Foundations of Financial Economics
Multi-period GE: Arrow-Debreu economy

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Multiperiod Arrow-Debreu economy

- In this lecture we extend the DSGE for an Arrow-Debreu economy from 2 to n periods
- Now all the variables are stochastic processes
- Including the AD price and the stochastic discount factor (SDF)
- We will also define a recursive stochastic discount factor, which is similar to what we found for the 2-period case

Topics

- Information structure
- Real part of the economy
- Market structure
- Definition of the DSGE
- Determination of the DGSE
- Characterization
- Recursive stochastic discount factor

Information structure

Information structure

- There is an information tree, with T periods,
- the information tree comprises a sequence of nodes $\{N_t\}_{t=1}^{T} = \{N_1, N_2, \ldots, N_s, \ldots, N_T\}$, where N_t is the number of nodes of the information tree at time t
- **Example**: for a binomial process $N_t = 2^t$
- there is a sequence of unconditional probabilities

$$\mathbb{P}^T \equiv \{\mathsf{P}_t\}_{t=1}^{T} = \{\mathsf{P}_1, \ldots, \mathsf{P}_t, \ldots, \mathsf{P}_T\}$$

  where $\mathsf{P}_t = \left(\pi_{t,1}, \ldots, \pi_{t,s}, \ldots, \pi_{t,N_t}\right)$

- for any process $\{X_t\}_{t=0}^{T} = \{X_0, X_1, \ldots X_t, \ldots, X_T\}$ we assume that X_t is $\mathcal{F}_t$-adapted (as we say in the slide "Introduction to stochastic processes")

Information structure

- **Example** for a binomial tree

$$\mathbb{P}^T = \left\{ 1, \begin{pmatrix} \pi_{1,1} \\ \pi_{1,2} \end{pmatrix}, \begin{pmatrix} \pi_{2,1} \\ \pi_{2,2} \\ \pi_{2,3} \\ \pi_{2,4} \end{pmatrix}, \begin{pmatrix} \pi_{3,1} \\ \cdots \\ \pi_{3,8} \end{pmatrix}, \ldots \right\}$$

- If the process is Markovian, with transition probabilities $\{p, 1-p\}$ for $0 < p < 1$

$$\mathbb{P}^T = \left\{ 1, \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} p^2 \\ p\,(1-p) \\ (1-p)\,p \\ (1-p)^2 \end{pmatrix}, \begin{pmatrix} p^3 \\ p^2(1-p) \\ \cdots \\ (1-p)^2\,p \\ (1-p)^3 \end{pmatrix} \ldots . \right\}$$

Real part of the economy

Real part of the economy: resources

- **Households are homogeneous** regarding information, technology (endowments) and preferences
- Therefore:
- **The information structure is common knowledge**
- **Endowments**: are exogenously given by the stochastic process

$$\mathbf{Y}^T \equiv \{ Y_t \}_{t=0}^T = \{y_0, Y_1, \ldots, Y_t, \ldots, Y_T\}$$

- where Y_t is $\mathcal{F}_t$- mensurable, such that

$$Y_t = \begin{pmatrix} y_{t,1} & \ldots & y_{t,N_t} \end{pmatrix}^\top$$

- **Example** for a binomial tree

$$\mathbf{Y}^T = \left\{ y_0, \begin{pmatrix} y_{1,1} \\ y_{1,2} \end{pmatrix}, \begin{pmatrix} y_{2,1} \\ y_{2,2} \\ y_{2,3} \\ y_{2,4} \end{pmatrix}, \begin{pmatrix} y_{3,1} \\ \ldots \\ y_{3,8} \end{pmatrix} \ldots \right\}$$

Real part of the economy: preferences

- **Preferences**; are represented by an intertemporal von-Neumman-Morgenstern functional

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) \right]$$

- where $\beta \in (0, 1)$ and $u(.)$ is increasing, concave and Inada
- Consumers choose a contingent-consumption sequence

$$\mathbf{C}^T \equiv \{C_t\}_{t=0}^{T} = \{c_0, C_1, \ldots, C_t, \ldots, C_T\}$$

where C_t is $\mathcal{F}_t$- mensurable,

Real part of the economy: preferences

- Observe that

$$
\begin{aligned}
\mathbb{E}_0\left[\sum_{t=0}^{T}\beta^t u(C_t)\right] &= \sum_{t=0}^{T}\beta^t \mathsf{P}_t u(C_t) = \\
&= u(C_0) + \ldots + \beta^t \mathsf{P}_t u(C_t) + \ldots + \beta^T \mathsf{P}_T u(C_T)
\end{aligned}
$$

- where

$$
\mathsf{P}_t u(C_t) = \sum_{s=1}^{N_t} \pi_{t,s} u(c_{t,s})
$$

and P_t are **unconditional** probability distributions, i.e., taken at time $t = 0$

