

Advanced macroeconomics 2021-2022

Problem set: consumption, savings, and asset accumulation

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1 Intertemporal utility

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- 1 Consider the following utility functional over a consumption path $c = (c(t))_{t \in [0, T]}$,

$$U[c] = \int_0^T \log(c(t)) e^{-\rho t} dt, \quad \rho > 0$$

- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
- (b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

- 2 Consider the following utility functional over a consumption path $c = (c(t))_{t \in [0, T]}$,

$$U[c] = \int_0^T -\frac{1}{\zeta} e^{-\zeta c(t)} e^{-\rho t} dt, \quad \rho > 0, \quad \zeta > 0.$$

- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
- (b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

- 3 Consider the following utility functional over a consumption path $c = (c(t))_{t \in [0, T]}$,

$$U[c] = \int_0^T \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt, \quad \rho > 0, \quad \theta > 0.$$

¹Questions are marked with asterisks depending on their degree of difficulty.

- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
- (b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

4 Consider the following utility functional over a consumption path $c = (c(t))_{t \in [0, T]}$,

$$U[c] = \int_0^T \left(c(t) - \frac{\beta}{2} c(t)^2 \right) e^{-\rho t} dt, \quad \rho > 0, \beta > 0.$$

- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
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- (b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

8* Consider the following utility functional over a consumption path $c = (c(t))_{t \in [0, T]}$,

$$U[c] = \int_0^T \log(c(t)) e^{\int_0^t \log(c(s)) ds} dt,$$

- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
- (b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

9* Consider the following utility functional over a consumption path $c = (c(t))_{t \in [0, T]}$,

$$U[c] = \int_0^T \log(c(t) - \zeta h(t)) e^{-\rho t} dt, \quad \rho > 0, \quad \zeta > 0,$$

where

$$\dot{h} = \eta(c - h)$$

and $h(0) = h_0$, where h_0 is given.

- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
- (b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

10* Consider the following utility functional over a consumption path $c = (c(t))_{t \in [0, T]}$,

$$U[c] = \int_0^T \log\left(\frac{c(t)}{h(t)^\zeta}\right) e^{-\rho t} dt, \quad \rho > 0, \quad \zeta > 0,$$

where

$$\dot{h} = \eta(c - h)$$

and $h(0) = h_0$, where h_0 is given.

- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
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- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
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$$U[c] = \int_0^T \log \left(\frac{c(t)}{h(t)^\zeta} \right) e^{-\rho t} dt, \quad \rho > 0, \quad \zeta > 0,$$

where

$$\dot{h} = \eta c - \delta h, \quad \text{for } 0 < \delta < 1$$

and $h(0) = h_0$, where h_0 is given.

- (a) Find the intertemporal marginal rate of substitution and the elasticity of intertemporal substitution, between time t_0 and $t_1 = t_0 + \tau$, for $\tau > 0$.
- (b) Characterize the intertemporal preferences (hint consider a stationary consumption). Provide an intuition.

2 Consumer problems

1 Assume that a consumer has utility functional

$$U[c] = \int_0^T \left(c(t) - \frac{\beta}{2} c(t)^2 \right) e^{-\rho t} dt, \quad \rho > 0, \quad \beta > 0.$$

and no constraints on consumer.

- (a) Find the optimal consumption function
- (b) Would the solution change if consumer had an initial wealth $w(0) = w_0$ and no further constraints on wealth ?
- (c) Would the solution change if consumer had an initial wealth $w(0) = w_0 > 0$ and had a constraint on wealth such that $w(t) \geq 0$?
- (d) Discuss the previous results.

2 Consider the problem

$$\begin{aligned} & \max_c \int_0^T \log(c(t)) e^{-\rho t} dt \\ & \text{subject to} \\ & \dot{a}(t) = r a - c(t), \text{ for } t \in T \\ & a(t) \in [\underline{a}, \infty), \text{ for every } t \in [0, T] \\ & a(0) = a_0 > \max\{0, \underline{a}\} \text{ given} \end{aligned}$$

- (a) Find the optimality conditions.
- (b) Find the solution to the problem. Under which conditions it is optimum to saturate the borrowing constraint at the terminal time T ?
- (c) Provide an intuition for your results.

