# Growth and human capital accumulation: the Uzawa-Lucas model

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# Stylized facts addressed by the model

Since the industrial revolution:

- the population growth rate is smaller than the rate of growth of the economies
- but human capital increase is a major source of long run growth
- there is a permanent increase in the wage rate
- this can only be possible if there is a permanent increase in labor productivity
- education, which became widespread, has been a major source of increase in human capital

# Recent (empirical) books on the importance of human capital

- The race between human capital and technology: are they substitutable or complementary ? (see Goldin and Katz (2008))
- Measurement of human capital: it is given by *quality* $\times$ *quantity*, the dimension which is more correlated with growth is quality (quantity) for the more (less) developed countries (see Hanushek and Woessmann (2015) )
- The recent slow down of growth (at least in the US) is essentially explained by the reduction in human capital (aging and reduction of hours worked, not compensated by schooling and on the job training, see Volrath (2020))

# The Uzawa- Lucas model

The economy has the following features:

1. there are **two reproducible** inputs: physical capital and human capital
2. there are **two sectors**: manufacturing and education (training)
   - the manufacturing good is used in consumption and investment
   - the education produces a service which is only used in production
3. consumption/savings are determined by a centralized planner (Ramsey planner)
4. there are versions of the model with or without externalities

# The Uzawa- Lucas model

- There are several versions
  - Some extend the $AK$ model: model with no externalities
  - Others extend the Romer model: versions with externalities
- Next we present only the first version (centralized economy with no externalities)

$$
\begin{tikzcd}
H_2(t) \arrow[r] & \boxed{\dot{H} = BH_2} \arrow[dl] & & \\
H(t) \arrow[u] \arrow[d] & & & \\
H_1(t) \arrow[r] & \boxed{F(H_1(t), K(t))} \arrow[r] & Y(t) \arrow[r] \arrow[d] & C(t) \\
 & \boxed{\dot{K}} \arrow[u] & I(t) \arrow[l] &
\end{tikzcd}
$$

# Assumptions

- the preference structure is analogous to the Ramsey and AK models;
- the education sector uses only human capital as an input and the manufacturing sector uses both factors (physical capital and labor) ;
- both sectors have production functions displaying constant returns to scale;
- there are no externalities;

# The model

## Variables in levels

- Intertemporal utility

$$\max_{C,K_1,H_1,H_2} \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

  assumption $\rho + B(\theta - 1) > 0$
- accumulation equations for stocks of physical and human capital

$$\dot{K} = Y_1(t) - C(t)$$
$$\dot{H} = Y_2(t)$$

- allocation constraints of the stocks between the two sectors

$$K(t) = K_1(t)$$
$$H(t) = H_1(t) + H_2(t)$$

- production functions for manufacturing and education

$$Y_1(t) = AK_1(t)^\alpha H_1(t)^{1-\alpha}$$
$$Y_2(t) = BH_2(t)$$

# Detrending

- We introduce the decomposition

$$K_j(t) = k_j(t)e^{\gamma t},\ H_j(t) = h_j(t)e^{\gamma t},\ j = 1,2$$

- $\gamma_k = \gamma_h = \gamma$ because a necessary condition for the existence of a balanced growth path is that the rates of growth are equal
- then

$$\frac{\dot{k}_j}{k_j} = \frac{\dot{K}_j}{K_j} - \gamma,\ \frac{\dot{h}_j}{h_j} = \frac{\dot{H}_j}{H_j} - \gamma\ j = 1,2$$

# The model

## Detrended variables

- Intertemporal utility

$$\max_{c,k_1,h_1,h_2} \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} e^{-(\rho-\gamma(1-\theta))t} dt$$

- accumulation equations for stocks of physical and human capital

$$\dot{k} = y_1(t) - c(t) - \gamma k(t) \quad (1)$$
$$\dot{h} = y_2(t) - \gamma h(t) \quad (2)$$

- allocation constraints of the stocks between the two sectors

$$k(t) = k_1(t),\ k(0) = k_0 \quad (3)$$
$$h(t) = h_1(t) + h_2(t),\ h(0) = h_0 \quad (4)$$

- production functions for manufacturing and education (because of linear homogeneity)

$$y_1(t) = Ak_1(t)^\alpha h_1(t)^{1-\alpha}$$
$$y_2(t) = Bh_2(t)$$

# Solving the model

- Observe that the model is an optimal control problem with:
  - four control variables: $c$, $h_1$, $h_2$, and $k_1$
  - two state variables: $k$ and $h$
  - two dynamic constrains (1), (2)
  - two static constrains (3), (4)
- the current-value Hamiltonian is

$$\mathcal{H} = \frac{c(t)^{1-\theta}}{1-\theta} + p_k \left( A k_1^{\alpha} h_1^{1-\alpha} - c - \gamma k \right) + p_h \left( B h_2 - \gamma h \right) + \\ + R(k - k_1) + W(h - h_1 - h_2) \quad (5)$$

