The household problem: consumption, savings and asset accumulation

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1 Introduction

This lecture note is dedicated to the household problem from the perspective of macroeconomics. This means that we are concerned with the allocation of household resources across time. It is thus a partial equilibrium exercise. The household problem is a building block of macroeconomic models usually featuring teh supply side of savings and the demand side in the asset markets.

In this note the analysis is conducted in continuous time and within a deterministic setting. That is, it is assumed that the household has perfect information.

After studying the main concepts of the intertemporal choice in a continuous-time setting we apply them to both intertemporal additive and intertemporally dependent preferences.

We review the household problem in several environments: without resource constraints, with an initial finite resource (cake eating problem), with borrowing constraints, with financial and non-financial income, and with a stochastic horizon. We distinguish between the effect of anticipated changes in income (both non-financial and financial) and non-anticipated income.

This allows us to provide background to the benchmark household problem in macroeconomics: the discounted infinite horizon problem in a complete market setting in which the rate of return is equal to the rate of time preference but in which the household has a solvability (or non-Ponzi game) constraint.

Although this model has a closed form (explicit) solution, we present the method of comparative statics applied to this case. This has an interest per se but also is the only tool available to characterize the solution to the intertemporally dependent household problem.

This model provides a relatively simple theory for one of the stylized facts in macroeconomics: the relatively more persistent behavior of consumption relative to aggregate income.

In section 2 we present the benchmark intertemporal additive utility case. In section 3 we present the household problem in the continuous time setting. In section 4.2 the method of continuous-time comparative dynamics is applied to the additive model. In section 5 the habit formation is presented and its comparative dynamics is studied, for an increase in non-financial income.

2 Intertemporaly additive utility functionals

In this section we provide some definitions regarding the intertemporal preference properties which are implicit in intertemporal utility functionals. We start with the case of additive utility functionals both in discrete and continuous time, in subsections 2.1 and 2.2. In section 5 we study, with the same concepts, a non-additive intertemporal utility functional arising in the habit formation model.[1]

2.1 Discrete time

Let measurements be taken at discrete time intervals $\mathrm{T} = \{\ 0, 1, \dots, t, \dots, T\}$. In this discrete time setting, the sequence of consumption $\{c\} = \{c_0, c_1, \dots, c_t, \dots, c_T\}$ is valued by the utility functional

$$\mathsf{U}\ [c] = \mathsf{U}[\{c_0, c_1, \dots, c_t, \dots, c_T\}].$$

The functional $\mathsf{U}[c]$ maps every consumption sequence into a number. The higher $\mathsf{U}[c]$ is the higher is the value of a consumption sequence.

Given the postulates from choice theory, we represent an order relationship between consumption sequences by a cardinal relationship between their intertemporal utilities. That is, given two consumption sequences $\{c'\} = \{c'_0, c'_1, \dots, c'_t, \dots, c'_T\}$ and $\{c''\} = \{c''_0, c''_1, \dots, c''_t, \dots, c''_T\}$ the household is indifferent between the two if and only if $\mathsf{U}[c'] = \mathsf{U}[c'']$ and the household prefers c' to c'' if and only if $\mathsf{U}[c'] > \mathsf{U}[c'']$.

In order to characterize the implicit properties introduced by the utility functional $\mathsf{U}[c]$ consider a given consumption sequence $\{c\}$ and introduce a change in consumption at any point in time t, i.e, consider the change from $\{c\}\ = \{c_0, c_1, \dots, c_t, \dots, c_T\}$ to $\{c + dc\} = \{c_0, c_1, \dots, c_t + dc_t, \dots, c_T\}$, where $\{dc\}\ = \{\ 0, \dots, 0, dc_t, 0, \dots 0\}$. The value of the consumtion sequence changes from $\mathsf{U}[c]$ to $\mathsf{U}[c + dc] = \mathsf{U}[c] + d\mathsf{U}[c; c_t]$ where

$$d\mathsf{U}[c; c_t] = \frac{\partial \mathsf{U}[c]}{\partial c_t}\ dc_t,$$

where $\dfrac{\partial \mathsf{U}[c]}{\partial c_t}$ is the partial simple derivative for consumption at time t. We call it the **marginal utility of consumption at any time** t and denote it as

$$\mathsf{U}_t\ [c]\ \equiv \frac{\partial \mathsf{U}[c]}{\partial c_t},\ \text{for any}\ \ t \in \mathrm{T}.$$

Observe that $\mathsf{U}_t\ [c]$ is also a functional. Therefore we can define the second partial derivatives derivatives

$$U_{t,t'}[c]\ \equiv\ \frac{\partial^2 \mathsf{U}[c]}{\partial c_t\, \partial c_{t'}}\, c_{t'}, \text{for any}\, t, t' \in \mathrm{T},$$

[1]For a thorough presentation of preference structures in macroeconomics, in discrete time, see Backus, Routledge, and Zin, 2004.

which yields the change in the marginal utility of consumption at time t, dc_t , for a change in consumption at time t', $dc_{t'}$, where $t' = t$, in which case we have own derivative, or $t' \neq t$, in which case we have cross derivative.

Now consider the following case: consider the initial sequence $c = \{c_0, c_1, \dots, c_{t_0}, \dots, c_{t_0+\tau}, \dots, c_T\}$ and perturb it by changing consumption in two periods t_0 and $t_1 = t_0+\tau$, to $c+dc = \{c_0, c_1, \dots, c_{t_0} + dc_{t_0}, \dots, c_{t_0+\tau} + dc_{t_0+\tau}, \dots, c_T\}$. The initial sequence has value $\mathsf{U}[c]$ and the perturbed sequence has value

$$\mathsf{U}[c+dc] = \mathsf{U}[c] + d\mathsf{U}[c] \ = \mathsf{U}[c] + \mathsf{U}_{t_0} \ dc_{t_0} + \mathsf{U}_{t_1} \ dc_{t_1}.$$

We define **intertemporal marginal rate of substitution**, between periods t_0 and $t_1 = t_0+\tau$, the change in consumption at $t_1 = t + \tau$, where $\tau > 0$, for one unit increase in consumption at t_0 such that it leaves intertemporal utility constant, i.e., $d\mathsf{U}[c] = 0$:

$$IMRS_{t_0,t_1} = -\frac{dc_{t_1}}{dc_{t_0}}\Big|_{\mathsf{U}=\text{constant}}.$$

But, as

$$d\mathsf{U}[c] \ = \mathsf{U}_{t_0} \ dc_{t_0} + \mathsf{U}_{t_1} \ dc_{t_1} \ = 0$$

then $IMRS_{t_0,t_1}$ is equivalent to the ratio between marginal utilities

$$IMRS_{t_0,t_1}(c) = \frac{\mathsf{U}_{t_0}}{\mathsf{U}_{t_1}}.$$

We say the utility functional displays **impatience** if, for a stationary consumption path, such that $c_t = \bar{c}$, for every t, the intertemporal marginal rate of substitution is greater than one: $IMRS_{t_0,t_1}(\bar{c}) \ > 1$.

This means that, given a time-independent stationary consumption sequence, the household attaches a higher value for consumption closer in time to the present than to consumption further away into the future. Equivalently, the household is willing to sacrifice a given amount of consumption in the present if a higher level of consumption will be given in the future, and that level increases with the time lag into the future in which it is made available.

The change in the marginal utility also provides an insight concerning the implicit consumption behavior which is formalized by the intertemporal utility functional $\mathsf{U}[c]$.

The **Uzawa-Allen elasticity of intertemporal substitution** of consumption in period t_0 and by consumption in period t_1 is defined by

$$\varepsilon_{t_0,t_1} = -\frac{\mathsf{U}_{t_0,t_1}[c]\ \ c_{t_0}}{U_{t_0}[c]}, \text{for any}\, t_0, t_1 \in \mathrm{T},$$

which measures the relative change in the marginal utility of consumption at time t_0 for a change in consumption at time t_1.

Considering a stationary consumption path, the utility functional displays one of the three following properties [2]:

- there is **intertemporal complementarity** if the marginal utility of consumption at time t_0 increases with consumption at time any future date t_1. In this case $\varepsilon_{t_0,t_1}(\bar{c})$ is negative;
- there is **intertemporal independence** if the marginal utility of consumption at time t_0 does not change with consumption at any future time t_1. In this case $\varepsilon_{t_0,t_1}(\bar{c})$ is equal to zero;
- or there is **intertemporal substitution** if the marginal utility of consumption at time t_0 decreases with consumption at time any future date t_1. In this case $\varepsilon_{t_0,t_1}(\bar{c})$ is positive.

In order to measure intertemporal substitution/complementarity the definition of **intertemporal elasticity of substitution** is introduced

$$EIS_{t_0,t_1} = \frac{\partial(c_{t_1}/c_{t_0})}{\partial IMRS_{t_0,t_1}}\,\frac{IMRS_{t_0,t_1}}{c_{t_1}/c_{t_0}}.$$

which measures the elasticity of the consumption ratio as regards the elasticity of the $IMRS$. Expanding the definition, we have the equivalent formula

$$EIS_{t_0,t_1} = \frac{c_{t_0}\,\mathsf{U}_{t_0} + c_{t_1}\,\mathsf{U}_{t_1}}{c_{t_0}\,\mathsf{U}_{t_0}\,\varepsilon_{t_1,t_1} - 2\,c_{t_0}\,\mathsf{U}_{t_0}\,\varepsilon_{t_0,t_1} + c_{t_1}\,\mathsf{U}_{t_1}\,\varepsilon_{t_0,t_0}}\ .$$

Example: The **benchmark utility functional** is the additive utility functional

$$\mathsf{U}[c] = \sum_{t=0}^{T} \beta^t u(c_t) \tag{1}$$

[2]We are referring to complementarity and substitutability in the Edgeworth-Pareto, or uncompensated, sense not in the Hicks-Allen, or compensated, sense. The issue of complementarity and substitutability should be dealt with care. This is discussed in the next subsection.

where $\beta \equiv \frac{1}{1+\rho}$ is the psychological discount factor and ρ is the rate of time preference. As $\rho > 0$ then $0 < \beta < 1$. It means that the value of a consumption sequence $c = \{c_t\}_{t \in \mathrm{X}}$ is equal to the present value of the sequence of period utilities of consumption. It is therefore linear on the utilities of consumption for every period.

Exercise: Prove that the utility functional (1) displays impatience, intertemporal independence, and, if we define the elasticity of marginal substitution by

$$\sigma(c) = -\frac{u''(c)\,c}{u'(c)}$$

then the elasticity of intertemporal substitution is equal to the inverse of the elasticity of marginal utility $EIS_{t_0,t_1}(\bar{c}) = \frac{1}{\sigma(\bar{c})}$

2.2 Continuous time

Let time be a real non-negative variable, $t \in \mathrm{T} \subseteq \mathbb{R}_+$. Consumption at time t is $c(t)$ and the flow of consumption is $c = \big(c(t)\big)_{t \in \mathrm{T}}$.

While the definition of marginal utility regarding intertemporal choice in discrete time is analogous to the static choice among several goods, the definition of marginal utility in a continuous time setting requires some background in integral derivatives, because the change in consumption at a point in time takes place within an infinitesimal time interval.

In this section we introduce the concepts of marginal utility, intertemporal marginal rate of substitution, Uzawa-Allen elasticities and elasticity of intertemporal substitution for the benchmark time additive utility functional. In section 5 we apply them to a particular type of intertemporally dependent preferences.

2.2.1 The benchmark utility functional

The benchmark **intertemporal utility functional** in continuous time is

$$\mathsf{U}\,[c] = \int_0^T u(c(t))e^{-\rho t}dt,\ \rho > 0 \tag{2}$$

where $e^{-\rho t}$ is the psychological discount factor and $\rho > 0$ is the rate of time preference. Again $\rho > 0$ implies $e^{-\rho t} \in (0, 1]$ if $t \in \mathbb{R}_+$. The **utility function** $u(\cdot)$ is assumed to be continuous, differentiable (and at least second-order differentiable in most of the rest of this paper), increasing and concave: that is $u''(c) < 0 < u'(c)$.

As we saw in last subsection, a crucial feature of the allocation of consumption across time is the intertemporal substitutability (or complementarity) of consumption. In a continuous time setting the utility functional $\mathsf{U}\,[c]$ is infinite-dimensional and consumption substitutability is related to the change in consumption in two points in time. Consider times t_0 and $t_0+\tau$, and the changes in consumption, $dc(t_0)$ and $dc(t_0+\tau)$, such that the utility functional $\mathsf{U}\,[c]$ remains constant. In other words, we want to compare the value of the flows $\left(c(t)\right)_{t\in\mathrm{T}}$ and $\left(\hat{c}\ (t)\right)_{t\in\mathrm{T}}$, where $\hat{c}\ (t)=c(t)$ if $t\neq\{\ t_0,t_0+\tau\}$ and $\hat{c}\ (t_0)=c(t_0)+dc(t_0)$ and $\hat{c}\ (t_0+\tau)=c(t_0+\tau)+dc(t_0+\tau)$, such that $\mathsf{U}[c]\ =\mathsf{U}[\hat{c}]$

There are two difficulties in dealing with intertemporal substitution/complementarity in continuous time.

The first difficulty is related to the economic definition of complementarity/substitutability. The concepts of complementarity and substitutability are not unequivocal. In a static microeconomic setting, preferences among different consumption bundles $\mathbf{c}=(c_1,\ldots c_n)$ are represented by the utility function $u(\mathbf{c})$, and the household's problem is $\max_{\mathbf{c}}\{\ u(\mathbf{c}):\mathbf{p}\cdot\mathbf{c}\leq y\}$, where $\mathbf{p}$ is the related price vector and y is income. In this setting we say two goods, indexed for instance by i and j, are substitutability in the Edgeworth-Pareto sense if $\dfrac{\partial^2 u(\mathbf{c})}{\partial c_i\, c_j}<0$ and they are substitutable in the Hicks-Allen sense if, at the optimum, the demand function is $\mathbf{c}=C(\mathbf{p},y)$ and $\dfrac{\partial c_i}{\partial p_j}>0$. The first type can be called uncompensated substitutability and the second compensated substitutability because it takes into account a normalizing effect of the budget constraint. Given the difficulty of defining an extension to the Hicksian sense of substitutability in continuous time [3] , the definition we present next is an extension of the Edgeworth-Pareto concept.[4]

The second difficulty is mathematical. Although continuous time makes analytical derivations of results easier, one has to specify the mathematical definition of intertemporal marginal rate of substitution between times t_0 and $t_0+\tau$, because the time variations become infinitesimal, i.e., variations have a measure zero.

As regards the utility functional (2), we can use the concept of a Gâteaux derivative for a "spike" variation of consumption at time t, introduced in the appendix A. Applying the definition of the integral derivative, provided there, to equation (14), we obtain the **marginal utility of**

[3]For a discussion see Biswas, 1976, and Ryder and Heal, 1973 and Heal and Ryder, 1976.

[4]There are issues here that are more profound than they look at first. There is a long tradition between ordinalists and cardinalists related comparative statics. The most recent approach to microeconomics, the monotone comparative statics approach, initiated by Milgrom and Shannon, 1994, associates substitutability defined in an ordinal way with the property of supermodularity. If a utility function is continuous and differentiable, the Edgeworth-Pareto substitutability criteriUm is equivalent to supermodularity.

consumption at a particular time $t = t_i \in \mathrm{T}$ is given by

$$\delta \mathsf{U}[c; t_i] \; = \mathsf{U}_{t_i} \equiv u'\big(c(t_i)\big)\, e^{-\rho t_i}, \text{ for any } t_i \in \mathrm{T},$$

where $\delta \mathsf{U}[c;t]$ is the integral derivative of the functional $\mathsf{U}[c]$ for a Dirac-δ variation in consumption centered at t_i, $\delta(t - t_i)$.

Therefore, the **intertemporal marginal rate of substitution** between consumption at time t_0 and $t_1 = t_0 + \tau$, for $\tau \geq 0$, if the intertemporal utility functional is (2) can be proved to be

$$\mathrm{IMRS}_{t_0,t_1} \; = \frac{\mathsf{U}_{t_0}}{\mathsf{U}_{t_1}} \; = \frac{u'\big(c(t_0)\big)}{u'\big(c(t_1)\big)} e^{\rho\, \tau}.$$

From our previous assumptions, we have $\mathrm{IMRS}_{t_0,t_1} \; > 0$ for any pair (t_0, t_1). If there is a satiation point, such that $u'(c_{t_s}) = 0$, then it is possible to have $\mathrm{IMRS}_{t_0,t_1} \; < 0$. We rule out this possibility by assuming that for any pair (t_0, t_1) there ia locally non-satiation.

