```
(* Calculus of Variations
   Discrete Time
 Solutions given in the problem set

 Paulo Brito: 22.12.2015
*)


(* 1.1.1 *)

F[t_, y_, y1_] := - (y1 - (1 / 2) y - 2) ^2
Factor[Simplify[D[F[t - 1, y[t - 1], y[t]], y[t]] + D[F[t, y[t], y[t + 1]], y[t]]]]
sol1 = RSolve[
   {D[F[t - 1, y[t - 1], y[t]], y[t]] + D[F[t, y[t], y[t + 1]], y[t]] == 0}, y[t], t]
ys[t_] := Evaluate[y[t] /. sol1]
ITC = Simplify[Solve[ {ys[0] == 1, ys[4] == 1}, {C[1], C[2]}]]
```

$\frac{1}{2}(4 + 2\,y[-1+t] - 5\,y[t] + 2\,y[1+t])$

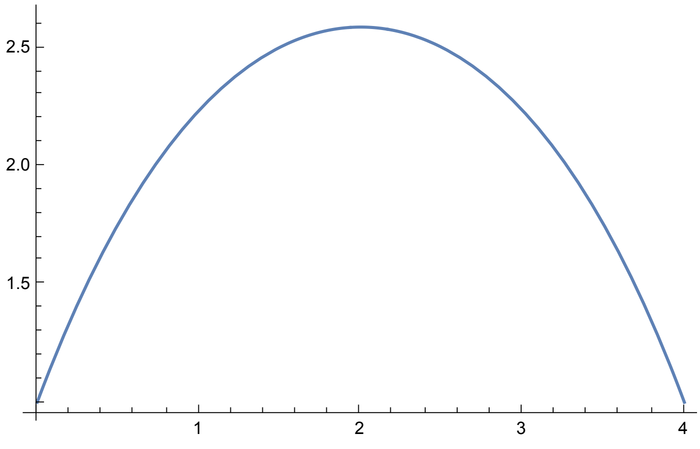
$\{\{y[t] \to 4 + 2^{-t}\,C[1] + 2^{t}\,C[2]\}\}$

$\{\{C[1] \to -\frac{48}{17}, C[2] \to -\frac{3}{17}\}\}$

$\{\{C[1] \to -\frac{48}{17}, C[2] \to -\frac{3}{17}\}\}$

```
y[t_] = Evaluate[y[t] /. sol1 /. C[1] → -48/17 /. C[2] → -3/17]
Plot[y[t], {t, 0, 4}]
```

$\{4 - \frac{3 \times 2^{4-t}}{17} - \frac{3 \times 2^{t}}{17}\}$

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(* 1.1.2 *)
```

```
ClearAll[y, y1, sol1, ITC]
F[t_, y_, y1_] := - (y1 - (1 / 2) y - 2) ^2
Factor[Simplify[D[F[t - 1, y[t - 1], y[t]], y[t]] + D[F[t, y[t], y[t + 1]], y[t]]]]
```

$$\frac{1}{2}\,(4 + 2\,y[-1+t] - 5\,y[t] + 2\,y[1+t])$$

```
D[F[T - 1, y[T - 1], y[T]], y[T]]
```

$$-2\left(-2 - \frac{1}{2}\,y[-1+T] + y[T]\right)$$

