

AME 2019-2020:  
Problem set 5: Stochastic differential equations

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## 1 General

1. The diffusion equation is

$$dX(t) = \gamma X(t)dt + \sigma X(t)dW(t)$$

for  $X(0) = x_0$ .

- (a) Prove that the solution is  $X(t) = x_0 e^{(\gamma - \sigma^2/2)t + \sigma W(t)}$
- (b) Find  $\mathbb{E}[X(t)|X(0) = x_0]$  and  $\mathbb{V}[X(t)|X(0) = x_0]$ .
- (c) Derive the backward Kolmogorov equation for the probability for  $X(T) \leq 2x$  assuming that  $X(t) = x$
- (d) Derive the forward Kolmogorov equation for the density associated to  $X(t) = x > 0$ , assuming that  $X(0) = 0$ .

2. Consider the diffusion equation

$$dX(t) = \gamma X(t)dt + \sigma dW(t)$$

where  $\{W(t)\}$  is a standard Brownian motion.

- (a) Let  $X(0) = x_0$  be known. Find the solution of the initial value problem.
- (b) Find  $\mathbb{E}[X(t)|X(0) = x_0]$  and  $\mathbb{V}[X(t)|X(0) = x_0]$ .
- (c) Write the forward Kolmogorov equation for the density associated to  $X(t) = x$ . Provide an intuition for this equation.

3. The Vasicek 1977 (or Ornstein-Uhlenbeck) process is the solution of the equation

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t)$$

for  $X(0) = x_0$ .

- (a) Prove that the solution is

$$X(t) = \mu + (x_0 - \mu)e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dW(s)$$

- (b) Find  $\mathbb{E}[X(t)|X(0) = x_0]$  and  $\mathbb{V}[X(t)|X(0) = x_0]$ .

## 2 Economic applications

1. In the stochastic Solow model assume that the population is deterministic and the production function is  $Y(t) = A(t)F(K(t), L(t))$  where productivity follows the process

$$dA(t) = \gamma A(t)dt + \sigma A(t)dW(t).$$

Determine the capital accumulation equation. Assuming a Cobb-Douglas equation find the asymptotic distribution of the capital stock.

2. Assume an AK model where  $Y = A(t)K(t)$  where

$$dA(t) = \gamma dt + \sigma dW(t).$$

Assuming an equilibrium equation  $dK(t) = sY(t)dt$ , and  $K(0) = K_0$  given, find an explicit solution for the capital stock. Determine the moment for the process of  $K(t)$ .

3. Solve the stochastic problem for a representative consumer assuming a log utility function.
4. Solve the stochastic Ramsey model assuming a log utility function.

## References

Vasicek, O. A. (1977). "An equilibrium characterization of the term structure". In: *Journal of Financial Economics* 5, pp. 177–88.