Advanced Mathematical Economics

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Lecture 4
14.4.2020

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6.1 Introduction

Assume the independent variable is time and the dependent variable y is governed by a ordinary differential equation depending upon a time-dependent exogenous variable $u(t)$,

$$\dot{y} = f(y, u(t)) \tag{6.1}$$

where $(y, u) : [t_0, t_1] \to \mathbb{R}^2$. Assume that $y(t_0)$ is known and $f(y, u)$ is continuous and differentiable. We can say that y is **controlled** by $u = (u(t))_{t=t_0}^{t_1}$, because the solution of the differential equation y depends on the path of u. As an example, assume the economy dynamics features a linear differential equations $\dot{y} = \lambda y + u(t)$ where $y(0)$ is known. As the solution of the differential equation is

$$y(t) = y(0)e^{\lambda t} + \int_0^t e^{\lambda(t-s)}u(s)ds$$

we clearly see that the value of $y(.)$ at time t depends on the path of control variable $\Big(u(s)\Big)_{s=0}^{t}$.

We say y is **optimally controlled** if the control u is determined in order to maximize some value or minimize some cost. The conventional approach is to say that y is optimally controlled if the control variable, $u(\cdot)$, maximizes (or minimizes) an objective functional

$$\mathcal{F}[y, u] \equiv \int_{t_0}^{t_1} F(t, y(t), u(t))dt, \tag{6.2}$$

where the dynamics of y is governed by equation (6.1). In theory, there is potentially an infinite number of solutions for equation (6.1), when we choose u arbitrarily. If there is an unique maximiser for functional (6.2), say $u^* = (u^*(t))_{t=t_0}^{t_1}$, then the equation

$$\dot{y} = f(y, u^*(t))$$

has a unique solution (if $f(.)$ has the appropriate properties), $y^* = (y^*(t))_{t=t_0}^{t_1}$.

Intuitively, while a function is a mapping of a number to a number, a functional is a mapping from a function to a number. This means that functional (6.1) provides a number, or an index, for every path of $(y, u) = ((y(t), u(t)))_{t=t_0}^{t_1}$. The optimal path would be the one which delivers the maximum score.

The question is: how to determine u^* allowing to determine y^* the optimal trajectory for y in the previous sense ?

There are basically three methods:

- the **calculus of variations** method, which can be applied if we write equation (6.1) implicitly as $u = g(\dot{y}, y)$, substitute in the integral and solve it for y;
- the **Pontriyagin maximum principle** which is an extension of the Lagrangean approach;
- the **principle of dynamic programming** which transforms the maximisation of the functional into a recursive problem.

For historical reasons, the literature distinguishes between calculus of variations and optimal control problems:[1]

Calculus of variations problem Let the set of independent variables by $T = [t_0, t_1] \subseteq \mathbb{R}$ and let Y belong to the space of $C^1(\mathbb{R})$ functions, which are continuous and continuously differentable. Our state variable $y \in \mathrm{Y}$ is a mapping $y : T \to Y \subseteq \mathbb{R}$, where the range Y is determined by the economic meaning of the problem. For example, if the variable y refers to a price Y is a subset of the set of non-negative numbers ($\mathbb{R}_+$). We want to find a function $y^* \in \mathrm{Y}$ that maximizes the functional [2]

$$\mathcal{J}[y] \equiv \int_{t_0}^{t_1} F(t, y(t), \dot{y}(t))dt \tag{6.3}$$

We call optimal value to the value of the functional evaluated at the optimal function y^*,

$$v(\varphi) = \max_{y \in \mathrm{Y}} \mathcal{J}[y]$$

[1]See Goldstine (1980).
[2]See the Appendix 6.A.2 for the definition of functional and functional derivative.

where $v(.)$ is a **function** of the initial value of y, and φ denotes other data of the problem, including information on t_0, t_1 and values of $y(.)$. As y is a function of an independent variable, time in the previous case, finding y^* is equivalent to finding an optimal path $y^* = (y^*(t))_{t_0 \leq t \leq t_1}$, starting from point y_0.

There are several **versions** of the problem: in addition to the initial condition, several other constraints can be imposed upon the initial time or the terminal time, t_0 and t_1, on the value of y at the initial or terminal time $y(t_0)$ or $y(t_1)$, on the form of the Lagrange function $F(.)$, and on the set of admissible states Y.

Optimal control problem Consider function $y \in \mathrm{Y}$, where Y is a space of $C^1(\mathbb{R})$ functions such that $y : [t_0, t_1] \to Y \subseteq \mathbb{R}$, and function $u \in \mathrm{U}$, where U is the space of PC^1 (piecewise-continuous) [3] such that $u : [t_0, t_1] \to U \subseteq \mathbb{R}^m$. We want to find functions, $y^* \in \mathrm{Y}$ and $u^* \in \mathrm{U}$, satisfying

$$\dot{y} = g(y(t), u(t), t), \text{ for } t \in [t_0, t_1] \tag{6.4}$$

and $y(0) = y_0$ is given, such that u^* maximises the functional

$$\mathcal{J}[y, u] \equiv \int_{t_0}^{t_1} F(t, y(t), u(t))dt. \tag{6.5}$$

In control theory y and u are called state and control variables , $F(.)$ is the objective function and $g(.)$ describes the working of the economy.

We call value function to the functional evaluated at the optimum u^*,

$$v(\varphi) = \max_{u \in \mathrm{U}} \mathcal{J}[y, u],$$

where φ refers to the data of the problem.

Intuition: this is equivalent to finding an optimal paths for the state and the control variables $y^* = (y^*(t))_{t_0 \leq t \leq t_1}$ and $u^* = (u^*(t))_{t_0 \leq t \leq t_1}$ that maximises the functional (6.28), starting from point y_0, which can be known or optimally determined, and satisfying the ODE (6.3.4).

There are several **versions** of the optimal control problem: several other constraints can be imposed upon the initial or terminal time, t_0 or t_1, the initial or the terminal value of the state variable,$y(t_0)$ or $y(t_1)$, and/or the terminal value of the control variable $u(t_1)$, the objective function $f(.)$, and the set of admissible states Y and or controls U.

We can see the CV problem as a particular type of OC problem where the constraint is $\dot{y} = u$.

Optimal control problems are an essential component of modern dynamic models in economics. They appear in several different categories of models:

[3] PC^1 functions are continuous in almost all the domain except in a small number of points in which they can be discontinuous.

- First, as dynamic microeconomic problems (such as the optimal investment and consumption problem, or as the production and investment problem for a representative firm, in dynamic marketing problems, political cycle problems, etc);
- Second, as the problem of the central planner in centralized economies, or as the equivalent equilibrium in decentralised economies without distortions in which the equilibrium is Pareto optimal (first welfare theorem);
- Third, as microeconomic problems for representative agents in DGE models, in which the general equilibrium is defined as the path that solves the dynamic optimization problem of the agents together with aggregate consistency conditions.
- Fourth, in dynamic game-theory models each player is characterized by an optimal control problem in which the objective function and the constraints depend on the state and control variables of the other players.

Next we present a brief introduction to dynamic optimization for the simplest problems with one state variable.

6.2 Calculus of variations: introduction

Two observations are important referring to the nature of the independent variable, t, and to its domain $T = [t_0, t_1]$.

First, from now on, as in most economic applications, t refers to time. However in some microeconomic problems or static macroeconomic problems with heterogeneity among agents, and, for example, information or searching frictions we are lead to solve optimal control problems in which the independent variable is not time a support belonging to a continuum. In time-dependent problems we call t_0 the initial time and t_1 the terminal time, or horizon, while for non-time-dependent models the designation depends on the context. For example in models in which t refers to the skill level t_0 refers to the lowest skill in the distribution and t_1 to the highest skill. Therefore, from now on we call t_0 the lower bound and t_1 the upper bound of T.

Second, in time-dependent problems we usually assume that t_0 and t_1 may be fixed (v.g., in macroeconomic models) or free (v.g., in microeconomic problems). If t refers to other type of variables t_0 and t_1 may refer to cutoff points which can be free and optimally determined. The other important reference to be made, which is particularly important in macroeconomics is related to the boundedness of T. We can consider t_1 to be bounded or unbounded $t_1 = \infty$. In the case in which t refers to time we have to distinguish between **finite or infinite horizon** cases.

In this section, we start with the simplest case, the case in which the boundary of T and the values of the state variables at that boundary are also known, i.e, t_0, t_1, $y(t_0)$ and $y(t_1)$ are known.

Next we consider the cases in which t_0 are t_1 known but $y(t_0)$ and $y(t_1)$ are free, the cases in which known $y(t_0)$ and $y(t_1)$ are known but t_0 are t_1 are free and the cases in which t_0, t_1, $y(t_0)$ and $y(t_1)$ are all free. Then we deal with two cases which are common to time-dependent models: the existence of terminal constraints and the infinite horizon problem.

6.2.1 The simplest CV problem

The simplest CV problem is the following: let the set of independent variables be closed and bounded $T = [t_0, t_1]$, where the limits t_0 and t_1 are known, find function $y^* \in \mathrm{Y}$, among functions $y : T \to Y \subseteq \mathbb{R}$, such that $y(t_0) = y_0$ and $y(t_1) = y_1$ are known, that maximizes the **objective functional**

$$\mathcal{J}[y] \equiv \int_{t_0}^{t_1} F(t, y(t), \dot{y}(t))dt \tag{6.6}$$

in which we call $F(.)$ the **objective function**.

We assume that $F_{\dot{y}}(t, y, \dot{y}) = \dfrac{\partial F(t, y, \dot{y})}{\partial \dot{y})} \neq 0$, except maybe in a subset of measure zero. This is the characteristic of function $F(.)$ which makes the problem a dynamic optimization problem, in the sense that the optimization involves a trade-off between the current state $y(t)$ and the change in the current state. if $F_{\dot{y}}(.) = 0$ globally the problem will degenerate to a static optimization.

We denote by $\varphi = (t_0, t_1, y_0, y_1, .)$ be vector of the data of the problem containing the lower and upper values of the independent variable, the associated values of the state function, and other parameters that might exist in function $F(.)$.

The **value function**, $V(\varphi) = \mathcal{J}[y^*]$, depends on the data of the problem, that is

$$V(t_0, t_1, y_0, y_1, .) = \mathcal{J}[y^*] \equiv \max_{y \in \mathrm{Y}} \int_{t_0}^{t_1} F(t, y(t), \dot{y}(t))dt,$$

where Y is the set of admissible solutions, that is

$$\mathrm{Y} \equiv \{\ y(t) \in Y : y(t_0) = y_0,\ y(t_1) = y_1\}$$

the set of functions which satisfy the lower and upper boundary data.

Proposition 1. ***First order necessary conditions for the simplest problem**: $y^* = (y^*(t))_{t \in T}$ is a solution of the simplest CV problem only if it satisfies the **Euler-Lagrange equation** [4]:*

$$F_y(t, y^*(t), \dot{y}^*(t)) = \frac{d}{dt}\left(F_{\dot{y}}(t, y^*(t), \dot{y}^*(t))\right),\ for\, t \in (t_0, t_1) \tag{6.7}$$

together with the boundary conditions

$$y^*(t_0) = y_0, and\, y^*(t_1) = y_1 \tag{6.8}$$

[4] We use the notation $F_y(t, y, \dot{y}) = \dfrac{\partial F(t, y, \dot{y})}{\partial y}$ and $F_{\dot{y}}(t, y, \dot{y}) = \dfrac{\partial F(t, y, \dot{y})}{\partial \dot{y}}$.

Proof. (Heuristic) Assume we know y^*. Then the maximum value for the functional is

$$\mathcal{J}[y^*] = \int_{t_0}^{t_1} F(t, y^*(t), \dot{y}^*(t))dt.$$

Function y^* is an optimum only if $\mathcal{J}[y^*] \geq \mathcal{J}[y]$ for any other admissible function $y \in \mathrm{Y}$. Take an admissible variation over y^*, $y = y^* + \delta y$ such that the variation is a *parameterized perturbation* of y^*, that is $\delta y = \varepsilon\eta$ where $\eta \in \mathrm{Y}$ and ε is a number. A variation to be admissible has to satisfy $y(t_1) = y^*(t_1) = y_1$ and $y(t_0) = y^*(t_0) = y_0$. Therefore, an admissible perturbation has to satisfy $\eta(t_0) = \eta(t_1) = 0$ and it can take arbitrary values $\eta(t) \in \mathrm{Y}$ for t in the interior of the domain T.

The value functional for y is

$$\mathcal{J}[y] = \mathcal{J}[y^* + \varepsilon\eta] \;\; = \int_{t_0}^{t_1} F\big(t, y^*(t) + \varepsilon\eta(t), \dot{y}^*(t) + \varepsilon\dot{\eta}(t)\big)dt.$$

A first-order expansion of the functional $\mathcal{J}[y]$ in a neighbourhood of y^*,

$$\mathcal{J}[y] \;\; = \mathcal{J}[y^*] + \delta\mathcal{J}[y^*](\eta)\varepsilon + o(\varepsilon)$$

Then $\mathcal{J}[y^*] \geq \mathcal{J}[y]$ only if the first integral derivative of J is zero: $\delta\mathcal{J}[y^*](\eta) = 0$. (see the appendix 6.A.1 for the $o(.)$ notation and the appendix 6.A.2 for the variation of a functional).

Because the Gâteaux derivative of a functional evaluated at y^* for the perturbation η is (see appendix 6.A.2)

$$\delta\mathcal{J}[y^*](\eta) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{J}[y^* + \varepsilon\eta]$$

we have

$$\delta\mathcal{J}[y^*](\eta) = \int_{t_0}^{t_1} \big(F_y(t, y^*(t), \dot{y}^*(t))\eta(t) + F_{\dot{y}}(t, y^*(t), \dot{y}^*(t))\dot{\eta}(t)\big)\, dt.$$

Intregrating by parts the second integral yields

$$\int_{t_0}^{t_1} F_{\dot{y}}(t, y^*(t), \dot{y}^*(t))\dot{\eta}(t)dt = \int_{t_0}^{t_1} F_{\dot{y}}(t, y^*(t), \dot{y}^*(t))\eta(t) - \int_{t_0}^{t_1} dF_{\dot{y}}(t, y^*(t), \dot{y}^*(t))\eta(t).$$

Then

$$\begin{aligned}\delta\mathcal{J}[y^*](\eta) = \int_{t_0}^{t_1} \Big(F_y(t, y^*(t), \dot{y}^*(t))\eta(t) - \eta(t)\frac{d}{dt}F_{\dot{y}}(t, y^*(t), \dot{y}^*(t))\Big)\, dt + \\ + F_{\dot{y}}(t_1, y^*(t_1), \dot{y}^*(t_1))\eta(t_1) - F_{\dot{y}}(t_0, y^*(t_0), \dot{y}^*(t_0))\eta(t_0).\end{aligned} \tag{6.9}$$

As, in this case with fixed boundary values for the variable y, the admissible perturbation satisfies $\eta(t_1) = \eta(t_0) = 0$ we get

$$\delta\mathcal{J}[y^*](\eta) = \int_{t_0}^{t_1} \Big(F_y(t, y^*(t), \dot{y}^*(t)) - \frac{d}{dt}F_{\dot{y}}(t, y^*(t), \dot{y}^*(t))\Big)\, \eta(t)dt$$

If $F(.)$ is a continuous function we can use the following result: if $h := [x_0, x_1] \to \mathbb{R}$ is a continuous function and $\int_{x_0}^{x_1} h(x)\eta(x)dx$ for all C^1 functions η and if $\eta(x_0) = \eta(x_1) = 0$ then $\int_{x_0}^{x_1} h(x)\eta(x)dx = 0$ if and only if $h(x) = 0$ for all $x \in (x_0, x_1]$.

