Mathematical Economics
Continuous time: optimal control problem

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Continuous time optimal control problem

- Find the **state** variable $x^* = (x^*(t))_{0 \le t \le T}$ and the **control** variable $u^* = (u^*(t))_{0 \le t \le T}$ that solve the problem:

$$\begin{aligned} \max_u \int_0^T & F(t, x(t), u(t))\, dt \\ & \text{subject to} \\ & \dot{x} = G(t, x(t), u(t)) \\ & x(0) = x_0 \text{ given} \\ & \text{given the horizon } T \\ & \text{constraints on the terminal value of } x(T) \end{aligned}$$

- We will consider the constraints on $x(T)$:

(P1) $x(T) = \phi_T$
(P2) $x(T)$ free
(P3) $h(T)x(T) \ge 0.$

Hamiltonian function

- We introduce the Hamiltonian function

$$H(t, x, u, \lambda) \equiv F(t, x, u) + \lambda\, G(t, x, u)$$

  where $\lambda(t)$ is the **co-state** or **adjoint** variable,

- its derivatives as regards the control variable

$$H_u(t, x, u, \lambda) \equiv \frac{\partial H(t, x, u, \lambda)}{\partial u} = F_u(t, x, u) + \lambda G_u(t, x, u)$$

- and the state variable

$$H_x(t, x, u, \lambda) \equiv \frac{\partial H(t, x, u, \lambda)}{\partial x} = F_x(t, x, u) + \lambda G_x(t, x, u)$$

Pontriyagin's maximum principle

Proposition (Necessary first order conditions)

Let (x^*, u^*) be a solution to the OC problem. Then there is a piecewise continuous function $\lambda(t)$ such that (x^*, u^*, λ) satisfy: Proof

- *the optimality condition*

$$H_u(t, x^*(t), u^*(t), \lambda(t)) = 0, \; 0 \leq t \leq T$$

- *the adjoint equation*

$$\dot{\lambda} = -H_x(t, x^*(t), u^*(t), \lambda(t)), \; 0 < t \leq T$$

- *the admissibility conditions:*

$$\begin{cases} \dot{x}^* = G(t, x^*(t), u^*(t)) & 0 < t \leq T \\ x^*(0) = \phi_0, & t = 0 \end{cases}$$

- *the terminal or transversality condition*

$$\text{(P1) } x(T) = \phi_T, \; \text{(P2) } \lambda(T) = 0, \; \text{(P3) } \lambda(T)x(T) = 0.$$

Example: the consumption-investment problem

- The problem: find $(a^*, c^*) = \left(a^*(t), c^*(t)\right)_{t=0}^{T}$ that solves

$$\max_{c(\cdot)} \int_0^T \ln\left(c(t)\right) e^{-\rho t}\, dt,\ \ \rho > 0$$

$$\begin{aligned}
&\text{subject to} \\
&\dot{a}(t) = ra(t) - c(t), \text{for } t \in (0, T] \\
&a(0) = a_0, \text{for } t = 0
\end{aligned}$$

- where: $c =$ consumption, $a =$ net financial wealth, $r =$ interest rate constant;
- and one of the alternative terminal conditions

$$\begin{aligned}
&\text{(P1)} \quad a(T) = a_T, \text{given} \\
&\text{(P2)} \quad a(T), \text{free} \\
&\text{(P3)} \quad e^{-rT} a(T) \geq 0
\end{aligned}$$

Example: the consumption-investment problem

- The Hamiltonian function is $H(a, c, \lambda, t) = \ln(c)\, e^{-\rho t} + \lambda\,(r\,a - c)$
- The first order conditions are

$$
\begin{aligned}
H_c = 0 &\;\Rightarrow\; \lambda(t)\, c(t) = e^{-\rho t} \\
\dot{\lambda} = -H_a &\;\Rightarrow\; \dot{\lambda} = -r\,\lambda \\
\dot{a} &= r\,a - c \\
a(0) &= a_0
\end{aligned}
$$

- Together with one of the following terminal conditions

$$
\begin{aligned}
&(\text{P1}) \quad a(T) = a_T, \text{given} \\
&(\text{P2}) \quad \lambda(T) = 0 \\
&(\text{P3}) \quad e^{-\rho T}\, \lambda(T)\, a(T) = 0
\end{aligned}
$$

Example: the consumption-investment problem (cont.)

- As $\dfrac{\dot{\lambda}}{\lambda} + \dfrac{\dot{c}}{c} = -\rho$ then we obtain the MHDS

$$\begin{aligned} \dot{a} &= r\,a - c \\ \dot{c} &= (r-\rho)\,c \\ a(0) &= a_0 \end{aligned}$$

- Together with one of the following terminal conditions

$$\begin{aligned} &(\text{P1}) \quad a(T) = a_T \\ &(\text{P2}) \quad \frac{e^{-\rho T}}{c(T)} = 0 \\ &(\text{P3}) \quad e^{-\rho T}\,\frac{a(T)}{c(T)} = 0 \end{aligned}$$

Example: the consumption-investment problem (cont.)

