Mathematical Economics
Introduction

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This part of the Math Econ course

- Deals with **dynamic optimization**
- Which is the **main** building block for modern macroeconomics, financial economics and growth economics.
- We will solve problems in both discrete time and continuous time
- And using three different methods

Syllabus

1. Discrete time problems
    1. Calculus of variations problem
    2. Optimal control problem: Pontriyagin maximum principle
    3. Optimal control problem with infinite horizon: dynamic programming
2. Continuous time
    1. Calculus of variations problem
    2. Optimal control problem: Pontriyagin maximum principle
    3. Optimal control problem with infinite horizon: dynamic programming

Options for the course

- We address the simplest problems, all with explicit solutions (most dynamic optimization problems don't have closed form solutions)
- Some heuristic proofs are provided.
- Pre-requisites: elementary calculus, difference and differential equations
  I assume you know how to solve scalar and planar linear difference and differential equations

Course material

- Slides, classnotes, and problem sets will be posted at `http://www.iseg.ulisboa.pt/~pbrito/cursos/mestrado/em/em_m_1819.html`
- Corrections may be introduced until 16th December: check the date of the document.

References

- Discrete time:

  de la Fuente, A (2000), *Mathematical Methods and Models for Economicts*, Cambridge

  Ljungqvist, L and T. J. Sargent (2004), *Recursive Macroeconomic Theory*, 2nd ed, MIT Press

- Continuous time:

  Kamien, M. I. and N. L. Schwartz (1991), *Dynamic Optimization*, 2nd ed, Elsevier

  Liberzon, D. (2012), *Calculus of Variations and Optimal Control Theory: A Concise Introduction*, Princeton.

- Other references in my classnotes.

Discrete time: calculus of variations problem

- Consider:
  - the set $\mathcal{T} = \{0, \dots, T\}$
  - the function $x : \mathcal{T} \to \mathbb{R}$, or, equivalently, the sequences $x \equiv \{x_0, \dots, x_t, \dots, x_T\}$
- Given:
  - T: the terminal time
  - $\mathcal{X}$: the set of **admissible** sequences $x = \{x_t\}_{t \in \mathcal{T}}$ verifying one initial condition $x_0 = \phi_0$ and/or terminal conditions at $t = T$
  - the **value functional**

$$J(x) \equiv \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$$

where $F_t = F(x_{t+1}, x_t, t)$ is called the **objective function**

- **CV problem**: find the sequence $x^* \equiv \{x_0^*, x_1^*, \dots, x_T^*\} \in \mathcal{X}$ that maximizes $J(x)$.
- The **value** of the optimal sequence x^* is a number:

$$J^* \equiv J(x^*) = \max_x \{J(x) : \; x \in \mathcal{X}\}$$

Discrete time: optimal control problem

- Consider:
  - the set $\mathcal{T} = \{0, \dots, T\}$
  - functions $x : \mathcal{T} \to \mathbb{R}$, and $u : \mathcal{T} \to \mathbb{R}$, or, equivalently, the sequences $x \equiv \{x_0, x_1, \dots, x_T\}$ and $u \equiv \{u_0, u_1, \dots, u_{T-1}\}$
- Given:
  - T: the terminal time
  - $\mathcal{D}$ the set of all **admissible** sequences $(x, u) = \{(x_t, u_t)\}_{t \in \mathcal{T}}$ verifying the difference equation

$$x_{t+1} = g(x_t, u_t, t),$$

    plus some initial or terminal conditions (over x_0 and/or x_T)
  - and the **value functional**

$$J(u, x) \equiv \sum_{t=0}^{T-1} F(t, u_t, x_t)$$

    where $F_t = F(t, u_t, x_t, t)$ is called the **objective function**
- **OC problem**: find the optimal sequences $u^* \equiv \{u_0^*, \dots, u_T^*\}$ and $x^* \equiv \{x_0^*, \dots, x_T^*\}$ that maximize $J(u, x)$.
- The **value** of the optimal sequence (u^*, x^*) is a number:

$$J^* \equiv J(x^*) = \max_u \{J(u, x) : \ (u, x) \in \mathcal{D}\}$$

Discrete time: economic applications

Cake eating problem

Find sequences $W \equiv \{W_t\}_{t=0}^{T}$ and $C \equiv \{C_t\}_{t=0}^{T-1}$ that solve the problem:

$$\max_{C} \sum_{t=0}^{T-1} \beta^t u(C_t)$$

subject to

$$\begin{cases} W_{t+1} = W_t - C_t, \quad t = 0, 1, \ldots, T-1 \\ \text{other conditions} \end{cases}$$

where

$W_t =$ size of the cake at time t (i.e, at the beginning of period t)

