The Ramsey and the benchmark DGE models in continuous time

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16.11.2022

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1 Introduction

In this note we present the simplest version of the dominant current view in macroeconomics, the dynamic general equilibrium framework.

We can distinguish between two views on macroeconomics, assuming that its birth, as a separate field in economics, started after the great depression of the late 1920's.

There is a dilemma in macroeconomic modelling between the consistency, at the micro level, of the actions of agents in all the markets they participate, and the aggregation and coordination of agents, at the macro level. This dilemma has been addressed in two different ways along the history of macroeconomic modelling [1]

The IS-LM model, first presented by John Hicks in the 1950's, stressed the aggregative aspect by sacrificing micro consistency. The model central core is based on market equilibrium relationships between the main macroeconomic aggregates, in which the behavioral functions are introduced separately. For instance, the consumption function is dependent on income, the investment function is dependent on the interest rate and the demand for money is dependent on income and the interest rate, but it does not consider that all those decisions are taken simultaneously and are tied down by financial constraints at the household level.

A different approach, which we now call the dynamic general equilibrium (DGE) approach, that emerged in the late 1960's and became dominant in the 1990's, is built around the idea of getting the micro consistency right, at the cost of modelling the behavior of an aggregate economy as analogous to the behavior of a single agent. The actions of agents in different markets are modelled by specific (dynamic) microeconomic models, which are made consistent at the aggregate level by market equilibrium conditions. That consistency was achieved at the cost of some unrealism by casting away complexities generated by aggregating heterogenous agents.

However, in the last decade, this approach is evolving by considering heterogeneities (in behavior, information and resources) among agents, and dealing with aggregation problems, at the cost of a huge increase in the mathematical complexity of the models.

In this note we provide a simple introduction to the simplest DGE modelsin section 2.

Section 3 presents the centralized version of the model. The decentralized version is presented in section 4. In both of the previous section the labor supply is exogenous. Section 5 presents a endogenous labor version of the model.

2 The basic DGE model and the stylised facts

A specialization has emerged between macroeconomics and growth economics, which is not clear in some macroeconomic textbooks (as Romer, 2019 or Alogoskoufis, 2019), but it is clear in growth textbooks (see Acemoglu, 2009).

For the purpose of this course, we want to make this distinction explicit. While growth economics deals with **trends** of the activity in the economy, in particular with average rates of growth for long periods of time, macroeconomics deals with the deviations from that trend, the **cycle** not necessarily regular. This means that while a growth model has to generate a theory on the **rate**

[1] This dilemma is not unique to economics.

of growth of the economy, a macromodel has to generate a theory on the **deviations of the economy relative to a stationary state**, i.e., the business cycle.

This implies that growth economics and macroeconomics look at different aspects of the economy. While growth economics addresses the effects of technological change, the dynamics of human capital, natural capital, and innovations, for instance, macroeconomics deals with the short run adjustments in consumption, in investment (particularly at its expenditure, financial and capacity building aspects), in the labor market, in prices, and stabilization policy issues.

A useful macroeconomic model should be consistent with (and provide one explanation of) the relevant stylized macroeconomic facts. In order to understand which they are, let us remember the basic instantaneous product market equilibrium equation

$$y(t) = c(t) + i(t) + g(t) + x(t) - m(t)$$

where y is income (or output), and c, i, x and m represent consumption, investment, exports, and import expemditures (or demand), which must be satisfied at every point in time t. The trade balance is $tb(t) = x(t) - m(t)$

Uribe and Schmitt-Grohé, 2017, ch 1 present several business cycle **stylized facts** regarding those quantities. In this note we are interested in the following [2]:

1. all the expenditure components are pro-cyclical (i.e., positively correlated with output) , with the exception of the trade balance which is counter-cyclical;

2. consumption and government expenditures are less volatile than output, while investment and trade balance are more volatile;

3. consumption is less volatile in developed than in less developed countries;

4. consumption is more persistent and investment less persistent than output.

All those variables have roughly a mean-reverting behavior, i.e, they fluctuate around their particular trends.

There are several issues a model has to explain, to allow for an understanding of those facts. In particular, we might want to understand what explains:

1. the stability mechanism that forces variables to adjust to their stationary value, or trend;

2. the shifters that generate deviations from their trend;

3. the persistent behavior of consumption;

4. the volatile behavior of investment;

5. the higher persistency of consumption as regards output.

The **simplest DGE model** provides some explanations for those questions (that we might check quantitatively):

[2] Cyclicality is measured by the correlation between every expenditure component and output, volatility is measured by the standard deviation of a variable, and persistency is measured by the autocorrelation of the variable time series.

1. the stability mechanism is provided by the decreasing marginal return of capital;
2. deviations from a stationary state are brought about by supply (increases in productivity) or demand shocks (preference to anticipate consumption);
3. the persistency in consumption is produced by intertemporal consumption substitution, that is, by a preference for smooth consumption trajectories, and by the possibility of using asset markets for intertemporal transfers of resources;
4. the volatility of investment is generated by the volatility of supply shocks (TFP) and of savings through its effect on asset prices;
5. the higher persistency of consumption dynamics, in comparison with output, is again explained by the existence of a mechanism for intertemporal allocation of goods together with a high elasticity of intertemporal substitution in consumption.

Other models provide other explanations for the different persistencies of consumption and output: life-cycle behavior in consumption, habit persistence, precautionary motives, just to name a few.

Next we address a model in which the dominant features are the existence of intertemporal substitution in consumption and the existence of decreasing marginal returns in production.

3 The Ramsey model

The Ramsey, 1928 model (in the version established by Cass, 1965 and Koopmans, 1965) has become the founding stone of the current dominant model in macroeconomics, that has been designated the DGE (dynamic general equilibrium) approach to macroeconomics.[3]

One of the main strong points, and also one of the main weaknesses, of this model relates to its nature: we can view it as providing a normative theory (how should the economy work ?) or a positive positive theory of an economy (how the economy actually works ?). Although the initial view of Ramsey was normative, some researchers use it as an acceptable positive description of the economy.[4]

Consider an economy populated by a constant number, N, of homogeneous households or agents. The economy has an initial endowment, denoted by k_0, of a durable good.

The stock of the durable good can be increased by production, which depends on the existing stock of the durable good, and can be consumed immediately or stored for future consumption. Then a natural question arises: how should consumption, and therefore the stock of the resource, evolve across time ? In order to answer that question we should find a way to value the state of the economy.

Therefore, the three basic elements of the model are the existence of:

[3]For a recent history of DSGE models see Gulan, 2018.

[4]Not wanting to delve into the philosophy of economics, maybe several economists (aware or unaware) follow Friedman, 1953's view that if a model, even if it looks very abstract, generates relationships between variables statistically similar to actual time series, is a good representation of the economy. For a recent debate on models and the macro-economy see Sargent, 2008 Nobel lecture, and for background in the philosophy of economics see https://plato.stanford.edu/archives/win2013/entries/economics/

1. the possibility of shifting the use of the resource over time;
2. constraints that condition those intertemporal transfers;
3. a technology allowing for the transformation of the existing stock of the resource into a different stock in the future;
4. a preference ordering over different future paths of consumption.

3.1 Environment and notation

In this economy there is one durable good, or a resource, that can be used in consumption or can be stored and used as an input in production. This allows us to distinguish between its stock and the flows of consumption, investment and production.

In the case of the Ramsey model the resource is considered to be physical capital, whose stock is denoted by K. The flow of output is denoted by Y and the flow of aggregate consumption is denoted by C. All those quantities are functions of time, t, which is assumed to be a non-negative real variable belonging to set $\mathrm{T} = \mathbb{R}_+$ (or $\mathrm{T} = \mathbb{R}_+ \cup \infty$). We take the present time as $t = 0$.

In this lecture it is important to distinguish: mappings, values and paths. Taking the example of consumption, we can have consumption as a **mapping** (a function in this case) $C(\cdot) : \mathrm{T} \to \mathbb{R}_+$, the **level** of consumption at a particular point in time $t \in \mathrm{T}$ is denoted by $C(t)$, and the **path** of consumption is the flow of the consumption level over time, $\mathbf{C} = \big(C(t)\big)_{t\in \mathrm{T}}$. Off course, a path $\mathbf{C}$ can be traced out if we know function $C(\cdot)$ and evaluate it at every point in time $t \in \mathrm{T}$, that is, $C(t)$.

Aggregate variables are denoted by uppercase symbols and per capita variables are denoted by lowercase symbols. Therefore $k \equiv K/N$, $y \equiv Y/N$ and $c \equiv C/N$ are, respectively, capital intensity, per-capita output and per-capita consumption. We assume all those variables take non-negative real values. We can denote accordingly per-capita consumption as a function, $c(\cdot)$, as a value at time t, $c(t)$, and as a path $\mathbf{c} = \big(c(t)\big)_{t\in \mathrm{T}}$ and analogously for the other variables.

3.2 Constraints

There are three constraints on the allocation of the good in the economy: first, the initial capital stock is given, it is known, and is positive, $k(0) = k_0 > 0$; second, the present-value of the terminal capital stock is non-negative $\lim_{t\to\infty} k(t)e^{-R(t)} \geq 0$, where $R(t)/t$ is the average rate of return on capital; and third, there is an **instantaneous budget constraint** $\dot{k}(t) = y(t) - c(t)$.

Let us assume we can write

$$\dot{k} = r(t)k(t) + z(t), \quad t \in [0, \infty) \tag{1a}$$

$$k(0) = k_0, \ t = 0 \tag{1b}$$

$$\lim_{t\to\infty} k(t)e^{-R(t)} \geq 0 \tag{1c}$$

where $r(t)$ is the instantaneous rate of return on capital, $e^{-R(t)} = e^{-\int_0^t r(s)ds}$ is the discount factor with a time-varying interest rate, and $z(t) = w(t) - c(t)$ is the savings out of non-financial income.

From those three constraints, we can derive an equivalent intertemporal budget constraint. The solution of the instantaneous budget constraint (1a) is

$$k(t) = e^{R(t)}\Big(k(0) + \int_0^t e^{-R(s)} z(s) ds\Big).$$

If we substitute (1b), and multiply by $e^{-R(t)}$, we obtain [5]

$$e^{-R(t)} k(t) = k_0 + \int_0^t e^{-R(s)} z(s) ds.$$

Taking the limit $t \to \infty$, and introducing the terminal constraint (1c) we have **intertemporal budget constraint**

$$k_0 \geq -\int_0^\infty e^{-R(s)}(w(s) - c(s)) ds$$

which means that the three conditions (1a)-(1c) together are equivalent to requiring that the initial capital stock is bigger (or equal) to the present value of the net uses in consumption of the good in the future. The human capital at time $t = 0$ is the present value of future wages, using the market interest rate as the deflator, is

$$h^\infty(0) \equiv \int_0^\infty e^{-R(s)} w(s) ds.$$

Because it is exogenous to the household we write $h^\infty(0) = h_0$, the intertemporal budget constraint is equivalent to

$$k_0 + h_0 \geq \int_0^\infty e^{-R(s)} c(s) ds,$$

which means that the present value of future consumption at time $t = 0$ should not be higher than the total capital (non-human and human).

As k_0 is bounded then the path of consumption $\mathbf{c} = \big(c(t)\big)_{t=0}^\infty$ should also be bounded in present value terms.

There are two alternative equivalent ways of seeing the instantaneous budget constraint. First, defining savings at time t as non-consumed output, $s(t) = y(t) - c(t)$, the instantaneous budget constraint is equivalent to investment equal savings: $\dot{k} = s(t)$. Second, as we assume that there is no capital depreciation, gross investment is equal to investment expenditures and is equal to net investment (or capital formation), $i(t) = \dot{k}$. Therefore, the budget constraint can be seen as a balance between the origin of output and its uses $y(t) = c(t) + i(t)$.

3.3 Assumptions

3.3.1 Technology

The technology of this economy is described by the production function,

$$Y = F(K, L)$$

[5] It is important to notice the conceptual difference between $k(0)$ and k_0. While the first is the level of a function of time evaluated at time $t = 0$, $k(0) = k(t)\big|_{t=0}$, k_0 is a known real number.

where aggregate output, Y, is produced by two inputs the stocks of physical capital, K and labor L. We assume there is no unemployment, then $L = N$. The production function is assumed to have three properties that allow to label it as **neo-classical**. [6]

First, there are positive marginal returns to both inputs, $F_K(\cdot) > 0$, $F_L(\cdot) > 0$. This means that increases in the quantity of any input increases production

Second, it is concave, i.e., $F_{KK}(\cdot) < 0$, $F_{LL}(\cdot) < 0$ and $F_{KK}(\cdot)\, F_{LL}(\cdot) - F_{KL}(\cdot)^2 > 0$, and usually the inputs are Edgeworth complementary $F_{KL}(\cdot) \geq 0$. This means that any input use is characterized by the existence of decreasing marginal returns for increases in its own use but increasing returns from the use of any other input, meaning that they are complements in the Edgeworth sense. [7]

Third, it is usually assumed that there are constant returns to scale. Mathematically the last property is translated by the requirement the $F(K, L)$ is a homogeneous of degree one (HDO) function. Thus if we increase every input by a scale factor λ the output also increases by the same factor $F(\lambda K, \lambda N) = \lambda\, F(K, L) = \lambda Y$. Linear homogenous functions have a nice property: Euler's lemma says that $Y = F_K(K, L)\, K + F_L(K, L) L$.

The three properties, display different roles in the model. Positive returns allows for a positive effect of the increase in resources on the possibility of having consumption in the future, without completely depleting the stock of capital. Concavity provides a stable dynamic mechanism, as we will see next. Constant returns to scale (CRS), allows to work with scale-free per capita variables: setting $\lambda = 1/N$ yields

$$y \equiv \frac{Y}{N} = \frac{F(K, N)}{N} \overset{(HDO)}{=} F\left(\frac{K}{N}, 1\right) = F(k, 1) \equiv f(k).$$

where $f''(k) < 0 < f'(k)$, from the properties of $F(\cdot)$. Then $Y = f(k)\, N$

3.3.2 Consumption preferences

Preferences over aggregate consumption, $\mathbf{C}$, are ranked according to a (cardinal) **social welfare functional**

$$\mathsf{U}[\mathbf{C}] = N \mathsf{U}[\mathbf{c}]$$

where $\mathsf{U}[\mathbf{c}]$ is the **intertemporal utility functional** for a single agent (or household) whose consumption is denoted by $\mathbf{c}$.

We assume an additive, discounted intertemporal utility functional

$$\mathsf{U}[\mathbf{c}] \equiv \int_0^\infty u(c(t))\, e^{-\rho t} dt,\ \rho > 0$$

There are three main features of the utility functional $\mathsf{U}[\mathbf{c}]$ I would like to emphasise: first, it displays impatience (because we are discounting future consumption by a time decreasing discount factor $e^{-\rho t}$); second, it displays additive separability because the value of consumption at each point

[6]In the sense that a common conception of the classical economists (Quesnay, Smith, Ricardo, and even Marx) is that marginal returns are decreasing.

