Foundations of Financial Economics
Revisions of utility theory

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Topics of the lecture

- Marginal concepts frequent in economics
- Basic utility theory

Marginalist concepts

Value function

- Consider a number of different objects **indexed** as $\mathbb{I} = \{1, \ldots, i, \ldots, n\}$
- The **quantity** of object i is denoted $x_i \in \mathbb{R}$
- We can represent a **bundle** of objects by the vector $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n$, where
- The **value** of a bundle is given by the (at least twice-) differentiable function

$$F = F(\mathbf{x}) = F(x_1, \ldots, x_i, \ldots, x_n)$$

- In economics usually $F(\cdot)$ represents is a utility or a production function
- Change in value is represented by the differential (under very weak conditions)

$$dF = F_1\, dx_1 + \ldots + F_i\, dx_i + \ldots = \nabla F \cdot d\mathbf{x}$$

  where ∇F is the gradient

$$\nabla F = \left(F_1, \ldots, F_i, \ldots, F_n\right)^\top$$

Marginalist concepts

Marginal values: goods

- Denote the partial derivative of object i by

$$F_i(\mathbf{x}) \equiv \frac{\partial F(\mathbf{x})}{\partial x_i}$$

- We say object i is a

$$\begin{cases} \textbf{good} & \text{if } F_i(\mathbf{x}) > 0 \text{ for any } \mathbf{x} \in \mathbb{R}^n \\ \textbf{saturated} & \text{if } F_i(\mathbf{x}) = 0 \text{ for any } \mathbf{x} \in \mathbb{R}^n \\ \textbf{bad} & \text{if } F_i(\mathbf{x}) < 0 \text{ for any } \mathbf{x} \in \mathbb{R}^n \end{cases}$$

- From now on we consider goods, i.e. $F_i > 0$ for any $i \in \mathbb{I}$
- We call **marginal contribution** of good i to the change in value brought about by dx_i

$$\text{(Definition) } M_i \equiv \frac{dF}{dx_i}$$

- For the bundle variation $d\mathbf{x} = (0, \ldots, 0, dx_i, 0, \ldots, 0)$ then $dF = F_i\, dX_i$ and therefore the marginal contribution is equal to the partial derivative

$$\text{(Implication) } M_i = F_i$$

therefore a good has a positive marginal contribution for value

Marginalist concepts

Relative marginal changes

- Observe that $M_i(\mathbf{x}) = F_i(\mathbf{x})$ because F_i is a function of $\mathbf{x}$
- If F is twice-differentiable we can calculate second-order derivatives

$$\text{(own) } F_{ii} \equiv \frac{\partial^2 F(\mathbf{x})}{\partial x_i^2} \quad \text{(crossed) } F_{ij} \equiv \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial_j}, \text{ for any } j \neq i \in \mathbb{I}$$

- The **marginal contribution** of i for a variation in x_i

$$\frac{\partial M_i}{\partial x_i} = F_{ii} = \begin{cases} > 0 & \textbf{increasing} \\ = 0 & \textbf{constant} \\ < 0 & \textbf{decreasing} \end{cases}$$

- **Pareto-Edgeworth** relationships: variation in M_i for a variation in any x_j:

$$\frac{\partial M_i}{\partial x_i} = F_{ii} = \begin{cases} > 0 & \textbf{complementarity} \\ = 0 & \textbf{independence} \\ < 0 & \textbf{substitutability} \end{cases}$$

- **Uzawa-Allen elasticities**: relative variation in M_i for a variation in any x_j

$$\text{(own) } \varepsilon_{ii} \equiv -\frac{F_{ii}\, x_i}{F_i} \quad \text{(crossed) } \varepsilon_{ij} \equiv -\frac{F_{ij}\, x_i}{F_i}$$

- If i is a good and its quantity is positive then $\varepsilon_{ii} > 0$ and it is complementary with (substitutable by) j if $\varepsilon_{ij} < 0$ ($\varepsilon_{ij} > 0$)

Marginalist concepts

Compensated variations

- The **marginal rate of substitution** of good i by good j is the variation in the quantity of good j by unit variation in good i

$$\textbf{(definition)}\ MRS_{ij} \equiv \frac{dx_j}{dx_i}$$

- Assume we want to know what would be dx_j if we change dx_i in such a way as to keep the value F constant, ie. if $d\mathbf{x} = (0, \ldots, 0, dx_i, 0, \ldots, dx_j, 0, \ldots, 0)$ such that $dF = 0$. That is

$$dF = \nabla F \cdot d\mathbf{x} = F_i\, dx_i + F_j\, dx_j = 0$$

- Then

$$\text{(Implication)}\ MRS_{ij}(\mathbf{x}) = -\frac{F_i(\mathbf{x})}{F_j(\mathbf{x})} \text{ for } F(\mathbf{x}) = \text{constant}$$

