Mathematical Economics
Problem set 2020/21

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1 Deterministic dynamic optimisation: discrete time

1.1 Optimal control and the Pontriyagin's principle

1.1.1 Consider the optimal control problem: $\max_u -\sum_{t=0}^{3}(2-u_t)^2$ subject to $y_{t+1} = 1/2 y_t + u_t$ such that $y_0 = y_4 = 1$.

(a) Determine first order conditions from the Pontriyagin's maximum principle.

(b) Determine the solution of the problem.

1.1.2 Consider the optimal control problem: $\max_u -\sum_{t=0}^{3}(2-u_t)^2$ subject to $y_{t+1} = 1/2 y_t + u_t$ such that $y_0 = 1$ and y_4 is free.

(a) Determine first order conditions from the Pontriyagin's maximum principle.

(b) Determine the solution of the problem.

1.1.3 Assume that there is a cake whose size at time $t \in \{0, 1, \ldots, T\}$, where T is finite, is W_t. A consumer wants to eat it in T periods. The initial size of the cake is $W_0 = \phi > 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

(a) Determine first order conditions from the Pontriyagin's maximum principle.

(b) Determine the solution of the problem.

1.1.4 Consider the optimal control problem: $\max_{\{u\}} \sum_{t=0}^{3} y_t - (2-u_t)^2$ subject to $y_{t+1} = 1/2(y_t - u_t)$ e a $y_0 = 0$ e $y_4 = 45/2$.

(a) Write the first order conditions according to Pontriyagin's principle.

(b) Solve the problem, that is determine the optimal sequences $\{y_t^*\}_{t=0}^4$ and $\{u_t^*\}_{t=0}^4$

1.1.5 Assume that there is a cake whose size at time $t \in \{0, 1, \ldots, T\}$, for a finite T, is W_t. A consumer wants to eat it in T periods; that is $W_T = 0$. The initial size of the cake is $W_0 = \phi > 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.6 Assume that there is a cake whose size at time $t \in \{0, 1, \ldots, \infty\}$ is W_t. A consumer wants to eat it with the terminal constraint $\lim_{t\to\infty} W_t \geq 0$. The initial size of the cake is $W_0 = \phi > 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.7 Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \ldots, T\}$. The horizon T is finite. The endowment evolves over time as $W_{t+1} = (1 + r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer wants to have $W_T = \phi$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is logarithmic.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.8 Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \ldots, \infty\}$. The endowment evolves over time as $W_{t+1} = (1+r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer wants to satisfy $\lim_{t\to\infty} e^{-rt} W_t \geq 0$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is logarithmic.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.9 Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \ldots, T\}$. The horizon T is finite. The endowment evolves over time as $W_{t+1} = (1 + r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer wants to have $W_T = \phi$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.10 Assume that that a consumer has an endowment denoted by W_t at time $t \in \{0, 1, \dots, \infty.\}$. The endowment evolves over time as $W_{t+1} = (1+r)W_t - C_t$, where C_t is the amount of the endowment consumed at time t and $r > 0$ is a parameter. Assume that $W_0 = \phi > 0$ and that the consumer has the terminal constraint $\lim_{t\to\infty} W_t \geq 0.$. The consumer has a psychological discount factor $0 < \beta < 1$ and the period utility function is isoelastic $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.11 Find the optimal investment sequence for a firm, $\{I_t\}_{t=0}^{\infty}$, that maximizes the value functional $\sum_{t=0}^{\infty}(1 + r)^{-t}\pi_t$, where $r > 0$ is the market interest rate. The *cash flow* in period t is $\pi_t = AK_t - I_t(1 + \xi I_t)$, where K_t is the capital stock, and $A > 0$ and$\xi > 0$ are productivity, and investment cost parameters, respectively. The restrictions of the problem are: the acumulation equation $K_{t+1} = I_t + (1 - \delta)K_t$, where $\delta \in [0, 1)$ is the rate of depreciation of capital, and the initial capital stock is given, $K_0 = \phi > 0$. Assume that $A > r + \delta$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.12 A representative consumer has the utility function $u = \ln(c_t)$ and has a constant intertemporal discount factor β^t, ith $0 < \beta < 1$, and a finite lifetime T, she/he has the budget constraint $a_{t+1} = y - c_t + (1 + r)a_t$, and has to bequeath the same wealth received at birth $a_0 = a_T = A$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.13 A representative consumer has the utility function

$$u = B - \zeta^{-1}e^{-\zeta C_t},$$

where $B > 0$ and $\zeta > 0$, and has a constant intertemporal discount factor β^t, with $0 < \beta < 1$, and a finite lifetime T, she/he has the budget constraint $A_{t+1} = A_t - C_t$, and $A_0 = \phi$ and $A_T = 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.14 A representative consumer has the utility function

$$u = B - \zeta^{-1} e^{-\zeta C_t},$$

where $B > 0$ and $\zeta > 0$, and has a constant intertemporal discount factor β^t, ith $0 < \beta < 1$, and a finite lifetime T, she/he has the budget constraint $A_{t+1} = Y - C_t + (1 + r)A_t$, and has to bequeath the same wealth received at birth $A_0 = A_T = A$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

