Growth and human capital accumulation

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Stylized facts addressed by the model

Since the industrial revolution:

- the population growth rate is smaller than the rate of growth of the economies
- but human capital increase is a major source of long run growth
- there is a permanent increase in the wage rate
- this can only be possible if there is a permanent increase in labor productivity
- education, which became widespread, has been a major source of increase in human capital

Recent (empirical) books on the importance of human capital

- Measurement of human capital: it is given by *quality* $\times$ *quantity*, the dimension which is more correlated with growth is quality (quantity) for the more (less) developed countries (see Hanushek and Woessmann (2015))
- The recent slow down of growth (at least in the US) is essentially explained by the reduction in human capital (aging and reduction of hours worked, not compensated by schooling and on the job training, see Vollrath (2020))
- The race between human capital and technology: are they substitutable or complements ? (see Goldin and Katz (2008))

Human capital in growth theory

- The papers Uzawa (1965) and Lucas (1988) are still the benchmark model
- This is the only endogenous growth model we will study in which there is both long run growth and transition dynamics. The reason is: there are two types of capital which are reproducible.

The Uzawa- Lucas model

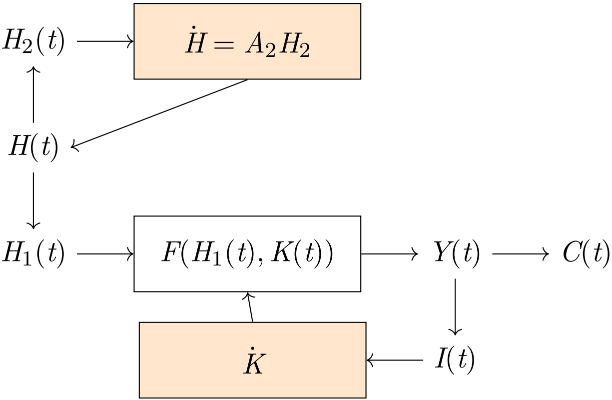
The economy has the following features:

1. there are **two reproducible** inputs: physical capital and human capital
2. there are **two sectors**: manufacturing and education (training)
   - the manufacturing good is used in consumption and investment
   - the education produces a service which is only used in production
3. consumption/savings are determined by a centralized planner (Ramsey planner)

The Uzawa- Lucas model

- There are several versions
  - Some extend the AK model: model with no externalities
  - Others extend the Romer model: versions with externalities (see Brito and Venditti (2010) for a general case)
- Next we present only the first version (centralized economy with no externalities)

The mechanics of the model



Assumptions

- the preference structure is analogous to the Ramsey and AK models: maximizing the present-value of the flow of utility derived from consumption;
- both sectors have production functions displaying **constant returns to scale** relative to their own capital;

$$\begin{aligned} Y_1 &= A_1\, K_1^{\alpha}\, H_1^{1-\alpha}, \text{ manufacturing} \\ Y_2 &= A_2\, H_2, \text{ education/training} \end{aligned}$$

- there is **no sector specific** human capital (i.e., human capital can be freely relocated between sectors);
- there are no externalities both in human and physical capital.

The model

Variables in levels

- Intertemporal utility

$$\max_{C,K_1,H_1,H_2} \int_0^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

  assumption $\rho + A_2(\theta - 1) > 0$

- accumulation equations for stocks of physical and human capital

$$\dot{K} = Y_1(t) - C(t)$$
$$\dot{H} = Y_2(t)$$

- allocation constraints of the stocks between the two sectors

$$K(t) = K_1(t)$$
$$H(t) = H_1(t) + H_2(t)$$

- production functions for manufacturing and education

$$Y_1(t) = A_1\, K_1(t)^{\alpha}\, H_1(t)^{1-\alpha}$$
$$Y_2(t) = A_2\, H_2(t)$$

Notation

- Y_j output of sector: manufacturing ($j = 1$) and education ($j = 2$)
- K_j physical capital allocated to sector $j = 1, 2$
- H_j human capital allocated to sector $j = 1, 2$
- A_j productibity parameter for sector $j = 1, 2$
- K aggregate stock of physical capital
- H aggregate stock of human capital
- $\alpha \in (0, 1)$ share of capital in manufacturing
- $\theta > 0$ inverse of the elasticity of intertemporal substitution for consumption
- $\rho > 0$ rate of time preference