- As the system is recursive we solve $\dot{c} = (r - \rho)\, c$,

$$c(t) = c(0)\, e^{(r-\rho)\, t}, \text{ where } c(0) \text{ is unknwon}$$

- The first ODE becomes $\dot{a} = ra - c(0)\, e^{(r-\rho)\, t}$. Solving

$$\begin{aligned} a(t) &= e^{rt}\Big(a(0) - c(0) \int_0^t e^{-rs}\, e^{(r-\rho)\, s}\, ds \Big) \\ &= e^{rt}\Big(a(0) - c(0) \int_0^t e^{-\rho\, s}\, ds \Big) \\ &= e^{rt}\Big(a(0) + \frac{c(0)}{\rho}\big(e^{-\rho t} - 1 \big) \Big) \end{aligned}$$

where $a(0)$ and $c(0)$ are unknwon (this is a general solution)

Example: the consumption-investment problem (cont.)

- To find the particular solutions, we use the initial condition and the terminal condition
- Problem (P1): $a(t)|_{t=0} = a_0$ and $a(t)|_{t=T} = a_T$. Then $a(0) = a_0$, and

$$e^{rT}\left(a_0 + \frac{c(0)}{\rho}\left(e^{-\rho T} - 1\right)\right) = a_T \Rightarrow c^*(0) = \rho\frac{a_0 - a_T\, e^{-rT}}{1 - e^{-\rho T}}$$

- then the solution to problem (P1) is

$$c^*(t) = \rho\frac{a_0 - a_T\, e^{-rT}}{1 - e^{-\rho T}}\, e^{(r-\rho)t},\ t \in [0, T]$$

$$a^*(t) = e^{rt}\left(a_0 - \frac{a_0 - a_T\, e^{-rT}}{1 - e^{-\rho T}}\left(1 - e^{-\rho t}\right)\right),\ t \in [0, T].$$

which only makes economic sense if $a_0 > a_T\, e^{-rT}$ implying $c^*(t) > 0$ for all $t \in [0, T]$.

Example: the consumption-investment problem (cont.)

- Problem (P2): $a(t)|_{t=0} = a_0$ and $\dfrac{e^{-\rho T}}{c(T)} = 0$.
- Then $a(0) = a_0$, and

$$\frac{e^{-\rho T}}{c(T)} = \frac{e^{-\rho T}}{c(0)\, e^{(r-\rho)T}} = \frac{e^{-rT}}{c(0)} = 0$$

- if r is finite, this solution can only occur if we could have $c(0) = \infty$, this would imply $a(t) = -\infty$ which means that the agent could borrow without limit. This does not occur in real economies.
- This is the reason for considering problem (P3) and the condition $e^{-rT} a(T) \geq 0$ is called non-Ponzi games condition implying that the transversality condition is a necessary (and sufficient condition) for an optimum $e^{-\rho T} \dfrac{a(T)}{c(T)} = 0$

Example: the consumption-investment problem (cont.)

- Problem (P3): $a(t)|_{t=0} = a_0$ and $e^{-\rho T}\dfrac{a(T)}{c(T)} = 0.$
- Then $a(0) = a_0$
- and

$$
\begin{aligned}
e^{-\rho T}\frac{a(T)}{c(T)} &= \frac{e^{(r-\rho)T}}{c(0)\, e^{(r-\rho)T}}\left(a_0 + \frac{c(0)}{\rho}\left(e^{-\rho T} - 1\right)\right) \\
&= \frac{a_0}{c(0)} + \frac{e^{-\rho T} - 1}{\rho} \\
&= 0 \Rightarrow c^*(0) = \frac{\rho\, a_0}{1 - e^{-\rho T}}
\end{aligned}
$$

- Then

$$
\begin{aligned}
c^*(t) &= \frac{\rho\, a_0\, e^{(r-\rho)\,t}}{1 - e^{-\rho T}},\ t \in [0, T] \\
a^*(t) &= a_0\, e^{rt}\frac{e^{-\rho t} - e^{-\rho T}}{1 - e^{-\rho T}},\ t \in [0, T].
\end{aligned}
$$

- Observe that $a^*(T) = 0$: it is optimal for the consumer to spend its initial net wealth and the income it generates along the time of the program.

Optimal control: autonomous discounted infinite horizon problem

Find (x^*, u^*) where $x^* = (x^*(t))_{0 \le t < \infty}$ and $u^* = (u^*(t))_{0 \le t < \infty}$ that solve the OCIH problem:

$$\max_u \int_0^\infty e^{-\rho t} f(x(t), u(t))\, dt$$

subject to

$$\dot{x}(t) = g(x(t), u(t))$$

- and $x(0) = \phi_0$
- alternative terminal conditions

$(P2)$ $\lim_{t \to \infty} x(t)$ free

$(P3)$ $\lim_{t \to \infty} h(t) x(t) \ge 0.$

Current-value Hamiltonian

- We define a **time-independent** current-value Hamiltonian function:

$$h(x, u, q) = f(x, u) + q\, g(x, u)$$

- as the capitalised value of the discounted Hamiltonian function

$$H(t, x(t), u(t), \lambda(t)) = e^{-\rho t}\, h(x(t), u(t), q(t))$$

- The **current-value co-state variable** is

$$q(t) = e^{\rho t} \lambda(t)$$

- By using the derived necessary conditions for problems (P2) and (P3) by taking $T \to \infty$, we find...

