Consumption, savings and asset accumulation

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1 Introduction

To be completed

2 Intertemporaly additive utility functional

In this section we provide some definitions regarding the intertemporal preference properties which are implicit in intertemporal utility functionals. We start with the case of additive utility functionals both in discrete and continuous time. In section 5 we study, with the same concepts, a non-additive intertemporal utility functional arising in the habit formation model.

2.1 Discrete time

Let measurements be taken at discrete time intervals $\mathrm{T} = \{0, 1, \ldots, t, \ldots, T\}$. In this discrete time setting, the sequence of consumption $\{c\} = \{c_0, c_1, \ldots, c_t, \ldots, c_T\}$ is valued by the utility functional

$$\mathsf{U}[c] = \mathsf{U}[\{c_0, c_1, \ldots, c_t, \ldots, c_T\}]$$

We can determine the marginal utility for a change consumption at any time t as a simple derivative

$$\mathsf{U}_t \equiv \frac{\partial \mathsf{U}[c]}{\partial c_t}, \text{ for any } t \in \mathrm{T}.$$

The own or crossed changes in the marginal utility are given by the second derivatives

$$U_{t,t'} \equiv \frac{\partial^2 \mathsf{U}[c]}{\partial c_t \, \partial c_{t'}} \, c_{t'}, \text{for any}\, t, t' \in \mathrm{T}.$$

The **intertemporal marginal rate of substitution**, between periods t_0 and $t_1 = t_0 + \tau$, is defined by the reduction in consumption at $t + \tau$ for one Unit increase in consumption at t such that it leaves intertemporal utility constant. Formally

$$IMRS_{t_0,t_1} = -\frac{dc_{t_1}}{dc_{t_0}}\Big|_{\mathsf{U}=\text{constant}}.$$

We consider the following perturbation: given the initial sequence $c = \{c_0, c_1, \dots, c_{t_0}, \dots, c_{t_0+\tau}, \dots, c_T\}$, with value $\mathsf{U}[c]$ we want to modify it to $c + dc = \{c_0, c_1, \dots, c_{t_0} + dc_{t_0}, \dots, c_{t_0+\tau} + dc_{t_0+\tau}, \dots, c_T\}$, which has value $\mathsf{U}[c + dc]$, such that $\mathsf{U}[c] = \mathsf{U}[c + dc]$. Making a first order approximation we $\mathsf{U}[c + dc] = \mathsf{U}[c] + d\mathsf{U}[c]$, we require

$$d\mathsf{U}[c] = \mathsf{U}_{t_0}\, dc_{t_0} + \mathsf{U}_{t_1}\, dc_{t_1} = 0.$$

Therefore

$$IMRS_{t_0,t_1}(c) = \frac{\mathsf{U}_{t_0}}{\mathsf{U}_{t_1}}$$

For a stationary consumption path, where $c_t = \bar{c}$, for any t we say the utility functional displays **impatience** if $IMRS_{t_0,t_1}(\bar{c}) > 1$. This means that the sacrifice of future consumption, to preserve value, should be higher than the present increase.

The Uzawa-Allen elasticity of intertemporal substitution, between consumption in period t_0 and t_1 is defined as

$$\varepsilon_{t_0,t_1} = -\frac{\mathsf{U}_{t_0,t_1}\, c_{t_0}}{U_{t_0}}, \text{for any } t_0, t_1 \in \mathrm{T}.$$

Again, taking a stationary consumption path we say the utility functional displays **intertemporal complementarity**, **intertemporal independence** or **intertemporal substitution** if $\varepsilon_{t_0,t_1}(\bar{c})$ is negative, zero, or positive, respectively.[1]

In order to measure intertemporal substitution the definition of **intertemporal elasticity of substitution** is introduced

$$EIS_{t_0,t_1} = \frac{\partial\big(c_{t_1}/c_{t_0}\big)}{\partial IMRS_{t_0,t_1}}\, \frac{IMRS_{t_0,t_1}}{c_{t_1}/c_{t_0}}.$$

which measures the elasticity of the consumption ratio as regards the elasticity of the $IMRS$. Expanding the definition, we have the equivalent formula

$$EIS_{t_0,t_1} = \frac{c_{t_0}\, \mathsf{U}_{t_0} + c_{t_1}\, \mathsf{U}_{t_1}}{c_{t_0}\, \mathsf{U}_{t_0}\, \varepsilon_{t_1,t_1} - 2\, c_{t_0}\, \mathsf{U}_{t_0}\, \varepsilon_{t_0,t_1} + c_{t_1}\, \mathsf{U}_{t_1}\, \varepsilon_{t_0,t_0}}.$$

[1] This is complementarity in the Edgeworth-Pareto, or Uncompensated, sense not in the Hicks-Allen, or compensated, sense. The issue of complementarity and substitutability should be dealt with care. This is discussed in the next subsection.

The most common utility functional, allowing to call it the **benchmark utility functional**

$$\mathsf{U}[c] = \sum_{t=0}^{T} \beta^t u(c_t) \tag{1}$$

where $\beta \equiv \dfrac{1}{1+\rho}$ is the psychological discount factor and ρ is the rate of time preference. It means that the value of a consumption sequence c is equal to the present value of the sequence of period Utilities of consumption. It is therefore linear on the Utilities of consumption for every period.

Exercise: Prove that the utility functional (1) displays impatience, intertemporal independence, and, if we define the elasticity of marginal substitution by

$$\sigma(c) = -\frac{u''(c)\, c}{u'(c)}$$

the $EIS_{t_0,t_1}(\bar{c}) = \dfrac{1}{\sigma(\bar{c})}$

2.1.1 Continuous time

Let time be a real non-negative variable, $t \in \mathrm{T} \subseteq \mathbb{R}_+$. Consumption at time t is $c(t)$ and the flow of consumption is $c = \big(c(t)\big)_{t\in\mathrm{T}}$.

Let Us consider first the benchmark **intertemporal utility functional**

$$\mathsf{U}[c] = \int_0^T u(c(t))e^{-\rho t}dt,\ \rho > 0 \tag{2}$$

where $e^{-\rho t}$ is the psychological discount factor and ρ is the rate of time preference, where the **utility function** $u(\cdot)$ is assumed to be continuous, differentiable (at least second-order differentiable), increasing and concave.

A crucial feature of the allocation of consumption through time is the intertemporal substitutability (or complementarity) of consumption. In a continuous time setting the utility functional $\mathsf{U}[c]$ is infinite-dimensional and consumption substitutability is related to the change in consumption in two periods of time, say t and $t+\tau$, by $dc(t_0)$ and $dc(t_0+\tau)$ such that the utility functional $\mathsf{U}[c]$ remains constant. That is we want to compare the flow $\big(c(t)\big)_{t\in\mathrm{T}}$ with the flow $\big(\hat{c}(t)\big)_{t\in\mathrm{T}}$ such that $\hat{c}(t) = c(t)$ if $t \neq \{t_0, t_0+\tau\}$ and $\hat{c}(t_0) = c(t_0) + dc(t_0)$ and $\hat{c}(t_0+\tau) = c(t_0+\tau) + dc(t_0+\tau)$.

There are two difficulties in dealing with intertemporal substitution/complementarity in continuous time. The first is economic. The concept of complementarity and substitutability are not Unequivocal. In a static microeconomic setting let preferences among different consumption bundles $\mathbf{c} = (c_1, \dots c_n)$ be represented by the utility function $U(\mathbf{c})$, and the consumer's problem be $\max_{\mathbf{c}}\{U(\mathbf{c}) : \mathbf{p} \cdot \mathbf{c} \leq y\}$, where $\mathbf{p}$ is the related price vector and y is income. In this setting we say two goods, indexed for instance by i and j, are substitutability in the Edgeworth-Pareto sense if $\dfrac{\partial^2 U(\mathbf{c})}{\partial c_i \, c_j} < 0$ and they are substitutable in the Hicks-Allen sense if, at the optimum demand function is $\mathbf{c} = C(\mathbf{p}, y)$ and $\dfrac{\partial c_i}{\partial p_j} > 0$. The first type can be called Uncompensated substitutability and the second compensated substitutability because it takes into account a normalizing effect of the budget constraint.

Given the difficulty of defining a an extension to the Hicksian sense of substitutability in continuous time [2] , the definition we present next is an extension to the Edgeworth-Pareto concept[3]

The second difficulty is mathematical. Although continuous time makes analytical derivations of results easier, one has to surmount some mathematical related to the fact that in the definition of the intertemporal marginal rate of substitution between times t_0 and $t_0 + \tau$, because the time variations become infinitesimal, i.e., variations with measure zero.

As regards the utility functional (2), we can use the concept of a Gâteaux derivative for a "spyke" variation of consumption at time t, introduced in the appendix A. Using, equation (13), we obtain the e marginal utility of consumption at any time t is given by

$$\mathsf{U}_t \equiv \delta \mathsf{U}[c; t] = u'\big(c(t)\big)\, e^{-\rho t}, \text{ for any } t \in \mathrm{T}.$$

Therefore, the intertemporal marginal rate of substitution between consumption at time t_0 and $t_1 = t_0 + \tau$, for $\tau \geq 0$ is

$$\mathrm{IMRS}_{t_0, t_1} = \frac{\mathsf{U}_{t_0}}{\mathsf{U}_{t_1}} = \frac{u'\big(c(t_0)\big)}{u'\big(c(t_1)\big)} e^{\rho\, \tau}.$$

Using the second order Gâteaux derivatives for a "spyke" variation at the same time and at a different time, we obtain

$$\mathsf{U}_{t_i t_i} \equiv \delta^2 \mathsf{U}[c; t_i, t_i] = u''\big(c(t_i)\big)\, e^{-\rho t_i}, \text{ for any } t \in \mathrm{T}.$$

[2]For a discussion see Biswas, 1976, and Ryder and Heal, 1973 and Heal and Ryder, 1976.

[3]There are issues here that are more profound than they look at first. There is a long tradition between ordinalists and cardinalists related comparative statics. The most recent approach to microeconomics, the monotone comparative statics approach, initiated by Milgrom and Shannon, 1994, associates substitutability defined in an ordinal way with the property of supermodularity. If a utility function is continuous and differentiable, the Edgeworth-Pareto substitutability criteriUm is equivalent to supermodularity.

and $\mathsf{U}_{t_i t_j} \equiv \delta^2 \mathsf{U}[c; t_i, t_j] = 0$ for $t_i \neq t_j$ both in T, which implies that the Uzawa-Allen elasticities [4] are

$$\epsilon_{t_i,t_i} = \sigma\big(c(t_i)\big) \equiv -\frac{u''\big(c(t_i)\big)\, c(t_i)}{u'\big(c(t_i)\big)}$$

where $\sigma\big(c(t_i)\big)$ is the elasticity of the marginal utility $u'(\cdot)$, and $\epsilon_{t_i,t_j} = 0$ if $t_i \neq t_j$.