Therefore $\delta\mathcal{J}[y^*](\eta) = 0$ if an only if $F_y(t, y^*(t), \dot{y}^*(t)) - \frac{d}{dt}F_{\dot{y}}(t, y^*(t), \dot{y}^*(t)) = 0$ for every $t \in T$. $\square$

The Euler-Lagrange equation is a 2nd order ODE (ordinary differential equation) if $F_{\dot{y}\dot{y}} \neq 0$: if we expand the right-hand-side we find

$$F^*_{\dot{y}\dot{y}}\ddot{y}^* + F^*_{\dot{y}y}\dot{y}^* \ + F^*_{\dot{y}t} - F^*_y = 0,\ 0 \leq t \leq \bar{t},$$

where the derivatives of $F(\cdot)$ are evaluated at the optimum $y^*(t)$: for instance $F^*_y = F_y(t, y^*(t), \dot{y}^*(t))$.

We can transform it into a system of first order ODE's if we define $y_1 = y$ and $y_2 = \dot{y}$ then

$$\begin{aligned}\dot{y}_1 &= y_2\\ F_{y_2y_2}(t, y_1, y_2)\dot{y}_2 &= F_{y_1}(t, y_1, y_2) - F_t(t, y_1, y_2) - F_{y_2y_1}(t, y_1, y_2)y_2.\end{aligned}$$

The first order necessary condition only allows for the determination of an extremum. In order to get the a necessary condition for a maximand we need a second order condition:

Proposition 2. ***Second order necessary conditions****: the solution to the CV problem y^* is a maximand only if it satisfies the Legendre-Clebsch condition*

$$F_{\dot{y}\dot{y}}(t, y^*(t), \dot{y}^*(t)) \leq 0 \tag{6.10}$$

Proof. (Heuristic but more complicated). Performing a second -order expansion of the functional $\mathcal{J}[x]$ in a neighbourhood of y^*, we obtain

$$\mathcal{J}[y] \ = \mathcal{J}[y^*] + \delta\mathcal{J}[y^*](\eta)\varepsilon + \frac{1}{2}\ \delta^2\mathcal{J}[y^*](\eta)\varepsilon^2 + o(\varepsilon^2),$$

where

$$\delta^2\mathcal{J}[y^*](\eta) = \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\mathcal{J}[y^* \ + \varepsilon\eta].$$

Because at the optimum for any admissible perturbation η we have $\delta\mathcal{J}[y^*](\eta) = 0$, and at a have a maximum $\mathcal{J}[y] \ \leq \mathcal{J}[y^*]$, a necessary condition is $\delta^2\mathcal{J}[y^*](\eta) \leq 0$.

The second-order functional derivative is

$$\delta^2\mathcal{J}[y^*](\eta) = \int_{t_0}^{t_1} \left(F_{yy}(t, y^*(t), \dot{y}^*(t))\eta(t)^2 + 2F_{y\dot{y}}(t, y^*(t), \dot{y}^*(t))\eta(t)\dot{\eta}(t) + F_{\dot{y}\dot{y}}(t, y^*(t), \dot{y}^*(t))(\dot{\eta}(t))^2\right) dt.$$

As

$$\int_{t_0}^{t_1} 2F^*_{y\dot{y}}(t)\eta(t)\dot{\eta}(t)dt = \int_{t_0}^{t_1} F^*_{y\dot{y}}(t)\frac{d}{dt}\ (\eta(t)^2)dt =$$
$$= F^*_{y\dot{y}}(t)\ \eta(t)^2\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\left(F^*_{y\dot{y}}\right)(\eta(t)^2)dt$$
$$= -\int_{t_0}^{t_1} \frac{d}{dt}\left(F^*_{y\dot{y}}\right)(\eta(t)^2)dt$$

because of the admissibility conditions $\eta(t_0) = \eta(t_1) = 0$. Then

$$\delta^2\mathcal{J}[y^*](\eta) = \int_{t_0}^{t_1}\left(\left(F^*_{yy}(t) - \frac{d}{dt}F^*_{y\dot{y}}(t)\right)\eta(t)^2 + F^*_{\dot{y}\dot{y}}(t)(\dot{\eta}(t))^2\right)dt.$$

Following (Liberzon, 2012, p.59-60)), it can be shown that $\delta^2\mathcal{J}[y^*](\eta) \leq 0$ only if condition (6.10) holds. □

Proposition 3. ***Sufficient conditions****: let $y^* \in \mathrm{Y}$ verify*

$$F_y\ (t, y^*, \dot{y}^*) = \frac{d}{dt}\ F_{\dot{y}}(t, y^*, \dot{y}^*)\ \ \textit{and}\ F_{\dot{y}\dot{y}}(t, y^*, \dot{y}^*) \leq 0$$

then (under some additional conditions on the trajectory of y) y^* is an optimiser to $\mathcal{J}[y]$.

Proof. See (Liberzon, 2012, p.62-68) □

Proposition 4. ***Necessary and sufficient conditions****: consider the simplest calculus of variations problem and assume that $F_{\dot{y}\dot{y}}(t, y(t), \dot{y}(t)) \leq 0$ for every $t \in [t_0, t_1]$ then equations* (6.2.6) *and* (6.8) *are necessary and sufficient conditions.*

6.2.2 Free boundary values for the state variable

Now we consider the problem: find function y^* among admissible functions $y \in \mathrm{Y}$ having the following properties: $y : T \to Y \subseteq \mathbb{R}$, where $T = [t_0, t_1]$ has known boundaries, t_0 and t_1, and such that $y(t_0)$ and/or $y(t_1)$ are free. The objective functional is again (6.6).

In this case the data of the problem is $\varphi = (t_0, t_1, .)$. and the value functional is

$$V(t_0, t_1, .) = \max_{y \in \mathrm{Y}} \int_{t_0}^{t_1} F(t, y(t), \dot{y}(t))dt.$$

Proposition 5. ***First order necessary conditions for the free terminal state problem:*** *$y^* \in \mathrm{Y}$ is the solution to the CV problem with free boundary values for the state variable and known terminal values for the independent variable, t_0 and t_1, only if it satisfies the Euler equation* (6.2.6) *and the boundary conditions:*

1. *if both boundary values are free*

$$F_{\dot{y}}(t_0, y^*(t_0), \dot{y}^*(t_0)) = 0, \text{ and } F_{\dot{y}}(t_1, y^*(t_1), \dot{y}^*(t_1)) = 0 \tag{6.11}$$

2. *if the lower boundary value is given by $y(t_0) = y_0$, and the upper boundary value is free*

$$y^*(t_0) = y_0, \text{ and } F_{\dot{y}}(t_1, y^*(t_1), \dot{y}^*(t_1)) = 0 \tag{6.12}$$

3. *if the upper boundary value is given by $y(t_1) = y_1$, and the lower boundary value is free*

$$F_{\dot{y}}(t_0, y^*(t_0), \dot{y}^*(t_0)) = 0, \text{ and } y^*(t_1) = y_1. \tag{6.13}$$

Proof. (Heuristic) Now the boundary values for perturbation are $\eta(t_0)$ and $\eta(t_1)$ can take any value, including zero if the associated boundary value for y is fixed. The proof follows the same steps as for the simplest CV problem. However, in (6.9), in order to get $\delta\mathcal{J}[y^*](\eta) = 0$, and after introducing the Euler-Lagrange condition, we should have

$$F_{\dot{y}}(t_j, y^*(t_j), \dot{y}^*(t_j))\eta(t_j) = 0, \text{ for } j = 0, 1. \tag{6.14}$$

Thus we have two cases, concerning the adjoint conditions at boundary t_j, for $j = 0, 1$, for an optimum. First, if the value of the state variable for the boundary t_j is known, i.e., $y(t_j) = y_j$, an admissible perturbation should verify $\eta(t_j) = 0$, implying that condition (6.14) holds automatically. This is the case in proposition 1. Second, if the value of the state variable for the boundary t_j is free, then the related perturbation value is arbitrary and $\eta(t_j) \neq 0$ in general. The optimally condition (6.14) holds if and only if $F_{\dot{y}}(t_j, y^*(t_j), \dot{y}^*(t_j)) = 0$ which provides one adjoint condition allowing for the determination of the optimal boundary value for the state variable $y^*(t_j)$. This is how we adjoint (6.11) to (6.13) depending on which boundary value for the state variable is free. □

In time-varying models in which the value of the state variable is known at time $t = 0$ and the terminal value of the state variable is endogenous we supplement the Euler-Lagrange with condition (6.12).

However, there are models in which the initial value of the state variable is unknown. This is the case, for instance, in optimal taxation models of the Mirrlees (1971) type in which the independent variable are skill values and the initial condition is related to the cutoff level of skill bellow which taxes should be zero. In this case condition (6.11) can be used.

Observation: as the Euler-Lagrange is a second-order differential equation, in order to fully solve a model we need to have information on the value of y at the two boundaries for $t = t_0$ and $t = t_1$.

6.2.3 Free boundary values for the independent variable

Now we consider the problem: find function $y^* \in \mathrm{Y}$ which is the set of functions $y : T^* \to \mathbb{R}$, where T^* has at least one unknown boundary, t_0^* and/or t_1^*, but such that the terminal values for the state variable are known. That is $T^* = [t_0^*, t_1]$ or $T^* = [t_0, t_1^*]$ or $T^* = [t_0^*, t_1^*]$ where t_j is known and t_j^* is free. If a boundary value for the independent variable is free the related boundary value for the state variable is known, that is $y(t_j^*) = y_j$.

The objective functional is again (6.6).

In this case the data of the problem is $\varphi = (y_0, y_1, .)$. and the value functional is

$$V(y_0, y_1, .) = \max_{y \in \mathrm{Y}} \int_{t_0}^{t_1} F(t, y(t), \dot{y}(t))dt = \int_{t_0^*}^{t_1^*} F(t, y^*(t), \dot{y}^*(t))dt$$

where we t_0^* and/or t_1^* are determined endogenously.

Proposition 6. ***First order necessary conditions for the free boundaries value problem:*** *$y^* \in \mathrm{Y}$ is the solution to the CV problem with known boundary values for the state variable, y_0 and y_1, and free terminal values for the independent variable, only if it satisfies the Euler equation* (6.2.6) *and the boundary conditions:*

1. *if both boundary values for the indepdendent variable are free*

$$F(t_0^*, y_0, \dot{y}^*(t_0^*)) - F_{\dot{y}}(t_0^*, y_0, \dot{y}^*(t_0^*))\, \dot{y}^*(t_0^*) = 0$$
$$\text{and } F(t_1^*, y_1, \dot{y}^*(t_1^*)) - F_{\dot{y}}(t_1^*, y_1, \dot{y}^*(t_1^*))\, \dot{y}^*(t_1^*) = 0 \quad (6.15)$$

2. *if the lower boundary value for the independent variable is known, $t_0^* = t_0$, and the upper boundary for the independent variable is free*

$$t_0^* = t_0, \text{ and } F(t_1^*, y_1, \dot{y}^*(t_1^*)) - F_{\dot{y}}(t_1^*, y_1, \dot{y}^*(t_1^*))\dot{y}^*(t_1^*) = 0 \quad (6.16)$$

3. *if the upper boundary value for the independent variable is known, $t_1^* = t_1$, and the lower boundary for the independent variable is free*

$$F(t_0^*, y_0, \dot{y}^*(t_0^*)) - F_{\dot{y}}(t_0^*, y_0, \dot{y}^*(t_0^*))\dot{y}^*(t_0^*) = 0, \text{ and } t_1^* = t_1. \quad (6.17)$$

Proof. (Heuristic) Let us assume that we know the solution $y^*(t)$ for $t \in [t_0^*, t_1^*]$, that is for all values of the independent variable contained between the two optimally chosen boundary values. In this case we have to introduce two types of perturbations: a perturbation to the state variable $y(t) = y^*(t) + \varepsilon\eta(t)$ and to the independent variable $t = t^* + \varepsilon\tau(t)$. If we denote $y_j^* = y^*(t_j^*)$, for $j = 0, 1$, the two terminal points for the independent and dependent variables are $P_j^* \equiv (t_j^*, y_j^*)$ for $j = 0, 1$ at the optimum. The related terminal points for the perturbed solution are written as $P_j = (t_j^* + \varepsilon\tau_j, y_j^* + \varepsilon\eta_j)$ for $j = 0, 1$.

Now,

$$\mathcal{J}[y^*; t^*] = \int_{t_0^*}^{t_1^*} F(t, y^*(t), \dot{y}^*(t))dt$$

and

$$\mathcal{J}[y^* + \varepsilon\eta; t^* + \varepsilon\tau] = \int_{t_0^*+\varepsilon\tau_0}^{t_1^*+\varepsilon\tau_1} F(t, y^*(t) + \varepsilon\eta(t), \dot{y}^*(t) + \varepsilon\dot{\eta}\ (t))dt.$$

Then, denoting $\Delta\mathcal{J}(\varepsilon) = \mathcal{J}[y^* + \varepsilon\eta; t^* + \varepsilon\tau] - \mathcal{J}[y^*; t^*]$ we have

$$\begin{aligned}
\Delta\mathcal{J}(\varepsilon) &= \int_{t_0^*+\varepsilon\tau_0}^{t_1^*+\varepsilon\tau_1} F(t, y^*(t) + \varepsilon\eta(t), \dot{y}^*(t) + \varepsilon\dot{\eta}\ (t))dt - \int_{t_0^*}^{t_1^*} F(t, y^*(t), \dot{y}^*(t))dt \\
&= \int_{t_0^*}^{t_1^*} \left(F(t, y^*(t) + \varepsilon\eta(t), \dot{y}^*(t) + \varepsilon\dot{\eta}\ (t)) - F(t, y^*(t), \dot{y}^*(t))\right) dt + \\
&\quad + \int_{t_1^*}^{t_1^*+\varepsilon\tau_1} F(t, y^*(t) + \varepsilon\eta(t), \dot{y}^*(t) + \varepsilon\dot{\eta}\ (t))dt - \\
&\quad - \int_{t_0^*+\varepsilon\tau_0}^{t_0^*} F(t, y^*(t) + \varepsilon\eta(t), \dot{y}^*(t) + \varepsilon\dot{\eta}\ (t))dt
\end{aligned}$$

Denoting $F^*(t) = F\left(t, y^*(t), \dot{y^*}(t)\right)$ and using the mean-value theorem,

$$\Delta\mathcal{J}(\varepsilon) = \varepsilon \int_{t_0^*}^{t_1^*} \left(F_y^*(t)\,\eta(t) + F_{\dot{y}}^*(t)\,\dot{\eta}(t)\right) dt + F(\tilde{t}_1)\,\varepsilon\,\tau_1 - F(\tilde{t}_0)\,\varepsilon\,\tau_0$$

where $\tilde{t}_1 \in (t_1^*, t_1^* + \varepsilon\,\tau_1)$ and $\tilde{t}_0 \in (t_0^*, t_0^* + \varepsilon\,\tau_0)$. Taking $\delta\mathcal{J}[y^*; t^*]\ (\eta, \tau) = \lim_{\varepsilon\to 0} \dfrac{\Delta\mathcal{J}(\varepsilon)}{\varepsilon}$, the functional derivative becomes

$$\delta\mathcal{J}[y^*; t^*]\ (\eta, \tau) = \int_{t_0^*}^{t_1^*} \left(F_y^*(t)\,\eta(t) + F_{\dot{y}}^*(t)\,\dot{\eta}(t)\right) dt + F^*(t)\big|_{t=t_1^*}\tau_1 - F^*(t)\big|_{t=t_0^*}\tau_0.$$