```
sol1 = RSolve[
  {D[F[t - 1, y[t - 1], y[t]], y[t]] + D[F[t, y[t], y[t + 1]], y[t]] == 0, y[0] == 1},
  y[t], t]
ys[t_] := Evaluate[y[t] /. sol1]
ITC = Simplify[Solve[ {-2 - 1/2 ys[3] + ys[4] == 0}, {C[1]}]]
```

$$\{\{y[t] \to -2^{-t}\left(3\times 2^{2\,t} - 2^{2+t} - C[1] + 2^{2\,t}\,C[1]\right)\}\}$$

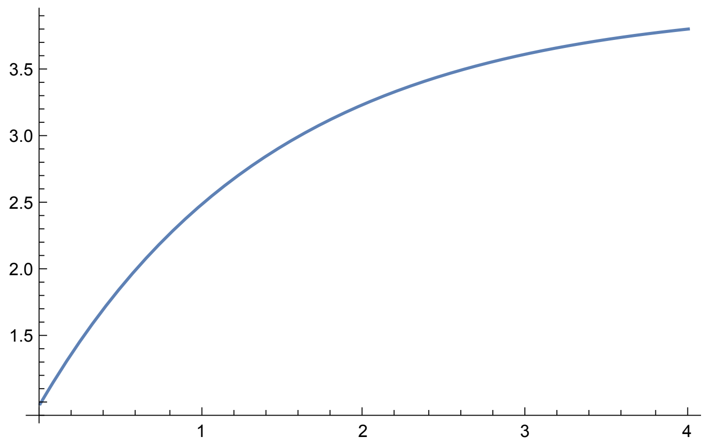
$$\{\{C[1] \to -3\}\}$$

```
y[t_] = Evaluate[ y[t] /.  sol1 /. C[1] → -3]
y[0]
y[4]
Plot[y[t], {t, 0, 4}]
```

$$\{-2^{-t}\left(3 - 2^{2+t}\right)\}$$

$$\{1\}$$

$$\left\{\frac{61}{16}\right\}$$




```
(* 1.1.3 *)

ClearAll
ClearAll[W, W1, Ws, sol1, ITC]
```

ClearAll

```
F[t_, W_, W1_] := (β^t) ((W - W1) ^ (1 - σ)) / (1 - σ)
Factor[Simplify[D[F[t - 1, W[t - 1], W[t]], W[t]] + D[F[t, W[t], W[t + 1]], W[t]] ,
  Assumptions → σ > 0 && β > 0 && β < 1 && ϕ > 0]]
```

$$-\beta^{-1+t}\,(W[-1+t] - W[t])^{-\sigma}\,\left(-\beta\,(W[-1+t] - W[t])^{\sigma} + (W[t] - W[1+t])^{\sigma}\right)\,(W[t] - W[1+t])^{-\sigma}$$

```
sol1 = RSolve[{-β^(-1/σ) (W[t] - W[1 + t]) + (W[-1 + t] - W[t]) == 0, W[0] == ϕ}, W[t], t]
```

$$\left\{\left\{W[t] \to \left(\beta^{\frac{1}{\sigma}}\right)^{t} \phi + C[1] - \left(\beta^{\frac{1}{\sigma}}\right)^{t} C[1]\right\}\right\}$$

```
Ws[t_] := Evaluate[W[t] /. sol1]
TC = Simplify[Solve[ {Ws[T] == 0}, {C[1]}]]
```

$$\left\{\left\{C[1] \to \frac{\left(\beta^{\frac{1}{\sigma}}\right)^{T} \phi}{-1 + \left(\beta^{\frac{1}{\sigma}}\right)^{T}}\right\}\right\}$$

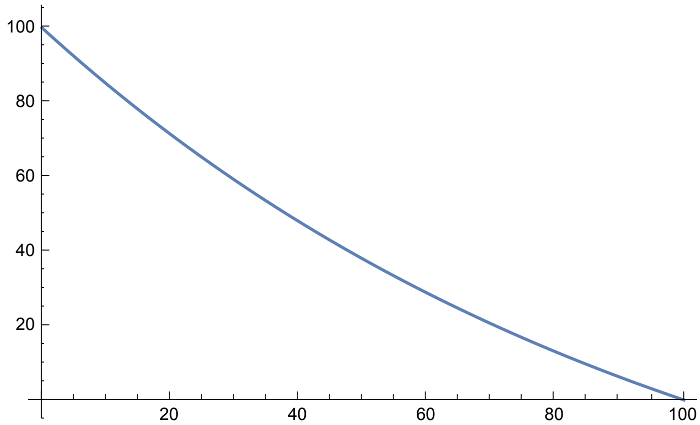
$$\texttt{Ww[t\_] := Evaluate}\left[\texttt{W[t] /. sol1 /. C[1]} \to \frac{\left(\beta^{\frac{1}{\sigma}}\right)^{T} \phi}{-1 + \left(\beta^{\frac{1}{\sigma}}\right)^{T}}\right]$$

