# Foundations of Financial Economics
## Two period financial markets

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# Topics

- Financial assets
- Financial market
- State prices
- Arbitrage opportunities
- Completeness
- Characterization of a financial market
- Asset pricing implicit probabilities and the stochastic discount factor
- Portfolios
- Arbitrage pricing theory
- Equity premium

# Financial contracts and assets

# Information

- Assume again that there is **complete** information regarding time $t = 0$, but **incomplete** information regarding time $t = 1$.
- Until now we have (mostly) assumed that the information for $t = 1$

$$\Omega = \{\omega_s\}_{s=1}^{N}, \text{ and } \mathbb{P} = \{\pi_s\}_{s=1}^{N}$$

  is provided by **nature**.
- But next we start assuming that information regarding time $t = 1$ is provided by the **financial market**. A financial market is defined by a collection of **contingent claims**.

The **general idea**: under some conditions, we can extract an **implicit probability distribution** of the states of nature from the elements of the financial market.

# Financial contracts or assets

## Prices and payoffs

**A financial contract or financial asset or contingent claim**: is defined by a **price and payoff** pair $(S_j, V_j)$ (for asset $j$) where:

- $S_j$ is a price which is **observed** (deterministic) at time $t = 0$
- $V_j$ is a payoff which is **state-contingent** (stochastic) at time $t = 1$,

$$V_j = (V_{j,1}, \dots, V_{j,s}, \dots, V_{j,N})^\top$$

$$V_{j,s} = \text{payoff of asset } j \text{ in the state of nature } s \in \{1, \dots, N\}$$

- This information refers to time $t = 0$

# Financial assets

## Returns and rates of return

- The return of asset $j$ in the state of nature $s$ is defined as the ratio between the payoff at state $s$ and the price

$$R_{j,s} = \frac{V_{j,s}}{S_j},\ s \in \{1, \ldots, N\}$$

- The rate of return $r_j$ of asset $j$ in the state of nature $s$ is defined from

$$r_{j,s} = \frac{V_{j,s} - S_j}{S_j},\ s \in \{1, \ldots, N\}$$

- Therefore the return and the rate of return are related as

$$R_{j,s} = 1 + r_{j,s},\ s \in \{1, \ldots, N\}$$

# Financial assets

## Returns and rates of return

- Therefore, in a two-period case, the return $R_j$ and rate of return $r_j$ of asset $j$ are two random variables which are related

$$R_j = \frac{V_j}{S_j} = \begin{pmatrix} \frac{V_{j,1}}{S_j} \\ \dots \\ \frac{V_{j,s}}{S_j} \\ \dots \\ \frac{V_{j,N}}{S_j} \end{pmatrix} = \begin{pmatrix} 1 + r_{j,1} \\ \dots \\ 1 + r_{j,s} \\ \dots \\ 1 + r_{j,N} \end{pmatrix}$$

- Or, more compactly

$$R_j = 1 + r_j,$$

# Financial assets

## Timing, information and flow of funds

Two alternative ways of representing (net) income flows (for a buyer) of asset $j$:

- as a **price-payoff** sequence: $\{S_j, V_j\}$

$$
\begin{array}{ccc}
 & & \begin{pmatrix} V_{j,1} \\ \dots \\ V_{j,s} \\ \dots \\ V_{j,N} \end{pmatrix} \\
-S_j & & \\
0 & \text{———————} & 1
\end{array}
$$

- as an **investment-return** sequence: $\{1, R_j\}$

$$
\begin{array}{ccc}
 & & \begin{pmatrix} R_{j,1} \\ \dots \\ R_{j,s} \\ \dots \\ R_{j,N} \end{pmatrix} \\
-1 & & \\
 & \text{———————} &
\end{array}
$$

# Financial assets

## Statistics

- From the information, at time $t = 0$, on the probabilities for the states of nature at time $t = 1$ we can compute:
- **Expected payoff** for asset $j$ at time $t = 1$, from the information at time $t = 0$

$$\mathbb{E}[V_j] = \sum_{s=1}^{N} \pi_s V_{j,s} = \pi_1 V_{j,1} + \ldots \pi_s V_{j,s} + \ldots + \pi_N V_{j,N}$$

- **Variance of the payoff** for asset $j$ at time $t = 1$, from the information at time $t = 0$

$$\mathbb{V}[V_j] = \sum_{s=1}^{N} \pi_s (V_{js} - \mathbb{E}[V_j])^2 = \pi_1 (V_{j,1} - \mathbb{E}[V_j])^2 + \ldots \pi_N (V_{j,N} - \mathbb{E}[V_j])^2$$

- Observation: an useful relationship

$$\mathbb{V}[V] = \mathbb{E}[V^2] - (\mathbb{E}[V])^2$$

# Financial assets

## Statistics

- From the information, at time $t = 0$, on the probabilities for the states of nature at time $t = 1$ we can compute:
- **Expected return** for asset $j$ at time $t = 1$, from the information at time $t = 0$

$$\mathbb{E}[R_j] = \sum_{s=1}^{N} \pi_s R_{j,s} = \pi_1 R_{j,1} + \ldots \pi_s R_{j,s} + \ldots + \pi_N R_{j,N}$$

- **Variance of the return** for asset $j$ at time $t = 1$, from the information at time $t = 0$

$$\mathbb{V}[R_j] = \sum_{s=1}^{N} \pi_s (R_{js} - \mathbb{E}[R_j])^2 = \pi_1 \left(R_{j,1} - \mathbb{E}[R_j]\right)^2 + \ldots \pi_N \left(R_{j,N} - \mathbb{E}[R_j]\right)^2$$

- Observation: an useful relationship

$$\mathbb{V}[R] = \mathbb{E}[R^2] - \left(\mathbb{E}[R]\right)^2$$

# Financial assets

## Statistics

- Relationship between the expected return and the expected payoff

$$\mathbb{E}[R_j] = \frac{\mathbb{E}[V_j] - S_j}{S_j}$$

- Relationship between the variance of the return and the variance of the payoff

$$\mathbb{V}[R_j] = \frac{\mathbb{V}[V_j]}{S_j^2}$$

# Classification of assets

## Risk classification

As regards **risk**:

