Foundations of Financial Economics
Revisions of utility theory

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Topics of the lecture

- Marginalist concepts which are frequent in economics
- Basic utility theory

Marginalist concepts

Valuation function

- Consider a number of different objects **indexed** by $\mathbb{I} = \{1, \ldots, i, \ldots, n\}$
- The **quantity** of object i is denoted $x_i \in \mathbb{R}$
- We can represent a **bundle** of objects by the vector $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n$
- The **valuation of a bundle** is represented by the (at least twice-) continuously differentiable function

$$F = F(\mathbf{x}) = F(x_1, \ldots, x_i, \ldots, x_n)$$

- In economics usually $F(\cdot)$ represents is a utility or a production function

Change in the valuation

- The **variation of the valuation of a bundel** is given by the differential (under very weak conditions)

$$dF = F_1(\mathbf{x})\, dx_1 + \ldots + F_i(\mathbf{x})\, dx_i + \ldots = \nabla F(\mathbf{x}) \cdot d\mathbf{x}$$

  where ∇F is the gradient (vector)

$$\nabla F(\mathbf{x}) = \big(F_1(\mathbf{x}), \ldots, F_i(\mathbf{x}), \ldots, F_n(\mathbf{x})\big)^\top$$

- where F_i is the partial derivative of i

$$F_i(\mathbf{x}) \equiv \frac{\partial F(\mathbf{x})}{\partial x_i}$$

- We say object i is a

$$\begin{cases} \textbf{good}(\text{more of it increases value of the bundle}) & \text{if } F_i(\mathbf{x}) > 0 \\ \textbf{saturated}(\text{more does not change the value of the bundle}) & \text{if } F_i(\mathbf{x}) = 0 \\ \textbf{bad}(\text{more does not change the value of the bundle}) & \text{if } F_i(\mathbf{x}) < 0 \end{cases}$$

- From now on we consider goods, i.e. objects $i \in \mathbb{I}$ such that $F_i > 0$

Marginal value for goods

- We call **marginal contribution** of good i to the change in value brought about by dx_i
$$\text{(Definition) } M_i \equiv \frac{dF}{dx_i}$$
- For the bundle variation $d\mathbf{x} = (0, \dots, 0, dx_i, 0, \dots, 0)$ then $dF = F_i\, dx_i$ and therefore the marginal contribution is equal to the partial derivative
$$\text{(Implication) } M_i = F_i$$
therefore a good has a positive marginal contribution for value.

Relative marginal variations

- Observe that $M_i(\mathbf{x}) = F_i(\mathbf{x})$ because F_i is a function of $\mathbf{x}$
- If F is twice-differentiable we can calculate second-order derivatives

$$\text{(own) } F_{ii} \equiv \frac{\partial^2 F(\mathbf{x})}{\partial x_i^2} \quad \text{(crossed) } F_{ij} \equiv \frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial_j}, \text{ for any } j \neq i \in \mathbb{I}$$

- The **marginal contribution** of i for a variation in x_i

$$\frac{\partial M_i}{\partial x_i} = F_{ii} = \begin{cases} > 0 & \textbf{increasing} \\ = 0 & \textbf{constant} \\ < 0 & \textbf{decreasing} \end{cases}$$

- **Pareto-Edgeworth** relationships: variation in M_i for a variation in any x_j:

$$\frac{\partial M_i}{\partial x_j} = F_{ij} = \begin{cases} > 0 & \textbf{complementarity} : \text{goods } i \text{ and } j \text{ are complements} \\ = 0 & \textbf{independence} \\ < 0 & \textbf{substitutability} : \text{goods } i \text{ and } j \text{ are substitutable} \end{cases}$$

- **Uzawa-Allen elasticities**: relative variation in M_i for a variation in any x_j

$$\text{(own) } \varepsilon_{ii} \equiv -\frac{F_{ii}\, x_i}{F_i} \quad \text{(crossed) } \varepsilon_{ij} \equiv -\frac{F_{ij}\, x_j}{F_i}$$

- If:
  if $\varepsilon_{ij} < 0$ then goods i and j are complements (in the PE sense)
  if $\varepsilon_{ij} = 0$ then goods i and j are independent (in the PE sense)
  if $\varepsilon_{ij} > 0$ then goods i and j are substitutable (in the PE sense)

