Mathematical Economics
Discrete time: calculus of variations problem

Paulo Brito

[1] pbrito@iseg.ulisboa.pt
University of Lisbon

November 23, 2018

Discrete time: intuition

- Assume we have a cake of initial size $W_0 = \phi$ and we want to eat it completely by time $t = T$, then $W_T = 0$
- The size of the cake evolution is:

$$W_{t+1} = W_t - C_t \Rightarrow W_t = W_0 - \sum_{s=0}^{t-1} C_s$$

- By imposing the two constraints $W_0 = \phi$ and $W_T = 0$ we have

$$\sum_{s=0}^{T-1} C_t = \phi$$

- There is an **infinite** number of paths $\{C_0, \ldots C_{T-1}\}$ verifying this condition
- Example: if we chose $C_t = \bar{C}$ constant for all t we get $\bar{C} = \frac{\phi}{T}$.
- Is this optimal ? Depends on the value functional

Discrete time dynamic optimization: Intuition

- A dynamic model is defined **over sequences** x **(or** (x, u)**)** in which there is some form of **intertemporal interaction**: actions today have an impact in the future;

  Types of time interactions:
  - intratemporal: iinteraction within one period
  - intertemporal: interaction across periods
- **Admissible sequences**: the set $\mathcal{X}$ contains a large number (possibly infinite) of admissible paths verifying a **given intratemporal relation** together with some **information regarding initial and/or terminal** data ;
- **Optimal sequences**: an **intertemporal optimality criterium** allows for choosing the best admissible sequence.

The discrete time calculus of variations problem (CV)

Calculus of variations problem: find $x = \{x_t\}_{t=0}^{T} \in \mathcal{X}$, i.e., a path belonging to the set of admissible paths $\mathcal{X}$, that maximises an intertemporal objective function (or value functional)

$$J(x) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t),\ x \in \mathcal{X}$$

where $F(t, x_t, x_{t+1})$ specifies an intratemporal relation for the value of the action upon the state variable, within period t (i.e., between times t and $t+1$).

flow		F_0		F_1				F_t				F_{T-2}		F_{T-1}		
period		0		1				t				$T{-}2$		$T{-}1$		
time	0		1		2		t		$t{+}1$		$T{-}2$		$T{-}1$		T	
state	x_0		x_1		x_2		x_t		x_{t+1}		x_{T-2}		x_{T-1}		x_T	

The **value for changing the state variable in period** t is

$$F_t = F(x_{t+1}, x_t, t).$$

The **value of the a given sequence of actions across** $T - 1$ **periods** is

$$J(x) = F(0, x_0, x_1) + \ldots + F(t, x_t, x_{t+1}) + \ldots + F(T - 1, x_{T-1}, x_T)$$

The **optimal sequence** x^* has value

$$J^* = J(x^*) = \max_x \{J(x) : \ x \in \mathcal{X}\}$$

The discrete time optimal control problem (OC)

Optimal control problem: find the pair $(x, u) = (\{x_t\}_{t=0}^{T}, \{u_t\}_{t=0}^{T-1}) \in \mathcal{D} = \mathcal{X} \times \mathcal{U}$ where $\mathcal{D}$ is defined by the sequence of intratemporal relations for periods 0 to $T-1$

$$x_{t+1} = G(x_t, u_t, t), \;\; t = 0, 1, \ldots, t, \ldots T-1$$

plus other conditions, that maximises an intertemporal objective functional

$$J(x, u) = \sum_{t=0}^{T-1} F(t, x_t, u_t)$$

Observation: $J(x)$ has terminal time $T-1$ because if we know the optimal x^*_{T-1} and u^*_{T-1}, we know the optimal x^*_T, because $x^*_T = G(x^*_{T-1}, u^*_{T-1}, T-1)$.