$p_k$, $p_h$: co-state variables (optimal asset prices)
$R$, $W$: Lagrange multipliers (optimal return on capital and wage rates)

# First order conditions for an interior solution

- optimal consumption

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \Leftrightarrow c^{-\theta} = p_k \tag{6}$$

- optimal allocation of human and physical capital between the two sectors, $r \equiv R/p_k$ and $w \equiv W/p_h$,

$$\frac{\partial \mathcal{H}}{\partial k_1} = 0 \quad \Leftrightarrow \quad \alpha y_1 = r k_1 \tag{7}$$

$$\frac{\partial \mathcal{H}}{\partial h_1} = 0 \quad \Leftrightarrow \quad (1-\alpha) p_k y_1 = w p_h h_1 \tag{8}$$

$$\frac{\partial \mathcal{H}}{\partial h_2} = 0 \quad \Leftrightarrow \quad w = B \tag{9}$$

- conditions for the Lagrange multipliers $R$ and $W$

$$\frac{\partial \mathcal{H}}{\partial R} = 0 \quad \Leftrightarrow \quad k = k_1 \tag{10}$$

$$\frac{\partial \mathcal{H}}{\partial W} = 0 \quad \Leftrightarrow \quad h = h_1 + h_2 \tag{11}$$

# First order conditions for an interior solution

(continuation)

- Euler equations

$$\dot{p}_k = p_k(\rho + \gamma(\theta - 1)) - \frac{\partial \mathcal{H}}{\partial k} = p_k(\rho + \gamma\theta - r) \quad (12)$$

$$\dot{p}_h = p_h(\rho + \gamma(\theta - 1)) - \frac{\partial \mathcal{H}}{\partial h} = p_h(\rho + \gamma\theta - B) \quad (13)$$

- transversality conditions

$$\lim_{t\to\infty} e^{-(\rho+\gamma(\theta-1))t}\,(p_k(t)k(t) + p_h(t)h(t)) = 0 \quad (14)$$

- admissibility conditions

$$\dot{k} = y_1(t) - c(t) - \gamma k(t) \quad (15)$$

$$\dot{h} = y_2(t) - \gamma h(t) \quad (16)$$

# Solution for returns and allocations

- Solving equations (7)-(11) for $k_1$, $h_1$, $h_2$, $r$ and $w$, we get

$$\begin{aligned}
r &= r(\pi) \equiv \left(\alpha_0 A(\pi/B)^{1-\alpha}\right)^{\frac{1}{\alpha}}, \text{for } \alpha_0 \equiv \alpha^{\alpha}(1-\alpha)^{(1-\alpha)} \\
w &= B \\
k_1 &= k \\
h_1 &= \left(\frac{r(\pi)}{\alpha A}\right)^{\frac{1}{1-\alpha}} k \\
h_2 &= h - h_1
\end{aligned}$$

- we define the relative prices as

$$\pi \equiv \frac{p_k}{p_h}$$

# Solution for sectoral outputs

- If we substitute in the production function of both sectors we get a linear system

$$\begin{aligned} y_1 &= a_1(\pi)k, \text{ where } a_1 = \frac{r(\pi)}{\alpha} > 0 \\ y_2 &= a_2(\pi)k + Bh, \text{ where } a_2 = -B\left(\frac{r(\pi)}{\alpha A}\right)^{\frac{1}{1-\alpha}} < 0 \end{aligned}$$

- then an increase in the relative price $\pi = p_k/p_h$ increases the output of manufactures and reduces the output of the educational sector

# Long run growth rate and factor returns

- From equation (13), setting $\dot{p}_h = 0$ we get the long-run growth rate
$$\bar{\gamma} = \frac{B - \rho}{\theta}$$
increases with the productivity of the educational sector
- From equation (13) and (12) we have a long-run arbitrage condition
$$\bar{r} = \bar{w} = B$$
- then the long-run relative price is
$$\bar{\pi} = \frac{\bar{p}_k}{\bar{p}_h} = \left(\frac{\alpha_0 A}{B}\right)^{\frac{1}{1-\alpha}}$$

# Other long run relationships

- ratio between the state variables

$$\frac{\bar{k}}{\bar{h}} = \eta \equiv -\frac{B-\bar{\gamma}}{\bar{a}_2} = \bar{\pi}\left(\frac{\bar{\gamma}-B}{B}\right)\left(\frac{\alpha}{1-\alpha}\right)$$

because

$$\bar{a}_2 = -B\left(\frac{B}{\alpha A}\right)^{\frac{1}{1-\alpha}} = -\frac{B}{\bar{\pi}}\left(\frac{1-\alpha}{\alpha}\right) < 0$$

