

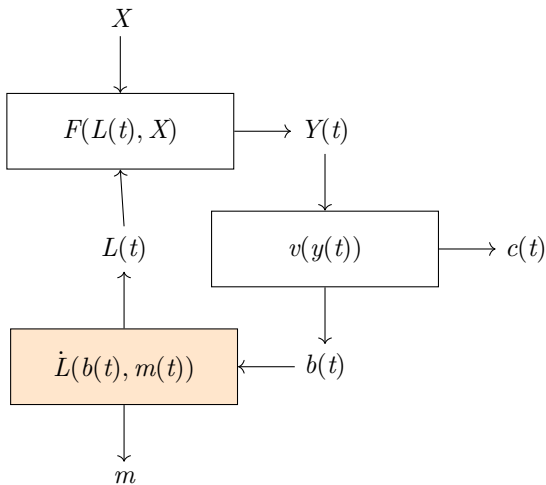
The Malthusian growth model

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Assumptions

- ▶ Production:
 - ▶ production uses two factors: labor and land
 - ▶ the production function has constant returns to scale
 - ▶ the only reproducible factor is labor, and it faces decreasing marginal returns
- ▶ Population:
 - ▶ fertility is endogenous and mortality is exogenous
- ▶ Farmers:
 - ▶ households are land-owners
 - ▶ they choose among consumption and child-rearing
 - ▶ there are no savings



where $v(y) = \max\{u(c, b) : c + \rho b \leq y\}$ and $y = \frac{Y}{L}$

The Malthus model as a growth model

- ▶ Defining the GDP per capita as

$$y(t) = \frac{Y(t)}{L(t)}$$

- ▶ we want to know what this theory implies for
 - ▶ the rate of growth of GDP $g(t) = \frac{\dot{y}(t)}{y(t)}$
 - ▶ the steady state level of GDP \bar{y}
 - ▶ and the dynamics: i.e. separating $g(t)$ into transition and long-run dynamics

The model

Production

- **Production function**

$$Y(t) = (AX)^{\alpha} L(t)^{1-\alpha}, \quad 0 < \alpha < 1$$

where: A productivity, X stock of land, L labor input

- displays constant returns to scale

$$(\lambda AX)^{\alpha} (\lambda L)^{1-\alpha} = \lambda Y$$

- implication: the Euler theorem holds

$$Y = \frac{\partial Y}{\partial L} L + \frac{\partial Y}{\partial X} X$$

The model

Production

- ▶ it has positive marginal returns

$$\frac{\partial Y}{\partial L} = (1 - \alpha) \frac{Y}{L} > 0, \frac{\partial Y}{\partial X} = \alpha \frac{Y}{X} > 0$$

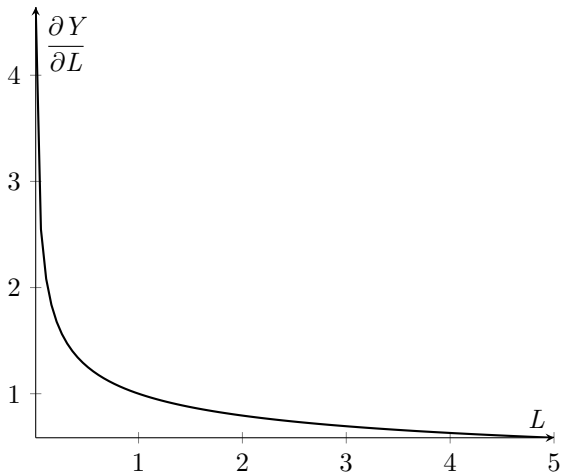
- ▶ but decreasing marginal returns

$$\frac{\partial^2 Y}{\partial L^2} = -\alpha(1 - \alpha) \frac{Y}{L} < 0, \frac{\partial^2 Y}{\partial X^2} = -\alpha(1 - \alpha) \frac{Y}{X} < 0$$