[7]The recent approach to microeconomics consider the property of supermodularity. Function $F(K, L)$ is supermodular if $F(\max\{K, L\}) + F(\min\{K, L\}) \geq F(K, K) + F(L, L)$. If we assume that $F(\cdot)$ is differentiable, then function F is supermodular if and only if $F_{KL}(\cdot) > 0$. In this case we say that K and L are (strategic) complements.

in time is not affected by the past consumption ($u(c(t))$ only depends on the level of consumption and not on the path of consumption); and third, there is intertemporal substitution in consumption, which can be measured by the relative concavity of $u(\cdot)$, or the inverse of the elasticity of intertemporal substitution

$$\sigma(c) \equiv -\frac{u''(c)\,c}{u'(c)}.$$

We assume implicitly that there is an institutional mechanism, represented by a social welfare functional, which measure the value of consumption allocations by summing up utility of every household (or dynasty), which is assumed to be infinitely living.

3.4 Optimal allocation

Gathering all the previous elements we can model the aggregate behavior of an homogeneous agent economy endowed with an efficient social allocation mechanism.

Definition 1 (Optimal allocation). *An optimal allocation* $(\mathbf{c}^*, \mathbf{k}^*) = \big(c^*(t), k^*(t)\big)_{t\in T}$ *is an allocation that solves the problem*

$$\max_{c(\cdot)} \int_0^\infty u(c(t))\, e^{-\rho t} dt \tag{2a}$$

$$\text{subject to}$$

$$\dot{k} = f(k) - c, \text{ for each } t \in [0,\infty) \tag{2b}$$

$$k(0) = k_0, \text{ for } t = 0 \tag{2c}$$

$$\lim_{t\to\infty} k(t)e^{-R(t)} \geq 0 \tag{2d}$$

$$c(\cdot) \in \mathbb{R}_{++} \tag{2e}$$

There are two approaches to solve this problem, based on two different (but equivalent) principles: the Pontryiagin's maximum principle (PMP) and the dynamic programming principle (DPP). [8]

Observe that the problem consists, in essence, to find the best way to use an initial resource, k_0, from now to the infinite future, knowing that if we don't consume it all at every point in time (that is, if $c(t) \leq f(k(t))$) there is a technology that allows for an increase in consumption in the future. The existence of a positive resource and the technology of production allows for several

[8] There is also the calculus of variations approach, which transforms the problem into

$$\max_{k(\cdot)} \int_0^\infty u\big(f(k(t)) - \dot{k}(t)\big)\, e^{-\rho t} dt$$

$$\text{subject to}$$

$$k(0) = k_0, \ t = 0$$

$$\lim_{t\to\infty} k(t)e^{-R(t)} \geq 0$$

possible paths of consumption in the future, and generates trade-offs between present and future consumption. Intertemporal preferences provide a criterium for choosing an optimum trade-off.

Therefore, the value of the initial endowment of the resource, k_0, can be measured by the optimal path of consumption it makes possible, which can be measured by the value function [9] for the initial resource

$$v(k_0) = \mathsf{U}[\mathbf{c}^*] = \int_0^\infty u(c^*(t))\, e^{-\rho t} dt.$$

This idea is used by the DPP, by implicitly obtaining and intertemporal valuation criterium. Instead, the PMP provides an optimality criterium together with that trade-off, for every point in time.

3.5 The solution according to the PMP

The Pontryiagin's maximum principle (PMP) is reminiscent from the Lagrangean approach to static optimization.

We introduce the (current-value) Hamiltonian function

$$H(k, q, c) = u(c) + q\,(f(k) - c)$$

which penalizes the utility of consumption, at every point in time, by the value of the foregone stock of capital, where q is a dynamic shadow value (which is called co-state variable or adjoint variable).

An optimal allocation $(\mathbf{c}^*, \mathbf{k}^*)$ satisfies the following optimality, adjoint and transversality conditions

$$u^{'}(c^*(t)) = q(t) \tag{4a}$$

$$\dot{q} = q\big(\rho - f'(k^*)\big) \tag{4b}$$

$$\lim_{t\to\infty} q(t)\, k^*(t)\, e^{-\rho t} = 0 \tag{4c}$$

together with the admissibility constraints (2b)-(2c), evaluated at $\big(k^*(t)\big)_{t=0}^{\infty}$. Off course $k^*(0) = k_0$ should hold. If the utility function is strictly concave, with $u''(c) < 0$ on all its domain, we can apply the implicit function theorem to obtain the adjoint variable as a decreasing function of consumption, $q = Q(c)$, with $Q'(c) < 0$. Taking time-derivatives,

$$\frac{d\big(u'(c(t))\big)}{dt} = u''\big(c(t)\big)\,\frac{dc(t)}{dt} = \frac{dq}{dt}$$

and substituting $\dot{q} = q\,(\rho - f'(k))$, yields the so-called Keynes-Ramsey rule (or Euler equation)

$$u''(c)\,\dot{c} = u'(c)\big(\rho - f'(k)\big).$$

Using the previously defined elasticity of the utility function, $\sigma(c) = -\dfrac{u''(c)c}{u'(c)}$, the modified Hamiltonian dynamic system (MHDS), together with the initial and the transversality condition,

[9] Mathematically it is a function, not a functional, because if we were able to find the optimal solution for consumption it would be of type $c^* = C(t, k_0)$. If we plugged it into the functional $\mathsf{U}[\mathbf{c}]$, and integrate over time, we would obtain a function of k_0. As we will see next, only in very special cases we can obtain explicitly function $C(t, k_0)$.

is

$$\dot{k}^* = f(k^*) - c^* \tag{5a}$$

$$\dot{c}^* = \frac{c^*}{\sigma(c^*)}\big(f'(k^*) - \rho\big) \tag{5b}$$

$$k^*(0) = k_0 \tag{5c}$$

$$\lim_{t\to\infty} u'(c^*(t))\, k^*(t)\, e^{-\rho t} = 0. \tag{5d}$$

The last condition forces the solution to converge asymptotically (i.e., when $T \to \infty$) to a constant or to a function of time that should grow at a non-positive rate.[10]

The PMP operates in the following way.

Let us assume that there is one unique solution to the problem. If this is the case, given an initial value for $(k(0), c(0))$ the system (5a)-(5b) traces out one single path $\big(c(t), k(t)\big)_{t=0}^{\infty}$. However, we only know $k(0) = k_0$ and for most values of $c(0)$ the solution will not satisfy the transversality condition (5d). On the other hand, uniqueness of solutions to the problem is equivalent to requiring that there will be only one value for $c(0)$ such that the solution satisfies (5d). If this condition only holds for constant values of k and c, this means that the unique solution converges to a steady state is of the saddle-point type.

From all this we learn two things: first, the crucial step in obtaining the solution to the problem is to find the initial value for consumption $c^*(0)$, and, second, this value will be determined backwards from the terminal conditions, which is usually a steady state of saddle-point type. This is only possible if there is some type of foresight, as is clear from the fact that the consumer values the present value of all the consumption path $(c(t))_{t\in[0,\infty)}$.

Therefore, an **optimal solution will contain both forward and backward mechanisms**. The first, represented by the instantaneous budget constraint, propagates the capital stock into the future through savings. The second, backward (anticipating) mechanism is related to the incentives for consumption, guided by both the return on production relative to impatience and the attitude of the consumer as regards intertemporal consumption substitution, which determines savings at every point in time.

Next we will show how to find the solution for a particular case.

3.6 The solution from the DPP approach

The DPP approach operates in a different way. It tries to find a rule that, given the observed capital stock at every point in time, would enable to find a time-independent optimal rule (i.e., a recursive mechanism) for consumption.

We call value function to

$$v(k(t)) = \mathsf{U}[\mathbf{c}^*; t] \equiv \max_{c(\cdot)} \int_t^{\infty} u(c^*(s)) e^{-\rho(s-t)}\, ds$$

which provides a valuation of the existing level of the capital stock as a resource allowing for consumption in the future. This notation highlights the fact that the value of the stock of the

[10] This is the case of endogenous growth models for which $f(k)$ is a linear function.

durable good at time t, $k(t)$, if used efficiently, allows us to attain the intertemporal utility level $\mathsf{U}[\mathbf{c}^*; t]$, at the optimum.

Applying the principle of dynamic programming, an optimal allocation $(\mathbf{c}^*, \mathbf{k}^*)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\rho\, v(k) = \max_c \big\{ u(c) + v'(k)\big(f(k) - c\big)\big\}, \tag{6}$$

where the optimal consumption satisfies

$$u'(c^*) = v'(k). \tag{7}$$

If again the utility function has no singularities, we can find implictly the **optimal policy function**

$$c^* = C(k) = (u')^{-1}(v'(k)) \tag{8}$$

which, substituting on the HJB equation, yields

$$\rho\, v(k) = u(C(k)) + v'(k)\big(f(k) - C(k)\big) \tag{9}$$

which is an implicit ordinary differential equation. If we are able to find the optimal policy function, we can determine the solution by solving

$$\begin{aligned} \dot{k} &= f(k) - C(k), \text{ for } t \in \mathrm{T} \\ k(0) &= k_0. \end{aligned} \tag{10}$$

3.7 Comparing the two approaches

Comparing the optimality conditions from the PMP, equation (4a), and from the DPP, equation (7), we observe that, in the infinite horizon case, the adjoint variable q of the first is formally equal to the marginal value of capital, $v'(k)$ of the second. This allows us to interpret q as a shadow-value of capital.

Remembering that the solution for c is forward looking, when using the PMP, we can interpret the policy function, in equation (8), as implicitly incorporating the transversality condition (5d). That is, it translates the backward solution of the adjoint equation together with the transversality condition, equations (5b) and (5d), into a single backward rule (8). This is why solving the problem by the DPP is said to be applying recursive methods[11]

There are advantages and disadvantages in applying the PMP or the DPP to solving the Ramsey-like types of models. The main difficulty in applying the PMP is related to finding the initial value of consumption by transforming the MHDS into a initial-terminal value problem. However, it allows for using the richness of results from dynamic systems theory for, at least, characterizing analytically the optimal solution. The main difficulty in applying the DPP is solving the implicit ordinary differential equation (9). However, the recursive structure of the solution provides a simpler interpretation of the characteristics of the solution.

By using the **envelop theorem** we can use the geometrical approach, analogous to the one from the PMP, after obtaining the first-order conditions through the DPP.

[11] Ljungqvist and Sargent, 2018 apply systematically this approach for discrete time versions of macromodels.

Taking derivatives as regards the state variable, k, of equation (9) we find

$$\rho\, v'(k) = u'(C(k))\, C'(k) + v''(k)\,\big(f(k) - C(k)\big) + v'(k)\,\big(f'(k) - C'(k)\big)$$

because, at the optimum $u'(C(k)) = v'(k)$ then this is equivalent to

$$\rho\, v'(k) = v''(k)\,\big(f(k) - C(k)\big) + v'(k)\, f'(k),$$

which is valid for any $k = k(t)$. Setting $q(t) = v'(k(t))$, for every $t \in (0, \infty)$ we have

$$\frac{dq(t)}{dt} = v''(k)\,\frac{dk(t)}{dt} = v''(k)\,\big(f(k(t)) - C(k(t))\big) = v'(k(t))\,\big(\rho - f'(k(t))\big).$$

Therefore

$$\frac{dq(t)}{dt} = q(t)\,\big(\rho - f'(k(t))\big).$$

which is the multiplier or adjoint condition from the PMP (also called the Ramsey-Keynes rule).

3.8 A benchmark application: the isoelastic-Cobb-Douglas case

In this section we solve the Ramsey model by introducing specific utility and production functions.

The Ramsey model in the general form just presented does not have an explicit solution because we have left unspecified both the utility function, $u(\cdot)$, and the production function, $f(\cdot)$.

Whatever the method we use, in general we cannot find explicitly the optimal allocation $(\mathbf{c}^*, \mathbf{k}^*)$, even if we consider explicit utility and production functions. Even if we specified those functions, in most cases we cannot obtain an explicit (i.e, exact) solution and we have to use approximate or numerical methods to find it. This is unfortunate, because a complete understanding of the nature of the solution can only be obtained by knowing the explicit solution.[12]

A benchmark particular case in which the utility function is a generalized logarithm[13]

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta},\ \text{for } \theta > 0$$

where $\sigma(c) = \theta$, is the inverse of the elasticity of intertemporal substitution, and the production function is Cobb-Douglas $Y = AK^\alpha L^{1-\alpha}$, yielding

$$y = f(k) = A\, k^\alpha,\ A > 0,\, 0 < \alpha < 1,$$

where A is the the total factor productivity (TFP) and α is the share of capital in national income.

If we want to determine or characterize the optimal allocation $(\mathbf{c}^*, \mathbf{k}^*)$, we have two alternative approaches we mentioned before:

[12]It is not uncommon to find in the literature inaccurate presentations of the solutions to the Ramsey problem.

[13]We can write it as $u(c) = \log_\theta(c)$. A particular case, when $\theta = 1$ is the logarithmic utility function $u(c) = \log_1(c) = \log(c)$. Prove this by nothing that $u(c) = \log(e^{u(c)})$ and by using the l'Hôpital rule.

We can use the PMP and solve the particular version of the MHDS system together with its initial and transversality conditions,

$$\dot{k}^* = A\,(k^*)^\alpha - c^* \tag{11a}$$

$$\dot{c}^* = \frac{c^*}{\theta}\big(\alpha\, A\,(k^*)^{\alpha-1} - \rho\big) \tag{11b}$$

$$k^*(0) = k_0 \tag{11c}$$

$$\lim_{t\to\infty} (c^*(t))^{-\theta}\, k^*(t)\, e^{-\rho t} = 0 \tag{11d}$$

We denote the rate of return of capital, or interest rate by

$$r(k) = f'(k) = \alpha\, A\, k^{\alpha-1}.$$

Because the initial stock of capital is given, in equation (11c), and, as we will see, consumption at time $t = 0$ is implicitly determined from the transversality condition (11d), we call k a **predetermined** variable and c a **non-predetermined** variable. If we are able to find $c(0)$ uniquely, we say that **the optimal path is determined**.

Alternatively, we can use the DPP, solve the particular version of the HJB equation (6)

$$\rho\, v(k) = \max_c \left\{ \frac{c^{1-\theta}-1}{1-\theta} + v'(k)\big(A\, k^\alpha - c\big) \right\} \tag{12}$$

obtain optimal consumption from the optimal policy function

$$c^* = C(k) = \big(v'(k)\big)^{-\theta},$$

and solve the ODE problem

$$\dot{k} = A\, k^\alpha - C(k)$$

together with the initial condition $k(0) = k_0$ to obtain $\mathbf{k}^* = \big(k^*(t)\big)_{t\in[0,\infty)}$, and substitute back in the policy function $C(k)$ to obtain $\mathbf{c}^* = \big(c^*(t)\big)_{t\in[0,\infty)}$

Then, we can answer two fundamental macroeconomic questions: First, how does consumption and output respond to demand and supply shocks, represented by shifts in the parameters A and ρ ? Second, does this type of response matches the stylized facts ?