Using the second order Gâteaux derivatives for a "spike" variation at the same time and at a different time, we obtain

$$\delta^2 \mathsf{U}[c; t_i, t_i] \; = \mathsf{U}_{t_i t_i} \equiv u''\big(c(t_i)\big)\, e^{-\rho t_i}, \text{ for any } t \in \mathrm{T}.$$

and $\delta^2 \mathsf{U}[c; t_i, t_j] = \mathsf{U}_{t_i t_j} = 0$ for $t_i \neq t_j$ both in T.

This implies that the **Uzawa-Allen elasticities** are [5] are

$$\epsilon_{t_i,t_i} \; = \sigma\big(c(t_i)\big) \equiv -\frac{u''\big(c(t_i)\big)\, c(t_i)}{u'\big(c(t_i)\big)}$$

where $\sigma\big(c(t_i)\big)$ is the elasticity of the marginal utility $u'(\cdot)$, and

$$\epsilon_{t_i,t_j} \; = 0 \text{ if } \; t_i \neq t_j.$$

[5] They are defined as

$$\epsilon_{t_i,t_j} \; \equiv -\frac{\delta^2 \mathsf{U}[c; t_i, t_j] \;\; c(t_i)}{\delta \mathsf{U}[c; t_i]} = -\frac{\mathsf{U}_{t_i t_j}\, c(t_i)}{\mathsf{U}_{t_i}} \;\; \text{for any } t_i, t_j \in \mathrm{T}.$$

Therefore the **elasticity of intertemporal substitution**[6]

$$
\begin{aligned}
IES_{t_0,t_1} &= \frac{d\log\left(c(t_0)/c(t_1)\right)}{d\log \mathrm{IMRS}_{t_0,t_1}} = \\
&= \frac{c_{t_0}\,\mathsf{U}_{t_0} + c_{t_1}\,\mathsf{U}_{t_1}}{c_{t_0}\,\mathsf{U}_{t_0}\,\epsilon_{t_1,t_1} - 2\,c_{t_0}\,\mathsf{U}_{t_0}\,\epsilon_{t_0,t_1} + c_{t_1}\,\mathsf{U}_{t_1}\,\epsilon_{t_0,t_0}}\ .
\end{aligned} \tag{3}
$$

Applying to the intertemporal utility functional (2) yields

$$
IES_{t_0,t_1} = \frac{c(t_0)\,u'(c(t_0))\,e^{-\rho t_0} + c(t_1)\,u'(c(t_1))\,e^{-\rho t_1}}{c(t_1)\,u'(c(t_1))\,e^{-\rho t_1}\,\sigma(c(t_0)) + c(t_1)\,u'(c(t_0))\,e^{-\rho t_0}\,\sigma(c(t_1))}.
$$

If we consider a stationary flow of consumption, such that $c(t) = \bar{c}$ for any $t \in \mathrm{T}$, which we interpret as establishing that the intertemporal utility functional $\mathsf{U}[c]$ displays: (1) **impatience** because $IMRS_{t_0,t_1}(\bar{c}) = e^{\rho\,\tau} > 1$; (2) **intertemporal independence** because $\epsilon_{t_0,t_1}(\bar{c}) = 0$; and (3) constant **intertemporal elasticity of substitution**, which is qual to the inverse of the elasticity of the marginal utility of consumption for a stationary consumption trajectory, because $IES_{t_0,t_1} = \dfrac{1}{\sigma(\bar{c})}$.

For the case in which the utility function is isoelastic (indeed a generalized logarithm),

$$
u(c) = \frac{c^{1-\theta} - 1}{1-\theta},
$$

we obtain $IES_{t_0,t_1} = \dfrac{1}{\theta}$, which led to the literature calling θ the **inverse of the elasticity of intertemporal substitution**.

2.2.2 Extensions

The benchmark utility functional in the form presented in equation (2) rests on three main elements:

1. the generalized indirect utility is a linear functional on the flow of utility across time, and displays non-satiation and intertemporal independence (in the Edgeworth-Pareto sense);
2. it displays a discount factor which is exponential, it is constant in time and it can be taken as a parameter

[6] Expanding the expression we obtain

$$
\frac{d\log\left(c(t_i)/c(t_j)\right)}{d\log \mathrm{IMRS}_{t_i,t_j}} = \frac{c(t_i)\,\mathsf{U}_{t_i} + c(t_j)\,\mathsf{U}_{t_j}}{c(t_j)\,\mathsf{U}_{t_j}\,\mathsf{U}_{t_i t_i} - 2\,c(t_i)\,\mathsf{U}_{t_i}\,\mathsf{U}_{t_i t_j} + c(t_i)\,\mathsf{U}_{t_i}\,\mathsf{U}_{t_j t_j}}.
$$

3. the time horizon is finite and known

4. the household only measures its welfare by the flow of consumption.

We will consider next some of those extensions.

3 The household problems in the continuous time setting

In this section we apply the previous concepts, in particular the continuous-time definition of marginal utility, to the household problem with several types of constraints. We assume a deterministic setting in which the household has perfect information.

We start by dealing with the meaning of maximizing utility in an unconstrained problem in subsection 3.1. In subsection 3.2 we consider the problem for a rentier, i.e., for an agent which consumes out of a an initial resource from which the only activity is to consume it through time. Next, in subsection 3.3 we consider the problem for a rentier which is also an investor. Section 3.4 considers the case in which the household has both financial and non-financial income. All those models consider a finite and known horizon.

In subsection 3.5 we present a justification for considering an infinite-horizon and solve the household problem. This model is the benchmark partial equilibrium model in intertemporal macroeconomics. It can also be seen as a simple model for a small open economy facing a perfect international capital market [7]

3.1 Maximizing utility without constraints

We address the first problem: finding the maximum consumption path in an unconstrained setting. In other words, we want to find a consumption trajectory $\left(c(t)\right)_{t\in \mathrm{T}}$, where $\mathrm{T} = [0, T]$, will the household choose in the case it has no constraints.

The problem is to find the maximum value of the utility aggregator (or intertemporal utility functional)

$$\mathsf{U}[c^*] \ = \max_{c(\cdot)} \int_0^T \ u(c(t))\, e^{-\rho t} dt \tag{P1}$$

where we denote $c^* = \left(c^*(t)\right)_{t\in\mathrm{T}}$ the maximum utility consumption path.

[7] For a discrete time version see Uribe and Schmitt-Grohé, 2017, ch 2.

If c^* is optimum, then any arbitrary perturbation φ will not change the intertemporal utility, that is $\mathsf{U}[c^* + \varphi] = \mathsf{U}[c^*]$. Therefore, at the optimum

$$d\mathsf{U}[c;\varphi] = \int_0^T u'(c^*(t))\, e^{-\rho t}\ \varphi(t)\, dt = 0$$

This is a linear functional which can be solved, because $\varphi(t) \neq 0$ for every $t \in \mathrm{T}$ [8] if and only if

$$u'(c^*(t))\, e^{-\rho t} = 0, \text{ for every } t \in \mathrm{T}. \tag{P1:foc}$$

The optimum level of consumption, because $e^{-\rho t} \in (0,1)$, depends on the properties of the utility function $u(\cdot)$:

1. if the utility function has the Inada property, i.e, if $\lim_{c\to\infty} u'(c) = 0$, then the optimum will be reached for $c^*(t) = \infty$ for any t;

2. however, if the utility function does not have the Inada property, then the optimum can be reached for a finite level of consumption. If there is a satiation point, that is a point $c^s \in \mathbb{R}_+$ such that $u'(c^s) = 0$, then the optimum will consist in consuming at that satiation point at every moment time $c^*(t) = c^s$ for any t.

Exercise: Prove that the utility function of the isoelastic type $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta}$, for $\theta > 0$ has the Inada property, and therefore the optimum consumption is $c^*(t) = \infty$ for every t.

Exercise: Prove that the quadratic utility function $u(c) = c - \dfrac{\beta}{2}\, c^2$, for $\beta > 0$ has a satiation point.

The intuition for this result is obvious: if there are no constrains on consumption and the utility function displays non-satiation, then optimal consumption would be infinite for every point in time, as in the Cockaigne land.

3.2 Maximizing utility for a rentier

Consider the problem for a rentier: it has an initial stock of net wealth, $a_0 > 0$ which is used to finance the purchase of consumption goods throughout lifetime. Assume prices are always equal to one.

Consider, again, any two moments t and $t + \tau$. Assuming that the rentier still possesses the stock level $a(t)$ at time t, and that she/he consumes a constant quantity $c(t)$ during any period

[8] See Gel'fand and Fomin, 1963, p.9

of length τ, what will be the stock of wealth remaining at time $t+\tau$? It is easy to see that $a(t+\tau) = a(t) - c(t)\,\tau$. If the time interval shrinks to zero, the infinitesimal change in his wealth is

$$\lim_{\tau \to 0} \frac{a(t+\tau) - a(t)}{\tau} = \dot{a}(t) = -c(t).$$

Solving the budget constraint $\dot{a}(t) = -c(t)$ we find that the level of net wealth at any point in time is equal to the initial net wealth minus the sum of consumption until time t,

$$a(t) = a_0 - \int_0^t c(s)\,ds.$$

From now on we assume that the initial value of wealth is finite, positive and known, that is $a(0) = a_0 > 0$.

Does the optimal consumption path changes as regards problem (P1), i.e., does existence of an initial finite resource constrains consumption ? The answer is, not necessarily, because it depends on the assumptions we introduce regarding the domain of a.

Consider a first case: the future net wealth level a **can take any real value**, that is the agent can borrow without limit. As a denotes the stock oh net wealth we assume that $-\infty < a(t) < \infty$ for any time $t > 0$.

The problem of the household becomes

$$\begin{aligned} &\max_{c(\cdot)} \int_0^T u(c(t))\, e^{-\rho t} dt \\ &\text{subject to} \\ &\dot{a}(t) = -c(t), \text{ for } t \in \mathrm{T} \\ &c(t) \in \mathbb{R}_+ \\ &a(0) = a_0 > 0 \text{ given} \\ &a(t) \in \mathbb{R} \cup \infty \end{aligned} \tag{P2}$$

We show in the appendix B that the optimality condition for this problem is the same as the solution of problem (P1), in equation (P1:foc):

$$u'(c^*(t))\, e^{-\rho t} = 0, \text{ for every } t \in [0, T]. \tag{P2:foc}$$

Again, if the utility has the Inada property (v.g, if it has a satiation point) then the household will want to consume an infinite amount at every point in time. This means that a becomes negative and unbounded just immediately after $t = 0$ and the initial constraint of the resource is not active.

A second, more realistic case is the one in which **the household faces a lower bound on** a, that we denote by $\underline{a}$. If he cannot borrow then $\underline{a} \geq 0$. If he can become a net debtor until some prescribed limit, then he faces a borrowing constraint $\underline{a} < 0$ which is finite. This can be seen as a case in which there is a financial friction.

We assume this constraint is active at every point in time, and the household has an initial net wealth level which is positive. The problem is now:

$$\begin{aligned}
&\max_{c(\cdot)} \int_0^T u(c(t))\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = -c(t), \text{ for } t \in \mathrm{T} \\
&c(t) \in \mathbb{R}_+ \\
&a(t) \in [\underline{a}, \infty), \text{ for every } t \in [0, T] \\
&a(0) = a_0 > \max\{0, \underline{a}\} \ \text{ given}
\end{aligned} \tag{P3}$$

Let us assume that the utility function $u(c)$ has the Inada property: $u''(c) < 0 < u'(c)$ and $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. In order to make the analysis clear let us assume that the utility function is $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta}$, for $\theta > 0$. This yields $u'(c) = c^{-\theta}$

The introduction of the borrowing constraint carries two important consequences: first, we assume that initial level of wealth is above the lower limit for a, $\underline{a}$ (which can be positive or negative); second, as the utility function would induce an unconstrained household to consume an infinite amount at every point in time, then the borrowing constraint will be active at some point in time, say $\underline{t} \in (0, T]$. The time of activation of the constraint will be higher than zero because the initial level of wealth is by assumption bigger than the lower limit, $a_0 > \underline{a}$ but it is **endogenous**. This raises a natural question: is it optimal to hit the constraint earlier or at the horizon T ?

The optimality conditions are

$$\begin{cases} u'(c(t))e^{-\rho t} = \lambda(t) \\ \dot{a} = -c(t) \\ \dot{\lambda} = -\eta(t) \\ \eta(t)(a(t) - \underline{a}) = 0, \quad \eta(t) \geq 0, \; a(t) \geq \underline{a} \text{ for } t \in (0, T] \\ \lambda(T^-) = \eta(T). \end{cases}$$

Therefore, we can divide the solution in two periods: the period $[0, \underline{t})$ in which the constraint is not active, and we have $\eta(t) = 0$ which implies $\lambda(t) = \lambda_0$ is a constant, consumption is positive and the net wealth diminishes through time, because we still have $a_0 > a(t) > \underline{a}$; and, the period $[\underline{t}, T]$ in which the constraint is active, and we have $\eta(t) > 0$, $\dot{\lambda} < 0$, consumption is equal to zero and net wealth is at the borrowing limit.

That is, we have the following solutions for consumption and net wealth:

1. if $\underline{t} < T$ hen

$$c(t) = \begin{cases} \lambda_0^{-\frac{1}{\theta}} e^{\gamma_c t} & \text{for } t \in [0, \underline{t}) \\ 0 & \text{for } t \in (\underline{t}, T] \end{cases}$$

here $\gamma_c = -\dfrac{\rho}{\theta} < 0$ is the (negative) rate of growth of consumption, and

$$a(t) = \begin{cases} a_0 + \dfrac{\lambda_0^{-\frac{1}{\theta}}}{\gamma_c}(1 - e^{\gamma_c t}) & \text{for } t \in [0, \underline{t}) \\ \underline{a} & \text{for } t \in (\underline{t}, T], \end{cases}$$

where λ_0 is unknown and can only be determined when $\underline{t}$ is found. At time $\underline{t}$ we have $c(\underline{t}^+) = c(\underline{t}^-) = 0$. This is only possible if $\lambda_0 = \infty$ which implies that $a(\underline{t}^-) = \underline{a} = a_0$. This is not possible for $\underline{t} < T$, because we have assumed that $\underline{a} < a_0$;

2. if $\underline{t} = T$ then, from $\underline{a} = a_0 + \dfrac{\lambda_0^{-\frac{1}{\theta}}}{\gamma_c}(1 - e^{\gamma_c T})$ we can determine λ_0 as

$$\lambda_0^{-\frac{1}{\theta}} = \gamma_c \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}}$$

and obtain the optimum consumption

$$c^*(t) = \gamma_c \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}} e^{\gamma_c t} \; t \in [0, T).$$

At time $t = T$, because we assumed that $\underline{a} < a_0$, there should exist a discontinuous jump in consumption because $c(T^-) = \gamma_c \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}} e^{\gamma_c T} > 0 \neq c(T) = 0$ (recall that $\gamma_c < 0$). The optimum net wealth stock evolves according to

$$a^*(t) = a_0 + \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}} \left(1 - e^{\gamma_c t}\right) \text{ for } t \in [0, T].$$

Therefore, if the household attributes an unlimited utility to consumption when it is close to zero, if the utility function has the Inada properties, if there is a financial constraint, and the household has perfect foresight, she/he will stay away from the borrowing constraint before the problem's horizon T. It finds optimal to exhaust his possibilities for borrowing only at the last moment.

Exercise If the utility function displays satiation what will be the solution to the problem ?

3.3 Maximizing utility for an investor

Usually, at any point in time, the stock of net wealth generates an income equal to $r(t)\, a(t)$ where r is the interest rate. The income flow is a source of earnings if the household is a net creditor (if $a(t) > 0$) but it is a source of expenditure is the household is a net debtor (if $a(t) < 0$). We assume that the interest rate is given to the household, and to simplify, that it is constant. Because we are dealing with real variables, r refers to the real interest rate.

Again we assume that the utility function has the Inada property and that, justified by the results in the last subsection, there is a terminal constraint on the level of net wealth.