Pontriyagin maximum principle

Proposition (Necessary conditions for the OCIHP)

Let (x^*, u^*) be the solution of the OCIH problem. Then there is a co-state variable $q(t)$ such that the solution $(x^*(t), u^*(t))_{t\in[0,\infty)}$ satisfies the following conditions:

- *the optimality condition*

$$h_u(x^*(t), u^*(t), q(t)) = 0, \ 0 \leq t < \infty$$

- *the adjoint equation*

$$\dot{q} = \rho q(t) - h_x(x^*(t), u^*(t), q(t)), \ 0 < t < \infty$$

- *the admissibility conditions:*

$$\begin{cases} \dot{x}^* = g(x^*(t), u^*(t)) & 0 < t < \infty \\ x^*(0) = \phi_0, & t = 0 \end{cases}$$

- *one of the transversality conditions*

(P2) $\lim_{t\to\infty} e^{-\rho t} q(t) = 0$
(P3) $\lim_{t\to\infty} e^{-\rho t} q(t) x(t) = 0.$

Example: consumption-investment problem

- The problem ((P3) case)

$$\max_c \int_0^\infty \ln(c(t))\, e^{-\rho t}\, dt,\ \rho > 0$$

$$\text{subject to}$$

$$\dot{a} = r\,a - c,\ t \in [0, \infty)$$

$$a(0) = a_0,\ \text{given},\ \{t = 0\}$$

$$\lim_{t\to\infty} e^{-r\,t}\, a(t) \geq 0,\ \{t = \infty\}$$

- Current-value Hamiltonian

$$h = \ln(c) + q\,(r\,a - c)$$

- First order conditions:

$$c(t) = 1/q(t)$$

$$\dot{q} = \big(\rho - r\big)\, q,\ \lim_{t\to\infty} e^{-\rho t} q(t) a(t) = 0$$

$$\dot{a} = ra - c,\ a(0) = a_0$$

Example: consumption-investment problem

The maximized Hamiltonian dynamic system (MHDS)

$$
\begin{aligned}
\dot{c} &= (r - \rho)\, c \\
\dot{a} &= r\, a - c \\
a(0) &= a_0 \\
\lim_{t \to \infty} e^{-\rho t} \frac{a(t)}{c(t)} &= 0
\end{aligned}
$$

Example: consumption-investment problem

- As the system is linear it has an explicit solution, which is something rare for generic optimal control problems
- As for the discrete case, there (at least) three potential methods to find a solution **when the MHDS is linear**
  - **method 1**: as the system is recursive, solve each equation independently, and use the initial and transversality conditions
  - **method 2**: introduce a transformation of variables reducing the system to a backward problem with a scalar ODE
  - **method 3**: solve the coupled ODE equations jointly to get a general solution and use the initial and transversality conditions (this is the only method available when the system is not recursive)
- By solution I mean the particular solution to the ODE problem.

Example: consumption-investment problem

Method 1

- First step: solve the Euler equation $\dot{c} = (r - \rho)\, c$:

$$c(t) = c(0)\, e^{(r-\rho)\, t}, \text{ where } c(0) \text{ is unknown}$$

- Second step: substitute in the budget constraint $\dot{a} = r\, a - c(0)\, e^{(r-\rho)\, t}$, and solve, knowing that $a(0) = a_0$

$$\begin{aligned} a(t) &= e^{r\,t} \left(a_0 - \int_0^t e^{-r\,s}\, c(s)\, ds \right) \\ &= e^{r\,t} \left(a_0 - c(0) \int_0^t e^{-\rho\, s}\, ds \right) \\ &= e^{r\,t} \left(a_0 + \frac{c(0)}{\rho} \left(e^{-\rho\, t} - 1 \right) \right) \end{aligned}$$

Example: consumption-investment problem

Method 1: continuation

- Third step: substitute in the transversality condition to find $c(0)$,

$$\begin{aligned}
\lim_{t\to\infty} e^{-\rho t}\frac{a(t)}{c(t)} &= \lim_{t\to\infty} e^{-\rho t}\frac{e^{(r-\rho)t}}{e^{(r-\rho)t}\,c(0)}\left(a_0 + \frac{c(0)}{\rho}\left(e^{-\rho\, t}-1\right)\right)\\
&= \lim_{t\to\infty}\left(\frac{a_0}{c(0)} - \frac{1}{\rho} + \frac{e^{-\rho t}}{\rho}\right)\\
&= \frac{a_0}{c(0)} - \frac{1}{\rho}\\
&= 0 \;\Rightarrow\; c^*(0) = \rho\, a_0
\end{aligned}$$

- Fourth step: substitute in general solutions for the budget constraint and in the Euler equation, to obtain the particular solutions

$$a^*(t) = a_0\, e^{(r-\rho)t},\; t\in[0,\infty)$$

$$c^*(t) = \rho\, a_0\, e^{(r-\rho)t},\; t\in[0,\infty)$$

Example: consumption-investment problem

Method 2

- First step: come up with a trial function $z(t) = \frac{a(t)}{c(t)}$. Then

$$\frac{\dot{z}}{z} = \frac{\dot{a}}{a} - \frac{\dot{c}}{c}$$

- Second step: substitute from the ODE's in the MHDS and obtain a backward problem

$$\begin{cases} \dot{z} = \rho z - 1 \\ \lim_{t \to \infty} e^{-\rho t} z(t) = 0 \end{cases}$$

- Third step: solve the backward problem. The general solution of the ODE is

$$z(t) = \bar{z} + (z(0) - \bar{z})\, e^{\rho t},$$

  where $z(0)$ is unknown and $\bar{z} = \frac{1}{\rho}$.

