Mathematical Economics
Discrete time: optimal control problem

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Optimal control problem

We present the optimality conditions for three problems:

- Simplest problem: x_0, x_T and T given
- Constrained terminal state problem: x_0 and T given and x_T constrained
- Discounted infinite horizon problem

Optimal control problem

Timing and value of the decisions

In discrete time models, it is important to distinguish between moments (time) and periods

- state variables, x, are stock variables and control variables, u, are flow variables
- the stock variable x_t are indexed to **time** t (ex: 31st December 2020)
- a flow variable u_t are indexed to **period** (ex: year 2020)
- F_t is indexed to period t
- **Warning: be careful and consistent as regards the timing you choose**

Optimal control problem

Timing and value of the decisions

value flow		F_0		F_1				F_t				F_{T-2}		F_{T-1}	
control		u_0		u_1				u_t				u_{T-2}		u_{T-1}	
period		0		1				t				$T-2$		$T-1$	
time	0		1		2	$\cdots$	t		$t+1$	$\cdots$	$T-2$		$T-1$		T
state	ϕ_0		x_1		x_2		x_t		x_{t+1}		x_{T-2}		x_{T-1}		ϕ_T

Optimal control problem

Timing and value of the decisions

- The **action in period** t:
  - in the beginning the state x_t is given,
  - during the period the control u_t is chosen,
  - at the end the state variable will be determined by $x_{t+1} = G(x_t, u_t, t)$
- The **value obtained by** u_t, given the state x_t, is

$$F_t = F(x_t, u_t, t) \text{ period } t = 0, 1, \ldots, T-1$$

Optimal control problem

Timing and value of the decisions

- A strategy :
  - is a **sequence of decisions** $u = \{u_0, u_1, \dots, u_{T-1}\}$ implying a sequence of states $x = \{x_1, \dots, x_T\}$
  - with **value**

$$J[u, x] = \sum_{t=0}^{T-1} F(u_t, x_t, t)$$
$$= F(0, x_0, u_0) + \dots + F(t, x_t, u_t) + \dots + F(T-1, x_{T-1}, u_{T-1})$$

  which is a functional: a mapping between a sequence and a number.

- The **optimal sequence** $u^* = \{u_0^*, u_1^*, \dots, u_{T-1}^*\}$ is the one that maximizes the value of the program
- The **value of the program** is the maximum value attained by choosing an that is

$$J^* = J[u^*] = \max_u \{J[x, u] : (x, u) \in \mathcal{X}\}$$

Optimal control: simplest problem

- **Problem OCP**: Find $x^* = \{x_0^*, x_1^*, \ldots, x_T^*\}$ and $u^* = \{u_0^*, u_1^*, \ldots, u_T^*\}$ that maximizes

$$\max_{\{u\}} \sum_{t=0}^{T-1} F(u_t, x_t, t)$$

subject to

$$\begin{cases} x_{t+1} = G(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T = \phi_T & t = T \end{cases} \quad \text{(Problem 1)}$$

T, ϕ_0 and ϕ_T given

Pontriyagin's maximum principle

Pontriyagin's maximum principle (PMP)

- This is one method for solving dynamic optimization problems.
- In order to use it, we define the Hamiltonian function

$$H_t = H(\psi_t, x_t, u_t, t) \equiv F(x_t, u_t, t) + \psi_t G(x_t, u_t, t)$$

where ψ_t is called the co-state variable at time t (obs: it has the same timing as u_t)

- Maximized Hamiltonian is a function

$$H_t^* = H^*(\psi_t, x_t^*, t) = \max_{u_t} H(\psi_t, x_t, u_t, t)$$

for the optimal control, $u_t^* = u^*(x_t, \psi_t, t)$.

PMP: necessary first order conditions

Proposition

- *If* $x^* = \{x_t^*\}_{t=0}^{T}$ *and* $u^* = \{u_t^*\}_{t=0}^{T-1}$ *are solutions to the OCP, then there is a sequence of the co-state variable* $\psi = \{\psi_t\}_{t=0}^{T-1}$ *such that the following conditions are satisfied:*
- *the optimality condition and the canonical equation*

$$\frac{\partial H_t^*}{\partial u_t} = 0, \textit{ for every } t = 0, \dots, T-1$$
$$\psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}, \textit{ for every } t = 0, \dots, T-1$$

- *and the admissibility conditions*

$$\begin{aligned} x_{t+1}^* &= G(x_t^*, u_t^*, t), \textit{ for every } t = 0, \dots, T-1 \\ x_0^* &= \phi_0, \textit{ for } t = 0 \\ x_T^* &= \phi_T, \textit{ for } t = T \end{aligned}$$

Proof.