1.1.15 Find the optimal investment sequence, $\{I_t\}_{t=0}^{T}$, that maximizes the value functional

$$\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t (pK_t - I_t(1 + (1/2)I_t))$$

where K_t is the capital stock, $r > 0$ is the market interest rate, and $p > 0$ is a productivity parameter. The restrictions of the problem are: the acumulation equation is $K_{t+1} = I_t + (1 - \delta)K_t$, where δ is the rate of depreciation of capital, and the initial and terminal capital stock is given by $K_0 = K_T = \phi > 0$. Assume that $p > r + \delta$ and $\delta \in [0, 1)$.

(a) Write the problem as a optimal problem and determine the optimality conditions from the Pontryiagin,'s maximum principle.

(b) Find an explicit solution for K_t. Justify and give an intuition for your the results.

1.1.16 A representative consumer wants to maximize the intertemporal utility functional $\sum_{t=0}^{\infty} \beta^t \ln (C_t^{\alpha} Z_t^{1-\alpha})$, where $0 < \alpha < 1$ and $0 < \beta < 1$, by using consumption C_t as a control variable. The variable Z_t denotes habits and is governed by the difference equation $Z_{t+1} = \delta(Z_t - C_t)$, where $\delta > 0$. The following initial and terminal conditions hold: $Z(0) = Z_0 > 0$, and $\lim_{t\to\infty} \beta^t Z(t) \geq 0$.

(a) Write the first order optimality conditions from the Pontyiagin's maximum principle.

(b) Solve the problem, and provide an intuition to your results.

1.2 Dynamic programming

1.2.1 Consider the infinite-horizon benchmark consumption-investment problem

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = (1+r)\,W_t - C_t,\ W_0 = \phi$$

where W_t is the stock of financial wealth in the beginning of period t, and C_t is consumption in period t, β the discount factor ($\beta \in (0,1)$), $r > 0$ is the market interest rate, and $\phi > 0$ is the initial stock of wealth.

(a) Write the HJB equation.

(b) Solve the HJB equation (hint: consider the trial function $V(W) = a + b\ln(W)$ where a and b are undetermined coefficients)

(c) Find the solution to the problem.

1.2.2 Consider the infinite-horizon cake-eating problem

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\theta} - 1}{1-\theta}, \text{ subject to } W_{t+1} = W_t - C_t,\ W_0 = \phi$$

where W_t is the stock of financial wealth in the beginning of period t, and C_t is consumption in period t, β the discount factor ($\beta \in (0,1)$), $\theta > 0$ is the inverse of the elasticity of intertemporal substitution, and $\phi > 0$ is the initial stock of wealth.

(a) Write the HJB equation.

(b) Solve the HJB equation (hint: consider the trial function $V(W) = a + b\,W^{1-\theta}$ where a and b are undetermined coefficients)

(c) Find the solution to the problem.

1.2.3 Consider the infinite-horizon benchmark consumption-investment problem

$$\max_{\{C_t\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\theta} - 1}{1-\theta}, \text{ subject to } W_{t+1} = (1+r)\,W_t - C_t,\ W_0 = \phi$$

where W_t is the stock of financial wealth in the beginning of period t, and C_t is consumption in period t, β the discount factor ($\beta \in (0,1)$), $\theta > 0$ is the inverse of the elasticity of intertemporal substitution, $r > 0$ is the market interest rate, and $\phi > 0$ is the initial stock of wealth.

(a) Write the HJB equation.

(b) Solve the HJB equation (hint: consider the trial function $V(W) = a + b\,W^{1-\theta}$ where a and b are undetermined coefficients)

(c) Find the solution to the problem.

