

# Toolkit for economic growth

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# Topics

1. Production functions: properties
2. Production functions and input incomes
3. Production functions and types of inputs
4. Production functions and technology bias
5. Extensions: continuum of inputs, multisector economies
6. Production function and input accumulation and dynamics
7. Types of growth dynamics and the production function

# Production function: properties



# Production function

- ▶ Production function

$$y = F(\mathbf{x}) \equiv F(x_1, \dots, x_n)$$

- ▶  $y$  = output of one good (can be used in consumption and/or investment)
- ▶  $\mathbf{x} = (x_1, \dots, x_n)$  bundle of inputs, vector assuming a discrete index set
- ▶  $x_i$  input of good  $i$  in production (intermediate good or final good)
- ▶  $F(\cdot)$  production function formalizes the **technology**: properties of the transformation of inputs into outputs
- ▶ relevant properties:
  - ▶ general properties: increasing, concave and homogeneous
  - ▶ at the input level: necessity, marginal variation, substitutability/complementarity

# Production: general properties

- ▶ Definition:  $F$  is **weakly increasing** if  $\mathbf{x}^* \geq \mathbf{x} \Rightarrow F(\mathbf{x}^*) \geq F(\mathbf{x})$
- ▶ Meaning: Increase in the quantity of inputs leads to an increase in production (bigger bundle increases output)
- ▶ Derivative:

$$\frac{\partial F(\mathbf{x})}{\partial x_i} = \lim_{\epsilon \rightarrow 0} \frac{F(\dots, x_i + \epsilon, \dots) - F(\dots, x_i, \dots)}{\epsilon}$$

- ▶ Gradient: vector of first derivatives

$$DF(\mathbf{x}) = \left( \frac{\partial F(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial F(\mathbf{x})}{\partial x_n} \right)^\top$$

- ▶ if  $DF(\mathbf{x}) \geq 0$  then  $F$  is weakly increasing

# Production: general properties

- ▶ Definition:  $F$  is **concave** if given any two bundles  $\mathbf{x}$  and  $\mathbf{x}^*$

$$F(\mathbf{x}) - F(\mathbf{x}^*) \leq +DF(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*)$$

(strictly concave if  $<$ )

- ▶ Meaning: increases in the quantity if inputs increases production less than linearly
- ▶ Hessian: matrix of second derivatives

$$D^2F(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 F(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 F(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 F(\mathbf{x})}{\partial x_n^2} \end{pmatrix}$$

- ▶ If  $F(\mathbf{x})$  is concave then  $D^2F(\mathbf{x})$  is negative semi-definite.
- ▶  $D^2F(\mathbf{x})$  is negative semi-definite if the principal minors of odd order are negative and the principal minors of even order are positive

## Production: general properties

- ▶ Definition:  $F$  is **homogeneous of degree  $\eta$**  if changing  $\mathbf{x}$  to a input of  $\lambda$ , where  $\lambda$  is a positive number changes output by a input of  $\lambda^\eta$

$$\lambda^\eta F(\mathbf{x}) - F(\lambda \mathbf{x}^*)$$

where  $\lambda \mathbf{x}^* = (\lambda x_1, \dots, \lambda x_n)^\top$

- ▶  $\eta$  measures the returns to scale
- ▶ Therefore, we say the production function displays
  - ▶ decreasing returns to scale if  $\eta < 1$
  - ▶ constant returns to scale if  $\eta = 1$
  - ▶ increasing returns to scale if  $\eta > 1$

- ▶ Euler theorem,

$$\eta F(\mathbf{x}) = DF(\mathbf{x}) \cdot \mathbf{x} = \sum_{i=1}^n \frac{\partial F(\mathbf{x})}{\partial x_i} dx_i$$

- ▶ A fundamental requirement for the existence of growth is that  $F(\cdot)$  is linearly homogeneous