Assuming an isoelastic utility function, the problem is

$$\begin{aligned}
&\max_{c(\cdot)} \int_0^T \frac{c(t)^{1-\theta} - 1}{1 - \theta}\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\, a - c(t), \text{ for } t \in \mathrm{T} \\
&a(0) = a_0 > \underline{a} \text{ given} \\
&a(T) \geq \underline{a}
\end{aligned} \tag{P4}$$

Observe that we are using what we have learned in the previous section and set the financial constraint at the terminal time, T, and not for every point in time.

In appendix D we prove that optimal consumption path is $(c^*\ (t))_{t\in[0,T]}$, where

$$c^*(t) = \Big(\frac{(r-\gamma_c)(a_0 - \underline{a}\, e^{-r\,T})}{1-e^{(\gamma_c - r)T}}\Big)\, e^{\gamma_c t}, \text{for every}\, t \in [0,T] \tag{4}$$

where the rate of growth of consumption is

$$\gamma_c \equiv \frac{r-\rho}{\theta}$$

and the optimal net wealth is generated by the function

$$a^*(t) = e^{r\,t}a_0 - \frac{a_0 - \underline{a}\, e^{-r\,T}}{1-e^{(\gamma_c - r)T}}\,(e^{r\,t} - e^{\gamma_c\, t}), \text{for every}\, t \in [0,T]. \tag{5}$$

The following observations can be made: first, as the sign of γ_c is the same as if $r-\rho$ then if $r > \rho$ consumption will grow across the lifetime, it will remain constant if $r = \rho$ and will diminish if $r < \rho$. Second, for any value of the parameters, the initial and the terminal values of net wealth are $a(0) = a_0$ and $a(T) = \underline{a}$, which means that the time profile of consumption (and savings) featuring more or less consumption (or less or more savings) in the beginning of the period or at the end of the period, depend on the relative value of the parameters, ρ and θ and on the market interest rate r. In particular, if θ is higher then the absolute value of γ_c will be lower.

In this model the path of income is $y(t) = r\,a(t)$ because there is only financial income.

3.4 Maximizing utility with non-financial income

In this subsection we assume that the household is entitled to a non-financial stream of income $(w(t))_{t\in\mathrm{T}}$, where $w(t) > 0$ for every t. It can be labor income or any other type of non-financial income. In the case of labor income, we assume that the household supplies inelastically a constant flow of working hours (or effort) normalized to one.

We consider now the following problem

$$\begin{aligned}
&\max_{c(\cdot)} \int_0^T \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt\\
&\text{subject to}\\
&\dot{a}(t) = r\,a + w(t) - c(t),\ \text{for}\ t \in \mathrm{T}\\
&a(0) = a_0 > \underline{a}\ \text{given}\\
&a(T) \geq \underline{a}
\end{aligned} \tag{P5}$$

Using the same approach as in the previous problem, we can prove that the solution is

$$c^*(t) = \Big(\frac{(r-\gamma_c)(a_0 + k_0 - \underline{a}\, e^{-r\,T})}{1-e^{(\gamma_c - r)T}}\Big)\, e^{\gamma_c t}, \text{for every}\, t \in [0,T] \tag{6}$$

and

$$a^*(t) = e^{r\,t} a_0 + k_0 - \frac{a_0 + k_0 - \underline{a}\, e^{-r\,T}}{1-e^{(\gamma_c - r)T}}\, (e^{r\,t} - e^{\gamma_c\, t}), \text{for every}\, t \in [0,T]. \tag{7}$$

where

$$k_0 = \int_0^T e^{-r\,t}\, w(t)\, dt$$

is the human capital of the household at time $t = 0$. Comparing with the previous problem, we see that while in the previous problem, consumption and savings essentially led net wealth to go from a_0 to $\underline{a}$, in this case, the initial total wealth is $a_0 + k_0$ and the terminal wealth is still $\underline{a}$. This means that the present flow of consumption in this problem is much higher than in the former problem.

Furthermore, if human wealth can be seen as a collateral for net borrowing, it is possible that the lower limit on net financial wealth $\underline{a}$ might be reduced by the existence of human wealth which could be offered as a collateral.

3.5 Infinite horizons

We saw that the determination of the initial level of consumption $c(0)$, and therefore, the level of consumption is dependent on the horizon of the problem, T, and on the terminal constraints on wealth. In this section we present the infinite-horizon case.

There are two main justifications for considering the infinite horizon case: first, the existence of incomplete information on the terminal time, and second, the existence of concern by the household of the utility of its family beyond the life of those living at time $t = 0$. We call the second case the dynastic model.

The uncertain horizon In order to study this case we assume that the lifetime T is stochastic. Let the cumulative distribution of lifetime be $F(T) = \int_0^T f(t)dt \in (0,1)$ for $T \in (0,\infty)$ and the density function $f(t)$ follow a Poisson process, with the instantaneous probability of death μ, $f(t) = \mu\, e^{-\mu t}$. Then $F(T) = 1 - e^{-\mu T}$. Off course $F^{'}(t) = f(t)$, $F(0) = 0$ and $F(\infty) = 1$.

Let $U(T) = \int_0^T u(c(t))\, e^{-\rho t}\, dt$ be the utility functional for an household having horizon T and let the intertemporal utility with a stochastic lifetime be given by the **expected utility functional**

$$\mathsf{U}\,[c] \;= \int_0^\infty \; f(T)\, U(T) \;\, dT = \int_0^\infty \; f(T) \int_0^T u(c(t))\, e^{-\rho t}\, dt\, dT.$$

We can prove that

$$\mathsf{U}\,[c] = \int_0^\infty u(c(t))\, e^{-(\rho+\mu)t}\, dt. \tag{8}$$

To prove this observe that

$$\mathsf{U}\,[c] \;= \int_0^\infty \; F'(T)\, U(T) \;\, dT.$$

Using integration by parts,[9] we have

$$\begin{aligned}
\mathsf{U}[c] \;&= F(T)\, U(T)\Big|_{T=0}^{\infty} - \int_0^\infty \; F(T)\, U'(T) \;\, dT \\
&= F(\infty)\, U(\infty) - F(0)\, U(0) - \int_0^\infty \; F(T)\, e^{-\rho T} u(c(T))\, dT \;\; \text{because } U^{'}(T) = \; e^{-\rho T} u(c(T)) \\
&= U(\infty) - \int_0^\infty (1 - e^{-\mu T}) e^{-\rho T}\, u(c(T))\, dT \\
&= \int_0^\infty u(c(T))\, e^{-\rho T}\, dT - \int_0^\infty (1 - e^{-\mu T}) e^{-\rho T}\, u(c(T))\, dT \\
&= \int_0^\infty u(c(T))\, e^{-\rho T}\, dT - \int_0^\infty u(c(T))\, e^{-\rho T}\, dT + \int_0^\infty e^{-(\rho+\mu)\, T}\, u(c(T))\, dT \\
&= \int_0^\infty u(c(T))\, e^{-(\rho+\mu)T}\, dT.
\end{aligned}$$

Therefore, interpreting the discount factor as the sum of the rate of time preference plus the instantaneous mortality rate for an uncertain lifetime model with a Poisson distribution of the time of death, justifies assuming an utility functional for a single agent with an infinite horizon, as in equation (8).

The dynastic model Another interpretation for an infinite horizon is the dynastic interpretation that is mathematically equivalent. In this case the economic agent can be seen as an household

[9] Assuming that some mathematical requirements are satisfied, implying the boundedness of $u(c)$.

which cares for the utility not only of the present but for all future generations. In this case we have

$$\int_0^\infty u(c(t))\, e^{-\rho t}\, dt$$

We consider now the following problem, in which the rate of time preference can include the mortality rate or not.

$$\begin{aligned}
&\max_{c(\cdot)} \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\,a + w(t) - c(t), \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&\lim_{t\to\infty} a(t)\, e^{-r\,t} \geq 0
\end{aligned} \tag{P6}$$

The terminal condition is called the non-Ponzi game condition, it means that in present value terms the household will not be a net debtor asymptotically.

Observe that the model with finite horizon and a Poisson distributed horizon reduces to this model if there is an annuity market in which the consumer could receive a rate or return equal to μ and its net wealth would be repossessed by the issuer in the event of death.[10]

Another interpretation is that the three constraints imply that the household consumption path is sustainable, or that it is solvent. In order to see this, solving the household budget constraint, $\dot{a}(t) = r\,a + w(t) - c(t)$, together with the initial condition, $a(0) = a_0$, yields

$$a(t) = e^{r\,t}\Big(a_0 + \int_0^t e^{-r\,s}\, w(s) - c(s) ds\Big).$$

Multiplying both sides by the discount factor $e^{-r\,t}$ and taking the limit to infinity, yields

$$\lim_{t\to\infty} e^{-r\,t} a(t) = a_0 + k_0 - \int_0^\infty e^{-r\,t}\; c(t) dt$$

where we used the definition of human capital at time $t = 0$,

$$k_0 \equiv \int_0^\infty e^{-r\,t}\, w(t) dt.$$

[10]This is the Yaari, 1965 model.

If the non-Ponzi game condition holds then we obtain the following intertemporal budget constraint

$$\int_0^\infty e^{-r\,t}\,c(t)\,dt \le a_0 + k_0$$

which means that the present value of consumption is smaller than the initial total net wealth which is equal to the sum of the financial and human wealth.

From now on, we assume that $r > 0$ and w is constant. Then $k_0 = \dfrac{w}{r}$.

By employing the methods already presented in the previous lecture[11] we can obtain explicitly the optimal policy function

$$c^* = C(a) = (r-\gamma)\Big(a + \frac{w}{r}\Big).$$

If $\theta \ge 1.$ the rate of growth of net wealth is smaller than the interest

$$\gamma \equiv \frac{r-\rho}{\theta} < r,$$

which implies implies that optimal consumption is a positive function of total net wealth, financial and human, and the propensity to save is endogenously determined as a function of the difference $r-\gamma$.

If we substitute optimal consumption in the budget constraint we have

$$\begin{aligned}\dot{a} &= r\,a + w - (r-\gamma)(a + \frac{w}{r})\\ &= \gamma\,(a + \frac{w}{r})\\ &= \gamma\,(a - \bar{a})\end{aligned}$$

where the steady state level of net financial wealth is the symmetric of human wealth

$$\bar{a} = -k_0 = -\frac{w}{r}.$$

Solving this differential equation, taking $a(0) = a_0$ yields

$$a^*(t) = \bar{a} + (a_0 - \bar{a})\,e^{\gamma\,t}, \text{for}\, t \in [0,\infty)$$

or, equivalently

$$a^*(t) = -k_0 + (a_0 + k_0)\,e^{\gamma\,t}, \text{for}\, t \in [0,\infty).$$

The optimal consumption trajectory is $(c^*\ (t))_{t\in[0,\infty)}$ where

$$c^*(t) = (r-\gamma)\,(a_0 - \bar{a})\,e^{\gamma\,t},\ \text{for each}\ \ t \in [0,\infty)$$

[11] See https://pmbbrito.github.io/cursos/phd/am/am2021_ramsey.pdf.

Therefore, the dynamics of both consumption and asset accumulation depends is determined by the rate of growth γ. There are three possible cases:

First, if the household is less patient than the market, that is if $r < \rho$ then $\gamma < 0$ and, assuming that the initial level of net wealth is above the natural debt limit $-k_0 = -w/r$ then its net asset position will evolve towards the natural debt limit $\lim_{t\to\infty} a^*(t) = -k_0$ and consumption will converge to zero, $\lim_{t\to\infty} c^*(t) = 0$. This is Case 1 in Figure 1. Observe that the optimal initial optimal consumption is such that the household has negative savings

$$c^*\ (0) = (r-\gamma)\,(a_0 + \frac{w}{r}) > r\,a_0 + w.$$

The household will have negative savings in the adjustment to its steady state position, i.e, $c^*\ (\infty) = 0$. This explains why the optimal adjustment path in Case 1 in Figure 1 is always above the isocline for a, i.e, the locus of points such that $c(t) = r\,a(t) + w$.

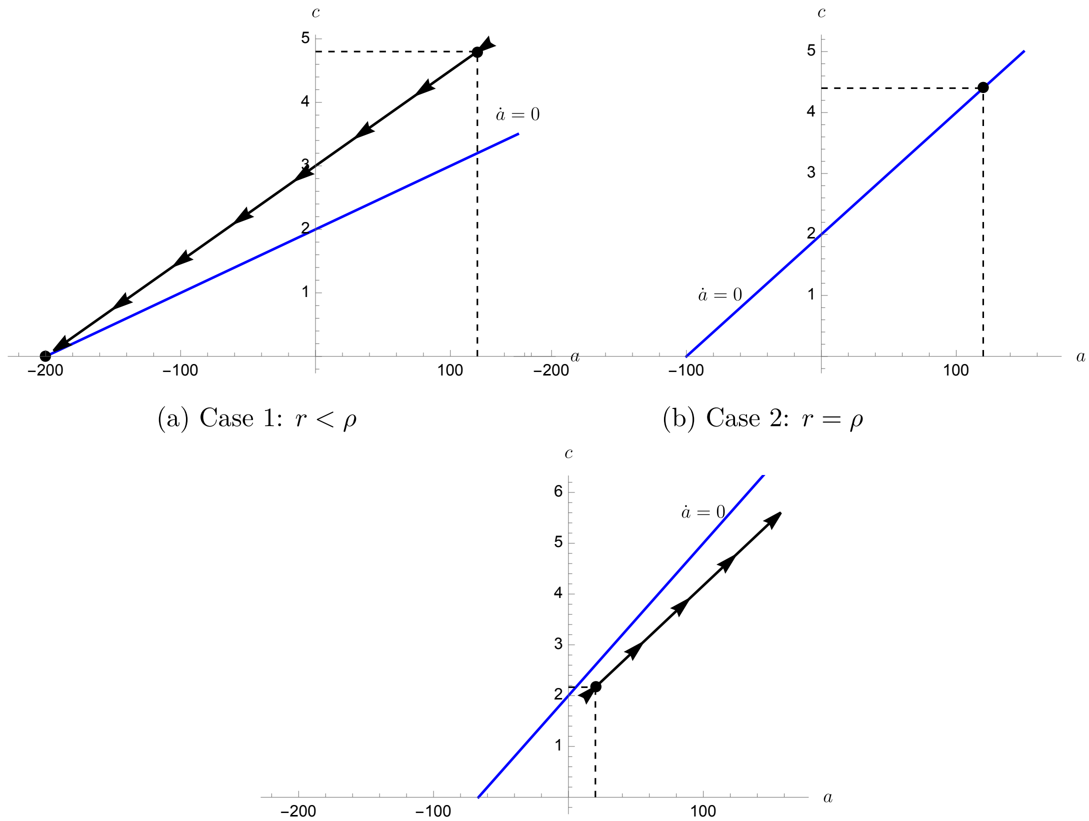
Second, if the household is more patient than the market, that is if $r > \rho$ then $\gamma > 0$ and, assuming that the initial level of net wealth is above the natural debt limit $-k_0 = -w/r$ then both the asymptotic level of wealth and consumption will be unbounded $\lim_{t\to\infty} a^*(t) = \lim_{t\to\infty} c^*(t) = +\infty$. This is Case 3 in Figure 1. Observe that the optimal initial optimal consumption is such that the household has positive savings at time $t = 0$

$$c^*\ (0) = (r-\gamma)\,(a_0 + \frac{w}{r}) < r\,a_0 + w,$$

and will keep having positive savings along the adjustment path. In this case, there is not a steady state.

At last, if the household is as patient as the market, that is if $r = \rho$ then $\gamma = 0$ and, assuming that the initial level of net wealth is above the natural debt limit $-k_0 = -w/r$, the solution is stationary $a^t\ = a_0$ and $c^t\ = r\,a_0 + w$ for all $t \in [0, \infty)$. This is Case 2 in Figure 1. This occurs because in this case the household has no incentives for having negative or positive savings, because the rate of return for savings is equal to the cost, measured by ρ.

What is the role of the parameter θ in the solution ? As we saw the higher θ is the lower is the EIS, which means that the cost of transfering consumption between moments in time, in utility terms, is higher. We see that although it does not affect the sign of γ, it reduces its absolute value. For any point in time it also increases the propensity to consume out of total net wealth, this means that savings will be smaller instantaneously and the transfer of consumption among periods will be smaller as well. Therefore, it tends to generate a smoother behavior of consumption.



