Foundations of Financial Economics
Multi-period GE: Arrow-Debreu economy

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Multiperiod Arrow-Debreu economy

Information structure

We consider an **homogeneous agent** economy, in which:

- there is an information tree, with T periods,
- the information tree comprises a sequence of nodes $\{N_t\}_{t=1}^{T} = \{N_1, N_2, \ldots, N_s, \ldots, N_T\}$, where N_t is the number of nodes of the information tree at time t
- **Example**: for a binomial process $N_t = 2^t$
- there is a sequence of unconditional probabilities

$$\mathbb{P}^T \equiv \{P_t\}_{t=1}^{T} = \{\mathsf{P}_1, \ldots, \mathsf{P}_t, \ldots, \mathsf{P}_T\}$$

  where $\mathsf{P}_t = \left(\pi_{t,1}, \ldots, \pi_{t,s}, \ldots, \pi_{t,N_t}\right)$

- for any process $\{X_t\}_{t=0}^{T} = \{X_0, X_1, \ldots X_t, \ldots, X_T\}$ we assume that X_t is $\mathcal{F}_t$-adapted (as we say in the slide "Introduction to stochastic processes")
- **The information structure is common knowledge**

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Real part of the economy: resources

- There is a **given** sequence of endowments

$$Y^T \equiv \{ Y_t \}_{t=0}^{T} = \{ Y_0, Y_1, \ldots, Y_t, \ldots, Y_T \}$$

- where Y_t is $\mathcal{F}_t$- mensurable, such that

$$Y_t = \begin{pmatrix} y_{t,1} \\ \ldots \\ y_{t,N_t} \end{pmatrix}$$

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Real part of the economy: preferences and distribution

- Consumers choose a contingent-consumption sequences belonging to the set

$$C^T \equiv \{C_t\}_{t=0}^{T} = \{C_0, C_1, \dots, C_t, \dots, C_T\}$$

  where C_t is $\mathcal{F}_t$- mensurable,

- through an intertemporal von-Neumman-Morgenstern functional

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) \right]$$

- where $\beta \in (0, 1)$ and $u(.)$ is increasing, concave and Inada

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Real part of the economy: preferences and distribution

- Observe that

$$
\begin{aligned}
\mathbb{E}_0\left[\sum_{t=0}^{T}\beta^t u(C_t)\right] &= \sum_{t=0}^{T}\beta^t \mathsf{P}_t u(C_t) = \\
&= u(C_0) + \ldots + \beta^t \mathsf{P}_t u(C_t) + \ldots + \beta^T \mathsf{P}_T u(C_T)
\end{aligned}
$$

- where

$$
\mathsf{P}_t u(C_t) = \sum_{s=1}^{N_t} \pi_{t,s} u(c_{t,s})
$$

and P_t are **unconditional** probability distributions, i.e., taken at time $t = 0$

Multiperiod Arrow-Debreu economy

Arrow-Debreu contingent claims

- There is a large number of **Arrow-Debreu contingent claims**, **traded** only at time $t = 0$, offering one unit of the good for **delivery** at every node of the information tree for $t = 1, \ldots, N_t$
- this means there is:
  1. one spot market taken as the numeraire: $Q_0 = 1$
  2. $\sum_{t=1}^{T} N_t = N_1 + \ldots + N_t + \ldots + N_T$ AD markets with prices

$$Q^T \equiv \{Q_t\}_{t=0}^{T} = \{Q_0, Q_1, \ldots, Q_t, \ldots, Q_T\}$$

where

$$Q_t = \begin{pmatrix} q_{t,1} \\ \ldots \\ q_{t,N_t} \end{pmatrix}, \text{ i.e. } Q_t \text{ is } \mathcal{F}_t\text{- mensurable}$$

Arrow-Debreu equilibrium

For an homogeneous economy

Definition: A Arrow-Debreu equilibrium is the process (C^T, Q^T), that is, it is the collection of $\mathcal{F}_t$−adapted processes for consumption $\{C_t\}_{t=0}^T$ and AD-prices $\{Q_t\}_{t=1}^T$ such that, given the $\mathcal{F}_t$−adapted process $Y^T = \{Y_t\}_{t=0}^T$:

1. consumers problem determine $\{C_t\}_{t=0}^T$ by solving

$$\max_{\{C_t\}_{t=0}^T} \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) \right] \text{ s.t. } \sum_{t=0}^{T} Q_t C_t \leq \sum_{t=0}^{T} Q_t Y_t$$

   given $\{Y_t\}_{t=0}^T$ and $\{Q_t\}_{t=1}^T$
2. and markets clear

$$C_t = Y_t, \quad t = 0, \ldots, T$$

- T can be finite or $T = \infty$

The budget constraint

Observe that:

