Foundations of Financial Economics
Multi-period GE: Arrow-Debreu economy

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Multiperiod Arrow-Debreu economy

▶ In this lecture we extend the DSGE for an Arrow-Debreu economy from 2 to n periods

▶ Now all the variables are stochastic processes

▶ Including the AD price and the stochastic discount factor (SDF)

▶ We will also define a recursive stochastic discount factor, which is similar to what we found for the 2-period case

Topics

- Information structure
- Real part of the economy
- Market structure
- Definition of the DSGE
- Determination of the DGSE
- Characterization
- Recursive stochastic discount factor

Information structure

Information structure

- There is an information tree, with T periods,

- the information tree comprises a sequence of nodes $\{N_t\}_{t=1}^T = \{N_1, N_2, \ldots, N_s, \ldots, N_T\}$, where N_t is the number of nodes of the information tree at time t

- **Example**: for a binomial process $N_t = 2^t$

- there is a sequence of unconditional probabilities

$$\mathbb{P}^T \equiv \{\mathsf{P}_t\}_{t=1}^T = \{\mathsf{P}_1, \ldots, \mathsf{P}_t, \ldots, \mathsf{P}_T\}$$

  where $\mathsf{P}_t = \left(\pi_{t,1}, \ldots, \pi_{t,s}, \ldots, \pi_{t,N_t}\right)$

- for any process $\{X_t\}_{t=0}^T = \{X_0, X_1, \ldots X_t, \ldots, X_T\}$ we assume that X_t is $\mathcal{F}_t$-adapted (as we say in the slide "Introduction to stochastic processes")

Information structure

- **Example** for a binomial tree

$$\mathbb{P}^T = \left\{ 1, \begin{pmatrix} \pi_{1,1} \\ \pi_{1,2} \end{pmatrix}, \begin{pmatrix} \pi_{2,1} \\ \pi_{2,2} \\ \pi_{2,3} \\ \pi_{2,4} \end{pmatrix}, \begin{pmatrix} \pi_{3,1} \\ \dots \\ \pi_{3,8} \end{pmatrix}, \dots \right\}$$

- If the process is Markovian, with transition probabilities $\{p, 1-p\}$ for $0 < p < 1$

$$\mathbb{P}^T = \left\{ 1, \begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} p^2 \\ p\,(1-p) \\ (1-p)\,p \\ (1-p)^2 \end{pmatrix}, \begin{pmatrix} p^3 \\ p^2(1-p) \\ \dots \\ (1-p)^2\,p \\ (1-p)^3 \end{pmatrix} \dots \right\}$$

Real part of the economy

Real part of the economy: resources

- **Households are homogeneous** regarding information, technology (endowments) and preferences
- Therefore:
- **The information structure is common knowledge**
- **Endowments**: are exogenously given by the stochastic process

$$\mathbf{Y}^T \equiv \{ Y_t \}_{t=0}^T = \{ y_0, Y_1, \ldots, Y_t, \ldots, Y_T \}$$

- where Y_t is $\mathcal{F}_t$- mensurable, such that

$$Y_t = \begin{pmatrix} y_{t,1} & \ldots & y_{t,N_t} \end{pmatrix}^\top$$

- **Example** for a binomial tree

$$\mathbf{Y}^T = \left\{ y_0, \begin{pmatrix} y_{1,1} \\ y_{1,2} \end{pmatrix}, \begin{pmatrix} y_{2,1} \\ y_{2,2} \\ y_{2,3} \\ y_{2,4} \end{pmatrix}, \begin{pmatrix} y_{3,1} \\ \ldots \\ y_{3,8} \end{pmatrix} \ldots \right\}$$

Real part of the economy: preferences

▶ **Preferences**; are represented by an intertemporal von-Neumman-Morgenstern functional

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) \right]$$

▶ where $\beta \in (0, 1)$ and $u(.)$ is increasing, concave and Inada

▶ Consumers choose a contingent-consumption sequence

$$\mathbf{C}^T \equiv \{C_t\}_{t=0}^{T} = \{c_0, C_1, \ldots, C_t, \ldots, C_T\}$$

where C_t is $\mathcal{F}_t$- mensurable,

Real part of the economy: preferences

▶ Observe that

$$\mathbb{E}_0\left[\sum_{t=0}^{T}\beta^t u(C_t)\right] = \sum_{t=0}^{T}\beta^t \mathsf{P}_t u(C_t) =$$
$$= u(C_0) + \ldots + \beta^t \mathsf{P}_t u(C_t) + \ldots + \beta^T \mathsf{P}_T u(C_T)$$

▶ where

$$\mathsf{P}_t u(C_t) = \sum_{s=1}^{N_t} \pi_{t,s} u(c_{t,s})$$

and P_t are **unconditional** probability distributions, i.e., taken at time $t = 0$

Real part of the economy: preferences

▶ **Example** for a binomial tree

▶ Consumption process

$$\mathbf{C}^T = \left\{ c_0, \begin{pmatrix} c_{1,1} \\ c_{1,2} \end{pmatrix}, \begin{pmatrix} c_{2,1} \\ c_{2,2} \\ c_{2,3} \\ c_{2,4} \end{pmatrix}, \begin{pmatrix} c_{3,1} \\ \dots \\ c_{3,8} \end{pmatrix} \dots \right\}$$