Maximized Hamiltonian dynamic system (MHDS)

- If $\frac{\partial^2 H_t}{\partial u_t^2} \neq 0$ then the optimality condition

$$\frac{\partial H_t^*}{\partial u_t} = \frac{\partial F(x_t^*, u_t^*, t)}{\partial u_t} + \psi_t \frac{\partial G(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

can be solved for the optimal control

$$u_t^* = U(x_t^*, \psi_t, t), \text{ for every } t = 0, \ldots, T-1$$

- Substituting into the f.o.c we get the **MHDS** in (x_t, ψ_t)

$$\begin{cases} x_{t+1}^* = G(x_t^*, \psi_t, t) \\ \psi_t = P(x_{t+1}^*, \psi_{t+1}, t+1) \end{cases}$$

where

$$\begin{aligned} G(x_t^*, \psi_t, t) &= G(x_t^*, U(x_t^*, \psi_t, t), t) \\ P(x_{t+1}, \psi_{t+1}) &\equiv \frac{\partial H_{t+1}^*}{\partial x_{t+1}}(x_{t+1}^*, U(x_{t+1}^*, \psi_{t+1}, t+1), \psi_{t+1}, t+1) \end{aligned}$$

Alternative MHDS

- Alternatively we can solve

$$\frac{\partial H_t^*}{\partial u_t} = \frac{\partial F(x_t^*, u_t^*, t)}{\partial u_t} + \psi_t \frac{\partial G(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

for $\psi_t = q_t(u_t^*, x_t^*, t)$ and we get an alternative **MHDS** in (x_t, u_t)

$$\begin{cases} x_{t+1}^* = G(x_t^*, u_t^*, t) \\ u_{t+1}^* = U(x_t^*, u_t^*, t, t+1) \end{cases}$$

- The system is characterized by a **forward** equation, $x_{t+1} = G(x_t, u_t, t)$, and a **backward** equation, $u_{t+1} = U(x_t, u_t, t, t+1))$
- The solution is the optimal path $\{u_t^*\}_{t=0}^{T-1}$ and $\{x_t^*\}_{t=0}^{T}$ that: (1) solves the the MHDS, (2) and **satisfies the two boundary conditions**

$$\begin{cases} x_0^* = \phi_0 \\ x_T^* = \phi_T \end{cases}$$

Application: resource depletion, or cake eating, problem

The problem

$$\max_{\{C\}} \sum_{t=0}^{T} \beta^t \ln(C_t),$$

$$\text{subject to}$$

$$W_{t+1} = W_t - C_t,$$

$$W_0 = \phi,$$

$$W_T = 0.$$

Application: resource depletion, or cake eating, problem

Applying the PMP:

- The Hamiltonian for this problem is

$$H_t = \beta^t \ln(C_t) + \psi_t(W_t - C_t)$$

- The first order conditions are

$$\begin{cases} \dfrac{\partial H_t^*}{\partial C_t} = \beta^t (C_t^*)^{-1} - \psi_t = 0, & t = 0, \ldots, T-1 \\ \psi_t = \dfrac{\partial H_{t+1}^*}{\partial W_{t+1}} = \psi_{t+1}, & t = 0, \ldots, T-1 \\ W_{t+1}^* = W_t^* - C_t^*, & t = 0, \ldots, T-1 \\ W_T^* = 0, & t = 0 \\ W_0^* = \phi, & t = T \end{cases}$$

Cake eating problem: MHDS

- As $C_t = \beta^t \psi_t$ and $\psi_{t+1} = \psi_t$ then $C_{t+1} = \beta^{t+1} \psi_{t+1} = \beta\, C_t$.
- Then the f.o.c in (W_t, C_t) are

$$
\begin{aligned}
W^*_{t+1} &= W^*_t - C^*_t, \ t = 0, \dots, T-1 && (1)\\
C^*_{t+1} &= \beta\, C^*_t && (2)\\
W^*_T &= 0 && (3)\\
W^*_0 &= \phi && (4)
\end{aligned}
$$

- To find the solution, C^*, W^*, we have to solve this problem.
- I am only interested to show which kind of solution is obtained. Next we will discuss other methods to solve an optimal control problem.

Cake eating problem: recursive solution

1 Solve the "Euler-equation" (2)

$$C_t = C_0\beta^t. \tag{5}$$

where C_0 is up to this point;

2 Substitute it in the constraint (1)

$$W_{t+1} = W_t - C_0\beta^t$$

3 Solve it to find

$$W_t = k - C_0 \sum_{s=0}^{t-1} \beta^s = k - C_0 \frac{1-\beta^t}{1-\beta} \tag{6}$$

Cake eating problem: recursive solution (cont.)

4 Evaluate the solution for W_t at the initial and terminal time

$$\begin{cases} W_0 = k \\ W_T = k - C_0 \frac{1-\beta^T}{1-\beta} \end{cases}$$

5 Remember the initial and terminal constraints (3) and (4): therefore

$$\begin{cases} W_0 = k = \phi \\ W_T = k - C_0 \frac{1-\beta^T}{1-\beta} = 0 \end{cases}$$

6 Solve the system for k and C_0 to get $C_0 = \frac{1-\beta}{1-\beta^T}\phi$ and $k = \phi$

Cake eating problem: recursive solution (cont.)