Marginalist concepts

Elasticity of substitution

- A fundamental concept here is the **elasticity of substitution** of good i by good j

$$\text{(definition) } ES_{ij}(\mathbf{x}) \equiv \frac{d\ln(x_j/x_i)}{d\ln MRS_{ij}(\mathbf{x})}$$

  intuition: relative change in the MRS_{ij} for a relative change in the ratio x_j/x_i.

- If F is twice differentiable, then the ES_{ij} is

$$\textbf{(Implication) } ES_{ij} = \frac{x_i\,F_i + x_j\,F_j}{x_j\,F_j\,\varepsilon_{ii} - 2\,x_i\,F_i\,\varepsilon_{ij} + x_i\,F_i\,\varepsilon_{jj}}$$

  where $x_i\,F_i\,\varepsilon_{ij} = x_j\,F_j\,\varepsilon_{ji}$ and $F_{ij} = F_{ji}$ if F is continuous.

Marginalist concepts

Elasticity of substitution: continuation

Sketch of the proof: remember we want to substitute j by i keeping the quantities of the other goods constant

$$d\ln(x_j/x_i) = d\ln x_j - d\ln x_i = \frac{dx_j}{x_j} - \frac{dx_i}{x_i} =$$

$$= -\frac{dx_i}{x_i\, x_j F_j}\Big(x_i F_i + x_j F_j\Big) \text{ (because } F_i\, dx_i + F_j\, dx_j = 0)$$

$$d\ln MRS_{ij} = d\ln\left(\frac{F_i(x_i,x_j)}{F_j(x_i,x_j)}\right) = d\ln F_i - d\ln F_j = \frac{dF_i}{F_i} - \frac{F_j}{F_j}$$

But

$$dF_i = F_{ii}dx_i + F_{ij}dx_j = dx_i\Big(F_{ii} + \frac{dx_j}{dx_i}F_{ij}\Big) = dx_i\Big(F_{ii} - \frac{F_i}{F_j}F_{ij}\Big)$$

$$dF_j = F_{ji}dx_i + F_{jj}dx_j = dx_i\Big(F_{ij} + \frac{dx_j}{dx_i}F_{jj}\Big) = dx_i\Big(F_{ij} - \frac{F_i}{F_j}F_{jj}\Big)$$

the rest of the proof is obtained by simplification and by using the definition of the Uzawa-Allen elasticities.

- **The problem**: consider an agent with a resource W that wants to **allocate it optimally** among two goods, 1 and 2, having (given) costs p_1 and p_2.
- The optimality criterium is $U(c_1, c_2)$, where the quantities of the two goods are c_1 and c_2.
- **Further assumptions**:
  - The utility function $U(\dot{})$ is: continuous, differentiable, increasing and concave.
  - The endowment is positive: $W > 0$
- Nominal expenditure $E \equiv E(c_1, c_2) = p_1 c_1 + p_2 c_2$

Optimal free allocation: definition

- Assume there are no other constraints with the exception of the resource constraint $E(c_1, c_2) = W$
- The problem is

$$V(W; p_1, p_2) = \max_{c_1, c_2} \left\{ U(c_1, c_2) : \; E(c_1, c_2) = W \right\}$$

- function $V(.)$ is called indirect utility or value function
- intuition: it gives the **value** of the endowment W in utility terms

Optimal free allocation: solution

- The Lagrangean
$$\mathcal{L} = u(c_1, c_2) + \lambda(W - E(c_1, c_2))$$
- The solution (which always exists) $(c_1^*, c_2^*, \lambda^*)$ satisfies the conditions
$$\begin{cases} U_{c_j}(c_1, c_2) - \lambda p_j = 0, \quad j = 1, 2 \\ W - E(c_1, c_2) = 0 \end{cases}$$
- We observe that, at the optimum that the $MRS_{1,2}$ is equalized to the relative prices
$$MRS_{1,2} = \frac{U_{c_1}(c_1^*, c_2^*)}{U_{c_2}(c_1^*, c_2^*)} = \frac{p_1}{p_2}$$
  and, in this case the resource is saturated
$$E(c_1^*, c_2^*) = p_1 c_1^* + p_2 c_2^* = W$$