▶ Utility process

$$\left\{ u(c_0), \begin{pmatrix} u(c_{1,1}) \\ u(c_{1,2}) \end{pmatrix}, \begin{pmatrix} u(c_{2,1}) \\ u(c_{2,2}) \\ u(c_{2,3}) \\ u(c_{2,4}) \end{pmatrix}, \begin{pmatrix} u(c_{3,1}) \\ \dots \\ u(c_{3,8}) \end{pmatrix} \dots \right\}$$

▶ Utility functional

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(C_t) \right] = u(c_0) + \beta \sum_{s=1}^{2} \pi_{1,s} \, u(c_{1,s}) + \beta^2 \sum_{s=1}^{4} \pi_{2,s} \, u(c_{2,s}) +$$
$$\beta^3 \sum_{s=1}^{8} \pi_{3,s} \, u(c_{3,s}) + \dots \quad (1)$$

Market structure

Arrow-Debreu contingent claims

▶ There is a large number of **Arrow-Debreu contingent claims**, **traded** only at time $t = 0$, offering one unit of the good for **delivery** at every node of the information tree for $t = 1, \ldots, N_t$

▶ this means there is:

1. one **spot** market taken as the numeraire: $Q_0 = 1$
2. $\sum_{t=1}^{T} N_t = N_1 + \ldots + N_t + \ldots + N_T$ **AD markets** with prices

$$\mathbf{Q}^T \equiv \{Q_t\}_{t=0}^{T} = \{q_0, Q_1, \ldots, Q_t, \ldots, Q_T\}$$

where

$$Q_t = \begin{pmatrix} q_{t,1} \\ \ldots \\ q_{t,N_t} \end{pmatrix}, \text{ i.e. } Q_t \text{ is } \mathcal{F}_t\text{- mensurable}$$

Dynamic stochastic general equilibrium

Arrow-Debreu equilibrium

For a representative household economy

Definition: An Arrow-Debreu equilibrium is the process $(\mathbf{C}^T, \mathbf{Q}^T)$, that is, it is the collection of $\mathcal{F}_t-$adapted processes for consumption $\{C_t\}_{t=0}^T$ and AD-prices $\{Q_t\}_{t=1}^T$ such that, given the $\mathcal{F}_t-$adapted process $Y^T = \{Y_t\}_{t=0}^T$:

1. consumers problem determine $\{C_t\}_{t=0}^T$ by solving

$$\max_{\{C_t\}_{t=0}^T} \mathbb{E}_0 \left[\sum_{t=0}^T \beta^t u(C_t) \right] \text{ s.t. } \sum_{t=0}^T Q_t C_t \leq \sum_{t=0}^T Q_t Y_t$$

given $\{Y_t\}_{t=0}^T$ and $\{Q_t\}_{t=1}^T$

2. and markets clear

$$C_t = Y_t, \quad t = 0, \ldots, T$$

▶ T can be finite or $T = \infty$

The budget constraint

Observe that:

▶ the budget constraint is equivalent to

$$\sum_{t=0}^{T} Q_t(Y_t - C_t) =$$

$$= Q_0(y_0 - c_0) + \ldots + Q_t(Y_t - C_t) + \ldots + Q_T(Y_T - C_T)$$

where

$$Q_t(Y_t - C_t) = \sum_{s=1}^{N_t} q_{t,s}(y_{t,s} - c_{t,s})$$

▶ If we define the 0-period **unconditional stochastic discount factor** for period t as

$$M_t \equiv Q_t/\mathsf{P}_t$$

where $M_t = (m_{t,1}, \ldots, m_{t,N_t})$

$$m_{t,s} = \frac{q_{t,s}}{\pi_{t,s}}, \ s = 1, \ldots, N_t$$

The budget constraint (cont)

- Then the **instantaneous budget constraint** at time $t = 0$, is equivalent to

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} M_t \left(Y_t - C_t \right) \right] \geq 0$$

- where

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} M_t \left(Y_t - C_t \right) \right] = m_0 \left(y_0 - c_0 \right) + \ldots + \mathsf{P}_T M_T (Y_T - C_T)$$

The solution of the consumer problem

▶ We can write the Lagrangean as

$$\mathcal{L} = \mathbb{E}_0 \Big[\sum_{t=0}^{T} \beta^t u(C_t) \Big] + \lambda \, \mathbb{E}_0 \Big[\sum_{t=0}^{T} M_t (Y_t - C_t) \Big]$$

$$= \mathbb{E}_0 \Big[\sum_{t=0}^{T} \beta^t u(C_t) + \lambda \, M_t (Y_t - C_t) \Big]$$

(linearity of the $\mathbb{E}_0$ operator)

▶ or equivalently

$$\mathcal{L} = \sum_{t=0}^{T} \sum_{s=1}^{N_t} \pi_{t,s} \left\{ \beta^t u(c_{t,s}) + \lambda m_{t,s} (y_{t,s} - c_{t,s}) \right\}$$