$C_t =$ consumption in period t

$T =$ horizon (terminal time)

Discrete time: economic applications

Consumption-investment problem

Find sequences $A \equiv \{A_t\}_{t=0}^{T}$ and $C \equiv \{C_t\}_{t=0}^{T-1}$ that solve the problem

$$\max_C \sum_{t=0}^{T-1} \beta^t u(C_t)$$

subject to

$$\begin{cases} A_{t+1} = Y_t + (1+r)A_t - C_t, \quad t = 0, 1, \ldots, T-1 \\ \text{other conditions} \end{cases}$$

where:
C_t = consumption in period t
$u_t = u(C_t)$ = value of consumption in period t
A_t = net financial wealth at time t
Y_t = non-financial flow of income in period t
r = interest rate

Discrete time: economic applications

Firm's investment problem

Find sequences $K \equiv \{K_t\}_{t=0}^{T}$ and $I \equiv \{I_t\}_{t=0}^{T-1}$ that solve the problem

$$\max_{I} \sum_{t=0}^{T-1} \left(\frac{1}{1+r}\right)^{t} \pi(K_t, I_t) + R(T, K_T)$$

subject to

$$\begin{cases} K_{t+1} = I_t - (1+\delta)K_t, \quad & t = 0, 1, \ldots, T-1 \\ \text{other conditions} \end{cases}$$

where:
$\pi_t = \pi(K_t, I_t) =$ firm's cash-flow in period t
$K_t =$ stock of capital at time t
$I_t =$ gross investment in period t
$r =$ interest rate (assumed to be constant)
$R(T, X_T) =$ scrap value

Discrete time: economic applications

Economic growth models

- **Endogenous growth model**: Find sequences $K \equiv \{K_t\}_{t=0}^{\infty}$ and $C \equiv \{C_t\}_{t=0}^{\infty}$ that solve

$$\max_{C} \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t),\ K_{t+1} = (1+A)K_t - C_t,\ \text{other conditions} \right\}$$

$K_t =$ stock of physical capital, $C_t =$ consumption

- **Ramsey model**: find sequences $K \equiv \{K_t\}_{t=0}^{\infty}$ and $C \equiv \{C_t\}_{t=0}^{\infty}$ that solve

$$\max_{C} \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t),\ K_{t+1} = K_t + F(K_t) - C_t,\ \text{other conditions} \right\}$$

$K_t =$ stock of physical capital, $C_t =$ consumption

Continuous time: calculus of variations problem

- Consider:
  - the set $\mathcal{T} = [0, T]$ or $[0, \infty)$
  - the function $x : \mathcal{T} \to \mathbb{R}$ of continuous and differentiable functions $t \mapsto x(t)$
- Given
  - T if $\mathcal{T}$ is finite
  - $\mathcal{X}$ the set of trajectories $x \equiv (x(t))_{t \in \mathcal{T}}$ verifying $x(0) = x_0$ and some terminal condition
  - the value **functional**

$$J(x) \equiv \int_0^T F(t, x(t), \dot{x}(t))\, dt$$

where $F(.)$ is given

- **CT CV problem**: find $x^* \equiv (x^*(t))_{t \in \mathcal{T}} \in \mathcal{X}$ that maximizes the functional $J(x)$
- The value for the optimal program is the number:

