Advanced Mathematical Economics

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Contents

Chapter 15

Stochastic optimal control

15.1 Introduction

In this chapter we define the stochastic optimal control problem as an optimal control problem of an Itô forward stochastic differential equation (FSDE) together with an initial condition on the state variable, and some cases in which there are terminal conditions. We deal with both the finite and the infinite horizon cases. We, again, present the simplest problems, heuristic proofs, and are mostly concerned with characterizing properties of solutions.

There are three approaches to solving the stochastic optimal control problem: (1) using the principle of dynamic programming (DP); (2) using the Pontriyagin maximum principle (PM); and (3) the convex duality method (see Pham (2009)).

The first method is the most well known (see Fleming and Rishel (1975) or Malliaris and Brock (1982) for applications in economics and finance) and leads to the solution of a parabolic PDE, or a second order ODE for infinite horizon problems. The second method is less well known and leads directly to a system of forward-backward stochastic differential equations (FBSDE). The third method is used in association to the Malliavin calculus and is still new. It is not presented in the following notes.

15.2 Stochastic dynamic programming

15.2.1 Finite horizon

Again we assume the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where a non-anticipating filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$, where $\mathcal{F}_t$ is the information is generated by past realizations of a Wiener process $(W(s))_{s\in(0,t)}$. This means that all the information is generated by past evants.

Let $X(t)$ denote the state variable and $U(t)$ the control variable at time t. We consider the stochastic optimal control problem, that consists in determining the value function, $V(.)$,

$$V(x_0) = \max_{(U(t))_{t\in[0,T]}} \mathbb{E}_0 \left[\int_0^T f(t, X(t), U(t))dt \right] \tag{15.1}$$

subjected to

$$dX(t) = g(t, X(t), U(t))dt + \sigma(t, X(t), U(t))dW(t) \tag{15.2}$$

given the initial distribution for the state variable $X(0) = x_0$. We assume that the objective, the drift and the volatility functions, $f(.)$, $g(.)$ and $\sigma(.)$ are known. Function $g(.)$ is assumed to be of class H and functions $f(.)$ and $\sigma(.)$ are of class N, that is bounded functions for any realization of the stochastic processses.

One important difference as regards deterministic optimal control is that while in this case the control variable, together with the transversality condition can be seen as a backward looking variable, in the stochastic case it should be a $\mathcal{F}_t$-adapted process. Therefore, some type of terminal condition should be imposed.

The stochastic dynamic programming principle is the analogue to the dynamic programming principle for the optimal control of ODE's. It gives a local necessary condition for optimality.

Proposition 1. ***Stochastic dynamic programming*** *Let the processes $(X^*(t), U^*(t))_{t\in[0,T]}$ be solution to the SOC problem* (15.1)-(15.2). *Then, at time t, every realizations of the state and control variables, $X^*(t) = x$ and $U^*(t) = u$, satisfy the* ***Hamilton-Jacobi-Bellman*** *equation*

$$-\frac{\partial V(t,x)}{\partial t} = \max_u \left(f(t,x,u) + g(t,x,u)\frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\sigma(t,x,u)^2 \frac{\partial^2 V(t,x)}{\partial x^2} \right). \tag{15.3}$$

Proof. (Heuristic) The value of the problem, at time $t = 0$ is

$$V(0, x_0) = \mathbb{E}_0\Big[\int_0^T f(t, X^*(t), U^*(t))\, dt\Big].$$

Equivalently, we have

$$\begin{aligned} V(0,x_0) &= \max_{(u(t))_{t\in[0,T]}} \mathbb{E}_0\Big[\int_0^T f(t,X(t),U(t))dt\Big] \\ &= \max_{(u(t))_{t\in[0,T]}} \mathbb{E}_0\Big[\int_0^{\Delta t} f(t,X(t),U(t))dt + \int_{\Delta t}^T f(t,X(t),U(t))dt\Big]. \end{aligned}$$

By the principle of the dynamic programming and the law of iterated expectations we have

$$V(x_0) = \max_{(u(t))_{t\in[0,\Delta t]}} \mathbb{E}_0\left[\int_0^{\Delta t} f(t,X(t),U(t))dt + \max_{(u(t))_{t\in[\Delta t,T]}} \mathbb{E}_{\Delta t}\left[\int_{\Delta t}^T f(t,X(t),U(t))dt\right]\right].$$