- Substitute C_0 and k into equations (6) and (5)
- We get the **solution to the optimal control problem**

$$W_t^* = \phi\left(\frac{\beta^t - \beta^T}{1-\beta^T}\right),\ t = 0, \ldots, T$$

$$C_t^* = \phi\left(\frac{1-\beta}{1-\beta^T}\beta^t\right),\ t = 0, \ldots, T-1.$$

Optimal control problems with free terminal state

Problem OCPTC: find $u = \{u_t\}_{t=0}^{T-1}$ and $x = \{x_t\}_{t=0}^{T}$ that solves

$$\max_u \sum_{t=0}^{T-1} F(u_t, x_t, t)$$

subject to

$$\begin{cases} x_{t+1} = G(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T \text{ free} & t = T \end{cases} \qquad \text{(Problem 2)}$$

T and ϕ_0 known

PMP for the free-terminal state problem

Proposition

- *If* $x^* = \{x_t^*\}_{t=0}^{T}$ *and* $u^* = \{u_t^*\}_{t=0}^{T-1}$ *are solutions of the OCP, there is a sequence* $\psi = \{\psi_t\}_{t=0}^{T-1}$ *such that*
- *the optimality conditions*

$$\begin{aligned} \frac{\partial H_t^*}{\partial u_t} &= 0,\ t = 0, 1, \dots, T-1 \\ \psi_t &= \frac{\partial H_{t+1}^*}{\partial x_{t+1}},\ t = 0, \dots, T-1 \end{aligned}$$

- *the admissibility conditions*

$$\begin{aligned} x_{t+1}^* &= G(x_t^*, u_t^*, t) \\ x_0^* &= \phi_0 \end{aligned}$$

- *and the transversality conditions*

$$\psi_{T-1} = 0$$

Proof.

Optimal control problems for the constrained terminal state

Problem OCPTC: find $u = \{u_t\}_{t=0}^{T-1}$ and $x = \{x_t\}_{t=0}^{T}$ that solves

$$\max_{u} \sum_{t=0}^{T-1} F(u_t, x_t, t)$$

subject to

$$\begin{cases} x_{t+1} = G(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T \geq \phi_T & t = T \end{cases} \qquad \text{(Problem 3)}$$

T, ϕ_0 and ϕ_T known

PMP for the constrained terminal state problem

Proposition

- *If* $x^* = \{x_t^*\}_{t=0}^{T}$ *and* $u^* = \{u_t^*\}_{t=0}^{T-1}$ *are solutions of the OCP, there is a sequence* $\psi = \{\psi_t\}_{t=0}^{T-1}$ *such that*
- *the optimality conditions*

$$\begin{aligned} \frac{\partial H_t^*}{\partial u_t} &= 0,\ t = 0, 1, \ldots, T-1 \\ \psi_t &= \frac{\partial H_{t+1}^*}{\partial x_{t+1}},\ t = 0, \ldots, T-1 \end{aligned}$$

- *the admissibility conditions*

$$\begin{aligned} x_{t+1}^* &= G(x_t^*, u_t^*, t) \\ x_0^* &= \phi_0 \end{aligned}$$

- *and the transversality conditions*

$$\psi_{T-1}(x_T^* - \phi_T) = 0$$

Proof

Discounted OCP with infinite horizon with a free terminal state

The next problem is very common in macroeconomics and growth theory.

- **Problem OCPIH**: find $(u, x) = \{(u_t, x_t)\}_{t=0}^{\infty}$ that solves

$$\max_{u} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t), \; 0 < \beta < 1$$

$$\text{subject to}$$

$$\begin{cases} x_{t+1} = g(x_t, u_t), & t = 0, 1, \dots \\ x_0 = \phi_0, \text{ given} & t = 0 \\ \lim_{t \to \infty} x_t \text{ is free} \end{cases} \qquad \text{(Problem 4)}$$

$$\phi_0 \text{ is known}$$

note this is a free endpoint problem ($T = \infty$ is undetermined)

Discounted OCP with infinite horizon with a constrained terminal state

The next problem is very common in macroeconomics and growth theory.

- **Problem OCPIH**: find $(u, x) = \{(u_t, x_t)\}_{t=0}^{\infty}$ that solves

$$\max_{u} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t),\ 0 < \beta < 1$$

subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t), & t = 0, 1, \dots \\ x_0 = \phi_0,\ \text{given} & t = 0 \\ \lim_{t\to\infty} x_t \geq 0 \end{cases} \tag{Problem 5}$$

ϕ_0 is known

note this is constrained endpoint problem ($T = \infty$ is undetermined)

Discounted OCP with infinite horizon (cont.)

