Advanced Mathematical Economics

Paulo B. Brito
PhD in Economics: 2022-2023
ISEG
Universidade de Lisboa
pbrito@iseg.ulisboa.pt

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Chapter 12

Scalar parabolic partial differential equations

12.1 Introduction

Parabolic partial differential equations involve a known function F depending on two independent variables (t, x), an unknown function of them $u(t, x)$, the first partial derivative as regards t and first and second partial derivatives as regards the "spatial" variable x:

$$F(t, x, u, u_t, u_x, u_{xx}) = 0$$

for $u : \mathrm{T} \times \mathrm{X} \to \mathbb{R}$, where $\mathrm{T} \subseteq \mathbb{R}_+$ and $\mathrm{X} \subseteq \mathbb{R}$.

In its simplest form, $F(u_t, u_{xx}) = 0$, the equation models the dynamics (in time) of a cross section distribution driven by dispersion. Dispersion if generated by spatial contact (think about the change in the distribution of a pollutant spreading within a lake where the water is completely still). Equation $F(u_t, u_x, u_{xx}) = 0$ features both dispersion and advection behaviors (think about the change in the distribution of a pollutant spreading within a flowing river). Equation $F(u_t, u, u_x, u_{xx}) = 0$ jointly displays dispersion, advection and growth or decay behaviors (think about the change in the distribution of a pollutant spreading within a flowing river, in which there is a permanent inflow or outflow of new pollutants). The independent terms, t and x, appear in function $F(.)$ if there are some time or spatial specific components.

We will also see in the next chapter that there is a close connection between partial differential equations and stochastic differential equations. This is the reason explaining the fact that continuous-time finance has been using parabolic PDE's since the beginning of the 1970's.

In economics and finance applications it is important to distinguish between **forward** (FPDE) and **backward** (BPDE) parabolic PDE's. While the first are complemented with an initial distribution and generate a flow of distributions forward in time, the latter are complemented with a terminal distribution and its solution generate a flow of distributions consistent with that terminal constraint. While for FPDE the terminal distribution is unknown, for FPDE the distribution at time $t = 0$ is unknown. For planar systems, we may have forward, backward or forward-backward (FBPDE) parabolic PDE's. The last case can be seen as a generalization of the saddle-path dynamics for ODE's.

In mathematical finance most applications, such as the Black and Scholes (1973) model, are PDE's of the backward type. In economics there is recent interest in PDE's related to the topical importance of distribution issues, and, in particular to spatial dynamics modelled by BPDE. Optimal control of PDE's and mean-field games usually lead to FBPDE's.

The body of theory and application of parabolic PDE's is huge. We only present next some very introductory results and applications. In particular, we deal with linear PDE's having explicit solutions, and that can be useful for studying the dynamics of the distributions for stochastic differential equations.

The rest of this chapter presents an overview for linear scalar PDEs in section 12.2. Section 12.3 contains the solutions for the simplest FPDEs. Sections 12.4 and 12.5 deal with homogeneous and non-homogenous general linear scalar parabolic PDE, respectively. Section 12.6 contains an introduction to forward non-autonomous equations. Next we briefly present the solutions to linear backward equations in section 12.7. The remaining sections present applications of previous results to solving some simple Fokker-Planck equations, in section 12.8, and to economics and finance in section 12.9.

12.2 A general overview of linear scalar parabolic PDE's

Consider function $u(t, x)$ where $(t, x) \in \mathrm{T} \times \mathrm{X} \subseteq \mathbb{R} \times \mathbb{R}_+$ and assume it is an at least $C^{1,2}(\mathbb{R}_+, \mathbb{R})$ function[1]. There are several types of linear parabolic PDE:

- if $F(.)$ is linear in the derivatives of u, but the coefficients can be non-linear functions of u, as
$$u_t = a(x, t, u)u_{xx} + b(x, t, u)u_x + c(x, t, u),$$
then it is called a **quasi-linear** parabolic PDE;
- it $F(.)$ is linear in the derivatives of u, and the coefficients are independent from u, as
$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(x, t, u)$$
then it is called a **semi-linear** parabolic PDE;
- if $F(.)$ is linear in u and all its derivatives, and the coefficients are independent from u, as
$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u + d(t, x)$$
then it is called a **linear** parabolic PDE.

[1] It is, at least, differentiable to the second order as regards x and to the first order as regards t.

A linear equation having constant coefficients, which are obviously independent from (t, x), is called an **autonomous** PDE. The general form of an **autonomous linear parabolic PDE** is

$$u_t = a u_{xx} + b u_x + c u + d. \tag{12.1}$$

There are two main particular cases, in analogy to linear scalar ODEs: if $d = 0$ the PDE is called **homogeneous** and if $d \neq 0$ it is **non-homogeneous**.

Equation (12.1) is sometimes called a diffusion equation with advection and growth. This is because, the time-behavior of u depends on three terms: a diffusion term, $a u_{xx}$, a transport term, $b u_x$, and a growth term $c\, u$ and a position term d. If $a > 0$ ($a < 0$) the equation is a **forward** PDE - FPDE (**backward** PDE - BPDE) because the diffusion operator works forward (backward) in time. The b coefficient introduces a behavior similar to the first-order PDE: it involves a transportation of the solution along the axis x, in the forward direction if $b < 0$ and in the backward direction if $b > 0$. The c coefficient generates a time behavior of the whole distribution $u(x, .)$ in a way similar to an ordinary differential equation, that is, it involves stability if $c < 0$ or instability if $c > 0$.

In the case of a parabolic PDE the stability or instability properties are related to the whole distribution: we have **stability in a distributional sense** if there is a solution $u(t, x) = \overline{u}(x)$ such that

$$\lim_{t \to \infty} u(t, x) = \overline{u}(x)$$

where $\overline{u}(x)$ is a stationary distribution.

An important element regarding the existence and characterization of the solution of PDE's is related to the characteristics of the support of the distribution X. The most common applications assume that the set X can be a line or a circle (a ring). In the first case, we can distinguish between three main cases:

- unbounded or infinite case $\mathrm{X} = (-\infty, \infty)$;
- the semi-bounded of semi-infinite case $\mathrm{X} = [\underline{x}, \infty)$ or $\mathrm{X} = (-\infty, \overline{x}]$;
- the bounded case $\mathrm{X} = (\underline{x}, \overline{x})$

where both $\underline{x}$ and $\overline{x}$ are finite numbers.

In order to define **problems involving parabolic PDE's** we have to supplement it with a distribution referred to a point in time (an initial distribution for the forward PDE or terminal distribution for a backward PDE), and possibly conditions involving known values for the values of $u(t, x)$ at the boundaries of X (called boundary conditions), i.e, values for $u(x, \cdot)$ for points $x \in \partial\mathrm{X}$, where $\partial\mathrm{X}$ denotes the boundary of X.

A **boundary-value problem** in defined by a PDE supplemented with conditions on their boundaries. The conditions on the boundaries can take several forms. In particular:

- we say we have Cauchy problem if the PDE is supplemented with a condition on the level of u at the boundaries: $u(t, \underline{x}) = \underline{u}$ and $u(t, \overline{x}) = \overline{u}$ where $\underline{u}$ and $\overline{x}$ are know numbers

- we say we haveNeumann problem if the PDE is supplemented with a condition on the spatial derivative of u at the boundaries: $u_x(t, \underline{x}) = \underline{u}$ and $u_x(t, \overline{x}) = \overline{u}$ where $\underline{u}$ and $\overline{x}$ are know numbers.

However, boundary conditions may not be specified, which is sometimes the case when $X = \mathbb{R}$, and there are other types boundary conditions.

A problem is said to be **well-posed** if there is a solution to the PDE that satisfies both the initial (or terminal) conditions, referring to time, and the ”spatial” boundary conditions, and it is continuous at those points. In this case we say we have a **classic solution**. If a problem is not well-posed it is **ill-posed**. In this case there are no solutions or classic solutions do not exist (but generalized or weak solutions can exist).

A necessary condition for a problem involving a FPDE to be well posed is that it is supplemented with an initial condition in time, and a necessary condition for a problem BPDE to be well-posed is that it involves a terminal condition in time.

12.3 The simplest forward PDE

This section presents the heat equation which is the simplest forward version of equation (12.1). In subsection 12.3.1 we show how the heat equation is obtained. In subsection 12.3.2 we introduce Fourier transforms, which allows us to transform some types of parabolic PDE into ODEs. In the remaining subsections several version of the (forward) heat equations are solved, depending on its domain: the infinite domain in subsection 12.3.3, the semi-infinite domain in section 12.3.4

12.3.1 The heat equation: derivation

The simplest linear parabolic PDE is the heat equation. Denoting the temperature at location $x \in \mathrm{X}$ at time t by $u(t, x)$, the heat equation formalizes the dynamics of the heat distribution across set X over time by the linear forward parabolic PDE[2]

$$u_t - u_{xx} = 0. \tag{12.2}$$

It describes the dynamics of the temperature distribution when spatial differences in temperature drive the change in spatial distribution of temperature over time. Consider a homogeneous rod with infinite width and let $u(t, x)$ be the temperature at point $x \in \mathrm{X} = (-\infty, \infty)$ at time $t \geq 0$. Consider a small segment of the rod between points x and $x + \Delta x$, where $\Delta x > 0$. The average temperature in the segment at time t is $\int_x^{x+\Delta x} u(t, z)\, dz$. and the instantaneous change of the average temperature in the segment is

$$\frac{d}{dt}\Big(\int_x^{x+\Delta x} u(t, z)\, dz \Big) = \int_x^{x+\Delta x} u_t(t, z)\, dz.$$

[2]The first formulation of the heat equation is attributed to Fourier in a presentation to the Institut de France, and in a book with title *Théorie de la Propagation de la Chaleur dans les Solides* both in 1807.

It is assumed that this change in temperature is generated by the heat flow through the segment, that is flowing across the two boundaries of the segment $(x, x+\Delta x)$. The difference in the temperature between the two boundaries of the segment, $u(t, x+\Delta x) - u(t,x)$, is by the mean value theorem

$$u(t, x+\Delta x) - u(t,x) = \int_x^{x+\Delta x} u_x(t,z)\, dz,$$

which can be seen as a measure of the average temperature in the segment $(x, x+\Delta x)$ at time t. If there is a hotter spot located outside the segment, for instance in a leftward region, and because the heat flows from hot to colder regions, then temperature in the segment $(x, x+\Delta x)$ is lower then in the leftward region, implying $u_x(t,x) < 0$, and it is higher than in the rightward region, implying $u_x(t, x+\Delta x) < 0$, and the gradient in the leftward boundary is higher in absolute terms that the rightward $u_x(t, x+\Delta x) - u_x(t,x) > 0$. Therefore, from the mean-value theorem, the heal flow is

$$u_x(t, x+\Delta x) - u_x(t,x) = \int_x^{x+\Delta x} u_{xx}\ (t,z)\, dz > 0.$$

If we **assume** that the instantaneous change in the segment's temperature is equal to the heat that flows through the segment, then

$$\frac{\partial}{\partial t}\Big(\int_x^{x+\Delta x} u(t,z)\, dz\Big) = \frac{\partial}{\partial x} u(t, x+\Delta x) - \frac{\partial}{\partial x} u(t,x),$$

that is

$$\int_x^{x+\Delta x} u_t(t,z)\, dz = \int_x^{x+\Delta x} u_{xx}\ (t,z)\, dz.$$

This is equivalent to

$$\int_x^{x+\Delta x} \big(u_t(t,z) - u_{xx}\ (t,z)\big)\, dz = 0,$$

which holds if and only if equation (12.2) is satisfied.

The simplest linear scalar parabolic PDE is a slight generalization of the forward partial differential equation $u_t(t,x) = a\, u_{xx}(t,x)$ where $a > 0$. The solution, and the solution methods, to this equation depends on our assumption regarding the domain X. For the equations, and the related problems we will be interested, a powerful method uses Fourier transforms. We present next a short introduction to Fourier transforms.