If we consider a small interval Δt we can approximate

$$V(0,x_0) \approx \max_{(u(t))_{t\in[0,\Delta t]}} \mathbb{E}_0\left[f(t,X(t),U(t))\Delta t + V(\Delta t, x(\Delta t))\right]$$

where $x(\Delta t) = x_0 + \Delta x$. If we consider any time t, and any realization fot the state variable $X(t) = x$, and if V is continuously differentiable of the second order, then from Itô's lemma then the variation of the value function is

$$V(t+dt, x+dX(t)) = V(t,x) + V_t(t,x)dt + V_x(t,x)dX(t) + \frac{1}{2}V_{xx}(t,x)(dX(t))^2 + h.o.t$$

where

$$\begin{aligned} dX(t) &= g(t, X(t), U(t))dt + \sigma(t, X(t), U(t))dW(t) \\ (dX(t))^2 &= g(t, X(t), U(t))^2(dt)^2 + 2\, g(t, X(t), U(t))\sigma(.)(dt)(dW(t)) + \\ &\quad +(\sigma(t, X(t), U(t)))^2(dW)^2 = (\sigma(t, X(t), U(t)))^2dt. \end{aligned}$$

Then

$$\begin{aligned} \frac{V(t+dt, x+dX^*(t)) - V(t,x)}{dt} &= \max_u \mathbb{E}\left[fdt + V + V_t dt + V_x g dt + V_x \sigma dW + \frac{1}{2}\sigma^2 V_{xx} dt\right]\frac{1}{dt} \\ &= \max_u \left[f + V_t + V_x g + \frac{1}{2}\sigma^2 V_{xx}\right] dt \end{aligned}$$

because $\mathbb{E}_0(dW) = 0$. At the optimum

$$\lim_{dt \to 0} \frac{V(t+dt, x+dX^*(t)) - V(t,x)}{dt} = 0,$$

which is satisfied only if HJB equation (15.3) holds.

□

15.2.2 Infinite horizon

The autonomous discounted infinite horizon problem is

$$V(x_0) = \max_u \mathbb{E}_0 \left[\int_0^\infty f\left(X(t), U(t)\right)\, e^{-\rho t} dt\right] \tag{15.4}$$

where $\rho > 0$, subject to

$$dX(t) = g\left(X(t), U(t)\right) dt + \sigma\left(X(t), U(t)\right) dW(t) \tag{15.5}$$

given the initial value of the state variable $X(0) = x_0$, and assuming the same properties for functions $f(.)$, $g(.)$ and $\sigma(.)$.

Applying, again, the Bellman's principle, now the HJB equation is the nonlinear second order ODE of the form

$$\rho\, v(x) = \max_u \left(f(x,u) + g(t,x,u)v'(x) + \frac{1}{2}\,\sigma(x,u)^2\, v''(x)\right). \tag{15.6}$$

Proof: Write $V(t,x) = e^{-\rho\, t}\ v(x)$. Therefore $V_t(t,x) = -\rho\, e^{-\rho\, t}\ v(x)$, $V_x(t,x) = e^{-\rho\, t}\ v'(x)$, and $V_{xx}(t,x) = e^{-\rho\, t}\ v''(x)$. Substitutimng in HJB equation equation (15.3) yields HJB equation (15.6).

15.2.3 Economic applications using stochastic dynamic programming

The representative agent problem

The Merton (1971) model is the standard micro model for the simultaneous determination of the strategies of consumption and portfolio investment. Next, we present a simplified version with one risky and one risk-free asset.

Assume that an agent can invest in two types of assets, a risk-free and a risky asset, whose prices are denoted by B and S, respectively. We denote by $\theta_0(t)$ and $\theta_1(t)$ the number of risk free and risky assets in the portfolio, and by $A(t)$ net financial wealth of the agent at time t, we have $A(t) = \theta_0(t)B(t) + \theta_1(t)S(t)$, for any $t \in [0, \infty)$. The agent can have a short position ($\theta_j(t) < 0$) or a long position (if $\theta_j(t) <> 0$) on any asset j at time t.