- The discounted-value Hamiltonian is

$$\begin{aligned} H_t &= \beta^t f(u_t, x_t) + \psi_t g(u_t, x_t) \\ &= \beta^t \left(f(u_t, x_t) + \beta^{-t} \psi_t g(u_t, x_t) \right) \\ &= \beta^t h_t \end{aligned}$$

- We define the current-value Hamiltonian

$$h_t \equiv h(x_t, \eta_t, u_t) = f(u_t, x_t) + \eta_t g(u_t, x_t)$$

where the current-value co-state variable is

$$\eta_t = \beta^{-t} \psi_t$$

PMP for the infinite horizon problems

Proposition

- *The solution of problem OCPIH verifies the following conditions:*

$$\begin{aligned}
\frac{\partial h_t^*}{\partial u_t} &= 0,\ t = 0, \dots, \infty && (7)\\
\eta_t &= \beta \frac{\partial h_{t+1}^*}{\partial x_{t+1}},\ t = 0, \dots, \infty && (8)\\
x_{t+1}^* &= g(x_t^*, u_t^*),\ t = 0, \dots, \infty && (9)\\
x_0^* &= \phi_0,\ t = 0 && (10)
\end{aligned}$$

- *plus: terminal values and transversality conditions*

$$\lim_{t\to\infty} x_t \, \textit{free},\ \lim_{t\to\infty} \beta^t \eta_t = 0 \tag{11}$$

or

$$\lim_{t\to\infty} x_t \geq 0 \ \lim_{t\to\infty} \beta^t \eta_t x_t^* = 0 \tag{12}$$

Solving the MHDS: methods

- In OPCIH problems the MHDS can be written as a system of autonomous difference equations

$$x^*_{t+1} = g(u^*_t, x^*_t)$$
$$u^*_{t+1} = k(u^*_t, x^*_t)$$

- **Main difficulty in solving the system**: is that we have only an initial condition for x (x_0). Another boundary condition should be obtained before we can find an explicit solution to the problem.

Solving the MHDS: methods

If the system is linear we can use the following rule of thumb to solve the system:

1. try to **reduce the dimensionality** of the system: this is the case, v.g.
   1. if the system is recursive (**method 1**): solve the scalar equation and substitute the solution in the other equation;
   2. find other type of reduction: if we can find a single variable like $z_t = \eta_t x_t$ and use the terminal constraint (**method 2**)
2. if we **cannot reduce the dimensionality** of the system: use the solution of planar linear difference equation (**method 3**)

Application: consumption-investment problem

- Find the optimal consumption-investment strategy that solves the problem: find $C = \{C_t\}_{t=0}^{\infty}$ that

$$\max_{C} \sum_{t=0}^{\infty} \beta^t \ln(C_t) \text{ (inter-temporal utility)}$$

- subject to the constraints:

$$\begin{cases} W_{t+1} = (1+r)\,W_t - C_t, \text{ (intra-temporal budget constraint)} \\ W_0 = \phi, \text{ (initial wealth given)} \\ \lim_{t\to\infty}(1+r)^{-t}\,W_t \geq 0, \text{ (Non-Ponzi game condition)} \end{cases}$$

where $r > 0$ is the (given and constant) interest rate.

Solving by using the PMP

- Discounted Hamiltonian

$$h_t = \ln(C_t) + \eta_t((1+r)W_t - C_t)$$

- PMP optimality conditions

$$\begin{cases} \dfrac{1}{C_t} = \eta_t \\ \eta_t = \beta(1+r)\eta_{t+1} \\ W_{t+1} = (1+r)W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t\to\infty} \beta^t \eta_t W_t = 0 \end{cases}$$

Optimality conditions

- Eliminating η, by substituting $\eta_t = \dfrac{1}{C_t}$ we obtain
- the maximized Hamiltonian dynamic system (MHDS)

$$\begin{aligned} C_{t+1} &= \beta(1+r)C_t && (13) \\ W_{t+1} &= (1+r)W_t - C_t && (14) \end{aligned}$$

- and the initial and transversality conditions

$$\begin{aligned} W_0 &= \phi_0 && (15) \\ \lim_{t\to\infty} \beta^t \frac{W_t}{C_t} &= 0 && (16) \end{aligned}$$

Solving the MHDS: method 1

1. Solve equation (13) (the solution of $x_{t+1} = \lambda\, x_t$ is $x_t = x_0\, \lambda^t$):

$$C_t = C_0 \beta^t (1+r)^t,\ t \in \{0, 1, \dots, \infty\}$$

   where C_0 is unknown

2. Substitute in equation (14)

$$W_{t+1} = (1+r)\, W_t - C_0 \beta^t (1+r)^t,\ t \in \{0, 1, \dots, \infty\} \tag{17}$$

3. Solve equation (17) (the solution of $x_{t+1} = \lambda\, x_t + b_t$ is $x_t = x_0\, \lambda^t + \sum_{s=0}^{t-1} \lambda^{t-1-s} b_s$):

$$\begin{aligned} W_t &= W_0 (1+r)^t - C_0 \sum_{s=0}^{t-1} (1+r)^{t-s-1} (1+r)^s \beta^s = \\ &= W_0 (1+r)^t - C_0 (1+r)^{t-1} \sum_{s=0}^{t-1} \beta^s = \\ &= (1+r)^t \left(W_0 - \frac{C_0}{1+r} \left(\frac{1-\beta^t}{1-\beta} \right) \right) \end{aligned}$$

Use the terminal conditions: cont.

