rho < \theta(A - \delta)$

Formally, the model is

$$\max_{[C(t)]_{t\geq 0}} \int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\begin{aligned}
\dot{K} &= AK(t) - C(t) - \delta K, \\
K(0) &= k_0, \text{ given, } \text{ t} = 0 \\
\lim_{t\to+\infty} e^{-At} K(t) &\geq 0.
\end{aligned}$$

With the former assumptions, we verify that $\partial \dot{K}/\partial K = A - \delta > 0$, which may imply that the transversality condition does not hold.

The current-value Hamiltonian is

$$H = \frac{C^{1-\theta} - 1}{1-\theta} + Q((A - \delta)K - C)$$

and the first order conditions (which are necessary and sufficient) according to the Pontryiagin's principle are

$$
\begin{aligned}
C^{-\theta} &= Q && (6.1)\\
\dot{Q} &= Q(\rho+\delta-A) && (6.2)\\
0 &= \lim_{t\to\infty} Q(t)K(t)e^{-\rho t} && (6.3)
\end{aligned}
$$

for admissible trajectories, i.e. verifying

$$
\begin{aligned}
\dot{K} &= AK - C - \delta K, && (6.4)\\
K(0) &= k_0, \text{ given, } \mathrm{t}=0. && (6.5)
\end{aligned}
$$

If we time-differentiate equation (6.1) and substitute equation (6.2), we get the equivalent version of the equilibrium representation in levels

$$
\begin{aligned}
\dot{C} &= C(A-\rho-\delta)/\theta && (6.6)\\
\dot{K} &= AK - C - \delta K, && (6.7)\\
0 &= \lim_{t\to\infty} C(t)^{-\theta}K(t)e^{-\rho t} && (6.8)
\end{aligned}
$$

An equilibrium (or the optimal solution for the central planner) is a trajectory $[(C(t), K(t))]_{t\geq 0}$ which is the solution of equations (6.6)-(6.8), for a given $K(0) = k_0$.

Let us assume that the variables have the representation:

$$K(t) = k(t)e^{\gamma_k t}, \quad C(t) = c(t)e^{\gamma_c t},$$

where C and K are the level variables, c and k are the trendless variables and γ_c and γ_k are the long run rates of growth.

Now, we want to determine an equivalent representation of the system (6.6)-(6.7) in trendless variables.

A necessary condition for this transformation is that the growth rates are related as

$$\gamma = \gamma_k = \gamma_c. \tag{6.9}$$

This is a necessary condition for transforming the differential equation (6.7) in an autonomous equation for k.

A balanced growth path is an equilibrium such that

$$\bar{K}(t) = \bar{k}e^{\gamma t}, \;\; \bar{C}(t) = \bar{c}e^{\gamma t}.$$

where $\bar{k}$, $\bar{c}$ and γ are determined from the fixed point of system (6.6)-(6.7).

The equivalent system, in trendless variables and with condition (6.9) is

$$\begin{aligned}
\dot{c} &= -\frac{c}{\theta}(\rho + \delta - A + \gamma\theta) \\
\dot{k} &= (A - \delta - \gamma)k - c && (6.10) \\
k(0) &= k_0 && (6.11) \\
0 &= \lim_{t\to+\infty} e^{-(\rho+\gamma(\theta-1))t}k(t)c(t)^{-\theta} && (6.12)
\end{aligned}$$

The fixed point of the system, allows for the determination of the long run (endogenous) growth rate and for the levels of the variables which are consistent with the BGP

$$\begin{aligned}
\bar{\gamma} &= \frac{A - \delta - \rho}{\theta} > 0 \\
\frac{\bar{c}}{\bar{k}} &= \beta, && (6.13)
\end{aligned}$$

where

$$\beta \equiv A - \delta - \bar{\gamma} = \frac{1}{\theta}\left((A - \delta)(\theta - 1) + \rho\right) > 0$$

from the previous assumptions.

If we substitute the endogenous growth rate, we get

$$\dot{c} = 0 \tag{6.14}$$

$$\dot{k} = \beta k - c \tag{6.15}$$

$$0 = \lim_{t\to+\infty} e^{-\beta t} k(t)c(t)^{-\theta} \tag{6.16}$$

because

$$\lim_{t\to+\infty} e^{-(\rho+\bar{\gamma}(\theta-1))t} k(t)c(t)^{-\theta} = \lim_{t\to+\infty} e^{-\beta t} k(t)c(t)^{-\theta}$$

which gives us the equilibrium behavior of the economy when it is not located along the BGP.

In this case, we can determine explicitly the solution, by solving the linear system (6.14)-(6.15) and by using both the initial and the transversality conditions.

From equation (6.14) we get $c(t) = B$ which is an arbitrary constant. If we substitute it into (6.15), and if we use the initial condition, we get the solution for the capital stock

$$k(t) = \left(k_0 - \frac{B}{\beta}\right) e^{\beta t} + \frac{B}{\beta}.$$

The arbitrary constant, B, can be determined from the transversality condition

$$\begin{aligned}
\lim_{t\to+\infty} e^{-\beta t} k(t)c(t)^{-\theta} &= \lim_{t\to+\infty} e^{-\beta t}\left[\left(k_0 - \frac{B}{\beta}\right) e^{\beta t} + \frac{B}{\beta}\right] B^{-\theta} = \\
&= \lim_{t\to+\infty} \left[k_0 - \frac{B}{\beta} + \frac{B}{\beta} e^{-\beta t}\right] B^{-\theta} = \\
&= \lim_{t\to+\infty} \left[k_0 - \frac{B}{\beta}\right] B^{-\theta} = \\
&= 0
\end{aligned}$$

if $B = \beta k_0$. Then we have $c(t) = \beta k_0$ and $k(t) = k_0$, which implies that there is no transition dynamics. That is, the solution of the of the problem is coincident to the BGP where

$$C(t) = \bar{C}(t) = \beta k_0 e^{\gamma t}, \quad K(t) = \bar{K}(t) = k_0 e^{\gamma t}$$

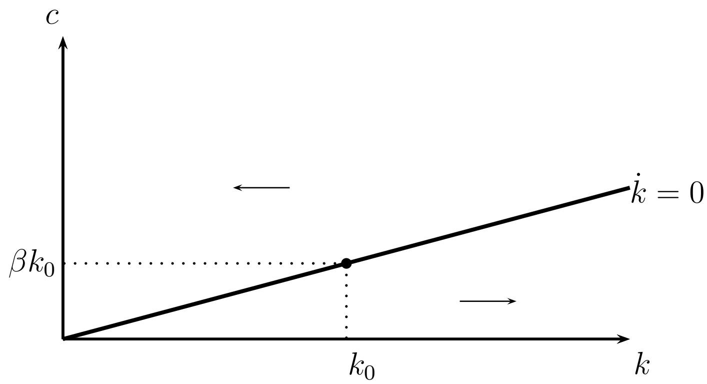

Figure 6.1: AK-model: phase diagram

and

$$Y(t) = \bar{Y}(t) = Ak_0 e^{\gamma t}$$

Figure 6.1 presents the phase diagram. Observe that as $\dot{c} = 0$ for any values of c and k then consumption will be constant, and as $\partial \dot{k}/\partial k = \beta > 0$ the capital stock will increase without bound for small deviations of the steady state $\bar{c} = \beta \bar{k}$. Then the transversality condition only holds if the system would stay permanently at the steady state, which as k_0 is given it should be $(\bar{k}, \bar{c}) = (k_0, \beta k_0)$. Then tehre are no transitional dynamics.

Conclusions

- the long run rate of growth is positive. It varies positively with the total factor productivity A: $\bar{\gamma} = \frac{A-\delta-\rho}{\theta} > 0$;
- the long run level of the product is also a positive function of productivity; $\bar{y}(t) = Ak_0$
- there is no transitional dynamics (which is counterfactual) $\lambda = 0$

References: Aghion and Howitt (2009):ch. 2, Acemoglu (2009):ch. 11

6.2 Externalities: the (k, K) Romer's model

Main idea:

- The productivity of capital, for any firm, depends on the environment in which the firm operates. A way to model this is by assuming that there are externalities which may be positive (a firm's productivity is a positive function of the number of firms which are around) or negative (congestion);
- in a centralized economy the central planner may internalize the externalities, but in a decentralized economy, the individual firms have no a priori incentives for internalizing the externalities.

Assumptions:

- preferences, the number or goods and markets are as in the previous models;
- technology is represented by a production function

  $$Y = F(K, \mathbf{K})$$

  where K is the own capital stock of the firm and $\mathbf{K}$ is the aggregate capital stock of the economy. There is decreasing marginal productivity for the own capital $\partial F_1/\partial K \geq 0$, and $\partial^2 F_1/\partial K^2 < 0$. Let us assume the production function is

  $$Y = AK^{\alpha}\mathbf{K}^{\beta},\ 0 < \alpha < 1,\ \beta > 0$$

  if $\alpha + \beta = 1$ though there is decreasing returns to scale as regards the firm's capital, there are constant returns to scale for the total capital which determines production by the firm (own + aggregate);

- let us first assume that the economy is centralized and next that the economy is decentralized.

Centralized economy

In this economy the central planner internalizes the externality and considers the total capital as a state variable, such that $\mathbf{K} = K$. Then the central planner solves the problem

$$\max_{[C(t)]_{t\geq 0}} \int_0^{\infty} \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{K} = AK(t)^{\alpha+\beta} - C(t) - \delta K(t),$$

with the initial and terminal conditions as in the previous models.

It is easy to see that a necessary condition for the existence of a BGP is that $\alpha + \beta = 1$. In this case we have the AK model which was studied in the last section.

Decentralized economy

In this economy there is no central planner and all the coordination of the agents actions is done through the market. Therefore, externalities are not internalized.

Let us assume that agents are homogeneous and that they solve jointly their consumption and production problems.

Definition: dynamic general equilibrium: It consists in the trajectories $[(C(t), K(t))]_{t\in[0,\infty)}$ such that

- the representative agent solves his problem

$$\max_{[C(t)]_{t\geq 0}} \int_0^{\infty} \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

  subject to

$$\dot{K} = AK^{\alpha}\mathbf{K}^{\beta} - C - \delta K,$$

  given the aggregate stock of capital $[\mathbf{K}(t)]_{t\in[0,\infty)}$

- the micro-macro consistency condition holds (the aggregate stock of capital is consistent with the agents decisions, at the macroeconomic level)

$$\mathbf{K}(t) = K(t).$$

The last condition means that though the individual firm takes the aggregate capital as given the aggregate capital is determined jointly by the behavior of all firms. As we assume firms are homogeneous then indeed $\mathbf{K}(t) = NK(t)$ where N is the number of firms which is exogenous, and therefore does not alter the results of the model

The first order conditions for the representative agent optimum, together with the micro-macro consistency condition, are

$$\dot{C} = \frac{C}{\theta}\left(\rho + \delta - \alpha A K^{\alpha-1}\mathbf{K}^{\beta}\right), \tag{6.17}$$

$$\dot{K} = AK^{\alpha}\mathbf{K}^{\beta} - C - \delta K, \tag{6.18}$$

$$K(0) = k(0) \tag{6.19}$$

$$0 = \lim_{t\to+\infty} C(t)^{-\theta}K(t)e^{-\rho t} \tag{6.20}$$

$$K(t) = \mathbf{K}(t). \tag{6.21}$$

The solution of this system generates the general equilibrium paths $[C(t), K(t)]_{t\in[0,\infty)}$.

Existence of a BGP: a necessary condition for the existence of a BGP is that $\alpha + \beta = 1$.

If we represent, again, the common rate of long run growth by γ, and set $Y(t) = y(t)e^{\gamma t}$, $C(t) = c(t)e^{\gamma t}$ and $K(t) = k(t)e^{\gamma t}$, then we get the ode in deviations as regards the BGP as

$$\begin{aligned}
\dot{c} &= -\frac{c}{\theta}(\rho + \delta - \alpha A + \gamma\theta) \\
\dot{k} &= (A - \delta - \gamma)k - c \\
k(0) &= k_0 \\
0 &= \lim_{t \to +\infty} e^{-(\rho+\gamma(\theta-1))t} k(t)c(t)^{-\theta}
\end{aligned}$$

In this case the long run growth rate becomes

$$\bar{\gamma}_d = \frac{\alpha A - \delta - \rho}{\theta} > 0, \tag{6.22}$$

$$\frac{\bar{c}}{\bar{k}} = \beta_d, \tag{6.23}$$

where

$$\beta_d \equiv A - \delta - \bar{\gamma}_d = \frac{1}{\theta}\left(A(\theta - \alpha) + \rho + \delta(1 - \theta)\right) > 0$$

Let us assume that $\rho + \gamma_d(\theta - 1) > 0$ in order to the transversality condition be verified .

If we compare with the centralized case (or the equivalent AK case), we have

$$\gamma_d < \gamma, \quad \beta_d > \beta$$

that is: the long run growth rate in the decentralised economy is smaller than in the centralised economy and the weight of consumption on total income along the BGP is higher.

Why ? Because agents only optimize as regards the private capital. There are no incentives for internalizing the externality.

If we use the same methodology as in the previous section we get

$$C(t) = \bar{C}(t) = \beta_d k_0 e^{\gamma_d t}, \quad K(t) = \bar{K}(t) = k_0 e^{\gamma_d t}$$

and

$$Y(t) = \bar{Y}(t) = Ak_0 e^{\gamma_d t}$$

Conclusions

- a necessary condition for the existence of endogenous growth is that $\alpha + \beta = 1$; that is: (1) there is endogenous growth if production displays constant return to scale as regards the total capital (private plus public); (2) we may have endogenous growth even in the case in which there are decreasing marginal returns for the private capital stock used in production;
- if there are non-internalized externalities then the rate of growth of the economy will be smaller and ratio of consumption to GDP higher, along the BGP, than in the case in which externalities are fully internalized.
- as in the AK model, in this simple version of the model, we still do not have transitional dynamics.
- This means that there are gains in terms of economic growth if there is an economic policy which allows for an internalization of externalities.

References: Acemoglu (2009):ch. 11

Decentralized economy: internalization of externalities

Assume that the government introduces the following fiscal policy:

- it introduces a distortionary component
$$T(t) = \tau Y(t)$$
which is a taxation if $0 \le \tau \le 1$ or a subsidy if $\tau < 0$
- and a lump sum component $G(t)$, which is a transfer if $G > 0$ or tax if $G < 0$
- and keeps a balanced budget
$$G(t) - T(t) = 0,\ t \in [0, \infty)$$

The budget constraint for the agent becomes

$$\dot{K} = (1-\tau)AK^{\alpha}\mathbf{K}^{\beta} - C + G - \delta K,$$

In this case, the general equilibrium paths for C and K are generated by equation

$$\dot{C} = \frac{C}{\theta}\left(\rho + \delta - (1-\tau)\alpha AK^{\alpha-1}\mathbf{K}^{\beta}\right), \tag{6.24}$$

and by equations (6.18)-(6.21).

The growth rate is

$$\bar{\gamma}_t = \frac{\alpha(1-\tau)A - \delta - \rho}{\theta} > 0, \tag{6.25}$$

instead of the growth rate in equation (6.22).

We can conclude that the equilibrium growth rate in this economy can be made equal to the growth rate of the centralized economy, $\bar{\gamma}_t = \bar{\gamma}$ if

$$\alpha(1-\tau) = 1$$

that is

$$\tau = -\frac{1-\alpha}{\alpha} < 0.$$

Therefore, the GE with distortions can allow for a Pareto efficient internalization of the externality if there is a distortionary subsidy (i.e, dependent on the level of output) financed by a lump sum taxation.

Chapter 7

Two-sector endogenous growth models

Main ideas:

- there are two sectors: one sector produces a good which can be used both for final consumption or investment and another sector which produces an investment good;
- in the case of the Uzawa (1964) - Lucas (1988) model the second sector produces human capital (can be the education sector and/or the professional training sector);
- the stylized fact according to which there is a permanent increase in the wage rate and the permanent increase in the labor productivity, for populations which increase less than the rate of growth of the economies, seems to lend support to the idea that human capital increase is a major source of long run growth;
- there are several versions of the model, with different technologies and institutional structures:
  - technology:
    the sector producing final goods, or manufacturing sector, uses two factors of production;
    the education sector uses only human capital as an input, or uses both factors

  – externalities: there are versions of the model with or without externalities

  – characteristics of the equilibrium:

    - when there are no externalities: the general equilibrium is Pareto optimal

    - when there are externalities: the externalities may be internalized or not. In the second case, which is the more interesting case, we may have long run growth with potentially smaller growth rates.

We will present next, in an abridge way, two versions of the model :

- Lucas (1988) model without externalities;

- Lucas (1988) model with externalities;;

7.1 The Uzawa-Lucas model without externalities

Assumptions:

- the preference structure is analogous to the previous models;

- the education sector uses only human capital as an input and the manufacturing sector uses both factors (physical capital and labor) ;

- there are no externalities;

- both sectors have production functions displaying constant returns to scale.

7.1.1 Original presentation: Uzawa (1964) and Lucas (1988)

As the equilibrium is Pareto optimal, the centralized and the decentralized versions are equivalent.

The centralized version of the model consists in the following dynamic problem for the central planner: choose that paths $\{(C(t), u(t), K(t), H(t),\ t \in [0, \infty)\}$ such that

$$\max_{C,u} \int_0^{\infty} \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

subject to,

$$\begin{aligned}
\dot{K} &= AK(t)^{\alpha}(u(t)H(t))^{1-\alpha} - C(t) \\
\dot{H} &= B(1 - u(t))H(t)
\end{aligned}$$

where, $0 \leq u(t) \leq 1$, plus the initial and the transversality conditions, where $0 < \alpha < 1$, u is the part of human capital which is used in the manufacturing sector, H is the stock of human capital, K is the stock of physical capital, and A and B are productivity parameters for both sectors. The initial stocks for physical and human capital are known: $K(0) = k_0$, and $H(0) = h_0$.

7.1.2 An alternative formulation: a two sector endogenous growth model

There is an alternative formulation of the problem that simplifies the algebra and makes the model more general.

Let Y_j, K_j and H_j is the production and the amount of physical capital and human capital used as inputs in sector $j = 1, 2$.

$$\max_{C,K_1,H_1,H_2} \int_0^{\infty} \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

subject to,

$$\begin{aligned}
\dot{K} &= Y_1(t) - C(t) \\
\dot{H} &= Y_2(t) \\
K(t) &= K_1(t) \\
H(t) &= H_1(t) + H_2(t)
\end{aligned}$$

where the production functions are

$$\begin{aligned} Y_1(t) &= AK_1(t)^{\alpha}H_1(t)^{1-\alpha} \\ Y_2(t) &= BH_2(t) \end{aligned}$$

We also assume that $\rho^* \equiv \rho + B(\theta - 1) > 0$.

As the utility function is homogeneous (check) and the production functions are linearly homogeneous, a necessary condition for the existence of a Balanced Growth path is that a decomposition as

$$K_j(t) = k_j(t)e^{\gamma t},\ H_j(t) = h_j(t)e^{\gamma t},$$

can be made.

Exercise: prove that a necessary condition for the existence of a BGP is that the rates of growth of all the stocks of human and physical capital are equal.

The model can be expressed in detrended variables as

$$\max_{c,k_1,h_1,h_2} \int_0^{\infty} \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-(\rho - \gamma(1-\theta))t} dt$$

subject to the accumulation equations

$$\begin{aligned} \dot{k} &= y_1(t) - c(t) - \gamma k(t) \\ \dot{h} &= y_2(t) - \gamma h(t), \end{aligned}$$

where

$$\begin{aligned} y_1(t) &= Ak_1(t)^{\alpha}h_1(t)^{1-\alpha} \\ y_2(t) &= Bh_2(t), \end{aligned}$$

and the following static constraints hold

$$k(t) = k_1(t) \quad (7.1)$$

$$h(t) = h_1(t) + h_2(t) \quad (7.2)$$

where all the quantity variables are detrended by the same rate of growth γ.

The current-value Hamiltonian is

$$\mathcal{H} = \frac{c^{1-\theta}-1}{1-\theta} + p_k\left(Ak_1^{\alpha}h_1^{1-\alpha} - c - \gamma k\right) + p_h\left(Bh_2 - \gamma h\right) + R(k-k_1) + W(h-h_1-h_2)$$

where the Lagrange multipliers, R and W, are related to the adding-up restrictions and can be interpreted as factor prices for capital and labour. This is an optimal control problem with four control variables (c, k_1, h_1, h_2), two state variables (k, h) and two static constraints, equations (7.1)-(7.2).