The prices of the assets are given to the agent and are assume to follow the exogenous processes

$$\begin{aligned} dB(t) &= rB(t)dt \\ dS(t) &= \mu S(t)dt + \sigma S(t)dW(t) \end{aligned}$$

where r is the risk-free interest rate, μ and σ are the constant rates of return and volatility for the risky asset. The change in financial income in the time interval dt, starting at time t, is therefore,

$$\theta_0(t)\, r\, B(t)\, dt + \theta_1(t)(\mu S(t)dt + \sigma S(t)dW(t)).$$

Assume that the agent is entitled to a deterministic endowment $(y(t))_{t\in\mathbb{R}}$ which adds to the financial income. Then the value of financial wealth at time t is

$$A(t) = A(0) + \int_0^t (r\theta_0(s)B(s) + \mu\theta_1(s)S(s) + Y(s) - C(s))\, ds + \int_0^t \sigma\mu\theta_1(s)S(s)dW(s),$$

where the process for consumption $(C(t))_{t\in\mathbb{R}}$ is endogenous. Denoting the shares of the equity and of the risk-free asset by $w = \dfrac{\theta_1 S}{A}$ and $1 - w = \dfrac{\theta_0 B}{A}$, the budget constraint is the Itô's stochastic differential equation

$$dA(t) = \Big[\Big(r\,(1 - w(t)) + \mu\, w(t)\Big)\, A(t) + Y(t) - C(t)\Big]\, dt + \sigma\, w(t)\, A(t)\, dW(t), \ \text{for}\, t \geq 0 \tag{15.7}$$

and the initial net wealth $A(0) = \theta_0(0)B(0) + \theta_1(0)S(0)$ is known. The rate of return on the total asset position $r^a(t) = r\,(1 - w(t)) + \mu\, w(t)$ is a weighted sum of the rates of return of the risk-free and the risky asset, and there is time-varying.

The problem for the consumer-investor is

$$\max_{c,w} \mathbb{E}_0 \left[\int_0^\infty u(c(t))e^{-\rho t}dt\right] \tag{15.8}$$

subject to the instantaneous budget constraint (15.7), given $A(0) = a_0$ and assuming that the utility function is increasing and concave.

This is a stochastic optimal control problem with infinite horizon, and has two control variables, c and w. We solve it by using proposition 1.

The Hamilton-Jacobi-Bellman equation (15.6) is

$$\rho\, v(A) = \max_{c,w} \left\{ u(c) + v^{'}(A)[(r(1-w) + \mu w)A + y - c] + \frac{1}{2} w^2\sigma^2 A^2 v^{''}(A) \right\}.$$

The first order necessary conditions allows us to get the optimal controls, i.e. the optimal policies for consumption and portfolio composition

$$
\begin{aligned}
u^{'}(c^*) &= v^{'}(A), & (15.9)\\
w^* &= W(A) = \frac{(\mu - r)}{\varepsilon_v(A)\,\sigma^2} & (15.10)
\end{aligned}
$$

where the $\frac{(\mu - r)}{\sigma}$ is the Sharpe index and $\varepsilon_v(A) \equiv -\frac{v^{'}(A)}{Av^{''}(A)}$ is the inverse of the elasticity of the value function.

If $u^{''}(.) < 0$ then the optimal policy function for consumption may be written as $c^* = C(A) \equiv (u^{'})^{-1}(v^{'}(A))$. Substituting the policy functions into the HJB equation, we get the differential equation over $v(A)$

$$
\rho\, v(A) = u(C(A)) + v^{'}(A)(y + rA - C(A)) + \frac{1}{2}\Big(\frac{r-\mu}{\sigma}\Big)^2 \frac{(v^{'}(A))^2}{v^{''}(A)}. \tag{15.11}
$$

In some cases, in particular when the utility function is a generalized mean and the constraint is a linear SDE, the HJB equation can be solved explicitly.

Example: the CRRA case In particular, let the utility function display constant relative risk aversion (CRRA)

$$
u(c) = \frac{c^{1-\eta} - 1}{1-\eta}, \text{ for } \eta > 0,
$$

and define total net wealth

$$
N = N(A) = \frac{y}{r} + A,
$$

as the sum of human wealth $(\frac{y}{r})$ and net financial wealth.

We can solve equation solve equation (15.11) by using the method of undetermined coefficients.

Conjecture that the solution for equation (15.11) is of type

$$
V(A) = \alpha + \theta\, N(A)^{1-\eta}
$$

where α and θ are arbitrary constants to be determined. If the functional form of this function is correct, by substituting in equation (15.11) the state variable, we obtain the HJB equation, at the optimum, containing only the unknowns α and θ. By finding a particular solution of that equation we find particular values for those two coefficients.