Therefore the elasticity of intertemporal substitution[5]

$$\begin{aligned} IES_{t_0,t_1} &= \frac{d \log\big(c(t_0)/c(t_1)\big)}{d \log \mathrm{IMRS}_{t_0,t_1}} = \\ &= \frac{c_{t_0}\, \mathsf{U}_{t_0} + c_{t_1}\, \mathsf{U}_{t_1}}{c_{t_0}\, \mathsf{U}_{t_0}\, \varepsilon_{t_1,t_1} - 2\, c_{t_0}\, \mathsf{U}_{t_0}\, \varepsilon_{t_0,t_1} + c_{t_1}\, \mathsf{U}_{t_1}\, \varepsilon_{t_0,t_0}}. \end{aligned} \tag{3}$$

We now have

$$IES_{t_0,t_1} = \frac{c(t_0)\, u'(c(t_0))\, e^{-\rho t_0} + c(t_1)\, u'(c(t_1))\, e^{-\rho t_1}}{c(t_1)\, u'(c(t_1))\, e^{-\rho t_1}\, \sigma(c(t_0)) + c(t_1)\, u'(c(t_0))\, e^{-\rho t_0}\, \sigma(c(t_1))}.$$

Taking a stationary flow of consumption, such that $c(t) = \bar{c}$ for any $t \in$ T, we obtain $IMRS_{t_0,t_1} = e^{\rho\,\tau} > 1$, $\epsilon_{t_0,t_1} = 0$ and $IES_{t_0,t_1} = \dfrac{1}{\sigma(\bar{c})}$, which we interpret as establishing that it displays **impatience**, **intertemporal independence** and an **intertemporal elasticity of substitution which is the inverse of the elasticity of the marginal utility** .

For the case in which the utility function is isoelastic (indeed a generalized logarithm),

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta},$$

we obtain $IES_{t_0,t_1} = \dfrac{1}{\theta}$, which led to the literature calling θ the **inverse of the elasticity of intertemporal substitution**.

[4] They are deined as

$$\epsilon_{t_i,t_j} = -\frac{\mathsf{U}_{t_i t_j}\, c(t_i)}{\mathsf{U}_{t_i}} \equiv -\frac{\delta^2 \mathsf{U}[c; t_i, t_j]\, c(t_i)}{\delta \mathsf{U}[c; t_i]} \text{ for any } t_i, t_j \in \mathrm{T}.$$

[5] Expanding the expression we obtain

$$\frac{d \log\big(c(t_i)/c(t_j)\big)}{d \log \mathrm{IMRS}_{t_i,t_j}} = \frac{c(t_i)\, \mathsf{U}_{t_i} + c(t_j)\, \mathsf{U}_{t_j}}{c(t_j)\, \mathsf{U}_{t_j}\, \mathsf{U}_{t_i t_i} - 2\, c(t_i)\, \mathsf{U}_{t_i}\, \mathsf{U}_{t_i t_j} + c(t_i)\, \mathsf{U}_{t_i}\, \mathsf{U}_{t_j t_j}}.$$

3 The consumer problem in the continuous time setting

In this section we apply the previous concepts, in particular the continuous-time definition of marginal utility to the consumer problem with several types of constraints. We assume a deterministic setting in which the consumer has perfect information.

We start by dealing with the meaning of maximizing utility in an unconstrained problem in subsection 3.1. Next we assume there is an initial (non-renewable) resource which can be consumed throughout time in subsection 3.2: a lazy rentier. Next we consider the problem for a rentier which is also an investor in subsection 3.3. Section 3.4 consider the case in which the consumer has non-financial income.

Subsection 3.5 justifies and deals with the infinite-horizon case.

3.1 Maximizing utility without constraints

We address the first problem: finding the maximum consumption path in a Unconstrained setting.

The problem is

$$\mathsf{U}[c^*] = \max_{c} \int_0^T u(c(t))\, e^{-\rho t} dt \tag{P1}$$

where we denote $c^* = \big(c^*(t)\big)_{t \in \mathrm{T}}$ the maximum utility consumption path.

If c^* is optimum, then any arbitrary perturbation φ, such that $\varphi(0) = \varphi(T) = 0$, will not change intertemporal utility, that is $\mathsf{U}[c^* + \varphi] = \mathsf{U}[c^*]$. Therefore, at the optimum

$$\int_0^T u'(c^*(t))\, e^{-\rho t}\, \varphi(t)\, dt = 0$$

This is a linear functional which holds [6] only if

$$u'(c^*(t))\, e^{-\rho t} = 0, \text{ for every } t \in [0, T] \tag{P1:foc}$$

If the utility function has the Inada property, i.e, if $\lim_{c \to 0} u'(c) = \infty$, because $e^{-\rho t} \in (0, 1)$, then the optimum will be reached for $c^*(t) = \infty$ for any t.

If the utility function does not have the Inada property, then the optimum can be reached for a finite level of consumption.

Exercise: Prove that this may not be the case for utility functions displaying satiation. In this case $c(t) = \bar{c}$ where $\bar{c}$ is the satiation point.

[6] See Gel'fand and Fomin, 1963, p.9

Exercise: Prove that the utility function of the isoelastic type $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta}$, for $\theta > 0$ has the Inada property, and therefore the optimum consumption is $c^*(t) = \infty$ for any t.

The message from this is clear: if there are no constrains on consumption and the utility function displays no satiation, then optimal consumption would be infinite for every point in time, as in Cockaigne.

3.2 Maximizing utility for a rentier

Now assume the problem of a lazy rentier: it has an initial stock of net wealth, $a_0 > 0$ which is used to finance the purchase of consumption goods throughout lifetime. Assume prices and equal always equal to one.

Taking, again, time moments t and $t+\tau$. Given that the rentier still has $a(t)$ at time t, that it consumes a constant quantity $c(t)$ in any period of length τ, what will be the stock of wealth remaining at time $t+\tau$? It is easy to see that $a(t+\tau) = a(t) - c(t)\,\tau$. If the time interval shrinks to zero, the infinitesimal change in his wealth is

$$\lim_{t \to 0} \frac{a(t+\tau) - a(t)}{\tau} = \dot{a}(t) = -c(t).$$

Does the optimal consumption path changes as regards problem (P1), i.e., does existence of an initial finite resource constraints ? The answer is, not necessarily, because depends on the assumptions we made regarding the domain of a.

First, we consider a problem in which the initial value is finite, positive and known, $a(0) = a_0 > 0$ Solving the budget constraint $\dot{a}(t) = -c(t)$ we know that

$$a(t) = a_0 - \int_0^t c(s)\,ds.$$

We introduce another assumption: the future net wealth level a can take any real value. This means that it is possible that $-\infty < a(t) < \infty$ for any $t > 0$, that is the agent can borrow and a represents the stock's net position.

The problem of the consumer becomes

$$\begin{aligned}
&\max_{c} \int_0^T u(c(t))\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = -c(t), \text{ for } t \in \mathrm{T} \\
&a(0) = a_0 > 0 \text{ given} \\
&a(t) \in \mathbb{R} \cup \infty
\end{aligned} \tag{P2}$$

We show in the appendix B that the optimality condition for this problem is the same as the solution of problem (P1), in equation (P1:foc):

$$u'(c^*(t))\, e^{-\rho t} = 0, \text{ for every}\, t \in [0, T]. \tag{P2:foc}$$

Again, if the utility has the Inada property (v.g, if it has a satiation point) then the consumer will want to consume an infinite amount at every point in time. This means that a becomes negative and Unbounded just immediately after $t = 0$ and the initial constraint of the resource is not active.

A second, more realistic case is the one in which the consumer faces a lower bound on a, that we denote by $\underline{\mathrm{a}}$. If he cannot borrow then $\underline{\mathrm{a}} \geq 0$ or if he faces a borrowing constraint $\underline{\mathrm{a}} < 0$ but finite. This is a source in which financial frictions can be introduced in the model. We assume this constraint is active in every moment and the consumer has an initial stock of goods which is positive.

The problem is now:

$$\begin{aligned}
&\max_{c} \int_0^T u(c(t))\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = -c(t), \text{ for } t \in \mathrm{T} \\
&a(t) \in [\underline{\mathrm{a}}, \infty), \text{ for every } t \in [0, T] \\
&a(0) = a_0 > \max\{0, \underline{\mathrm{a}}\} \text{ given}
\end{aligned} \tag{P3}$$

Let us assume that the utility function $u(c)$ has the Inada property: $u''(c) < 0 < u'(c)$ and $\lim_{c \to 0} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$. In order to make the analysis clear let us assume that the utility function is $u(c) = \dfrac{c^{1-\theta} - 1}{1 - \theta}$, for $\theta > 0$. This yields $u'(c) = c^{-\theta}$

Two important observations are raised by introduction of the borrowing constraint: first, we assume that initial level of wealth is above the lower limit for a, $\underline{a}$ (which can be positive or negative); second, as the utility function induces a unconstrained consumer to consume an infinite amount every point in time, the borrowing constraint will "bite" at some point in time, say $\underline{t} \in (0, T]$. The time of activation of the constraint will be higher than zero because the initial level of wealth is by assumption bigger than the lower limit, $a_0 > \underline{a}$ but it is **endogenous**. The raises a natural question: is it optimal to hit the constraint earlier or at the horizon T ?

The optimality conditions are

$$\begin{cases} u'(c(t))e^{-\rho t} = \lambda(t) \\ \dot{a} = -c(t) \\ \dot{\lambda} = -\eta(t) \\ \eta(t)(a(t) - \underline{a}) = 0, \quad \eta(t) \geq 0,\ a(t) \geq \underline{a} \text{ for } t \in (0, T] \\ \lambda(T^-) = \eta(T). \end{cases}$$

Therefore, we can divide the solution in two periods: before the constraint hits, for $t \in [0, \underline{t})$ we have $\eta = 0$, which implies $\lambda(t) = \lambda_0$ is a constant, and Therefore

That is for

$$c(t) = \begin{cases} \lambda_0^{-\frac{1}{\theta}} e^{\gamma_c t} & \text{for } t \in [0, \underline{t}) \\ 0 & \text{for } t \in (\underline{t}, T] \end{cases}$$

for $\gamma_c = -\dfrac{\rho}{\theta} < 0$ and

$$a(t) = \begin{cases} a_0 + \lambda_0^{-\frac{1}{\theta}} \left(1 - e^{\gamma_c t}\right) & \text{for } t \in [0, \underline{t} \\ \underline{a} & \text{for } t \in (\underline{t}, T] \end{cases}$$

At time $\underline{t}$ we have $c(\bar{t}^+) = c(\bar{t}^-) = 0$ which is only possible if $\lambda = \infty$ which implies $a(\underline{t}) = \underline{a} = a_0$, which is not possible for $\bar{t} < T$.

If we set, $\bar{t} = T$ from $\underline{a} = a_0 + \lambda_0^{-\frac{1}{\theta}} \left(1 - e^{\gamma_c T}\right)$ we can determine λ_0 and obtain the optimum consumption as

$$c^*(t) = \gamma_c \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}} e^{\gamma_c t}\ t \in [0, T)$$

at $c(T^-) = 0$, there is a discontinuous jump in consumption, and the net optimum wealth stock is

$$a^*(t) = a_0 + \gamma_c \frac{\underline{a} - a_0}{1 - e^{\gamma_c T}} \left(1 - e^{\gamma_c t}\right) \text{ for } t \in [0, T].$$

Therefore, because the consumer attributes an unlimited utility to consumption when it is close to zero, if the utility function has the Inada properties, if there is a financial constraint, and perfect foresight, it will stay away from it before the problem's horizon t.

Exercise If the utility function displays satiation what will be the solution to the problem ?

3.3 Maximizing utility for an investor

Next we assume that the household can extract a financial income $r\,a$ from the ownership of the asset. However, if the household is a net debtor it would represent an extra expenditure, instead. To simplify, we assume that the interest rate r is constant.