- risk-less or **risk-free** asset: payoff is state-independent (non-contingent)
$$V^\top = (v, \ldots, v)^\top$$

- risky asset: payoff is state-dependent (contingent)
$$V^\top = (v_1, \ldots, v_N)^\top$$
with at least two different elements, i.e. there are at least two elements, $v_i$ and $v_j$ such that $v_i \neq v_j$ for $i \neq j$

# Classification of assets

## Types of assets

Particular **types** of assets as regards the income flows:

- one period **bonds** with unit facial value:

$$S = \frac{1}{1+i}, \ V^\top = (1, 1, \ldots, 1)^\top$$

  $i$ yield to maturity (or risk-free interest rate)

- **deposits** or banking credit:

$$S = 1, \ V^\top = (1+i, 1+i, \ldots, 1+i)^\top$$

- **equity**: the payoff are state-dependent dividends

$$S_e, \ V_e^\top = (d_1, \ldots, d_N)^\top$$

# Classification of assets

## Types of assets

**Derivatives** over an **underlying** asset with payoff $V^\top = (v_1, \ldots, v_N)^\top$:

- **forward** contract on an underlying asset with offered price $p$:

$$S_f,\ V_f^\top = (v_1 - p, \ldots, v_N - p)^\top$$

- **european call option** with exercise price $p$:

$$S_c,\ V_c^\top = (\max\{v_1 - p, 0\}, \ldots, \max\{v_N - p, 0\})^\top$$

- **european put option** with exercise price $p$:

$$S_p,\ V_p^\top = (\max\{p - v_1, 0\}, \ldots, \max\{p - v_N, 0\})^\top$$

# Financial market

# Financial market

**Definition 1**

*A **financial market** is a collection of $K$ traded assets. it can be **characterized** by the structure of prices and payoffs of all $K$ assets: i.e by the pair $(\mathbf{S}, \mathbf{V})$.*

**Information**: the participants observe $\mathbf{S}$ and have common beliefs $\mathbf{V}$

# Financial market

**Characterization**: :

- vector of observed prices, at time $t = 0$ (we use row vectors for prices)
$$\underset{(1\times K)}{\mathbf{S}} = (S_1, \ldots, S_K),$$
- and a matrix of contingent (i.e., uncertain) payoffs, at time $t = 1$
$$\underset{(N\times K)}{\mathbf{V}} = \begin{pmatrix} V_{11} & \ldots & V_{K1} \\ \vdots & & \vdots \\ V_{1N} & \ldots & V_{KN} \end{pmatrix}$$

**Realization**: ex-post only one row of $\mathbf{V}$ will be realized (i.e, will be cast by nature)

# Financial market

The payoff matrix contains the following information:

- each column represents the beliefs for the payoff of **asset** $j = 1, \dots, K$

$$\mathbf{V} = \begin{pmatrix} V_{11} & \dots & \mathbf{V_{j1}} & \dots & V_{K1} \\ \vdots & & \vdots & & \vdots \\ V_{1s} & \dots & \mathbf{V_{js}} & \dots & V_{Ks} \\ \vdots & & \vdots & & \vdots \\ V_{1N} & \dots & \mathbf{V_{jN}} & \dots & V_{KN} \end{pmatrix}$$

- each row represents the outcomes (in terms of payoffs) for **state** of nature $s = 1, \dots, N$

$$\mathbf{V} = \begin{pmatrix} V_{11} & \dots & V_{j1} & \dots & V_{K1} \\ \vdots & & \vdots & & \vdots \\ \mathbf{V_{1s}} & \dots & \mathbf{V_{js}} & \dots & \mathbf{V_{Ks}} \\ \vdots & & \vdots & & \vdots \\ V_{1N} & \dots & V_{jN} & \dots & V_{KN} \end{pmatrix}$$

# Financial market

- Equivalently, we can charaterize a financial market by the matrix of returns

$$\underset{(N\times K)}{\mathbf{R}} = \begin{pmatrix} R_{11} & \dots & R_{K1} \\ \vdots & & \vdots \\ R_{1N} & \dots & R_{KN} \end{pmatrix}$$

  where $R_{j,s} = \dfrac{V_{j,s}}{S_j} = 1 + r_{j,s}$

- Therefore

$$\underset{(N\times K)}{\mathbf{R}} = \underset{(N\times K)}{\mathbf{V}} \underset{(K\times K)}{\big(\mathrm{diag}(\mathbf{S})\big)}^{-1} \iff \mathrm{diag}(\mathbf{S})\,\mathbf{R} = \mathbf{V}$$

  where

$$\underset{(K\times K)}{\mathrm{diag}(\mathbf{S})} = \begin{pmatrix} S_1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & S_j & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & S_K \end{pmatrix}$$

# State prices

# State prices

**Implicit state valuation**: the modern approach to financial economics characterizes asset markets by the prices of the states of nature **implicit** in the relationship between present prices $S$ and future payoffs $V$, i.e.

$$Q := \{x : \mathbf{S} = x\mathbf{V}\}$$

- if $Q$ is unique it is a vector

$$\underset{(1\times N)}{Q} = (q_1, \ldots, q_N)$$

satisfying

$$\underset{(1\times K)}{\mathbf{S}} = \underset{(1\times N)}{Q}\underset{(N\times K)}{\mathbf{V}} \iff \underset{(K\times 1)}{\mathbf{S}^\top} = \underset{(K\times N)}{\mathbf{V}^\top}\underset{(N\times 1)}{Q^\top}$$

# State prices

**Definition 2**
*$Q$ is a state price vector if is it positive, i.e. $q_s > 0$ for all $s \in \{1, \ldots, N\}$. In this case $q_s$ is the price of state of nature $s$.*

# Arbitrage opportunities

# Arbitrage opportunities

**Definition 2**

*If there is at least one $q_s \leq 0$, $s = 1, \ldots, N$ then we say* ***there are arbitrage opportunities***

**Definition 3**

*We say* ***there are no arbitrage opportunities*** *if $q_s > 0$, for all $s = 1, \ldots, N$*

**Intuition**:

- existence of arbitrage (opportunities) means there are free or negatively valued states of nature
- absence of arbitrage means every state of nature is costly and therefore, positively priced.