Integration by parts yields

$$\begin{aligned}
\delta\mathcal{J}[y^*; t^*]\ (\eta, \tau) &= \int_{t_0^*}^{t_1^*} \left(F_y^*(t) - \frac{d}{dt}F_{\dot{y}}^*(t)\right)\eta(t)dt + \\
&\quad + F_{\dot{y}}^*(t)\eta(t)\big|_{t=t_1^*} - F_{\dot{y}}^*(t)\eta(t)\big|_{t=t_0^*} + F^*(t)\big|_{t=t_1^*}\tau_1 - F^*(t)\big|_{t=t_0^*}\tau_0.
\end{aligned}$$

We only know the perturbations for the state variables at the perturbed boundaries t_0 and t_1 and not at t_0^* and t_1^*, which inhibits the computation of the integral in the last equation. In order to find $\eta(t_j^*)$ the following approximation is introduced

$$\eta(t_j^*) \approx \eta_j - \dot{y}(t_j^*)\tau_t, \text{ for } j = 0, 1.$$

Therefore,

$$\delta\mathcal{J}[y^*; t^*]\ (\eta, \tau) = \int_{t_0^*}^{t_1^*} \left(F_y^*(t) - \frac{d}{dt} F_{\dot{y}}^*(t) \right) \eta(t) dt + F_{\dot{y}}^*(t_1^*)\eta_1 - F_{\dot{y}}^*(t_0^*)\eta_0 + \\ + \left(F^*(t) - F_{\dot{y}}^*(t)\dot{y}(t) \right)\Big|_{t=t_1^*} \tau_1 - \left(F^*(t) - F_{\dot{y}}^*(t)\dot{y}(t) \right)\Big|_{t=_0^*} \tau_0$$

As the terminal values of the state variables, $y(t_0^*) = y_0$ and $y(t_1^*) = y_1$, are known then the terminal perturbation for the independent variable should satisfy $\eta_0 = \eta_1 = 0$. Therefore, $\delta\mathcal{J}[y^*; t^*]\ (\eta, \tau) = 0$ if and only if the Euler-Lagrange equation holds and $\left(F^*(t) - F_{\dot{y}}^*(t)\dot{y}(t) \right)\Big|_{t=t_1^*} \tau_1 = 0$ and/or $\left(F^*(t) - F_{\dot{y}}^*(t)\dot{y}(t) \right)\Big|_{t=_0^*} \tau_0 = 0$. This encompasses the three cases reported in equations (6.15), (6.16) and (6.17). □

6.2.4 Free boundaries for both independent and dependent variables

The most general problem is: find function $y^* \in \mathrm{Y}$ among functions $y : T^* \to \mathbb{R}$, where T^* has at least one unknown boundary, t_0^* and/or t_1^*, as in the previous subsection, and the terminal values for the state variables, $y(t_0^*)$ and/or $y(t_1^*)$ are also free. The objective functional is again (6.6).

In this case the data of the problem, $\varphi = (.)$, only involves parameters that may be present in function $F(.)$. The value functional is

$$V(.) = \max_{y \in \mathrm{Y}} \int_{t_0}^{t_1} F(t, y(t), \dot{y}(t)) dt = \int_{t_0^*}^{t_1^*} F(t, y^*(t), \dot{y}^*(t)) dt$$

where we t_0^* and/or t_1^* and $y^*(t_0^*)$ and/or $y^*(t_1^*)$ are determined endogenously.

Proposition 7. ***First order necessary conditions for the free terminal boundary problem:*** *$y^* \in \mathrm{Y}$ is the solution to the CV problem with free boundary values for the state variable and for the independent variable, only if it satisfies the Euler equation* (6.2.6) *and the boundary conditions:*

1. *if both values for lower boundary are free*

$$F_y(t_0^*, y^*(t_0^*), \dot{y}^*(t_0^*)) = F_{\dot{y}}(t_0^*, y^*(t_0^*), \dot{y}^*(t_0^*)) = 0 \tag{6.18}$$

2. *if both values for upper boundary are free*

$$F_y(t_1^*, y^*(t_1^*), \dot{y}^*(t_1^*)) = F_{\dot{y}}(t_1^*, y^*(t_1^*), \dot{y}^*(t_1^*)) = 0 \tag{6.19}$$

Proof. We use the previous proof and, in equation (6.18), we consider $\eta_0 \neq 0$, $\eta_1 \neq 0$, $\tau_0 \neq 0$ and $\tau_1 \neq 0$. □

Table 6.1 assembles all the previous results. Observe that if we consider all the possible combinations of the information on both boundaries we have **16 possible cases**.

Table 6.1: Adjoint conditions for bounded domain CV problems

data		optimum	
t_j	$y(t_j)$	t_j^*	$y^*(t_j^*)$
fixed	fixed	t_j	y_j
fixed	free	t_j	$F_{\dot{y}}(t_j, y^*(t_j), \dot{y}^*(t_j)) = 0$
free	fixed	$F(t_j^*, y_j, \dot{y}^*(t_j^*)) - \dot{y}^*(t_j^*) F_{\dot{x}}(t_j^*, y_j, \dot{y}^*(t_j^*)) = 0$	y_j
free	free	$F\left(t_j^*, y^*(t_j^*), \dot{y}^*(t_j^*)\right) = 0$	$F_{\dot{y}}(t_j^*, y^*(t_j^*), \dot{y}^*(t_j^*)) = 0$

The index refers to the lower boundary when $j = 0$ and to the upper boundary when $j = 1$

6.2.5 The constrained terminal value problem

We consider a problem in which the lower boundary is known, i.e., both t_0 and $y(t_0) = y_0$ are known, the upper value of the independent variable, t_1, is known but the state should satisfy the condition $R(t_1, y(t_1)) \geq 0$.

Proposition 8. ***First order necessary conditions for the constrained terminal state problem:*** *$y^* \in \mathrm{Y}$ is the solution to the CV problem with a free terminal state only if it satisfies the Euler-Lagrange equation* (6.2.6)*, the initial condition $y^*(t_0) = y_0$ and the boundary condition*

$$F_{\dot{y}}(t_1, y^*(t_1), \dot{y}^*(t_1)) R(t_1, y^*(t_1)) = 0 \tag{6.20}$$

Proof. In this case we consider the functional

$$L[y] = \int_{t_0}^{t_1} F(t, y(t), \dot{y}(t)) dt + \mu R(t_1, y(t_1))$$

where μ is a Lagrange multiplier (a number). We proceed as previously to get the optimality conditions for a perturbation $\eta \in \mathrm{Y}$ over the optimal function y^*. The first order necessary condition is

$$\delta L[y^*](\eta) = \int_{t_0}^{t_1} \left(F_y^*(t) - \frac{d}{dt} F_{\dot{y}}^*(t) \right) \eta(t) dt + \left(F_{\dot{y}}^*(t_1) + \mu H_y^*(t_1) \right) \eta(t_1) = 0$$

where $F^*(t) = F(t, y^*(t), \dot{y}^*(t))$. Because of the free terminal state, admissible perturbations are such that $\eta(t_1) \neq 0$. Therefore $\delta L[y^*](\eta) = 0$ requires that the adjoint condition $F_{\dot{y}}^*(t_1) + \mu H_y^*(t_1) = 0$ holds.

Due to the existence of a static inequality constraint at the boundary t_1, the Karush-Kuhn-Tucker (KKT) complementarity slackness conditions are also necessary:

$$\mu H^*(t_1) = 0,\ \mu \geq 0 \text{ and } \ H^*(t_1) \geq 0$$

where $H^*(t_1) = R(t_1, y^*(t_1))$. Multipying the adjoint condition by $H^*(t_1)$ we obtain an equivalent condition

$$H^*(t_1)F^*_{\dot{y}}(t_1) + \mu H^*(t_1)H^*_y(t_y) = 0,$$

which is equivalent to $F^*_{\dot{y}}(t_1)H^*(t_1) = 0$, after considering the KKT condition. Therefore $\delta L[y^*](\eta) = 0$ if the Euler-Lagrange equation (6.2.6) and adjoint boundary condition (6.20) hold. □

6.2.6 Calculus of variations: discounted infinite horizon

The most common problem in macroeconomics and growth theory has three main common features. First, time is the independent variable, and assumes that the initial time and values are known, usually $t_0 = 0$ and $y(0) = y_0$, and an unbounded value for the terminal time, $t_1 \to \infty$. Second, the objective function is of type $F(t, y, \dot{y}) = f(y, \dot{y})e^{-\rho t}$, where $e^{-\rho t}$ is a discount factor with a time-independent rate of discount $\rho \geq 0$, and the current value objective function $f(y, \dot{y})$ is time -independent. Third, there are two main versions to the problem depending on the terminal value of the state variable, that can be free or constrained.

Free asymptotic state

Find function $y^* \in \mathrm{Y}$ which is the set of functions $y : [0, \infty) \to \mathbb{R}$ such that $y(0) = y_0$ where y_0 are given that maximizes

$$\mathcal{J}[y] \equiv \int_0^\infty f(y(t), \dot{y}(t))e^{-\rho t}dt,\ \rho \geq 0. \tag{6.21}$$

This can be treated as a problem with a fixed initial time and value for the state variable, a fixed terminal time but a free terminal value for the state variable.

Proposition 9. ***First order necessary conditions for the discounted infinite horizon problem with free terminal state:*** *$y^* \in \mathrm{Y}$ is the solution to the discounted infinite horizon CV problem with a known initial data, $\varphi = (y_0, \rho, .)$, and with a free terminal state only if it satisfies the Euler-Lagrange equation*

$$\frac{d}{dt}\left(f_{\dot{y}}(y^*(t), \dot{y}^*(t))\right) = f_y(y^*(t), \dot{y}^*(t)) + \rho f_{\dot{y}}(y^*(t), \dot{y}^*(t)),\ \textit{for}\ \ t \in [0, \infty), \tag{6.22}$$

the so-called transversality condition

$$\lim_{t\to\infty} f_{\dot{y}}(y^*(t), \dot{y}^*(t))e^{-\rho t} = 0 \tag{6.23}$$

and the initial condition $y^*(0) = y_0$

Proof. In the proof for the free boundaries value problem we extend $t_1 \to \infty$ and take it as fixed but let $\lim_{t\to\infty} y^*(t)$ be free. In this discounted problem the Euler-Lagrange equation $F^*_y(t) = \frac{d}{dt}F^*_{\dot{y}}(t)$ is equivalent to

$$e^{-\rho t} f_y(y^*, \dot{y}^*) = \frac{d}{dt}\left(e^{-\rho t} f_{\dot{y}}(y^*, \dot{y}^*)\right),$$

and the terminal condition (6.23) is obtained fro the boundary condition $\lim_{t\to\infty} F^*_{\dot{y}}(t) = 0$. □

Observe that the Euler-Lagrange is again a 2nd order non-linear autonomous ODE

$$f_y(y^*, \dot{y}^*) + \rho f_{\dot{y}}(y^*, \dot{y}^*) - f_{\dot{y}y}(y^*, \dot{y}^*)\dot{y} - f_{\dot{y}\dot{y}}(y^*, \dot{y}^*)\ddot{y} = 0.$$

The constrained terminal state problem

In several problems in economics the former condition can lead to an asymptotic state which does not make economic sense (v.g, a negative level for a capital stock).

The most common discounted infinite horizon model is usually the following: find function $y^* \in \mathrm{Y}$ which is the set of functions $y : [0, \infty) \to \mathbb{R}$ such that $y(0) = y_0$ and $\lim_{t\to\infty} R(t, y(t)) \geq 0$, where $t_0 = 0$ and $y(t_0) = y_0$ are given, that maximizes the objective functional (6.21)

Proposition 10. ***First order necessary conditions for the discounted infinite horizon problem with constrained terminal state****:* $y^* \in \mathrm{Y}$ *is the solution to the discounted infinite horizon CV problem with a known initial data,* $(t_0, y(t_0)) = (0, y_0)$*, and with a terminal state constrained by* $\lim_{t\to\infty} R(t, y(t)) \geq 0$ *only if it satisfies the Euler-Lagrange equation* (6.22)*, the initial condition* $y^*(0) = y_0$*, and the (so-called) transversality condition*

$$\lim_{t\to\infty} f_{\dot{y}}(y^*(t), \dot{y}^*(t))R(t, y^*(t))e^{-\rho t} = 0 \tag{6.24}$$

Exercise: prove this. Observe that as the terminal constraint is $\lim_{t\to\infty} y(t) \geq 0$ we have to introduce a Lagrange multiplier associated to the terminal time.

6.2.7 Applications

The resource depletion problem

Assume we have a resource W (v.g., a cake) with initial size W_0 and we want to consume it along period $[0,\bar{t}]$. If $C(t)$ denotes the consumption at time t we evaluate the consumption of the resource by the functional $\int_0^{\bar{t}} \ln(C(t))e^{-\rho t}dt$. Several properties: (1) we are impatient (we discount time at a rate $\rho > 0$); (2) the felicity at every point in time is only a function of the instantaneous consumption (preferences are inter temporally additive); (3) more consumption means more felicity but at a decreasing rate (the increase in utility for big bites is smaller than for small bites); and (4) there is no satiation (there is not a bite with a zero or negative marginal utility): consumption is always good.

Cake eating problem with the terminal state given CE problem: find $C^* = (C^*(t))_{0\leq t\leq \bar{t}}$ that

$$\max_C \int_0^{\bar{t}} \ln(C(t))e^{-\rho t}dt$$

subject to

$$\dot{W}(t) = -C(t), \text{ for } t \in [0,T]$$

given $W(0) = W_0$ and $W(\bar{t}) = 0$.

Formulated as a CV problem: find $W^* = (W^*(t))_{0\leq t\leq \bar{t}}$ such that

$$V(W_0,\bar{t},\rho) = \max_W \mathcal{J}[W] \;= \max_W \int_0^{\bar{t}} \ln(-\dot{W}(t))e^{-\rho t}dt$$

given $W(0) = W_0$ and $W(\bar{t}) = 0$. The data of the problem is the vector of constants $\varphi = (0,\bar{t},W_0,0,\rho)$

The solution of the problem, $(W^*(t))_{t=0}^{\bar{t}}$, is obtained from

$$\begin{cases} \ddot{W}^* + \rho\dot{W}^* = 0, & 0 < t < \bar{t} \\ W^*(0) = W_0, & t = 0 \\ W^*(T) = 0, & t = T \end{cases}$$

The solution of the Euler equation is [5]

$$W(t) = W(0) - \frac{k}{\rho}(1 - e^{-\rho t})$$

[5] Hint: setting $z = \dot{W}$ we get a first-order ODE $\dot{z} = -\rho z$ with solution $\dot{z} = ke^{-\rho t}$. As $dW(t) = z(t)dt$, if we integrate we have $\int_{W(0)}^{W(t)} dW = \int_0^t z(s)ds = \int_0^t ke^{-\rho s}ds$.

where k is an arbitrary constant. Using the adjoint conditions $W^*(\bar{t}) = 0$ and $W^*(0) = W_0$ we find the solution

$$W^*(t) = \frac{e^{-\rho t} - e^{-\rho \bar{t}}}{1 - e^{-\rho \bar{t}}} \; W_0, \text{ for } t \in [0, \bar{t}].$$

The value of the cake is

$$\begin{aligned} V(\varphi) &= \int_0^{\bar{t}} \ln\left(-\dot{W}^*(t)\right) e^{-\rho t} dt = \\ &= \frac{1}{\rho}\left[\left(1 + \ln\left(\frac{1 - e^{-\rho \bar{t}}}{\rho W_0}\left(e^{-\rho \bar{t}} - 1\right)\right)\right)\right] + \bar{t} e^{-\rho \bar{t}} \end{aligned}$$

if the consumer is rational this should be equal its reservation price for the cake. If $\rho = 0.01$ and the cake lasts for one week and the calorie content is $W_0 = 1000$ then the reservation price for should be $V(10, 0.01, 1/52) \approx 0.12$ per 100 calories.