Example: consumption-investment problem

Method 2: continuation

- Third step (continuation): To get the particular solution substitute in the transversality condition

$$\begin{aligned} \lim_{t\to\infty} e^{-\rho t} z(t) &= \lim_{t\to\infty} e^{-\rho t}\left(\bar{z} + (z(0)-\bar{z})\, e^{\rho t}\right) \\ &= \lim_{t\to\infty} e^{-\rho t}\,\bar{z} + (z(0)-\bar{z}) \\ &= z(0)-\bar{z} = 0 \;\Rightarrow z(0) = \bar{z} = \rho^{-1} \end{aligned}$$

Then $c(t) = \frac{a(t)}{z(t)} = \rho\, a(t)$

- Fourth step: substitute in the budget constraint and solve the initial-value problem

$$\begin{cases} \dot{a} = (r-\rho)\, a, & t \in [0,\infty) \\ a(0) = a_0, & t = 0 \end{cases}$$

We obtain the same solution.

Example: consumption-investment problem

Method 3: general method for linear MHDS

- First step: observe that the MHDS is a linear ODE system. Defining

$$\mathbf{X}(t) = \begin{pmatrix} a(t) \\ c(t) \end{pmatrix}$$

it can be written in matrix form

$$\dot{\mathbf{X}} = \mathbf{A}\,\mathbf{X}, \text{ where } \mathbf{A} = \begin{pmatrix} r & -1 \\ 0 & r-\rho \end{pmatrix}$$

- Second step: find the general solution of this system we know it is

$$\mathbf{X}(t) = \mathbf{e}^{\mathbf{A}\,t}\,\mathbf{X}(0), \text{ for } t \in [0,\infty)$$

where the vector $\mathbf{X}(0) = \begin{pmatrix} a(0) \\ c(0) \end{pmatrix} = \begin{pmatrix} a_0 \\ c(0) \end{pmatrix}$ where $c(0)$ is unknwon.

Example: consumption-investment problem

Method 3: continuation

- Third step: the hard part is finding $\mathbf{e}^{\mathbf{A}t}$. In optimal control problems we usually have

$$\mathbf{e}^{\mathbf{A}t} = \mathbf{P} \begin{pmatrix} e^{\lambda_- \, t} & 0 \\ 0 & e^{\lambda_+ \, t} \end{pmatrix} \mathbf{P}^{-1}$$

  where $\lambda_\pm$ are the eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is the associated eigenvector matrix.

- To find $\mathbf{e}^{\mathbf{A}t}$ we need to find the eigenvalues and eigenvectors of matrix $\mathbf{A}$.

Example: consumption-investment problem

Method 3: continuation

- To determine the eigenvalues, by finding the roots of the characteristic polynomial equation

$$c(\lambda) = \lambda^2 - \mathrm{Trace}(\mathbf{A})\,\lambda + \mathrm{Det}(\mathbf{A}) = 0$$

  that is

$$\lambda_{\mp} = \frac{\mathrm{Trace}(\mathbf{A})}{2} \pm \sqrt{\left(\frac{\mathrm{Trace}(\mathbf{A})}{2}\right)^2 - \mathrm{Det}(\mathbf{A})}$$

- To determine the eigenvalues the associated eigenvectors, which are the solutions of the homogeneous equations

$$\Big(\mathbf{A} - \lambda_{-}\mathbf{I}\Big)\,\mathbf{P}^{-} = \mathbf{0} \text{ yields } \mathbf{P}^{-}$$

  where $\mathbf{I}$ is the (2×2) identity matrix and

$$\Big(\mathbf{A} - \lambda_{+}\mathbf{I}\Big)\,\mathbf{P}^{+} = \mathbf{0} \text{ yields } \mathbf{P}^{+}$$

- the eigenvector matrix is obtained by concatenating the two eigenvectors

$$\mathbf{P} = \mathbf{P}^{-} | \mathbf{P}^{+}$$

Example: consumption-investment problem

Method 3: continuation

- Third step (continuation): in our problem we obtain

$$\lambda_- = r - \rho, \ \lambda_+ = r$$

  and

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ \rho & 0 \end{pmatrix}$$

- The exponential matrix is

$$\mathbf{e}^{\mathbf{A}\,t} = \begin{pmatrix} 1 & 1 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} e^{(r-\rho)\,t} & 0 \\ 0 & e^{r\,t} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\rho} \\ 1 & -\frac{1}{\rho} \end{pmatrix} = \begin{pmatrix} e^{r\,t} & \frac{1}{\rho}\left(e^{(r-\rho)\,t} - e^{r\,t}\right) \\ 0 & e^{(r-\rho)\,t} \end{pmatrix}$$

- Therefore the general solution to the MHDS is

$$\begin{pmatrix} a(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} a(0)\,e^{r\,t} + \frac{c(0)}{\rho}\left(e^{(r-\rho)\,t} - e^{r\,t}\right) \\ c(0)\,e^{(r-\rho)\,t} \end{pmatrix}$$

Example: consumption-investment problem

Method 3: continuation

- Fourth step: to find $a(0)$ we set $a(t)|_{t=0} = a_0$, and find $a(0) = a_0$ and to find $c(0)$ we use the transversality condition

$$\begin{aligned}
\lim_{t\to\infty} e^{-\rho t}\frac{a(t)}{c(t)} &= \lim_{t\to\infty}\left(\frac{a_0}{c(0)}\,e^{\rho t} + \frac{1}{\rho}\Big(1 - e^{\rho t}\Big)\right) \\
&= \frac{a_0}{c(0)} - \frac{1}{\rho} \\
&= 0
\end{aligned}$$

  where we find again $c^*(0) = \rho\, a_0$.