Compensated variations

- The **marginal rate of substitution** of good i by good j is the variation in the quantity of good j for a unit variation in good i such that the value of the bundle does not change

$$\textbf{(definition)}\ MRS_{ij} \equiv -\left.\frac{dx_j}{dx_i}\right|_{dF=0}$$

- Assume we want to know what would be dx_j if we change dx_i in such a way as to keep the value F constant

$$d\mathbf{x} = (0, \ldots, 0, dx_i, 0, \ldots, dx_j, 0, \ldots, 0) \text{ such that } dF = 0$$

That is

$$dF = \nabla F \cdot d\mathbf{x} = F_i\, dx_i + F_j\, dx_j = 0$$

- Then

$$\text{(Implication)}\ MRS_{ij}(\mathbf{x}) = \frac{F_i(\mathbf{x})}{F_j(\mathbf{x})} \text{ for } F(\mathbf{x}) = \text{constant}$$

Elasticity of substitution

- A fundamental concept here is the **elasticity of substitution** of good i by good j such that the value of the bundle remains constant

$$\text{(definition) } ES_{ij}(\mathbf{x}) \equiv \frac{d\ln(x_j/x_i)}{d\ln MRS_{ij}(\mathbf{x})}$$

  intuition: relative change in the MRS_{ij} for a relative change in the ratio x_j/x_i.

- If F is twice differentiable, then the ES_{ij} is

$$\textbf{(Implication) } ES_{ij} = \frac{x_i\,F_i + x_j\,F_j}{x_j\,F_j\,\varepsilon_{ii} - 2\,x_i\,F_i\,\varepsilon_{ij} + x_i\,F_i\,\varepsilon_{jj}}$$

  where $x_i\,F_i\,\varepsilon_{ij} = x_j\,F_j\,\varepsilon_{ji}$ and $F_{ij} = F_{ji}$ if F is continuous.

Elasticity of substitution: continuation

- Sketch of the proof: remember we have $F_1\,dx_1 + F_2\,dx_2 = 0$.
- The numerator is

$$d\ln(x_j/x_i) = d\ln x_j - d\ln x_i = \frac{dx_j}{x_j} - \frac{dx_i}{x_i} =$$
$$= -\frac{dx_i}{x_i\,x_jF_j}\Big(x_iF_i + x_jF_j\Big)\text{ (because } F_i\,dx_i + F_j\,dx_j = 0)$$

- The denominator is

$$d\ln MRS_{ij} = d\ln\left(\frac{F_i(x_i,x_j)}{F_j(x_i,x_j)}\right) = d\ln F_i - d\ln F_j = \frac{dF_i}{F_i} - \frac{F_j}{F_j}$$

where

$$dF_i = F_{ii}dx_i + F_{ij}dx_j = dx_i\Big(F_{ii} + \frac{dx_j}{dx_i}F_{ij}\Big) = dx_i\Big(F_{ii} - \frac{F_i}{F_j}F_{ij}\Big)$$
$$dF_j = F_{ji}dx_i + F_{jj}dx_j = dx_i\Big(F_{ij} + \frac{dx_j}{dx_i}F_{jj}\Big) = dx_i\Big(F_{ij} - \frac{F_i}{F_j}F_{jj}\Big)$$

- simplify and use the definition of the Uzawa-Allen elasticities.

Example: Cobb-Douglas function

- The Coob-Douglas function is a geometric mean: for $\mathbf{x} = (x_1, x_2)$

$$F = F(\mathbf{x}) = x_1^{\alpha}\, x_2^{1-\alpha}, \text{ for } 0 < \alpha < 1,\ x_1 > 0, x_2 > 0$$

- First derivatives: both variables 1 and 2 are goods

$$F_1 = \alpha \frac{F}{x_1} > 0,\ F_2 = (1-\alpha)\frac{F}{x_2} > 0$$

- Second derivatives: they have decreasing marginal valuations and are Pareto-Edgeworth complements (but usually are substitutable in the Hicksian sense, i.e., when we consider their cost)

$$F_{11} = -\alpha(1-\alpha)\frac{F}{(x_1)^2} < 0,\ F_{22} = -\alpha(1-\alpha)\frac{F}{(x_2)^2} < 0,$$

$$F_{12} = F_{21} = \alpha(1-\alpha)\frac{F}{x_1\, x_2} > 0$$

Example: Cobb-Douglas function

- The Hicks-Allen elasticities are

$$\varepsilon_{11} = 1 - \alpha > 0,\ \varepsilon_{22} = \alpha > 0,\ \varepsilon_{12} = -(1-\alpha) < 0$$

