Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
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Exam: **Época Normal**
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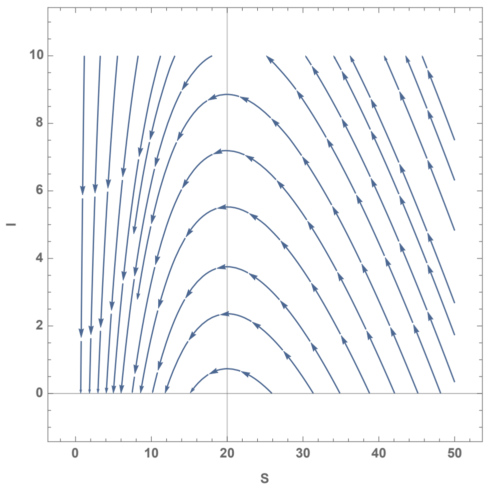
1. (a) We have $N(t) = S(t) + I(t) + R(t)$. Population is constant if and only if $\dot{N} = 0$. Then $\dot{S} + \dot{I} = -\dot{R} = 0$. Substituting, we find $-\beta SI + \beta SI - \gamma I = -\gamma I = -\dot{R}$. Then, the last equation is redundant.

   (b) The steady state is $\bar{I} = 0$, for any value of S. Then there is an infinite number of steady states, for any $S \in (0., \infty)$. The Jacobian, evaluated at the steady state is

$$J = \begin{pmatrix} 0 & -\beta S \\ 0 & \beta S - \gamma \end{pmatrix}$$

   Then $\det(J) = 0$ and $\mathrm{trace}(J) = \beta S - \gamma$. Therefore, if $S < \frac{\gamma}{\beta}$ the steady states are asymptotically stable (although degenerate) and if $S > \frac{\gamma}{\beta}$ they are unstable.
   The phase diagram is



   (c) For $\beta = \beta_0$ both S and I are decreasing to a steady state $\bar{S} < \frac{\gamma}{\beta_0}$ and $\bar{I} = 0$. After the shock, from β_0 to $\beta_1 > \beta_0$, I will increase and S will decrease until S reaches the point $S(\tau) = \frac{\gamma}{\beta_1}$, at time $t = \tau$. From that point on both S and I will decrease to a steady state such that $\bar{I} = 0$ and $\bar{S} < \frac{\gamma}{\beta_1}$. This can only be determined numerically

2. (a) The current-value Hamiltonian is $h(I,K,Q) = AK - \frac{I^2}{2} + Q(I - \delta K)$. The f.o.c are

$$
\begin{aligned}
I &= Q \\
\dot{Q} &= (r+\delta)Q - A \\
\dot{K} &= I - \delta K \\
0 &= \lim_{t\to\infty} Q(t)K(t)e^{-rt}
\end{aligned}
$$

The phase diagram is

(c) Optimal K,

$$K(t) = k_0 e^{-\delta t} + \frac{\bar{I}}{\delta}\left(1 - e^{-\delta t}\right)$$

where $\bar{I} = \frac{A}{r+\delta}$.

(d) The HJB equation

$$rV(K) = \max_I \left\{AK - \frac{I^2}{2} + V^K(I - \delta K)\right\}$$

The policy function is $I^* = V^{'}(K)$. Therefore the HJB equation becomes, at the optimum,

$$rV(K) = AK + \left(\frac{V^{'}(K)}{2}\right)^2 - \delta\, K\, V^{'}(K)$$

(e) Trial function $V(K) = \alpha_0 + \alpha_1\, K + \alpha_2\, K^2$. We find there are two candidate solutions

$$V(K) = \bar{I}\left(\frac{\bar{I}}{4r} + K\right) \tag{1}$$

or

$$V_2(K) = \frac{1}{4r}\left(\frac{A}{\delta}\right)^2 - \frac{A}{\delta}K + (r+2\delta)K^2 \tag{2}$$

As $\lim_{K\to\infty} V^{'}(K) = \bar{I}$ is bounded and $\lim_{K\to\infty} V_2^{'}(K)$ is unbounded we take the first, in equation (1), as the value function.