The first order conditions, for an interior optimum, are:

1. for the optimal consumption

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \;\Leftrightarrow\; c^{-\theta} = p_k$$

2. for the allocation of the stocks of human and physical capital between the two sectors, where $r \equiv R/p_k$ and $w = W/p_h$,

$$\frac{\partial \mathcal{H}}{\partial h_1} = 0 \;\Leftrightarrow\; (1-\alpha)p_k y_1 = w p_h h_1$$
$$\frac{\partial \mathcal{H}}{\partial h_2} = 0 \;\Leftrightarrow\; w = B$$
$$\frac{\partial \mathcal{H}}{\partial k_1} = 0 \;\Leftrightarrow\; \alpha y_1 = r k_1$$

3. the conditions regarding the "Lagrange multipliers" R and W

$$\frac{\partial \mathcal{H}}{\partial R} = 0 \;\Leftrightarrow\; k = k_1$$
$$\frac{\partial \mathcal{H}}{\partial W} = 0 \;\Leftrightarrow\; h = h_1 + h_2$$

4. the Euler conditions

$$\begin{aligned}
\dot{p}_k &= = p_k\left(\rho + \gamma(\theta - 1)\right) - \frac{\partial \mathcal{H}}{\partial k} = p_k(\rho + \theta\gamma - r)\\
\dot{p}_h &= = p_h\left(\rho + \gamma(\theta - 1)\right) - \frac{\partial \mathcal{H}}{\partial h} = p_h(\rho + \theta\gamma - B)
\end{aligned}$$

5. the conditions for admissibility

$$\begin{aligned}
\dot{k} &= Ak_1^{\alpha}h_1^{1-\alpha} - c - \gamma k\\
\dot{h} &= Bh_2 - \gamma h
\end{aligned}$$

6. the transversality conditions

$$\lim_{t\to\infty} e^{-(\rho-\gamma(1-\theta))t} p_k(t)k(t) = \lim_{t\to\infty} e^{-(\rho-\gamma(1-\theta))t} p_h(t)h(t) = 0.$$

are associated to the terminal conditions for the two state variables.

If we solve the static equations for the allocations of the inputs and for the factor prices, k_1, h_1, h_2, r and w, as functions of the variables which are explicitly dynamic, p_k, p_h, k and h, we get

$$\begin{aligned}
r &= r(\pi) \equiv \left(\alpha_0 A B^{\alpha-1}\pi^{1-\alpha}\right)^{1/\alpha},\ \alpha_0 \equiv \alpha^{\alpha}(1-\alpha)^{1-\alpha}\\
w &= B\\
k_1 &= k\\
h_1 &= \left(\frac{r}{\alpha A}\right)^{1/(1-\alpha)} k\\
h_2 &= h - \left(\frac{r}{\alpha A}\right)^{1/(1-\alpha)} k
\end{aligned}$$

where we define

$$\pi = \frac{p_k}{p_h}.$$

If we substitute in the production function of both sectors, we see that they are linear functions of the aggregate capital stocks

$$\begin{aligned} y_1 &= a_{11}(\pi)k, \text{ where } a_{11} = r(\pi)/\alpha > 0 \\ y_2 &= a_{21}(\pi)k + Bh, \end{aligned}$$

where

$$a_{21} \equiv -B \left(\frac{r(\pi)}{\alpha A}\right)^{1/(1-\alpha)} < 0.$$

At the optimum there is a linear input output structure

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a_{11}(\pi(t)) & 0 \\ a_{21}(\pi(t)) & B \end{pmatrix} \begin{pmatrix} k(t) \\ h(t) \end{pmatrix}$$

where the coefficients regarding the allocation of capital between the two sectors are determined by the change in the relative prices.

The maximized Hamiltonian detrended dynamic system (MHDDS) is

$$\dot{p}_k = p_k(\rho + \theta\gamma - r(p_k/p_h)) \tag{7.3}$$

$$\dot{p}_h = p_k(\rho + \theta\gamma - B) \tag{7.4}$$

$$\dot{k} = (a_{11}(r(p_k/p_h)) - \gamma)\, k - c(p_k) \tag{7.5}$$

$$\dot{h} = a_{21}(r(p_k/p_h))k - (B - \gamma)h. \tag{7.6}$$

7.1.3 The balanced growth path (BGP)

Along the BGP the stocks of physical and human capital will be

$$\bar{K}(t) = \bar{k}e^{\gamma t},\ \bar{H}(t) = \bar{h}e^{\gamma t}.$$

We determine the levels $\bar{k}$ and $\bar{h}$, the rate of growth γ and the levels for the co-state variables $\bar{p}_k$ and $\bar{p}_h$ by solving the MHDDS for the steady state

Evaluating at the steady state, we find, from equations $\dot{p}_k = \dot{p}_k = 0$ that:

- the long run growth rate is
$$\bar{\gamma} = \frac{B-\rho}{\theta}$$
and, therefore, it is a positive function of the productivity of the educational sector;

- long-run arbitrage conditions hold
$$\bar{r} = \bar{w} = B.$$

- the long run relative price is
$$\bar{\pi} \equiv \frac{\bar{p}_1}{\bar{p}_2} = \left(\frac{\alpha_0 A}{B}\right)^{\dfrac{1}{1-\alpha}}$$

- there is a linear relationship between the stock of physical and human capital
$$\bar{k} = \eta \bar{h}$$
where
$$\eta \equiv -\frac{B-\bar{\gamma}}{\bar{a}_{21}}$$
for
$$\bar{a}_{21} = -B\left(\frac{B}{\alpha A}\right)^{\dfrac{1}{1-\alpha}} = -\frac{B}{\bar{\pi}}\left(\frac{\alpha_0}{\alpha}\right)^{\dfrac{1}{1-\alpha}}$$

- that ratio is positive if the transversality condition holds
$$\rho + \bar{\gamma}(\theta-1) = \frac{B(\theta-1)+\rho}{\theta} = B - \bar{\gamma} > 0$$
which holds if $\rho + B(\theta - 1) > 0$; from now on we let
$$\mu \equiv B - \bar{\gamma} > 0,$$

- at last, we can obtain $\bar{p}_k$ from
$$c(p_k) = \beta\,\bar{k}.$$
where
$$\beta \equiv a_{11}(\bar{\pi}) - \bar{\gamma} = \frac{B(\theta-\alpha)+\rho}{\theta} > 0.$$

If we substitute $\gamma = \bar{\gamma}$ we obtain the MHDDS in the neighborhood of a balanced growth path

$$\begin{aligned}
\dot{p}_k &= p_k(B - r(p_k/p_h)) && (7.7)\\
\dot{p}_h &= 0 && (7.8)\\
\dot{k} &= \left(a_{11}(r(p_k/p_h)) - \bar{\gamma}\right)k - c(p_k) && (7.9)\\
\dot{h} &= a_{21}(r(p_k/p_h))k - (B-\bar{\gamma})h && (7.10)
\end{aligned}$$

which represents the transitional dynamics together with the initial and the transversality conditions.

Taking the previous relationships between the parameters, the transversality conditions become
$$\lim_{t\to\infty} e^{-\mu t} p_k(t)k(t) = \lim_{t\to\infty} e^{-\mu t} p_h(t)h(t) = 0,$$
where $\mu > 0$.

If we can find a solution to the MHDDS (7.7)-(7.10) the solution to the Uzawa-Lucas model will be
$$K(t) = k(t)e^{\bar{\gamma}t},\ H(t) = h(t)e^{\bar{\gamma}t}$$
where $p_k(t)$ and $p_h(t)$ will have no trend. That solution should be co-incident or converge to the BGP model will be
$$\bar{K}(t) = \bar{k}e^{\bar{\gamma}t},\ \bar{H}(t) = \bar{h}e^{\bar{\gamma}t}$$
and $p_k(t) = \bar{p}_k$ and $p_h(t) = \bar{p}_h$.

It is clear that there is an indeterminacy related to the levels of the state variables $(\bar{k}, \bar{h})$: at the BGP they should verify $\bar{k} = \eta\bar{h}$. If we fix $\bar{h} = h_0$ and $k_0 = \eta h_0$ then the economy will be in a BGP and the physical and human capital will be given by

$$K(t) = \bar{K}(t) = \eta h_0 e^{\bar{\gamma}t},\ H(t) = \bar{H}(t) h_0 e^{\bar{\gamma}t}.$$

However, if $k_0 \neq \eta h_0$ then the economy will not be in the BGP and there would be both long run and transitional dynamics, with reallocations of capital driven by the dynamics of the relative prices π.

In the AK model, which has only one state variable (k) we were able to resolve this by finding a closed form solution to the MHDS and using the initial condition and the transversality condition to find $K(t) = k_0 e^{\bar{\gamma}t}$. Therefore: the indeterminacy is only apparent and the detrended level of capital (but not the growth rate) displays path dependency because it depended on the initial level of capital $k(t) = k_0$.

Unfortunately, the MHDDS (7.7)-(7.10), which is non-linear in the co-state variables p_k and p_h, does not seem to have a closed form solution. Therefore, we have to resort to approximate (linearization) methods to study transitional dynamics in the neighborhood of the BGP.

7.1.4 Transitional dynamics

Using methods similar to the ones used in previous models, next we we may also conclude that:

- there is transitional dynamics;
- there is no indeterminacy (that is, the dimension of the stable manifold is one);

- there is path dependency regarding the detrended variables.

The variational system for the MHDDS (7.7)-(7.10), in a neighborhood of the BGP, will provide an approximation to $(p_k(t), p_h(t), k(t), h(t))$ in a neighborhood of $(\bar{p}_k, \bar{p}, \bar{k}, \bar{h})$, is a linear four-dimensional system with coefficient matrix given by the Jacobian of (7.7)-(7.10),

$$\begin{pmatrix} \dot{p}_k \\ \dot{p}_h \\ \dot{k} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{p}_k}{\partial p_k} & \frac{\partial \dot{p}_k}{\partial p_h} & \frac{\partial \dot{p}_k}{\partial k} & \frac{\partial \dot{p}_k}{\partial h} \\ \frac{\partial \dot{p}_h}{\partial p_k} & \frac{\partial \dot{p}_h}{\partial p_h} & \frac{\partial \dot{p}_h}{\partial k} & \frac{\partial \dot{p}_h}{\partial h} \\ \frac{\partial \dot{k}}{\partial p_k} & \frac{\partial \dot{k}}{\partial p_h} & \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial h} \\ \frac{\partial \dot{h}}{\partial p_k} & \frac{\partial \dot{h}}{\partial p_h} & \frac{\partial \dot{h}}{\partial k} & \frac{\partial \dot{h}}{\partial h} \end{pmatrix} \begin{pmatrix} p_k - \bar{p}_k \\ p_h - \bar{p}_h \\ k - \bar{k} \\ h - \bar{h} \end{pmatrix} \tag{7.11}$$

where the Jacobian $\mathbf{J}$, is evaluated at the BGP levels $(\bar{p}_k, \bar{p}_h, \bar{k}, \bar{h})$. We obtain

$$\bar{\mathbf{J}} = \begin{pmatrix} \mu - \beta & \bar{\pi}(\beta - \mu) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta(\theta - \alpha) - \alpha\mu)\bar{k}}{\alpha\theta\bar{p}_k} & -\frac{(\beta - \mu)\bar{k}}{\alpha\bar{p}_h} & \beta & 0 \\ -\frac{\mu\bar{h}}{\alpha\bar{p}_k} & \frac{\mu\bar{h}}{\alpha\bar{p}_h} & -\frac{\mu}{\eta} & \mu \end{pmatrix}$$

where we have used the fact

$$B\left(\frac{1-\alpha}{\alpha}\right) = \beta - \mu > 0.$$

The characteristic polynomial of the Jacobian, $\bar{\mathbf{J}}$, is

$$C(\mathbf{J}, \lambda) = \lambda\left(\lambda - (\mu - \beta)\right)\left(\lambda - \beta\right)\left(\lambda - \mu\right),$$

which implies that the eigenvalues (which are the roots of the characteristic equation $c(J, \lambda) = 0$) are

$$\lambda_1 = \mu - \beta < 0,\ \lambda_2 = 0,\ \lambda_3 = \beta > 0,\ \lambda_4 = \mu > 0$$

Therefore, the stable manifold is of dimension 1 and there is transitional dynamics, that is, there is one negative eigenvalue. Observe that that eigenvalue is equal to the derivative of p_k, evaluate along the BGP

$$\lambda_1 = \frac{\partial \dot{p}_1}{\partial p_k}\bigg|_{BGP} = \mu - \beta = -B\left(\frac{1-\alpha}{\alpha}\right) < 0$$

which means that the stabilizing force is related to the behavior of consumption and of the relative prices $\pi = p_k/p_h$.

We provide in Appendix 7.A the solution to equation (7.11) converging to a BGP. In particular we find

$$h(t) = h_\infty + (h_0 - h_\infty)e^{(\mu-\beta)t} \tag{7.12}$$

where

$$h_\infty = \frac{k_0\mu(\theta-\alpha) + \eta h_0(\beta(\alpha+\theta) - \mu\theta)}{\eta(\beta(\alpha+\theta) - \alpha\mu)}$$

and

$$k(t) = \eta h_\infty + (k_0 - \eta h_\infty)e^{(\mu-\beta)t}. \tag{7.13}$$

For the co-state variables we obtain [1]

$$p_k(t) = \bar{p}_k\left[1 + \theta\left(\frac{\bar{h} - h_\infty}{\bar{h}}\right) + \frac{\theta\alpha(2\beta-\mu)}{\mu(\theta-\alpha)}\left(\frac{h_0 - h_\infty}{\bar{h}}\right)e^{(\mu-\beta)t}\right] \tag{7.14}$$

and

$$p_h(t) = \bar{p}_h\left[1 + \theta\left(\frac{\bar{h} - h_\infty}{\bar{h}}\right)\right]. \tag{7.15}$$

In Figure we depict the phase diagram in the (h, k) space, and the trajectories for $p_k(t)$ the rate of growth of K, $\dfrac{\dot{K}(t)}{K(t)}$ and for the same variables along a BGP, taking arbitrary initial values for (k_0, h_0) such that $k_0 \neq \eta h_0$. We see there is transitional dynamics. If the $k_0 < \eta h_0$ we see that the initial level of the co-state variable $p_k(0)$ is relatively high and $\pi(0) > \bar{\pi}$ is

[1] The non-linearity of the system 7.7-7.10 is clearly displayed in the equations for the co-state variables because they depend on the arbitrary steady state level $\bar{h}$. As we saw $\bar{p}_k$ and $\bar{p}_h$ are functions of $\bar{h}$. We could pick $\bar{h} = h_0$ to approximate the solutions to the co-state variables.

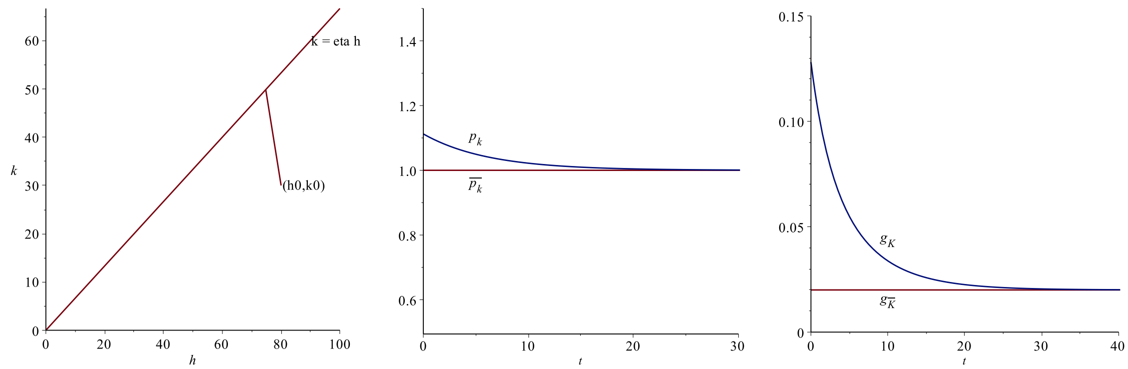

Figure 7.1: Uzawa-Lucas model: phase diagram, trajectories for p_k and for the rate of growth of K and $\bar{K}$. Parameter values: $\rho = 0.02$, $\alpha = 0.3$, $\theta = 2$, $A = 0.2$ and $B = 0.06$.

also relatively high. This tends to increase k relative to h implying a relative increase in the output of manufactures and to an increase in the ratio $k(t)/h(t)$. But this also implies that $r(\pi) > B$ which implies a negative rate of growth of π. This implies that the rate of return $r(\pi)$ converges to B which reduces the incentives for physical capital accumulation until the ratio $k(t)/h(t)$ converges to η.

7.1.5 Endogenous growth dynamics

The trajectories for the manufacturing output is

$$Y_1(t) = y_1(t)e^{\bar{\gamma}t}$$

where

$$y_1(t) = A\left(\eta h_\infty + (k_0 - \eta h_\infty)e^{(\mu-\beta)t}\right)^\alpha \left(h_\infty + (h_0 - h_\infty)e^{(\mu-\beta)t}\right)^{1-\alpha}.$$

Taking the limit to the detrended level we get an approximation $\lim_{t\to\infty} y_1(t) = A\eta^\alpha h_\infty$. Therefore the solution for the GDP in manufacturing converges to the BGP path

$$\bar{Y}_1(t) \approx A\eta^\alpha h_\infty e^{\bar{\gamma}t}.$$

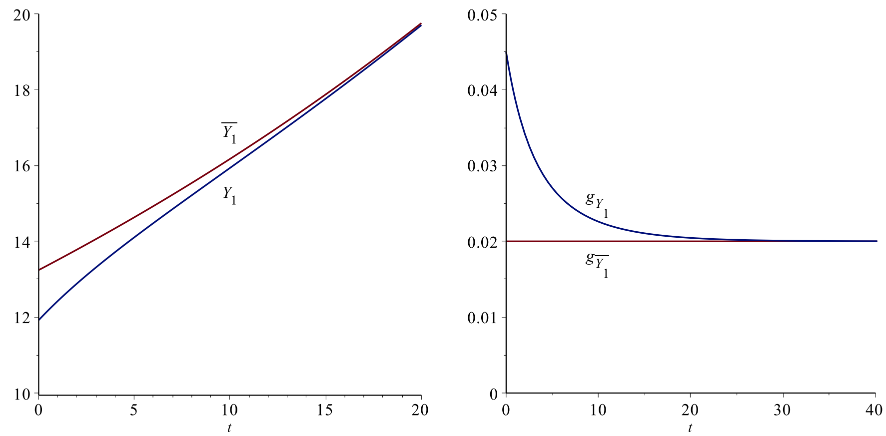

Figure 7.2: Uzawa-Lucas model: trajectories for levels and rates of growth of Y_1 and $\bar{Y}_1$

In Figure we present, in the left panel, $Y_1(t)$ and $\bar{Y}_1(t)$, and, in the right panel, their growth rates.

7.2 Uzawa-Lucas model with externalities

It is a general equilibrium model where:

- agents determine $[(C(t), u(t), K(t), H(t))]_{t\in[0,\infty)}$

  $$\max_{C,u} \int_0^{\infty} \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

  subject to , where $0 \le u(t) \le 1$,

  $$\begin{aligned} \dot{K} &= AK(t)^{\alpha}(u(t)H(t))^{1-\alpha}\mathbf{H}(t)^{\psi} - C(t) \\ \dot{H} &= B(1-u(t))H(t) \end{aligned}$$

  given a trajectory for the aggregate level of the human capital stock $[\mathbf{H}(t)]_{t\in[0,\infty)}$;

- the micro-macro consistency condition holds

  $$\mathbf{H}(t) = H(t)$$

If we solve the model, we may conclude that:

- the rate of long run growth is

$$\gamma = \frac{(1-\alpha)(B-\rho)}{\theta(1-\alpha+\psi)-\beta}$$

- there is transitional dynamics;

- if $\rho < B$ there is no indeterminacy (that is, the dimension of the stable manifold is not higher than the number of state variables);

- if $\rho > B$ we may have:
  - indeterminacy (that is, the dimension of the stable manifold is higher than one). In this case, there is an infinity of trajectories which converge to the BGP, that is, verifying the transversality condition;
  - endogenous fluctuations (i.e., oscillatory convergence towards the BGP) may exist for particular values of θ and ψ.

