Mathematical Economics

Deterministic dynamic optimization

Discrete time

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1 Introduction

In this note we introduce deterministic dynamic optimization problems and three methods for solving them.

Deterministic dynamic programming deals with finding deterministic sequences $x = \{x_0, x_1, \dots, x_T\}$ that maximize (or minimise) a given intertemporal criterium such that some known conditions are verified.

1.1 Deterministic and optimal sequences

Consider the time set $\mathcal{T} = \{0, 1, \dots, t, \dots, T\}$ where T can be finite or $T = \infty$. We denote the value of variable at time t, by x_t. That is x_t is a mapping $x : \mathcal{T} \to \mathbb{R}$.

The timing of the variables differ: if x_t can be measured at **instant** t we call it a state variable, if u_t takes values in **period** t, which takes place between instants t and $t+1$ we call it a control variable.

Usually, stock variables (both prices and quantities) refer to instants and flow variables (prices and quantities) refer to periods.

A dynamic model is characterised by the fact that sequences have some form of intertemporal time-interaction. We distinguish intratemporal from intertemporal relations. Intratemporal, or period, relations take place within a single period and intertemporal relations involve trajectories.

A trajectory or path for state variables starting at $t = 0$ with the horizon $t = T$ is denoted by $x = \{x_0, x_1, \dots, x_T\}$. We denote the trajectory starting at $t > 0$ by ${}^t x = \{x_t, x_{t+1}, \dots, x_T\}$, and the whole trajectory up until time ${}^t x = \{x_0, x_1, \dots, x_t\}$.

We consider two types of problems:

1. calculus of variations problems: feature sequences of state variables and evaluate these

sequences by an intertemporal objective function directly

$$J(x) = \sum_{t=0}^{T-1} F(t, x_t, x_{t+1})$$

2. optimal control problems: feature sequences of state and control variables, which are related by a sequence of intratemporal relations

$$x_{t+1} = G(x_t, u_t, t) \tag{1}$$

and evaluate these sequences by an intertemporal objective function over sequences (x, u)

$$J(x, u) = \sum_{t=0}^{T-1} F(t, x_t, u_t)$$

From equation (1) and the value of the state x_t at some points in time we could also determine an intertemporal relation[1].

In a deterministic dynamic model there is full information over the state x_t or the path x^t for any $t > s$ if we consider information at time s.

In general we have some conditions over the value of the state at time $t = 0$, x_0 and we may have other restrictions as well. The set of all trajectories x verifying some given conditions is denoted by $\mathcal{X}$. In optimal control problems the restrictions may involve both state and control sequence, x and u. In this case we denote the domain of all trajectories by $\mathcal{D}$

Usually $\mathcal{X}$, or $\mathcal{D}$, have infinite number of elements. Deterministic dynamic optimisation problems consist in finding the optimal sequences $x^\star \in \mathcal{X}$ (or $(x^\star, u^\star) \in \mathcal{D}$).

[1] In economics the concept of sustainability is associated to meeting those types of intertemporal relations.

1.2 Some history

The calculus of variations problem is very old: Dido's problem, brachistochrone problem (Galileo), catenary problem and has been solved in some versions by Euler and Lagrange (XVII century) (see Liberzon (2012)). The solution of the optimal control problem is due to Pontryagin et al. (1962). The dynamic programming method for solving the optimal control problem has been first presented by Bellman (1957).

1.3 Types of problems studied next

The problems we will study involve maximizing an intertemporal objective function (which is mathematically a **functional**) subject to some restrictions:

1. the **simplest calculus of variations problem**: we want to find a path $\{x_t\}_{t=0}^{T}$, where T, and both the initial and the terminal values of the state variable are known, $x_0 = \phi_0$ and $x_T = \phi_T$, such that it maximizes the functional $\sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$. Formally, the problem is: find a trajectory for the state of the system, $\{x_t^*\}_{t=0}^{T}$, that solves the problem

$$\max_{\{x_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t), \ s.t. \ x_0 = \phi_0, \ x_T = \phi_T$$

   where ϕ_0, ϕ_T and T are given;

2. **calculus of variations problem with a free endpoint**: this is similar to the previous problem with the difference that the terminal state x_T is free. Formally:

$$\max_{\{x_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t), \ s.t. \ x_0 = \phi_0, \ x_T \text{ free}$$

   where ϕ_0 and T are given;

3. the **optimal control problem with given terminal state**: we assume there are two types of variables, control and state variables, represented by u and x which are

related by the difference equation $x_{t+1} = g(x_t, u_t)$. We assume that the initial and the terminal values of the state variable are known $x_0 = \phi_0$ and $x_T = \phi_T$ and we want to find an optimal trajectory joining those two states such that the functional $\sum_{t=0}^{T-1} F(u_t, x_t, t)$ is maximized by choosing an optimal path for the control.

Formally, the problem is: find a trajectories for the control and the state of the system, $\{u_t^*\}_{t=0}^{T-1}$ and $\{x_t^*\}_{t=0}^{T}$, which solve the problem

$$\max_{\{u_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} F(u_t, x_t, t),\ s.t.\ x_{t+1} = g(x_t, u_t),\ t = 0, \dots T-1,\ x_0 = \phi_0,\ x_T = \phi_T$$

where ϕ_0, ϕ_T and T are given;

4. the **optimal control problem with free terminal state**: find a trajectories for the control and the state of the system, $\{u_t^*\}_{t=0}^{T-1}$ and $\{x_t^*\}_{t=0}^{T}$, which solve the problem

$$\max_{\{u_t\}_{t=0}^{T}} \sum_{t=0}^{T-1} F(u_t, x_t, t),\ s.t.\ x_{t+1} = g(x_t, u_t),\ t = 0, \dots T-1,\ x_0 = \phi_0,\ x_T = \phi_T$$

   where ϕ_0 and T are given.

5. in macroeconomics the **infinite time discounted optimal control problem** is the most common: find a trajectories for the control and the state of the system, $\{u_t^*\}_{t=0}^{\infty}$ and $\{x_t^*\}_{t=0}^{\infty}$, which solve the problem

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{T-1} \beta^t f(u_t, x_t),\ s.t.\ x_{t+1} = g(x_t, u_t),\ t = 0, \dots \infty,\ x_0 = \phi_0,$$

   where $\beta \in (0, 1)$ is a discount factor and ϕ_0 is given. The terminal condition $\lim_{t\to} \eta^t x_t \geq 0$ is also frequently introduced, where $0 < \eta < 1$.

There are three methods for finding the solutions: (1) calculus of variations, for the first two problems, which is the reason why they are called calculus of variations problems, and (2) maximum principle of Pontriyagin and (3) dynamic programming, which can be used for all the five types of problems.

1.4 Some economic applications

The cake eating problem : let W_t be the size of a cake at instant t. If we eat C_t in period t, the size of the cake at instant $t+1$ will be $W_{t+1} = W_t - C_t$. We assume we know that the cake will last up until instant T. We evaluate the bites in the case by the intertemporal utility function featuring impatience, positive but decreasing marginal utility

$$\sum_{t=0}^{T-1} \beta^t u(C_t)$$

If the initial size of the cake is ϕ_0 and we want to consume it all until the end of period $T-1$ what will be the best eating strategy ?

The consumption-investment problem : let W_t be the financial wealth of a consumer at instant t. The intratemporal budget constraint in period t is

$$W_{t+1} = Y_t + (1+r)W_t - C_t,\ t = 0, 1, \ldots, T-1$$

where Y_t is the labour income in period t and r is the asset rate of return. The consumer has financial wealth W_0 initially. The consumer wants to determine the optimal consumption and wealth sequences $\{C_t\}_{t=0}^{T-1}$ and $\{W_t\}_{t=0}^{T}$ that maximises his intertemporal utility function

$$\sum_{t=0}^{T-1} \beta^t u(C_t)$$

where T can be finite or infinite.

The AK model growth model: let K_t be the stock of capital of an economy at time and consider the intratemporal aggregate constraint of the economy in period t

$$K_{t+1} = (1+A)K_t - C_t$$

where $F(K_t) = AK_t$ is the production function displaying constant marginal returns. Given the initial capital stock K_0 the optimal growth problem consists in finding the trajectory

$\{K_t\}_{t=0}^{\infty}$ that maximises the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to a boundedness constraint for capital. The Ramsey (1928) model is a related model in which the production function displays decreasing marginal returns to capital.

2 Calculus of Variations

A **calculus of variations problem** consists in finding a trajectory, or a sequence, for a state variable that maximizes or minimizes a value functional. There are several versions of the problem depending on additional information, for instance, regarding the level of the state variable, the timing of the problem, and constraints on the state variable.

In most applications in economics, for instance macroeconomics and growth theory, time is the independent variable. Time t belongs to the set $\mathcal{T} = \{0, 1, \dots, T\}$, where T can be finite or infinite. We have to distinguish **time**, or moments in time, from periods. A **period** t is the interval between times t and $t+1$. This distinction is important, in discrete time optimization models, because the state variables are usually stock variables, that are measured in time (instants), while flow variables are measured in periods.

The level of the **state** variable at time t is denoted by x_t. The history of the state variable until time t is denoted by $x^t = \{x_0, x_1, \dots, x_t\}$, the future path of the state variable is denoted by ${}^t x = \{x_t, x_{t+1}, \dots, x_T\}$ and the whole **trajectory** is denoted by $x = \{x_t\}_{t=0}^{T} \equiv \{x_0, x_1, \dots, x_t, \dots, x_T\}$.

The main ingredient that makes **dynamic** the optimization problem is the fact that the valuation of the state variable is made within periods. This means that we take into account the **change** of the state variable within a period, or between the beginning and the end of a period. The outcome of an action, consisting in changing the value of the state in period t, is measured by the **value function for period** t, $F_t = F(x_{t+1}, x_t, t)$, where we the valuation

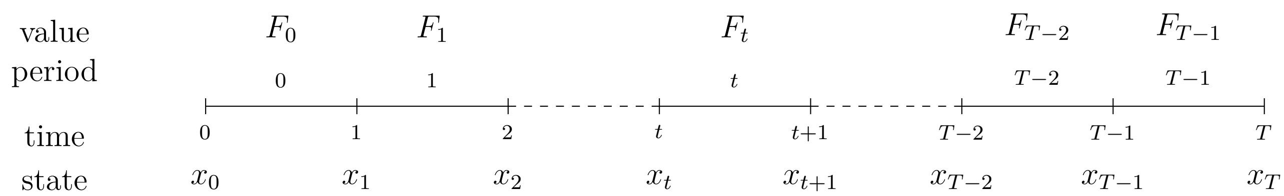

Figure 1: Timing for the state variables and the objective function where $F_t = F(x_{t+1}, x_t, t)$

can be period specific. Figure 1 shows the relationship between timing, the level of the state variable, and the value of an action over the state variable.

In a calculus of variations problem we are interested in finding the **best trajectory**, $x = \{x_0, \dots, x_T\}$. In order to be able to formalize the problem we have to put forward a measure for valuing trajectories: a **value functional**. A functional is a mapping from trajectories to numbers, that is, it assigns one number to a specific trajectory[2] This allows for establishing an order relationship between trajectories and allows for choosing the best one.

In this note we consider the simplest value functional, which is a benchmark in Economics, in which there is **additive separability** across time. That is, we value a trajectory by summing the period-specific values for actions over the state variable

$$V[x] = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t). \tag{2}$$

Therefore, our goal is to find the best trajectory for the state variable, $x^* = \{x^*_0, x^*_1, \dots, x^*_T\}$ that has the **maximum value**, i.e.

$$V(x^*) = \max_x \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$$

Throughout this note we assume that the value function, $F(x^{'}, x)$, is continuous, differentiable in $(x^{'}, x)$, and, almost always, concave.

Next we start with the simplest calculus of variations (CV) problem, in section 2.1,

[2]By th eway, a function $y = f(x)$ is a mapping between a number, x, and another number, y.

2.1 The simplest problem

The **simplest calculus of variations problem** starts by assuming that the initial and the terminal times, $t = 0$ and $t = T$, are known, and the levels of the state variable at those times, x_0 and x_T, and the problem consists in finding the best trajectory between those two levels.

Definition 1. ***The simplest CV problem*** *Find $x = \{x_t\}_{t=0}^T$ that solves*

$$\max_x \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \text{ subject to } x_0 = \phi_0 \text{ and } x_T = \phi_T \tag{CV1}$$

where T, ϕ_0 and ϕ_T are given.

Observations: First, expanding the value functional in equation (2) we have

$$\begin{aligned} V[x] &= \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \\ &= F(x_1, x_0, 0) + F(x_2, x_1, 1) + \ldots + F(x_t, x_{t-1}, t-1) + F(x_{t+1}, x_t, t) + \ldots \\ &\quad \ldots + F(x_{T-1}, x_{T-2}, T-2) + F(x_T, x_{T-1}, T-1). \end{aligned}$$

Second, the time index of the functional refers to periods, which means that if T is the last time, the upper limit of the functional should be $T - 1$, if we chose to index periods by the initial time.

We denote the solution of the calculus of variations problem by $\{x_t^*\}_{t=0}^T$. Therefore, if we know the solution the **optimal value** is a number

$$V^* \equiv V[x^*] = \sum_{t=0}^{T-1} F(x_{t+1}^*, x_t^*, t) = \max_x \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t),$$

In addition $V^* = V(\phi_0, \phi_T)$, that is, it is a function of the fixed values of the state variable.

Proposition 1. *(First order necessary condition for optimality) Let $\{x_t^*\}_{t=0}^T$ be a solution of problem* (CV1)*. Then it satisfies the Euler-Lagrange condition*

$$\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} = 0, \quad t = 1, 2, \ldots, T-1 \tag{3}$$

and the initial and the terminal conditions

$$\begin{cases} x_0^* = \phi_0, & t = 0 \\ x_T^* = \phi_T, & t = T. \end{cases}$$

Proof. Assume that we know the optimal solution $x^* = \{x_t^*\}_{t=0}^T$. Therefore, we also know the optimal value $V[x^*] = \sum_{t=0}^{T-1} F(x_{t+1}^*, x_t^*, t)$. Consider an alternative candidate path as a solution of the problem, $\{x_t\}_{t=0}^{T-1}$ such that $x_t = x_t^* + \varepsilon_t$. We denote the perurbation by $\varepsilon = \{\varepsilon\}_{t=0}^T$.

In order to be admissible, the perturbation has to verify the restrictions of the problem: $\varepsilon_0 = \varepsilon_T = 0$. All the other values are arbitrary, which means that we choose $\varepsilon_t \neq 0$ for $t = 1, \ldots, T-1$. Therefore, the alternative candidate solution has the same initial and terminal values as the optimal solution, although following a different path between the initial and the terminal fixed levels, $x_0^* = \phi_0$ and $x_T^* = \phi_T$.