12.3.2 Fourier transforms

There are several methods for solving linear parabolic PDE's. A general method consist in transforming the PDE into a parameterized ODE. There are several possible transformations - sine, cosine, Laplace, Mellin or Fourier transforms. The choice of the method usually depends on the domain of the "spatial" variable, X.

When the domain of the independent variable x is $\mathrm{X} = (-\infty, \infty)$, the most direct method to find a solution is to use Fourier and inverse Fourier transforms (see Appendix 12.10 Tables 12.1 and 12.2).

The method of obtaining a solution follows three steps: first, we transform function $u(t,x)$ into function $U(t.,\omega)$, where $\omega \in \Omega = \mathbb{R}$ represents frequencies, such that the PDE over $u(t,x)$ is transformed into a parameterized ordinary differential equation over $U(t\omega)$; second we solve the ODE over $U(\cdot)$; and finally we transform back to the original function $u(t,x)$. When the domain of x is not the double-infinity we may have to adapt this method.

The (spatial) **Fourier transform** of the integrable function $u(t,x)$ over x, taking t as a parameter, is [3]

$$U(t,\omega) = \mathcal{F}[u(t,x)](\omega) \equiv \int_{-\infty}^{\infty} u(t,x) e^{-2\pi i \omega x} dx \tag{12.3}$$

where $i^2 = -1$ and the **inverse Fourier transform** is

$$u(t,x) = \mathcal{F}^{-1}[U(t,\omega)](x) \equiv \int_{-\infty}^{\infty} U(t,\omega) e^{2\pi i \omega x} d\omega. \tag{12.4}$$

By representing the primitive function $u(t,x)$ as

$$u(t,x) = \int_{-\infty}^{\infty} U(t,\omega) e^{2\pi i \omega x} d\omega$$

we can transform the PDE $f(u_t, u, u_x, u_{xx}) = 0$ into an ODE $F(U_t, U) = 0$, because the derivatives of the original variable have the following representations

$$u_t(t,x) = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} U(t,\omega) e^{2\pi i \omega x} d\omega = \int_{-\infty}^{\infty} U_t(t,\omega) e^{2\pi i \omega x} d\omega,$$

and

$$u_x(t,x) = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} U(t,\omega) e^{2\pi i \omega x} d\omega = \int_{-\infty}^{\infty} 2\pi\omega i \, U(t,\omega) e^{2\pi i \omega x} d\omega,$$

and the second-order derivative is

$$\begin{aligned} u_{xx}(t,x) &= \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} U(t,\omega) e^{2\pi i \omega x} d\omega \\ &= \frac{\partial}{\partial x} \int_{-\infty}^{\infty} 2\pi\omega i \, U(t,\omega) e^{2\pi i \omega x} d\omega \\ &= \int_{-\infty}^{\infty} (2\pi\omega i)^2 U(t,\omega) e^{2\pi i \omega x} d\omega \\ &= -\int_{-\infty}^{\infty} (2\pi\omega)^2 U(t,\omega) e^{2\pi i \omega x} d\omega. \end{aligned}$$

If function $f(u_t, u, u_x, u_{xx})$ is linear then $F(U_t, U)$ will be linear as well. Assume that the solution of the ODE $F(U_t, U) = 0$ is a product of two Fourier transforms,

$$U(t,\omega) = U(0,\omega) \, G(t,\omega)$$

[3] There are different definitions of Fourier transforms, we use the definition by, v.g., Kammler (2000).

where $U(0,\omega) = \mathcal{F}[u(0,x)](\omega)$ and $G(t,\omega) = \mathcal{F}[g(t,x)](\omega)$. If we are able to find the inverse Fourier transforms, $u(0,x)$ and $g(t,x)$ then we can obtain $u(t,x)$ as the inverse Fourier transform of $U(t,\omega)$, as the convolution

$$u(t,x) = \mathcal{F}^{-1}[U(t,\omega)](x) = \mathcal{F}^{-1}[U(0,\omega)\, G(t,\omega)](x) = u(0,x) * g(t,x).$$

If function $u(t,x)$ is a convolution of functions $u(0,x)$ and $g(t,x)$, assumed to be integrable functions in the domain of x then it satisfies

$$u(t,x) = u(0,x) * g(t,x) \equiv \int_{-\infty}^{\infty} u(0,\xi)\, g(t, x-\xi)\, d\xi.$$

12.3.3 The forward heat equation in the infinite domain

In this subsection we solve the slightly more general version of equation (12.2) in the infinite domain for an arbitrary bounded initial condition, and for a given initial condition. The last two are versions of Cauchy problems in which the side conditions refer to $t = 0$.

General solution

The simplest forward linear PDE for an infinite domain $X = (-\infty, \infty) = \mathbb{R}$ is

$$u_t - a u_{xx} = 0, \; (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{12.5}$$

where $a > 0$.

Proposition 1. *Let $u(0,x)$ be an arbitrary but bounded function with domain in $\mathbb{R}$, i.e. satisfying $\int_{-\infty}^{\infty} |u(0,x)| dx < \infty$. Then the solution to PDE* (12.5) *is*

$$u(t,x) = \begin{cases} u(0,x), & (t,x) \in \{t=0\} \;\times \mathbb{R} \\ \dfrac{1}{2\sqrt{\pi a t}} \displaystyle\int_{-\infty}^{\infty} u(0,\xi) e^{-\frac{(x-\xi)^2}{4at}}\, d\xi, & (t,x) \in \mathbb{R}_{++} \times \mathbb{R} \end{cases} \tag{12.6}$$

Proof. The proof proceeds in three steps. First step: transform PDE (12.5) into a parameterized ODE. Applying the previous definition of Fourier transform to the PDE (12.5) we have

$$u_t - a u_{xx} = \int_{-\infty}^{\infty} \left(U_t(t,\omega) + a (2\pi\omega)^2 U(t,\omega) \right) e^{2\pi i \omega x} d\omega = 0.$$

Defining $\lambda(\omega) \equiv -a\,(2\pi\omega)^2$ (called an eigenfunction), the equation is satisfied if and only if $U(\cdot,\omega)$ solves the characteristic equation

$$U_t(t,\omega) = \lambda(\omega)\, U(t,\omega),$$

which is a linear ODE with time as the independent variable, and ω as a parameter. Second step: the characteristic equation is

$$U(t,\omega) = U(0,\omega)\, G(t,\omega)$$

where $U(0,\omega)$ is an arbitrary function of ω, and

$$G(\omega,t) \equiv \begin{cases} 1, & t = 0 \\ e^{\lambda(\omega)\,t}, & t > 0 \end{cases}$$

is called the **Gaussian kernel**. Third step: we transform back from $U(t,\omega)$ to $u(t,x)$ using the inverse Fourier transform, yielding

$$u(t,x) = \mathcal{F}^{-1}[U(t,\omega)](x) = \mathcal{F}^{-1}[U(0,\omega)\, G(t,\omega)] = u(0,x) * g(t,x)$$

where $u(0,x) = \mathcal{F}^{-1}[U(0,\omega)](x)$ and $g(t,x) = \mathcal{F}^{-1}[G(t,\omega)](x)$ are inverse Fourier transforms, and

$$u(0,x) * g(t,x) = \int_{-\infty}^{\infty} u(0,\xi)\, g(t,x-\xi) d\xi$$

Using the table 12.2 in the Appendix, for $g(x,t) = \mathcal{F}^{-1}[G(t,\omega)](x)$ the Gaussian kernel in the initial variable is

$$g(t,x) = \begin{cases} \delta(x), & t = 0 \\ \dfrac{e^{-\frac{x^2}{4at}}}{2\sqrt{\pi\, a\, t}}, & t > 0 \end{cases}$$

where $\delta(.)$ is the Dirac's delta function. Then we obtain the general solution (12.6) where $u(0,x)$ **is an arbitrary but bounded function**, i.e. satisfying $\int_{-\infty}^{\infty} |u(0,x)| dx < \infty$, because $\int_{-\infty}^{\infty} u(0,\xi)\delta(x-\xi)d\xi = u(0,x)$. □

Two observations can be made concerning the solution of this PDE in equation (12.6):

First, applying the Fourier transform, we change from a distribution in the original variables x to a frequency distribution ω.

The transformed PDE becomes a linear ODE where the coefficient is eigenfunction

$$\lambda(\omega) = -a\,(2\pi\omega)^2, \text{ for } \omega \in \mathbb{R}$$

which is a real and non-positive function of any $\omega \in \mathbb{R}$ such that: $\lambda(0) = 0$ and $\lambda(\omega) < 0$ for $\omega \neq 0$ and, $\lim_{\omega\to\pm\infty} \lambda(\omega) = -\infty$. This means that $\lim_{\omega\to\pm\infty} U(t,\omega) = 0$ for any t if $K(\omega)$ is bounded.

Second, associated to the previous property, the solution $u(t,x)$ is a time-varying expected value of the arbitrary function where the density function is a Gaussian density function with average 0 and variance $2\,a\,t$.

Initial value problems Similarly to ODE's we define an **initial value problem**

$$\begin{cases} u_t = a u_{xx}, & (t,x) \in (0,\infty) \times (-\infty,\infty) \\ u(0,x) = u_0(x) & (t,x) \in \{t=0\} \times (-\infty,\infty) \end{cases} \tag{12.7}$$

where $a > 0$ and $u_0(x)$ is a **known function** (and not a known number as in an ODE). The problem is **well-posed** if $u_0(x)$ is bounded, which means that the solution to the problem exists and is continuous in time.

Substituting the arbitrary function $u(0,x) = u_0(x)$, because $u_0(x)$ is a known function, in the general solution of the PDE, equation (12.6), yields the particular solution to the initial problem (12.7)

$$u(t,x) = \begin{cases} u_0(x), & (t,x) \in \{t=0\} \times \mathbb{R} \\ \displaystyle\int_{-\infty}^{\infty} \frac{u_0(\xi)}{2\sqrt{\pi\, a\, t}}\; e^{-\dfrac{(x-\xi)^2}{4at}}\; d\xi, & (t,x) \in \mathbb{R}_{++} \times \mathbb{R}. \end{cases} \tag{12.8}$$

Next we present several examples, for different initial-value functions.

Example 1: Dirac's delta initial distribution Let $a = 1$ and assume that the initial distribution is concentrated at point $x = 0$. Then $u_0(x) = \delta(x)$, satisfying $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$. Using equation (12.8) yields

$$u(t,x) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\, s\pi\, t}}, \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

This solution has two properties: first, it satisfies a conservation law at every point in time

$$\int_{-\infty}^{\infty} u(t,x)\, dx = 1, \text{for every}\, t \geq 0,$$

and it decays over time to a spatially homogeneous distribution

$$\lim_{t\to\infty} u(t,x) = 0, \forall x \in (-\infty,\infty)$$

meaning that the variance will tend to infinity.

Example 2: A Gaussian initial distribution Let again $a = 1$, but assume instead that $u_0(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$. Equation (12.8) becomes

$$u(t,x) = \frac{1}{\sqrt{\pi(1+4\,t)}}\, e^{-\dfrac{x^2}{1+4\,t}}.$$

Figure 12.1 illustrates the behavior of this solution. This solution has the same properties as the previous example: it satisfies a conservation law, and decays over time and converges to a flat distribution.

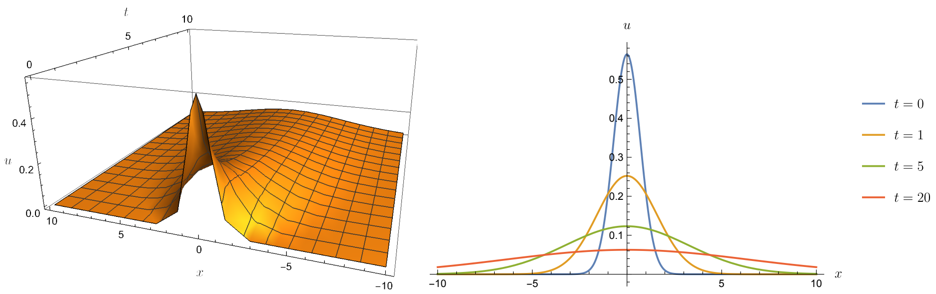


Figure 12.1: Solution for the initial value problem for the heat equation with $a = 1$ and $u_0(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$.

