EMA 2019-2020:
Problem set 1: linear ODE's

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1 Linear scalar ODE's

1.1 Autonomous ODE's

1.1.1 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following scalar ordinary differential equations and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -\frac{1}{2}y$;

(b) $\dot{y} = \frac{1}{2}y$;

(c) $\dot{y} = 2y$;

(d) $\dot{y} = -2y$;

(e) $\dot{y} = 0$;

(f) $\dot{y} = 2$;

(g) $\dot{y} = -2$;

1.1.2 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following scalar ordinary differential equations and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -\frac{1}{2}y + 1$;

(b) $\dot{y} = \frac{1}{2}y - 1$;

(c) $\dot{y} = 2y - 2$;

(d) $\dot{y} = -2y + 2$;

(e) $\dot{y} = ay - 2$ for $a \in (-2, 2)$

(f) $\dot{y} = y + b$ for $b \in (-1, 1)$

1.1.3 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following initial value problems and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -0.5y + 1$, for $t \geq 0$ and $y(0) = 1$ for $t = 0$;

(b) $\dot{y} = 0.5y - 1$, for $t \geq 0$ and $y(0) = 1$ for $t = 0$.

1.1.4 Let $y : \mathbb{R}_+ \to \mathbb{R}$. Solve the following terminal value problems and characterise the solutions analytically and geometrically:

(a) $\dot{y} = -0.5y + 1$, for $t \geq 0$ and $\lim_{t\to\infty} y(t) = \overline{y}$, where $\overline{y}$ is the steady state;

(b) $\dot{y} = 0.5y - 1$, for $t \geq 0$ and $\lim_{t\to\infty} e^{-0.5t} y(t) = 0$.

1.1.5 Perform a bifurcation analysis to the following equation $\dot{y} = ay + b$ for $a \in [-2, 2]$ and $b \in (-1, 1)$.

1.1.6 Let $y = y(t)$ is a function, $y : \mathbb{R}_+ \to \mathbb{R}$. Consider the terminal value problem

$$\begin{cases} \dot{y} = gy + b & t \geq 0 \\ \lim_{t\to\infty} y(t) = \overline{y} \end{cases}$$

where $\overline{y}$ is the steady state, and g and $b \neq 0$ are constants.

(a) Assume that $g < 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

(b) Assume that $g > 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

1.1.7 Consider the following problem

$$\begin{cases} \dot{y} = \lambda\big(y - \bar{y}\big) & \text{for } t \in \mathbb{R}_+ \\ \int_0^\infty y(t)\, \phi(t)\, dt = \bar{y} \end{cases}$$

where $\lambda > 0$ and $\phi(t) = \lambda\, e^{-\lambda t}$. Observe that $\phi(t)$ is a distribution.

(a) Solve the problem.

(b) Provide an intuition for the problem and its solution.

1.2 Non-autonomous ODE's

1.2.1 Consider the scalar ODE

$$\dot{y} = ay + b(t),\ y : [0, \infty) \to \mathbb{R}$$

where

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^*, \\ b_1 & \text{if } t^* \leq t < \infty. \end{cases}$$

(a) Assume that $a \neq 0$ and $y(0) = y_0$ is given. Solve the initial value problem.

(b) Assume that $a > 0$ and $\lim_{t\to\infty} y(t)e^{-at} = 0$. Solve the terminal-value problem.

1.2.2 Consider the scalar ODE

$$\dot{y} = ay + b(t),\ y : [0, \infty) \to \mathbb{R}$$

where

$$b(t) = \begin{cases} b & \text{if } 0 \le t < t^*, \\ b + \Delta b & \text{if } t^* \le t < t^* + \Delta t, \\ b & \text{if } t^* + \Delta t \le t < \infty, \end{cases}$$

where $\Delta t > 0$.

(a) Assume that $a \neq 0$ and $y(0) = y_0$ is given. Solve the initial value problem.

(b) Assume that $a > 0$ and $\lim_{t\to\infty} y(t)e^{-at} = 0$. Solve the terminal-value problem.

1.2.2 Consider the scalar ODE

$$\frac{y^{'}(x)\, x}{y(x)} = \mu,\ x \in \mathbb{R}$$

where μ is a constant.

(a) Prove that the general solution follows a power law.

(b) Impose conditions on the parameter and an initial value such that the solution satisfies

$$\int_{x_0}^{\infty} y(x)dx = 1$$

1.2.3 Consider the scalar ODE

$$\frac{y^{'}(x)}{y(x)} = -x,\ x \in \mathbb{R}.$$

(a) Prove that the general solution follows a standard Gaussian probability distribution.