```
W[β_, σ_, ϕ_, t_, T_] = Factor[Simplify[Ww[t]]]
Simplify[Ww[0]]
Simplify[Ww[T]]
Plot[W[1 / 1.02, 2, 100, t, 100], {t, 0, 100}]
```

$$\left\{\frac{\left(-\left(\beta^{\frac{1}{\sigma}}\right)^{t} + \left(\beta^{\frac{1}{\sigma}}\right)^{T}\right) \phi}{-1 + \left(\beta^{\frac{1}{\sigma}}\right)^{T}}\right\}$$

$\{\phi\}$

$\{0\}$




```
(* 1.1.5 *)
ClearAll
ClearAll[W, W1, Ws, Ww, sol1, ITC]
F[t_, W_, W1_] := (β^t) Log[((1 + r) W - W1)]
Factor[Simplify[D[F[t - 1, W[t - 1], W[t]], W[t]] + D[F[t, W[t], W[t + 1]], W[t]] ,
  Assumptions → β > 0 && β < 1 ]]
sol1 = RSolve[{D[F[t - 1, W[t - 1], W[t]], W[t]] + D[F[t, W[t], W[t + 1]], W[t]] == 0,
   W[0] == ϕ}, W[t], t]
```

ClearAll

$$\left(\beta^{-1+t} \left(\beta W[-1+t] + 2 r \beta W[-1+t] + r^{2} \beta W[-1+t] - W[t] - r W[t] - \beta W[t] - r \beta W[t] + W[1+t]\right)\right) \Big/ \left((W[-1+t] + r W[-1+t] - W[t]) (W[t] + r W[t] - W[1+t])\right)$$

$$\left\{\left\{W[t] \to (1+r)^{t} \phi - (1+r)^{t} C[1] + ((1+r) \beta)^{t} C[1]\right\}\right\}$$

```
Ws[t_] := Evaluate[W[t] /. sol1]
TC = Simplify[Solve[ {Ws[T] == ϕ}, {C[1]}]]
```

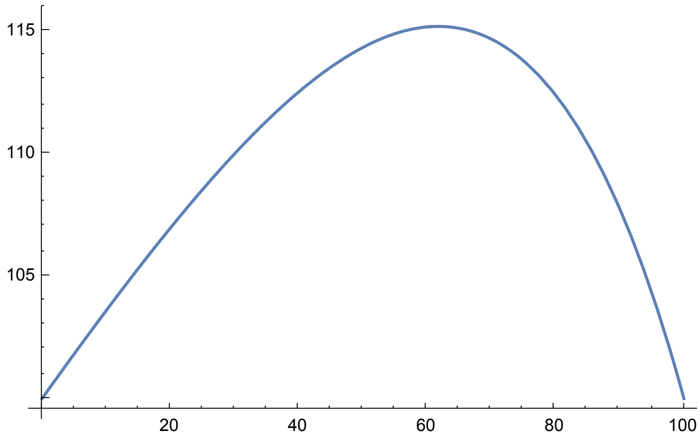
$$\left\{\left\{C[1] \to \frac{\left(-1 + (1+r)^T\right) \phi}{(1+r)^T - ((1+r)\ \beta)^T}\right\}\right\}$$

```
Ww[t_] := Evaluate[ W[t] /. sol1 /. C[1] → (-1 + (1 + r)^T) ϕ / ((1 + r)^T - ((1 + r) β)^T)]
W[β_, r_, ϕ_, t_, T_] = Factor[Simplify[Ww[t]]]
Simplify[Ww[0]]
Simplify[Ww[T]]
Plot[W[1 / 1.02, 0.025, 100, t, 100], {t, 0, 100}]
```

$$\left\{\frac{\left((1+r)^t - ((1+r)\ \beta)^t + (1+r)^T\ ((1+r)\ \beta)^t - (1+r)^t\ ((1+r)\ \beta)^T\right)\ \phi}{(1+r)^T - ((1+r)\ \beta)^T}\right\}$$

$\{\phi\}$

$\{\phi\}$