(a) Case 1: $r < \rho$

(b) Case 2: $r = \rho$

(c) Case 3: $r > \rho$

Figure 1: Phase diagrams for the dynastic household problem.

The previous results contrast with those we obtained for the finite horizon case. In particular, it is not possible to have an unbounded net asset position. The reason for this is related to the fact that the terminal condition refers to the infinity, and given the fact that it involves discounting at a rate which is higher, in absolute value, than the rate of growth of the wealth position, it allows for an unbounded evolution of net financial wealth. On the other hand, if the agent is more impatient than the market, i.e, if $\rho > r$, the solution for consumption looks similar to the problem for an agent which depletes a given stock of wealth (i.e, to problem (P2)).

The previous results were obtained in a partial equilibrium setting. The only case which can be valid in both partial equilibrium and general equilibrium settings is the last one. This is the case of a representative agent dynamic general equilibrium (DGE).

In a DGE setting the situation in which an agent could have an unbounded asset position would be impossible, unless in the economy there is an infinite aggregate net supply of assets, which is not a realistic situation. Therefore, in a DGE setting, we would expect that an unbounded level of assets would lead the interest rate to be reduced and possibly converge to the rate of time preference or to an average rate of time preference. Symmetrically, if a big mass of agents would carry on having negative savings an increase of the interst rate would be expected.

4 Expected an expected changes in income

In this section we continue to assume the time additive dynastic model and study the effects of an anticipated and unanticipated changes in income. In the first case, we assume a deterministic setting, and assume the time of the switch is given to the household.

4.1 Future changes in income

We consider the problem

$$\begin{aligned} &\max_{c(\cdot)} \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt \\ &\text{subject to} \\ &\dot{a}(t) = \begin{cases} r\,a + w_1 - c(t), & \text{for } 0 \le t < \tau \\ r\,a + w_2 - c(t), & \text{for } \tau \le t < \infty \end{cases} \\ &a(0) = a_0 \text{ given} \\ &\lim_{t\to\infty} a(t)\, e^{-r\,t} \ge 0 \end{aligned} \tag{P7}$$

where teh switching time, τ, such that $0 < \tau < \infty$, is known. This can be the case if there are two stages in lifetime: work/retirement, employed/unemployed. We could also consider the case in which a change of the interest rates is forecasted.

The optimal path for consumption $\left(c^*(t)\right)_{t\in[0,\infty)}$ is such that

$$c^*(t) = (r-\gamma)\left(a_1^* + \frac{w_2}{r}\right) e^{\gamma\,(t-\tau)}, \text{ for } t \in [0,\infty),$$

where again $\gamma \equiv \frac{r-\rho}{\theta}$, and there optimal level on net wealth at the switching time is

$$a^*(\tau) = a_1^* = e^{\gamma\,\tau}\left(a_0 + \frac{w_1}{r}\,\left(1 - e^{-r\,\tau}\right) + \frac{w_2}{r}\,\left(e^{-r\,\tau} - e^{-\gamma\,\tau}\right)\right).$$

and the optimal path for the net asset position is $\Big(a^*(t)\Big)_{t\in[0,\infty)}$ for

$$a^*(t)=\begin{cases}-\frac{w_1}{r}+\left(a_1^*+\dfrac{w_2}{r}\right)e^{\gamma(t-\tau)}+\left(\dfrac{w_1}{r}-\dfrac{w_2}{r}\right)e^{r(t-\tau)}, & \text{for } t\in[0,\tau)\\ -\frac{w_2}{r}+\left(a_1^*+\dfrac{w_2}{r}\right)e^{\gamma(t-\tau)}, & \text{for } t\in[\tau,\infty).\end{cases}$$

The proof is in appendix G. [12] Figure 2 illustrates the possible dynamics for the case in which $w_2<w_1$, and can be compared to the analog phase diagrams in Figure 1.

The following remarks are important:

First, although there is a jump in the non-financial income w, both paths for consumption and net asset position are continuous at τ, i.e. $c^*(\tau^-)=c^*(\tau)$ and $a^*(\tau^-)=a^*(\tau)$. This is the case because the Euler equation is jointly solved with the transversality condition (for $t\to\infty$), which means that teh consumption decision from time $t=0$ fully anticipates the shift in income, which implies that the dynamics of asset accumulation, which is given by the budget constraint, also responds to the that anticipation.

Second, while consumption is continuously differentiable at the switching time $t=\tau$, the net asset position is not, $\dfrac{d}{dt}a^*(\tau^-)\neq\dfrac{d}{dt}a^*(\tau)$. The reason for this is that although consumption fully anticipates the changes in income across lifetime, when there is a switch in income, there will be an immediate effect in savings, and therefore in the **change** in net asset accumulation.

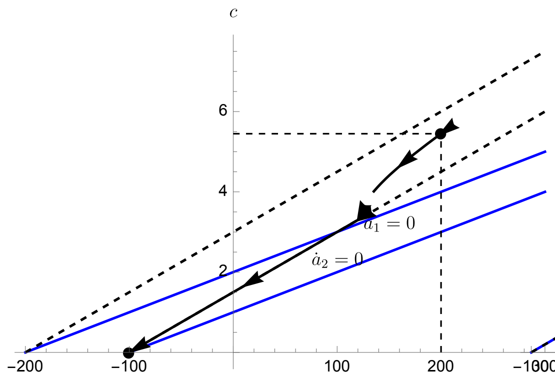
Third, the initial consumption level will be somewhere between the initial consumption level for cases in which the non-financial income will be permanently equal to w_1 or w_2. If we assume that $w_1>w_2$ then

$$c(0)_2\ <c^*(0)<c_1\ (0)\iff(r-\gamma)(a_0+\frac{w_2}{r})<c^*\ (0)<(r-\gamma)(a_0+\frac{w_1}{r}).$$

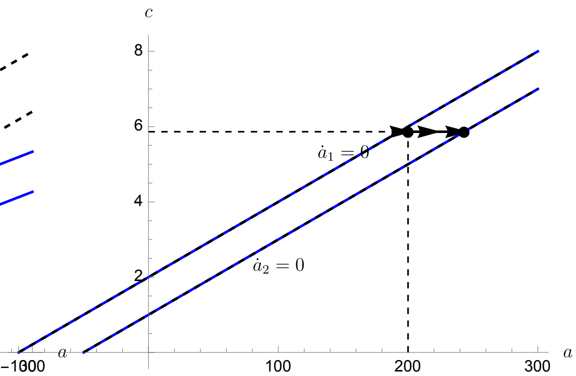
The initial consumption will be determined by some average of the optimal level of consumption at time zero for the cases in which the household will have a constant non-financial income equal to w_1 (denoted by $c_1\ (0)$) and w_2 (denoted by $c_2\ (0)$).

At last, the solution will tends asymptotically to the solution of a problem in which the non-financial income would be permanently equal to w_2 as in the cases depicted in Figure 1. Observe that, even in the case in which $r=\rho$, the solution will not be stationary, because savings will have to adjust. However consumption will be stationary.

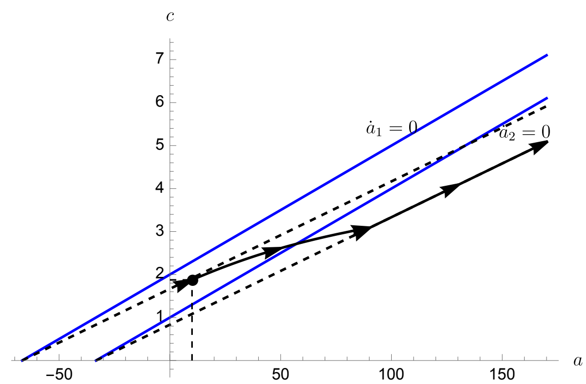
[12]This is a two-stage optimal control problem and the first-order necessary conditions and its heuristic derivation can be found in https://pmbbrito.github.io/cursos/phd/ame/ame2122/ame2021_lecture5.pdf.



(a) Case 1: $r < \rho$



(b) Case 2: $r = \rho$



(c) Case 3: $r > \rho$

Figure 2: Phase diagrams for the dynastic household problem. The upper isocline is drawn assuming that $w = w_1$ and the lower isocline is drawn assuming that $w = w_2 < w_1$.

4.2 Unexpected changes in income and comparative dynamics

In aggregate terms, in actual economies, one would expect that no agent would have an unbounded net wealth level, both as a net creditor or as a net debtor, because the individual agent interest rate would eventually be endogenous to its level of wealth. This means that it will become a large agent, if it is a creditor, or eventually it will face a borrowing constraint, if it is a debtor.

This means that if we consider problem (P6) as representative of an aggregate economy the natural case to take is the second, that is the case in which $r = \rho$. This is indeed the simplest model for a small open economy facing perfect international capital markets.

In addition to the those already presented, another reason for considering infinite horizons is related to the use of this model as (or within) a business cycle model in which we are interested in studying deviations from a stationary trend. We can show that the infinite horizon version of the previous models produces stationary solutions, i.e. steady states and deviation from steady states.

In this section we study the comparative dynamics for a shock in the non-financial income w, in order to discuss the theory provided by this model on the relationship between consumption and income. In addition, we want to have a comparison with the same type of results for the habit formation model that we present in the next section 5.

There are two types of changes in income with a bearing on the solution of this model: anticipated and non-anticipated changes. Anticipated changes are already incorporated in the solution of the model, as we saw in the previous subsection. Non-anticipated changes can be seen as shocks occurring at time $t = 0$ and we can adapt the results of the previous section to study that case.

However, we dedicate this section to the study of non-anticipated changes by a **comparative dynamics exercise**. In the rest of this subsection we consider again problem (P6) in the case in which $r = \rho$. The optimality conditions are

$$\dot{a} = \rho\, a + w - C(q) \tag{9a}$$

$$\dot{q} = 0 \tag{9b}$$

$$c(t) = C(q) \equiv q(t)^{-\frac{1}{\theta}} \tag{9c}$$

$$a(0) = a_0, \tag{9d}$$

which comprise the instantaneous budget constraint, the adjoint equation, the static optimality condition and the initial level for the net wealth.

From equation (9d) as net wealth a is given initially, we call it pre-determined variable, and the adjoint variable q, or consumption c, which is monotonously related to q by equation (9c), as is a non-predetermined variable.

Equation (9b) implies that q is constant, which implies that c is constant as well, as we saw in the last section. Therefore, there are potentially an infinite number of steady states, from equation (9a), comprising all combinations of a and q that satisfy

$$c = \rho\, a + w$$

However, from the fact that a is pre-determined we can tie down the steady state which interests us to be

$$\bar{c} = C(\bar{q}) = \rho\, a_0 + w.$$

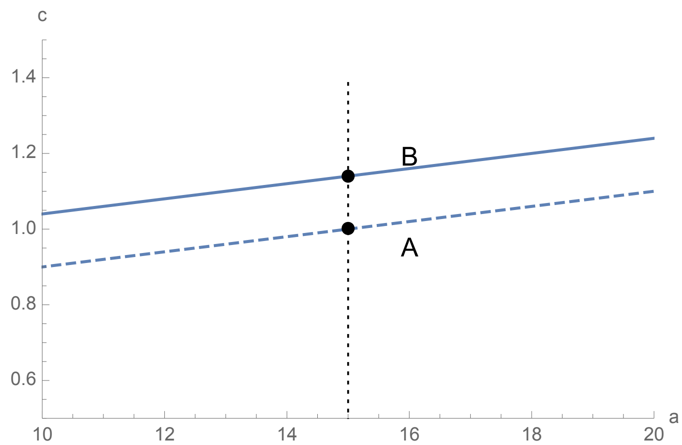


Figure 3: Effect of a non-anticipated increase in income. The line corresponds to the isocline for a, $c = w + \rho\, a$. A shock in income, w, moves the isocline up, leading the economy to jump from point A to point B.

This steady state value for c is dependent on the value of non-financial income w, which lead us to write $\bar{c} = \bar{c}(w)$.

A comparative dynamics exercise asks the following question: given an initial value of w,say w_0 what is the effect on the solution to the problem if w_0 increases to $w_1 = w_0 + dw$, starting from a steady state ?

In the appendix E we prove that the multipliers are

$$dc(t) = dw, \text{for any } t \in (0, \infty)$$
$$da(t) = 0, \text{for any } t \in (0, \infty)$$

that is, consumption immediately and completely adjusts to innovations in income, which means that they are perfectly correlated. There is no transitional dynamics.

A phase diagram is presented in Figure 3.

Therefore, in the benchmark infinite horizon model of the household behavior, in a deterministic setting, non-anticipated changes in income will be immediately spent in consumption, which means that this model displays a counterfactual perfect correlation between consumption and income and a potentially volatile behavior of consumption.

This is a consequence of the assumption $r = \rho$, which is what one would expect to be the case for a representative household in the long run.

Therefore we need some mechanism allowing for an incomplete translation of income shocks to consumption expenditures which could generate a consumption adjustment closer to stylized facts. Unfortunately this comes at a cost of increasing the dimension of the model.

5 Intertemporally dependent preferences

There are several ways to introduce intertemporally dependent preferences. [13] In this section we consider the habit formation model.

There are two types of models dealing with habit formation: the internal habit formation (also called habit persistence) model and the external habit formation model. In the first type of models household have an internal pattern of consumption that only changes marginally and in the second they follow an external pattern of consumption. While in the first the pattern of consumption, or habit, is built throughout time internally, in the second type of models it is an externality. This is why the second type of habits involve ”going along with the Joneses' (see Abel, 1990). In this sense, classifying the two types of models within the same category can be misleading

Next we present the preferences under (internal) habit formation, in subsection 5.1 and in subsection 5.2 we extend the previous household consumption model with habit formation.

5.1 Preferences under habit formation

We saw, in section 2, that if the intertemporal utility functional is additive, in the utility of consumption for different moments in time, it displays intertemporally independent preferences, in the sense that the history of consumption until time t, i.e., $c^t = \big(c(s)\big)_{s=0}^{t}$, does not influence the valuation of consumption at time t, $u(c(t))$. We also saw that this leads to large shifts in consumption after innovations, which is counter-factual. In the **habit formation** utility functional past consumption affects the evaluation of consumption for every point in time.

Assume that the household has a consumption pattern, that we call habit, and denote by $h(t)$. Habits are formed from past consumption. Therefore, in this model, current consumption $c(t)$ has two effects: first, it is an immediate source of utility, and, second, it marginally changes the pattern of consumption. In this sense there is a ”technology” in changing habits through current consumption which we assume to be linear and have a parameter η. However, habits record the past consumption history with some rate of decay which we assume to be also equal to η.

[13] For the relationship between the habit preference model and alternative specifications of intertemporally dependent preferences see Shi and Epstein, 1993. On the relationship between addiction and satiation see Iannaccone, 1986.

The following version is a simplified version of the model: [14]

$$\mathsf{U}[c] = \int_0^T u(c(t), h(t))\, e^{-\rho\, t}\, dt$$

where $h(t)$ is the habits at time t, which are

$$h(t) = e^{-\eta\, t}\Big(h_0 + \eta \int_0^t e^{\eta\, s}\, c(s)\, ds\Big)$$

where $\eta > 0$ represents both the effect present consumption in the stock of habits and the rate of decay (or forgetting) of habits. This is a stock (pre-determined) variable which can be seen as the solution to the problem

$$\begin{cases} \dot{h} = \eta\,(c(t) - h) & \text{for } t \in \mathrm{T} \\ h(0) = h_0 & \text{for } t = 0. \end{cases}$$

In this case the utility at time t is a function of the consumption and the habits, $u(t) = u(c(t), h(t))$, where we assume that the marginal utility of consumption is positive $u_c(c,h) \equiv \dfrac{\partial u(c,h)}{\partial c} > 0$ but the marginal utility of habits is negative $u_h(c,h) \equiv \dfrac{\partial u(c,h)}{\partial h} < 0$. We also assume that the utility function is continuous and smooth.