value flow	F_0	F_1		F_t		F_{T-2}	F_{T-1}	
control	u_0	u_1		u_t		u_{T-2}	u_{T-1}	
period	0	1		t		$T-2$	$T-1$	
time	0	1	2	t	$t+1$	$T-2$	$T-1$	T
state	x_0	x_1	x_2	x_t	x_{t+1}	x_{T-2}	x_{T-1}	x_T

The **value associated to choosing the control** u_t, given the state x_t, throughout period t is

$$F_t = F(x_t, u_t, t).$$

Choosing the control u_t, generates a the value for the state variable at the end of period t, $x_{t+1} = g(x_t, u_t, t)$

The **value of a given sequence of controls across** $T - 1$ **periods** is

$$J(u, x) = F(0, x_0, u_0) + \ldots + F(t, x_t, u_t) + \ldots + F(T - 1, x_{T-1}, u_{T-1})$$

The **optimal sequence** u^* has value

$$J^* = J(u^*) = \max_u \{ J(x, u) : (x, u) \in \mathcal{D} \}$$

Calculus of variations problems

We consider the following **problems**:

- **Simplest problem**: $\mathcal{X} = \{T, x_0, \text{ and } x_T \text{ given}\}$
- **Free terminal state problem**: $\mathcal{X} = \{x_0, \text{ and } T \text{ given}\}$, x_T free
- **Constrained terminal state problem**: $\mathcal{X} = \{x_0, \text{ and } T, \text{ and } h(x_T) \geq 0 \text{ given}\}$
- **Discounted infinite horizon problems**: $T = \infty$ and $\lim_{t\to\infty} x_t$ **free** or **constrained**

Calculus of variations: simplest problem

- **The problem** : Find $x^* = \{x_t^*\}_{t=0}^{T}$ that maximizes

$$J(x) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t), \text{ s. t. } x_0 = \phi_0,\ x_T = \phi_T \tag{1}$$

  where T, ϕ_0 and ϕ_T are known.

- **First order necessary conditions**

Proposition

if $x^* \equiv \{x_0^*, x_1^*, \ldots, x_T^*\}$ is a solution of problem (1) , it verifies the Euler-Lagrange equation and the admissibility conditions

$$\begin{cases} \frac{\partial F}{\partial x_t}(x_t^*, x_{t-1}^*, t-1) + \frac{\partial F}{\partial x_t}(x_{t+1}^*, x_t^*, t) = 0, & t = 1, 2, \ldots, T-1 \\ x_0^* = \phi_0, & t = 0 \\ x_T^* = \phi_T, & t = T \end{cases}$$

Proof.

Application: cake eating problem

- The problem

$$\max_C \left\{ \sum_{t=0}^{T} \beta^t \ln(C_t) : \; W_{t+1} = W_t - C_t, \; W_0 = \phi, \; W_T = 0 \right\}$$

- **Assumptions:** $0 < \beta < 1$, $T \geq 1$, $\phi > 0$
- Intuition:
  - at time $t = 0$, we have a cake of initial size ϕ, and we want to consume it completely until time T (known)
  - by choosing a sequence of bites such that:
    (1) we value independently each bite (the value functional is a sum);
    (2) the instantaneous pleasure of each bite is increasing with the size of the bite, but at a decreasing rate (the utility function $u(C_t)$ is concave);
    (3) we are impatient: when we plan to eat the cake we value more the immediate bites rather than future bites (there is a discount factor β^t which decreases with time).
  - How should we eat the cake ?