$$J^* \equiv J(x^*) = \max_x \{J(x) : \; x \in \mathcal{D}\}$$

Continuous time: optimal control problem

- Consider:
  - the set $\mathcal{T} = [0, T]$ or $[0, \infty)$
  - the continuous and differentiable functions $x : \mathcal{T} \to \mathbb{R}$ and the piecewise-continuous function $u : \mathcal{T} \to \mathbb{R}^m$, $m \geq 1$
- Given:
  - T the terminal time if $\mathcal{T}$ is finite
  - the set $\mathcal{D}$ of trajectories $(x, u) = ((x(t), u(t))_{t \in \mathcal{T}}$ verifying

$$\dot{x} = g(t, u(t), x(t))$$

    and one initial condition $x(0) = x_0$ and one terminal condition

    the functional

$$J(x, u) \equiv \int_0^T F(t, x(t), u(t))\, dt$$

- **CT OC problem**: find $u^* \equiv (u^*(t))_{t \in \mathcal{T}}$ and $x^* \equiv (x^*(t))_{t \in \mathcal{T}}$, belonging to $\mathcal{D}$, that maximize the functional $J(x, u)$
- The optimal value for the program is:

$$J^* \equiv J(x^*) = \max_u \{J(x, u) : (x, u) \in \mathcal{D}\}$$

Continuous time: economic applications

Cake eating problem

Find flows $W \equiv \left(W(t)\right)_{t=0}^{T}$ and $C = \equiv \left(C(t)\right)_{t=0}^{T}$ that solve the problem

$$\max_{C} \int_{t=0}^{T} u(C(t))\, e^{-\rho t} dt$$

subject to

$$\begin{cases} \dot{W}(t) = -C(t), & t \in [0, T] \\ \text{other conditions} \end{cases}$$

where:

$C(t)$ = consumption at time t

$u(t) = u(C(t))$ = value of consumption at time t

$W(t)$ size of the cake at time t,

$\dot{W}(t) = \frac{dW(t)}{dt}$ instantaneous change in $W(t)$

Continuous time: economic applications

Consumption-investment problem

Find flows $A \equiv (A(t))_{t=0}^{T}$ and $C \equiv (C_t)_{t=0}^{T}$ that solve the problem

$$\max_{C} \int_{t=0}^{T} u(C(t))e^{-\rho t}dt$$

subject to

$$\begin{cases} \dot{A} = Y(t) + rA(t) - C(t), & t \in [0, T] \\ \text{other conditions} \end{cases}$$

where:
$C(t) =$ consumption at time t
$u(t) = u(C(t)) =$ value of consumption at time t
$A(t) =$ net financial wealth at time t $Y(t) =$ non-financial flow of income at time t
$r =$ interest rate

Continuous time: economic applications

Firm's investment problem

Find flows $K \equiv (K(t))_{t=0}^{T}$ and $I \equiv (I(t))_{t=0}^{T}$ that solve the problem

$$\max_{I} \int_{0}^{T} \pi(K(t), I(t)) e^{-rt} dt + R(T, K(T))$$

subject to

$$\begin{cases} \dot{K}(t) = I(t) - \delta K(t), & t \in [0, T] \\ \text{other conditions} \end{cases}$$

where:

$\pi(t) = \pi(K(t), I(t)) =$ firm's cash-flow at time t

$K(t) =$ stock of capital at time t

$I(t) =$ gross investment at time t

$r =$ interest rate (assumed to be constant)

$R(T, K(T)) =$ scrap value

Continuous time: economic applications

Economic growth

- **Simple endogenous growth model**: find flows $K \equiv (K(t))_{t=0}^{T}$ and $C \equiv (C_t)_{t=0}^{T}$ that solve

$$\max_{C} \left\{ \int_{t=0}^{\infty} u(C(t))e^{-\rho t}dt : \ \dot{K} = AK(t) - C(t) - \delta K(t), \text{ other conditions} \right\}$$

  $K(t)$ stock of physical capital, $C(t)$ consumption flow at time t

- **Ramsey model**: find flows $K \equiv (K(t))_{t=0}^{T}$ and $C \equiv (C_t)_{t=0}^{T}$ that solve

$$\max_{C} \left\{ \int_{t=0}^{\infty} u(C(t))e^{-\rho t}dt : \dot{K} = F(K(t)) - C(t) - \delta K(t), \right.$$

$$\left. \text{plus other conditions} \right\}$$