(b) Impose conditions on the initial value such that the solution satisfies

$$\int_{-\infty}^{\infty} y(x)dx = 1$$

1.3 Applications

1.3.1 The simplest model of population dynamics assumes that the rate of population growth is deterministic, age-independent, and constant:

$$\dot{N} = \nu N.\ N : \mathbb{R}_+ \to \mathbb{R}_+, \tag{1}$$

where $N(t)$ is the population at time t and $\nu \equiv \beta - \mu$ is the net rate of growth, β is the fertility rate and μ is the mortality rate. We assume that $N(0) = N_0 \geq 0$ is given. (References Banks (1994) see also `http://en.wikipedia.org/wiki/Exponential_growth`)

(a) Solve equation (1).

(b) Solve the initial value problem.

(c) Characterize the dynamics.

1.3.2 The stock-flow dynamics is generically represented by an equation of type,

$$\dot{A} = \pi + rA, \; A : \mathbb{R}_+ \to \mathbb{R} \tag{2}$$

where A is the stock of an asset at time t, π is net income and r is the rate of return. Assume that $r > 0$

(a) Solve equation (2) and characterise qualitatively the dynamics.

(b) Assuming we know $A(0) = A_0$, solve the initial value problem.

(c) Assuming we introduce a solvability requirement $\lim_{t\to\infty} A(t)e^{-rt} = 0$, determine the initial level of $A(0)$.

1.3.3 Sargent and Wallace (1973) is one of the first papers to deal with perfect foresight dynamics. The main equation of the paper was

$$\dot{p} = \beta(p - m(t)), \; p : \mathbb{R}_+ \to \mathbb{R} \tag{3}$$

where p and m are the logs of the price index and nominal money supply and $\beta > 0$

(a) Solve equation (3).

(b) Setting $p(0) = p_0$, where p_0 is known, solve the initial value problem. Does the solution to this problem makes economic sense (hint: recall the expected relationship between increases in the money supply and the price evolution) ?

(c) Let m is constant. Assuming there are no speculative bubbles, i.e, $\lim_{t\to\infty} p(t)e^{-\beta t} = 0$, determine $p(0)$.

(d) Modify the previous results assuming that there is an anticipated (to time $\tau > 0$ and finite) monetary shock.

1.3.4 The government budget constraint, in nominal variables, is

$$\dot{B} = D + iB,$$

where $B(t)$ is the stock of government bonds at time t, (where $B : \mathbb{R}_+ \to \mathbb{R}$), D is the primary deficit, and i is the interest rate on government bonds. Assume that the GDP, Y, follows the process $\dot{Y} = gY$. All variables are in nominal terms.

(a) Let $b \equiv B/Y$ and $d \equiv D/Y$. Which types of dynamic behavior for b one should expect ?

(b) Assuming we know $b(0) = b_0$, solve the initial value problem.

(c) If we introduce a solvability requirement such that $\lim_{t\to\infty} b(t)e^{-rt} = 0$, determine the initial level of $b(0)$, assuming that $r \equiv i - g > 0$.

1.3.5 Let the government budget constraint be $\dot{b} = -\tau(t) + rb(t)$ where $b(t)$ is the government debt and $\tau(t)$ is the time-varying primary surplus, at time $t \geq 0$, and $r > 0$ is the interest rate on the government debt. Assume that the government adopts a fiscal rule taking the form $\dot{\tau} = \gamma b(t) - \xi\tau(t)$ where $\gamma > 0$. Assume that the initial level of the debt is given $b(0) = b_0$.

(a) If we assume that $r > \xi$, under which conditions on the parameters of the fiscal rule can the government reach the following goal: $\lim_{t\to\infty} b(t) = 0$?

(b) Assuming the previous condition determine the paths for the government debt and primary surplus.

(c) What should be the initial surplus $\tau(0)$? Provide an intuition for this result.

2 Planar ODE's

2.1 General

2.1.1 Let $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{A}$ can take one of the following values

$$\text{a)} \begin{pmatrix} -3 & 1 \\ -1 & -5 \end{pmatrix}, \text{ b)} \begin{pmatrix} -3 & 2 \\ -1 & -6 \end{pmatrix}, \text{ c)} \begin{pmatrix} -4 & 4 \\ -2 & -4 \end{pmatrix},$$

(a) Solve the planar ODE's and characterise analytically and geometrically the solutions for each case.

(b) Let $\mathbf{B} = (1,1)$. Solve the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, for the three cases, and characterise analytically and geometrically the solutions.

