Foundations of Financial Economics *2020/21*
Problem set 1: revisions of consumer demand theory

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Disclaimer: This problem set is provided as a help for self study, in an open academic spirit of providing potentially interesting problems. Solving them is not mandatory, however it is advisable because exams' questions will be in a large part similar to some of them. The instructor does not commit himself to provide the solutions of them all, but is available to help solving specific difficulties arising in efforts to **actually** solving them. Not all questions have been completely verified. However, the solutions that will be provided (and the questions arising in exams) will or have been carefully verified.

1. Consider the next utility functions. For every utility function solve the following:

   (a) Find the marginal rate of substitution, $MRS_{1,2}$.

   (b) Find the elasticity of substitution, $IES_{1,2}$

   (c) Characterize the properties two goods and the relationship between the goods from the point of view of the consumer (regarding marginal utilities, variation in marginal utilities, and the Edgeworth substitutability/complementarity between the two goods.

   (d) Consider the problem

$$\begin{aligned} V(\mathbf{p}, Y) = \max_{x_1, x_2} & \; U(x_1, x_2) \\ & \text{subject to} \\ & p_1\, x_1 + p_2\, x_2 = Y \\ & x_i \geq 0, \; \text{for}\, i = 1, 2 \end{aligned}$$

   in which $\mathbf{p} = (p_1, p_2)$ is the vector of prices of the two goods and Y is nominal income. Find the demand functions $x_i = X_i(\mathbf{p}, Y)$, and the indirect utility function $V(\mathbf{p}, Y)$. Characterize the Hickian substitutability/complementarity between the two goods. Under which circumstances the constraints will entail a welfare loss ?

(e) Consider the problem

$$
\begin{aligned}
V(\mathbf{p}, Y) = \max_{x_1, x_2} & \; U(x_1, x_2) \\
& \text{subject to} \\
& p_1\, x_1 + p_2\, x_2 = Y \\
& x_1 \geq 0 \\
& 0 \leq x_2 \leq \frac{Y}{2}.
\end{aligned}
$$

Find the solution to the problem. Under which circumstances the constraints will entail a welfare loss ?

(f) Consider the problem

$$
\begin{aligned}
V(\mathbf{p}, Y) = \max_{x_1, x_2} & \; U(x_1, x_2) \\
& \text{subject to} \\
& p_1\, x_1 + p_2\, x_2 = Y \\
& 0 \leq x_1 \leq x_2.
\end{aligned}
$$

Find the solution to the problem. Under which circumstances would the last constraint entail a welfare loss ?

(a) generalized sums

$$U(x_1, x_2) = \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2), \text{ for } \alpha_1 > 0,\ \alpha_2 > 0 \tag{1}$$

$$U(x_1, x_2) = \alpha_1 \frac{x_1^{1-\theta} - 1}{1-\theta} + \alpha_2 \frac{x_2^{1-\theta} - 1}{1-\theta}, \text{ for } \alpha_1 > 0,\ \alpha_2 > 0,\ \theta > 0 \tag{2}$$

$$U(x_1, x_2) = \alpha_1 \frac{e^{-\zeta x_1}}{\zeta} + \alpha_2 \frac{e^{-\zeta x_2}}{\zeta}, \text{ for } \alpha_1 > 0,\ \alpha_2 > 0,\ \zeta > 0 \tag{3}$$

(b) generalized averages

$$U(x_1, x_2) = x_1^{\alpha}\, x_2^{\alpha}, \text{ for } 0 < \alpha < 1 \tag{4}$$

$$U(x_1, x_2) = \alpha x_1 + \alpha_2 x_2, \text{ for } \alpha_1 > 0,\ \alpha_2 > 0 \tag{5}$$

$$U(x_1, x_2) = \left(\alpha_1 x_1^2 + \alpha_2 x_2^2\right)^{\frac{1}{2}}, \text{ for } \alpha_1 > 0,\ \alpha_2 > 0 \tag{6}$$

$$U(x_1, x_2) = \max\{\alpha_1 x_1, \alpha_2 x_2\}, \text{ for } \alpha_1 > 0,\ \alpha_2 > 0 \tag{7}$$

$$U(x_1, x_2) = \min\{\alpha_1 x_1, \alpha_2 x_2\}, \text{ for } \alpha_1 > 0,\ \alpha_2 > 0 \tag{8}$$

If you are brave (or curious) enough after reaching this point, maybe you would be willing to take a look at (Bullen, 2003, p. 175-177)[1] to discover that utility functions (4)-(8) are all particular cases of a generalized mean

$$U(\mathbf{x}) = \Big[\sum_{i=1}^{n} \alpha_i \, x_i^{\eta}\Big]^{\frac{1}{\eta}}$$

for $\eta \in \{-\infty, \dots, \infty\}$, that economists have christened by different names depending on the value of the parameter η: Cobb- Douglas when $\eta = 0$ and $\sum_{i=1}^{n} = 1$, CES for constant elasticity of substitution when $\eta < 1$, Leontieff when $\eta = -\infty$, linear when $\eta = 1$. A nice exercise is to show that

(a) for all of them $MRS_{ij} = \dfrac{\alpha_i}{\alpha_j}\left(\dfrac{x_j}{x_i}\right)^{1-\eta}$ for any pair i, j

(b) for all of them $ES_{ij} = \dfrac{1}{1-\eta}$ for any pair i, j.

References

P.S. Bullen. *Handbook of Means and Their Inequalities (Mathematics and Its Applications).* Mathematics and Its Applications", 560. Springer, 2nd edition, 2003. ISBN 1402015224,9781402015229.

[1] See also `https://en.wikipedia.org/wiki/Generalized_mean`