- The marginal rate of substitution is

$$MRS_{12} = \frac{F_1}{F_2} = \frac{\alpha\, x_2}{(1-\alpha)\, x_1}$$

- The elasticity of substitution is

$$ES_{12} = \frac{x_1 F_1 + x_2 F_2}{x_2 F_2 \varepsilon_{11} - 2 x_1 F_1 \varepsilon_{12} + x_1 F_1 \varepsilon_{22}} = \frac{F}{F} = 1$$

Basic utility theory

Utility theory

The problem: optimal allocation

- **The problem**: consider an agent with a resource W that wants to **allocate it optimally** among two goods, 1 and 2, having (given) costs p_1 and p_2.
- The optimality criterium is $U(c_1, c_2)$, where the quantities of the two goods are c_1 and c_2.
- **Further assumptions**:
  - The utility function $U(\cdot)$ is: continuous, differentiable, increasing and concave.
  - The endowment is positive: $W > 0$
- Nominal expenditure $E \equiv E(c_1, c_2) = p_1 c_1 + p_2 c_2$

Free allocation: optimality

- Assume there are no other constraints with the exception of the resource constraint $E(c_1, c_2) = W$
- The problem is

$$V(W; p_1, p_2) = \max_{c_1, c_2} \left\{ U(c_1, c_2) : \; E(c_1, c_2) = W \right\}$$

- function $V(.)$ is called indirect utility or value function
- intuition: it gives the **value** of the endowment W when it can be spent in the two goods and used in consumption.

Optimal free allocation: solution

- The Lagrangean
$$\mathcal{L} = u(c_1, c_2) + \lambda(W - E(c_1, c_2))$$
where λ is the Lagrange multiplier
- The solution (which always exists) $(c_1^*, c_2^*, \lambda^*)$ satisfies the conditions
$$\begin{cases} U_{c_j}(c_1, c_2) - \lambda p_j = 0, \quad j = 1, 2 \\ W - E(c_1, c_2) = 0 \end{cases}$$
- We observe that, at the optimum that the $MRS_{1,2}$ is equalized to the relative prices
$$MRS_{1,2} = \frac{U_{c_1}(c_1^*, c_2^*)}{U_{c_2}(c_1^*, c_2^*)} = \frac{p_1}{p_2}$$
and, in this case the resource is saturated
$$E(c_1^*, c_2^*) = p_1 c_1^* + p_2 c_2^* = W$$

Optimal free allocation: solution

- When there is free allocation, the solution is characterized by the equations,

$$
\begin{aligned}
p_2\, U_{c_1}(c_1^*, c_2^*) &= p_1\, U_{c_2}(c_1^*, c_2^*) && (1)\\
E(c_1^*, c_2^*) &= W && (2)
\end{aligned}
$$

- Equation (1) is a first-order partial differential equation with solution (check this)

$$U(c_1^*, c_2^*) = V\left(\frac{p_1\, c_1^* + p_2\, c_2^*}{p_1}\right)$$

- from equation (2), in the optimum we have

$$U(c_1^*, c_2^*) = V(w),\ w \equiv \frac{W}{p_1} \text{ (real resources deflated } p_1\text{)}$$

- if the utility function is strictly concave then with very weak conditions (differentiability) we have an unique interior optimum

Optimal free allocation: graphical representation

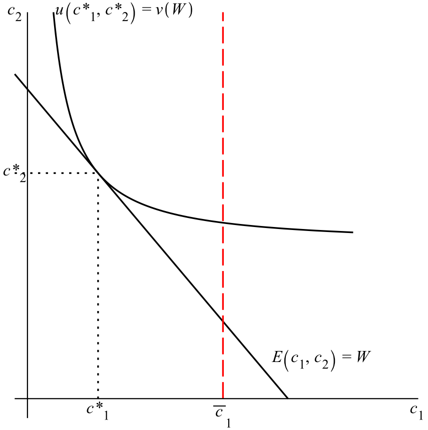

Figure: Interior optimum for a log utility function
$U(c_1, c_2) = \ln c_1 + b \ln c_2$