Application: cake eating as a CV problem

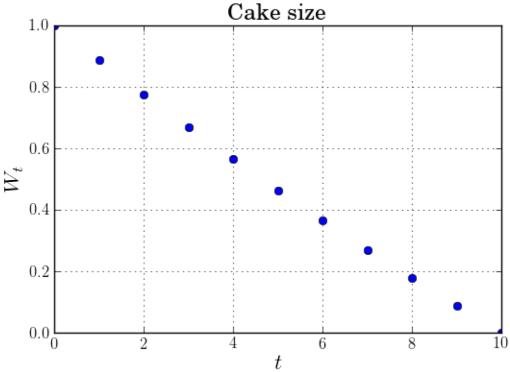
- We can transform the previous problem into a calculus of variations problem

$$\max_{W} \left\{ \sum_{t=0}^{T} \beta^t \ln(W_t - W_{t+1}) : \; W_0 = \phi, \; W_T = 0 \right\}$$

Application: cake eating problem - solution

The optimal cake size is $W^* = \{\phi, \dots, W_t^*, \dots, W_{T-1}^*, 0\}$ where

$$W_t^* = \left(\frac{\beta^t - \beta^T}{1 - \beta^T}\right)\phi, \;\; t = 0, 1, \dots T$$



It slightly bends down from a linear path because of time discounting, as $0 < \beta < 1$ Proof.

Calculus of variations: free terminal state problem

- Find $x^* = \{x_t^*, \}_{t=0}^{T}$ that maximizes

$$J(x) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \text{ s.t. } x_0 = \phi_0 \tag{2}$$

  where ϕ_0 and T is given and x_T is free.
- First order necessary conditions:

Proposition

if x^* is a solution of problem (2) , it verifies the Euler-Lagrange equation the admissibility condition and the transversality condition

$$\begin{cases} \dfrac{\partial F}{\partial x_t}(x_t^*, x_{t-1}^*, t-1) + \frac{\partial F}{\partial x_t}(x_{t+1}^*, x_t^*, t) = 0, & t = 1, 2, \ldots, T-1 \\ x_0^* = \phi_0, & t = 0 \\ \frac{\partial F}{\partial x_T}(x_T^*, x_{T-1}^*, T-1) = 0, & t = T \end{cases}$$

Proof.

Cake eating problem: free terminal state

- Problem

$$\max_{\{C\}} \sum_{t=0}^{T} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = W_t - C_t,\ W_0 = \phi,\ W_T \text{free}$$

- The problem is **ill-posed**: there is no solution with economic meaning Proof.
- Reason: the transversality condition only holds if $C_0 = \infty$
- this is intuitive: if we had no restriction on the terminal size of the case (or if could borrow freely) we would overeat.
- We have to redefine the problem to obtain a reasonable solution.

Calculus of variations: constrained terminal state

- Find $x^* = \{x_t^*\}_{t=0}^{T}$ that maximizes

$$J(x) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \text{ s.t. } x_0 = \phi_0,\ x_T \geq \phi_T \tag{3}$$

  where T, ϕ_0 and ϕ_T are given.
- First order necessary conditions:

Proposition

if x^* is a solution of problem (3), it verifies

$$\begin{cases} \dfrac{\partial F}{\partial x_t}(x_t^*, x_{t-1}^*, t-1) + \dfrac{\partial F}{\partial x_t}(x_{t+1}^*, x_t^*, t) = 0, & t = 1, 2, \ldots, T-1 \\ x_0^* = \phi_0, & t = 0 \\ \dfrac{\partial F}{\partial x_T}(x_T^*, x_{T-1}^*, T-1) \cdot (\phi_T - x_T^*) = 0, & t = T \end{cases}$$

Proof.

Cake eating problem: constrained terminal state

- Problem

$$\max_{\{C\}} \sum_{t=0}^{T} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = W_t - C_t,\ W_0 = \phi,\ W_T \geq 0$$

- F.o.c

$$\begin{cases} W^*_{t+2} = (1+\beta)\, W^*_{t+1} - \beta\, W^*_t, & t = 0, 1, \ldots, T-2 \\ W^*_0 = \phi, & t = 0 \\ \dfrac{\beta^{T-1}}{W^*_T - W^*_{T-1}}\, W^*_T = 0, & t = T \end{cases}$$

- has the same *formal* solution as the simplest problem: $W^*_T = 0$ is determined endogenously not by assumption. Proof.