# Arbitrage opportunities

### Proposition 1

*Given* $(\mathbf{S}, \mathbf{V})$, *there are no arbitrage opportunities if and only if* $Q$ *is a vector of state prices.*

# Completeness

# Completeness of a financial market

**Definition 4**

*If $Q$ **is unique**, for a given $(\mathbf{S}, \mathbf{V})$, then we say markets are **complete***

**Definition 5**

*if $Q$ **is not unique**, for a given $(\mathbf{S}, \mathbf{V})$, then we say markets are **incomplete***

Intuition:

- completeness: there is an unique valuation for each state of nature (i.e, every state of nature can be uniquely priced).
- incompleteness: there is not an unique valuation for every state of nature (i.e, there are states of nature whose price is uncertain).

# Completeness of a financial market

## Conditions for completeness

We can determine the completeness of a market by looking at the **number of assets with independent payoffs** and the **number of states of nature**:

1. if $K = N$ and $\det(\mathbf{V}) \neq 0$ then markets are complete and all assets are independent;
2. if $K > N$ and $\det(\mathbf{V}) \neq 0$ then markets are complete and there are $N$ independent and $K - N$ **redundant** assets;
3. if $K < N$ or $K \geq N$ and $\det(V) = 0$ then markets are incomplete.

# Completeness of a financial market

## Conditions for completeness

### Proposition 2

*Given* $(\mathbf{S}, \mathbf{V})$*, markets are complete if and only if* $dim(\mathbf{V}) = N$

$\dim(\mathbf{V})$ = number of linearly independent columns (i.e., assets)

# Characterization of financial markets

# Possible cases

We will study next the following cases:

1. $K = N$ and $\det \mathbf{V} \neq 0$
2. $K < N$
3. $K = N$ and $\det \mathbf{V} = 0$
4. $K > N$ and $\det \mathbf{V} \neq 0$

# Characterization of a financial market

## Exact completeness

Case 1: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K = N \text{ and } \det(\mathbf{V}) \neq 0$$

- We defined state prices from $\mathbf{S} = Q\mathbf{V}$
- As $K = N$ and $\det(\mathbf{V}) \neq 0$ then $\mathbf{V}^{-1}$ exists and is unique
- Therefore, we obtain uniquely

$$\boxed{\underset{(1\times N)}{Q} = \underset{(1\times K)}{\mathbf{S}}\,\underset{(K\times N)}{\mathbf{V}^{-1}}}$$

that is

$$(q_1, \ldots, q_N) = (S_1, \ldots, S_K) \begin{pmatrix} V_{1,1} & \ldots & V_{1,K} \\ \ldots & \ldots & \ldots \\ V_{N,1} & \ldots & V_{N,K} \end{pmatrix}^{-1}$$

# Characterization of a financial market

## Exact completeness

First alternative computation:

- because

$$\mathbf{V}^\top Q^\top = \mathbf{S}^\top$$

- If $\det(\mathbf{V}) \neq 0$ then

$$\boxed{\underset{(N\times 1)}{Q^\top} = \underset{(N\times K)}{\left(\mathbf{V}^\top\right)^{-1}} \underset{(K\times 1)}{\mathbf{S}^\top}}$$

that is

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} V_{1,1} & \dots & V_{N,1} \\ \dots & \dots & \dots \\ V_{1,K} & \dots & V_{N,K} \end{pmatrix}^{-1} \begin{pmatrix} S_1 \\ \vdots \\ S_K \end{pmatrix}$$

# Characterization of a financial market

## Exact completness

Second alternative computation:

- As $R_{js} = V_{js}/S_j$ equivalently,
- we can write

$$\underset{(1\times N)}{Q}\ \underset{(N\times K)}{\mathbf{R}} = \underset{(1\times K)}{\mathbf{1}^\top}$$

expanding

$$(q_1, \dots, q_N) \begin{pmatrix} R_{1,1} & \dots & R_{1,K} \\ \dots & \dots & \dots \\ R_{N,1} & \dots & R_{N,K} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

- then

$$\boxed{\underset{(1\times N)}{Q} = \underset{(1\times K)}{\mathbf{1}^\top}\ \underset{(K\times N)}{\mathbf{R}^{-1}}}$$

# Characterization of a financial market

## Exact completness

Third alternative computation:

- Transposing the last equation $\mathbf{R}^\top Q^\top = \mathbf{1}$
- then

$$\boxed{\underset{(N\times 1)}{Q^\top} = \underset{(N\times K)}{(\mathbf{R}^\top)^{-1}} \underset{(K\times 1)}{\mathbf{1}}} \tag{1}$$

matricially

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} R_{1,1} & \dots & R_{1,K} \\ \dots & \dots & \dots \\ R_{N,1} & \dots & R_{N,K} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

# Characterization of a financial market

## Arbitrage opportunities and completeness

Case 1: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K = N \text{ and } \det(\mathbf{V}) \neq 0$$

- Because $Q$ is unique, we say markets are **complete**
- But the value of $Q$ matters:
  - if $Q \gg \mathbf{0}$ we say there are **no arbitrage opportunities**: (meaning: all states of nature are costly to insure)
  - if $Q$ has a zero or negative element we say there **are arbitrage opportunities** (meaning: there are states that have no cost or are pathological (there is a benefit to insure)
- The equation $Q^\top = (\mathbf{R}^\top)^{-1}\,\mathbf{1}$ conveys an important message: the value of $Q$ is related to the structure of $\mathbf{R}$), i.e., the relationship between beliefs on $\mathbf{V}$ related to observed prices $\mathbf{S}$