First, as

$$
V^{'}(A) = \theta\,(1-\eta)\,N^{-\eta}, \text{ and } V^{''}(A) = -\theta\,\eta\,(1-\eta)\,N^{-\eta-1}
$$

then the optimal policy functions are: for consumption is

$$
C(A) = (\theta(1-\eta))^{-\frac{1}{\eta}}\, N(A)
$$

which requires that $\theta\,(1-\eta) > 0$ to be a real number, and for the portfolio composition is

$$
W(A) = \frac{(\mu - r)}{\sigma^2}\;\frac{N}{\eta\, A}.
$$

Substituting in (15.11), we obtain

$$\rho\Big(\alpha+\theta N^{1-\eta}\Big) \;=\; \frac{1}{1-\eta}\Big(\left(\theta\,(1-\eta)\right)^{\frac{\eta-1}{\eta}}N^{1-\eta}-1\Big)+ \\ +\Big(\theta\,(1-\eta)\,N^{1-\eta}\Big)\Big(r-\left(\theta\,(1-\eta)\right)^{\frac{-1}{\eta}}-\frac{1}{2\,\eta}\Big(\frac{\mu-r}{\sigma}\Big)^2\Big).$$

If we set $\alpha\,\rho\,(1-\eta)+1=0$, we can eliminate $N^{1-\eta}$ and obtain an equation in θ. Solving it, yields

$$\theta=\theta^* \;\equiv \frac{1}{1-\eta}\left[\frac{\rho+r(1-\eta)}{\eta}+\frac{(1-\eta)}{2\eta^2}\left(\frac{\mu-r}{\sigma}\right)^2\right]^{-\eta}$$

Then

$$V(A)=\frac{1}{1-\eta}\Bigg\{\;\left[\frac{\rho-r(1-\eta)}{\eta}+\frac{(1-\eta)}{2\eta^2}\left(\frac{\mu-r}{\sigma}\right)^2\right]^{-\eta}N(A)^{1-\eta}-\frac{1}{\rho}\Bigg\}.$$

Then the optimal consumption is

$$C^*=\left(\frac{\rho+r(\eta-1)}{\eta}+\frac{(1-\eta)}{2\eta^2}\left(\frac{\mu-r}{\sigma}\right)^2\right)\,N,$$

and the share of the risky asset in the portfolio is again

$$w^* \;=-\frac{(r-\mu)}{\sigma^2}\;\frac{N}{\eta\,A}.$$

In the deterministic analogue, with only the risk-free asset, optimal consumption would be

$$C^*=\frac{\rho+r(\eta-1)}{\eta}\,N,$$

which means that if $\eta>1$ consumption will be smaller in the stochastic environment than in the stochastic one.

We see that the consumer cannot eliminate risk, in general. If we write $c^*=\chi N$, where $\chi\equiv\frac{\rho-r(1-\eta)}{\eta}+\frac{(1-\eta)}{2\eta^2}\left(\frac{\mu-r}{\sigma}\right)^2$, then the optimal net wealth is stochastic and follows a geometric Brownian motion

$$dN(t)=\Big(\mu_n dt+\sigma_n dW(t)\Big)\;N(t),$$

where

$$\begin{aligned}\mu_n \;&=\; r+\left(\frac{\mu-r}{\sigma}\right)^2\left(\frac{1-\eta}{\eta}\right)-\chi\\ \sigma_n \;&=\; \frac{1-\eta}{\eta}\left(\frac{\mu-r}{\sigma}\right).\end{aligned}$$

Given the initial wealth $n(0)=\dfrac{y}{r}+a_0$, and using the results in the previous chapter, we find that the probability density of a realization $A(t)=a/a_0$ follows a log-normal distribution.

As $C^*=C(N)$, the optimal consumption is also stochastic. If we apply Itô's lemma,

$$dC=\chi dN=C\left(\mu_c dt+\sigma_c dW(t)\right)$$

where

$$
\begin{aligned}
\mu_c &= \frac{r-\rho}{\eta} \\
\sigma_c &= \frac{r-\eta\rho}{\eta} + \cdot\frac{1}{2}\Big(\frac{\mu-r}{\sigma}\frac{1-\eta}{\eta}\Big)^2
\end{aligned}
$$

The stochastic differential equation has the solution

$$C(t) = c(0)\exp\left\{\left(\mu_c - \frac{\sigma_c^2}{2}\right)t + \sigma_c W(t)\right\}$$

where

$$c(0) = (1-\eta)(\theta^*)^{\frac{1}{\eta}} n(0) = (1-\eta)(\theta^*)^{\frac{1}{\eta}}\Big(\frac{y+ra_0}{r}\Big).$$

The unconditional expected value for consumption at time t

$$\mathbb{E}_0[C(t)] = c(0)\, e^{\mu_c t}.$$

The value function follows a stochastic process which is a monotonous function for wealth. The optimal strategy for consumption follows a stochastic process which is a linear function of the process for wealth and the fraction of the risky asset in the optimal portfolio is a direct function of the premium of the risky asset relative to the riskless asset and is a inverse function of the volatility.

