Advanced Mathematical Economics

Paulo B. Brito

PhD in Economics: 2022-2023

ISEG

Universidade de Lisboa

pbrito@iseg.ulisboa.pt

Lecture 5

16.11.2022

Contents

Chapter 12

Scalar parabolic partial differential equations

12.1 Introduction

Parabolic partial differential equations involve a known function F depending on two independent variables (t, x), an unknown function of them $u(t, x)$, the first partial derivative as regards t and first and second partial derivatives as regards the "spatial" variable x:

$$F(t, x, u, u_t, u_x, u_{xx}) = 0$$

where $u : \text{T} \times \text{X} \to \mathbb{R}$, where $\text{T} \subseteq \mathbb{R}_+$ and $\text{X} \subseteq \mathbb{R}$.

In its simplest form, $F(u_t, u_{xx}) = 0$, the equation models the dynamics (in time) of a cross section distribution driven by dispersion. Dispersion if generated by spatial contact (think about the change in the distribution of a pollutant spreading within a lake where the water is completely still). Equation $F(u_t, u_x, u_{xx}) = 0$ features both dispersion and advection behaviors (think about the change in the distribution of a pollutant spreading within a flowing river). Equation $F(u_t, u, u_x, u_{xx}) = 0$ jointly displays dispersion, advection and growth or decay behaviors (think about the change in the distribution of a pollutant spreading within a flowing river, in which there is a permanent inflow or outflow of new pollutants). The independent terms, t and x, appear in function $F(.)$ if there are some time or spatial specific components.

We will also see in the next chapter that there is a close connection between partial differential equations and stochastic differential equations. This is the reason explaining the fact that continuous-time finance has been using parabolic PDE's since the beginning of the 1970's.

In economics and finance applications it is important to distinguish between **forward** (FPDE) and **backward** (BPDE) parabolic PDE's. While the first are complemented with an initial distribution and generate a flow of distributions forward in time, the latter are complemented with a terminal distribution and its solution generate a flow of distributions consistent with that terminal

constraint. While for FPDE the terminal distribution is unknown, for BPDE the distribution at time $t = 0$ is unknown. For planar systems, we may have forward, backward or forward-backward (FBPDE) parabolic PDE's. The last case can be seen as a generalization of the saddle-path dynamics for ODE's.

In mathematical finance most applications, such as the Black and Scholes (1973) model, are PDE's of the backward type. In economics there is recent interest in PDE's related to the topical importance of distribution issues, and, in particular to spatial dynamics modelled by BPDE. Optimal control of PDE's and mean-field games usually lead to FBPDE's.

The body of theory and application of parabolic PDE's is huge. We only present next some very introductory results and applications. In particular, we deal with linear PDE's having explicit solutions, and that can be useful for studying the dynamics of the distributions for stochastic differential equations.

The rest of this chapter presents an overview for linear scalar PDEs in section 12.2. Section 12.3 contains the solutions for the simplest FPDEs.

12.2   A general overview of linear scalar parabolic PDE's

Consider function $u(t, x)$ where $(t, x) \in \mathrm{T} \times \mathrm{X} \subseteq \mathbb{R} \times \mathbb{R}_+$ and assume it is an at least $C^{1,2}(\mathbb{R}_+, \mathbb{R})$ function[1]. There are several types of linear parabolic PDE:

- if $F(.)$ is linear in the derivatives of u, but the coefficients can be non-linear functions of u, as

$$u_t = a(x, t, u)u_{xx} + b(x, t, u)u_x + c(x, t, u),$$

  then it is called a **quasi-linear**   parabolic PDE;

- it $F(.)$ is linear in the derivatives of u, and the coefficients are independent from u, as

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(x, t, u)$$

  then it is called a **semi-linear**   parabolic PDE;

- if $F(.)$ is linear in u and all its derivatives, and the coefficients are independent from u, as

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u + d(t, x)$$

  then it is called a **linear**   parabolic PDE.

A linear equation having constant coefficients, which are obviously independent from (t, x), is called an **autonomous** PDE. The general form of an **autonomous linear parabolic PDE**   is

$$u_t = au_{xx} + bu_x + cu + d. \tag{12.1}$$

---

[1]It is, at least, differentiable to the second order as regards x and to the first order as regards t.

There are two main particular, analogously to linear ODEs: if $d = 0$ the PDE is called **homogeneous** and if $b \neq 0$ it is **non-homogeneous**.

Equation (12.1) is sometimes called a diffusion equation with advection and growth. This is because, the time-behavior of u depends on three terms: a diffusion term, au_{xx}, a transport term, bu_x, and a growth term $c\,u$ and a position term d. If $a > 0$ ($a < 0$) the equation is a **forward** PDE - FPDE (**backward** PDE - BPDE) because the diffusion operator works forward (backward) in time. The b introduces a behavior similar to the first-order PDE: it involves a transportation of the solution along the axis x, in the forward direction if $b > 0$ and in the backward direction if $b < 0$. The c generates a time behavior of the whole distribution $u(x, .)$ in a way similar to an ordinary differential equation, that is, it involves stability if $c < 0$ or instability if $c > 0$.

In the case of a parabolic PDE the stability or instability properties are related to the whole distribution: we have **stability in a distributional sense** if there is a solution $u(t, x) = \overline{u}(x)$ such that

$$\lim_{t \to \infty} u(t, x) = \overline{u}(x)$$

where $\overline{u}(x)$ is a stationary distribution.

An important element regarding the existence and characterization of the solution of PDE's is related to the characteristics of the support of the distribution X. The most common applications assume that the set X can be a line or a circle (a ring). In the first case, we can distinguish between three main cases:

- unbounded or infinite case $X = (-\infty, \infty)$

- the semi-bounded of semi-infinite case $X = [0, \infty)$ or $X = (-\infty, 0]$, where 0 can be substituted by any finite number

- the bounded case $X = (\underline{x}, \overline{x})$ where both limits are finite.

In order to define **problems involving parabolic PDE's** we have to supplement it with a distribution referred to a point in time (an initial distribution for the forward PDE or terminal distribution for a backward PDE), and possibly conditions involving known values for the values of $u(t, x)$ at the boundaries of X (called boundary conditions), i.e, values for $u(x, \cdot)$ for points $x \in \partial X$, where ∂X denotes the boundary of X.

A problem is said to be **well-posed** if there is a solution to the PDE that satisfies jointly the initial (or terminal) conditions refering to time, and the "spatial" boundary conditions, and that is continuous at those points. In this case we say we have a **classic solution**. If a problem is not well-posed it is **ill-posed**. In this case there are no solutions or classic solutions do not exist (but generalized or weak solutions can exist).

A necessary condition for a problem involving a FPDE to be well posed is that it is supplemented with an initial condition in time, and a necessary condition for a problem BPDE to be well-posed is that it involves a terminal condition in time.

Next we will present the solutions for some simple equations and problems.

12.3   The simplest forward PDE

In this section we deal with the simplest version of equation (12.1): the heat equation. In subsection 12.3.1   we show how to derive the heat equation. In subsection **??** we introduce Fourier transforms. In

12.3.1   The heat equation: derivation

The simplest linear parabolic PDE is the heat equation. Denoting the temperature at location $x \in X$ at time t by $u(t, x)$, the heat equation formalizes the dynamics of the heat distribution across set X over time by the linear forward parabolic PDE[2]

$$u_t - u_{xx} = 0. \tag{12.2}$$

It describes the dynamics of the temperature distribution when spatial differences in temperature drive the change in spatial distribution of temperature over time. Consider a homogeneous rod with infinite width and let $u(t, x)$ be the temperature at point $x \in X = (-\infty, \infty)$ at time $t \geq 0$. Consider a small segment of the rod between points x and $x + \Delta x$, where $\Delta x > 0$. The difference in the temperature between the two boundaries of the segment, $u(t, x + \Delta x) - u(t, x)$, is by the mean value theorem

$$u(t, x + \Delta x) - u(t, x) = \int_x^{x+\Delta x} u_x(t, z) \, dz,$$

which can be seen as a measure of the average temperature in the segment $(x, x + \Delta x)$ at time t. The instantaneous change in average temperature in the segment is

$$\frac{d}{dt}\Big(\int_x^{x+\Delta x} u(t, z) \, dz \Big) = \int_x^{x+\Delta x} u_t(t, z) \, dz$$

If there is a hotter spot located outside the segment, for instance in a leftward region, and because the heat flows from hot to colder regions, then temperature in the segment Δx is lower then in the leftward region, implying $u_x(t, x) < 0$, and it is higher than in the rightward region, implying $u_x(t, x + \Delta x) < 0$, and the gradient in the leftward boundary is higher in absolute terms that the rightward $u_x(t, x) - u_x(t, x + \Delta x) < 0$. Therefore, the temperature change is

$$u_x(t, x + \Delta x) - u_x(t, x) = \int_x^{x+\Delta x} u_{xx}(t, z) \, dz.$$

If we **assume** that the instantaneous change in the segment's temperature is equal to the heat that flows through the segment, that is

$$\frac{\partial}{\partial t}\Big(u(t, x + \Delta x) - u(t, x) \Big) = \frac{\partial}{\partial x} u(t, x + \Delta x) - \frac{\partial}{\partial x} u(t, x)$$

---

[2]The first formulation of the heat equation is attributed to Fourier in a presentation to the Institut de France, and in a book with title *Théorie de la Propagation de la Chaleur dans les Solides* both in 1807.

then

$$\int_x^{x+\Delta x} u_t(t,z)\,dz = \int_x^{x+\Delta x} u_{xx}\,(t,z)\,dz.$$

This is equivalent to

$$\int_x^{x+\Delta x} u_t(t,z) - u_{xx}\,(t,z)\,dz = 0,$$

which holds if and only if equation (12.2) is satisfied.

The simplest linear scalar parabolic is a slight generalization of the forward partial differential equation $u_t(t,x) = a\,u_{xx}(t,x)$ where $a > 0$. The solution, and the solution methods, to this equation depends on our assumption regarding the domain X. For the equations, and the related problems we will be interested, an powerful method uses Fourier transforms. We present next a short introduction to Fourier transforms.

12.3.2   Fourier transforms

There are several methods for solving linear parabolic PDE's. When the domain of the independent variable x is $(-\infty, \infty)$, the most direct method to find a solution is to using Fourier and inverse Fourier transforms (see Appendix 12.10).