The value of the perturbed trajectory is

$$V[x] = V[x^* + \varepsilon] = \sum_{t=0}^{T-1} F(x_{t+1}^* + \varepsilon_{t+1}, x_t^* + \varepsilon_t, t).$$

The change in value by perturbing the solution to the problem, that is variation of the value functional introduced by the perturbation is

$$\Delta V[\varepsilon] = V[x^* + \varepsilon] - V[x^*] = \sum_{t=0}^{T-1} \left[F(x_{t+1}^* + \varepsilon_{t+1}, x_t^* + \varepsilon_t, t) - F(x_{t+1}^*, x_t^*, t) \right].$$

At the optimum $\Delta V[\varepsilon] = 0$.

Next, we start by determining the change in value for any perturbation, then we determine the change in value for an admissible perturbation, and, at last we find the conditions such that $\Delta V[\varepsilon] = 0$. As $F(.)$ is differentiable, we can use a first order Taylor approximation, evaluated along the trajectory $\{x_t^*\}_{t=0}^T$, to find the variation in the value functional for any perurbation

$$\begin{aligned}
\Delta V = & \frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_0}(x_0 - x_0^*) + \left(\frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_1} + \frac{\partial F(x_2^*, x_1^*, 1)}{\partial x_1}\right)(x_1 - x_1^*) + \dots \\
& \dots + \left(\frac{\partial F(x_{T-1}^*, x_{T-2}^*, T-2)}{\partial x_{T-1}^*} + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_{T-1}}\right)(x_{T-1} - x_{T-1}^*) + \\
& + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T}(x_T - x_T^*) = \\
& = \frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_0}\varepsilon_0 + \left(\frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_1} + \frac{\partial F(x_2^*, x_1^*, 1)}{\partial x_1}\right)\varepsilon_1 + \dots \\
& \dots + \left(\frac{\partial F(x_{T-1}^*, x_{T-2}^*, T-2)}{\partial x_{T-1}^*} + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_{T-1}}\right)\varepsilon_{T-1} + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_{T-1}}\varepsilon_T
\end{aligned}$$

Then, for any perturbation the variation is

$$\Delta V = \frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_0}\varepsilon_0 + \sum_{t=1}^{T-1}\left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t}\right)\varepsilon_t + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_{T-1}}\varepsilon_T. \tag{4}$$

For an admissible perturbation, such that $\varepsilon_0 = \varepsilon_T = 0$ we have

$$\Delta V = \sum_{t=1}^{T-1}\left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t}\right)\varepsilon_t. \tag{5}$$

Therefore, $\{x_t\}_{t=0}^{T-1}$ is an optimal trajectory only if $V(x) - V(x^*) = 0$, which holds if and only if (3) is satisfied. □

Interpretation: equation (3) is an **intratemporal arbitrage condition** for period t. The optimal sequence has the property that at every period marginal benefits (from increasing one unit of x_t) are equal to the marginal costs (from sacrificing one unit of x_{t+1}):

Observations

- equation (3) is a non-linear difference equation of the second order: if we set

$$\begin{cases} y_{1,t} = & x_t \\ y_{2,t} = & x_{t+1} = y_{1,t+1}. \end{cases}$$

then the Euler Lagrange equation can be written as a planar equation in $\mathbf{y}_t = (y_{1,t}, y_{2,t})$

$$\begin{cases} y_{1,t+1} = y_{2,t} \\ \frac{\partial}{\partial y_{2,t}} F(y_{2,t}, y_{1,t}, t-1) + \frac{\partial}{\partial y_{2t}} F(y_{2,t+1}, y_{2,t}, t) = 0 \end{cases}$$

- if we have a **minimization problem** we can transform it into a maximization problem, like problem (CV1), by taking the symmetric to the value function

$$\min_y \sum_{t=0}^{T-1} F(y_{t+1}, y_t, t) = \max_y \sum_{t=0}^{T-1} -F(y_{t+1}, y_t, t)$$

Sufficient conditions The optimality conditions in Proposition 1 are only necessary. They are also sufficient if the value function, $F(x_{t+1}, x_t, t)$, is concave in (x_{t+1}, x_t).

Example 1: Let $F(x_{t+1}, x_t) = -(x_{t+1} - x_t/2)^2$, the terminal time be $T = 4$, and the state constraints $x_0 = x_4 = 1$. Solve the calculus of variations problem.

Solution: If we apply the Euler-Lagrange equation we get a second order difference equation which is verified by the optimal solution

$$\frac{\partial}{\partial x_t}\left[-\left(x_t - \frac{x_{t-1}}{2}\right)^2\right] + \frac{\partial}{\partial x_t}\left[-\left(x_{t+1} - \frac{x_t}{2}\right)^2\right] = 0,$$

that is

$$-2x_t + x_{t-1} + x_{t+1} - \frac{x_t}{2} = 0$$

If we introduce a time-shift, we get the equivalent Euler equation

$$x_{t+2} = \frac{5}{2}x_{t+1} - x_t,\ t = 0, \dots, T-2$$

Therefore, the solution to the problem, $\{x_t^*\}_{t=0}^4$, solves the following initial-terminal value problem

$$\begin{cases} x_{t+2}^* = \frac{5}{2}x_{t+1}^* - x_t^*,\ t = 0, \dots, 2 \\ x_0 = 1 \\ x_4 = 1. \end{cases} \tag{6}$$

Example 1: Let $F(x_{t+1}, x_t) = -(x_{t+1} - x_t/2 - 2)^2$, the terminal time $T = 4$, and the state constraints $x_0 = x_4 = 1$. Solve the calculus of variations problem.

Solution: If we apply the Euler-Lagrange equation we get a second order difference equation which is verified by the optimal solution

$$\frac{\partial}{\partial x_t}\left[-\left(x_t - \frac{x_{t-1}}{2} - 2\right)^2\right] + \frac{\partial}{\partial x_t}\left[-\left(x_{t+1} - \frac{x_t}{2} - 2\right)^2\right] = 0,$$

evaluated along $\{x_t^*\}_{t=0}^4$.

Then, we get

$$-2x_t^* + x_{t-1}^* + 4 + x_{t+1}^* - \frac{x_t^*}{2} - 2 = 0$$

If we introduce a time-shift, we get the equivalent Euler equation

$$x_{t+2}^* = \frac{5}{2}x_{t+1}^* - x_t^* - 2,\ t = 0, \ldots, T-2$$

which together with the initial condition and the terminal conditions constitutes a mixed initial-terminal value problem,

$$\begin{cases} x_{t+2}^* = \frac{5}{2}x_{t+1}^* - x_t^* - 2,\ t = 0, \ldots, 2 \\ x_0 = 1 \\ x_4 = 1. \end{cases} \tag{7}$$

In order to solve problem (7) we follow the method:

1. First, solve the Euler equation, whose solution is a function of two unknown constants (k_1 and k_2 next)

2. Second, we determine the two constants (k_1, k_2) by using the initial and terminal conditions.

First step: solving the Euler equation We can use one of the next two methods for solving the Euler equation: (1) by direct methods, using equation (**??**) in the Appendix, or (2) solve it generally by transforming it to a first order difference equation system.

Method 1: applying the solution for the second order difference equation (??) **(see the Appendix)** :

Applying the results we derived in the Appendix for the second order difference equations we get:

$$x_t = 4 + \left(-\frac{1}{3}2^t + \frac{4}{3}\left(\frac{1}{2}\right)^t\right)(k_1 - 4) + \left(\frac{2}{3}2^t - \frac{2}{3}\left(\frac{1}{2}\right)^t\right)(k_2 - 4). \tag{8}$$

Method 2: general solution for the second order difference equation We follow the method:

1. First, we transform the second order equation into a planar equation by using the transformation $y_{1,t} = x_t$, $y_{2,t} = x_{t+1}$. The solution will be a known function of two arbitrary constants, that is $y_{1,t} = \varphi_t(k_1, k_2)$.

2. Second, we apply the inverse transformation $x_t = y_{1,t} = \varphi_t(k_1, k_2)$ which is function of two constants (k_1, k_2)

The equivalent planar system in $y_{1,t}$ and $y_{2,t}$ is

$$\begin{cases} y_{1,t+1} = y_{2,t} \\ y_{2,t+1} = \frac{5}{2}y_{2,t} - y_{1,t} - 2 \end{cases}$$

which is equivalent to a planar system of type $\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}$ where

$$\mathbf{y}_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 5/2 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

The solution of the planar system is $\mathbf{y}_t = \overline{\mathbf{y}} + \mathbf{P}\mathbf{\Lambda}^t\mathbf{P}^{-1}(k - \overline{\mathbf{y}})$ where $\overline{\mathbf{y}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ that is

$$\overline{y} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

and

$$\mathbf{\Lambda} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \; \mathbf{P} = \begin{pmatrix} 1/2 & 2 \\ 1 & 1 \end{pmatrix}, \; \mathbf{P}^{-1} = \begin{pmatrix} -2/3 & 4/3 \\ 2/3 & -1/3 \end{pmatrix}.$$

Then

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}2^t & 2\left(\frac{1}{2}\right)^t \\ 2^t & \left(\frac{1}{2}\right)^t \end{pmatrix} \begin{pmatrix} -\frac{2}{3}(k_1-4) + \frac{4}{3}(k_2-4) \\ \frac{2}{3}(k_1-4) - \frac{1}{3}(k_2-4) \end{pmatrix}$$

If we substitute in the equation for $x_t = y_{1,t}$ and take the first element we have, again, the general solution of the Euler equation (8).

Second step: particular solution In order to determine the (particular) solution of the CV problem we take the general solution of the Euler equation (8), and determine k_1 and k_2 by solving the system $x_t|_{t=0} = 1$ and $x_t|_{t=4} = 1$:

$$\begin{aligned} 4 + 1 \times (k_1 - 4) + 0 \times (k_2 - 4) &= 1 && (9) \\ 4 + \left(-\frac{1}{3}2^4 + \frac{4}{3}\left(\frac{1}{2}\right)^4\right)(k_1-4) + \left(\frac{2}{3}2^4 - \frac{2}{3}\left(\frac{1}{2}\right)^4\right)(k_2-4) &= 1 && (10) \end{aligned}$$

Then we get $k_1 = 1$ and $k_2 = 38/17$. If we substitute in the solution for x_t, we get

$$x_t^* = 4 - \frac{3}{17}2^t - \frac{48}{17}(1/2)^t$$

Therefore, the solution for the calculus of variations problem is the sequence

$$x^* = \{x^*\}_{t=0}^4 = \left\{1, \frac{38}{17}, \frac{44}{17}, \frac{38}{17}, 1\right\}.$$

Example 2: The cake eating problem Assume that there is a cake whose size at the beginning of period t is denoted by W_t and there is a muncher who wants to eat it until the beginning of period T. The initial size of the cake is $W_0 = \phi$ and, off course, $W_T = 0$ and the eater takes bites of size C_t at period t. The eater evaluates the utility of its bites through a logarithmic utility function and has a psychological discount factor $0 < \beta < 1$. What is the optimal eating strategy ?

Formally, the problem is to find the optimal paths $C^* = \{C_t^*\}_{t=0}^{T-1}$ and $W^* = \{W_t^*\}_{t=0}^{T}$ that solve the problem

$$\max_{\{C\}} \sum_{t=0}^{T-1} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = W_t - C_t,\ W_0 = \phi,\ W_T = 0. \tag{11}$$

This problem can be transformed into the calculus of variations problem, because $C_t = W_t - W_{t+1}$,

$$\max_{W} \sum_{t=0}^{T-1} \beta^t \ln(W_t - W_{t+1}), \text{ subject to } W_0 = \phi,\ W_T = 0.$$

The Euler-Lagrange condition is:

$$-\frac{\beta^{t-1}}{W_{t-1}^* - W_t^*} + \frac{\beta^t}{W_t^* - W_{t+1}^*} = 0.$$

Then, the first order conditions are:

$$\begin{cases} W_{t+2}^* = (1+\beta)W_{t+1}^* - \beta W_t^*,\ t = 0, \dots T-2 \\ W_0 = \phi \\ W_T = 0 \end{cases}$$

In the appendix we find the solution of this linear second order difference equation (see equation (54))

$$W_t^* = \frac{1}{1-\beta}\left(-\beta k_1 + k_2 + (k_1 - k_2)\beta^t\right),\ t = 0, 1 \dots, T \tag{12}$$

which depends on two arbitrary constants, k_1 and k_2. We can evaluate them by using the initial and terminal conditions

$$\begin{cases} W_0^* &= \frac{1}{1-\beta}\left(-\beta k_1 + k_2 + (k_1 - k_2)\right) = \phi \\ W_T^* &= -\beta k_1 + k_2 + (k_1 - k_2)\beta^T = 0. \end{cases}$$

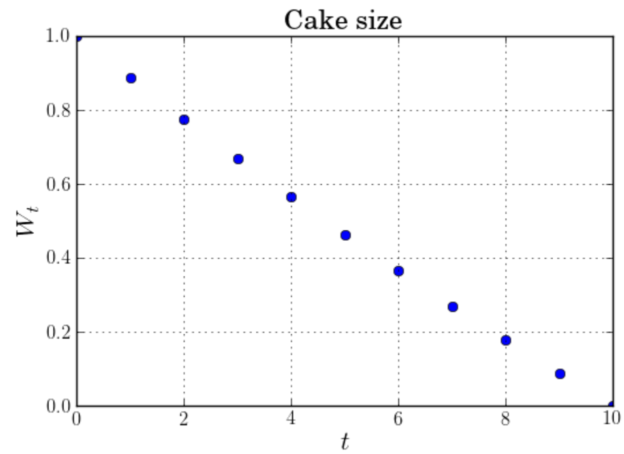

Figure 2: Solution to the cake eating problem with $T = 10$, $\phi_0 = 1$, $\phi_T = 0$ and $\beta = 1/1.03$

Solving this linear system for k_1 and k_2, we get:

$$k_1 = \phi, \quad k_2 = \frac{\beta - \beta^T}{1 - \beta^T}\phi$$

Therefore, the solution for the cake-eating problem $\{C^*\}$, $\{W^*\}$ is generated by

$$W_t^* = \left(\frac{\beta^t - \beta^T}{1 - \beta^T}\right)\phi,\ t = 0, 1, \dots T \tag{13}$$

and, as $C_t^* = W_t^* - W_{t+1}^*$

$$C_t^* = \left(\frac{1 - \beta}{1 - \beta^T}\right)\beta^t\phi,\ t = 0, 1, \dots T - 1. \tag{14}$$

2.2 Free terminal state problem

Now let us consider the problem

Definition 2. ***The free terminal state CV problem*** *Find $x = \{x_t\}_{t=0}^T$ that solves*

$$\max_x \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \textit{subject to } x_0 = \phi_0 \textit{ and } x_T \textit{ is free} \tag{CV2}$$

where T, and ϕ_0 are given.