Example 3: Piecewise-constant initial condition Let the initial distribution be a discontinuous function defined over the infinite domain

$$u_0(x) = \begin{cases} \phi_0, & \text{if } x \in [\underline{x}, \overline{x}] \\ 0 & \text{if } x \notin [\underline{x}, \overline{x}) \end{cases}$$

where $\underline{x} < \overline{x}$ are both finite and ϕ_0 is a constant. In this case, applying solution (12.8) yields the solution

$$u(t,x) = \phi_0 \left[\Phi\left(\frac{x - \underline{x}}{\sqrt{2at}} \right) - \Phi\left(\frac{x - \overline{x}}{\sqrt{2at}} \right) \right] \tag{12.9}$$

where $\Phi(z)$ is the cumulative distribution function (CDF) of the standard normal probability distribution

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}} dz.$$

Observe that $\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$ then $\Phi(y) \in (0,1)$.

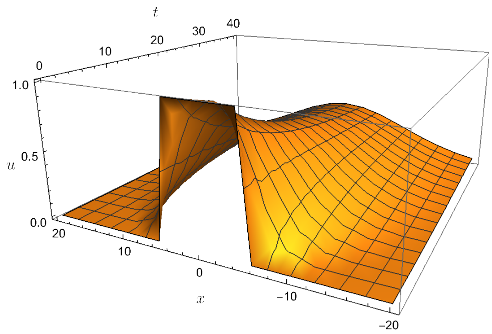


Figure 12.2: Solution for the initial value problem for the heat equation with $a = 1$ and piecewise initial condition.

To prove this result, with apply the general solution, presented in equation (12.6). This yields the solution of the initial-value problem

$$u(t,x) = \frac{\phi_0}{2\sqrt{\pi a t}} \int_{\underline{x}}^{\overline{x}} e^{-\frac{(x-\xi)^2}{4at}} d\xi.$$

To simplify the expression, we make the transformation of variables $z \equiv (x-\xi)/\sqrt{2at}$, and denote $\overline{z} \equiv (\overline{x}-\xi)/\sqrt{2at}$ and $\underline{z} \equiv (\underline{x}-\xi)/\sqrt{2at}$. Then, because $dz = -\frac{1}{\sqrt{2at}}\, d\xi$, the solution simplifies to [4]

$$\begin{aligned}
\frac{1}{\sqrt{4\pi at}} \int_{\underline{x}}^{\overline{x}} e^{-(x-\xi)^2/4at} d\xi &= -\frac{\sqrt{2at}}{\sqrt{4\pi at}} \int_{(x-\underline{x})/\sqrt{2at}}^{(x-\overline{x})/\sqrt{2at}} e^{-z^2/2} dz \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{(x-\underline{x})/\sqrt{2at}} e^{-z^2/2} dz - \int_{-\infty}^{(x-\overline{x})/\sqrt{2at}} e^{-z^2/2} dz \right) = \\
&= \Phi\left(\frac{x-\underline{x}}{\sqrt{2at}}\right) - \Phi\left(\frac{x-\overline{x}}{\sqrt{2at}}\right).
\end{aligned}$$

The solution (12.9) is illustrated in Figure 12.2.

12.3.4 The forward linear equation in the semi-infinite domain

Now consider the equation defined on the semi-infinite domain for x, that is $\mathrm{X} = \mathbb{R}_+$. This case is more interesting for economic applications in which the independent variable can only take non-negative values, for instance when x refers to a price or a physical stock variable. This case should not be confused with the previous case in which we had an initial condition that could take the value zero for a subset of the infinite space X.

The FPDE we consider is

$$u_t(t,x) - a u_{xx}(t,x) = 0,\ (t,x) \in \mathbb{R}^2_+ \tag{12.10}$$

where $a > 0$.

Proposition 2. *The solution to equation* (12.10) *is*

$$u(t,x) = \frac{1}{2\sqrt{\pi\, a\, t}} \begin{cases} \int_0^\infty u(0,\xi) \left(e^{-\frac{(x-\xi)^2}{4\,a\,t}} - e^{-\frac{(x+\xi)^2}{4\,a\,t}} \right)\, d\xi, & \text{if } u(0,\cdot) \text{ is an odd function} \\ \int_0^\infty u(0,\xi) \left(e^{-\frac{(x-\xi)^2}{4\,a\,t}} + e^{-\frac{(x+\xi)^2}{4\,a\,t}} \right)\, d\xi, & \text{if } u(0,\cdot) \text{ is an even function} \end{cases} \tag{12.11}$$

where $u(0,x)$ *is defined over the semi-infinite space* $\mathrm{X} = \mathbb{R}_+$.

Proof. We solve this equation by using the **method of images**, which consists in extending the arbitrary function $u(0,x)$, which has the domain $\mathrm{X} = \mathbb{R}_+$ to set $\mathbb{R}$, which is the domain of the general solution (12.6). If $f(x)$, where $x \in \mathbb{R}$, is an odd function it has the property $f(x) = -f(-x)$, and if it is an even function it has the property $f(x) = f(-x)$.

[4] Recalling the formula for integration by substitution of variables, if we set $z = \varphi(\xi)$ and $\xi \in (a,b)$ then

$$\int_{\varphi(a)}^{\varphi(b)} f(z) dz = \int_a^b f(\varphi(\xi))\, \varphi'(\xi)\, d\xi.$$

Therefore, if $u(0,x)$, where $x \in \mathbb{R}_+$, we can make the following extensions to functions $\tilde{u}(0,x)$, where $x \in \mathbb{R}$, for an odd function

$$\tilde{u}(0,x) = \begin{cases} u(0,x), & \text{if } x \geq 0 \\ -u(0,-x) & \text{if } x < 0, \end{cases}$$

or for an even function

$$\tilde{u}(0,x) = \begin{cases} u(0,x), & \text{if } x \geq 0 \\ u(0,-x) & \text{if } x < 0. \end{cases}$$

Substituting in equation (12.6), for $t > 0$, yields, for an odd function

$$\begin{aligned} u(t,x) &= \frac{1}{2\sqrt{\pi a t}} \int_{-\infty}^{\infty} \tilde{u}\ (0,\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi = \\ &= \frac{1}{2\sqrt{\pi a t}} \left(\int_{-\infty}^{0} \tilde{u}\ (0,\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi + \int_{0}^{\infty} \tilde{u}(0,\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi \right) = \\ &= \frac{1}{2\sqrt{\pi a t}} \left(-\int_{0}^{\infty} u\ (0,\xi) e^{-\frac{(x+\xi)^2}{4at}}\ d\xi + \int_{0}^{\infty} u(0,\xi) e^{-\frac{(x-\xi)^2}{4at}}\ d\xi \right) \end{aligned}$$

where the last step involves integration by substitution: i.e., if we define $u = -x$ for $x \in [0,\infty)$ then $\int_{-\infty}^{0} f(u) du = -\int_{\infty}^{0} f(-x) dx = \int_{0}^{\infty} f(-x) dx$. Then the solution of equation (12.10) for an odd arbitrary function in equation (12.11) is obtained. The solution for an arbitrary even function is obtained in an analogous way. □

The solution to the initial-value problem

$$u(t,x) = \frac{1}{2\sqrt{\pi\, a\, t}} \begin{cases} \int_0^\infty u_0(\xi) \left(e^{-\frac{(x-\xi)^2}{4\,a\,t}} - e^{-\frac{(x+\xi)^2}{4\,a\,t}} \right)\, d\xi, & \text{if } u_0(\cdot) \text{ is a known odd function} \\ \int_0^\infty u_0(\xi) \left(e^{-\frac{(x-\xi)^2}{4\,a\,t}} + e^{-\frac{(x+\xi)^2}{4\,a\,t}} \right)\, d\xi, & \text{if } u_0(\cdot) \text{ is a known even function} \end{cases} \tag{12.12}$$

We obtain this result by direct application of equation (12.11).

Example Assume the initial distribution is log-normal, that is,

$$u_0(x) = \frac{e^{-\dfrac{\ln(x) - \mu}{2\,\sigma^2}}}{\sigma\, x\, \sqrt{2\,\pi}}, \text{ for } x \in \mathbb{R}_+,$$

is an even function because $u_0(x) = u_0(-y)$ if $x > 0$ and $y = -x < 0$. Substituting in equation (12.11) yields the solution to the initial-value problem which we plot in Figure 12.4 for several moments in time.

We observe that the solution is conservative, i.e. the integral $\int_0^\infty u(t,x)\, dx = 1$ for every $t > 0$.

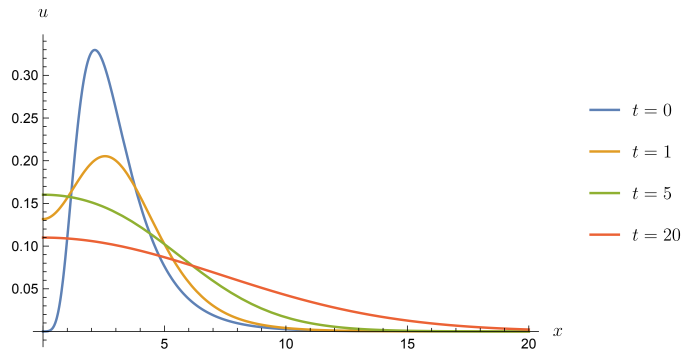


Figure 12.3: Solution for the initial value problem for the heat equation in the semi-infinite line with $a = 1$ and an initial log-normal density.

12.4 The forward homogeneous linear PDE

The general forward linear homogeneous parabolic PDE in the infinite domain, $\mathrm{X} = (-\infty, \infty)$ is

$$u_t(t,x) = a\,u_{xx}(t,x) + b\,u_x(t,x) + c\,u(t,x),\ (t,x) \in (0,\infty) \times \mathrm{X} \tag{12.13}$$

where $a > 0$, and b and c are real numbers. It contains three terms: a diffusion term, if $a \neq 0$, a transport term, if $b \neq 0$, and a growth or decay term if $c > 0$ or $c < 0$.

In order to solve the equation, we can follow one of two alternative methods:

1. apply the Fourier transform method to transform the PDE into a parameterized ODE, solve it, and apply inverse Fourier transforms.

2. transform the PDE over the original function into a heat equation over a transformed function, solve this heat equation for the transformed function, and transform back to the original function.

Next, we use the first method to prove the following result

Proposition 3. *The solution to equation* (12.16) *is*

$$u(t,x) = \int_{-\infty}^{\infty} u(0,s)\,\frac{1}{\sqrt{4\pi a t}}\exp\left(-\frac{(x-s)^2 + 2\,b\,(x-s)\,t + (b^2 - 4ac)\,t^2}{4\,a\,t}\right) ds,\ (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{12.14}$$

where $u(0,x)$ is an arbitrary bounded function mapping $u(0,\cdot) : \mathrm{X} \to \mathbb{R}$.