Real part of the economy: preferences

Example for a binomial treeConsumption process

$$\mathbf{C}^T = \left\{ c_0, \begin{pmatrix} c_{1,1} \\ c_{1,2} \end{pmatrix}, \begin{pmatrix} c_{2,1} \\ c_{2,2} \\ c_{2,3} \\ c_{2,4} \end{pmatrix}, \begin{pmatrix} c_{3,1} \\ \dots \\ c_{3,8} \end{pmatrix} \dots \right\}$$

Utility process

$$\left\{ u(c_0), \begin{pmatrix} u(c_{1,1}) \\ u(c_{1,2}) \end{pmatrix}, \begin{pmatrix} (c_{2,1}) \\ u(c_{2,2}) \\ u(c_{2,3}) \\ u(c_{2,4}) \end{pmatrix}, \begin{pmatrix} u(c_{3,1}) \\ \dots \\ u(c_{3,8}) \end{pmatrix} \dots \right\}$$

Utility functional

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) \right] = u(c_0) + \beta \sum_{s=1}^{2} \pi_{1,s}\, u(c_{1,s}) + \beta^2 \sum_{s=1}^{4} \pi_{2,s}\, u(c_{2,s}) + \beta^3 \sum_{s=1}^{8} \pi_{3,s}\, u(c_{3,s}) + \dots \quad (1)$$

is a scalar

- Market structure

Arrow-Debreu contingent claims

- There is a large number of **Arrow-Debreu contingent claims**, **traded** only at time $t = 0$, offering one unit of the good for **delivery** at every node of the information tree for $t = 1, \ldots, N_t$
- this means there is:
  1. one **spot** market taken as the numeraire: $Q_0 = 1$
  2. $\sum_{t=1}^{T} N_t = N_1 + \ldots + N_t + \ldots + N_T$ **AD markets** with prices

$$\mathbf{Q}^T \equiv \{Q_t\}_{t=0}^T = \{q_0, Q_1, \ldots, Q_t, \ldots, Q_T\}$$

where

$$Q_t = \begin{pmatrix} q_{t,1} \\ \ldots \\ q_{t,N_t} \end{pmatrix}, \text{ i.e. } Q_t \text{ is } \mathcal{F}_t\text{- mensurable}$$

Dynamic stochastic general equilibrium

Arrow-Debreu equilibrium

For a representative household economy

Definition: An Arrow-Debreu equilibrium is the process $(\mathbf{C}^T, \mathbf{Q}^T)$, that is, it is the collection of $\mathcal{F}_t-$adapted processes for consumption $\{C_t\}_{t=0}^T$ and AD-prices $\{Q_t\}_{t=1}^T$ such that, given the $\mathcal{F}_t-$adapted process $Y^T = \{Y_t\}_{t=0}^T$:

1. consumers problem determine $\{C_t\}_{t=0}^T$ by solving

$$\max_{\{C_t\}_{t=0}^T} \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) \right] \text{ s.t. } \sum_{t=0}^{T} Q_t C_t \leq \sum_{t=0}^{T} Q_t Y_t$$

   given $\{Y_t\}_{t=0}^T$ and $\{Q_t\}_{t=1}^T$
2. and markets clear

$$C_t = Y_t, \quad t = 0, \ldots, T$$

- T can be finite or $T = \infty$

The budget constraint

Observe that:

- the budget constraint is equivalent to

$$\sum_{t=0}^{T} Q_t(Y_t - C_t) =$$

$$= Q_0(Y_0 - C_0) + \ldots + Q_t(Y_t - C_t) + \ldots + Q_T(Y_T - C$$

  where

$$Q_t(Y_t - C_t) = \sum_{s=1}^{N_t} q_{t,s}(y_{t,s} - c_{t,s})$$

- If we define the 0-period **unconditional stochastic discount factor** for period t as

$$M_t \equiv Q_t / \mathsf{P}_t$$

  where $M_t = (m_{t,1}, \ldots, m_{t,N_t})$

$$m_{t,s} = \frac{q_{t,s}}{\pi_{t,s}}, \; s = 1, \ldots, N_t$$

The budget constraint (cont)

- Then the **instantaneous budget constraint** at time $t = 0$, is equivalent to

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} M_t \left(Y_t - C_t \right) \right] \geq 0$$

- where

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} M_t \left(Y_t - C_t \right) \right] = M_0 (Y_0 - C_0) + \ldots + \mathsf{P}_T M_T (Y_T - C_T)$$

The solution of the consumer problem

- We can write the Lagrangean as

$$\begin{aligned}\mathcal{L} &= \mathbb{E}_0\Big[\sum_{t=0}^{T} \beta^t u(C_t)\Big] + \lambda\,\mathbb{E}_0\Big[\sum_{t=0}^{T} M_t(Y_t - C_t)\Big] \\ &= \mathbb{E}_0\Big[\sum_{t=0}^{T} \beta^t u(C_t) + \lambda\, M_t(Y_t - C_t)\Big]\end{aligned}$$

(linearity of the $\mathbb{E}_0$ operator)

- or equivalently

$$\mathcal{L} = \sum_{t=0}^{T}\sum_{s=1}^{N_t} \pi_{t,s}\left\{\beta^t u(c_{t,s}) + \lambda m_{t,s}(y_{t,s} - c_{t,s})\right\}$$