```
(* 1.1.7 *)
ClearAll
ClearAll[W, W1, Ws, sol1, sol1x, ITC]
```

ClearAll

```
F[t_, W_, W1_] := (β^t) (((1 + r) W - W1) ^ (1 - σ)) / (1 - σ)
Factor[Simplify[D[F[t - 1, W[t - 1], W[t]], W[t]] + D[F[t, W[t], W[t + 1]], W[t]] ,
  Assumptions → σ > 0 && β > 0 && β < 1 ]]
```

$$\beta^{-1+t}\ ((1+r)\ W[-1+t] - W[t])^{-\sigma}$$
$$\left(\beta\ ((1+r)\ W[-1+t] - W[t])^{\sigma} + r\ \beta\ ((1+r)\ W[-1+t] - W[t])^{\sigma} - ((1+r)\ W[t] - W[1+t])^{\sigma}\right)$$
$$((1+r)\ W[t] - W[1+t])^{-\sigma}$$

```
sol1x = RSolve[
  {D[F[t - 1, W[t - 1], W[t]], W[t]] + D[F[t, W[t], W[t + 1]], W[t]] == 0, W[0] == ϕ},
  W[t], t]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\mathrm{RSolve}\left[\left\{-\beta^{-1+t}\left((1+r)\,W[-1+t]-W[t]\right)^{-\sigma}+(1+r)\,\beta^{t}\left((1+r)\,W[t]-W[1+t]\right)^{-\sigma}==0,\ W[0]==\phi\right\},\ W[t],\ t\right]$$

```
sol1 =
 RSolve[{- ((1 + r) W[-1 + t] - W[t]) + ((1 + r) β) ^ (-1 / σ) ((1 + r) W[t] - W[1 + t]) == 0,
   W[0] == ϕ}, W[t], t]
```

$$\left\{\left\{W[t]\to\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{t}\phi+(1+r)^{t}\,C[1]-\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{t}C[1]\right\}\right\}$$

```
Ws[t_] := Evaluate[W[t] /. sol1]
TC = Simplify[Solve[ {Ws[T] == ϕ}, {C[1]}]]
```

$$\left\{\left\{C[1]\to\frac{\phi-\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{T}\phi}{(1+r)^{T}-\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{T}}\right\}\right\}$$

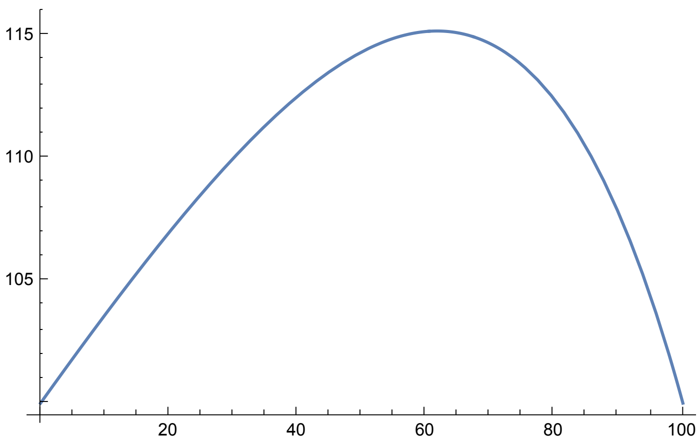
$$\mathbf{Ww[t\_] := Evaluate\left[W[t]\ /.\ sol1\ /.\ C[1]\to\frac{\phi-\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{T}\phi}{(1+r)^{T}-\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{T}}\right]}$$

```
W[β_, σ_, r_, ϕ_, t_, T_] = Factor[Simplify[Ww[t]]]
Simplify[Ww[0]]
Simplify[Ww[T]]
Plot[W[1 / 1.02, 1, 0.025, 100, t, 100], {t, 0, 100}]
```

$$\left\{\frac{\left((1+r)^{t}-\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{t}+(1+r)^{T}\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{t}-(1+r)^{t}\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{T}\right)\phi}{(1+r)^{T}-\left(\left((1+r)\,\beta\right)^{\frac{1}{\sigma}}\right)^{T}}\right\}$$

