Foundations of Financial Economics
Multi-period finance economies

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Topics for today

- The multiperiod finance economy: structure
- Arbitrage asset pricing
- Equilibrium asset pricing for a homogeneous economy
- Equity premium puzzle

Multiperiod Finance economy

Information structure

We consider an **homogeneous agent** economy, in which:

- there is an information tree, with T periods,
- the information tree comprises a sequence of nodes $\{N_t\}_{t=1}^{T} = \{N_1, N_2, \ldots, N_s, \ldots, N_T\}$, where N_t is the number of nodes of the information tree at time t
- **Example**: for a binomial process $N_t = 2^t$
- there is a sequence of unconditional probabilities

$$\mathbb{P}^T \equiv \{P_t\}_{t=1}^{T} = \{\mathsf{P}_1, \ldots, \mathsf{P}_t, \ldots, \mathsf{P}_T\}$$

  where

$$\mathsf{P}_t = \begin{pmatrix} \pi_{t,1} \\ \ldots \\ \pi_{t,s} \\ \ldots \\ \pi_{t,N_t} \end{pmatrix}$$

- for any process $\{X_t\}_{t=0}^{T} = \{X_0, X_1, \ldots X_t, \ldots, X_T\}$ we assume that X_t is $\mathcal{F}_t$-adapted

Arbitrage asset pricing

The structure of the asset market

- there are K assets traded in every period $t = 0, \ldots, T-1$ (not just at $t = 0$ as before)
- there is a payoff process $\{V_t\}_{t=1}^{T} = \{V_1, V_2, \ldots, V_t, \ldots, V_T\}$, where

$$V_t = (V_t^1, \ldots, V_t^K), \text{ for } V_t^j = \begin{pmatrix} v_{t,1}^j \\ \ldots \\ v_{t,s}^j \\ \ldots \\ v_{t,N_t}^j \end{pmatrix}$$

- or, expanding, the possible realizations for the payoff at time t are

$$V_t = \begin{pmatrix} v_{t,1}^1 & \ldots & v_{t,1}^j & \ldots & v_{t,1}^K \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_{t,s}^1 & \ldots & v_{t,s}^j & \ldots & v_{t,s}^K \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_{t,N_t}^1 & \ldots & v_{t,N_t}^j & \ldots & v_{t,N_t}^K \end{pmatrix}$$

Arbitrage asset pricing

The structure of the asset market

- there is a process for asset prices $\{S_t\}_{t=0}^{T-1} = \{S_0, S_1, \dots, S_t, \dots, S_{T-1}\}$, where

$$S_t = (S_t^1, \dots, S_t^K), \text{ for } S_t^j = \begin{pmatrix} s_{t,1}^j \\ \dots \\ s_{t,s}^j \\ \dots \\ s_{t,N_t}^j \end{pmatrix}$$

- or, expanding, the possible realizations for the payoff at time t are

$$S_t = \begin{pmatrix} s_{t,1}^1 & \dots & s_{t,1}^j & \dots & s_{t,1}^K \\ \dots & \dots & \dots & \dots & \dots \\ s_{t,s}^1 & \dots & s_{t,s}^j & \dots & s_{t,s}^K \\ \dots & \dots & \dots & \dots & \dots \\ s_{t,N_t}^1 & \dots & s_{t,N_t}^j & \dots & s_{t,N_t}^K \end{pmatrix}$$

Arbitrage asset pricing

The structure of the asset market: example

Example: assume we have two assets denoted by a and b and we have a two-state binomial tree and $T = 3$. The asset market is defined by the sequences

- at $t = 0$

$$S_0 = (S_0^a, S_0^b)$$

- at $t = 1$

$$V_1 = \begin{pmatrix} V_{1,1}^a & V_{1,1}^b \\ V_{1,2}^a & V_{1,2}^b \end{pmatrix}, \ S_1 = \begin{pmatrix} S_{1,1}^a & S_{1,1}^b \\ S_{1,2}^a & S_{1,2}^b \end{pmatrix}$$

- at $t = 2$

$$V_2 = \begin{pmatrix} V_{2,1}^a & V_{2,1}^b \\ V_{2,2}^a & V_{2,2}^b \\ V_{2,3}^a & V_{2,3}^b \\ V_{2,4}^a & V_{2,4}^b \end{pmatrix}, \ S_2 = \begin{pmatrix} S_{2,1}^a & S_{2,1}^b \\ S_{2,2}^a & S_{2,2}^b \\ S_{2,3}^a & S_{2,3}^b \\ S_{2,4}^a & S_{2,4}^b \end{pmatrix}$$