Detrending

- We introduce the temporal decomposition of variables: transition plus long run components

$$K_j(t) = k_j(t)e^{\gamma t},\ H_j(t) = h_j(t)e^{\gamma t},\ j = 1, 2$$

- assuming a **necessary condition for the existence of a balanced growth path**: the rates of growth are equal

$$\gamma_k = \gamma_h = \gamma$$

- then the rates of growth of the detrended variables are

$$\frac{\dot{k}_j}{k_j} = \frac{\dot{K}_j}{K_j} - \gamma,\ \frac{\dot{h}_j}{h_j} = \frac{\dot{H}_j}{H_j} - \gamma\ j = 1, 2$$

The model in detrended variables

Central planner's problem

- Intertemporal utility

$$\max_{c,k_1,h_1,h_2} \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{-(\rho-\gamma(1-\theta))t} dt$$

- accumulation equations for stocks of physical and human capital

$$\begin{aligned} \dot{k} &= y_1(t) - c(t) - \gamma k(t) && (1) \\ \dot{h} &= y_2(t) - \gamma h(t) && (2) \end{aligned}$$

- allocation constraints of the stocks between the two sectors

$$\begin{aligned} k(t) &= k_1(t),\ k(0) = k_0 && (3) \\ h(t) &= h_1(t) + h_2(t),\ h(0) = h_0 && (4) \end{aligned}$$

- production functions for manufacturing and education (because of linear homogeneity)

$$y_1(t) = A_1\, k_1(t)^\alpha h_1(t)^{1-\alpha}$$

$$y_2(t) = A_2\, h_2(t)$$

Solving the model

Characterizing the model

- Observe that the model is an optimal control problem with:
  - four control variables: c, h_1, h_2, and k_1
  - two state variables: k and h
  - two dynamic constrains (1), (2)
  - two static constrains (3), (4)

Solving the model

Applying the Pontriyagin's principlel

- the current-value Hamiltonian is

$$\mathcal{H} = \frac{c(t)^{1-\theta}}{1-\theta} + p_k\left(A_1 k_1^{\alpha} h_1^{1-\alpha} - c - \gamma k\right) + p_h\left(A_2 h_2 - \gamma h\right) + \\ + R(k - k_1) + W(h - h_1 - h_2) \quad (5)$$

p_k, p_h: co-state variables (optimal asset prices)
R, W: Lagrange multipliers (optimal return on capital and wage rates)

- defining the optimal real rate of return of capital $r \equiv R/p_k$ and the real wage $w \equiv W/p_h$, yields
- the current-value Hamiltonian becomes

$$\mathcal{H} = \frac{c(t)^{1-\theta}}{1-\theta} + p_k\Big(\left(A_1 k_1^{\alpha} h_1^{1-\alpha} - c - \gamma k\right) + r\big(k - k_1\big)\Big) \\ + p_h\Big(\left(A_2 h_2 - \gamma h\right) + w\big(h - h_1 - h_2\big)\Big) \quad (6)$$

First order conditions for an interior solution

- optimal consumption

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \Leftrightarrow c^{-\theta} = p_k \tag{7}$$

- optimal allocation of human and physical capital between the two sectors,

$$\frac{\partial \mathcal{H}}{\partial k_1} = 0 \quad \Leftrightarrow \quad \alpha y_1 = r k_1 \tag{8}$$

$$\frac{\partial \mathcal{H}}{\partial h_1} = 0 \quad \Leftrightarrow \quad (1-\alpha) p_k y_1 = w p_h h_1 \tag{9}$$

$$\frac{\partial \mathcal{H}}{\partial h_2} = 0 \quad \Leftrightarrow \quad w = A_2 \tag{10}$$

- conditions for the Lagrange multipliers r and w

$$\frac{\partial \mathcal{H}}{\partial r} = 0 \quad \Leftrightarrow \quad k = k_1 \tag{11}$$

$$\frac{\partial \mathcal{H}}{\partial w} = 0 \quad \Leftrightarrow \quad h = h_1 + h_2 \tag{12}$$

First order conditions for an interior solution
(continuation)

- Euler equations (recall that the discount rate in the detrended problem is $\rho + \gamma(\theta - 1)$)

$$\dot{p}_k = p_k(\rho + \gamma(\theta - 1)) - \frac{\partial \mathcal{H}}{\partial k} = p_k(\rho + \gamma\theta - r) \quad (13)$$

$$\dot{p}_h = p_h(\rho + \gamma(\theta - 1)) - \frac{\partial \mathcal{H}}{\partial h} = p_h(\rho + \gamma\theta - A_2) \quad (14)$$

- transversality conditions

$$\lim_{t\to\infty} e^{-(\rho+\gamma(\theta-1))t} (p_k(t)k(t) + p_h(t)h(t)) = 0 \quad (15)$$