The best way to answer those questions would involve obtaining exact (also called closed form) solutions. Even in this isoelastic-Cobb-Douglas case, for generic values of the parameters, there is no known explicit solution, which explains why researchers resort to approximate analytical of numerical methods to solve it.

However, only if $\theta = \alpha$ we can obtain an explicit solution. Although this relationship is counterfactual [14] because most empirical research, using macro and micro data, finds a realistic value for θ to be in an interval centered at $\theta = 2$, this case is important because it allows for an understanding of our previous discussion.

[14] References benchmark parameter values: Basu and Fernald, 1997, Hall, 1988.

3.8.1 The case $\theta = \alpha$

If we have any hope of getting an explicit solution, the hard problem is solving the backward looking part, i.e., obtaining consumption as a function of the capital stock: $c = C(k)$. We prove in the appendix[15] that both methods yield **one unique solution**

$$c = C(k) \equiv \frac{\rho}{\alpha}\,k.$$

Observe that this consumption function is different from the Keynesian consumption function. As $k = \left(\frac{y}{A}\right)^{\frac{1}{\alpha}}$ we will have

$$c = \frac{\rho}{\alpha}\left(\frac{y}{A}\right)^{\frac{1}{\alpha}}$$

which is different from a Keynesian consumption function which is linear in y, for instance $c = \beta\,y$ for $0 < \beta < 1$. As $\alpha \in (0, 1)$ this means that consumption in the Ramsey model will change more than linearly with income, because $\frac{1}{\alpha} > 1$. [16]

The solution for capital is obtained by solving forward in time [17]

$$\begin{cases} \dot{k}^* = A\,(k^*)^{\alpha} - \dfrac{\rho}{\alpha}\,k^* & \text{for}\, t \in (0, \infty) \\ k^*(0) = k_0 \end{cases} \tag{13}$$

which is

$$k^*(t) = \left[\bar{k}^{1-\alpha} + \left(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\right)e^{-\rho\left(\frac{1-\alpha}{\alpha}\right)t}\right]^{\frac{1}{1-\alpha}},\ \text{for}\, t \in [0, \infty), \tag{14}$$

where

$$\bar{k} = \left(\frac{\alpha\,A}{\rho}\right)^{\frac{1}{1-\alpha}} \tag{15}$$

is the steady state level of the capital stock.

From this result we can get explicit solutions for consumption $c^*(t) = \dfrac{\rho}{\alpha}k^*(t)$, output $y^*(t) = A\left(k^*(t)\right)^{\alpha}$, savings $s(t) = y(t) - c(t)$ and the interest rate $r(t) = \alpha\,A\left(k^*(t)\right)^{\alpha-1}$. For instance,

$$y^*(t) = \left[\bar{y}^{\frac{1-\alpha}{\alpha}} + \left(y_0^{\frac{1-\alpha}{\alpha}} - \bar{y}^{\frac{1-\alpha}{\alpha}}\right)e^{-\rho\left(\frac{1-\alpha}{\alpha}\right)t}\right]^{\frac{\alpha}{1-\alpha}},\ \text{for}\, t \in [0, \infty) \tag{16a}$$

$$r^*(t) = \left[\frac{1}{\rho} + \left(\frac{1}{r(0)} - \frac{1}{\rho}\right)e^{-\rho\left(\frac{1-\alpha}{\alpha}\right)t}\right]^{-1},\ \text{for}\, t \in [0, \infty). \tag{16b}$$

where the steady state output is

$$y(\infty) = \bar{y} = \bar{c} = \left(A\left(\frac{\alpha}{\rho}\right)^{\alpha}\right)^{\frac{1}{1-\alpha}} = \frac{\rho}{\alpha}\,\bar{k} \tag{17}$$

and the steady state interest rate is equal to the rate of time preference $r(\infty) = \bar{r} = \rho$.

We only have the following type of dynamic adjustments (see Figure 1):

[15] See appendix A.

[16] This flags out a potential counterfactual property of this model, in the sense that it displays a higher volatility of consumption than income, differently from stylized facts.

[17] Observe, in the initial condition, that while $k^*(0)$ refers to the optimal solution $k^*(t)$ evaluated at $t = 0$, which is not known at this point, k_0 refers to an observed level of the percapita capital stock which we can obtain from published statistics.

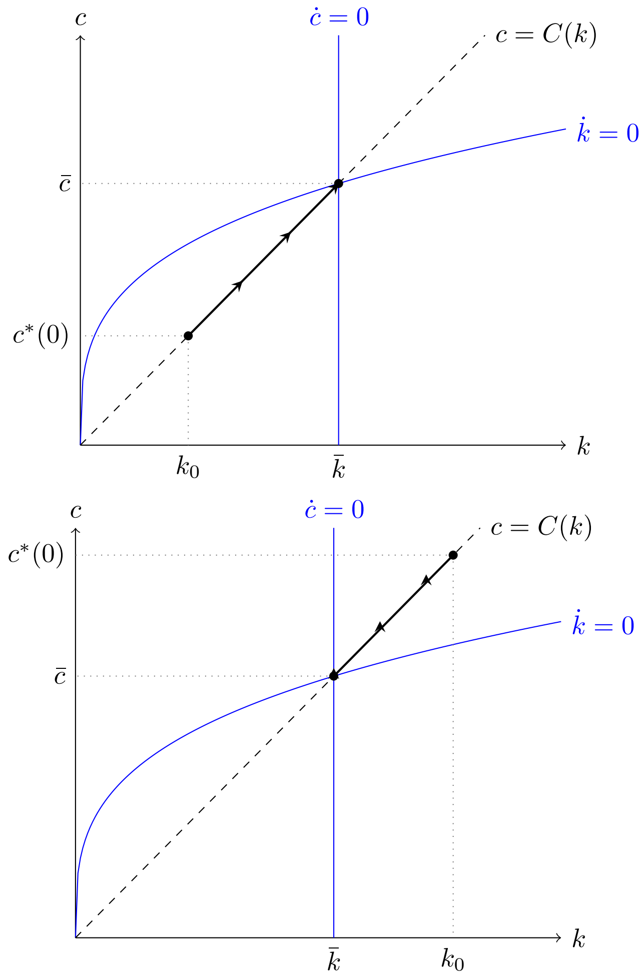

Figure 1: Phase diagrams for the exact optimal adjustments for c and k for the $\theta = \alpha$ case. The adjustment depends on the relationship between the initial level of capital, k_0 and the steady state level $\bar{k}$: if $k_0 < \bar{k}$, as in the upper panel, there will be both an increase in consumption and the stock of capital; and if $k_0 > \bar{k}$, as in the lower panel, both will decrease.

- If $k_0 = \bar{k}$ the solution is stationary, that is time-invariant: consumption and the capital stock, and therefore output will not change in time, if there is no anticipated shock in any parameter. In this case the rate of return on capital, and therefore, the real interest rate is equal to the rate of time preference $r(\bar{k}) = \rho$.
- If $k_0 < \bar{k}$ both consumption and the capital stock will increase in time: if the level of consumption is too low given the level of production, there is positive savings, implying there is investment and capital accumulation. This increases output and allows for an increase in consumption in the future. There is substitution of present consumption by future consumption. The incentive for increasing consumption continues as long as the interest rate is higher than the rate of time preference $r(k(t)) > \rho$. Because the interest rate decreases with the process of capital accumulation, this reduces the incentives for savings and therefore for capital accumulation. This adjustment is slowed down by the fact that there are decreasing marginal returns: increasing the production capacity does not increase output commensurably. Stability is brought about by the fact that the rate of return of capital converges to ρ from above ($\lim_{k \to \bar{k}^-} r(k) = \rho$).
- If $k_0 > \bar{k}$ both consumption and the capital stock will decrease in time: if the level of consumption is higher than output, part of the durable good is de-accumulated, which decreases the capital stock and therefore output. This forces consumption to reduce as well. In this case, because there is too much capital the rate of return is below the rate of time preference $r(k) < \rho$, i.e., the economy has too much capital from the perspective of the utility maximizing consumer. Stability is brought about by the fact that the decrease in the stock of capital increases the rate of return until it converges to ρ from below ($\lim_{k \to \bar{k}^+} r(k) = \rho$).

3.8.2 The general case with $\theta \neq \alpha$

In this general, more realistic case, we cannot find an exact solution to the optimal allocation by using any of the two previous approaches, the PMP and DPP.

The most common approach to solving the model is to resort to a linear approximation. In this case, the PMP allows for a simpler derivation of an approximative solution[18]. We proceed as follows: first, we obtain a non-zero steady state of the model (which in this case is unique); second, we perform a first-order Taylor approximation of the MHDS (11a)-(11b) in the neighborhood of the steady state , obtaining a variational MHDS; third, we solve the variational MHDS by requiring that: first, it converges to the steady state, and its initial value is given by (11c). As we require that the solution converges to a constant, then the solution satisfies the transversality condition (11d).

This steady state is given by the positive pair (k, c) such that $r(k) = \rho$ and $c = f(k)$, that is

$$(\bar{k}, \bar{c}) = \Big\{ (k, c) \in \mathbb{R}^2_{++} : \alpha\, A\, k^{\alpha - 1} = \rho,\ c = A\, k^{\alpha} \Big\},$$

that is

$$\bar{k} = \Big(\frac{\alpha\, A}{\rho} \Big)^{\frac{1}{1-\alpha}},\ \bar{c} = \Big(A \Big(\frac{\alpha}{\rho} \Big)^{\alpha} \Big)^{\frac{1}{1-\alpha}} = \frac{\rho}{\alpha} \bar{k}.$$

[18] However, some people find the DPP to be more convenient to reach a numerical solution to the problem.

The steady state is the same as in the previous case (see equations (15) and (17)). Again, in the steady state $\bar{c} = \bar{y}$. This means that the parameter θ does not influence the steady state, but influences the optimal path out of the steady state.

The **variational MHDS**, in the neighborhood of the steady state $(\bar{k}, \bar{c})$, is represented by the linear-ODE

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \mathbf{J} \begin{pmatrix} k(t) - \bar{k} \\ c(t) - \bar{c} \end{pmatrix}, \text{where } \mathbf{J} \equiv \begin{pmatrix} \rho & -1 \\ \frac{\bar{c}r'(\bar{k})}{\theta} & 0 \end{pmatrix} \tag{18}$$

As we prove in the Appendix B the solution is

$$k(t) = \bar{k} + (k(0) - \bar{k})e^{\lambda_s t} \tag{19a}$$

$$c(t) = \bar{c} + \lambda_u(k(0) - \bar{k})e^{\lambda_s t} \tag{19b}$$

where $\lambda_s < 0 < \lambda_u$ are the eigenvalues of the Jacobian matrix $\mathbf{J}$, satisfying the properties $\lambda_s + \lambda_u = \rho > 0$ and $\lambda_s \lambda_u = \det(J) < 0$, where the the determinant of the Jacobian $\mathbf{J}$ is dependent on the parameters of the model

$$\det(\mathbf{J}) = -\Big(\frac{1-\alpha}{\alpha}\Big)\frac{\rho^2}{\theta} < 0.$$

Therefore

$$\lambda_s = \frac{\rho}{2}\left(1 - \Big(1 + \Big(\frac{1-\alpha}{\alpha}\Big)\frac{4}{\theta}\Big)^{\frac{1}{2}}\right), \; \lambda_u = \frac{\rho}{2}\left(1 + \Big(1 + \Big(\frac{1-\alpha}{\alpha}\Big)\frac{4}{\theta}\Big)^{\frac{1}{2}}\right)$$

both λ_s and λ_u depend upon the parameters ρ, α and θ, that is on the substitution between labor and capital in production, on the rate of time preference (which measures impatience) and on the intertemporal substitution in consumption. If $\theta \to \infty$, meaning that there is no intertemporal substitution in consumption we have $\lim_{\theta\to\infty} \lambda_s = 0$ and $\lim_{\theta\to\infty} \lambda_u = \rho$ which means that there will be no savings and consumption will be equal to output. Therefore the smaller is θ the higher (positive or negative) savings will be and consumption will be less instantaneously correlated with income.

Several other observations can be made. First, if $k(0) \neq \bar{k}$ then the solution converges asymptotically to the steady state $\lim_{t\to\infty} k(t) = \bar{k}$, because $\lim_{t\to\infty} e^{-\lambda_s t} = 0$. The speed of convergence can be measured by the half-life of the adjustment [19] which is, in this case $\tau = -\frac{\log(1/2)}{\lambda_s} > 0$. This means the higher in absolute value λ_s is the quicker the adjustment will be. Again, a small θ will speed convergence to the steady state.

Second, as in the previous case the solution is unique, or **determinate**: given any initial value for k, $k(0)$, there is only one value for c, $c(0)$, which is determined endogenously such that $\lim_{t\to\infty} c(t) = \bar{c}$. The exact solution for consumption throughout time is tangent to a linear approximation

$$c(t) - \bar{c} = \lambda_u(k(t) - \bar{k}),$$

this line is represented in the lower panel in Figure 2 by the label E^s and defines the stable eigenspace (i.e., the space tangent to the stable manifold which is the geometrical analog of the exact relationship between c and k for the solution of the problem).

[19] The half-life is defined by $\tau = \{t : k(t) - k_0 = \frac{\bar{k} - k_0}{2}\}$. It is the required time to travel half of the distance between the initial, k_0, and the steady state level for the stock of capital, $\bar{k}$.