- the budget constraint is equivalent to

$$\sum_{t=0}^{T} Q_t(Y_t - C_t) =$$

$$= Q_0(Y_0 - C_0) + \ldots + Q_t(Y_t - C_t) + \ldots + Q_T(Y_T - C$$

  where

$$Q_t(Y_t - C_t) = \sum_{s=0}^{N_t} q_{t,s}(y_{t,s} - c_{t,s})$$

- If we define the 0-period **unconditional stochastic discount factor** for period t as

$$M_t \equiv Q_t/\mathsf{P}_t$$

  where $M_t = (m_{t,1}, \ldots, m_{t,N_t})$

$$m_{t,s} = \frac{q_{t,s}}{\pi_{t,s}},\ s = 1, \ldots, N_t$$

The budget constraint (cont)

- Then the **instantaneous budget constraint** at time $t = 0$, is equivalent to

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} M_t \left(Y_t - C_t \right) \right] \geq 0$$

- where

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} M_t \left(Y_t - C_t \right) \right] = M_0 (Y_0 - C_0) + \ldots + \mathsf{P}_T M_T (Y_T - C_T)$$

The solution of the consumer problem

- We can write the Lagrangean as

$$\mathcal{L} = \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) + M_t(Y_t - C_t) \right]$$

- or equivalently

$$\mathcal{L} = \sum_{t=0}^{T} \sum_{s=1}^{N_t} \pi_{t,s} \left\{ \beta^t u(c_{t,s}) + \lambda m_{t,s}(y_{t,s} - c_{t,s}) \right\}$$

First order conditions

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_{t,s}} &= \mathbf{0},\ s=1,\dots N_t,\ t=0,\dots T,\ \left(\sum_{t=0}^{T} N_t \text{dimensional}\right)\\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 0\ \text{(1 dimensional)}\end{aligned}$$

Solution of the consumer's problem

First-order conditions for optimality

$$
\begin{array}{rcl}
u'(c_0^*) & = & \lambda \ \text{(1 equation)} \\
\beta u'(c_{1,s}^*) & = & \lambda m_{1,s}, \ s = 1, \dots N_1 \ \ (N_1 \text{ equations}) \\
 & \dots & \\
\beta^t u'(c_{t,s}^*) & = & \lambda m_{t,s}, \ s = 1, \dots N_t \ \ (N_t \text{ equations}) \\
 & \dots & \\
\beta^T u'(c_{T,s}^*) & = & \lambda m_{T,s}, \ s = 1, \dots N_T \ \ (N_T \text{ equations}) \\
\sum_{t=0}^{T} \sum_{s=1}^{N_t} \pi_{t,s} m_{t,s} c_{t,s}^* & = & H_0 \equiv \sum_{t=0}^{T} \sum_{s=1}^{N_t} \pi_{t,s} m_{t,s} y_{t,s} \ \ \text{(1 equation)}
\end{array}
$$

where H_0 is human wealth equal to the expected discounted present value (in market prices) of the future stream of endowments.

Equilibrium conditons for a homogeneous agent economy

- The **Euler equation for consumption** is, because $u^{'}(c_0^*) = \lambda$

$$m_{t,s} u^{'}(c_0^*) = \beta^t u^{'}(c_{t,s}^*), \quad s = 1, \ldots N_t, \quad t = 0, \ldots, T.$$

- The equilibrium conditions are (**in this homogeneous-agent model**)

$$c_{t,s}^* = y_{t,s}, \quad s = 1, \ldots N_t, \quad t = 0, \ldots, T.$$

Equilibrium stochastic discount factor

- Then the **equilibrium stochastic discount factor** (SDF) is a stochastic process $\{M_t\}_{t=0}^{T}$ such that $M_0 = m_0 = 1$ and $M_t = (m_{t,1}, \ldots, m_{t,N_t})^\top$ where

$$\boxed{M_t^* = \beta^t \frac{u^{'}(Y_t)}{u^{'}(Y_0)},\ t = 0, \ldots T}$$

$$M_t^* = \begin{pmatrix} m_{t,1} \\ \ldots \\ m_{t,N_t} \end{pmatrix}$$

- or, equivalently the possible realizations of the unconditional stochastic discount factor are

$$\boxed{m_{t,s}^* = \beta^t \frac{u^{'}(y_{ts})}{u^{'}(y_0)},\quad s = 1, \ldots, N_t,\quad t = 0, \ldots T}$$

Equilibrium stochastic discount factor

Definition: **recursive stochastic discount factor** for period $t+1$ conditional on period t

$$M_{t+1|t} = \frac{M_{t+1}}{M_t}$$

where

$$M_{t+1|t} = \begin{pmatrix} \mu_{t+1|t,1} \\ \cdots \\ \mu_{t+1|t,s} \\ \cdots \\ \mu_{t+1|t,N_{t,t+1}} \end{pmatrix}$$

Equilibrium stochastic discount factor

- The **equilibrium recursive stochastic discount factor** (RSDF) for period $t+1$ conditional on period t is

$$\boxed{M^*_{t+1|t} = \beta \frac{u^{'}(Y_{t+1})}{u^{'}(Y_t)}}$$

- Has possible realizations

$$\boxed{\mu_{t+1} = m^*_{t+1|t,s} = \beta \frac{u^{'}(y_{t+1,s})}{u^{'}(y_t)},\ s = 1 \ldots N_{t,t+1}}$$

- These relations hold for T finite or infinite
- Observation: this RSDF is similar to what we have studied for the two-period case.