Proof. A Fourier representation of the PDE yields

$$u_t(t,x) - a u_{xx}(t,x) - b u_x - c u(t,x) = \int_{-\infty}^{\infty} e^{2\pi i \omega x}\left[\,U_t(t,\omega) + \lambda(\omega) U(t,\omega)\right]\,d\omega,$$

where the eigenfunction is a complex-valued function of ω

$$\lambda(\omega) \equiv -\Big(a(2\pi\omega)^2 - c - b\,2\pi\,\omega\,i\Big),\ \text{for}\quad \in\in \mathbb{R}$$

where $i = \sqrt{-1}$. The equation Fourier representation is equal to zero if and only if $U(t, \omega)$ solves the parameterized linear ODE,

$$U_t(t, \omega) = \lambda(\omega)\, U(t, \omega),\ (t, \omega) \in \mathbb{R}_+ \times \mathbb{R},$$

which is called a characteristic equation. The solution is

$$U(t, \omega) = U(0, \omega)\, G(t, \omega),\ \text{for} t \in [0, \infty)$$

where $G(t, \omega)$ is the Gaussian kernel

$$G(t, \omega) = e^{-\lambda(\omega) t},\ \text{for } t > 0.$$

We obtain the solution of problem (12.16) by applying the inverse Fourier transform

$$u(t, x) = \mathcal{F}^{-1}\left[U(t, \omega)\right]\ (x) = \mathcal{F}^{-1}\left[U(0, \omega)\, G(t, \omega)\right]\ (x) = \int_{-\infty}^{\infty} u(0, s) g(t, x - s) ds$$

where (see the Appendix 12.10 Table 12.2)

$$g(t, y) = \mathcal{F}^{-1}\left[\ e^{\lambda(\omega) t}\right]\ = \frac{1}{\sqrt{4\pi a t}} \exp\left(-\frac{y^2 + 2\, b\, t\, y + (b^2 - 4\, a\, c)\, t^2}{4at}\right), \tag{12.15}$$

because $a\, t > 0$, and $u(0, x) = \mathcal{F}^{-1}\left[U(0, \omega)\right]\ (x)$. □

The initial value problem for a general linear homogeneous (forward) diffusion equation is

$$\begin{cases} u_t(t, x) = a\, u_{xx}(t, x) + b\, u_x(t, x) + c\, u(t, x), & (t, x) \in (0, \infty) \times (-\infty, \infty) \\ u(0, x) = u_0(x), & (t, x) \in \{t = 0\} \times (-\infty, \infty), \end{cases} \tag{12.16}$$

where $a > 0$, $b \neq 0$ and $c \neq 0$ and $u_0(x)$ is a known bounded function defined on $X = \mathbb{R}$.

The solution to this problem is

$$u(t, x) = \int_{-\infty}^{\infty} u_0(\xi)\, \frac{1}{\sqrt{4\pi a t}} \exp\left(-\frac{(x - \xi)^2 + 2\, b\, (x - \xi)\, t + (b^2 - 4ac)\, t^2}{4\, a\, t}\right) d\xi. \tag{12.17}$$

12.4.1 Particular case without transport term

If the linear forward PDE does not contain a transport term, that is when $b = 0$, we can illustrate the second method to solve the PDE, that was already mentioned. The PDE is thus $u_t(t, x) = a\, u_{xx}(t, x) + c\, u(t, x)$, and, to solve it, follow this steps: First, define $v(t, x) = e^{-c\, t} u(t, x)$, which has derivatives $v_t = -c e^{-c\, t} u + e^{-c\, t} u_t$ and $v_{xx} = e^{-c t} u_{xx}$. Second, equation (12.16) is equivalent to the simplest linear equation $v_t = a v_{xx}$ which has solution (12.6). Third, as $u(t, x) = e^{c\, t}\, v(t, x)$ we obtain the solution

$$u(t, x) = e^{c\, t} \int_{-\infty}^{\infty} u_0(\xi) \frac{e^{-\frac{(x - \xi)^2}{4\, a\, t}}}{\sqrt{4\, \pi\, a\, t}}\, d\xi, (t, x) \in \mathbb{R}_{++} \times \mathbb{R},$$

for a bounded initial distribution $u_0(x)$ with domain $x \in \mathbb{R}$.

We see that the dynamics of the solution depends crucially on the sign of c:

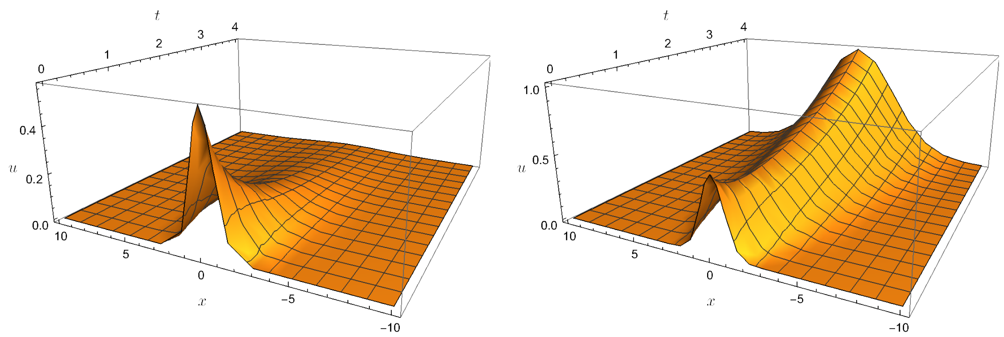
$$\lim_{t \to \infty} u(t,x) = \begin{cases} 0 & \text{if } c < 0 \\ \infty & \text{if } c > 0 \end{cases}$$

If $u_0(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$ we obtain the solution of the initial value problem,

$$u(t,x) = \frac{e^{c\,t - \dfrac{x^2}{4\,a\,t}}}{2\sqrt{\pi\,a\,t}},$$

plotted in Figure 12.4 for the cases in which $c < 0$ and $c > 0$. In both cases we see that the long-time behavior of the solution is commanded by $e^{c\,t}$: if $c < 0$ then $\lim_{t\to\infty} u(t,x) = 0$, for any $x \in \mathbb{R}$, and if $c > 0$ then $\lim_{t\to\infty} u(t,x) \propto \lim_{t\to\infty} e^{c\,t} = \infty$, for any $x \in \mathbb{R}$.

This means that the diffusion equation display **asymptotic stability** if $c < 0$ and **instability** if $c > 0$, both in a distributional sense. In the first case the solution $u(t,x)$ is asymptotically bounded and in the second case it is aymptotically unbounded.



(a) Case $c = -0.5$ (b) Case $c = 0.5$

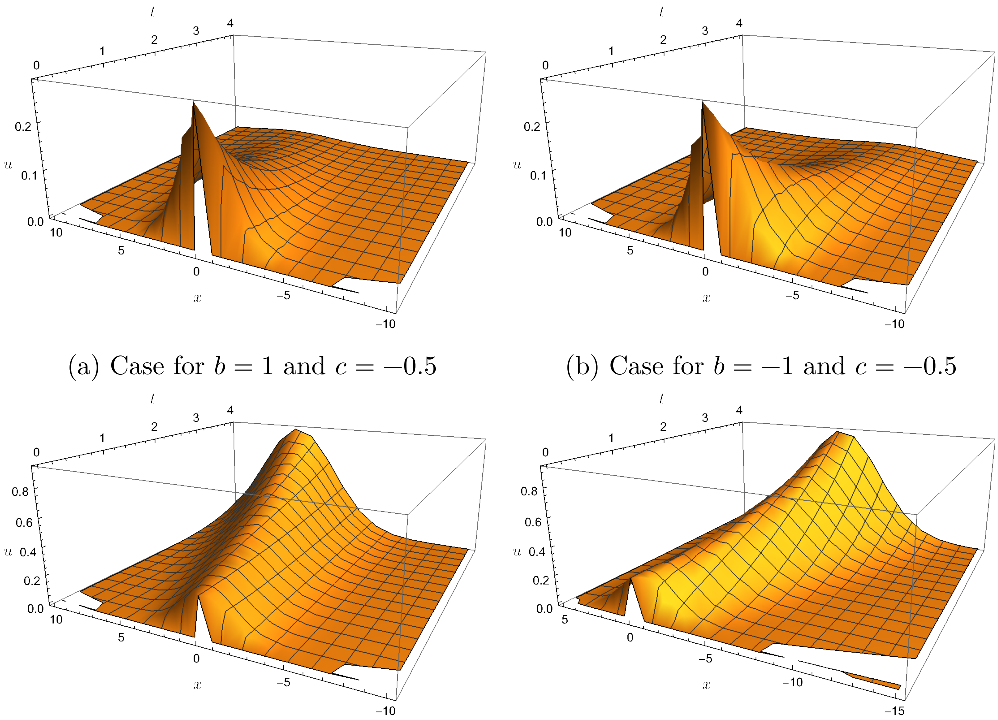
Figure 12.4: Solutions for the initial value problem for the heat equation with $a = 1$ and $u_0(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$.

12.4.2 Particular case with transport term

If we assume that $b \neq 0$ and $u_0(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$ we obtain the solution of the initial value problem,

$$u(t,x) = \frac{e^{c\,t - \dfrac{(x+b\,t)^2}{4\,a\,t}}}{2\sqrt{\pi\,a\,t}},$$

and plot it in Figure 12.5. It illustrates the solution to the initial value problem for positive (in panels (a) and (c)) and for negative values of b (in panels (b) and (d)). As in Figure 12.4 panels (a) and (b) are for negatives values of c and panels (c) and (d) for positive values for c. We observe that parameter b introduces a sliding across the spatial domain X - a transportation - in the positive direction , if $b < 0$, or in the negative direction, if $b > 0$. As in Figure 12.4 , c is associated to the stability properties over time of the whole distribution.



(a) Case for $b = 1$ and $c = -0.5$

(b) Case for $b = -1$ and $c = -0.5$

(c) Case for $b = 1$ and $c = 0.5$

(d) Case for $b = -1$ and $c = 0.5$

Figure 12.5: Solutions for the initial value problem linear PDE for $a = 1$ and $u_0(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$.

12.5 The non-homogeneous forward linear PDE

The general forward linear non-homogeneous parabolic PDE in the infinite domain, $\mathrm{X} = (-\infty, \infty)$ is

$$u_t(t, x) = a\, u_{xx}(t, x) + b\, u_x(t, x) + c\, u(t, x) + d, \;\; (t, x) \in (0, \infty) \times \mathrm{X} \tag{12.18}$$

where $a > 0$, and b, c and d are real numbers. We add a new constant term to PDE (12.16).

Proposition 4. *The solution to equation* (12.18) *is*

$$u(t, x) = \int_{-\infty}^{\infty} u(0, \xi)\, \frac{e^{\gamma(x-\xi, t)}}{\sqrt{4\pi a t}}\, d\xi + \int_0^t \int_{-\infty}^{\infty} \frac{d\, e^{\gamma(x-\xi, t-s)}}{\sqrt{4\pi a (t-s)}}\, d\xi\, ds, \;\; (t, x) \in \mathbb{R}_+ \times \mathbb{R} \tag{12.19}$$

where

$$\gamma(t, x) \equiv -\frac{(x^2 + 2\, b\, x\, t + (b^2 - 4ac)\, t^2}{4\, a\, t}$$

where $u(0,x)$ is an arbitrary bounded function mapping $u_0 : \mathrm{X} \to \mathbb{R}$.

Proof. We will solve this problem using the Fourier transform representation of equation $u_t - (au_{xx} + bu_x + cu + d) = 0$. Using inverse Fourier transforms yields

$$u_t(t,x) - au_{xx}(t,x) - bu_x - cu(t,x) - d = \int_{-\infty}^{\infty} e^{2\pi i \omega x} \left[\, U_t(t,\omega) + \lambda(\omega) U(t,\omega) - d\,\delta(\omega)\right] \, d\omega = 0.$$

where again $\lambda(\omega) \equiv -a(2\pi\omega)^2 + 2\pi\, b\,\omega\, i + c$. Therefore, the PDE (12.16) has the characteristic equation

$$U_t(t,\omega) = \lambda(\omega)\, U(t,\omega) + d\,\delta(\omega),\ (t,\omega) \in \mathbb{R}_+ \times \mathbb{R},$$

which has the explicit solution

$$U(t,\omega) = U(0,\omega)\, G(t,\omega) + \int_0^t \, d\,\delta(\omega)\, G(t-s,\omega)\, ds,\ \text{for} t \in [0,\infty)$$

where $G(t,\omega)$ is the Gaussian kernel.