- ▶ and is Inada

The model

Inada property



The model

Production efficiency

- ▶ production efficiency:

$$\max_{L,X} \{ Y(L, X) - wL - RX \}$$

where w is the wage rate and R are is land rent

- ▶ and competitive markets lead to

$$w(L, X) = \frac{\partial Y}{\partial L} = (1 - \alpha) \frac{Y}{L} > 0$$

$$R(L, X) = \frac{\partial Y}{\partial X} = \alpha \frac{Y}{X} > 0$$

- ▶ with the properties

$$w_L < 0, w_X > 0; R_X < 0, R_L > 0$$

Farmers' problem

Endogenous rate of population growth

- ▶ There are L farmers; who get percapita income from farming and decide which part to consume and which part allocate to raising offspring, by deciding the number of offspring (Beckerian model)
- ▶ **Household's (farmer's) problem**

$$\max_{c(t), b(t)} \{c(t)^{1-\gamma} b(t)^\gamma : c(t) + \rho b(t) = y(t)\}$$

ρ = relative cost of raising children

- ▶ solution

$$c(t) = (1 - \gamma)y(t)$$

$$b(t) = \frac{\gamma}{\rho}y(t)$$

The model

Population dynamics

► Population growth

$$\dot{L} \equiv \frac{dL(t)}{dt} = (b(t) - m)L(t)$$

- where the fertility rate $b(t)$
- the mortality rate is exogenous m
- the initial level of population is assumed to be given by number L_0

$$L(t)|_{t=0} = L(0) = L_0$$

The Malthusian model

Endogenous rate of population growth

- ▶ Then

$$\dot{L} = \left(\frac{\gamma}{\rho} y(t) - m \right) L(t)$$

- ▶ where the per capita GDP is

$$y(t) \equiv \frac{Y(t)}{L(t)} = \left(\frac{AX}{L(t)} \right)^\alpha$$

- ▶ There are two approaches to **solving the model**
 - ▶ Approach 1: solve the differential equation for L and substitute in y to get the dynamics of growth
 - ▶ Approach 2: get a differential equation in y and solve it

Detour

Per-capita rate of growth arithmetics

- ▶ taking log-derivatives w.r.t time we have

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L}$$

- ▶ that we denote by

$$g(t) = g_Y(t) - n(t)$$

- ▶ as the per capita GDP is

$$y(t) \equiv \frac{Y(t)}{L(t)} = \left(\frac{AX}{L(t)} \right)^\alpha$$

- ▶ Then

$$g(t) = \frac{\dot{y}}{y} = -\alpha \frac{\dot{L}}{L}$$

Solving the Malthusian model

Approach 1: solving for L

- ▶ If we substitute y in the dynamic equation for L we have the initial value problem

$$\begin{cases} \dot{L} = \left(\frac{\gamma}{\rho} \left(\frac{AX}{L(t)} \right)^{\alpha} - m \right) L(t), & t \geq 0 \\ L(0) = L_0 \text{ given} & t = 0 \end{cases}$$

- ▶ we can solve it
- ▶ then differentiate it and substitute in

$$g(t) = -\alpha \frac{\dot{L}}{L}$$

to get model's explanation for the per-capita growth of the economy

Solving the Malthusian model

Approach 2: solving for y directly

► from

$$\frac{\dot{y}}{y} = -\alpha \frac{\dot{L}}{L}$$

► we get the dynamic equation for the GDP per capita

$$\dot{y} = -\alpha \left(\frac{\gamma}{\rho} y(t) - m \right) y(t) \quad (1)$$

together with the initial value

$$y(0) = y_0 = (AX)^\alpha L_0^{1-\alpha}$$

Solving the Malthusian model

Explicit solution for y

- ▶ Equation (1) has two steady states $y^* = \{0, \bar{y}\}$ where

$$\bar{y} = \frac{m\rho}{\gamma}$$

- ▶ we can re-write the growth equation as

$$\dot{y} = \alpha \frac{\gamma}{\rho} (\bar{y} - y(t)) y(t)$$

Explicit solution for y

- ▶ This is a Bernoulli differential equation with has an explicit solution [appendix](#)

$$y(t) = \left[\frac{1}{\bar{y}} + \left(\frac{1}{y(0)} - \frac{1}{\bar{y}} \right) e^{-\alpha m t} \right]^{-1}, \text{ for } 0 \leq t < \infty$$