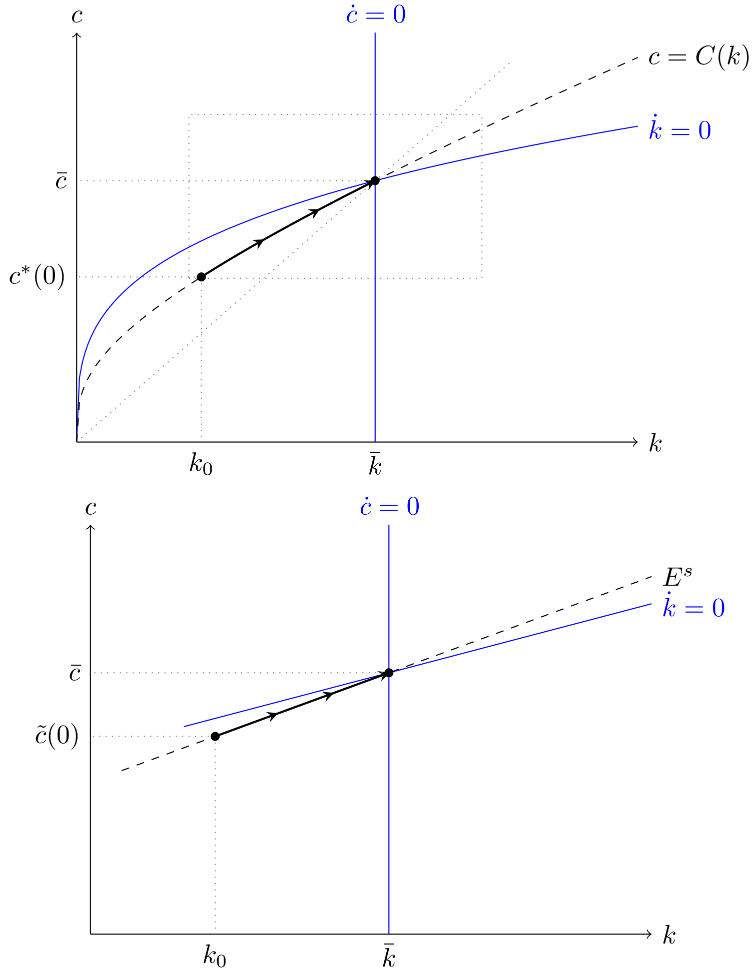

Figure 2: Phase diagrams for the exact optimal adjustments for c and k: the upper panel is the exact solution and the lower panel is the linear approximation

3.8.3 Approximation to the optimal path: linear approximation

The difference between the exact solution and the linear approximation can be seen in Figure 2, where the exact solution is depicted in the upper panel and the approximate solution is depicted in the lower panel). The linear approximation captures the qualitative dynamics of the optimal trajectory, although it can quantitatively be slightly different if the initial value of k_0 is far away from the steady state (or the economy is perturbed by an unanticipated shock in any parameter.

There are, in the literature, some misunderstandings regarding the nature of the solution. Sometimes people use the fact that the steady state is a saddle-point, in which the stable manifold passing through the steady state is unique, by using this fact to derive the solution. This can be done as a device to characterize the solution, but introduces a potential misunderstanding on the mathematical meaning of the uniqueness of the solution to an optimal control problem. If we look at the derivation of the exact solution for the $\theta = \alpha$ case makes this clear: the uniqueness of the policy function is associated to the existence of a stable-manifold of dimension one, meaning that the solution of the problem is unique (and it will never wander around the phase diagram as might be understood if we draw the complete vector field).

From a dynamic systems perspective, the solution traces out an heteroclinic trajectory [20] linking $(0,0)$ to $(\bar{k},\bar{c})$ and following a smooth trajectory for (k,c) higher than $(\bar{k},\bar{c})$. That is, the policy function is geometrically equivalent to that heteroclinic trajectory. For the case $\theta=\alpha$ the line $c=\frac{\rho}{\alpha}k$ corresponds to that heteroclinic trajectory (which we saw is the only locus where the solution lies). If $\theta>\alpha$ the policy function will be "trapped" between the same line, $c=\frac{\rho}{\alpha}k$, and the isocline for k, $c=f(k)=A\,k^{\alpha}$. If $0<\theta<\alpha$ the "trapping" area should be different: between $c=0$, $k=\bar{k}$ and $c=\frac{\rho}{\alpha}\,k$.

3.8.4 Approximation to the optimal path: heteroclinic approximation

Given the fact that the stable manifold is an heteroclinic orbit joining the origin to the steady state, a better approximation to the policy function (or to the stable manifold in the language of the dynamic systems) would be to approximate it by an average between the two curves defining a trapping area:

$$c=S(\omega,k)=\omega\frac{\rho}{\alpha}\,k+(1-\omega)\,A\,k^{\alpha}$$

where ω is a weighting factor. We can determine ω by requiring that the slope of curve $S(k)$ in the neighborhood of the steady state is the same as the slope of the stable eigenvector, which we saw was λ_u. We can do this by requiring that $S(k)$ is tangent to the linear approximation in the neighborhood of the steady state. Therefore,

$$\omega\equiv\{w:\partial_k S(w,\bar{k})=\lambda_u\}.$$

As $\partial_k S(\omega,\bar{k})=\omega\,\frac{\rho}{\alpha}+(1-\omega)\,\rho$ then

$$\omega=\frac{\alpha\,(\lambda_u-\rho)}{\rho(1-\alpha)}.$$

Therefore a "heteroclinic" approximation of the policy function is

$$c=S(k)=\frac{\lambda_u-\rho}{1-\alpha}\,k+\frac{\rho-\alpha\,\lambda_u}{\rho(1-\alpha)}\,A\,k^{\alpha}.$$

This function can be seen as a weighting of the Keynesian and of the Ramsey consumption functions, if we write it as

$$c=S(k)=\frac{\lambda_u-\rho}{1-\alpha}\,k+\frac{\rho-\alpha\,\lambda_u}{\rho(1-\alpha)}\,y.$$

To interpret geometrically this result, observe that the slope of the isocline $\dot{k}=0$ in the neighborhood of the steady state is equal to ρ, which means that if $\lambda_u\to\rho^-$ then the optimal trajectory would approach the curve $c=f(k)$, which means there will be no savings and output will be completely consumed. We will have this case if $\theta\to\infty$, meaning that the consumer elasticity of intertemporal substitution approaches zero.

Therefore the distance between the policy function $c=C(k)$ and the curve $c=f(k)$ is governed by the elasticity of substitution θ: the further apart they are the higher savings is, and the lower the consumption correlation with output will be. This is consistent with stylized facts.

[20] An heteroclinic trajectory is a path linking, in a dynamic system, two steady states.

3.8.5 Consequences of supply and demand shocks

Assuming we depart from a steady state Figure 3 presents the **geometry** of the dynamic adjustments following a non-anticipated, permanent and constant supply shock (i.e., an increase in productivity from A to $A' > A$) and a non-anticipated, permanent and constant demand shock (i.e., an increase in the rate of time preference from ρ to $\rho' > \rho$).

In the case of a positive supply shock, starting from point A (in the upper panel of Figure 3) consumption, responding to the new anticipated path for the rate of return, increases discontinuously to point B. As there is also an increase in output, which is greater than the increase in consumption, there is positive savings which, through investment, sets in motion a process of further capital accumulation. This process of increases in consumption, and while there is positive savings, drives a transition dynamic process towards the new steady state, point C, in which both consumption, output and the capital stock will be greater than before the shock.

In the case of a positive demand shock, there is also an immediate increase in consumption, because the implicit relationship between present and future consumption becomes unbalanced at point A (in the lower panel): consumers prefer consuming now than in the future. Because output does not change, this generates negative savings (i.e, the durable good is used more for consumption rather than investment) which reduces the stock of capital and output, forcing a reduction in consumption. The process is stabilized by the fact that further reduction in the stock of capital raises the rate of return of capital until we have $r(\bar{k}') = \rho'$.

This type of dynamics justifies labelling capital as predetermined and consumption as non-pretedermined: capital moves continuously when there is savings, while consumption responds to news regarding the future evolution of the economy.

The most common approach to obtain an analytical derivation of the whole process is through a **comparative dynamics exercise**. As $\det \mathbf{J} \neq 0$ we could approximate the response function for the model for a change in parameter φ by taking derivatives to equation (19a) yielding [21]

$$\partial_\varphi k(t,\varphi) = \partial_\varphi \bar{k}(\varphi)\,(1 - e^{\lambda_s t})\; t \in [0,\infty)$$

where $\partial_\varphi \bar{k}(\varphi)$ is the long run multiplier and $\partial_\varphi k(t,\varphi)$ is the short run multiplier for the stock of capital for a permanent, non-anticipated shock in the parameter φ. For consumption, from equation (19b), one would get

$$\partial_\varphi c(t,\varphi) = \partial_\varphi \bar{c}(\varphi)\, - \lambda_u\, \partial_\varphi \bar{k}(\varphi)\, e^{\lambda_s t}\; t \in [0,\infty).$$

We call impact multiplier to the short run multiplier evaluated at time $t = 0$. Performing the calculation we have

$$\partial_\varphi k(0,\varphi) = 0,\; \partial_\varphi c(0,\varphi) = \partial_\varphi \bar{c}(\varphi)\, - \lambda_u\, \partial_\varphi \bar{k}(\varphi)$$

where we have an analytical confirmation to the fact that k only changes continuously and that the initial change in c responds to the long-run anticipated behavior of the economy.

In order to calculate the long-run multipliers we use the implicit function theorem. Looking to the MHDS system, in particular to the dynamic equations (11a)-(11b) we see that at the steady

[21] We use the following notation. Let a function be $y = f(x,\varphi)$ we write $\partial_x y = \dfrac{\partial f(x,\varphi)}{\partial x}$.

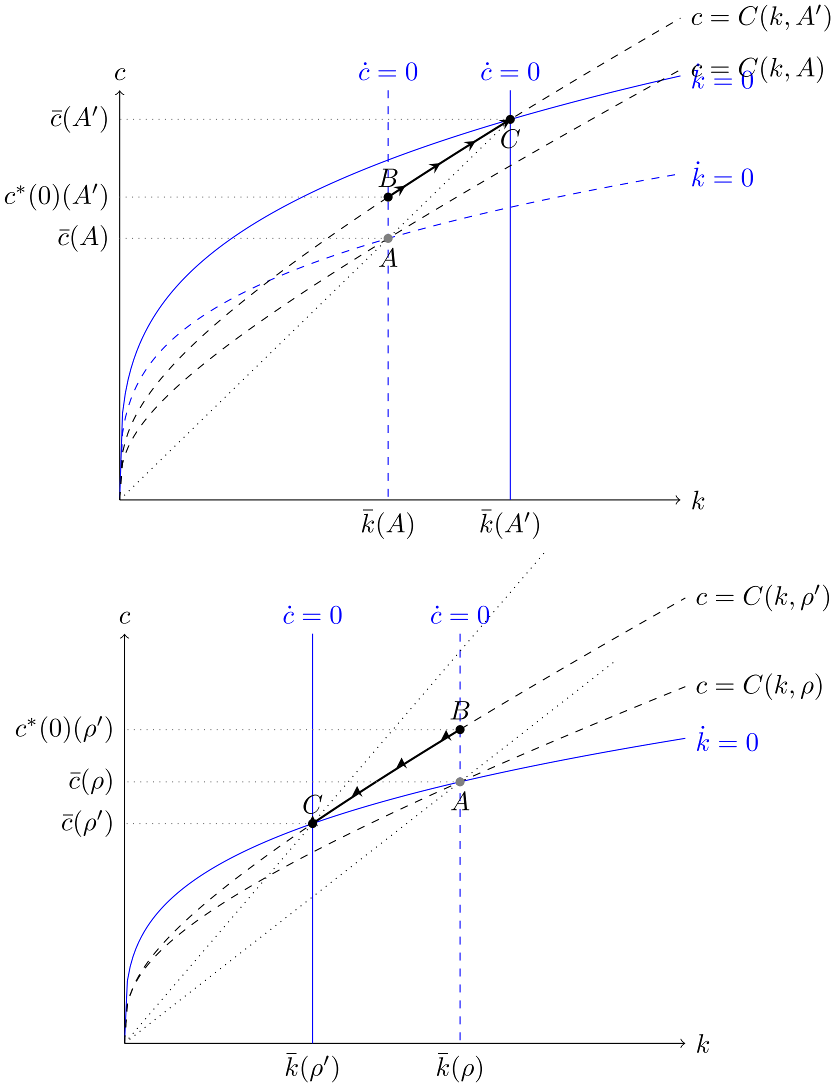

Figure 3: Dynamic adjustments for non-anticipated, permanent and constant supply (upper panel) and a demand (lower panel) shocks. In both cases I assume that the economy is in the steady state (point A). Immediately after the shock, becomes off-balance and there is an immediate jump in consumption (point B) to the new curve $C(k)$ (or the stable eigenspace in the linearized version). Across time it converges asymptotically to the new steady state (point C). We see that while a supply shock generates a pro-cyclical adjustment, the demand shock has a transient countercyclical adjustment.

state

$$\begin{cases} y(A,k) - c = 0 \\ \dfrac{c}{\theta}\big(r(A,k) - \rho\big) = 0 \end{cases}$$

Computing differentials for both the endogenous variables and the parameters, evaluating the derivatives at the steady state, we have, for a change in A

$$\begin{pmatrix} \partial_k y(\bar{k}) & -1 \\ \dfrac{\bar{c}\,\partial_k r(\bar{k})}{\theta} & \dfrac{r(\bar{k}) - \rho}{\theta} \end{pmatrix} \begin{pmatrix} dk \\ dc \end{pmatrix} + \begin{pmatrix} \partial_A\, y(\bar{k}) \\ \dfrac{\bar{c}\partial_A\, r(\bar{k})}{\theta} \end{pmatrix} dA$$

where, at the steady state $r(\bar{k}) - \rho$. As the first Jacobian matrix is matrix $\mathbf{J}$ and we already found that it has a non-zero determinant, then we can invert it to get the expressions for the long-run multipliers

$$\begin{pmatrix} \partial_A \bar{k} \\ \partial_A \bar{c} \end{pmatrix} = \begin{pmatrix} \dfrac{dk}{dA}\Big|_{k=\bar{k}} \\ \dfrac{dc}{dA}\Big|_{c=\bar{c}} \end{pmatrix} = -\begin{pmatrix} \partial_k\, y(\bar{k}) & -1 \\ \dfrac{\bar{c}\partial_k\, r(\bar{k})}{\theta} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \partial_A\, y(\bar{k}) \\ \dfrac{\bar{c}\,\partial_A\, r(\bar{k})}{\theta} \end{pmatrix}.$$

For the case of a shock in A we find comparative dynamics exercise for the cases depicted in Figure 3, performing simplifications and evaluating the derivatives at the steady state, we find the long run multipliers

$$\begin{pmatrix} \partial_A \bar{k} \\ \partial_A \bar{c} \end{pmatrix} = -\begin{pmatrix} \dfrac{\bar{c}}{A} \\ \dfrac{\rho\, c}{\theta\, A} \end{pmatrix} = \begin{pmatrix} \dfrac{\bar{k}}{(1-\alpha)\,A} \\ \dfrac{\bar{c}}{(1-\alpha)\,A} \end{pmatrix},$$

which are both positive. Therefore, the impact multipliers for consumption is

$$\partial_A c(0) = \Big(\frac{\rho - \alpha\,\lambda_u}{\alpha}\Big)\frac{\bar{k}}{(1-\alpha)\,A}$$

which is positive if and only if $\rho - \alpha\,\lambda_u = \lambda_s + (1-\alpha)\lambda_u > 0$.

3.9 Empirical implementation

If we would like to take this model to data we can fix parameters $\alpha = 0.3$, $\theta = 2$ and $\rho = 0.02$ but the parameter A can only be obtained by calibration.

To calibrate A, assume that an economy is at a steady state and obtain relevant statistics. For example, from[22] we observe that the GDP per capita in Portugal in 2019 is $y = 23.7$ and the capital output ratio is around $k/y = 3.43$. Then we could calibrate $A = 6.33484$. However, this produces a high rate of return: $r \approx 8.7\%$.