References Merton (1971), Merton (1990), Duffie (1996) Cvitanić and Zapatero (2004)

The stochastic Ramsey model

Let K denote the stock of physical capital and L the labor input which is equal to the population (no unemployment, diseases, etc). The economy is represented by the the differential equations

$$
\begin{aligned}
dK(t) &= (F(K(t), L(t)) - C(t))dt \\
dL(t) &= \mu L(t)dt + \sigma L(t)dW(t)
\end{aligned}
$$

where we assume that $F(K, L)$ is linearly homogeneous, given the (deterministic) initial stock of capital and labor $K(0) = K_0$ and $L(0) = L_0$. The growth of the labor input (or its productivity) is stochastic.

If we define the variables in intensity terms,

$$k(t) \equiv \frac{K(t)}{L(t)},\;\; c(t) \equiv \frac{C(t)}{L(t)},$$

we can get an equivalent representation of the economy by a single stochastic differential equation over k. Using the Itô's lemma yields

$$dk = \left(f(k) - c - (\mu - \sigma^2)k\right)dt - \sigma^2 k dW(t) \tag{15.12}$$

where the production function in intensity terms is $f(k) = F\left(\frac{K}{L}, 1\right)$.

There is a central planner who wants to find the optimal path of consumption $(c^*(t))_{t\geq 0}$ that maximizing the intertemporal utility functional

$$\mathbb{E}_0\left[\int_0^{\infty} u(c(t))e^{-\rho t}dt\right]$$

subject to the budget constraint (15.12).

We use the stochastic dynamic programming principle to solve the problem. The HJB equation, (15.6), is

$$\rho V(k) = \max_c \left\{u(c) + V^{'}(k)\left(f(k) - c - (\mu - \sigma^2)k\right) + \frac{1}{2}(k\sigma)^2 V^{"}(k)\right\}$$

the optimality condition is again

$$u^{'}(c) = V^{'}(k)$$

and, substituting in the HJB equation yields an implicit second-order ODE

$$\rho V(k) = u(h(k)) + V^{'}(k)\left(f(k) - h(k) - (\mu - \sigma^2)k\right) + \frac{1}{2}(k\sigma)^2 V^{"}(k).$$

Again, we assume the benchmark **particular case**: $u(c) = \dfrac{c^{1-\theta}}{1-\theta}$ and $f(k) = k^{\alpha}$. Then the optimal policy function becomes

$$c^* = V^{'}(k)^{-\frac{1}{\theta}}$$

and the HJB becomes

$$\rho V(k) = \frac{\theta}{1-\theta} V^{'}(k)^{\frac{\theta-1}{\theta}} + V^{'}(k)\left(k^{\alpha} - (\mu - \sigma^2)k\right) + \frac{1}{2}(k\sigma)^2 V^{"}(k).$$

This equation does not seem to have a closed form solution.

However, to illustrate how a solution would be obtained in the case in which a closed-form solution would be obtained, we consider the (unrealistic) case $\theta = \alpha$. Again we conjecture that the solution if of the form

$$V(k) = B_0 + B_1 k^{\alpha}$$

Using the same methods as before we get

$$\begin{aligned} B_0 &= (1-\alpha)\frac{B_1}{\rho} \\ B_1 &= \frac{1}{1-\alpha}\left[\frac{(1-\alpha)\theta}{(1-\theta)(\rho - (1-\alpha)^2\sigma^2)}\right]^{\alpha}. \end{aligned}$$

Then

$$V(k) = B_1\left(\frac{1-\alpha}{\rho} + k^{1-\alpha}\right)$$

and

$$c^* = c(k) = \left(\frac{(1-\theta)(\rho - (1-\alpha)^2\sigma^2)}{(1-\alpha)\theta}\right) k \equiv \varrho k$$

as we see an increase in volatility decreases consumption for every level of the capital stock.

Then the optimal dynamics of the per capita capital stock is the SDE

$$dk^*(t) = \left(f(k^*(t)) - (\mu + \varrho - \sigma^2)k^*(t)\right) dt - \sigma^2 k^*(t) dW(t).$$

In this case we can not solve it explicitly as in the deterministic case.