Exercise: (1) Find the first order conditions, from the Pontriyagin's maximum principle of the agents' problem; (2) determine the dynamic system which represent the dynamic general equilibrium for this economy; (3) determine the system in detrended variables; (4) find the long run growth rate and the level variables along the BGP.

7.3 Recent assessment of the Uzawa-Lucas model

See grossman&helpman&oberfield&sampson2017

7.3.1 Measuring the human capital

Human capital is not easy to measure. Initial empirical measures used school attendance (see barro&lee1993 and barro&lee2013). hanushek&woessmann2015 show that this indicator has a poor correlation to economic growth. They propose instead a measure they call knowledge capital which measures the quality of human capital. In order to do that they use a several existing indicators on the quality of schooling measured by international test scores (like PISA). They find a much higher correlation between rates of growth and initial level of knowledge capital.

Appendix

7.A Solution to system (7.11)

Because we MHDS of the Uzawa-Lucas model is non-linear we obtain the approximate (linearized) solution converging to the BGP

$$\begin{pmatrix} p_k(t) \\ p_h(t) \\ k(t) \\ h(t) \end{pmatrix} = \begin{pmatrix} \bar{p}_k \\ \bar{p}_h \\ \bar{k} \\ \bar{h} \end{pmatrix} + z_1 \mathbf{P}^1 e^{\lambda_1 t} + z_2 \mathbf{P}^2 \tag{7.16}$$

where $\mathbf{P}^1$ and $\mathbf{P}^2$ are the eigenvector of matrix $\bar{\mathbf{J}}$ associated to the eigenvalues λ_1 and λ_2 and z_1 and z_2 are obtained such that the solution verifies $k(0) = k_0$ and $h(0) = h_0$ for any given fixed values k_0 and h_0 for the aggregate physical and human capital.

Recall that the eigenvector $\mathbf{P}^j$ solves the equation

$$(\mathbf{J} - \lambda_j \mathbf{I})\, \mathbf{P}^j = \mathbf{0},\ j \in \{1, 2, 3, 4\}.$$

and we find the arbitrary constants z_1 and z_2 from solving

$$\begin{pmatrix} k_0 - \bar{k} \\ h_0 - \bar{h} \end{pmatrix} = \begin{pmatrix} P_3^1 & P_3^2 \\ P_4^1 & P_4^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The Jacobian has the following structure

$$\mathbf{J} = \begin{pmatrix} j_{11} & j_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ j_{31} & j_{32} & j_{33} & 0 \\ j_{41} & j_{42} & j_{43} & j_{44} \end{pmatrix}$$

where $\bar{\pi} j_{11} + j_{12} = \bar{\pi} j_{41} + j_{42} = \bar{\pi} j_{31} + j_{32} - \dfrac{\beta \bar{k}}{\theta \bar{p}_k} = 0$. We find

$$\mathbf{P}^1 = \begin{pmatrix} \dfrac{(j_{11}-j_{33})(j_{11}-j_{44})}{j_{31}j_{43}+j_{41}(j_{11}-j_{33})} \\ 0 \\ \dfrac{j_{31}(j_{11}-j_{44})}{j_{31}j_{43}+j_{41}(j_{11}-j_{33})} \\ 1 \end{pmatrix}, \text{ and } \mathbf{P}^2 = \begin{pmatrix} -\dfrac{j_{12}j_{44}j_{33}}{j_{43}(j_{11}j_{32}-j_{12}j_{31})} \\ \dfrac{j_{11}j_{44}j_{33}}{j_{33}(j_{11}j_{32}-j_{12}j_{31})} \\ -\dfrac{j_{44}}{j_{33}} \\ 1 \end{pmatrix}.$$

After substituting the values for the coefficients and simplifying we find

$$\mathbf{P}^1 = \begin{pmatrix} \dfrac{\alpha\theta(2\beta-\mu)\bar{p}_k}{\mu(\theta-\alpha)\bar{h}} \\ 0 \\ \dfrac{\eta(\beta(\theta-\alpha)-\mu\theta)}{\mu(\theta-\alpha)} \\ 1 \end{pmatrix}, \text{and } \mathbf{P}^2 = \begin{pmatrix} -\dfrac{\theta\bar{p}_k}{\bar{h}} \\ -\dfrac{\theta\bar{p}_h}{\bar{h}} \\ \eta \\ 1 \end{pmatrix}$$

Then

$$\begin{pmatrix} k_0-\bar{k} \\ h_0-\bar{h} \end{pmatrix} = \begin{pmatrix} \dfrac{\eta(\beta(\theta-\alpha)-\mu\theta)}{\mu(\theta-\alpha)} & \eta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Solving this system we obtain, after simplification, and noting that $\bar{k} = \eta\bar{h}$,

$$z_1 = -\frac{\mu(\theta-\alpha)\,(k_0-\eta h_0)}{\eta\,(\beta(\theta+\alpha)-\mu\alpha)} \tag{7.17}$$

$$z_2 = h_0 - \bar{h} - z_1. \tag{7.18}$$

Substituting in equation (7.16) we find

$$\begin{aligned}
p_k(t) &= \bar{p}_k\left\{1-\theta\left(\frac{h_0-\bar{h}-z_1}{\bar{h}}\right)+z_1\frac{\alpha\theta(2\beta-\mu)}{\mu(\theta-\alpha)\bar{h}}e^{(\mu-\beta)t}\right\}\\
p_h(t) &= \bar{p}_h\left[1-\theta\left(\frac{h_0-\bar{h}-z_1}{\bar{h}}\right)\right]\\
k(t) &= \eta(h_0-z_1)+\eta z_1\frac{\beta(\alpha+\theta)-\mu\theta}{\mu(\alpha-\theta)}e^{(\mu-\beta)t}\\
h(t) &= h_0-z_1+z_1e^{(\mu-\beta)t}.
\end{aligned}$$

Because $\mu-\beta<0$ then $\lim_{t\to\infty}h(t)=h_0-z_1\equiv h_\infty$, we can simplify the solution to obtain (7.12), (7.13), (7.14) and (7.15).

Chapter 8

Technology and endogenous growth

The main reason for stressing the role of technology: the accumulation of physical capital is not enough to explain aggregate growth rates (i.e, the rate of GDP growth per capita is higher than the rate of physical capital accumulation if we consider a given technology).

There are several channels for introducing technology:

- basic research or R&D: technical progress may be the outcome of basic research which is not conducted as a profitable venture or may be the outcome of R&D which is conducted for profit;
- horizontal innovation: technical progress takes the form of an expansion in product variety (seen as consumer or producer goods or both) ;
- vertical innovation: technical progress takes the form of an expansion in product quality (seen as consumer or producer goods or both);
- biased innovation: technical may introduce biases in the use and return of factors (if we see them as heterogeneous in productivity);

- application: technical progress may be directed to final products or to intermediate products.

Technical progress generates several puzzles:

1. ideas are non-rival: once ideas or inventions are made they cannot be ”des-invented”, which implies that the developer may not recover fully the costs of research or R&D;
2. the market for producing R&D tends to be imperfectly competitive: either there are barriers to their adoption or there should be some monopoly power over inventions, otherwise there will not exist incentives to pay the cost to do R&D.
3. this implies there are externalities which are not internalized, implying that the general equilibrium growth path is not Pareto (i.e, the non-rivalry introduces a distortion);
4. who performs R&D ?: it can be the incumbent in an industry, as a device to keep a monopolist position, or can be the entrant as a way to dislodge an incumbent.

In models in which R&D is considered, there are, therefore, several modeling options in the literature:

1. type of R&D: expanding variety or quality ladders' models;
2. application: final goods or intermediate goods;
3. type of entry: horizontal or vertical (by an entrant or an incumbent);
4. technology of R&D: lab-equipment models (R&D uses capital) or knowledge-driven (R&D uses skilled labour) models;

Stylized facts

1. the proportion of R&D expenditures is a constant fraction of GDP, in the long run;

2. R&D expenditures are countercyclical;

3. entry in industries is done by smaller firms.

Alternative models generate different rationale for the existence of endogenous growth.

8.1 Expanding product varieties

Next we present a benchmark expanding variety model in two versions, for a decentralized economy and for a centralized economy.

Main ideas:

- technical progress, or innovations, takes the form of an expansion in the variety of products, which are used as intermediate goods by final producers;
- the R&D activities are made by entrants in a new intermediate product sector. Each entrant creates a new input and remains a monopolist in the market for that intermediate product;
- however, after entry, intermediate product firms operate in a competitive monopolist frameworks, because they compete on prices with the (monopolist) producers of other inputs;
- there is free entry in the market for intermediate goods but there is a cost of entry;
- there is no physical capital.

Summing up: the engine of growth is the increasing in the number of varieties, which are used as intermediate products. R&D activities take place in a decentralized but non-competitive environment which creates an externality which is not internalised.

8.1.1 Decentralized economy

Main assumptions:

- there are three agents: consumers, firms which produce the good used in final consumption (there is no accumulation of capital in this simple version of the model) and firms that engage in R&D activities;
- the model is dynamic general equilibrium model.

Consumers

They earn labor and capital incomes, consume the final product, save and lend capital to firms.

Problem:

$$\max_{C} V[C] = \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{W} = \omega(t)L + r(t)W(t) - C(t)$$

given $W(0)$, such that the no-Ponzi game condition holds $\lim_{t\to\infty} e^{-R(t)t}W(t) \geq 0$, $R(t) = (1/t)\int_0^t r(s)ds$ is the average interest rate, and where:

C per capita consumption;

W financial wealth composed by equity from firms;

ω wage rate ;

L labor supply assumed to be constant.

The first order conditions, are, from Pontryiagin's maximum principle

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r(t) - \rho) \tag{8.1}$$

$$\dot{W} = \omega(t)L + r(t)W(t) - C(t) \tag{8.2}$$

and $\lim_{t\to\infty} e^{-\rho t}W(t)C(t)^{\theta} = 0$.

Producers of the final good

They use labor and intermediate goods, which are supplied by R&D firms and supply the final good in a competitive market. Firms are homogeneous. They follow the dixit&stiglitz1977 model assumptions.

Production function: increasing, concave and displays constant returns to scale

$$Y(t) = F(L, N, [X(j)]_{j=0}^{N}) = AL^{1-\alpha}\int_0^{N(t)} X(j,t)^{\alpha}dj, \;\; 0 < \alpha < 1$$

where:

A total factor productivity;

$X(j)$ intermediate input where j is a variety index;

N number of existing varieties of the intermediate inputs.

Profits for the representative firm (competitive in its market)

$$\pi(t) = Y(t) - \omega(t)L - \int_0^{N(t)} P(j,t)X(j,t)dj$$

Problem:

$$\max_{L,[X(j)]_{j=0}^{N}} \pi(t)$$

where

$$Y(t) = F(L, N, \{X(j)\})$$

Then

$$\pi(t) = AL^{1-\alpha}\int_0^{N(t)} X(j,t)^{\alpha}dj - \omega(t)L - \int_0^{N(t)} P(j,t)X(j,t)dj$$

To maximize π for L we determine $\partial\mathcal{L}/\partial L = 0$, for each moment in time, t, where

$$\frac{\partial\mathcal{L}}{\partial L} = (1-\alpha)AL^{-\alpha}\int_0^N X(j)^\alpha dj - \omega = (1-\alpha)\frac{Y}{L} - \omega$$

and, to maximize for $X(j)$ we determine $\delta\mathcal{L}/\delta X(j) = 0$, which is a functional derivative,

$$\frac{\delta\mathcal{L}}{\delta X(j)} = AL^{1-\alpha}\alpha X(j)^{\alpha-1} - P(j) = 0.$$

Therefore, the first order conditions are

$$\omega(t) = (1-\alpha)\frac{Y(t)}{L} \tag{8.3}$$

$$P(j,t) = \alpha AL^{1-\alpha}X(j,t)^{\alpha-1},\ j\in[0,N(t)] \tag{8.4}$$

As firms that produce the final product are price takers in the market for intermediate goods, we get their demand for intermediate input j

$$X(j,t) = \left(\frac{\alpha AL^{1-\alpha}}{P(j,t)}\right)^{1/(1-\alpha)}. \tag{8.5}$$

Then, total demand for the intermediate goods is

$$X(t) = \int_0^{N(t)} X(j,t)dj = (\alpha A)^{1/(1-\alpha)}L\int_0^{N(t)} P(j,t)^{1/(\alpha-1)}dj$$

and the total supply for the final good is

$$Y(t) = (\alpha A)^{\alpha/(1-\alpha)}L\int_0^{N(t)} P(j,t)^{\alpha/(\alpha-1)}dj.$$

Next, we will determine $P(j,t)$, which are determined by producers of intermediate inputs.

Producers of R&D

When they produce a new variety they are monopolists in the market for that variety. The monopoly is permanent, but there is free entry. They have a two phase decision process:

1. first, they decide on the amount of resources they will use in the innovation process. As there is free entry, the equilibrium entry is determined by the equality between the present value of future profits and the cost of developing a new patent;
2. second, after the decision of entry, they determine the price for the new variety in which R&D was incorporated.

Assumptions:

- the cost of production of a unit $X(j)$ is equal to 1;
- the cost of development of a new variety is proportional to the mean output per variety,
$$\eta \frac{Y}{N}, \eta > 0.$$
  This assumption eliminates the existence of scale effects, which are counterfactual, and is in accordance with the stylized facts that the proportion of R&D expenditures in the GDP is constant.

As the decision to entry depends on profits which are obtained from producing each variety, we solve the problem backwards.

2nd phase: Price decision if there is entry

The benefits from introducing a new variety j, $V^*(j)$, are determined from the present value of the cash flow generated by innovation.

The problem of the R&D producer is:

$$V^*(j,t) = \max_P V[P(j,t)](t) = \int_t^\infty \pi(P(j,s)) e^{-\int_t^s r(l)dl} ds$$

where $r(t)$ is the (endogenous) rate of return of financial assets, or interest rate. The instantaneous profit function, for every moment t (with a unit cost of production and a monopolist behavior in the market) is

$$\pi(j,t) = (P(j,t) - 1)\, X(j,t)$$

where the firm knows the demand function of variety j by the producers of the final good (see equation (8.5))

$$X(j,t) = L\left(\frac{\alpha A}{P(j,t)}\right)^{1/(1-\alpha)}.$$

Then

$$V^*(j,t) = \max_P \int_t^\infty [P(j,s) - 1] L\left(\frac{\alpha A}{P(j,s)}\right)^{1/(1-\alpha)} e^{-\int_t^s r(l)dl} ds$$

If we use, again, the concept of functional derivative, we have the first order condition for a change in prices at any time $t \leq s < \infty$ [1]

$$\frac{\delta V(j,t)}{\delta P(j,s)} = \frac{\partial \pi(s)}{\partial P(j,s)} = 0,\ t \leq s < \infty.$$

Then prices are time-independent and symmetric across varieties

$$P(j,t) = \frac{1}{\alpha} = P,\ \text{for any}(j,t) \tag{8.6}$$

because it is independent from j, meaning that :

- we have a constant mark up, equal to $1/\alpha$, over the production cost which is equal to 1;
- the price is equal for all the varieties;
- and it is constant through time.

[1] In the appendix we prove this result. Observe that δ denotes a functional derivative and ∂ denotes the derivative of a function. We get

$$\frac{\partial \pi}{\partial P(j)} = L(\alpha A)^{1/(1-\alpha)} P^{1/(\alpha-1)} \left(1 + \frac{P(j) - 1}{\alpha - 1} \frac{1}{P(j)}\right) = 0$$

if $(\alpha - 1)P(j) + P(j) - 1 = 0$ because $(\alpha - 1)P(j) \neq 0$.

Now, we can determine the optimal profits for producer j

$$\pi^*(j,t) = \pi^* = LA^{1/(1-\alpha)} \left(\frac{1-\alpha}{\alpha}\right) \alpha^{2\alpha/(1-\alpha)}.$$

which is constant and symmetric across varieties. That is, we have $\pi^*(j,t) = \pi^*$, which is constant for all pairs (j,t).

Therefore,

$$V^*(j,t) = V^*(t) = \pi^* \int_t^\infty e^{-\int_t^s r(l)dl} ds,$$

for any j.

If we differentiate as regards t, we get after applying the Leibniz' rule (see the Appendix)

$$\dot{V}^* = -\pi^* + r(t)V^*(t)$$

which is equivalent to

$$r(t) = \frac{\pi^* + \dot{V}^*}{V^*(t)}. \tag{8.7}$$

This is an arbitrage condition which means that the rate of return from innovation activities should be equal to investing in financial capital, which earns the market's real interest rate. This gives the supply condition for intermediate goods.

1st phase: Entry decision when there is free entry

The free entry condition is for the market for variety j is :

$$V^*(j,t) = \eta \frac{Y(j,t)}{N(j,t)}$$

meaning that the benefits and costs of producing new varieties balance.

Given the symmetry for different varieties, we have, equivalently

$$V^*(t) = \eta \frac{Y(t)}{N(t)} \tag{8.8}$$

Aggregation The demand for intermediate goods is, from equations (8.5) and (8.6))

$$X(t) = \int_0^{N(t)} X^*(j,t)dj = A^{1/(1-\alpha)}\alpha^{2/(1-\alpha)}LN(t)$$

from which the output of the final good results

$$Y(t) = AL^{1-\alpha}\int_0^{N(t)} X^*(j)^\alpha dj = \phi N(t)$$

where

$$\phi \equiv A^{1/(1-\alpha)}\alpha^{2\alpha/(1-\alpha)}L$$

is a constant. Then we have an linear aggregate production

$$Y(t) = \phi N(t)$$

where ϕ is a constant.

Observe that we also have the total demand of intermediate goods as a linear function of the number of varieties,

$$X(t) = \alpha^2\phi N(t) = \alpha^2 Y(t).$$

Equilibrium entry From the above, we conclude that the cost of creating new varieties is also constant $\eta Y/N = \eta\phi$.

Then the equilibrium in the market for intermediate goods (and for innovations) becomes (see equation (8.8))

$$V^* = \phi\eta.$$

where V^* is a constant which implies that $\dot{V} = 0$.

Therefore, as $\pi^* = \alpha(1-\alpha)Y(t)/N(t) = \alpha(1-\alpha)\phi$, we get the interest rate (see equation (8.7)

$$r(t) = r = \frac{\alpha(1-\alpha)}{\eta}$$

which is also constant.

General equilibrium

The equilibrium is represented by the paths of $[(C(t), N(t))]_{t\in[0,\infty)}$ such that: (1) consumers, final producers and intermediate producers solve their problems; (2) consistency conditions are met; (3) markets clear.

As consumers are the owners of capital (as they are the only agent that save) the wealth of consumers is composed of equity which finance producers. As the only firms which have "capital" are the producers of R&D, then consumers wealth is

$$W(t) = \int_0^{N(t)} V^*(j,t)dj = V^*N(t) = \eta\phi N(t)$$

because, as we saw, V^* is constant across firms and across time, then

$$\dot{W} = \eta\phi\dot{N}.$$

The budget constraint of the consumer

$$\dot{W} = rW + \omega L - C$$

at the aggregate level, after income is distributed as wage, $\omega L = (1-\alpha)Y = (1-\alpha)\phi N$, $rW = \alpha(1-\alpha)\phi N$, is equivalent to

$$\eta\phi\dot{N} = (1-\alpha)(1+\alpha)\phi N - C = (1-\alpha^2)\phi N - C$$

Alternatively, we get the same expression from the equilibrium equation for the market of the final good

$$Y = I + X + C$$

where $X = \alpha^2\phi N$, $I = \eta\phi\dot{N}$ and $Y = \phi N$.

Then the DGE is represented by the system

$$\begin{aligned} \frac{\dot{C}}{C} &= \frac{1}{\theta}(r-\rho) \\ \eta\phi\dot{N} &= (1-\alpha^2)\phi N - C \end{aligned}$$

Balanced growth path

If we represent consumption as

$$C(t) = c(t)e^{\gamma t},\ N(t) = n(t)e^{\gamma t},\ Y(t) = \phi n(t)e^{\gamma t} = y(t)e^{\gamma t}$$

we get the systems in detrended variables

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\theta}(r-\rho-\theta\gamma) \\ \dot{n} &= \left(\frac{(1-\alpha^2)}{\eta} - \gamma\right) n - \frac{c}{\eta\phi} \end{aligned}$$

where $r = \dfrac{\alpha(1-\alpha)}{\eta}$.