Proposition 2. *(Necessary condition for optimality for the free end point problem* (CV2)*)* *Let $\{x_t^*\}_{t=0}^T$ be a solution for the problem (CV2). Then it verifies the Euler-Lagrange condition*

$$\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} = 0, \quad t = 1, 2, \ldots, T-1 \tag{15}$$

and the initial and the transversality conditions

$$\begin{aligned} x_0^* &= \phi_0, \ t = 0 \\ \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} &= 0, \ t = T. \end{aligned} \tag{16}$$

Proof. Again we assume that we know $x^* = \{x_t^*\}_{t=0}^T$ and $V(x^*)$, and we use the same method as in the proof for the simplest problem. However, instead of equation (5) the variation introduced by the perturbation $\{\varepsilon_t\}_{t=0}^T$ is

$$V(x) - V(x^*) = \sum_{t=1}^{T-2} \left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} \right) \varepsilon_t + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} \varepsilon_T$$

because $x_T = x_T^* + \varepsilon_T$ and $\epsilon_T \neq 0$ because the terminal state is not given. Then $V(x) - V(x^*) = 0$ if and only if the Euler and the transversality (16) conditions are verified. □

Condition (16) is called the **transversality condition**. Its meaning is the following: if the terminal state of the system is free, it would be optimal if there is no gain in changing the solution trajectory as regards the horizon of the program. If $\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} > 0$ then we could improve the solution by increasing x_T^* (remember that the utiity functional is additive along time) and if $\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} < 0$ we have an non-optimal terminal state by excess.

Example 1 (bis) Consider Example 1 and take the same objective function and initial state but assume instead that x_4 is free. In this case we have the terminal condition associated to the optimal terminal state,

$$2x_4^* - x_3^* - 4 = 0.$$

If we substitute the values of x_4 and x_3, from equation (8), we get the equivalent condition $-32 + 8k_1 + 16k_2 = 0$. This condition together with the initial condition, equation equation (9), allow us to determine the constants k_1 and k_2 as $k_1 = 1$ and $k_2 = 5/2$. If we substitute in the general solution, equation (8), we get $x_t = 4 - 3(1/2)^t$. Therefore, the solution for the problem is $\{1, 5/2, 13/4, 29/8, 61/16\}$, which is different from the path $\{1, 38/17, 44/17, 38/17, 1\}$ that we have determined for the fixed terminal state problem. □

However, in free endpoint problems we need sometimes an additional terminal condition in order to have a meaningful solution. To convince oneself, consider the following problem.

Cake eating problem with free terminal size . Consider the previous cake eating example where T is known but assume instead that W_T is free. The first order conditions from proposition (17) are

$$\begin{cases} W_{t+2} = (1+\beta)W_{t+1} - \beta W_t, \ t = 0, 1, \dots, T-2 \\ W_0 = \phi \\ \frac{\beta^{T-1}}{W_T - W_{T-1}} = 0. \end{cases}$$

If we substitute the solution of the Euler-Lagrange condition, equation (12), the transversality condition becomes

$$\frac{\beta^{T-1}}{W_T - W_{T-1}} = \frac{\beta^{T-1}}{\beta^T - \beta^{T-1}} \frac{1-\beta}{k_1 - k_2} = \frac{1}{k_1 - k_2}$$

which can only be zero if $k_2 - k_1 = \infty$. If we look at the transversality condition, the last condition only holds if $W_T - W_{T-1} = \infty$, which does not make sense. □

2.3 Free terminal state problem with a terminal constraint

In some cases, when reducing the state variable x has utility, the free terminal state problem, as in the cake eating problem, can lead to a solution which would been meaningless in economic terms. That problem was misspecified.

One way to solve this, and which is very important in applications to economics is to introduce a terminal constraint forcing the state variable to be higher than a given constant, as in the next problem.

Definition 3. ***The free terminal state CV problem*** *Find $x = \{x_t\}_{t=0}^{T}$ that solves*

$$\max_{x} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \text{subject to } x_0 = \phi_0 \text{ and } x_T \geq \phi_T \tag{CV3}$$

where T, ϕ_0 and ϕ_T are given.

Proposition 3. *(Necessary condition for optimality for the free end point problem with terminal constraints)*
*Let $\{x_t^*\}_{t=0}^{T}$ be a solution for the problem defined by equations (2) and (**??**). Then it verifies the Euler-Lagrange condition*

$$\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} = 0, \quad t = 1, 2, \ldots, T-1 \tag{17}$$

and the initial and the transversality condition

$$\begin{cases} x_0^* = \phi_0, & t = 0 \\ \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T^*}(\phi_T - x_T^*) = 0, & t = T. \end{cases}$$

Proof. Now we write $V(\{x\})$ as a Lagrangian

$$V(\{x\}) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) + \lambda(\phi_T - x_T)$$

where λ is a Lagrange multiplier. Using again the variational method with $\epsilon_0 = 0$ and $\epsilon_T \neq 0$ the different between the perturbed candidate solution and the solution becomes

$$\begin{aligned} V(x) - V(x^*) &= \sum_{t=1}^{T-2} \left(\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} \right) \varepsilon_t + \\ &\quad + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} \varepsilon_T + \lambda(\phi_T - x_T^* - \varepsilon_T) \end{aligned}$$

From the Kuhn-Tucker conditions, we have the conditions, regarding the terminal state,

$$\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} - \lambda = 0,\ \lambda(\phi_T - x_T^*) = 0.$$

□

The cake eating problem again Now, if we introduce the terminal condition $W_T \geq 0$, the first order conditions are

$$\begin{cases} W_{t+2}^* = (1+\beta)W_{t+1}^* - \beta W_t,\ t = 0, 1, \ldots, T-2 \\ W_0^* = \phi \\ \frac{\beta^{T-1} W_T^*}{W_T^* - W_{T-1}^*} = 0. \end{cases}$$

If T is finite, the last condition only holds if $W_T^* = 0$, which means that it is optimal to eat all the cake in finite time. The solution is, thus formally, but not conceptually, the same as in the fixed endpoint case.

2.4 Infinite horizon problems

The most common problems in macroeconomics is the discounted infinite horizon problem.

We consider two problems, without or with terminal conditions.

Definition 4. ***The infinite horizon CV problem with free terminal state*** *Find $x = \{x_t\}_{t=0}^{\infty}$ that solves*

$$\max_x \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_t) \text{ subject to } x_0 = \phi_0 \text{ and } \lim_{t\to\infty} x_t \text{ free} \tag{CV4}$$

where, $0 < \beta < 1$ and ϕ_0 are given.

Proposition 4. *(Necessary condition for optimality for the infinite horizon problem* (CV4)*) Let $\{x_t^*\}_{t=0}^{\infty}$ be a solution for the problem defined by equation (CV4). Then it verifies the*

Euler-Lagrange condition

$$\frac{\partial F(x^*_t, x^*_{t-1})}{\partial x_t} + \beta \frac{\partial F(x_{t+1}, x_t)}{\partial x_t} = 0,\ t = 0, 1, \dots$$

and

$$\begin{cases} x^*_0 = x_0, \\ \lim_{t\to\infty} \beta^{t-1} \frac{\partial F(x^*_t, x^*_{t-1})}{\partial x_t} = 0, \end{cases}$$

Proof We can see this problem as a particular case of the free terminal state problem (CV3) for $T = \infty$. □

With terminal conditions

Definition 5. ***The infinite horizon CV problem with constrained terminal state*** *Find $x = \{x_t\}_{t=0}^{\infty}$ that solves*

$$\max_x \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_t) \text{ subject to } x_0 = \phi_0 \text{ and } \lim_{t\to\infty} x_t \geq 0 \tag{CV5}$$

where, $0 < \beta < 1$ and ϕ_0 are given.

Proposition 5. *(Necessary condition for optimality for the infinite horizon problem* (CV5)*) Let $\{x^*_t\}_{t=0}^{\infty}$ be a solution for the problem defined by equation (CV4). Then it verifies the Euler-Lagrange condition*

$$\frac{\partial F(x^*_t, x^*_{t-1})}{\partial x_t} + \beta \frac{\partial F(x_{t+1}, x_t)}{\partial x_t} = 0,\ t = 0, 1, \dots$$

and

$$\begin{cases} x^*_0 = x_0, \\ \lim_{t\to\infty} \beta^{t-1} \frac{\partial F(x^*_t, x^*_{t-1})}{\partial x_t} x^*_t = 0. \end{cases}$$

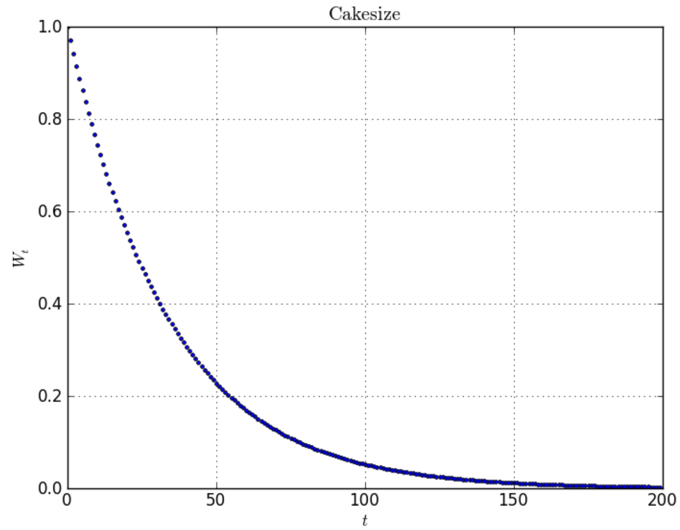

Figure 3: Solution for the cake eating problem with $T = \infty$, $\beta = 1/1.03$ and $\phi_0 = 1$

Exercise: the discounted infinite horizon cake eating problem The solution of the Euler-Lagrange condition was already derived as

$$W_t^* = \frac{1}{1-\beta}\left(-\beta k_1 + k_2 + (k_1 - k_2)\beta^t\right),\ t = 0, 1 \ldots, \infty$$

If we substitute in the transversality condition for the infinite horizon problem without terminal conditions, we get

$$\lim_{t\to\infty} \beta^{t-1} \frac{\partial \ln(W_{t-1}^* - W_t^*)}{\partial W_t} = \lim_{t\to\infty} \beta^{t-1} (W_t^* - W_{t-1}^*)^{-1} = \lim_{t\to\infty} \frac{\beta^{t-1}}{\beta^t - \beta^{t-1}} \frac{1-\beta}{k_1 - k_2} = \frac{1}{k_2 - k_1}$$

which again ill-specified because the last equation is only equal to zero if $k_2 - k_1 = \infty$.

If we consider the infinite horizon problem with a terminal constraint $\lim_{t\to\infty} x_t \geq 0$ and substitute, in the transversality condition for the infinite horizon problem without terminal conditions, we get

$$\lim_{t\to\infty} \beta^{t-1} \frac{\partial \ln(W_{t-1} - W_t)}{\partial W_t} W_t = \lim_{t\to\infty} \frac{W_t^*}{k_1 - k_2} = \frac{-\beta k_1 + k_2}{(1-\beta)(k_2 - k_1)}$$

because $\lim_{t\to\infty}\beta^t = 0$ as $0 < \beta < 1$. The transversality condition holds if and only if $k_2 = \beta k_1$. If we substitute in the solution for W_t, we get

$$W_t^* = \frac{k_1(1-\beta)}{1-\beta}\beta^t = k_1\beta^t,\ t = 0, 1\ldots, \infty.$$

The solution verifie the initial condition $W_0 = \phi_0$ if and only if $k_1 = \phi_0$. Therefore the solution for the infinite horizon problem is $\{W_t^*\}_{t=0}^{\infty}$ where

$$W_t^* = \phi_0\beta^t.$$

Table 2.4 gathers the results obtained thus far

Problem	Given		Optimality contitions	
	T	x_T	T^*	x_T^*
(CV1)	fixed	fixed	T	x_T
(CV2)	fixed	free	T	$\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} = 0$
(CV3)	fixed	$x_T \geq \phi_T$	T	$\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T}(\phi_T - x_T^*) = 0$
(CV4)	∞	free	∞	$\lim_{t\to\infty}\beta^{t-1}\frac{\partial F(x_t^*, x_{t-1}^*)}{\partial x_t} = 0$
(CV5)	∞	$x_\infty \geq 0$	∞	$\lim_{t\to\infty}\beta^{t-1}\frac{\partial F(x_t^*, x_{t-1}^*)}{\partial x_t}x_t^* = 0$

Table 1: Terminal conditions for the calculus of variations problems

3 Optimal Control and the Pontriyagin's principle

The optimal control problem is a generalization of the calculus of variations problem. It involves two variables, the **control** and the **state** variables and consists in maximizing a functional over functions of the state and control variables subject to a difference equation over the state variable, which characterizes the system we want to control. Usually the initial state is known and there could exist or not additional terminal conditions over the state.

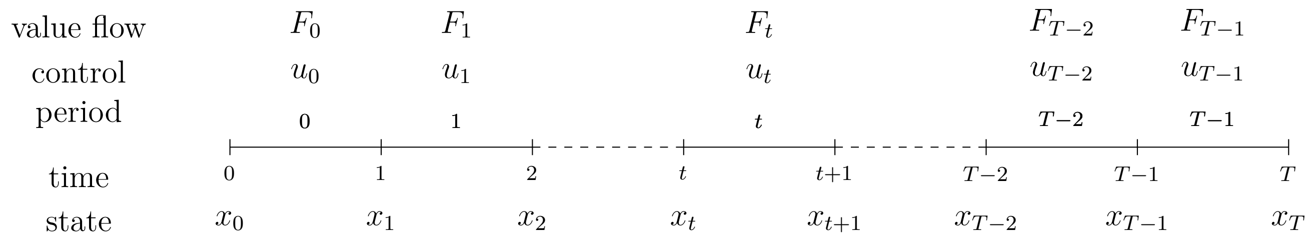

Figure 4: Timing for the state variables and the objective function where $F_t = f(u_t, x_t, t)$ and $x_{t+1} = g(x_t, u_t, t)$

The trajectory (or orbit) of the state variable, $x \equiv \{x_t\}_{t=0}^T$, characterizes the state of a system, and the control variable path $u \equiv \{u_t\}_{t=0}^T$ allows us to control its evolution.

3.1 The simplest problem

Let T be finite. The simplest optimal control problem consist in finding the optimal paths $(\{u^*\}, \{x^*\})$ such that the value functional is maximized by choosing an optimal control,

$$\max_{\{u\}} \sum_{t=0}^{T-1} f(x_t, u_t, t), \tag{18}$$

subject to the constraints of the problem

$$\begin{cases} x_{t+1} = g(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T = \phi_T & t = T \end{cases} \tag{19}$$

where ϕ_0, ϕ_T and T are given.

We assume that certain conditions hold: (1) differentiability of f; (2) concavity of g and f; (3) regularity [3]

[3] That is, existence of sequences of $x = \{x_1, x_2, ..., x_T\}$ and of $\overline{u} = \{\overline{u}_1, \overline{u}_2, ..., \overline{u}_T\}$ satisfying $\overline{x}_{t+1} = \frac{\partial g}{\partial x}(x_t^0, u_t^0)\overline{x}_t + g(x_t^0, \overline{u}_t) - g(x_t^0, u_t^0)$.

Define the **Hamiltonian function**

$$H_t = H(\psi_t, x_t, u_t, t) = f(x_t, u_t, t) + \psi_t g(x_t, u_t, t)$$

where ψ_t is called the **co-state variable** and $\{\psi\} = \{\psi_t\}_{t=0}^{T-1}$ is the co-state variable path.