References: Brock and Mirman (1972), Merton (1975), Merton (1990)

15.3 The stochastic PMP

In order to find the necessary optimality conditions by using the stochastic version of the Pontriyagin maximum principle (SPMP) it is useful to distinguish the case in which the volatility component depends on the control variable, as in equation (15.2), from the case in which it does not, as in equation

$$dX(t) = g(t, X(t), U(t))dt + \sigma(t, X(t))dW(t). \tag{15.13}$$

The reason for this is, again, related to the fact that the control variable should be $\mathcal{F}_t$ adapted.

Assume again the optimal control problem with value function (15.1) with a finite horizon, i.e, $t \in [0, T]$ for a finite T.

15.3.1 Volatility function independent of the control variable

Proposition 2. ***Stochastic PMP*** *Let the processes* $(X^*(t), U^*(t))_{t\in[0,T]}$ *be solution to the SOC problem* (15.1)-(15.13)*. Then, there are two processes* $(p(t), q(t))_{t\in[0,T]}$ *satisfying the adjoint equation and a terminal condition*

$$\begin{cases} dp(t) = -\Big\{ f_x(t, X^*(t), U^*(t)) + p(t)g_x(t, X^*(t), U^*(t)) + q(t)\sigma_x(t, X^*(t))\Big\}\, dt + q(t)dW(t) \\ p(T) = 0 \end{cases}$$

and, defining the Hamiltonian function by

$$H(t, x, u, p, q) = f(t, x, u) + p\, g(t, x, u) + q\, \sigma(t, x),$$

the optimal control satisfies for the realizations of the state and the control variables $X^*(t) = x$ *and* $U^*(t) = u$,

$$H(t, x^*, u^*, p, q) = \max_u H(t, x^*, u, p, q)$$

The proof is in (Yong and Zhou, 1999, p.123-137)

If function $H(\cdot)$ is differentiable for every u, then a necessary condition for the maximum is

$$\frac{\partial H(t, x^*, u^*, p, q)}{\partial u} = 0.$$

15.3.2 Volatility dependent on the control variable

Proposition 3. ***Stochastic PMP*** *Let the processes* $(X^*(t), U^*(t))_{t\in[0,T]}$ *be solution to the SOC problem* (15.1)-(15.2)*. Then, there are four processes* $(p(t), q(t), P(t), Q(t))_{t\in[0,T]}$ *satisfying the two adjoint equations and associated terminal conditions*

$$\begin{cases} dp(t) = -\Big\{ f_x(t, X^*(t), U^*(t)) + p(t)g_x(t, X^*(t), U^*(t)) + q(t)\sigma_x(t, X^*(t), U^*(t))\Big\}\, dt + q(t)dW(t) \\ p(T) = 0 \end{cases}$$

and

$$\begin{cases} dP(t) & = -\Big\{ f_{xx}(t, X^*(t), U^*(t)) + 2P(t)g_x\ (t, X^*(t), U^*(t)) + P(t)\left(g_x(t, X^*(t), U^*(t))\right)^2 + \\ & +2Q(t)\sigma_x(t, X^*(t), U^*(t))\Big\}\, dt + Q(t)dW(t) \\ P(T) & = 0 \end{cases}$$

and, defining the Generalized Hamiltonian function

$$G(t, x, u, p, P) = f(t, x, u) + pg(t, x, u) + \frac{1}{2}\sigma^2(t, x, u)P$$

the optimal control satisfies locally $X^*(t) = x^*$ *and* $U^*(t) = u^*$ *such that defining*

$$\mathcal{H}(t, x^*, u) = G(t, x^*, u, p, P) + \sigma(t, x^*, u)\left(q - P\sigma(t, x^*, u^*)\right)$$

it satisfies

$$\mathcal{H}(t, x^*, u^*) = \max_u \mathcal{H}(t, x^*, u)$$

The proof is in (Yong and Zhou, 1999, p.123-137)

15.3.3 Economic applications using stochastic maximum principle

We present next two applications of the stochastic PMP: a stochastic endogenous growth model and, again, the Merton model. In the first case the control variable does not affect the volatility term and in the second it does. This means that we use Proposition 2 in the first case and Proposition 3 in the second.

Application: the stochastic AK model

This is a stochastic version of the simplest endogenous growth model:

$$\max_{C(.)} \int_0^T \ln\left(C(t)\right)e^{-\rho t}dt$$

subject to

$$dK(t) = \left(\mu K(t) - C(t)\right)dt + \sigma K(t)dW(t) \tag{15.14}$$

$$K(0) = k_0$$

Observe that, as in this case the volatility term is independent of the control variable, C, we use proposition 2.