The long run growth rate, along the BGP is

$$\bar{\gamma} = \frac{1}{\theta}(r-\rho) = \frac{1}{\theta}\left(\frac{\alpha(1-\alpha)}{\eta} - \rho\right)$$

and the levels along the BGP are

$$\bar{c} = \frac{\eta\phi}{\theta}\left(\rho + \frac{(\theta(1+\alpha)-\alpha)(1-\alpha)}{\eta}\right)\bar{n} = \beta\bar{n}$$

which is positive if $\theta > \alpha/(1+\alpha)$.

Then, in the neighborhood of the BGP the system becomes (substituting $\gamma = \bar{\gamma}$)

$$\begin{aligned} \frac{\dot{c}}{c} &= 0 \\ \dot{n} &= \frac{\beta n - c}{\eta\phi} \end{aligned}$$

Which clearly does not display transitional dynamics [2].

[2] An alternative perspective: W can be seen as counterpart of the stock of capital of the economy, that is $K(t) = W(t) = \eta\phi N(t)$. Then we can recast the model as an AK model, where GE representation for the variables with trend is $\dot{C} = C(t)(r-\rho)/\theta$, and $\dot{K} = rK(t) - C(t)$.

Exercise : proof this (suggestion: the model is similar to the AK model.)

Conclusions

- expansion of varieties, for a fixed stock of capital generates long-run growth;
- the long run growth rate is a negative function of η, i.e, of the cost of entry into R&D activities (and θ and ρ as in the AK model);
- there is no transitional dynamics;
- η does not affects the long run level of the product $\bar{y}$.

8.1.2 Centralized economy: Pareto optimum

Main ideas:

- technical progress takes the form of an expansion in the variety of products;
- the economy is efficient.

The problem for the optimal planer is to maximize the intertemporal utility function Problem

$$\max_{C} V[C] = \int_0^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to the aggregate budget constraint

$$Y(t) = C(t) + I(t) + X(t)$$

which we saw is equivalent to

$$\dot{N} = \frac{1}{\eta}\left((1-\alpha^2)N - \frac{1}{\phi}C\right)$$

This is an optimal control problem in which the number of varieties is the state variable.

The current-value Hamiltonian is

$$H = \frac{C^{1-\theta}}{1-\theta} + \frac{Q}{\eta}\left((1-\alpha^2)N - \frac{1}{\phi}C\right)$$

where Q is the co-state variable.

The first order conditions are

$$\begin{aligned} C^{-\theta} &= \frac{Q}{\eta\phi} \\ \dot{Q} &= Q\left(\rho - \frac{(1-\alpha^2)}{\eta}\right) \\ \dot{N} &= \frac{1}{\eta}\left((1-\alpha^2)N - \frac{1}{\phi}C\right) \end{aligned}$$

plus the initial and transversality conditions.

If we consider the same decomposition as before, $C(t) = c(t)e^{\gamma t}$, $N(t) = n(t)e^{\gamma t}$, we get the equivalent representation of the first order conditions on the detrended variables,

$$\begin{aligned}\frac{\dot{c}}{c} &= \frac{1}{\theta}(r_g - \rho - \theta\gamma_g)\\ \eta\phi\dot{n} &= [(1-\alpha^2)\phi - \eta\phi\gamma_g]n - c.\end{aligned}$$

where

$$r_g = (1-\alpha^2)/\eta = (1+\alpha)(1-\alpha)/\eta$$

is the rate of return which is analogous to r in a decentralized economy.

Then we get the long-run growth rate, for the centralized economy,

$$\gamma_g = \frac{r_g - \rho}{\theta}$$

It is easy to see that

$$\gamma_g > \gamma \;\text{ because }\; r_g = \frac{(1+\alpha)(1-\alpha)}{\eta} > r = \frac{\alpha(1-\alpha)}{\eta}.$$

The wedge between the social and the private rates of return is $r_g - r = (1-\alpha)/\eta > 0$.

Conclusions

- the rate of growth in the decenralized version is not Pareto optimal: this is a result of the imperfect competition in the R&D production sector;
- there is an externality which is created by the R&D activity which is not internalized in a competitive economy.

References Barro and Sala-i-Martin (2004):chapter 6, Aghion and Howitt (2009):chapter 3, Acemoglu (2009):chapter 13

8.2 Quality ladders (Schumpeterian models)

Main ideas

- the economy produces a final good and a number of different varieties of intermediate products, which are non-durable goods;
- the number of types of the intermediate products, N, is constant, differently from the previous model, but there may exist improvements in their quality;
- better quality is reflected in improvements in the productivity in the production of the final good when they use quality-upgraded intermediate goods;
- the emergence of quality improvements, for every intermediate input, is the consequence of purposeful R&D activity;
- the success of the R&D activity is random, and the probability that a new level of quality is introduced follows a Poisson process, where the probability of success depends on the R&D expenditures which are incurred for its invention;
- there is creative destruction: when a better technique is invented the goods from a previous generation are abandoned. That is, only the better quality inputs for every intermediate good are used. Therefore, there is a quality ladder structure for the distribution of the quality for every input;
- innovations are introduced by entrants;
- the behavior of the inventor of an input belonging to a higher quality ladder is different depending on the structure of the economy. In the decentralized version, the inventor

has a temporary monopoly, in the window of time in which that input is in the top of the quality ladders.

Summing up: creative destruction is the engine of growth for the economy. In the decentralized version of the model the non-competitive structure of the innovation market allows for the introduction of a role for technological policy.

References: schumpeter1934 and aghion&howitt1992.

8.2.1 Decentralized version of the model

We have a dynamic general equilibrium model (DGE) in which there are three types of agents: consumers, producers of the final product and producers of the intermediate goods. The producers of the intermediate goods in order to enter have to perform R&D activities. Only the producer of the best quality grade enters. Therefore, the DGE is defined by the paths of consumption, production of the final good and of the intermediate goods, costs and benefits of the R&D activity, and wages and rate of return of capital such that agents optimize and the markets are in equilibrium.

Consumers

As in the previous model. There is a representative consumer who gets income from ownership of firms and from wages and spend in the final good and save (or dissave). Their problem is

$$\max_{C} V[C] = \int_0^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{K} = \omega(t)L + r(t)K(t) - C(t) + \Pi(t)$$

given $K(0)$, where K is financial wealth, in real terms, composed by equity from firms, C is per capita consumption, ω is the wage rate, $r(t)$ is the rate of return on capital, $\Pi(t)$ are the profits generated by the imperfect competition in the intermediate sector, and L labor supply assumed to be constant [3].

[3] We could distinguish, as in the previous model between equity W, which is a liability of firms, and physical capital, K, which is a productive asset of firms. However, as in the previous case, they are equivalent, as there are no costs of adjustment of capital and asset markets are perfect.

The first order conditions, are, from Pontryiagin's maximum principle

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r(t) - \rho) \tag{8.9}$$

$$\dot{K} = \omega(t)L + r(t)K(t) - C(t) \tag{8.10}$$

Technology and innovations: general presentation

The technology, as regards the intermediate inputs, has the following characteristics:

- improvements in technology are equivalent to improvements in the quality of the intermediate inputs;
- innovation is developed in a R&D sector, which has as many components as intermediary sectors, $j \in [0, N]$;
- innovation in quality is represented as a sequence of technological ladders which are followed by sector j is
$$1, l, l^2, \ldots, l^{\nu_j}$$
- in every sector, only inputs with the maximum quality are produced, l^{ν_j};
- we assume that every R&D firm only produces one particular type of input. If an innovation occurs the existing firm is replaced by the most recent innovator. This means that, when l^{ν_j} arises the firm which produces $l^{\nu_j - 1}$ ceases its activity. We call this creative destruction;
- the firm has a monopoly in the period in which its innovation is the most recent, $T_{\nu_j} = t_{\nu_j + 1} - t_{\nu_j} > 0$
- the probability for an innovation to be at the top of the technological scale depends on the R&D effort.

Producers of the final good

The producers of the final good, use labour and a continuum of intermediate inputs, each intermediate input has a specific productivity depending on its quality. The producer of the final good is price-taker in all the markets in which he participates.

The technology of production of the final good is represented by the production function which is increasing, concave and with constant returns to scale

$$F(L, [\tilde{x}(j)]_{j\in[0,N]}) = AL^{1-\alpha}\int_0^N \tilde{x}(j,t)^\alpha dj, \quad 0<\alpha<1$$

where: A total factor productivity; N (constant) number of existing varieties; $\tilde{X}(j)$ intermediate input in efficiency units, that is adjusted by quality, where ν_j is an index of quality such that

$$\tilde{x}(j) = l^{\nu_j}x(j)$$

where l^{ν_j} is the maximum technological ladder which is reached by intermediate sector j (highest quality ladder).

The profit, in real terms, of the representative firm is

$$\pi(t) = Y(t) - \omega(t)L - \int_0^N P(j,t)x(j,t)dj$$

The firm seeks to maximize profits, by choosing labour and intermediate good inputs. As the firm is competitive in the market of its product it takes prices as given. Equivalently the profit can be written as

$$\pi(t) = AL^{1-\alpha}\int_0^N l^{\alpha\nu_j}x(j,t)^\alpha dj - \omega(t)L - \int_0^N P(j,t)x(j,t)dj$$

and the first order conditions are $\partial\pi/\partial L = 0$ and $\delta\pi/\delta x(j) = 0$, for $j\in[0,N]$, which is equivalent to [4] :

$$\omega(t) = (1-\alpha)\frac{Y(t)}{L} \tag{8.11}$$

$$P(j,t) = \alpha AL^{1-\alpha}l^{\alpha\nu_j}X(j,t)^{\alpha-1},\ j\in[0,N] \tag{8.12}$$

[4]Again the derivative as regards the intermediate inputs is a functional derivative

Therefore, the demand function for the j intermediate good is

$$x(j,t) = L\left(\frac{\alpha A}{P(j,t)}\right)^{1/(1-\alpha)} q(j)$$

where $q(\nu_j) \equiv l^{\frac{\alpha\nu_j}{1-\alpha}}$ is the quality level of the intermediate inputs used in sector j.

R&D sectors

As producer of an intermediate good j can only exist after developing a successful innovation, through R&D activities. We call it a R&D producer.

The R&D producer has, as in the variety-expansion model, a two-phase decision process:

1. first, there is a decision to make or not R&D (or, how much to invest in R&D activities). The investment in R&D determines $p(\nu_j)$, the probability of success of an invention of intermediate input of grade ν_j. There if free entry in the market for inventions, and the firm enters if returns are equal to costs. The returns are measured from profits accruing from successful R&D;

2. second, successful R&D firms (i.e., those which are in the top of the quality ladder) produce and, as they have a temporary monopoly, they determine the price of the intermediate goods to be sold to the final good producers.

Again, in order to determine the supply of intermediate goods we operate backwards:

Second phase: determination of the price of intermediate goods

The ingredients of the firm problem are the following:

1. successful firms have an instantaneous profit $\pi(\nu_j, t)$ by selling the intermediate good at price $P(j)$ and having a unit production cost. As they have monopoly power in the market for intermediate good j, they take the demand by final producers, $x(j)$ as given;

2. however, their monopoly power is only temporary as it lasts for an interval $[t_{\nu_j}, t_{\nu_j}+T_{\nu_j}]$. The profits are zero outside this interval. Their problem, is therefore intrinsically dynamic;

3. but it is also stochastic, as the probability of success, $g(\nu_j, t)$, is random (because the inventive activity may not produce a positive result or another inventor comes up before with a successful invention with level $\nu_j + 1$). Therefore the objective of the firm is to maximize the present value of the expected value of a successful of the introduction of the innovation of grade ν_j.

The instantaneous profit when ν_j is the maximal technological level in sector j, is

$$\begin{aligned}\pi(\nu_j, t) &= [P(j,t) - 1]X(j,t) = \\ &= [P(j,t) - 1]\left(\frac{\alpha A L^{1-\alpha}}{P(j,t)}\right)^{1/(1-\alpha)} q(\nu_j),\ t_{\nu_j} \le t \le t_{\nu_j} + T_{\nu_j}\end{aligned}$$

where we assume again a unit production cost for input j.

The profit is maximized, at every point in time by

$$\begin{aligned}\frac{\partial \pi(j,t)}{\partial P(j,t)} &= \left(\alpha A L^{1-\alpha}\right)^{1/(1-\alpha)} q(\nu_j) P(j,t)^{1/(\alpha-1)}\left(1 + \frac{P(j,t)-1}{P(j,t)(\alpha-1)}\right) = \\ &= \pi(j,t)\left(\frac{\alpha P(j,t) - 1}{(P(j,t)-1)(\alpha-1)P(j,t)}\right) = 0\end{aligned}$$

Then

$$P(j,t) = \frac{1}{\alpha} = P,\ \text{for all } j \in [0, N],\ t \in [0, \infty).$$

Again, we have a symmetric equilibrium.

Then we get the equilibrium output for the producer of the input of quality level ν_j

$$\begin{aligned} x^*(\nu_j, t) &= x^*(\nu_j) = \\ &= L\left(\alpha^2 A\right)^{\frac{1}{1-\alpha}} q(\nu_j) \end{aligned}$$

the profit is linear in the quality level

$$\pi^*(\nu_j, t) = \pi^*(\nu_j) = \pi_0 L q(\nu_j)$$

where $\pi_0 \equiv \left(\frac{1-\alpha}{\alpha}\right)(\alpha^2 A)^{1/(1-\alpha)}$. Note that $\pi^*(\nu_j, t) = \frac{1-\alpha}{\alpha} X^*(\nu_j, t)$

Therefore:

1. the price is equalized among all sectors, j and is constant;
2. the quantity of the input is constant and depends on the quality level of the sector;
3. the maximum profit is constant and depends on the quality level of the sector.

The value of a successful R&D firm, i.e., of the one which sells the input with superior quality in sector j, ν_j, during the period $t \in [t_{\nu_j}, t_{\nu_j} + T(\nu_j)]$ is

$$V(\nu_j, T_{\nu_j}) = \int_{t_{\nu_j}}^{t_{\nu_j}+T_{\nu_j}} \pi(\nu_j, t).e^{-\int_{t_{\nu_j}}^{t} r(s)ds} dt$$

However, as the ex-ante the firm does not know if it would be successful, its decision is based upon the mathematical expectation for the value of a firm which produces the input with quality ν_j is

$$E[V(\nu_j)] = \int_0^\infty V(\nu_j, T_{\nu_j}) g(\nu_j, T_{\nu_j}) dT_{\nu_j}.$$

Let $p(\nu_j)$ be the instantaneous probability of survival of a technology level ν_j in sector j. We assume that:

1. the probability of success is invariant in time ;

2. the survival is governed by a Poisson process, with the probability density function decreasing in time

$$g(\nu_j, t) = p(\nu_j - 1)e^{-p(\nu_j-1)t},\ t = 0, \dots, \infty.$$

As the firms have a monopoly power at the market of input j the R&D firm's post-entry expected value is then

$$E[V(\nu_j)]^* = \max_{P(j,t)} E[V(\nu_j)].$$

once we know the probability $p(\nu_j)$, which is endogenously determined in the first phase.

If we assume that T_{ν_j} is small enough to allow for a constant rate of interest in that period, assume that $z = t_j + T_{\nu_j}$, then the value of a invention in sector j, if is is successful

$$\begin{aligned} V^*(\nu_j) &= \pi^*(\nu_j)\int_{t_{\nu_j}}^{t_{\nu_j}+T_{\nu_j}} e^{-r(t-t_{\nu_j})}dt = \\ &= \frac{\pi^*(\nu_j)}{r}\left(1 - e^{-rT_{\nu_j}}\right) \end{aligned}$$

assuming that $r > 0$, has the mathematical expected

$$\begin{aligned} E[V^*(\nu_j)] &= \frac{\pi^*(\nu_j)}{r}\int_0^\infty \left(1 - e^{-rt}\right)p(\nu_j)e^{-p(\nu_j)T_{\nu_j}}dT_{\nu_j} = \\ &= \frac{\pi^*(\nu_j)}{r + p(\nu_j)} \end{aligned}$$

assuming that $p > 0$, which may be time-independent or not, depending on the behavior of r.

This is the value of incumbency: i.e., of being a monopolist in the market for input j when the technological level is ν_j.

First phase : determination of the R&D effort

The success probability for an innovation when the technological level is ν_j, $p(\nu_j)$, depends on the R&D effort in sector j. The effort of technological innovation, when the level of quality is ν_j, will allow, with a probability $p(\nu_j)$, to control an innovation which will generate the new technology $\nu_j + 1$.

Assumptions:

1. the probability of success is a linear function of the R&D expenditure

$$p(\nu_j) = Z(\nu_j)\psi(\nu_j)$$

   where:

   $Z(\nu_j)$ is the R&D expenditure in sector j when the technological level is ν_j, which allows for the progression of the technological level from ν_j to $\nu_j + 1$;

   $\psi(\nu_j)$ is the effect of the expenditure over the success probability, $p(\nu_j)$, per time unit;

2. efficiency depends inversely on the level of the output of the final good which can be attributable to the intermediate good ν_j,

$$\psi(\nu_j) = \frac{1}{\zeta Y(\nu_j)}$$

   where

$$\begin{aligned} Y(\nu_j) &= AL^{1-\alpha} l^{\alpha(\nu_j+1)} X(\nu_j)^{\alpha} = \\ &= A_Y L q(\nu_j), \end{aligned}$$

   where $q(\nu_j) \equiv l^{\alpha\nu_j/(1-\alpha)}$ and

$$A_Y \equiv \left(\alpha^{2\alpha} A\right)^{1/(1-\alpha)}$$

   is the total factor productivity which is homogeneous across sectors. Then $Y(\nu_j) = X^*(\nu_j)\alpha^{-2}$;

3. we assume free entry in the R&D sector.

The research effort, when the technological level is ν_j is only worthwhile, that is $Z(\nu_j) > 0$, only if

$$p(\nu_j)E[V(\nu_j)] \geq Z(\nu_j)$$

that is, when the expected benefit is at least as big as the cost.

Free entry in the R& D sector implies that the net expected income is zero

$$p(\nu_j)E[V(\nu_j)] = Z(\nu_j)$$

as $Z(\nu_j)/p(\nu_j) = 1/\psi(\nu_j)$ then

$$E[V(\nu_j)] = \zeta Y(\nu_j) \tag{8.13}$$

But as the innovation will have a value consistent with the expected optimal value determined on phase two,

$$E[V^*(\nu_j)] = \frac{\pi^*(\nu_j)}{r + p(\nu_j)}$$

then the free entry condition, equation (8.13) is equivalent to

$$\frac{1-\alpha}{\alpha} X^*(\nu_j) = (r + p(\nu_j))\zeta Y(\nu_j)$$

Then there is an arbitrage condition for entry

$$r + p(\nu_j) = r_0$$

where $r_0 \equiv \dfrac{\alpha(1-\alpha)}{\zeta}$ is constant We determine $p(.)$ from this equation,

$$p(\nu_j) = r_0 - r(t) = p(t), \text{ for all } \nu_j \ j \in [0, N], \tag{8.14}$$

which means that the probability of success is independent from the technological level or from the position on the technological ladder, $p(\nu_j) = p$, and takes place only if $r_0 > r$.

Aggregate evolution of quality

Let us define the aggregate quality index as

$$Q \equiv \int_0^N q(\nu_j)dj$$

Evolution of quality:

(1) we know that given the quality of inputs in a given moment l^{ν_j}, it tends to increase to l^{ν_j+1} with probability p per unit of time;

(2) we also know, from equation (8.14) that the probability of variation is independent from the technological level.

Then, the expected variation of Q in the period dt is

$$\begin{aligned} \frac{E[dQ]}{dt} &= \int_0^N p(\nu_j)\left(l^{(\nu_j+1)\alpha/(1-\alpha)} - l^{\nu_j\alpha/(1-\alpha)}\right)dj = \\ &= \int_0^N pq(\nu_j)\left(l^{\alpha/(1-\alpha)} - 1\right)dj = \\ &= \Xi pQ. \end{aligned}$$

where $\Xi \equiv l^{\alpha/(1-\alpha)} - 1$ is the "quality jump" generated by technological innovation. If N is very large, and if the time interval dt is very small, then we can approximate it by a non-stochastic relationship

$$\dot{Q} = p\Xi Q$$

or

$$\frac{\dot{Q}}{Q} = \Xi(r_0 - r).$$

The behavior of Q drives all the other aggregate magnitudes: the aggregate output,

$$Y(t) = \int_0^N Y(\nu_j)dj = \int_0^N A_Y Lq(\nu_j)dj = A_Y LQ(t).$$

and the aggregate demand for intermediate goods

$$X(t) = \int_0^N X^*(\nu_j)dj = \alpha^2 \phi Q(t) = \alpha^2 Y(t)$$

are linear on the aggregate quality index.