We obtain the solution of problem (12.17) by applying the inverse Fourier transform

$$\begin{aligned} u(t,x) &= \mathcal{F}^{-1}\left[U(t,\omega)\right]\ (x) = \mathcal{F}^{-1}\left[U_t(\omega)\, G(t,\omega)\right]\ (x) \\ &= u(0,x) * g(t,x) + \int_0^t \, d * g(t-s,x)\, ds \\ &= \int_{-\infty}^{\infty} \, u(0,x)\, g(t,x-\xi)\, d\xi + d\, \int_0^t \int_{-\infty}^{\infty} \, g(t-s,x-\xi)\, d\xi\, ds. \end{aligned}$$

□

If we assume that $b \neq 0$ and $u_0(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$ we obtain the solution of the initial value problem,

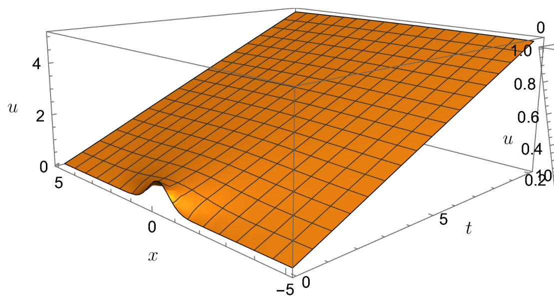
$$u(t,x) = \frac{e^{c\,t - \dfrac{(x+b\,t)^2}{4\,a\,t}}}{2\,\sqrt{\pi\,a\,t}} - \frac{d}{c}\ \left(1 - e^{c\,t}\right)$$

and Figure 12.6 illustrates some cases. We see in panel (a) that if $b = c = 0$ and $d > 0$ then the solution becomes unbounded and converges to a homogenous distribution with linear growth. In panel (b), where $c < 0$ the distribution also converges asymptotically to a homogeneous distribution with positive values. This behavior is also observable in panel (c), where $b > 0$, but the transition time profile is slight different, with a temporary transport dynamics.

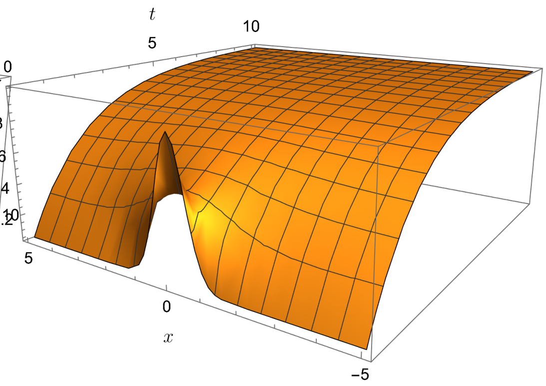
12.6 Non-autonomous forward linear equations

Non-autonomous scalar linear parabolic equations are linear in the state variable $u(\cdot)$ and on its derivatives, as the autonomous equations, but its coefficients are functions of the independent variables variables (t,x),

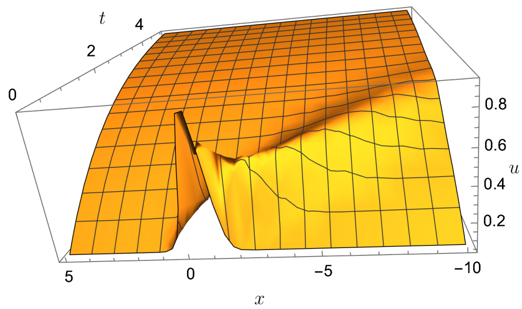
$$u_t = a(t,x)\, u_{xx}\ + b(t,x)\, u_x + c(t,x)\, u + d(t,x). \tag{12.20}$$



(a) For $b = 0$, $c = 0$ and $d = 0.5$



(b) For $b = 0$, $c = -0.5$ and $d = 0.5$



(c) For $b = 5$, $c = -0.5$ and $d = 0.5$

Figure 12.6: Illustration of equation (12.19) for $a = 1$ and $u_0(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$.

This structure covers a large number of cases, some of then will be covered in the next sections. In the rest of this section we assume that a, b and c are constant and $d(t, x)$ is a specified function of the independent variables

$$u_t = a\, u_{xx} + b\, u_x + c\, u + d(t, x). \tag{12.21}$$

Following similar steps as in the derivation of equation (12.17), we find the general solution (12.21) to the forward PDE

$$u(t, x) = \int_{-\infty}^{\infty} u(0, \xi)\, \frac{e^{\gamma(x-\xi, t)}}{\sqrt{4\pi a t}}\, d\xi + \int_0^t \int_{-\infty}^{\infty} \frac{e^{\gamma(x-\xi, t-s)}}{\sqrt{4\pi a (t-s)}}\, d(s, \xi)\, d\xi\, ds, \text{ for } (t, x) \in \mathbb{R}_+ \times \mathbb{R} \tag{12.22}$$

where

$$\gamma(t, x) \equiv -\frac{(x^2 + 2\, b\, x\, t + (b^2 - 4ac)\, t^2}{4\, a\, t}.$$

Next we illustrate two cases which some qualitative dynamics effects of this modification.

Example 1 : non-homogeneous heat equation We consider the problem:

$$\begin{cases} u_t = u_{xx} \ + d\,\delta(x - x_0) \\ u_0(x) = e^{-x^2}\pi^{-\frac{1}{2}}, \end{cases} \tag{12.23}$$

where d is a constant. The solution is plotted in Figure 12.7. Although the independent term is a degenerate distribution centered at $x = x_0$ (where $x_0 = 2$ in the figure), the diffusion mechanism implies the solution tends to $\lim_{t\to\infty} u(t,x) = \infty$ for any $x \in \mathrm{X}$.

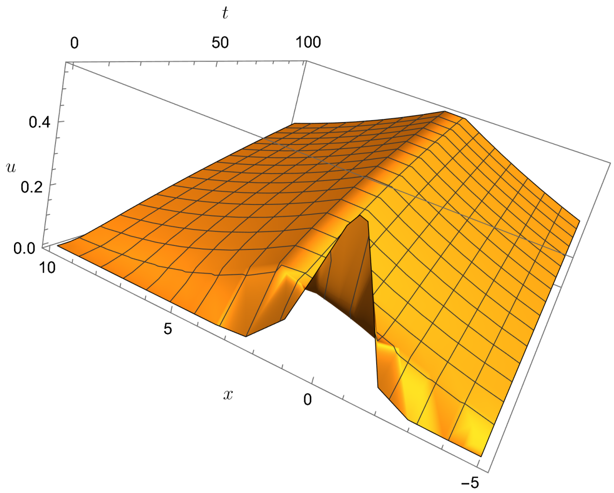


Figure 12.7: Solution to the non-autonomous equation problem (12.23) with $x_0 = 2$.

Example 2 : non-homogeneous linear equation We consider the problem:

$$\begin{cases} u_t = u_{xx} \ + cu + e^{-(x-x_0)^2}\pi^{-\frac{1}{2}} \\ u_0(x) = \ \delta(x - x_0). \end{cases} \tag{12.24}$$

The solution is plotted in Figure 12.8

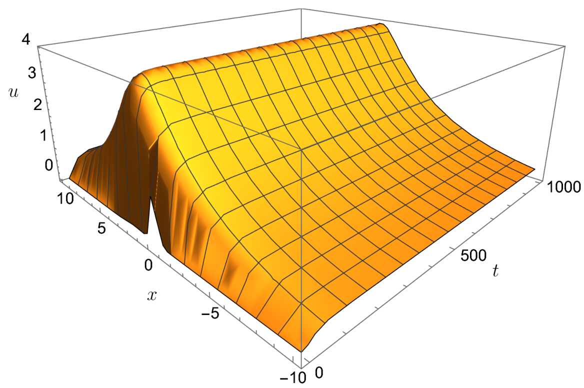


Figure 12.8: Solution to the non-autonomous equation problem (12.24) with $c < 0$ and $x_0 = 2$.

In this case the existence of a negative coefficient for u (i.e, $c < 0$) leads to a convergence of the solution to a bounded non-degenerate but non-flat distribution.

12.7 Linear backward PDE

12.7.1 The backward heat equation

In finance applications, and in Euler equations for optimal control problems, we sometimes need to solve backward parabolic PDE.

The simplest parabolic BPDE equation in the infinite domain for x and a bounded domain for t is

$$u_t + au_{xx} = 0,\ (t,x) \in [0,T] \times (-\infty, \infty) \tag{12.25}$$

where $a > 0$.

Proposition 5. *Consider the BPDE equation* (12.25)*. The solution is*

$$u(t,x) = \frac{1}{\sqrt{4\pi a(T-t)}} \int_{-\infty}^{\infty} u(T,\xi) e^{-\frac{(x-\xi)^2}{4a(T-t)}}\ d\xi, t \in (0,T) \tag{12.26}$$

Proof. In order to solve it we introduce a change in variables $\tau = T - t$ and denote $v(\tau, x) = u(t(\tau), x)$ where $t(\tau) = T - \tau$. As

$$v_\tau(\tau,x) = -u_t(t(\tau),x),\ \text{and}\ v_{xx}(\tau,x) = u_{xx}(t(\tau),x)$$

Then $u_t(t,x) = -au_{xx}(t,x)$ is equivalent to

$$v_\tau(\tau,x) = av_{xx}(\tau,x).$$

Using the solution already found in equation (12.6) we get

$$v(\tau,x) = \begin{cases} v(0,x), & \tau = 0 \\ \displaystyle\int_{-\infty}^{\infty} v(0,\xi)\,(4\pi a\tau)^{-1/2}\, e^{-\frac{(x-\xi)^2}{4a\tau}}\ d\xi, & \tau \in (0,T). \end{cases}$$

Transforming back to $u(t,x)$, and observing that $t = T$ if $\tau = 0$, yields solution (12.26). □

A problem involving a backward PDE is only well posed if together with the PDE we have a terminal condition, for example $u(T,x) = u_T(x)$, that is known. In this case the solution at time $t = 0$, $u(0,x)$, becomes endogenous.

The **terminal-value problem**

$$\begin{cases} u_t = -au_{xx}, & (t,x) \in (-\infty,\infty) \times (0,T] \\ u(T,x) = u_T(x) & (t,x) \in (-\infty,\infty) \times \{\ t = T\}, \end{cases} \tag{12.27}$$

has the solution, for $a > 0$,

$$u(t,x) = \begin{cases} u_T(x), & (t,x) \in \{t = T\} \times \mathbb{R} \\ \dfrac{1}{\sqrt{4\pi a(T-t)}} \displaystyle\int_{-\infty}^{\infty} u_T(\xi) e^{-\frac{(x-\xi)^2}{4a(T-t)}}\, d\xi, & (t,x) \in (0,T) \times \mathbb{R} \end{cases} \tag{12.28}$$

The initial distribution can be obtained by setting $t = 0$

$$u(0,x) = \frac{1}{\sqrt{4\,\pi\,a\,T}} \int_{-\infty}^{\infty} u_T(\xi) e^{-\frac{(x-\xi)^2}{4\,a\,T}}\, d\xi.$$

Example Assuming a terminal distribution $u_T(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$, solution (12.28) simplifies to

$$u(t,x) = \frac{e^{-\frac{x^2}{1+4\,a\,(T-t)}}}{\sqrt{\pi\,(1+4\,a\,(T-t))}}$$

and is illustrated in Figure 12.9, which can be compared with the forward PDE in Figure 12.1. In this case the initial distribution depends on the terminal time T,

$$u(0,x) = \frac{e^{-\frac{x^2}{1+4\,a\,T}}}{\sqrt{\pi\,(1+4\,a\,T)}}.$$

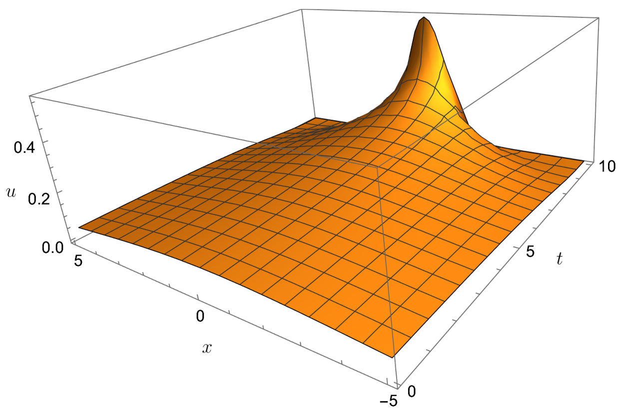


Figure 12.9: Solutions for backward ODE $a = 1$ and $u_T(x) = \dfrac{e^{-x^2}}{\sqrt{\pi}}$.