3** Consider the problem

$$\begin{aligned} & \max_c \int_0^\infty \log(c(t)) e^{-\rho t} dt \\ & \text{subject to} \\ & \dot{a}(t) = \begin{cases} r_1 a + y - c(t), & \text{for } 0 \leq t < t_s \\ r_2 a + y - c(t), & \text{for } t_s \leq t < \infty \end{cases} \\ & a(0) = a_0 > \text{ given} \\ & \lim_{t \rightarrow \infty} e^{-r_2 t} a(t) \geq 0 \end{aligned}$$

where $y > 0$ and $r_1 > \rho > r_2$.

- (a) Find the optimality conditions.
- (b) Find the solution to the problem.
- (c) Draw the phase diagram.
- (d) Provide an intuition for your results.

3 Comparative dynamics

1 Consider the problem

$$\begin{aligned} & \max_c \int_0^\infty \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \\ & \text{subject to} \\ & \dot{a}(t) = (1-\tau)(r a + w) - c(t), \text{ for } t \in \mathbb{R}_+ \\ & a(0) = a_0 \text{ given} \\ & \lim_{t \rightarrow \infty} a(t) e^{-r t} \geq 0 \end{aligned}$$

where $0 < \tau < 1$ is the income tax rate.

- (a) Find the optimality conditions.
- (b) Study the comparative dynamics effects for an anticipated, permanent and constant increase in the income tax rate τ .
- (c) Provide an intuition for your results.

2 Consider the problem

$$\begin{aligned} & \max_c \int_0^\infty -\frac{e^{-\xi c(t)}}{\xi} e^{-\rho t} dt, \xi > 0 \\ & \text{subject to} \\ & \dot{a}(t) = (1-\tau)(r a + w) - c, \text{ for } t \in \mathbb{R}_+ \\ & a(0) = a_0 \text{ given} \\ & \lim_{t \rightarrow \infty} a(t) e^{-r t} \geq 0, \end{aligned}$$

where $0 < \tau < 1$ is the income tax rate.

- (a) Find the optimality conditions.
- (b) Study the comparative dynamics effects for an anticipated, permanent and constant increase in the income tax rate τ .
- (c) Provide an intuition for your results.

4 Habit formation

1 Consider the problem

$$\begin{aligned} & \max_c \int_0^\infty \log (c(t) - \zeta h(t)) e^{-\rho t} dt \\ & \text{subject to} \\ & \dot{a}(t) = r a + w - c(t), \text{ for } t \in \mathbb{R}_+ \\ & \dot{h}(t) = \eta (c - h) \text{ for } t \in \mathbb{R}_+ \\ & a(0) = a_0 \text{ given} \\ & h(0) = h_0 \text{ given} \\ & \lim_{t \rightarrow \infty} a(t) e^{-r t} \geq 0 \end{aligned}$$

for $\rho > 0$, $0 < \zeta < 1$ and $\eta > 0$.

- (a) Find the first order conditions
- (b) Under which conditions there will be transitional dynamics
- (c) . Discuss the dynamics of consumption and income for a non-anticipated, permanent and constant increase in non-financial income w .
- (d) Is consumption response perfectly correlated with income ? Why ?

2 Consider the problem

$$\begin{aligned} & \max_c \int_0^\infty \log (c(t)h(t)^{-\zeta}) e^{-\rho t} dt \\ & \text{subject to} \\ & \dot{a}(t) = r a + w - c(t), \text{ for } t \in \mathbb{R}_+ \\ & \dot{h}(t) = \eta (c - h) \text{ for } t \in \mathbb{R}_+ \\ & a(0) = a_0 \text{ given} \\ & h(0) = h_0 \text{ given} \\ & \lim_{t \rightarrow \infty} a(t) e^{-r t} \geq 0 \end{aligned}$$

for $\rho > 0$, $0 < \zeta < 1$ and $\eta > 0$.

- (a) Find the first order conditions
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- (c) . Discuss the dynamics of consumption and income for a non-anticipated, permanent and constant increase in non-financial income w .
- (d) Is consumption response perfectly correlated with income ? Why ?