# Production function: specific properties

- ▶ Necessity: input  $x_i$  is **necessary** if  $x_i = 0 \Rightarrow f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0$  (zero output if it is not used)
- ▶ Variation in production: measured by the differential

$$dy = DF(\mathbf{x}) \cdot \mathbf{x} = \sum_{i=1}^n F_i(\mathbf{x}) dx_i$$

- ▶ Marginal product (or productivity):

$$MP_i = F_i = \frac{\partial F(\mathbf{x})}{\partial x_i}$$

meaning: variation in production if input  $i$  is increased by one unit

$$dy = \sum_{i=1}^n F_i(\mathbf{x}) dx_i = F_i, \text{ if } d\mathbf{x} = (0, \dots, 0, 1, 0, \dots, 0, \dots, 0)$$

- ▶ we say input  $i$  is **productive** if  $F_i > 0$ .
- ▶ If  $F(\mathbf{x})$  is strictly increasing then all inputs are productive.

# Production function: technology

- Change in the marginal product:

$$MP_{ij} = F_{ij} = \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial j}$$

- The marginal product is
  - decreasing if  $MP_{ii} < 0$
  - constant if  $MP_{ii} = 0$
  - increasing if  $MP_{ii} > 0$
- If the technology is concave (among all other conditions) we have:

$$F_{ii}(\mathbf{x}) \leq 0 \text{ for all } i = 1, \dots, n \text{ (decreasing MP)}$$

$$F_{ii}(\mathbf{x}) F_{jj}(\mathbf{x}) - F_{ij}(\mathbf{x})^2 \geq 0, \text{ for all pairs } i \neq j = 1, \dots, n$$

# Production function: technology

- ▶ Allen-Uzawa elasticities

$$\epsilon_{ij} = -\frac{F_{ij}(\mathbf{x})x_j}{F_i(\mathbf{x})}, \quad i, j = 1, \dots, n$$

- ▶ If  $MP_i$  is decreasing then the "own" elasticity is positive

$$\epsilon_{ii} = -\frac{F_{ii}(\mathbf{x})x_i}{F_i(\mathbf{x})}, \quad i = 1, \dots, n$$

- ▶ If  $MP_i$  is decreasing then the "own" elasticity is positive
- ▶ For "crossed" elasticities, we say there is:
  - ▶ gross substitutability if  $\epsilon_{ij} > 0$
  - ▶ gross independence if  $\epsilon_{ij} = 0$
  - ▶ gross complementarity if  $\epsilon_{ij} < 0$

# Production function: technology

- ▶ Compensated changes in two inputs: variations in inputs  $i$  and  $j$  such that the output is constant

$$dy = F_i(\mathbf{x})dx_i + F_j(\mathbf{x})dx_j = 0$$

- ▶ **Marginal rate of substitution** between inputs  $i$  and  $j$ : compensated change in  $j$  for a unit change in  $i$

$$MRS_{ij} \equiv -\frac{dx_j}{dx_i} \Big|_{dy=0}$$

- ▶ Then: it is equal to the ratio of marginal products

$$MRS_{ij} = \frac{F_i(\mathbf{x})}{F_j(\mathbf{x})}, \quad i \neq j = 1, \dots, n$$

- ▶ Elasticity of substitution between inputs  $i$  and  $j$

$$ES_{ij} \equiv \frac{d \ln(x_j/x_i)}{d \ln MRS_{ij}(\mathbf{x})} \Big|_{dy=0}$$

# Production function: the benchmark case

The benchmark production function is the generalized mean

$$y = M_{\sigma}(\mathbf{x}) = \left( \sum_{i=1}^n \alpha_i x_i^{\sigma} \right)^{\frac{1}{\sigma}}, \quad \sigma \in [-\infty, \infty]$$

where  $0 \leq \alpha_i \leq 1$  is the share of input  $i$  and

$$\sum_{i=1}^n \alpha_i = 1$$

- We readily see that it is an homogeneous function, for any value of  $\sigma$

$$M_{\sigma}(\lambda \mathbf{x}) = \lambda M_{\sigma}(\mathbf{x})$$

thus it displays a constant returns to scale technology.