$\{\phi\}$

$\{\phi\}$



```
(* 1.1.11 *)
ClearAll
ClearaAll[A, A1, As, Aw]
```

```
ClearAll
```

```
ClearaAll[A, A1, As, Aw]
```

```
F[t_, A_, A1_] := (β^t) (B - ξ^(-1) Exp[-ξ (A - A1)])
Factor[Simplify[D[F[t - 1, A[t - 1], A[t]], A[t]] + D[F[t, A[t], A[t + 1]], A[t]] ,
  Assumptions → ξ > 0 && β > 0 && β < 1]]
```

$\beta^{-1+t}\left(-e^{\xi(-A[-1+t]+A[t])}+e^{\xi(-A[t]+A[1+t])}\beta\right)$

```
sol1 = RSolve[
  {D[F[t - 1, A[t - 1], A[t]], A[t]] + D[F[t, A[t], A[t + 1]], A[t]] == 0, A[0] == ϕ},
  A[t], t]
```

Solve::ifun: Inverse functionsare being used by Solve, so
some solutionsmay not be found; use Reduce for completesolutioninformation ≫

$\left\{\left\{A[t]\to\phi+\text{Log}\left[\left(\beta^{-1/\xi}\right)^{\frac{1}{2}(-3+t)\,t}\,C[1]^{t}\right]\right\}\right\}$

```
As[t_] := Evaluate[A[t] /. sol1]
TC = Simplify[Solve[ {As[T] == 0}, {C[1]}]]
```

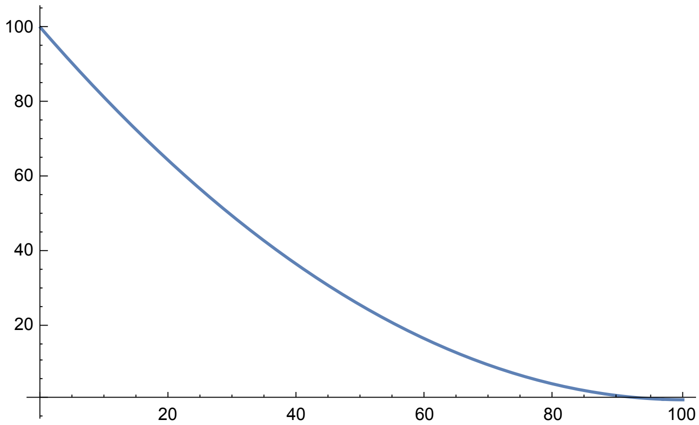
$\left\{\left\{C[1]\to e^{-\frac{\phi}{T}}\beta^{\frac{-3+T}{2\xi}}\right\}\right\}$

```
Aw[t_] := Expand[Evaluate[ A[t] /. sol1 /. C[1] → e^(-ϕ/T) β^((-3+T)/(2 ξ))]]
A[β_, ξ_, ϕ_, t_, T_] = Factor[Simplify[Aw[t]]]
Simplify[Aw[0]]
Factor[Expand[Aw[T]]]
Plot[A[1 / 1.02, 1, 100, t, 100], {t, 0, 100}]
```

$\left\{\phi+\text{Log}\left[\left(\beta^{-1/\xi}\right)^{\frac{1}{2}(-3+t)\,t}\left(e^{-\frac{\phi}{T}}\beta^{\frac{-3+T}{2\xi}}\right)^{t}\right]\right\}$

{A[0]}

{A[T]}



```
(* 1.1.14 *)
ClearAll
ClearAll[y, y1, ys, yw]
F[t_, y_, y1_] := - (y1 - y - 1) ^2
Factor[Simplify[D[F[t - 1, y[t - 1], y[t]], y[t]] + D[F[t, y[t], y[t + 1]], y[t]]]]
sol1 = RSolve[
  {D[F[t - 1, y[t - 1], y[t]], y[t]] + D[F[t, y[t], y[t + 1]], y[t]] == 0}, y[t], t]
ys[t_] := Evaluate[y[t] /. sol1]
ITC = Simplify[Solve[ {ys[0] == 1, ys[T] == 1 + T}, {C[1], C[2]}]]
```

```
ClearAll
```