Cake eating problem: infinite horizon If we assume an infinite horizon and the terminal condition $\lim_{t\to\infty} W(t) \geq 0$, meaning that we cannot have a negative level of resource asymptotically. The first order conditions are:

$$\begin{cases} \rho \dot{W}^*(t) + \ddot{W}^*(t) = 0 \\ W^*(0) = W_0 \\ -\lim_{t\to\infty} e^{-\rho t} \frac{W^*(t)}{\dot{W}^*(t)} = 0 \end{cases}$$

We already found

$$W(t) = W_0 - \frac{k}{\rho}\left(1 - e^{-\rho t}\right)$$

then

$$\dot{W}(t) = -k e^{-\rho t}$$

Solution (as $k = \rho W_0$)

$$W^*(t) = W_0 e^{-\rho t}, \; t \in \mathbb{R}_+$$

Again $\lim_{t\to\infty} W^*(t) = 0$.

The benchmark representative problem

The benchmark representative consumer problem in macroeconomics is to find optimal consumption and asset holdings (C, A) such that $C : \mathbb{R}_+ \to \mathbb{R}_+$ and $A : \mathbb{R}_+ \to \mathbb{R}$ that maximize the value functional

$$U[C] = \int_0^{\infty} u(C(t)) e^{-\rho t} dt$$

subject to the instantaneous budget constraint

$$\dot{A} = Y - C + rA$$

given $A(0) = A_0$ and the non-Ponzi game condition $\lim_{t\to} A(t)e^{-rt} \geq 0$. In the above equations Y and r denote, respectively the non-financial income and the interest rate, and are both positive. The following assumptions on utility are standard: $u(0) = 0$, $u^{'}(C) > 0$ and $u^{''}(C) < 0$.

The inverse of the elasticity of intertemporal substitution can be proved to be

$$\theta(C) = -\frac{u^{''}(C)C}{u^{'}(C)} > 0.$$

Assumption: the elasticity of intertemporal substitution $\theta(C) = \theta$ is constant and

$$\theta > \frac{r-\rho}{r} > 0.$$

We can transform it into a CV problem by observing that consumption is a function of the both wealth and savings, $\dot{A}$,

$$C = C(A, \dot{A}) \equiv Y + rA - \dot{A}.$$

Therefore, the problem becomes a CV problem with value functional

$$\mathcal{J}[A] = \int_0^\infty u\left(Y + rA(t) - \dot{A}(t)\right) e^{-\rho t} dt$$

where $f(A(t), \dot{A}(t)) = u\left(Y + rA(t) - \dot{A}(t)\right)$. The optimality conditions (which are necessary and sufficient in this case) are

$$\begin{cases} (r-\rho)u^{'}(C(A,\dot{A})) + (r\dot{A} - \ddot{A})u^{''}(C(A,\dot{A})) = 0 \\ A(0) = A_0 \\ -\lim_{t\to\infty} e^{-\rho t}u^{'}(C(A,\dot{A}))A(t) \ = 0 \end{cases}$$

Observing that $\dot{C} = r\dot{A} - \ddot{A}$ and using the definition of the inverse intertempopral elasticity of substitution we can transform the Euler equation into

$$\dot{C} = \gamma C, \text{ for } \gamma \equiv \frac{r-\rho}{\theta} > 0.$$

This allows us to find a general solution for optimal consumption

$$C(t) = C(0)\, e^{\gamma t},$$

where $C(0)$ is an arbitrary unknown admissible level for consumption, i.e., it should be non-negative. In order to find that value we use the transversality condition. But for this we need to determine admissible values for A. The asset dynamics is then governed by

$$\dot{A} = Y + rA - ke^{\gamma t}, \text{for } t > 0, \ A(0) = A_0, \text{ for } t = 0$$

The solution to this initial value problem is

$$A(t) = -\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right)e^{rt} + \frac{C(0)}{r-\gamma}\left(e^{rt} - e^{\gamma t}\right).$$

With the previous assumption we have $r > \gamma$. As $u^{'}(C) = C^{-\theta}$ with an isoelastic utility function we find

$$\begin{aligned}
\lim_{t\to\infty} u^{'}(C(t))A(t)e^{-\rho t} &= \lim_{t\to\infty} \left(C(0)e^{\gamma t}\right)^{-\theta} A(t)e^{-\rho t} = \\
&= \lim_{t\to\infty} C(0)^{-\theta}e^{-rt}\left[-\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right)e^{rt} + \frac{C(0)}{r-\gamma}\left(e^{rt} - e^{\gamma t}\right)\right] = \\
&= \lim_{t\to\infty} C(0)^{-\theta}\left[A_0 + \frac{Y}{r} + \frac{C(0)}{r-\gamma} - \frac{C(0)}{r-\gamma}e^{(\gamma - r)t}\right] = \\
&= C(0)^{-\theta}\left[A_0 + \frac{Y}{r} + \frac{C(0)}{r-\gamma}\right] = 0
\end{aligned}$$

if and only if $C(0) = (r-\gamma)\left(A_0 + \dfrac{Y}{r}\right)$. Therefore the optimal consumption and asset holdings are

$$C^*(t) = (r-\gamma)\left(A_0 + \frac{Y}{r}\right)e^{\gamma t},\ t \in [0, \infty) \tag{6.25}$$

$$A^*(t) = -\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right)e^{\gamma t},\ t \in [0, \infty). \tag{6.26}$$

Observations: First, if we define human capital as the present value, at rate r, of the non-financial income

$$H(t) = \int_t^{\infty} Y e^{r(t-s)} ds$$

we find $H(0) = \dfrac{Y}{r}$. Therefore the solution is a linear function of the total capital, financial and non-financial

$$C^*(t) = (r-\gamma)(A_0 + H(0))e^{\gamma t},\ A^*(t) = -H(0) + (A_0 + H(0))e^{\gamma t}$$

Second, because $\gamma > 0$ then the asymptotic value of the optimal A becomes unbounded. However, it still satisfies that boundary condition $\lim_{t\to\infty} A^*(t)e^{-rt} = 0$ because, by assumption, $r > \gamma$. What matters is not the absolute level of A but its level in present-value terms.

6.3 Optimal control: the Pontriyagin's maximum principle

The Pontriagin's maximum principle provides first order necessary conditions for the optimal control problem. We consider the same cases as we did for the calculus of variations problem

We denote again the independent variable by t and assume it has the domain $T = [t_0, t_1]$ where $t_0 < t_1$ are not necessarily known, and t_1 can be bounded or unbounded. The optimal control problem contains two variables: the **state variable**, denoted by $y(t)$ and the **control variable**, denoted by $u(t)$.

The **optimal control problem** consists in finding functions $y \in \mathrm{Y}$ and $u \in \mathrm{U}$, where $\mathrm{Y} \in C^1(\mathbb{R})$, the set of continous and continuously differentiable functions $y : T \to \mathcal{Y} \subseteq \mathbb{R}$, and $\mathrm{U} \in PC^1(\mathbb{R})$, the set of piecewise continuous functions $u : T \to U \subseteq \mathbb{R}$ such that

$$\dot{y} = G(y(t), u(t), t), \text{ for } t \in [t_0, t_1] \tag{6.27}$$

that maximize the functional

$$\mathcal{J}[y, u] \equiv \int_{t_0}^{t_1} F(t, y(t), u(t))dt \tag{6.28}$$

with additional data is given. The additional data is related to the information concerning the boundary values of the independent variable t_0 and t_1 and the boundary values for the state variable $y(t_0)$ and $y(t_1)$.

The necessary conditions for an optimum according to the **Pontriyagin's maximum principle** consider the **Hamiltonian** function, defined as

$$H(t, y, u, \lambda) = F(t, y, u) + \lambda G(t, y, u).$$

where λ is called the co-state variable.

6.3.1 Bounded domain optimal control problems

In this subsection we assume that the data of the problem includes the boundary values for the independent variable: i.e., t_0 and t_1 are known. The optimal control problem is to find an optimal control $(u^*(t))_{t \in [t_0, t_1]}$ that maximizes the functional (6.28) subject to ODE constraint (6.3.4) and, possibly additional information for the state variables at the boundary values for the independent variable.

We can consider the following cases:

(P1) both boundary values are known $y(t_0) = y_0$ and $y(t_1) = y_1$;

(P2) the lower boundary values is known $y(t_0) = y_0$ but $y(t_1)$ is free

(P3) the upper boundary values is known $y(t_1) = y_1$ but $y(t_0)$ is free

(P4) both boundary values $y(t_0)$ and $y(t_1)$ are free.

Proposition 11. *[**First order necessary conditions for fixed boundary values of the independent variable**] Let (y^*, u^*) be a solution to the OC problem. Then there is a piecewise continuous function $\lambda : [t_0, t_1] \to \mathbb{R}$, called co-state variable, such that (y^*, u^*, λ) satisfy the following conditions:*

- *the optimality condition:*

$$H_u(t, y^*(t), u^*(t), \lambda(t)) = 0, \quad t \in [t_0, t_1] \tag{6.29}$$

- *the multiplier equation*

$$\dot{\lambda} = -H_y(t, y^*(t), u^*(t), \lambda(t)), \quad t \in (t_0, t_1) \tag{6.30}$$

- *the constraint of the problem:*

$$\dot{y}^* = G(t, y^*(t), u^*(t)), \; t \in (t_0, t_1) \tag{6.31}$$

- *and the adjoint conditions associated to the boundary conditions (P1) to (P4)*
  - *for problem (P1)*

$$y^*(t_0) = y_0 \, for \, t = t_0, \; and \; y^*(t_1) = y_1 \, for \, t = t_1, \tag{6.32}$$

  - *for problem (P2)*

$$y^*(t_0) = y_0 \, for \, t = t_0, \; and \; \lambda(t_1) = 0 \, for \, t = t_1, \tag{6.33}$$

  - *for problem (P3)*

$$\lambda(t_0) = 0 \, for \, t = t_0, \; and \; y^*(t_1) = y_1 \, for \, t = t_1, \tag{6.34}$$

  - *for problem (P4)*

$$\lambda(t_0) = 0 \, for \, t = t_0, \; and \; \lambda(t_1) = 0 \, for \, t = t_1. \tag{6.35}$$

Proof. (Heuristic) Let u^* be an optimal control and let y^* be the associated state. The value of the problem.

$$\mathcal{J}[y^*, u^*] = \int_{t_0}^{t_1} F(t, y^*(t), u^*(t))dt.$$

it is an optimiser if $\mathcal{J}[y^*, u^*] \geq \mathcal{J}[y, u]$ for any other admissible pair of functions (u, y).

It is convenient to write

$$\begin{aligned}
\mathcal{J}[y^*, u^*] &= \int_{t_0}^{t_1} F(t, y^*(t), u^*(t))dt = \\
&= \int_{t_0}^{t_1} [F(t, y^*(t), u^*(t)) + \lambda(t)(G(t, y^*(t), u^*(t)) - \dot{y}^*(t))]\, dt = \\
&= \int_{t_0}^{t_1} (H(t, y^*(t), u^*(t), \lambda(t)) - \dot{y}^*(t)\lambda(t))\ \ dt
\end{aligned}$$

Again we introduce a perturbation on the optimal state-control pair $(y, u) = (y^*, u^*) + \varepsilon$, where ε is a constant and $\ = (\eta_y, \eta_u)$. The admissible perturbations differ for the different versions of the problem: for (P1) we should have $\eta_y(t_0) = \eta_y(t_1) = 0$, for (P2) we should have $\eta_y(t_0) = 0$ and $\eta_y(t_1) \neq 0$, for (P3) we should have $\eta_y(t_0) \neq 0$ and $\eta_y(t_1) = 0$, and for (P4) we should have $\eta_y(t_0) \neq 0$ and $\eta_y(t_1) \neq 0$.

The first-order Taylor approximation of the functional is

$$\mathcal{J}[y, u] = \mathcal{J}[y^*, u^*] + \delta\mathcal{J}[y^*, u^*](\ )\epsilon + o(\epsilon)$$

where

$$\begin{aligned}
\delta\mathcal{J}[y^*, u^*](\ ) = &\int_{t_0}^{t_1} \big(H_u(t, y^*(t), u^*(t), \lambda(t))\eta_u(t) + H_y(t, y^*(t), u^*(t), \lambda(t))\eta_y(t) - \lambda(t)\dot{\eta}_y(t)\big)\, dt = \\
= &\int_{t_0}^{t_1} \Big(H_u(t, y^*(t), u^*(t), \lambda(t))\eta_u(t) + (H_y(t, y^*(t), u^*(t), \lambda(t)) + \dot{\lambda}(t))\eta_y(t)\Big)\, dt + \\
&+ \lambda(t_0)\eta_y(t_0) - \lambda(t_1)\eta_y(t_1).
\end{aligned}$$

Then $\mathcal{J}[y, u] \leq \mathcal{J}[y^*, u^*]$ only if $\delta\mathcal{J}[y^*, u^*](\ ) = 0$, which, using similar arguments as to the case of the CV problem, is equivalent to the Pontriyagin's conditions: $H_u(.) = \dot{\lambda} - H_y(.) = 0$. The adjoint constraints should verify $\lambda(t_0)\eta_y(t_0) = \lambda(t_1)\eta_y(t_1) = 0$. From this and the admissibility values for $\eta_y(t_0)$ and $\eta_y(t_1)$ then the adjoint constraints are as in equations (6.32) to (6.35) □

6.3.2 Free domain and fixed boundary state variable optimal control problems

In this subsection we consider the case in which one or both bounds of the space of independent variables can be optimally choses, i.e $t \in T^* = [t_0^*, t_1^*]$, where one or both t_j^*, for $j = 0, 1$ are free, but the boundary values for the state variable are fixed: i.e. $y(t_0^*) = y_0$ and/or $y(t_1^*) = y_1$ are fixed. The optimal control problem is to find the optimal cut-off values for the indepdendent variable, t_0^* and/or t_1^* and an optimal control $(u^*(t))_{t\in[t_0^*, t_1^*]}$ that maximizes the functional (6.28) subject to ODE constraint (6.3.4).

We can consider the following cases:

(P5) the lower boundary cut-off t_0 is known but the upper boundary t_1 is free

(P6) the upper boundary cut-off t_1 is known but the lower boundary t_0 is free

(P7) both boundary cut-off values t_0 and t_1 are free.