Again we assume that the utility function has the Inada property and that, from the results of the last subsection, there is a terminal constraint on the level of net wealth.

Assuming an isoelastic utility function, the problem is

$$
\begin{aligned}
&\max_{c} \int_0^T \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\,a - c(t), \text{ for } t \in \mathrm{T} \\
&a(0) = a_0 > \text{\underline{a} given} \\
&a(T) \geq \underline{a}
\end{aligned}
\tag{P4}
$$

In appendix D we prove that the solution is

$$
c^*(t) = \Big(\frac{(r-\gamma_c)\big(a_0 - \underline{a}\, e^{-r\,T}\big)}{1-e^{(\gamma_c - r)T}}\Big)\, e^{\gamma_c t} \tag{4}
$$

and

$$
a^*(t) = e^{r\,t} a_0 - \frac{a_0 - \underline{a}\, e^{-r\,T}}{1-e^{(\gamma_c - r)T}}\, \big(e^{r\,t} - e^{\gamma_c\, t}\big). \tag{5}
$$

The following observations can be made: first, as the sign of γ_c is the same as if $r-\rho$ then if $r>\rho$ consumption will grow across the lifetime, it will remain constant if $r=\rho$ and will diminish if $r<\rho$. Second, for any value of the parameters, $a(0)=a_0$ and $a(T)=\underline{a}$, which means that the dynamics of the consumption of saving more or less in the beginning of the period rather than to the end will depend on the relative value of the parameters, ρ and θ and on the market interest rate.

3.4 Maximizing utility with non-financial income

In this subsection we assume that the household is entitled to a non-financial stream of income $\big(w(t)\big)_{t\in \mathrm{T}}$. It can be labor income or any other type of non-financial income. In the case of labor income, we assume that the household supplies inelastically a constant flow of hours worked normalized to one.

We consider now the following problem

$$\begin{aligned}
&\max_{c} \int_0^T \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\,a + w(t) - c(t), \text{ for } t \in \mathrm{T} \\
&a(0) = a_0 > \underline{a} \text{ given} \\
&a(T) \geq \underline{a}
\end{aligned} \tag{P5}$$

Using the same approach as in the previous problem, we can prove that the solution is

$$c^*(t) = \Big(\frac{(r-\gamma_c)\big(a_0 + h(0) - \underline{a}\, e^{-r\,T}\big)}{1-e^{(\gamma_c - r)T}}\Big)\, e^{\gamma_c t} \tag{6}$$

and

$$a^*(t) = e^{r\,t} a_0 + h(0) - \frac{a_0 + h(0) - \underline{a}\, e^{-r\,T}}{1-e^{(\gamma_c - r)T}} \left(e^{r\,t} - e^{\gamma_c\,t}\right). \tag{7}$$

where

$$h(0) = \int_0^T e^{-r\,t}\, w(t)\, dt$$

is the human capital of the household at time $t = 0$. Comparing with the previous problem, we see that while in the previous problem, consumption and savings essentially led net wealth to go from a_0 to $\underline{a}$, in this case, the initial total wealth is $a_0 + h(0)$ and the terminal wealth is still $\underline{a}$. This means that the present flow of income in this problem is much higher than in the former problem.

Furthermore, if human wealth can be seen as a collateral for net borrowing, it is possible that the lower limit on net financial wealth $\underline{a}$ might be reduced by the existence of human wealth which could be offered as a collateral.

3.5 Infinite horizons

We saw that the determination of the initial level of consumption $c(0)$, and therefore, the level of consumption is dependent on the horizon of the problem and on the terminal constraints on wealth. In this section we present the infinite-horizon case.

There are two main justifications for considering the inifinite horizon case: first, the existence of incomplete information on the terminal time, and second, the existence of some consideration for the household beyond the life of those living at time $t = 0$. We call the second case the dynastic model.

The uncertain horizon In order to study this case we assume that the lifetime T is stochastic. Let the cumulative distribution of lifetime be $F(T) = \int_0^T f(t)dt \in (0,1)$ for $T \in (0,\infty)$ and the density function $f(t)$ follows a Poisson process, with the instantaneous probability of death μ, $f(t) = \mu\, e^{-\mu t}$. Then $F(T) = 1 - e^{-\mu T}$. Off course $F^{'}(t) = f(t)$, $F(0) = 0$ and $F(\infty) = 1$.

Let $\mathsf{U}(T) = \int_0^T u(c(t))\, e^{-\rho t}\, dt$ and the intertemporal utility with stochastic lifetime

$$\mathsf{U}[c] = \int_0^\infty f(T)\, \mathsf{U}(T)\, dT = \int_0^\infty f(T) \int_0^T u(c(t))\, e^{-\rho t}\, dt\, dT.$$

Using integration by parts, we find [7]

$$\begin{aligned}
\mathsf{U}[c] &= F(T)\, U(T)\Big|_{T=0}^{\infty} - \int_0^\infty f(T) e^{-\rho T} u(c(T))\, dT \quad \text{because}\, U^{'}(T) = e^{-\rho T} u(c(T)) \\
&= \int_0^\infty u(c(t))\, e^{-\rho t}\, dt - \int_0^\infty \left(1 - e^{-\mu T}\right) e^{-\rho T}\, u(c(T))\, dT \\
&= \int_0^\infty u(c(t))\, e^{-(\rho+\mu)t}\, dt
\end{aligned}$$

Therefore, properly interpreted Uncertain lifetime with a Poisson distribution of the time of death, is consistent with a utility functional for a single agent with an infinite horizon

$$\int_0^\infty u(c(t))\, e^{-(\rho+\mu)t}\, dt.$$

The dynastic model Another interpretation for an infinite horizon is the dynastic interpretation is mathematically equivalent. In this case the economic agent can be seen as an household which cares for the utility not only of the present but for all future generations. In this case we have

$$\int_0^\infty u(c(t))\, e^{-\rho t}\, dt$$

[7] Assuming that some mathematical requirements are satisfied, implying the boundedness of $u(c)$.

We consider now the following problem, in which the rate of time preference can include the mortality rate or not.

$$\begin{aligned}
&\max_{c} \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta}\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\, a + w(t) - c(t), \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&\lim_{t\to\infty} a(t)\, e^{-r\,t} \geq 0
\end{aligned} \tag{P6}$$

The terminal condition is called the non-Ponzi game condition, it means that in present value terms the household will not be a net debtor asymptotically.

Another interpretation is that the three constraints imply that the household consumption path is sustainable, or that it is solvent. In order to see this, solving the household budget constraint together with the initial condition, yields

$$a(t) = e^{r\,t}\Big(a_0 + \int_0^t e^{-r\,s}\, w(s) - c(s) ds\Big).$$

Multiplying both sides by the discount factor $e^{-r\,t}$ and taking the limit to infinity, yields

$$\lim_{t\to\infty} e^{-r\,t} a(t) = a_0 + h(0) - \int_0^\infty e^{-r\,t}\ c(t) dt$$

where we used the definition of human capital at time $t = 0$,

$$h(0) \equiv \int_0^\infty e^{-r\,t}\, w(t) dt.$$

If the transversality condition holds then the following intertemporal budget constraint holds,

$$\int_0^\infty e^{-r\,t}\ c(t) dt \leq a_0 + h(0)$$

which means that the present value of consumption is smaller than the initial total wealth which is equal to the sum of the financial and human wealth.

From now on, we assume that $r > 0$ and w is constant. Then $h(0) = \dfrac{w}{r}$.

By employing the methods already presented in the previous lecture[8] we can obtain explicitly the optimal policy function

$$c^* = C(a) = (r - \gamma_a)\Big(a + \frac{w}{r}\Big)$$

[8] See https://pmbbrito.github.io/cursos/phd/am/am2021_ramsey.pdf.

where

$$\gamma_a \equiv \frac{r-\rho}{\theta} < r$$

if $\theta \geq 1$. Optimal consumption is a positive function of total net wealth, financial and human, and the propensity to save is endogenously determined as a function of the difference $r - \gamma_a$. The last term is the rate of growth of the net financial wealth of the agent, because if we substitute optimal consumption in the budget constraint and solve the differential equation yields

$$a^*(t) = \bar{a} + (a_0 - \bar{a})\, e^{\gamma_a\, t}, \text{for}\, t \in [0, \infty)$$

where

$$\bar{a} = -h(0) = -\frac{w}{r}.$$

Therefore, the solution can be written as

$$a^*(t) = -h(0) + (a_0 + h(0))\, e^{\gamma_a\, t}, \text{for}\, t \in [0, \infty)$$

Therefore, the following dynamics are possible

1. if $r > \rho$ then $\gamma_a > 0$ and $\lim_{t\to\infty} a(t) = +\infty$ if the initial wealth is positive $(a_0 + h(0) > 0)$ it is equal to $-h(0)$ if $a_0 + h(0) = 0$ and it is $-\infty$ if the initial wealth is negative;

2. if $r = \rho$ then $\gamma_a = 0$ and the financial wealth is stationary $a(t) = a_0$ for any t;

3. if $r < \rho$ then $\gamma_a < 0$ and $\lim_{t\to\infty} a(t) = -h(0)$ the agent will be a net debtor.

What is the role of the parameter θ in the solution ? As we saw the higher θ is the lower is the EIS, which means that the cost of transfering consumption between moments in time, in utility terms, is higher. We see that although it does not affect the sign of γ_a, it reduces its absolute value. For any point in time it also increase the propensity to consume out of total net wealth, this means that savings will be smaller instantaneously and the transfer of consumption among periods will be smaller as well. Therefore, it tends to generate a smoother behavior of consumption.

The previous results contrast with those we obtained for the finite horizon case. In particular, it is not possible to have a net asset position to be unbounded. The reason for this is related to the fact that the terminal condition refers to the infinity, and given the fact that it involves discounting at a rate which is higher, in absolute value than the rate of growth of the wealth position, it allows for an unbounded evolution of net financial wealth. On the other hand, if the agent is more impatient than the market, i.e, if $\rho > r$, the solution for consumption looks similar to the problem for an agent which depletes a given stock of wealth (i.e, to problem (P2)).

Another fundamental aspect of introducing an infinite horizon is related to the existence of stationary solutions, i.e., solutions which do not change in time, as we will see in the next sections.

3.6 Conclusion

We discussed previously the effect of existence of satiation or no satiation in maximizing consumption utility, the effect of the constraints given by initial wealth or borrowing constraints, how constraints would have an effect on optimal solution when the consumer has no satiation, the justification and the effect of assuming an infinite horizon.

We found that when there are constraints, initial or otherwise, there is intertemporal substitution in consumption and the path of consumption is responsive to the elasticity of intertemporal substitution.

4 Comparative dynamics for the time additive model

In aggregate terms, in actual economies, one would expect that no agent would have an unbounded solution both as a net creditor or as a net debtor, because the individual agent interest rate would eventually be endogenous to the level of wealth.

This means that if we consider problem (P6) as representative of an aggregate economy the natural case to take is the second, that is the case in which $r = \rho$. This is indeed the simplest model for a small open economy.

Another reason for considering infinite horizons is related to the use of this models as providing explanations for the business cycle, i.e, deviations from a stationary trend. We can show that the infinite horizon version of the previous models produces stationary solutions, i.e. steady states and deviation from steady states.

In this section we study the comparative dynamics for a shock in the non-financial income w, in order to discuss the theory provided by this model on the relationship between consumption and income and as to have a comparison with the same type of results for the habit formation model we present in the next section 5.