The **maximized** Hamiltonian

$$H_t^*(\psi_t, x_t^*) = \max_u H_t(\psi_t, x_t, u_t)$$

is obtained by substituting in H_t the optimal control, $u_t^* = u^*(x_t, \psi_t)$.

Proposition 6. *(Maximum principle)*
If x^* and u^* are solutions of the optimal control problem (18)-(19) and if the former differentiability and regularity conditions hold, then there is a sequence $\{\psi\} = \{\psi_t\}_{t=0}^{T-1}$ such that the following conditions hold

$$\frac{\partial H_t^*}{\partial u_t} = 0, \; t = 0, 1, \ldots, T-1 \tag{20a}$$

$$\psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}, \; t = 0, \ldots, T-1 \tag{20b}$$

$$x_{t+1}^* = g(x_t^*, u_t^*, t) \tag{20c}$$

$$x_T^* = \phi_T \tag{20d}$$

$$x_0^* = \phi_0 \tag{20e}$$

Proof. Assume that we know the solution (u^*, x^*) for the problem. Then the optimal value of value functional is $V^* = V(x^*) = \sum_{t=0}^{T-1} f(x_t^*, u_t^*, t)$.

Consider the Lagrangean

$$\begin{aligned} L &= \sum_{t=0}^{T-1} f(x_t, u_t, t) + \psi_t(g(x_t, u_t, t) - x_{t+1}) \\ &= \sum_{t=0}^{T-1} H_t(\psi_t, x_t, u_t, t) - \psi_t x_{t+1} \end{aligned}$$

where Hamiltonian function is

$$H_t = H(\psi_t, x_t, u_t, t) \equiv f(x_t, u_t, t) + \psi_t(g(x_t, u_t, t). \tag{21}$$

Define

$$G_t = G(x_{t+1}, x_t, u_t, \psi_t, t) \equiv H(\psi_t, , x_t, u_t, t) - \psi_t x_{t+1}.$$

Then

$$L = \sum_{t=0}^{T-1} G(x_{t+1}, x_t, u_t, \psi_t, t)$$

If we introduce again a variation as regards the solution $\{u^*, x^*\}_{t=0}^{T}$, $x_t = x_t^* + \epsilon_t^x$, $u_t = u_t^* + \epsilon_t^u$ and form the variation in the value function and apply a first order Taylor approximation, as in the calculus of variations problem,

$$L - V^* = \sum_{t=1}^{T-1} \left(\frac{\partial G_{t-1}}{\partial x_t} + \frac{\partial G_t}{\partial x_t} \right) \epsilon_t^x + \sum_{t=0}^{T-1} \frac{\partial G_t}{\partial u_t} \epsilon_t^u + \sum_{t=0}^{T-1} \frac{\partial G_t}{\partial \psi_t} \epsilon_t^{\psi}.$$

Then, get the optimality conditions

$$\begin{aligned}
&\frac{\partial G_t}{\partial u_t} = 0,\ t = 0, 1, \ldots, T-1 \\
&\frac{\partial G_t}{\partial \psi_t} = 0,\ t = 0, 1, \ldots, T-1 \\
&\frac{\partial G_{t-1}}{\partial x_t} + \frac{\partial G_t}{\partial x_t} = 0,\ t = 1, \ldots, T-1
\end{aligned}$$

where all the variables are evaluated at the optimal path.

Evaluating these expressions for the same time period $t = 0, \ldots, T-1$, we get

$$\begin{aligned}
\frac{\partial G_t}{\partial u_t} &= \frac{\partial H_t}{\partial u_t} = \frac{\partial f(x_t^*, u_t^*, t)}{\partial u} + \psi_t \frac{\partial g(x_t^*, u_t^*, t)}{\partial u} = 0, \\
\frac{\partial G_t}{\partial \psi_t} &= \frac{\partial H_t}{\partial \psi_t} - x_{t+1} = g(x_t^*, u_t^*, t) - x_{t+1} = 0,
\end{aligned}$$

which is an admissibility condition

$$\begin{aligned}
\frac{\partial G_t}{\partial x_{t+1}} + \frac{\partial G_{t+1}}{\partial x_{t+1}} &= \frac{\partial (H_t - \psi_t x_{t+1})}{\partial x_{t+1}} + \frac{\partial H_{t+1}}{\partial x_{t+1}} \\
&= -\psi_t + \frac{\partial f(x_{t+1}^*, u_{t+1}^*, t+1)}{\partial x} + \psi_{t+1} \frac{\partial g(x_{t+1}^*, u_{t+1}^*, t+1)}{\partial x} = 0.
\end{aligned}$$

Then, setting the expressions to zero, we get, equivalently, equations (20a)-(20e) $\square$

This is a version of the **Pontriyagin's maximum principle**. The first order conditions define a mixed initial-terminal value problem involving a planar difference equation.

If $\partial^2 H_t/\partial u_t^2 \neq 0$ then we can use the inverse function theorem on the static optimality condition

$$\frac{\partial H_t^*}{\partial u_t} = \frac{\partial f(x_t^*, u_t^*, t)}{\partial u_t} + \psi_t \frac{\partial g(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

to get the optimal control as a function of the state and the co-state variables as

$$u_t^* = h(x_t^*, \psi_t, t)$$

if we substitute in equations (20b) and (20c) we get a non-linear planar ode in (ψ, x), called the **canonical system**,

$$\begin{cases} \psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}(x_{t+1}^*, h(x_{t+1}^*, \psi_{t+1}, t+1), t+1), \psi_{t+1}, t+1) \\ x_{t+1}^* = g(x_t^*, h(x_t^*, \psi_t, t), t) \end{cases} \tag{22}$$

where

$$\frac{\partial H_{t+1}^*}{\partial x_{t+1}} = \frac{\partial f(x_{t+1}^*, h(x_{t+1}^*, \psi_{t+1}, t+1), t+1)}{\partial x_{t+1}} + \psi_{t+1} \frac{\partial g(x_{t+1}^*, h(x_{t+1}^*, \psi_{t+1}, t+1), t+1)}{\partial x_{t+1}}$$

The first order conditions, according to the Pontriyagin principle, are then constituted by the canonical system (23) plus the initial and the terminal conditions (20d) and (20e).

Alternatively, if the relationship between u and ψ is monotonic, we could solve condition $\partial H_t^*/\partial u_t = 0$ for ψ_t to get

$$\psi_t = q_t(u_t^*, x_t^*, t) = -\frac{\frac{\partial f(x_t^*, u_t^*, t)}{\partial u_t}}{\frac{\partial g(x_t^*, u_t^*, t)}{\partial u_t}}$$

and we would get an equivalent (implicit or explicit) canonical system in (u, x)

$$\begin{cases} q_t(u_t^*, x_t^*, t) = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}(x_{t+1}^*, u_{t+1}^*, q_{t+1}(u_{t+1}^*, x_{t+1}^*, t+1), t+1) \\ x_{t+1}^* = g(x_t^*, u_t^*, t) \end{cases} \tag{23}$$

which is an useful representation if we could isolate u_{t+1}, which is the case in the next example.

Exercise: cake eating Consider again problem (11) and solve it using the maximum principle of Pontriyagin. The present value Hamiltonian is

$$H_t = \beta^t \ln(C_t) + \psi_t(W_t - C_t)$$

and from first order conditions from the maximum principle

$$\begin{cases} \frac{\partial H_t^*}{\partial C_t} = \beta^t (C_t^*)^{-1} - \psi_t = 0, \ t = 0, 1, \ldots, T-1 \\ \psi_t = \frac{\partial H_{t+1}^*}{\partial W_{t+1}} = \psi_{t+1}, \ t = 0, \ldots, T-1 \\ W_{t+1}^* = W_t^* - C_t^*, \ t = 0, \ldots, T-1 \\ W_T^* = 0 \\ W_0^* = \phi. \end{cases}$$

From the first two equations we get an equation over C, $C_{t+1}^* \beta^t = \beta^{t+1} C_t^*$, which is sometimes called the Euler equation. This equation together with the admissibility conditions, lead to the canonical dynamic system

$$\begin{cases} C_{t+1}^* = \beta C_t^* \\ W_{t+1}^* = W_t^* - C_t^*, \ t = 0, \ldots, T-1 \\ W_T^* = 0 \\ W_0^* = \phi. \end{cases}$$

There are two methods to solve this mixed initial-terminal value problem: recursively or jointly.

First method: we can solve the problem recursively. First,we solve the Euler equation to get

$$C_t = k_1 \beta^t.$$

Then the second equation becomes

$$W_{t+1} = W_t - k_1 \beta^t$$

which has solution

$$W_t = k_2 - k_1 \sum_{s=0}^{t-1} \beta^s = k_2 - k_1 \frac{1-\beta^t}{1-\beta}.$$

In order to determine the arbitrary constants, we consider again the initial and terminal conditions $W_0 = \phi$ and $W_T = 0$ and get

$$k_1 = \frac{1-\beta}{1-\beta^T}\phi,\ k_2 = \phi$$

and if we substitute in the expressions for C_t^* and W_t^* we get the same result as in the calculus of variations problem, equations (14)-(13).

Second method: we can solve the canonical system as a planar difference equation system. The first two equations have the form $\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t$ where

$$\mathbf{A} = \begin{pmatrix} \beta & 0 \\ -1 & 1 \end{pmatrix}$$

which has eigenvalues $\lambda_- = 1$ and $\lambda_+ = \beta$ and the associated eigenvector matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & 1-\beta \\ 1 & 1 \end{pmatrix}.$$

The solution of the planar equation is of type $\mathbf{y}_t = \mathbf{P}\mathbf{\Lambda}^t\mathbf{P}^{-1}k$

$$\begin{aligned}\begin{pmatrix} C_t^* \\ W_t^* \end{pmatrix} &= \frac{1}{1-\beta}\begin{pmatrix} 0 & 1-\beta \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta^t \end{pmatrix}\begin{pmatrix} -1 & 1-\beta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \\ &= \begin{pmatrix} k_1\beta^t \\ k_2 - k_1\frac{1-\beta^t}{1-\beta} \end{pmatrix}.\end{aligned}$$

3.2 Free terminal state

Again, let T be finite. This is a slight modification of the simplest optimal control problem which has the objective functional (18) subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \end{cases} \tag{24}$$

where ϕ_0 is given.

The Hamiltonian is the same as in the former problem and the first order necessary conditions for optimality are:

Proposition 7. *(Maximum principle)*
If $\{x^*\}_{t=0}^{T}$ and $\{u^*\}_{t=0}^{T}$ are solutions of the optimal control problem (18)-(24) and if the former assumptions on f and g hold, then there is a sequence $\{\psi\} = \{\psi_t\}_{t=0}^{T-1}$ such that for $t = 0, 1, ..., T-1$

$$\frac{\partial H_t^*}{\partial u_t} = 0, \; t = 0, 1, \ldots, T-1 \tag{25a}$$

$$\psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}, \; t = 0, \ldots, T-1 \tag{25b}$$

$$x_{t+1}^* = g(x_t^*, u_t^*, t) \tag{25c}$$

$$x_0^* = \phi_0 \tag{25d}$$

$$\psi_{T-1} = 0 \tag{25e}$$

Proof. The proof is similar to the previous case, but now we have for $t = T$

$$\frac{\partial G_{T-1}}{\partial x_T} = \psi_{T-1} = 0.$$

□

3.3 Free terminal state with terminal constraint

Again let T be finite and assume that the terminal value for the state variable is non-negative. This is another slight modification of the simplest e simplest optimal control problem which has the objective functional (18) subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T \geq 0 & t = T \end{cases} \tag{26}$$

where ϕ_0 is given.

The Hamiltonian is the same as in the former problem and the first order necessary conditions for optimality are

Proposition 8. *(Maximum principle)*
If $\{x^*\}_{t=0}^T$ and $\{u^*\}_{t=0}^T$ are solutions of the optimal control problem (18)-(26) and if the former conditions hold, then there is a sequence $\psi = \{\psi_t\}_{t=0}^{T-1}$ such that for $t = 0, 1, ..., T-1$ satisfying equations (25a)-(25d) and

$$\psi_{T-1} x_T^* = 0 \tag{27}$$

The cake eating problem Using the previous result, the necessary conditions according to the Pontryiagin's maximum principle are

$$\begin{cases} C_t = \beta^t / \psi_t \\ \psi_t = \psi_{t+1} \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \psi_{T-1} = 0 \end{cases}$$

This is equivalent to the problem involving the canonical planar difference equation system

$$\begin{cases} C_{t+1} = \beta C_t \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \frac{\beta^{T-1}}{C_{T-1}} = 0 \end{cases}$$

whose general solution was already found. The terminal condition becomes

$$\frac{\beta^{T-1}}{C_{T-1}} = \frac{\beta^{T-1}}{\beta^{T-1}k_1} = \frac{1}{k_1}$$

which can only be zero if $k_1 = \infty$, which does not make sense.

If we solve instead the problem with the terminal condition $W_T \geq 0$, then the transversality condition is

$$\psi_{T-1}W_T = \beta^{T-1}\frac{W_T}{C_{T-1}} = 0$$

If we substitute the general solutions for C_t and W_t we get

$$\beta^{T-1}\frac{W_T}{C_{T-1}} = \frac{1}{1-\beta}\left[\frac{-k_1 + (1-\beta)k_2}{k_1} + \frac{k_1}{k_1}\beta^T\right]$$

which is equal to zero if and only if

$$k_2 = k_1\frac{1-\beta^T}{1-\beta}.$$

We still have one unknown k_1. In order to determine it, we substitute in the expression for W_t

$$W_t = k_1\frac{\beta^t - \beta^T}{1-\beta},$$

evaluate it at $t = 0$, and use the initial condition $W_0 = \phi$ and get

$$k_1 = \frac{1-\beta}{1-\beta^T}\phi.$$

Therefore, the solution for the problem is the same as we got before, equations (14)-(13).

3.4 The discounted infinite horizon problem

The discounted infinite horizon optimal control problem consist on finding (u^*, x^*) such that

$$\max_u \sum_{t=0}^{\infty} \beta^t f(x_t, u_t),\ 0 < \beta < 1 \tag{28}$$

subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t) & t = 0, 1, \ldots \\ x_0 = \phi_0 & t = 0 \end{cases} \tag{29}$$

where ϕ_0 is given.

Observe that the functions $f(.)$ and $g(.)$ are now autonomous, in the sense that time does not enter directly as an argument, but there is a discount factor β^t which weights the value of $f(.)$ along time.

The **current-value Hamiltonian** is

$$h_t = h(x_t, \eta_t, u_t) \equiv f(u_t, y_t) + \eta_t g(y_t, u_t) \tag{30}$$

where η_t is the discounted co-state variable.