(c) Consider the ODE's of the last question. Let $\mathbf{y}(0) = (0,0)$. Solve the initial value problems. Characterise analytically and geometrically the solutions of the initial value problem.

2.1.2 Let $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{A}$ can take one of the following values

$$\text{a)} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \text{ b)} \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, \text{ c)} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \text{d)} \begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}, \text{e)} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$

(a) Solve the planar ode and characterise analytically and geometrically the solutions

(b) Let $\mathbf{B} = (1,1)$. Solve the planar ode $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$ and characterise analytically and geometrically the solutions

(c) Consider the ODE's of the last question. Let $\mathbf{y}(0) = (0, 0)$. Solve the initial value problems. Characterise analytically and geometrically the solutions of the initial value problem.

2.1.3 Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ a & -2 \end{pmatrix},$$

for $a \in \mathbb{R}$. Let a take any value on its domain. Determine the different solutions and characterise them.

2.1.4 Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} a & 1 \\ 1 & -3a \end{pmatrix},$$

for $a \in \mathbb{R}$. Let a take any value on its domain. Determine the different solutions and characterise them.

2.1.5 Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

(a) Solve the ODE.

(b) Draw the phase diagram and characterize it.

(c) Let $\mathbf{y}(0) = (0, 1)$. Solve the initial value problem.

2.1.6 Let $y = y(t)$ is a function, $y : \mathbb{R}_+ \to \mathbb{R}^2$. Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}.$$

(a) Solve the ODE.

(b) Draw the phase diagram and characterize it.

(c) Let $\mathbf{y}(0) = (-1, 1)$. Solve the initial value problem.

2.1.7 Consider the planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ 0 & -b \end{pmatrix}.$$

where a and b are arbitrary constants.

(a) Which type of dynamics one would have.

(b) Let $a + b \neq 0$. Solve the ODE.

(c) Let a and b be strictly positive. Find, and characterize (as regards existence and uniqueness) the solutions converging asymptotically to $\mathbf{y} = (0,0)^\top$.

2.1.8 Consider the two planar ODE $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

and

$$\mathbf{A} = \begin{pmatrix} \lambda & \beta \\ \beta & \lambda \end{pmatrix}.$$

where λ and β are arbitrary constants.

(a) Solve the two ODE's, assuming that $\lambda \neq 0$.

(b) Perform a bifurcation analysis.

(c) Let $\lambda < 0$ and $\beta < 0$. Draw the phase diagrams and characterize them.

2.2 Applications

2.2.1 Consider a continuous time version of a two-state Markov process $\dot{y} = My$, where the transition matrix is

$$M = \left(\begin{array}{cc} p-1 & 1-p \\ 1-q & q-1 \end{array} \right)$$

for $0 < p < 1$ and $0 < q < 1$

(a) solve the differential equation;

(b) let $y(0) = (1,2)$. Solve the initial value problem;

(c) draw the phase diagram.

2.2.2 Consider a continuous time version of a two-state Markov process $\dot{y} = My$, where the transition matrix is

$$M = \left(\begin{array}{cc} -\pi_1 & \pi_1 \\ \pi_2 & -\pi_2 \end{array} \right)$$

for $0 < \pi_1 < 1$ and $0 < \pi_2 < 1$

(a) solve the differential equation;

(b) let $y(0) = (0,1)$ and solve the initial value problem;

(c) draw the phase diagram associated to the initial value problem.

2.2.3 Consider the wage-price dynamics for an economy in which there is perfect foresight in the product market and in which wages do not fully adjust to imbalances in the labour market. The economy is represented by a planar ODE

$$\begin{aligned} \dot{p} &= \lambda(p - m) \\ \dot{w} &= \gamma(p - w - N) \end{aligned}$$

where λ and γ are positive parameters and m and N are exogenous variables (money and population respectively). Assume that $w(0) = w_0$ is given and that prices verify $\lim_{t\to\infty} p(t) = \overline{p}$:

(a) determine the fixed point;

(b) solve the ODE;

(c) solve the mixed initial-terminal value problem;

(d) draw the phase diagram;

(e) which consequences will arise from an increase in the money supply ?