- at $t = 3$

$$\begin{pmatrix} V_{3,1}^a & V_{3,1}^b \\ \end{pmatrix}$$

Arbitrage asset pricing

Stochastic discount factor: intertemporal form

Definition: a **stochastic discount factor** (SDF) is a process $\{M_t\}_{t=0}^{T-1}$, such that, **for any asset** $j = 1, \ldots, K$:

1. M_t is $\mathcal{F}_t$-measurable,
2. $M_0 = m_0 = 1$
3. such that the value of the asset at time t is equal to the (conditional) expected value of the present value of future payoffs

$$\boxed{M_t S_t^j = \mathbb{E}_t \left[\sum_{\tau=t+1}^{T} M_\tau \, V_\tau^j \right], \text{ for } t = 0, \ldots, T-1}$$

Arbitrage asset pricing

Stochastic discount factor: intertemporal form

Observations:

1. We say this is SDF definition in the **intertemporal form**
2. the meaning of the conditional expectation $\mathbb{E}_t[.]$

$$M_t S_t^j = \mathbb{E}_t\left[\sum_{\tau=t+1}^{T} M_\tau\, V_\tau^j\right] = \mathbb{E}\left[\sum_{\tau=t+1}^{T} M_\tau\, V_\tau^j \,\middle|\, S^t,\, V^t\right]$$

where $S^t = \{S_0, S_1, \ldots S_t\}$ and $V^t = \{V_1, \ldots V_t\}$ are the histories of the asset prices and payoffs up until time t

Arbitrage asset pricing

Stochastic discount factor: recursive form

- **Proposition**: the stochastic discount factor can be equivalently defined in the **recursive from**

$$\boxed{M_t S_t^j = \mathbb{E}_t\left[M_{t+1}(S_{t+1}^j + V_{t+1}^j)\right]}$$

- **Intuition**: the stochastic discount factor is a stochastic process $\{M_t\}$, that equalizes the **value** of the asset price in period t with the conditional expected value of the **value** of the income in period $t+1$ (the income is equal to the payoff plus the anticipated market price)

Arbitrage asset pricing

Stochastic discount factor: recursive form

Proof:

- using the definition of intertemporal form and expanding

$$M_t S^j_t = \mathbb{E}_t \left[M_{t+1}\, V^j_{t+1} + \sum_{\tau=t+2}^{T} M_\tau\, V^j_\tau \right]$$

- by the law of iterated expectations

$$\mathbb{E}_t \left[M_{t+1}\, V^j_{t+1} + \sum_{\tau=t+2}^{T} M_\tau\, V^j_\tau \right] = \mathbb{E}_t \left[M_{t+1}\, V^j_{t+1} + \mathbb{E}_{t+1} \left[\sum_{\tau=t+2}^{T} M_\tau\, V^j_\tau \right] \right]$$

- but

$$M_{t+1} S^j_{t+1} = \mathbb{E}_{t+1} \left[\sum_{\tau=t+2}^{T} M_\tau\, V^j_\tau \right],$$

- then $M_t S^j_t = \mathbb{E}_t \left[M_{t+1}\, V^j_{t+1} \right] + \mathbb{E}_t \left[M_{t+1} S^j_{t+1} \right]$

Arbitrage asset pricing

Transactions strategy

Definition: A **transactions strategy** is a sequence of portfolios $\{\theta_{t+1}\}_{t=0}^{T-1}$, with $\theta_{t+1} = (\theta_{t+1}^1 \dots \theta_{t+1}^K)$, where θ_{t+1}^j is $\mathcal{F}_t$-measurable.

- generating an **income** stream $\{Z_t^\theta\}_{t=0}^T$

$$
\begin{aligned}
Z_0^\theta &= -\theta_1 S_0 = -\sum_{j=1}^K \theta_1^j S_0^j \\
&\dots \\
Z_t^\theta &= \theta_t (S_t + V_t) - \theta_{t+1} S_t = \sum_{j=1}^K \left(\theta_t^j (S_t^j + V_t^j) - \theta_{t+1}^j S_t^j \right), \\
Z_T^\theta &= \theta_T V_T = \sum_{j=1}^K \theta_T^j V_T^j
\end{aligned}
$$

where $Z_t^\theta \in \mathbb{R}^{N_t}$ is $\mathcal{F}_t$-measurable.

- **Defenition**: if $Z_0^\theta = \dots = Z_t^\theta = \dots = Z_T^\theta = 0$ we say the transactions strategy is **self-financed**.

Arbitrage asset pricing

Absence of arbitrage opportunities

- **Definition**: there is **absence of arbitrage opportunities** if there is a positive process $\{M_t\}_{t=0}^{T-1}$ such that the income stream $\{Z_t^\theta\}_{t=0}^{T}$, generated by the transaction strategy $\{\theta_{t+1}\}_{t=0}^{T-1}$, verifies

$$\boxed{\mathbb{E}_0\left[\sum_{t=0}^{T} M_t Z_t^\theta\right] = 0}$$

- **Intuition**: there are no arbitrage opportunities if, with a zero initial investment, the expected value of the present value of any transaction strategy is zero, if the discount factor is positive.