There are two types of changes in income with a bearing on the solution of this model: anticipated and non-anticipated changes. Anticipated changes are already incorporated in the solution of the model. If at a certain point in time consumption is changing it is because it is not at a steady state. Non-anticipated changes involve time changes because they alter the steady state of the problem.

The optimality conditions for problem (P6) are

$$\dot{a} = \rho\, a + w - C(q) \tag{8a}$$

$$\dot{q} = 0 \tag{8b}$$

$$c(t) = C(q) \equiv q(t)^{-\frac{1}{\theta}} \tag{8c}$$

$$a(0) = a_0 \tag{8d}$$

From equation (8d) we can label financial wealth a as a pre-determined variable and the adjoint variable q, or consumption c, which is monotonously related to q by equation (8c), as a non-predetermined variable.

Equation (8b) implies that q is constant, which implies that c is constant as well, as we saw in the last section. Therefore, there are potentially an infinite number of steady states, from equation (8a), comprising all combinations of a and q that satisfy

$$c = r\, a + w$$

However, from the fact that a is pre-determined we can tie down the steady state which interests us to be

$$\bar{c} = C(\bar{q}) = r\, a_0 + w.$$

This steady state value for c is dependent on the value of non-financial income w, which lead us to write $\bar{c} = \bar{c}(w)$.

A comparative dynamics exercise asks the following question: given an initial value of w,say w_0 what is the effect on the solution to the problem if w_0 increases to $w_1 = w_0 + dw$, starting from a steady state ?

In the appendix E we prove that the multipliers are

$$dc(t) = dw, \text{for any } t \in (0, \infty)$$
$$da(t) = 0, \text{for any } t \in (0, \infty)$$

that is, consumption immediately and completely adjusts to innovations in income, which means that they are (counterfactually) perfectly correlated. There is no transitional dynamics.

A phase diagram is presented in Figure 1.

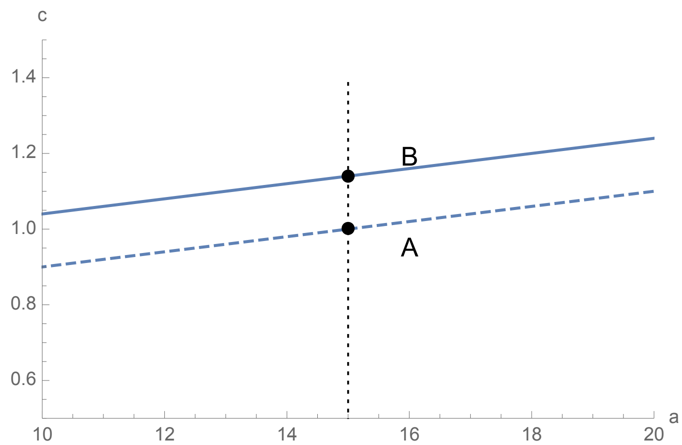

Figure 1: Effect of a non-anticipated increase in income. The line corresponds to the isocline for a, $c = w + \rho\, a$. A shock in income, w, moves the isocline up, leading the economy to jump from point A to point B.

5 Intertemporally dependent preferences

There are two types of models dealing with habit formation: the internal habit formation model and the external habit formation model. IN the first type of models consumer have an internal pattern of consumption that only changes marginally and in the second they follow an external pattern of consumption. While in the first the pattern of consumption, or habit, is built throughout time internally, in the second type of models it is an externality. This is why the second type of habits involve "going along with the Joneses' (see Abel, 1990). In this sense, classifying the two types of models within the same category can be misleading

Next we present the preferences under (internal) habit formation, in subsection 5.1 and in subsection 5.2 we extend the previous household consumption model with habit formation.

5.1 Preferences under habit formation

We just saw that if intertemporal utility functional is additive in the utility of consumption for different moments in time it displays intertemporally independent preferences, in the sense that the history of consumption Until time t, i.e., $c^t = \big(c(s)\big)_{s=0}^{t}$, does not influence the valuation of consumption at time t, by $u(c(t))$. As we will see this leads to large time shifts in consumption to innovations, which is counter-factual. There are ways to introduce intertemporally dependent preferences. Next we present the **habit formation** utility functional, in which past consumption affects the evaluation of consumption for every point in time. In this model, current consumption $c(t)$ has two effects: first, it is an immediate source of utility, and, second, it marginally changes the pattern of consumption, which we call habit. Habits record the past consumption history with some rate of decay η.

The following version is a simplified version of the model: [9]

$$\mathsf{U}[c] = \int_0^T u\big(c(t), h(t)\big)\, e^{-\rho\, t}\, dt$$

where $h(t)$ is the habits at time t, which are

$$h(t) = e^{-\eta\, t}\Big(h_0 + \eta \int_0^t e^{\eta\, s}\, c(s)\, ds\Big)$$

where $\eta > 0$ represents both the effect present consumption in the stock of habits and the rate of decay (or forgetting) of habits. This is a stock variable which can be seen as the solution to the problem

$$\begin{cases} \dot{h} = \eta\, (c(t) - h) & \text{for } t \in \mathrm{T} \\ h(0) = h_0 & \text{for } t = 0. \end{cases}$$

In this case the utility at time t is a function of the consumption and the habits, $u_t = u\big(c(t), h(t)\big)$, where we assume that the marginal utility of consumption is positive $u_c(c, h) \equiv \dfrac{\partial u(c, h)}{\partial c} > 0$ but the marginal utility of habits is negative $u_h(c, h) \equiv \dfrac{\partial u(c, h)}{\partial h} < 0$. We also assume that the utility function is continuous and smooth.

Using the same definition as before, we obtain the marginal utility for consumption at time t

$$\mathsf{U}_t \equiv \delta\mathsf{U}[c; t] = e^{-\rho t}\Big(u_c\big(c(t), h(t)\big) + \eta \int_t^{\bar{t}} e^{-(\eta+\rho)\,(s-t)} u_h\big(c(s), h(s)\big)\, ds\Big), \text{ for any } t \in \mathrm{T}.$$

where $\bar{t} = \max\ \mathrm{T}$. The intertemporal marginal rate of substitution between t_0 and $t_1 = t_0 + \tau$ is now

$$IMRS_{t_0,t_1} = \frac{e^{\rho\,\tau}\Big(u_c\big(c(t_0), h(t_0)\big) + \eta \int_{t_0}^{\bar{t}} e^{-(\eta+\rho)\,(s-t_0)} u_h\big(c(s), h(s)\big)\, ds\Big)}{u_c\big(c(t_1), h(t_1)\big) + \eta \int_{t_1}^{\bar{t}} e^{-(\eta+\rho)\,(s-t_1)} u_h\big(c(s), h(s)\big)\, ds}.$$

For a stationary consumption path, such that $c(t) = \bar{c}$ and $h(0) = \bar{c}$, implying $h(t) = \bar{c}$, for any $t \in \mathrm{T}$, we find [10]

$$\bar{\mathsf{U}}_t = \delta\mathsf{U}[\bar{c}; t] = e^{-\rho t}\, \bar{\mathsf{U}}'$$

[9] The first papers are Wan, 1970 and Ryder and Heal, 1973.

[10] Setting $h_0 = \bar{c}$, we find $h(t) = \bar{c}$ for any $t \in \mathrm{T}$ and therefore, we write $u_c(\bar{c}) = u_c(\bar{c}, \bar{c})$ and $u_h(\bar{c}) = u_h(\bar{c}, \bar{c})$.

where we define

$$\bar{\mathsf{U}}' \equiv u_c(\bar{c}) + \frac{\eta}{\eta+\rho} u_h(\bar{c})$$

and assume that $\bar{\mathsf{U}}_1 > 0$. For any other moment in time $t' \neq t$ we have the marginal utility $\bar{\mathsf{U}}_{t'} = e^{-\rho t'}\, \bar{\mathsf{U}}'$.

This implies that the $IMRS_{t_0,t_1} = e^{\rho\,\tau}$ is the same as in the additive model, which means that preferences with habit formation display impatience as well.

The change in marginal utility is now given by the second-order functional derivative

$$\begin{aligned}
\mathsf{U}_{t_i,t_j} &\equiv \delta^2 \mathsf{U}[c; t_i, t_j] = \\
&= \eta\, e^{-\rho t_i} \Big(\int_{t_i} e^{-(\eta+\rho)(t-t_i)}\, U_{hc}\big(c(t), h(t)\big)\, dt + \eta \int_{t_j} e^{-(\eta+\rho)(t-t_i)-\eta(t-t_j)}\, U_{hh}\big(c(t), h(t)\big)\, dt \Big)
\end{aligned}$$

and for a stationary consumption path, we find

$$\bar{\mathsf{U}}_{t_i,t_j} = \eta\, e^{-\rho\, t_j - \eta(t_j - t_i)}\, \bar{\mathsf{U}}''$$

where we define

$$\bar{\mathsf{U}}'' \equiv U_{hc}(\bar{c}) + \frac{\eta}{2\eta+\rho} U_{hh}(\bar{c}).$$

The Uzawa-Allen elasticities, associated to a stationary consumption path, depend only on the time difference between t and t'

$$\epsilon_{t,t'} = -\frac{\bar{\mathsf{U}}_{t,t'}\, \bar{c}}{\bar{\mathsf{U}}_t} = e^{-(\rho+\eta)\,(t'-t)} \sigma_h(\bar{c})$$

where we define

$$\sigma_h(\bar{c}) = -\frac{\eta\, \bar{\mathsf{U}}''\, \bar{c}}{\bar{\mathsf{U}}'}.$$

As $e^{-(\rho+\eta)\,(t'-t)}$ then we have intertemporal substitutability if $\bar{\mathsf{U}}'' < 0$, implying $\sigma_h(\bar{c}) > 0$, independence if $\bar{\mathsf{U}}'' = 0$, implying $\sigma_h(\bar{c}) = 0$ and intertemporal complementarity if $\bar{\mathsf{U}}'' > 0$, implying $\sigma_h(\bar{c}) < 0$.

As, in general $\sigma_h(\bar{c}) \neq 0$ then, using the definition of the intertemporal elasticity of substitution given in equation (3), yields

$$IES_{t_0,t_1} = \frac{1}{\sigma_h(\bar{c})} \Big(\frac{1 + e^{-\rho\tau}}{1 + e^{-\rho\tau} - 2\, e^{-(\rho+\eta)\tau}} \Big).$$

Comparing with the additive model, in this case the IES can have any sign depending on the elasticity $\sigma_h(\bar{c})$ and its magnitude depends in the lag between the two moments, τ. If there is intertemporal substitution we see that $\lim_{\tau\to 0} IES_{t,t+\tau} = +\infty$ and if $\lim_{\tau\to\infty} IES_{t,t+\tau} = \dfrac{1}{\sigma_h(\bar{c})}$, which means that for any $0 < \tau < \infty$ $IES_{t,t+\tau} > \dfrac{1}{\sigma_h(\bar{c})}$.

There are two benchmark utility functions displaying habit formation in the literature: the additive habits model (see Constantinides, 1990),

$$u(c,h) = v\big(c - \zeta\, h\big), \text{ for } \zeta > 0$$

and the multiplicative habits model (see Carroll, 2000)

$$u(c,h) = v\big(c\, h^{-\zeta}\big), \text{ for } 0 < \zeta < 1$$

where ζ measures the force of habits, i.e, the relative weight of habits as regards present consumption. In both models the utility of consumption is measured against the change of a proportion of habits.