Proposition 12 (First order necessary conditions for free domain and fixed boundary state variable optimal control problems). *Let (y^*, u^*) be a solution to the OC problem where $y(t_0) = y_0$ and $y(t_1) = y_1$ are fixed. Then there is an optimal domain for the independent variable $T^* = [t_0^*, t_1^*] \subset \mathbb{R}$, a piecewise continuous function $\lambda : T^* \to \mathbb{R}$, called co-state variable, such that (y^*, u^*, λ) satisfy the optimality condition* (6.29)*, the multiplier equation* (6.30) *and the ODE constraint of the problem* (6.31)*, all for $t \in int(T^*)$ and the adjoint conditions associated to the boundary conditions (P5) to (P7)*

- *for problem (P5): $y^*(t_0) = y_0$ and $y^*(t_1^*) = y_1$ and*

$$t_0^* = t_0 \text{ and } H(t_1^*, y_1, u^*(t_1^*)) - \dot{y}^*(t_1^*)\lambda(t_1^*) = 0, \tag{6.36}$$

- *for problem (P6): $y^*(t_0^*) = y_0$ and $y^*(t_1) = y_1$ and*

$$H(t_0^*, y_0, u^*(t_0^*)) - \dot{y}^*(t_0^*)\lambda(t_0^*) = 0 \text{ and } t_1^* = t_1, \tag{6.37}$$

- *for problem (P7): $y^*(t_0^*) = y_0$ and $y^*(t_1^*) = y_1$ and*

$$H(t_0^*, y_0, u^*(t_0^*)) - \dot{y}^*(t_0^*)\lambda(t_0^*) = 0 \text{ and } H(t_1^*, y_1, u^*(t_1^*)) - \dot{y}^*(t_1^*)\lambda(t_1^*) = 0. \tag{6.38}$$

Proof. Using the same method for finding perturbations we used in the proof of propositions 6 and 11, the obtain the Gâteaux derivative

$$\begin{aligned}\delta\mathcal{J}[y^*, u^*; t^*]\ (\eta, \tau) = &\int_{t_0^*}^{t_1^*} \big(H_u(t, y^*(t), u^*(t), \lambda(t))\eta_u(t) + H_y(t, y^*(t), u^*(t), \lambda(t))\eta_y(t) - \lambda(t)\dot{\eta}_y(t)\big)\, dt+ \\ &+ H(t, y^*(t), u^*(t), \lambda(t))\big|_{t=t_1^*} \tau_1 - H(t, y^*(t), u^*(t), \lambda(t))\big|_{t=t_0^*} \tau_0\end{aligned}$$

Setting $H^*(t) = H(t, y^*(t), u^*(t), \lambda(t))$, integrating by parts,

$$\begin{aligned}\delta\mathcal{J}[y^*, u^*; t^*]\ (\eta, \tau) = &\int_{t_0^*}^{t_1^*} \Big(H_u^*(t)\eta_u(t) + \big(H_y^*(t) + \dot{\lambda}(t)\big)\,\eta_y(t)\Big)\, dt + \lambda(t_0^*)\eta_y(t_0^*) - \lambda(t_1^*)\eta_y(t_1^*)+ \\ &+ H^*(t_1^*)\tau_1 - H^*(t_0^*)\tau_0.\end{aligned}$$

Using the same approximation as in the proof of Proposition 6 yields the analogue to equation (6.18)

$$\begin{aligned}\delta\mathcal{J}[y^*,u^*;t^*]\ (\eta,\tau) = \int_{t_0^*}^{t_1^*} \left(H_u^*(t)\eta_u(t) + \left(H_y^*(t)+\dot\lambda(t)\right)\eta_y(t)\right)dt + \lambda(t_0^*)\eta_0 - \lambda(t_1^*)\eta_1 + \\ + \left(H^*(t_1^*) - \dot y^*(t_1^*)\lambda(t_1^*)\right)\tau_1 - \left(H^*(t_0^*) - \dot y^*(t_0^*)\lambda(t_0^*)\right)\tau_0\end{aligned} \tag{6.39}$$

The adjoint necessary conditions for the optimum, because $\eta_1 = \eta_0 = 0$, are presented, for the different versions of the problem, in equations (6.36) to (6.38). □

6.3.3 Free domain and boundary state variable optimal control problems

This is a general case that encompasses combinations of all the previous cases: we assume both the domains of the independent variables and the boundary values of the state variables are free. That is t_0 and/or t_1 are unknown and $y(t_0)$ and/or $y(t_1)$ are also unknown and should be optimized. The optimal control problem is to find the optimal cut-off values for the indepdendent variable, t_0^* and/or t_1^* and an optimal control $(u^*(t))_{t\in[t_0^*,t_1^*]}$ that maximizes the functional (6.28) subject to ODE constraint (6.3.4) and having free boundary values for the state variable.

The necessary conditions include the optimality condition (6.29), the multiplier equation (6.30) and the ODE constraint of the problem (6.31), all for $t \in \text{int}(T^*)$. To get the adjoint condition associated to the terminal values of the state variable, when they need to be optimized, are obtained by setting in equation (6.39), $\eta_0 \neq 0$ and $\eta_1 \neq 0$. Therefore, the adjoint condition associated to $y^*(t_j^*)$ and $\lambda(t_j^*) = 0$, implying that the adjoint condition associated to the optimal boundary value of the independent variable, t_j^* is $H^*(t_j^*) = 0$, for $j = 0, 1$.

The adjoint conditions presented in Table 6.2 cover the same cases as the ones in Table 6.1 for the calculus of variations problem.

Table 6.2: Adjoint conditions for bounded domain OC problems

data		optimum	
t_j	$y(t_j)$	t_j^*	$y^*(t_j^*)$
fixed	fixed	t_j	y_j
fixed	free	t_j	$\lambda(t_j) = 0$
free	fixed	$H(t_j^*, y_j, u^*(t_j^*)) - \dot y^*(t_j^*)\lambda(t_j^*) = 0$	y_j
free	free	$H(t_j^*, y^*(t_j^*), u^*(t_j^*)) = 0$	$\lambda(t_j^*) = 0$

The index refers to the lower boundary when $j = 0$ and to the upper boundary when $j = 1$

6.3.4 Constrained terminal state problem

A common problem in macroeconomics is the following: the set of independent variables is known such as $t_0 = 0$ and $t_1 = \bar{t}$, the initial value of the state value is fixed, $y(0) = y_0$, the structure of the economy given by the ODE (6.28), and value functional , but now we assume that the terminal value for the state variable is constrained by $R(\bar{t}, y(\bar{t})) \geq 0$ is free

Proposition 13. ***1st order necessary conditions for the constrained terminal value problem*** *The previous necessary conditions for optimality* (6.29) *and* (6.31) *still hold, with the exception of the transversality condition in* (6.33) *which is now*

$$\lambda(\bar{t})R(\bar{t}, y^*(\bar{t})) = 0.$$

Proof. In this case the value at the optimum is

$$\mathcal{J}[y^*, u^*] = \int_0^{\bar{t}} \left(H(t, u^*(t), y^*(t)) - \dot{y}^*(t)\lambda(t)\right) \, dt + \psi R(\bar{t}, y(\bar{t}))$$

where ψ is a Lagrange multiplier. The functional derivative for an arbitrary perturbation $(\delta y, \delta u) = \varepsilon$ around (y^*, u^*) is now

$$\delta\mathcal{J}[y^*, u^*](\,) = \int_0^{\bar{t}} \left[H_u(t, y^*(t), u^*(t), \lambda(t))\eta_u(t) + (H_y(t, y^*(t), u^*(t), \lambda(t)) + \dot{\lambda}(t))\eta_y(t)\right] dt +$$
$$+ \lambda(0)\eta_y(0) + \left(\psi R_y(\bar{t}, y^*(\bar{t})) - \lambda(\bar{t})\right)\eta_y(\bar{t}),$$

where admissible perturbations satisfy $\eta_y(0) = 0$ and $\eta_y(\bar{t}) \neq 0$. Given the inequality constraint, the KKT conditions

$$R(\bar{t}, y^*(\bar{t})) \geq 0, \;\; \psi \geq 0, \;\; \psi R(\bar{t}, y^*(\bar{t})) = 0,$$

are also necessary for an optimum. Setting $H_u^*(t) = \dot{\lambda}\,(t) - H_y^*(t) = \eta_y(0) = 0$, and because $\eta_y(\bar{t}) \neq 0$, the remaining necessary condition for an optimum is

$$\psi R_y(\bar{t}, y^*(\bar{t})) - \lambda(\bar{t}) = 0,$$

which, multiplying both terms by $R(\bar{t}, y^*(\bar{t}))$ and using the KKT condition yields the remaining adjoint condition $\lambda(\bar{t})R(\bar{t}, y^*(\bar{t})) = 0$. □

6.3.5 Infinite horizon problems

The benchmark problem in macroeconomics and growth theory is the (autonomous) **discounted infinite horizon problem** has the constraint

$$\dot{y} = g(y(t), u(t)), \;\text{for } t \in [0, \infty) \tag{6.40}$$

instead of (6.28), and $y(0) = y_0$ and alternative boundary conditions

$$\lim_{t \to \infty} y(t) \text{ is free or } \lim_{t \to \infty} y(t) \geq 0$$

The value functional is

$$\mathcal{J}[y, u] \equiv \int_0^\infty e^{-\rho t} f(y(t), u(t)) dt.$$

Now, we define the **current-value Hamiltonian** function

$$\begin{aligned} h(y(t), u(t), q(t)) &= f(y(t), u(t)) + q(t) g(y(t), u(t)) = \\ &= e^{-\rho t} H(t, y(t), u(t), \lambda(t)). \end{aligned}$$

where $q(t) = e^{\rho t} \lambda(t)$ is the current-value co-state variable. Consistently with the previous definitions we call **discounted Hamiltonian** and **discounted co-state variable** to $H(t, y, u, \lambda)$ and λ.

Observe that the current-value Hamiltonian is time-independent.

Proposition 14 (**First order necessary conditions: Pontriyagin maximum principle**). *Let (y^*, u^*) be the optimal state and control pair. Then there is a PC^1 continuous co-state variable q such that the following conditions hold:*

- *the optimality condition*

$$h_u(y^*(t), u^*(t), q(t)) = 0, \; t \in [0, \infty)$$

- *the multipliers equation for the current co-state variable (also called canonical equation)*

$$\dot{q} = \rho q - h_y(y^*(t), u^*(t), q(t)), \; t \in [0, \infty)$$

- *the transversality condition for the free or the constrained terminal state*

$$\lim_{t \to \infty} e^{-\rho t} q(t) = 0, \; \textit{or} \lim_{t \to \infty} e^{-\rho t} q(t) y^*(t) = 0$$

- *and the admissibility conditions*

$$\begin{cases} \dot{y}^* = g(y^*(t), u^*(t)), & t \in [0, \infty) \\ y^*(0) = y_0 & t = 0 \end{cases}$$

6.3.6 Applications

We consider the same problems as in the calculus of variations section.

Example 1: Resource depletion problem

The (non-renewable) resource depletion problem can now be solved by using the Pontriyagin's principle. Recall that the problem is

$$\max_{C} \int_0^{\infty} e^{-\rho t} \ln (C(t))dt, \ \rho > 0$$

subject to

$$\begin{cases} \dot{W}(t) = -C(t), \ t \in [0, \infty) \\ W(0) = W_0, \ \text{given} \\ \lim_{t\to\infty} W(t) \geq 0. \end{cases}$$

What is the best path for consumption-depletion ?

For applying the Pontryiagin maximum principle we write the current-value Hamiltonian

$$h = ln(C) - qC.$$

The first order conditions are

$$\begin{aligned} C(t) &= 1/q(t) \\ \dot{q} &= \rho q(t) \\ \lim_{t\to\infty} e^{-\rho t} q(t) W(t) &= 0 \\ \dot{W} &= -C(t) \\ W(0) &= W_0 : \end{aligned}$$

and can be written as a planar differential equation in (W, C), together with the initial and the transversality condition is

$$\begin{aligned} \dot{W} &= -C(t) \\ \dot{C} &= -\rho C(t) \\ W(0) &= W_0 \\ \lim_{t\to\infty} e^{-\rho t} \frac{W(t)}{C(t)} &= 0 \end{aligned}$$

If we want to find the solution we must solve the system, together with the conditions on time.

There are several ways to solve it. Here is a simple one. First, define $z(t) \equiv W(t)/C(t)$. Time-differenting and substituting, we get the scalar terminal-value problem

$$\begin{cases} \dot{z} = -1 + \rho z \\ \lim_{t\to\infty} e^{-\rho t} z(t) = 0 \end{cases}$$

which has a constant solution $z(t) = \frac{1}{\rho}$ for every $t \in [0, \infty)$. Second, substitute $C(t) = W(t)/z(t) = \rho W(t)$. therefore,

$$\begin{cases} \dot{W} = -C(t) = -\rho W(t) \\ W(0) = W_0 \end{cases}$$

Then $W^*(t) = W_0 e^{-\rho t}$ for $t \in [0, \infty)$ and $C^*(t) = \rho W^*(t)$.

Characterization of the solution: there is asymptotic extinction

$$\lim_{t\to\infty} W^*(t) = 0,$$

at a speed given by the half-life of the process

$$\tau \equiv \left\{ t : \ W^*(t) = \frac{W(0) - W^*(\infty)}{2} \right\} = -\frac{\ln(1/2)}{\rho}$$

if $\rho = 0.02$ then $\tau \approx 34.6574$ years.

Example 2: the consumption-savings problem

Problem: find the (A, C) pair that maximizes the functional

$$\max_C \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} dt, \ \rho > 0$$

subject to

$$\begin{cases} \dot{A}(t) = Y - C(t) + rA, \ t \in [0, \infty) \\ A(0) = A_0, \text{ given} \\ \lim_{t\to\infty} A(t)^{-rt} \geq 0. \end{cases}$$

The current value Hamiltonian is

$$h(A, C, Q) = \frac{C^{1-\theta}}{1-\theta} + Q(Y - C + rA)$$

and the first order conditions according to the Pontiyagin's principle are

$$\begin{cases} C(t)^{-\theta} = Q(t) \\ \dot{Q} = Q(\rho - r) \\ \dot{A} = Y - C + rA \\ A(0) = A_0 \\ \lim_{t\to\infty} Q(t)A(t)e^{-\rho t} = 0 \end{cases}$$

As

$$\frac{\dot{Q}}{Q} = -\theta\frac{\dot{C}}{C}$$

we can obtain the solution by solving the mixed initial-terminal value problem for ODE's

$$\begin{cases} \dot{A} = Y - C + rA \\ \dot{C} = \gamma\, C \\ A(0) = A_0 \\ \lim_{t\to\infty} C(t)^{-\theta}A(t)e^{-\rho t} = 0 \end{cases}$$

where again $\gamma \equiv \dfrac{r-\rho}{\theta}$. If we assume that $\gamma < r$ we get exactly the same solution as the equivalent CV problem.

6.3.7 The Modified Hamiltonian Dynamic System

In regular cases we can have a geometric interpretation for the solution of an optimal control problem. Recall that the necessary conditions for the infinite-horizon discounted optimal control problem, feature a differential-algebraic system:

$$\begin{aligned} \dot{y} &= g(y,u) \\ \dot{q} &= \rho q - h_y(u,y,q). \\ 0 &= h_u(u,y,q) \end{aligned} \tag{6.41}$$

where $h(u,y,q) = f(u,y) + qg(u,y)$. Therefore, $h_q(u,y,q) = g(u,y)$.