Constrained allocation: optimality

- Let us assume that the agent is constrained in the allocation of resources to good 1. For instance, assume that $c_i \in [0, \bar{c}_1]$
- The problem is now

$$V(W; p_1, p_2, \bar{c_1}) = \max_{c_1, c_2} \{ U(c_1, c_2) : \; E(c_1, c_2) = W, 0 \leq c_1 \leq \bar{c}_1 \}$$

- Most models of financial frictions introduce constraints of this type
- More generally we could assume there are restrictions in allocation resources to the two goods.
- The problem would become

$$V(W; p_1, p_2, \bar{c_1}, \bar{c_2}) = \max_{c_1, c_2} \{ U(c_1, c_2) : \; E(c_1, c_2) = W, 0 \leq c_j \leq \bar{c}_j, j = 1, 2 \}$$

Constrained allocation: optimality

- The Lagrangean is now

$$\mathcal{L} = u(c_1, c_2) + \lambda(W - E(c_1, c_2)) -$$
$$- \eta_1 \, c_1 - \eta_2 \, c_2 + \zeta_1 \, (\bar{c}_1 - c_1) + \zeta_2 \, (\bar{c}_2 - c_2)$$

  where λ is the Lagrange multiplier and η_i and ζ_i are marginal valuation of the constraints;

- The solution (which always exists) $(c_1^*, c_2^*, \lambda^*, \eta_1^*, \eta_2^*, \zeta_1^*, \zeta_2^*)$ satisfies the Karush-Kuhn-Tucker conditions:
  the optimality conditions

$$\begin{cases} U_{c_j}(c_1, c_2) - \lambda p_j - \eta_j - \zeta_j = 0, & j = 1, 2 \\ \lambda(W - E(c_1, c_2)) = 0, \ \lambda \geq 0, \ E(c_1, c_2) \leq W & \end{cases}$$

  together with the complementarity slackness conditions

$$\begin{cases} \eta_j c_j = 0, \ \eta_j \geq 0, \ c_j \geq 0, & j = 1, 2 \\ \zeta_j(\bar{c}_j - c_j) = 0, \ \zeta_j \geq 0, \ c_j \leq \bar{c}_j, & j = 1, 2 \end{cases}$$

Optimal constrained allocation: solution

Corner solution: lower $c_1 = 0$

- Let $c_1^* = 0$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- FOC: $\eta_1^* > 0$ and $\eta_2^* = \zeta_1^* = \zeta_2^* = 0$, and

$$\begin{aligned} p_2 \, U_{c_1}(c_1^*, c_2^*) &= p_1 \, U_{c_2}(c_1^*, c_2^*) - p_2 \eta_1 && (3) \\ E(c_1^*, c_2^*) &= W && (4) \end{aligned}$$

- Now, the MRS is smaller than the relative price

$$MRS_{12} = \frac{U^*_{c_1}}{U^*_{c_2}} = \frac{p_1}{p_2} - \frac{\eta_1}{U^*_{c_2}} < \frac{p_1}{p_2}$$

  i.e., there is a "wedge" between relative prices and the MRS_{12}
- Equation (3) is a first-order partial differential equation with solution

$$U(c_1^*, c_2^*) = \frac{\eta_1 \, c_2^*}{p_1} + V\left(\frac{p_1 \, c_1^* + p_2 \, c_2^*}{p_1}\right)$$

- if we use equation (6) in the optimum we have

$$U(c_1^*, c_2^*) = -\eta_1^* \, w + V(w) < V(w)$$

Optimal constrained allocation: figure

Corner solution 1

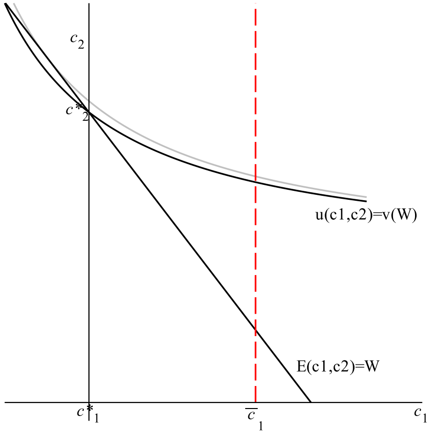

Figure: Corner solution: the indirect utility level is smaller than for the unconstrained case