It is obtained from the discounted Hamiltonian as follows:

$$\begin{aligned} H_t &= \beta^t f(u_t, x_t) + \psi_t g(x_t, u_t) \\ &= \beta^t \left(f(u_t, y_t) + \eta_t g(y_t, u_t) \right) \\ &\equiv \beta^t h_t \end{aligned}$$

where the co-state variable (η) relates with the actualized co-state variable (ψ) as $\psi_t = \beta^t \eta_t$. The Hamiltonian h_t is independent of time in discounted autonomous optimal control problems. The maximized current value Hamiltonian is

$$h_t^* = \max_u h_t(x_t, \eta_t, u_t).$$

Proposition 9. *(Maximum principle)*
If $x^* = \{x^*\}_{t=0}^{\infty}$ and $\{u^*\}_{t=0}^{\infty}$ is a solution of the optimal control problem (28)-(29) and if the former regularity and continuity conditions hold, then there is a sequence $\{\eta\} = \{\eta_t\}_{t=0}^{\infty}$ such that the optimal paths verify

$$\frac{\partial h_t^*}{\partial u_t} = 0, \; t = 0, 1, \ldots, \infty \tag{31a}$$

$$\eta_t = \beta \frac{\partial h_{t+1}^*}{\partial x_{t+1}}, \; t = 0, \ldots, \infty \tag{31b}$$

$$x_{t+1}^* = g(x_t^*, u_t^*, t) \tag{31c}$$

$$\lim_{t\to\infty} \beta^t \eta_t = 0 \tag{31d}$$

$$x_0^* = \phi_0 \tag{31e}$$

Proof. Exercise. □

Again, if we have the terminal condition

$$\lim_{t\to\infty} x_t \geq 0$$

the transversality condition is

$$\lim_{t\to\infty} \beta^t \eta_t x_t^* = 0 \tag{32}$$

instead of (31d).

The necessary first-order conditions are again represented by the system of difference equations. If $\partial^2 h_t / \partial u_t^2 \neq 0$ then we can use the inverse function theorem on the static optimality condition

$$\frac{\partial h_t^*}{\partial u_t} = \frac{\partial f(x_t^*, u_t^*, t)}{\partial u_t} + \eta_t \frac{\partial g(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

to get the optimal control as a function of the state and the co-state variables as

$$u_t^* = h(x_t^*, \eta_t)$$

if we substitute in equations (31b) and (31c) we get a non-linear autonomous planar difference equation in (η, x) (or (u, x), if the relationship between u and η is monotonic)

$$\begin{cases} \eta_t = \beta \left(\frac{\partial f(x^*_{t+1}, h(x^*_{t+1}, \eta_{t+1}))}{\partial x_{t+1}} + \eta_{t+1} \frac{\partial g(x^*_{t+1}, h(x^*_{t+1}, \eta_{t+1}))}{\partial x_{t+1}} \right) \\ x^*_{t+1} = g(x^*_t, h(x^*_t, \eta_t)) \end{cases}$$

plus the initial and the transversality conditions (31d) and (31e) or (32).

Exercise: the cake eating problem with an infinite horizon The current-value Hamiltonian is

$$h_t = \ln(C_t) + \eta_t (W_t - C_t)$$

and the f.o.c are

$$\begin{cases} C_t = 1/\eta_t \\ \eta_t = \beta \eta_{t+1} \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t\to\infty} \beta^t \eta_t W_t = 0 \end{cases}$$

This is equivalent to the planar difference equation problem

$$\begin{cases} C_{t+1} = \beta C_t \\ W_{t+1} = W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t\to\infty} \beta^t \frac{W_t}{C_t} = 0 \end{cases}$$

If we substitute the solutions for C_t and W_t in the transversality condition, we get

$$\lim_{t\to\infty} \beta^t \frac{W_t}{C_t} = \lim_{t\to\infty} \frac{-k_1 + (1-\beta)k_2 + k_1 \beta^t}{(1-\beta)k_1} = \frac{-k_1 + (1-\beta)k_2}{(1-\beta)k_1} = 0$$

if and only if $k_1 = (1-\beta)k_2$. Using the same method we used before, we finally reach again the optimal solution

$$C_t^* = (1-\beta)\phi\beta^t,\ W_t^* = \phi\beta^t,\ t = 0, 1, \ldots, \infty.$$

Exercise: the consumption-savings problem with an infinite horizon Assume that a consumer has an initial stock of financial wealth given by $\phi > 0$ and gets a financial return if s/he has savings. The intratemporal budget constraint is

$$W_{t+1} = (1+r)W_t - C_t,\ t = 0, 1, \ldots$$

where $r > 0$ is the constant rate of return. Assume s/he has the intertemporal utility functional

$$J(C) = \sum_{t=0}^{\infty} \beta^t \ln(C_t),\ 0 < \beta = \frac{1}{1+\rho} < 1,\ \rho > 0$$

and that the non-Ponzi game condition holds: $\lim_{t\to\infty} W_t \geq 0$. What are the optimal sequences for consumption and the stock of financial wealth ?

We next solve the problem by using the Pontriyagin's maximum principle. The current-value Hamiltonian is

$$h_t = \ln(C_t) + \eta_t\left((1+r)W_t - C_t\right)$$

where η_t is the discounted co-state variable. The f.o.c. are

$$\begin{cases} C_t = 1/\eta_t \\ \eta_t = \beta(1+r)\eta_{t+1} \\ W_{t+1} = (1+r)W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t\to\infty} \beta^t \eta_t W_t = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} C_{t+1} = \beta C_t \\ W_{t+1} = (1+r)W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t\to\infty} \beta^t \frac{W_t}{C_t} = 0 \end{cases}$$

If we define and use the first two and the last equation

$$z_t \equiv \frac{W_t}{C_t}$$

we get a boundary value problem

$$\begin{cases} z_{t+1} = \frac{1}{\beta}\left(z_t - \frac{1}{1+r}\right) \\ \lim_{t\to\infty} \beta^t z_t = 0. \end{cases}$$

The difference equation for z_t has the general solution [4]

$$z_t = \left(k - \frac{1}{(1+r)(1-\beta)}\right)\beta^{-t} + \frac{1}{(1+r)(1-\beta)}.$$

We can determine the arbitrary constant k by using the transversality condition:

$$\begin{aligned} \lim_{t\to\infty} \beta^t z_t &= \lim_{t\to\infty} \beta^t \left[\left(k - \frac{1}{(1+r)(1-\beta)}\right)\beta^{-t} + \frac{1}{(1+r)(1-\beta)}\right] \\ &= k - \frac{1}{(1+r)(1-\beta)} + \lim_{t\to\infty} \beta^t \left(\frac{1}{(1+r)(1-\beta)}\right) = \\ &= k - \frac{1}{(1+r)(1-\beta)} = 0 \end{aligned}$$

which is equal to zero if and only if

$$k = \frac{1}{(1+r)(1-\beta)}.$$

[4]The difference equation is of type $x_{t+1} = ax_t + b$, where $a \neq 1$ and has solution

$$x_t = \left(k - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$$

where k is an arbitrary constant.

Then, $z_t = 1/\left((1+r)(1-\beta)\right)$ is a constant. Therefore, as $C_t = W_t/z_t$ the average and marginal propensity to consume out of wealth is also constant, and

$$C_t^* = (1+r)(1-\beta)W_t.$$

If we substitute in the intratemporal budget constraint and use the initial condition

$$\begin{cases} W_{t+1}^* = (1+r)W_t^* - C_t^* \\ W_0^* = \phi \end{cases}$$

we can determine explicitly the optimal stock of wealth for every instant

$$W_t^* = \phi\left(\beta(1+r)\right)^t = \phi\left(\frac{1+r}{1+\rho}\right)^t,\ t = 0, 1, \ldots, \infty$$

and

$$C_t^* = \phi(1+r)(1-\beta)\left(\frac{1+r}{1+\rho}\right)^t,\ t = 0, 1, \ldots, \infty.$$

We readily see that the solution depends crucially upon the relationship between the rate of return on financial assets, r and the rate of time preference ρ:

1. if $r > \rho$ then $\lim_{t\to\infty} W_t^* = \infty$: if the consumer is more patient than the market s/he optimally tends to have an abounded level of wealth asymptotically;

2. if $r = \rho$ then $\lim_{t\to\infty} W_t^* = \phi$: if the consumer is as patient as the market it is optimal to keep the level of financial wealth constant. Therefore: $C_t^* = rW_t = r\phi$;

3. if $r < \rho$ then $\lim_{t\to\infty} W_t^* = 0$: if the consumer is less patient than the market s/he optimally tends to end up with zero net wealth asymptotically.

The next figures illustrate the three cases

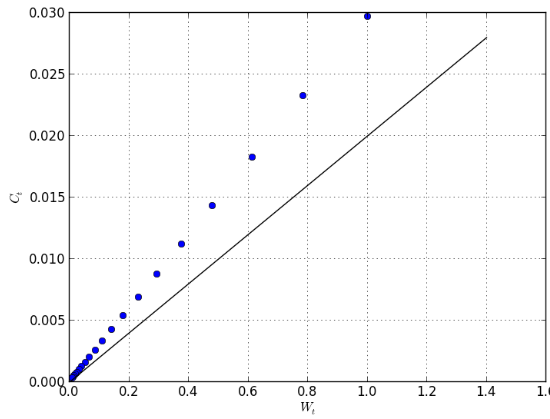

Figure 5: Phase diagram for the case in which $\phi > r$

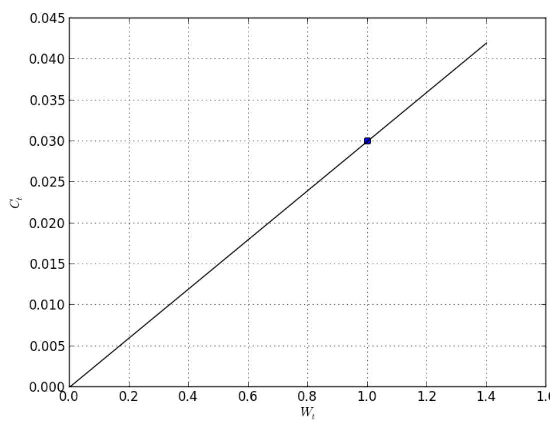

Figure 6: Phase diagram for the case in which $\phi = r$

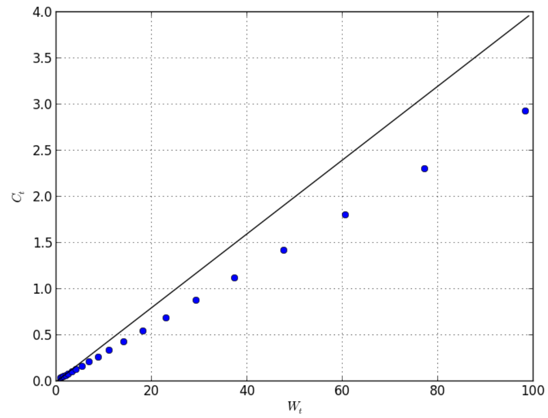

Figure 7: Phase diagram for the case in which $\phi < r$

Observe that although s/he may have an infinite level of wealth and consumption, asymptotically, the optimal value of the problem is bounded

$$\begin{aligned} J^* &= \sum_{t=0}^{\infty} \beta^t \ln\left(C_t^* = \right. \\ &= \sum_{t=0}^{\infty} \beta^t \ln\left(\phi(1+r)(1-\beta)\left(\beta(1+r)\right)^t,\right) = \\ &= \sum_{t=0}^{\infty} \beta^t \ln\left(\phi(1-\beta)\right) + \sum_{t=0}^{\infty} \beta^t \ln\left(\beta^t(1+r)^{t+1}\right) = \\ &= \frac{1}{1-\beta} \ln\left(\phi(1-\beta)\right) + \frac{\beta\ln\beta + \ln(1+r)}{(1-\beta)^2} = \end{aligned}$$

then

$$J^* = \frac{1}{1-\beta} \ln\left[\phi\left((1+r)(1-\beta)^{1-\beta}\beta^{\beta}\right)^{\frac{1}{1-\beta}}\right]$$

which is always bounded.

Table 3.4 presents the terminal conditions for the problems studied.

Problem	Given		Optimality contitions	
	T	x_T	T^*	x^*_T
(CV1)	fixed	fixed	T	x_T
(CV2)	fixed	free	T	$\psi_{T-1}=0$
(CV3)	fixed	$x_T \geq \phi_T$	T	$\psi_{T-1}(\phi_T - x^*_{T-1} = 0$
(CV4)	∞	free	∞	$\lim_{t\to\infty} \beta^{t-1}\psi_{t-1} = 0$
(CV5)	∞	$x_\infty \geq 0$	∞	$\lim_{t\to\infty} \beta^{t-1}\psi_{t-1}x^*_t = 0$

Table 2: Terminal conditions for the optimal control problems for the Pontriyagin maximum princeiple

4 Optimal control and the dynamic programming principle

Consider the discounted finite horizon optimal control problem which consists in finding (u^*, x^*) such that

$$\max_u \sum_{t=0}^{T} \beta^t f(x_t, u_t),\ 0 < \beta < 1 \tag{33}$$

subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \end{cases} \tag{34}$$

where ϕ_0 is given.

The principle of dynamic programming allows for an alternative method of solution.

According to the **Principle of the dynamic programming** (Bellman (1957)) an optimal trajectory has the following property: in the beginning of any period, take as given values of the state variable and of the control variables, and choose the control variables optimally for the rest of period. Apply this methods for every period.

4.1 The finite horizon problem

We start by the finite horizon problem, i.e. T finite.

Proposition 10. *Consider problem (33)-(34) with T finite. Then given an optimal solution the problem (x^*, u^*) satisfies the* ***Hamilton-Jacobi-Bellman equation***

$$V_{T-t}(x_t) = \max_{u_t} \left\{ f(x_t, u_t) + \beta V_{T-t-1}(x_{t+1}) \right\}, \ t = 0, \dots, T-1. \tag{35}$$

Proof. Define **value function** at time τ

$$V_{T-\tau}(x_\tau) = \sum_{t=\tau}^{T} \beta^{t-\tau} f(u_t^*, x_t^*) = \max_{\{u_t\}_{t=\tau}^T} \sum_{t=\tau}^{T} \beta^{t-\tau} f(u_t, x_t)$$

Then, for time $\tau = 0$ we have

$$\begin{aligned}
V_T(x_0) &= \max_{\{u_t\}_{t=0}^T} \sum_{t=0}^{T} \beta^t f(u_t, x_t) = \\
&= \max_{\{u_t\}_{t=0}^T} \big(f(x_0, u_0) + \beta f(x_1, u_1) + \beta^2 f(x_2, u_2) + \dots \big) = \\
&= \max_{\{u_t\}_{t=0}^T} \left(f(x_0, u_0) + \beta \sum_{t=1}^{T} \beta^{t-1} f(x_t, u_t) \right) = \\
&= \max_{u_0} \left(f(x_0, u_0) + \beta \max_{\{u_t\}_{t=1}^T} \sum_{t=1}^{T} \beta^{t-1} f(x_t, u_t) \right)
\end{aligned}$$

by the principle of dynamic programming. Then

$$V_T(x_0) = \max_{u_0} \left\{ f(x_0, u_0) + \beta V_{T-1}(x_1) \right\}$$

We can apply the same idea for the value function for any time $0 \le t \le T$ to get the equation (35) which holds for feasible solutions, i.e., verifying $x_{t+1} = g(x_t, u_t)$ and given x_0. $\square$

Intuition: we transform the maximization of a functional into a recursive two-period problem. We solve the control problem by solving the HJB equation. To do this we have to find $\{V_T, \ldots, V_0\}$, through the recursion

$$V_{t+1}(x) = \max_u \{f(x,u) + \beta V_t(g(x,u))\} \tag{36}$$

Exercise: cake eating In order to solve the cake eating problem by using dynamic programming we have to determine a particular version of the Hamilton-Jacobi-Bellman equation (35). In this case, we get

$$V_{T-t}(W_t) = \max_{C_t} \{\ln(C_t) + \beta V_{T-t-1}(W_{t+1})\},\ t = 0, 1, \ldots, T-1,$$

To solve it, we should take into account the restriction $W_{t+1} = W_t - C_t$ and the initial and terminal conditions.