# Production function: the benchmark case

For different values of  $\sigma$  economists call different names

- ▶ Linear case: if  $\sigma = 1$

- ▶ Cobb-Douglas if  $\sigma = 0$ . Proof:

$$\ln M_0 = \lim_{\sigma \rightarrow 0} \frac{\ln \sum_{i=1}^n \alpha_i x_i^\sigma}{\sigma} = \lim_{\sigma \rightarrow 0} \frac{\sum \alpha_i x_i^\sigma \ln(x_i)}{\sum \alpha_i x_i^\sigma} = \sum \alpha_i \ln(x_i)$$

- ▶ Constant elasticity of substitution (CES) if  $\sigma < 1$  and is finite

- ▶  $M_\infty = x_{\max} = \max\{x_1, \dots, x_n\}$

$$\text{Proof } \lim_{\sigma \rightarrow \infty} M_\sigma = x_{\max} \lim_{\sigma \rightarrow \infty} \sum_i \alpha_i \left( \frac{x_i}{x_{\max}} \right)^\sigma \Big)^{\frac{1}{\sigma}} = x_{\max} \lim_{\sigma \rightarrow \infty} \text{const}^{\frac{1}{\sigma}} = x_{\max} \text{ where } \text{const} \in (0, 1)$$

- ▶ Leontieff if  $M_{-\infty}(\mathbf{x}) = x_{\min} = \min\{x_1, \dots, x_n\}$

$$\text{Proof: } M_{-\infty}(\mathbf{x}) = \frac{1}{M_\infty(1/\mathbf{x})}$$

# Production function: the benchmark case

Applying our previous concepts we find:

- ▶ Marginal product of input  $i$ : all inputs are productive

$$MP_i(\mathbf{x}) = F_i(\mathbf{x}) = \alpha_i \left( \frac{F(\mathbf{x})}{x_i} \right)^{1-\sigma} \geq 0$$

- ▶ Allen-Uzawa elasticities

- ▶ input  $i$  has decreasing  $MP_i$  if  $\sigma \leq 1$

$$\epsilon_{ii} = (1 - \sigma) \left( 1 - \frac{\alpha_i x_i^\sigma}{F(\mathbf{x})^\sigma} \right) \geq 0 \text{ if } \sigma < 1$$

- ▶ inputs  $i$  and  $j$  are
  - ▶ gross substitutes if  $\sigma < 1$
  - ▶ gross complements if  $\sigma > 1$

because

$$\epsilon_{ij} = (\sigma - 1) \frac{\alpha_j x_j^\sigma}{F(\mathbf{x})^\sigma}$$

# Production function: the benchmark case

Applying our previous concepts we find:

- ▶ Marginal rate of substitution depends only on the quantities of the two inputs

$$MRS_{ij}(\mathbf{x}) = \frac{\alpha_i}{\alpha_j} \left( \frac{x_j}{x_i} \right)^{1-\sigma}$$

- ▶ Elasticity of substitution is constant, for any pair of inputs

$$ES_{ij}(\mathbf{x}) = \frac{1}{1-\sigma}$$

then  $i$  and  $j$  are

- ▶ gross substitutes implies  $ES_{ij}(\mathbf{x}) > 0$
- ▶ gross complements implies  $ES_{ij}(\mathbf{x}) < 0$



# Production function and input prices

# Problem for a competitive firm

- ▶ **Assumption:** competitive product and input markets
- ▶ Total cost:

$$C(\mathbf{x}, \mathbf{w}) = \mathbf{w} \cdot \mathbf{x} = \sum_{i=1}^n w_i x_i$$