# Example: case 1 - no arbitrage opportunities

- Let $\mathbf{S} = (1, 1)$ and $\mathbf{V} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- As $\det \mathbf{V} = 2$ then $K = N = 2$
- Using the third approach we determine

$$\mathbf{R} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \mathbf{R}^\top$$

- Therefore, using equation (1)

$$Q^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

- Conclusion: markets are complete ($Q$ is unique) and there are no arbitrage opportunities ($Q \gg \mathbf{0}$)

# Example case 1 - existence of arbitrage opportunities

- Let $\mathbf{S} = (1, 1)$ and $\mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- then
$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \ \mathbf{R}^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
- Therefore
$$Q^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Conclusion: markets are complete but **there are arbitrage opportunities**. The cost to ensure state $s = 2$ is zero. Why ?

# Characterization of a financial market

## Market incompleteness because of insufficient assets

Case 2: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K < N \text{ assume that } \det(\mathbf{V}_1) \neq 0$$

in the partition

$$\underset{(K\times N)}{\mathbf{V}^\top} = \left( \underset{(K\times K)}{\mathbf{V}_1^\top} \mid \underset{(N-K\times K)}{\mathbf{V}_2^\top} \right)$$

- We defined state prices from $\mathbf{V}^\top Q^\top = \mathbf{S}^\top$
- But now we have

$$\begin{pmatrix} \mathbf{V}_1^\top & \mathbf{V}_2^\top \end{pmatrix} \begin{pmatrix} Q_1^\top \\ Q_2^\top \end{pmatrix} = \mathbf{S}^\top$$

where $Q_1$ is $(1 \times K)$ and $Q_2$ is $(1 \times N - K)$.

# Characterization of a financial market

## Market incompleteness

- Then

$$\mathbf{V}_1^\top Q_1^\top + \mathbf{V}_2^\top Q_2^\top = \mathbf{S}^\top$$

- Because $\det(\mathbf{V}_1) \neq 0$ we can make

$$Q_1^\top = (\mathbf{V}_1^\top)^{-1}\left(\mathbf{S}^\top - \mathbf{V}_2^\top Q_2^\top\right)$$

- There are only $K$ independent prices: i.e, $Q$ is indeterminate and the degree of indeterminacy is $N - K$,
- As $Q$ is not uniquely determined (i.e, we can fix arbitrarilty $N - K$ state prices) the market is **incomplete**

# Example case 2: market incompleteness because of insufficient number of assets

- Let $\mathbf{S} = (1)$ and $\mathbf{V} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- then

$$\mathbf{R} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

- then we also have

$$(q_1, q_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \iff 2q_1 + q_2 = 1$$

- has an infinite number of solutions

$$Q = \begin{pmatrix} \frac{1-k}{2} \\ k \end{pmatrix}$$

for any $k$, then markets are incomplete and if $0 < k < 1$ there are no arbitrage opportunities (but they cannot be ruled out)

# Characterization of a financial market

## Market incompleteness because of insufficient independent assets assets

Case 3: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K = N \text{ assume that } \det(\mathbf{V}) = 0$$

- If there are $\tilde{K}$ independent assets (i.e., $\text{char}(\mathbf{V}) = \tilde{K} < N$, then we can apply the previous method by extracting data from the original $\mathbf{V}$ such that $\tilde{\mathbf{V}}$ are the independent columns of $\mathbf{V}$ and having the partition

$$\underset{(\tilde{K}\times N)}{\tilde{\mathbf{V}}^\top} = \left( \underset{(\tilde{K}\times K)}{\tilde{\mathbf{V}}_1^\top} \,\middle|\, \underset{(\tilde{K}\times N-K)}{\mathbf{V}_2^\top} \right)$$

  where $\det(\tilde{\mathbf{V}}_1) \neq 0$

- We obtain the state prices from $Q_1$ from

$$\begin{pmatrix} \tilde{\mathbf{V}}_1^\top & \tilde{\mathbf{V}}_2^\top \end{pmatrix} \begin{pmatrix} Q_1^\top \\ Q_2^\top \end{pmatrix} = \tilde{\mathbf{S}}^\top$$

  where $Q_1$ is $(1 \times \tilde{K})$ and $Q_2$ is $(1 \times N - \tilde{K})$.

# Characterization of a financial market

## Market incompleteness

- Then

$$\tilde{\mathbf{V}}_1^\top \, Q_1^\top + \tilde{\mathbf{V}}_2^\top \, Q_2^\top = \tilde{\mathbf{S}}^\top$$

- Because $\det(\tilde{\mathbf{V}}_1) \neq 0$ we can make

$$Q_1^\top = (\tilde{\mathbf{V}}_1^\top)^{-1} \left( \tilde{\mathbf{S}}^\top - \tilde{\mathbf{V}}_2^\top \, Q_2^\top \right)$$

- There are only $\tilde{K}$ independent prices: i.e, $Q$ is indeterminate and the degree of indeterminacy is $N - \tilde{K}$,
- As $Q$ is not uniquely determined (i.e, we can fix arbitrarilty $N - \tilde{K}$ state prices) the market is **incomplete**

# Example case 3: market incompleteness with non-independent prices

- Let $\mathbf{S} = (1, 2)$ and $\mathbf{V} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ and $\det \mathbf{V} = 0$
- then $\mathbf{R} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{R}^\top = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ two assets have the same returns
- then $\mathbf{R}^\top Q^\top = \mathbf{1}^\top$ becomes

  $$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  has an infinite number of solutions

  $$Q^\top = \begin{pmatrix} (1-k)/2 \\ k \end{pmatrix}$$

  for an arbitrary $k$, then markets are incomplete and if $0 < k < 1$ there are no arbitrage opportunities;