Calculus of variations: discounted infinite horizon 1

- The problem: find the infinite sequence $x^* = \{x_t^*\}_{t=0}^{\infty}$

$$\max_x \sum_{t=0}^{\infty} \beta^t f(x_{t+1}, x_t), \text{ s.t. } x_0 = \phi_0 \tag{4}$$

  where, $0 < \beta < 1$, and ϕ_0 are given and the terminal state is free (i.e., $\lim_{t\to\infty} x_t$ is free).

- The first order conditions are:

$$\begin{cases} \dfrac{\partial f}{\partial x_t}(x_t^*, x_{t-1}^*) + \beta \dfrac{\partial f}{\partial x_t}(x_{t+1}, x_t) = 0, & t = 0, 1, \ldots \\ x_0^* = x_0, & t = 0 \\ \lim_{t\to\infty} \beta^{t-1} \frac{\partial f(x_t^*, x_{t-1}^*)}{\partial x_t} = 0, & t \to \infty \end{cases}$$

Calculus of variations: discounted infinite horizon 2

- The problem: find the infinite sequence $x^* = \{x^*_t\}_{t=0}^{\infty}$

$$\max_x \sum_{t=0}^{\infty} \beta^t f(x_{t+1}, x_t), \text{ s.t. } x_0 = \phi_0, \quad \lim_{t\to\infty} x_t \geq 0 \tag{5}$$

given β and ϕ_0

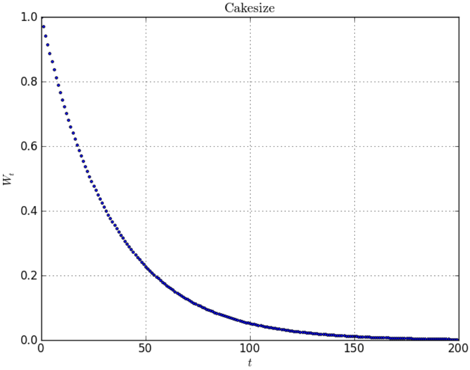
- F.o.c.

$$\begin{cases} \dfrac{\partial f}{\partial x_t}(x^*_t, x^*_{t-1}) + \beta \dfrac{\partial f}{\partial x_t}(x_{t+1}, x_t) = 0, & t = 0, 1, \ldots \\ x^*_0 = x_0, \\ \lim_{t\to\infty} \beta^{t-1} \dfrac{\partial f}{\partial x_t}(x^*_t, x^*_{t-1}) \cdot x^*_t = 0, \end{cases}$$

Cake eating problem: infinite horizon

- Prove that if the terminal state is free there is no finite solution
- Prove that with the terminal condition $\lim_{t\to\infty} W_t \geq 0$ the solution is generated by

$$W_t^* = \phi_0 \beta^t, \ t = 0, 1, \ldots, \infty$$



Terminal conditions

Problem	Given		Optimality contitions	
	T	x_T	T^*	x_T^*
(CV1)	fixed	fixed	T	x_T
(CV2)	fixed	free	T	$\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} = 0$
(CV3)	fixed	$x_T \geq \phi_T$	T	$\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T}(\phi_T - x_T^*) = 0$
(CV4)	∞	free	∞	$\lim_{t\to\infty} \beta^{t-1} \frac{\partial F(x_t^*, x_{t-1}^*)}{\partial x_t} = 0$
(CV5)	∞	$x_\infty \geq 0$	∞	$\lim_{t\to\infty} \beta^{t-1} \frac{\partial F(x_t^*, x_{t-1}^*)}{\partial x_t} x_t^* = 0$

Proofs

Proof of proposition 1

Let $x^* = \{x_t^*\}_{t=0}^{T}$ be an optimal path. Then the optimal value is

$$J(x^*) = \sum_{t=0}^{T-1} F(x_{t+1}^*, x_t^*, t)$$

Introduce an admissible perturbation $x_t = x_t^* + \varepsilon_t$ where $\varepsilon_0 = \varepsilon_T = 0$, and $\varepsilon_t \neq 0$ for any $t \in \{1, \ldots, T-1\}$. The value for the perturbed path is

$$J(x) = \sum_{t=0}^{T-1} F(x_{t+1}^* + \varepsilon_{t+1}, x_t^* + \varepsilon_t, t)$$

