

Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2017-2018

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Exam: **Época de Recurso**
30.1.2018 (18.00h-20.30h)

Closed book exam.

1. [2 points] Consider the terminal value problem $\dot{y} = gy + b$, for $t \geq 0$ and $\lim_{t \rightarrow \infty} y(t) = \bar{y}$, where \bar{y} is the steady state. Let $b \neq 0$:
 - (a) assume that $g < 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically;
 - (b) assume that $g > 0$. Solve the terminal value problem and characterize the solutions analytically and geometrically.

2. [3 points] Consider the planar ODE, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

- (a) Solve the ODE.
 - (b) Draw the phase diagram and characterize it.
 - (c) Let $\mathbf{y}(0) = (0, 1)$. Solve the initial value problem.
3. [3 points] Consider the following ODE:

$$\begin{aligned}\dot{y}_1 &= y_1(1 - y_2^2), \\ \dot{y}_2 &= y_2(y_1 - 1)\end{aligned}$$

- (a) Find the equilibrium points and characterize them.
 - (b) Prove that the equation has the first integral: $V(y_1, y_2) = \log(y_1 y_2^2) - (y_1 + y_2^2)$.
 - (c) Draw the phase diagram and provide an intuition for it.
4. [4 points] Sidrausky Sidrauski (1967) wrote an influential article on money and economic growth. Assume a representative agent economy in which the stock of financial wealth in real terms, a , is the sum of the stock of real money and equity, k , $a = k + m$. The budget constraint is $\dot{a} = y + (\mu - \pi)m - c$, where c is real consumption, y is the income from labor and dividends, μm are nominal money transfers (subsidies minus taxes), and π is the inflation rate. Investment in equity produces a flow of income $y = f(k)$. The intertemporal utility functional is

$$J(c, m) = \int_0^{+\infty} u(c, m) e^{-\delta t} dt,$$

where we assume that the utility of cash holdings is related to the reduction of time devoted to transactions of goods. The control variables are the consumption flow and the stock of money.

Assume that the production function, $f(k)$ is increasing, concave and Inada, and the utility function, $u(c, m)$ is increasing, concave, Inada and separable (i.e. $u_{cm}(c, m) = u_{mc}(c, m) = 0$). Assume that that $a(0) = a_0 > 0$ and that $\lim_{t \rightarrow \infty} e^{-(\mu - \pi)t} a(t) \geq 0$.

- (a) Write the problem as an optimal control problem and the first order conditions according to the Pontryagin's principle (Hint: use a as the state variable). Are those conditions necessary and sufficient ?
 - (b) Determine the steady state(s) and the local dynamic properties
 - (c) Depict the phase diagram.
 - (d) Find the short-run and the long run multipliers for a permanent increase in the rate of growth of money (μ). Provide a geometric representation
5. [3 points] Consider the first-order partial differential equation $y_t(t, x) + ax y_x(t, x) = 0$, where $y = y(t, x)$ and $(t, x) \in (0, \infty) \times (-\infty, \infty)$.
- (a) Find the solution (hint use the method of characteristics).
 - (b) Let $y(0, x) = \delta(x - x_0)$, where $\delta(\cdot)$ is Dirac's delta function and $x_0 > 0$. Find the solution to the initial-value problem.
6. [3 points] Consider the parabolic partial differential equation $u_t - u_{xx} = 0$, where $u = u(x, t)$ and $(x, t) \in (-\infty, \infty) \times (0, \infty)$.
- (a) Find the solution to the PDE.
 - (b) If $u(x, 0) = \delta(x - x_0)$, where $\delta(\cdot)$ is Dirac's delta "function" and $x_0 > 0$ find the solution to the initial-value problem.
7. [2 points] Consider the diffusion equation

$$dX(t) = \gamma X(t)dt + \sigma dW(t)$$

where $\{W(t)\}$ is a standard Brownian motion.

- (a) Let $X(0) = x_0$ be known. Find the solution of the initial value problem.
- (b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.
- (c) Derive the forward Kolmogorov equation for the density associated to $X(t) = x > 0$, assuming that $X(0) = 0$.

References

Sidrauski, M. (1967). Rational choices and patterns of growth in a monetary economy. *American Economic Review*, 57:534–544.