Optimal constrained allocation: solution

Corner solution: upper constraint $c_1 = \bar{c}_1$

- Let $c_1^* = \bar{c}_1$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- then $\zeta_1^* > 0$ and $\eta_1^* = \eta_2^* = \zeta_1^* = \zeta_2^* = 0$
- In addition

$$\begin{aligned} p_2\, U_{c_1}(c_1^*, c_2^*) &= p_1\, U_{c_2}(c_1^*, c_2^*) + p_2\zeta_1 \quad (5) \\ E(c_1^*, c_2^*) &= W \quad (6) \end{aligned}$$

- There is again a "wedge" between the MRS_{12} and the relative price, but now

$$MRS_{12} = \frac{U^*_{c_1}}{U^*_{c_2}} = \frac{p_1}{p_2} + \frac{\zeta_1}{U^*_{c_2}} > \frac{p_1}{p_2}$$

- Equation (5) is a first-order partial differential equation with solution

$$U(c_1^*, c_2^*) = -\frac{\zeta_1\, c_2^*}{p_1} + V\left(\frac{p_1\, c_1^* + p_2\, c_2^*}{p_1}\right)$$

- if we use equation (6) in the optimum we have

$$U(c_1^*, c_2^*) = -\frac{\zeta_1\, p_1 (w - \bar{c}_1)}{p_2} + V(w) < V(w)$$

Consumer problem

Corner solution 2

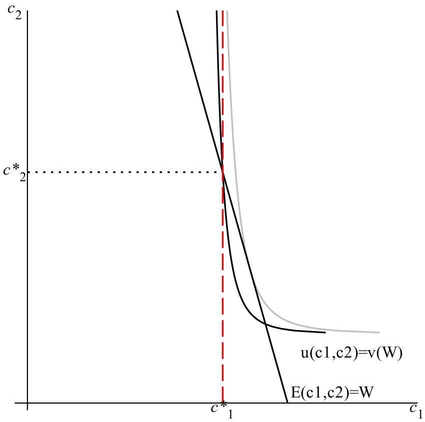

Figure: Corner solution: the indirect utility level is smaller than for the unconstrained case

Equivalent interpretation

- Let the value function in which there are constraints on the consumer be denoted by $\tilde{V}(w)$
- Looking at the previous cases we can write

$$\tilde{v}(w) = V(w) - \delta(w)$$

where $\delta(w) \geq 0$ measures the welfare loss introduced by the constraint $c_1 \in [0, \bar{c}_1]$.

- We could obtain a similar solution for the consumer problem is instead of considering the endowment level w we consider the resource level

$$\tilde{w} = \{x : \ (\tilde{v}^{-1})(x) = 0\} < w$$

that is a **smaller** level for the endowment.

Take away

- If there are **no constraints** in the allocation of an endowment to the purchase of two goods, at the optimum the **internal relative valuation (MRS) of the two goods is equal to the relative market valuation** (provided by their prices)
- If there are **constraints** on the free allocation of an endowment to the purchase of two goods
  1. there is a **wedge between the the** MRS **and the relative prices** (i.e., between the internal and the market valuation)
  2. there is **an welfare loss** relative to the free allocation

- this gives a general benchmark on the mechanism of transfer of resources, through time and states of nature, an the effects of constraints (financial or otherwise) on the operation of that mechanism of transfer.

Example

1. Assume the utility function is of Cobb-Douglas type

$$U = U(c_1, c_2) = c_1^{\alpha}\, c_2^{1-\alpha}, \text{ for } 0 < \alpha < 1$$

2. Case 1: Assume that (c_1, c_2) are only constrained by the budget constraint $p_1\, c_1 + p_2\, c_2 = W$
3. Case 2: in addition to the budget constraint impose the constraint $c_1 > 0$
4. Case 3: in addition to the budget constraint impose the constraint $c_1 \leq \beta W/p_1$ with $0 < \beta < \alpha$
5. Observe that

$$U_1 = \frac{\partial U}{\partial c_1} = \alpha \frac{U}{c_1} > 0, \text{and } U_2 = \frac{\partial U}{\partial c_2} = (1-\alpha)\frac{U}{c_2} > 0$$