The adjoint equation is

$$\begin{cases} dp(t) = -\left(\mu p(t) + \sigma q(t)\right) dt + q(t) dW(t), & t \in (0, T) \\ p(T) = 0 \end{cases}$$

and the Hamiltonian is

$$H(t, c, k, p, q) = \ln(c) e^{-\rho t} + p(\mu k - c) + q\sigma k.$$

We determine optimal consumption such that $C^* = c^*$ by making $\dfrac{\partial H}{\partial c} = 0$. Therefore,

$$C^*(t) = \left(p(t) e^{\rho t}\right)^{-1}.$$

Consumption is a stochastic process, depending on p. Using Itô's lemma yields

$$\begin{aligned} dC^*(t) &= -\rho \frac{e^{-\rho t}}{p(t)} dt - \frac{e^{-\rho t}}{p(t)^2} dp(t) + \frac{e^{-\rho t}}{p(t)^3} (dp(t))^2 \\ &= C^*(t) \left(-\rho dt - \frac{dp(t)}{p(t)} + \left(\frac{dp(t)}{p(t)} \right)^2 \right) \\ &= C^*(t) \left[\left(\mu - \rho + \sigma \frac{q(t)}{p(t)} + \left(\frac{q(t)}{p(t)} \right)^2 \right) dt - \frac{q(t)}{p(t)} dW(t) \right] \end{aligned}$$

We have a stochastic differential equation for $p(.)$ but we do not have one equation allowing for the determination of $q(.)$. Based on our knowledge of the related deterministic model, we introduce a trial relationship

$$C(t) = \phi\, K(t)$$

where ϕ is a constant to be determined. Applying the Itô's lemma we have

$$\begin{aligned} dC(t) &= \phi\, dK(t) \\ &= \phi\, \left(\left(\mu K(t) - C(t)\right) dt + \sigma K(t) dW(t) \right) \end{aligned}$$

If we match the deterministic and the stochastic components of the two equations for C, we have, for any realization of $C(t) = c$, $K(t) = k$, $p(t) = p$, and $q(t) = q$

$$\begin{cases} c \left(A - \rho + \sigma \dfrac{q}{p} + \left(\dfrac{q}{p} \right)^2 \right) = \phi \left(\mu\, k - c \right) \\ -c \dfrac{q}{p} = \phi\, \sigma\, k \end{cases}$$

that would hopefully allow for the determination of the two unknowns, the realization q and the parameter ϕ. Solving the system we get $q = -\sigma p$ and $\phi = \rho$. Therefore,

$$C^*(t) = \rho K^*(t)$$

substituting in equation (15.14) yields

$$dK^*(t) = K^*(t)\left((\mu-\rho)dt + \sigma dW(t)\right)$$

Therefore

$$K^*(t) = k_0 e^{(\mu-\rho-\frac{1}{2}\sigma^2)t+\sigma W(t)}$$

and

$$C^*(t) = \rho k_0 e^{(\mu-\rho-\frac{1}{2}\sigma^2)t+\sigma W(t)}$$

meaning that:

1. consumption and capital accumulation are perfectly correlated;
2. they both follow a log-normal process with mean, where
$$\mathbb{E}[K(t)] \ = k_0 e^{(\mu-\rho-\frac{1}{2}\sigma^2)t}$$
3. meaning that there wil be long-run growth if $\mu-\rho-\frac{1}{2}\sigma^2 > 0$ that is if volatility does not affect much total factor productivity.

The Merton (1990) model

Next we consider again the problem of maximizing the intertemporal utility functional (15.8) subject to the stochastic differential equation (15.7). Differently from the previous presentation of the Merton's model, we now assume that there is no non-financial income, that is $y=0$ and the utility function is logarithmic.

We consider the problem

$$\max_{C.w} \mathbb{E}_0\left[\ \int_0^\infty \ln\left(C(t)\right)e^{-\rho t}dt\right]$$

subject to budget constraint, represents the dynamics of financial net wealth N,

$$dN(t) = [\ \left(r+(\mu-r)w\right)N - C]\ \ dt + \sigma w N dW(t)$$

and $N(0)=n_0$ is given and perfectly observed.

In this case there are two control variables, C and w, but one control variable, w, affects the volatility term. Therefore, we have to apply Proposition 3.