The total level of aggregate expenditures on R&D is

$$\begin{aligned} Z(t) &= \int_0^N Z(\nu_j)dj = \int_0^N p\zeta Y(\nu_j)dj = \\ &= l^{\alpha/(1-\alpha)} \int_0^N p\zeta Y(\nu_j)dj = \\ &= \zeta l^{\alpha/(1-\alpha)} pY(t) \end{aligned}$$

Consumers own all the capital, and therefore own the R&D firms. Therefore, the stock of financial wealth is given by

$$\begin{aligned} K(t) &= \int_0^N E[V^*(\nu_j)]dj = \\ &= \int_0^N \frac{\pi^*(\nu_j)}{r+p}dj = \frac{1}{r+p}\left(\frac{1-\alpha}{\alpha}\right)\int_0^N X^*(\nu_j)dj = \\ &= \zeta Y(t) \end{aligned}$$

General equilibrium

The equilibrium condition for the final good market is[5]

$$Y(t) = C(t) + X(t) + Z(t)$$

which is equivalent to

$$Y(t)\left(1 - \alpha^2 - \zeta l^{\alpha/(1-\alpha)} p(\nu_j)\right) = C(t).$$

[5] We can prove that this condition is equivalent to the aggregate micro-macro consistency condition $\dot{K} = r(t)K(t) + w(t)Y(t) - C(t) + \Pi(t)$ if the variables are evaluated at equilibrium values. Observe that $Z = \zeta\phi(\dot{Q} + pQ) = \dot{K} + pK$.

From this equation we can determine the probability of introducing successful innovations

$$p = \frac{1}{l^{\alpha/(1-\alpha)}}\left[\frac{1-\alpha^2}{\zeta} - \frac{C}{K}\right]$$

Then the (endogenous) endogenous rate of return of capital is

$$r(t) = \frac{1}{l^{\alpha/(1-\alpha)}}\left[\frac{1-\alpha^2}{\zeta} - \frac{C}{K}\right] - r_0 \tag{8.15}$$

Therefore the general equilibrium path, $[(C(t), Y(t), X(t), Q(t), Z(t), K(t), r(t)]_{t\in[0,\infty)}$ is represented by the ODE system

$$\dot{C} = \frac{C}{\theta}\left(r(C,K) - \rho\right) \tag{8.16}$$

$$\dot{K} = \left(r_0 - r(C,K)\right)\Xi K. \tag{8.17}$$

where r is given in equation (8.15). We use that fact $\dot{K}/K = \dot{Q}/Q$. The initial $K(0) = k_0$ and the transversality $\lim_{t\to\infty} C(t)^{-\theta}K(t)e^{-\rho t} = 0$ conditions should also hold.

Balanced growth path

If a BGP exists, then we have an equilbrium such we can perform the decomposition

$$C(t) = c(t)e^{\gamma t}, \quad K(t) = k(t)e^{\gamma t}.$$

The representation of the general equilibrium for the trendless variables,

$$\begin{aligned}\frac{\dot{c}}{c} &= \frac{1}{\theta}(r(c,k) - \rho - \gamma\theta)\\ \frac{\dot{k}}{k} &= \Xi\left(r_0 - r(c,k)\right) - \gamma\end{aligned}$$

In the steady state the endogenous long run growth rate has the familiar form:

$$\gamma = \frac{\bar{r} - \rho}{\theta}.$$

We determine it explicitly from equations $r-\rho=\gamma\theta$ and $\Xi(r_0-r)=\gamma$. Then the stationary equilibrium real interest rate is

$$\bar{r}=\frac{\rho+\theta r_0\Xi}{1+\theta\Xi}.$$

and the endogenus growth rate becomes

$$\gamma=\frac{r_0-\rho}{\Xi^{-1}+\theta}.$$

where $\bar{p}=r_0-\rho$ can be interpreted as a probability of occurrence of innovations in the long run, that is, when the rate of discount is equal to the rate of time preference, and Ξ is the quality jump generated by innovations.

Conclusions

- the growth rate is a positive function of the height of the quality jumps, Ξ, if $r_0>\rho$, because

$$\frac{\partial\gamma}{\partial\Xi}=\frac{r_0-\rho}{(1+\theta\Xi)^2}>0$$

- the growth rate is a positive function of the probability of occurrence of the innovations, $\bar{p}$;

- the rate of growth and the rate of occurrence of the innovations are a negative function of the costs of innovation, ζ, because $r_0=\alpha(1-\alpha)/\zeta$. We can interpret ζ as a parameter associated to barriers to entry.

References Barro and Sala-i-Martin (2004):chpter 7, Aghion and Howitt (2009):chapter 4 Acemoglu (2009):chapter 14, gil&brito&afonso2013

8.3 Technology adoption and institutional barriers

Some institutional frameworks can create barriers to the adoption of new technologies in developing countries. Barriers can be defined by the some aspects of governmental regulation, corruption, violence threats, union actions, etc.

Those barriers tend to generate differences in the levels of development and not necessarily differences among the rates of long run growth.

Next we present a simplified adaptation of parente&prescott1994, parente1995 and parente&prescott1

Assume the production function:

$$Y(t) = A(t)L$$

where $A(t)$ is the domestic technological level, which depends upon the technological level of the technological leaders, $W(t)$.

The world technological level is exogenous and is given by

$$W(t) = W_0 e^{\gamma^* t}$$

where γ^* is the leader's rate of technological progress.

Consider the case in which the economy wants to increase the domestic technological level, from $A(t)$ to $A(t+\epsilon)$, where $A(t+\epsilon) > A(t)$. We assume that

$$A(t+\epsilon) - A(t) = \frac{X(t)}{\pi}\left(\frac{W(t)}{A(t)}\right)^{\alpha}\epsilon$$

where $X(t)$ represents the expenditure incurred, π represents the height of the institutional barriers to the adoption of the world technological level and $\alpha > 0$. This means that the increase in the domestic technological level is a positive function of the expenditure and of the world technological level and it is a negative function of the barriers to technological adoption, and of the initial level of technology. If $0 < \alpha < 1$ this translates the idea that it is harder to improve the technological level that higher it already is. [6] If we let the interval tend to zero we have an accumulation equation for the domestic technological level

$$\dot{A} = \lim_{\epsilon \to 0} \frac{A(t+\epsilon) - A(t)}{\epsilon} = \frac{X(t)}{\pi} W(t)^{\alpha} A(t)^{-\alpha}.$$

We define the stock of technological capital which is used by the firms in the economy by

$$Z(t) \equiv \frac{\pi}{1+\alpha} A(t)^{1+\alpha} W(t)^{-\alpha}.$$

As

$$\frac{\dot{Z}}{Z} = (1+\alpha)\frac{\dot{A}}{A} - \alpha \frac{\dot{W}}{W}$$

where $\frac{\dot{W}}{W} = \gamma$, then Z verifies

$$\dot{Z} = X(t) - \alpha\gamma Z(t).$$

We assume a centralized economy in which the central planner solves the following problem

$$\max \int_0^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to

$$\begin{aligned} Y(t) &= \left((1+\alpha)\pi^{-1} Z(t) W(t)^{\alpha}\right)^{1/(1+\alpha)} L = C(t) + X(t) && (8.18)\\ \dot{Z} &= X(t) - \alpha\gamma Z(t) && (8.19)\\ \dot{W} &= \gamma^{*} W(t) && (8.20) \end{aligned}$$

Let us consider that along the BGP

$$\bar{Z}(t) = \bar{z} e^{\gamma t},\ \bar{C}(t) = \bar{c} e^{\gamma t}$$

[6] Equivalently, we could set the expenditure in technology as $X(t) = \pi \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{A(t)}^{A(t+\epsilon)} \left(\frac{S}{W(t)}\right)^{\alpha} dS$. As $\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{A(t)}^{A(t+\epsilon)} f(s) ds = \frac{dA(t)}{dt} f(A(t)) = \dot{A}(t) f(A(t))$, we have $\frac{d}{dt}\left(A(t)^{1+\alpha}\right) = \frac{1+\alpha}{\pi} X(t) W(t)^{\alpha}$

and the deviations from the BGP

$$Z(t) = z(t)e^{\gamma t},\ C(t) = c(t)e^{\gamma t}.$$

Given the linearly homogeneous property of the production function, as a function of (Z, W), and the homogeneity of the instantaneous utility function, and the fact that $W(t) = W_0 e^{\gamma^* t}$ is exogenous to the economy, a necessary condition for the existence of a BGP is that

$$\gamma = \gamma^*$$

the long run rate of growth of the economy is equal to the rate of technological progress of the technological leader.

Therefore, we can recast the the planner's problem in deviations from the trend

$$\max \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{-(\rho-\gamma^*(1-\theta))t} dt$$

where we assume that $\rho - \gamma(1-\theta) > 0$ subject to

$$\dot{z} = y(z(t)) - c(t) - (1+\alpha)\gamma^* z(t) \tag{8.21}$$

where

$$y(t) = \left((1+\alpha)\pi^{-1} z(t) W_0^\alpha\right)^{1/(1+\alpha)} L.$$

The first order conditions become

$$\dot{c} = \frac{c}{\theta}\left(r - \rho - \gamma^*(\alpha+\theta)\right) \tag{8.22}$$

$$\dot{z} = y(z) - c - (1+\alpha)\gamma^* z \tag{8.23}$$

where

$$r = \frac{\partial y}{\partial z} = \frac{1}{1+\alpha}\left(\frac{y}{z}\right).$$

As the long run rate of growth is given, the BGP levels for the interest rate are endogenously determined from

$$\bar{r} = \rho + \gamma^*(\alpha+\theta) \tag{8.24}$$

$$\bar{c}/\bar{z} = (1+\alpha)\left(\rho - \gamma^*(1-\theta) + \gamma^*\alpha\right) \tag{8.25}$$

where $\bar{z} = Z(0)$ given. Therefore there is no transitional dynamics and the GDP along the BGP is

$$\bar{Y}(t) = \bar{y}e^{\gamma^* t}$$

where

$$\bar{y} = (1+\alpha)^2(\rho + \gamma(\alpha+\theta))^{-1/\theta} W_0 L^{(1+\alpha)/\alpha} \pi^{-1/\alpha}$$

Conclusions

The barriers to technological adoption have the following consequences:

- they do not affect the long run growth rate;
- but they affect negatively the level of the product because

$$\frac{\partial \bar{y}}{\partial \pi} = -\frac{\bar{y}}{\alpha\pi} < 0$$

8.4 O-ring theory of development

kremer1993b

8.5 Directed technological change

There is evidence that technical progress in the last century, differently from what happened in the eighteen and the nineteen centuries is skill-biased. This shows up in the positive correlation between w_H/w_L and L_H/L_L where w_j and L_j refer to the wages and numbers of skilled (H) versus unskilled (L) workers. This led some researchers to argue that the technical progress is biased towards skilled workers (otherwise, that correlation would be inverted).

The basic model has the following features:

- it extends the expansions of variety model by dividing the intermediate good sectors into two: sectors producing machines which are complements to skilled labour and sectors producing machines which are complements to unskilled labour;
- R&D are performed by potential entrants. Entrants have to decide if they enter in the sectors which produce machines which are complements to the skilled or unskilled labour.

In the context of this model we may measure the bias in technological change by the relative growth of the number of varieties of skilled-labour complementary machines versus the unskilled-labour complementary machines which is induced by induced by R&D activities.

Next, we present the model in its decentralized version.

Consumers

Problem:

$$\max_{C} V[C] = \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{W} = \omega_L(t) L_L + \omega_H(t) L_H + r(t) W(t) - C(t)$$

given $W(0)$.

The first order conditions, are, from Pontryiagin's maximum principle

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r(t) - \rho) \tag{8.26}$$

$$\dot{W} = \omega_L(t) L_L + \omega_H(t) L_H + r(t) W(t) - C(t) \tag{8.27}$$

Producers of the final good

They use two types of intermediate goods, produced with skill-biased technologies, are price takers. Their problem is, for every moment t

$$\max_{Y_L, Y_H} \pi(t),\ \pi(t) \equiv Y(t) - P_L(t) Y_L(t) - P_H(t) Y_H(t)$$

where P_j, $j = L, H$ are the prices for the two types of inputs and the production function is CES (constant elasticity of substitution)

$$Y(t) = \left[A_L Y_L(t)^{\dfrac{\varepsilon - 1}{\varepsilon}} + A_H Y_H(t)^{\dfrac{\varepsilon - 1}{\varepsilon}} \right]^{\dfrac{\varepsilon}{\varepsilon - 1}}$$

The first order conditions are

$$Y_j(t) = \left(\frac{P_j(t)}{A_j} \right)^{-\varepsilon} Y(t),\ \text{for } j = L, H$$

and the following condition should hold (for zero profits)

$$\left[A_L^{\varepsilon} P_L(t)^{1-\varepsilon} + A_H^{\varepsilon} P_H(t)^{1-\varepsilon}\right]^{\dfrac{1}{1-\varepsilon}} = 1 \tag{8.28}$$

Producers of intermediate goods

There are two producers, one produces machines by using other machines that are complementary to skilled or unskilled labour. Each producer, using labour and a continuum of machines. We assume that producers are price-takers

The production function for intermediate good $j = L, H$ is

$$Y_j(t) = \frac{1}{1-\beta}\left(\int_0^{N_j(t)} x_j(v,t)^{1-\beta} dv\right) L_j^{\beta},\ j = L, H$$

where $0 < \beta < 1$, $N_j(t)$ and $x_j(v,t)$ are the number and the level of machines of variety $v \in [0, N_j(t)]$, which are are complements to factor L_j, for $j = L, H$.

The problem for producer of intermediate good $j = L, H$ is

$$\max_{L_j, [x_j(v,t)]_{v\in[0,N_j(t)]}} \pi_j(t),$$

where

$$\pi_j(t) = P_j(t)Y_j(t) - W_j(t)L_j - \int_0^{N_j(t)} p_j^x(v,t)x_j(v,t)dv,\ j = L, H.$$

The first order conditions are: firms equalize the real wage to the marginal product of the type of labour they employ

$$\omega_j(t) = \frac{W_j(t)}{P_j(t)} = \beta\frac{Y_j(t)}{L_j},\ j = L, H$$

and the demand for the machines of type $v \in [0, N_j(t)]$ for sector $j = L, H$ is a linear function of labour

$$x_j(v,t) = \left(\frac{P_j(t)}{p_j^x(v,t)}\right)^{1/\beta} L_j,\ v \in [0, N_j(t)],\ j = L, H. \tag{8.29}$$

Producers of skill-complementary machines and R&D performers

Again producers of machines $v \in [0, N_j(t)]$ have a monopoly power, but have to engage in R&D activities before they start to produce. They enter each market according to a free entry condition. They have, as in the previous models, a two-stage decision process, that we solve backwards.

Production phase The problem

$$\max_{p_j^x(v,t)} \pi_j(v,t) \; v \in [0, N_j(t)], \; j = L, H$$

where $\pi_j(v,t) = (p_j^x(v,t) - \psi)x_j(v,t)$ where it is assumed for simplicity that the costs of production are $\psi = 1 - \beta$ and equal to all producers. The first order conditions are

$$x_j(v,t) + (p_j^x(v,t) - \psi)\frac{x_j(v,t)}{p_j^x(v,t)} = 0$$

Then,

$$p_j^x(v,t) = \frac{\psi}{1-\beta} = 1, \; v \in [0, N_j(t)], \; j = L, H.$$

prices for machines are symmetric.

If we substitute in equation (8.29), we get the production of intermediate R&D products

$$x_j(v,t) = x_j(t) = P_j(t)^{1/\beta} L_j, \; v \in [0, N_j(t)], \; j = L, H,$$

which is symmetric across varieties. Then the profits of the intermediate producers

$$\pi_j(v,t) = \pi_j(t) = \beta P_j(t)^{1/\beta} L_j, \; v \in [0, N_j(t)], \; j = L, H, \tag{8.30}$$

and the output of j-complementary intermediate products

$$Y_j(t) = \frac{1}{1-\beta} P_j(t)^{(1-\beta)/\beta} N_j(t) L_j, \; j = L, H,$$

are also symmetric across varieties.

Entry phase The technology for introducing the innovation is of the lab-equipment type, that is

$$\dot{N}_j = \eta_j Z_j(t),\ j = L, H$$

that is an increase in the number of varieties increases instantaneously as a linear unction of the expenditures Z_j, where η_j is a productivity parameter for the R&D activity.

The value of introducing a new variety of machines which is j-complementary, when the monopoly power lasts indefinitely, and entering into the market is

$$V_j(v,t) = \int_t^{\infty} \pi_j(v,s) e^{-\int_t^s r(\tau)d\tau} ds = \int_t^{\infty} \pi_j(t) e^{-\int_t^s r(\tau)d\tau} ds,\ v \in [0, N_j(t)],\ j = L, H,$$

or, differentiating

$$r(t)V_j(t) - \dot{V}_j(t) = \pi_j(t),\ j = L, H.$$

This implies

$$r(t) = \frac{\pi_L(t) + \dot{V}_L(t)}{V_L(t)} = \frac{\pi_H(t) + \dot{V}_H(t)}{V_H(t)} \tag{8.31}$$

because the real interest rate, $r(t)$ is determined for the aggregate economy, and is common to all sectors. The free entry condition is assumed to be

$$V_j(t) < \frac{Z_j(t)}{\dot{N}_t} \text{ if } Z_j(t) = 0$$

then there is no entry, and

$$V_j(t) = \frac{Z_j(t)}{\dot{N}_t} \text{ if } Z_j(t) > 0$$

then there is entry, where $Z_j(t)/\dot{N}_t(t)$ are the costs for introducing a new variety. Equivalently $\eta_j V_j(t) \leq 1$ if $Z_j(t) \geq 0$

$$\eta_j V_j(t) = 1 \text{ if } Z_j(t) > 0,\ j = L, H.$$

If there are expenditures on R&D in both sectors, $Z_L(t) > 0$ and $Z_H(t) > 0$, then:

1. the market clearing condition should hold

$$\eta_L V_L(t) = \eta_H V_H(t) = 1$$

which is equivalent to $\eta_L \pi_L(t) = \eta_H \pi_H(t)$ for every t. Then $\dot{V}_j = 0$ for $j = L, H$.

2. Then from equations (??) and (8.31)

$$r = \frac{\pi_L}{V_L} = \eta_L \pi_L = \eta_L L_L P_L^{1/\beta}$$

and

$$r = \frac{\pi_H}{V_H} = \eta_H \pi_H = \eta_H L_H P_H^{1/\beta}$$

Then

$$P_j(t) = P_j = \left(\frac{r}{\eta_j L_j}\right)^{\beta},\ j = L, H$$

3. This equation together with equation (8.28) allows us to determine the interest rate as a constant

$$r = \beta\left(A_L^{\varepsilon}(\eta_L L_L)^{\sigma-1} + A_H^{\varepsilon}(\eta_H L_H)^{\sigma-1}\right)^{1/(\sigma-1)} \tag{8.32}$$

where

$$\sigma \equiv 1 + (\varepsilon - 1)\beta$$

is the elasticity of substitution between the two factors. Observe that $\sigma > 1$ if $\varepsilon > 1$.

The relative price of the skilled-complementary relative to the unskilled-complementary input

$$p(t) = \frac{P_H(t)}{P_L(t)} = \left(\frac{\eta_H L_H}{\eta_L L_L}\right)^{-\beta}.$$

Then, profits are constant and the $j-$complementary intermediate outputs are of type

$$Y_j(t) = \phi_j N_t(t),\ j = L, H.$$

where

$$\phi_j = \frac{1}{1-\beta}\left(\frac{r}{\eta_j}\right)^{1-\beta} L_j^{\beta} N_j,\ j = L, H.$$

General equilibrium

The equilibrium in the market for the final good is

$$Y(t) = C(t) + Z(t) + X(t)$$

where total expenditure in R&D is

$$Z(t) = Z_L(t) + Z_H(t)$$

and the total expenditure on intermediate goods is

$$\begin{aligned} X(t) &= \psi\left(\int_0^{N_L(t)} x_L(v,t)dv + \int_0^{N_H(t)} x_H(v,t)dv\right) = \\ &= (1-\beta)\left(P_L(t)^{1/\beta} L_L N_L(t) + P_H(t)^{1/\beta} L_H N_H(t)\right). \end{aligned}$$

Wealth for consumers is

$$W(t) = N_L(t)V_L(t) + N_H(t)V_H(t)$$

If we time differentiate W [7] and substitute the equilibrium condition for the final product market, we get macroeconomic restriction

$$\dot{W} = rW + \omega_L(t)L_L + \omega_H(t)L_H - C$$

which is consistent with the restriction on the consumers problem.