- admissibility conditions

$$\dot{k} = y_1(t) - c(t) - \gamma k(t) \quad (16)$$

$$\dot{h} = y_2(t) - \gamma h(t) \quad (17)$$

Solution for returns and allocations

- Solving equations (8)-(12) for k_1, h_1, h_2, r and w, we get the optimal rate of return, and the optimal allocation of physical capital and human capital between sectors
- the optimal returns are

$$r^* = r(\pi) \equiv \left(\alpha_0 A_1 \left(\frac{\pi}{A_2}\right)^{1-\alpha}\right)^{\frac{1}{\alpha}}, \text{for } \alpha_0 \equiv \alpha^{\alpha}(1-\alpha)^{(1-\alpha)}$$

$$w^* = A_2$$

- where we define the relative price of physical capital relative to human capital (prices of stocks) as

$$\pi \equiv \frac{p_k}{p_h}$$

- then r increases with A_1 and π and decreases with A_2 and w is equal to the productivity of education

Solution for returns and allocations

- the optimal allocation of physical and human capital are

$$k_1^* = k$$

$$h_1^* = \left(\frac{r(\pi)}{\alpha A_1}\right)^{\frac{1}{1-\alpha}} k$$

$$h_2^* = h - h_1^* = h - \left(\frac{r(\pi)}{\alpha A_1}\right)^{\frac{1}{1-\alpha}} k$$

- the allocations are linear functions of the aggregate stocks of capital and human labor and are non-linear functions of the relative price π
- an increase in the rate of return of capital shifts human capital from education to manufacturing

Solution for sectoral outputs

- If we substitute in the production function of both sectors we get the optimal allocation of production between manufacturing and education

$$
\begin{aligned}
y_1^* &= a_1(\pi)k, \text{ where } a_1 = \frac{r(\pi)}{\alpha} > 0 \\
y_2^* &= a_2(\pi)k + A_2 h, \text{ where } a_2 = -A_2 \left(\frac{r(\pi)}{\alpha A_1}\right)^{\frac{1}{1-\alpha}} < 0
\end{aligned}
$$

- the outputs have a linear input-output structure (the manufacturing sector has an AK structure)
- then an increase in the relative price π increases the r.o.r of capital which increases output of manufactures and reduces the output of the educational sector

Solving the model

From now in we follow the following method:

1. Solve the steady state obtaining: γ, π, k and h
2. Obtain the MHDS substituting $\gamma = \bar{\gamma}$
3. Check under which conditions economy can be in a balanced growth path
4. Linearize and solve the MHDS to obtain the transitional dynamics

Long run growth rate and factor returns

- From equation (14), setting $\dot{p}_h = 0$ we get the **long-run growth rate**

$$\boxed{\bar{\gamma} = \frac{A_2 - \rho}{\theta}}$$

  increases with the productivity of the educational sector (similar to the AK model)

- From equation (14) and (13) we have a long-run arbitrage condition

$$\boxed{\bar{r} = \bar{w} = A_2}$$

- then the long-run relative asset price is

$$\boxed{\bar{\pi} = \frac{\bar{p}_k}{\bar{p}_h} = \left(\frac{\alpha_0 A_1}{A_2}\right)^{\frac{1}{1-\alpha}}}$$

Other long run relationships

- ratio between the state variables (setting $\dot{h} = 0$ in equation (17))
$$\frac{\bar{k}}{\bar{h}} = \eta \equiv -\frac{A_2 - \bar{\gamma}}{\bar{a}_2} = \bar{\pi}\left(\frac{\bar{\gamma} - A_2}{A_2}\right)\left(\frac{\alpha}{1-\alpha}\right)$$
because
$$\bar{a}_2 = -A_2\left(\frac{A_2}{\alpha A_1}\right)^{\frac{1}{1-\alpha}} = -\frac{A_2}{\bar{\pi}}\left(\frac{1-\alpha}{\alpha}\right) < 0$$
- the ratio $\frac{\bar{k}}{\bar{h}}$ is positive because of the transversality condition holds if and only if
$$\rho + \bar{\gamma}(\theta - 1) = \frac{\rho + A_2(\theta - 1)}{\theta} = A_2 - \bar{\gamma} > 0$$
- BGP consumption (for $\dot{k} = 0$ in equation (16))
$$\bar{c} = c(p_k) = \beta\,\bar{k},\ \beta \equiv \frac{A_2}{\alpha} - \bar{\gamma} > 0$$