- ▶ Marginal cost of input  $i$

$$MC_i = \frac{\partial C(\mathbf{x}, \mathbf{w})}{\partial x_i}$$

- ▶ Profit:  $\pi(\mathbf{x}, \mathbf{w}) = F(\mathbf{x}) - C(\mathbf{x}, \mathbf{w}) = F(x_1, \dots, x_n) - \sum_{i=1}^n w_i x_i$
- ▶ The firm's primal problem (simpler problem)  
 $\pi^*(\mathbf{w}) = \max_{\mathbf{x}} \pi(\mathbf{x}, \mathbf{w})$  s.t.  $F(\mathbf{x}) \geq y$
- ▶ Optimum (interior) conditions: demand functions

$$\mathbf{x}^* = \{\mathbf{x} : F_i(\mathbf{x}) = w_i\} = X(\mathbf{w})$$

$$\text{or } MP_i = MC_i$$

# MRS and input prices

- ▶ Then, at the optimum the *MRS* is equal to the ratio of input returns

$$MRS_{ij} = \frac{F_i(\mathbf{x}^*)}{F_j(\mathbf{x}^*)} = \frac{w_i}{w_j}$$

- ▶ For the benchmark case

$$x_i^*(\mathbf{w}) = \left( \frac{w_i}{\alpha_i} \right)^{\frac{1}{\sigma-1}} F(\mathbf{x}^*)$$

- ▶ Then

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j} \left( \frac{x_j^*}{x_i^*} \right)^{1-\sigma} = \frac{w_i}{w_j}$$

## Production functions: types of inputs

# Types of inputs

We can distinguish between:

- ▶ intermediary goods and factors of production
- ▶ produced inputs and non-produced inputs
- ▶ exogenous inputs and endogenous inputs
- ▶ private and aggregate inputs

# Intermediate goods

- ▶ in a given production function: intermediate inputs enter as flows and factors enter as stocks in production functions

$$y = F(\mathbf{x}, \mathbf{z})$$

where

- ▶ intermediate inputs are products of other sectors and use factors of production: example  $x_i = f_i(\mathbf{z})$
- ▶ usually for the final use sector they are private goods (i.e., firms pay the full price for their use)
- ▶ they can be produced in a competitive or non-competitive market

# Factors of production

- ▶ in a given production function: factors enter as stocks in production functions

$$y = F(\mathbf{x}, \mathbf{z})$$

- ▶ factors of production are usually exogenous to the firm
- ▶ but they can be exogenous or endogenous to the economy
- ▶ when factors of production are produced, their output is a flow which generates a stock-flow dynamics

$$\dot{z}_i = \frac{dz_i(t)}{dt} = G(\mathbf{z})$$

- ▶ durable goods entail necessarily a dynamic mechanism (ex: capital stock)

# Factors of production

- ▶ The provision of factor of production can be internal or external to the firm
- ▶ The firm's level production function can be

$$y = f(\mathbf{z}, \mathbf{Z})$$

where  $\mathbf{z}$  are private factors and  $\mathbf{Z}$  are external factors

- ▶ their use faces different incentives:
  - ▶ if the factor of production is private the firm has to pay the price  $w_i$  for its use
  - ▶ if the factor of production is an externality the firm does not have to pay for its use
- ▶ their existence introduces a distinction between production functions at the firm level  $y = f(\mathbf{z}, \mathbf{Z})$  and at the aggregate level

$$y = F(\mathbf{Z}) = f(\mathbf{Z}, \mathbf{Z})$$

- ▶ then the previous properties can be different at the firm's level (related to the incentives) and at the aggregate level



# Production function and technological bias

# Production function in input intensity form

- ▶ Production function in efficiency form

$$y = F(\mathbf{A}, \mathbf{x}) \equiv F(A_1 x_1, \dots, A_n x_n)$$

where  $A_i$  input augmenting index measuring the specific productivity of input  $i$ ,

- ▶ In growth models:
  - ▶  $A_i$  measures specific or aggregate productivity increases
  - ▶  $A_i(t)$ , where  $t$  is time measures technical progress (or decay),
  - ▶  $A_i$  can be exogenous or endogenous (learning-by-doing, R&D)
- ▶ Here we introduce a first take to the subject

# Types of technical progress

- ▶ Consider the benchmark production function in intensity form

$$y = F(\mathbf{A}, \mathbf{x}) = \left( \sum_{i=1}^n \alpha_i (A_i x_i)^\sigma \right)^{\frac{1}{\sigma}}$$