[7] With the former assumptions $d(N_L(t)V_L(t)+N_H(t)V_H(t))/dt = \dot{N}_L V_L + \dot{N}_H V_H = Z_L(t)+Z_H(t) = Z(t)$.

The BGP growth rate is obtained from the detrended Euler equation, (8.26), as

$$\gamma^* = \frac{r^* - \rho}{\theta}$$

where $r^* = r(L_H, L_L)$ is given in equation (8.32).

It is also possible to get the following relative magnitudes, along the BGP:

1. the ratio of relative technologies

$$n \equiv \frac{N_H}{N_L} = \left(\frac{\eta_H}{\eta_L}\right)^{\sigma} \left(\frac{A_H}{A_L}\right)^{\varepsilon} \left(\frac{L_H}{L_L}\right)^{\sigma-1}$$

2. the relative wage premium for skilled workers

$$\omega \equiv \frac{\omega_H}{\omega_L} = \left(\frac{\eta_H}{\eta_L}\right)^{\sigma-1} \left(\frac{A_H}{A_L}\right)^{\varepsilon} \left(\frac{L_H}{L_L}\right)^{\sigma-2}$$

Exercise: prove the last two results.

Conclusions

Let us assume that $\sigma > 1$, which can be interpreted as the elasticity of substitution between the two factors:

1. an increase in L_H/L_L always induces a relatively H-biased technological change, in the sense that N_H/N_L increases, there are more innovations in the H-sector;

2. an increase in L_H/L_L induces an increase in the relative wage of H-skilled workers if $\sigma > 2$;

3. we already saw that an L_H/L_L reduces the relative price P_H/P_L, for any value of σ;

4. an increase in both L_L and L_H increases the growth rate (there are scale effects in this version of the model).

References Acemoglu (2009):chapter 15

Chapter 9

Growth and the environment

9.1 Introduction

Let us assume an economy in which the only inputs are natural resources.

There are two types of natural resources, as far as the dynamic properties of their effects on the economy are concerned: non-renewable and renewable resources. The first are in limited, finite and constant supply, while the latter vary with time and are potentially unbounded (at least in a time frame smaller than millions of years).

For both types of resources, a fundamental issue is related to what has been recently termed as the sustainability issue: under which conditions natural resources are not completely depleted ? A related question is: given the fact that natural resources are an input in production, under which conditions can we have both growth and sustainability ? Another question is: what are the consequences of ”greenery” ?

Next, we present a very simple model dealing with these issues:

1. production only uses a natural resource as an input, but this use depletes the stock natural resources;

2. there is technical progress which takes the form of dematerialisation;

3. the natural resource is renewable;

4. the natural resource has an amenity value for the consumer.

There are no externalities and no other distortions in the economy.

There are two sources of unbounded growth (in consumption and utility): the growth of the renewable resource and the technical progress in dematerialization.

We conclude that the feasible rate of growth should lie between the rate of technical progress and the sum of the rate of technical progress and the natural rate of regeneration, in order to reach sustainability.

Therefore, even in the case in which the physical rate of renewal is small, this will allow for unbounded growth.

9.2 The model

The economy produces a single good with the production function

$$Y(t) = A(t)P(t)$$

where productivity grows exponentially at an exogenous growth rate γ_A,

$$A(t) = A(0)e^{\gamma_A t}$$

and $P(t)$ is the flow of polution. The natural resource accumulates as a result of two counteracting processes, the natural regeneration which takes place at a rate μ and the deplection of resources as a result of the productive activity,

$$\dot{N}(t) = \mu N(t) - P(t), \tag{9.1}$$

and $N(0) = N_0$ is given.

The equilibrium condition for the the goods markets is

$$Y(t) = C(t)$$

as there is no investment. We assume implicitly that if there is an environmentalist policy, it is directly performed by firms by controling $P(.)$.

We assume that the representative agent has the intertemporally independent utility function

$$V([C(t)]_{t\in[0,\infty)}, [N(t)]_{t\in[0,\infty)}) = \int_0^\infty u(C(t), N(t))e^{-\rho t}dt$$

where $\rho > 0$ is the psychological discount rate and the atemporal utility function, $u(.)$, is increasing and concave in both its arguments. The consumer derives utility not only from the consumption of the manufactured good but also from the amenity services produced by nature. A concave utility function means that there is some degree of substitutability.

There are several necessary conditions for the existence of a balanced growth path (BGP).

1. First, the levels of consumption and natural resources should be written as

$$C(t) = c(t)e^{\gamma_c t},$$

and

$$N(t) = n(t)e^{\gamma_n t},$$

where c and n are the detrended variables and γ_c and γ_n are the long run growth rates.

2. Second, from equation (9.1), we see that the growth rates of the stock of natural resources and of pollution should be the equal. Therefore $P(t) = p(t)e^{\gamma_n t}$.

3. Third, the equilibrium condition in the goods market should hold. Then,

$$\gamma_c = \gamma_n + \gamma_A = \gamma$$

where γ is the growth rate of the output, and $c(t) = A(0)p(t)$.

Then, the detrended resource accumulation equation, becomes

$$\dot{n}(t) = (\mu - \gamma_n)n(t) - \alpha c(t) \tag{9.2}$$

where $\alpha \equiv A(0)^{-1}$.

4. The fourth condition for a BGP, is that the utility function should be homogeneous. As we have a state variable in the utility function and the rate of growth of the two variables is not equal when there is technical progress, we assume that the utility function can be written in the form

$$u(C(t), N(t)) = e^{\gamma_u t} u(c(t), n(t)), \tag{9.3}$$

A utility function that verifies all those properties is

$$u(c, n) = \frac{(cn^{\varphi})^{1-\sigma}}{1-\sigma}$$

where φ measures the relative utility from the amenity services produced by natural capital as regards services from consumption of material goods. In this case, in equation (9.3) we have

$$\gamma_u = (1-\sigma)(\gamma_c + \varphi\gamma_n) = (1-\sigma)(\gamma_A + (1+\varphi)\gamma_n).$$

The marginal utilities are

$$u_c = (1-\sigma)u(c, n)/c, \; u_n = \varphi(1-\sigma)u(c, n)/n$$

and verify the relationship

$$\frac{u_n}{u_c} = \varphi\frac{c}{n}.$$

Therefore, the intertemporal optimization problem for the centralized version of this economy is

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} \frac{\left(c(t)n(t)^{\varphi}\right)^{1-\sigma}}{1-\sigma} e^{-\rho^* t} dt \tag{9.4}$$

where

$$\begin{aligned} \rho^* &= \rho - \gamma_u \\ &= \rho - (1-\sigma)(\gamma_A + (1+\varphi)\gamma_n) > 0 \end{aligned}$$

subject to equation (9.2) and given $n(0) = n_0$.

The intuition behind this problem is the following. The growth rate of consumption depends on the growth rate of natural resources and the growth rate of technical progress. In our setting, technical progress means dematerialization, i.e., the possibility of producing more with a decreasing use of raw materials. In this first approximation, we assume that technical progress is exogenous, but it could be endogenized. Some natural resources should be used in order to consume manufactured goods, but this decreases the amenity services produced by nature. Therefore, the optimal rate of growth of natural resources should belong to the interval $(0, \mu)$ and can be determined by the trade-off between consumption of manufactured goods and consumption of amenities. Note that the growth rate of the natural resource is different from the rate of natural renewal. The difference is $\gamma_n - \mu$.

Assumption 1

1. $\sigma = 1$ and $\varphi\mu < \rho < (1+\varphi)\mu$

2. $\sigma \neq 1$ and

$$(1-\sigma)(1+\varphi)\mu < \rho - (1-\sigma)\gamma_A < (1+\varphi)\mu. \tag{9.5}$$

This assumption will guarantee that $\rho^* > 0$ and that there is sustainability. In our case sustainability holds if the stock of natural capital is positive.

In the next sections we determine the (optimal) balanced growth path and study next the dynamics around the BGP.

9.3 The balanced growth path

The (optimal) balanced growth path, is defined by the paths of consumption and of the stock of natural resources, $\{[\overline{C}(t)]_{t\in[0,\infty)}, [\overline{N}(t)]_{t\in[0,\infty)}\}$, where $\overline{C}(t) = \bar{c}e^{\gamma t}$ and $\overline{N}(t) = \overline{n}e^{\gamma t}$, such that the endogenous growth rate γ and $\bar{c}$ and $\overline{n}$ are jointly determined from the steady state solution of the problem for the centralized economy.

Given the curvature properties of the utility function and of the equation for the accumulation of the natural resource the first order conditions are both necessary and sufficient.

The current value Hamiltonian is

$$H(c, n, q) = u(c, n) + q((\mu - \gamma_n)n - \alpha c),$$

and the first order conditions are

$$u_c(c^*(t), n(t)) = \alpha q(t)$$

$$\dot{q}(t) = (\rho^* - \mu + \gamma_n)q(t) - u_n(c^*(t), n(t))$$

for every admissible trajectories verifying equation (9.2) and the initial condition and the transversality condition

$$\lim_{t\to\infty} e^{-\rho^* t} q(t)n(t) = 0.$$

Proposition 1. If assumption 1 holds then the long-run endogenous growth rate for the natural resource is

$$\overline{\gamma}_n = \frac{1}{\sigma}\left(\frac{(1+\varphi)\mu + (1-\sigma)\gamma_A - \rho}{1+\varphi}\right) \tag{9.6}$$

such that

$$0 < \overline{\gamma}_n < \mu,$$

and the steady state values for the detrended variables verify

$$\frac{\bar{c}}{\overline{n}} = A(0)(\mu - \overline{\gamma}_n) \tag{9.7}$$

Proof. As

$$\begin{aligned} u_c &= c^{-\sigma} n^{\varphi(1-\sigma)} \\ u_n &= \varphi \left(c^{1-\sigma} n^{\varphi(1-\sigma)-1} \right) \end{aligned}$$

Using the condition $u_c(c,n) = \alpha q$ then we determine

$$c = \left(\alpha q n^{\varphi(\sigma-1)} \right)^{-\dfrac{1}{\sigma}}$$

then substituting in $u_n(c,n)$ we get

$$u_n = \varphi \alpha^{1-\frac{1}{\sigma}} q^{1-\frac{1}{\sigma}} n^{\varphi \frac{1-\sigma}{\sigma}-1} = q\varphi\Gamma(q,n)$$

where we define

$$\Gamma(q,n) \equiv \alpha^{1-\frac{1}{\sigma}} q^{-\frac{1}{\sigma}} n^{\frac{\varphi(1-\sigma)}{\sigma}-1}.$$

Then, because we have

$$\alpha c = \Gamma(q,n) n$$

the modified Hamiltonian dynamic system can be written as

$$\dot{q}(t) = q(t) \left(\rho^* - \mu + \gamma_n - \varphi\, \Gamma(q(t), n(t)) \right) \tag{9.8}$$

$$\dot{n}(t) = n(t) \left(\mu - \gamma_n - \Gamma(q(t), n(t)) \right). \tag{9.9}$$

In the steady state $\dot{q} = \dot{n} = 0$ we can determine the endogenous rate of growth γ_n by setting

$$\Gamma(\bar{q}, \bar{n}) = \mu - \gamma_n = \frac{\rho^* - \mu + \gamma_n}{\varphi}$$

and solving for γ_n the equation

$$(\mu - \gamma_n)(1 + \varphi) = \rho^* = \rho - (1 - \sigma)(\gamma_A + (1 + \varphi)\gamma_n)$$

We call the solution $\bar{\gamma}_n$ and write it in equation (9.6). As $\rho^* > 0$ then $\bar{\gamma}_n < \mu$. and $\bar{\gamma}_n > 0$ if assumption 1 holds.

$$\dot{q}(t) = \varphi q(t) \left(\mu - \bar{\gamma}_n - \Gamma(q(t), n(t))\right) \tag{9.10}$$

$$\dot{n}(t) = n(t) \left(\mu - \bar{\gamma}_n - \Gamma(q(t), n(t))\right) \tag{9.11}$$

Therefore, we get $\frac{d\ln(q(t))}{dt} = \varphi \frac{d\ln(n(t))}{dt}$, which means that the dynamics of the system will tend to be degenerate.

Defining

$$z \equiv \frac{c}{n} = \frac{1}{\alpha}\Gamma(q, n)$$

as $\dot{\Gamma} = -(1+\varphi)\left(\mu - \bar{\gamma}_n - \alpha c/n\right)$ then we easily get

$$\dot{z}(t) = \frac{1}{\alpha}\dot{\Gamma} = (1+\varphi)(z(t) - \overline{z})z(t) \tag{9.12}$$

where

$$\overline{z} = \frac{\mu - \bar{\gamma}_n}{\alpha} = \frac{A(0)}{\sigma(1+\varphi)}\left(\rho - (1-\sigma)(\gamma_A + (1+\varphi)\mu\right) \tag{9.13}$$

□

9.4 Dynamics

Proposition 2. There are no transitional dynamics,

$$C(t) = \overline{C}(t) = \overline{z}n(0)e^{\gamma t} \tag{9.14}$$

$$N(t) = \overline{N}(t) = n(0)e^{\gamma_n t} \tag{9.15}$$

Proof. The general solution for $z(t)$ is

$$\begin{aligned} z(t) &= \overline{z}\left(1 + k_z \overline{z} e^{\alpha(1+\varphi)\overline{z}t}\right)^{-1} \\ &= \overline{z}\left(1 + k_z \overline{z} e^{\rho^* t}\right)^{-1}, \end{aligned} \tag{9.16}$$

where k_z is a constant of integration. We will determine k_z such that the transversality condition holds. But as

$$\dot{n}(t) = n(t)(\mu - \bar{\gamma}_n - \alpha z(t))$$

then the solution for $n(t)$ becomes

$$\begin{aligned} n(t) &= k_n \exp\left\{ \int_0^t (\mu^* - \alpha z(s)) ds \right\} = \\ &= k_n \exp\left\{ \mu^* t - \alpha \overline{z} \left[\left(s - \frac{1}{\rho^*} \ln\left(1 + \overline{z} k_z e^{\rho^* s}\right) \right) \Bigg|_0^t \right] \right\} = \\ &= k_n \left(\frac{1 + \overline{z} k_z e^{\rho^* t}}{1 + \overline{z} k_z} \right)^{\dfrac{1}{1+\varphi}}. \end{aligned}$$

We can determine the constant of integration k_n by using the data on n at time $t = 0$. Then we get $w(0) = k_n = w_0$ which is given. Therefore

$$\begin{aligned} \lim_{t\to\infty} e^{-\rho^* t} q(t) n(t) &= \lim_{t\to\infty} e^{-\rho^* t} \alpha^{-1} z(t)^{-\sigma} n(t)^{(1+\varphi)(1-\sigma)} = \\ &= \lim_{t\to\infty} \alpha^{-1} \overline{z}^{-\sigma} n(0)^{(1+\varphi)(1-\sigma)} e^{-\rho^* t} \left(1 + k_z \overline{z} e^{\rho^* t}\right)^{\sigma} \left(\frac{1 + k_z \overline{z} e^{\rho^* t}}{1 + k_z \overline{z}} \right)^{1-\sigma} = \\ &= \lim_{t\to\infty} e^{-\rho^* t} \left(1 + k_z \overline{z} e^{\rho^* t}\right) = \\ &= \lim_{t\to\infty} \left(e^{-\rho^* t} + k_z \overline{z}\right) = \\ &= k_z \overline{z} \end{aligned}$$

which is equal to 0 if $k_z = 0$. Therefore $z(t) = \overline{z}$ and $n(t) = n_0$ are the solutions for the centralized model. □

9.5 GDP growth

As there are no transitional dynamics the level of the product is always along a BGP, which is

$$Y(t) = \bar{Y}(t) = A(0) n(0) (\mu - \gamma_n) e^{\bar{\gamma} t},$$

where the long run growth of rate is

$$\bar{\gamma} = \gamma_A + \gamma_n = \frac{1}{\sigma}\left(\frac{(1+\varphi)\mu + (1+\sigma\varphi)\gamma_A - \rho}{1+\varphi}\right) \tag{9.17}$$

which, from the constraints on γ_n, is also bounded by the rate of growth of natural resources:

$$\gamma_A < \gamma < \gamma_A + \mu.$$

9.6 Conclusions

1. the rate of growth of the utilization of the renewable resource is endogenous and depends on the rate of growth of the technical progress, on the rate of growth of the renewable resource, and on preferences parameters;

2. in order for the growth of the GDP to be sustainable, it should be bounded by both sides: there is a lower bound in the rate of growth which is equal to the rate of growth of the renewable resources; and the upper bound is given by the sum of the rate of growth of the renewable resources and the technical progress;

3. observe that the rate of growth of technical progress γ_A may have a negative effect on γ_n, but always has a positive effect over γ;

4. the difference $\mu - \gamma_A$ has also a positive level effect over the product.

References tahvonen&kuuluvainen93, smulders&gradus96, li&lofgren2000.

Chapter 10

Growth in open economies

The main difference between closed and small open economies is that in the second type of economies the real rate of return is partly or totally exogenous. If the ”border” does not exist (from both the institutional or behavioural points of view) then the real rate of return is exogenous.

To illustrate this, we present a model for a small open economy, both in the good and asset markets, where the law of one price applies. The trade balance may not clear in the short run, therefore, the economy may accumulate a net balance of foreign assets. As the international capital markets are perfect and the economy is small, the rate of return on foreign bonds, r, is given and constant. The economy produces a tradeable good by using only capital by means of a linear technology, of the AK type. We assume that there are convex adjustment costs in investment.

10.1 The model

In an open economy the equilibrium in the product market is

$$Y = C + IE + TB$$

where IE is the investment expenditure, and TB is the balance of trade, exports minus imports of goods and services.

We assume that the technology is of the AK , which implies that the supply of the domestic good is

$$Y(t) = AK(t)$$

and the gross investment is

$$I = \dot{K} + \delta K$$

We assume that the expenditure on investment, IE involves adjustment costs

$$IE = I\left(1 + \frac{\zeta}{2}\frac{I}{K}\right),$$

meaning that the economy in order to increase the gross investment by I has to incur a cost $\zeta I^2/2K$. That is, Tobin's q is different from 1. Because the domestic and foreign goods are perfectly substitutable, the law of one price prevails, and we can determine the trade balance as

$$TB = Y - C - IE = AK - C - I\left(1 + \frac{\zeta}{2}\frac{I}{K}\right)$$

The financial constraint for the economy is

$$TB + IB + KB = 0$$

where IB is the incomes balance and BK is the long term capital balance (there is no flows of foreign currency).

Let B the stock of foreign assets in the portfolio of the economy. This means that if $B > 0$ the economy is a net creditor and if $B < 0$ the economy is a net debtor. We assume that international capital markets are perfect and the economy is small. This implies that

$$IB = rB$$

where r is independent from B (implicitly the economy may borrow or lend at any level without changes in the rate of interest). At last, the capital balance is equal to the symmetric of the flow of foreign assets.

$$KB = -\dot{B}$$

Therefore, the financial constraint of the economy is

$$\dot{B} = AK(t) - C(t) - I(t)\left(1 + \frac{\zeta}{2}\frac{I(t)}{K(t)}\right) + rB(t),\ t \geq 0$$

As the markets are perfects and there are no externalities, our economy is Pareto optimal.

We assume a central planer who choses paths of consumption, $[C(t)]_{t\geq 0}$, investment, $[I(t)]_{t\geq 0}$, stock of capital, $[K(t)]_{t\geq 0}$ and net external position $[B(t)]_{t\geq 0}$, for a small open economy, such that the representative agent maximizes the intertemporal utility function

$$V([C],[i]) \equiv \int_0^\infty \frac{C(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \tag{10.1}$$

subject to the following

$$\begin{aligned}
\dot{K} &= I - \delta K && (10.2)\\
\dot{B} &= AK - C - I\left(1 + \frac{\zeta}{2}\frac{I}{K}\right) + rB && (10.3)\\
K(0) &= K_0 \ \text{given} && (10.4)\\
B(0) &= B_0 \ \text{given} && (10.5)\\
\lim_{t\to\infty} B(t)e^{-rt} &\geq 0 && (10.6)\\
\lim_{t\to\infty} K(t)e^{-At} &\geq 0 && (10.7)
\end{aligned}$$

where all the parameters, σ, δ, ρ, A, and ζ are strictly positive.