If we assume multiplicative habits such that

$$u(c,h) = \frac{1}{1-\theta}\Big(\big(c\, h^{-\zeta}\big)^{1-\theta} - 1\Big) \tag{9}$$

and evaluate it at a stationary path such that $h(t) = c(t) = \bar{c}$, for any t, then we obtain

$$\bar{\mathsf{U}}' = \Big(\frac{\eta\,(1-\zeta)+\rho}{\eta+\rho}\Big)\,\bar{c}^{\zeta\,(\theta-1)-\theta} > 0$$

$$\bar{\mathsf{U}}'' = \zeta\Big(\frac{(\theta-1)\big(\rho+\eta\,(2-\zeta)\big)+\eta}{2\,\eta+\rho}\Big)\,\bar{c}^{\zeta\,(\theta-1)-\theta-1} > 0$$

if $\theta \geq 1$ and $0 < \zeta < 1$. Therefore,

$$\sigma_h(\bar{c}) = -\eta\,\zeta\,\frac{(\eta+\rho)\Big(\dfrac{(\theta-1)\big(\rho+\eta\,(2-\zeta)\big)+\eta}{2\,\eta+\rho}\Big)}{(2\,\eta+\rho)\,(\eta\,(1-\zeta)+\rho)} < 0$$

which means that this model displays **intertemporal complementarity**.

Exercise Prove this.

Exercise Find the IES of the additive habit formation model where

$$u(c,h) = \frac{1}{1-\theta}\Big(\big(c - \zeta\, h\big)^{1-\theta} - 1\Big)$$

and find its intertemporal dependence properties.

5.2 Partial equilibrium under habit formation

The extension of the household problem eq:problem6 with habit formation becomes:

$$
\begin{aligned}
&\max_{c} \int_0^{\infty} u(c(t), h(t))\, e^{-\rho t} dt \\
&\text{subject to} \\
&\dot{a}(t) = r\, a + w(t) - c(t), \text{ for } t \in \mathbb{R}_+ \\
&\dot{h}(t) = \eta\, \big(c - h\big) \text{ for } t \in \mathbb{R}_+ \\
&a(0) = a_0 \text{ given} \\
&h(0) = h_0 \text{ given} \\
&\lim_{t\to\infty} a(t)\, e^{-r\, t} \geq 0
\end{aligned}
\tag{P7}
$$

where, if we assume the multiplicative habit formation model, we have $u_c(c, h) > 0$, $u_h(c, h) < 0$, $u_{cc}(c, h) < 0$, $u_{ch}(c, h) = u_{hc}(c, h) > 0$ and $u_{hh}(c, h)$ has an ambiguous sign.

Exercise Compute the those derivatives for the general case and for the case in which $c = h$.

Observe that we now have two state variables, a and h and just one control variable c. This means that we have two initial conditions for both of them, $a(0) = a_0$ and $h(0) = h_0$.

Assume from now on that $r = \rho$. This implies, using the intuition from subsection 3.5 that the consumption path is bounded, which implies that the stock of habits is bounded as well.

The current-value Hamiltonian function is now

$$
\mathcal{H} = u(c, h) + q_a(\rho\, a + w - c) + q_h\, \eta\, (c - h)
$$

where q_a is the adjoint variable associated to the stock of net wealth a and q_h is the adjoint variable associated with the stock of habits h.

Using the Pontriyagin's maximum principle the optimality conditions for problem (P7) are

$$
\dot{a} = \rho\, a + w - c, \tag{10a}
$$

$$
\dot{h} = \eta\, (c - h), \tag{10b}
$$

$$
\dot{q}_a = 0, \tag{10c}
$$

$$
\dot{q}_h = (\rho + \eta)\, q_h - u_h(c, h), \tag{10d}
$$

$$
u_c(c(t), h(t)) = q_a(t) - \eta\, q_h(t), \tag{10e}
$$

$$
a(0) = a_0, h(0) \qquad = h_0 \tag{10f}
$$

and the transversality conditions.

As are interested in comparing the dynamic comparative statics properties of this model with the non-habit formation model we deal with the case in which the initial condition is a steady state, and introduce a perturbation in non-financial income from $w = w_0$ to $w = w_1 = w_0 + dw$.

As in that model equation (10c) implies that the steady state exists but there is potentially an infinite number of steady states. Furthermore, equation (10b), evaluated at the steady state yields $\bar{c} = \bar{h}$.

Anchoring again the steady state by the initial value of financial wealth a_0, a steady state only exists if the initial value of habits satisfies $\bar{h} = h_0 = \rho\, a_0 + w_0$, which we assume is the case from now on.

Therefore, the steady state, for $w = w_0$, is determined from the equations

$$\bar{a} = a_0 \tag{11a}$$

$$\bar{h} = h_0 = \rho\, a_0 + w_0 \tag{11b}$$

$$\bar{c}(w_0) = \rho\, a_0 + w_0 \tag{11c}$$

$$\bar{q}_h(w_0) = \frac{u_h(\bar{c}(w_0), h_0)}{\rho + \eta} \tag{11d}$$

$$\bar{q}_a(w_0) = u_c(\bar{c}(w_0), h_0) + \frac{\eta}{\rho + \eta}\, u_h(\bar{c}(w_0), h_0) \tag{11e}$$

This steady state, projected in the space (a, c) is shown by point A in Figure 2.

Exercise Find the steady state for the multiplicative habits model.

Exercise Find the steady state for the additive habits model.

It can be shown that if there is an increase in the wage rate by $dw > 0$ the (linearly approximate) dynamics that unfolds is the following: at the time of the shock consumption increases discontinuously from point A to point B; this introduces a change in the stock of net wealth but also a change in the stock of habits; however, as the stock of habits only changes slowly, the increase in wage is not completely used in the purchase of goods, which generates positive savings; changes in savings increases the stock of net wealth which increases further consumption, the stock of habits and savings.

A new steady state, depicted as point C will only be reached when we have again $\bar{c}(w_1) = \bar{h}(w_1) = \bar{a}(w_1)$. The stability mechanism is brought about by the fact that there is a decay mechanism in habit formation (in equation $\dot{h} = \eta\,(c - h)$) and the degenerate nature of the model is solved not by anchoring the solution to a_0 as in the non-habit formation model, but by a steady state relationship between the stock of habits and net wealth by $\bar{h} = \rho\,\bar{a} + w$.

The lower diagram in Figure 2 show the (approximate) trajectories for income $y(t) = \rho a(t) + w$ and consumption after the shock in non-financial income. As can be seen their behavior is positively correlated, but consumption has a slower adjustment, for the reasons just explained.

In Appendix F we provide the proofs

6 Refences

Important initial papers Wan, 1970, Ryder and Heal, 1973, Ravn, Schmitt-Grohé, and Uribe, 2006

Policy implications Fuhrer, 2000, Lettau and Uhlig, 2000

Literature on habit formation and finance: Sundaresan, 1989, Constantinides, 1990, Detemple and Zapatero, 1991, Chapman, 1998.

Habit formation and growth: Carroll, 2000

Cursory observations in Romer, 2019

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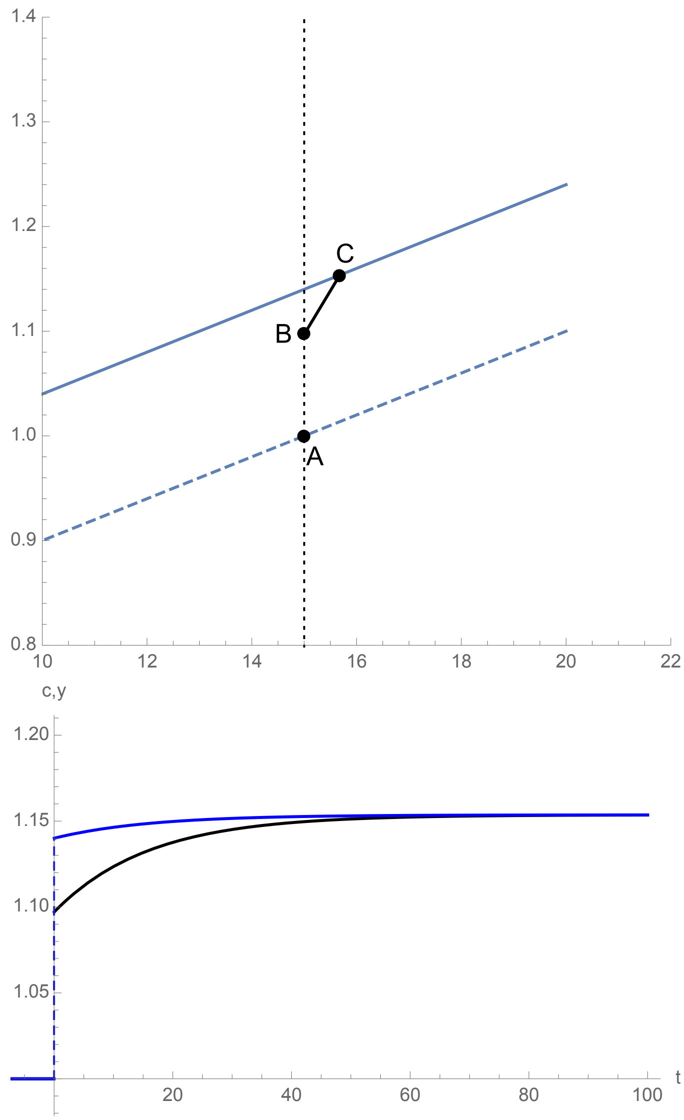

Figure 2: Effect of a non-anticipated increase in income in the habit formation model

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A functional derivatives

There are several different ways to present the idea of the derivative of a functionaldeal, some more mathematically correct and some more informal, as is the case in physics or mechanics. We follow Gel'fand and Fomin, 1963 which presents a good compromise between those two approaches.

Assume we have the space of functions $\mathcal{F}$, i.e., a collection of functions sharing some common property, for instance, continuity, differentiability, boundedness, etc. Every element of $\mathcal{F}$, for instance f, is a mapping between a space X and a subset of the space of real numbers, that is $f : \mathrm{X} \subseteq \mathbb{R} \to \mathbb{R}$. A **functional** is a mapping between the space of functions and the set of real numbers $\mathsf{F} : \mathcal{F} \to \mathbb{R}$.

The following functionals are common examples in economics:

$$\mathsf{F}_1[f] = \int_{\mathrm{X}} f(x)\, dx,$$

$$\mathsf{F}_2[f] = \int_{\mathrm{X}} u(f(x), x)\, dx$$

where $u(\cdot)$ is a function, or

$$\mathsf{F}_3[f] = \int_{\mathrm{X}} u(f'(x), f(x), x)\, dx$$

where $f'(\cdot)$ is the derivative of function $f(\cdot)$.

There are two concepts of derivatives related to functionals.

Consider a perturbation in function f from $f \mapsto f + \delta f$. A Frechet derivative of functions $\mathsf{F}[f]$, denoted by $\delta\mathsf{F}[f]$ is defined from

$$\lim_{\delta f \to 0} \frac{||\mathsf{F}[f + \delta f] - \mathsf{F}[f] - \delta\mathsf{F}[f]||}{\delta f} = 0$$

where $|| \cdot ||$ is the norm of space $\mathcal{F}$.