- ▶ Assume the benchmark production function

$$MP_i = \alpha_i A_i^\sigma \left( \frac{F(\mathbf{A}, \mathbf{x})}{x_i} \right)^{1-\sigma}$$

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j} \left( \frac{A_i}{A_j} \right)^\sigma \left( \frac{x_j}{x_i} \right)^{1-\sigma}$$

- ▶ The effect of the technical progress on production depends on:
  - ▶ the vector  $\mathbf{A}$ , in particular if it is equal for all inputs: Hicks neutral technical progress  $A_i = A$  for all  $i$
  - ▶ on the substitutability properties of the production function, if  $\mathbf{A}$  is heterogeneous

# Types of technical progress

There are several concepts of neutrality and bias in technical progress. Here we consider

- ▶ **neutral** technical progress: if the change in any  $A_i$  leaves the  $MRS_{ij}$  unchanged
- ▶ **biased** technical progress: if the change in any  $A_i$  changes the  $MRS_{ij}$ .
- ▶ If we consider input prices

$$MRS_{ij} = \frac{\alpha_i}{\alpha_j} \left( \frac{A_i}{A_j} \right)^\sigma \left( \frac{x_j}{x_i} \right)^{1-\sigma} = \frac{w_i}{w_j}$$

- ▶ In the above sense
  - ▶ if  $\sigma = 0$  then the technical progress is neutral in this sense
  - ▶ if  $\sigma \neq 0$  then the technical progress is biased
- ▶ However, observe that the demand functions  $x_i = X_i(\mathbf{A}, \mathbf{w})$  can depend on  $\mathbf{A}$ . If  $\sigma = 0$  we can interpret this as an income effect.

# Types of technical progress

- ▶ We can also write

$$\frac{\alpha_i}{\alpha_j} \left( \frac{A_i x_i}{A_j x_j} \right)^\sigma = \frac{w_i x_i}{w_j x_j}$$

- ▶ Assume that the expenditures in inputs (or factor shares in national income) is constant and  $\sigma \neq 0$ , then:
  - ▶ if the technical progress is **neutral** the ratio of the inputs remains constant;
  - ▶ if the technical progress is biased an increase in  $A_i/A_j$  the technical progress is ***i*-saving**, i.e, there is a reduction in its quantity; the ration of the two inputs changes.

## Extensions 1: continuum of inputs

# Continuum of inputs

- ▶ Dixit-Stiglitz production functions consider a continuum of inputs which makes  $y$  a functional

$$y = F[x] = \left( \int_0^N \alpha(i) x(i)^\sigma di \right)^{\frac{1}{\sigma}}$$

- ▶ How to calculate  $MP(i)$  ?
- ▶ we use the functional derivative

$$MP(i) = \frac{\delta F[x]}{\delta x(i)} = \alpha(i) \left( \frac{F[x]}{x(i)} \right)^{1-\sigma}$$

- ▶ all the concepts of production theory can be adapted to this case.
- ▶ in particular

$$MRS(i, j) = \frac{\alpha(i)}{\alpha(j)} \left( \frac{x(j)}{x(i)} \right)^{1-\sigma}$$

and

$$ES(i, j) = \frac{1}{1-\sigma}$$

## Extensions 2: multisector economies



# Multisector economies

- If we consider the existence of  $m$  production sectors, in this case we have an input-output structure

$$\mathbf{y} = \mathbf{F}(\mathbf{x})$$

where now  $\mathbf{x}$  is a  $(m \times n)$  vector

$$\begin{pmatrix} y_1 \\ \dots \\ y_m \end{pmatrix} = \begin{pmatrix} F_1(x_{11}, \dots, x_{1n}) \\ \dots \\ F_m(x_{m1}, \dots, x_{mn}) \end{pmatrix} =$$

- In these economies a vector of prices is also required,  $\mathbf{p}(p_1, \dots, p_m)^\top$ . We can all the previous concepts component-wise, that is for every sector

# Multisector economies

- We can find the optimal allocation as a solution of the problem

$$\max_{x_{11}, \dots, x_{mn}} \sum_{i=1}^n p_i y_i(\mathbf{x}_i) - \sum_{j=1}^n w_j x_j, \text{ st } \sum_{i=1}^n x_{ij} = x_j$$

- under some conditions, we can find, at the optimum a relationship as

$$y_j = F^j(\mathbf{x})$$

a supply function for every sector as a function of the aggregate input.