2.2.4 This is inspired in the Calvo (1983) model. Assume an imperfectly competitive economy in which the firms have the following rule for setting prices: $x(t) = \delta \int_t^{+\infty} (p^*(s)+\beta y^*(s))e^{-\delta(t-s)}ds$, for $\beta > 0$ and $\delta > 0$, where x is the price set by each firm, p is the aggregate price index, y is the aggregate level of activity, and δ denotes the (constant) probability for price revisions. All the variables are logarithms and the star represents expectations. Differentiating, we have equivalently $\dot{x} = \delta(x - p^* - \beta y^*)$. We assume that the aggregate price level is a weighted average of the prices set by all firms and it is given by $p(t) = \delta \int_{-\infty}^{t} x(s)e^{-\delta(t-s)}ds$, or equivalently $\dot{p} = \delta(x - p)$.

Equilibria in the goods and monetary markets implies that the following reduced form equation holds $y(t) = a(m(t) - p(t)) + b\pi^*(t)$, where m is the (log) of the money stock and $\pi \equiv \dot{p}$ is the inflation rate. The nominal money stock m is constant and exogenous. At last, assume that in this economy agents have perfect foresight (i.e, $p^* = p$). All the parameters are positive.

a) Obtain a planar ODE over (p, x), representing this economy

b) Perform a qualitative analysis of the local dynamics. Assume that $\beta b < 1 < \beta(a + b)$.

c) Assume there is a non-anticipated and permanent shock in m. Study the comparative dynamics assuming that x is non-predetermined and p is pre-determined.

d) Discuss the goodness of the choice of x as a non-predetermined variable, versus the alternative in which p is non-predetermined. Does it makes sense to assume that both variables are non-predetermined ? What would be the comparative dynamics in this case ?

2.2.5 The Dornbusch (1976) was, possibly, the most popular macroeconomic model between the second half of the seventies and most part of the eighties (at least). It is representative of the "rational expectations revolution" before the DGSE models became the benchmark model in dynamic macroeconomics. It is is a model for an open economy with following features: (1) there is free international movements of capital and the domestic interest rate i adjusts to (exogenous) international interest rate i^* through an open-Fisher equation; (2) there is rational expectations concerning the rate of depreciation, $x = \dot{e}$, where x is the expected depreciation rate and $\dot{e}$ is the market rate of depreciation; (3) the only policy instrument is the aggregate money supply m; (4) the aggregate supply is given ($\bar{y}$) but the prices (p) adjust sluggishly to the excess demand in the product market. The equations of the model assume typically Keynesian behavioral assumptions:

$$\begin{aligned} i &= i^* + x \\ x &= \dot{e} \\ m - p &= -\xi i + \phi y \\ d &= g + \delta(e + p^* - p) + \gamma y - \sigma i,\ 0 < \gamma < 1 \\ y &= \bar{y} \\ \dot{p} &= \pi(d - y), \end{aligned}$$

all the parameters are positive and all the variables are in logs.

(a) Prove that those equations reduce to a linear planar ODE

$$\begin{aligned} \dot{e} &= \frac{1}{\xi}\Big(p - m + \phi y\Big) - i^*, \\ \dot{p} &= \pi\left[\delta e - \Big(\delta + \frac{\sigma}{\xi}\Big)p + \frac{\sigma}{\xi}m + \Big(\gamma - 1 - \frac{\phi\sigma}{\xi}\Big)y + \delta p^* + g\right]. \end{aligned}$$

(b) Find the steady state and study the local dynamics

(c) Draw the phase diagram and explain.

(d) Solve the model assuming that $p(0) = p_0$ is given and $\lim_{t\to\infty} e(t) = \bar{e}$.

(e) Perform a dynamic comparative analysis of a permanent, non-anticipated, positive shock in m.

2.2.5 The AK model (see Jones and Manuelli (2005) for a survey) is at the same time the simplest and the benchmark endogenous growth model. The endogenous variables are consumption C and the stock of capital K and the optimal conditions are a planar ordinary differential equation

$$\begin{aligned} \dot{C} &= C\Big(\frac{A - \delta - \rho}{\theta}\Big) \\ \dot{K} &= AK - C - \delta K \end{aligned}$$

together with the initial conditions

$$K(0) = K_0 \text{ and } \lim_{t\to\infty} \frac{K(t)}{C(t)^{\theta}} e^{-\rho t} = 0$$

where $A > 0$ is the total factor productivity, $\delta > 0$ is the depreciation rate and $\rho > 0$ is the rate of time preference. We assume that state space is $\mathbb{R}^2_{++}$.

(a) Study the existence and uniqueness of steady states and its (their) local dynamics properties.

(b) Draw the phase diagram.

(c) Find the explicit solution to the planar ODE.

(d) Find the explicit solution of the problem.

(e) The aggregate income be $Y(t) = AK(t)$. Discuss the characteristics of the growth process generated by this model.

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