Proposition: A necessary condition for the absence of arbitrage opportunities is that:

1. $M_T S_T = 0$ if T is finite;
2. ruling-out speculative bubbles condition holds: $\lim_{t\to\infty} M_t S_t = 0$ if $T = \infty$

Arbitrage asset pricing

Absence of arbitrage opportunities

Proof (assuming $K = 1$):

- use the definition of stochastic discount factor (in the recursive form)
$$-M_0 Z_0^\theta = M_0 \theta_1 S_0 = \mathbb{E}_0 \left[M_1 \theta_1 (S_1 + V_1) \right]$$
- use a little trick, introducing $\pm M_1 \theta_2 S_1$;
$$\begin{aligned} \mathbb{E}_0 \left[M_1 \theta_1 (S_1 + V_1) \right] &= \mathbb{E}_0 \left[M_1 \theta_1 (S_1 + V_1) \pm M_1 \theta_2 S_1 \right] = \\ &= \mathbb{E}_0 \left[M_1 Z_1^\theta + M_1 \theta_2 S_1 \right] \end{aligned}$$
- use the definition of stochastic discount factor and the law of iterated expectations
$$\begin{aligned} \mathbb{E}_0 \left[M_1 Z_1^\theta + M_1 \theta_2 S_1 \right] &= \mathbb{E}_0 \left[M_1 Z_1^\theta + \mathbb{E}_1 \left[M_2 \theta_2 (S_2 + V_2) \right] \right] \\ &= \mathbb{E}_0 \left[M_1 Z_1^\theta + M_2 \theta_2 (S_2 + V_2) \right] \end{aligned}$$

Arbitrage asset pricing

Absence of arbitrage opportunities

Proof (assuming $K = 1$ continuation):

- by repeatedly using the previous steps we arrive at

$$-M_0 Z_0^\theta = \mathbb{E}_0 \left[\sum_{t=1}^{T} M_t Z_t^\theta + M_T \theta_{t+1} S_T \right]$$

- then

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} M_t Z_t^\theta \right] = 0$$

  only if $M_T S_T = 0$

Application 1: zero payoffs

Absence of arbitrage opportunities

- **Zero payoffs (or no dividends case)**: Assume that there are no dividends, i.e., $V_t = \mathbf{0}$ for any $t = 1, \ldots, T$.
- If there are no arbitrage opportunities then

$$M_t S_t^j = \mathbb{E}_t \left[M_{t+1} S_{t+1}^j \right]$$

- therefore: the process $\{M_t S_t\}_{t=0}^{T-1}$ is a **martingale under measure** $\mathbb{P}$,

Arbitrage asset pricing

Fundamental theorem

Proposition: for a zero-dividend asset, absence of arbitrage opportunities is equivalent to the existence of an equivalent probability measure $\mathbb{Q}$ such that $\{S_t\}_{t=0}^{T-1}$ is a martingale under $\mathbb{Q}$, that is

$$S_t^j = \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

Arbitrage asset pricing

Fundamental theorem

Sketch of proof

- Let us define the **conditional** stochastic discount factor

$$M_{t+1|t} \equiv \frac{M_{t+1}}{M_t}$$

- Then if there are no arbitrage opportunities (because M_t is $\mathbb{F}_t$-measurable)

$$S^j_t = \mathbb{E}_t[M_{t+1|t} S^j_{t+1}]$$

- This is valid for the degenerate process $\{\mathbf{1}\}_{t=0}^T = \{\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}\}$, then

$$1 = \mathbb{E}_t[M_{t+1|t}]$$

$$S^j_t = \mathbb{E}_t[M_{t+1|t} S^j_{t+1}] = \frac{\mathbb{E}_t[M_{t+1|t} S^j_{t+1}]}{\mathbb{E}_t[M_{t+1|t}]} = \mathbb{E}^{\mathbb{Q}}_t[S^j_{t+1}]$$

- from the Radon-Nikodym theorem $\mathbb{Q}$ is an equivalent martingale measure.