We get the optimal policy function for consumption by deriving the right hand side for C_t and setting it to zero

$$\frac{\partial}{\partial C_t}\{\ln(C_t) + \beta V_{T-t-1}(W_{t+1})\} = 0$$

From this, we get the optimal policy function for consumption

$$C_t^* = \left(\beta V'_{T-t-1}(W_{t+1})\right)^{-1} = C_t(W_{t+1}).$$

Then the HJB equation becomes

$$V_{T-t}(W_t) = \ln(C_t(W_{t+1})) + \beta V_{T-t-1}(W_{t+1}),\ t = 0, 1, \ldots, T-1 \tag{37}$$

which is a partial difference equation.

In order to solve it we make the **conjecture** that the solution is of the type

$$V_{T-t}(W_t) = A_{T-t} + \left(\frac{1-\beta^{T-t}}{1-\beta}\right)\ln(W_t),\ t = 0, 1, \ldots, T-1$$

where A_{T-t} is arbitrary. We apply the method of the undetermined coefficients in order to determine A_{T-t}.

With that trial function we have

$$C_t^* = \left(\beta V'_{T-t-1}(W_{t+1})\right)^{-1} = \left(\frac{1-\beta}{\beta(1-\beta^{T-t-1})}\right) W_{t+1},\ t = 0, 1, \ldots, T-1$$

which implies. As the optimal cake size evolves according to $W_{t+1} = W_t - C_t^*$ then

$$W_{t+1} = \left(\frac{\beta - \beta^{T-t}}{1-\beta^{T-t}}\right) W_t. \tag{38}$$

which implies

$$C_t^* = \left(\frac{1-\beta}{1-\beta^{T-t}}\right) W_t,\ t = 0, 1, \ldots, T-1.$$

This is the same optimal policy for consumption as the one we got when we solve the problem by the calculus of variations technique. If we substitute back into the equation (37) we get an equivalent HJB equation

$$\begin{aligned} A_{T-t} \ &+ \ \left(\frac{1-\beta^{T-t}}{1-\beta}\right) \ln W_t = \\ &= \ \ln\left(\frac{1-\beta}{1-\beta^{T-t}}\right) + \ln W_t + \beta \left\{ A_{T-t-1} + \left(\frac{1-\beta^{T-t-1}}{1-\beta}\right) \left[\ln\left(\frac{\beta-\beta^{T-t}}{1-\beta^{T-t}}\right) + \ln W_t\right] \right\} \end{aligned}$$

As the terms in $\ln W_t$ cancel out, this indicates (partially) that our conjecture was right. Then, the HJB equation reduces to the difference equation on A_t, the unknown term:

$$A_{T-t} = \beta A_{T-t-1} + \ln\left(\frac{1-\beta}{1-\beta^{T-t}}\right) + \left(\frac{\beta-\beta^{T-t}}{1-\beta}\right) \ln\left(\frac{\beta-\beta^{T-t}}{1-\beta^{T-t}}\right)$$

which can be written as a non-homogeneous difference equation, after some algebra,

$$A_{T-t} = \beta A_{T-t-1} + z_{T-t} \tag{39}$$

where

$$z_{T-t} \equiv \ln\left(\left(\frac{1-\beta}{1-\beta^{T-t}}\right)^{\frac{1-\beta^{T-t}}{1-\beta}} \left(\frac{\beta-\beta^{T-t}}{1-\beta}\right)^{\frac{\beta-\beta^{T-t}}{1-\beta}} \right)$$

In order to solve equation (39), we perform the change of coordinates $\tau = T - t$ and oberve that $A_{T-T} = A_0 = 0$ because the terminal value of the cake should be zero. Then, operating by recursion, we have

$$\begin{aligned} A_\tau &= \beta A_{\tau-1} + z_\tau = \\ &= \beta\left(\beta A_{\tau-2} + z_{\tau-1}\right) + z_\tau = \beta^2 A_{\tau-2} + z_\tau + \beta z_{\tau-1} = \\ &= \dots \\ &= \beta^\tau A_0 + z_\tau + \beta z_{\tau-1} + \dots + \beta^\tau z_0 \\ &= \sum_{s=0}^{\tau} \beta^s z_{\tau-s}. \end{aligned}$$

Then

$$A_{T-t} = \sum_{s=0}^{T-t} \beta^s \ln\left(\left(\frac{1-\beta}{1-\beta^{T-t-s}}\right)^{\frac{1-\beta^{T-t-s}}{1-\beta}} \left(\frac{\beta-\beta^{T-t-s}}{1-\beta}\right)^{\frac{\beta-\beta^{T-t-s}}{1-\beta}}\right).$$

If we use terminal condition $A_0 = 0$, then the solution of the HJB equation is, finally,

$$\begin{aligned} V_{T-t}(W_t) &= \ln\left(\prod_{s=0}^{T-t}\left(\frac{1-\beta}{1-\beta^{T-t-s}}\right)^{\frac{\beta^s-\beta^{T-t}}{1-\beta}} \left(\frac{\beta-\beta^{T-t-s}}{1-\beta}\right)^{\frac{\beta^{s+1}-\beta^{T-t}}{1-\beta}}\right) + \\ &\quad + \left(\frac{1-\beta^{T-t}}{1-\beta}\right)\ln(W_t),\ t = 0, 1, \dots, T-1 \end{aligned} \tag{40}$$

We already determined the optimal policy for consumption (we really do not need to determine the term A_{T-t} if we only need to determine the optimal consumption)

$$C_t^* = \left(\frac{1-\beta}{1-\beta^{T-t}}\right) W_t = \left(\frac{1-\beta}{1-\beta^T}\right)\beta^t\phi,\ t = 0, 1, \dots, T-1,$$

because, in equation (38) we get

$$\begin{aligned} W_t &= \beta\left(\frac{1-\beta^{T-t}}{1-\beta^{T-(t-1)}}\right) W_{t-1} = \\ &= \beta\left(\frac{1-\beta^{T-t}}{1-\beta^{T-(t-1)}}\right)\beta\left(\frac{1-\beta^{T-(t-1)}}{1-\beta^{T-(t-2)}}\right) W_{t-2} = \beta^2\left(\frac{1-\beta^{T-t}}{1-\beta^{T-(t-2)}}\right) W_{t-2} = \\ &= \dots \\ &= \beta^t\left(\frac{1-\beta^{T-t}}{1-\beta^T}\right) W_0 \end{aligned}$$

and $W_0 = \phi$.

4.2 The infinite horizon problem

For the infinite horizon discounted optimal control problem, the limit function $V = \lim_{j\to\infty} V_j$ is independent of j so the Hamilton Jacobi Bellman equation becomes

$$V(x) = \max_u \{f(x,u) + \beta V[g(x,u)]\} = \max_u H(x,u)$$

Properties of the value function: it usually hard to get the properties of $V(.)$. In general continuity is assured but not differentiability (this is a subject for advanced courses on DP, see Stokey and Lucas (1989)).

If some regularity conditions hold, we may determine the optimal control through the **optimality condition**

$$\frac{\partial H(x,u)}{\partial u} = 0$$

if $H(.)$ is C^2 then we get the **policy function**

$$u^* = h(x)$$

which gives an optimal rule for changing the optimal control, given the state of the economy. If we can determine (or prove that there exists such a relationship) then we say that our problem is **recursive**.

In this case the HJB equation becomes a non-linear functional equation

$$V(x) = f(x, h(x)) + \beta V[g(x, h(x))].$$

Solving the HJB: means finding the value function $V(x)$. Methods: analytical (in some cases exact) and mostly numerical (value function iteration).

Example 1: the cake eating problem with infinite horizon Consider the problem

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \, \ln{(C_t)},$$

subject to

$$W_{t+1} = W_t - C_t, \text{ for } t \in \{0, \dots, \infty\}$$

$$W_0 = \phi \text{ given for } t = 0$$

The HJB equation is

$$V(W) = \max_C \left\{ \ln{(C)} + \beta V(W_1) \right\},$$

where $W_1 = W - C$. We say we solve the problem if we can find the unknown function $V(W)$.

In order to do this, first, we find the policy function $C^* = C(W)$, from the optimality condition

$$\frac{\partial \left\{ \ln{(C)} + \beta V(W-C) \right\}}{\partial C} = \frac{1}{C} - \beta V^{'}(W-C) = 0.$$

Then

$$C^* = \frac{1}{\beta V'(W-(C))},$$

which, if V is differentiable, yields $C^* = C(W))$.

Then $W_1 = W - C^{'}(W) = W_1(W)$ and the HJB becomes a functional equation

$$V(W) = \ln{(C^*(W))} + \beta V[W_1(W)].$$

Next, we try to solve the HJB equation by introducing a trial solution

$$V(W) = a + b\ln(W)$$

where the coefficients a and b are unknown, but we try to find them by using the **method of the undetermined coefficients**.

First, observe that

$$\begin{aligned} C &= \frac{1}{1+b\beta}W \\ W_1 &= \frac{b\beta}{1+b\beta}W \end{aligned}$$

Substituting in the HJB equation, we get

$$a + b\ln(W) = \ln(W) - \ln(1+b\beta) + \beta\left(a + b\ln\left(\frac{b\beta}{1+b\beta}\right) + b\ln(W)\right),$$

which is equivalent to

$$(b(1-\beta)-1)\ln(W) = a(\beta-1) - \ln(1+b\beta) + \beta b\ln\left(\frac{b\beta}{1+b\beta}\right).$$

We can eliminate the coefficients of $ln(W)$ if we set

$$b = \frac{1}{1-\beta}.$$

Then the HJB equation becomes

$$0 = a(\beta-1) - \ln\left(\frac{1}{1-\beta}\right) + \frac{\beta}{1-\beta}\ln(\beta)$$

then

$$a = \frac{\ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta)}{1-\beta} = \frac{\Psi}{1-\beta}$$

where $\Psi \equiv \left(\beta^{\beta}(1-\beta)^{1-\beta}\right)^{1/(1-\beta)}$. Then the value function is

$$V(W) = \frac{1}{1-\beta}\ln(\Psi W),$$

and $C^* = (1-\beta)W$, that is

$$C_t^* = (1-\beta)W_t,$$

which yields the optimal cake size dynamics as

$$W_{t+1}^* = W_t - C_t^* = \beta W_t^*$$

which has the solution, again, $W_t^* = \phi\beta^t$.

Example 2: the consumption-investment problem with infinite horizon Consider the problem

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \, \frac{C_t^{1-\theta}-1}{1-\theta},$$

$$\text{subject to}$$

$$W_{t+1} = (1+r)\,W_t - C_t, \text{ for } t \in \{0,\ldots,\infty\}$$

$$W_0 = \phi \text{ given for } t = 0$$

where $\varepsilon = 1/\theta > 0$ is the elasticity of intertemporal substitution, $0 < \beta < 1$ and $r > 0$.

The HJB equation is

$$V(W) = \max_{C} \left\{ \frac{C_t^{1-\theta}-1}{1-\theta} + \beta V(W_1) \right\}, \tag{41}$$

where $W_1 = W_1(C,W) \equiv (1+r)\,W - C$. We say we solve the problem if we can find the unknown function $V(W)$.

The optimality condition is

$$C^{-\theta} + \beta \, \frac{\partial V(W_1(C,W))}{\partial C} = 0 \tag{42}$$

which, if $V(\cdot)$ is sufficiently smooth would allow us to determine optimal consumption as a function of financial wealth $C^* = C(W)$.

Again, we conjecture that the value function has the form

$$V(x) = a + b\,x^{1-\theta}$$

where a and b are undetermined coefficients. With this value function, equation (42) becomes

$$C^{-\theta} = \beta\, b\,(1-\theta) \Big((1+r)\,W - C \Big)^{-\theta}.$$

Writing $\xi \equiv \big(\beta\, b\,(1-\theta)\big)^{-\frac{1}{\theta}}$, we can solve this equation for C yielding

$$C = \frac{\xi}{1+\xi}(1+r)W \tag{43}$$

which substituting in W_1, yields

$$W_1 = \frac{1}{1+\xi}(1+r)W.$$

Substituting the trial function, $V(W)$, and the previous expressions for the optimal C and W_1 in equation (41) we obtain an equation on the two unknown coefficients a and b

$$a + b\,W^{1-\theta} = \frac{1}{1-\theta}\left(\left(\frac{\xi}{1+\xi}(1+r)W\right)^{1-\theta} - 1\right) + \beta\left(a + b\left(\frac{1}{1+\xi}(1+r)W\right)^{1-\theta}\right).$$

Separating the term involving $W^{1-\theta}$, recalling the definition of ξ, and after some simplification, we obtain an equivalent equation

$$1 + (1-\theta)\,(1-\beta)\,a = \left\{(1+r)^{1-\theta}\left(\frac{\xi}{1+\xi}\right)^{\theta} - b\,(1-\theta)\right\} W^{1-\theta}$$

setting both sides to zero, and simplifying, we find

$$a = -\frac{1}{(1-\theta)(1-\beta)}$$
$$b = \frac{(1+r)^{1-\theta}}{1-\theta}\left(1 - \left(\beta\,(1+r)^{1-\theta}\right)^{\frac{1}{\theta}}\right)^{-\theta},$$

because

$$\frac{1+\xi}{\xi} = \left(1 - \left(\beta\,(1+r)^{1-\theta}\right)^{\frac{1}{\theta}}\right)^{-1}.$$

Therefore, the value function is

$$V^*(W) = \frac{1}{1-\theta}\left\{\left(1 - \left(\beta\,(1+r)^{1-\theta}\right)^{\frac{1}{\theta}}\right)^{-\theta}\left((1+r)\,W\right)^{1-\theta} - \frac{1}{1-\beta}\right\},$$

and the optimum policy function for consumption is

$$C_t^* = \left(1 + r - \left(\beta\,(1+r)\right)^{\varepsilon}\right) W_t$$

where (recall) $\varepsilon = 1/theta$ is the elasticity of intertemporal substitution.