12.8 Application: Fokker-Planck-Kolmogorov equation for a linear diffusion

Stochastic differential equations, under some conditions, have a time-varying probability distribution that follows a particular parabolic PDE, called the Fokker-Planck-Kolmogorov equation.

This equation is having an increased application in economics, as a model for processes satisfying a conservation law.

Assume we have a diffusion process for $\left(X(t)\right)_{t\in\mathbb{R}_+}$ that is generated by the stochastic differential equation (SDE)

$$dX(t) = b(t, X(t))\, dt + a(t, X(t))\, dW(t),$$

where $\left(W(t)\right)_{t\in\mathbb{R}_+}$ is a Wiener process. Assume that at time $t = 0$ we observe the initial value of the process to be x_0, i.e, $X(0) = x_0$. The probability that the realization of the process at time $t > 0$ is a number x, i.e., $X(t) = x$ is denoted by $p(t, x)$. Therefore $p(t, x) = \mathbb{P}\ [X(t) = x|X(0) = x_0]$ and, for every t, we have $p(t, x) \in (0, 1)$ and $\int_{\mathrm{X}}\ p(t, x)dx = 1$, where X is the set of all possible realizations of the process $\left(X(t)\right)_{t\in\mathbb{R}_+}$. It can be proved that the probability process $\left(p(t, x)\right)_{(t,x)\in\mathrm{T}\times\mathrm{X}}$, where $p(t, x) : \mathrm{T} \times \mathrm{X} \to (0, 1)$, satisfies the Fokker-Planck-Kolmogorov equation together with an initial condition

$$\partial_t p(t, x) = \frac{1}{2}\partial_{xx}\ \Big(a(t, x)^2\, p(t, x)\Big) - \partial_x\Big(b(t, x)\, p(t, x)\Big), \tag{12.29}$$

where we assume $p(0, x)$ is known and satisfies

$$\int_X p(0, x)dx = 1.$$

As

$$\int_{\mathrm{X}} p(t, x)dx = 1,\ \text{for every}\ \ t \in \mathrm{T},$$

the probability process satisfies a conservation law.

In applications resulting from stochastic differential equations, the initial state is known $x = x_0$ and the dynamics of a probability distribution is given by Kolmogorov forward equation (or Fokker-Planck equation) and the initial condition $p(0, x) = \delta(x-x_0)$ where $\delta(\cdot)$ is Dirac's delta generalized function.

If we find the solution to the problem we can obtain the dynamics for the average, variance, and other statistics, as

$$\mathbb{E}[X(t)]\ = \int_{\mathrm{X}}\ x\, p(t, x)\, dx,\ \mathbb{V}[X(t)]\ = \int_{\mathrm{X}}\ \left(x - \mathbb{E}[X(t)]\ \right)^2 p(t, x)\, dx$$

12.8.1 The simplest problem

The simplest model has constant coefficients $b(t, x) = \mu$ and $a(t, x) = \sigma$ and a Dirac delta, centered in x_0, as an initial condition:

$$\begin{cases} \partial_t p(t, x) = \dfrac{\sigma^2}{2}\ \partial_{xx} p(t, x) - \mu\partial_x p(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ p(0, x) = \delta(x - x_0) & (t, x) \in \{\ t = 0\}\ \times \mathbb{R} \end{cases} \tag{12.30}$$

The solution is a Gamma probability density

$$\begin{aligned} p(t,x) &= \Gamma\big(-\mu t; \frac{\sigma^2 t}{2}, x - x_0\big) \\ &= \frac{e^{-\left(\frac{x - x_0 - \mu t}{2\sigma^2 t}\right)^2}}{\sqrt{2\pi\sigma^2 t}}, \text{ for } \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \end{aligned}$$

This equation is a particular case of equation (12.16). Figure 12.10 presents an illustration of the solution

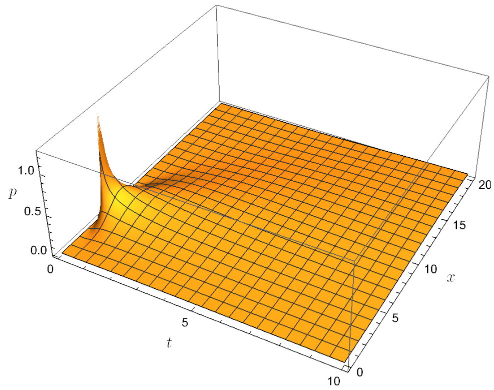


Figure 12.10: Solution for (12.30) for $x_0 = 5$, $\mu = 1$ and $\sigma = 0.5$.

12.8.2 The distribution associated to the Ornstein Uhlenbeck equation

An important stochastic process has a probability distribution represented by a linear PDE with (non-autonomous) coefficients $b(t,x) = \mu_0 + \mu_1 x$ and $a(t,x) = \sigma$ and a Dirac delta initial distribution:

$$\begin{cases} \partial_t p(t,x) = -\partial_x \Big((\mu_0, + \mu_1 x)\, p(t,x) \Big) + \frac{1}{2}\, \partial_{xx} \Big(\sigma^2\, p(t,x) \Big), & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ p(0,x) = \delta(x - x_0) & (t,x) \in \{\, t = 0\} \times \mathbb{R} \end{cases} \tag{12.31}$$

The solution is a Gaussian density function

$$p(t,x) \sim N\left(\frac{\mu_0}{\mu_1} - \left(\frac{\mu_0}{\mu_1} + x_0 \right) e^{\mu_1 t} , \frac{\sigma^2}{2\,\mu_1} (e^{2\mu_1 t} - 1\,) \right),$$

because

$$p(t,x) = \left(\pi\, \frac{\sigma^2}{\mu_1}\, (e^{2\,\mu_1\, t} - 1) \right)^{-\frac{1}{2}} \exp\left\{ - \frac{\left(x + \frac{\mu_0}{\mu_1} - \left(\frac{\mu_0}{\mu_1} + x_0 \right) \right)^2}{\frac{\sigma^2}{\mu_1}\, (e^{2\,\mu_1\, t} - 1)} \right\} \; (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{12.32}$$

Exercise: To prove function (12.32) is the solution of problem (12.31) we follow the following steps. First, apply Fourier transforms. Using the results in the Appendix, Table 12.1, we find

$$\mathcal{F}[\partial_t p(t,x)]\;(\omega) = \partial_t P(t,\omega),$$

$$\mathcal{F}\Big[\,-\partial_x\Big((\mu_0, +\mu_1 x)\,p(t,x)\Big)+\frac{1}{2}\,\partial_{xx}\Big(\sigma^2\,p(t,x)\Big)\Big]\;(\omega) = -\mu_0 2\pi\omega i\,P(t,\omega)+\mu_1\omega\partial_w P(t,\omega)-2(\pi\sigma\omega)^2 P(t,\omega)$$

and

$$\mathcal{F}[p(0,x)]\;(\omega) = e^{-2\pi\omega i x_0}.$$

Therefore, the characteristic equation is a first-order PDE in the Fourier transforms

$$\begin{cases} \partial_t P(t,\omega) - \mu_1\omega\,\partial_w P(t,\omega) = \lambda(\omega)\,P(t,\omega) & (t,\omega)\in\mathbb{R}_+\;\times\mathbb{R} \\ P(0,\omega) = e^{-2\pi\omega i x_0} & (t,\omega)\in\{t=0\}\;\times\mathbb{R} \end{cases}$$

where the eigenfunction $\lambda(\omega)$ is a complex function of ω

$$\lambda(\omega) = -(2(\pi\omega\sigma)^2 + 2\pi\omega\mu_0 i).$$

Second, we can solve this PDE by using the method of characteristics: along a characteristic let $\omega = W(t)$, for any $t\in[0,\infty)$ and write $\hat{P}(t) = P(t,W(t))$ and $\Gamma(t) = \lambda(W(t))$. Taking the time derivative to $\hat{P}(t)$ we have

$$\frac{d\hat{P}(t)}{dt} = \partial_t P(t,W(t)) + \partial_w P(t,W(t))\frac{dW(t)}{dt}$$

which is consistent with PDE in the transformed variables along a characteristic if it is the solution of the ODE system

$$\begin{cases} \dfrac{dW(t)}{dt} = -\mu_1\,W(t), \\ \dfrac{d\hat{P}(t)}{dt} = \Gamma(t)\,\hat{P}(t), \text{for } \hat{P}(0) = P(0,W(0)) = e^{-2\pi W(0) i x_0}. \end{cases}$$

Solving the first ODE yields $W(t) = W(0)e^{-\mu_1 t}$, and as $W(t)=\omega$, we have, along a characteristic, $W(0) = \omega\,e^{\mu_1 t}$.

Solving the second initial-value problem, and because $P(t,\omega) = \hat{P}(t)$ then

$$\begin{aligned} P(t,\omega) &= \hat{P}(0)\,e^{\int_0^t \Gamma(s)ds} \\ &= \exp\left\{-2\,\pi\,i\,x_0\,W(0) - 2\,(\pi\,\sigma\,W(0))^2\int_0^t e^{-2\mu_1 s}ds - 2\,\pi\,i\,\mu_0\,W(0)\int_0^t e^{-\mu_1 s}ds\right\} \\ &= \exp\left\{-\frac{(\pi\sigma)^2}{\mu_1}\,W(0)^2\left(1-e^{-2\mu_1 t}\right) + 2\,\pi\,i\,W(0)\left(\frac{\mu_0}{\mu_1}(e^{-\mu_1 t}-1) - x_0\right)\right\} \end{aligned}$$

substituting $W(0) = \omega\,e^{\mu_1 t}$ yields the function

$$P(t,\omega) = \exp\Big\{-a\,(2\,\pi\,\omega)^2 + b\,(2\,\pi\,i\,\omega)\Big\}$$

where

$$a \equiv \frac{\sigma^2}{4\mu_1}\left(e^{2\mu_1 t}-1\right) \text{ and } b \equiv \frac{\mu_0}{\mu_1}-\left(\frac{\mu_0}{\mu_1}+x_0\right)e^{\mu_1 t}.$$

Transforming back (using the inverse Fourier transform in Appendix 12.10 Table 12.2) , that is making $p(t,x)=\mathcal{F}^{-1}[P(t,\omega)]\ (x)$ we find function (12.32).

Characterizing the solution: We see that if $\mu_1<0$ then

$$\lim_{t\to\infty} p(t,x) \sim N\left(-\frac{\mu_0}{\mu_1}, -\frac{\sigma^2}{2\mu_1}\right),$$

then the distribution is ergodic: for any initial value x_0 it tends asymptotically to a normal distribution (see Figure 12.11).

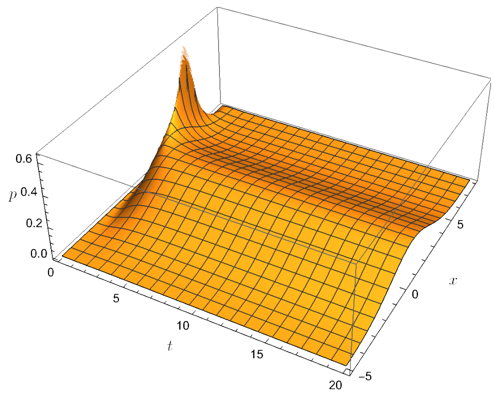


Figure 12.11: Solution for (12.30) for $x_0=5$, $\mu_0=1$, $\mu_1=-1$ and $\sigma=0.5$.