First order conditions

$$\frac{\partial \mathcal{L}}{\partial c_{t,s}} = \mathbf{0}, \ s = 1, \dots N_t, \ t = 0, \dots T, \ \left(\sum_{t=0}^{T} N_t \text{dimensional} \right)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \ (1 \text{ dimensional})$$

Solution of the consumer's problem

First-order conditions for optimality

$$
\begin{aligned}
u'(c_0^*) &= \lambda \ (1 \text{ equation}) \\
\beta u'(c_{1,s}^*) &= \lambda m_{1,s}, \ s = 1, \ldots N_1 \ (N_1 \text{ equations}) \\
&\cdots \\
\beta^t u'(c_{t,s}^*) &= \lambda m_{t,s}, \ s = 1, \ldots N_t \ (N_t \text{ equations}) \\
&\cdots \\
\beta^T u'(c_{T,s}^*) &= \lambda m_{T,s}, \ s = 1, \ldots N_T \ (N_T \text{ equations}) \\
\sum_{t=0}^{T} \sum_{s=1}^{N_t} \pi_{t,s} m_{t,s} c_{t,s}^* &= H_0 \equiv \sum_{t=0}^{T} \sum_{s=1}^{N_t} \pi_{t,s} m_{t,s} y_{t,s} \ (1 \text{ equation})
\end{aligned}
$$

where H_0 is human wealth equal to the expected discounted present value (in market prices) of the future stream of endowments.

Characterization

Equilibrium conditons for a homogeneous agent economy

▶ The **Euler equation for consumption** is, because $u^{'}(c_0^*) = \lambda$

$$m_{t,s} u^{'}(c_0^*) = \beta^t u^{'}(c_{t,s}^*), \quad s = 1, \dots N_t, \quad t = 0, \dots, T.$$

▶ The equilibrium conditions are (**in this homogeneous-agent model**)

$$c_{t,s}^* = y_{t,s}, \quad s = 1, \dots N_t, \quad t = 0, \dots, T.$$

Equilibrium stochastic discount factor

► Then the **equilibrium unconditional stochastic discount factor** (SDF) is a stochastic process $\{M_t\}_{t=0}^{T}$ such that $M_0 = m_0 = 1$ and $M_t = (m_{t,1}, \ldots, m_{t,N_t})^{\top}$ where

$$M_t^* = \beta^t \frac{u^{'}(Y_t)}{u^{'}(y_0)}, \; t = 0, \ldots T$$

$$M_t^* = \begin{pmatrix} m_{t,1} \\ \ldots \\ m_{t,N_t} \end{pmatrix}$$

► or, equivalently the possible realizations of the unconditional stochastic discount factor are

$$m_{t,s}^* = \beta^t \frac{u^{'}(y_{ts})}{u^{'}(y_0)}, \;\; s = 1, \ldots, N_t, \;\; t = 0, \ldots T$$

Equilibrium stochastic discount factor

Definition: **recursive stochastic discount factor** for period $t+1$ conditional on period t

$$M_{t+1|t} = \frac{M_{t+1}}{M_t}$$

where

$$M_{t+1|t} = \begin{pmatrix} m_{t+1|t,1} \\ \dots \\ m_{t+1|t,s} \\ \dots \\ m_{t+1|t,N_{t,t+1}} \end{pmatrix} = \begin{pmatrix} \mu_{t+1,1} \\ \dots \\ \mu_{t+1|s} \\ \dots \\ \mu_{t+1|N_{t,t+1}} \end{pmatrix}$$

where $N_{t,t+1}$ is the number of nodes taken an time t for all the subsequent nodes at time $t+1$

Equilibrium stochastic discount factor

▶ The **equilibrium recursive stochastic discount factor** (RSDF) for period $t+1$ conditional on period t is

$$M^*_{t+1|t} = \beta \frac{u'(Y_{t+1})}{u'(y_t)}$$

where the realization of Y_t at time t is $Y_t = y_t$

▶ Has possible realizations

$$\mu_{t+1} = m^*_{t+1|t,s} = \beta \frac{u'(y_{t+1,s})}{u'(y_t)}, \ s = 1 \ldots N_{t,t+1}$$

▶ These relations hold for T finite or infinite

▶ Observation: this RSDF is similar to what we have studied for the two-period case.

Equilibrium stochastic discount factor: statistics

▶ Given the probability and the endowment process (assuming that they are both adapted stochastic processes)

$$\mathbf{P} = \{1, \mathsf{P}_1, \ldots, \mathsf{P}_t, \ldots\} \text{ and } \mathbf{Y} = \{y_0, Y_1, \ldots, Y_t, \ldots\}$$

▶ we can calculate statistics for the stochastic discount factor

$$\mathbb{E}_t\big[M_{t+1}\big] = \beta \, \mathbb{E}_t\Big[\frac{u'(Y_{t+1})}{u'(y_t)}\Big] = \frac{\beta}{u'(y_t)} \, \mathbb{E}_t\Big[u'(Y_{t+1})\Big]$$