4 Use the initial condition (15),

$$W_t = (1+r)^t \left(\phi - \frac{C_0}{1+r} \left(\frac{1-\beta^t}{1-\beta} \right) \right)$$

5 Use the transversality condition (16) to determine C_0

$$\begin{aligned} \lim_{t\to\infty} \beta^t \frac{W_t}{C_t} &= \frac{1}{C_0} \left(\phi - \frac{C_0}{(1+r)(1-\beta)} + \lim_{t\to\infty} \frac{C_0 \beta^t}{(1+r)(1-\beta)} \right) = \\ &= \frac{1}{C_0} \left(\phi - \frac{C_0}{(1+r)(1-\beta)} \right) = 0 \end{aligned}$$

if and only if

$$C_0 = \phi(1+r)(1-\beta)$$

1 Then the solution is

$$\begin{aligned} W_t^* &= \phi \beta^t (1+r)^t, \ t = 0, 1, \ldots, \infty \\ C_t^* &= (1+r)(1-\beta) W_t^* \ t = 0, 1, \ldots, \infty \end{aligned} \tag{18}$$

Solving the MHDS: method 2

1. Introduce a transformation of variables $z_t \equiv W_t/C_t$ (suggestion: use the transversality condition)
2. We get a scalar linear difference equation equivalent to equations (13) and (14)

$$z_{t+1} = \frac{W_{t+1}}{C_{t+1}} = \frac{(1+r)W_t - C_t}{\beta(1+r)C_t} = \frac{1}{\beta}\left(z_t - \frac{1}{1+r}\right)$$

3. Jointly with condition (16) we have a simpler boundary value problem for z_t

$$\begin{cases} z_{t+1} = \frac{1}{\beta}\left(z_t - \frac{1}{1+r}\right) \\ \lim_{t\to\infty} \beta^t z_t = 0. \end{cases}$$

4. The general solution for z_t is

$$z_t = \bar{z} + (k - \bar{z})\beta^{-t}k.$$

where

$$\bar{z} = \frac{1}{(1-\beta)(1+r)}$$

Solving the MHDS: method 2, cont

4 We use equation (16) to determine k

$$\lim_{t\to\infty} \beta^t z_t = \lim_{t\to\infty} \beta^t \bar{z} + k - \bar{z} = k - \bar{z} = 0$$

if and only if $k = \bar{z}$. Then z_t is time-independent

$$z_t = \bar{z} = \frac{1}{(1+r)(1-\beta)},\ t = 0, 1, \ldots, \infty$$

5 Because $C_t^* = (1-\beta)(1+r)\,W_t^*$, substituting in equation (14) and using condition (15) we can solve the initial value problem

$$\begin{cases} W_{t+1}^* = (1+r)\,W_t^* - C_t^* = \beta(1+r)\,W_t^*,\ t = 0, 1, \ldots \\ W_0^* = \phi \end{cases}$$

6 Which, after solving, yields the same solution (18)

Solving the MHDS: method 3

1 We write equations (13) and (14) in matrix notation

$$\begin{pmatrix} C_{t+1} \\ W_{t+1} \end{pmatrix} = \begin{pmatrix} \beta(1+r) & 0 \\ -1 & 1+r \end{pmatrix} \begin{pmatrix} C_t \\ W_t \end{pmatrix}$$

2 The general solution of this planar equation has the form Planar.

$$\begin{pmatrix} C_t \\ W_t \end{pmatrix} = h_+ \mathbf{P}^+ \lambda_+^t + h_- \mathbf{P}^- \lambda_-^t \tag{19}$$

3 Compute the eigenvalues $\lambda_\pm$. The characteristic polynomial is

$$\begin{aligned} c(\lambda) &= \lambda^2 - (1+r)(1+\beta)\lambda + \beta(1+r)^2 = \\ &= (\lambda - (1+r))\,(\lambda - \beta(1+r)) \end{aligned}$$

it happens to factorize (if not use the general formula). Then

$$\lambda_+ = 1+r,\ \lambda_- = \beta(1+r)$$

Solving the MHDS: method 3, cont

4 Compute the eigenvectors $\mathbf{P}^+$ and $\mathbf{P}^-$

$$\begin{pmatrix} (1+r)(\beta-1) & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{P}^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ -1 & (1+r)(1-\beta) \end{pmatrix} \begin{pmatrix} p_1^- \\ p_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{P}^- = \begin{pmatrix} (1+r)(1-\beta) \\ 1 \end{pmatrix}$$

5 Substituting in equation (19) we get

$$\begin{pmatrix} C_t \\ W_t \end{pmatrix} = h_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+r)^t + h_- \begin{pmatrix} (1+r)(1-\beta) \\ 1 \end{pmatrix} \beta^t (1+r)^t$$

Then the general solution for C_t and W_t is

$$C_t = h_- (1-\beta)(1+r)^{1+t}\beta^t, \;\; W_t = (1+r)^t \left(h_+ + h_- \beta^t\right)$$

Solving the MHDS: method 3, cont

5 To obtain the particular solution, we use the transversality condition

$$\lim_{t\to\infty} \beta^t \frac{W_t}{C_t} = \lim_{t\to\infty} \frac{h_+ + h_- \beta^t}{(1+r)(1-\beta)h_-} = \frac{h_+}{(1+r)(1-\beta)h_-} = 0.$$

Condition (16) holds if and only if $h_+ = 0$.

