Universidade de Lisboa
Instituto Superior de Economia e Gestão
Departamento de Economia

Master in Economics
Growth Economics
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Exam: **First exam** (Época Normal)
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Solutions: only analytical questions

Part 1 1(a) $\dot{L} = \left(\psi\, L^{-\beta} - m\right) L$ where $\psi \equiv \dfrac{\theta A X^{\beta}}{\mu(1-\theta)}$

1(b) $L(t) = \left[\dfrac{\psi}{m} + \left(L_0^{\beta} - \dfrac{\psi}{m}\right) e^{-\beta m t}\right]^{\frac{1}{\beta}}$ for $t \in [0, \infty)$

1(c) Growth facts: (1) there is no long run growth; (2) there is transition dynamics; (3) the long-run level is $y(\infty) = \bar{y} \equiv \dfrac{m\mu(1+\theta)}{\theta}$, and $\dfrac{\partial \bar{y}}{\partial m} > 0$, $\dfrac{\partial \bar{y}}{\partial \mu} > 0$, $\dfrac{\partial \bar{y}}{\partial \theta} < 0$, and $\dfrac{\partial \bar{y}}{\partial A} = \dfrac{\partial \bar{y}}{\partial X} = 0$.

2(a) The MHDS is

$$\begin{aligned}
\dot{L} &= \frac{A X^{\beta}}{\mu}\, L^{1-\beta} - \frac{C}{\mu} - mL \\
\dot{C} &= \frac{C}{\sigma}\Big(\frac{(1-\beta) A X^{\beta} L^{-\beta}}{\mu} - (\rho + m)\Big)
\end{aligned}$$

together with $\lim_{t\to\infty} \mu C(t)^{-\sigma}\, L(t)\, e^{-\rho t} = 0$ and $L(0) = L_0$.

2(c) The steady state GDP per capita is

$$\bar{y} \;= A^{\frac{1}{\beta}} \left(\frac{1-\beta}{\mu(\rho+m)}\right)^{\frac{1-\beta}{\beta}} X$$

Comparison with 1(c). As in 1(c) there is no long run growth and there is transition dynamics. Differently from 1(c): there is a positive effect of A and X (scale effect) on $\bar{y}$, and effects of μ and m have the opposite sign. There is no love-for-children then population is like an accumulation asset with decreasing marginal returns.

Part 2 1(a) $\dot{y} = \alpha \left(s\,(1+m)^{\frac{1-\alpha}{\alpha}}\, y^{\frac{\alpha-1}{\alpha}} - (n+\delta)\right) y$

1(b) $\dot{y} = \bar{y} + \big(y(0) - \bar{y}\big)\, e^{-(1-\alpha)(n+\delta)t}$, where $\bar{y} = (1+m)\Big(\dfrac{s}{n+\delta}\Big)^{\frac{\alpha}{1-\alpha}}$

1(c) There is no long-run growth. The long-run level $\bar{y}$ is positively related to the (exogenous) number of robots.

2(a) The joint dynamics of (k, m) is driven by the system

$$\begin{aligned}
\dot{k} &= s\,\rho\,(1+m)^{1-\alpha}\, k^{\alpha} \;\, - (n+\delta)\, k \\
\dot{m} &= s\,(1-\rho)\,(1+m)^{1-\alpha}\, k^{\alpha} \;\, - (n+\delta)\, m.
\end{aligned} \tag{1}$$

2(b) The steady state is $(\bar{k}, \bar{m}) = \left(\frac{\tilde{k}}{1-\tilde{m}}, \frac{\tilde{m}}{1-\tilde{m}}\right)$ where $\tilde{k} \equiv \left(\frac{s\rho}{n+\delta}\right)^{\frac{1}{1-\alpha}}$, $\tilde{m} \equiv \left(\frac{1-\rho}{\rho}\right)\tilde{k}$. Linearizing the system (1) in the neighborhood of the steady state, we get the linear ODE

$$\begin{pmatrix} \dot{k} \\ \dot{m} \end{pmatrix} = \begin{pmatrix} -(1-\alpha)\,(n+\delta) & (1-\alpha)\,(n+\delta)\,\tilde{k} \\ \alpha(n+\delta)\dfrac{\tilde{m}}{\tilde{k}} & (n+\delta)\big((1-\alpha)\tilde{m}-1\big) \end{pmatrix} \begin{pmatrix} k-\bar{k} \\ m-\bar{m} \end{pmatrix}. \tag{2}$$