The method of obtaining a solution follows three steps: first, we transform function $u(t,x)$ into function $U(t.,\omega)$, where $\omega \in \Omega = \mathbb{R}$ represents frequencies, such that the PDE over $u(t,x)$ is transformed into a parameterized ordinary differential equation over $U(t\omega)$; second we solve the ODE over $U(\cdot)$; and finally we transform back to the original function $u(t,x)$. When the domain of x is not the double-infinity we may have to adapt this method.

There are several possible transformations: sine, cosine, Laplace, Mellin or Fourier transforms. Next we use the Fourier transform approach.

The **Fourier transform** of the integrable $u(t,x)$, taking t as a parameter, is [3]

$$U(t,\omega) = \mathcal{F}[u(t,x)](\omega) \equiv \int_{-\infty}^{\infty} u(t,x)e^{-2\pi i \omega x}dx \tag{12.3}$$

where $i^2 = -1$ and the **inverse Fourier transform** is

$$u(t,x) = \mathcal{F}^{-1}[U(t,\omega)](x) \equiv \int_{-\infty}^{\infty} U(t,\omega)e^{2\pi i \omega x}d\omega. \tag{12.4}$$

Derivatives also have Fourier transform representations: first-order derivative representations are

$$u_t(t,x) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} U(t,\omega)e^{2\pi i \omega x}d\omega = \int_{-\infty}^{\infty} U_t(t,\omega)e^{2\pi i \omega x}d\omega,$$

and

$$u_x(t,x) = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} U(t,\omega)e^{2\pi i \omega x}d\omega = \int_{-\infty}^{\infty} 2\pi\omega i\,U(t,\omega)e^{2\pi i \omega x}d\omega,$$

---

[3]There are different definitions of Fourier transforms, we use the definition by, v.g., Kammler (2000).

and the second-order derivative is

$$
\begin{aligned}
u_{xx}(t,x) &= \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} U(t,\omega) e^{2\pi i \omega x} d\omega \\
&= \frac{\partial}{\partial x} \int_{-\infty}^{\infty} 2\pi \omega i \, U(t,\omega) e^{2\pi i \omega x} d\omega \\
&= \int_{-\infty}^{\infty} (2\pi \omega i)^2 U(t,\omega) e^{2\pi i \omega x} d\omega \\
&= -\int_{-\infty}^{\infty} (2\pi \omega)^2 U(t,\omega) e^{2\pi i \omega x} d\omega.
\end{aligned}
$$

Fourier transforms share the associative and distributive properties of integrals of functions defined by sums, but not of multiplications of functions. However, there is a useful relationship between between a convolution of functions and the multiplication of Fourier transforms.

Function $u(t,x)$ is a convolution if it can be written as

$$
u(t,x) = v(t,x) * y(t,x) \equiv \int_{-\infty}^{\infty} v(t,\xi) y(t, x - \xi) d\xi,
$$

where $v(t,x)$ and $y(t,x)$ are integrable functions in the domain $\mathbb{R}_+ \times \mathbb{R}$. Let Fourier transform of $u(t,x)$ be written as a product of two Fourier transforms,

$$
U(t,\omega) = \mathcal{F}[u(t,x)](\omega) = V(t,\omega)\, Y(t,\omega)
$$

where $V(t,\omega) = \mathcal{F}[v(t,x)](\omega)$ and $Y(t,\omega) = \mathcal{F}[y(t,x)](\omega)$. Then $u(t,x)$ is the inverse Fourier transform of $U(t,\omega)$ if and only if $u(t,x)$ is the convolution of the inverse Fourier transforms of $V(\cdot)$ and $Y(\cdot)$,

$$
u(t,x) = \mathcal{F}^{-1}[U(t,\omega)](x) = \mathcal{F}^{-1}[V(t,\omega)Y(t,\omega)](x) = v(t,x) * y(t,x).
$$

12.3.3   The forward heat equation in the infinite domain

In this subsection we solve the slightly more general version of equation (12.2) in the infinite domain for an arbitrary bounded initial condition and for a given initial conditions. The last two are versions of Cauchy problems in which the side conditions refer to $t = 0$.

Free but bounded initial condition

The simplest linear PDE for an infinite domain $X = \mathbb{R}$

$$
u_t - a u_{xx} = 0, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{12.5}
$$

where $a > 0$ makes it a forward PDE.

Proposition 1. *Let $u_0(x)$ be an arbitrary but bounded function, i.e. satisfying $\int_{-\infty}^{\infty} |u_0(x)| dx < \infty$. Then the solution to PDE (12.5) is*

$$u(t,x) = \begin{cases} u_0(x), & (t,x) \in \{t=0\} \times \mathbb{R} \\ \dfrac{1}{2\sqrt{\pi a t}} \displaystyle\int_{-\infty}^{\infty} u_0(\xi) e^{-\frac{(x-\xi)^2}{4at}} \, d\xi, & (t,x) \in \mathbb{R}_{++} \times \mathbb{R} \end{cases} \qquad (12.6)$$

Proof. The proof proceeds in three steps. First step: transform PDE (12.5) into a parameterized ODE. Applying the previous definition of Fourier transform to the PDE (12.5) we have

$$u_t - a u_{xx} = \int_{-\infty}^{\infty} \left(U_t(t,\omega) + a(2\pi\omega)^2 U(t,\omega) \right) e^{2\pi i \omega x} d\omega = 0.$$

Defining $\lambda(\omega) \equiv -a\,(2\pi\omega)^2$ (called an eigenfunction), the equation holds iff and only if $U(\cdot, \omega)$ solves the characteristic equation

$$U_t(t,\omega) = \lambda(\omega)\, U(t,\omega),$$

which is an ODE with time as the independent variable and ω is treated as a parameter. Second step: solving the characteristic equation yields

$$U(t,\omega) = U_0(\omega)\, G(t,\omega)$$

where function $U_0(\omega)$ is arbitrary and

$$G(\omega, t) \equiv \begin{cases} 1, & t = 0 \\ e^{\lambda(\omega)\,t}, & t > 0 \end{cases}$$

is called the **Gaussian kernel**. Third step: we transform back from $U(t,\omega)$ to $u(t,x)$ using the inverse Fourier transform, yielding

$$u(t,x) = \mathcal{F}^{-1}[U(t,\omega)](x) = \mathcal{F}^{-1}[U_0(\omega)\, G(t,\omega)] = u_0(x) * g(t,x)$$

where $u_0(x) * g(t,x)$ is a convolution, that is

$$u_0(x) * g(t,x) = \int_{-\infty}^{\infty} u_0(\xi) g(t, x - \xi) d\xi.$$

Using the tables in the Appendix, for $g(x,t) = \mathcal{F}^{-1}[G(t,\omega)](x)$ the Gaussian kernel in the initial variable is

$$g(t,x) = \begin{cases} \delta(x), & t = 0 \\ \dfrac{e^{-\frac{x^2}{4at}}}{2\sqrt{\pi a t}}, & t > 0 \end{cases}$$

where $\delta(.)$ is the Dirac's delta function.

Therefore, because and $u_0(x) = \mathscr{F}^{-1}[U_0(t\omega)](x)$,

$$u(t,x) = \begin{cases} u_0(x), & t = 0 \\ \dfrac{1}{2\sqrt{\pi a t}} \displaystyle\int_{-\infty}^{\infty} u_0(\xi) e^{-\frac{(x-\xi)^2}{4at}} \, d\xi, & t > 0 \end{cases} \tag{12.7}$$

where $u_0(x)$ **is an arbitrary but bounded function**, i.e. satisfying $\int_{-\infty}^{\infty} |u_0(x)| dx < \infty$, and because $\int_{-\infty}^{\infty} u_0(\xi)\delta(x-\xi)d\xi = u_0(x)$. $\square$

Two observations can be made concerning the solution of this PDE.

First, applying the Fourier transform, we change from a distribution in the original variables x to a frequency distribution ω.

The transformed PDE becomes a linear ODE the coefficient is eigenfunction

$$\lambda(\omega) = -a\,(2\pi\omega)^2, \text{ for } \omega \in \mathbb{R}$$

which is real and non-positive for any $\omega \in \mathbb{R}$: $\lambda(0) = 0$ and $\lambda(\omega) < 0$ for $\omega \neq 0$ and, $\lim_{\omega \to \pm\infty} \lambda(\omega) = -\infty$. This means that $\lim_{\omega \to \pm\infty} U(t,\omega) = 0$ for any t if $K(\omega)$ is bounded.

Second, associated to the previous property is the solution of $u(t,x)$ is an expected value of the arbitrary function where the density function is a Gaussian density function with average 0 and variance $2\,a\,t$.

Initial value problem Now we consider a well-posed linear FPDE. Assume we know the distribution at time $t = 0$, then we have an **initial value problem**

$$\begin{cases} u_t = a u_{xx}, & (t,x) \in (0,\infty) \times (-\infty,\infty) \\ u(0,x) = \phi(x) & (t,x) \in \{t=0\} \times (-\infty,\infty) \end{cases} \tag{12.8}$$

where $a > 0$ and $\phi(x)$ is a **known** bounded function. Applying (12.7), the solution is

$$u(t,x) = \begin{cases} \phi(x), & (t,x) \in \{t=0\} \times \mathbb{R} \\ \displaystyle\int_{-\infty}^{\infty} \frac{\phi(\xi)}{2\sqrt{\pi a t}} \, e^{-\dfrac{(x-\xi)^2}{4at}} \, d\xi, & (t,x) \in \mathbb{R}_{++} \times \mathbb{R}. \end{cases}$$

Example Figure 12.1 illustrates the behavior of the solution for $a = 1$ and $\phi(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$, which is simplified to

$$u(t,x) = \frac{1}{\sqrt{\pi(1+4\,t)}} \, e^{-\dfrac{x^2}{1+4\,t}}.$$

As can be seen, the solution decays over time and converges to a homogeneous distribution

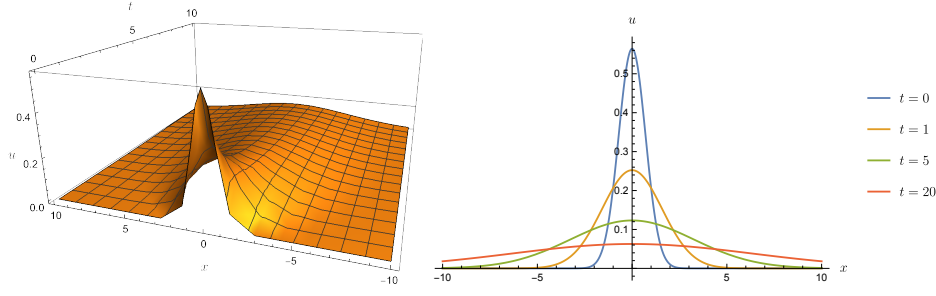$$\lim_{t \to \infty} u(t,x) = 0, \forall x \in (-\infty,\infty)$$

Figure 12.1: Solution for the initial value problem for the heat equation with $a = 1$ and $\phi(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$.

