AME 2021-2022:
Problem set 6: parabolic PDE's parabolic

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1 Parabolic partial differential equations

1.1 General

1. Consider the following parabolic PDE $u_t(t,x) = au_{xx}(t,x) + b$, where $a > 0$. Solve the following problems:

   a) For $(t,x) \in [0,\infty) \times (-\infty,\infty)$.
   b) For $(t,x) \in [0,\infty) \times [0,\infty)$.
   c) For $(t,x) \in [0,\infty) \times [\underline{x},\bar{x}]$.

2. Consider the following parabolic PDE $u_t(t,x) = au_{xx}(t,x) + b(t)$, where $a > 0$. Solve the following problems:

   a) For $(t,x) \in [0,\infty) \times (-\infty,\infty)$.
   b) For $(t,x) \in [0,\infty) \times [0,\infty)$.

3. Assume that $(t,x) \in [0,\infty) \times [0,\infty)$ and $u = u(t,x)$ and consider the initial-value problem

$$\begin{cases} u_t = au_{xx} + bu, & (x,t) \in [0,\infty)^2 \\ u(0,x) = \phi(x), & (x,t) \in [0,\infty) \times \{t=0\}. \end{cases}$$

   a) Find the general solution to the initial-value problem.
   b) Let $\phi(x) = \delta(x)$, where $\delta(.)$ is Dirac's delta "function". Find the solution to the problem. Provide an intuition for the solution.

4. Consider the parabolic partial differential equation $u_t - u_{xx} = 0$, where $u = u(t,x) \in \mathbb{R}$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

   a) Find the solution to the PDE.

b) If $u(0,x) = \delta(x - x_0)$, where $\delta(.)$ is Dirac's delta "function" and $x_0 > 0$ find the solution to the initial-value problem. Provide an intuitive characterization of the solution.

5. Assume that $(t,x) \in [0,T] \times [0,\infty)$ and $u = u(t,x)$ and consider the initial-value problem

$$\begin{cases} u_t + u_{xx} + bu, & (t,x) \in [0,T] \times [0,\infty) \\ u(T,x) = \phi_T(x), & (t,x) \in \{t = T\} \times [0,\infty). \end{cases}$$

a) Classify the PDE.

b) Find the general solution to the PDE.

c) Find the solution to the problem. Provide an intuition for the solution.

6. Assume that $(t,x) \in [0,\infty) \times [0,\infty)$ and $u = u(t,x)$ and consider the initial-value problem

$$\begin{cases} u_t = u_{xx} + u_x, & (x,t) \in [0,\infty)^2 \\ u(0,x) = \phi(x), & (x,t) \in [0,\infty) \times \{t = 0\}. \end{cases}$$

a) Find the general solution to the PDE.

b) Find the general solution to the initial-value problem.

c) Let $\phi(x) = \delta(x)$, where $\delta(.)$ is Dirac's delta "function". Find the solution to the problem. Provide an intuition for the solution.

7. Assume that $(t,x) \in [0,\infty) \times [0,\infty)$ and $u = u(t,x)$ and consider the initial-value problem

$$\begin{cases} u_t = u_{xx} + u_x + b(x), & (x,t) \in [0,\infty)^2 \\ u(0,x) = \phi(x), & (x,t) \in [0,\infty) \times \{t = 0\}. \end{cases}$$

a) Find the general solution to the PDE.

b) Find the general solution to the initial-value problem.

c) Let $\phi(x) = \delta(x)$, where $\delta(.)$ is Dirac's delta "function" and $b(x) = x^2$. Find the solution to the problem. Provide an intuition for the solution.

1.2 Applications

1. Let $k(t,x)$ be the capital stock located at time $t \in [0,\infty)$ at location $x \in (-\infty,\infty)$. If capital flows freely between locations, and if it is locally driven by the difference in spatial concentration, it can be proved that the distribution of capital can be modelled by the PDE

$$k_t(t,x) = k_{xx}(t,x) + f(k(t,x)) - c(t,x)$$

where $f(.)$ is the production function and c is consumption. Let $c = (1-s)f(k)$ where $0 < s < 1$ is the savings rate, and $f(k) = Ak$, where $A > 0$. Assume that the initial distribution of the capital stock is $k(0,x) = e^{-(k-k_0)^2}$ where $k_0 > 0$.

a) Solve the initial-value problem

b) Provide an intuition to the solution, and, in particular, to the long-run behavior $\lim_{t\to\infty} k(t,x)$.

2. Consider the parabolic PDE

$$C_t(t,x) = -C_{xx}(t,x) + (r-\rho)C(t,x),\ (t,x) \in [0,\infty) \times [0,\infty)$$

where r and ρ are both positive, subject to the constraint $\lim_{t\to\infty} e^{-\rho t}C(t,x) = e^{-x^2}$.

a) Classify the PDE and discuss the well-posedness of the problem.

b) Solve the equation