Observe that $\{M_t\}$ is also a martingale, under measure $\mathbb{Q}$ because

$$M_t = \mathbb{E}_t^{\mathbb{Q}}[M_{t+1}]$$

Application 2: positive payoffs

- **Positive payoffs (or positive dividends case)**: if asset j pays a positive dividend, that is $V_t^j \geq \mathbf{0}$ is a positive vector, then

$$\mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j + V_{t+1}^j] \geq \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

- Then $\{S_t\}$ is a **submartingale** under measure $\mathbb{Q}$

$$S_t^j \geq \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

Application 3: Existence of a risk-free asset

- Consider a bond, issued at every time $t = 0, \ldots, T-1$, with the maturity of one period and paying a (deterministic) payoff with unit face value
- Then

$$S_t^f = \frac{1}{1+r_{t+1}}, \; V_{t+1}^f = \mathbf{1}, \; V_{t+2}^f = \mathbf{0}, \ldots \, V_T^f = \mathbf{0}$$

- If there are no arbitrage opportunities then

$$\frac{1}{1+r_{t+1}} = \mathbb{E}_t \left[M_{t+1|t} \right].$$

Application 3: Existence of a risk-free asset

- For any other risky asset, j, we can write

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left[S_{t+1}^j + V_{t+1}^j\right] &= \frac{\mathbb{E}_t\left[M_{t+1|t}\left(S_{t+1}^j + V_{t+1}^j\right)\right]}{\mathbb{E}_t\left[M_{t+1|t}\right]} = \\ &= \left(1 + r_{t+1}\right)\mathbb{E}_t\left[M_{t+1|t}\left(S_{t+1}^j + V_{t+1}^j\right)\right] = \\ &= \left(1 + r_{t+1}\right)S_t^j \end{aligned}$$

- Then

$$S_t^j = \frac{1}{1 + r_{t+1}}\mathbb{E}_t^{\mathbb{Q}}\left[S_{t+1}^j + V_{t+1}^j\right]$$

- Using the definition of return for asset j this is equivalent to

$$R_{t+1}^f = \mathbb{E}_t^{\mathbb{Q}}\left[R_{t+1}^j\right]$$

there are no arbitrage opportunities if there is a probability process $\mathbb{Q}$ such that the expected return for a risky asset is equal to the return of the riskless asset ($R_t^f = 1 + r_t$)

- It can also be proved that

$$S_t^j = \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{\tau=t+1}^{T} D_{t+1,\tau} \, V_\tau^j \right]$$

the asset price at time t is the conditional expected value of the present value of the future payoffs;

- where the discount factor is

$$D_{t+1,\tau} = \prod_{h=t+1}^{\tau} \frac{1}{1+r_h}, \ \tau \geq t+1.$$

- Exercise: prove this.

Equilibrium asset pricing

Real part of the economy: resources

- There is a **given** sequence of endowments

$$Y^T \equiv \{ Y_t \}_{t=0}^{T} = \{ Y_0, Y_1, \dots, Y_t, \dots, Y_T \}$$

- where Y_t is $\mathcal{F}_t$- mensurable, such that

$$Y_t = \begin{pmatrix} y_{t,1} \\ \dots \\ y_{t,N_t} \end{pmatrix}$$

Equilibrium asset pricing

Real part of the economy: preferences and distribution

- Consumers choose a contingent-consumption sequence belonging to the set

$$C^T \equiv \{C_t\}_{t=0}^T = \{C_0, C_1, \ldots, C_t, \ldots, C_T\}$$

where C_t is $\mathcal{F}_t$- mensurable,

- through an intertemporal von-Neumman-Morgenstern functional

$$\mathbb{E}_0\left[\sum_{t=0}^{T} \beta^t u(C_t)\right]$$

- expansion of the utility functional

$$\begin{aligned}\mathbb{E}_0\left[\sum_{t=0}^{T} \beta^t u(C_t)\right] &= \sum_{t=0}^{T} \beta^t \mathsf{P}_t u(C_t) = \\ &= u(C_0) + \beta \mathsf{P}_1 u(C_1) + \ldots + \beta^t \mathsf{P}_t u(C_t) + \ldots + \beta^T \mathsf{P}\end{aligned}$$

where

$$\mathsf{P}_t u(C_t) = \sum_{s=1}^{N_t} \pi_{t,s} u(c_{t,s})$$

Equilibrium asset pricing: Zero initial wealth

Assumption: the level of initial net wealth is zero.