A more useful concept is the concept is the Gâteaux derivative. It is an extension of the directional derivative for functions. Assume that we introduce a perturbation on function $f \mapsto f + \varepsilon\varphi$ in which φ is a function, $\varphi : X \to \mathbb{R}$ and $\varepsilon > 0$ is a number.

The first variation of a functional is

$$d\mathsf{F}[f; \varphi] = \mathsf{F}[f + \eta\, \varepsilon] - \mathsf{F}[f]$$

and the functional derivative (in the GâteUx sense) is analogous to the concept of derivative for functions:

$$\delta\mathsf{F}[f; \varphi] = \lim_{\varepsilon \to 0} \frac{d\mathsf{F}[f; \varphi]}{\epsilon}.$$

If the functional involves more than one function we may write $\delta_{f_1}\mathsf{F}[f_1, f_2; \varphi_1, \varphi_2]$.

In regular cases, in order apply this concept, we perform a first order Taylor approximation

$$\mathsf{F}[f + \epsilon\varphi] = \mathsf{F}[f] + \delta\mathsf{F}[f; \varphi]\,\varepsilon + o(\varphi)$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi)||}{\varepsilon} = 0$.

Performing a second order Taylor approximation we find

$$\mathsf{F}[f + \epsilon\,\varphi] = \mathsf{F}[f] + \delta\mathsf{F}[f; \varphi]\,\varepsilon + \delta^2\mathsf{F}[f; \varphi^2]\,\varepsilon^2 + o(\varphi^2).$$

However, we can distinguish between the "own" second order functional derivative

$$\delta\mathsf{F}[f + \varepsilon\,\varphi; \varphi] = \delta\mathsf{F}[f; \varphi] + \delta^2\mathsf{F}[f; \varphi^2]\,\varepsilon^2 + o(\varphi^2)$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi^2)||}{\varepsilon} = 0$, and "crossed" second order derivative

$$\delta\mathsf{F}[f + \varepsilon\,\varphi'; \varphi] = \delta\mathsf{F}[f; \varphi] + \delta^2\mathsf{F}[f; \varphi, \varphi']\,\varepsilon^2 + o(\varphi\,\varphi')$$

where $\lim_{\varepsilon\to 0} \dfrac{||o(\varphi\,\varphi')||}{\varepsilon} = 0$

For the previous examples, and assuming that $\varphi(x) = 0$ for $x \in \partial\mathrm{X}$, where $x \in \partial\mathrm{X}$ is the boundary of X, we have

$$\delta\mathsf{F}_1[f; \varphi] = \int_X f(x)\,\varphi(x)\,dx,$$

$$\delta\mathsf{F}_2[f; \varphi] = \int_X \frac{\partial U\big(f(x), x\big)}{\partial f}\,\varphi(x)\,dx$$

and, if f belongs to a space of differentiable functions, which means that so φ does,

$$\delta\mathsf{F}_3[f; \varphi] = \int_X \frac{\partial U\big(f'(x), f(x), x\big)}{\partial f'}\,\varphi(x) + \frac{\partial U\big(f'(x), f(x), x\big)}{\partial f}\,\varphi'(x)\,dx.$$

For the second case we have the second order functional derivative

$$\delta^2\mathsf{F}_3[f; \varphi] = \int_X \frac{\partial^2 U\big(f(x), x\big)}{\partial f^2}\,\varphi^2(x)\,dx.$$

A particular case of perturbation is the "spike" variation. In this case the functional derivative is sometimes called the Volterra derivative.

Consider an element of the domain of function f, say x' and introduce the variation $df(x) = 0$ if $x \neq x'$ and $df(x) = \varepsilon$ if $x = x'$. We can write this as a perturbation $f \mapsto f + \varepsilon\,\delta(x - x')$ where δ is Dirac's delta generalized function. It has the property $\int_{\mathrm{X}} \delta(x - x')\, f(x)\, dx = f(x')$.

Applying to the previous examples, we find

$$\delta \mathsf{F}_1[f; x'] = \int_X f(x)\,\delta(x - x')\,dx = f(x') \tag{12}$$

$$\delta \mathsf{F}_2[f; x'] = \int_X \frac{\partial U\big(f(x), x\big)}{\partial f}\,\delta(x - x')\,dx = \frac{\partial U\big(f(x), x\big)}{\partial f}\Big|_{x=x'}. \tag{13}$$

B Necessary conditions for problem (P2)

We introduce the **functional**

$$\mathsf{U}([c], a_0) = \int_0^T u(c(t))\,e^{-\rho t} - \lambda(t)\big(\dot{a}(t) + c(t)\big)\,dt$$

where we introduce an adjoint multiplier $\lambda : \mathrm{T} \to \mathbb{R}_+$. Its introduction involves a penalization associated to the reduction in value brought about by the budget constraint. Because are now constrained by the initial value of the stock, we call value **function** to

$$V(a_0) = \max_c \mathsf{U}([c], a_0) = \mathsf{U}([c^*], a_0) = \int_0^T u(c^*(t))\,e^{-\rho t} - \lambda(t)\big(\dot{a}^*(t) + c^*(t)\big)\,dt.$$

Assume we know the optimal path $\big(c^*(t), a^*(t)\big)_{t \in \mathrm{T}}$.

We introduce now perturbations in both functions $c^*(t) \to c^*(t) + \varphi_c(t)$ and $a^*(t) \to a^*(t) + \varphi_a(t)$, such that $\varphi_a(0) = 0$, because $a^*(0) = a_0$ is not free. The value functional is now

$$\mathsf{U}([c^* + \varphi_c], a_0) = \int_0^T u(c^*(t) + \varphi_c(t))\,e^{-\rho t} - \lambda(t)\big(c^*(t) + \varphi_c(t) + \dot{a}^*(t) + \dot{\varphi}_a(t)\big)\,dt.$$

The first variation becomes,

$$\begin{aligned}
\delta \mathsf{U}([c^*]) &= \int_0^T \big(u'(c^*(t)\,e^{-\rho t} - \lambda(t)\big)\,\varphi_c(t) - \lambda(t)\,\dot{\varphi}_a(t)\,dt \\
&= \int_0^T \big(u'(c^*(t)\,e^{-\rho t} - \lambda(t)\big)\,\varphi_c(t)\,dt - \int_0^T \lambda(t)\,\dot{\varphi}_a(t)dt \\
&= \int_0^T \big(u'(c^*(t)\,e^{-\rho t} - \lambda(t)\big)\,\varphi_c(t)\,dt - \lambda(t)\varphi_a(t)\Big|_{t=0}^{T} + \int_0^T \dot{\lambda}(t)\varphi_a(t)dt \\
&= \int_0^T \big(u'(c^*(t))\,e^{-\rho t} - \lambda(t)\big)\,\varphi_c(t)\,dt - \lambda(T)\varphi_a(T) + \int_0^T \dot{\lambda}(t)\varphi_a(t)dt
\end{aligned}$$

Using integration by parts and the fact that $\varphi_a(0) = 0$. At the optimum $\big(c^*(t), a^*(t)\big)_{t\in \mathrm{T}}$ is such that $\delta\mathsf{U}([c^*]) = 0$. The first-order conditions are thus: $u'(c^*(t)\, e^{-\rho t} - \lambda(t) = \dot{\lambda}(t) = 0$, for $t \in [0, T]$ and $\lambda(T) = 0$. Integrating $\dot{\lambda}(t) = 0$ we find $\lambda(t) =$ constant for every $t \in [0, T]$. but as $\lambda(T) = 0$ then $\lambda(t) = 0$ for every $t \in [0, T]$. Therefore $u'(c^*(t)\, e^{-\rho t} = 0$ for every $t \in [0, T]$ as in (P1:foc).

C Necessary conditions for problem (P2)

The (penalized) utility functional is

$$\mathsf{U}([c], a_0) = \int_0^T u(c(t))\, e^{-\rho t} - \lambda(t)\big(\dot{a}(t) + c(t)\big) + \eta(t)(a(t) - \underline{\mathrm{a}})\, dt$$

where we introduce a multiplier $\eta : \mathrm{T} \to \mathbb{R}_+$ associated to the instantaneous constraint on a such that the complementary slackness conditions hold

$$\eta(t) \geq 0,\ \eta(t)(a(t) - \underline{\mathrm{a}}) = 0,\ \text{for every}\, t \in \mathrm{T}.$$

Using the same method as in section B the perturbed value functional is

$$\mathsf{U}([c^*{+}\varphi_c], a_0) = \int_0^T u(c^*(t){+}\varphi_c(t))\, e^{-\rho t}{-}\lambda(t)\big(c^*(t){+}\varphi_c(t){+}\dot{a}^*(t){+}\dot{\varphi}_a(t)\big) + \eta(t)(a^*(t){+}\varphi_a(t){-}\underline{\mathrm{a}})\, dt.$$

Then, using the same procedure as before

$$\delta\mathsf{U}([c^*]) = \int_0^T \big(u'(c^*(t))\, e^{-\rho t} - \lambda(t)\big)\, \varphi_c(t)\, dt - \lambda(T)\varphi_a(T) + \int_0^T \big(\dot{\lambda}(t) + \eta(t)\big)\varphi_a(t) dt.$$

Therefore the f.o.c are $u'(c^*(t))\, e^{-\rho t} = \lambda(t)$, $\dot{\lambda}(t) = -\eta(t)$, for $t \in [0, T]$, together with the complementary slackness conditions. However at $t = T$ we have also $\lambda(T) \geq 0$ and $\lambda(T)(a(T) - \underline{a}) = 0$, which is only possible if $\lambda(T^-) = \eta(T)$, or there is a discontinuity on $\lambda(t)$ at $t = T$.

D Necessary conditions for problem (P2)

We can find the necessary (in this case also sufficient) optimality conditions for problem (P4) by using the Pontriyagin maximum Principle. As the Hamiltonian function is

$$H = \frac{c^{1-\theta} - 1}{1 - \theta} + q\, (r\, a - c)$$

we have

$$\begin{cases} \dot{a} = r\,a - c & \text{for } t \in [0,T] \\ \dot{c} = \gamma_c\,c & \text{for } t \in [0,T] \\ a(0) = a_0 \text{ given} & \text{for } t = 0 \\ c(T)^{-\theta}(a(T) - \underline{a}) = 0 & \text{for } t = T \end{cases}$$

where the rate of growth of consumption is $\gamma_c \equiv \dfrac{r-\rho}{\theta}$. Solving the Euler equation we have $c(t) = c(0)\,e^{\gamma_c t}$. Substituting in the budget constraint, together with the initial condition yields

$$a(t) = e^{rt}\Big(a_0 + \frac{c(0)}{\gamma_c - r}\big(1 - e^{\gamma_c - r)\,t}\big)\Big), \text{ for } t \in [0,T]$$

If $c(0) > 0$ and finite, then $c(T) > 0$ and finite, which implies that the transversality constraint only holds if $a(T) = \underline{a}$. Therefore, we can find $c(0)$ by solving the equation

$$\underline{a}\,e^{rT} = a_0 + \frac{c(0)}{\gamma_c - r}\big(1 - e^{\gamma_c - r)\,T}\big)$$

Then we obtain equations (4) and (5).

E Comparative dynamics

Assume we have a non-linear dynamic system

$$\dot{X} = F(X, \varphi)$$

where φ is an exogenous variable and $X = (x_1, x_2)$ in which x_1 is pre-determined and x_2 is non-predetermined.