Case 4: Consider a financial market $(\mathbf{S}, \mathbf{V})$, such that

$$K > N \text{ assume that } \det(\mathbf{V}) \neq 0$$

- As $K > N$ we can partition $\mathbf{V}$ as

$$\underset{(N\times K)}{\mathbf{V}} = \left( \underset{(N\times N)}{\mathbf{V}_1} \,\middle|\, \underset{(N\times K-N)}{\mathbf{V}_2} \right)$$

  and $\det(\mathbf{V}_1) \neq 0$

- We defined state prices from $\mathbf{V}^\top Q^\top = \mathbf{S}^\top$
- But now we have

$$\begin{pmatrix} \mathbf{V}_1^\top \\ \mathbf{V}_2^\top \end{pmatrix} (Q^\top) = \begin{pmatrix} \mathbf{S}_1^\top \\ \mathbf{S}_2^\top \end{pmatrix}$$

  where $\mathbf{S}^1$ is $(1 \times N)$ and $\mathbf{S}^2$ is $(1 \times K - N)$.

# Characterization of a financial market

## Market completeness with redundant assets

- Then

$$\begin{cases} \mathbf{V}_1^\top \, Q^\top = \mathbf{S}_1^\top \\ \mathbf{V}_2^\top \, Q^\top = \mathbf{S}_2^\top \end{cases}$$

- Because $\det\left(\mathbf{V}_1\right) \neq \mathbf{0}$ we can determine $Q$ uniquely

$$Q^\top = (\mathbf{V}_1^\top)^{-1}\mathbf{S}_1^\top$$

- Which implies that the prices of the remaining $K - N$ assets can be obtained from the prices $\mathbf{S}_1$

$$\mathbf{S}_2^\top = \mathbf{V}_2^\top \, Q^\top = \mathbf{V}_2^\top \, \left((\mathbf{V}_1^\top)^{-1}\mathbf{S}_1^\top\right)$$

- There are $K - N$ **redundant assets** (i.e, they do not add new information on $Q$)
- As $Q$ is uniquely determined the market is **complete**

# Example case 4: market completeness and redundant assets

- Let $\mathbf{S} = (1, 1, 2)$ and $\mathbf{V} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
- then $\mathbf{R} = \begin{pmatrix} 2 & 1 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} \end{pmatrix}$ and $\mathbf{R}^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}$
- we pick all the combinations of two assets

$$Q^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$Q^\top = \begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} \\ -1 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

and

$$Q^\top = \begin{pmatrix} 1 & 2 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{4}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

then markets are complete, there are no arbitrage opportunities and there is one redundant asset

# Characterization of a financial market

## Suggestion on how to apply this theory

- First: obtain $\mathbf{R}^T$ which is a $(K \times N)$ matrix
- Second: check the number of assets, $K$ and the number of states of nature $N$
  - if $N > K$ then markets are incomplete
  - if $N \geq K$ check the number of independent rows of $R$
    - if $\dim(\mathbf{V}) < N$ then markets are incomplete
    - if $\dim(\mathbf{V}) \geq N$ then markets are complete
- Third: obtaining the state prices $Q$
  - if $K = N$ and $\det \mathbf{R} \neq 0$ compute $Q^\top = (\mathbf{R}^\top)^{-1}\mathbf{1}$
  - if $K > N$ and $\tilde{\mathbf{R}}$ a $N \times N$ partition of $\mathbf{R}$ has $\det \tilde{\mathbf{R}} \neq 0$ then compute $Q^\top = (\tilde{\mathbf{R}}^\top)^{-1}\mathbf{1}$
  - if $K = N$ but $\det \mathbf{R} = 0$, then, solve the independent equations in $\mathbf{R}^\top Q^\top = \mathbf{1}$
  - if $K < N$ solve the independent equations in $\mathbf{R}^\top Q^\top = \mathbf{1}$.

# Asset pricing implicit probabilities and the stochastic discount factor

# Implicit market probabilities: risk-free probabilities

- Let $Q = (q_1, \ldots, q_N)$ be a state-price vector
- If there are no arbitrage opportunities then $q_s > 0$, i.e $Q \gg 0$

### Definition 6

*A* ***Radon-Nikodyn derivative*** *is defined as*

$$\boxed{\pi_s^Q = \frac{q_s}{\bar{q}}}$$

*where* $\bar{q} = \sum_{s=1}^{N} q_s$.

- Therefore, if there are no arbitrage opportunities then

$$0 < \pi_s^Q < 1, \text{ and} \sum_{s=1}^{N} \pi_s^Q = 1$$

that is $\mathbb{P}^Q = \{\pi_s^Q\}_{s=1}^N$ **is a probability measure**

- This is a probability measure implicit in the financial market $(\mathbf{S}, \mathbf{V})$

# Implicit market probabilities: risk-free probabilities

Therefore:

- In complete markets the probability measure $\mathbb{P}^Q$ is unique
- in incomplete the probability measure $\mathbb{P}^Q$ is not unique

# And asset pricing

- Using the definition for $Q$ (from $\mathbf{S}^\top = \mathbf{V}^\top \, Q^\top$)

$$S_j = \sum_{s=1}^{N} q_s \; V_{sj} = \sum_{s=1}^{N} \bar{q} \frac{q_s}{\bar{q}} \; V_{sj}$$

then, for any asset

$$\boxed{S_j = \bar{q} \, \mathbb{E}^Q \Big[ V_j \Big]}$$

- This means **asset prices are proportional to the expected value of the payoff, using an implicit market probability**.