- Fifth step: we substitute again $c(0) = \rho\, a_0$ in the general solution to get the same (particular) solution to our problem

Solving non-linear optimal control problems

- **Most optimal control problems do not have explicit solutions**
- However, in sufficiently smooth cases qualitative results on the solution can be obtained
- Consider again the infinite-horizon problem

$$
\begin{aligned}
\max_{u(\cdot)} \int_0^\infty & f(u(t), x(t))\, e^{-\rho t}\, dt, \ \text{where}\ \rho > 0 \\
& \text{subject to} \\
& \dot{x} = g(u, x) \\
& x(0) = x_0 \\
& \lim_{t \to \infty} x(t) \ \text{is bounded}
\end{aligned}
$$

- We can obtain a **qualitative solution** to the problem if the solution converges to a steady state.

Solving non-linear optimal control problems (cont.)

- The Hamiltonian function is

$$h(u, x, q) = f(x, u) + q\, g(x, u)$$

- the f.o.c are

$$\begin{aligned} &h_u(u, x, q) = 0 \\ &\dot{q} = \rho\, q - h_x(u, x, q) \\ &\dot{x} = g(u, x) \\ &x(0) = x_0 \\ &\lim_{t \to \infty} e^{-\rho t} q(t) x(t) = 0 \end{aligned}$$

- Assumption: $h_{uu}(u, x, q) = \frac{\partial^2 h}{\partial u^2} \neq 0$.
- From the implicit function theorem, from $h_u(u, x, q) = 0$ we can obtain uniquely

$$u = U(x, q)$$

  at the optimum

Solving non-linear optimal control problems (cont.)

- The maximized Hamiltonian is

$$h^*(x, q) = h(U(x, q), x, q)$$

- Then we get the modified Hamiltonian dynamic system (MHDS):

$$\begin{cases} \dot{x} = \dot{k}(q, x) \equiv g(x, U(x, q)) \\ \dot{q} = \dot{q}(q, x) \equiv \rho q - h_x(x, U(x, q)) \end{cases}$$

- Assume the MHDS has a fixed point $(\bar{q}, \bar{x})$ such that $\dot{q} = \dot{k} = 0$.
- In the neighbourhood of $(\bar{x}, \bar{q})$ we can approximate the MHDS by the linear system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial q} \\ \frac{\partial \dot{q}}{\partial x} & \frac{\partial \dot{q}}{\partial q} \end{pmatrix} \begin{pmatrix} x(t) - \bar{x} \\ q(t) - \bar{q} \end{pmatrix} = \mathbf{J} \begin{pmatrix} x(t) - \bar{x} \\ q(t) - \bar{q} \end{pmatrix}$$

Solving non-linear optimal control problems (cont.)

- The Jacobian becomes

$$\mathbf{J} = \begin{pmatrix} h^*_{qx}(\bar{x}, \bar{q}) & h^*_{qq}(\bar{x}, \bar{q}) \\ -h^*_{xx}(\bar{x}, \bar{q}) & \rho - h^*_{xq}(\bar{x}, \bar{q}) \end{pmatrix}$$

- It can be proven that $h^*_{xq} = h^*_{qx}$ which implies that the Jacobian has the structure

$$\mathbf{J} = \begin{pmatrix} a & b \\ c & \rho - a \end{pmatrix}$$

- having trace and determinant

$$\mathrm{tr}(\mathbf{J}) = \rho > 0, \; \det(\mathbf{J}) = a(\rho - a) - bc < 0$$

- this implies the eigenvalues of $\mathbf{J}$ are real and satisfy $\lambda_- < 0 < \lambda_+$
- Interpretation: - the equilibrium point $(\bar{x}, \bar{q})$ is a saddle point. The stable manifold associated with $(\bar{x}, \bar{q})$ is the solution set of the OC problem.
  - this means that the solution to the OC problem is unique.

Solving non-linear optimal control problems (cont.)

- Solving the approximate system we find

$$\begin{pmatrix} x(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{q} \end{pmatrix} + h_- \, \mathbf{P} e^{\lambda_- \, t} + h_+ \, \mathbf{P} e^{\lambda_+ \, t}$$

- where h_- and h_+ are arbitrary constants.
- From what we concluded regarding the signs of the eigenvalues, then

$$\lim_{t \to \infty} e^{\lambda_- \, t} = 0, \text{and } \lim_{t \to \infty} e^{\lambda_+ \, t} = +\infty$$

- We find the two constants:
  - by forcing the solution to converge to the steady state by making $h_+ = 0$
  - by making it satisfy the initial value of the state variable, by solving for h_-

$$x_0 = x(0) = \bar{x} + h_- \, \mathbf{P}_1^-$$

Solving non-linear optimal control problems (cont.)

- Therefore the **approximate solution** is

$$\begin{pmatrix} x(t) \\ q(t) \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{q} \end{pmatrix} + (x_0 - \bar{x}) \begin{pmatrix} 1 \\ \frac{\mathbf{P}_2^-}{\mathbf{P}_1^-} \end{pmatrix} e^{\lambda_- t}$$

- As required we find

$$\lim_{t \to \infty} \begin{pmatrix} x(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{q} \end{pmatrix}$$

- and the initial values for the state and the co-state variables

$$\begin{pmatrix} x(t) \\ q(t) \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} x(0) \\ q(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ \bar{q} + (x_0 - \bar{x}) \frac{\mathbf{P}_2^-}{\mathbf{P}_1^-} \end{pmatrix}$$