Let the exogenous variable takes the value φ_0, and let the associated steady state be $\bar{X}(\varphi_0)$.

Now consider a variation in the exogenous variable from φ_0 to $\varphi_1 = \varphi_0 + d\varphi$. If the syatem is at the steady state $\bar{X}(\varphi_0)$ it will be perturbed away from it. Let $dX(t) = X(t) - \bar{X}(\varphi_0)$ be the variation of X when away from the steady state.

The perturbation of can be studied from the solutions of the **variational system**

$$\dot{X} = \bar{F}_x(\varphi_0)\,dX(t) + \bar{F}_\varphi(\varphi_0)\,d\varphi.$$

where the Jacobians are

$$\bar{F}_x(\varphi_0) \equiv F_x\big(\bar{X}(\varphi_0), \varphi_0\big),\ \bar{F}_\varphi(\varphi_0) \equiv F_\varphi\big(\bar{X}(\varphi_0), \varphi_0\big)$$

The **comparative dynamics multipliers** are the solutions, $dX(t)$ to this system, which is a linear ordinary differential equation.

In MHDS systems the Jacobian $barF_x(\varphi_0)$ usually has eigenvalues that satisfy $\lambda_s \leq 0 < \lambda_u$.

Two cases can occur, which have consequences on the method for determining $dX(t)$, depending on the Jacobian having a non-zero or a zero determinant. In the first case, we have $\lambda_s < 0$ and the dynamics will not depend on $x_1(0)$, the initial value of the pre-determined variable and in the second case $\lambda_s = 0$ and the dynamics will depend on $x_1(0)$.

Next we deal with the two cases separately.

Non-zero eigenvalues case

As det $\big(\bar{F}_x(\varphi_0)\big) < 0$ then there is a classic inverse for the Jacobian, which allows us to determine the **long-run multipliers**

$$d\bar{X} = -\bar{F}_x(\varphi_0)^{-1}\,\bar{F}_\varphi(\varphi_0)\,d\varphi = X_\varphi(\varphi_0)\,d\varphi$$

and the general solution to the variational system is

$$dX(t) = d\bar{X} + k_s\,P^s\,e^{\lambda_s\,t} + k_u\,P^u\,e^{\lambda_u\,t}$$

where P^s and P^u are the eigenvectors associated to the eigenvalues $\lambda_s < 0$ and $\lambda_u > 0$ and k_s and k_u are two arbitrary constants.

The two arbitrary constants are determined such that the solution converges to a new steady state $\bar{X}(\varphi_1)$ and the pre-determined variable x_1 is continuous at the time of the shock, that is $dx_1(0) = x_1(0) - \bar{x}_1(\varphi_0) = 0$.

We obtain the first type of behavior by setting $k_u = 0$, which yields $dX(t) = d\bar{X} + k_s\,P^s\,e^{\lambda_s\,t}$. Expanding, we have

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} d\bar{x}_1 \\ d\bar{x}_2 \end{pmatrix} + k_s\,\begin{pmatrix} P_1^s \\ P_2^s \end{pmatrix}\,e^{-\lambda_s\,t}.$$

At time $t = 0$, setting $dx_1(t) = 0$ yields $k_s = -\dfrac{d\bar{x}_1}{P_1^s}$.

Therefore, the **comparative dynamics variations** , are

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} d\bar{x}_1\big(1 - e^{-\lambda_s\,t}\big) \\ d\bar{x}_2 - d\bar{x}_1\Big(\dfrac{P_2^s}{P_1^s}\Big)\,e^{-\lambda_s\,t} \end{pmatrix}.$$

If we want to trace-out the adjustment paths of the variables, we find

$$x_1(t) = \bar{x}_1(\varphi_0) + d\bar{x}_1\big(1 - e^{-\lambda_s\,t}\big)$$
$$x_2(t) = \bar{x}_2(\varphi_0) + d\bar{x}_2 - d\bar{x}_1\left(\frac{P^s_2}{P^s_1}\right) e^{-\lambda_s\,t}.$$

Evaluating for $t \to \infty$ yields the **long-run multipliers**

$$\frac{X(\infty) - \bar{X}(\varphi_0)}{d\varphi} = \frac{d\bar{X}}{d\varphi}$$

and evaluating at $t = 0$ we obtain the **impact multiplier for the non-predetermined variable**

$$\frac{x_2(0) - \bar{x}_2(\varphi_0)}{d\varphi} = \frac{P^s_1\,d\bar{x}_2 - d\bar{x}_1\,P^s_2}{P^s_1\,d\varphi}.$$

Also, if we write $d\bar{x}_i = \bar{x}_i(\varphi_1) - \bar{x}_i(\varphi_0)$, for $i = 1, 2$, that is the difference between the initial and the (approximated) final steady state after the shock in φ, we see that along the adjustment we have the following relationship

$$P^s_1\,\big(x_2(t) - \bar{x}_2(\varphi_1)\big) = P^s_2\,\big(x_1(t) - \bar{x}_1(\varphi_1)\big).$$

In the presence of a zero eigenvalue

When $\det\big(\bar{F}_x(\varphi_0)\big) = 0$ the Jacobian has eigenvalues $\lambda_s = 0 < \lambda_u$ and there is a not a classic inverse for the Jacobian, We can use the Moore-Penrose inverse to determine the long run multipliers

$$d\bar{X} = -\bar{F}_x(\varphi_0)^{+}\bar{F}_\varphi(\varphi_0)\,d\varphi + \Big(I - \bar{F}_x(\varphi_0)^{+}\,\bar{F}_x(\varphi_0)\Big)\,Z$$

where I is the identity matrix, $Z = (z_1, z_2)^\top$ is a vector of constants and we use

$$\bar{F}_x(\varphi_0) = P\,\Lambda\,P^{-1}\;\;\bar{F}_x(\varphi_0)^{+} = P\,\Lambda^{+}\,P^{-1}$$

where the Jordan form, the Moore-Penrose inverse and the eigenvector matrices are

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_u \end{pmatrix},\; \Lambda^{+} = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{\lambda_u} \end{pmatrix},\; P = \begin{pmatrix} P^s_1 & P^u_1 \\ P^s_2 & P^u_2 \end{pmatrix}.$$

Because, differently from the classic inverse $\bar{F}_x(\varphi_0)^{+}\,\bar{F}_x(\varphi_0) \neq I$ then the expression for the multipliers would allows us to obtain a linear equation in one of the elements of vector Z, say z_2. We can determine it by using the predetermine nature of x_1 by setting $d\bar{x}_1 = 0$. This yields

$$d\bar{X} = \begin{pmatrix} 0 \\ d\bar{x}_2 \end{pmatrix}$$

The solution to the variational system is now

$$dX(t) = d\bar{X} + k_s\, P^s + k_u\, P^u\, e^{\lambda_u t}$$

depending on two arbitrary constants k_s and k_u. To eliminate unbounded trajectories, we set again $k_u = 0$ and determine k_s such that $dx_1(0) = 0$. This yields the variations

$$dx_1(t) = 0, \text{ for all } t \in [0, \infty)$$
$$dx_2(t) = d\bar{x}_2, \text{ for all } t \in [0, \infty)$$

where all the variation is absorbed by x_2. This means that the values of the perturbed variables are

$$x_1(t) = x_{1,0}, \text{ for all } t \in [0, \infty)$$
$$x_2(t) = \bar{x}_2(\varphi_1), \text{ for all } t \in (0, \infty)$$

where we set $d\bar{x}_2 = \bar{x}_2(\varphi_1, x_{1,0}) - \bar{x}_2(\varphi_0, , x_{1,0})$ because, as we saw in the main text that the value of the steady state for the non-predetermined variable depends on the initial value of the predetermined variable. This means that the non-predetermined immediately "jumps" to the new steady state.

Application

For the problem having first-order conditions in equations (8a)-(8d) we have the initial steady state $\bar{X}(w_0) = (\bar{a}_0, \bar{q}_0)^\top = \big(a_0, (\rho a_0 + w_0)^{-\theta})\big)^\top$, and the Jacobian for an increase in the wage rate $dw = w_1 - w_0$

$$\begin{pmatrix} \dot{a} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \rho & -C'(\bar{q}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} da(t) \\ q(t) \end{pmatrix} + \begin{pmatrix} dw \\ 0 \end{pmatrix}$$

The first Jacobian has eigenvalues $\lambda_s = 0$ and $\lambda_u = \rho$ and eigenvector matrix

$$P = \begin{pmatrix} C'(\bar{q}_0) & 1 \\ \rho & 1 \end{pmatrix}$$

which implies

$$\bar{F}_x(w_0)^+ = P\, \Lambda^+\, P^{-1} = \begin{pmatrix} C'(\bar{q}_0) & 1 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\lambda_u} \end{pmatrix} \frac{1}{\rho} \begin{pmatrix} 0 & 1 \\ \rho & -C'(\bar{q}_0) \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} 1 & -\frac{C'(\bar{q}_0)}{\rho} \\ 0 & 0 \end{pmatrix}$$

and

$$I - \bar{F}_x(w_0)^+\, \bar{F}_x(w_0) = \begin{pmatrix} 0 & -\frac{C'(\bar{q}_0)}{\rho} \\ 0 & 0 \end{pmatrix}$$

Therefore, the general expression for the long run multipliers is

$$\begin{pmatrix} d\bar{a} \\ d\bar{q} \end{pmatrix} = \begin{pmatrix} -\dfrac{dw}{\rho} + \frac{C'(\bar{q}_0)}{\rho}\, z_2 \\ z_2 \end{pmatrix}$$

As we require $d\bar{a} = 0$ then $z_2 = \dfrac{dw}{C'(\bar{q}_0)}$ then

$$\begin{pmatrix} d\bar{a} \\ d\bar{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix}.$$

We obtain the short run variations from

$$\begin{pmatrix} da(t) \\ dq(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix} + k_s \begin{pmatrix} C'(\bar{q}_0) \\ \rho \end{pmatrix}$$

Setting again $da(t) = 0$ yields $k_s = 0$, and, therefore, the short-run variations are

$$\begin{pmatrix} da(t) \\ dq(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{dw}{C'(\bar{q}_0)} \end{pmatrix}.$$

As $dc(t) = d\bar{c} = C'(\bar{q}_0)\, dq(t) = dw = w_1 - w_0$, and writing $\bar{c}(w_1) = d\bar{c} + \bar{c}(w_0)$ yields the linear approximation for the behavior of consumption after the shock $c(t) = \bar{c}(w_0) + d\bar{c} = \bar{c}(w_1) = \rho\, a_0 + w_1$. We conclude that consumption changes discontinuously from $\bar{c}(w_0) = \rho\, a_0 + w_0$ to $\bar{c}(w_1) = \rho\, a_0 + w_1$.

F Comparative dynamics for the habit formation model

As in the non-habit formation model we take the static arbitrage condition (10e) to find, implicitly, the optimal consumption function , as

$$c = C(h, q_a, q_h),$$

which has the partial derivatives

$$C_h = -\frac{u_{ch}(c,h)}{u_{cc}(c,h)},\ C_{q_a} = dfrac1u_{cc}(c,h),\ C_{q_h} = -\frac{|eta}{}(c,h)u_{cc}(c,h).$$

We start from the steady state in equations (11a)-(11e), for the initial level of the exogenous variable $w = w_0$ and consider a positive change to $w_1 = w_0 + dw$.