$2\,(y[-1+t] - 2\,y[t] + y[1+t])$

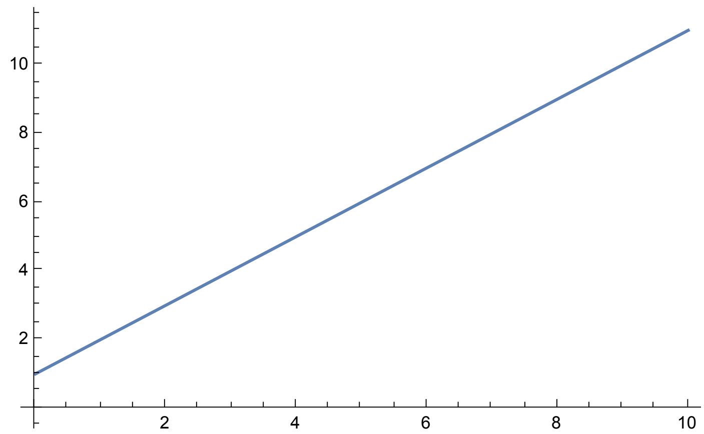
$\{\{y[t] \to C[1] + t\,C[2]\}\}$

$\{\{C[1] \to 1,\ C[2] \to 1\}\}$

```
y[t_] = Evaluate[ y[t] /. sol1 /. C[1] → 1 /. C[2] → 1]
y[t]
Plot[y[t], {t, 0, 10}]
```

$\{1 + t\}$

$\{1 + t\}$




```
ClearAll
ClearAll[b, b1]
```

```
ClearAll
```

```
F[t_, b_, b1_] := β^(t) (- ((1 + r) b - b1) ^2)
Factor[Simplify[D[F[t - 1, b[t - 1], b[t]], b[t]] + D[F[t, b[t], b[t + 1]], b[t]]]]
sol1 = RSolve[
  {D[F[t - 1, b[t - 1], b[t]], b[t]] + D[F[t, b[t], b[t + 1]], b[t]] == 0}, b[t], t]
```

$-2\,\beta^{-1+t}\left(-b[-1+t] - r\,b[-1+t] + b[t] + \beta\,b[t] + 2\,r\,\beta\,b[t] + r^2\,\beta\,b[t] - \beta\,b[1+t] - r\,\beta\,b[1+t]\right)$

$\left\{\left\{b[t] \to \left(\frac{1}{(1+r)\,\beta}\right)^t C[1] + (1+r)^t\,C[2]\right\}\right\}$

```
bs[t_] := Evaluate[b[t] /. sol1]
ITC = Simplify[Solve[ {bs[0] == ϕ, bs[T] == 0}, {C[1], C[2]}]]
```

$$\left\{\left\{C[1] \to \frac{(1+r)^T \phi}{(1+r)^T - \left(\frac{1}{\beta + r\,\beta}\right)^T},\ C[2] \to \frac{\left(\frac{1}{\beta + r\,\beta}\right)^T \phi}{-(1+r)^T + \left(\frac{1}{\beta + r\,\beta}\right)^T}\right\}\right\}$$

```
b[t_, T_, r_, β_, ϕ_] :=
```

$$\text{Evaluate}\left[\, b[t]\ /.\ \text{sol1}\ /.\ C[1] \to \frac{(1+r)^T \phi}{(1+r)^T - \left(\frac{1}{\beta + r\,\beta}\right)^T}\ /.\ C[2] \to \frac{\left(\frac{1}{\beta + r\,\beta}\right)^T \phi}{-(1+r)^T + \left(\frac{1}{\beta + r\,\beta}\right)^T}\right]$$

```
Factor[Simplify[b[t, T, r, β, ϕ]]]
```

$$\left\{\frac{\left((1+r)^T \left(\frac{1}{\beta + r\,\beta}\right)^t - (1+r)^t \left(\frac{1}{\beta + r\,\beta}\right)^T\right) \phi}{(1+r)^T - \left(\frac{1}{\beta + r\,\beta}\right)^T}\right\}$$

```
Plot[b[t, 10, 0.025, 0.02, 1], {t, 0, 10}]
```