Optimal free allocation: solution

- When there is free allocation, the solution is characterized by the equations,

$$
\begin{aligned}
p_2\, U_{c_1}(c_1^*, c_2^*) &= p_1\, U_{c_2}(c_1^*, c_2^*) && (1)\\
E(c_1^*, c_2^*) &= W && (2)
\end{aligned}
$$

- Equation (1) is a first-order partial differential equation with solution (check this)

$$U(c_1^*, c_2^*) = V\left(\frac{p_1\, c_1^* + p_2\, c_2^*}{p_1}\right)$$

- from equation (2), in the optimum we have

$$U(c_1^*, c_2^*) = V(w),\ \ w \equiv \frac{W}{p_1}\ \text{(real resources deflated } p_1\text{)}$$

- if the utility function is strictly concave then with very weak conditions (differentiability) we have an unique interior optimum

Optimal free allocation: graphical representation

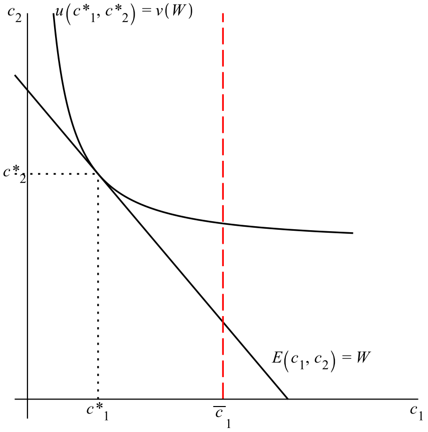

Figure: Interior optimum for a log utility function
$U(c_1, c_2) = \ln c_1 + b \ln c_2$

Utility theory

Optimal constrained allocation: definition

- Let us assume that the agent is constrained in the allocation of resources to good 1. For instance, assume that $c_i \in [0, \bar{c}_1]$
- The problem is now

$$V(W; p_1, p_2, \bar{c_1}) = \max_{c_1, c_2} \{ U(c_1, c_2) : \; E(c_1, c_2) = W, 0 \leq c_1 \leq \bar{c}_1 \}$$

- Most models of financial frictions introduce constraints of this type
- More generally we could assume there are restrictions in allocation resources to the two goods.
- The problem would become

$$V(W; p_1, p_2, \bar{c_1}, \bar{c_2}) = \max_{c_1, c_2} \{ U(c_1, c_2) : \; E(c_1, c_2) = W, 0 \leq c_j \leq \bar{c}_j, j = 1, 2 \}$$

Utility theory

Optimal constrained allocation: optimality

- The Lagrangean is now

$$
\begin{aligned}
\mathcal{L} = {} & u(c_1, c_2) + \lambda(W - E(c_1, c_2)) - \\
& - \eta_1 c_1 - \eta_2 c_2 + \zeta_1(\bar{c}_1 - c_1) + \zeta_2(\bar{c}_2 - c_2)
\end{aligned}
$$

- The solution (which always exists) $(c_1^*, c_2^*, \lambda^*, \eta_1^*, \eta_2^*, \zeta_1^*, \zeta_2^*)$ satisfies the Karush-Kuhn-Tucker conditions

$$
\begin{cases}
U_{c_j}(c_1, c_2) - \lambda p_j - \eta_j - \zeta_j = 0, & j = 1, 2 \\
\eta_j c_j = 0,\ \eta_j \geq 0,\ c_j \geq 0, & j = 1, 2 \\
\zeta_j(\bar{c}_j - c_j) = 0,\ \zeta_j \geq 0,\ c_j \leq \bar{c}_j, & j = 1, 2 \\
\lambda(W - E(c_1, c_2)) = 0,\ \lambda \geq 0,\ E(c_1, c_2) \leq W &
\end{cases}
$$

Optimal constrained allocation: solution

Corner solution: lower $c_1 = 0$

- Let $c_1^* = 0$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- FOC: $\eta_1^* > 0$ and $\eta_2^* = \zeta_1^* = \zeta_2^* = 0$, and

$$
\begin{aligned}
p_2 \, U_{c_1}(c_1^*, c_2^*) &= p_1 \, U_{c_2}(c_1^*, c_2^*) - p_2 \eta_1 && (3) \\
E(c_1^*, c_2^*) &= W && (4)
\end{aligned}
$$