- ▶ satisfies $\lim_{t \rightarrow \infty} y(t) = \bar{y}$

Explicit solution for g

- ▶ the GDP growth rate is

$$g(t) = \alpha m \left[1 - \left(1 + \left(\frac{\bar{y}}{y(0)} - 1 \right) e^{-\alpha m t} \right)^{-1} \right], \text{ for } 0 \leq t < \infty$$

Malthusian model

Properties

- ▶ there is **no long run growth**, because $\lim_{t \rightarrow \infty} g(t) = 0$
- ▶ the **long run level** of GDP per capita

$$\bar{y} = \frac{m\rho}{\gamma}$$

increases with the mortality rate, the cost of rearing children and the "moral restraint" (no productivity effects)

- ▶ there is **only transitional dynamics**
 - ▶ if the initial GDP $y(0)$ is small then there is an increase in time of the GDP $g(t) > 0$
 - ▶ if the initial GDP $y(0)$ is large then there is a decrease in time of the GDP $g(t) < 0$

Malthusian model

Mechanics of the model

- ▶ if $y(0)$ is large so is the wage rate $w(0) = (1 - \alpha)y(0)$
- ▶ this implies that the initial fertility rate is higher, $b(0) = \frac{\gamma}{\rho}y(0)$
- ▶ population increases, which increases output,
- ▶ but decreases the rate of growth of GDP

$$g(t) = -\alpha n(t)$$

because there are decreasing marginal returns due to the fact that X is fixed.

Malthusian model

Trajectories: y , L and w

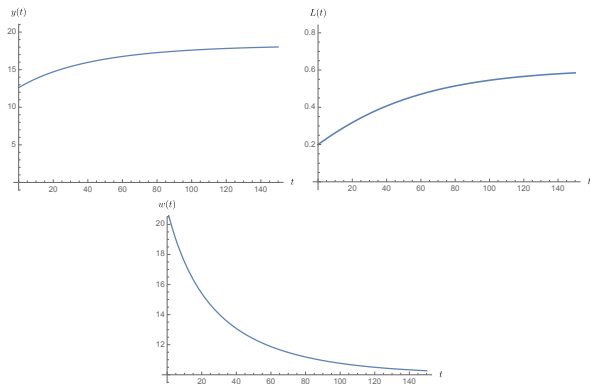


Figure: Parameter values: $\alpha = 2/3$, $m = 0.03$, $\gamma = 0.01$, $\rho = 10$, $A = 1$, $X = 100$, $y(0) < \bar{y}$

Malthusian model

Exponential increase in land productivity

Can increases in land-productivity generate long-run growth?

- ▶ now $Y(t) = (A(t)X)^\alpha L(t)^{1-\alpha}$
- ▶ where $\dot{A} = g_A A$, $g_A > 0$
- ▶ this implies

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{A}}{A} + (1 - \alpha) \frac{\dot{L}}{L} - \frac{\dot{L}}{L} = \alpha \left(m + g_A - \frac{\gamma}{\rho} y(t) \right)$$

- ▶ there is **no increase in the long run growth rate**

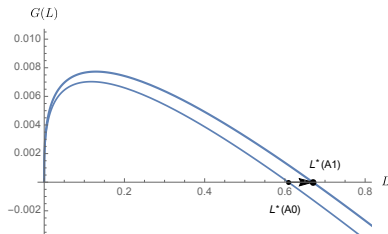
$$\lim_{t \rightarrow \infty} g(t) = 0$$

- ▶ there is **an increase in GDP level**

$$\bar{y} = \frac{(g_A + m)\rho}{\gamma}$$

Malthusian model and land productivity

Phase diagram for an increase in A



Can an increase in the productivity of labor generate long-run growth ?