However, a conservation law holds,

$$\int_{-\infty}^{\infty} u(t, x)\, dx = 1, \text{ for each } t \geq 0$$

Piecewise-constant initial condition We consider again the simple linear defined in the infinite domain but with the initial condition

$$\phi(x) = \begin{cases} \phi_0, & \text{if } x \in [\underline{x}, \overline{x}] \\ 0 & \text{if } x \notin [\underline{x}, \overline{x}) \end{cases}$$

where $\underline{x} < \overline{x}$ are both finite and ϕ_0 is a constant. In this case, the solution to the problem is

$$u(t, x) = \phi_0 \left[\Phi\left(\frac{x - \underline{x}}{\sqrt{2at}} \right) - \Phi\left(\frac{x - \overline{x}}{\sqrt{2at}} \right) \right] \tag{12.9}$$

where $\Phi(z)$ is the standard normal distribution function

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}}\, dz.$$

Observe that $\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz = \sqrt{2\pi}$ then $\Phi(y) \in (0, 1)$.

To prove this result, with apply the general solution, presented in equation (12.7). This yields the solution of the initial-value problem

$$u(t, x) = \frac{\phi_0}{2\sqrt{\pi a t}} \int_{\underline{x}}^{\overline{x}} e^{-\frac{(x - \xi)^2}{4at}}\, d\xi.$$

To simplify the expression, we make the transformation of variables $z \equiv (x - \xi)/\sqrt{2at}$, and denote
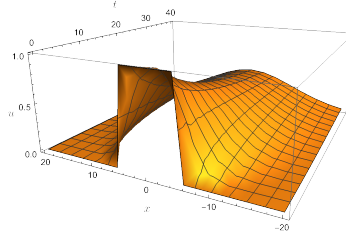
Figure 12.2: Solution for the initial value problem for the heat equation with $a = 1$ and piecewise initial condition.

$\overline{z} \equiv (\overline{x} - \xi)/\sqrt{2at}$ and $\underline{z} \equiv (\underline{x} - \xi)/\sqrt{2at}$. Then, because $dz = -1/\sqrt{2at}d\xi$ the solution simplifies [4]

$$
\begin{aligned}
\frac{1}{\sqrt{4\pi at}} \int_{\underline{x}}^{\overline{x}} e^{-(x-\xi)^2/4at}d\xi &= -\frac{\sqrt{2at}}{\sqrt{4\pi at}} \int_{(x-\underline{x})/\sqrt{2at}}^{(x-\overline{x})/\sqrt{2at}} e^{-z^2/2}dz \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{(x-\underline{x})/\sqrt{2at}} e^{-z^2/2}dz - \int_{-\infty}^{(x-\overline{x})/\sqrt{2at}} e^{-z^2/2}dz \right) = \\
&= \Phi \left(\frac{x - \underline{x}}{\sqrt{2at}} \right) - \Phi \left(\frac{x - \overline{x}}{\sqrt{2at}} \right).
\end{aligned}
$$

The solution (12.9) is illustrated in Figure 12.2.

12.3.4 The forward linear equation in the semi-infinite domain

Now consider the equation defined on the semi-infinite domain for x, that is $X = \mathbb{R}_+$. This case is more interesting for economic applications in which the independent variable can only take non-negative values, for instance when x refers to a stock.

The FPDE we consider is

$$
u_t - au_{xx} = 0, \ (t,x) \in \mathbb{R}_+^2 \tag{12.10}
$$

where $a > 0$.

Proposition 2. *The solution to equation* (12.12) *is*

$$
u(t,x) = \frac{1}{2\sqrt{\pi a t}} \int_0^\infty u_0 \left(\xi \right) \left(e^{-\frac{(x-\xi)^2}{4at}} - e^{-\frac{(x+\xi)^2}{4at}} \right) \ d\xi, \ t > 0. \tag{12.11}
$$

where $u_0(x) : X \to \mathbb{R}_+$ is an arbitrary bounded function.

---

[4]Recalling the formula for integration by substitution of variables, if we set $z = \varphi(\xi)$ and $\xi \in (a,b)$ then

$$
\int_{\varphi(a)}^{\varphi(b)} f(z)dz = \int_a^b f(\varphi(\xi)) \, \varphi'(\xi) \, d\xi.
$$

Proof. We solve this equation by using the **method of images**. It consists in introducing the following extension to the arbitrary function $k(x)$

$$\tilde{u}_0(x) = \begin{cases} u_0(x), & \text{if } x \geq 0 \\ -u_0(-x) & \text{if } x < 0 \end{cases}$$

where $k(.)$ is an odd function satisfying $k(-x) = -k(x)$. Using the solution (12.7) for $t > 0$ we have

$$u(t,x) = \frac{1}{2\sqrt{\pi a t}} \int_{-\infty}^{\infty} \tilde{u}_0(\xi) e^{-\frac{(x-\xi)^2}{4at}} \, d\xi =$$

$$= \frac{1}{2\sqrt{\pi a t}} \left(\int_{-\infty}^{0} \tilde{u}_0(\xi) e^{-\frac{(x-\xi)^2}{4at}} \, d\xi + \int_{0}^{\infty} \tilde{u}_0(\xi) e^{-\frac{(x-\xi)^2}{4at}} \, d\xi \right) =$$

$$= \frac{1}{2\sqrt{\pi a t}} \left(-\int_{0}^{\infty} u_0(\xi) e^{-\frac{(x+\xi)^2}{4at}} \, d\xi + \int_{0}^{\infty} u_0(\xi) e^{-\frac{(x-\xi)^2}{4at}} \, d\xi \right)$$

where the last step involves integration by substitution: i.e., if we define $u = -x$ for $x \in [0, \infty)$ then $\int_{-\infty}^{0} f(u) du = -\int_{\infty}^{0} f(-x) dx = \int_{0}^{\infty} f(-x) dx$. Then the solution of equation (12.12) is equation (12.13). □

The solution to the initial-value problem

$$\begin{cases} u_t - a u_{xx} = 0, & \text{for } a > 0, \ (t,x) \in \mathbb{R}_+^2 \\ u(0,t) = u_0(x), & \text{for } (t,x) \in \{t = 0\} \times \mathbb{R}_+ \end{cases} \tag{12.12}$$

is

$$u(t,x) = \frac{1}{2\sqrt{\pi a t}} \int_{0}^{\infty} u_0(\xi) \left(e^{-\frac{(x-\xi)^2}{4at}} - e^{-\frac{(x+\xi)^2}{4at}} \right) \, d\xi, \ t > 0. \tag{12.13}$$

We obtain this result by direct application of equation (12.13).

Example Consider the initial-value problem in which the initial distribution is log-normal

$$u_0(x) = \frac{e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}}{2\sqrt{\pi x^2 \sigma^2}}$$

if we substitute in equation (12.13) we have the solution depicted in Figure 12.5 for several moments in time.

We observe that the solution is not conservative, i.e. the integral $U(t) = \int_{0}^{\infty} u(t,x) \, dx$ decays in time such that $\lim_{t \to \infty} U(t) = 0$.

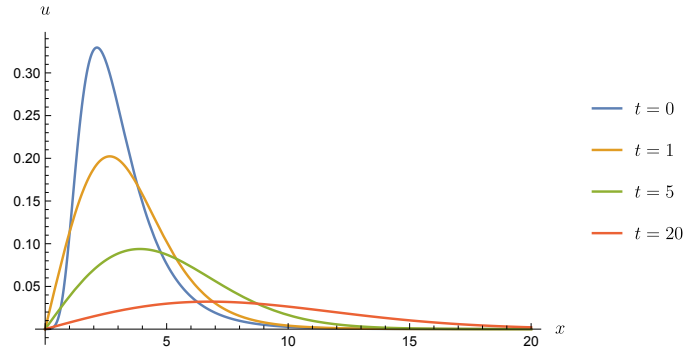## 12.4   The forward homogeneous linear PDE

Figure 12.3: Solution for the initial value problem for the heat equation in the semi-infinite line with $a = 1$ and an initial log-normal density.

The general forward linear homogeneous parabolic PDE in the infinite domain, $X = (-\infty, \infty)$ is

$$u_t(t, x) = a\, u_{xx}(t, x) + b\, u_x(t, x) + c\, u(t, x), \ (t, x) \in (0, \infty) \times X \tag{12.14}$$

where $a > 0$, and b and c are real numbers. It contains three terms: a diffusion term, if $a \neq 0$, a transport term, if $b \neq 0$, and a growth or decay term if $c > 0$ or $c < 0$.

In order to solve the equation, we can follow one of two alternative methods:

1. apply the Fourier transform method to transform the PDE into a parameterized ODE, solve it, and apply inverse Fourier transforms.

2. transform the equation into a heat equation, solve the heat equation and transform back to the initial variables.

Next, we use the first method to prove the following result

Proposition 3. *The solution to equation* (12.17) *is*

$$u(t, x) = \int_{-\infty}^{\infty} u_0(s)\, \frac{1}{\sqrt{4\pi a t}} \exp\left(-\frac{(x - s)^2 + 2\, b\, (x - s)\, t + (b^2 - 4ac)\, t^2}{4\, a\, t}\right) ds, \ (t, x) \in \mathbb{R}_+ \times \mathbb{R} \tag{12.15}$$

where $u_0(x)$ is an arbitrary bounded function mapping $u_0 : X \to \mathbb{R}$.