which means that the objects indexed by 1 and 2 are both goods

Example

Case 1: free allocations

- the first order conditions are

$$\begin{cases} p_2 U_1 = p_1 U_2 \\ p_1 c_1 + p_2 c_2 = W \end{cases} \Leftrightarrow \begin{cases} (1-\alpha) p_1 c_1 - \alpha p_2 c_2 = 0 \\ p_1 c_1 + p_2 c_2 = W \end{cases}$$

then the **optimal consumption allocation** is, therefore

$$\begin{cases} c_1^* = \alpha \dfrac{W}{p_1} \\ c_2^* = (1-\alpha) \dfrac{W}{p_2} \end{cases}$$

- **Properties**: as

$$c_1^* = c_1^*(p_1, W),\ c_2^* = c_2^*(p_2, W)$$

1. Each type of consumption is proportional to nominal wealth deflated by its price
2. there is no complementarity or substitutability in the Hicksian sense, i.e. their cross-derivatives relative to the price of the other good are zero

$$\frac{\partial c_1^*}{\partial p_2} = \frac{\partial c_2^*}{\partial p_1} = 0.$$

Example

Case 1: free allocations

1. Substituting in the utility function we get the value of the resource W

$$V(W) = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} W =$$
$$= \chi(\alpha)\frac{W}{P}$$

   where $P \equiv p_1^{\alpha}\, p_2^{1-\alpha}$ is the consumers price index

2. The value of the resource W, assuming there is an optimal free allocation among the two goods, is proportional to the real value of the resource deflated by the consumer's own price index (which is a geometrical mean whose weights are given by those of the utility function.

Example

Case 2: positive allocations to good 1

- In this case we require that $c_1 \geq 0$.
- As we saw in the free allocation case that $c^* = \alpha W/p_1 > 0$ then the optimum will be interior
- This means that the constraint is not binding.
- Therefore the solution is the same as in case 1

$$\begin{cases} c_1^* = \alpha \dfrac{W}{p_1} \\ c_2^* = (1-\alpha)\dfrac{W}{p_2} \end{cases}$$

Example

Case 3: upper bound on the allocations to good 1

- In this case we require that $c_1 \leq \bar{c}_1$ and $\bar{c}_1 = \beta W/p_1$, for $\beta < \alpha$
- As we saw in the free allocation case that $c^* = \alpha W/p_1 > \bar{c}_1$ which means that this solution is not admissible.
- The first order conditions are now (5) and (6) with $c_1 = \bar{c}_1$

$$\begin{cases} \alpha p_2 c_2 = (1-\alpha) p_1 \bar{c}_1 + p_2 \bar{c}_1 c_2 \zeta_1 \\ p_1 \bar{c}_1 + p_2 c_2 = W \end{cases}$$

  that we need to solve for c_2 and ζ_1.
- The solution is

$$c_1^* = \bar{c}_1 = \beta \frac{W}{p_1} < \alpha \frac{W}{p_1}$$

$$c_2^* = (1-\beta) \frac{W}{p_2} > (1-\alpha) \frac{W}{p_2}$$

$$\zeta_1 = \frac{(\alpha - \beta) p_1}{\beta (1-\beta) W} > 0$$

  Therefore: the consumption of good 1 (2) will smaller (larger) than in the free allocation case

Example

Case 3: upper bound on the allocations to good 1

- However, **there will be a loss in value**.
- To see this observe that the value of the resource is now

$$V(W) = \left(\frac{\beta}{p_1}\right)^{\alpha}\left(\frac{1-\beta}{p_2}\right)^{1-\alpha} W =$$

$$= \beta^{\alpha}(1-\beta)^{\alpha}\frac{W}{P} =$$

$$X(\beta)\chi(\alpha)\frac{W}{P} < \chi(\alpha)\frac{W}{P}$$

  which is smaller than for the free allocation case.
- To prove this let

$$X(\beta) \equiv \left(\frac{\beta}{\alpha}\right)^{\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha} > 0$$

  and remember that we assume that $\beta < \alpha$
- and show that $X(\alpha) = 1$ and that

$$\frac{\partial X}{\partial \beta} = \left(\frac{\alpha - \beta}{\beta(1-\beta)}\right) X > 0$$

  Then $X(\beta) < 1$ for $\beta < \alpha$.