The current value Hamiltonian is

$$H = \frac{C^{1-\sigma}}{1-\sigma} + Q_k(I - \delta K) + Q_b\left[AK - C - I\left(1 + \frac{\zeta}{2}\frac{I}{K}\right) + rB\right]$$

where Q_k and Q_b are the instantaneous values for the co-state variables associated to the stock of capital and external net asset position.

The optimal instantaneous levels for consumption and investment are determined from $\partial H/\partial C = 0$ and $\partial H/\partial I = 0$, that is

$$
\begin{aligned}
C^{-\sigma} &= Q_k && (10.8)\\
I &= \frac{1}{\zeta}\left(\frac{Q_k}{Q_b} - 1\right) K && (10.9)
\end{aligned}
$$

The Euler equations are

$$
\begin{aligned}
\dot{Q}_b &= Q_b(\rho - r) && (10.10)\\
\dot{Q}_k &= Q_k(\rho + \delta) - Q_b \frac{\zeta}{2}\left(\frac{I}{K}\right)^2. && (10.11)
\end{aligned}
$$

Let us define the relative value of physical relative to the the external position

$$
q(t) \equiv \frac{Q_k(t)}{Q_b(t)}. \tag{10.12}
$$

Then we can express the first order conditions equivalently as

$$
\begin{aligned}
\dot{C} &= C\left(\frac{r-\rho}{\sigma}\right) && (10.13)\\
\dot{q} &= (r+\delta)q - \frac{(q-1)^2}{2\zeta} - A && (10.14)\\
\dot{K} &= \left(\frac{q-1}{\zeta} - \delta\right) K && (10.15)\\
\dot{B} &= \left(A - \frac{(q+1)(q-1)}{2\zeta}\right) K - C + rB && (10.16)\\
0 &= \lim_{t\to\infty} B(t)C(t)^{-\sigma}e^{-\rho t} && (10.17)\\
0 &= \lim_{t\to\infty} q(t)C(t)^{-\sigma}K(t)e^{-\rho t} && (10.18)
\end{aligned}
$$

for $B(0) = B_0$ and $K(0) = K_0$ given.

10.2 The balanced growth path

We say that a BGP exists if the model admits a solution, for a particular choice of parameters, such that the the economy converges to an unbounded solution which verifies the transversality conditions.

In our case, the BGP is defined by the orbits such that we have, for every $t \geq 0$,

$$\overline{C}(t) = \bar{c}e^{\gamma t},\ \overline{K}(t) = \bar{k}e^{\gamma t},\ \overline{B}(t) = \bar{b}e^{\gamma t},\ \bar{q}(t) = \bar{q}$$

Proposition (Necessary conditions for the existence of a BGP)
Let

$$\gamma = \frac{r - \rho}{\sigma} \tag{10.19}$$

and

$$A^* \equiv \delta + r + \frac{\zeta}{2}(\delta + \gamma)(\delta + 2r - \gamma).$$

If $A = A^*$ and $r > \gamma$ then there exists a BGP such that

$$\overline{C}(t) = \left((r - \gamma)(\bar{q}K_0 + B_0)\right) e^{\gamma t} \tag{10.20}$$

$$\bar{q}(t) = 1 + \zeta(\delta + \gamma) \tag{10.21}$$

$$\overline{K}(t) = K_0 e^{\gamma t} \tag{10.22}$$

$$\overline{B}(t) = B_0 e^{\gamma t} \tag{10.23}$$

Proof. In order to prove the existence of a BGP, we assume that the variables have the representation as $X(t) = x(t)\exp\gamma t$. Then, the system in detrended variables comes

$$\begin{aligned}
\dot{c} &= c\left(\frac{r-\rho}{\sigma}-\gamma\right) && (10.24)\\
\dot{q} &= (r+\delta)q - \frac{(q-1)^2}{2\zeta} - A && (10.25)\\
\dot{k} &= \left(\frac{q-1}{\zeta}-(\delta+\gamma)\right)k && (10.26)\\
\dot{b} &= \left(A - \frac{(q+1)(q-1)}{2\zeta}\right)k - c + (r-\gamma)b && (10.27)\\
\lim_{t\to\infty} b(t)c(t)^{-\sigma}e^{(\gamma-r)t} &= 0 && (10.28)\\
\lim_{t\to\infty} q(t)c(t)^{-\sigma}k(t)e^{(\gamma-r)t} &= 0 && (10.29)
\end{aligned}$$

A BGP exists if and only if there is a solution such that $\lim_{t\to\infty} x(t) = \overline{x}$. The long-run growth rate, see equation (10.19), is taken from equation (10.24). From equation (10.26), we get

$$\overline{q} = 1 + \zeta(\delta+\gamma),$$

which should also be the equilibrium point for equation (10.25). This will only be true if and only if $A = A^*$. If we substitute γ and A in equation (10.27), then the equilibrium values for c, k, q and b, if they exist, should verify the relationship

$$\overline{c} = (r-\gamma)(\overline{q}\overline{k} + \overline{b}),$$

which is a two-dimensional linear manifold. In order to determine those values, we have to try to solve explicitly the system and use the transversality conditions. This can be done because the system is recursive. From equations (10.24) and (10.19) then $c(t) = c(0) = x_1$ for any value of t. Equation (10.25), with $A = A^*$ has the explicit general solution

$$q(t) = 1 + \zeta(r+\delta) - \zeta \mid r-\gamma| i \tan\left(\frac{\mid r-\gamma\mid}{2} i(t+x_2)\right)$$

where $i = \sqrt{-1}$ and x_2 is an arbitrary constant. Note that, for any finite value of the arbitrary constant x_2 we get

$$\lim_{t\to\infty} q(t) = \overline{q}.$$

If we write $\phi(t) = \dfrac{q(t) - \overline{q}}{\zeta}$ then equation (10.26) may be written as $\dot{k} = \phi(t)k$ which has the general solution, where $k(0) = K_0$ is given

$$k(t) = k(0)e^{\int_0^t \phi(s)ds}.$$

If we substitute in the transversality condition (10.29) we have

$$\overline{q}x_1^{-\sigma}k_0 \lim_{t\to\infty} e^{\int_0^t \phi(s)ds-(r-\gamma)t}$$

as

$$\lim_{t\to\infty} e^{\int_0^t \phi(s)ds-(r-\gamma)t} = \begin{cases} 0 & \text{if} \quad x_2 = -t \\ \infty & \text{if} \quad x_2 \neq -t \end{cases}$$

then the transversality condition will only hold if $x_2 = -t$. In this case then $q(t) = \overline{q}$, $\phi(t) = 0$ and $k(t) = K_0$ for any $t \geq 0$. If we substitute those values in the equation for b, (10.27), we get

$$\dot{b} = \overline{q}(r-\gamma)K_0 - x_1 + (r-\gamma)b$$

which has the general solution

$$b(t) = B_0e^{(r-\gamma)t} - \frac{\overline{q}(r-\gamma)K_0 - x_1}{r-\gamma}\left(1 - e^{(r-\gamma)t}\right).$$

To get the arbitrary constant x_1 we substitute in the the transversality condition (10.28). As

$$\begin{aligned} x_1^{-\sigma}b(t)e^{(\gamma-r)t} &= \frac{x_1^{-\sigma}}{r-\gamma}\left[x_1 - \overline{q}(r-\gamma)K_0 + ((r-\gamma)(\overline{q}K_0 + B_0) - x_1)\, e^{(r-\gamma)t}\right] e^{(\gamma-r)t} = \\ &= \frac{x_1^{-\sigma}}{r-\gamma}\left[(x_1 - \overline{q}(r-\gamma)K_0)\, e^{(\gamma-r)t} + (r-\gamma)(\overline{q}K_0 + B_0) - x_1\right] = \\ &= 0 \end{aligned}$$

if $r > \gamma$ and $x_1 = (r-\gamma)(\overline{q}K_0 + B_0)$. □

The growth rate is positive and the transversality conditions are verified if:

1. if $\sigma \geq 1$ and $\rho < r$: that is if the elasticity of intertemporal substitution is low and the economy is more patient than the world financial markets;
2. if $0 < \sigma < 1$ and $r(1-\sigma) < \rho < r$: that is if the elasticity of intertemporal substitution is high and the economy is more patient, with constraints, than the world financial markets.

10.3 Transitional dynamics

Proposition There is no transitional dynamics.

Proof. We have implicitly proved this because we solved the model explicitly. Alternatively, if we perform a qualitative study in the neighborhood of the BGP, after substituting γ and $A = A^*$ in the system (10.24)-(10.27), we get the jacobian,

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r-\gamma & 0 & 0 \\ 0 & \dfrac{\overline{k}}{\zeta} & 0 & 0 \\ -1 & -\dfrac{\overline{k}\overline{q}}{\zeta} & \overline{q}(r-\gamma) & r-\gamma \end{pmatrix}$$

As the characteristic equation is

$$c(\lambda) = \lambda^2 \left(\lambda^2 - 2(r-\gamma)\lambda + (r-\gamma)^2\right),$$

then the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = r - \gamma$. As $r > \gamma$ for the transversality condition to hold, then the zero eigenvalues are related to the dynamics of the state variables, and there is no transitional dynamics. □

Therefore, the dynamics of the GDP along the BGP is

$$Y(t) = \bar{Y}(t) = A^* K_0 e^{\gamma t}$$

given the "knife-edge" structure of the model, increases in productivity A have only level effects is they are positively correlated to the world rate of return, which should be a positive function of the world's productivity.

10.4 Conclusions

Main comments:

1. to get all the previous results we assume that the economy is always small. If the transversality does not hold for B, then the economy will become big (a big debtor or a big creditor). In this case, it would influence the world interest rate r;

2. the non-existence of transitional dynamics should be no surprise as there are two well know sources of degenerate dynamics: one related to the existence of a BGP and the other to the dynamics of small open economies in perfect international markets;

3. even in the case in which $r = \rho$ the existence of equilibrium is only possible if $A = \delta + r + \frac{\zeta\delta}{2}(\delta + 2r)$ if there are costs of adjustment and a linear production function.

4. however, under a "knife-edge" condition there is positive growth for a small open economy if it is more patient than the world financial markets.

References : Barro and Sala-i-Martin (2004):sec. 3.3, 3.4

Chapter 11

Growth and public debt

At the macroeconomic level the ”technology” for production of some goods involve externalities, for example the provision of infrastructures, the provision of health services, or the functioning of a legal system making possible the enforcement of contracts.

There is an institutional side to this provision related to the concrete historical developments which lead to the onset of the modern state and with the characteristics of the institutions that shape the way the decisions are made, the decisors are chosen. This is a wide and important issue and is dealt in several areas in economic thought and theory (and empirics, off course).

Next we only deal with some simple consequences on the rate of growth from financing the expenditures related to the provision of public goods through distortionary taxation and government debt, by extending a simple model presented in barro1990.

11.1 The model

The representative private consumer-producer maximizes the intertemporal utility functional

$$\int_0^\infty \frac{C(t)^{1-\sigma}}{1-\sigma} e^{-\rho t} dt$$

where C is consumption subject to the budget constraint

$$\dot{K} + \dot{B} = (1 - \tau(t))\,(r(t)B(t) + Y(t)) - C(t) \tag{11.1}$$

where K and B are the stocks of physical capital and government debt, τ is the income-tax rate r is the rate of interest on government bonds and Y is the income from production.

We assume that the rate of time preference is positive, $\rho > 0$ and the intertemporal elasticity of substitution is below unity $\sigma \geq 1$. The initial levels for the two stocks are given and the are non-negative present value terms $\lim_{t\to\infty}(K(t) + B(t))e^{\int_0^t r(s)ds} \geq 0$.

There is only one good in this economy, which is produced by a technology using public goods provided by the government G. We posit a Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha}G(t)^{1-\alpha}$$

where $0 < \alpha < 1$ is the share of private capital. A non-arbitrage condition holds such that the rate of interest on government bonds is instantaneously equalized to the marginal productivity of capital

$$r(t) = \alpha K(t)^{\alpha-1}G(t)^{\alpha-1}.$$

From the household optimality conditions we derive the Euler equation

$$\frac{\dot{C}}{C(t)} = \frac{(1 - \tau(t))r(t) - \rho}{\sigma} \tag{11.2}$$

and the transversality condition $\lim_{t\to\infty} C(t)^{-\sigma}(K(t) + B(t))e^{-\rho t} = 0$ should hold.

The government finances expenditures by taxing and issuing bonds. The government budget constraint is

$$\dot{B} = r(t)B(t) + \tau(t)\,(r(t)B(t) + Y(t)) - G(t) \tag{11.3}$$

where B is the stock of outstanding government bonds (indeed net debt).

Consolidating the government and the household budget constraints we get the good's market clearing equation

$$\dot{K} = Y(t) - C(t) - G(t). \tag{11.4}$$

In order to close the model, and because we have three fiscal instruments, taxes, government expenditures and debt financing we assume that the government has a fiscal rule consisting in keeping the debt-GDP ratio constant: formally

$$B(t) = \bar{b}Y(t) \tag{11.5}$$

The dynamic general equilibrium (DGE) is defined by the paths $(K(t), C(t), B(t), G(t), r(t))_{t\in[0,\infty)}$ verifying equations (11.2), (11.4) and (11.3) together with the fiscal rule (11.5) and the transversality and initial conditions ($K(0) = K_0$ and $B(0) = B_0$ given).

An equilibrium balanced growth path (BGP) is a particular DGE such that the rates of growth for K, B and C are asymptotically constant.

As the necessary conditions for the existence BGP hold, next we address their existence and multiplicity and characterise their properties, including the transitional dynamics: we reduce the dimension of the dynamic system by detrending the system.

This is an alternative technique to get the long run endogenous growth rate and to determine if there is transitional dynamics introduced by mulligan&salaimartin1993.

11.2 The DGE in detrended variables

In the rest of the paper we denote the ratios of government debt and government expenditures over the GDP by $b \equiv B/Y$ and $g \equiv G/Y$, respectively.

Introducing g in the production function we get a linear function on the capital stock $Y = A(g)K$ where the productivity is an increasing function of g, as well: $A(g) \equiv g^{\frac{1-\alpha}{\alpha}}$. Then, from the arbitrage condition between returns on private capital and government bonds, the interest rate is also an increasing function of the ratio of government expenditures over the GDP: $r = r(g) = \alpha A(g)$.

Then the rate of growth of consumption is also an increasing function of the government expenditure over the GDP

$$\frac{\dot{C}}{C} = \gamma(g) \equiv \frac{(1-\tau)r(g) - \rho}{\sigma} \tag{11.6}$$

Defining $z \equiv C/K$ we get the rate of growth for private capital as a function of g and z

$$\frac{\dot{K}}{K} = \gamma_K(g,z) \equiv (1-g)A(g) - z \tag{11.7}$$

Therefore

$$\frac{\dot{z}}{z} = \gamma_z(g,z) = \gamma(g) - \gamma_K(g,z) = z - z(g) \tag{11.8}$$

where

$$z(g) = \frac{\left(\sigma(1-g) - \alpha(1-\tau)\right) A(g) + \rho}{\sigma} \tag{11.9}$$

Function $z(g)$ is sometimes interpreted as a Laffer-curve associating government expenditures to the rate of growth of the economy. It has an inverted U form, starting from $z(0) = \rho/\sigma > 0$, reaching a maximum at a point $g = (1-\alpha)(\sigma - \alpha(1-\tau))/\sigma > 0$ and decreasing for higher values of g (possibly becoming negative for large g). The increasing part is related to positive effect that productive government expenditures have on output thus unceasing the rate of capital accumulation. This positive effect is eventually overcome by the effect of the increase in the tax returns generated by the increase in the interest rate.

From the definition of b, the rate of growth of the debt ratio is equal to the difference in the rates of growth of the government debt minus the rate of growth of the GDP

$$\frac{\dot{b}}{b} = \frac{\dot{B}}{B} - \frac{\dot{Y}}{Y} = \gamma_B - \gamma_Y$$

where

$$\frac{\dot{B}}{B} = \gamma_B(g) = (1-\tau)r(g) + \frac{G - \tau Y}{B} = (1-\tau)r(g) + \frac{g-\tau}{b}$$

and

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} + \frac{1-\alpha}{\alpha}\frac{\dot{g}}{g}$$

because $Y = KA(g)^{(1-\alpha)/\alpha}$. On the other hand the fiscal rule is equivalent to setting $b(t) = \bar{b}$ constant, and therefore $\dot{b} = 0$.

Then the model in detrended variables becomes

$$\dot{z} = z\left(z - z(g)\right) \tag{11.10}$$

$$\dot{g} = g\frac{\alpha}{1-\alpha}\left(z - \zeta(g)\right) \tag{11.11}$$

where we define

$$\zeta(g) \equiv (1-g)A(g) - (1-\tau)r(g) + \frac{\tau - g}{\bar{b}}.$$

The following conditions should be verified: $g \in (0,1)$, $z > 0$, and the transversality condition should hold.

11.2.1 Steady state rate of growth

If we denote the (g^*, z^*) the steady state values of (g, z) of system (11.10)-(11.11), then the long run rates of growth for the output, capital stock and the level is debt are

$$\gamma^* = \gamma(g^*) = \frac{(1-\tau)r(g^*) - \rho}{\gamma}$$

$$\gamma_K^* = \gamma_K(g^*, z^*) = (1-g^*)A(g^*) - z^* = \gamma^*$$

implied by the steady state condition $\dot{z} = 0$ in equation (??) and

$$\gamma_B^* = \gamma^*$$

implied by the fiscal rule.

A sufficient condition for the verification of the transversality condition is

$$\rho + (\sigma - 1)\gamma^* > 0.$$

To determine the steady state value for g we set $z(g) = \zeta(g)$ to get the implicit equation

$$\Phi(g, \tau, \bar{b}) \equiv \frac{(1-\tau)(\sigma-1)}{\sigma} r(g) + \frac{g-\tau}{\bar{b}} + \frac{\rho}{\sigma}$$

We assume that $\bar{b} < (\tau\sigma)/\rho$, which with the usual values for the parameters ($\tau \approx 0.3$, $\sigma \approx 2$ and $\rho \approx 0.02$ gives a huge number around 30) holds. Under this assumption, although we cannot determine g^* explicitly the steady state g^* exists and is unique. The steady state is the solution of $\Phi(g) = 0$.

On the other hand it is easy to see that $g^* < \tau - \rho\bar{b}/\sigma$ and we always have $g^* < 1$.

From the solution of equation $\Phi(g) = 0$ for g we determine $g^* = g^*(\tau, \bar{b})$. Therefore the rate of long run growth is also a function of $(\tau, \bar{b})$. We obtain the following comparative statics results at the steady state

$$\left.\frac{dg}{d\bar{b}}\right|_{g=g^*} = \frac{(g^*-\tau)\sigma}{\bar{b}\left[(1-\tau)(\sigma-1)r'(g^*)\bar{b}+\sigma\right]} < 0 \tag{11.12}$$

$$\left.\frac{dg}{d\tau}\right|_{g=g^*} = \frac{(\sigma-1)r(g^*)\bar{b}+\sigma}{\bar{b}\left[(1-\tau)(\sigma-1)r'(g^*)\bar{b}+\sigma\right]} > 0 \tag{11.13}$$

$$\tag{11.14}$$

The effects on the long run growth rate are

$$\frac{\partial\gamma^*}{\partial\bar{b}} = \frac{(1-\tau)r'(g^*)}{\sigma}\left.\frac{dg}{d\bar{b}}\right|_{g=g^*} < 0 \tag{11.15}$$

$$\frac{\partial\gamma^*}{\partial\bar{b}} = \frac{(1-\tau)r'(g^*) - r(g^*)}{\bar{b}\left[(1-\tau)(\sigma-1)r'(g^*)\bar{b}+\sigma\right]} \tag{11.16}$$

has an ambiguous sign.

Therefore, in this simple model while debt financing reduces the long run growth rate, tax finance may reduce or not the long run growth rate.

11.2.2 Transitional dynamics

Figure 11.1 presents a representative phase diagram.