Proof. We will solve this problem using the Fourier transform representation of equation $u_t - (au_{xx} + bu_x + cu) = 0$. Using inverse Fourier transforms yields

$$u_t(t, x) - au_{xx}(t, x) - bu_x - cu(t, x) = \int_{-\infty}^{\infty} e^{2\pi i\omega x} \left[\, U_t(t, \omega) + \lambda(\omega)U(t, \omega)\right]\ d\omega = 0.$$

where the eigenfunction is a complex-valued function of ω [5]

$$\lambda(\omega) \equiv a(2\pi\omega)^2 - c - 2\pi\, b\, \omega\, i, \ \text{where} \ i \equiv \sqrt{-1}.$$

---

[5]The advection term, involving the first derivative has a complex-valued Fourier transform representation

$$u_x(t, x) = \frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty} U(t, \omega)e^{2\pi i x\omega}d\omega\right) = \int_{-\infty}^{\infty} 2\pi\omega i\, U(t, \omega)e^{2\pi i x\omega}d\omega.$$

.

Therefore, the PDE (12.17) has the characteristic equation

$$U_t(t,\omega) = -\lambda(\omega)\,U(t,\omega), \ (t,\omega) \in \mathbb{R}_+ \times \mathbb{R},$$

which has the explicit solution

$$U(t,\omega) = U_0(\omega)\,G(t,\omega), \ \text{for} t \in [0,\infty)$$

where $G(t,\omega)$ is the Gaussian kernel

$$G(t,\omega) = e^{-\lambda(\omega)t}, \ \text{for } t > 0.$$

We obtain the solution of problem (12.17) by applying the inverse Fourier transform

$$u(t,x) = \mathscr{F}^{-1}\left[U(t,\omega)\right](x) = \mathscr{F}^{-1}\left[U_0(\omega)\,G(t,\omega)\right](x) = \int_{-\infty}^{\infty} u_0(s)g(t,x-s)ds$$

where (see the Appendix 12.10)

$$g(t,y) = \mathscr{F}^{-1}\left[\,e^{-\lambda(\omega)t}\,\right] = \frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{y^2 + 2bty + (b^2 - 4ac)t^2}{4at}\right), \tag{12.16}$$

because $a\,t > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

12.4.1   Initial value problem

The initial value problem for a a general linear homogeneous (forward) diffusion equation is

$$\begin{cases} u_t(t,x) = a\,u_{xx}(t,x) + b\,u_x(t,x) + c\,u(t,x), & (t,x) \in (0,\infty) \times (-\infty,\infty) \\ u(0,x) = \phi(x), & (t,x) \in \{t=0\} \times (-\infty,\infty), \end{cases} \tag{12.17}$$

where $a > 0$, $b \neq 0$ and $c \neq 0$ and $\phi(x)$ is a known bounded function defined on $X = \mathbb{R}$.

The solution to this problem is

$$u(t,x) = \int_{-\infty}^{\infty} \phi(s)\,\frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{(x-s)^2 + 2\,b\,(x-s)\,t + (b^2 - 4ac)\,t^2}{4\,a\,t}\right) ds.$$

Figure 12.4 illustrates the solution to the initial value problem associated to equation (12.20) for negative (left figures) and positive (right figures) values for b and negative (upper figures) and positive (lower figures) values of c. It is clear that while b introduces a transportation in the positive direction, if $b < 0$, or in the negative direction, if $b > 0$, c is associated to the time stability of the whole distribution.

12.4.2   Particular case: no transport term

If the linear forward PDE does not contain a transport term, we have

$$u_t = a\,u_{xx} + c\,u, \ (t,x) \in (0,\infty) \times (-\infty,\infty) \tag{12.18}$$
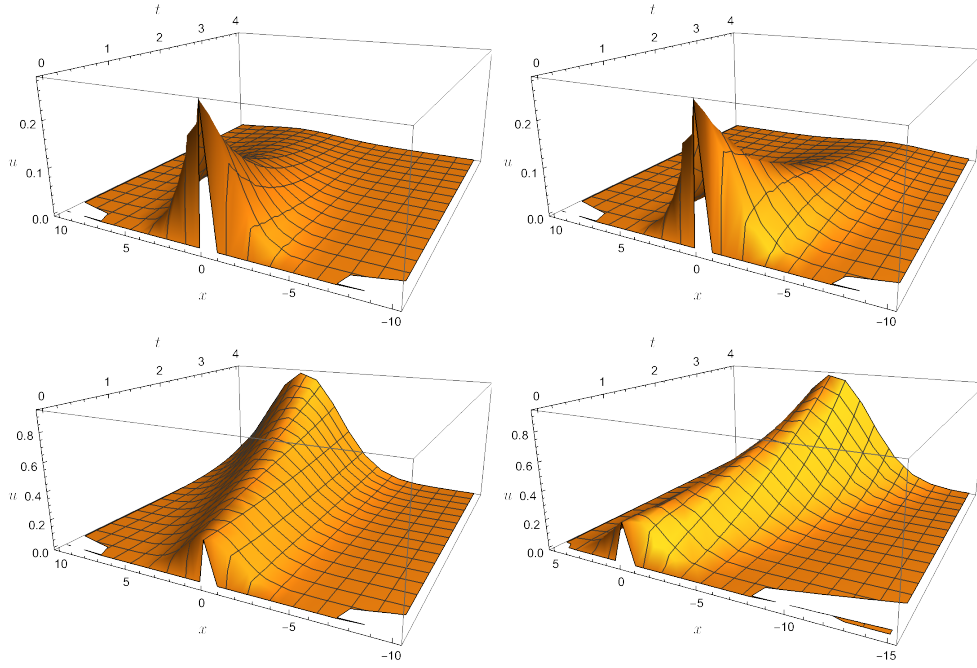
Figure 12.4: Solution for the initial value problem linear PDE for $a = 1$, $b = 1$ and $b = -1$ and $c = -0.5$ and $b = 1$ and $b = -1$ and $c = 0.5$.

where $a > 0$ and $c \neq 0$, which has solution, for a known bounded function $\phi(x)$

$$
u(t,x) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \phi(\xi)\delta(x - \xi)d\xi = \phi(x), & (t,x) \in \{t = 0\} \times \mathbb{R} \\ \displaystyle e^{ct} \int_{-\infty}^{\infty} \phi(\xi)\frac{1}{\sqrt{4\,\pi\,a\,t}}e^{-\frac{(x-\xi)^2}{4\,a\,t}}\ d\xi, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \end{cases}
$$

To find the solution, we use the second method. First, define $v(t,x) = e^{-ct}u(t,x)$, which has derivatives $v_t = -ce^{-ct}u + e^{-ct}u_t$ and $v_{xx} = e^{-ct}u_{xx}$. Second, equation (12.18) is equivalent to the simplest linear equation $v_t = av_{xx}$ which has solution (12.7). Third, as $u(t,x) = e^{ct}\,v(t,x)$ we obtain the solution

Figure 12.5 illustrates the cases in which $c < 0$ and $c > 0$. In both cases we see that the long-time behavior of the solution is commanded by e^{ct}: if $c < 0$ then $\lim_{t\to\infty} u(t,x) = 0$, for any $x \in \mathbb{R}$, and if $c > 0$ then $\lim_{t\to\infty} u(t,x) \propto \lim_{t\to\infty} e^{ct} = \infty$, for any $x \in \mathbb{R}$.

We see that the dynamics of the solution depends crucially on the sign of c:

$$
\lim_{t\to\infty} u(t,x) = \begin{cases} 0 & \text{if } c < 0 \\ \infty & \text{if } c > 0 \end{cases}
$$

This means that the diffusion equation display **asymptotic stability** if $c < 0$ and **instability** if $c > 0$, both in a distributional sense. In the first case the solution $u(t,x)$ is bounded and in the second case it is unbounded.
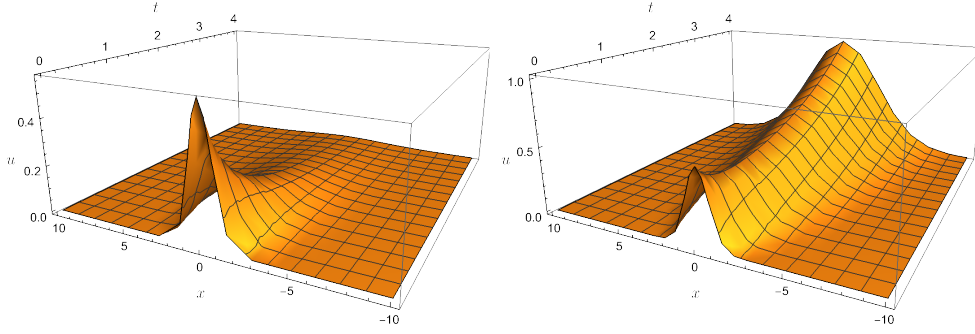
Figure 12.5: Solution for the initial value problem for the heat equation with $a = 1$ and $\phi(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$ and $c = -0.5$ and $c = 0.5$.

12.5 The non-homogeneous linear PDE

The general forward linear non-homogeneous parabolic PDE in the infinite domain, $X = (-\infty, \infty)$ is

$$u_t(t, x) = a\, u_{xx}(t, x) + b\, u_x(t, x) + c\, u(t, x) + d, \ (t, x) \in (0, \infty) \times X \tag{12.19}$$

where $a > 0$, and b, c and d are real numbers. We had a new constant term to PDE (12.17).

Proposition 4. *The solution to equation* (12.19) *is*

$$u(t, x) = \int_{-\infty}^{\infty} u_0(\xi)\, \frac{e^{\gamma(x, t, \xi)}}{\sqrt{4\pi a t}}\, d\xi + \int_0^t \int_{-\infty}^{\infty} \frac{d\, e^{\gamma(x, t-s, \xi)}}{\sqrt{4\pi a (t - s)}}\, d\xi\, ds, \ (t, x) \in \mathbb{R}_+ \times \mathbb{R} \tag{12.20}$$

where

$$\gamma(t, x, \xi) \equiv -\frac{(x - \xi)^2 + 2\, b\, (x - \xi)\, t + (b^2 - 4ac)\, t^2}{4\, a\, t}$$

where $u_0(x)$ is an arbitrary bounded function mapping $u_0 : X \to \mathbb{R}$.