Consequence: we can transform the consumer problem in a finance economy into the consumer problem in an equivalent AD economy

Non-zero initial wealth: we have to apply other methods for solving the consumer-investor problem (v.g, dynamic programming or optimal control, see next)

Equilibrium asset pricing: Zero initial wealth

Radner or sequential general equilibrium

Definition *The* **Radner or sequential general equilibrium** *is defined by the processes* $\{C_t\}_{t=0}^T$, $\{\theta_t\}_{t=1}^T$ *and* $\{S_t\}_{t=0}^{T-1}$ *such that,* ***given*** *the processes of endowments* $\{Y_t\}_{t=0}^T$ *and payoffs* $\{V_t\}_{t=1}^T$:
(1) the consumer solves his ***consumption-portfolio problem****, with rational expectations regarding future asset prices, and*
(2) the ***markets clear****,*

$$\begin{aligned} C_t &= Y_t,\ t = 0, \dots, T \\ \theta_t &= 0,\ t = 1, \dots, T. \end{aligned}$$

Observations:

- there is a information space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^T$
- C_t, Y_t and V_t are $\mathcal{F}_t$-adapted
- $\theta_t$is $\mathcal{F}_t$-predictable

Equilibrium asset pricing: Zero initial wealth

The (sequential) consumer-investor problem

Find the process for consumption $\{C_t\}_{t=0}^{T}$ and a transactions' strategy $\{\theta_t\}_{t=1}^{T}$

- that maximizes the value functional

$$V_0(\{C_t\}, \{\theta_t\}) \equiv \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) \right]$$

- subject to the sequential budget constraints

$$\begin{aligned} C_0 + \sum_{j=1}^{K} \theta_1^j S_0^j &\leq Y_0 \\ &\cdots \\ C_t + \sum_{j=1}^{K} \theta_{t+1}^j S_t^j &\leq Y_t + \sum_{j=1}^{K} \theta_t^j (S_t^j + V_t^j),\ t = 1, \ldots, T-1\ (\mathcal{F}_t - \text{adapt} \\ &\cdots \\ C_T &\leq Y_T + \sum_{j=1}^{K} \theta_T^j V_T^j\ (\mathcal{F}_T - \text{adapted}) \end{aligned}$$

Equilibrium asset pricing: Zero initial wealth

The (sequential) consumer problem

- We can write the sequence of budget constraints equivalently as

$$\begin{aligned} C_0 &\leq Y_0 + Z_0^\theta \\ &\cdots \\ C_t &\leq Y_t + Z_t^\theta,\ t = 1, \ldots, T-1\ (\mathcal{F}_t - \text{adapted}) \\ &\cdots \\ C_T &\leq Y_T + Z_T^\theta\ (\mathcal{F}_T - \text{adapted}) \end{aligned}$$

  where Z_t^θ is the income generated at time t by the transaction strategy $\{\theta_t\}_{t=1}^T$.

- If the utility function $u(.)$ displays no-satiation the constraints hold with equality in the optimum.

Equilibrium asset pricing: Zero initial wealth

Equivalent simultaneous consumer problem

- **If there are no arbitrage opportunities**, then there is stochastic discount factor process $\{M_t\}_{t=0}^{T-1}$, such that

$$-\mathbb{E}_0\left[\sum_{t=0}^{T} M_t Z_t^{\theta}\right] = \mathbb{E}_0\left[\sum_{t=0}^{T} M_t(Y_t - C_t)\right] = 0.$$

- Then, the consumer's problem is the same as in the AD economy

$$\max_{\{C_t\}_{t=0}^{T}} \mathbb{E}_0\left[\sum_{t=0}^{T} \beta^t u(C_t)\right] \text{ s.t } \mathbb{E}_0\left[\sum_{t=0}^{T} M_t(Y_t - C_t)\right] = 0$$

- We already found the f.o.c

$$u^{'}(C_0)M_t = \beta^t u^{'}(C_t),\ (\mathcal{F}_t - \text{adapted})$$

Equilibrium asset pricing: Zero initial wealth

Equilibrium stochastic discount factor

- The consumer arbitrage condition and the market equilibrium conditions

$$\begin{cases} u'(C_0)M_t = \beta^t u'(C_t) & t = 1, \dots, T \\ C_t = Y_t & t = 0, \dots, T \end{cases}$$

- imply that, at equilibrium, as in the AD economy

$$\boxed{M_t = \beta^t \frac{u'(Y_t)}{u'(Y_0)} \ (\mathcal{F}_t - \text{adapted})}$$

- In terms of the possible realizations

$$M_t = \begin{pmatrix} m_{t,1} \\ \dots \\ m_{t,N_t} \end{pmatrix}, \ t = 0, \dots, T-1$$

where

$$m_{ts} = \beta^t \frac{u'(y_{ts})}{u'(y_0)}, \ \text{for } s = 1, \dots, N_t, \text{and}, \ t = 0, \dots T-1.$$

Equilibrium asset pricing: Zero initial wealth

- If there are no arbitrage opportunities, we proved that, for any asset j

$$M_t S_t^j = \mathbb{E}_t \left[M_{t+1} (V_{t+1}^j + S_{t+1}^j) \right], \quad j = 1, \ldots, K$$

- Then the **GE equilibrium** asset pricing is

$$\boxed{u^{'}(Y_t) S_t^j = \beta \mathbb{E}_t \left[u^{'}(Y_{t+1}) (V_{t+1}^j + S_{t+1}^j) \right], \quad j = 1, \ldots, K}$$

- determines asset price process $\{S_t^j\}$ given the processes $\{V_t^j\}$ and $\{Y_t\}$.