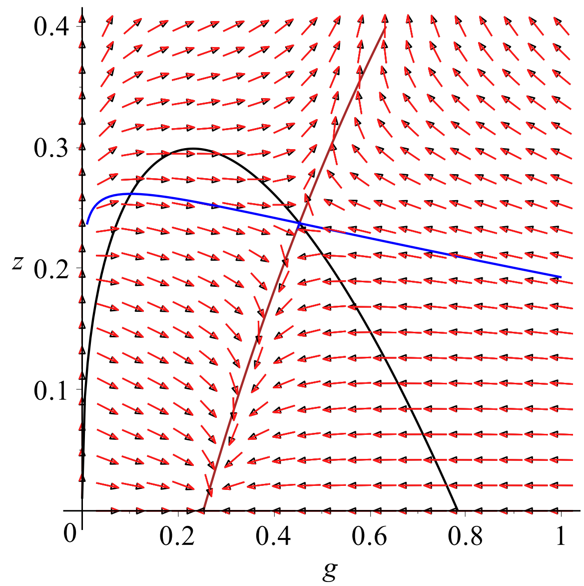

Figure 11.1: Transitional dynamics for $\sigma = 1$, $\alpha = 0.7$, $\rho = 0.02$, $\tau = 0.3$ and $\bar{b} = 0.6$.

11.3 Conclusions

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Appendix A

Appendix

A.1 Homogeneous functions

Let $y = f(x)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$. The function $f(.)$ is homogenous of degree n if

$$f(\lambda x) = f(\lambda x_1, \ldots, \lambda x_n) = \lambda^n y = \lambda^n f(x_1, \ldots, x_n) = \lambda^n f(x)$$

If $f(.)$ is homogeneous of order $n = 1$ thin it is said to be linearly homogeneous.

Homogeneous functions verify the Euler's theorem

$$y = n\left(\frac{\partial f}{\partial x_1} x_1 + \ldots + \frac{\partial f}{\partial x_n} x_n\right)$$

A.2 Static optimization

Let functions $f(x)$ and $g(x)$ be of class $C^2(\mathbb{R}^n)$, i.e. continuous and continuously differentiable up to order 2.

Consider the problem

$$\max_x f(x)$$

and let $x^* = \text{argmax} f(x)$, that is, the value of x which maximizes $f(x)$. A necessary condition for optimality is that

$$\left.\frac{\partial f(x)}{\partial x_i}\right|_{x=x^*} = 0,\ i = 1, \dots, n$$

Consider the problem

$$\max_x \{f(x) :\ g(x) \leq g\}$$

In this case we build the Lagrangean

$$L(x, \lambda) = f(x) + \lambda(g - g(x))$$

the f.o.c are

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= \frac{\partial f}{\partial x_i} - \sum_{j=1}^{n} \lambda_j \frac{\partial g_j}{\partial x_i} = 0,\ i = 1, \dots, n \\ \frac{\partial L}{\partial \lambda_i} &= g_i(x_1, \dots, x_n) - g_i = 0,\ \ i = 1, \dots, n. \end{aligned}$$

A.3 Leibniz integral rule

Let

$$F(t) = \int_{b(t)}^{a(t)} f(x, t) dx$$

The Leibniz integral rule is

$$\frac{dF(t)}{dt} = \int_{b(t)}^{a(t)} \frac{\partial f(x, t)}{\partial t} dx + f(b(t), t) \frac{db(t)}{dt} - f(a(t), t) \frac{da(t)}{dt}.$$

A.4 Functional derivatives

Let us consider the functional over the path $[x(t)]_{t_0}^{t_1}$,

$$V([x]) = \int_{t_0}^{t_1} f(x(t)) dt,$$

where $f(.)$ is a differentiable function. Let us consider a new path $[x_\epsilon(t)]$ in which a "spike" perturbation has been introduced over $[x]$ at time $t_0 \leq \tau \leq t_1$ such that

$$x_\epsilon(t) = x(t) + \varepsilon\delta(\tau - t)$$

where $\delta(.)$ is the Dirac's delta function, which has the properties

$$\delta(\tau - t) = \begin{cases} +\infty, & \text{if } t = \tau \\ 0, & \text{if } t \neq \tau \end{cases}$$

$\int \delta(t)dt = 1$ and $\int g(t)\delta(\tau - t)dt = g(\tau)$. The functional for the perturbed path is

$$V([x_\epsilon]) = \int_{t_0}^{t_1} f(x_\epsilon(t))dt = \int_{t_0}^{t_1} f(x(t) + \epsilon\delta(\tau - t))dt$$

A functional derivative for a change in $x(\tau)$ is defined as

$$\frac{\delta V([x])}{\delta x(\tau)} = \frac{d}{d\varepsilon} V([x_\epsilon]) = \lim_{\epsilon \to 0} \frac{V[x_\epsilon] - V[x]}{\epsilon}.$$

Then

$$\frac{\delta V([x])}{\delta x(\tau)} = \int_{t_0}^{t_1} f^{'}(x(t))\delta(\tau - t))dt = f^{'}(x(\tau))$$

Example Let $f(x) = x^\alpha$ where α is a constant. Then $V([x]) = \int_{t_0}^{t_1} (x(t))^\alpha dt$ and

$$\frac{\delta V([x])}{\delta x(t)} = \alpha(x(t))^{\alpha - 1}$$

for any $t_0 \leq t \leq t_1$.

A.5 Ordinary differential equations

An ordinary differential equation (of first order) is

$$\dot{x} = f(x)$$

where $x = x(t) \in \mathbb{R}^n$ is a function of one independent variable $t \in \mathbb{R}$, generally representing time.

A fixed point (or equilibrium point) of the ordinary differential equation (ODE) is defined as

$$x^* = \{x : \; f(x) = 0\}, \; x^* \in \mathbb{R}^n.$$

A.5.1 Scalar ODE

Let $x(t) \in \mathbb{R}$. An initial value problem is defined by an ODE plus an initial condition

$$\begin{cases} \dot{x} = f(x(t)), & \text{for } t \geq 0 \\ x(0) = x_0, & \text{for } t = 0 \end{cases}$$

where m_0 is given. The solution of the IVP are the flows $x(t, x_0)$, for $t \geq 0$ which solve the problem.

Linear ODE If the ODE is linear and homogeneous, that is

$$\dot{x} = \lambda x$$

it has the general solution

$$x(t) = ke^{\lambda t}, \; t \geq 0$$

where k is an arbitrary constant. Then the solution of the IVP is

$$x(t) = x_0 e^{\lambda t}, \; t \geq 0.$$

For solving the non-homogenous differential equation $\dot{y} = -b + ay$, observe that, if $a \neq 0$ it has the fixed point $\overline{y} = b/a$, we can introduce the transformation $x \equiv y - \overline{y}$ and get equivalently $\dot{x} = \dot{y} = a(y - b/a) = a(y - \overline{y}) = ax$. This equation has the form $\dot{x} = \lambda x$.

Then using the solution for the homogeneous ode we get the solution for the IVP involving a non-homogeneous ode

$$\begin{cases} \dot{y} = -b + ay, & \text{for } t \geq 0 \\ y(0) = y_0, & \text{for } t = 0 \end{cases}$$

as

$$y(t) - \overline{y} = (y_0 - \overline{y})e^{at}.$$

Explicit solutions

Solving a Bernoulli ODE: the Bernoulli ode is an equation of the form

$$\dot{y} = \alpha y^{\beta} + \gamma y \tag{A.1}$$

Define a transformed variable as $x(t) = y(t)^{1-\beta}$. If we time differentiate, we get

$$\dot{x} = (1-\beta)y^{-\beta}\dot{y} = (1-\beta)\left(\alpha + \gamma y^{1-\alpha}\right) = (1-\beta)\left(\alpha + \gamma x\right)$$

which is a linear non-homogeneous equation. Therefore, its solution is

$$y(t) = \left[-\frac{\alpha}{\gamma} + \left(\frac{\alpha}{\gamma} + y(0)^{1-\beta}\right)e^{(1-\beta)\gamma t}\right]^{1/(1-\beta)}.$$

.

Qualitative theory

Consider the non-linear ode $\dot{x} = g(x)$, such that $x \in X$, where X is the state space (i.e, the set of point in which the ode is defined. If g is smooth then a solution to the ode exists, $\varphi(t)$, which is a smooth function of time. Given any point in $x^{'} \in X$ the solution of the ode generates two sets of points belonging to X, the positive orbit $\gamma_{+}(x^{'}) \equiv \{\varphi(t, x^{'}) \in X : \varphi(0) = x^{'}, t \in [0, \infty)\}$ and a negative orbit $\gamma_{-}(x^{'}) \equiv \{\varphi(t, x^{'}) \in X : \varphi(0) = x^{'}, t \in (-\infty, 0]\}$.

Assume that the ode has at least one equilibrium point $x^{*} \equiv \{x \in X : g(x) = 0\}$. The equilibrium point is called asymptotically stable if $\lim_{t\to\infty} \varphi(t, x^{'}) = x^{*}$, (local stability if this holds if $x^{'}$ belongs to a small neighborhood of x^*, global stability if this holds for any $x^{'} \in X$) . In this case $\gamma_+ = [x^{'}, x^*]$. The equilibrium point is unstable if $\lim_{t\to\infty} \varphi(t, x^{'}) = \pm\infty$ is $x^{'} \neq x^*$. In this case $\lim_{t\to-\infty} \varphi(t, x^{'}) = x^*$.

The local asymptotic stability properties may be determined from a linear approximation of the ode in the neighborhood of an equilibrium point. Let $\left.\frac{\partial g}{\partial x}\right|_{x=x^*} \neq 0$. Then, the equilibrium point x^* is locally asymptotically stable if

$$\left.\frac{\partial g}{\partial x}\right|_{x=x^*} < 0.$$

and it is locally unstable if

$$\left.\frac{\partial g}{\partial x}\right|_{x=x^*} > 0.$$

A.5.2 Planar ordinary differential equations

Planar ODE's, in this case autonomous and homogeneous ordinary differential equations are equations of type

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2). \end{aligned}$$

A linear autonomous homogeneous ode is an equation of type

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$$

where $\mathbf{x} = (x_1, x_2)$ and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We find next a solution for the linear planar ODE which will show up in the models presented in the main text.

First, observe that the characteristic polynomial of matrix $\mathbf{A}$ is

$$c(\mathbf{A}, \lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

where $\mathbf{I}$ is the (2×2) identity matrix, and the trace and the determinant of $\mathbf{A}$ are

$$\begin{aligned} \text{tr}(\mathbf{A}) &= a_{11} + a_{22} \\ \det(\mathbf{A}) &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of the characteristic equation $c(\mathbf{A}, \lambda) = 0$,

$$\begin{aligned} \lambda_1 &= \frac{\text{tr}(\mathbf{A})}{2} + \Delta(\mathbf{A})^{1/2} \\ \lambda_2 &= \frac{\text{tr}(\mathbf{A})}{2} - \Delta(\mathbf{A})^{1/2} \end{aligned}$$

where

$$\Delta(\mathbf{A}) = \left(\frac{\text{tr}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A})$$

is the discriminant of $\mathbf{A}$. Then:

1. if $\Delta(\mathbf{A}) > 0$ the two roots are real and distinct with $\lambda_1 > \lambda_2$;

2. if $\Delta(\mathbf{A}) = 0$ the two roots are real and equal with $\lambda_1 = \lambda_2 = \lambda = \text{tr}(\mathbf{A})/2$;

3. if $\Delta(\mathbf{A}) < 0$ the two roots are complex conjugate $\lambda_{1,2} = \alpha \pm \beta i$, where $i^2 = -1$.

It is easy to see that the roots verify the relationship, which is sometimes usefull,

$$\begin{aligned} \lambda_1 + \lambda_2 &= \text{tr}(\mathbf{A}) \\ \lambda_1\lambda_2 &= \det(\mathbf{A}). \end{aligned}$$

Second: any $\mathbf{A}$ matrix there are two matrices, $\mathbf{\Lambda}$ and an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{\Lambda}\mathbf{P} \Leftrightarrow \mathbf{\Lambda} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$$

Matrix $\mathbf{\Lambda}$ has only three forms, called Jordan normal forms:

1. if the eigenvalues are real and distinct, that is, if $\Delta(\mathbf{A}) > 0$ then

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

2. if the eigenvalues are real and equal, that is, if $\Delta(\mathbf{A}) = 0$ then

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

3. if the eigenvalues are complex conjugate, that is, if $\Delta(A) < 0$ then

$$\mathbf{\Lambda} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

We will only have models of the first type, therefore we will concentrate on solutions of the differential equation where matrix A verifies

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathbf{PAP}^{-1}$$

matrix P is the eigenvector matrix.

Now given the initial differential equation $\dot{\mathbf{x}} = \mathbf{Ax}$ if we define a new vector $\mathbf{w} = (w_1, w_2)^\top$ by

$$\mathbf{w} \equiv \mathbf{P}^{-1}\mathbf{x},$$

then we can transform the initial ODE system into

$$\dot{\mathbf{w}} = \mathbf{Jw}$$

that is

$$\begin{aligned} \dot{w}_1 &= \lambda_1 w_1 \\ \dot{w}_2 &= \lambda_2 w_2. \end{aligned}$$

That is, we transformed ths system with two coupled differential equations into a system with two uncoupled scalar linear differential equations. Using the results from the previous section, we have the solution

$$\begin{aligned} w_1(t) &= h_1 e^{\lambda_1 t} \\ w_2(t) &= h_2 e^{\lambda_2 t}. \end{aligned}$$

for two arbitrary constants h_1 and h_2.

Going back to the original variables, we get, from $\mathbf{x} = \mathbf{P}\mathbf{w}$

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{P}^1 w_1(t) + \mathbf{P}^2 w_2(t) \\ &= h_1 \mathbf{P}^1 e^{\lambda_1 t} + h_2 \mathbf{P}^2 e^{\lambda_2 t} \end{aligned}$$

where $\mathbf{P}^j$ is the eigenvector associated to the eigenvalue λ_j, for $j = 1, 2$. We choose h_1 and h_2 from the, initial and/or terminal conditions of the problem.

References: hale&kocak1991

A.6 Optimal control: the discounted infinite horizon problem

Consider the optimal control problem defined over the pair of functions $(x(t), u(t))$ with (positive) orbits for the state variable $x = \{x(t),\ t \in [0, \infty)\}$ and for the control variable $u = \{u(t),\ t \in [0, \infty)\}$,

$$\max_u \int_0^\infty f(x(t), u(t)) e^{-\rho t} dt$$

subject to

$$\begin{cases} \dot{x} &= g(x, u),\ t \geq 0 \\ x(0) &= x_0,\ \text{given},\ t = 0. \end{cases}$$

If $f(x,u)$ and $g(x,u)$ are concave in (x,u), the necessary and sufficient conditions for an optimum are the following: define the current-value Hamiltonian,

$$H(u,x,q) = f(x,u) + qg(x,u)$$

where $q(t)$ is the co-state variable. The solution paths for the problem, (u^*, x^*), verifies the following conditions, according to the Pontryiagin's maximum principle

$$\begin{aligned}
\left.\frac{\partial H}{\partial u}\right|_{(x,u)=(x^*,u^*)} &= 0\\
\dot{q} &= \rho q - \left.\frac{\partial H}{\partial x}\right|_{(x,u)=(x^*,u^*)}\\
\dot{x} &= g(x,u)|_{(x,u)=(x^*,u^*)}\\
x^*(0) &= x_0\\
0 &= \lim_{t\to\infty} q(t)e^{-\rho t}
\end{aligned}$$

the last condition is called the transversality condition.

If we have the alternative problem where the restrictions are

$$\begin{cases}
\dot{x} & = g(x,u),\ t \geq 0\\
x(0) & = x_0,\ \text{given},\ t = 0,\\
0 & \leq \lim_{t\to\infty} h(t)x(t),
\end{cases}$$

the first order conditions are the same, except for the transversality condition, which becomes

$$\lim_{t\to\infty} q(t)x^*(t)e^{-\rho t} = 0.$$

A.7 Comparative dynamics

Consider the scalar ODE $\dot{x} = f(x,\varphi)$ where φ is a parameter and assume there is one steady state $\bar{x} = \{x : \ f(x,\varphi) = 0\}$. This is true if the implicit function theorem holds: i.e., $\frac{\partial f}{\partial x}(\bar{x},\varphi) \neq 0$.

If function $f(x, \varphi)$ is monotone in φ we would have $\bar{x} = g(\varphi)$. Now consider one steady state associated to a particular value of φ, for instance $\varphi = \varphi_0$ and denote the associated steady state by $\bar{x}_0 = g(\varphi_0)$. Off course, we have $f(\bar{x}_0, \varphi_0) = 0$ and therefore locally $\dot{x} = 0$.

Next we introduce a permanent change in φ, $\delta\varphi$ and call $\varphi_1 = \varphi_0 + \delta\varphi$. Clearly $f(\bar{x}_0, \varphi_1) \neq 0$. However if we set $\bar{x}_1 = g(\varphi_1) \approx \bar{x}_0 + \delta\bar{x}$ then $f(\bar{x}_1, \varphi_1) \approx 0$.

Comparative dynamics corresponds to tracing out the evolution of $x(t)$ after that change in the parameter assuming we start from $(\bar{x}_0, \varphi_0)$ such that it finishes at the point $(\bar{x}_1, \varphi_1)$.

Two cases are possible: (1) if $\frac{\partial f}{\partial x}(\bar{x}_0, \varphi_0) < 0$ then x will change continuously in time; or if (2) if $\frac{\partial f}{\partial x}(\bar{x}_0, \varphi_0) > 0$ then x will change discontinuously.

To see this let $\delta x(t) = x(t) - \bar{x}_0$ be the change in x resulting from the change in the parameter $\delta\varphi$. Using a first-order Taylor approximation (recall that we are assuming that $\frac{\partial f}{\partial x}(\bar{x}_0, \varphi_0) \neq 0$) we have

$$
\begin{aligned}
\dot{x} &= f(x, \varphi) = \\
&= f(\bar{x}_0, \varphi_0) + \frac{\partial f}{\partial x}(\bar{x}_0, \varphi_0)\delta x(t) + \frac{\partial f}{\partial \varphi}(\bar{x}_0, \varphi_0)\delta\varphi = \\
&= \frac{\partial f}{\partial x}(\bar{x}_0, \varphi_0)\delta x(t) + \frac{\partial f}{\partial \varphi}(\bar{x}_0, \varphi_0)\delta\varphi
\end{aligned}
\tag{A.2}
$$

Noting that $\frac{d\delta x(t)}{dt} = \dot{x}$ and denoting the multiplier as

$$
y(t) = \frac{\delta x(t)}{\delta\varphi}
$$

we can write equation (A.2) as

$$
\dot{y} = \lambda_x y + \lambda_\varphi \tag{A.3}
$$

where $\lambda_x \equiv \frac{\partial f}{\partial x}(\bar{x}_0, \varphi_0)$ and $\lambda_\varphi \equiv \frac{\partial f}{\partial \varphi}(\bar{x}_0, \varphi_0)$. The comparative dynamics exercise is equivalent to solving equation (A.3) together with the conditions $y(0) = \delta x(0) = 0$ and

$$
y(\infty) = \bar{y} \equiv -\frac{\lambda_\varphi}{\lambda_x} = -\frac{\frac{\partial f}{\partial \varphi}(\bar{x}_0, \varphi_0)}{\frac{\partial f}{\partial x}(\bar{x}_0, \varphi_0)} = \frac{\delta\bar{x}}{\delta\varphi}
$$

which we call long-run multiplier. Two cases are possible,

1. if $\lambda_x < 0$ then $y(t)$ can be continuous at time $t = 0$ and the solution to the problem is

$$y(t) = \bar{y}\left(1 - e^{\lambda_x t}\right)$$

   which verifies our terminal condition $\lim_{t\to\infty} y(t) = \bar{y}$. Therefore, the short-run multipliers are

$$\frac{\delta x(t)}{\delta \varphi} = \frac{\delta \bar{x}}{\delta \varphi}\left(1 - e^{\lambda_x t}\right) \text{ for } t \in [0, \infty)$$

2. if $\lambda_x > 0$ there is no continuous solution converging to $\bar{y}$. In this case, at time $t = 0$ the multiplier should "jump" discontinuously to $\bar{y}$. That is $y(t) = y(0^+) = \bar{y}$. In this case

$$\frac{\delta x(t)}{\delta \varphi} = \frac{\delta \bar{x}}{\delta \varphi} \text{ for } t \in (0, \infty)$$

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