Proof. We will solve this problem using the Fourier transform representation of equation $u_t - (au_{xx} + bu_x + cu + d) = 0$. Using inverse Fourier transforms yields

$$u_t(t, x) - au_{xx}(t, x) - bu_x - cu(t, x) - d = \int_{-\infty}^{\infty} e^{2\pi i \omega x} \left[U_t(t, \omega) + \lambda(\omega) U(t, \omega) - d\, \delta(\omega) \right]\, d\omega = 0.$$

where again $\lambda(\omega) \equiv a(2\pi\omega)^2 - c - 2\pi\, b\, \omega\, i$. Therefore, the PDE (12.17) has the characteristic equation

$$U_t(t, \omega) = -\lambda(\omega)\, U(t, \omega) + d\, \delta(\omega), \ (t, \omega) \in \mathbb{R}_+ \times \mathbb{R},$$

which has the explicit solution

$$U(t, \omega) = U_0\, G(t, \omega) + \int_0^t d\, \delta(\omega)\, G(t - s, \omega)\, ds, \ \text{for}\, t \in [0, \infty)$$

where $G(t, \omega)$ is the Gaussian kernel.

We obtain the solution of problem (12.17) by applying the inverse Fourier transform

$$u(t, x) = \mathcal{F}^{-1}\left[U(t, \omega)\right](x) = \mathcal{F}^{-1}\left[U_t(\omega)\, G(t, \omega)\right](x)$$

$$= u_0(x) * g(t, x) + \int_0^t d * g(t - s, x)\, ds.$$

$\square$

12.6　Linear backward PDE

12.6.1　The backward heat equation

In finance applications and associated to Euler equation in optimal control problems, we sometimes need to solve backward parabolic PDE.

The simplest parabolic BPDE equation in the infinite domain for x and for the semi-infinite domain for t is

$$u_t + au_{xx} = 0, \ (t, x) \in [0, T] \times (-\infty, \infty) \tag{12.21}$$

where $a > 0$.

Proposition 5. *Consider the BPDE equation* (12.21). *The solution is*

$$u(t, x) = \begin{cases} k(x), & t = T \\ \frac{1}{\sqrt{4\pi a(T-t)}} \int_{-\infty}^{\infty} k(\xi) e^{-\frac{(x-\xi)^2}{4a(T-t)}}\, d\xi, & t \in (0, T) \end{cases}$$

Proof. In order to solve it we introduce a change in variables $\tau = T - t$ and denote $v(\tau, x) = u(t(\tau), x)$ where $t(\tau) = T - \tau$. As

$$v_\tau(\tau, x) = -u_t(t(\tau), x), \text{ and } v_{xx}(\tau, x) = u_{xx}(t(\tau), x)$$

Then $u_t(t, x) = -au_{xx}(t, x)$ is equivalent to

$$v_\tau(\tau, x) = av_{xx}(\tau, x).$$

Using the solution already found in equation (12.7) we get

$$v(\tau, x) = \begin{cases} k(x), & \tau = 0 \\ \int_{-\infty}^{\infty} k(\xi)\, (4\pi a\tau)^{-1/2}\, e^{-\frac{(x-\xi)^2}{4a\tau}}\, d\xi, & \tau \in (0, T). \end{cases}$$

Transforming back to $u(t, x)$ we have solution. $\square$

A problem involving a backward PDE is only well posed if together with the PDE we have a terminal condition, for example $u(T, x) = \phi_T(x)$. In this case the value of the variable at time $t = 0$ becomes endogenous.

Consider the **terminal-value problem**

$$
\begin{cases}
u_t = -au_{xx}, & (t, x) \in (-\infty, \infty) \times (0, T] \\
u(T, x) = \phi_T(x) & (t, x) \in (-\infty, \infty) \times \{\, t = T\}.
\end{cases}
$$

The solution is

$$
u(t, x) = \begin{cases}
\phi_T(x), & (t, x) \in \{t = T\} \times \mathbb{R} \\
\dfrac{1}{\sqrt{4\pi a(T - t)}} \displaystyle\int_{-\infty}^{\infty} \phi_T(\xi) e^{-\frac{(x-\xi)^2}{4a(T-t)}} \, d\xi, & (t, x) \in (0, T) \times \mathbb{R}
\end{cases}
$$

The initial distribution can be obtained by setting $t = 0$

$$
u(0, T) = \frac{1}{\sqrt{4\pi a(T - t)}} \int_{-\infty}^{\infty} \phi_T(\xi) e^{-\frac{(x-\xi)^2}{4aT}} \, d\xi.
$$

12.7 Non-autonomous linear equations

Next we consider two non-autonomous equations in which there is one term depending on the independent variables (t, x)

Non-autonomous diffusion equation

A non-autonomous linear PDE in which the independent term is dependent on the "spatial" variable can be

$$
u_t = au_{xx} + bu + d(x), \ (t, x) \in (-\infty, \infty) \times (0, \infty)
$$

where $a > 0$. It can be proved (see Exercise 1) that the solution for $t > 0$ is

$$
u(t, x) = \frac{e^{bt}}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} \phi(\xi) e^{-\frac{(x-\xi)^2}{4at}} \, d\xi + \frac{1}{\sqrt{4\pi a(t - \tau)}} \int_0^t e^{b(t-\tau)} \int_{-\infty}^{\infty} d(\xi) e^{-\frac{(x-\xi)^2}{4a(t-\tau)}} \, d\xi d\tau
$$

Non-homogeneous heat equation A non-homogeneous and non-autonomous (forward) heat equation in which the independent term is dependent on (t, x) can be

$$
u_t - au_{xx} - b(t, x) = 0, \ (t, x) \in (0, \infty) \times (-\infty, \infty) \tag{12.22}
$$

this equation has a component which is not affected by u, although it changes with (t, x).

In order to solve it, we again use inverse Fourier transforms to get an equivalent ODE in transformed variables $U(t, \omega)$,

$$U(t, \omega) = -\lambda(\omega)U(t, \omega) + B(t, \omega)$$

where $B(t, \omega) = \mathcal{F}[b(t, x)](\omega)$ and $\lambda(\omega) = (2\pi\omega)^2 a$. The solution to equation (12.22)is

$$U(t, \omega) = K(\omega)G(t, \omega) + \int_0^t B(s, \omega)\, G(t - s, \omega)ds,$$

where $G(t, \cdot)$ is a Gaussian kernel. Applying inverse Fourier transforms yields

$$u(t, x) = k(x) * g(t, x) + \int_0^t b(s, x) * g(t - s, x)ds.$$

Therefore, the solution to the parabolic PDE (12.22) is, for $t > 0$,

$$u(t, x) = \frac{1}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} k(\xi)e^{-\frac{(x-\xi)^2}{4at}}\, d\xi + \int_0^t \frac{1}{\sqrt{4\pi a(t - s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4a(t-s)}} b(s, \xi)d\xi ds.$$

The solution can converge to a spatially non-homogenous distribution.

12.8   Fokker-Planck-Kolmogorov equation

We will see in the chapter on stochastic differential equations, that the probability distribution of a diffusion   process follows a particular parabolic PDE, called the Fokker-Planck-Kolmogorov equation. This equation is having an increase attention, also in economics, as a model for processes satisfying a conservation law.

Assume a process $\big(p(t, x)\big)_{(t,x)\in T\times X}$ such where $p(t, x) : T \times X \to (0, 1)$, and

$$\int_X p(t, x)dx = 1, \text{ for every } t \in T,$$

which is a conservation law. The Fokker-Planck-Kolmogorov equation for this process can have the form

$$\partial_t p(t, x) = \frac{1}{2}\partial_{xx}\Big(a(t, x)^2\, p(t, x)\Big) - \partial_x\Big(b(t, x)\, p(t, x)\Big) \tag{12.23}$$

where we assume $p(0, x)$ is known and satisfies

$$\int_X p(0, x)dx = 1.$$

In applications resulting from stochastic differential equations, the initial state is known $x = x_0$ and the dynamics of a probability distribution is given by Kolmogorov forward equation (or Fokker-Planck equation) and the initial condition $p(0, x) = \delta(x - x_0)$ where $\delta(\cdot)$ is Dirac's delta generalized function.

12.8.1   The simplest problem

The simplest model has constant coefficients $b(t,x) = \mu$ and $a(t,x) = \sigma$ and a Dirac delta initial function:

$$\begin{cases} \partial_t p(t,x) = \dfrac{\sigma^2}{2}\,\partial_{xx} p(t,x) - \mu\partial_x p(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ p(0,x) = \delta(x - x_0) & (t,x) \in \{\, t = 0\,\}\, \times \mathbb{R} \end{cases} \tag{12.24}$$

The solution is a Gamma probability density

$$p(t,x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2\,t}}\,\exp-\Big(\frac{x - x_0 - \mu\,t}{2\,\sigma^2\,t}\Big)^2$$

$$= \Gamma\big(-\mu t; \frac{\sigma^2}{2}, x - x_0\big)\,(t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

This equation is a particular case of equation (12.17).

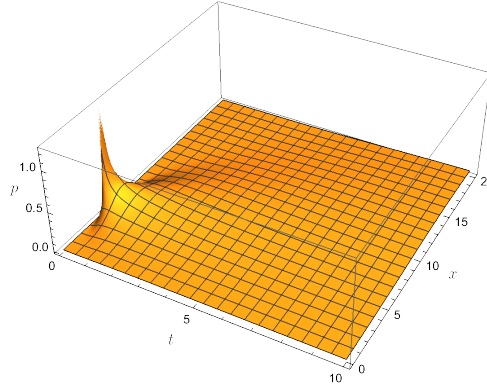Figure 12.6   presents an illustration of the solution

Figure 12.6: Solution for (12.24)   for $x_0 = 5$, $\mu = 1$ and $\sigma = 0.5$.