The Ramsey model

- **The problem**: find the optimal allocation of savings through time in order to maximize the time aggregate of the discounted **value** of consumption (in utility terms), when there is a technology of production displaying decreasing marginal returns:

$$\max_{c} \int_0^{\infty} e^{-\rho t} u(c(t))dt, \ \rho > 0,$$

$$\begin{aligned}
&\text{subject to} \\
&\dot{k} = f(k) - c, \ t \in [0, \infty) \\
&k(0) = k_0, \ \text{given} \\
&\lim_{t \to \infty} e^{-\rho t} k(t) \geq 0
\end{aligned}$$

- Utility and production functions, $u(c)$ and $f(k)$; are increasing, concave and Inada :

$$u''(.) \leq 0 < u'(.), \ u'(0) = \infty, \ u'(\infty) = 0$$

$$f''(.) \leq 0 < f'(.), \ f'(0) = \infty, \ f'(\infty) = 0$$

The Ramsey model: optimality conditions

- The current-value Hamiltonian

$$h(c, k, q) = u(c) + q\,(f(k) - c)$$

- The Pontriyagin's f.o.c

$$\begin{aligned} u'(c(t)) &= q(t) \\ \dot{q} &= q(t)\left(\rho - f'(k(t))\right) \\ \lim_{t\to\infty} e^{-\rho t} q(t) k(t) &= 0 \\ \dot{k} &= f(k(t)) - c(t) \\ k(0) &= k_0 \end{aligned}$$

The Ramsey model: the non-linear MHDS

- The MHDS

$$
\begin{aligned}
\dot{c} &= \frac{c}{\sigma(c)}\,(r(k)-\rho) \\
\dot{k} &= f(k) - c \\
k(0) &= k_0 > 0 \\
0 &= \lim_{t\to\infty} e^{-\rho t} u'(c(t))\,k(t)
\end{aligned}
$$

  where
  $r(k) \equiv f'(k)$ is the rate of return of capital
  $\sigma(c) \equiv -\frac{u''(c)c}{u'(c)}$ is the inverse of the elasticity of intertemporal substitution

- The MHDS has no explicit solution: we can only use qualitative methods:
  - determine the steady state(s)
  - linearize the system around the candidate steady states
  - solve the linearized MHDS

The Ramsey model: the linearized MHDS

- The steady state (if $k > 0$ and $c > 0$)

$$\begin{aligned} r(\bar{k}) &= \rho \Rightarrow \bar{k} = (r)^{-1}(\rho) \\ \bar{c} &= f(\bar{k}) \end{aligned}$$

  is unique from the Inada property of $f(k)$ implying $r(k) \in (0, \infty)$
- The linearized MHDS is

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \psi & 0 \end{pmatrix} \begin{pmatrix} k(t) - \bar{k} \\ c(t) - \bar{c} \end{pmatrix}$$

  where $\psi \equiv \frac{\bar{c}}{\sigma(\bar{c})} r'(\bar{k}) < 0$ because of the concavity of $f(\cdot)$
- The jacobian J has trace and determinant:

$$\mathrm{tr}(J) = \rho, \ \det(J) = \psi < 0$$

  then $(\bar{k}, \bar{c})$ is a saddle point

The Ramsey model: solving the linearized MHDS

The general solution of the linearized MHDS

- The general solution is

$$\begin{pmatrix} k(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \end{pmatrix} + h_- \mathbf{P}^- e^{\lambda_- t} + h_+ \mathbf{P}^+ e^{\lambda_+ t}$$

  where $\lambda_\pm$ are the eigenvalues and $\mathbf{P}^\mp$ are the associated eigenvectors of matrix $\mathbf{J}$

- The eigenvalues of $\mathbf{J}$ are

$$\lambda_- = \frac{\rho}{2} - \sqrt{\Delta} < 0, \; \lambda_+ = \frac{\rho}{2} + \sqrt{\Delta} > \rho$$

  where the discriminant of J is $\Delta = \left(\frac{\rho}{2}\right)^2 - \psi > \left(\frac{\rho}{2}\right)^2$

- The eigenvector matrix is

$$\mathbf{P} = (\mathbf{P}^- | \mathbf{P}^+) = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}$$

- Then the general solution to the approximate MHDS is Then

$$\begin{pmatrix} k(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \end{pmatrix} + h_- \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} e^{\lambda_- t} + h_+ \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix} e^{\lambda_+ t}$$

The Ramsey model: solving the linearized MHDS

The particular solution of the linearized MHDS

- To find the particular solution, we determine the constants: h_- and h_+ such that the the solution converges to the steady state and the initial value for $k(0) = k_0$ is satisfied:
  - convergence to the steady state

$$\lim_{t\to\infty} \begin{pmatrix} c(t) \\ k(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \end{pmatrix} \Leftrightarrow h_+ = 0$$

  - initial value for the state variable is satisfied if

$$k(t)|_{t=0} = \bar{k} + h_- = k_0 \Leftrightarrow h_- = k_0 - \bar{k}$$

The Ramsey model: the approximate solution

- The **approximate** solution to the Ramsey model is, therefore,
  Therefore, the linearized solution is

$$\begin{pmatrix} k^*(t) \\ c^*(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \end{pmatrix} + (k_0 - \bar{k}) \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} e^{\lambda_- t}$$

- Solution at $t = 0$

$$\begin{pmatrix} k^*(0) \\ c^*(0) \end{pmatrix} = \begin{pmatrix} k_0 \\ \bar{c} + \lambda_+(k_0 - \bar{k}) \end{pmatrix}$$

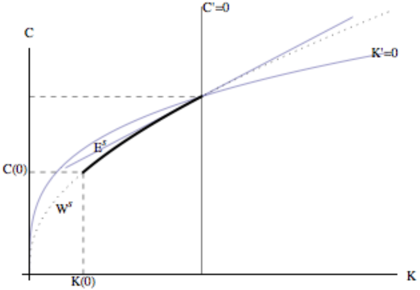
  the initial value of consumption is determined endogenously

- Asymptotic solution

$$\lim_{t \to \infty} \begin{pmatrix} k^*(t) \\ c^*(t) \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \end{pmatrix}$$

  the solution tends to the fixed point of the MHDS.