4 The simplest dynamic general equilibrium (DGE) model

4.1 The model

Instead of the existence of a central planner as a coordinating device, in a market economy the coordination is make through market transactions. We assume production is done by firms which

[22] See https://www.ine.pt/xportal/xmain?xpid=INE&xpgid=ine_destaques&DESTAQUESdest_boui=306571350&DESTAQUESmodo=2.

are owned by households. This allows us to distinguish between financial capital, a and physical capital k. Firms are a device to transform financial capital into physical capital. They distribute not only capital income but also wage income to households as a result of the production process. Firms are also on the supply side of the output market where demand comes from consumption and investment expenditures.

Furthermore, we assume:

1. preferences: households behave like in the dynastic model and have an intertemporally additive utility functional, and supply one unit of labor inelastically;

2. technology: firms use a neoclassical production function and adjust instantaneously both factor inputs, labor and physical capital and there are no financial frictions.

Households As owners of firms, household also have to finance investment expenditures through savings. We assume that the utility functional is time additive

$$\mathsf{U}[c] = \int_0^\infty u(c(t))\, e^{-\rho\, t}\, dt$$

The household problem is

$$v(a_0) = \max_c \mathsf{U}[c] \tag{20a}$$

subject to

$$\dot{a} = r(t)\, a + w(t)\ell(t) - c(t), \text{ for } t \in (0, \infty) \tag{20b}$$

$$a(0) = a_0 \tag{20c}$$

$$\lim_{t\to\infty} a(t) e^{-\int_0^t r(s) ds} \geq 0 \tag{20d}$$

The budget constraint (20b) has the following meaning: consumers' income comprise capital income $r\, a$ and labor income w, where it is assumed that consumers supply inelastically one unit of labor $\ell(t) = 1$; the difference between total income and consumption are equal to savings $s(t) = r(t)\, a + w(t)\ell(t) - c(t)$ which, if positive, it represents an increase in the ownership financial assets, which in this economy we assume it is deterministic. Differently from k in the Ramsey model $a(t)$, at time t, can be positive, if the household is a net creditor, or negative, if the household is a net debtor.

The last condition is called in the literature the **non-Ponzi game condition** (NPG). It essentially means that households can not expect to be a net debtor asymptotically. This can only be a net debtor in the short run. [23] Taken together, the constraints to the household mean that it cannot expect to use more than its initial asset position a_0, if it is a net creditor (i.e., if $a_0 > 0$) or has to repay its initial level of indebtedness, if it is an initial net debtor (i.e, if $a_0 < 0$).

In order to see this, integrate equation (20b) together with the initial condition (20c) to get

$$a(t) = e^{\int_0^t r(s) ds} \Big(a_0 + \int_0^t e^{-\int_0^s r(z) dz} \big(w(s) - c(s) \big)\, ds \Big)$$

[23] Sometimes people call this condition the transversality condition, which is a misnomer.

multiply both terms by the discount factor $e^{-\int_0^t r(s)ds}$

$$e^{-\int_0^t r(s)ds}\, a(t) = a_0 + \int_0^t e^{-\int_0^s r(z)dz}\, w(s) - c(s)\, ds$$

pass to the limit as $t \to \infty$

$$\lim_{t\to\infty} e^{-\int_0^t r(s)ds}\, a(t) = a_0 + \int_0^\infty e^{-\int_0^s r(z)dz}\, w(s) - c(s)\, ds.$$

Introduce the NPG condition,

$$\int_0^\infty e^{-\int_0^t r(s)ds}\, c(t)\, dt \leq a_0 + h^\infty(0)$$

where

$$h^\infty(0) \equiv \int_0^\infty e^{-\int_0^t r(s)ds}\, w(t)\, dt$$

is the lifetime human capital, i.e, the present value (at the time of planning) of the future wages discounted by the market interest rate.

Therefore the three constraints are equivalent to requiring that the present value of consumption does not exceed the initial financial and human capital, measured by the present value of future wage earnings.

Firms This capital represents participation in firms. Therefore, the firms balance sheet takes the form of $K(t) = N\, a(t)$, where we assume $K(t)$ to be aggregate capital and N population (or the number of households) assumed to be homogeneous and constant.

The firms' problem is static: at every moment in time,

$$\max_{K(t),L(t)} \Pi(K(t), L(t)) = F(K(t), L(t)) - r(t)K(t) - w(t)L(t)$$

where we assume that $F(\cdot)$ is homogeneous of degree one. We assume firms are price takers in all the markets and that there is no unemployment, that is $L(t) = N$.

Because of linear homogeneity, the firms problem may be simplified to

$$\max_{k(t)} \pi(k(t)) = f(k(t)) - r(t)k(t) - w(t). \tag{21}$$

Definition 2 (DGE). *A dynamic general equilibrium is an allocation* $\left(c^{eq}(t), k^{eq}(t), y^{eq}(t), w^{eq}(t), r^{eq}(t)\right)_{t\in T}$ *such that*

1. *every household solves problem* (20a)-(20d) *taking the interest rate,* r*, and the wage rate,* w*, as given but having perfect foresight on their determination;*
2. *firms solve problem* (21)*;*
3. *The compatibility condition, which takes the form of a balance sheet, holds:* $a(t) = k(t)$*.*
4. *Labor, capital and goods markets clear. In particular, the product market equilibrium condition holds:*

$$y(t) = c(t) + \dot{k}(t), \textit{for every}\, t \in T.$$

Using the PMP yield the first order optimality conditions for the household

$$\dot{c} = \frac{c}{\sigma(c)}\big(r(t) - \rho\big) \tag{22a}$$

$$\lim_{t\to\infty} u'(c(t))\, a(t) e^{-\rho t} = 0 \tag{22b}$$

together with the constraints (20a)-(20c). Equation (22a) provides an arbitrage condition between consumption and increasing the investment in financial assets. Equation (22b) is the transversality condition, which is a dual condition associated to the NPG condition (20d) by requiring that the asset position has no value in present-value terms.

The solution to the firm's problem together with the zero profit condition (because markets are competitive) yield the equations

$$f'(k(t)) = r(t) \tag{23a}$$

$$f\big(k(t)\big) - k(t)\, f'\big(k(t)\big) = w(t). \tag{23b}$$

The first equation means that the optimal the return on capital (obtained from production) is equal, at the optimum, to the interest rate, and the second means that the marginal return from employing labor is equal to the wage rate.

Because labor is supplied inelastically to firms and households supply capital to firms, equations (23a) and (23b) are identically market clearing conditions for capital and labor market. As $y(t) = f(k(t)) = f'(k(t))k(t) + w(t) = r(t)k(t) + w(t)$ then we get the equilibrium total income to households.

From the compatibility condition we have $a(t) = k(t)$. Then, household's budget constraint is formally identical to the market equilibrium condition $\dot{k} = y(t) - c(t)$.

Therefore, we can find the DGE $\big(c^{eq}(t), k^{eq}(t), y^{eq}(t), w^{eq}(t), r^{eq}(t)\big)_{t\in\mathrm{T}}$ by solving a dynamic system which is formally identical to the first optimum conditions for an optimal allocation $\big(c^*(t), k^*(t)\big)_{t\in\mathrm{T}}$ of the Ramsey model (5a)-(5d).

In the case of the isoelastic-Cobb-Douglas case the DGE satisfies the following dynamic system

$$\dot{k}^{eq} = A\,(k^{eq})^{\alpha} - c^{eq} \tag{24a}$$

$$\dot{c}^{eq} = \frac{c^{eq}}{\theta}\big(\alpha\, A\,(k^{eq})^{\alpha-1} - \rho\big) \tag{24b}$$

$$k^{eq}(0) = k_0 \tag{24c}$$

$$\lim_{t\to\infty} (c^{eq}(t))^{-\theta}\, k^{eq}(t)\, e^{-\rho t} = 0 \tag{24d}$$

This **means that the DGE is Pareto optimal**, and all our previous analysis can be reinterpreted as a result of a decentralized market allocation.

4.2 Interpreting the DGE: over-saving and perfect foresight

We have used the PMP to obtain the first order conditions for an optimum. There is a potential pitfall in following a recursive approach, by using the DPP, which allows for a discussion of one important feature of the information agents are ascribed in a DGE model.

Assume that the utility function is $u(c) = \dfrac{c^{1-\theta}-1}{1-\theta}$ and consider the consumer problem (20a)-(20d) **taking the rate of return and the wage rate as constants** $r(t) = r$ and $w(t) = w$. The HJB equation is

$$\rho\, v(a) = \max_c \Big\{ \frac{c^{1-\theta}-1}{1-\theta} + v'(a)\big(r\, a + w - c\big) \Big\}. \tag{25}$$

The policy function is $c = C(a) = (v'(a))^{-\frac{1}{\theta}}$. Substituting in the HJB equation yields the implicit ODE

$$\rho\, v(a) = \frac{1}{1-\theta}\Big(\theta\,(v'(a))^{-\frac{\theta-1}{\theta}} - 1\Big) + v'(a)\,\big(r\, a + w\big)$$

Using a similar method as the one used in Appendix B we explicitly find the value function

$$v(a) = \frac{1}{1-\theta}\left[\Big(\frac{\rho + r\,(\theta-1)}{\theta}\Big)^{-\theta}\Big(\frac{r\,a+w}{r}\Big)^{1-\theta} - \frac{1}{\rho}\right]$$

which implies that the optimal consumption is given by the policy function

$$c = C(a) = \bar{\gamma}\Big(\frac{r\,a+w}{r}\Big),\ \text{where}\ \bar{\gamma} \equiv \frac{\rho + r\,(\theta-1)}{\theta}. \tag{26}$$

If we consider the optimality conditions for the firm, the consistency condition $a = k$ and the market equilibrium condition we see that the DGE becomes a recursive system, where

$$\dot{k} = A\, k^{\alpha} - C(k)$$

and

$$C(k) = \frac{\rho}{\alpha\,\theta}\, k + \frac{\theta-1}{\theta}\, A\, k^{\alpha}, \tag{27}$$

is consumer's policy function. If we solve the differential equation we obtain

$$k(t) = \Big(\bar{k}^{1-\alpha} + \big(k_0^{1-\alpha} - \bar{k}^{1-\alpha}\big)\, e^{-\tilde{\lambda}\, t}\Big)^{\frac{1}{1-\alpha}}$$

where $\bar{k}$ is the same as in equation (15) with

$$\tilde{\lambda} = \frac{\rho}{\theta}\,\Big(\frac{1-\alpha}{\alpha}\Big).$$

If we compare this solution to the Ramsey case dealt in section 3.8.2, we observe: first, now we obtain a closed form solution while for the Ramsey model we could not obtain a closed form solution; second, we obtain an explicit solution to solution path (27) while this was not possible for the centralized model; third, both consumption and the capital stock converge to the same steady state; and fourth, the main difference is related to the speed of adjustment, which is now exactly $\tilde{\lambda}$ and it was approximated by λ_s in the Ramsey case.

Looking to equation (27) we observe that, as in our discussion to the "heteroclinic" approach to approximating the optimal path for the centralized problem, it is an average of the same two schedules $c = \dfrac{\rho}{\alpha} k$ and $c = A\, k^{\alpha}$, but now we are able to find the explicit weight $\omega = \frac{1}{\theta}$. This means that a linear solution along curve $c = \dfrac{\rho}{\alpha} k$ will occur for $\theta = 1$ and not $\theta = \alpha$ as in the Ramsey model.

This means that compared to the model in subsection 3.8.2 the equilibrium trajectory determined in this way is shifted down, which means that now **there is over-saving**. We show next the reason for over-saving: solving the model this way implicitly assumes that the consumer is myopic, and takes the factor prices as constants, and does not take into consideration that by saving now it depresses the future interest rate because of the existence of decreasing marginal returns.

It can be proved (see the proof in Appendix C) that the optimal consumption satisfies

$$c^{eq}(t) = \frac{a(t) + h^{\infty}(t)}{\int_t^{\infty} e^{-\int_t^s \gamma(z)dz} ds}, \quad \text{for } t \in (0, \infty) \tag{28}$$

where

$$\gamma(t) \equiv \frac{(\theta - 1)\, r(t) + \rho}{\theta}$$

and

$$h^{\infty}(t) \equiv \int_t^{\infty} e^{-\int_t^s r(s)ds}\, w(t)\, dt$$

is the human capital at time t. In this case the rate of consumption growth is endogenous and time varying, because the interest rate is a function of the capital stock $r(t) = \alpha\, A\, k(t)^{1-\alpha}$ which depends on consumption. If the interest rate were constant, i.e, $r(t) = r$ for time t onward, we would obtain the policy function in equation (26).

This clarifies why the equilibrium in this case is different from the one derived from using the PMP together with the market equilibrium condition for the rate of return as in the previous subsection: if the consumer takes both the interest rate and the wage rates as constants it does not incorporate the effect of the increase in savings on capital accumulation which will increase wages and decrease the rate of return on capital. It is as an externality which is not internalized.

4.3 DGE and the HJB equation

We can obtain a dynamic representation of the DGE dynamics while using the DPP for solving the household problem and using the envelop theorem as we see next.

First, after finding the HJB equation (25) we can take derivatives as regards the state variable a, evaluated at the optimum. by taking into account that fact that the policy function is a function of the state variable $c = C(a)$,

$$\rho\, v'(a) = u'(c)\, C'(a) + v''(a)\, \big(r\, a + w - c\big) + v'(a)\, \big(r - C'(a)\big).$$

As the optimality condition $u'(c) = v'(a)$ should hold then

$$\big(\rho - r\big)\, v'(a) = v''(a)\, \big(r\, a + w - c\big) = v''(a)\, \dot{a} = u'(c)\, \dot{c}$$

because both c and a are function of time, which implies $u'(c)\, \dot{c} = v''(a)\, \dot{a}$.

Second, using the balance sheet condition for firms, $k(t) = a(t)$, for every $t \in \mathbb{R}_+$ and the first order condition (23a) for firms , we have the equilibrium condition (33a).

Third, the balance sheet condition for firms together with the returns for capital and labor from firms, that are distributed to households as returns of capital and labor yield $ra + w = f'(k)\, k + (f(k) - f'(k)\, k) = f(k)$, which is total income of households at the equilibrium. This together with the fact that savings take the form of change in equity that finances capital accumulation, $f(k) - c = \dot{a} = \dot{k}$, yields the equilibrium condition (33b).

Summing up, through this approach we recover all the dynamic system which represents both the Ramsey model, and the simple DGE model given in equations (33b)-(24d), with the exception of the transversality condition.