12.8.2   The distribution associated to the Ornstein Uhlenbeck equation

An important stochastic process has a probability distribution represented by a linear PDE with (non-autonomous) coefficients $b(t,x) = \mu_0 + \mu_1 x$ and $a(t,x) = \sigma$ and a Dirac delta initial distribution:

$$\begin{cases} \partial_t p(t,x) = -\partial_x\Big((\mu_0, +\mu_1 x)\,p(t,x)\Big) + \dfrac{1}{2}\,\partial_{xx}\Big(\sigma^2\,p(t,x)\Big), & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ p(0,x) = \delta(x - x_0) & (t,x) \in \{\, t = 0\,\}\, \times \mathbb{R} \end{cases} \tag{12.25}$$

The solution is a Gaussian density function

$$p(t,x) \sim N\Big(\frac{\mu_0}{\mu_1} - \Big(\frac{\mu_0}{\mu_1} +\ x_0\Big)\,e^{\mu_1 t}\,,\,\frac{\sigma^2}{2\,\mu_1}\,(e^{2\mu_1 t} - 1\,)\Big),$$

because

$$p(t,x) = \left(\pi \frac{\sigma^2}{\mu_1}\left(e^{2\mu_1 t} - 1\right)\right)^{-\frac{1}{2}} \exp\left\{-\frac{\left(x + \frac{\mu_0}{\mu_1} - \left(\frac{\mu_0}{\mu_1} + x_0\right)\right)^2}{\frac{\sigma^2}{\mu_1}\left(e^{2\mu_1 t} - 1\right)}\right\} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \qquad (12.26)$$

Exercise: To prove function (12.26) is the solution of problem (12.25) we follow the following steps. First, apply Fourier transforms. Using the results in the appendix, we find

$$\mathcal{F}[\partial_t p(t,x)](\omega) = \partial_t P(t,\omega),$$

$$\mathcal{F}\left[-\partial_x\left((\mu_0, +\mu_1 x)\, p(t,x)\right) + \frac{1}{2}\,\partial_{xx}\left(\sigma^2\, p(t,x)\right)\right](\omega) = -\mu_0 2\pi\omega i\, P(t,\omega) + \mu_1\omega\partial_w P(t,\omega) - 2(\pi\sigma\omega)^2 P(t,\omega)$$

and

$$\mathcal{F}[p(0,x)](\omega) = e^{-2\pi\omega i x_0}.$$

Therefore, the characteristic equation is a first-order PDE in the Fourier transforms

$$\begin{cases} \partial_t P(t,\omega) - \mu_1\omega\,\partial_w P(t,\omega) = \lambda(\omega)\, P(t,\omega) & (t,\omega) \in \mathbb{R}_+ \times \mathbb{R} \\ P(0,\omega) = e^{-2\pi\omega i x_0} & (t,\omega) \in \{t = 0\} \times \mathbb{R} \end{cases}$$

where the eigenfunction $\lambda(\omega)$ is a complex function of ω

$$\lambda(\omega) = -(2(\pi\omega\sigma)^2 + 2\pi\omega\mu_0 i).$$

Second, we can solve this PDE by using the method of characteristics: along a characteristic let $\omega = W(t)$, for any $t \in [0,\infty)$ and write $\hat{P}(t) = P(t, W(t))$ and $\Gamma(t) = \lambda(W(t))$. Taking the time derivative to $\hat{P}(t)$ we have

$$\frac{d\hat{P}(t)}{dt} = \partial_t P(t, W(t)) + \partial_w P(t, W(t))\frac{dW(t)}{dt}$$

which is consistent with PDE in the transformed variables along a characteristic if it is the solution of the ODE system

$$\begin{cases} \dfrac{dW(t)}{dt} = -\mu_1\, W(t), \\ \dfrac{d\hat{P}(t)}{dt} = \Gamma(t)\,\hat{P}(t), \text{ for } \hat{P}(0) = P(0, W(0)) = e^{-2\pi W(0) i x_0}. \end{cases}$$

Solving the first ODE yields $W(t) = W(0)e^{-\mu_1 t}$, and as $W(t) = \omega$, we have, along a characteristic, $W(0) = \omega\, e^{\mu_1 t}$.

Solving the second initial-value problem, and because $P(t,\omega) = \hat{P}(t)$ then

$$P(t,\omega) = \hat{P}(0)\, e^{\int_0^t \Gamma(s)ds}$$

$$= \exp\left\{-2\pi i x_0 W(0) - 2(\pi\sigma W(0))^2 \int_0^t e^{-2\mu_1 s}ds - 2\pi i \mu_0 W(0) \int_0^t e^{-\mu_1 s}ds\right\}$$

$$= \exp\left\{-\frac{(\pi\sigma)^2}{\mu_1}W(0)^2\left(1 - e^{-2\mu_1 t}\right) + 2\pi i W(0)\left(\frac{\mu_0}{\mu_1}(e^{-\mu_1 t} - 1) - x_0\right)\right\}$$

substituting $W(0) = \omega \, e^{\mu_1 t}$ yields the function

$$P(t, \omega) = \exp\left\{ -a \, (2\,\pi\,\omega)^2 + b \, (2\,\pi\,i\,\omega) \right\}$$

where

$$a \equiv \frac{\sigma^2}{4\mu_1}\left(e^{2\mu_1 t} - 1\right) \text{ and } b \equiv \frac{\mu_0}{\mu_1} - \left(\frac{\mu_0}{\mu_1} + x_0\right) e^{\mu_1 t}.$$

Transforming back (using the inverse Fourier transform in Appendix 12.10) , that is making $p(t, x) = \mathcal{F}^{-1}[P(t, \omega)] \, (x)$ we find function (12.26).

Characterizing the solution: We see that if $\mu_1 < 0$ then

$$\lim_{t \to \infty} p(t, x) \sim N\big(-\frac{\mu_0}{\mu_1}, -\frac{\sigma^2}{2\mu_1} \big)$$

this means that the distribution is ergodic: for any initial value x_0 it tends asymptotically to a normal distribution (see Figure 12.7).
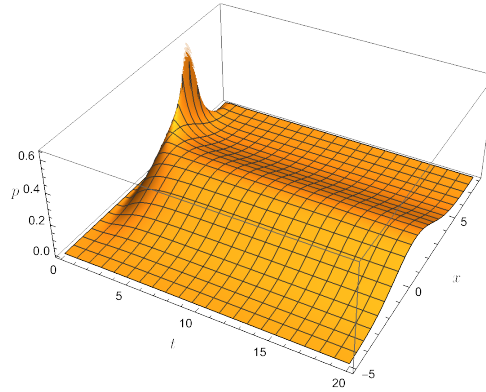

Figure 12.7: Solution for (12.24) for $x_0 = 5$, $\mu_0 = 1$, $\mu_1 = -1$ and $\sigma = 0.5$.

12.9   Economic applications

12.9.1   The distributional Solow model

In Brito (2004) we prove that in an economy in which the capital stock is distributed in an heterogeneous way between regions, $K(t, x)$, if there is an infinite support, and there are free capital flows between regions, the budget constraint for the location x can be represented by the parabolic PDE.

Consider the accounting balance between savings and internal and external investment for a region x at time t

$$I(t,x) + T(t,x) = S(t,x)$$

where $I(t,x)$ and $S(t,x)$ is investment and domestic savings of location x at time t and $T(t,x)$ is the savings flowing to other regions.

Assume that the capital flow for a region of length Δx is symmetric to the capital distribution gradient to neighboring regions:

$$T(t,x)\Delta x = -\left(K_x(x+\Delta x, t) - K_x(t,x)\right)$$

that is capital flows proportionaly and in a reverse direction to the "spatial gradient" of the capital distribution: regions with high capital intensity will tend to "leak" capital to other regions. If $\Delta x \to 0$ leads to $T(t,x) = -K_{xx}(t,x)$.

If there is no depreciation then $I(t,x) = K_t(t,x)$. If the technology is AK and the savings rate is determined as in the Solow model then $S(t,x) = sAK(t,x)$ where $0 < s < 1$ and A is assume to be spatially homogeneous.

Therefore we obtain a distributional Solow equation for an economy composed by heterogenous regions

$$K_t = K_{xx} + sAK, (t,x) \in (-\infty, \infty) \times (0,\infty)$$

We can define a spatially-homogenous balanced growth path (BGP) as

$$\overline{K}(t) = \overline{K}e^{\gamma t}$$

where $\gamma = sA$.

Then, if we define the deviations as regards the BGP as $k(t,x) = K(t,x)e^{-\gamma t}$, we observe that the transitional dynamics is given by the solution of the equation

$$k_t = k_{xx}$$

which is the heat equation. Therefore, given the initial distribution of the capital stock $K(x,0) = k_0(x)$ the solution for this spatial AK model is

$$K(t,x) = \begin{cases} k_0(x), & t = 0 \\ e^{\gamma t} \displaystyle\int_{-\infty}^{\infty} k_0(\xi)\,(4\pi t)^{-1/2}\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi, & t > 0 \end{cases}$$

and the solution is similar to the case depicted in Figure 12.2 when $b > 0$.

The main conclusion is that: (1) there is long run growth; (2) , if there are homogenous technologies and preferences the asymptotic distribution will become spatially homogeneous. That is: the so-called β- and σ- convergences can be made consistent !

12.9.2   The option pricing model

The Black and Scholes (1973) model is a case in which a research paper had an immense impact on the operation of the economy. It is related to the onset of derivative markets and basically gave birth to stochastic finance[6].

It provides a formula (the so called Black-Scholes formula) for the value of an European call option when there are two assets, a riskless asset with interest rate r and a underlying asset whose price, S which follows a diffusion process (in a stochastic sense): $dS = \mu S dt + \sigma S dB$ where dB is the standard Brownian motion (see next chapter). An European call option offers the right to buy the underlying asset at time T for a price K fixed at time $t = 0$, which is conventioned to be the moment of the contract.

Under the assumption that there are no arbitrage opportunities Black and Scholes (1973) proved that the price of the option $V = V(t, S)$ is a a function of time, $t \in (0, T)$ and the price of an underlying asset $S \in (0, \infty)$ follows the backward parabolic PDE and has a terminal constraint

$$\begin{cases} V_t(t,S) & = -\frac{\sigma^2 S^2}{2} \, V_{SS}(t,S) - rSV_S(t,S) + rV(t,S), \ (t,S) \in [0,T] \times (0,\infty) \\ V(T,S) & = \max\{ \, S - K, 0 \, \} \, , \ (t,S) \in \{t = T\} \times (0,\infty). \end{cases} \tag{12.27}$$

The first equation is valid for any financial option having the same underlying asset dynamics, and the terminal constraint is characteristic of the European call option: because the writer sells the right, but not the obligation, to purchase the underlying asset at the price K at time $t = T$, the buyer is only interested in that purchase if he can sell it at the prevailing market price $S = S(T)$ when that price is higher than the exercise price K. In this case the payoff will be $S(T) - K$. Otherwise he will not execute the option and the terminal payoff will be zero.