- Now, the MRS is smaller than the relative price

$$
MRS_{12} = \frac{U^*_{c_1}}{U^*_{c_2}} = \frac{p_1}{p_2} - \frac{\eta_1}{U^*_{c_2}} < \frac{p_1}{p_2}
$$

  i.e., there is a "wedge" between relative prices and the MRS_{12}
- Equation (3) is a first-order partial differential equation with solution

$$
U(c_1^*, c_2^*) = \frac{\eta_1 \, c_2^*}{p_1} + V\left(\frac{p_1 \, c_1^* + p_2 \, c_2^*}{p_1}\right)
$$

- if we use equation (6) in the optimum we have

$$
U(c_1^*, c_2^*) = -\eta_1^* \, w + V(w) < V(w)
$$

Optimal constrained allocation: figure

Corner solution 1

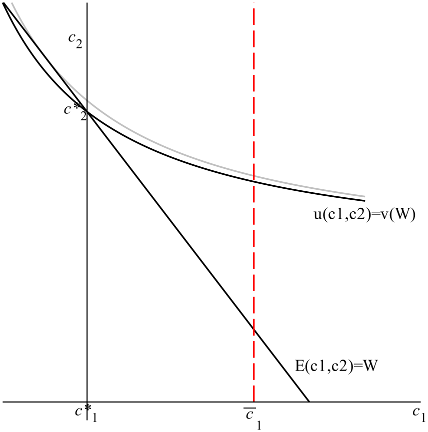

Figure: Corner solution: the indirect utility level is smaller than for the unconstrained case

Optimal constrained allocation: solution

Corner solution: upper constraint $c_1 = \bar{c}_1$

- Let $c_1^* = \bar{c}_1$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- then $\zeta_1^* > 0$ and $\eta_1^* = \eta_2^* = \zeta_1^* = \zeta_2^* = 0$
- In addition

$$\begin{aligned} p_2 \, U_{c_1}(c_1^*, c_2^*) &= p_1 \, U_{c_2}(c_1^*, c_2^*) + p_2 \zeta_1 & (5) \\ E(c_1^*, c_2^*) &= W & (6) \end{aligned}$$

- There is again a "wedge" between the MRS_{12} and the relative price, but now

$$MRS_{12} = \frac{U^*_{c_1}}{U^*_{c_2}} = \frac{p_1}{p_2} + \frac{\zeta_1}{U^*_{c_2}} > \frac{p_1}{p_2}$$

- Equation (5) is a first-order partial differential equation with solution

$$U(c_1^*, c_2^*) = -\frac{\zeta_1 \, c_2^*}{p_1} + V\left(\frac{p_1 \, c_1^* + p_2 \, c_2^*}{p_1}\right)$$

- if we use equation (6) in the optimum we have

$$U(c_1^*, c_2^*) = -\frac{\zeta_1 \, p_1 (w - \bar{c}_1)}{p_2} + V(w) < V(w)$$

Consumer problem

Corner solution 2

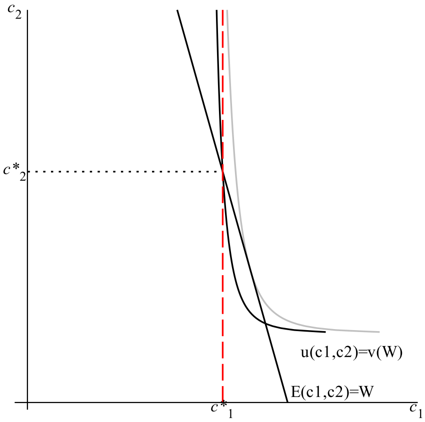

Figure: Corner solution: the indirect utility level is smaller than for the unconstrained case

Equivalent interpretation

- Let the value function in which there are constraints on the consumer be denoted by $\tilde{V}(w)$
- Looking at the previous cases we can write

$$\tilde{v}(w) = V(w) - \delta(w)$$

  where $\delta(w) \geq 0$ measures the welfare loss introduced by the constraint $c_1 \in [0, \bar{c}_1]$.
- We could obtain a similar solution for the consumer problem is instead of considering the endowment level w we consider the resource level

$$\tilde{w} = \{x : \ (\tilde{v}^{-1})(x) = 0\} < w$$

  that is a **smaller** level for the endowment.

Conclusion

Constraints on the free allocation of resources between the two consumption goods

1. create a (algebraic) wedge between the the MRS and the relative prices
2. generate welfare losses
3. this gives a rough idea on the effects of constraints in the intertemporal or intra-state of nature allocation of resources (at least for a benchmark model)