Ramsey model: phase diagram



Exact solution (stable manifold - W^s), linearized solution (stable eigenspace - E^s).
Close to the steady state W^s has slope equal to to the slope of E^s, and they are higher than the slope of the isocline $\dot{k}(c,k) = 0$

$$\left.\frac{dc}{dk}\right|_{W^s} = \left.\frac{dc}{dk}\right|_{E^s} = \lambda_+ > \left.\frac{dc}{dk}\right|_{\dot{k}} = f'(\bar{k}) = \rho$$

Dynamic Programming

Principle of the Dynamic Programming

- Consider again the infinite horizon discounted optimal control problem

$$\max_{u(\cdot)} \int_0^{\infty} e^{-\rho t} f(x(t), u(t))\, dt$$

subject to

$$\begin{cases} \dot{x} = g(x, u), & 0 \leq t < \infty \\ x(0) = \phi, & t = 0 \\ \lim_{t \to \infty} h(t) x(t) \geq 0 \end{cases}$$

Principle of the Dynamic Programming

Proposition

Let (x^*, u^*) be a solution of the discounted infinite horizon optimal control problem. Then for every $t \in [0, \infty)$, $x^*(t) = x$, and $u^*(t) = u$ satisfy the Hamilton-Jacobi-Bellman equation

$$\rho V(x) = \max_u \left[f(x,u) + V'(x)\, g(x,u) \right]$$

- We call

$$u^* = U\big(x, V'(x)\big)$$

the policy function, where $u^* = \arg\max_u \{f(x,u) + V^{'}(x) g(x,u)\}$.

Solving optimal control problems using DP

- First, write the HJB equation: $\rho V(x) = \max_u \left[f(x,u) + V'(x)\, g(x,u)\right]$
- Second, find the policy function: $u = U(x, V'(x))$
- Third, substitute in the HJB equation to obtain

$$\rho V(x) = f\Big(x, U\big(x, V'(x)\big)\Big) + V'(x)\, g\Big(x, U\big(x, V'(x)\big)\Big)$$

- Fourth, In some rare cases we are able to solve this equation for $V(x)$ (see next)
- Fifth, if we know $V(x)$, we can take its derivative $V'(x)$ and can find the optimal control as a function of the state variable $u^* = U^*(x)$
- Sixth, then we can substitute in the constraints

$$\begin{cases} \dot{x} = g(x, U^*(x)) \\ x(0) = x_0 \end{cases}$$

  and solve it to get the optimal state trajectory $\Big(x^*(t)\Big)_{t=0}^{\infty}$.
- Seventh, at last we can obtain the optimal path for the control variable, $\Big(u^*(t)\Big)_{t=0}^{\infty}$. by using $u^*(t) = U^*(x^*(t))$.

Application: the consumption-investment problem

A representative household (or economy) wants to find the trajectories for consumption and asset holdings, $(C(t))_{t=0}^{\infty}$, and $(W(t))_{t=0}^{\infty}$ that solve the problem:

$$\max_{c(\cdot)} \int_0^{\infty} e^{-\rho t} \ln(c(t))\, dt, \ \rho > 0$$

subject to

$$\begin{cases} \dot{a} = ra - c & \text{(budget constraint)} \\ a(0) = a_0 & \text{(initial wealth} \\ \lim_{t\to\infty} e^{-r\,t}\, a(t) \geq 0 & \text{(NPG constraint)} \end{cases}$$

Application: (cont.)

- **First step:** Write the HJB equation

$$\rho V(w) = \max_{c} \left[\ln(c) + V^{'}(a)(ra - c) \right]$$

- **Second step:** find the policy function for consumption

$$\frac{1}{c^*} - V^{'}(a) = 0 \Leftrightarrow c^* = \left(V^{'}(a) \right)^{-1}$$

- **Third step:** substitute it into the HJB equation

$$\rho V(a) = -\ln(V^{'}(a)) + V^{'}(a)ra - 1$$

Application: (cont.)

- **Fourth step:** solving the HJB equation by using the **method of undermined coefficients** (i.e., α and β)
  - put forward a **trial function**

$$V(a) = \alpha + \beta \ln(W) \Rightarrow V'(a) = \frac{\beta}{a}$$

  - substitute in the HJB equation

$$\rho(\alpha + \beta \ln(W)) = -\ln(\beta) + \ln(a) + r\beta - 1$$

  - separate terms, isolating $\ln(a)$,

$$(\rho\beta - 1)\ln(a) = -\ln(\beta) + r\beta - 1 - \rho\alpha$$

  - determine the coefficients, α and β, such that both sides are equal to zero:

$$\beta = \frac{1}{\rho} \text{ and } \alpha = \frac{1}{\rho}\left(\ln\rho + \frac{r}{\rho} - 1\right)$$

  - Then our conjecture was correct and the value function is

$$V(a) = \frac{1}{\rho}\left(\frac{r-\rho}{\rho} + \ln\rho + \ln(a)\right)$$

Application: (cont.)