Using the concepts introduced in section 2, the marginal utility for consumption at time t is now

$$\mathsf{U}_t \equiv \delta \mathsf{U}[c;t] = e^{-\rho t}\Big(u_c(c(t), h(t)) + \eta \int_t^{\bar{t}} e^{-(\eta+\rho)\,(s-t)} u_h(c(s), h(s))\, ds\Big)\ ,\ \text{for any } t \in \mathrm{T}.$$

where $\bar{t} = \max\ \mathrm{T}$. The intertemporal marginal rate of substitution between t_0 and $t_1 = t_0 + \tau$ is now

$$IMRS_{t_0,t_1} = \frac{e^{\rho\,\tau}\Big(u_c(c(t_0), h(t_0)) + \eta \int_{t_0}^{\bar{t}} e^{-(\eta+\rho)\,(s-t_0)} u_h(c(s), h(s))\, ds\Big)}{u_c(c(t_1), h(t_1)) + \eta \int_{t_1}^{\bar{t}} e^{-(\eta+\rho)\,(s-t_1)} u_h(c(s), h(s))\, ds}.$$

For a stationary consumption path, such that $c(t) = \bar{c}$ and $h(0) = \bar{c}$, implying $h(t) = \bar{c}$, for any $t \in \mathrm{T}$, we find [15]

$$\bar{\mathsf{U}}_t = \delta \mathsf{U}[\bar{c};t] = e^{-\rho t}\ \bar{\mathsf{U}}'$$

where we define

$$\bar{\mathsf{U}}' \equiv u_c(\bar{c}) + \frac{\eta}{\eta+\rho} u_h(\bar{c})$$

[14] The first papers are Wan, 1970 and Ryder and Heal, 1973.

[15] Setting $h_0 = \bar{c}$, we find $h(t) = \bar{c}$ for any $t \in \mathrm{T}$ and therefore, we write $u_c(\bar{c}) = u_c(\bar{c}, \bar{c})$ and $u_h(\bar{c}) = u_h(\bar{c}, \bar{c})$.

and assume that $\bar{\mathsf{U}}' > 0$. For any other moment in time $t' \neq t$ we have the marginal utility $\bar{\mathsf{U}}_{t'} = e^{-\rho t'}\ \bar{\mathsf{U}}'$.

This implies that the $IMRS_{t_0,t_1} = e^{\rho\,\tau}$ is the same as in the additive model, which means that preferences with habit formation display **impatience** as in the additive model.

The change in marginal utility is now given by the second-order functional derivative

$$\begin{aligned}\mathsf{U}_{t_i,t_j} \equiv\ & \delta^2\mathsf{U}[c;t_i,t_j]\ = \\ &= \eta\, e^{-\rho t_i}\ \Big(\int_{t_i} e^{-(\eta+\rho)(t-t_i)}\, u_{hc}(c(t),h(t))\,dt + \eta\int_{t_j} e^{-(\eta+\rho)(t-t_i)-\eta(t-t_j)}\, u_{hh}(c(t),h(t))\,dt\Big)\end{aligned}$$

and for a stationary consumption path, we find

$$\bar{\mathsf{U}}_{t_i,t_j} = \eta\, e^{-\rho\, t_j - \eta(t_j-t_i)}\ \bar{\mathsf{U}}''$$

where we define

$$\bar{\mathsf{U}}'' \equiv u_{hc}(\bar{c}) + \frac{\eta}{2\eta+\rho}\ \ u_{hh}(\bar{c}).$$

The Uzawa-Allen elasticities, associated to a stationary consumption path, depend only on the time difference between t and t'

$$\epsilon_{t,t'} = -\frac{\bar{\mathsf{U}}_{t,t'}\ \bar{c}}{\bar{\mathsf{U}}_t} = e^{-(\rho+\eta)\,(t'-t)}\sigma_h(\bar{c})$$

where we define

$$\sigma_h(\bar{c}) = -\frac{\eta\,\bar{\mathsf{U}}''\,\bar{c}}{\bar{\mathsf{U}}'}.$$

As $e^{-(\rho+\eta)\,(t'-t)} \in (0,1)$ then we have intertemporal substitutability if $\bar{\mathsf{U}}'' < 0$, implying $\sigma_h(\bar{c}) > 0$, independence if $\bar{\mathsf{U}}'' = 0$, implying $\sigma_h(\bar{c}) = 0$, and intertemporal complementarity if $\bar{\mathsf{U}}'' > 0$, implying $\sigma_h(\bar{c}) < 0$.

As, in general $\sigma_h(\bar{c}) \neq 0$ then, using the definition of the intertemporal elasticity of substitution given in equation (3), yields

$$IES_{t_0,t_1} = \frac{1}{\sigma_h(\bar{c})}\ \Big(\frac{1+e^{-\rho\tau}}{1+e^{-\rho\tau} - 2\,e^{-(\rho+\eta)\tau}}\Big).$$

Comparing with the additive model, in this case the IES can have any sign, depending on the elasticity $\sigma_h(\bar{c})$, and its magnitude is a function of the lag between the two moments, τ. If there is intertemporal substitution we see that $\lim_{\tau\to 0} IES_{t,t+\tau} = +\infty$ and if $\lim_{\tau\to\infty} IES_{t,t+\tau} = \dfrac{1}{\sigma_h(\bar{c})}$, which means that for any $0 < \tau < \infty$, the intertemporal elasticity of substitution will be smaller than the inverse of $\sigma_h(\bar{c})$, $IES_{t,t+\tau} > \dfrac{1}{\sigma_h(\bar{c})}$.

There are two benchmark utility functions displaying habit formation in the literature: the additive habits model (see Constantinides, 1990),

$$u(c,h) = v(c - \zeta\, h),\ \text{for}\ \zeta > 0$$

and the multiplicative habits model (see Carroll, 2000)

$$u(c,h) = v(c\, h^{-\zeta}),\ \text{for}\ 0 < \zeta < 1$$

where ζ measures the force of habits, i.e, the relative weight of habits as regards present consumption. In both models the utility of consumption is measured against the change of a monotonous function of habits.

If we assume multiplicative habits such that

$$u(c,h) = \frac{1}{1-\theta}\Big(\big(c\, h^{-\zeta}\big)^{1-\theta} - 1\Big) \tag{10}$$

and evaluate it at a stationary path such that $h(t) = c(t) = \bar{c}$, for any t, then we obtain

$$\bar{\mathsf{U}}' = \Big(\frac{\eta\,(1-\zeta) + \rho}{\eta + \rho}\Big)\,\bar{c}^{\zeta\,(\theta-1)-\theta} > 0$$

$$\bar{\mathsf{U}}'' = \zeta\Big(\frac{(\theta-1)(\rho + \eta\,(2-\zeta)) + \eta}{2\,\eta + \rho}\Big)\,\bar{c}^{\zeta\,(\theta-1)-\theta-1} > 0$$

if $\theta \geq 1$ and $0 < \zeta < 1$. Therefore,

$$\sigma_h(\bar{c}) = -\eta\,\zeta\,\frac{(\eta+\rho)\Big((\theta-1)(\rho + \eta\,(2-\zeta)) + \eta\Big)}{(2\,\eta + \rho)\,(\eta\,(1-\zeta) + \rho)} < 0$$

which means that this model displays **intertemporal complementarity** (in the Edgeworth-Pareto sense).

Exercise Prove this.

Exercise Find the IES of the additive habit formation model where

$$u(c,h) = \frac{1}{1-\theta}\Big(\big(c - \zeta\, h\big)^{1-\theta} - 1\Big)$$

and find its intertemporal dependence properties.

5.2 Partial equilibrium under habit formation

The extension of the household problem (P6) with habit formation becomes:

$$
\begin{aligned}
&\max_{c(\cdot)} \int_0^{\infty} u(c(t), h(t))\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\, a + w(t) - c(t), \text{ for } t \in \mathbb{R}_+ \\
&\dot{h}(t) = \eta\,(c - h) \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&h(0) = h_0 \text{ given} \\
&\lim_{t\to\infty} a(t)\, e^{-r\,t} \geq 0
\end{aligned}
\tag{P7}
$$

where, if we assume the multiplicative habit formation model, we have $u_c(c, h) > 0$, $u_h(c, h) < 0$, $u_{cc}(c, h) < 0$, $u_{ch}(c, h) = u_{hc}(c, h) > 0$ and $u_{hh}(c, h)$ has an ambiguous sign.

Exercise Compute those derivatives for the general case and for the case in which $c = h$.

Observe that we now have two state variables, a and h and just one control variable c. This means that we have two initial conditions , one for each pre-determined (or state) variable, $a(0) = a_0$ and $h(0) = h_0$.

Assume from now on that $r = \rho$. This implies, using the intuition from subsection 3.5, that the consumption path is bounded, and therefore the stock of habits is bounded as well.

The current-value Hamiltonian function is now

$$
\mathcal{H} = u(c, h) + q_a(\rho\, a + w - c) + q_h\, \eta\,(c - h)
$$

where q_a is the adjoint variable associated to the stock of net wealth a and q_h is the adjoint variable associated with the stock of habits h.

Using the Pontriyagin's maximum principle the optimality conditions for problem (P7) are

$$\dot{a} = \rho\, a + w - c, \tag{11a}$$
$$\dot{h} = \eta\,(c - h), \tag{11b}$$
$$\dot{q}_a = 0, \tag{11c}$$
$$\dot{q}_h = (\rho + \eta)\, q_h - u_h(c, h), \tag{11d}$$
$$u_c(c(t), h(t)) = q_a(t) - \eta\, q_h(t), \tag{11e}$$
$$a(0) = a_0, \tag{11f}$$
$$h(0) = h_0 \tag{11g}$$

and the transversality conditions.

As in problem (P6), with an intertemporally additive utility functional, the static arbitrage condition (11e) can be implicitly solved for optimal consumption. Consumption is not only a function of shadow value of net financial asset, as in problem (P6), but it is also a function of the stock of habits and of its shadow value

$$c = C(h, q_a, q_h),$$

with partial derivatives

$$C_h = -\frac{u_{ch}\ (c, h)}{u_{cc}\ (c, h)} > 0,\ C_{q_a}\ = \frac{1}{u_{cc}\ (c, h)} < 0,\ C_{q_h} = -\frac{\eta(c, h)}{u_{cc}\ (c, h)} > 0.$$

Therefore consumption decreases with the shadow value of the financial asset but increases with both the level and the value of the stock of habit.

As we are interested in comparing the dynamic comparative statics properties of this model with the non-habit formation model we deal with the case in which the initial condition is a steady state, and introduce a perturbation in non-financial income from $w = w_0$ to $w = w_1 = w_0 + dw$.

As in that model equation (11c) implies that the steady state exists but there is potentially an infinite number of steady states. Furthermore, equation (11b), evaluated at the steady state yields $\bar{c} = \bar{h}$.

Anchoring again the steady state by the initial value of financial wealth a_0, a steady state only exists if the initial value of habits satisfies $\bar{h} = h_0 = \rho\, a_0 + w_0$, which we assume is the case from now on.

Therefore, the steady state, for $w = w_0$, is determined from the equations

$$\bar{a} = a_0 \tag{12a}$$

$$\bar{h} = h_0 = \rho\, a_0 + w_0 \tag{12b}$$

$$\bar{c}(w_0) = \rho\, a_0 + w_0 \tag{12c}$$

$$\bar{q}_h(w_0) = \frac{u_h(\bar{c}(w_0), h_0)}{\rho + \eta} \tag{12d}$$

$$\bar{q}_a(w_0) = u_c(\bar{c}(w_0), h_0) + \frac{\eta}{\rho + \eta} \quad u_h(\bar{c}(w_0), h_0) \tag{12e}$$

This steady state, projected in the space (a, c) is shown by point A in Figure 4.

Assuming a multiplicative habits model with utility function

$$u(c, h) = \frac{\left(c\, h^{\zeta}\right)^{1-\theta} - 1}{1 - \theta}$$

we find the steady state values for q_a and q_h

$$\bar{q}_a = \frac{\rho + \eta\,(1 - \zeta)}{\eta + \rho}\left(w_0 + \rho a_0\right)^{-\theta(1-\zeta)-\zeta}$$

$$\bar{q}_a = -\frac{\zeta}{\eta + \rho}\left(w_0 + \rho a_0\right)^{-\theta(1-\zeta)-\zeta}$$

Exercise Prove this.

Exercise Find the steady state for the additive habits model.

The Jacobian of the MHDS, evaluated at the steady state is a four dimensional matrix which can be found in Appendix F. This Jacobian has four eigenvalues, one is equal to zero, another is equal to ρ (as in the additive model), and we have two more eigenvalues

$$\lambda_s = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 - S},\ \lambda_u = \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}\right)^2 - S}$$

If $S < 0$ then $\lambda_s < 0 < \lambda_u$ [16] and the model can display transition dynamics converging to a steady state.

[16] These two eigenvalues also satisfy $\lambda_s + \lambda_u = \rho$ and $\lambda_s\, \lambda_u = S$.

In the appendix F we prove that

$$S \equiv -\eta\Big(\frac{(\rho+\eta)\bar{u}_{cc} + (2\eta+\rho)(\bar{u}_{hc} + \eta\bar{u}_{hh})}{\bar{u}_{cc}}\Big).$$

Observe that, using the notation same notation as in subsection 5.1 we can write

$$S = -\frac{\eta\,(2\eta+\rho)}{\bar{u}_{cc}}\,\left(\frac{\rho+\eta}{2\eta+\rho}\bar{u}_{cc} + \bar{\mathsf{U}}''\right)$$

which means that S is negative if consumption is intertemporally substitutable or independent, i.e., $\bar{\mathsf{U}}'' \le 0$, or if it is intertemporally complementary, i.e., $\bar{\mathsf{U}}'' > 0$, the concavity of the utility function $\bar{u}_{cc}$ dominates the intertemporal complementarity effect.

For the multiplicative case we found that there is intertemporal complementarity, and, furthermore, we also have

$$S = -\frac{\zeta\,(\rho+\eta(1-\zeta))\,(\theta\,(1-\zeta)+\zeta)}{\theta}\ < 0$$

which means that we have the last case: although there is intertemporal complementarity, if $0 < \zeta < 1$, it is dominated by the decreasing marginal utility relative to instantaneous consumption.

It can be shown that if there is an increase in the wage rate by $dw > 0$ the (linearly approximate) dynamics that unfolds is the following:

1. at the time of the shock consumption increases discontinuously from point A to point B;
2. this introduces a change in the stock of net wealth but also a change in the stock of habits;
3. however, as the stock of habits only changes slowly, the increase in wage is not completely used in the purchase of goods, which generates positive savings;
4. changes in savings increases the stock of net wealth which increases further consumption, the stock of habits and savings;
5. eventually, a new steady state will be reached. A new steady state, depicted as point C will only be reached when we have again $\bar{c}(w_1) = \bar{h}(w_1) = \bar{a}(w_1)$.

There are two noteworthy dynamic features of this model. First, there is a mechanism for stability wihich takes the form of a decay mechanism in habit formation (in equation $\dot{h} = \eta\,(c - h)$). Second, this model still has a degenerate nature related to the fact that the interest rate is independent of the level of net wealth of the agent and we have assumed it is equal to the rate of time preference. We deal with the degenerate nature of the model by ”anchoring” the solution.

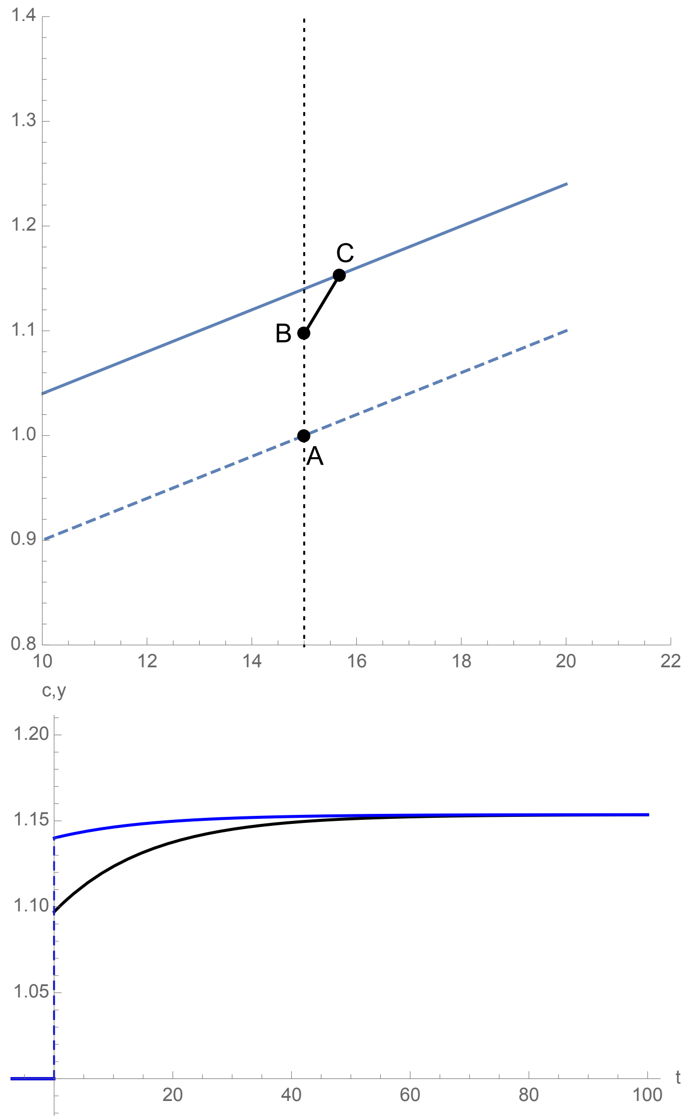


Figure 4: Effect of a non-anticipated increase in income in the habit formation model

Differently from the the time-additive case, we do not anchor the solution to a_0 but by a steady state relationship between the stock of habits and net wealth by $\bar{h} = \rho\, \bar{a} + w$.