6 Therefore

$$W_t = h_- (1+r)^t \beta^t$$

and condition (15) holds if and only if $h_- = \phi$.

7 We get the same solution (18)

Optimal consumption-saving: characterization of the solution

- As

$$C_t^* = (1+r)(1-\beta)\, W_t^*$$

the dynamics of consumption is monotonously related to financial wealth

- The optimal stock of financial wealth is

$$W_t^* = \phi \left(\beta(1+r)\right)^t = \phi \left(\frac{1+r}{1+\rho}\right)^t, \ t = 0, 1, \ldots, \infty$$

where

$$\beta = \frac{1}{1+\rho}$$

$\rho =$ rate of time preference

- Characterisation of the solution
  - if $r > \rho$ then $\lim_{t\to\infty} W_t^* = \infty$ and $\lim_{t\to\infty} C_t^* = \infty$
  - if $r = \rho$ then $\lim_{t\to\infty} W_t^* = \phi$ and $\lim_{t\to\infty} C_t^* = \rho\phi$
  - if $r < \rho$ then $\lim_{t\to\infty} W_t^* = 0$ and $\lim_{t\to\infty} C_t^* = 0$

- Even though wealth and consumption may be unbounded (if $\rho < r$) the value functional is bounded
- The value of the intertemporal utility for the optimal consumption path is

$$
\begin{aligned}
J^* &= \sum_{t=0}^{\infty} \beta^t \ln(C_t^*) = \sum_{t=0}^{\infty} \beta^t \ln((1+r)(1-\beta)W_t^*) = \\
&= \sum_{t=0}^{\infty} \beta^t \ln\left((1+r)(1-\beta)\phi\,(\beta(1+r))^t\right) = \\
&= \dots \\
&= \frac{1}{1-\beta} \ln\left(\left[(1+r)(1-\beta)^{1-\beta}\beta^{\beta}\right]^{1/(1-\beta)} \phi\right)
\end{aligned}
$$

- is bounded if ϕ is bounded for any r and ρ
- This is a consequence of the transversality condition: what matters is boundedness in present value terms not at the asymptotic levels of the variables.

Dynamic programming

Infinite-horizon discounted problem

- We present an alternative approach to solving Problems 4 and 5 **Problem OCPIH** Consider the infinite-horizon discounted optimal control problem: find $(x^*, u^*)_{t=0}^{\infty}$ that solves the problem

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t)$$
$$\text{subject to}$$
$$x_{t+1} = g(x_t, u_t),$$
$$x_0 = \phi_0$$

  where $0 < \beta < 1$ and ϕ_0 is given.

- The approach:
  - find the optimal policy function $u^* = U(x)$
  - substitute in the constraint and solve

$$\begin{cases} x_{t+1} = g\big(x_t, U(x_t)\big) & t \in \{0, \dots, \infty\} \\ x_0 = .\phi_0 & t = 0 \end{cases}$$

Infinite-horizon discounted problem

Proposition

Let (x^*, u^*) be a solution to problem OCPIH: then it satisfies the Hamilton-Jacobi-Bellman condition

$$V(x) = \max_u \{f(x,u) + \beta V(g(x,u))\}$$

for any admissible $x_t^* = x$ for $t \in \{0, \ldots \infty\}$. Proof.

HJB equation: properties

- The properties of V are hard to determine: in general it is continuous, but differentiability is not assured
- if $H(x, u) \equiv f(x, u) + \beta\, V(g(x, u))$ has second order derivatives for u then we can determine the optimal control through the **optimality condition**

$$\frac{\partial H(x, u)}{\partial u} = 0.$$

  If $\dfrac{\partial^2 H(x, u)}{\partial u^2} \neq 0$ we obtain the **policy function**

$$u^* = U(x)$$

- Substituting in the HJB equation yields a non-linear functional equation

$$V(x) = f(x, h(x)) + \beta\, V[g(x, U(x))].$$

- V has an explicit solution only in very rare cases.

Application: the cake strikes again

- The problem

$$\max_{\{C\}} \left\{ \sum_{t=0}^{\infty} \beta^t \ln(C_t) : \text{subject to } W_{t+1} = W_t - C_t,\ W_0 = \phi \right\}$$