- ▶ now $Y(t) = X^\alpha (hL(t))^{1-\alpha}$
- ▶ where $\dot{h} = g_h h$, $g_h > 0$
- ▶ this implies

$$\frac{\dot{y}}{y} = (1 - \alpha) \left(\frac{\dot{h}}{h} + \frac{\dot{L}}{L} \right) - \frac{\dot{L}}{L} = \alpha \left[\frac{(1 - \alpha)}{\alpha} g_h + m - \frac{\gamma}{\rho} y(t) \right] y(t)$$

- ▶ there is **no increase in the long run growth rate**

$$\lim_{t \rightarrow \infty} g(t) = 0$$

- ▶ there is **an increase in GDP level**

$$\bar{y} = \frac{((1 - \alpha)g_h + \alpha m)\rho}{\alpha \gamma}$$

Can learning by doing generate long-run growth ?

- ▶ now $Y(t) = (A(t)X)^\alpha L(t)^{1-\alpha}$
- ▶ from $A(t) = \beta \int_{-\infty}^t e^{-\mu(t-s)} A(s)y(s)ds$ we get

$$\dot{A} = (\beta y(t) - \mu) A(t)$$

- ▶ meaning: past production experience increases productivity
- ▶ the dynamic equation for per-capita GDP becomes

$$\frac{\dot{y}}{y} = \left(\beta - \alpha \frac{\gamma}{\rho} \right) y(t) + \alpha m - \mu$$

Can learning by doing generate long-run growth ? Cont.

- ▶ If we assume $\beta = \alpha \frac{\gamma}{\rho}$ then

$$\dot{y} = (\alpha m - \mu)y$$

- ▶ there is **long run growth** if $\alpha m > \mu$ because

$$g(t) = \alpha m - \mu > 0, \text{ for all } t > 0$$

- ▶ the GDP level is exogenous

$$y(t) = y_0 e^{(\alpha m - \mu)t}$$

Conclusions

- ▶ the existence of decreasing marginal returns to the reproducible factor of production (labor, L) implies that the Malthusian model does not feature long-run growth: there is only transitional dynamics (if initial population is too high, wages will be too low, which generates a fall in fertility and therefore a decrease in population until population is constant)
- ▶ **exogenous** permanent increases in productivity will only increase the long-run **GDP level** but will not generate long-run growth
- ▶ however, **endogenous** increases in productivity (v.g, generated by learning-by-doing) may generate long run growth (but in this case there is not transition dynamics)

References

- ▶ (Galor, 2011, ch 2, 3)

Oded Galor. *Unified Growth Theory*. Princeton University Press, 2011.

Appendix

Solving a linear ODE's

- The linear ODE

$$\dot{x} = \lambda(x(t) - \bar{x})$$

has an exact solution

$$x(t) = \bar{x} + (k - \bar{x})e^{\lambda t}$$

where k is an arbitrary constant

- The initial value problem

$$\begin{cases} \dot{x} = \lambda(x(t) - \bar{x}) \\ x(0) = x_0 \text{ given} \end{cases}$$

has the exact solution

$$x(t) = \bar{x} + (x_0 - \bar{x})e^{\lambda t}$$

Appendix

The linear and Bernoulli ODE's

- ▶ The Bernoulli equation is

$$\dot{x} = \alpha x(t) - \beta x^\eta$$

- ▶ If we set $z(t) = x(t)^{1-\eta}$ and differentiate

$$\begin{aligned}\dot{z} &= (1 - \eta)x(t)^{-\eta}\dot{x} = \\ &= (1 - \eta)x(t)^{-\eta}(\alpha x(t) - \beta x^\eta) = \\ &= \lambda(z(t) - \bar{z})\end{aligned}$$

- ▶ is a linear ODE with solution with $\lambda = (1 - \eta)\alpha$ and $\bar{z} = \frac{\beta}{\alpha}$

$$z(t) = \bar{z} + (k_z - \bar{z})e^{\alpha(1-\eta)t}$$

- ▶ transforming back by making $x(t) = z(t)^{\frac{1}{1-\eta}}$

$$x(t) = \left(\frac{\beta}{\alpha} + \left(x(0)^{1-\eta} - \frac{\beta}{\alpha} \right) e^{\alpha(1-\eta)t} \right)^{\frac{1}{1-\eta}}$$