Using a Taylor expansion we get first variation in the value functional

$$\begin{aligned} J(x) - J(x^*) = {} & \frac{\partial F(x_1^*, x_0^*, 0)}{\partial x_0} \varepsilon_0 + \sum_{t=1}^{T-1} \left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} \right) \varepsilon_t + \\ & + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} \varepsilon_T \\ = {} & \sum_{t=1}^{T-1} \left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} \right) \varepsilon_t \end{aligned}$$

The path x is optimal if $J(x) - J(x^*) = 0$. Then the EL is obtained.

Return.

Solution of the cake eating problem 1

The Euler-Lagrange equation

- Observe that in this problem the utility function, evaluated at the optimum, is

$$F_t^* = F(W_{t+1}^*, W_t^*, t) = \beta^t \ln(W_t^* - W_{t+1}^*)$$

and the Euler-Lagrange condition is

$$\frac{\partial F_{t-1}^*}{\partial W_t} + \frac{\partial F_t^*}{\partial W_t} = 0$$

- applying to our utility function we have

$$\begin{aligned} \frac{\partial}{\partial W_t}\left(\beta^{t-1}\ln(W_{t-1}^* - W_t^*)\right) + \frac{\partial}{\partial W_t}\left(\beta^t \ln(W_t^* - W_{t+1}^*)\right) = \\ = -\frac{\beta^{t-1}}{W_{t-1}^* - W_t^*} + \frac{\beta^t}{W_t^* - W_{t+1}^*} = 0 \end{aligned} \quad (6)$$

- is a linear second-order difference equation

$$W_{t+1}^* - W_t^* - \beta(W_t^* - W_{t-1}^*),\ t = -1, \dots, T-3$$

Solution of cake eating problem 1

- If W^* is a solution of the problem then it verifies the first order conditions

$$\begin{cases} W^*_{t+2} = (1+\beta)\, W^*_{t+1} - \beta\, W^*_t, \; t = 0, \ldots T-2 \\ W^*_0 = \phi \\ W^*_T = 0 \end{cases}$$

- To find the solution either we transform to a planar system or are able to transform to a simpler problem (we follow the second strategy)

Solution of cake eating problem 1

Solving the problem: recursive approach

- Because $W_{t+1} - W_t = -C_t$, the EL equation is equivalent to

$$C_{t+1} = \beta C_t$$

- This is a first-order DE which has solution

$$C_t = C_0 \beta^t, \text{ where } C_0 \text{ unknown}$$

- Therefore

$$W_{t+1} = W_t - C_0 \beta^t, \text{ for } t = 0, 1, \ldots, T-1$$

- This is a unit-root equation with solution

$$W_t = \phi - C_0 \sum_{s=0}^{t-1} \beta^s = \phi - C_0 \left(\frac{1-\beta^t}{1-\beta} \right)$$

  because $W_0 = \phi$.

Solution of cake eating problem 1

- However, to be optimal, W_t should be admissible, which means satisfying

$$\begin{cases} W_0^* = \phi \\ W_T^* = 0 \end{cases}$$

But

$$\begin{cases} W_t|_{t=0} = \phi = \phi \\ W_t|_{t=T} = \phi - C_0 \left(\dfrac{1-\beta^T}{1-\beta} \right) \end{cases}$$

- Setting $W_t|_{t=T} = 0$ yields $C_0^* = \phi \left(\dfrac{1-\beta}{1-\beta^T} \right)$
- Then **the solution to the cake eating problem** is generated by

$$W_t^* = \phi \left(1 - \frac{(1-\beta)}{(1-\beta^T)} \frac{(1-\beta^t)}{(1-\beta)} \right) = \phi \left(\frac{\beta^t - \beta^T}{1-\beta^T} \right)$$

Return.