# Stochastic discount factor

- Let $Q = (q_1, \ldots, q_N)$
- Assume we have a (objective or subjective) probability distribution for the states of nature

$$\mathbb{P} = (\pi_1, \ldots, \pi_N)$$

  such that $0 < \pi_s < 1$ and $\sum_{s=1}^{N} \pi_s = 1$

- The stochastic discount factor for state $s$ is

$$\boxed{m_s = \frac{q_s}{\pi_s}, \;\; s = 1, \ldots, N}$$

- The stochastic discount factor is the random variable

$$\underset{(1\times N)}{M} = (m_1, \ldots, m_s, \ldots, m_N)$$

# And asset pricing

- Using the definition for $Q$ (from $\mathbf{S}^\top = \mathbf{V}^\top\, Q^\top$)

$$S_j = \sum_{s=1}^{N} q_s\, V_{sj} = \sum_{s=1}^{N} \pi_s \frac{q_s}{\pi_s}\, V_{sj}$$

  then, for any asset

$$\boxed{S_j = \mathbb{E}\Big[M\, V_j\Big]}$$

- This means that the stochastic discount factor combines both market and fundamental information.

# The state price approach to asset markets

Summing up:

- The state price translates the information structure implicit in financial transactions
- The relevant information is related to:
  - how costly is the insurance against the states of nature
  - the uniqueness of that insurance cost
  - suggests a method for pricing new assets: **pricing by redundancy** (this is the approah followed in the arbitrage pricing theory (APT))

# Portfolios

# Portfolios

**Definition 7**

*A **portfolio** is a vector specifying the positions, $\theta$, in all the assets in the market*

$$\theta^\top = (\theta_1, \dots, \theta_K)^\top \in \mathbb{R}^K$$

If $\theta_j > 0$ there is a **long** position in asset $j$

if $\theta_j < 0$ there is a **short** position in asset $j$

# Portfolios

- The stream of income generated by a position $\theta_j$ in asset $j$ is

$$
\begin{array}{ccc}
 & & \begin{pmatrix} \theta_j V_{j,1} \\ \dots \\ \theta_j V_{j,s} \\ \dots \\ \theta_j V_{j,N} \end{pmatrix} \\
-S_j\theta_j & & \\
\vdash & \text{———————} & \dashv \\
0 & & 1
\end{array}
$$

  - long position ($\theta_j > 0$): pay $S_j\theta_j$ at time $t = 0$ and receive the contingent payoff $\theta_j V_j$ at time $t = 1$
  - short position ($\theta_j < 0$): receive $S_j\theta_j$ at time $t = 0$ and pay the contingent payoff $\theta_j V_j$ at time $t = 1$

# Portfolios

A portfolio generates a **stochastic sequence of income** $\{z_0^\theta, Z_1^\theta\}$

$$
\begin{aligned}
z_0^\theta &= -C(\theta, \mathbf{S}) = -\underset{(1\times 1)}{\mathbf{S}\,\theta} = -\sum_{j=1}^{K} S_j\,\theta_j \\
Z_1^\theta &= \underset{(N\times 1)}{\mathbf{V}\,\theta} = \sum_{j=1}^{K} V_j\,\theta_j
\end{aligned}
$$

where

$$
Z_1^\theta = \begin{pmatrix} z_{1,1}^\theta \\ \dots \\ z_{1,s}^\theta \\ \dots \\ z_{1,N}^\theta \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{K} V_{j,1}\,\theta_j \\ \dots \\ \sum_{j=1}^{K} V_{j,s}\,\theta_j \\ \\ \sum_{j=1}^{K} V_{j,N}\,\theta_j \end{pmatrix}
$$

# Portfolios

The flow of income generated by a portfolio $\theta$ is:

$$\begin{array}{ccc} z_0^\theta & & \begin{pmatrix} z_{1,1}^\theta \\ \dots \\ z_{1,s}^\theta \\ \dots \\ z_{1,N}^\theta \end{pmatrix} \\ \vdash & \!\!\text{———————}\!\! & \dashv \\ 0 & & 1 \end{array}$$

# Portfolios and arbitrage opportunities

## Proposition 3

*Assume there are **arbitrage opportunities**. Then there exists at least one portfolio $\vartheta$ such that*

$$z_0^{\vartheta} = 0 \textit{ and } Z_1^{\vartheta} > \mathbf{0}$$

*or*

$$z_0^{\vartheta} > 0 \textit{ and } Z_1^{\vartheta} = \mathbf{0}.$$

(Obs. A positive vector $X > 0$ has non-negative elements, and has at least one equal to zero. A strictly positive vector $X \gg 0$ has only positive elements )

**Intuition**
with a **zero cost** we can get a **positive income** in at least one state of nature.
If we have an **initial income**, we will pay a **zero cost** in the future, in every state of nature.

# Example

- We consider Case 1 example 2 again: $\mathbf{S} = (1, 1)$ and $\mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- We can build a portfolio $\vartheta = (\vartheta_1, \vartheta_2)^\top$ that has a zero cost

$$z_0^\vartheta = -\mathbf{S}\,\vartheta = -(\vartheta_1 + \vartheta_2) = 0 \implies \vartheta_1 = -x, \vartheta_2 = x$$

- The return at time $t = 1$ is

$$Z_1^\vartheta = \mathbf{V}\,\vartheta = \begin{pmatrix} -x + x \\ -x + 2x \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

then if we take a long position in the risky asset $(x > 0)$ we have a **certain** positive income at $t = 1$ with a zero investment.

# Portfolios and arbitrage opportunities

### Proposition 4

*Assume there are **no arbitrage opportunities**. Then a portfolio with zero cost, $z_0^\vartheta = 0$, generates a **nonpositive** income at time 1, $Z_1^\vartheta$.*

- Non-positive $Z_1^\vartheta$ means that there is at least one state of nature, $s$ such that $z_{1,s}^\theta < 0$.

  **Intuition**: with a zero cost although we can get a positive income in several states of nature, there will be **at least one state of nature** in which we will have a negative income.

# Example 2

- Consider the previous case 1 example 1: $\mathbf{S} = (1, 1)$ and $\mathbf{V} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- We can build a portfolio $(\vartheta_1, \vartheta_2)^\top$ such that $z_0^\vartheta = -S\vartheta = 0$:

$$\vartheta_1 = -x, \vartheta_2 = x$$

- The return at time $t = 1$ is

$$Z_1^\vartheta = \mathbf{V}\,\vartheta = \begin{pmatrix} -2x + x \\ -x + 2x \end{pmatrix} = \begin{pmatrix} -x \\ x \end{pmatrix}$$

then whatever the position, long $(x > 0)$ or short $(x < 0)$, in the risky asset we will **not have positive income** at $t = 1$ (in some states of the nature we win, but we will lose in others).