Equilibrium asset pricing: Zero initial wealth

Equivalent representations:

1. The **equilibrium rate of return** for asset j is determined from

$$\boxed{\mathbb{E}_t\left[M_{t+1|t}R^j_{t+1}\right] = 1}$$

where the **equilibrium recursive stochastic discount factor** is

$$M_{t+1|t} \equiv \beta \frac{u'(Y_{t+1})}{u'(Y_t)}$$

and the return is

$$R^j_{t+1} = \frac{V^j_{t+1} + S^j_{t+1}}{S^j_t}$$

2. or, equivalently

$$\boxed{u'(Y_0)S^j_0 = \mathbb{E}_0\left[\sum_{t=1}^{T}\beta^t u'(Y_t)V^j_t\right], \quad j = 1, \dots, K.}$$

Infinite horizon case, $T = \infty$

- The arbitrage condition is, off course, still valid.
- Fundamental equilibrium arbitrage condition: if we rule out speculative bubbles, then the price for asset j verifies

$$S^j_t = \mathbb{E}_t \left[\sum_{\tau=1}^{\infty} \beta^{\tau} \frac{u^{'}(C_{t+\tau})}{u^{'}(C_t)} V^j_{t+\tau} \right], \quad j = 1, \ldots, K, \quad t \in [0, \infty) \qquad (1)$$

Proof:

$$\begin{aligned} u^{'}(C_t) S^j_t &= \beta \mathbb{E}_t \left[u^{'}(C_{t+1})(S^j_{t+1} + V^j_{t+1}) \right] = \\ &= \lim_{k \to \infty} \beta^k \mathbb{E}_t \left[u^{'}(C_{t+k}) S^j(t+k) \right] + \\ & \quad + \mathbb{E}_t \left[\sum_{\tau=1}^{\infty} \beta^{\tau} u^{'}(C_{t+\tau}) V^j(t+\tau) \right] \end{aligned}$$

and if we rule out speculative bubbles, that is

$$\lim_{k \to \infty} \beta^k \mathbb{E}_t \left[u^{'}(C_{t+k}) S^j_{t+k} \right] = 0$$

equation (1)

Risky and risk-free assets

- For a risky asset
$$\mathbb{E}_t\left[M_{t+1|t}R^j_{t+1}\right] = 1$$

- For a riskless asset with return $R^f_t = 1 + r^f_t$ we have
$$\mathbb{E}_t\left[M_{t+1|t}\right]R^f_{t+1} = 1$$

- Then, for any asset j
$$\mathbb{E}_t\left[M_{t+1|t}R^j_{t+1}\right] = \mathbb{E}_t\left[M_{t+1|t}\right]R^f_{t+1}$$

Equilibrium equity premium: example

Equilibrium risk premium for a Markovian case

- **Assumptions:**
  1. Homogeneous agent finance economy
  2. CRRA Bernoulli utility function
  3. growth factor for the return is Markovian following an iid log-normal distribution
  4. there is one riskless and one risky asset such that the return is Markovian following an iid log-normal distribution
- Problem: **Derive the distribution for the multiplicative risk premium for the risky asset** R^j/R^f
- Solution: the risk premium for asset j, verifies

$$\boxed{\ln \mathbb{E}_t[R^j_{t+1}] = \ln R^f_{t+1} + \zeta \text{Cov}_t \left[\ln(1+\gamma_{t+1}), \ln R^j_{t+1}\right],\ j = 1, \dots, K}$$

Auxiliary: log-normal distributions

Some properties

Assume two random variables X and Y following log-normal distributions. Then $\ln X$ and $\ln Y$ are normally distrbuted. Then:

$$\ln \mathbb{E}[X] = \mathbb{E}[\ln X] + \frac{1}{2}\mathbb{V}[\ln X]$$

$$\ln \mathbb{E}[\alpha X] = \ln \alpha + \mathbb{E}[\ln X] + \frac{1}{2}\mathbb{V}[\ln X],\ \alpha \text{ constant}$$

$$\ln \mathbb{E}[\alpha X^{\beta}] = \ln \alpha + \beta\mathbb{E}[\ln X] + \frac{\beta^2}{2}\mathbb{V}[\ln X],\ \alpha, \beta,\ \text{constants}$$

$$\ln \mathbb{E}[XY] = \mathbb{E}[\ln X] + \mathbb{E}[\ln Y] + \frac{1}{2}\left\{\mathbb{V}[\ln X] + \mathbb{V}[\ln Y] - 2\,Cov(\ln X, \ln Y)\right\}$$

$$\ln \mathbb{E}[X^{\beta} Y] = \beta\mathbb{E}[\ln X] + \mathbb{E}[\ln Y] + \\ + \frac{1}{2}\left\{\beta^2\mathbb{V}[\ln X] + \mathbb{V}[\ln Y] - 2\beta\,Cov(\ln X, \ln Y)\right\}$$

because $Cov[\beta X, Y] = \beta\, Cov[XY]$.