Proof of proposition 2

Using the same method of the proof of Proposition 1 we have

$$
\begin{aligned}
J(x) - J(x^*) = {} & \frac{\partial F(x_1^*, x_0^*, 0)}{\partial x_0}\varepsilon_0 + \sum_{t=1}^{T-1}\left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t}\right)\varepsilon_t + \\
& + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T}\varepsilon_T \\
= {} & \sum_{t=1}^{T-1}\left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t}\right)\varepsilon_t + \\
& + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T}\varepsilon_T
\end{aligned}
$$

because the admissibility condition only implies that $\varepsilon_0 = 0$. Return.

Cake eating problem: free terminal state

- The first-order conditions are

$$\begin{cases} W^*_{t+2} = (1+\beta)\, W^*_{t+1} - \beta\, W^*_t, & t = 0, 1, \ldots, T-1 \\ W_0 = \phi, & t = 0 \\ \frac{\beta^{T-1}}{W^*_T - W^*_{T-1}} = 0. & t = T \end{cases}$$

- Using the same transformation as in the fixed-terminal state problem we have

$$\begin{cases} C_{t+1} = \beta\, C_t, & t = 0, 1, \ldots, T-1 \\ \frac{\beta^{T-1}}{C_{T-1}} = 0. & t = T \end{cases}$$

- But we already know that the solution to the first equation is $C_t = C_0 \beta^t$. This implies that the transversality constraint becomes

$$\frac{\beta^{T-1}}{C_{T-1}} = \frac{\beta^{T-1}}{C_0 \beta^{T-1}} = \frac{1}{C_0}$$

which can only be zero if $C_0 = \infty$. This means that $W_T \to -\infty$ which does not make sense. The problem is ill-posed.

Return.

Proof of proposition 3

Using the same method of the proof of Proposition 3 but introducing the terminal constraint

$$J(x) = \sum_{t=0}^{T-1} F(x_{t+1}^* + \varepsilon_{t+1}, x_t^* + \varepsilon_t, t) + \lambda(\phi_T - x_T^* - \varepsilon_T)$$

where λ is a Lagrange multiplier. Then the variation of the value of the problem is

$$J(x) - J(x^*) = \sum_{t=1}^{T-1} \left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} \right) \varepsilon_t + \\ + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} \varepsilon_T + \lambda(\phi_T - x_T^* - \varepsilon_T)$$

Proof of proposition 3

Then $J(x) = J(x^*)$ if the EL equation holds,

$$\left(\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} - \lambda \right) \varepsilon_T = 0$$

where ε_T is arbitrary, and the Karush-Kuhn-Tucker conditions hold

$$\lambda(\phi_T - x_t^*) = 0,\ \lambda \geq 0,\ x_t^* \geq \phi_T.$$

Therefore $\frac{\partial F_{T-1}^*}{\partial x_T} - \lambda = 0$ but

$$\frac{\partial F_{T-1}^*}{\partial x_T} - \lambda = \left(\frac{\partial F_{T-1}^*}{\partial x_T} - \lambda \right) (\phi_T - x_t^*) = \frac{\partial F_{T-1}^*}{\partial x_T} (\phi_T - x_t^*) = 0$$

which is the transversality condition. Return.

Solution to the cake eating problem with constrained terminal state

Now the transversality condition is

$$-W_T \frac{\beta^{T-1}}{W_T - W_{T-1}} = 0$$

Using our previous transformation we have

$$-W_T \frac{\beta^{T-1}}{W_T - W_{T-1}} = -W_T \frac{\beta^{T-1}}{C_{T-1}} = -W_T \frac{\beta^{T-1}}{\beta^{T-1} C_0} = -\frac{W_T}{C_0} = 0$$

only if $W_T = 0$ for any finite and positive C_0. Return.