This means that this approach does not provide a terminal condition for the DGE dynamics, as the one in which the optimality for the household are derived by using the PMP approach. However the two approached lead to the same equilibrium solution if the equilibrium path is unique, that is, if it is a saddle path. [24]

4.4 Final remarks

If we assume perfect foresight, the PMP approach allows for the right determination of the DGE path. The DPP approach seems simpler to use if the technology of production is linear (as in endogenous growth models) or in models in which there is not perfect foresight. Using the DPP approach to solve the household problem can lead to the same dynamic representation of the GDE as the one which uses the PMP if the equilibrium conditions are obtained by using the envelope theorem

5 Endogenous labor supply

In the previous model the labor input is constant along the adjustment to the steady state, meaning that we should expect the labor input (number of hours of works per worker times the number of workers) would stay constant along the business cycle. In this case the wage rate is determined by labor demand (I.e, by firms) and the labor input by supply (i.e, by households).

However, there is a stylized fact that although hours worked is less volatile than the GDP the total employment is almost as volatile as the GDP. This means that total labor effort should be treated as endogenous.

A benchmark way to introduce the endogeneity of employment is to consider the supply of labor as endogenous. Next we follow the neoclassical theory of labor supply by considering labor as a substitute to consumption. [25]

In this competitive equilibrium DGE framework this can be done by assuming that workers face a trade-off between work and leisure: while labor has a disutility because it reduces leisure, therefore reducing utility, if generates wage earnings which allow for increases in consumption, which increases utility.

5.1 A generic DGE model with endogenous labor supply

In the previous simple benchmark version of the model, we assumed that every household supplied elastically one unit of labor supply, i.e. $\ell = 1$, where ℓ represents work effort (hours worked plus

[24] Ekeland, 2010 discusses the the terminal conditions for this case.

[25] See Cahuc and Zylberberg, 2004, ch. 1.

effort per hour). Now we assume that labor effort, [26] ℓ, is a new variable which can take any positive number.

Household's problem: The utility functional of households depends now on the paths of both consumption and work effort $\big(c(t), \ell(t)\big)_{t\in\mathbb{R}_+}$,

$$\mathsf{U}[c,\ell] = \int_0^\infty u(c(t),\ell(t))\, e^{-\rho\, t}\, dt$$

The household's problem now is

$$v(a_0) = \max_{c,\ell} \mathsf{U}[c,\ell] \tag{29a}$$

subject to

$$\dot{a} = r(t)\, a + w(t)\ell - c(t), \ \text{for}\, t \in (0,\infty) \tag{29b}$$

$$a(0) = a_0 \tag{29c}$$

$$\lim_{t\to\infty} a(t) e^{-\int_t^\infty r(s)ds} \geq 0 \tag{29d}$$

(29e)

Firms' problem The firms' problem is the same as in the previous DGE model.

DGE In a homogeneous agent perfect foresight environment with competitive product and factor markets, the DGE is Pareto optimal and is equivalent to the following centralized problem.

5.2 A centralized version with endogenous labor supply

A central planner chooses optimally consumption and the labor effort in order to maximize an utilitarian social welfare functional subject to the budget constraint of the economy, given the initial capital stock, such that an intertemporal budget (or a sustainability) constraint is satisfied:

$$v(k_0) = \max_{c,\ell} \int_0^\infty u(c(t),\ell(t))\, e^{-\rho\, t}\, dt \tag{30a}$$

subject to

$$\dot{k} = f\big(k(t),\ell(t)\big) - c(t), \ \text{for}\, t \in (0,\infty) \tag{30b}$$

$$k(0) = k_0 \tag{30c}$$

$$\lim_{t\to\infty} k(t) e^{-\int_t^\infty r(s)ds} \geq 0 \tag{30d}$$

The model is said to be neo-classical if the utility function, $u(c,\ell)$ is increasing and concave in c and decreasing and convex in ℓ, and the production function is increasing in both inputs, displays decreasing marginal returns and is concave, and both functions are sufficiently smooth such that there are singularities are ruled out. The last property allows us to invoke the implicit function theorem to determine uniquely the marginal utilities and marginal productivities.

Defining the current-value Hamiltonian function by

$$H(k,q,c,\ell) = u(c,\ell) + q\big(f(k,\ell) - c\big)$$

[26] We consider labor effort and not the number of hours worked to avoid the discontinuities which are introduced by the fact that there is a maximum number of hours that can be worked, differently from the case of work effort.

the necessary conditions for an optimum $\big(c^*(t), \ell^*(t), k^*(t)\big)_{t\in\mathbb{R}_+}$ are [27]

$$u_c(c,\ell) = q \tag{31a}$$

$$u_\ell(c,\ell) = q\, f_\ell(k,\ell) \tag{31b}$$

$$\dot{q} = q\big(\rho - f_k(k,\ell)\big) \tag{31c}$$

$$\lim_{t\to\infty} q(t)\, k(t)\, e^{-\rho t} = 0 \tag{31d}$$

together with the admissibility conditions (30b) and (30c), where the partial derivatives of the utility function are denoted by $u_j(c,\ell) \equiv \frac{\partial u(c,\ell)}{\partial j}$, for $j = c,\ell$ and the marginal productivities are denoted by $f_j(k,\ell) \equiv \frac{\partial f(k,\ell)}{\partial j}$, for $j = k,\ell$.

There are two main versions of the model, depending on the separability of the utility function:

1. the additive utility case, in which the utility function is

$$u(c,\ell) = u(c) - v(\ell)$$

   satisfying $u''(c) < 0 < u'(c)$ and $v'(\ell) > 0$ and $v''(\ell) > 0$, and consumption and labor are Edgeworth independent $u_{c\ell}(c,\ell) = 0$;

2. non-additively separable cases comprise several possibilities:

   (a) the Cobb-Douglas case

$$u(c,\ell) = u(v(c,\ell)) = \frac{\big(c^\gamma (1-\ell)^{1-\gamma}\big)^{1-\theta} - 1}{1-\theta}$$

   (b) the Greenwood et al., 1988, or GHH case

$$u(c,\ell) = u(c - v(\ell))$$

   (c) the King et al., 1988 or KPR case

$$u(c,\ell) = u(c - v(c,\ell))$$

The strategy to solving the model is different depending on the the assumption regarding the separability of the utility function.

[27] From now, although we are refering to optimal allocations, we do not introduce the notation * in cases in which there is no ambiguity.

5.3 Additively separable utility

If the utility function is additively separable, then $u_c(c,\ell) = u'(c)$ and $u_\ell(c,\ell) = -v'(\ell)$. Then we can use equation (31a) to eliminate q, yielding the arbitrage equation for labor, equation (31b),

$$v'(\ell) = u'(c)\,w(k,\ell). \tag{32}$$

This arbitrage equation is for a centralized economy. But we can interpret it as an equilibrium condition in the labor market for a decentraized economy, in which the supply for labor is implicitly determined from $v'(\ell) = u'(c)\,w$, and the demand from labor is determined from the firms' optimality condition $w = f_\ell(k,\ell)$. The equilibrium in the labor market is depicted in Figure 4

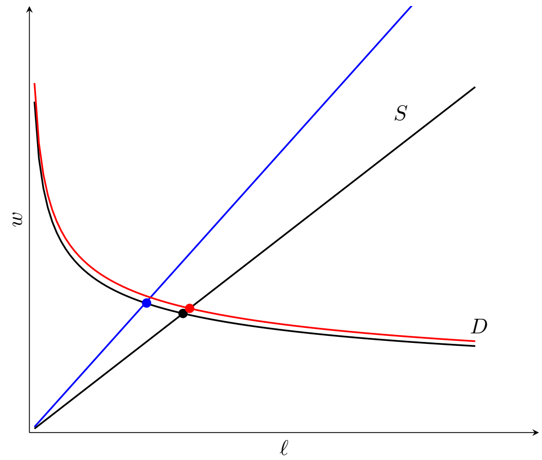

Figure 4: Equilibrium in labor supply. Perturbations: increase in the demand for labor k (red); increase in the supply of labor c (blue)

If there is an increase in the stock of capital the demand curve, $w = f_\ell(k,\ell)$, moves up which causes an increase in both the equilibrium level for ℓ and the wage rate, w. If there is an increase in consumption the supply curve, $w = u'(c)/v'(\ell)$, moves up causing an increase in wages but a decrease in the equilibrium effort level.

In centralized setting both supply and demand are jointly determined by the aggregate optimal allocation of consumption and labor effort. The condition is the same as in a market economy in there are no externalities.

If the utility and the production functions are sufficiently smooth, we can apply the implicit function theorem to equation (32) to obtain the optimal work effort as a function of consumption and the stock of capital

$$\ell = L(c,k),$$

which corresponds to the equilibrium point in Figure 4. Taking the differential to equation (32) we find $d\ell = L_c\,dc + L_k\,dk$, where the partial derivatives are

$$L_c(c,k) = \frac{u''(c)\,f_\ell(\ell,k)}{v''(\ell) - u'(c)\,f_{\ell\ell}(k,\ell)} < 0$$

$$L_k(c,k) = \frac{u'(c)\,f_{\ell k}(\ell,k)}{v''(\ell) - u'(c)\,f_{\ell\ell}(k,\ell)} > 0,$$

which correspond to the change ℓ in Figure 4.

This means that the optimal work effort unambiguously decreases with consumption, meaning that consumption and labor effort are Hicksian substitutable, and increases with expansion of the stock of capital, because it increases the wage rate $w_k(k,\ell) = f_{\ell k}(\ell,k) > 0$.

As we are able to isolate q, and we are able to obtain the optimal labor effort uniquely (if there are no singularities) then the MHDS becomes

$$\dot{k} = F(k,c) - c \tag{33a}$$

$$\dot{c} = \frac{c}{\sigma(c)}\big(R(k,c) - \rho\big) \tag{33b}$$

$$k(0) = k_0 \tag{33c}$$

$$\lim_{t\to\infty} u'(c(t))\,k(t)\,e^{-\rho t} = 0 \tag{33d}$$

where the rate of return of capital is

$$R(k,c) = r\big(k, L(c,k)\big) = f_k\big(k, L(c,k)\big)$$

and output is

$$Y(k,c) = y\big(k, L(c,k)\big) = f\big(k, L(c,k)\big).$$

These two functions now depend not only on the stock of capital (as in the inelastic labor supply model) but also on consumption. This is because the work effort is now endogenous and depends on the leisure-consumption arbitrage at the level of the household.

Clearly, the reduced form rate of return

$$R_c(c,k) = f_{k\ell}(k,\ell)\,L_c(c,k) < 0$$
$$R_k(k,c) = f_{kk}(k,\ell) + f_{k\ell}(k,\ell)\,L_k(c,k)$$

is a decreasing function of consumption and an ambiguously signed function of the capital stock. In the last case, there is a negative direct effect (as in the exogenous labor version of the Ramsey model), and a positive indirect effect of the increase in the equilibrium labor effort, working through the positive income effect of labor at the level of the consumer. However, as a consequence of the concavity of the production function $f(\cdot)$ and of the convexity of $v(\cdot)$ the direct effect dominates and the rate of return reduces with an increase in the capital stock

$$R_k(k,c) = \frac{v''(\ell)f_{kk}(k,\ell) - u'(c)\left(f_{kk}(\ell,k)f_{\ell\ell}(\ell,k) - (f_{k\ell}(\ell,k)^2)\right)}{v''(\ell) - u'(c)\,f_{\ell\ell}(k,\ell)} < 0.$$

In any case, the marginal reduction of the rate of return for an increase in one unit in the stock of capital is smaller than in the exogenous labor case.

Furthermore, we find that output decreases with consumption

$$Y_c(c,k) = f_\ell(k,\ell)L_c(c,k) < 0,$$

because of the Hicksian substitutability between consumption and labor effort, and it is a positive function of the stock of capital

$$Y_k(c,k) = f_k(k,\ell) + f_\ell(k,\ell)L_k(c,k) > 0.$$

In the last equation an increase in the capital stock has both a direct effect (as in the exogenous labor version of the model) and an indirect effect through an increase in the equilibrium labor input working through an income effect at the level of the consumer.

5.3.1 Particular case

Assume the following utility and production functions:

$$u(c,\ell) = \frac{c^{1-\theta}-1}{1-\theta} - \psi\,\frac{\ell^{1+\xi}}{1+\xi},\ \text{where}\,\theta>0,\ \psi>0,\ \xi>0$$
$$f(k,\ell) = A\,k^{\alpha}\,\ell^{1-\alpha},\ \text{where}\ 0<\alpha<1,\ A>0$$

The parameter ξ is called Frisch elasticity and ψ controls the relative utility from consumption and leisure.

The optimal labor effort is

$$\ell^* = L(k,c) \equiv \left(\frac{A\,(1-\alpha)}{\psi}\,c^{-\theta}\,k^{\alpha}\right)^{\frac{1}{\alpha+\xi}}.$$

This implies that output is

$$y^* = Y(k,c) = \left(A^{1+\xi}\left(\frac{1-\alpha}{\psi}\right)^{1-\alpha} c^{-\theta\,(1-\alpha)}\,k^{\alpha\,(1+\xi)}\right)^{\frac{1}{\alpha+\xi}},$$

and the marginal productivity of capital (which is equal to the real rate of return)

$$r^* = R(k,c) = \alpha\left(A^{1+\xi}\,\left(\frac{1-\alpha}{\psi}\right)^{1-\alpha} c^{-\theta\,(1-\alpha)}\,k^{-\xi\,(1-\alpha)}\right)^{\frac{1}{\alpha+\xi}},$$

and they satisfy the same relationship as in the exogenous labor model $\alpha\,Y(k,c) = R(k,c)\,k$.

If we constrain the solution to positive values for (k,c), the steady state is the unique point $(\bar{k},\bar{c})$ satisfying jointly $R(k,c)=\rho$ and $Y(k,c)=c$. From the previous relationship between output and the rate of return, we find that $\alpha\,\bar{c} = \rho\,\bar{k}$. Therefore

$$\bar{k} = \left(A^{1+\xi}\left(\frac{1-\alpha}{\psi}\right)^{1-\alpha}\left(\frac{\alpha}{\rho}\right)^{\alpha+\xi+\theta\,(1-\alpha)}\right)^{\frac{1}{(1-\alpha)(\theta+\xi)}} \tag{34a}$$
$$\bar{c} = \left(A^{1+\xi}\left(\frac{1-\alpha}{\psi}\right)^{1-\alpha}\left(\frac{\alpha}{\rho}\right)^{\alpha\,(1+\xi)}\right)^{\frac{1}{(1-\alpha)(\theta+\xi)}}. \tag{34b}$$

Substituting in function $L(k,c)$ we obtain the steady state labor effort

$$\bar{\ell} = \left(A^{1-\theta}\left(\frac{1-\alpha}{\psi}\right)^{1-\alpha}\left(\frac{\alpha}{\rho}\right)^{\alpha\,(1-\theta)}\right)^{\frac{1}{(1-\alpha)(\theta+\xi)}}.$$

Looking at equations (34a) and (34b) we readily see that: first, an increase in A (TFP) increases the steady state levels for the capital stock, consumption and the labor effort, and, consequently, of output; second, a demand shock in which agents decrease their relative preference for leisure (i.e, if ψ increases) has the opposite effect; and, an increase in the rate of time preference (an increase in ρ) reduces capital and consumption but increases the labor effort in the steady state. The last result is natural because an increase in ρ reduces savings which forces a substitution of capital to labor in production.