3. (a) The HJB equation is

$$rV(K) = \max_I \left\{AK - \frac{I^2}{2} + V^K(I - \delta K) + \frac{\sigma^2}{2}V^{''}(K)\right\}$$

The policy function is again $I^* = V^{'}(K)$. Therefore the HJB equation becomes, at the optimum,

$$rV(K) = AK + \left(\frac{V^{'}(K)}{2}\right)^2 - \delta\, K\, V^{'}(K) + \frac{\sigma^2}{2}V^{''}(K)$$

(b) Using a trial function $V(K) = \alpha_0 + \alpha_1 K$, the value function is again given by equation (1). If we use a quadratic trial function we could also rule it out because of non boundedness of $V^{'}(K)$.

(c) As $I^* = V^{'}(K) = \bar{I}$ then capital accumulation is driven by the SDE

$$dK(t) = \left(\bar{I} - \delta K(t)\right)dt + \sigma\, dW(t) \tag{3}$$

where $K(0) = k_0$ given.

(d) Observe that equation (3) is generates a Ornstein-Uhlenbeck process. Therefore, the solution is

$$K(t) = k_0 e^{-\delta t} + \frac{\bar{I}}{\delta}\big(1 - e^{-\delta t}\big) + \sigma \int_0^t e^{-\delta(t-s)} dW(s)$$

(e) For $\bar{K} = \frac{\bar{I}}{\delta}$ the statistics are

$$\mathbb{E}\Big[K(t)|K(0) = k_0\Big] = \bar{K} + (k_0 - \bar{K})e^{-\delta t}$$
$$\mathbb{V}\Big[K(t)|K(0) = k_0\Big] = \frac{\sigma^2}{2\delta}\Big(1 - e^{-2\delta t}\Big)$$

(g) Writing $p(t,k) = \mathbb{P}[K(t) = k \,|\, K(0) = k_0]$ the FPK equation is

$$\partial_t p(t,k) = -\partial_k\Big(\big(\bar{I} - \delta k\big)\, p(t,k)\Big) + \frac{1}{2}\partial_{kk}\Big(\sigma^2\, p(t,k)\Big)$$

(h) The FPK is a semi-linear parabolic PDE. Letting $\delta = 0$ and solving the equation with the initial condition $p(0, k_0) = \delta(k - k_0)$ we find

$$p(t,k) = \frac{1}{\sqrt{2\pi\sigma}} \exp\Big\{-\Big(\frac{(k - k_0 + \bar{I}t)^2}{2\sigma^2 t}\Big)\Big\}$$

(a) (a) $y(t,x) = f\big(xe^{-at}\big)$ for a differentiable arbitrary function $f(\cdot)$

(b) $y(t,x) = \exp\Big\{-\big(xe^{-at}\big)^2\Big\}$.

Linear 1 (a) General solution: $y(x) = \bar{y} + \big(k - \bar{y}\big)\, e^{-\mu\, x}$ for $\bar{y} = \frac{\beta}{\mu}$.

(b) Particular solution $y(x) = \bar{y} + \mu\, e^{-\mu x}$.

Linear 2 (a) Effects of parameters on dynamics: α and β determine the dynamic properties (the type of phase diagram) and the bifurcations; γ determines the value of the steady state. The possible phase diagrams are: saddle point if $\beta < 0 < \alpha$ or $\alpha < 0 < beta$, stable node if $\alpha < 0$ and $\beta < 0$, unstable node if $\alpha > 0$ and $\beta > 0$, unstable degenerate node (or saddle-node unstable) if $\alpha = 0 < \beta$ or $\beta = 0 < \alpha$; stable degenerate node (or saddle-node stable) node if $\alpha = 0 > \beta$ or $\beta = 0 > \alpha$, and degenerate saddle-node if $\alpha = \beta = 0$.

(c) $y_1(t) = 0$ and $y_2(t) = \frac{\gamma}{\beta}\Big(1 - e^{\beta t}\Big)$ for $t \in [0, \infty)$.