# Production function and input accumulation and dynamics

# Investment and savings

- ▶ Assume there is only one factor of production and there are no intermediate goods

$$y = f(x)$$

- ▶ Assume that the good produce is durable and can be used both for consumption and investment, then

$$y = c + \dot{x}$$

- ▶ Let savings be a function of  $x$ ,  $s = s(x) = y - c$
- ▶ Then we have a stock-flow dynamics where

$$\dot{x} = \frac{dx(t)}{dt} = s(x(t))$$

- ▶ The solution to this differential equation, gives

$$x(t) = x_0 + \int_0^t s(x(\tau)) d\tau$$

# Types of growth dynamics and the production function

# Growth dynamics

- ▶ From this solution we can obtain the dynamics of product from

$$y(t) = f(x(t))$$

- ▶ We can also set directly an ODE on the GDP

$$\dot{y} \equiv \frac{dy(t)}{dt} = \mu(y)$$

- ▶ We say **there is long run growth** if the solution to this equation tends asymptotically to an exponential

$$\lim_{t \rightarrow \infty} y(t) \propto e^{\gamma t}, \quad \gamma > 0$$

- ▶ we will see that this requires the **technology to be linear at the aggregate level**. V.g:  $y = Ak$

# Growth theory: on the exponential structure of growth

- ▶ Long run growth exits for a particular mathematical structure of  $\mu(y)$ 
  - ▶ **logistic** growth:  $\mu(y) = \alpha y(\beta - y)$ ,
  - ▶ **exponential** growth:  $\mu(y) = \gamma y$ ,
  - ▶ **power law** growth:  $\mu(y) = y^\phi$  for  $\phi > 1$ ,
- ▶ **razor edge property of growth models**: although the exponential case is very particular it this the structure underlying (almost) all growth theories

# Growth theory: on the exponential structure of growth

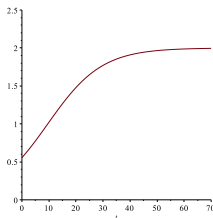


Figure: Logistic growth  $\mu(y) = \alpha y(\beta - y)$

- ▶ there is short run (transition) growth
- ▶ but there is no long-run growth



# Growth theory: on the exponential structure of growth

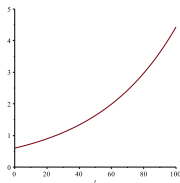


Figure: Exponential growth  $\mu(y) = \gamma y$

- ▶ there is no short run (transition) growth
- ▶ but there is long-run growth
- ▶ gdp becomes infinite ( $y(t) \rightarrow \infty$ ) in **infinite** time

# Growth theory: on the exponential structure of growth

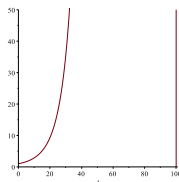


Figure: Power law growth  $\mu(y) = y^\phi$  for  $\phi > 1$

- ▶ GDP becomes infinite ( $y(t) \rightarrow \infty$ ) in **finite** time
- ▶ and collapses afterwards

# Growth and transition dynamics

- In order to have both long run growth and transition dynamics we need to have at least two durable goods

$$\dot{x}_1 = s_1(x_1, x_2)$$

$$\dot{x}_2 = s_2(x_1, x_2)$$

- We have long run growth if we can find a **balanced growth path**, i.e., a solution of type

$$x_1(t) = \phi_1(t) e^{\gamma t},$$

$$x_2(t) = \phi_2(t) e^{\gamma t},$$

where  $\gamma$  is the long run growth rate and  $\phi_i(\phi_1(t), \phi_2(t))$  are the transition components;

- This also requires a particular structure of the production functions: they should be CRS (at the aggregate level) for every sector.