12.9 Economic applications

12.9.1 The distributional Solow model

In Brito (2022) we prove that in an economy in which the capital stock is distributed in an heterogeneous way between regions, $K(t,x)$, if there is an infinite support, and there are free capital flows between regions, the budget constraint for the location x can be represented by the parabolic PDE.

Consider the accounting balance between savings and internal and external investment for a region x at time t

$$I(t,x)+T(t,x)=S(t,x)$$

where $I(t,x)$ and $S(t,x)$ is investment and domestic savings of location x at time t and $T(t,x)$ is the savings flowing to other regions.

Assume that the capital flow for a region of length Δx is symmetric to the capital distribution gradient to neighboring regions:

$$T(t,x)\Delta x = -\left(K_x(x+\Delta x,t) - K_x(t,x)\right)$$

that is capital flows proportionaly and in a reverse direction to the "spatial gradient" of the capital distribution: regions with high capital intensity will tend to "leak" capital to other regions. If $\Delta x \to 0$ leads to $T(t,x) = -K_{xx}(t,x)$.

If there is no depreciation then $I(t,x) = K_t(t,x)$. If the technology is AK and the savings rate is determined as in the Solow model then $S(t,x) = sAK(t,x)$ where $0 < s < 1$ and A is assume to be spatially homogeneous.

Therefore we obtain a distributional Solow equation for an economy composed by heterogenous regions

$$K_t = K_{xx} + sAK, (t,x) \in (-\infty,\infty) \times (0,\infty)$$

We can define a spatially-homogenous balanced growth path (BGP) as

$$\overline{K}(t) = \overline{K}e^{\gamma t}$$

where $\gamma = sA$.

Then, if we define the deviations as regards the BGP as $k(t,x) = K(t,x)e^{-\gamma t}$, we observe that the transitional dynamics is given by the solution of the equation

$$k_t = k_{xx}$$

which is the heat equation. Therefore, given the initial distribution of the capital stock $K(x,0) = k_0(x)$ the solution for this spatial AK model is

$$K(t,x) = \begin{cases} k_0(x), & t = 0 \\ e^{\gamma t}\displaystyle\int_{-\infty}^{\infty} k_0(\xi)\,(4\pi t)^{-1/2}\,e^{-\frac{(x-\xi)^2}{4t}}\,d\xi, & t > 0 \end{cases}$$

and the solution is similar to the case depicted in Figure 12.2 when $c > 0$.

The main conclusion is that: (1) there is long run growth; (2) , if there are homogenous technologies and preferences the asymptotic distribution will become spatially homogeneous. That is: the so-called β- and σ- convergences can be made consistent !

12.9.2 The option pricing model

The Black and Scholes (1973) model is a case in which a research paper had an immense impact on the operation of the economy. It is related to the onset of derivative markets and basically gave birth to stochastic finance[5].

[5] Myron Scholes was awarded the Nobel prize in 1997, together with Robert Merton another important contributer to stochastic finance, precisely for this formula. Fisher Black was deceased at the time.

It provides a formula (the so called Black-Scholes formula) for the value of an European call option when there are two assets, a riskless asset with interest rate r and a underlying asset whose price, S which follows a diffusion process (in a stochastic sense): $dS = \mu S dt + \sigma S dB$ where dB is the standard Brownian motion (see next chapter). An European call option offers the right to buy the underlying asset at time T for a price K fixed at time $t = 0$, which is conventioned to be the moment of the contract.

Under the assumption that there are no arbitrage opportunities Black and Scholes (1973) proved that the price of the option $V = V(t, S)$ is a a function of time, $t \in (0, T)$ and the price of an underlying asset $S \in (0, \infty)$ follows the backward parabolic PDE and has a terminal constraint

$$\begin{cases} V_t(t,S) & = -\frac{\sigma^2 S^2}{2} V_{SS}(t,S) - rSV_S(t,S) + rV(t,S), \; (t,S) \in [0,T] \times (0,\infty) \\ V(T,S) & = \max\{ S - K, 0\} \;, \; (t,S) \in \{t = T\} \times (0,\infty). \end{cases} \tag{12.33}$$

The first equation is valid for any financial option having the same underlying asset dynamics, and the terminal constraint is characteristic of the European call option: because the writer sells the right, but not the obligation, to purchase the underlying asset at the price K at time $t = T$, the buyer is only interested in that purchase if he can sell it at the prevailing market price $S = S(T)$ when that price is higher than the exercise price K. In this case the payoff will be $S(T) - K$. Otherwise he will not execute the option and the terminal payoff will be zero.

The two boundary constraints

$$\begin{aligned} V(t,0) = 0, &\quad (t,S) \in [0,T] \times \{S = 0\} \\ \lim_{S \to \infty} V(t,S) = S, &\quad (t,S) \in [0,T] \times \{S \to \infty\}, \end{aligned}$$

are sometimes referred to, but they are redundant.

The same structure occurs in the Merton's model (see Merton (1974)) which is a seminal paper on the pricing of default bonds. It was the first model on the so-called structural approach to modelling credit risk.[6] In essence, this model assumes that the value of the firm follows a linear diffusion process and it considers the issuance of a bond with an expiring date T whose indenture gives it absolute priority on the value of the firm at the expiry date. This means that either if the value of the firm is smaller that the face value of the bond the creditor takes possession of the firm and in the opposite case it recovers the face value. In this case, we can interpret the position of the equity owner as holding an European call option over the value of the firm with strike price equal to the face value of the debt and the creditor as having an European put option security.

The price of the European call option [7], given the former assumptions is given by

$$V(t,S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \quad t \in [0,T] \tag{12.34}$$

[6]This model is the inspiration of credit risk models used by rating agencies (see Duffie and Singleton (2003)).

[7]For the credit risk model S would be the value of assets of a firm, K would be the face value of loan, and T the term of the loan.

where $\Phi(.)$ is cumulative Gaussian density function such that $\Phi(d) = \mathbb{P}(x \le d)$ where

$$d_1 = \frac{\ln(S/K) + (T-t)\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \tag{12.35}$$

$$d_2 = \frac{\ln(S/K) + (T-t)\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \tag{12.36}$$

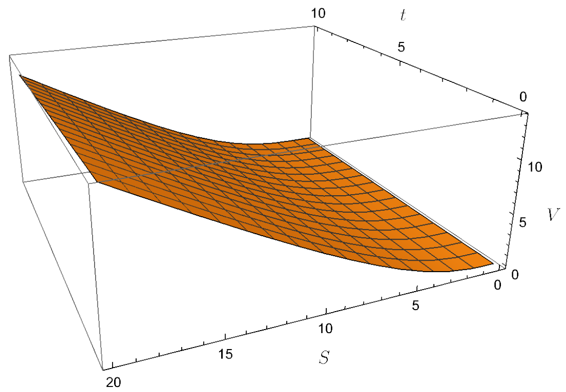


Figure 12.12: Solution for the Black and Scholes model, for $r = 0.02$, $T = 20$, $\sigma = 0.2$, and $K = 10$.

Proof. In order to solve the B-S PDE, which is a non-linear backward parabolic PDE, we transform it to to a quasi-linear parabolic forward PDE, by applying the transformations: $t(\tau) = T - \tau$ and $S = Ke^x$ and setting $u(\tau, x) = V(t(\tau), S(x))$. We can transform the option-pricing problem to the equivalent initial-value problem PDE equivalent to (12.33)

$$\begin{cases} u_\tau = \frac{\sigma^2}{2} u_{xx} + \left(r - \frac{\sigma^2}{2}\right) u_x - ru, & (\tau, x) \in [0, T] \times (-\infty, \infty) \\ u(0, x) = u_0(x) \end{cases} \tag{12.37}$$

where

$$u_0(x) = \begin{cases} 0, & \text{if } x \le 0 \\ K\left(e^x - 1\right), & \text{if } x > 0 \end{cases}$$

The PDE is a particular example of equation (12.16), which implies that the solution is

$$\begin{aligned} u(\tau, x) &= \int_{-\infty}^{0} 0\, g(\tau, x-s) ds + K \int_0^\infty \left(e^s - 1\right) g(\tau, x-s) ds \\ &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \int_0^\infty \left(e^s - 1\right) e^{h(\tau, x-s)} ds \end{aligned}$$

where (from equation (12.15))

$$h(\tau, y) \equiv -\frac{y^2 + 2\tau\left(r - \frac{\sigma^2}{2}\right) y + \left(r + \frac{\sigma^2}{2}\right)^2 \tau^2}{2\tau\sigma^2}.$$

Then

$$
\begin{aligned}
u(\tau, x) &= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \left(\int_0^\infty e^{s+h(\tau,x-s)} ds - \int_0^\infty e^{h(\tau,x-s)} ds \right) \\
&= \frac{K}{\sqrt{2\pi\sigma^2\tau}} \left(I_1 - I_2 \right).
\end{aligned}
$$

In order to simplify the integrals, it is useful to remember the formulas for the error function, $\mathrm{erf}(x)$, and of the Gaussian cumulative distribution $\Phi(x)$,

$$
\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^x e^{-z^2} dz, \; \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz
$$

which are related as

$$
\Phi(x) = \frac{1}{2} \left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}} \right) \right].
$$

After some algebra we obtain

$$
\begin{aligned}
s + h(\tau, x - s) &= x - \frac{1}{2}(\delta_1(s))^2 \\
h(\tau, x - s) &= -r\tau - \frac{1}{2}(\delta_2(s))^2
\end{aligned}
$$

where

$$
\delta_1(s) \equiv \frac{x - s + \left(r + \frac{\sigma^2}{2} \right)}{\sigma\sqrt{\tau}}, \text{ and } \delta_2(s) \equiv \frac{x - s + \left(r - \frac{\sigma^2}{2} \right)}{\sigma\sqrt{\tau}}.
$$

Then, integration by transformation of variables $z = \varphi(s)$,

$$
\begin{aligned}
I_1 &= e^x \int_0^\infty e^{-\frac{1}{2}(\delta_1(s))^2} ds = \\
&= -\sigma\sqrt{\tau}e^x \int_{d_1}^{-\infty} e^{-\frac{1}{2}\delta_1^2} d\delta_1 = \\
&= \sqrt{\sigma^2\tau}e^x \int_{-\infty}^{d_1} e^{-\frac{1}{2}\delta_1^2} d\delta_1 = \\
&= \sqrt{2\pi\sigma^2\tau}e^x \Phi(d_1)
\end{aligned}
$$

where $d_1 = \delta_1(0)$ as in equation (12.35) for $\tau = T - t$, and also, writing that $d_2 = \delta_2(0)$, as in equation (12.36) for $\tau = T - t$,

$$
\begin{aligned}
I_2 &= e^{-r\tau} \int_0^\infty e^{-\frac{1}{2}(\delta_2(s))^2} ds = \\
&= -\sigma\sqrt{\tau}e^{-r\tau} \int_{d_2}^{-\infty} e^{-\frac{1}{2}\delta_2^2} d\delta_2 =, \\
&= \sqrt{\sigma^2\tau}e^{-r\tau} \int_{-\infty}^{d_2} e^{-\frac{1}{2}\delta_2^2} d\delta_2 = \\
&= \sqrt{2\pi\sigma^2\tau}e^{-r\tau} \Phi(d_2)
\end{aligned}
$$

Thus

$$u(\tau, x) = K\left(e^{x}\Phi(d_1) - e^{-r\tau}\Phi(d_2)\right)$$

and transforming back $V(t, S) = u\left(T - t, \ln\left(S/K\right)\right)$ we get equation (12.34). $\square$

Observe this is a backward parabolic PDE, which implies that the terminal condition determines the particular solution.