In order to study the transition dynamics, we linearize the MHDS (33a) -(33b) in the neighborhood of the steady state $(\bar{k},\bar{c})$. The Jacobian evaluated at the steady state is

$$\mathbf{J} = \begin{pmatrix} \dfrac{\rho\,(1+\xi)}{\alpha+\xi} & -1-\dfrac{\theta\,(1-\alpha)}{\alpha+\xi} \\ -\dfrac{\rho^2\,\xi\,(1-\alpha)}{\alpha\,\theta\,(\alpha+\xi)} & -\dfrac{\rho\,(1-\alpha)}{\alpha+\xi} \end{pmatrix}$$

We readily observe that the steady state is a saddle point because

$$\text{trace}(\mathbf{J}) = \rho > 0 \text{ and } \det(\mathbf{J}) = -\frac{\rho^2 (1-\alpha)(\theta+\xi}{\alpha\,\theta\,(\alpha+\xi)} < 0$$

and the eigenvalues are

$$\lambda_s = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 + \frac{\rho^2 (1-\alpha)(\theta+\xi}{\alpha\,\theta\,(\alpha+\xi)}} < 0 < \rho < \lambda_u = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 + \frac{\rho^2 (1-\alpha)(\theta+\xi}{\alpha\,\theta\,(\alpha+\xi)}}.$$

This means that the solution to the centralized problem is unique, and that the associated DGE is determinate: there is a unique trajectory starting from $k(0) = k_0$ and converging to the steady state.

The representations of the eigenvector associated to the negative eigenvalue, λ_s, we adopt is

$$\mathbf{P}^s = \begin{pmatrix} 1 \\ P_2^s \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{\xi\,(\alpha+\xi)\,\lambda_s\,\lambda_u}{(\theta+\xi)\,(\lambda_s\,(\alpha+\xi)+\rho\,(1-\alpha))} \end{pmatrix}$$

if we use the generic relationships $\text{trace}(\mathbf{J}) = \lambda_s + \lambda_u = \rho$ and $\det(\mathbf{J}) = \lambda_s\,\lambda_u$. Because the denominator of second element of the eigenvector has an ambiguous sign, this implies that the slope of the eigenspace is ambiguous as well.

In order to see this closely, we study the short run mutipliers for a permanent and unanticipated TFP shock. In order to obtain them, we first take the Jacobian for the MHDS (33a) -(33b), evaluated at the steady state

$$\mathbf{J}_A = \begin{pmatrix} 1 \\ \frac{\rho}{\theta} \end{pmatrix} \frac{\rho\,(1+\xi)\,\bar{k}}{\alpha\,\theta\,(\alpha+\xi)\,A}.$$

Solving $-\mathbf{J}^{-1}\,\mathbf{J}_A$ we obtain the long-run multipliers for a shock in A

$$\frac{\partial\bar{k}}{\partial A} = \frac{(1+\xi)\,\bar{k}}{(1-\alpha)\,(\theta+\xi)\,A} > 0 \tag{35a}$$

$$\frac{\partial\bar{c}}{\partial A} = \frac{\rho}{\alpha}\,\frac{\partial\bar{k}}{\partial A} > 0 \tag{35b}$$

$$\tag{35c}$$

The short run multipliers are

$$\frac{\partial k(t)}{\partial A} = \frac{\partial\bar{k}}{\partial A}\left(1 - e^{\lambda_s\,t}\right), \text{ for } t \in [0,\infty)$$
$$\frac{\partial c(t)}{\partial A} = \frac{\partial\bar{c}}{\partial A} - P_2^s\,\frac{\partial\bar{k}}{\partial A}\,e^{\lambda_s\,t}, \text{ for } t \in [0,\infty).$$

Therefore along the linear approximation to the stable manifold we have

$$\frac{\partial c(t)}{\partial A} = \frac{\partial\bar{c}}{\partial A} + P_2^s\left(\frac{\partial k(t)}{\partial A} - \frac{\partial\bar{k}}{\partial A}\right)$$

where the slope has an ambiguous sign

$$P_2^s = \frac{\xi\,(\alpha+\xi)\,\lambda_s\,\lambda_u}{(\theta+\xi)\,\big(\lambda_s\,(\alpha+\xi)+\rho\,(1-\alpha)\big)}$$

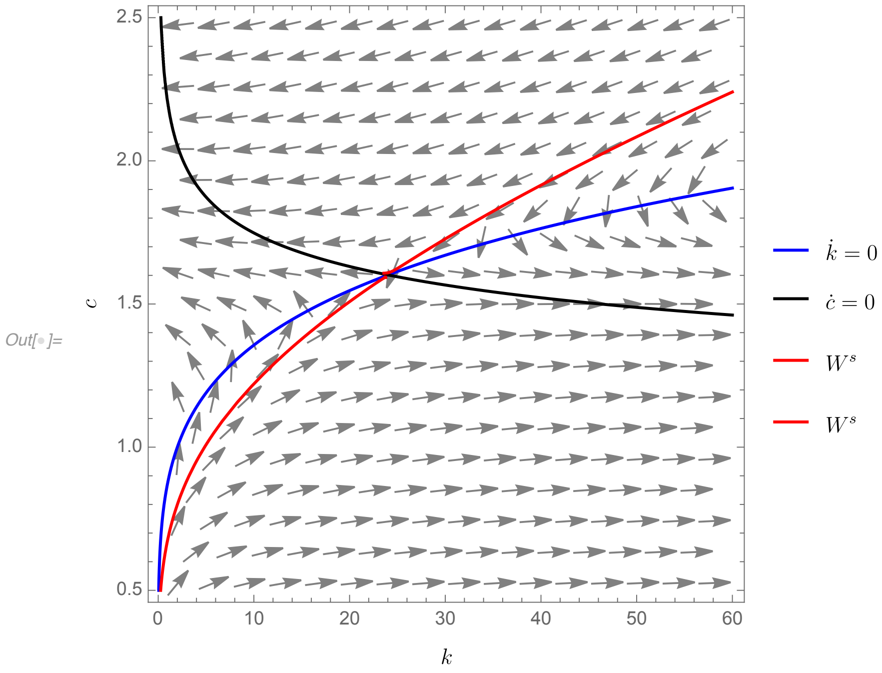

Figure 5: Phase diagram for the endogenous labor model with an additive separable utility function: $u(c,\ell) = \dfrac{c^{1-\theta}-1}{1-\theta} - \psi\,\dfrac{\ell^{1+\xi}}{1+\xi}$.

because the numerator is negative and the denominator can have any sign. However, $P_2^s > 0$ if and only if $0 > \lambda_s\,(\alpha+\xi) > -\rho\,(1-\alpha)$.

Figure 5 shows the phase diagram for a particular choice of parameters yielding a positive P_2^s. We see that, when comparing with the exogenous labor case, the isocline $\dot{c}=0$ is negatively sloped in the diagram (k,c) (and not vertical sloped) and the isocline $\dot{k}=0$ is positively sloped (as in the previous case).

We also observe in the phase diagram (see Figure 5) that the stable manifold is positively sloped, which means that along the optimal paths consumption and the capital stock are positively related. Given the properties of function $L(c,k)$ this implies that the adjustment of the labor effort is ambiguous, because of its responds positively to the increase in wages but negatively to the increase in consumption.

Figure 6 represents the multipliers for an increase in TFP: the impact multipliers for consumption is positive, $\frac{\partial c(0)}{\partial A} > 0$, there is a transition increase both in consumption and the capital stock, $\frac{\partial c(t)}{\partial A} > 0$ and $\frac{\partial k(t)}{\partial A} > 0$, towards reaching the steady state multipliers, $\frac{\partial c(\infty)}{\partial A} = \frac{\partial \bar{c}}{\partial A} > 0$ and $\frac{\partial k(\infty)}{\partial A} = \frac{\partial \bar{k}}{\partial A} > 0$.

What are the effects on labor effort and output ?

The short-run multipliers for labor effort are determined from

$$\frac{\partial \ell(t)}{\partial A} = \frac{\partial \bar{\ell}}{\partial A} + \frac{\partial \bar{\ell}}{\partial k}\,\frac{\partial k(t)}{\partial A} + \frac{\partial \bar{\ell}}{\partial c}\,\frac{\partial c(t)}{\partial A}$$

where

$$\frac{\partial \ell}{\partial A} = \frac{\ell}{(\alpha+\xi)\,A},\quad \frac{\partial \ell}{\partial k} = \frac{\alpha\,\ell}{(\alpha+\xi)\,k},\quad \frac{\partial \ell}{\partial c} = -\frac{\theta\,\ell}{(\alpha+\xi)\,c}$$

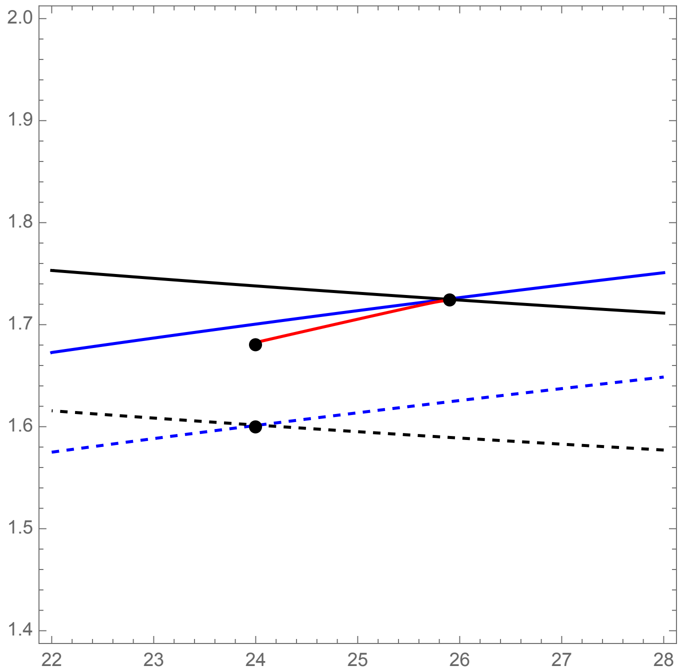

Figure 6: Phase diagram a positive TFP shock.

Therefore the impact multiplier for labor effort,

$$\frac{\partial \ell(0)}{\partial A} = \frac{\overline{\ell}}{(\alpha+\xi)\,A} - \frac{\theta\,\overline{\ell}}{(\alpha+\xi)\,\overline{c}}\,\frac{\partial c(0)}{\partial A},$$

is ambiguous, because the first term is positive and the second is negative. The long-run multiplier is negative

$$\begin{aligned}\frac{\partial \ell(\infty)}{\partial A} &= \frac{\partial \overline{\ell}}{\partial A} + \frac{\partial \overline{\ell}}{\partial k}\,\frac{\partial \overline{k}}{\partial A} + \frac{\partial \overline{\ell}}{\partial c}\,\frac{\partial \overline{c}}{\partial A}\\ &= \frac{(\alpha-\theta)\,\overline{\ell}}{(1-\alpha)(\theta+\xi)\,A} < 0\end{aligned}$$

meaning that an increase in TFP induces a long-run substitution of labor with capital.

We can determine in an analogous way the multipliers for output:

$$\frac{\partial y(t)}{\partial A} = \frac{\partial \overline{y}}{\partial A} + \frac{\partial \overline{y}}{\partial k}\,\frac{\partial k(t)}{\partial A} + \frac{\partial \overline{y}}{\partial c}\,\frac{\partial c(t)}{\partial A}$$

where

$$\frac{\partial y}{\partial A} = \frac{(1+\xi)\,y}{(\alpha+\xi)\,A},\ \frac{\partial y}{\partial k} = \frac{\alpha\,(1+\xi)\,y}{(\alpha+\xi)\,k},\ \frac{\partial y}{\partial c} = -\frac{\theta\,(1-\alpha)\,y}{(\alpha+\xi)\,c}.$$

Again we find that the impact multiplier is ambiguous

$$\frac{\partial y(0)}{\partial A} = \frac{(1+\xi)\,\overline{y}}{(\alpha+\xi)\,A} - \frac{\theta\,(1-\alpha)\,\overline{y}}{(\alpha+\xi)\,\overline{c}}\,\frac{\partial c(0)}{\partial A}$$

but the long run multiplier is unambiguously positive

$$\begin{aligned}\frac{\partial y(\infty)}{\partial A} &= \frac{\partial \overline{y}}{\partial A} + \frac{\partial \overline{y}}{\partial k}\,\frac{\partial \overline{k}}{\partial A} + \frac{\partial \overline{y}}{\partial c}\,\frac{\partial \overline{c}}{\partial A}\\ &= \frac{(1+\xi)\,\overline{y}}{(\alpha+\xi)\,A}\left(\frac{\alpha+\xi+\theta\,(1-\alpha)^2}{(1-\alpha)\,(\theta+\xi)}\right) > 0.\end{aligned}$$

We say **TFP shocks are pro-cyclical**: although there is a reduction in labor effort, via substitution of labor for capital in production, the direct effect of the productivity shock and the increase in the capital stock more than compensate that effect.

5.4 Non-additively separable utility

If the utility function is non-separable, we cannot separate the effect of consumption over q. This means that the marginal utility of consumption responds to both changes in consumption and labor effort. In this case, we can use the implicit function theorem in equation (31a) and (31b) to solve both consumption and labor effort as a function of the adjoint variable q, $c = C(q,k)$ and $\ell = L(q,k)$. In appendix D we prove that all the partial derivatives in functions $C(\cdot)$ and $L(\cdot)$ have ambiguous signs. However, it we expect that $L_k > 0$ and $C_q < 0$ The MHDS becomes

$$\dot{q} = q\big(\rho - f_k\big(k, L(q,k)\big)\big) \tag{36a}$$

$$\dot{k} = y\big(k, L(q,k)\big) - C(q,k) \tag{36b}$$

$$\lim_{t\to\infty} q(t)\, k(t)\, e^{-\rho t} = 0 \tag{36c}$$

We provide in the problem set 2 several particular cases of this model.

6 References

The original Ramsey model is presented in Ramsey, 1928 and was rediscovered by Cass, 1965 and Koopmans, 1965.

There are several textbook presentations of this model: in continuous time, recent presentations can be found in Heijdra, 2009, sec. 14.5 Acemoglu, 2009, ch.8, Romer, 2019, ch 2 and Alogoskoufis, 2019, ch. 4.

The model with endogenous labor can be found in Wickens, 2008, sec 4.6 (discrete time version) and Heijdra, 2009, ch. 7.

