

# Economic Growth Theory:

## Problem set 2: Ramsey models

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### Ramsey models

1. Consider a version of the Ramsey model with increasing population  $\dot{N} = nN$  and  $N(0) = 1$ . The objective utility functional for the central planner is:

$$\max_c \int_0^\infty \ln(c(t)) e^{nt} e^{-\rho t} dt$$

where  $\rho > n > 0$ , subject to

$$\dot{k} = Ak^\alpha - c(t) - nk(t), \quad 0 < \alpha < 1$$

where  $c$  and  $k$  are the percapita consumption and capital stock. Let  $k(0)$  be given.

- (a) Apply the Pontryagin's principle and determine the optimality conditions as a dynamic system in  $(c, k)$ .
- (b) Determine the steady state, and study their stability properties and draw the phase diagram;
- (c) Discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics.

(d) Determine the effects of an increase in productivity,  $A$ .

2. Consider a version of the Ramsey model with increasing population  $\dot{N} = nN$  and  $N(0) = 1$ . The objective utility functional for the central planner is:

$$\max_c \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{nt} e^{-\rho t} dt$$

where  $\rho > n > 0$  and  $\theta > 0$ , subject to

$$\dot{k} = Ak^\alpha - c(t) - nk(t), \quad 0 < \alpha < 1$$

where  $c$  and  $k$  are the percapita consumption and capital stock. Let  $k(0)$  be given.

- (a) Apply the Pontryagin's principle and determine the optimality conditions as a dynamic system in  $(c, k)$ .
  - (b) Determine the steady state, and study their stability properties and draw the phase diagram;
  - (c) Discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics.
  - (d) Determine the effects of an increase in productivity,  $A$ .
3. Consider a version of the Ramsey model with increasing population  $\dot{N} = nN$  and  $N(0) = 1$ . The objective utility functional for the central planner is:

$$\max_c -\frac{1}{\eta} \int_0^\infty e^{-\eta c(t)} e^{nt} e^{-\rho t} dt$$

where  $\rho > n > 0$  and  $\eta > 0$ , subject to

$$\dot{k} = Ak^\alpha - c(t) - nk(t), \quad 0 < \alpha < 1$$

where  $c$  and  $k$  are the per capita consumption and capital stock. Let  $k(0)$  be given.

- (a) Apply the Pontryagin's principle and determine the optimality conditions as a dynamic system in  $(c, k)$ .
- (b) Determine the steady state, and study their stability properties and draw the phase diagram;

- (c) Discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics.
  - (d) Determine the effects of an increase in productivity,  $A$ .
4. Consider a version of the Ramsey model with increasing population  $\dot{N} = nN$  and  $N(0) = 1$ . The objective utility functional for the central planner is:

$$\max_c \int_0^\infty c(t) e^{nt} e^{-\rho t} dt$$

where  $\rho > n > 0$ , subject to

$$\dot{k} = Ak^\alpha - c(t) - nk(t), \quad 0 < \alpha < 1$$

where  $c$  and  $k$  are the per capita consumption and capital stock. Let  $k(0)$  be given.

- (a) Determine the optimal path for the capital stock and consumption <sup>1</sup> Under which conditions are those paths feasible ?
  - (b) Under conditions that ensure feasibility, discuss the existence of a BGP, and the properties of the model regarding long run growth and transition dynamics.
5. Consider a version of the Ramsey model with constant population where the objective utility functional for the central planner is:

$$\max_c \int_0^\infty \ln(c(t) - \bar{c}) e^{-\rho t} dt$$

where  $\rho > 0$  and  $\bar{c} > 0$  is a minimum level of consumption, subject to

$$\dot{k} = Ak^\alpha - c(t) - \delta k(t), \quad 0 < \alpha < 1, \quad \delta > 0$$

where  $c$  and  $k$  are the per capita consumption and capital stock. Let  $k(0)$  be given.

- (a) Apply the Pontryagin's principle and determine the optimality conditions as a dynamic system in  $(c, k)$ .

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<sup>1</sup>Transform the problem into a calculus of variations problem  $\max \int_0^\infty F(x(t), \dot{x}) e^{-\delta t} dt$ . The optimal path  $[x(t)]_{t \geq 0}$  should verify the dynamic optimality condition is given by the Euler equation  $F_x + \delta F_{\dot{x}} - F_{\ddot{x}} \dot{x} - F_{\dot{x}\dot{x}} \ddot{x} = 0$  and the initial and terminal conditions.

- (b) Draw the phase diagram.
- (c) Determine the steady states and study their local stability properties.
- (d) Discuss the properties of the solution to the planner's problem.
- (e) What are the effects of an increase in productivity,  $A$  ?

6. Consider a version of the Ramsey model with constant population where the objective utility functional for the central planner is:

$$\max_c \int_0^\infty \ln(c(t) - \bar{c}) e^{-\rho t} dt,$$

where  $\rho > 0$  and  $\bar{c} > \frac{\rho}{\alpha}$  is a minimum level of consumption, subject to

$$\dot{k} = Ak(t)^\alpha - c(t), \quad 0 < \alpha < 1$$

where  $c$  and  $k$  are the per capita consumption and capital stock. We also assume that  $k(0) = k_0$  is given and that the stock of capital is bounded.

- (a) Apply the Pontryagin's principle and determine the optimality conditions as a dynamic system in  $(c, k)$ .
- (b) Draw the phase diagram.
- (c) Determine the steady states and study their local stability properties.
- (d) Find an approximate solution to the problem in the neighborhood of the steady state associated with a maximum consumption.
- (e) Determine the effects of a permanent increase in productivity,  $A$ .

7. Consider a version of the Ramsey model with constant population where the objective utility functional for the central planner is:

$$\max_C \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

where  $\rho > 0$  and  $\theta \geq 1$ , subject to

$$\dot{K} = A(t)^{1-\alpha} K(t)^\alpha - C(t) - \delta K(t), \quad 0 < \alpha < 1, \delta > 0$$

where  $C$  and  $K$  denote (aggregate) consumption and capital stock. Let  $K(0)$  be given. There labor-augmenting technical progress where labour productivity  $A(t)$  grows at the rate  $\mu$  and  $A(0) = 1$ .

- (a) Recast the variables in efficiency terms by setting  $c(t) \equiv C(t)/A(t)$  and  $k(t) \equiv K(t)/A(t)$ . Prove that the Ramsey problem can be equivalently defined as

$$\max_c \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{-\tilde{\rho}t} dt$$

where  $\tilde{\rho} = \rho + \mu(\theta - 1)$

$$\dot{k} = k(t)^\alpha - c(t) - (\delta + \mu)k(t).$$

- (b) Apply the Pontryagin's principle and determine the optimality conditions as a dynamic system in  $(c, k)$ .
- (c) Draw the phase diagram.
- (d) Determine the steady states and study their local stability properties.
- (e) Characterize the growth properties of the solution to the Ramsey problem.