If functions $f(.)$ and $g(.)$ are sufficiently smooth we may qualitative characterize the optimal path for (y,q) (or for (u,y)).

If $\partial^2 h/\partial u^2 \neq 0$, the implicit function theorem allows for obtaining from the optimality condition for u, $h_u(u,y,q) = 0$, an implicit representation of the control as a function of the state and co-state variables $u = u(y,q)$. If we substitute this control representation in the differential equations of

(6.41) we obtain the **modified Hamiltonian dynamic system** (MHDS) as a non-linear planar ODE,

$$\begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \mathbf{M}\ (y,q) \equiv \begin{pmatrix} g(u(y,q),y) \\ \rho q - h_y(u(y,q),y,q). \end{pmatrix} \tag{6.42}$$

Assume there is one steady state for the MHDS, $(\bar{y},\bar{q}) = \{(y,q) : \dot{y} = \dot{q} = 0\}$. In the neighbourhood of $(\bar{y},\bar{q})$ we can approximate the non-linear MHDS (6.42) by the linear system

$$\begin{pmatrix} \dot{y}(t) \\ \dot{q}(t) \end{pmatrix} = D_{(y,q)}\mathbf{M}(\bar{y},\bar{q}) \begin{pmatrix} y(t)-\bar{y} \\ q(t)-\bar{q} \end{pmatrix}$$

where the Jacobian, evaluated at the steady state $(\bar{y},\bar{q})$ is the matrix of constants

$$D_{(q,y)}\mathbf{M}(\bar{y},\bar{q}) = \begin{pmatrix} \frac{\partial \dot{y}(\bar{y},\bar{q})}{\partial y} & \frac{\partial \dot{y}(\bar{y},\bar{q})}{\partial q} \\ \frac{\partial \dot{q}(\bar{y},\bar{q})}{\partial y} & \frac{\partial \dot{q}(\bar{y},\bar{q})}{\partial q} \end{pmatrix}.$$

If functions $f(.)$ and $g(.)$ have no singularities we can obtain a generic characterization of the dynamics of the MHDS, and, therefore, of the solution to the optimal control problem.

Proposition 15. *Let there be a steady state for the MHDS system. It can never be locally a stable node or focus, and if there is transitional dynamics it can only be a saddle-point.*

Proof. The differential of the current value Hamiltonian, $h = f(u,y) + qg(u,y)$, is

$$dh = (f_u + qg_u)du + (f_y + qg_y)dy + gdq.$$

At the optimum $h_u(u,y,q) = f_u(u,y) + qg_u(u,y) = 0$. Taking the differential to this static optimality condition, we have

$$h_{uu}du + h_{uy}dy + g_u dq = 0$$

and if $h_{uu} \neq 0$, by the implicit function theorem, function $u = u(y,q)$ has derivatives

$$u_y = -\frac{h_{uy}}{h_{uu}}\ ,\ u_q = -\frac{g_u}{h_{uu}}.$$

Now, we can take compute the Jacobian for matrix $\mathbf{M}$, evaluated at any optimum pair (y,q). The differential of the first row of $\mathbf{M}$ is

$$dg(y,u(y,q)) = \left(g_y - g_u\frac{h_{uy}}{h_{uu}}\right)dy - g_u\frac{g_u}{h_{uu}}dq$$

and the differential of the second row is

$$\rho dq - dh_y(y,u(y,q)) = -\left(h_{yy} - h_{yu}\frac{h_{uy}}{h_{uu}}\right)dy + \left(\rho - g_y + h_{yu}\frac{g_u}{h_{uu}} +\right)dq.$$

Evaluating the derivatives at the steady state $(\bar{y}, \bar{q})$, with $\bar{u} = u(\bar{y}, \bar{q})$, we find that the Jacobian matrix has the following structure

$$D_{(y,q)}\mathbf{M}\ (\bar{y}, \bar{q}) = \begin{pmatrix} \bar{g}_y - \dfrac{\bar{h}_u \bar{h}_{uy}}{\bar{h}_{uu}} & -\dfrac{(\bar{g}_u)^2}{\bar{h}_{uu}} \\ -\bar{h}_{yy} + \dfrac{(\bar{h}_{uy})^2}{\bar{h}_{uu}} & \rho - \bar{g}_y + \dfrac{\bar{g}_u \bar{h}_{uy}}{\bar{h}_{uu}} \end{pmatrix}$$

where $\bar{g}_y = g(u(\bar{y}, \bar{q}), \bar{y})$, etc[6]. Observe that the Jacobian matrix has a particular structure

$$D_{(y,q)}\mathbf{M}(\bar{y}, \bar{q}) = \begin{pmatrix} a & b \\ c & \rho - a \end{pmatrix}.$$

impiying that the trace is equal to the rate of time preference,

$$Trace(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})) = \rho > 0$$

and is always positive and

$$\det\left(D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})\right) = a(\rho - a) - bc.$$

This implies that, if there is a steady state, it can never a stable node or focus, and if there is transitional dynamics it can only be of the saddle-point type. □

Then we can conclude the following:

1. in generic cases the equilibrium point $(\bar{y}, \bar{q})$ is a saddle point. The stable manifold associated with $(\bar{y}, \bar{q})$
$$W^s = \{\ (y, q) : \lim_{t \to} (y(t), q(t)) = (\bar{y}, \bar{q})\}$$
passing through point $y(0) = y_0$ **is the solution set of the OC problem**;

2. this means that the solution to the OC problem is unique;

3. the optimal trajectory is asymptotically tangent to the stable eigenspace E^s associated to Jacobian $D_{(q,y)}\mathbf{M}(\bar{y}, \bar{q})$

Example 3: the Ramsey (1928) model The Ramsey problem:

$$\max_{C} \int_0^{\infty} e^{-\rho t} u(C(t))dt,\ \rho > 0,$$

[6]because if $h(.)$ is continuous then $h_{uy}(.) = h_{yu}(.)$.

subject to

$$\dot{K}(t) = F(K(t)) - C(t),\ t \in [0, \infty)$$

$K(0) = K_0$ given and $\lim_{t\to\infty} e^{-\rho t} K(t) \geq 0$. We also assume that $(K, C) : \mathbb{R}_+ \to \mathbb{R}_+^2$.

The utility and the production functions, $u(C)$ and $F(K)$, are usually assumed to have the following properties: Increasing, concave and Inada :

$$U^{'}(.) > 0,\ U^{''}(.) < 0, F^{'}(.) > 0,\ F^{''}(.) < 0$$

$$U^{'}(0) = \infty,\ U^{'}(\infty) = 0,\ F^{'}(0) = \infty,\ F^{'}(\infty) = 0$$

Although we do not have explicit utility and production functions we can still characterize the optimal consumption-accumulation process (we are using the Grobman-Hartmann theorem).

The current-value Hamiltonian is

$$h(C, K, Q) = U(C) + Q(F(K) - C)$$

Pontriyagin's f.o.c

$$\begin{aligned}
U^{'}(C(t)) &= Q(t) \\
\dot{Q} &= Q(t)\left(\rho - F^{'}(K(t))\right) \\
\lim_{t\to\infty} e^{-\rho t} Q(t) K(t) &= 0 \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0
\end{aligned}$$

The MHDS and the initial and transversality conditions become

$$\begin{aligned}
\dot{C} &= \frac{C(t)}{\theta(C(t))}\left(F^{'}(K(t)) - \rho\right) \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0 > 0 \\
0 &= \lim_{t\to\infty} e^{-\rho t} U^{'}(C(t)) K(t)
\end{aligned}$$

where $\theta(C) = -\frac{U^{''}(C)C}{U^{'}(C)} > 0$ is the inverse of the elasticity of intertemporal substitution.

The MHDS has no explicit solution (it is not even explicitly defined) : we can only use **qualitative methods**. They consist in:

- determining the steady state(s): $(\bar{C}, \bar{K})$

- characterizing the linearised dynamics (it is useful to build a phase diagram).

The steady state (if $K > 0$) is

$$\begin{aligned} F^{'}(\bar{K}) &= \rho \Rightarrow \bar{K} = (F^{'})^{-1}(\rho) \\ \bar{C} &= F(\bar{K}) \end{aligned}$$

The linearized MHDS is

$$\begin{pmatrix} \dot{C} \\ \dot{K} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\bar{C}}{\theta(\bar{C})} F^{''}(\bar{K}) \\ -1 & \rho \end{pmatrix} \begin{pmatrix} C(t) - \bar{C} \\ K(t) - \bar{K} \end{pmatrix}$$

where we denote $D\mathbf{M}$ the Jacobian matrix. The jacobian J has trace and determinant:

$$\mathrm{tr}(D\mathbf{M}) = \rho, \ \det(D\mathbf{M}) = \frac{\bar{C}}{\theta(\bar{C})} F^{''}(\bar{K}) < 0$$

Again $(\bar{C}, \bar{K})$ is a saddle point. The eigenvalues of $D\mathbf{M}$ are

$$\lambda_s = \frac{\rho}{2} - \sqrt{\Delta} \ < 0, \ \lambda_u = \frac{\rho}{2} + \sqrt{\delta} \ > \rho > 0$$

where the discriminant is

$$\Delta = \ \left(\frac{\rho}{2}\right)^2 - \frac{\bar{C}}{\theta(\bar{C})} F^{''}(\bar{K}) > \left(\frac{\rho}{2}\right)^2.$$

and the eigenvector matrix of $D\mathbf{M}$ is

$$\mathbf{P} \ = (\mathbf{P}^s \mathbf{P}^u) = \begin{pmatrix} \lambda_u & \lambda_s \\ 1 & 1 \end{pmatrix}$$

Then the approximate solution for the Ramsey problem, in the neighbourhood of the steady state, is

$$\begin{pmatrix} C^*(t) \\ K^*(t) \end{pmatrix} = \begin{pmatrix} \bar{C} \\ \bar{K} \end{pmatrix} + K_0 \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix} e^{\lambda_s t}, \ t \in [0, \infty) \tag{6.43}$$

Then the local stable manifold has slope higher than the isocline $\dot{K}(C, K) = 0$

$$\left.\frac{dC}{dK}\right|_{W^s} = \lambda_u > \left.\frac{dC}{dK}\right|_{\dot{K}} = F^{'}(\bar{K}) = \rho$$

Geometrically (see figure 6.1) the **approximate** solution (6.43) belongs to the stable sub space E^s

$$E^s = \{ \ (K, C) : (C - \bar{C}) = \lambda_u (K - \bar{K})\}$$

while the **exact** solution belongs to the stable manifold W^s (which cannot be determined explicitly). Observe that while the slope of the isocline in the neighborhood of the steady is flatter then the slope of the stable manifold

$$\frac{dC}{dK}\Big|_{\dot{k}=0} = F^{'}(\bar{K}) = \rho < \frac{dC}{dK}\Big|_{W^s} = \lambda_u$$

meaning that the solution approaches the steady state by accumulating (reducing) capital is the initial capital level is smaller (bigger) than the steady state level.

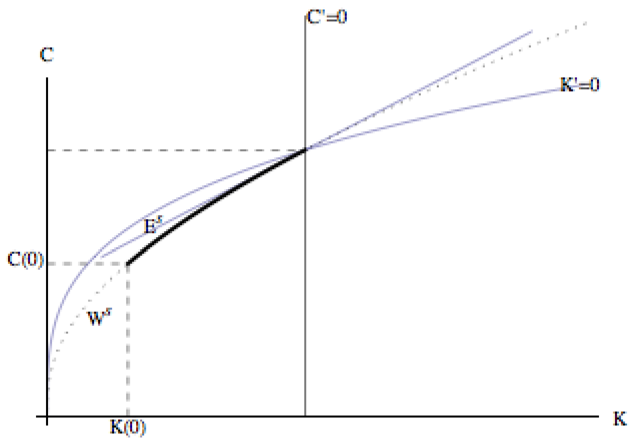

Figure 6.1: The phase diagram for the Ramsey model: it depicts the isoclines $\dot{C} = 0$ and $\dot{K} = 0$, the stable manifold W^s and the stable eigenspace, E^s, which is tangent asymptotically to the stable manifold. The exact solution follows along the stable manifold, but we have determined just the approximation along the stable eigenspace.

6.3.8 Unbounded solutions

We saw that the consumer-saver problem may have an unbounded solution.

If we write the MHDS in the (A, Q) space, we have

$$\begin{cases} \dot{A} = Y - Q^{-\frac{1}{\theta}} + rA \\ \dot{Q} = Q(\rho - r) \end{cases}$$

the solution of the optimal control problem are the solutions of that MHDS together with the inicial and transversality conditions

$$A(0) = a_0, \ \lim_{t\to\infty} Q(t)A(t)e^{-\rho t} = 0.$$

There are two interesting cases. First, if $r = \rho$ then there is an infinity of stationary solutions satisfying $Q^{-\frac{1}{\theta}} = Y + rA$. Second, if $r \neq \rho$ it has no steady state in $\mathbb{R}$. To see this note that, $\dot{Q} = 0$ if and only if $Q = 0$ but then $\dot{A} = 0$ can only be reached asymptotically when $A \to \infty$.

We can have a clearer characterization if we recast the problem in the (A, C) spac. Recall that in this case we have the MHDS

$$\begin{cases} \dot{A} = Y - C + rA \\ \dot{C} = \gamma C, \end{cases}$$

where

$$\gamma \equiv \frac{r - \rho}{\theta},$$

which, for the moment, we assume has an ambiguous sign.

The solution of the optimal control problem are the solutions of that MHDS together with the inicial and transversality conditions

$$\begin{cases} A(0) = a_0, \\ \lim_{t\to\infty} C(t)^{-\frac{1}{\theta}} A(t) e^{-\rho t} = 0. \end{cases}$$

The MHDS is linear planar ODE with coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} r & -1 \\ 0 & \gamma \end{pmatrix}$$

that has eigenvalues

$$\lambda_- = \gamma, \ \lambda_+ = r > 0.$$

and has eigenvector matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ r - \gamma & 0 \end{pmatrix}$$

The solution to the MHDS is, for $\gamma \neq 0$

$$\begin{pmatrix} A(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} -\dfrac{Y}{r} \\ 0 \end{pmatrix} + h_- \begin{pmatrix} 1 \\ r - \gamma \end{pmatrix} e^{\gamma t} + h_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{rt}.$$

For later use, observe that the trajectories starting from $A(0) = a_0$ and travelling along the eigenspace associated to eigenvalue λ_- are

$$\begin{pmatrix} A(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} -\dfrac{Y}{r} \\ 0 \end{pmatrix} + (A_0 + \frac{Y}{r}) \begin{pmatrix} 1 \\ r - \gamma \end{pmatrix} e^{\gamma t}.$$

that is

$$\mathbb{E}^- = \left\{ (A, C) \in \mathbb{R} \times \mathbb{R}_+ : \ C = (r - \gamma)\left(A + \frac{Y}{r}\right) \right\}.$$

We saw that the only requirement for the transversality condition to be me, and therefore for the optimal control problem to have a solution was $r > \gamma$. Even if we keep this assumption, three cases are possible

1. if $r < \rho$ then $\lambda_{-} = \gamma < 0$ and and the steady state $(\bar{A}, \bar{C}) = (-Y/r, 0)$ is a saddle-point. The solution of the optimal control problem, which lies along the stable manifold converges to $C^*(\infty) = 0$ and $A^*(\infty) = -Y/r < 0$. The steady state is a saddle point. The intuition is: the consumer is more impatient than the market and therefore will be asymptotically a debtor to a point in which it can collateralize the debt by its human capital $A(\infty) + H(0) = 0$;

2. if $\gamma < r = \rho$ then $\lambda_{-} = 0$ and the solution is constant $C^*(t) = Y + rA_0$ and $A^*(t) = A_0$ for all $t \in [0, \infty)$. This was the case corresponding to the existence of an infinite number of equilibria when the characterization is conducted in the (A, Q) space;

3. if $r > \rho$ then $\lambda_{-} = \gamma > 0$ and the steady state $(\bar{A}, \bar{C}) = (-Y/r, 0)$ is an unstable node. In this case, there are admissible solutions only if $A_0 \geq -Y/r$, otherwise consumption would be negative. However, if $A_0 > -Y/r$ there is an admissible solution to the optimal control problem but it is unbounded.