The lower diagram in Figure 4 show the (approximate) trajectories for income $y(t) = \rho a(t) + w$ and consumption after the shock in non-financial income. As can be seen they are positively but not perfectly correlated, and consumption has a slower adjustment, for the reasons just explained.

This type of behavior replicates more closely the stylized facts than the additive model.

6 Bibliography

Important initial papers Wan, 1970, Ryder and Heal, 1973, Ravn, Schmitt-Grohé, and Uribe, 2006

Policy implications Fuhrer, 2000, Lettau and Uhlig, 2000

Literature on habit formation and finance: Sundaresan, 1989, Constantinides, 1990, Detemple and Zapatero, 1991, Chapman, 1998.

Habit formation and growth: Carroll, 2000

Surveys on intertemporal substitution: Thimme, 2017 Cursory observations in Romer, 2019

References

Backus, D. K., Routledge, B. R., & Zin, S. E. (2004). Exotic preferences for macroeconomists. In M. Gertler & K. Rogoff (Eds.), *NBER Macroeconomics Annual 2004* (Vol. 19, pp. 319–414). MIT Press.

Biswas, T. (1976). A note on complementarity over time. *The Review of Economic Studies*, *43*(1), 179–181.

Ryder, H. E. J., & Heal, G. M. (1973). Optimal growth with intertemporally dependent preferences. *Review of Economic Studies*, *40*(1), 1–32.

Heal, G., & Ryder, H. E. (1976). Reply to "A Note on Complementarity Over Time". *Review of Economic Studies*, *43*(1), 183–184.

Milgrom, P., & Shannon, C. (1994). Monotone comparative statics. *Econometrica*, *62*(1), pp. 157–180.

Uribe, M., & Schmitt-Grohé, S. (2017). *Open economy macroeconomics*. Princeton University Press.

Gel'fand, I. M., & Fomin, S. V. (1963). *Calculus of variations*. Dover.

Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of consumer. *Review of Economic Studies*, *32*(2), 137–50.

Shi, S., & Epstein, L. (1993). Habits and time preference. *International Economic Review*, *34*, 61–84.

Iannaccone, L. (1986). Addiction and satiation. *Economics Letters*, *21*(1), 95–99.

Abel, A. (1990). Asset prices under habit formation and catching up with the Joneses. *American Economic Review, Papers and Proceedings*, *80*, 38–42.

Wan, J., Henry Y. (1970). Optimal Saving Programs under Intertemporally Dependent Preferences. *International Economic Review*, *11*(3), 521–47.

Constantinides, G. M. (1990). Habit formation: A resolution of the equity premium puzzle. *Journal of Political Economy*, *98*(3), 519–543.

Carroll, C. (2000). Solving consumption models with multiplicative habits. *Economic Letters*, *68*, 67–77.

Ravn, M., Schmitt-Grohé, S., & Uribe, M. (2006). Deep habits. *Review of Economic Studies*, *73*, 195–218.

Fuhrer, J. C. (2000). Habit Formation in Consumption and its Implications for Monetary Policy. *American Economic Review*, *90*, 367–390.

Lettau, M., & Uhlig, H. (2000). Can Habit Formation Be Reconciled with Business Cycle Facts. *Review of Economic Dynamics*, *3*, 79–99.

Sundaresan, S. M. (1989). Intertemporally dependent preferences and the volatility of consumption and wealth. *Review of Financial Studies*, *2*(1), 73–89.

Detemple, J. B., & Zapatero, F. (1991). Asset prices in an exchange economy with habit formation. *Econometrica*, *59*(6), 1633–57.

Chapman, D. A. (1998). Habit formation and aggregate consumption. *Econometrica*, *66*(5), 1223–1230.

Thimme, J. (2017). Intertemporal substitution in consumption: a literature review. *Journal of Economic Surveys*, *31*(1), 226–257.

Romer, D. (2019). *Advanced macroeconomics* (Fifth). McGraw-Hill.

A functional derivatives

There are several different ways to present the idea of the derivative of a functionaldeal, some more mathematically correct and some more informal, as is the case in physics or mechanics. We follow Gel'fand and Fomin, 1963 which presents a good compromise between those two approaches.

Assume we have the space of functions $\mathcal{F}$, i.e., a collection of functions sharing some common property, for instance, continuity, differentiability, boundedness, etc. Every element of $\mathcal{F}$, for instance f, is a mapping between a space X and a subset of the space of real numbers, that is $f : \mathrm{X} \subseteq \mathbb{R} \to \mathbb{R}$. A **functional** is a mapping between the space of functions and the set of real numbers $\mathsf{F} : \mathcal{F} \to \mathbb{R}$.

The following functionals are common examples in economics:

$$\mathsf{F}_1[f] \; = \int_{\mathrm{X}} f(x)\, dx,$$

$$\mathsf{F}_2[f] \; = \int_{\mathrm{X}} u(f(x), x)\, dx$$

where $u(\cdot)$ is a function, or

$$\mathsf{F}_3[f] \; = \int_{\mathrm{X}} u(f'(x), f(x), x)\, dx$$

where $f'(\cdot)$ is the derivative of function $f(\cdot)$.

There are two concepts of derivatives related to functionals.

Consider a perturbation in function f from $f \mapsto f + \delta f$. A Frechet derivative of functions $\mathsf{F}[f]$, denoted by $\delta\mathsf{F}[f]$ is defined from

$$\lim_{\delta f \to 0} \frac{||\mathsf{F}[f + \delta f] \; - \mathsf{F}[f] \; - \delta\mathsf{F}[f]||}{\delta f} = 0$$

where $|| \cdot ||$ is the norm of space $\mathcal{F}$.

A more useful concept is the concept is the Gâteaux derivative. It is an extension of the directional derivative for functions. Assume that we introduce a perturbation on function $f \mapsto f + \varepsilon\varphi$ in which φ is a function, $\varphi : X \to \mathbb{R}$ and $\varepsilon > 0$ is a number.

The first variation of a functional is

$$d\mathsf{F}[f; \varphi] \; = \mathsf{F}[f + \eta\, \varepsilon] \; - \mathsf{F}[f]$$

and the functional derivative (in the GâteUx sense) is analogous to the concept of derivative for functions:

$$\delta\mathsf{F}[f; \varphi] = \lim_{\varepsilon \to 0} \frac{d\mathsf{F}[f; \varphi]}{\epsilon}.$$

If the functional involves more than one function we may write $\delta_{f_1}\mathsf{F}[f_1, f_2; \varphi_1, \varphi_2]$.

In regular cases, in order apply this concept, we perform a first order Taylor approximation

$$\mathsf{F}\ [f + \epsilon\varphi]\ = \mathsf{F}[f]\ + \delta\mathsf{F}[f;\varphi]\ \varepsilon + o(\varphi)$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi)||}{\varepsilon} = 0$.

Performing a second order Taylor approximation we find

$$\mathsf{F}\ [f + \epsilon\,\varphi]\ = \mathsf{F}[f]\ + \delta\mathsf{F}[f;\varphi]\ \varepsilon + \delta^2\mathsf{F}[f;\varphi^2]\ \varepsilon^2 + o(\varphi^2).$$

However, we can distinguish between the "own" second order functional derivative

$$\delta\mathsf{F}[f + \varepsilon\,\varphi;\varphi] = \delta\mathsf{F}[f;\varphi] + \delta^2\mathsf{F}[f;\varphi^2]\ \varepsilon^2 + o(\varphi^2)$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi^2)||}{\varepsilon} = 0$, and "crossed" second order derivative

$$\delta\mathsf{F}[f + \varepsilon\,\varphi';\varphi] = \delta\mathsf{F}[f;\varphi] + \delta^2\mathsf{F}[f;\varphi,\varphi']\ \varepsilon^2 + o(\varphi\,\varphi')$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi\,\varphi')||}{\varepsilon} = 0$

For the previous examples, and assuming that $\varphi(x) = 0$ for $x \in \partial\mathrm{X}$, where $x \in \partial\mathrm{X}$ is the boundary of X, we have

$$\delta\mathsf{F}_1[f;\varphi]\ = \int_X f(x)\,\varphi(x)\ dx,$$

$$\delta\mathsf{F}_2[f;\varphi]\ = \int_X \frac{\partial U(f(x),x)}{\partial f}\ \varphi(x)\ dx$$

and, if f belongs to a space of differentiable functions, which means that so φ does,

$$\delta\mathsf{F}_3[f;\varphi]\ = \int_X \frac{\partial U(f'(x),f(x),x)}{\partial f'}\ \varphi(x)\ + \frac{\partial U(f'(x),f(x),x)}{\partial f}\,\varphi'(x)\,dx.$$

For the second case we have the second order functional derivative

$$\delta^2\mathsf{F}_3\ [f;\varphi]\ = \int_X \frac{\partial^2 U(f(x),x)}{\partial f^2}\ \varphi^2(x)\ dx.$$

A particular case of perturbation is the "spike" variation. In this case the functional derivative is sometimes called the Volterra derivative.

Consider an element of the domain of function f, say x' and introduce the variation $df(x) = 0$ if $x \neq x'$ and $df(x) = \varepsilon$ if $x = x'$. We can write this as a perturbation $f \mapsto f + \varepsilon\, \delta(x - x')$ where δ is Dirac's delta generalized function. It has the property $\int_{\mathrm{X}} \delta(x - x')\, f(x)\, dx = f(x')$.

Applying to the previous examples, we find

$$\delta \mathsf{F}_1[f; x'] = \int_X f(x)\, \delta(x - x')\ \ dx = f(x') \tag{13}$$

$$\delta \mathsf{F}_2[f; x']\ = \int_X \frac{\partial U(f(x), x)}{\partial f}\ \ \delta(x - x')\ \ dx = \frac{\partial U(f(x), x)}{\partial f}\ \Big|_{x = x'}. \tag{14}$$

B Necessary conditions for problem (P2)

We introduce the **functional**

$$\mathsf{U}([c], a_0) = \int_0^T u(c(t))\, e^{-\rho t} - \lambda(t)\big(\dot{a}(t) + c(t)\big)\, dt$$

where we introduce an adjoint multiplier $\lambda : \mathrm{T} \to \mathbb{R}_+$. Its introduction involves a penalization associated to the reduction in value brought about by the budget constraint. Because are now constrained by the initial value of the stock, we call value **function** to

$$V(a_0) = \max_{c(\cdot)} \mathsf{U}([c], a_0) = \mathsf{U}([c^*], a_0) = \int_0^T u(c^*(t))\, e^{-\rho t} - \lambda(t)\big(\dot{a}^*(t) + c^*(t)\big)\, dt.$$

Assume we know the optimal path $\big(c^*(t), a^*(t)\big)_{t \in \mathrm{T}}$.

We introduce now perturbations in both functions $c^*(t) \to c^*(t) + \varphi_c(t)$ and $a^*(t) \to a^*(t) + \varphi_a(t)$, such that $\varphi_a(0) = 0$, because $a^*(0) = a_0$ is not free. The value functional is now

$$\mathsf{U}([c^* + \varphi_c], a_0) = \int_0^T u(c^*(t) + \varphi_c(t))\, e^{-\rho t} - \lambda(t)\big(c^*(t) + \varphi_c(t) + \dot{a}^*(t) + \dot{\varphi}_a(t)\big)\, dt.$$

The first variation becomes,

$$\begin{aligned}
\delta \mathsf{U}([c^*]) &= \int_0^T \big(u'(c^*(t)\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t) - \lambda(t)\, \dot{\varphi}_a(t)\, dt \\
&= \int_0^T \big(u'(c^*(t)\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t)\, dt - \int_0^T \lambda(t)\, \dot{\varphi}_a(t) dt \\
&= \int_0^T \big(u'(c^*(t)\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t)\, dt - \lambda(t) \varphi_a(t)\Big|_{t=0}^T + \int_0^T \dot{\lambda}\ (t) \varphi_a(t) dt \\
&= \int_0^T \big(u'(c^*(t))\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t)\, dt - \lambda(T) \varphi_a(T) + \int_0^T \dot{\lambda}\ (t) \varphi_a(t) dt
\end{aligned}$$

Using integration by parts and the fact that $\varphi_a(0) = 0$. At the optimum $\left(c^*(t), a^*(t)\right)_{t\in \mathrm{T}}$ is such that $\delta \mathsf{U}([c^*]) = 0$. The first-order conditions are thus: $u'(c^*(t)\, e^{-\rho t} - \lambda(t) = \dot{\lambda}(t) = 0$, for $t \in [0, T]$ and $\lambda(T) = 0$. Integrating $\dot{\lambda}(t) = 0$ we find $\lambda(t) =$ constant for every $t \in [0, T]$. but as $\lambda(T) = 0$ then $\lambda(t) = 0$ for every $t \in [0, T]$. Therefore $u'(c^*(t)\, e^{-\rho t} = 0$ for every $t \in [0, T]$ as in (P1:foc).

C Necessary conditions for problem (P3)

The (penalized) utility functional is

$$\mathsf{U}([c], a_0) = \int_0^T u(c(t))\, e^{-\rho t} - \lambda(t)\big(\dot{a}(t) + c(t)\big) + \eta(t)(a(t) - \underline{a})\, dt$$

where we introduce a multiplier $\eta : \mathrm{T} \to \mathbb{R}_+$ associated to the instantaneous constraint on a such that the complementary slackness conditions hold

$$\eta(t) \geq 0,\ \eta(t)(a(t) - \underline{a}) = 0,\ \text{for every}\, t \in \mathrm{T}.$$

Using the same method as in section B the perturbed value functional is

$$\mathsf{U}([c^* + \varphi_c], a_0) = \int_0^T u(c^*(t) + \varphi_c(t))\, e^{-\rho t} - \lambda(t)\big(c^*(t) + \varphi_c(t) + \dot{a}^*(t) + \dot{\varphi}_a(t)\big) + \eta(t)(a^*(t) + \varphi_a(t) - \underline{a})\, dt.$$

Then, using the same procedure as before

$$\delta \mathsf{U}([c^*]) = \int_0^T \left(u'(c^*(t))\, e^{-\rho t} - \lambda(t)\right) \varphi_c(t)\, dt - \lambda(T)\varphi_a(T) + \int_0^T (\dot{\lambda}\ (t) + \eta(t))\varphi_a(t) dt.$$

Therefore the f.o.c are $u'(c^*(t))\, e^{-\rho t} = \lambda(t)$, $\dot{\lambda}(t) = -\eta(t)$, for $t \in [0, T]$, together with the complementary slackness conditions. However at $t = T$ we have also $\lambda(T) \geq 0$ and $\lambda(T)(a(T) - \underline{a}) = 0$, which is only possible if $\lambda(T^-) = \eta(T)$, or there is a discontinuity on $\lambda(t)$ at $t = T$.

