AME 2018-2019:
Problem set 5: Stochastic differential equations

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1 General

1. The diffusion equation is

$$dX(t) = \gamma X(t)dt + \sigma X(t)dW(t)$$

for $X(0) = x_0$.

(a) Prove that the solution is $X(t) = x_0 e^{(\gamma-\sigma^2/2)t+\sigma W(t)}$

(b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

(c) Derive the backward Kolmogorov equation for the probability for $X(T) \leq 2x$ assuming that $X(t) = x$

(d) Derive the forward Kolmogorov equation for the density associated to $X(t) = x > 0$, assuming that $X(0) = 0$.

2. Consider the diffusion equation

$$dX(t) = \gamma X(t)dt + \sigma dW(t)$$

where $\{W(t)\}$ is a standard Brownian motion.

(a) Let $X(0) = x_0$ be known. Find the solution of the initial value problem.

(b) Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

(c) Derive the forward Kolmogorov equation for the density associated to $X(t) = x > 0$, assuming that $X(0) = 0$.

3. The **vasicek1977** (or Ornstein-Uhlenbeck) process is the solution of the equation

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t)$$

for $X(0) = x_0$. Prove that the solution is

$$X(t) = \mu + (x_0 - \mu)e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)}dW(s)$$

Find $\mathbb{E}[X(t)|X(0) = x_0]$ and $\mathbb{V}[X(t)|X(0) = x_0]$.

2 Economic applications

1. In the stochastic Solow model assume that the population is deterministic and the production function is $Y(t) = A(t)F(K(t), L(t))$ where productivity follows the process
$$dA(t) = \gamma A(t)dt + \sigma A(t)dW(t).$$
Determine the capital accumulation equation. Assuming a Cobb-Douglas equation find the asymptotic distribution of the capital stock.

2. Assume an AK model where $Y = A(t)K(t)$ where
$$dA(t) = \gamma dt + \sigma dW(t).$$
Assuming an equilibrium equation $dK(t) = sY(t)dt$, and $K(0) = K_0$ given, find an explicit solution for the capital stock. Determine the moment for the process of $K(t)$.

3. Solve the stochastic problem for a representative consumer assuming a log utility function.

4. Solve the stochastic Ramsey model assuming a log utility function.