12.10 Bibiography

- Mathematics of PDE's: introductory Olver (2014), Salsa (2016) and (Pinsky, 2003, ch 5). Advanced (Evans, 2010, ch 3).
- Applications to economics (with more advanced material) : Achdou et al. (2014)
- Applications to growth theory Brito (2022)

12.A Appendix: Fourier transforms

Consider a function $f(x)$ such that $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} |f(x)|dx < \infty$. We can define a pair of generalized functions, the Fourier transform of $f(x)$, $F(s) = \mathcal{F}[f(x)](s)$ and the inverse Fourier transform $\mathcal{F}^{-1}[F(s)]\ (x) = f(x)$ (using the definition of Kammler (2000)), where

$$F(s) = \mathcal{F}[f(x)] \equiv \int_{-\infty}^{\infty} f(x)\, e^{-2\,\pi\,i\,s\,x}\, dx$$

where $i^2 = -1$ and

$$f(x) = \mathcal{F}^{-1}[F(s)] \equiv \int_{-\infty}^{\infty} F(s) e^{2\pi i s x} ds.$$

Table 12.1: Fourier and inverse Fourier transforms: properties

	$f(x)$ for $-\infty < x < \infty$	$F(s)$ for $-\infty < s < \infty$	obs
1	a	$a\,\delta(s)$	$a \in \mathbb{C}$ complex number, $\delta(\dot{})$ Dirac's delta
2	$a\,\delta(x)$	a	$a \in \mathbb{C}$ complex number,
3	$a\,f(x)$	$a\,F(s)$	$a \in \mathbb{C}$ complex number
4	$a\,f(x) + b\,g(x)$	$a\,F(s) + b\,G(s)$	$a, b\mathbb{C}$ complex numbers
5	$f(x) * g(x)$	$F(s)\,G(s)$	$f(x) * g(x)$ is a convolution
6	x	$-\dfrac{\delta'(s)}{2\,\pi\,i}$	$\delta'(\cdot)$ is the first derivative of $\delta(\cdot)$
7	x^2	$\dfrac{\delta''(s)}{2\,\pi}$	$\delta''(\cdot)$ is the second derivative of $\delta(\cdot)$
8	$x\,f(x)$	$-\dfrac{F'(s)}{2\,\pi\,i}$	
9	$f(t, x)$	$F(t, s)$	t is a real number
10	$f'(x)$	$2\,\pi, i\,s\,F(s)$	
11	$x\,f'(x)$	$-(F(s) + s\,F'(s))$	if $s \in \mathbb{R}$
12	$f''(x)$	$-(2\,\pi\,s)^2\,F(s)$	
13	$x\,f''(x)$	$\dfrac{2\,\pi\,s\,(2\,F(s) + s\,F'(s))}{i}$	
14	$x^2\,f''(x)$	$-s^2\,F''(s)$	

There are some useful properties of the Fourier transform that we use in the main text:

1. the Fourier transform of a constant: let $a \in \mathbb{C}$ be a number

$$F(a) \ = \mathcal{F}[a] \ = a \int_{-\infty}^{\infty} e^{-2\,\pi\,i\,s\,x}\, dx = a \int_{-\infty}^{\infty} \delta(s - x)\, dx = a\,\delta(s)$$

and

$$\mathcal{F}^{-1}[a\delta(s)] \ = \int_{-\infty}^{\infty} a\,\delta(s)\, e^{2\,\pi\,i\,s\,x}\, ds = a;$$

2. the Fourier transform of $a \in \mathbb{C}$ constant times a Dirac's delta generalized function:

$$F(a\,\delta(x)) \;= \mathcal{F}[a\,\delta(x)] \;= a \int_{-\infty}^{\infty} \delta(x)\, e^{-2\,\pi\,i\,s\,x}\, dx = a,$$

and

$$\mathcal{F}^{-1}[a] \;= \int_{-\infty}^{\infty} a, e^{2\,\pi\,i\,s\,x}\, ds = a\,\delta(x);$$

3. the Fourier transform preserves multiplication by a complex number $a \in \mathbb{C}$:

$$\mathcal{F}[a\,f(x)] \;= a\,F(s), \text{ and } \mathcal{F}^{-1}[aF(s)] \;= af(x),$$

Proof.

$$\mathcal{F}[a\,f(x)] \;= \int_{-\infty}^{\infty} a\,f(x)\,e^{-2\,\pi\,i\,s\,x}\,dx = a\int_{-\infty}^{\infty} f(x)\,e^{-2\,\pi\,i\,s\,x}\,dx = a\,F(s),$$

and

$$\mathcal{F}^{-1}[a\,F(s)] \equiv \int_{-\infty}^{\infty} a\,F(s)e^{2\pi isx}ds = a\int_{-\infty}^{\infty} F(s)e^{2\pi isx}ds = a\,f(x).$$

□

4. the Fourier transform preserves linearity:

$$\mathcal{F}[a\,f(x) + b\,g(x)] \;= a\,F(s) + b\,G(s), \text{ and } \mathcal{F}^{-1}[a\,F(s) + b\,G(s)] \;= a\,f(x) + b\,g(x)$$

5. the Fourier transform does not preserve multiplication of two functions. However, there is a relationship between convolution of functions and multiplication of Fourier transforms. A **convolution** between two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy.$$

The inverse Fourier transform of a product of two Fourier transforms is a convolution,

$$f(x) * g(x) = \mathcal{F}^{-1}[F(s)\,G(s)] \;= \int_{-\infty}^{\infty} F(s)\,G(s)\,e^{2\pi isx}ds$$

6. $\mathcal{F}[x] = -\dfrac{1}{2\,\pi\,i}\delta'(s)$, where $\delta(x)$ is Dirac's delta

Proof. Observe that $\int_{-\infty}^{\infty} e^{2\pi i s x}\delta(s)ds = 1$ and

$$\int_{-\infty}^{\infty} e^{2\,\pi\,i\,s\,x}\,\delta(s) = 0.$$

Then

$$\begin{aligned} x &= x\int_{-\infty}^{\infty} e^{2\pi i s x}\delta(s)ds \\ &= -\frac{1}{2\,\pi\,i}\,\Big(\int_{-\infty}^{\infty} e^{2\,\pi\,i\,s\,x}\,\delta(s) - \int_{-\infty}^{\infty} 2\,\pi\,i\,x\,e^{2\,\pi\,i\,s\,x}\delta(s)\,ds\ \Big) \\ &= -\frac{1}{2\,\pi\,i}\int_{-\infty}^{\infty} e^{2\,\pi\,i\,s\,x}\delta'(s)\,ds \\ &= -\frac{1}{2\,\pi\,i}\mathcal{F}^{-1}\Big[\ \frac{1}{2\,\pi\,i}\,\delta'(s)\Big] \end{aligned}$$

□

7. $\mathcal{F}[x^2] = \dfrac{1}{(2\,\pi)^2}\delta''(s)$

8. $\mathcal{F}[x\,f(x)]\ = -\dfrac{1}{2\,\pi\,i}\,F'(s)$

   Proof.

$$\begin{aligned} \mathcal{F}[x\,f(x)]\ &= \int_{-\infty}^{\infty} x\,f(x)\,e^{-2\,\pi\,i\,s\,x}\,dx \\ &= -\frac{1}{2\,\pi\,i}\ \int_{-\infty}^{\infty} -2\pi\,i\,x\,f(x)\,e^{-2\,\pi\,i\,s\,x}\,dx \\ &= -\frac{1}{2\,\pi\,i}\ \int_{-\infty}^{\infty} f(x)\,\frac{d}{ds}\Big(e^{-2\,\pi\,i\,s\,x}\Big)\,dx \\ &= -\frac{1}{2\,\pi\,i}\ \frac{d}{ds}F(s) = \frac{d}{ds}\left[\ \int_{-\infty}^{\infty} f(x)\,e^{-2\,\pi\,i\,s\,x}\,dx\right] \\ &= -\frac{1}{2\,\pi\,i}\,F'(s) \end{aligned}$$

   Alternative proof:

$$\mathcal{F}[x\,f(x)]\ = \mathcal{F}[x] * \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} -\frac{1}{2\,\pi\,i}\,\delta'(y)\,F(s-y)dy = -\frac{1}{2\,\pi\,i}\,F'(s)$$

   if $f = f(x,t)$ where t is a real variable then $F(s,t) = \mathcal{F}[f(x,t)]$ and $f(x,t) =\ \mathcal{F}^{-1}[F(s,t)]$. Also $F_t(s,t) = \mathcal{F}[f_t(x,t)]$ and $f_t(x,t) =\ \mathcal{F}^{-1}[F_t(s,t)]$ □

9. $\mathcal{F}[f'(x)]\ = 2\,\pi\,i\,s\,F(s)$

Proof.

$$
\begin{aligned}
\mathcal{F}[f'(x)] &= \int_{-\infty}^{\infty} f'(x)\, e^{-2\,\pi\, i\, s\, x}\, dx \\
&\text{integration by parts} \\
&= \int_{-\infty}^{\infty} f(x)\, e^{-2\,\pi\, i\, s\, x} - \int_{-\infty}^{\infty} f(x)\, \frac{\partial}{\partial x}\Big(e^{-2\,\pi\, i\, s\, x}\Big)\, dx \\
&\text{because } e^{-2\,\pi\, i\, s\, x} \text{ is symmetric the first integral is equal to zero} \\
&= 2\,\pi\, i\, s \int_{-\infty}^{\infty} f(x)\, e^{-2\,\pi\, i\, s\, x}\, dx \\
&= 2\,\pi\, i\, s\, F(s)
\end{aligned}
$$

□

10. $\mathcal{F}[xf'(x)] \;= -\big(F(s) + sF'(s)\big)$ if $s \in \mathbb{R}$

Proof.

$$
\begin{aligned}
\mathcal{F}[x\, f'(x)] &= \mathcal{F}[x] * \mathcal{F}[f'(x)] \\
&= \int_{-\infty}^{\infty} \Big(- \frac{1}{2\,\pi\, i}\, \delta'(y)) \Big(2\,\pi\, i\,(s-y)\, F(s-y)\Big) dy \\
&= -\int_{-\infty}^{\infty} \delta'(y)\,(s-y)\, F(s-y) dy \\
&= -s \int_{-\infty}^{\infty} \delta'(y)\, F(s-y) dy + \int_{-\infty}^{\infty} \delta'(y)\, y\, F(s-y) dy \\
&= -sF'(s) + \int_{-\infty}^{\infty} \delta(y)\, y\, F(s-y) - \int_{-\infty}^{\infty} \delta(y)\, F(s-y) dy \\
&= sF'(s) - F(s)
\end{aligned}
$$

□

11. $\mathcal{F}[f''(x)] \;= -4\,\pi^2\, s^2\, F(s)$
12. $\mathcal{F}[x\, f''(x)] \;= \dfrac{2\,\pi\, s}{i}\Big(2\, F(s) + s\, F'(s)\Big)$
13. $\mathcal{F}[x^2\, f''(x)] \;= -s^2\, F''(s)$

References: `https://dlmf.nist.gov/1.14`

Table 12.2: Fourier and inverse Fourier transforms of some functions

$f(x)$ for $-\infty < x < \infty$	$F(s)$ for $-\infty < s < \infty$	obs
$\delta(x-a)$	$e^{-2\pi i s a}$	
$\frac{1}{\sqrt{4\pi a}}e^{-\frac{x^2}{4a}}$	$e^{-a(2\pi s)^2}$	$a>0$
$\frac{1}{\sqrt{4\pi a}}e^{-\frac{(x+b)^2}{4a}}$	$e^{-a(2\pi s)^2+b(2\pi i s)}$	$a>0, \quad b\in\mathbb{R}$
$\frac{1}{\sqrt{4\pi a}}e^{c-\frac{x^2}{4a}}$	$e^{-a(2\pi s)^2+c}$	$a>0, \quad c\in\mathbb{R}$
$\frac{1}{\sqrt{4\pi a}}e^{c-\frac{(x+b)^2}{4a}}$	$e^{-a(2\pi s)^2+b(2\pi i s)+c}$	$a>0, \quad (b,c)\in\mathbb{R}^2$

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