- the ratio $\frac{\bar{k}}{\bar{h}}$ is positive because of the transversality condition holds if and only if

$$\rho + \bar{\gamma}(\theta-1) = \frac{\rho + B(\theta-1)}{\theta} = B - \bar{\gamma} > 0$$

- the long run level of consumption is

$$\bar{c} = c(p_k) = \beta\bar{k},\ \beta \equiv \frac{B}{\alpha} - \bar{\gamma} > 0$$

# The MHDS

- substituting $\gamma = \bar{\gamma}$ the MHDS becomes

$$\begin{aligned} \dot{p}_k &= p_k(B - r(p_k/p_h)) && (17) \\ \dot{p}_h &= 0 && (18) \\ \dot{k} &= (a_1 r(p_k/p_h)) - \bar{\gamma})\, k - c(p_k) && (19) \\ \dot{h} &= a_2(r(p_k/p_h))k - (B - \bar{\gamma})h && (20) \end{aligned}$$

# Initial conditions and the BGP

- If the initial conditions verifies

$$k_0 = \eta h_0$$

- then the economy will evolve along the BGP such that

$$\bar{K}(t) = \eta h_0 e^{\bar{\gamma} t}, \bar{H}(t) = h_0 e^{\bar{\gamma} t}$$

- If the initial conditions verifies

$$k_0 \neq \eta h_0$$

- then there will be transitional dynamics

# Transitional dynamics

- The system (17)-(20) is non-linear. A linear approximation in the neighborhood of the BGP is

$$\begin{pmatrix} \dot{p}_k \\ \dot{p}_h \\ \dot{k} \\ \dot{h} \end{pmatrix} = J \begin{pmatrix} p_k - \bar{p}_k \\ p_h - \bar{p}_h \\ k - \bar{k} \\ h - \bar{h} \end{pmatrix}$$

where

$$\bar{\mathbf{J}} = \begin{pmatrix} \mu - \beta & \bar{\pi}(\beta - \mu) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\beta(\theta-\alpha)-\alpha\mu)\bar{k}}{\alpha\theta\bar{p}_k} & -\frac{(\beta-\mu)\bar{k}}{\alpha\bar{p}_h} & \beta & 0 \\ -\dfrac{\mu\bar{h}}{\alpha\bar{p}_k} & \dfrac{\mu\bar{h}}{\alpha\bar{p}_h} & -\dfrac{\mu}{\eta} & \mu \end{pmatrix}$$

# Local dynamics in the neighborhood of the BGP

- The characteristic polynomial of the Jacobian $J$ is

$$C(\mathbf{J},\lambda) = \lambda\left(\lambda - (\mu - \beta)\right)\left(\lambda - \beta\right)\left(\lambda - \mu\right),$$

- Then the eigenvalues are

$$\lambda_1 = \mu - \beta < 0,\ \lambda_2 = 0,\ \lambda_3 = \beta > 0,\ \lambda_4 = B - \bar{\gamma} > 0$$

- there is transitional dynamics: because $\lambda_1 < 0$ which is

$$\lambda_2 = \left.\frac{\partial \dot{p}_k}{p_k}\right|_{BGP} = \mu - \beta < 0$$

# Local dynamics in the neighborhood of the BGP

- solving the system (see my revised notes chapter 7)

$$
\begin{aligned}
h(t) &= h_\infty + (h_0 - h_\infty)\, e^{(\mu-\beta)t} \\
k(t) &= \eta h_\infty + (k_0 - \eta h_\infty)\, e^{(\mu-\beta)t} \\
p_k(t) &= \bar{p}_k \left[ 1 + \frac{\theta\alpha(2\beta-\mu)}{\mu(\theta-\alpha)} \left( \frac{h_0 - h_\infty}{h_\infty} \right) e^{(\mu-\beta)t} \right] \\
p_h(t) &= \bar{p}_h
\end{aligned}
$$

where

$$
h_\infty = \frac{k_0 \mu(\theta-\alpha) + \eta h_0 (\beta(\alpha+\theta) - \mu\theta)}{\eta(\beta(\alpha+\theta) - \alpha\mu)}
$$

if we take $\bar{h} = h_\infty$. This implies $\bar{p}_k = (\beta\eta h_\infty)^{-\theta}$ and $\bar{p}_h = \bar{\pi}\bar{p}_k$.