First order conditions

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{t,s}} &= \mathbf{0},\ s = 1, \ldots N_t,\ t = 0, \ldots T,\ \left(\sum_{t=0}^{T} N_t \text{dimensional} \right) \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 0\ \text{(1 dimensional)}
\end{aligned}$$

Solution of the consumer's problem

First-order conditions for optimality

$$
\begin{aligned}
u^{'}(c_0^*) &= \lambda \ \text{(1 equation)} \\
\beta u^{'}(c_{1,s}^*) &= \lambda m_{1,s}, \ s = 1, \ldots N_1 \ \ (N_1 \text{ equations}) \\
& \ldots \\
\beta^t u^{'}(c_{t,s}^*) &= \lambda m_{t,s}, \ s = 1, \ldots N_t \ \ (N_t \text{ equations}) \\
& \ldots \\
\beta^T u^{'}(c_{T,s}^*) &= \lambda m_{T,s}, \ s = 1, \ldots N_T \ \ (N_T \text{ equations}) \\
\sum_{t=0}^{T} \sum_{s=1}^{N_t} \pi_{t,s} m_{t,s} c_{t,s}^* &= H_0 \equiv \sum_{t=0}^{T} \sum_{s=1}^{N_t} \pi_{t,s} m_{t,s} y_{t,s} \ \ \text{(1 equation)}
\end{aligned}
$$

where H_0 is human wealth equal to the expected discounted present value (in market prices) of the future stream of endowments.

Characterization

Equilibrium conditons for a homogeneous agent economy

- The **Euler equation for consumption** is, because $u^{'}(c_0^*) = \lambda$

$$m_{t,s}u^{'}(c_0^*) = \beta^t u^{'}(c_{t,s}^*), \quad s = 1, \ldots N_t, \quad t = 0, \ldots, T.$$

- The equilibrium conditions are (**in this homogeneous-agent model**)

$$c_{t,s}^* = y_{t,s}, \quad s = 1, \ldots N_t, \quad t = 0, \ldots, T.$$

Equilibrium stochastic discount factor

- Then the **equilibrium unconditional stochastic discount factor** (SDF) is a stochastic process $\{M_t\}_{t=0}^{T}$ such that $M_0 = m_0 = 1$ and $M_t = (m_{t,1}, \dots, m_{t,N_t})^\top$ where

$$\boxed{M_t^* = \beta^t \frac{u'(Y_t)}{u'(Y_0)}, \; t = 0, \dots T}$$

$$M_t^* = \begin{pmatrix} m_{t,1} \\ \dots \\ m_{t,N_t} \end{pmatrix}$$

- or, equivalently the possible realizations of the unconditional stochastic discount factor are

$$\boxed{m_{t,s}^* = \beta^t \frac{u'(y_{ts})}{u'(y_0)}, \quad s = 1, \dots, N_t, \quad t = 0, \dots T}$$

Equilibrium stochastic discount factor

Definition: **recursive stochastic discount factor** for period $t+1$ conditional on period t

$$M_{t+1|t} = \frac{M_{t+1}}{M_t}$$

where

$$M_{t+1|t} = \begin{pmatrix} m_{t+1|t,1} \\ \dots \\ m_{t+1|t,s} \\ \dots \\ m_{t+1|t,N_{t,t+1}} \end{pmatrix} = \begin{pmatrix} \mu_{t+1,1} \\ \dots \\ \mu_{t+1|s} \\ \dots \\ \mu_{t+1|N_{t,t+1}} \end{pmatrix}$$

where $N_{t,t+1}$ is the number of nodes taken an time t for all the subsequent nodes at time $t+1$

Equilibrium stochastic discount factor

- The **equilibrium recursive stochastic discount factor** (RSDF) for period $t+1$ conditional on period t is

$$\boxed{M^*_{t+1|t} = \beta \frac{u'(Y_{t+1})}{u'(y_t)}}$$

where the realization of Y_t at time t is $Y_t = y_t$

- Has possible realizations

$$\boxed{\mu_{t+1} = m^*_{t+1|t,s} = \beta \frac{u'(y_{t+1,s})}{u'(y_t)}, \; s = 1 \dots N_{t,t+1}}$$

- These relations hold for T finite or infinite
- Observation: this RSDF is similar to what we have studied for the two-period case.

Equilibrium stochastic discount factor: statistics

- Given the probability and the endowment process (assuming that they are both adapted stochastic processes)

$$\mathbf{P} = \{1, \mathsf{P}_1, \ldots, \mathsf{P}_t, \ldots\} \text{ and } \mathbf{Y} = \{y_0, Y_1, \ldots, Y_t, \ldots\}$$

- we can calculate statistics for the stochastic discount factor

$$\mathbb{E}_t\big[M_{t+1}\big] = \beta\, \mathbb{E}_t\Big[\frac{u'(Y_{t+1})}{u'(y_t)}\Big] = \frac{\beta}{u'(y_t)}\, \mathbb{E}_t\Big[u'(Y_{t+1})\Big]$$