Foundations of Financial Economics *2017/18*
Problem set 4: two-period APT and GEAP

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1 Two-period finance economy model

1. Consider a finance economy in which (assets in the columns)

$$S = (1, s)\,, \quad V = \begin{pmatrix} r+\epsilon & r(1-\epsilon) \\ r-\epsilon & r(1+\epsilon) \end{pmatrix}$$

   where $0 < r < 1$, $s > 0$ and ϵ can take any (real) value:

   (a) Under which conditions arbitrage opportunities exist ?

   (b) Can we have absence of arbitrage opportunities and market incompleteness ?

   (c) Compute the market price for an Arrow-Debreu contingent claim, under the assumption of absence of arbitrage opportunities. What is replicating transactions' strategy ? Explain.

2. Consider a finance economy in which

$$S = \left(1, \frac{1}{1+r}\right)\,, \quad V = \begin{pmatrix} r+\epsilon & 1 \\ r-\epsilon & 1 \end{pmatrix}$$

   where $r > 0$ and ϵ can take any (real) value.

   (a) Under which conditions arbitrage opportunities exist ?

   (b) Compute the state prices, by assuming the appropriate existence conditions.

   (c) Under the conditions that you imposed in previous point, can there be market completeness ? How ?

   (d) Compute the market price for an Arrow-Debreu contingent claim, under the assumption of absence of arbitrage opportunities. What is replicating transactions' strategy ? Explain.

3. Assume an asset market represented by the pair of payoff prices (S, V) and consider the linear pricing relationship $S = qV$. Consider a simple case in which the number of assets is equal to the number of states of nature and both equal to 2:
   (a) determine an equivalent expression involving a stochastic discount factor and asset returns;
   (b) determine a sufficient condition for the existence of arbitrage in terms of the asset returns;
   (c) assume that there are no arbitrage opportunities, determine conditions for the existence of completeness in terms of the asset returns;
   (d) assume that there are no arbitrage opportunities, determine conditions for the existence of incompleteness in terms of the asset returns;
   (e) what are the consequences of the existence of incompleteness on the expected value of asset returns, variances and the covariance with the stochastic discount factor ?

4. Consider the following return matrices $(K \times N)$

$$\begin{pmatrix} 1.1 & 1.2 \\ 1.02 & 1.01 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 1.111 & 1.122 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 1.111 & 1.122 \\ 0.99 & 0.95 \end{pmatrix}, \begin{pmatrix} 1.01 & 1.02 \\ 0.99 & 1.02 \end{pmatrix}.$$

   Characterize the asset market as regards the existence of arbitrage opportunities and completeness

5. Consider two (alternative) financial markets, A and B, characterized by the return matrices $(N \times K)$

$$R = \begin{pmatrix} 1 & 1+a & b \\ 1 & 1-a & b \end{pmatrix}, \text{ for financial market } A, \text{ and } R = \begin{pmatrix} 1 & 1 \\ 1+a & 1-a \\ 1-b & 1+b \end{pmatrix}, \text{ for financial market } B$$

   where a and b can take any real value.
   (a) Under which conditions there are no arbitrage opportunities and there is market completeness in the financial market A
   (b) Under which conditions there are no arbitrage opportunities and there is completeness in the financial market B

6. Assume there is a financial market with two assets, one risky asset with return $1 + r$ and one riskless asset with return $1 + i$. Assume there are no arbitrage opportunities. Prove that the Sharpe index verifies

$$\left| \frac{E[r-i]}{\sigma[r]} \right| \leq \frac{\sigma[m]}{E[m]}$$

   where m is the stochastic discount factor.

7. Assume there is a financial market with two assets, one risky asset and one riskless asset with prices and payoffs
$$S = \begin{pmatrix} \frac{1}{1+i} & s \end{pmatrix},\ V = \begin{pmatrix} 1 & d_1 \\ 1 & d_2 \end{pmatrix},$$
where $i > 0$ and $0 < d_1 < d_2$. Introduce an european call option with exercise price $d_1 < p < d_2$. Prove that its price, if there are absence of arbitrage opportunities is
$$S_o = \frac{(r_1 - i)(d_2 - p)}{(1+i)(r_1 - r_2)}$$

8. Consider a finance economy in which there are three assets with the vector of prices and payoff matrix given by
$$S = \left(p, 1, \frac{1}{1+r}\right),\quad V = \begin{pmatrix} r-\epsilon & r+\epsilon & 1 \\ r+\epsilon & r-\epsilon & 1 \end{pmatrix}$$
where we assume that $r > 0$, $p > 0$ and $\epsilon \geq 0$.
    (a) Under which conditions there are no arbitrage opportunities ?
    (b) Under the conditions that you imposed in the answer to the previous point, what would be the meaning of market completeness in this economy ? Determine the conditions for the existence of market completeness.
    (c) Compute the price for the first asset, p, by building a replicating transactions' strategy. Explain.