Next we introduce the notation for the partial derivatives $\bar{u}_i = u_i\big(\bar{c}(w_0), \bar{h}(w_0)\big)$ for $i = c, h$ and $\bar{u}_{ij} = u_{ij}\big(\bar{c}(w_0), \bar{h}(w_0)\big)$ for $i, j = c, h$. We define accordingly $\bar{C}_h$, $\bar{C}_{q_a}$ and $\bar{C}_{q_h}$.

Using this notation for the partial derivatives, evaluated at the initial steady state, and the notation in section D for the Jacobians we obtain the Jacobian

$$\bar{F}_x(w_0) = \begin{pmatrix} \rho & -\bar{C}_h & -\bar{C}_{q_a} & -\bar{C}_{q_h} \\ 0 & \eta\big(\bar{C}_h - 1\big) & \eta\,\bar{C}_{q_a} & \eta\,\bar{C}_{q_h} \\ 0 & 0 & 0 & 0 \\ 0 & -\big(\bar{u}_{hc}\,\bar{C}_h + \bar{u}_{hh}\big) & -\bar{u}_{hc}\,\bar{C}_{q_a} & \rho + \eta - \bar{u}_{hc}\,\bar{C}_{q_h} \end{pmatrix}$$

This Jacobian has the characteristic equationdet $\big(\bar{F}_x(w_0) - \lambda\, I\big) = 0$. Expanding, yields the polynomial equation

$$\lambda\,(\lambda - \rho)\big(\lambda^2 - \rho\,\lambda + S\big) = 0.$$

This is because $\mu\,\bar{C}_h = \bar{u}_{hc}\,\bar{C}_{q_h}$ and we have

$$\begin{aligned} S &= \eta\,\Big(\big(\bar{C}_h - 1\big)\,\big(\rho + \eta - \bar{u}_{hc}\,\bar{C}_{q_h}\big) + \bar{C}_{q_h}\big(\bar{u}_{hc}\,\bar{C}_h + \bar{u}_{hh}\big) \\ &= \eta\,(\rho + \eta)\,\Big(\bar{C}_h - 1 + \frac{\bar{C}_{q_h}}{\rho + \eta}\big(\bar{u}_{hc} + \bar{u}_{hh}\big)\Big) \\ &= -\eta\Big(\frac{(\rho + \eta)\bar{u}_{cc} + (2\eta + \rho)\big(\bar{u}_{hc} + \eta\bar{u}_{hh}\big)}{\bar{u}_{cc}}\Big) \end{aligned}$$

There are four real eigenvalues $\{\lambda_s, 0, \rho, \lambda_u\}$ where

$$\begin{aligned} \lambda_s &= \frac{\rho}{2} - \sqrt{\Big(\frac{\rho}{2}\Big)^2 - S} \\ \lambda_u &= \frac{\rho}{2} + \sqrt{\Big(\frac{\rho}{2}\Big)^2 - S} \end{aligned}$$

If $S < 0$ then $\lambda_s < 0 < \rho < \lambda_u$ and the steady state is a degenerate saddle-point. Additionally we have $\lambda_s + \lambda_u = \rho$ and $\lambda_s\,\lambda_u = S$.

Looking at the expression for S, we can write it as

$$S = -\frac{\eta\pi(\underline{u})}{\bar{u}_{cc}}$$

where $\pi(\underline{u}) \equiv (\rho+\eta)\bar{u}_{cc} + (2\eta+\rho)\big(\bar{u}_{hc} + \eta\bar{u}_{hh}\big) = (\rho+\eta)\bar{u}_{cc} + (2\eta+\rho)\bar{\mathsf{U}}''$. Then, $S < 0$ if and only if $\pi(\underline{u}) < 0$ which requires $\bar{\mathsf{U}}'' < -\dfrac{\rho + \eta}{2\eta + \rho}\bar{u}_{cc}$ which only holds if consumption is intertemporally substitutable or independent and, if there is intertemporal complementarity, it is not to large compared with the concavity as regards consumption c.

From now on we assume this condition holds.

As we have a zero eigenvalue we can adapt the method explained in the last section. The generalized long-run multipliers are

$$d\bar{X} = -\bar{F}_x(w_0)^+ \bar{F}_w(w_0)\, dw + \Big(I - \bar{F}_x(w_0)^+ \, \bar{F}_x(w_0)\Big)\, Z$$

where the Jacobian for the exogenous variable is

$$\bar{F}_w(w_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

and the Moore-Penrose inverse is $\bar{F}_x(w_0)^+ = P\,\Lambda^+\,P^{-1}$ where

$$\Lambda^+ = \begin{pmatrix} \frac{1}{\lambda_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda_u} \end{pmatrix},$$

and the eigenvector matrix concatenating the eigenvectors in the same order as in the Jordan matrix Λ, i,e, $P = [P^s P^0 P^\rho P^u]$ is

$$P = \begin{pmatrix} -\frac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u H(\bar{u})} & 1 & 1 & \frac{\bar{u}_{hc}\left(\eta\bar{u}_{hc} + (\lambda_u+\eta) baru_{cc}\right) - \eta H(\bar{u})}{\bar{u}_{cc}\, H(\bar{u})\lambda_s} \\ -\frac{\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u H(\bar{u})} & \rho & 0 & \frac{\eta\bar{u}_{hc} + (\lambda_u+\eta) baru_{cc}}{H(\bar{u})} \\ 0 & \frac{\rho\,\chi(\bar{u})}{\rho+\eta} & 0 & 0 \\ 1 & \frac{\rho\,(\bar{u}_{hc}+\bar{u}_{hh})}{\rho+\eta} & 0 & 1 \end{pmatrix},$$

where $H(\bar{u}) = \bar{u}_{cc}\,\bar{u}_{hh} - \bar{u}_{hc}^2$ and $\chi(\underline{u}) \equiv (\rho+\eta)\bar{u}_{cc} + \bar{u}_{hc}(2\eta+\rho) + \eta\bar{u}_{hh}$

Performing the calculations yields the generalized variation

$$d\bar{X} = \begin{pmatrix} d\bar{a} \\ d\bar{h} \\ d\bar{q}_a \\ d\bar{q}_h \end{pmatrix} = \begin{pmatrix} -\dfrac{dw}{\rho} + \dfrac{(\rho+\eta)}{\rho\,\chi(\bar{u})}\,z_3 \\ +\dfrac{(\rho+\eta)}{\chi(\bar{u})}\,z_3 \\ z_3 \\ \dfrac{\bar{u}_{hc}+\bar{u}_{hh}}{\chi(\bar{u})}\,z_3 \end{pmatrix}$$

We set $d\bar{a} = 0$ to find the value for z_3 and substituting back we obtain the particular long-run variation

$$d\bar{X} = \begin{pmatrix} d\bar{a} \\ d\bar{h} \\ d\bar{q}_a \\ d\bar{q}_h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \dfrac{\chi(\bar{u})}{\rho+\eta} \\ \dfrac{\bar{u}_{hc}+\bar{u}_{hh}}{\rho+\eta} \end{pmatrix} dw$$

The short run variation $dX(t) = X(t) - \bar{X}(w_0)$, introduced by the perturbation in w can be obtained from the general solution of the variational system,

$$dX(t) = d\bar{X} + k_s\,P^s\,e^{\lambda_s t} + k_0 P^0 + k_\rho P^\rho e^{\rho t} + k_u P^u e^{\lambda_u t}$$

where k_s, k_0, k_ρ and k_u are arbitrary constants.

Eliminating the explosive components by setting $k_\rho = k_u = 0$ and solving for k_s and k_0 such that $da(0) = 0$ and $dh(0) = 0$, yields

$$\bar{k}_s = -\frac{(\eta+\lambda_u)\,\bar{u}_{hc} + \eta\bar{u}_{hh}}{(\eta+\lambda_u)\big(\lambda_u\bar{u}_{cc} + \rho\bar{u}_{hc}\big) + \eta\rho\bar{u}_{hh}}\,dw$$

$$\bar{k}_0 = -\frac{\lambda_u H(\bar{u})}{(\eta+\lambda_u)\big(\lambda_u\bar{u}_{cc} + \rho\bar{u}_{hc}\big) + \eta\rho\bar{u}_{hh}}\,dw$$

Therefore the short run variation is

$$
\begin{aligned}
da(t) &= -\frac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})}(1-e^{\lambda_s t})\,dw, \text{for } t \in [0,\infty)\\
dh(t) &= \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)}{\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})}(1-e^{\lambda_s t})\,dw, \text{for } t \in [0,\infty)\\
dq_a(t) &= \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)\pi(\bar{u})}{(\rho+\eta)\Big(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)}dw, \text{for } t \in [0,\infty)\\
dq_h(t) &= \frac{\lambda_u\,\bar{u}_{hc}\big((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh}\big)}{(\rho+\eta)\Big(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)}\times\\
&\quad\times\Big((\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2 - (\rho+\eta)H(\bar{u})e^{\lambda_s t}\Big)\,dw, \text{for } t \in [0,\infty).
\end{aligned}
$$

The variation in consumption can be obtained as

$$dc(t) = \bar{C}_h\,dh(t) + \bar{C}_{q_a}\,dq_a(t) + \bar{C}_{q_h}\,dq_h(t).$$

In the expressions for the variations of the state variables, we see the effect of the existence of a zero eigenvalue: we find that

$$\big(\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}\big)\,dh(t) + \lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)\,da(t) = 0.$$

We also find that

$$
\begin{aligned}
da(0) &= 0\\
dh(0) &= 0\\
dq_a(0) = dq_a(t) &= \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)\pi(\bar{u})}{(\rho+\eta)\Big(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)}dw\\
dq_h(0) &= \frac{\lambda_u\,\bar{u}_{hc}\big((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh}\big)\Big((\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2 - (\rho+\eta)H(\bar{u})\Big)}{(\rho+\eta)\Big(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)}\,dw.
\end{aligned}
$$

and

$$
\begin{aligned}
da(\infty) &= -\frac{\eta\bar{u}_{hh} + (\eta+\lambda_u)\bar{u}_{hc}}{\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})}\, dw, \text{for } t \in [0,\infty) \\
dh(\infty) &= \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)}{\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})}(1 - e^{\lambda_s t})\, dw, \text{for } t \in [0,\infty) \\
dq_a(\infty) &= \frac{\lambda_u\big(\eta\bar{u}_{hc} + (\eta+\lambda_u)\bar{u}_{cc}\big)\pi(\bar{u})}{(\rho+\eta)\Big(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)} dw, \text{for } t \in [0,\infty) \\
dq_h(\infty) &= \frac{\lambda_u\, \bar{u}_{hc}\big((\eta+\lambda_u)\bar{u}_{hc} + \eta\bar{u}_{hh}\big)\Big((\bar{u}_{cc}\bar{u}_{hh}(\eta+\lambda_u) + \eta\bar{u}_{hc}^2\Big)}{(\rho+\eta)\Big(\lambda_u\big((\eta+\lambda_u)\bar{u}_{cc} + (\rho+\eta)\bar{u}_{hc}\big) + \rho + \eta(\bar{u}_{hc} + \bar{u}_{hh})\Big)}\, dw.
\end{aligned}
$$

which are the long run multipliers. To determine the levels of the new steady state after the shock we can write $\bar{X}(w_1) = dX(\infty) + \bar{X}(w_0)$. This is point C shown in figure 2.