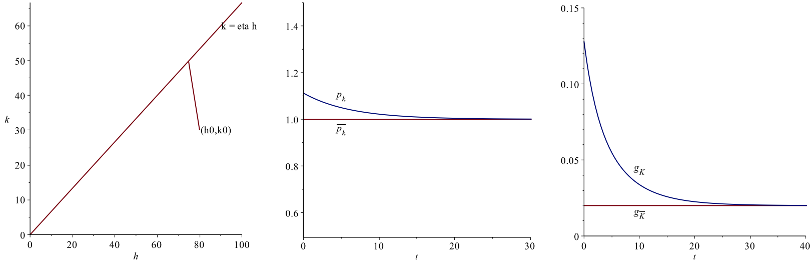


Figure: Uzawa-Lucas model: phase diagram, trajectories for $p_k$ and for the rate of growth of $K$ and $\bar{K}$. Parameter values: $\rho = 0.02$, $\alpha = 0.3$, $\theta = 2$, $A = 0.2$ and $B = 0.06$.

# Trajectory for the GDP

- the GDP for the manufacturing sector is

$$Y_1(t) = y_1(t)e^{\bar{\gamma}t}$$

  where

$$y_1(t) = y_{1,\infty}\left(1 + \left(\frac{k_0}{\eta h_\infty} - 1\right)e^{(\mu-\beta)t}\right)^{\alpha}\left(1 + \left(\frac{h_0}{h_\infty} - 1\right)e^{(\mu-\beta)t}\right)^{1-\alpha}$$

- Taking $\lim_{t\to\infty} y_1(t) = y_{1,\infty} \equiv A\eta^{\alpha}h_\infty$ we get the BGP

$$\bar{Y}_1(t) \approx y_{1,\infty}e^{\bar{\gamma}t}.$$

# Growth implications

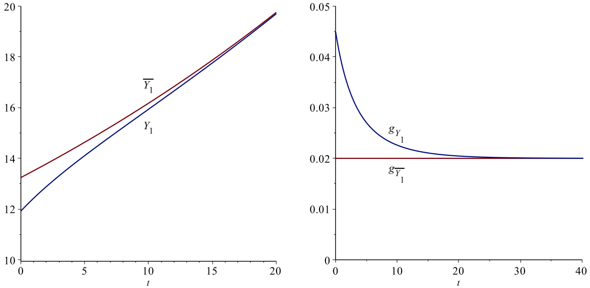


Figure: Uzawa-Lucas model: trajectories for levels and rates of growth of $Y_1$ and $\bar{Y}_1$

# Conclusions

- there is long run growth and the growth rate is a positive function of $B$
- the long run level of GDP depends on the initial levels of $k$ and $h$

$$\bar{y}_1 = y_{1,\infty} = A\eta^{\alpha-1}\left(\frac{k_0\mu(\theta-\alpha)+\eta h_0(\beta(\alpha+\theta)-\mu\theta)}{\beta(\alpha+\theta)-\alpha\mu}\right)$$

- if $k_0 = \eta h_0$ the economy will be at a BGP with $\bar{y}_1 = A\eta^{\alpha} h_0$;
- there is transitional dynamics (if $k_0 \neq \eta h_0$) with

$$y_1(t) - \bar{y}_1 \approx e^{(\mu-\beta)t}$$

- the GDP path in levels is

$$Y_1(t) = y_1(t)e^{ga\bar{m}mat}$$

# Conclusions

- The driving force for transitional dynamics if $\dot{\pi}/\pi$

$$\dot{\pi}/\pi = B - r(\pi)$$

- if initial capital $k_0$ is too low relative to $\eta h_0$ then $\pi(0) > \bar{\pi}$ and two effects will occur
  - because $a_1^{'}(\pi) > 0$ and $a_2^{'}(\pi) < 0$ there will be an increase in the ratio $k(t)/h(t)$
  - because $r(\pi) > B$ then $\dot{\pi}/\pi < 0$;
- the adjustment of $\pi$ will eliminate through time the both the divergences $B - r(\pi)$ and $k(t) - \eta h(t)$ leading to convergence to the BGP.

# Conclusions: effect of an increase in $B$

A positive shock in $B$ (from $B_0$ to $B_1 > B_0$), will produce the following effects (starting from a BGP)

- an increase in the long-run growth rate $\bar{\gamma}(B_1) > \bar{\gamma}(B_0)$
- an increase in $\eta$ (because $\frac{\partial \eta}{\partial B} > 0$)
- if before the shock $k_0 = \eta(B_0)h_0$, then after the shock $k_0 < \eta(B_1)h_0$ which means physical capital becomes "too low" relative to human capital
- then the process just described unfolds: $\pi(0) > \bar{\pi}(B_0)$, the interest rate becomes higher then $B_1$, $k$ accumulates faster than $h$ but $\pi$ starts to decrease to eliminate the "excess" $k$.

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