The AK model

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The AK model

- This is the simplest endogenous growth model
- The economy has the following features:
  1. population is constant and normalised to one $N = 1$
  2. there is one reproducible input: physical capital
  3. there is one good which is produced by a CRS technology (using capital)
  4. the good is used in consumption and investment (it is a closed economy)
  5. the consumer solves an intertemporal optimization problem (savings for consumption smoothing)

The AK model

Two versions of the model

- Decentralized version: there is no state and the allocation of capital through time is determined by market equilibrium
- Centralized version: there is a central planner ("benevolent dictator") that determines the optimal allocation of capital by maximizing the intertemporal social welfare
- As there are no externalities or other distortions, the two versions are equivalent: in this case we say that the **equilibrium allocations are Pareto optimal**
- We will see that when there are externalities the two economies lead to different allocations: then **equilibrium allocations are not Pareto optimal**

Assumptions

- Technology: production function:

$$Y = AK$$

- Economy's constraint: capital accumulation equation

$$\dot{K} = Y - \delta K - C$$

- Preferences: utility functional

$$\int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

The model

Centralized version

- The central planner determines the optimal paths $(C(t), K(t))_{t\in[0,\infty)}$ by solving the problem

$$\max_{[C(t)]_{t\geq 0}} \int_0^\infty \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$$

  subject to

$$\begin{aligned} \dot{K} &= AK(t) - C(t) - \delta K, \\ K(0) &= k_0, \ \text{given}, \ \ \text{t} = 0 \\ \lim_{t\to+\infty} e^{-At}K(t) &\geq 0. \end{aligned}$$

- assumption: $A > \delta$

The MHDS

- We determine the growth facts on $Y(t) = AK(t)$ as a solution of the MHDS (maximised hamiltonian dynamic system) solving
- two dynamic equations

$$\begin{aligned} \dot{C} &= C(A-\rho-\delta)/\theta && (1) \\ \dot{K} &= AK - C - \delta K, && (2) \\ & && (3) \end{aligned}$$

- initial and the transversality conditions

$$\begin{aligned} 0 &= \lim_{t\to\infty} C(t)^{-\theta} K(t) e^{-\rho t} && (4) \\ K(0) &= K_0, \text{ given} && (5) \end{aligned}$$

Growth in the AK model

- The capital stock solution is

$$K(t) = \bar{K}(t) = k_0 e^{\gamma t},\ t \in [0, \infty)$$

- which implies that the output is

$$Y(t) = \bar{Y}(t) = A k_0 e^{\gamma t},\ t \in [0, \infty)$$

- Conclusion (growth facts):
  - the (endogenous) long run rate of growth is

$$\boxed{\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0}$$

  - the long run level is $\bar{y} = A k_0$
  - there is no transitional dynamics $\lambda = 0$ (this is counterfactual)

Solution method

for endogenous growth models

1. Write the level of variables as: $X(t) = x(t)e^{\gamma_x t}$
   (level = short run $\times$ trend)
2. Rewrite the MHDS for the detrended variables by introducing assumptions on the rates of growth (call it detrended MHDS) such that it is an autonomous ODE
3. Determine the long run growth rate from the steady state of the detrended MHDS
4. Introduce the long run growth rate in the detrended MHDS and solve for the detrended variables, k and $y = Ak$
5. Get the final solution for K and, therefore, for $Y = AK$

Step 1 : detrending variables

- Separation of transition, (k, c), and long-run trend $(e^{\gamma_k t}, e^{\gamma_c t})$

$$K(t) = k(t)\,e^{\gamma_k t}, \quad C(t) = c(t)\,e^{\gamma_c t},$$

- Then

$$\frac{\dot{c}}{c} = \frac{\dot{C}}{C} - \gamma_c$$
$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \gamma_k$$

Step 2 : building the detrended MHDS

- Substituting $\dot{C}/C$ and $\dot{K}/K$ we get

$$\frac{\dot{c}}{c} = \frac{A-\rho-\delta}{\theta} - \gamma_c$$
$$\frac{\dot{k}}{k} = A-\delta-\frac{c}{k}e^{(\gamma_c-\gamma_k)t}\gamma_k$$

- A necessary condition for the MHDS to be autonomous (time-independent) is

$$\boxed{\gamma = \gamma_k = \gamma_c}$$

- Therefore, the detrended MHDS is

$$\dot{c} = c\left(\frac{A-\rho-\delta}{\theta} - \gamma\right)$$
$$\dot{k} = (A-\delta-\gamma)k - c$$

Step 3 : the long-run growth rates

- Setting $\dot{c} = 0$ we get the long run growth rate

$$\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0$$

- Setting $\dot{k} = 0$ we get the long run ratio

$$\frac{\bar{c}}{\bar{k}} = \beta,$$

where

$$\beta \equiv A - \delta - \bar{\gamma} = \frac{1}{\theta}\left((A - \delta)(\theta - 1) + \rho\right) > 0$$

Step 4: solving the detrended MHDS

- if we substitute the rate of growth γ in the detrended MHDS we have

$$\begin{aligned} \dot{c} &= 0 && (6) \\ \dot{k} &= \beta k - c && (7) \\ 0 &= \lim_{t \to +\infty} e^{-\beta t} k(t) c(t)^{-\theta} && (8) \end{aligned}$$

because

$$\lim_{t \to +\infty} e^{-(\rho + \bar{\gamma}(\theta - 1))t} k(t) c(t)^{-\theta} = \lim_{t \to +\infty} e^{-\beta t} k(t) c(t)^{-\theta}$$

Step 4: solving the detrended MHDS (cont.)

- the solution of equation (**??**) is

$$c(t) = B$$

  B is an arbitrary constant

- substituting c and solving equation (**??**) we find

$$k(t) = \left(k_0 - \frac{B}{\beta}\right) e^{\beta t} + \frac{B}{\beta}.$$

- to determine B we substitute in the TVC

$$\begin{aligned}
\lim_{t\to+\infty} e^{-\beta t} k(t) c(t)^{-\theta} &= \lim_{t\to+\infty} e^{-\beta t} \left[\left(k_0 - \frac{B}{\beta}\right) e^{\beta t} + \frac{B}{\beta}\right] B^{-\theta} = \\
&= \lim_{t\to+\infty} \left[k_0 - \frac{B}{\beta}\right] B^{-\theta} = \\
&= 0
\end{aligned}$$

  if $B = \beta k_0$

Step 5: the solution to the AK model

- The BGP is
$$\bar{K}(t) = \bar{k}e^{\gamma t}, \quad \bar{C}(t) = \bar{c}e^{\gamma t}.$$
- where $\gamma = \bar{\gamma}$ is determined from the steady state of the detrended MHDS
- the endogenous rate of growth is
$$\bar{\gamma} = \frac{A - \delta - \rho}{\theta} > 0$$
- we get additionally the ratio of the levels along the BGP
$$\frac{\bar{c}}{\bar{k}} = \beta,$$
  where
$$\beta \equiv A - \delta - \bar{\gamma} = \frac{1}{\theta}\left((A - \delta)(\theta - 1) + \rho\right) > 0$$
- Observe that there is an indeterminacy here: we have two equations ($\dot{c} = 0$ and $\dot{k} = 0$) and three variables (γ, c, k))
- this is a typical property of the endogenous growth models.

References

- (**?**, ch.11.1)