The question the last case poses is the following. First, if we look at the MHDS as a dynamical system we would say that it is unstable but most of the qualitative theory of ODE characterizes the dynamics close to a steady state. But we already found that this case is indeed a solution to the optimal control problem. How can we reconcile the two points ?

A way to deal with the last type of behavior is to consider convergence of the solution to a kind of invariant structure and to consider convergence to that structure. An approach which is used in the literature is to consider convergence to an exponential solution, some×called **balanced growth path**, such that the initial and the transversality conditions hold.

The method proceeds along five steps.

First step define the variables using multiplicative deviations along an exponential trends with proportional growth rates. In our case we try the case in which the rates of growth are equal

$$A(t) = a(t)e^{gt}, \ C(t) = c(t)e^{gt}$$

Second, obtain the dynamic system for the detrended variables (a, c), observing that

$$\frac{\dot{a}}{a} = \frac{\dot{A}}{A} - g, \ \frac{\dot{c}}{c} = \frac{\dot{C}}{C} - g$$

We get

$$\begin{cases} \dot{a} = Ye^{-gt} - c + (r - g)a \\ \dot{c} = (\gamma - g)c \end{cases}$$

Third, obtain g from a stationary solution to the system in detrended variables. In our case setting $g = \gamma$ transforms the previous system in

$$\begin{cases} \dot{a} = Ye^{-\gamma t} - c + (r-\gamma)a \\ \dot{c} = 0 \end{cases}$$

which implies that $c(t) = \bar{c}$ which is an unknown constant. Setting $a(0) = A_0$ and $c(t) = \bar{c}$ we can solve the equation for the detrended asset holdings

$$a(t) = \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1 - e^{-rt}\right) \right) e^{(r-\gamma)t} + \frac{\bar{c}}{r-\gamma}.$$

Fourth, we can determine $\bar{c}$ from the transversality condition

$$\begin{aligned} \lim_{t\to\infty} (C(t))^{-\theta} A(t) e^{-\rho t} &= \lim_{t\to\infty} \bar{c}^{-\theta} e^{(\gamma(\theta-1)-\rho)t} a(t) = \\ &= \lim_{t\to\infty} \bar{c}^{-\theta} e^{(\gamma(\theta-1)-\rho+r-\gamma)t} \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1-e^{-rt}\right) + \frac{\bar{c}}{r-\gamma} e^{-(r-\gamma)t} \right) = \\ &= \lim_{t\to\infty} \bar{c}^{-\theta} \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r}\left(1-e^{-rt}\right) + \frac{\bar{c}}{r-\gamma} e^{-(r-\gamma)t} \right) = \\ &= \bar{c}^{-\theta} \left(A_0 - \frac{\bar{c}}{r-\gamma} + \frac{Y}{r} \right) = 0 \end{aligned}$$

if and only if $\bar{c} = c^* = (r-\gamma)\left(A_0 + \dfrac{Y}{r}\right)$.

At last we get the solution

$$C^*(t) = c^* e^{\gamma t},\ A^*(t) = a^*(t) e^{\gamma t}$$

where

$$c^* = (r-\gamma)\left(a_0 + \frac{Y}{r}\right),\ a^*(t) = A_0 + \frac{Y}{r}\left(1 - e^{-\gamma t}\right).$$

We see that

$$C^*(t) = (r-\gamma)\left(A^*(t) + \frac{Y}{r}\right), \text{for } t \in [0,\infty)$$

which means that the solution to the optimal control problem evolves along the eigenspace associated to the eigenvalue λ_- (see figure 6.2).

If $r < \rho$, and therefore $\gamma < 0$, the solution evolves along the eigenspace associated to λ_- butit converges to the steady state in which $A(\infty) = -Y/r$. In this case $\mathbb{E}^- = \mathbb{E}^s$ that is this is the stable eigenspace (which as the model is linear is the stable manifold).

From this we have a geometrical interpretation of the solution to the optimal control problem: if $r \neq \rho$ the solution will belong to the eigenspace $\mathbb{E}^-$, and it converges to the steady state if $r < \rho$ and diverges from it if $r > \rho$.

This illustrates, and reinforces, the fact that interpreting phase diagrams for MHDS of optimal control problems should be done with care: if the optimal control problem has a single solution, the geometrical analog of it is also unique.

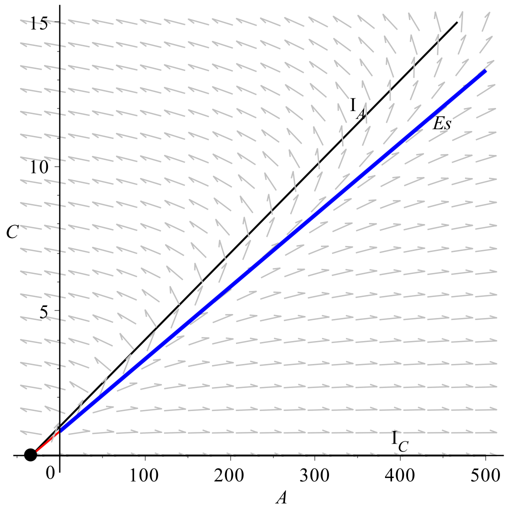

Figure 6.2: Phase diagram for the benchmark consumer problem for the case $r > \gamma$.

6.4 Optimal control: dynamic programming principle

Consider again the free terminal state optimal control problem: among functions $y \in \mathrm{Y}$ and $u \in U$ satisfying

$$\dot{y} = G(y(t), u(t), t), \text{ for } t \in [0, \bar{t}] \tag{6.44}$$

and $y(0) = y_0$ find the pair (y^*, u^*) that maximize the functional

$$\mathcal{J}[y, u] \equiv \int_0^{\bar{t}} F(t, y(t), u(t))dt \tag{6.45}$$

where $\bar{t}$ is given and $y^*(\bar{t})$ is free.

6.4.1 First order necessary conditions: dynamic programming principle

Proposition 16 (**Necessary conditions according to the principle of dynamic programming).** *Consider the optimal state and control functions $y^* \in \mathrm{Y}$ and $u^* \in U$ for the optimal control problem with free terminal state. Then the* ***Hamilton-Jacobi-Bellman*** *equation must hold*

$$-V_t(t, y) = \max_{u \in \mathcal{U}} \left\{ F(t, y, u) + V_y(t, y)G(t, y, u) \right\} \tag{6.46}$$

for all $t \in [0, \bar{t})$ and all $y \in Y \subseteq \mathbb{R}$.

Proof. (heuristic) We define the functional over the state and control functions continuing from an arbitrary time $t \geq 0$: $(y, u) : [t, \bar{t}] \to Y \times U \subseteq \mathbb{R}^2$

$$\mathcal{J}[y, u](t) = \int_t^{\bar{t}} F(s, y(s), u(s))ds.$$

and call **value function** to

$$V(t, y(t)) \equiv \max_{(u(s)|s \in [t, \bar{t}])} \mathcal{J}[y, u; t]$$

for $y(t) \in Y$.

The **Principle of dynamic programming optimality** states the following: for every $(t, y) \in [0, \bar{t}] \times Y$ and every $\Delta t \in (0, \bar{t} - t]$ the value function satisfies

$$V(t, y(t)) = \max_{(u(s)|s \in [t, t+\Delta t])} \left\{ \int_t^{t+\Delta t} F(s, y(s), u(s))ds + V(t + \Delta t, y(t + \Delta t)) \right\}$$

where

$$y(t + \Delta t) = y(t) + G(t, y(t), u(t))\Delta t + o(\Delta t).$$

Performing a first-order Taylor expansion we get

$$V(t + \Delta t, y(t + \Delta t)) = V(t, y(t)) + V_t(t, y(t))\Delta t + V_y(t, y(t))G(t, y(t), u(t))\Delta t + o(\Delta t)$$

(this requires that V is C^1). If the interval Δt is sufficiently small we can use the mean-value theorem

$$\int_t^{t+\Delta t} F(s, y(s), u(s))ds = F(t, y(t), u(t))\Delta t$$

Then

$$V(t, y(t)) = \max_{(u(s)|s \in [t, t+\Delta t])} \left\{ F(t, y(t), u(t))\Delta t + V(t, y(t)) + V_t(t, y(t))\Delta t + V_y(t, y)g(t, y(t), u(t))\Delta t + o(\Delta t) \right\}.$$

Cancelling out $V(t, y(t))$, dividing by Δt, taking $\Delta t \to 0$ and observing that the pair $(t, y(t))$ is an arbitrary element of $T \times Y$ we get the HJB equation (6.46). □

For solving the optimal control problem, while the Pontriagyin's principle provides necessary conditions in a form of a initial-terminal value problem for a planar ODE, the principle of the dynamic programming provides a formula for evaluating the value of our resource in a recursive way and independent of time.

The HJB equation (6.46) is a PDE (partial differential equation).

6.4.2 Infinite horizon discounted optimal control problem

The infinite horizon discounted optimal control problem is, again, to find functions $u^* \in \mathrm{U}$ and $u^* \in \mathrm{Y}$ satisfying

$$\begin{cases} \dot{y} = g(y(t), u(t)), & t \in [0, \infty) \\ y(0) = y_0, \\ \lim_{t\to\infty} h(t)y(t) \geq 0 \end{cases}$$

that maximize the objective functional

$$\mathcal{J}[y, u] \equiv \int_0^{\infty} e^{-\rho t} f(y(t), u(t))dt$$

Proposition 17 (Necessary conditions according to the principle of dynamic programming for the infinite horizon problem). *Let (y^*, u^*) be the solution to the discounted infinite horizon problem. Then it satisfies the HJB equation*

$$\rho v(y) = \max_u \left\{ f(y, u) + v^{'}(y)g(y, u) \right\} \tag{6.47}$$

Proof. For $y(t) = y$ the value function is

$$V(t, y) \equiv \int_t^{\infty} e^{-\rho s} f(y^*(s), u^*(s))ds$$

Multiplying by a inverse of the discount factor, the value function becomes independent of the initial time,

$$e^{\rho t} V(t, y) = \int_t^{\infty} e^{-\rho(s-t)} f(y^*(s), u^*(s))ds = v(y).$$

Then we can write

$$V(t, y) = e^{-\rho t} v(y)$$

and upon substituting in equation (6.46) we get equation (6.47). □

In the case of the discounted infinite horizon the HJB equation is not a PDE but an ODE in implicit form. In order to see this we need to determine another important element of the DP approach: the policy function.

If we define the function $h(u, y) \equiv f(y, u) + v^{'}(y)g(y, u)$ the HJB equation (6.47) can be written as

$$\rho v(y) = \max_u h(u, v).$$

We can obtain the optimal control from the first-order condition

$$\frac{\partial h(u, y)}{\partial u} = 0.$$

If function $h(u, v)$ is monotonic as regards u, by appealing to the implicit function theorem, we can obtain the optimal control as a function of the state variable, $u^* = \pi(y)$. Function $\pi(.)$ in the DP literature is called **policy function**. It gives the optimal control as a function of the state variable. This is why it is called a **feedback control** problem.

The reason for this is the following. If we substitute the policy function in equation (6.47) we finally obtain the HJB equation as an ODE in implicit form

$$\rho v(y) = f(\pi(y), y) + v^{'}(y) g(\pi(y), y)$$

where the state variable y is the independent variable and the value function, $v(y)$, is the unknown function.

If we are able to determine a solution to this equation, we can usually specify the utility function, which means that we are able to obtain the optimal control as a function of the state variable. We can obtain the solution to the optimal control problem by substituting in the ODE constraint to get

$$\dot{y} \ = g(y, \pi(y)),\ t \in [0, \infty)$$

which, together with the initial condition $y(0) = y_0$, would, hopefully, allow for the determination of the solution for the state variable.

If we can find the policy function, then obtaining the optimal dynamics for y reduces to solving an initial-value problem instead of a mixed initial-terminal value problem (or two-point boundary value problem) as is the case when we use the calculus of variations of the Pontriyagin's principle approaches.

However, only in a very small number of cases we can obtain closed form solutions to the HJB equation. Next we show some cases in which this is possible.

6.4.3 Applications

Example 1: The resource depletion problem

We solve again resource-depletion problem for an infinite horizon

$$\max_C \int_0^\infty e^{-\rho t} \ln{(C(t))} dt, \text{ s.t } \dot{W} = -C,\ W(0) = W_0$$

by using the DP principle.

The HJB equation is

$$\rho v(W) = \max_C \left[\ \ln{(C)} + v^{'}(W)(-C) \right]$$

Policy function

$$\frac{1}{C^*} - v^{'}(W) = 0 \Leftrightarrow C^* = \left(v^{'}(W)\right)^{-1}$$

Then the HJB becomes

$$\rho v(W) = -\ \ln(v^{'}(W)) - 1$$

The textbook method for solving the HJB equation through is by using the **method of undetermined coefficients** after we make a conjecture over the form of the value function (no constructive way here).

Assume the trial function

$$v(W) = a + b\ln(W)$$

As $v^{'}(W) = b/W$ and substituting and collecting terms we get

$$\rho a + 1 + \ln(b) = \ln(W)(1 - \rho b)$$

then $b = 1/\rho$ and $a = (\ln\rho - 1)/\rho$.

Then:

$$v(W) = \frac{\ln\rho - 1 + \ln(W)}{\rho},\ C^* = (v^{'}(W))^{-1} = \rho W$$

A second method: the HJB equation is an ODE, where W is the independent variable, so we can try to solve it (this is a constructive method).

The HJB is equivalent to

$$v^{'}(W) = e^{-(1+\rho v(W))}$$

ODE $y^{'}(x) = e^{(a+by(x))}$ has the closed form solution

$$y(x) = \frac{1}{b}\left(-a + \ln\left(-\frac{1}{b(k+x)}\right)\right)$$

where k is an arbitrary constant. Then we determine

$$v(W) = -\frac{1}{\rho}\left(1 + \ln\left(\frac{1}{\rho(W+k)}\right)\right)$$

and

$$C^* = \left(V^{'}(W)\right)^{-1} = \rho(W+k)$$

Substituting in the constraint $\dot{W} = -C = -\rho(W+k)$, we get the solution

$$W(t) = -k + (W(0)+k)e^{-\rho t}.$$

The problem is somewhat incompletely specified, which reveals a potential problem when using the DP approach.