D Necessary conditions for problem (P4)

We can find the necessary (in this case also sufficient) optimality conditions for problem (P4) by using the Pontriyagin maximum Principle. As the Hamiltonian function is

$$H = \frac{c^{1-\theta} - 1}{1 - \theta} + q\,(r\, a - c)$$

we have

$$\begin{cases} \dot{a} = r\,a - c & \text{for } t \in [0,T] \\ \dot{c} = \gamma_c\,c & \text{for } t \in [0,T] \\ a(0) = a_0 \text{ given} & \text{for } t = 0 \\ c(T)^{-\theta}(a(T) - \underline{a}) = 0 & \text{for } t = T \end{cases}$$

where the rate of growth of consumption is $\gamma_c \equiv \dfrac{r-\rho}{\theta}$. Solving the Euler equation we have $c(t) = c(0)\,e^{\gamma_c t}$. Substituting in the budget constraint, together with the initial condition yields

$$a(t) = e^{rt}\Big(a_0 + \frac{c(0)}{\gamma_c - r}(1 - e^{\gamma_c - r)\,t})\Big),\ \text{for}\, t \in [0,T]$$

If $c(0) > 0$ and finite, then $c(T) > 0$ and finite, which implies that the transversality constraint only holds if $a(T) = \underline{a}$. Therefore, we can find $c(0)$ by solving the equation

$$\underline{a}\ \ e^{rT} = a_0 + \frac{c(0)}{\gamma_c - r}(1 - e^{\gamma_c - r)\,T})$$

Then we obtain equations (4) and (5).

E Comparative dynamics

Assume we have a non-linear dynamic system

$$\dot{X} = F(X, \varphi)$$

where φ is an exogenous variable and $X = (x_1, x_2)$ in which x_1 is pre-determined and x_2 is non-predetermined.

Let the exogenous variable takes the value φ_0, and let the associated steady state be $\bar{X}(\varphi_0)$.

Now consider a variation in the exogenous variable from φ_0 to $\varphi_1 = \varphi_0 + d\varphi$. If the syatem is at the steady state $\bar{X}(\varphi_0)$ it will be perturbed away from it. Let $dX(t) = X(t) - \bar{X}(\varphi_0)$ be the variation of X when away from the steady state.

The effects resulting from the perturbation $d\varphi$ can be studied from the solutions of the **variational system**. Taking a time derivative, and observing that $d\dot{X}(t) = \dot{X}$ (because $\dot{\bar{X}} = 0$, yields the linear ordinary differential equation

$$\dot{X} = \bar{F}_x(\varphi_0)\,dX(t) + \bar{F}_\varphi(\varphi_0)\,d\varphi.$$

where the Jacobians are

$$\bar{F}_x(\varphi_0) \equiv F_x(\bar{X}(\varphi_0), \varphi_0),\ \bar{F}_\varphi\ (\varphi_0) \equiv F_\varphi(\bar{X}(\varphi_0), \varphi_0).$$

The **comparative dynamics multipliers** are the solutions, $dX(t)$ to this system.

For MHDS systems with only one pre-determined variable (or state variable) the Jacobian $\bar{F}_x(\varphi_0)$ has two eigenvalues, λ_s and λ_u, that satisfy the relationship $\lambda_s \le 0 < \lambda_u$. This means that the steady state is a saddle point, if $\lambda_s < 0$ or an unstable saddle-node, if $\lambda_s = 0$.

Therefore, two generic cases can occur, which have consequences on the method for determining $dX(t)$, depending on the Jacobian having a non-zero or a zero determinant. In the first case, we have $\lambda_s < 0$ and the dynamics will not depend on $x_1(0)$, the initial value of the pre-determined variable and in the second case $\lambda_s = 0$ and the dynamics will depend on $x_1(0)$.

Next we deal with the two cases separately.

Non-zero eigenvalues case

As $\det\left(\bar{F}_x(\varphi_0)\right) < 0$ then the Jacobian has a classic inverse, $\bar{F}_x(\varphi_0)^{-1}$, which allows us to determine the **long-run multipliers** as

$$d\bar{X} = -\bar{F}_x(\varphi_0)^{-1}\,\bar{F}_\varphi(\varphi_0)\,d\varphi = X_\varphi(\varphi_0)\,d\varphi$$

and the general solution to the variational system is

$$dX(t) = d\bar{X} + k_s\,P^s\,e^{\lambda_s\,t} + k_u\,P^u\,e^{\lambda_u\,t} \tag{15}$$

where P^s and P^u are the eigenvectors associated to the eigenvalues $\lambda_s < 0$ and $\lambda_u > 0$, respectively, and k_s and k_u are two arbitrary constants.

The two arbitrary constants provide us with two degrees of freedom allowing us to introduce two properties in the solution: first, we can force it converge to a new steady state $\bar{X}(\varphi_1)$, and, second, to make the pre-determined variable x_1 be continuous at the time of the shock, such that $dx_1(0) = x_1(0) - \bar{x}_1(\varphi_0) = 0$. The state variable, while being constant at the time of the shock, will start to change as a consequence of the shock, thus $d\dot{x}_1(0) \neq 0$.

The first condition is satisfied if we set set $k_u = 0$, which yields $dX(t) = d\bar{X} + k_s\,P^s\,e^{\lambda_s\,t}$, or, invector notation,

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} d\bar{x}_1 \\ d\bar{x}_2 \end{pmatrix} + k_s \begin{pmatrix} P^s_1 \\ P^s_2 \end{pmatrix} e^{\lambda_s\,t}.$$

The second condition is satisfied if, at time $t = 0$, we set $dx_1(0) = 0$, or equivalently, $d\bar{x}_1 + k_s\,P^s_1 = 0$. Solving for the other arbitrary constant yields $k_s = -\dfrac{d\bar{x}_1}{P^s_1}$.

Substituting both constants in equation (15) allows us to obtain the **comparative dynamics variations**

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} d\bar{x}_1(1 - e^{\lambda_s t}) \\ d\bar{x}_2 - d\bar{x}_1 \left(\frac{P_2^s}{P_1^s} \right) e^{\lambda_s t} \end{pmatrix}.$$

Recalling that $dX(t) = X(t) - \bar{X}(\varphi_0)$ we have equivalently

$$x_1(t) = \bar{x}_1(\varphi_0) + d\bar{x}_1(1 - e^{\lambda_s t})$$
$$x_2(t) = \bar{x}_2(\varphi_0) + d\bar{x}_2 - d\bar{x}_1 \left(\frac{P_2^s}{P_1^s} \right) e^{\lambda_s t}.$$

The meaning of this formula is the following: assuming that at time $t = 0$ the economy is at a steady state associated to the level of the exogenous variable φ_0, $(\bar{x}_1(\varphi_0), \bar{x}_2(\varphi_0))$, a change in the exogenous variable to level $\varphi_1 = \varphi_0 + d\varphi$ changes the steady state by $(d\bar{x}_1, d\bar{x}_2)$; as the variable x_1 is pre-determined the adjustment is not immediate; the variables $(x_1(t), x_2(t))$ trace out the path of the economy following that shock.

Evaluating for $t \to \infty$ yields the **long-run multipliers**

$$\frac{X(\infty) - \bar{X}(\varphi_0)}{d\varphi} = \frac{d\bar{X}}{d\varphi} = \begin{pmatrix} \frac{d\bar{x}_1}{d\varphi} \\ \frac{d\bar{x}_2}{d\varphi} \end{pmatrix}, \tag{16}$$

and evaluating at $t = 0$ we obtain the **impact multiplier for the non-predetermined variable**

$$\frac{x_2(0) - \bar{x}_2(\varphi_0)}{d\varphi} = \frac{P_1^s \, d\bar{x}_2 - d\bar{x}_1 \, P_2^s}{P_1^s \, d\varphi}.$$

The difference between the initial and the (approximated) final steady state after the shock in φ, which we denote by $d\bar{x}_i = \bar{x}_i(\varphi_1) - \bar{x}_i(\varphi_0)$, for $i = 1, 2$, for the two variables have a close relationship,

$$P_1^s \, (x_2(t) - \bar{x}_2(\varphi_1)) = P_2^s \, (x_1(t) - \bar{x}_1(\varphi_1)),$$

which is given by the slope of the eigenspace associated to the negative eigenvalue (the stable eigenspace).

In the presence of a zero eigenvalue

When $\det\left(\bar{F}_x(\varphi_0)\right) = 0$ the Jacobian has eigenvalues $\lambda_s = 0 < \lambda_u$ and there is a not a classic inverse for the Jacobian. We can use the Moore-Penrose inverse to determine the long run multipliers

$$d\bar{X} = -\bar{F}_x(\varphi_0)^+ \ \bar{F}_\varphi(\varphi_0) \, d\varphi + \left(I - \bar{F}_x(\varphi_0)^+ \ \bar{F}_x(\varphi_0) \right) Z$$

where I is the identity matrix, $Z = (z_1, z_2)^\top$ is a vector of constants and we use

$$\bar{F}_x(\varphi_0) = P\,\Lambda\,P^{-1}\ \bar{F}_x(\varphi_0)^+ = P\,\Lambda^+\,P^{-1}$$

where the Jordan form, the Moore-Penrose inverse and the eigenvector matrices are

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_u \end{pmatrix},\ \Lambda^+ = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{\lambda_u} \end{pmatrix},\ P = \begin{pmatrix} P_1^s & P_1^u \\ P_2^s & P_2^u \end{pmatrix}.$$

Because, differently from the classic inverse $\bar{F}_x(\varphi_0)^+\ \bar{F}_x(\varphi_0) \neq I$ then the expression for the multipliers would allows us to obtain a linear equation in one of the elements of vector Z, say z_2. We can determine it by using the predetermine nature of x_1 by setting $d\bar{x}_1 = 0$.

The solution to the variational system is now

$$dX(t) = d\bar{X} + k_s\,P^s + k_u\,P^u\,e^{\lambda_u t}$$

where

$$d\bar{X} = \begin{pmatrix} 0 \\ d\bar{x}_2 \end{pmatrix}$$

which again contains two arbitrary constants, k_s and k_u. To eliminate unbounded trajectories, we set again $k_u = 0$ and determine k_s such that $dx_1(0) = 0$. This yields the variations

$$dx_1(t) = 0,\ \text{for all}\ \ t \in [0, \infty)$$
$$dx_2(t) = d\bar{x}_2,\ \text{for all}\ \ t \in [0, \infty)$$

where the shock in φ is completely absorbed by x_2. This means that the values of the perturbed variables are

$$x_1(t) = x_{1,0},\ \text{for all}\ \ t \in [0, \infty)$$
$$x_2(t) = \bar{x}_2(\varphi_1),\ \text{for all}\ \ t \in (0, \infty)$$

where we set $d\bar{x}_2 = \bar{x}_2(\varphi_1, x_{1,0}) - \bar{x}_2(\varphi_0, x_{1,0})$ because, as we saw in the main text that the value of the steady state for the non-predetermined variable depends on the initial value of the predetermined variable. This means that the non-predetermined immediately "jumps" to the new steady state.

Application to problem P6

For the problem having first-order conditions in equations (9a)-(9d) we have the initial steady state

$$\bar{X}(w_0) = \begin{pmatrix} \bar{a}_0 \\ \bar{q}_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ (\rho a_0 + w_0)^{-\theta}) \end{pmatrix}$$

and the Jacobian for an increase in the wage rate $dw = w_1 - w_0$ is

$$\begin{pmatrix} \dot{a} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \rho & -C'(\bar{q}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} da(t) \\ q(t) \end{pmatrix} + \begin{pmatrix} dw \\ 0 \end{pmatrix}.$$

The first Jacobian has eigenvalues $\lambda_s = 0$ and $\lambda_u = \rho$, which means that we have to use the formulas derived for the case in which there is one zero eigenvalue.

In order to find the long run variation introduced by the shock in w, from equation (16), we have to do some preliminary work: we find the

$$P = \begin{pmatrix} C'(\bar{q}_0) & 1 \\ \rho & 0 \end{pmatrix}$$

the Moore-Penrose inverse of the Jacobian

$$\bar{F}_x(w_0)^+ = P\,\Lambda^+\,P^{-1} = \begin{pmatrix} C'(\bar{q}_0) & 1 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\lambda_u} \end{pmatrix} \frac{1}{\rho} \begin{pmatrix} 0 & 1 \\ \rho & -C'(\bar{q}_0) \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} 1 & -\frac{C'(\bar{q}_0)}{\rho} \\ 0 & 0 \end{pmatrix}$$

and

$$I - \bar{F}_x(w_0)^+\,\bar{F}_x(w_0) = \begin{pmatrix} 0 & -\frac{C'(\bar{q}_0)}{\rho} \\ 0 & 0 \end{pmatrix}$$

Therefore, the general expression for the long run multipliers is

$$\begin{pmatrix} d\bar{a} \\ d\bar{q} \end{pmatrix} = \begin{pmatrix} -\dfrac{dw}{\rho} + \frac{C'(\bar{q}_0)}{\rho} & z_2 \\ & z_2 \end{pmatrix}$$

where z_2 is an arbitrary constant.

As we require $d\bar{a} = 0$ then $z_2 = \dfrac{dw}{C'(\bar{q}_0)}$ then

$$\begin{pmatrix} d\bar{a} \\ d\bar{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix}.$$

We obtain the short run variations from

$$\begin{pmatrix} da(t) \\ dq(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix} + k_s \begin{pmatrix} C'(\bar{q}_0) \\ \rho \end{pmatrix}$$

Setting again $da(t) = 0$ yields $k_s = 0$, and, therefore, the short-run variations are

$$\begin{pmatrix} da(t) \\ dq(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix}.$$

As

$$dc(t) = d\bar{c} = C'(\bar{q}_0)\, dq(t) = \frac{C'(\bar{q}_0)}{C'(\bar{q}_0)}\, dw \;= dw$$

where $dw = w_1 - w_0$.

Writing $\bar{c}(w_1) = d\bar{c} + \bar{c}(w_0)$ yields the linear approximation for the behavior of consumption after the shock $c(t) = \bar{c}(w_0) + d\bar{c} = \bar{c}(w_1) = \rho\, a_0 + w_1$. We conclude that consumption changes discontinuously from $\bar{c}(w_0) = \rho\, a_0 + w_0$ to $\bar{c}(w_1) = \rho\, a_0 + w_1$, as shown in Figure 3.

F Comparative dynamics for the habit formation model

We start from the steady state in equations (12a)-(12e), for the initial level of the exogenous variable $w = w_0$ and consider a positive change to $w_1 = w_0 + dw$.

Next we introduce the notation for the partial derivatives $\bar{u}_i = u_i(\bar{c}(w_0), \bar{h}(w_0))$ for $i = c, h$ and $\bar{u}_{ij} = u_{ij}(\bar{c}(w_0), \bar{h}(w_0))$ for $i, j = c, h$. We define accordingly $\bar{C}_h$, $\bar{C}_{q_a}$ and $\bar{C}_{q_h}$.

Using this notation for the partial derivatives, evaluated at the initial steady state, and the notation in section D for the Jacobians we obtain the Jacobian

$$\bar{F}_x(w_0) = \begin{pmatrix} \rho & -\bar{C}_h & -\bar{C}_{q_a} & -\bar{C}_{q_h} \\ 0 & \eta(\bar{C}_h - 1) & \eta\, \bar{C}_{q_a} & \eta\, \bar{C}_{q_h} \\ 0 & 0 & 0 & 0 \\ 0 & -(\bar{u}_{hc}\, \bar{C}_h + \bar{u}_{hh}) & -\bar{u}_{hc}\, \bar{C}_{q_a} & \rho + \eta - \bar{u}_{hc}\, \bar{C}_{q_h} \end{pmatrix}$$

This Jacobian has the characteristic equationdet $\left(\bar{F}_x(w_0) - \lambda\, I\right) = 0$. Expanding, yields the polynomial equation

$$\lambda\,(\lambda - \rho)(\lambda^2 - \rho\,\lambda + S) = 0.$$

This is because $\mu\, \bar{C}_h = \bar{u}_{hc}\, \bar{C}_{q_h}$ and we have

$$\begin{aligned} S &= \eta\left((\bar{C}_h - 1)\,(\rho + \eta - \bar{u}_{hc}\, \bar{C}_{q_h}) + \bar{C}_{q_h}\,(\bar{u}_{hc}\, \bar{C}_h + \bar{u}_{hh}) \right. \\ &= \eta\,(\rho + \eta)\left(\bar{C}_h - 1 + \frac{\bar{C}_{q_h}}{\rho + \eta}(\bar{u}_{hc} + \bar{u}_{hh}) \right) \\ &= -\eta\Big(\frac{(\rho + \eta)\bar{u}_{cc} + (2\eta + \rho)(\bar{u}_{hc} + \eta\bar{u}_{hh})}{\bar{u}_{cc}} \Big) \end{aligned}$$

There are four real eigenvalues $\{\lambda_s, 0, \rho, \lambda_u\}$ where

$$\lambda_s = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 - S}$$
$$\lambda_u = \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}\right)^2 - S}$$

If $S < 0$ then $\lambda_s < 0 < \rho < \lambda_u$ and the steady state is a degenerate saddle-point. Additionally we have $\lambda_s + \lambda_u = \rho$ and $\lambda_s\,\lambda_u = S$.