Deterministic dynamic optimisation: continuous time

Optimal control: Pontryiagin's principle

2.1.1 A representative consumer has the utility functional $\int_0^T e^{-\rho t} \ln(C(t))dt$, where $\rho > 0$, and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial and terminal wealth $A(0) = A(T) = A_0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.2 A representative consumer has the utility functional $\int_0^T e^{-\rho t} \ln(C(t))dt$, where $\rho > 0$, and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial wealth $A(0) = A_0$ and $A(T) \geq 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.3 A representative consumer has the utility functional $\int_0^\infty e^{-\rho t} \ln(C(t))dt$, where $\rho > 0$, and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial wealth $A(0) = A_0$ and $\lim_{t\to\infty} A(t) \geq 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.4 A representative consumer has the utility functional $\int_0^T e^{-\rho t}(1-\sigma)^{-1}(C(t))^{1-\sigma}dt$, where $\rho > 0$ and $\sigma > 0$ (but $\sigma \neq 1$), and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial and terminal wealth $A(0) = A(T) = A_0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.5 A representative consumer has the utility functional $\int_0^T e^{-\rho t} (1-\sigma)^{-1}(C(t))^{1-\sigma}dt$, where $\rho > 0$, and T is finite. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial wealth $A(0) = A_0$ and $A(T) \geq 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.6 A representative consumer has the utility functional $\int_0^\infty e^{-\rho t}(1-\sigma)^{-1}(C(t))^{1-\sigma}dt$, where $\rho > 0$. She/he has the budget constraint $\dot{A}(t) = -C(t) + rA(t)$, and has initial wealth $A(0) = A_0$ and $\lim_{t\to\infty} A(t) \geq 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.7 Solve $\max_C \int_0^\infty e^{-\rho t} \;\left(B - \zeta \exp -(C(t)/\zeta\;\right) dt$, $\zeta > 0$ subject to $\dot{A} = rA - C$ where $A(0) = A_0$ given and $\lim_{t\to\infty} e^{-rt}A(t) \geq 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.8 Solve $\max_\pi \int_0^T -(\pi(t))^2\, e^{-\rho t} dt$ subject to $\dot{\pi} = \pi - \bar{\pi}$, where $\rho > 0$, subject to $\pi(0) = 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.9 Solve $\max_I \int_0^\infty e^{-rt} \left(pK(t) - qI(t)^2\right) dt$ subject to $\dot{K} = I(t) - \delta K(t)\;\, K(0) = k_0 > 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.10 Solve $\max_I \int_0^\infty e^{-rt} \left(pK(t) - qI(t)(1 + \xi I(t)/K(t))\right) dt$ subject to $\dot{K} = I(t) - \delta K(t)$ $K(0) = k_0 > 0$.

(a) Write the first order conditions from the Pontryiagin's maximum principle.

(b) Solve the problem.

2.1.11 Consider the following endogenous growth model:

$$\max_C \int_0^\infty \frac{1}{1-\sigma}\; C(t)^{1-\sigma} e^{-\rho t} dt, \text{ subject to } \;\dot{K} = Y(t) - C(t)$$

together with $K(0) = K_0$ given and $\lim_{t\to\infty} e^{-At}K(t) \geq 0$. The production function is linear $Y(t) = AK(t)$ and the parameters verify: $\rho > 0$, $\sigma > 1$ and $A > 0$.

(a) Write the first order conditions according to the maximum principle of Pontriyagin.

(b) Solve the problem. Under which conditions the solution displays unbounded growth ?

2.1.12 A representative consumer wants to maximize the intertemporal utility functional $\int_0^\infty e^{-\rho t} \ln\left(C(t)\right) dt$, where $\rho > 0$, by using consumption $C(.)$ as a control variable. She/he has initial wealth $A(0) = A_0$, and the instantaneous budget constraint is $\dot{A}(t) = (1-\tau)(Y + rA(t)) - C(t)$, where income Y is constant and positive, and the income tax rate verifies $0 < \tau < 1$. The non-Ponzi game condition $\lim_{t\to\infty} e^{-rt}A(t) \geq 0$ holds.

(a) Write the first order optimality conditions from the Pontryiagin's maximum principle.

(b) Solve the problem, and supply an intuition for your results.

2.1.13 Assuming that $x(.)$ is a state variable and $u(.)$ is a control variable, consider the optimal control problem

$$\max_{(u(t))_{t=0}^{\infty}} \int_0^{\infty} (x(t)^2 + u(t)^2) e^{-\rho t} dt$$

subject to $\dot{x} = \alpha(x-u)$ and $x(0) = \phi$ and $\lim_{t\to\infty} x(t)e^{-\rho t} = 0$. Assume that $0 < \rho < 2\alpha$ and that $\phi > 0$

(a) Determine the optimality conditions from the Pontryiagin,'s maximum principle.

(b) Find an explicit solution for the optimal state variable $x(.)$. Justify.

Dynamic programming

2.2.1 Consider the optimal control problem $\max_u -\int_0^4 (2-u)^2 dt$ subject to $\dot{y} = 1/2y(t)+u(t)$ for $t \in [0,4]$ and $y(0) = 1$.