Substituting in the budget constraint yields the optimum dynamics for wealth accumulation

$$\begin{cases} W_{t+1} = \big(\beta\,(1+r)\big)^{\varepsilon}\,W_t, & \text{for}\,t \in \{0,\cdots,\infty\} \\ W_0 = \phi & \text{for}\,t = 0 \end{cases}.$$

Solving this initial-value problem, we obtain the solution to the consumption-investmnet problem

$$\begin{aligned} W^*_t &= \big(\beta\,(1+r)\big)^{\varepsilon\,t}\,\phi, \\ C^*_t &= \Big(1+r-\big(\beta\,(1+r)\big)^{\varepsilon}\Big)\,\big(\beta\,(1+r)\big)^{\varepsilon\,t}\,\phi, \end{aligned} \quad \text{for}\,t \in \{0,\cdots,\infty\} \tag{44}$$

Observe that

- the value of $\beta(1+r) = \dfrac{1+r}{1+\rho}$ determines the qualitative dynamics of the solution: it is increasing in time ir $r > \theta$, it is constant in time if $r = \rho$ and it tends to zero if $r < \rho$;

- the elasticity of intertemporal substitution ε does not determine the qualitative dynamics of the solution, but has an effect on the quantitative features. If θ is high then the elasticity ε is low which means that the initial adjustment of wealth is slower, which means that consumption initial value C^*_0 is lower, and the dynamic adjustment will be higher.

5 Bibliographic references

(Ljungqvist and Sargent, 2004, ch. 3, 4) (de la Fuente, 2000, ch. 12, 13)

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A Auxiliary results on linear difference equations

Let $t \in \mathcal{T} \subseteq \mathbb{N}_0$ which is the set of non-negative integers $\mathbb{N}_0 = \{0, 1, \ldots\}$. We take t as an independent variable and x_t as an unknown mapping indexed by t.

A.1 Scalar difference equations of first order

A linear scalar autonomous difference equation is an equation of type

$$x_{t+1} = a x_t + b$$

where a and b are real numbers and x_t is an unknown mapping between the set $\mathcal{T}$ and a real number: $x_t : \mathcal{T} \to \mathbb{R}$.

A linear scalar non-autonomous equation is an equation

$$x_{t+1} = a_t x_t + b_t$$

where a_t and b_t are known functions of t. Next we present the solutions for both equations.

A.1.1 Autonomous equations

Homogeneous equation The simplest autonomous difference equation (DE) is the **homogeneous** equation

$$x_{t+1} = a x_t \tag{45}$$

where a is a constant. The solution for equation (45), called general solution, is

$$x_t = \begin{cases} k a^t, & \text{if } a \neq 1 \\ k, & \text{if } a = 1 \end{cases} \tag{46}$$

where k is an arbitrary number. The equation with $a = 1$ is sometimes called unit-root equation, and from (46) we see that it has a stationary, or time-independent solution. If

$a \neq 1$ the solution is time-dependent. The DE is unstable if $|a| > 1$ and displays asymptotic stability, converging towards zero, if $|a| < 1$. If $a = -1$ it oscillates fluctuates between k and $-k$.

The value of k can be determined, depending on the additional information we have on the value of x_t at one point in time. If we know x_t at time $t = 0$, say ϕ_0, then we have an **initial-value problem**. An initial-value problem is defined by the difference equation (45) and the fixed value of x_0

$$\begin{cases} x_{t+1} = a x_t, & t \in \mathcal{T} \\ x_0 = \phi_0, & t = 0 \end{cases}$$

where ϕ_0 is given. The solution

$$x_t = \begin{cases} \phi_0 a^t, & \text{if } a \neq 1 \\ \phi_0 & \text{if } a = 1 \end{cases}$$

is called **particular** solution. Sometimes in economics and finance we have a terminal-value problem, for instance,

$$\begin{cases} x_{t+1} = a x_t, & t \in \mathcal{T} \\ b^{-T} x_T = \phi_T, & t = T \end{cases}$$

In this case the particular solution, for $a \neq 1$ is

$$x_t = \phi_T b^T a^{t-T}.$$

If $a = 1$ there are only solutions for terminal-value problems in which the terminal condition is time-independent, such as

$$\begin{cases} x_{t+1} = x_t, & t \in \mathcal{T} \\ x_T = \phi_T, & t = T. \end{cases}$$

In this case the solution is

$$x_t = x_T, \text{for any } t \in \mathcal{T}$$

In the case of the initial value problem x_0 is given, but in the case of the terminal value problem it is determined endogenously. Therefore, from now on we stick to the form (46) for the general solution of the difference equations we present.

Non-homogeneous equation The autonomous scalar linear **non-homogenous** equation is

$$x_{t+1} = a x_t + b \tag{47}$$

where a and b are arbitrary real numbers. In order to find a solution, we have to distinguish between two cases: the case in which $a \neq 1$, and the case in which $a = 1$.

If $a \neq 1$ there is one unique steady state (or stationary solution) for equation (47). It is determined by setting $x_{t+1} = x_t = \bar{x}$. Performing this substitution in (47) we find

$$\bar{x} = \frac{b}{1-a}.$$

Let us define the deviations from the steady state by $y_t = x_t - \bar{x}$. Then $y_{t+1} = x_{t+1} - \bar{x}$ and if we substitute x_{t+1} from equation (47) and simplify, we readily obtain $y_{t+1} = a y_t$ which has solution $y_t = k_y a^t$. Performing the inverse transformation, we readily obtain the solution for the non-homogeneous equation

$$x_t = \bar{x} + (k - \bar{x})\, a^t, \text{ if } a \neq 1$$

where k is again an arbitrary real number (belonging to the range of x). Again k can be obtained if we have additional information on x_0 or x_T.

Now consider the unit root case for (47) where $a = 1$. From our previous discussion it is easy to conclude that if $b \neq 0$ the equation has no steady state and if $b = 0$ there is an infinite number of steady states, in fact there is no dynamics. If $b \neq 0$ the solution to equation (47) is

$$x_t = k + b\, t \text{ for } t \in \mathcal{T}$$

We can show this by recursion: let $x_0 = k$, then applying (47) we have

$$
\begin{aligned}
x_0 &= k\\
x_1 &= x_0 + b = k + b\\
x_2 &= x_1 + b = k + b + b = k + 2b\\
&\dots\\
x_t &= x_{t-1} + b = k + (t-1)b + b = k + bt
\end{aligned}
$$

Therefore, letting $b \neq 0$, equation (47) has the general equation

$$
x_t = \begin{cases} \bar{x} + (k - \bar{x})a^t, & \text{if } a \neq 1, \text{ for } \bar{x} = \dfrac{b}{1-a}, \\ k + bt, & \text{if } a = 1. \end{cases} \tag{48}
$$

A.1.2 Non-autonomous equations

A first scalar linear non-homogenous and non-autonomous equation is

$$
x_{t+1} = a x_t + b_t
$$

where a is a constant and b_t is a time-dependent known function. The general solution of equation (A.1.2) is, for $a \neq 1$

$$
x_t = k a^t + \sum_{s=0}^{t-1} a^{t-1-s} b_s.
$$

Again, we can obtain the solution by recursion.

Now the unit root case, where $a = 1$ has solution

$$
x_t = k + \sum_{s=0}^{t-1} b_s.
$$

A second scalar linear first-order non-homogenous and non-autonomous equation

$$
x_{t+1} = a_t x_t
$$

where a_t is a known function of time. In this case the general solution is

$$x_t = \left(\prod_{s=0}^{t-1} a^s\right) k$$

where

$$\prod_{s=0}^{t-1} a^s = 1 + a + a^2 + \ldots + a^{t-1}.$$

Finally the most general scalar linear non-homogenous and non-autonomous equation is

$$x_{t+1} = a_t x_t + b_t$$

where both a_t and b_t are known time-dependent functions. The general solution to equation (A.1.2) is

$$x_t = \prod_{s=0}^{t-1} a^s \left(k + \sum_{\tau=0}^{t-1} b_\tau \left(\prod_{s=0}^{\tau} a_s\right)^{-1} \right).$$

A.2 Second order linear difference equations

The Euler-Lagrange condition for the calculus of variations problem is, in general, a non-linear second-order difference equation. However, all our examples feature linear equations. Here we gather some useful results on the solution of second-order linear difference equations.

Second order scalar linear difference equations can be solved after being transformed into a particular first order planar linear difference equation. However, in order to understand the problem posed by unit roots for the general planar problem let us consider these equations as a particular case of a planar linear first-order difference equation.

Next, we only consider autonomous equations, starting with homogenous equations, in subsection A.2.1, and then dealing with non-homogeneous equations, in subsection A.3

A.2.1 Homogeneous equation

Let homogeneous linear second order difference equation be written as

$$x_{t+2} = a_1 x_{t+1} - a_0 x_t, \tag{49}$$

where a_0 and a_1 are real constants and $a_0 \neq 0$. If $a_0 = 0$ this equation is a first-order scalar equation.

The general solution to equation (49) is

$$x_t = \begin{cases} k_1 + k_0 \left(\dfrac{1-a_0^t}{1-a_0}\right) & \text{if } 1-a_1+a_0 = 0 \\ \left(\dfrac{\lambda_+ k_0 - k_1}{\lambda_+ - \lambda_-}\right)\lambda_-^t + \left(\dfrac{k_1 - \lambda_- k_0}{\lambda_+ - \lambda_-}\right)\lambda_+^t, & \text{if } 1-a_1+a_0 \neq 0, \text{ and } \left(\frac{a_1}{2}\right)^2 - a_0 > 0 \end{cases} \tag{50}$$

where k_0 and k_1 are arbitrary real numbers, and

$$\lambda_- = \frac{a_1}{2} - \left[\left(\frac{a_1}{2}\right)^2 - a_0\right]^{1/2} \tag{51a}$$

$$\lambda_+ = -\frac{a_1}{2} + \left[\left(\frac{a_1}{2}\right)^2 - a_0\right]^{1/2}. \tag{51b}$$

Case $1 - a_1 + a_0 = 0$.

In this case, we if we define $z_t \equiv x_{t+1} - x_t$ then, equation (49) is equivalent to $z_{t+1} = a_0 z_t$. This equation has the general solution $z_t = k_0\, a_0^t$. Therefore, this reveals that in this case we have a unit-root equation with a non-autonomous term

$$x_{t+1} = x_t + z_t = x_t + k_0\, a_0^t$$

which has solution

$$x_t = k_1 + k_0 \sum_{s=0}^{t-1} a_0^t = k_1 + k_0 \left(\frac{1-a_0^t}{1-a_0}\right)$$

Case $1 - a_1 + a_0 \neq 0$.

To solve the second order equation (49) we can transform it into an equivalent linear planar difference equation of the first order, by seting $y_{1,t} \equiv x_t$ and $y_{2,t} \equiv x_{t+1}$, and observe that $y_{1,t+1} = y_{2,t}$ and equation (49) can be written as $y_{2,t+1} = -a_0 y_{1,t} + a_1 y_{2,t}$.

Written in matrix form we have the equivalent first order planar equation

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t \tag{52}$$

where

$$\mathbf{y}_t \equiv \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} \text{ and } \mathbf{A} \equiv \begin{pmatrix} 0 & 1 \\ -a_0 & a_1 \end{pmatrix}.$$

Equation (52) has the unique solution

$$\mathbf{y}_t = \mathbf{P}\mathbf{\Lambda}^t \mathbf{h} \text{ for } \mathbf{h} \equiv \mathbf{P}^{-1}\mathbf{k} \tag{53}$$

where $\mathbf{P}$ and $\mathbf{\Lambda}$ are the eigenvector and Jordan form associated to $\mathbf{A}$, and $\mathbf{k} = (k_1, k_2)^\top$ is a vector of arbitrary constants.

Matrix $\mathbf{A}$ has eigenvalues λ_-, in equation (51a), and λ_+, in equation (51b).

It is important to note two facts regarding those eigenvalues:

1. they are both real and distinct (which is usually the case in when we have a calculus of variations problem) if and only if the discriminant $\Delta > 0$, for

$$\Delta \equiv \left(\frac{a_1}{2}\right)^2 - a_0$$

2. if $1 - a_1 + a_0 = 0$, then the discriminant becomes

$$\Delta = \left(\frac{a_1}{2} - 1\right)^2$$

   Then $\sqrt{\Delta} = \left|\frac{a_1}{2} - 1\right|$. Therefore $\lambda_- = 1$ and $\lambda_+ = a_1 - 1 = a_0$ if$a_1 > 2$ or $\lambda_- = a_1 - 1 = a_0$ and $\lambda_+ = 1$ if$a_1 < 2$. That is, we always have a unit root.

From now on we assume that $\Delta > 0$. If this is the case the Jordan canonical form of matrix $\mathbf{A}$ is diagonal

$$\Lambda = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}$$

and

$$\mathbf{\Lambda}^t = \begin{pmatrix} \lambda_-^t & 0 \\ 0 & \lambda_+^t \end{pmatrix}$$

The eigenvector matrix is

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ \lambda_- & \lambda_+ \end{pmatrix},$$

then

$$\mathbf{P}^{-1} = \frac{1}{\lambda_+ - \lambda_1} \begin{pmatrix} \lambda_+ & -1 \\ -\lambda_- & 1 \end{pmatrix}$$

Therefore the solution for the planar equation (52), associated to the second order linear equation (49), is

$$\begin{pmatrix} x_t \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_- & \lambda_+ \end{pmatrix} \begin{pmatrix} \lambda_-^t & 0 \\ 0 & \lambda_+^t \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} \lambda_-^t & \lambda_+^t \\ \lambda_-^{t+1} & \lambda_+^{t+1} \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}$$

where

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \frac{1}{\lambda_+ - \lambda_1} \begin{pmatrix} \lambda_+ & -1 \\ -\lambda_- & 1 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = \frac{1}{\lambda_+ - \lambda_1} \begin{pmatrix} \lambda_+ k_0 - k_1 \\ -\lambda_- k_0 + k_1 1 \end{pmatrix}$$

Then we get equation (50).

A.3 Non-homogeneous equation

Now consider the homogeneous linear second order difference equation

$$x_{t+2} = a_1 x_{t+1} - a_0 x_t + b \tag{54}$$

where a_0, a_1 and b are real constants and $a_0 \neq 0$.

The general solution to equation (54) is

$$x_t = \begin{cases} k_1 + \left(k_0 - \frac{b}{2}(t-1)\right) t, & \text{if } a_0 = 1,\ a_2 = 2 \\ k_1 + \left(\frac{b}{1-a_0}\right) t + \left(k_0 - \frac{b}{1-a_0}\right)\left(\frac{1-a_0^t}{1-a_0}\right) & \text{if } 1 - a_1 + a_0 = 0,\ \text{and } a_0 \neq 1 \\ \frac{b}{1-a_1+a_0} + \left(\frac{\lambda_+ k_0^b - k_1^b}{\lambda_+ - \lambda_-}\right)\lambda_-^t + \left(\frac{k_1^b - \lambda_- k_0^b}{\lambda_+ - \lambda_-}\right)\lambda_+^t, & \text{if } 1 - a_1 + a_0 \neq 0,\ \text{and } \left(\frac{a_1}{2}\right)^2 - a_0 > 0 \end{cases} \tag{55}$$

for

$$k_0^b = k_0 - \frac{b}{1-a_1+a_0},\ k_1^b = k_1 - \frac{b}{1-a_1+a_0}$$

where k_0 and k_1 and λ_- and λ_+ are given in equations (51a) and (51b), respectively.