Equilibrium equity premium example: proof

solution

- The risky asset j follows a iid log-normal distribution: then

$$\ln \mathbb{E}_t[R^j_{t+1}] = \mathbb{E}_t[\ln R^j_{t+1}] + \frac{1}{2}\mathbb{V}_t[\ln R^j_{t+1}]$$

- the endowment process verifies $Y_{t+1} = (1+\gamma_{t+1})\, Y_t$, where the growth factor follows also a iid log-normal distribution: then

$$\ln \mathbb{E}_t[1+\gamma_{t+1}] = \mathbb{E}_t[\ln(1+\gamma_{t+1})] + \frac{1}{2}\mathbb{V}_t[\ln(1+\gamma_{t+1})]$$

- the utility function is CRRA $u(C) = (1-\zeta)^{-1}\, C^{1-\zeta}$ then the stochastic discount factor is

$$M_{t+1|t} = \beta(1+\gamma_{t+1})^{-\zeta}$$

- then, for any asset, the arbitrage condition holds as

$$1 = \beta\mathbb{E}_t\left[(1+\gamma_{t+1})^{-\zeta}\, R^j_{t+1}\right]$$

Equilibrium equity premium example: proof

solution (cont.)

- for the riskless asset, after taking logs to the arbitrage condition, we have

$$
\begin{aligned}
0 &= \ln\beta + \ln\mathbb{E}_t\left[(1+\gamma_{t+1})^{-\zeta} R^f_{t+1}\right] \\
&= \ln\beta + \mathbb{E}_t\left[\ln\left((1+\gamma_{t+1})^{-\zeta} R^f_{t+1}\right)\right] + \\
&\quad + \frac{1}{2}\mathbb{V}_t\left[\ln\left((1+\gamma_{t+1})^{-\zeta} R^f_{t+1}\right)\right] \\
&= \ln\beta - \zeta\mathbb{E}_t[\ln(1+\gamma_{t+1})] + \ln R^f_{t+1} + \frac{\zeta^2}{2}\mathbb{V}_t[\ln(1+\gamma_{t+1})]
\end{aligned}
$$

Equilibrium equity premium example: proof

solution (cont.)

- for the risky asset j, after taking logs to the arbitrage condition, we have

$$
\begin{aligned}
0 &= \ln\beta + \ln\mathbb{E}_t\left[(1+\gamma_{t+1})^{-\zeta} R^j_{t+1}\right] = \\
&= \ln\beta - \zeta\mathbb{E}_t[\ln(1+\gamma_{t+1})] + \mathbb{E}_t[\ln R^j_{t+1}] + \\
&\quad + \frac{1}{2}\Big\{\zeta^2\mathbb{V}_t[\ln(1+\gamma_{t+1})] + \mathbb{V}_t[\ln R^j_{t+1}] - \\
&\quad -2\zeta\mathrm{Cov}_t\left[\ln(1+\gamma_{t+1}), \ln R^j_{t+1}\right]\Big\} = \\
&= -\ln R^f_{t+1} + \mathbb{E}_t[\ln R^j_{t+1}] + \frac{1}{2}\mathbb{V}_t[\ln R^j_{t+1}] - \\
&\quad -\zeta\mathrm{Cov}_t\left[\ln(1+\gamma_{t+1}), \ln R^j_{t+1}\right] = \\
&= -\ln R^f_{t+1} + \ln\mathbb{E}_t\left[R^j_{t+1}\right] - \zeta\mathrm{Cov}_t\left[\ln(1+\gamma_{t+1}), \ln R^j_{t+1}\right].
\end{aligned}
$$

(end of proof)

Equilibrium equity premium

Hansen-Jaganathan bounds

- Let us write the **Equity premium** for asset risky j as: $R_{t+1}^{j} - R_{t+1}^{f}$
- Expected premium and standard deviation

$$\mathbb{E}_t\left[R_{t+1}^{j} - R_{t+1}^{f}\right],\ \sigma_t\left[R_{t+1}^{j} - R_{t+1}^{f}\right]$$