- **Fifth step:** obtain the optimal consumption rule

$$c^* = (V'(a))^{-1} = \rho a$$

- **Sixth step:** substitute in the budget constraint and define the optimal forward equation (back to the time domain)

$$\begin{cases} \dot{a} = (r - \rho)\, a \\ a(0) = a_0 \end{cases}$$

- **Seventh step:** solve the initial value problem (for a), and substitute in the policy function (for c) to obtain the solution

$$a^*(t) = a_0\, e^{(r-\rho)t},\ t \in [0, \infty)$$

$$c^*(t) = \rho\, a_0\, e^{(r-\rho)t},\ t \in [0, \infty)$$

Application: characterizing the solution

- **Eigth step:** (optional) make sure the NPG is satisfied

$$\lim_{t\to\infty} e^{-r\,t}a^*(t) = \lim_{t\to\infty} e^{-\rho\,t}a_0 = 0$$

- **This holds for any values of r and ρ.** We already know that we may have the following solutions
  - if $r < \rho$ then the financial wealth and consumption converge to zero
  $$\lim_{t\to\infty} a^*(t) = \lim_{t\to\infty} a^*(t) = 0$$
  - if $r > \rho$ then the financial wealth and consumption grow without bounds
  $$\lim_{t\to\infty} a^*(t) = \lim_{t\to\infty} c^*(t) = \infty$$
  - if $r = \rho$ then the solution is stationary

$$a^*(t) = a_0,\;\; c^*(t) = \rho a_0,\;\text{for every } t \in [0,\infty)$$

Proofs

Proof of proposition 1

- The value functional is for any paths (x, u)

$$
\begin{aligned}
V(x) &= \int_0^T f(u(t), x(t), t)\, dt = \text{(definition of } H \text{ function)} \\
&= \int_0^T H(u(t), x(t), t) - \lambda(t)\dot{x}(t)\, dt = \text{(integration by parts)} \\
&= \int_0^T \Big(H(u(t), x(t), \lambda(t), t) + \dot{\lambda}(t)x(t) \Big)\, dt + \lambda(0)x(0) - \lambda(T)x(T)
\end{aligned}
$$

- The value at the optimum is

$$
\begin{aligned}
V(x^*) &= \int_0^T f(u^*(t), x^*(t), t)\, dt = \\
&= \int_0^T \Big(H(u^*(t), x^*(t), \lambda(t), t) + \dot{\lambda}(t)x^*(t) \Big)\, dt + \lambda(0)x^*(0) - \lambda(T)x^*(T) \\
&\quad (+\mu\, h(T)x^*(T) \text{ (for case P3)})
\end{aligned}
$$

Proof of proposition 1 (cont.)

- Now we introduce perturbations in the state and co-state variables $x(t) = x^* + \epsilon\, d_x(t)$ and $u(t) = u^* + \epsilon\, d_u(t)$
- The perturbations are admissible if $d_x(0) = 0$ and, for (P1) $d_x(T) = 0$, and $d_x(T)$ is free for (P2) and (P3).
- The optimal should satisfy

$$\delta V(x^*) = \lim_{\epsilon \to 0} \frac{V(x^* + \epsilon d_x) - V(x^*)}{\epsilon} = \frac{dV(x^*)}{d\epsilon} = 0.$$

Proof of proposition 1 (cont.)

But, writing $H^*(t) = H(u^*(t), x^*(t), \lambda(t), t)$ we have:

- For case (P1) , where $d_x(0) = d_x(T) = 0$

$$\frac{dV(x^*)}{d\epsilon} = \int_0^T \left[\frac{\partial H^*(t)}{\partial u} d_u(t) + \left(\frac{\partial H^*(t)}{\partial x} + \dot{\lambda}(t) \right) d_x(t) \right] dt$$

  then

$$\delta V(x^*) = 0 \Leftrightarrow \frac{\partial H_t^*}{\partial u} = \frac{\partial H_t^*}{\partial x} + \dot{\lambda} = 0$$

- For case (P2) , where $d_x(0) = 0$ and $d_x(T)$ is free

$$\frac{dV(x^*)}{d\epsilon} = \int_0^T \left[\frac{\partial H^*(t)}{\partial u} d_u(t) + \left(\frac{\partial H^*(t)}{\partial x} + \dot{\lambda}(t) \right) d_x(t) \right] dt - \lambda(T) d_x(T)$$

  then

$$\delta V(x^*) = 0 \Leftrightarrow \frac{\partial H_t^*}{\partial u} = \frac{\partial H_t^*}{\partial x} + \dot{\lambda} = \lambda(T) = 0$$

Proof of proposition 1 (cont.)

- For case (P3), where $d_x(0) = 0$ and $d_x(T)$ is free

$$\frac{dV(x^*)}{d\epsilon} = \int_0^T \left[\frac{\partial H^*(t)}{\partial u} d_u(t) + \left(\frac{\partial H^*(t)}{\partial x} + \dot{\lambda}(t) \right) d_x(t) \right] dt + \\ + (\mu h(T) - \lambda(T))\, d_x(T)$$

  and the Kuhn-Tucker condition $\mu h(T) x^*(T) = 0$ for $\mu \geq 0$ should also hold, then

$$\delta V(x^*) = 0 \Leftrightarrow \frac{\partial H_t^*}{\partial u} = \frac{\partial H_t^*}{\partial x} + \dot{\lambda} = \lambda(T)\, x^*(T) = 0$$

Return.