# Arbitrage pricing theory

# Replicating portfolios

### Definition 9

*Assume we have a contingent income $W = (w_1, \ldots, w_N)^\top$. To the portfolio, $\vartheta$, build using the assets in the market, generating the same (contingent) income*

$$\vartheta \equiv \{\theta : \; Z_1 = \mathbf{V}\,\theta = \; W\}$$

*we call **replicating portfolio**.*

The **cost** of the replicating portfolio is

$$C(\vartheta, \mathbf{S}) = \mathbf{S}\vartheta$$

# Arbitrage asset pricing theory (APT)

- Deals with the determination of asset prices **through replication**.
- APT vs GEAP (general equilibrium asset pricing):
  - in APT we take $(\mathbf{S}, \mathbf{V})$ as given ($\mathbf{S}$ is exogenous)
  - in GEAP we take $\mathbf{V}$ as given and want to determine $\mathbf{S}$ from the fundamentals ($\mathbf{S}$ is endogenous)

Both theories have three equivalent formulations:

- using state prices $Q$ (micro view)
- using the stochastic discount factor $M$ (macro view)
- using market or risk-neutral probabilities $\mathbb{P}^Q$ (finance view)

# Arbitrage asset pricing

## Using state prices

### Proposition 5

*Assume there is a financial market* $(\mathbf{S}, \mathbf{V})$ *such that there are no arbitrage opportunities. Then, given a new asset with payoff* $V_k$ *its price can be determined by redundancy by using the expression*

$$S_k = Q V_k$$

*where* $Q = \mathbf{S}\,\mathbf{V}^{-1} \gg 0$ *is determined from the market pair* $(\mathbf{S}, \mathbf{V})$.

That is

$$S_k = \sum_{s=1}^{N} q_s V_{ks}$$

# Arbitrage asset pricing

## Using state prices

From what we saw previously, there are two important results

### Proposition 6

*If markets are **complete** then the value, $S_k$, is **unique**. If markets are **incomplete** then $S_k$ is **not unique**.*

### Proposition 7

*Assume a financial market $(\mathbf{S}, \mathbf{V})$ is complete and there are $K - N$ redundant assets. Then the **price of any asset is equal to the cost of its replicating portfolio build with $N$ independent assets**.*

# Arbitrage asset pricing

## Using stochastic discount factors

- If $Q$ is a state price vector and $\pi$ is the vector or probabilities, we define the stochastic discount factor (SDF) for state $s$ by

$$m_s \equiv \frac{q_s}{\pi_s}$$

### Proposition 8

*Assume there is a financial market $(\mathbf{S}, \mathbf{V})$ such that there are no arbitrage opportunities. Then, given a new asset with payoff $V_k$ its price can be determined by redundancy by using the expression*

$$S_k = \mathbb{E}[MV_k]$$

*where $M = (m_1, \ldots, m_N)^\top$ is the stochastic discount factor.*

# Arbitrage asset pricing

## Using stochastic discount factors

- Proof: as $S_k = \sum_{s=1}^{N} q_s V_{ks}$ and using the definition of SDF we get

$$S_k = \sum_{s=1}^{N} q_s V_{ks} = \sum_{s=1}^{N} \pi_s m_s V_{ks}$$

- **Intuition**: if there are no arbitrage opportunities the price of an asset is the expected value of the future payoffs discounted by the stochastic discount factor (which is the market discount for uncertain payoffs)

# Arbitrage asset pricing

## Using risk-neutral probabilities

- Therefore,

$$S_k = \sum_{s=1}^{N} q_s \, V_{ks} = \bar{q} \sum_{s=1}^{N} \pi_s^Q \, V_{ks} = \bar{q} \, \mathbb{E}^Q[\, V_k]$$

  or compactly

$$\boxed{S_k = \bar{q} \, \mathbb{E}^Q[\, V_k]}$$

- **Intuition**: if there are no arbitrage opportunities there is an equivalent probability measure such that the price of an asset is proportional of the expected value of the future payoffs
- Although $\bar{q}$ is mysterious we can calculate it a intuitive way.

# Arbitrage asset pricing

## Existence of risk-free asset

### Proposition 9

*Assume there are no arbitrage opportunities and there is a risk-free bond with price $1/(1+i)$ and face value $1$. Then the price of any asset satisfies the relationship*

$$S_j = \frac{1}{1+i}\mathbb{E}^Q[V_j]$$

Exercise: prove this.

## Proposition 10

*Assume there are* ***no arbitrage opportunities*** *and there is a risk-free asset. Then* ***there is a (market) probability measure*** *such that the expected returns of every asset is equal to the return of the risk-free asset.*

- Proof. From Proposition 8 we have

$$1 + i = \mathbb{E}^Q[R_j], \text{ for any, } j = 1, \ldots, K$$

As this holds for any asset, therefore $\pi^Q$ has the property

$$1 + i = \mathbb{E}^Q[R_1] = \ldots = \mathbb{E}^Q[R_K]$$

# Application

## Arbitrage and completeness with two assets

- Assume that $N = 2$ and there is a risky and a risk-free asset

$$\mathbf{S} = \begin{pmatrix} \frac{1}{1+i} & p \end{pmatrix}, \ \mathbf{V} = \begin{pmatrix} 1 & d_1 \\ 1 & d_2 \end{pmatrix}$$

- and that $r_1 < i < r_2$.

  Then markets are complete and there are no arbitrage opportunities.