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A Solving the Ramsey problem for $\theta = \alpha$

A.1 Using the PMP

Looking at the Euler equation (11b) we can write it $\dot{c} = \gamma_c(t)c(t)$, where γ_c is the (variable growth rate of consumption. It has a solution of type

$$c(t) = c(0)e^{\int_0^t \gamma(s)\,ds}, \text{ where } \gamma(t) \equiv Ak^{\alpha-1}(t) - \frac{\rho}{\alpha}$$

where we do no know $c(0)$. The idea is to obtain it from the transversality condition (11d). There is another difficulty related to the fact that γ is a function if k and the transversality equation also depends on k. In order to circumvent both difficulties define

$$z(t) \equiv \frac{k(t)}{c(t)}$$

Therefore, taking log derivatives of time and substituting equations (11a) and (11b) we find ´

$$\frac{\dot{z}}{z} = \frac{\dot{k}}{k} - \frac{\dot{c}}{c} = A\,k^{\alpha-1} - \frac{c}{k} - \Big(Ak^{\alpha-1}(t) - \frac{\rho}{\alpha}\Big) = -\frac{1}{z} + \frac{\rho}{\alpha}$$

which is a linear ODE (without time varying coefficients)

$$\dot{z} = \frac{\rho}{\alpha}\,z - 1.$$

This ODE has the solution

$$z(t) = \frac{\alpha}{\rho} + \Big(z(0) - \frac{\alpha}{\rho}\Big)e^{\frac{\rho}{\alpha}\,t}$$

where we do not know $z(0) = k(0)/c(0)$ because $c(0)$ is unknown. We find $c(0)$ by using the transversality condition (11d). Substituting in the transversality condition (11d) yields

$$\begin{aligned}
\lim_{t\to\infty} c(t)^{-\alpha}\,k(t)\,e^{-\rho\,t} &= \lim_{t\to\infty} c(t)^{1-\alpha}\,z(t)\,e^{-\rho\,t} \\
&= \lim_{t\to\infty} c(t)^{1-\alpha}\Big[\frac{\alpha}{\rho}\,e^{-\rho\,t} + \Big(\frac{k(0)}{c(0)} - \frac{\alpha}{\rho}\Big)e^{\rho\left(\frac{1-\alpha}{\alpha}\right)\,t}\Big] = \\
&\text{(because } z(0) = k(0)/c(0)\text{)} \\
&= 0 + \lim_{t\to\infty} c(t)^{1-\alpha}\Big[\Big(\frac{k(0)}{c(0)} - \frac{\alpha}{\rho}\Big)e^{\rho\left(\frac{1-\alpha}{\alpha}\right)\,t}\Big] \\
&\text{(because } \lim_{t\to\infty} e^{-\rho\,t} = 0\text{)}.
\end{aligned}$$

It is equal to zero if and only if $c(0) = \dfrac{\rho}{\alpha}k(0)$ because $c(t) > 0$ and $\lim_{t\to\infty} e^{\rho\left(\frac{1-\alpha}{\alpha}\right)\,t} = \infty$. Then, $z(t) = \bar{z} = \dfrac{\alpha}{\rho}$ is a constant, which means that $c(t) = \dfrac{\rho}{\alpha}k(t)$ for all $t \in [0, \infty)$.

Then, if we choose the optimal consumption rule, we can obtain the optimal path for capital by solving problem (13). This is a Bernoulli ODE than can be transformed into a linear ODE. Define $x(t) \equiv k(t)^{1-\alpha}$. Then

$$\frac{\dot{x}}{x} = (1-\alpha)\,\frac{\dot{k}}{k} = (1-\alpha)\Big(Ak^{\alpha-1} - \frac{\rho}{\alpha}\Big) = (1-\alpha)\Big(\frac{A}{x} - \frac{\rho}{\alpha}\Big),$$

which is equivalent to the linear ODE

$$\dot{x} = (1-\alpha)\Big(A - \frac{\rho}{\alpha}\,x\Big)$$

which has the solution

$$x(t) = \frac{\alpha\,A}{\rho} + \Big(x(0) - \frac{\alpha\,A}{\rho}\Big)\,e^{-\rho\left(\frac{1-\alpha}{\alpha}\right)\,t}.$$

By transforming back to k we find the optimal capital accumulation as a function of time (14).

A.2 Using the DPP

The HJB equation, if $\theta = \alpha$, is

$$\rho\,v(k) = \max_{c}\Big\{\frac{c^{1-\alpha}-1}{1-\alpha} + v'(k)\big(A\,k^{\alpha} - c\big)\Big\}$$

The optimality condition for consumption is

$$c^{-\alpha} = v'(k) \Rightarrow C(k) = (v'(k))^{-\frac{1}{\alpha}}$$

After substitutiong $C(k)$ back in the HJB equation yields the implicit differencial equation for $v(k)$ over k,

$$\rho\,v(k) = \frac{\alpha}{1-\alpha}(v'(k))^{\frac{\alpha-1}{\alpha}} + v'(k)\,A\,k^{\alpha} - \frac{1}{1-\alpha}$$

Although this is a highly non-linear equation we can obtain an explicit solution by using the method of undetermined coefficients. Unfortunately, it does not provide a constructive way to obtain the solution: we conjecture a functional form of the equation, depending on unknown parameters; if the functional form is right, by substituting in the HJB equation, we would obtain the values of those parameters. Of course that depends on our ability to conjecture the right explicit functional form (assuming that it exists).

Let us conjecture that the solution is of type

$$v(k) = \beta_0 + \beta_1\,k^{1-\alpha}$$

where β_0 and β_1 are arbitrary real coefficients. The derivative of this function is

$$v'(k) = \beta_1\,(1-\alpha)\,k^{-\alpha}.$$

Substituting in the HJB equation, we have

$$\rho\big(\beta_0 + \beta_1\,k^{1-\alpha}\big) = \frac{\alpha}{1-\alpha}\Big(\beta_1\,(1-\alpha)\Big)^{\frac{\alpha-1}{\alpha}}k^{1-\alpha} + A\,\beta_1\,.(1-\alpha) - \frac{1}{1-\alpha}$$

We can find β_0 and β_1 by matching the term depending on $k^{1-\alpha}$ and the constant term on both sides of the equation, which means solving the system of equations

$$\begin{cases}\rho\,\beta_0 = A\,\beta_1\,.(1-\alpha) - \dfrac{1}{1-\alpha}, \\ \rho\,\beta_1 = \dfrac{\alpha}{1-\alpha}\Big(\beta_1\,(1-\alpha)\Big)^{\frac{\alpha-1}{\alpha}}.\end{cases}$$

Therefore the solution of the HJB equation is

$$v(k) = \frac{1}{\rho}\left(A\left(\frac{\alpha}{\rho}\right)^{\alpha} - \frac{1}{1-\alpha}\right) + \frac{1}{1-\alpha}\left(\frac{\alpha}{\rho}\right)^{\alpha} k^{1-\alpha}$$

and the optimal policy function is ´

$$c^* = C(k) = v'(k)^{-\frac{1}{k}} = \frac{\rho}{\alpha}\, k$$

which is the same as the one we obtained by using the PMP. We follow the same approach to find the optimal capital accumulation function $k^*(t)$ and $c^*(t)$.

B Solving the Ramsey problem for the case $\theta \neq \alpha$

Consider the variational MHDS (18). The jacobian matrix, J, has a positive trace, $\mathrm{tr}(J) = \rho > 0$ and a negative determinant, because

$$\det(J) = \frac{\bar{c}r^{'}(\bar{k})}{\theta} = \frac{\bar{c}\alpha(\alpha-1)A\bar{k}^{\alpha-2}}{\theta} < 0$$

Then, the eigenvalues are both real and

$$\lambda_s = \frac{\rho}{2} - \left[\left(\frac{\rho}{2}\right)^2 - \det(J)\right]^{1/2} < 0,\ \lambda_u = \frac{\rho}{2} + \left[\left(\frac{\rho}{2}\right)^2 - \det(J)\right]^{1/2} > 0.$$

Therefore the steady state $(\bar{c}, \bar{k})$ is a saddle point.

The eigenvector matrix associated to J [28] is

$$\begin{pmatrix} 1 & 1 \\ \lambda_u & \lambda_s \end{pmatrix}$$

and therefore, the general solution is

$$\begin{pmatrix} k(t) - \bar{k} \\ c(t) - \bar{c} \end{pmatrix} = h_1 \begin{pmatrix} 1 \\ \lambda_u \end{pmatrix} n e^{\lambda_s t} + h_2 \begin{pmatrix} 1 \\ \lambda_s \end{pmatrix} e^{\lambda_u t},$$

where h_1 and h_2 are arbitrary constants. We determine them by requiring that the solution will converge asymptotically to a steady state and the predetermined variable k satisfies $k(0) = k_0$. As the explosive dynamics is generated by $e^{\lambda_u\, t}$ we set $h_2 = 0$ and by setting $h_1 = k(0) - \bar{k}$ the initial condition is satisfied. Then we obtain the approximate solution (**??**).

C Proof of equation (28)

Solving the household's Euler equation (22a) for the isoelastic case we get

$$c(t) = c(0)e^{\int_0^t \gamma_c(s)ds},\ \text{for } \gamma_c(t) = \frac{r(t) - \rho}{\gamma}$$

[28] We determine the column P^j by solving the homogeneous system $(J - \lambda_j I)P^j = 0$, where I is the (2×2) identity matrix, for non-zero solutions.

on the other hand integrating the budget constraint we get and substituting consumption yields

$$a(t) = e^{\int_0^t r(s)ds} a_0 + \int_0^t e^{\int_s^t r(z)dz} w(s)ds - c(0) \int_0^t e^{\int_s^t r(z)dz} e^{\int_0^s \gamma_c(z).dz} ds$$

Therefore, after some algebra we have

$$c(t)^{-\theta}\, a(t)\, e^{-\rho t} = c(0)^{-\theta} \left[a_0 + h^t(0) - c(0) \int_0^t e^{-\int_0^s \gamma(z)dz} ds \right]$$

where $\gamma(t) = \dfrac{(\theta-1)r(t)+\rho}{\theta}$ and $h^t(0) = \int_0^t e^{-\int_0^s r(z)\,dz}\, w(s)\, ds$. The transversality condition, $\lim_{t\to\infty} c(t)^{-\theta}\, a(t)\, e^{-\rho t} = 0$, holds if and only if

$$c^*(0) = \frac{a_0 + h^\infty(0)}{\int_0^\infty e^{-\int_t^s \gamma(z)dz} ds}$$

which is the optimal initial level of consumption from the point of view of the household equation.

Therefore, households consumption for any time $t \in (0, \infty)$ becomes

$$c^*(t) = \frac{\big(a_0 + h^\infty(0)\big)\, e^{\int_0^t \gamma_c(s)ds}}{\int_0^\infty e^{-\int_t^s \gamma(z)dz} ds}$$

and the optimal level of net wealth at time t is

$$\begin{aligned}
a^*(t) &= e^{\int_0^t r(s)ds} \left(a_0 + h^t(0) - c^*(0) \int_0^t e^{-\int_0^s (r(z)-\gamma_c(z))dz} ds \right) = \\
&= e^{\int_0^t r(s)ds} \left(a_0 + h^\infty(0) - h^\infty(t) e^{-\int_0^t r(s)ds} - c^*(0) \int_0^t e^{-\int_0^s \gamma(z))dz} ds \right) = \\
&= -h^\infty(t) + e^{\int_0^t r(s)ds} \left(c^*(0) \int_0^\infty e^{-\int_0^t \gamma(s)ds} dt - c^*(0) \int_0^t e^{-\int_0^s \gamma(z))dz} ds \right) = \\
&= -h^\infty(t) + c^*(0)\, e^{\int_0^t r(s)ds} \left(\int_0^\infty e^{-\int_0^t \gamma(s)ds} dt - \int_0^t e^{-\int_0^s \gamma(z))dz} ds \right) = \\
&= -h^\infty(t) + c^*(t)\, e^{\int_0^t \gamma(s)ds} \left(\int_0^t e^{-\int_0^s \gamma(s)ds} dt + e^{-\int_0^t \gamma(s)ds} \int_t^\infty e^{-\int_t^s \gamma(z)dz} ds - \int_0^t e^{-\int_0^s \gamma(z))dz} ds \right) = \\
&= -h^\infty(t) + c^*(t) \int_t^\infty e^{-\int_t^s \gamma(z)dz} ds
\end{aligned}$$

which solving for $c^*(t)$ is equation (28). We have used the fact that $h^\infty(0) = h^t(0) + e^{-\int_0^t r(s)ds} h^\infty(t)$.

D Behavioral functions for the non-additive separable model

Differencing the optimality conditions, equation (31a) and (31b), we have

$$\begin{pmatrix} u_{cc}(c,\ell) & u_{c\ell}(c,\ell) \\ u_{c\ell}(c,\ell) & u_{\ell\ell} - q\, f_{\ell\ell}(k,\ell) \end{pmatrix} \begin{pmatrix} d\,c \\ d\,\ell \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f_\ell(k,\ell) & q\, f_{\ell k}(k,\ell) \end{pmatrix} \begin{pmatrix} dq \\ dk \end{pmatrix}.$$

Writing

$$\begin{pmatrix} \frac{dc}{dq} & \frac{dc}{dk} \\ \frac{d\ell}{dq} & \frac{d\ell}{dk} \end{pmatrix} = \begin{pmatrix} C_q(q,k) & C_k(q,k) \\ L_q(q,k) & L_k(q,k) \end{pmatrix}$$

we have

$$\begin{aligned}
\begin{pmatrix} C_q & C_k \\ L_q & L_k \end{pmatrix} &= \begin{pmatrix} u_{cc}(c,\ell) & u_{c\ell}(c,\ell) \\ u_{c\ell}(c,\ell) & u_{\ell\ell} - q\, f_{\ell\ell}(k,\ell) \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ f_{\ell}(k,\ell) & q f_{\ell k}(k,\ell) \end{pmatrix} \\
&= \frac{1}{D} \begin{pmatrix} u_{\ell\ell} - q\, f_{\ell\ell}(k,\ell) & -u_{c\ell}(c,\ell) \\ -u_{c\ell}(c,\ell) & u_{cc}(c,\ell) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f_{\ell}(k,\ell) & q f_{\ell k}(k,\ell) \end{pmatrix} \\
&= \frac{1}{D} \begin{pmatrix} u_{\ell\ell} - q\, f_{\ell\ell}(k,\ell) - u_{c\ell}(c,\ell) f_{\ell}(k,\ell) & -u_{c\ell}(c,\ell)\, q\, f_{\ell k}(k,\ell) \\ -u_{c\ell}(c,\ell) + u_{cc}(c,\ell) f_{\ell}(k,\ell) & u_{cc}(c,\ell)\, q\, f_{\ell k}(k,\ell) \end{pmatrix}
\end{aligned}$$

where $D \equiv u_{cc} u_{\ell\ell} - u_{c\ell}^2 - q u_{cc} f_{\ell\ell}$ has an ambiguous sign.