The MHDS

- substituting $\gamma = \bar{\gamma}$ the MHDS becomes

$$\begin{aligned}
\dot{p}_k &= p_k(A_2 - r(p_k/p_h)) && (18)\\
\dot{p}_h &= 0 && (19)\\
\dot{k} &= (a_1 r(p_k/p_h)) - \bar{\gamma})\, k - c(p_k) && (20)\\
\dot{h} &= a_2(r(p_k/p_h))k + (A_2 - \bar{\gamma})h && (21)
\end{aligned}$$

Initial conditions and the BGP

- If the initial conditions satisfies

$$k_0 = \eta h_0$$

- then the economy will evolve along the BGP such that

$$\bar{K}(t) = \eta h_0 e^{\bar{\gamma} t}, \bar{H}(t) = h_0 e^{\bar{\gamma} t}$$

- alternatively, if the initial conditions satisfies

$$k_0 \neq \eta h_0$$

- then there will be transitional dynamics

Transitional dynamics

- The system (18)-(21) is non-linear. A linear approximation in the neighborhood of the BGP is

$$\begin{pmatrix} \dot{p}_k \\ \dot{p}_h \\ \dot{k} \\ \dot{h} \end{pmatrix} = J \begin{pmatrix} p_k - \bar{p}_k \\ p_h - \bar{p}_h \\ k - \bar{k} \\ h - \bar{h} \end{pmatrix}$$

where ($\mu \equiv A_2 - \bar{\gamma} > 0$)

$$\bar{\mathbf{J}} = \begin{pmatrix} \mu - \beta & \bar{\pi}(\beta - \mu) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta(\theta-\alpha)-\alpha\mu)\bar{k}}{\alpha\theta\bar{p}_k} & -\frac{(\beta-\mu)\bar{k}}{\alpha\bar{p}_h} & \beta & 0 \\ -\dfrac{\mu\bar{h}}{\alpha\bar{p}_k} & \dfrac{\mu\bar{h}}{\alpha\bar{p}_h} & -\dfrac{\mu}{\eta} & \mu \end{pmatrix}$$

Local dynamics in the neighborhood of the BGP

- The characteristic polynomial of the Jacobian J is

$$C(\mathbf{J}, \lambda) = \lambda \left(\lambda - (\mu - \beta)\right) (\lambda - \beta) (\lambda - \mu) \,,$$

- Then the eigenvalues are

$$\lambda_1 = \mu - \beta < 0,\ \lambda_2 = 0,\ \lambda_3 = \beta > 0,\ \lambda_4 = A_2 - \bar{\gamma} > 0$$

- there is **transitional dynamics** : because $\lambda_1 < 0$ which is

$$\lambda_2 = \left. \frac{\partial \dot{p}_k}{p_k} \right|_{BGP} = \mu - \beta < 0$$

Local dynamics in the neighborhood of the BGP

- solving the system (see my revised notes chapter 7)

$$
\begin{aligned}
h(t) &= h_\infty + (h_0 - h_\infty) e^{(\mu-\beta)t} \\
k(t) &= \eta h_\infty + (k_0 - \eta h_\infty) e^{(\mu-\beta)t} \\
p_k(t) &= \bar{p}_k \left[1 + \frac{\theta\alpha(2\beta-\mu)}{\mu(\theta-\alpha)} \left(\frac{h_0 - h_\infty}{h_\infty} \right) e^{(\mu-\beta)t} \right] \\
p_h(t) &= \bar{p}_h
\end{aligned}
$$

where

$$h_\infty = \frac{k_0\mu(\theta-\alpha) + \eta h_0(\beta(\alpha+\theta) - \mu\theta)}{\eta(\beta(\alpha+\theta) - \alpha\mu)}$$

if we take $\bar{h} = h_\infty$. This implies $\bar{p}_k = (\beta\eta h_\infty)^{-\theta}$ and $\bar{p}_h = \bar{\pi}\bar{p}_k$.

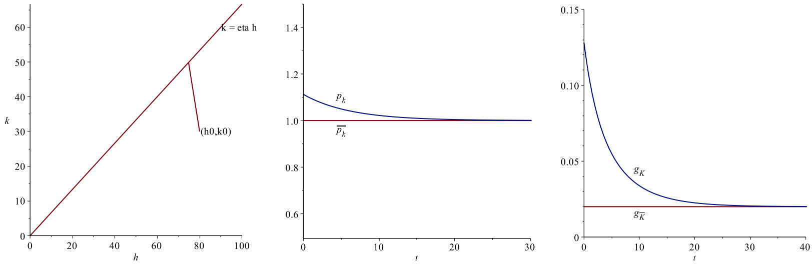


Figure: Uzawa-Lucas model: phase diagram, trajectories for p_k and for the rate of growth of K and $\bar{K}$. Parameter values: $\rho = 0.02$, $\alpha = 0.3$, $\theta = 2$, $A_1 = 0.2$ and $A_2 = 0.06$.