The two boundary constraints

$$V(t,0) = 0, \quad (t,S) \in [0,T] \times \{S = 0\}$$
$$\lim_{S \to \infty} V(t,S) = S, \quad (t,S) \in [0,T] \times \{S \to \infty\},$$

are sometimes referred to, but they are redundant.

The same structure occurs in the Merton's model (see Merton (1974)) which is a seminal paper on the pricing of default bonds. It was the first model on the so-called structural approach to modelling credit risk.[7] In essence, this model assumes that the value of the firm follows a linear diffusion process and it considers the issuance of a bond with an expiring date T whose indenture gives it absolute priority on the value of the firm at the expiry date. This means that either if the value of the firm is smaller that the face value of the bond the creditor takes possession of the firm and in the opposite case it recovers the face value. In this case, we can interpret the position of the

---

[6]Myron Scholes was awarded the Nobel prize in 1997, together with Robert Merton another important contributer to stochastic finance, precisely for this formula. Fisher Black was deceased at the time.

[7]This model is the inspiration of credit risk models used by rating agencies (see Duffie and Singleton (2003)).

equity owner as holding an European call option over the value of the firm with strike price equal to the face value of the debt and the creditor as having an European put option security.

The price of the European call option [8], given the former assumptions is given by

$$V(t,S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \quad t \in [0,T] \tag{12.28}$$

where $\Phi(.)$ is cumulative Gaussian density function such that $\Phi(d) = \mathbb{P}(x \le d)$ where

$$d_1 = \frac{\ln(S/K) + (T-t)\left(r + \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \tag{12.29}$$

$$d_2 = \frac{\ln(S/K) + (T-t)\left(r - \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \tag{12.30}$$
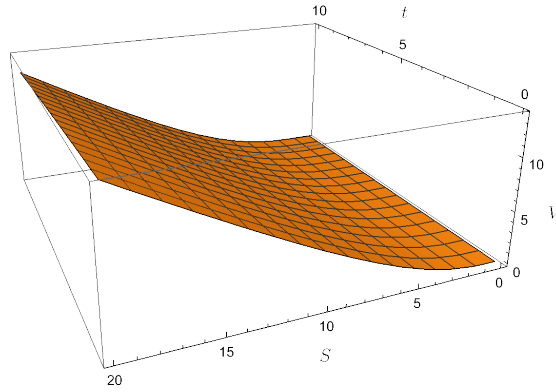

Figure 12.8: Solution for the Black and Scholes model, for $r = 0.02$, $T = 20$, $\sigma = 0.2$, and $K = 10$.

Proof. In order to solve the B-S PDE, which is a non-linear backward parabolic PDE, we transform it to to a quasi-linear parabolic forward PDE, by applying the transformations: $t(\tau) = T - \tau$ and $S = Ke^x$ and setting $u(\tau, x) = V(t(\tau), S(x))$. We can transform the option-pricing problem to the equivalent initial-value problem PDE equivalent to (12.27)

$$\begin{cases} u_\tau = \dfrac{\sigma^2}{2}u_{xx} + \left(r - \dfrac{\sigma^2}{2}\right)u_x - ru, & (\tau, x) \in [0,T] \times (-\infty, \infty) \\ u(0,x) = u_0(x) \end{cases} \tag{12.31}$$

where

$$u_0(x) = \begin{cases} 0, & \text{if } x \le 0 \\ K(e^x - 1), & \text{if } x > 0 \end{cases}$$

---

[8]For the credit risk model S would be the value of assets of a firm, K would be the face value of loan, and T the term of the loan.

The PDE is a particular example of equation (12.17), which implies that the solution is

$$u(\tau, x) = \int_{-\infty}^{0} 0\, g(\tau, x - s) ds + K \int_{0}^{\infty} (e^s - 1)\, g(\tau, x - s) ds$$

$$= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \int_{0}^{\infty} (e^s - 1)\, e^{h(\tau, x - s)} ds$$

where (from equation (12.16))

$$h(\tau, y) \equiv -\frac{y^2 + 2\tau\left(r - \dfrac{\sigma^2}{2}\right) y + \left(r + \dfrac{\sigma^2}{2}\right)^2 \tau^2}{2\tau\sigma^2}.$$

Then

$$u(\tau, x) = \frac{K}{\sqrt{2\pi\sigma^2\tau}} \left(\int_{0}^{\infty} e^{s + h(\tau, x - s)} ds - \int_{0}^{\infty} e^{h(\tau, x - s)} ds \right)$$

$$= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \left(I_1 - I_2 \right).$$

In order to simplify the integrals, it is useful to remember the formulas for the error function, $\text{erf}(x)$, and of the Gaussian cumulative distribution $\Phi(x)$,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-z^2} dz, \ \ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2} z^2} dz$$

which are related as

$$\Phi(x) = \frac{1}{2} \left[1 + \text{erf}\left(\frac{x}{\sqrt{2}} \right) \right].$$

After some algebra we obtain

$$s + h(\tau, x - s) = x - \frac{1}{2}(\delta_1(s))^2$$

$$h(\tau, x - s) = -r\tau - \frac{1}{2}(\delta_2(s))^2$$

where

$$\delta_1(s) \equiv \frac{x - s + \left(r + \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}, \ \text{and} \ \delta_2(s) \equiv \frac{x - s + \left(r - \dfrac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}.$$

Then [9]

$$I_1 = e^x \int_0^\infty e^{-\frac{1}{2}(\delta_1(s))^2} ds =$$

$$= -\sigma\sqrt{\tau}e^x \int_{d_1}^{-\infty} e^{-\frac{1}{2}\delta_1^2} d\delta_1 =$$

$$= \sqrt{\sigma^2\tau}e^x \int_{-\infty}^{d_1} e^{-\frac{1}{2}\delta_1^2} d\delta_1 =$$

$$= \sqrt{2\pi\sigma^2\tau}e^x\Phi(d_1)$$

where $d_1 = \delta_1(0)$ as in equation (12.29) for $\tau = T - t$, and also, writing that $d_2 = \delta_2(0)$, as in equation (12.30) for $\tau = T - t$,

$$I_2 = e^{-r\tau} \int_0^\infty e^{-\frac{1}{2}(\delta_2(s))^2} ds =$$

$$= -\sigma\sqrt{\tau}e^{-r\tau} \int_{d_2}^{-\infty} e^{-\frac{1}{2}\delta_2^2} d\delta_2 =,$$

$$= \sqrt{\sigma^2\tau}e^{-r\tau} \int_{-\infty}^{d_2} e^{-\frac{1}{2}\delta_2^2} d\delta_2 =$$

$$= \sqrt{2\pi\sigma^2\tau}e^{-r\tau}\Phi(d_2)$$

Thus

$$u(\tau, x) = K\left(e^x\Phi(d_1) - e^{-r\tau}\Phi(d_2)\right)$$

and transforming back $V(t, S) = u\left(T - t, \ln(S/K)\right)$ we get equation (12.28). ☐

Observe this is a backward parabolic PDE, which implies that the terminal condition determines the particular solution.

12.10   Bibiography

- Mathematics of PDE's: introductory Olver (2014), Salsa (2016) and (Pinsky, 2003, ch 5). Advanced (Evans, 2010, ch 3).

- Applications to economics (with more advanced material) : Achdou et al. (2014)

- Applications to growth theory Brito (2004) and Brito (2011) and the references therein.

---

[9]We use integration by transformation of variables: if we define $z = \varphi(s)$ where $\varphi : [a, b] \to \mathcal{J}$ and $f : \mathcal{J} \to \mathbb{R}$ we have that

$$\int_{\varphi(a)}^{\varphi(b)} f(z)dz = \int_a^b f(\varphi(s))\varphi'(s)ds.$$

12.A Appendix: Fourier transforms

Consider a function $f(x)$ such that $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. We can define a pair of generalized functions, the Fourier transform of $f(x)$, $F(s) = \mathcal{F}[f(x)](s)$ and the inverse Fourier transform $\mathcal{F}^{-1}[F(s)](x) = f(x)$ (using the definition of Kammler (2000)), where

$$F(s) = \mathcal{F}[f(x)] \equiv \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx$$

where $i^2 = -1$ and

$$f(x) = \mathcal{F}^{-1}[F(s)] \equiv \int_{-\infty}^{\infty} F(s) e^{2\pi i s x} ds.$$

Table 12.1: Fourier and inverse Fourier transforms: properties

	$f(x)$ for $-\infty < x < \infty$	$F(s)$ for $-\infty < s < \infty$	obs
1	a	$a\, \delta(s)$	$a \in \mathbb{C}$ complex number, $\delta()$ Dirac's delta
2	$a\, \delta(x)$	a	$a \in \mathbb{C}$ complex number,
3	$a\, f(x)$	$a\, F(s)$	$a \in \mathbb{C}$ complex number
4	$a\, f(x) + b\, g(x)$	$a\, F(s) + b\, G(s)$	$a, b\,\mathbb{C}$ complex numbers
5	$f(x) * g(x)$	$F(s)\, G(s)$	$f(x) * g(x)$ is a convolution
6	x	$-\dfrac{\delta'(s)}{2\pi i}$	$\delta'(\cdot)$ is the first derivative of $\delta(\cdot)$
7	x^2	$\dfrac{\delta''(s)}{2\pi}$	$\delta''(\cdot)$ is the second derivative of $\delta(\cdot)$
8	$x\, f(x)$	$-\dfrac{F'(s)}{2\pi i}$	
9	$f(t, x)$	$F(t, s)$	t is a real number
10	$f'(x)$	$2\pi, i\, s\, F(s)$	
11	$x\, f'(x)$	$-(F(s) + s\, F'(s))$	if $s \in \mathbb{R}$
12	$f''(x)$	$-(2\pi s)^2\, F(s)$	
13	$x\, f''(x)$	$\dfrac{2\pi s\, (2\, F(s) + s\, F'(s))}{i}$	
14	$x^2\, f''(x)$	$-s^2\, F''(s)$	

There are some useful properties of the Fourier transform that we use in the main text:

1. the Fourier transform of a constant: let $a \in \mathbb{C}$ be a number

$$F(a) = \mathcal{F}[a] = a \int_{-\infty}^{\infty} e^{-2\pi i s x}\, dx = a \int_{-\infty}^{\infty} \delta(s - x)\, dx = a\, \delta(s)$$

and

$$\mathcal{F}^{-1}[a\delta(s)] = \int_{-\infty}^{\infty} a\, \delta(s)\, e^{2\pi i s x}\, ds = a;$$