- **Equilibrium equity premium** for risky asset j verifies, under the assumptions of the model:

$$\mathbb{E}_t\left[M_{t+1|t}\left(R_{t+1}^{j} - R_{t+1}^{f}\right)\right] = 0$$

- Then,

$$\boxed{\left|\frac{\mathbb{E}_t\left[R_{t+1}^{j} - R_{t+1}^{f}\right]}{\sigma_t\left[R_{t+1}^{j}\right]}\right| \leq \frac{\sigma_t[M_{t+1|t}]}{\mathbb{E}_t[M_{t+1|t}]}} \qquad (2)$$

the l.h.s is called the Sharpe ratio and r.h.s. the Hansen-Jaganathan bounds

Equilibrium equity premium

Hansen-Jaganathan bounds

- Proof:
- From a standard result on the covariance between two random variables

$$
\begin{aligned}
& \mathbb{E}_t\left[M_{t+1|t}\left(R_{t+1}^j - R_{t+1}^f\right)\right] = \\
= \ & \mathbb{E}_t\left[M_{t+1|t}\right]\mathbb{E}_t\left[R_{t+1}^j - R_{t+1}^f\right] + \mathrm{Cov}_t\left[M_{t+1|t}, \left(R_{t+1}^j - R_{t+1}^f\right)\right] = \\
= \ & 0
\end{aligned}
$$

- But

$$
\begin{aligned}
\mathrm{Cov}_t & \left[M_{t+1|t}, \left(R_{t+1}^j - R_{t+1}^f\right)\right] \\
& = \rho_{M_{t+1|t}, R_{t+1}^j - R_{t+1}^f}\,\sigma_t(M_{t+1|t})\,\sigma_t\left(R_{t+1}^j - R_{t+1}^f\right) \qquad (3)
\end{aligned}
$$

where $\rho_{M_{t+1|t}, R_{t+1}^j - R_{t+1}^f}$ is the correlation coefficient between $M_{t+1|t}$ and $R_{t+1}^j - R_{t+1}^f$

Equilibrium equity premium

Hansen-Jaganathan bounds (cont.)

- Then

$$\frac{\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right]}{\sigma_t\left[R^j_{t+1}\right]} = \rho_{M_{t+1|t}, R^j_{t+1} - R^f_{t+1}} \frac{\sigma_t[M_{t+1|t}]}{\mathbb{E}_t[M_{t+1|t}]}$$

- We use the fact $|\rho_{M_{t+1|t}, R^j_{t+1} - R^f_{t+1}}| \in [0, 1]$

Equilibrium equity premium

Example

- If we assume that $Y_{t+1} = (1+\gamma_{t+1})\, Y_t$, the utility function is homogeneous, and $R^f_{t+1} \approx 1/\beta$ then

$$\left| \frac{\mathbb{E}_t\left[R^j_{t+1} - R^f_{t+1}\right]}{\sigma_t\left[R^j_{t+1}\right]} \right| \leq \sigma_t\left[u^{'}(1+\gamma_{t+1})\right]$$

- if the utility function is homogeneous, from the equilibrium arbitrage condition

$$\beta \mathbb{E}_t[u^{'}(1+\gamma_{t+1})] R^f_{t+1} = 1$$

- if $R^f_{t+1} \approx 1/\beta$ then

$$\mathbb{E}_t[u^{'}(1+\gamma_{t+1})] = 1$$

Equilibrium equity premium: example

- If we ssume a CRRA utility function

$$u(C) = \frac{C^{1-\zeta}}{1-\zeta}$$

Then

$$M_{t+1|t} = \beta(1+\gamma_{t+1})^{-\zeta}$$

$$\sigma_t\left[u^{'}(1+\gamma_{t+1})\right] = \sigma_t\left[(1+\gamma_{t+1})^{-\zeta}\right]$$

The higher η the lower $\sigma_t[M_{t+1|t}]$ is.

Equilibrium equity premium puzzle

- **Equity premium puzzle**: if we set $\zeta \approx 2$, we find excessive risk premium in the data:

$$\text{Sharpe ratio} = 0.37 > \frac{\sigma_t[M_{t+1|t})]}{\mathbb{E}_t[M_{t+1|t}]} \approx \frac{0.002}{0.96}$$

- This means that the data displays a higher risk premium than the model would predict (or consumption displays a lower relative volatility than the model predicts)

Equilibrium equity premium puzzle

- This has led to a whole research program (still going on) for macro finance: see `http://academicwebpages.com/preview/mehra/pdf/FIN200201.pdf` for a survey, by introducing in the model:
  - changes in preferences: habit formation, non-additive preferences concerning risk
  - transactions costs, taxes, etc
  - distributions
  - imperfectly competitive environments
- The basic change we have to introduce should do the following: consumption (and investment) should have a smoother behaviour than the model predicts, which means that the reaction of portfolios to changes in asset prices is more rigid, which implies a higher variation in prices to unpredicted shocks.