- The HJB equation

$$V(W) = \max_C \{\ln(C) + \beta V(W - C)\},$$

- Finding the optimal control: the optimality condition

$$\frac{\partial \{\ln(C) + \beta V(W - C)\}}{\partial C} = 0.$$

- The best we can do is to say that optimal consumption is a function of the size of the cake

$$\frac{1}{C} - \beta V'(W - C) = 0 \Leftrightarrow C^* = C(W)$$

- and that the HJB has the form

$$V(W) = \ln(C(W)) + \beta V[W - C(W)].$$

The cake problem: solution

- Step 1: solve the HJB equation explicitly
  1. we use a trial function of W depending upon some undetermined coefficients;
  2. if the form of the function is right, then we use the method of the undetermined coefficients (try to get the unknown coefficients by substituting in the HJB equation)
  3. we get an explicit solution for C as a function of W
- Step 2: substitute $C(W)$ in the constraints of the problem to get

$$W_{t+1} = W_t - C(W_t)$$

- Step 3: solve the difference equation with $W_0 = \phi$

The cake problem: solving the HJB equation

- Recall: The HJB equation

$$V(W) = \max_{C} \{\ln(C) + \beta V(W - C)\},$$

- Conjecture: the solution is of the form

$$V(W) = a + b\ln(W)$$

  where a and b are unknown constants;

- Policy function:

$$\frac{1}{C} = \beta V'(W - C) = \frac{\beta b}{W - C} \Rightarrow C = \frac{W}{1 + b\beta}$$

- Substituting in the HJB equation

$$a + b\ln(W) = \ln(W) - \ln(1 + b\beta) + \beta\left(a + b\ln\left(\frac{b\beta}{1 + b\beta}W\right)\right),$$

- collecting terms

$$(b(1-\beta) - 1)\ln(W) = a(\beta - 1) - \ln(1 + b\beta) + \beta b\ln\left(\frac{b\beta}{1 + b\beta}\right).$$

Step 1: solving the HJB equation

- then we determine (by setting both setting both sides to zero)

$$\begin{cases} b = \dfrac{1}{1-\beta}, \\ a = \dfrac{1}{1-\beta}\Big(\ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta) \Big) = \dfrac{\ln(\Psi)}{1-\beta} \end{cases}$$

  where $\Psi \equiv \left(\beta^{\beta}(1-\beta)^{1-\beta}\right)^{\frac{1}{1-\beta}}$

- Therefore, **our conjecture was right** and the value function is

$$V(W) = \frac{1}{1-\beta}\ln\Big(\Psi\, W\Big).$$

- Then the optimal policy function is

$$C^* = \frac{W^*}{1+b\,\beta} = (1-\beta)\,W^*$$

Step 2: optimal budget constraint

- Substituting the policy function in the intratemporal budget constraint we get

$$W_{t+1}^{*} = W_{t}^{*} - (1-\beta)\, W_{t}^{*} = \beta\, W_{t},\ t = 0, 1, \ldots, \infty$$

- given

$$W_0 = \phi,\ \text{given}$$

Step 3: solution for the cake-eating problem

The infinite horizon cake eating problem has the solution:

- the optimal sequence of cake size $W^* = \{W_t^*\}_{t=0}^{\infty}$ is generated by

$$W_t^* = \phi\beta^t, \ t = 0, 1, \ldots, \infty$$

- the optimal sequence of cake consumption $C^* = \{C_t^*\}_{t=0}^{\infty}$ is generated by

$$C_t^* = \phi(1-\beta)\beta^t, \ t = 0, 1, \ldots, \infty$$

Proofs

Proof of proposition 1

- Assume we know the solution $(u^*, x^*) = \{x_t^*, u_t^*\}_{t=0}^{T}$ for the problem.
- The optimal value is

$$J^* = \sum_{t=0}^{T-1} F(x_t^*, u_t^*, t)$$

- We write the Lagrangean

$$\begin{aligned} L &= \sum_{t=0}^{T-1} F(x_t, u_t, t) + \psi_t(G(x_t, u_t, t) - x_{t+1}) \\ &= \sum_{t=0}^{T-1} H_t(\psi_t, x_t, u_t, t) - \psi_t x_{t+1} = \\ &= \sum_{t=0}^{T-1} \ell(\psi_t, x_t, u_t, x_{t+1}, t) \end{aligned}$$

Proof of proposition 1

- Consider an arbitrary perturbation away from the solution to the problem, such that $x_t = x_t^* + \epsilon_t^x$. The perturbation is admissible if $\epsilon_0^x = \epsilon_T^x = 0$, and $u_t = u_t^* + \epsilon_t^u$. It induces the variation in value

$$L - J^* = \frac{\partial H_0}{\partial x_0}\epsilon_0^x + \sum_{t=1}^{T-1}\left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t}\right)\epsilon_t^x - \psi_{T-1}\epsilon_T^x +$$

$$\sum_{t=0}^{T-1}\frac{\partial H_t}{\partial u_t}\epsilon_t^u + \sum_{t=0}^{T-1}\left(\frac{\partial H_t}{\partial \psi_t} - x_{t+1}\right)\epsilon_t^\psi =$$

$$= \sum_{t=1}^{T-1}\left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t}\right)\epsilon_t^x + \sum_{t=0}^{T-1}\frac{\partial H_t}{\partial u_t}\epsilon_t^u + \sum_{t=0}^{T-1}\left(G(x_t, u_t, t) - x_{t+1}\right)\epsilon_t^\psi$$

- Then $L = J^*$ only if $\psi_{t-1} - \frac{\partial H_t}{\partial x_t} = \frac{\partial H_t}{\partial u_t} = x_{t+1} - G(x_t, u_t, t) = 0$.