(a) Write the HJB equation.

(b) Determine the optimal policy function.

2.2.2 Assume that there is an endowment whose size at time $t \in [0,T]$, where T is finite, is $W(t)$. A consumer wants to consume until time T. That is $W(T) = 0$. The initial size of the cake is $W(0) = \phi > 0$. The consumer has a psychological rate of time preference $\rho > 0$ and a static logarithmic utility function. Determine the optimal consumption strategy using the principle of dynamic programming.

(a) Write the HJB equation.

(b) Determine the optimal policy function.

Solutions for discrete time problems

1.1.4 (a) $y_t^* = (3/2)(-1+2^t)$ and $u_t^* = (3/2)(1-3(2^t))$ for $t = 0,1,2,3,4$ or $y_t^* = \{0, 3/2, 9/2, 21/2, 45/2\}$ and $u_t^* = \{-3, -15/2, -33/2, -69/2\}$.

1.1.15 (a) The first order conditions are $\eta_{t+1} = ((1+r)\eta_t - p)\frac{1}{1-\delta}$ and $K_{t+1} = \eta_t - 1 + (1-\delta)K_t$ for $t \in \{0, 1, \ldots, T\}$, $K_0 = \phi = K_T$. (b) The problem has the unique solution

$$
\begin{aligned}
K_t &= k^* + (\phi - k^*)\left(\frac{((1+r^*)^T - 1)(1-\delta)^t + (1-(1-\delta)^T)(1+r^*)^t}{(1+r^*)^T - (1-\delta)^T}\right) \\
\eta_t &= \eta^* + \frac{(r^*+\delta)(\phi - k^*)(1-(1-\delta)^T)}{(1+r^*)^T - (1-\delta)^T}(1+r^*)^t
\end{aligned}
$$

for $t \in \{0, 1, \ldots, T\}$, where $1 + r^* = \frac{1+r}{1-\delta}$, $k^* = \frac{p-(r+\delta)}{\delta(r+\delta)}$ and $\eta^* = \frac{p}{r+\delta}$.

1.1.16 (a) F.o.c. $C_t^* = (\alpha)/\delta)\frac{1}{\eta_t}$, $\eta_{t+1} = \frac{\eta_t}{\beta\delta} - \frac{1-\alpha}{\delta Z_{t+1}^*}$, $Z_{t+1}^* = \delta(Z_t^* - C_t^*)$, $Z_0^* = Z_0$ and $\lim_{t\to\infty} \beta^t \eta_t Z_t^* = 0$; (b) the solution $Z_t^* = Z_0\left(\frac{\beta\delta}{\alpha+(1-\alpha)\beta}\right)^t$ and $C_t^* = \frac{\alpha(1-\alpha)}{\alpha+\beta(1-\alpha)}Z_t^*$

Solutions for continuous-time problems

2.1.12 (a) The first order conditions include the optimality conditions $1/C(t) = Q(t)$, $\dot{Q}(t) = (\rho - r(1-\tau))Q(t)$ and $\lim_{t\to\infty} Q(t)A(t)e^{-\rho t} = 0$, plus the admissibility conditions $\dot{A}(t) = (1-\tau)(Y + rA(t)) - C(t)$ and $A(0) = A_0$. (b) The problem has the unique solution (where $\gamma \equiv r(1-\tau) - \rho$), $C(t) = \rho\,(A_0 + Y/r)\,e^{\gamma t}$ and $A(t) = -Y/r + (A_0 + Y/r)\,e^{\gamma t}$ for $t \in [0, \infty)$. Then: if $\gamma < 0$ then $\lim_{t\to\infty}(C(t), A(t) = (0, -Y/r)$, if $\gamma = 0$ then $\lim_{t\to\infty}(C(t), A(t)) = (\rho\,(A_0 + Y/r)\,, -Y/r)$ or if $\gamma > 0$ then $\lim_{t\to\infty}(C(t), A(t) = (+\infty, +\infty)$.

2.1.13 (a) $u^*(t) = \frac{\alpha}{2}\, q(t)$, $\dot{q}(t) = (\alpha - \alpha)q(t) - 2x^*(t)$, $\dot{x}^*(t) = \alpha(x^*(t) - u^*(t))$, $x^*(0) = \phi$ and $\lim_{t\to\infty} x^*(t)e^{-\rho t} = 0$; (b) $x^*(t) = \phi e^{\lambda_s t}$ where $\lambda_s = \frac{\rho}{2} - \left(\left(\frac{\rho}{2}\ - \alpha(\rho - 2\alpha)\right)^2\right)^{1/2} < 0$