Write this system as $\dot{X} = J(X - \bar{X})$. Two ways to find solutions to the problem:

First As the steady state satisfies $\bar{m} = \frac{1-\rho}{\rho}\bar{k}$ and we require $m(0) = \frac{1-\rho}{\rho}k(0)$ then the solution will satisfy $m(t) = \frac{1-\rho}{\rho}k(t)$ for any t. Therefore, $\dot{m} = \frac{1-\rho}{\rho}\dot{k}$. Using the system (2) we have

$$\begin{aligned} \dot{k} &= j_{11}(k-\bar{k}) + \; j_{12}(m-\bar{m}) \\ \left(\frac{1-\rho}{\rho}\right)\dot{k} &= j_{21}(k-\bar{k}) + \; j_{22}(m-\bar{m}) \end{aligned}$$

Eliminating $(m-\bar{m})$ in the two equations we obtain

$$\dot{k} = \gamma(k-\bar{k}),$$

where

$$\gamma = \frac{\rho\,(j_{11}j_{22} - j_{12}j_{21})}{\rho j_{21} - (1-\rho)j_{12}} = -(1-\alpha)(n+\delta)(1-\tilde{m}).$$

The solution to this ODE is

$$k(t) = \bar{y} + \Big(k_0 - \bar{y}\Big)\,e^{\gamma\,t},\; t \in [0,\infty)$$

as in the questionaire.

Second Use the solution of a linear system of two differential equations: $X(t) = \bar{X} + e^{Jt}(X(0) - \bar{X})$ for $e^{Jt} = Ve^{\Lambda t}V^{-1}$, where V is the eigenvector matrix and Λ is the diagonal matrix with the eigenvalues of J. The solution of this system is

$$\begin{pmatrix} k(t) \\ m(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{m} \end{pmatrix} + h_1 \begin{pmatrix} -\dfrac{1-\alpha}{\alpha}\,\tilde{k} \\ 1 \end{pmatrix} e^{-(n+\delta)\,t} + h_2 \begin{pmatrix} \dfrac{\rho}{1-\rho} \\ 1 \end{pmatrix} e^{\gamma\,t}$$

where

$$\gamma \equiv (1-\alpha)(n+\delta)(\tilde{m}-1),$$

and $h_1 = -\dfrac{\alpha}{1+\alpha\,(1-\tilde{m})}\Big(\big(\frac{1-\rho}{\rho}\big)k(0) - m(0)\Big)$ and $h_2 = \big(\frac{1-\rho}{\rho}\big)\Big(k(0) - \bar{k}\Big)$. As we assumed that $m(0) = \big(\frac{1-\rho}{\rho}\big)k(0)$ then the solution is, again as in the questionnaire

$$\begin{aligned} k(t) &= \bar{y} + \Big(k_0 - \bar{y}\Big)\,e^{\gamma\,t},\; t \in [0,\infty) \\ m(t) &= \Big(\frac{1-\rho}{\rho}\Big)k(t),\; t \in [0,\infty). \end{aligned}$$

2(c) With the assumption that $\tilde{m} > 1$ then $\lim_{t\to\infty} e^{\gamma\,t} = \infty$ and the model displays long-run growth, because in the long run $g_y(t) \to \alpha\,g_k(t) + (1-\alpha)\,g_m(t) = \gamma > 0$. There are two accumulating factors, which given the fact that the production function displays constant returns to scale means that, in the long run, GDP becomes exponential. There is though a stark conclusion: the share of labor in GDP tends to zero !