  The return matrix is

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\frac{1}{1+i}} & \frac{d_1}{p} \\ \frac{1}{\frac{1}{1+i}} & \frac{d_2}{p} \end{pmatrix} = \begin{pmatrix} 1+i & 1+r_1 \\ 1+i & 1+r_2 \end{pmatrix}$$

# Application

## Arbitrage asset pricing

- If there are risk-free assets and absence of arbitrage opportunities, then any asset with payoff $V_k$ can be priced by redundancy as

$$S_k = q_1 V_{k,1} + q_2 V_{k,2}$$

where

$$q_1 = \frac{i - r_2}{(1+i)(r_1 - r_2)},\ q_2 = \frac{r_1 - i}{(1+i)(r_1 - r_2)}.$$

- Proof: assume there is a risk-free and a risky asset such that $Q\mathbf{R} = \mathbf{1}^\top$ becomes

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1+i & 1+i \\ 1+r_1 & 1+r_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and if there are no arbitrage opportunities $r_1 < i < r_2$ implying $q_1 > 0$ and $q_2 > 0$.

# Application

## Simple Black and Scholes (1973) equation

- Consider an european call option with exercise price $p$ on an underlying asset with payoff

$$V^{\top}_{underlying} = (d_1, d_2)$$

- Then the contingent payoff of the option is

$$V^{\top}_{option} = (\max\{d_1 - p, 0\}, \max\{d_2 - p, 0\})$$

- Question: assuming there are no arbitrage opportunities and there is a risk-free asset what is the market price of the option ?
- Answer: the price for a call option with exercise price $p$ is

$$S_{option} = \frac{(r_2 - i)\max\{d_1 - p, 0\} + (r_1 - i)\max\{d_2 - p, 0\}}{(1+i)(r_1 - r_2)}$$

(prove this)

Equity premium

# Equity premium

## Definition 10

*We call* *equity premium* *to the difference in the rates of return between the risky and a risk-free asset*

$$r - i = \begin{pmatrix} r_1 - i \\ r_2 - i \end{pmatrix}$$

## Definition 11

*The* *Sharpe index* *is defined as*

$$\frac{\mathbb{E}[r-i]}{\sigma[r]} = \frac{\sum_{s=1}^{2} \pi_s (r_s - i)}{\sqrt{\sum_{s=1}^{2} \pi_s ((r_s - i) - \mathbb{E}[r-i])^2}}$$

Prove that $\sigma[r-i] = \sigma[r]$ where is the standard deviation of the

# Risk neutral probabilities

## Proposition 11

*If there are no arbitrage opportunities then there is a (market) risk neutral probability distribution such that the expected value of the equity risk premium is zero,*

$$E^Q[r - i] = 0$$

This is the reason why $\pi_s^Q$ are called risk neutral probabilities: $\mathbb{P}^Q = (\pi_1^Q, \dots \pi_N^Q)$ is a probability measure such that the expected value of the equity premium is zero.

# Risk neutral probabilities

- Proof: Let
$$\mathbf{R}^\top = \begin{pmatrix} 1+i & 1+i \\ 1+r_1 & 1+r_2 \end{pmatrix}$$

- Then, because $\mathbf{R}^\top Q^\top = \mathbf{1}$
$$\begin{cases} (1+i)q_1 + (1+i)q_2 = 1 \\ (1+r_1)q_1 + (1+r_2)q_2 = 1 \end{cases}$$

- Then $(1+r_1)q_1 + (1+r_2)q_2 = (1+i)q_1 + (1+i)q_2$.
- Therefore
$$q_1(r_1 - i) + q_2(r_2 - i) = 0$$

# Risk neutral probabilities

- Proof (cont)Define $\bar{q} = q_1 + q_2$

$$\frac{q_1}{\bar{q}}(r_1 - i) + \frac{q_1}{\bar{q}}(r_2 - i) = 0$$

- Define again $\pi_s^Q = \frac{q_s}{\bar{q}}$. If there are no arbitrage opportunities, then $q_s > 0$ and $\pi_s^Q > 0$. Because $\sum_{s=1}^{2} \pi_s^Q = 1$ then $\pi_s^Q$ are probabilities.
- At last

$$\pi_1^Q(r_1 - i) + \pi_2^Q(r_2 - i) = 0$$

# The Sharpe index and the SDF

## Proposition 12

*Assume there are no arbitrage opportunities and there is a risk-free asset. Then the Sharpe index satisfies the relationship*

$$\frac{\mathbb{E}[r-i]}{\sigma[r]} = -\rho_{r-i,m}\left(\frac{\mathbb{E}[M]}{\sigma[M]}\right)^{-1}$$

*where $\rho_{r-i,m}$ is the coefficient of correlation between the equity premium and the stochastic discount factor.*

# The Sharpe index and the SDF

- Proof: Using our previous definition of the stochastic discount factor $m_s = q_s/\pi_s$ then

$$\mathbb{E}[M(r-i)] = 0$$

- But $\mathbb{E}[M(r-i)] = Cov[M(r-i)] + \mathbb{E}[M]\,\mathbb{E}[r-i]$
- Using the definition of correlation

$$\rho_{r-i,m} = \frac{Cov[M(r-i)]}{\sigma[M]\,\sigma[r-i]} \in (-1,1)$$

- Then $\rho_{r-i,m}\,\sigma[M]\,\sigma[r-i] + \mathbb{E}[M]\,\mathbb{E}[r-i] = 0$

# The equity premium and DGE models

- **Intuition**: the Sharpe index is equal symmetric to the product between coefficient of variation of the stochastic discount factor and correlation coefficient between the equity premium and the stochastic discount factor, $\rho_{r-i,m}$
- The coefficient of variation of the stochastic discount factor contains the aggregate market value of risk.
- $\mathbb{E}[M]/\sigma[M]$ can be derived from simple DSGE models (as we will see next)
- Observe that the Sharpe index satisfies

$$\frac{\mathbb{E}[R - R^f]}{\sigma[R]}$$

where $R^f = 1 + i$ is the risk-free return and $R$ can be taken as a return on a market index. (see
`http://web.stanford.edu/~wfsharpe/art/sr/SR.htm`