Trajectory for the GDP

- the GDP for the manufacturing sector is

$$Y_1(t) = y_1(t) e^{\bar{\gamma} t}$$

  where

$$y_1(t) = y_{1,\infty} \left(1 + \left(\frac{k_0}{\eta h_\infty} - 1\right) e^{(\mu-\beta)t}\right)^{\alpha} \left(1 + \left(\frac{h_0}{h_\infty} - 1\right) e^{(\mu-\beta)t}\right)^{1-\alpha}$$

- Taking $\lim_{t\to\infty} y_1(t) = y_{1,\infty} \equiv A_1 \eta^{\alpha} h_\infty$ we get the BGP

$$\bar{Y}_1(t) \approx y_{1,\infty} e^{\bar{\gamma} t}.$$

Growth implications

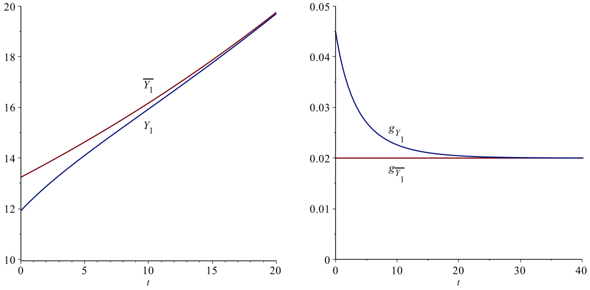


Figure: Uzawa-Lucas model: trajectories for levels and rates of growth of Y_1 and $\bar{Y}_1$

Conclusions

- **there is long run growth** and the growth rate is a positive function of A_2
- the **long run level of GDP** depends on the initial levels of k and h

$$\bar{y}_1 = y_{1,\infty} = A_1 \eta^{\alpha-1} \left(\frac{k_0 \mu(\theta - \alpha) + \eta h_0(\beta(\alpha+\theta) - \mu\theta)}{\beta(\alpha+\theta) - \alpha\mu} \right)$$

  if $k_0 = \eta h_0$ then $\bar{y}_1 = A_1 \eta^{\alpha} h_0 = A_1 k_0^{\alpha} h_0^{1-\alpha}$ the economy will be at a BGP
- there is transitional dynamics (if $k_0 \neq \eta h_0$) with

$$y_1(t) - \bar{y}_1 \approx e^{(\mu-\beta)t}$$

- the GDP path in levels is

$$Y_1(t) = y_1(t) e^{\bar{\gamma} t}$$

Conclusions

- The driving force for transitional dynamics if $\dot{\pi}/\pi$

$$\dot{\pi}/\pi = A_2 - r(\pi)$$

- if initial **capital** k_0 **is too low relative to** ηh_0 then $\pi(0) > \bar{\pi}$ and two effects will occur
  - because $a_1^{'}(\pi) > 0$ and $a_2^{'}(\pi) < 0$ there will be an increase in the ratio $k(t)/h(t)$
  - because $r(\pi) > A_2$ then $\dot{\pi}/\pi < 0$;
- the adjustment of π will eliminate through time both the divergences $A_2 - r(\pi)$ and $k(t) - \eta h(t)$ leading to convergence to the BGP.

Conclusions: effect of an increase in A_2

A positive permanent shock in A_2 (from $A_{2,0}$ to $A_{2,1} > A_{2,0}$), will produce the following effects (starting from a BGP)

- an increase in the long-run growth rate $\bar{\gamma}(A_{2,1}) > \bar{\gamma}(A_{2,0})$
- an increase in $\dfrac{\bar{k}}{\bar{h}} = \eta$ (because $\frac{\partial \eta}{\partial A_2} > 0$)
- if before the shock $k_0 = \eta(A_{2,0})h_0$, then after the shock $k_0 < \eta(A_{2,1})h_0$ which means physical capital becomes "too low" relative to human capital
- then the process just described unfolds: $\pi(0) > \bar{\pi}(A_{2,0})$, the interest rate becomes higher then $A_{2,1}$, k accumulates faster than h but π starts to decrease to eliminate the "excess" k.

Bibliography

- Barro and Sala-i-Martin (2004) (Acemoglu, 2009, ch.10) and (Aghion and Howitt, 2009, ch.13)
- The original papers Uzawa (1965) and Lucas (1988)

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