Again we distinguish the unit-root case, in which $1 - a_1 + a_0 = 0$, from the general case, in which $1 - a_1 + a_0 \neq 0$.

Case $1 - a_1 + a_0 = 0$.

Let us define again $z_t \equiv x_{t+1} - x_t$ then, equation (54) is equivalent to $z_{t+1} = a_0 z_t + b$. Now, we have to consider two cases:

- If $a_0 \neq 1$ then this equation is a scalar non-homogeneous equation, similar to (47). This equation has one steady state $\bar{z} = \dfrac{b}{1-a_0}$, and the solution is

$$z_t = \bar{z} + (k_0 - \bar{z})a_0^t.$$

  Therefore, because $x_{t+1} = x_t + z_t$,

$$\begin{aligned} x_t &= k_1 + \sum_{s=0}^{t-1} z_s \\ &= k_1 + \sum_{s=0}^{t-1} \left(\bar{z} + (k_0 - \bar{z})a_0^t\right) \\ &= k_1 + \bar{z}t + (k_0 - \bar{z})\frac{1-a_0^t}{1-a_0}. \end{aligned}$$

- If $a_0 = 1$ then $z_{t+1} = z_t + b$ has solution $z_t = k_0 + b\,t$. Therefore

$$
\begin{aligned}
x_t &= k_1 + \sum_{s=0}^{t-1} z_s \\
&= k_1 + \sum_{s=0}^{t-1} (k_0 + b\,t) \\
&= k_1 + k_0 t + \frac{b}{2} t(t-1).
\end{aligned}
$$

This equation has the general solution $z_t = k_0\,a_0^t$. Therefore, this reveals that in this case we have a unit-root equation with a non-autonomous term

$$x_{t+1} = x_t + z_t = x_t + k_0\,a_0^t$$

which has solution

$$x_t = k_1 + k_0 \sum_{s=0}^{t-1} a_0^t = k_1 + k_0 \left(\frac{1 - a_0^t}{1 - a_0} \right)$$

Case: $1 - a_1 - a_0 \neq 0$

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B} \tag{56}$$

where vector $\mathbf{y}_t$ and matrix $\mathbf{A}$ are as in equation (52), and vector $\mathbf{B}$ is

$$\mathbf{B} \equiv \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

It can be readily proved that, in this case $\det(\mathbf{I} - \mathbf{A}) \neq 0$. To see this observe that

$$\det(\mathbf{I} - \mathbf{A}) = \det \begin{pmatrix} 1 & -1 \\ a_0 & 1 - a_1 \end{pmatrix} = 1 - a_1 + a_0 \neq 0$$

from our assumption. Therefore, equation (56) has one unique steady state

$$\bar{\mathbf{y}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \frac{1}{1 - a_1 + a_0} \begin{pmatrix} 1 - a_1 & 1 \\ -a_0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = \frac{1}{1 - a_1 + a_0} \begin{pmatrix} b \\ b \end{pmatrix}$$

then, as expected

$$\bar{y}_1 = \bar{y}_2 = \bar{x} = \frac{b}{1 - a_1 + a_0}$$

Therefore, because $\det(\mathbf{I} - \mathbf{A}) \neq 0$, equation (56) has the unique solution

$$\mathbf{y}_t = \bar{\mathbf{y}} + \mathbf{P}\mathbf{\Lambda}^t\mathbf{h} \text{ for } \mathbf{h} \equiv \mathbf{P}^{-1}(\mathbf{k} - \bar{\mathbf{y}}) \tag{57}$$

where $\mathbf{P}$ and $\mathbf{\Lambda}$ are the same eigenvector matrix and Jordan form, associated to $\mathbf{A}$, as in equation (49), and $\mathbf{k} = (k_1, k_2)^\top$ is a vector of arbitrary constants.

This can be simply proved: define $\mathbf{z}_t = \mathbf{y}_t - \bar{\mathbf{y}}$. Then, from equation (56)

$$\begin{aligned} \mathbf{z}_{t+1} &= \mathbf{y}_{t+1} - \bar{\mathbf{y}} \\ &= \mathbf{A}\mathbf{y}_t + \mathbf{B} - \bar{\mathbf{y}} \\ &= \mathbf{A}(\mathbf{y}_t - \bar{\mathbf{y}}) + \mathbf{B} - (\mathbf{I} - \mathbf{A})\bar{\mathbf{B}} \\ &= \mathbf{A}\mathbf{z}_t. \end{aligned}$$

This equation has solution $\mathbf{z}_t = \mathbf{P}\mathbf{\Lambda}^t\mathbf{P}^{-1}\mathbf{k}_z$. Then, performing the back transformation we have $\mathbf{y}_t = \mathbf{z}_t + \bar{y}$ we have solution (55) for the non-unit root case.

A.4 General planar linear difference equations

Now consider the non-autonomous planar linear autonomous equation

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{58}$$

or in matrix form

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}. \tag{59}$$

The general solution of equation (58) is

$$\mathbf{x}_t = \bar{\mathbf{x}} + \mathbf{P}\Lambda^t\mathbf{P}^{-1}(\mathbf{k} - \bar{\mathbf{x}}). \tag{60}$$

where $\bar{\mathbf{x}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, where $\mathbf{I}$ is the identity matrix, Λ is the Jordan form associated to matrix $\mathbf{A}$, $\mathbf{P}$ is the eigenvector matrix and $\mathbf{k}$ is a vector of constants.

Observe that we can obtain steady states from the equation

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{B} \tag{61}$$

Then the eigenvalues of matrix $\mathbf{A}$ are

$$\lambda_{\pm} = \frac{\text{trace}(\mathbf{A})}{2} \pm \sqrt{\Delta(\mathbf{A})}$$

where the discriminant is

$$\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A})$$

where

$$\text{trace}(\mathbf{A}) = a_{11} + a_{22}$$

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

The following relationship holds between the eigenvalues and trace($\mathbf{A}$) and $\det(\mathbf{A})$:

$$\lambda_- + \lambda_+ = \text{trace}(\mathbf{A}) \tag{62a}$$

$$\lambda_-\lambda_+ = \det(\mathbf{A}). \tag{62b}$$

It can be shown that, as in the previous case, the existence of unit roots depends on the value of $\det(\mathbf{I} - \mathbf{A})$.

Because

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{pmatrix}$$

then

$$\begin{aligned} \det(\mathbf{I} - \mathbf{A}) &= (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} = \\ &= 1 - (a_{11} + a_{22}) + a_{11}a_{22} - a_{12}a_{21} = \\ &= 1 - \text{trace}(\mathbf{A}) + \det(\mathbf{A}) \end{aligned}$$

Therefore

$$\det(\mathbf{I}-\mathbf{A}) = 0 \text{ if and only if } 1 - \text{trace}(\mathbf{A}) + \det(\mathbf{A}) = 0,$$

or, equivalently $\text{trace}(\mathbf{A}) = 1 + \det(\mathbf{A})$. If we substitute in equations (62a)- (62b), yields

$$\lambda_- + \lambda_+ = \text{trace}(\mathbf{A}) = 1 + \det(\mathbf{A}) = 1 + \lambda_-\lambda_+$$

which holds only if there is at least one eigenvalue, λ_-, or λ_+ or both that is equal to one.

Therefore there are no unit roots if

$$1 - \text{trace}(\mathbf{A}) + \det(\mathbf{A}) \neq 0.$$

Next we obtain the general solution for the case in which there are unit roots and for the case in which there are no unit roots.

The unit root case Now, assume that $1 - \text{trace}(\mathbf{A}) + \det(\mathbf{A}) = 0$ and there is at last one unit root, which is the case if, furthermore, $\det(\mathbf{A}) \neq 1$.

There are two consequences from the existence of unit roots.

First, there will be either no steady states or there is an infinite number of steady states. Remembering that

$$(\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{\det(\mathbf{I}-\mathbf{A})}\text{adj}(\mathbf{I}-\mathbf{A})^\top$$

because $\det(\mathbf{I}-\mathbf{A}) = 0$ we write equation (61) as

$$\text{adj}(\mathbf{I}-\mathbf{A})^\top\mathbf{B} = \det(\mathbf{I}-\mathbf{A})\mathbf{x}.$$

Then, we have two cases: First, if $\text{adj}(\mathbf{I}-\mathbf{A})^\top\mathbf{B} = \mathbf{0}$, or equivalently

$$(1 - a_{22} - a_{21})b_1 = (1 - a_{11} - a_{12})b_2$$

then there is an infinite number of steady states. This means that the two lines defined by equations (61) are co-incident. Second, if $\text{adj}(\mathbf{I}-\mathbf{A}))^\top\mathbf{B} \neq \mathbf{0}$, or equivalently

$$(1 - a_{22} - a_{21})b_1 \neq (1 - a_{11} - a_{12})b_2$$

then there are no steady states. This means that the two lines defined by equations (61) are parallel.

Second, the Jordan canonical form of $\mathbf{A}$ is diagonal and one of the elements is equal to one, for instance

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \tag{63}$$

where ℓ is the other eigenvalue. We can obtain a general solution for the difference equation (58) in this case.

We know that every matrix $\mathbf{A}$ has a similar (i.e., with equal eigenvalues) matrix Λ such that there is a matrix $\mathbf{P}$ with $\det(\mathbf{P}) \neq 0$ such that

$$\mathbf{A} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1} \Leftrightarrow \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \boldsymbol{\Lambda}.$$

Let we write the state variable as $\mathbf{x}_t = \mathbf{P}\mathbf{z}_t$. As $\mathbf{P}$ is non-singular (i.e., has a single classic inverse) then there is an unique $\mathbf{z}_t = \mathbf{P}|^{-1}\mathbf{x}_t$. Therefore

$$\begin{aligned} \mathbf{z}_{t+1} &= \mathbf{P}^{-1}\mathbf{x}_{t+1} = \\ &= \mathbf{P}^{-1}\left(\mathbf{A}\mathbf{x}_t + \mathbf{B}\right) = \\ &= \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z}_t + \tilde{\mathbf{B}} = \\ &= \Lambda\mathbf{z}_t + \tilde{\mathbf{B}} \end{aligned}$$

where

$$\tilde{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix}.$$

Therefore, in the transformed coordinates, $\mathbf{z}$, we have an equivalent recursive system system, i.e., with two scalar first order difference equations

$$\begin{pmatrix} z_{1,t+1} \\ z_{2,t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} + \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix}$$

The system has the general solution

$$\begin{aligned} z_{1,t} &= k_1 + \tilde{b}_1 t \\ z_{2,t} &= \frac{\tilde{b}_2}{1-\ell} + \left(k_2 - \frac{\tilde{b}_2}{1-\ell}\right)\ell^t. \end{aligned}$$

Going back to the original variables, we have the general solution

$$\mathbf{x}_t = \mathbf{P}_1\left(k_1 + \tilde{b}_1 t\right) + \mathbf{P}_\ell\left(\frac{\tilde{b}_2}{1-\ell} + \left(k_2 - \frac{\tilde{b}_2}{1-\ell}\right)\ell^t\right) \tag{64}$$

where $\mathbf{P}_1$ is the eigenvector, of matrix $\mathbf{A}$, associated with the unit eigenvalue and $\mathbf{P}_\ell$ is the eigenvecrtor associated with the other eigenvalue which is different from one.

Non-existence of unit roots There are two consequences from the existence of unit roots.

First there will one unique steady state. As $\det(\mathbf{I} - \mathbf{A}) \neq 0$ then we can solve uniquely equation (61) for $\mathbf{x}$ to obtain

$$\bar{\mathbf{x}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

Second, the Jordan canonical form of $\mathbf{A}$ may not be diagonal. However, we can still obtain a general solution.

Define

$$\mathbf{z}_t \equiv \mathbf{P}^{-1}(\mathbf{x}_t - \bar{\mathbf{x}}) \Leftrightarrow \mathbf{x}_t = \bar{\mathbf{x}} + \mathbf{P}\mathbf{z}_t.$$

Then

$$\begin{aligned} \mathbf{z}_{t+1} &= \mathbf{P}^{-1}(\mathbf{x}_{t+1} - \bar{\mathbf{x}}) \\ &= \mathbf{P}^{-1}(\mathbf{A}\mathbf{x}_t + \mathbf{B} - \bar{\mathbf{x}}) \\ &= \mathbf{P}^{-1}(\mathbf{A}\mathbf{P}\mathbf{z}_t + \mathbf{A}\bar{\mathbf{x}} + \mathbf{B} - \bar{\mathbf{x}}) \\ &= \mathbf{\Lambda}\mathbf{z}_t + \mathbf{P}^{-1}(\mathbf{B} - (\mathbf{I} - \mathbf{A})\bar{\mathbf{x}}) \\ &= \mathbf{\Lambda}\mathbf{z}_t \end{aligned}$$

The solution of the transformed equation is

$$\mathbf{z}_t = \mathbf{\Lambda}^t \mathbf{k}_z$$

where $\mathbf{k}_z$ is a vector of arbitrary constants, in the transformed coordinates.

Transforming back to the original variables [5], the general solution (58), when there are no unit roots, is

$$\mathbf{x}_t = \bar{\mathbf{x}} + \mathbf{P}\mathbf{\Lambda}^t \mathbf{h}$$

where

$$\mathbf{h} = \mathbf{P}^{-1}\left(\mathbf{k} - \bar{\mathbf{x}}\right)$$

where again $\mathbf{k}$ is a vector of arbitrary numbers.

If $\Delta \geq 0$ [6], then the two eigenvalues are real and distinct and the Jordan canonical form is a diagonal matrix of type

$$\Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

and the power matrix is

$$\Lambda^t = \begin{pmatrix} \lambda_+^t & 0 \\ 0 & \lambda_-^t \end{pmatrix}.$$

This implies that the solution to the planar difference equation is

$$\mathbf{x}_t = \bar{\mathbf{x}} + h_+ \mathbf{P}^+ \lambda_+^t + h_- \mathbf{P}^- \lambda_-^t$$

where

$$\mathbf{h} = \begin{pmatrix} h_+ \\ h_- \end{pmatrix} = \mathbf{P}^{-1}\left(\mathbf{k} - \bar{\mathbf{x}}\right)$$

in applications it easier to determine $\mathbf{h}$ rather than $\mathbf{k}$, from the initial and terminal conditions.

[5] And noticing that $\mathbf{k}_z = \mathbf{P}^{-1}\left(\mathbf{k} - \bar{\mathbf{x}}\right)$.

[6] If $\Delta(\mathbf{A}) < 0$ the eigenvalues are complex conjugate and therefore the eigenvectors have complex elements and the Jordan canonical form is not diagonal.

A.4.1 References

See Galor (2007).