In our case, as it is natural to assume that $\lim_{t\to\infty} W(t) = 0$ we would obtain $k = 0$ and therefore we would get the same solution as from using the CV and Pontriyagin's approaches:

$$C^*(t) = \rho W_0 e^{-\rho t},\ W^*(t) = W_0 e^{-\rho t},\ \text{for } t \in [0, \infty).$$

Example 2: The benchmark consumption-savings problem

Applying the HJB equation (6.47) to our problem we have

$$\rho v(A) = \max_C \left\{ \frac{C^{1-\theta} - 1}{1-\theta} + v^{'}(A)(Y - C + rA) \right\}.$$

Defining,

$$h(C, A) = \frac{C^{1-\theta} - 1}{1-\theta} + v^{'}(A)(Y - C + rA)$$

we find the optimal policy function as

$$C^* = \left\{ C : \frac{\partial h(C, A)}{\partial C} = 0 \right\}.$$

The optimal control is

$$C^* = \pi(A) \equiv \left(v^{'}(A)\right)^{-\frac{1}{\theta}}.$$

meaning that the policy function, $\pi(A)$, is obtained explicitly. If we substitute back in the HJB function we obtain a non-linear ODE in implicit form

$$\rho v(A) = \frac{1}{1-\theta}\left(\left(v^{'}(A)\right)^{\frac{\theta-1}{\theta}} - 1\right) + v^{'}(A)\left(Y + rA - \left(v^{'}(A)\right)^{-\frac{1}{\theta}}\right),$$

or, equivalently,

$$v(A) = \frac{\theta}{1-\theta}\left(v^{'}(A)\right)^{\frac{\theta-1}{\theta}} - \frac{1}{1-\theta} + rv^{'}(A)\left(\frac{Y}{r} + A\right) \tag{6.48}$$

Using again the method of undetermined coefficients we try a solution

$$v(A) = a + bH(A)^{1-\theta},$$

where $H(A) \equiv \dfrac{Y}{r} + A$ and the parameters a and b are arbitrary. If this trial function has the right form, the by substitution, we can determine coefficient b. As

$$v^{'}(A) = b(1-\theta)H(A)^{-\theta}$$

after substitution in equation (6.48) we get

$$a\rho + \frac{1}{1-\theta} = H(A)^{1-\theta}\, b\, \theta \Big[\ (1-\theta)^{-1/\theta} - (r-\gamma)\Big]$$

where we have again $\gamma \equiv (r-\rho)/\sigma$. Setting both sides to zero, yields

$$a = \frac{1}{\rho(\theta-1)} \text{ and } b = \frac{(r-\gamma)^{-\theta}}{1-\theta}$$

Then

$$v(A) = \frac{1}{1-\theta}\Big[\ (r-\gamma)^{-\theta}\left(\frac{Y}{r}+A\right)^{1-\theta} - \frac{1}{\rho}\Big].$$

Taking the derivative as regards A and substituting in the policy function for C, we find the optimal consumption in feedback form

$$C^*(A) = (r-\gamma)\left(\frac{Y}{r}+A\right)$$

which only makes sense if $r > \gamma$.

We can get the optimal asset path by substituting optimal consumption in the budget constraint

$$\dot{A}^* = Y + rA - C^*(A) = \gamma\left(\frac{Y}{r}+A\right).$$

Solving this equation with $A(0) = A_0$ we get the optimal paths for asset holdings

$$A^*(t) = -\frac{Y}{r} + \left(\frac{Y}{r}+A_0\right)e^{\gamma t},\ \text{ for } t \in [0,\infty),$$

and consumption

$$C^*(t) = (r-\gamma)\left(\frac{Y}{r}+A_0\right)e^{\gamma t},\ \text{ for } t \in [0,\infty).$$

Exercise Prove, by setting $\theta = 1$ that the value function is

$$V(A) = \frac{1}{\rho}\Big[\ \frac{r-\rho}{\rho} + \ln\big(\frac{Y}{r}+A\big)\ \Big].$$

Hint: use the property $f(x) = \exp(\ln f(x))$ and the l'Hôpital theorem.

The utility function is a generalized logarithm $u(C) = \ln_\theta(C) = \dfrac{C^{1-\theta}-1}{1-\theta}$. Sometimes in the literature people write

$$u(C) = \begin{cases} \dfrac{C^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1 \\ \ln(C) & \textit{if}\ \theta = 1 \end{cases}$$

The problem with this formulation is that if we cannot obtain the value function for the logarithm utility by setting the limit of $\theta = 1$ for the general case $\theta = 1$, which is

$$v(A) = \frac{(r-\gamma)^{-\theta}}{1-\theta}\left(\frac{Y}{r}+A\right)^{1-\theta}.$$

Example 3: The Ramsey and AK models

The HJB for the Ramsey model is

$$\rho v(K) = \max_C \left\{ u(C) + v^{'}(K)\left(F(K) - C\right)\right\}$$

does not have a closed form solution even for an iso-elastic utility function and for a Cobb-Douglas production function.

However, if we assume an isoelastic utility and a constant return production function, as $Y = AK$, we get the AK model, which has an HJB function

$$\rho v(K) = \max_C \left\{ \frac{C^{1-\theta}}{1-\theta} + v^{'}(K)\left(AK - C\right)\right\}$$

Using the same steps as before, we get

$$\rho v(K) = \frac{\theta}{1-\theta}\left(v^{'}(K)\right)^{\frac{\theta-1}{\theta}} + v^{'}(K)AK \tag{6.49}$$

To solve the equation we use again the method of undetermined coefficients and find

$$v(K) = \frac{\left((A-\gamma)K\right)^{1-\theta}}{1-\theta}.$$

where

$$\gamma = \frac{A-\rho}{\theta}.$$

The consumption, in the feedback form is,

$$C^{*}(K) = (A-\gamma)K$$

and the budget constraint of the economy is

$$\dot{K}^{*} = AK^{*} - C^{*}(K) = \gamma K^{*}.$$

Considering the given initial level for capital $K(0) = K_0$ we get the optimal paths for capital and output

$$K^{*}(t) = K_0 e^{\gamma t},\ Y^{*}(t) = AK_0 e^{\gamma t},\ \text{for } t \in [0, \infty).$$

6.5 Bibiography

- Introductory Kamien and Schwartz (1991), Chiang (1992)
- A little less introductory Weber (2011) Liberzon (2012)
- Complete reference, particularly for optimal control Grass et al. (2008)
- The seminal contributions: optimal control Pontryagin et al. (1962), the dynamic programming principle Bellman (1957)
- Applications: Kamien and Schwartz (1991) (management) , Grass et al. (2008) (economics, terror and drugs). Historical contributions: Arrow and Kurz (1970), Intriligator (2002), Brock and Malliaris (1989). Growth applications: Acemoglu (2009).

6.A Appendix

6.A.1 Approximating functions

Assume we have a function $f(x)$ for $x \in \mathcal{X} \subseteq \mathbb{R}$. If function $f(.)$ is non-linear, or non explicitly specified, we need sometimes to compare it with another function whose behaviour can be simpler, or we need to know only about how it grows or decays without worrying about the details. The notations $O(.)$ and $o(.)$ are useful for this context.

Big O notation A function is of *constant order* if there is a non-zero constant c such that we can write, equivalently

$$\lim_{x\to\infty} \frac{f(x)}{c} = 1 \text{ or } \lim_{x\to\infty} f(x) = c$$

We say in this case that

$$f(x) \in O(1)$$

More generally a function $f(x)$ *is big-O of function* $g(x)$, written as $f(x) \in O(g(x))$ (sometimes written $f(x) = O(g(x))$ if $f(x)$ is of the same order than $g(x)$ (they grow or decay at the same rate)

$$f(x) \in O(g(x)) \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = c$$

or $\frac{f(x)}{g(x)} \in O(1)$.

Examples:

- $ax + b \in O(x)$ if $a \neq 0$

- $ax^2 + bx + c \in O(x^2)$ if $a \neq 0$
- $ax^{-2} + bx^{-1} \in O(x^{-1})$

Little o notation There is an associate function, the *little-o* notation that means that a functions is asymptotically smaller than another function. Function $f(x)$ is asymptotically smaller than a non-zero constant c if

$$\lim_{x\to\infty} \frac{f(x)}{c} = 0, \text{ or} f(x) \in o(1)$$

In general, function $f(x)$ *is little-o of function* $g(x)$, written as of $f(x) \in o(g(x))$ if for $n \to \infty$ there is a number N such that

$$|f(x)| < \epsilon|x| \text{ for } n > N$$

If $g(x) \neq 0$ in all its domain them $f(x)$ is a small-o of function $g(x)$ if and only if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

Examples:

- $2x = o(x^2)$
- $e^x = 1 + x + \frac{1}{2}x^2 + o(x^3)$,
- $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^4)$
- $\ln(x) \in o(x)$
- Let $0 < \alpha < 1$ then x^α if $o(x)$

If $f(x) \in o(g(x))$ then $f(x) \in O(g(x))$ but the converse may not be true.
There are several properties for $O(.)$ and $o(.)$ functions, this is a small sample:

- If $f(x) \in O(g(x))$ then $c\, f(x) \in O(g(x))$ for any constant c,
- If $f_1(x)$ and $f_2(x)$ are both $O(g(x))$ then $f_1(x) + f_2(x) \in O(g(x))$,
- If $f_1(x) \in O(g(x))$ and $f_2(x) \in o(g(x))$ then $f_1(x) + f_2(x) \in O(g(x))$,
- If $f_1(x) \in O(f_2(x))$ and $f_2(x) \in o(g(x))$ then $f_1(x) \in o(g(x))$,
- $x^n o(x^m) = o\left(x^{n+m}\right)$
- $o\left(x^n\right) o\left(x^m\right) = o\left(x^{n+m}\right)$
- $x^m = o\left(x^n\right)$ if $n < m$

Notation	Limit definition (a)	Limit definition (b)
$f \in o(g)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$\limsup \frac{f(x)}{g(x)} = 0$
$f \in O(g)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$\limsup \frac{f(x)}{g(x)} \infty$

Calculus with big-O and little-o The function f has a strong derivative at x if

$$f(x+\epsilon) = f(x) + f^{'}(x)\epsilon + O(\epsilon^2)$$

for ϵ sufficiently small.

Weak derivative

$$f(x+\epsilon) = f(x) + f^{'}(x)\epsilon + o(\epsilon)$$

The Lagrange formula can be written as

$$f(x+\epsilon) = f(x) + f^{'}(x)\epsilon + \frac{f^{''}(x)}{2}\epsilon^2 + ... + \frac{f^{(n)}(x)}{n}\epsilon^n + o(\epsilon^n)$$

6.A.2 Functionals and integral derivatives

Consider function $f \in \mathcal{F}$ the space of C^1 functions $f : \mathbb{R} \to \mathcal{F} \subseteq \mathbb{R}$ and $\mathbf{f} : \mathbb{R} \to \mathcal{F} \subseteq \mathbb{R}^n$, $x \mapsto f(x)$

A *functional* is a mapping from a function of one variable or from a set of variables into a number $F : f \to \mathbb{R}$ or $F : \mathbf{f} \to \mathbb{R}$, i.e..

$$F : f(x) \mapsto F[f], \qquad \mathbf{f}(x) \mapsto F[\mathbf{f}]$$

Examples:

- A definite integral over a continuous function $f : (x_0, x_1) \to \mathbb{R}$

$$F[f] = \int_{x_0}^{x_1} f(x)dx$$

- Or if $w(.)$ is a weighting function

$$F[f] = \int_{x_1}^{x_2} w(x)f(x)dx$$

- It can also map a function into a point

$$F[f] = f(x^*), \text{ for } x^* \in (x_0, x_1)$$

  which can be represented by

$$F_\delta[f](x^*) = \int_{x_0}^{x_1} \delta(x - x^*) f(x) dx$$

- the value functional in the calculus of variations

$$J[y] = \int_{t_0}^{t_1} F(t, y(t), \dot{y}(t)) dt$$

  where $y : [t_0, t_1] \rightarrow \mathbb{R}$, that is $t \mapsto y(t)$.

- the utility functional in the Dixit and Stiglitz (1977) model

$$U[x] = \left(\int_0^N x(i)^\gamma di \right)^{\frac{1}{\gamma}}$$

  where $(x(i))_{i=0}^N$ are different varieties of a product.

Taylor expansion of a functional A first variation of a functional is

$$\delta F[f] = F[f + \delta f] - F[f]$$

Introduce a perturbation $f \rightarrow f + \varepsilon\eta$ where $\eta \in \mathcal{F}$ and ε is a real number, and define the variation as

$$\delta f = \varepsilon\eta.$$

A first order Taylor expansion around f for a perturbation proportional to η

$$F[f + \varepsilon\eta] = F[f] + \delta F[f](\eta)\varepsilon + o(\varepsilon)$$

where $\lim_{\varepsilon\rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0$, is a function of the parameter ε.

The *first variation of the functional* around f for a perturbation η is then

$$\delta F[f](\eta) = \lim_{\varepsilon\rightarrow 0} \frac{F(f + \varepsilon\eta) - F(f)}{\varepsilon} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F[f + \varepsilon\eta]$$

This is also called a Gâteaux derivative of a functional or integral derivative.

A second-order Taylor expansion around f for a perturbation η

$$F[f + \varepsilon\eta] = F[f] + \delta F[f](\eta)\varepsilon + \delta^2 F[f](\eta)\varepsilon^2 + o(\varepsilon^2)$$

where $\lim_{\varepsilon\rightarrow 0} \frac{o(\varepsilon^2)}{\varepsilon^2} = 0$

Maximizer of a functional Function f^* is a maximiser only if for any admissible perturbation $f^* + \varepsilon\eta$, the following conditions hold

1. the first variation is equal to zero
$$\delta F[f^*](\eta) = 0$$

2. the second variation is negative (or the Hessian is negative definite)
$$\delta^2 F[f^*](\eta) < 0$$

Sufficient condition: assume there is a function $f = f^* + \eta\varepsilon$ such that

$$\delta F[f^*] = 0, \text{ and } \delta^2 F[f^*](\eta) < 0$$

then f^* is a maximiser of the functional $F[f]$

This is a local maximizer for an unconstrained domain. If there are constraints $D \in \mathcal{F}$ we have to adapt the conditions.

Functional of a vector Consider a functional on a vector $F[\mathbf{f}]$ where $\mathbf{f} \in \mathcal{F}$ is a vector of functions of x. Again $F[\mathbf{f}]$ is a scalar.

We can define the first variation

$$\delta F[\mathbf{f}] = F[\mathbf{f} + \delta\mathbf{f}] - F[\mathbf{f}]$$

and again
$$\delta\mathbf{f} = \epsilon$$

where $\ \in \mathcal{F}$.

Again, a first-order Taylor approximation for a perturbation is

$$F[\mathbf{f} \ + \varepsilon \ ] = F[\mathbf{f}] + \delta F[\mathbf{f}](\ )\varepsilon + o(\varepsilon)$$

where the integral derivative is

$$\delta F[\mathbf{f}](\ ) = \lim_{\varepsilon \to 0} \frac{F(\mathbf{f} + \varepsilon\ ) - F(\mathbf{f})}{\varepsilon} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F[\mathbf{f} + \varepsilon\ ].$$

Also, if $\mathbf{f}^*$ is a maximizer if for any admissible perturbation by the first variation is equal to zero,

$$\delta F[\mathbf{f}^*](\ ) = 0.$$

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