2. the Fourier transform of $a \in \mathbb{C}$ constant times a Dirac's delta generalized function:

$$F(a\,\delta(x)) \ = \mathcal{F}[a\,\delta(x)] \ = a \int_{-\infty}^{\infty} \delta(x)\,e^{-2\,\pi\,i\,s\,x}\,dx = a,$$

and

$$\mathcal{F}^{-1}[a] \ = \int_{-\infty}^{\infty} a,e^{2\,\pi\,i\,s\,x}\,ds = a\,\delta(x);$$

3. the Fourier transform preserves multiplication by a complex number $a \in \mathbb{C}$:

$$\mathcal{F}[a\,f(x)] \ = a\,F(s), \text{ and } \mathcal{F}^{-1}[aF(s)] \ = af(x),$$

Proof.

$$\mathcal{F}[a\,f(x)] \ = \int_{-\infty}^{\infty} a\,f(x)\,e^{-2\,\pi\,i\,s\,x}\,dx = a\int_{-\infty}^{\infty} f(x)\,e^{-2\,\pi\,i\,s\,x}\,dx = a\,F(s),$$

and

$$\mathcal{F}^{-1}[a\,F(s)] \equiv \int_{-\infty}^{\infty} a\,F(s)e^{2\pi isx}ds = a\int_{-\infty}^{\infty} F(s)e^{2\pi isx}ds = a\,f(x).$$

$\square$

4. the Fourier transform preserves linearity:

$$\mathcal{F}[a\,f(x)+b\,g(x)] \ = a\,F(s)+b\,G(s), \text{ and } \mathcal{F}^{-1}[a\,F(s)+b\,G(s)] \ = a\,f(x)+b\,g(x)$$

5. the Fourier transform does not preserve multiplication of two functions. However, there is a relationship between convolution of functions and multiplication of Fourier transforms. A **convolution** between two functions $f(x)$ and $g(x)$ is defined as

$$f(x)*g(x) = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy.$$

The inverse Fourier transform of a product of two Fourier transforms is a convolution,

$$f(x)*g(x) = \mathcal{F}^{-1}[F(s)\,G(s)] \ = \int_{-\infty}^{\infty} F(s)\,G(s)\,e^{2\pi isx}ds$$

6. $\mathcal{F}[x] = -\dfrac{1}{2\,\pi\,i}\delta'(s)$, where $\delta(x)$ is Dirac's delta

Proof.   Observe that $\int_{-\infty}^{\infty} e^{2\pi i s x}\delta(s)ds = 1$ and

$$\int_{-\infty}^{\infty} e^{2\pi i s x}\,\delta(s) = 0.$$

Then

$$x = x\int_{-\infty}^{\infty} e^{2\pi i s x}\delta(s)ds$$

$$= -\frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} e^{2\pi i s x}\,\delta(s) - \int_{-\infty}^{\infty} 2\pi i x\, e^{2\pi i s x}\delta(s)\,ds\right)$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{2\pi i s x}\delta'(s)\,ds$$

$$= -\frac{1}{2\pi i}\mathcal{F}^{-1}\!\left[\frac{1}{2\pi i}\delta'(s)\right]$$

□

7. $\mathcal{F}[x^2] = \dfrac{1}{(2\pi)^2}\delta''(s)$

8. $\mathcal{F}[x\,f(x)] = -\dfrac{1}{2\pi i}F'(s)$

Proof.

$$\mathcal{F}[x\,f(x)] = \int_{-\infty}^{\infty} x\,f(x)\,e^{-2\pi i s x}\,dx$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} -2\pi i x\,f(x)\,e^{-2\pi i s x}\,dx$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(x)\,\frac{d}{ds}\!\left(e^{-2\pi i s x}\right)dx$$

$$= -\frac{1}{2\pi i}\frac{d}{ds}F(s) = \frac{d}{ds}\left[\int_{-\infty}^{\infty} f(x)\,e^{-2\pi i s x}\,dx\right]$$

$$= -\frac{1}{2\pi i}F'(s)$$

Alternative proof:

$$\mathcal{F}[x\,f(x)] = \mathcal{F}[x]*\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} -\frac{1}{2\pi i}\delta'(y)\,F(s-y)dy = -\frac{1}{2\pi i}F'(s)$$

if $f = f(x,t)$ where t is a real variable then $F(s,t) = \mathcal{F}[f(x,t)]$ and $f(x,t) = \mathcal{F}^{-1}[F(s,t)]$. Also $F_t(s,t) = \mathcal{F}[f_t(x,t)]$ and $f_t(x,t) = \mathcal{F}^{-1}[F_t(s,t)]$ □

9. $\mathcal{F}[f'(x)] = 2\pi i s\,F(s)$

Proof.

$$\mathcal{F}[f'(x)] = \int_{-\infty}^{\infty} f'(x)\, e^{-2\pi i s x}\, dx$$

integration by parts

$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x} - \int_{-\infty}^{\infty} f(x)\, \frac{\partial}{\partial x}\left(e^{-2\pi i s x}\right) dx$$

because $e^{-2\pi i s x}$ is symmetric the first integral is equal to zero

$$= 2\pi i s \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx$$

$$= 2\pi i s\, F(s)$$

$\square$

10. $\mathcal{F}[xf'(x)] = -(F(s) + sF'(s))$ if $s \in \mathbb{R}$

Proof.

$$\mathcal{F}[x\, f'(x)] = \mathcal{F}[x] * \mathcal{F}[f'(x)]$$

$$= \int_{-\infty}^{\infty} \left(-\frac{1}{2\pi i}\, \delta'(y)\right)\left(2\pi i\, (s-y)\, F(s-y)\right) dy$$

$$= -\int_{-\infty}^{\infty} \delta'(y)\, (s-y)\, F(s-y) dy$$

$$= -s \int_{-\infty}^{\infty} \delta'(y)\, F(s-y) dy + \int_{-\infty}^{\infty} \delta'(y)\, y\, F(s-y) dy$$

$$= -sF'(s) + \int_{-\infty}^{\infty} \delta(y)\, y\, F(s-y) - \int_{-\infty}^{\infty} \delta(y)\, F(s-y) dy$$

$$= sF'(s) - F(s)$$

$\square$

11. $\mathcal{F}[f''(x)] = -4\pi^2 s^2\, F(s)$

12. $\mathcal{F}[x\, f''(x)] = \dfrac{2\pi s}{i}\left(2F(s) + s\, F'(s)\right)$

13. $\mathcal{F}[x^2\, f''(x)] = -s^2\, F''(s)$

References: `https://dlmf.nist.gov/1.14`

Table 12.2: Fourier and inverse Fourier transforms of some functions

$f(x)$ for $-\infty < x < \infty$	$F(s)$ for $-\infty < s < \infty$	obs
$\delta(x-a)$	$e^{-2\pi i s a}$	
$\frac{1}{\sqrt{4\pi a}} e^{-\frac{x^2}{4a}}$	$e^{-a(2\pi s)^2}$	$a > 0$
$\frac{1}{\sqrt{4\pi a}} e^{-\frac{(x+b)^2}{4a}}$	$e^{-a(2\pi s)^2 + b(2\pi i s)}$	$a > 0, \quad b \in \mathbb{R}$
$\frac{1}{\sqrt{4\pi a}} e^{c-\frac{x^2}{4a}}$	$e^{-a(2\pi s)^2 + c}$	$a > 0, \quad c \in \mathbb{R}$
$\frac{1}{\sqrt{4\pi a}} e^{c-\frac{(x+b)^2}{4a}}$	$e^{-a(2\pi s)^2 + b(2\pi i s) + c}$	$a > 0, \quad (b,c) \in \mathbb{R}^2$

Bibliography

Achdou, Y., Buera, F. J., Lasry, J.-M., Lions, P.-L., and Moll, B. (2014). Partial differential equation models in macroeconomics. *Philosophical Transactions of the Royal Society*, 372.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–659.

Brito, P. (2004). The dynamics of growth and distribution in a spatially heterogeneous world. Working Papers of the Department of Economics, ISEG-UTL http://ideas.repec.org/p/ise/isegwp/wp142004.html.

Brito, P. (2011). Global endogenous growth and distributional dynamics. MPRA Paper 41653, University Library of Munich, Germany.

Butkovskiy, G. (1969). *Distributed Control Systems*. American Elsevier, New-York.

Carlson, D. A., Haurie, A. B., and Leizarowitz, A. (1996). *Infinite Horizon Optimal Control*. Springer-Verlag, 2nd edition edition.

Derzko, N. A., Sethi, S. P., and Thompson, G. L. (1984). Necessary and sufficient conditions for optimal control of quasilinear partial differential equations systems. *Journal of Optimization Theory and Applications*, 43(1):89–101.

Duffie, D. and Singleton, K. J. (2003). *Credit Risk. Pricing, Measurement and Management*. Princeton Series in Finance. Princeton University Press, Princeton and Oxford.

Evans, L. C. (2010). *Partial Differential Equations*, volume 19 of *Graduate Series in Mathematics*. American Mathematical Society, Providence, Rhode Island, second edition.

Fattorini, H. O. (1999). *Infinite Dimensional Optimization and Control Theory*. Cambridge.

Kammler, D. W. (2000). *A First Course in Fourier Analysis*. Prentice-Hall.

Lions, J. L. (1971). *Optimal Control of Systems Governed by Partial Differential Equations*. Springer-verlag, New-York.

Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470.

Olver, P. J. (2014). *Introduction to Partial Differential Equations.* Undergraduate Texts in Mathematics. Springer International Publishing, 1 edition.

Pinsky, M. A. (2003). *Partial Differential Equations and Bounday-value Problems with Applications.* Waveland, 3rd edition.

Salsa, S. (2016). *Partial Differential Equations in Action: From Modelling to Theory.* Number 99 in Unitext. Springer International Publishing, third edition.

Tröltzsch, F. (2010). *Optimal Control of Partial Differential Equations.* Number 112 in Graduate Studies in Mathematics. American Mathematical Society.

Youn, H., De Leenheer, P., and Tremblay, V. (2016). Output diffusion of the monopolist over time and space. *Journal of Optimization Theory and Applications*, 169:290–298.