The adjoint equations are

$$\begin{cases} dp(t) = -\left[\left(r+(\mu-r)w(t)\right)p(t) + \sigma w(t) q(t)\right]dt + q(t)dW(t) \\ \lim_{t\to\infty} p(t) = 0 \end{cases}$$

and

$$\begin{cases} dP(t) = -\left[2\Big(r+(\mu-r)w(t)\Big)P(t) + \Big(r+(\mu-r)w(t)\Big)^2 P(t) + 2\sigma w(t) Q(t)\right]dt + Q(t)dW(t) \\ \lim_{t\to\infty} P(t) = 0. \end{cases}$$

To find the optimal controls we write the generalized Hamiltonian

$$G(t, N, C, w, p, P) = e^{-\rho t} \ln(C) + p\left[\ (r + (\mu - r)w)\, N - C\right]\ + \frac{1}{2}\sigma^2 w^2 N^2 P$$

and

$$\mathcal{H}(t, N, C, w) = G(t, N, C, w, p, P) + \sigma w N\left(q - P\sigma w^* N\right).$$

The optimal controls, C^* and w^* are found by maximizing function $\mathcal{H}(t, N, C, w)$ for C and w. Therefore, we find

$$C^*(t) = e^{-\rho t} p(t)^{-1} \tag{15.15}$$

and the condition

$$p(t)(\mu - r)N^*(t) + w^*(t)\sigma^2 N^*(t)^2 P(t) + \sigma N^*(t)\left(q(t) - \sigma w^*(t) N^*(t) P(t)\right) = 0$$

which is equivalent to

$$p(t)(\mu - r)N^*(t) + \sigma q(t) N^*(t) = 0.$$

Observe that the dual variables P and Q do not influence the behavior of p and q, and therefore do not influence the solution of C. From the last equation, we find

$$q(t) = -p(t)\left(\frac{\mu - r}{\sigma}\right),$$

and, substituting in the adjoint equation,

$$dp(t) = -p(t)\left(r dt + \left(\frac{\mu - r}{\sigma}\right) dW(t)\right).$$

Observe that the structure of the model is such that the shadow value of volatility functions P and Q have no effect in the shadow value functions associated with the drift component p and q, which simplifies the solution.

Applying the Itô's formula to consumption (15.15), and using this expression for the adjoint variable q, we find

$$\begin{aligned}
dC(t) &= -\rho C(t)dt - \frac{C(t)}{p(t)} dp(t) + \frac{C(t)}{p^2(t)}(dp(t))^2 = \\
&= -\rho C(t)dt + C(t)\left(r dt + \left(\frac{\mu - r}{\sigma}\right) dW(t)\right) + C(t)\ \left(\frac{\mu - r}{\sigma}\right)^2 dt = \\
&= C(t)\left\{\left(r - \rho + \left(\frac{\mu - r}{\sigma}\right)^2\right) dt + \left(\frac{\mu - r}{\sigma}\right) dW(t)\right\}.
\end{aligned}$$

Now, we **conjecture** that consumption is a linear function of net wealth $C = \xi N$. If this is the case this would allow us to obtain the optimal portfolio composition w^*. If the conjecture is right then we will also have

$$\begin{aligned}
dC(t) &= \xi dN(t) \\
&= \xi N(t)\left[\ (r + (\mu - r)w - \xi)\ dt + \sigma w dW(t)\right] \\
&= C(t)\left[\ (r + (\mu - r)w - \xi)\ dt + \sigma w dW(t)\right]
\end{aligned}$$

This is only consistent with the previous derivation if

$$\begin{cases} r - \rho + \left(\frac{\mu - r}{\sigma}\right)^2 = r + (\mu - r)w - \xi \\ \frac{\mu - r}{\sigma} = \sigma w \end{cases}$$

Solving for ξ and w we obtain the optimal controls

$$C^*(t) = \rho N^*(t) \tag{15.16}$$

$$w^*(t) = \frac{\mu - r}{\sigma^2} \tag{15.17}$$

Substituting in the budget constraint we have the optimal net wealth process

$$\frac{dN^*(t)}{N^*(t)} = \mu_n \, dt + \sigma_n dW(t)$$

where

$$\mu_n = r - \rho + \left(\frac{\mu - r}{\sigma}\right)^2 \tag{15.18}$$

$$\sigma_n = \frac{\mu - r}{\sigma} \tag{15.19}$$

which can be explicitly solved with the initial condition $N^*(0) = n_0$. We also find that

$$\frac{dC^*(t)}{C^*(t)} = \mu_n \, dt + \sigma_n dW(t)$$

the rates of return for consumption and wealth are perfectly correlated.

15.4 References

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