Looking at the expression for S, we can write it as

$$S = -\frac{\eta\pi(\underline{u})}{\bar{u}_{cc}}$$

where $\pi(\underline{u}) \equiv (\rho+\eta)\bar{u}_{cc} + (2\eta+\rho)(\bar{u}_{hc} + \eta\bar{u}_{hh}) = (\rho+\eta)\bar{u}_{cc} + (2\eta+\rho)\bar{\mathsf{U}}''$. Then, $S < 0$ if and only if $\pi(\underline{u}) < 0$ which requires $\bar{\mathsf{U}}'' < -\dfrac{\rho+\eta}{2\eta+\rho}\ \bar{u}_{cc}$ which only holds if consumption is intertemporally substitutable or independent and, if there is intertemporal complementarity, it is not to large compared with the concavity as regards consumption c.

From now on we assume this condition holds.

As we have a zero eigenvalue we can adapt the method explained in the last section. The generalized long-run multipliers are

$$d\bar{X} = -\bar{F}_x(w_0)^+\ \bar{F}_w(w_0)\,dw + \Big(I - \bar{F}_x(w_0)^+\ \bar{F}_x(w_0)\Big)\,Z$$

where the Jacobian for the exogenous variable is

$$\bar{F}_w(w_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

and the Moore-Penrose inverse is $\bar{F}_x(w_0)^+ = P\,\Lambda^+\,P^{-1}$ where

$$\Lambda^+ = \begin{pmatrix} \frac{1}{\lambda_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda_u} \end{pmatrix},$$

and the eigenvector matrix concatenating the eigenvectors in the same order as in the Jordan matrix Λ, i,e, $P = [P^s P^0 P^\rho P^u]$ is

$$P = \begin{pmatrix} -\frac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u H(\bar{u})} & 1 & 1 & \frac{\bar{u}_{hc}\left(\eta\bar{u}_{hc} + (\lambda_u+\eta) baru_{cc}\right) - \eta H(\bar{u})}{\bar{u}_{cc} H(\bar{u})\lambda_s} \\ -\frac{\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u H(\bar{u})} & \rho & 0 & \frac{\eta\bar{u}_{hc} + (\lambda_u+\eta) baru_{cc}}{H(\bar{u})} \\ 0 & \frac{\rho\,\chi(\bar{u})}{\rho+\eta} & 0 & 0 \\ 1 & \frac{\rho\,(\bar{u}_{hc}+\bar{u}_{hh})}{\rho+\eta} & 0 & 1 \end{pmatrix},$$

where $H(\bar{u}) = \bar{u}_{cc}\,\bar{u}_{hh} - \bar{u}_{hc}^2$ and $\chi(\underline{u}) \equiv (\rho+\eta)\bar{u}_{cc} + \bar{u}_{hc}(2\eta+\rho) + \eta\bar{u}_{hh}$

Performing the calculations yields the generalized variation

$$d\bar{X} = \begin{pmatrix} d\bar{a} \\ d\bar{h} \\ d\bar{q}_a \\ d\bar{q}_h \end{pmatrix} = \begin{pmatrix} -\frac{dw}{\rho} + \frac{(\rho+\eta)}{\rho\,\chi(\bar{u})}\ z_3 \\ +\frac{(\rho+\eta)}{\chi(\bar{u})}\ z_3 \\ z_3 \\ \frac{\bar{u}_{hc}+\bar{u}_{hh}}{\chi(\bar{u})}\ z_3 \end{pmatrix}$$

We set $d\bar{a} = 0$ to find the value for z_3 and substituting back we obtain the particular long-run variation

$$d\bar{X} = \begin{pmatrix} d\bar{a} \\ d\bar{h} \\ d\bar{q}_a \\ d\bar{q}_h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{\chi(\bar{u})}{\rho+\eta} \\ \frac{\bar{u}_{hc}+\bar{u}_{hh}}{\rho+\eta} \end{pmatrix} dw$$

The short run variation $dX(t) = X(t) - \bar{X}(w_0)$, introduced by the perturbation in w can be obtained from the general solution of the variational system,

$$dX(t) = d\bar{X} + k_s\,P^s\,e^{\lambda_s t} + k_0 P^0 + k_\rho P^\rho e^{\rho t} + k_u P^u e^{\lambda_u t}$$

where k_s, k_0, k_ρ and k_u are arbitrary constants.

Eliminating the explosive components by setting $k_\rho = k_u = 0$ and solving for k_s and k_0 such that $da(0) = 0$ and $dh(0) = 0$, yields

$$\bar{k}_s = -\frac{(\eta+\lambda_u)\,\bar{u}_{hc} + \eta\bar{u}_{hh}}{(\eta+\lambda_u)(\lambda_u\bar{u}_{cc} + \rho\bar{u}_{hc}) + \eta\rho\bar{u}_{hh}}\,dw$$

$$\bar{k}_0 = -\frac{\lambda_u H(\bar{u})}{(\eta+\lambda_u)(\lambda_u\bar{u}_{cc} + \rho\bar{u}_{hc}) + \eta\rho\bar{u}_{hh}}\,dw$$

Therefore the short run variation is

$$
\begin{aligned}
da(t) &= -\frac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})}(1 - e^{\lambda_s t})\, dw, \text{for } t \in [0, \infty)\\
dh(t) &= \frac{\lambda_u(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc})}{\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})}(1 - e^{\lambda_s t})\, dw, \text{for } t \in [0, \infty)\\
dq_a(t) &= \frac{\lambda_u(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc})\pi(\bar{u})}{(\rho+\eta)\Big(\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)} dw, \text{for } t \in [0, \infty)\\
dq_h(t) &= \frac{\lambda_u\,\bar{u}_{hc}((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh})}{(\rho+\eta)\Big(\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)} \times\\
&\quad \times \Big((\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2 - (\rho+\eta)H(\bar{u})e^{\lambda_s t}\Big)\, dw, \text{for } t \in [0, \infty).
\end{aligned}
$$

The variation in consumption can be obtained as

$$
dc(t) = \bar{C}_h\, dh(t) + \bar{C}_{q_a}\, dq_a(t) + \bar{C}_{q_h}\, dq_h(t).
$$

In the expressions for the variations of the state variables, we see the effect of the existence of a zero eigenvalue: we find that

$$
(\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc})\, dh(t) + \lambda_u(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc})\, da(t) = 0.
$$

We also find that

$$
\begin{aligned}
da(0) &= 0\\
dh(0) &= 0\\
dq_a(0) = dq_a(t) &= \frac{\lambda_u(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc})\pi(\bar{u})}{(\rho+\eta)\Big(\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)} dw\\
dq_h(0) &= \frac{\lambda_u\,\bar{u}_{hc}((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh})\Big((\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2 - (\rho+\eta)H(\bar{u})\Big)}{(\rho+\eta)\Big(\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)}\ dw.
\end{aligned}
$$

and

$$
\begin{aligned}
da(\infty) &= -\frac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})}\, dw, \text{for } t \in [0,\infty)\\
dh(\infty) &= \frac{\lambda_u(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc})}{\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})}(1-e^{\lambda_s t})\, dw, \text{for } t \in [0,\infty)\\
dq_a(\infty) &= \frac{\lambda_u(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc})\pi(\bar{u})}{(\rho+\eta)\left(\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})\right)} dw, \text{for } t \in [0,\infty)\\
dq_h(\infty) &= \frac{\lambda_u\,\bar{u}_{hc}((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh})\left((\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2\right)}{(\rho+\eta)\left(\lambda_u((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}) + \rho + \eta(\bar{u}_{hc}+\bar{u}_{hh})\right)}\ dw.
\end{aligned}
$$

which are the long run multipliers. To determine the levels of the new steady state after the shock we can write $\bar{X}(w_1) = dX(\infty) + \bar{X}(w_0)$. This is point C shown in figure 4.

G Two-stage problem

Consider a dynastic household problem in which the non-financial income has a switch at a known time $\tau \in (0,\infty)$ such that

$$
y(t) = \begin{cases} y_1, & \text{for } \ 0 \le t < \tau \\ y_2, & \text{for } \ \tau \le t < \infty, \end{cases}
$$

where $y_1 \neq y_2$. This can translate income associated to employed/unemployed or work/retirement. The household problem is

$$
\begin{aligned}
\max_{c(\cdot)} \ & \int_0^\infty \ u(c(t))\, e^{-\rho\, t} \ dt \\
& \text{subject to} \\
& \dot{a} \ = r\, a + y(t) - c, \ t \in [0,\infty) \\
& a(0) = a_0 \, \text{given} \\
& \lim_{t\to\infty} \ e^{-r\,t} \ a(t) \ge 0
\end{aligned}
\tag{17}
$$

where $r > 0$ is the interest rate, $\rho > 0$ is the rate of time preference. We assume that the utility function is iso-elastic $u(c) = (1-\theta)^{-1} \left(c^{1-\theta} - 1\right)$, for $\theta > 0$ and also assume that $(\sigma-1)\,r + \sigma\rho > 0$.

As referred, we solve the problem backward the problem in two steps and obtain the final solution by using the using the matching condition

First step: terminal stage We solve the problem for the interval $t \in [t_1, \infty)$ assuming we know the initial value of the state variable $a(\tau) = a_\tau$. The problem is

$$
\begin{aligned}
\max_{c(\cdot)} \int_\tau^\infty & u(c(t))\, e^{-\rho\, t}\ dt \\
& \text{subject to} \\
& \dot{a}\ = r\, a + y_2 - c,\ t \in [\tau, \infty) \\
& a(\tau) = a_\tau \text{ assumed to be given} \\
& \lim_{t\to\infty}\ e^{-r\, t}\ a(t) \geq 0.
\end{aligned}
$$

The Hamiltonian for this problem is

$$
H_2(t) = H_2(a(t), c(t), \lambda_2(t)) = u(c(t))\, e^{-\rho t}\ + \lambda_2(t)(ra(t) + y_2 - c(t),\ \text{for}\ t \in [\tau, \infty)
$$

and the necessary (and in this case sufficient) conditions for an optimum are

$$
\begin{aligned}
c(t)^{-\theta}\ e^{-\rho t}\ &= \lambda_2(t),\ t \in [\tau, \infty) \\
\dot{\lambda}_2 &= -r\, \lambda_2,\ t \in [\tau, \infty) \\
\dot{a}\ &= r\, a + y_2 - c, \\
a(\tau) &= a_1,\ t = \tau \\
\lim_{t\to\infty}\ \lambda_2(t)\, a(t) &= 0.
\end{aligned}
$$

Solving the Euler equation and using the optimality condition we find

$$
\begin{aligned}
c(t) &= c(\tau)\, e^{\gamma(t-\tau)} \\
\lambda_2(t) &= \ c(\tau)^{-\theta}\ e^{-\rho(t-\tau)}.
\end{aligned}
$$

The solution of the budget constraint is

$$
a(t) = e^{r(t-\tau)}\ \left[a_1 + e^{r\,\tau}\ \left(\int_\tau^t\ (y_2 - c(s))ds\right)\right]
$$

Substituting the solution for c this is equivalent to

$$
e^{-r(t-\tau)}\ a(t) = a_1 + e^{r\,\tau}\ \left(\int_\tau^t\ (y_2 - c(\tau)\, e^{\gamma(s-\tau)})ds\right)
$$

where $c(\tau)$ is unknown. Multiplying by the solution to $\lambda_2(t)$ and using the transversality condition, and assuming that $r > 0$ and $r - \gamma > 0$, [17] we find the optimal consumption at the time of the

[17] If this condition is not satisfied, there will be no solution to the problem.

switch

$$c^*(\tau) = (r-\gamma)\left(a_1 + \frac{y_2}{r}\right).$$

Therefore, after the switch we obtain

$$\begin{aligned} c^*(t) &= (r-\gamma)\left(a_1 + \frac{y_2}{r}\right) e^{\gamma(t-\tau)},\ \tau \le t < \infty \\ a^*(t) &= -\frac{y_2}{r} \ + \left(a_1 + \frac{y_2}{r}\right) e^{\gamma(t-\tau)},\ \tau \le t < \infty \end{aligned}$$

and the value for the co-state variable at the time of the switch is

$$\lambda_2(\tau) = \left((r-\gamma)\left(a_1 + \frac{y_2}{r}\right)\right)^{-\theta} \ e^{-\rho\tau}.$$

where $a^*(\tau) = a_1$ is unknown at this state.

Second step: initial stage

The problem is

$$\begin{aligned} \max_{c(\cdot)} \ & \int_0^\tau \ u(c(t))\, e^{-\rho\, t} \ dt \\ & \text{subject to} \\ & \dot{a} \ = r\, a + y_1 - c,\ t \in [0,\tau) \\ & a(0) = a_0 \,\text{given} \\ & a(\tau^-) = a_1 \,\text{assumed to be given} \end{aligned}$$

where $a(\tau^-) = \lim_{t\uparrow\tau} \ a(t)$.

The Hamiltonian for this problem is

$$H_1(t) = H_1(a(t), c(t), \lambda_1(t)) = u(c(t))\, e^{-\rho t} \ + \lambda_1(t)(ra(t) + y_1 - c(t),\ \text{for} \ \ t \in [0,\tau)$$

and the necessary (and in this case sufficient) conditions for an optimum are

$$\begin{aligned} c(t)^{-\theta} \ e^{-\rho t} \ &= \lambda_1(t),\, t \in [0,\tau) \\ \dot{\lambda}_1 &= -r\, \lambda_1,\ t \in (0,\tau) \\ \dot{a} \ &= r\, a + y_1 - c, \\ a(0) &= a_0,\ t = 0. \end{aligned}$$

Solving the Euler equation, substituting for consumption, and solving the budget constraint, subject to the initial net asset position a_0 we find

$$c(t) = c(0)\, e^{\gamma t},\ t \in [0, \tau)$$
$$a(t) = e^{r\,t}\ \left(a_0 + \int_0^t e^{-rs}\ \left(y_1 - c(0)e^{\gamma\, s}\right) ds\right)$$

where $c(0)$ is unknown, and

$$\lambda_1(\tau) = c(0)^{-\sigma}\, r^{-r\,\tau}.$$

Final step

Up until this phase, we have two unknowns: $c(0)$ and a_1. We can determine them by using the matching condition for the co-state variable and by using using the solution for the net asset position obtained in the last step. That is, we can find them by solving jointly

$$\begin{cases} \lambda_1(\tau^-)\ = \lambda_2(\tau), \\ a(\tau^-) = a_1. \end{cases}$$

Solving the first equation we find

$$c^*\ (0) = (r - \gamma)\left(a_1 + \frac{y_2}{r}\right) e^{-\gamma\,\tau}$$

which, upon substitution in the second equation yields

$$a_1^* = e^{\gamma\,\tau}\left(a_0 + \frac{y_1}{r}\ \left(1 - e^{-r\,\tau}\right) + \frac{y_2}{r}\ \left(e^{-r\,\tau}\ - e^{-\gamma\,\tau}\right)\right). \tag{18}$$

Solution

We obtain the optimal consumption path $\left(c^*(t)\right)_{t\in[0,\infty)}$, where

$$c^*(t) = (r - \gamma)\left(a_1^* + \frac{y_2}{r}\right) e^{\gamma\,(t-\tau)},\ \text{for}\ \ t \in [0, \infty),$$

and substituting a_1^* in the solution for $a(t \in [0, \tau))$ we obtain the optimal net asset position path $\left(a^*(t)\right)_{t\in[0,\infty)}$, where

$$a^*(t) = \begin{cases} -\frac{y_1}{r} + \left(a_1^* + \frac{y_2}{r}\right) e^{\gamma\,(t-\tau)} + \left(\frac{y_1}{r} - \frac{y_2}{r}\right) e^{r(t-\tau)}, & \text{for}\ \ t \in [0, \tau) \\ -\frac{y_2}{r} + \left(a_1^* + \frac{y_2}{r}\right) e^{\gamma\,(t-\tau)}, & \text{for}\ \ t \in [\tau, \infty) \end{cases}$$