2 Two-period equilibrium asset prices

1. Consider a financial economy characterized by (S, V) where
$$S = (1/(1+i), s),\ V = \begin{pmatrix} 1 & v_1 \\ 1 & v_2 \end{pmatrix}$$
for $v_1 < 1 < v_2$. Agents are homogeneous and we consider an endowment economy with an endowment process $\{y_0, y_1\}$ where $y_{1,s} = (1+\gamma_s)y_0$.
    (a) characterize the asset market as regards the existence of arbitrage opportunities and completeness;
    (b) define the equilibrium, determine the asset prices, and interpret the results obtained, for the following utility functions:
        i. a logaritmic utility function, $u(c) = \ln(c)$;
        ii. a quadratic utility function, $u(c) = ac - b/2c^2$, $a > 0$;
        iii. an exponential utility function, $u(c) = -\frac{e^{-\lambda x}}{\lambda}$, $\lambda > 0$;

iv. a power utility function, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, $\theta > 0$;

(c) assume that an european call option is issued, at time 0 for exercise at time 1 with the price k, such that $v_1 < k < v_2$. Determine a replicating portfolio, and find the equilibrium price for option.

2. Consider a financial economy characterized by (S, V) where

$$S = (s_1, s_2),\ V = \begin{pmatrix} 2v_1 & v_1 \\ 2v_2 & v_2 \end{pmatrix}$$

Agents are homogeneous and we consider an endowment economy with an endowment process $\{y_0, y_1\}$ where $y_{1,s} = (1 + \gamma_s)y_0$.

(a) characterize the asset market as regards the existence of arbitrage opportunities and completeness;

(b) define the equilibrium, determine the asset prices, and interpret the results obtained, for the following utility functions:

i. a logaritmic utility function, $u(c) = \ln(c)$;

ii. a quadratic utility function, $u(c) = ac - b/2c^2$, $a > 0$;

iii. an exponential utility function, $u(c) = -\frac{e^{-\lambda x}}{\lambda}$, $\lambda > 0$;

iv. a power utility function, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, $\theta > 0$;

3. Let information be given by a two-period binomial tree, with equal probabilities for the two states of nature, and assume an endowment finance economy in which there are no arbitrage opportunities and agents are homogeneous. Further, assume that agents have a logarithmic Bernoulli utility function and assume that the endowment at time 1 is $Y_1 = ((1-\gamma)\,y_0, (1+\gamma)\,y_0)$ where $0 < \gamma < 1$. The financial market is characterized by the existence of a bond with unit face value (to be paid at time $t = 1$) and a risky asset with price S and payoff $(2\,v\,S, v\,S)$.

(a) Find the stochastic discount factor.

(b) Find the Sharpe index. Justify your reasoning.

(c) Find the Hansen-Jagannathan bound. Explain its meaning.

(d) For which values of γ the equity premium puzzle holds ? Provide an intuition for your result.

4. Consider a finance economy in which there is a risk-free asset with return $1 + i$, for $i > 0$, and a risky asset with return $R = (1 + i + \varepsilon, 1 + i - \varepsilon)$ for $0 < \varepsilon < 1$. Assume that the representative consumer has a logarithmic Bernoulli utility function and the endowment process is $Y = \{y_0, Y_1\}$ where $Y_1 = ((1+\gamma)y_0, (1-\gamma)y_0)^\top$ for $0 < \gamma < 1$.

(a) Find the probabilities for the two states of nature assuming that there are no arbitrage opportunities (Hint: use the condition $\mathbb{E}[M(R - R^f]] = 0$).

(b) Find the Sharpe index for the risk premium. Justify.

(c) Find the Hansen-Jaganathan bound. Justify.

(d) Does the equity premium puzzle holds in this case ? Provide an intuition.