Return.

Proof of proposition 1 b

- From the previous proof, because the terminal state is free, an admissible is such that $\epsilon_0^x = 0$ but ϵ_T^x is free. Therefore,

$$L - J^* = \sum_{t=1}^{T-1} \left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t} \right) \epsilon_t^x +$$

$$+ \sum_{t=0}^{T-1} \frac{\partial H_t}{\partial u_t} \epsilon_t^u + \sum_{t=0}^{T-1} \left(G(x_t, u_t, t) - x_{t+1} \right) \epsilon_t^\psi -$$

$$- \psi_{T-1} \epsilon_T^x$$

- Then $L = J^*$ only if $\psi_{t-1} - \frac{\partial H_t}{\partial x_t} = \frac{\partial H_t}{\partial u_t} = x_{t+1} - G(x_t, u_t, t) = 0$ and $\psi_{T-1} = 0$

Return.

Proof of proposition 1c

- Because of the terminal constraint the Lagrangean is

$$L = \sum_{t=0}^{T-1} H_t(\psi_t, x_t, u_t, t) - \psi_t x_{t+1} + \mu_T(\phi_T - x_T)$$

- where $\mu_T(\phi_T - x_T) = 0$.
- Taking the previous proof, because the terminal state is free, an admissible is such that $\epsilon_0^x = 0$ but ϵ_T^x is free. Therefore,

$$\begin{aligned} L - J^* = & \sum_{t=1}^{T-1} \left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t} \right) \epsilon_t^x + \\ & + \sum_{t=0}^{T-1} \frac{\partial H_t}{\partial u_t} \epsilon_t^u + \sum_{t=0}^{T-1} \left(G(x_t, u_t, t) - x_{t+1} \right) \epsilon_t^\psi - \\ & - (\psi_{T-1} + \mu_T) \epsilon_T^x \end{aligned}$$

- Then $L = J^*$ only if $\psi_{t-1} - \frac{\partial H_t}{\partial x_t} = \frac{\partial H_t}{\partial u_t} = x_{t+1} - G(x_t, u_t, t) = 0$ and $\psi_{T-1} = -\mu_T$. Then $\mu_T(\phi_T - x_T) = \psi_{T-1}(x_T - \phi_T) = 0$.

Return.

Proof of proposition 2

The value functional for x_t^* is

$$
\begin{aligned}
V(x_t^*) &= \sum_{s=t}^{\infty} \beta^{s-t} f(x_s^*, u_s^*) = \\
&= \max_{\{u_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} f(x_s, u_s) = \\
&= \max_{\{u_s\}_{s=t}^{\infty}} \left\{ f(x_t, u_t) + \beta \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} f(x_s, u_s) \right\} \\
&\text{(by the principle of dynamic programming)} \\
&= \max_{u_t} \left\{ f(x_t, u_t) + \beta \left(\max_{\{u_s\}_{s=t+1}^{\infty}} \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} f(x_s, u_s) \right) \right\} \\
&= \max_{u_t} \{ f(x_t, u_t) + \beta V(x_{t+1}^*) \}
\end{aligned}
$$

Proof of proposition 2, cont.

- But to be admissible $x^*_{t+1} = g(x^*_t, u^*_t)$, and the previous equation should hold for any $t \in \{0, \dots, \infty\}$ and for any admissible value for $x^*_t = x$,

$$V(x) = \max_u \{f(x, u) + \beta V(g(x, u))\}$$

Return.

Solution of planar linear difference equations

- Consider a linear difference equation $\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}$ and assume that $\det(\mathbf{I} - \mathbf{A}) \neq 0$

$$\begin{pmatrix} y_{1,t+1} \\ y_{2,t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- The solution of this equation can be written as

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + h_+ \mathbf{P}^+ \lambda_+^t + h_- \mathbf{P}^- \lambda_-^t$$

  where:

  - $\bar{\mathbf{y}}$ is the steady state of the planar equation
  - $\lambda_\pm$ are the eigenvalues of matrix $\mathbf{A}$
  - vectors $\mathbf{P}^+$ and $\mathbf{P}^-$ are the eigenvectors associated to λ_+ and λ_-,
  - the arbitrary constants h_+ and h_- are determined by using the initial and the terminal or tranversality conditions

The components of the solution:

- The steady state is
$$\bar{\mathbf{y}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

- the eigenvalues $\lambda_{\pm}$ of matrix $\mathbf{A}$ which are the roots of the characteristic equation
$$C(\lambda) = \lambda^2 - \text{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

- the eigenvectors vectors $\mathbf{P}^+$ and $\mathbf{P}^-$, associated to λ_+ and λ_-, are determined from the homogeneous equation
$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{P}^i = \mathbf{0}, \text{ for, } i = +, -$$

  where $\mathbf{I}$ is the identity matrix and $\mathbf{0} = (0, 0)^\top$

Return.