# Foundations of Financial Economics
# Two period GE: heterogeneous agents

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# Topics for today

- Sources of heterogeneity
- AD equilibrium with heterogeneous agent economies
- Aggregate and idiosyncratic uncertainty

# Heterogeneity in AD economies

## Heterogeneity: sources and types

**Sources of heterogeneity**:
there is heterogeneity if there are at least two agents $j$ and $l$ such that they differ in:

- **information**: their probability spaces may be different $(\Omega_j, P_j) \neq (\Omega_l, P_l)$
- **preferences**: their degree of impatience, and/or attitudes towards risk may differ: $\beta^j \neq \beta_l$, $u^j(.) \neq u^l(.)$
- **endowments**: their wealth may differ: $y^j = \{y_0^j, Y_1^j\} \neq y^l = \{y_0^l, Y_1^l\}$

# Heterogeneity in AD economies

## Heterogeneity: sources and types

**Types of uncertainty**: related with state-dependency

- If $Y_1^j \neq Y_1^l$ we say there is **idiosyncratic uncertainty**
- If $Y_1 = \sum_{i=1}^{I} Y_1^i$ **is state-independent**, i.e., $y_{1,s} = \bar{y}_1$ for all $s = 1, \ldots, N$ then there is **aggregate certainty**,
- If $Y_1 = \sum_{i=1}^{I} Y_1^i$ **is state-dependent**, i.e., there is a pair of components of $Y_1$ such that $y_{1,s} \neq y_{1,s'}$ for all $s, s^{'} = 1, \ldots, N$ then we say there is **aggregate uncertainty**

# Heterogeneity in AD economies

## Heterogeneity: sources and types

Then, we can have:

- **idiosyncratic and aggregate certainty**: the GE is deterministic (both consumption at $t = 1$ and discount factor are deterministic) (This was the case studied in chapter 2)
- **idiosyncratic and aggregate uncertainty**: the GE is stochastic (both consumption at $t = 1$ and the stochastic discount factor are stochastic)
- **idiosyncratic uncertainty and aggregate certainty**: the GE is partially stochastic (consumption at $t = 1$ can be stochastic or deterministic and the stochastic discount factor is deterministic)

- Therefore:

  In an **homogeneous** agent economy idiosyncratic and aggregate uncertainty are undistinguishable.
  In a **heterogeneous** agent economy they differ.

# GE for an AD economy with heterogeneous agents

**Definition**: General equilibrium (GE):

- is the sequence of **distributions** $\{(c_0^{*1}, \dots c_0^{*I}), (C_1^{*1}, \dots C_1^{*I})\}$ and prices $q$ such that:
  1. every consumer $i = 1, \dots, I$ determines the optimal sequence $\{c_0^i, C_1^i\}$ by solving the problem

$$\max_{\{c_0^i, C_1^i\}} \mathbb{E}^i{}_0 \left[ u^i(c_0^i) + \beta^i u^i(C_1^i) \right]$$

$$c_0^i - y^i{}_0 + q(C_1^i - Y_1^i) = 0$$

given $q$ and $\{y_0^i, Y_1^i\}$;

  2. the good market clears in every period:

$$C_t = Y_t, t = 0, 1$$

where **aggregate** consumption and endowments are

$$C_t = \sum_{i=1}^{I} C_t^i, \;\; Y_t = \sum_{i=1}^{I} Y_t^i, \;\; t = 0, 1$$

# Case 1: General equilibrium with heterogeneity in endowments

**Assumptions**: logarithmic preferences, and **idiosyncratic** uncertainty as regards endowments $Y_1^i$.

**Question**: what are the properties of the equilibrium stochastic discount factor ?

**Method of determination**: we may have to solve explicitly the consumers' problems (exception: if the only source of inequality is related to endowments)

# Case 1: General equilibrium with heterogeneity in endowments

## Determination

1. household' $i \in 1, \ldots, I$ problem

$$\max_{c_0^i, c_{11}^i, \ldots, c_{1N}^i} \ln\left(c_0^i\right) + \beta \sum_{s=1}^{N} \pi_s \ln\left(c_{1s}^i\right)$$

subject to

$$c_0^i + \sum_{s=1}^{N} q_s c_{1s}^i \leq h^i \equiv y_0^i + \sum_{s=1}^{N} q_s y_{1s}^i$$

where $q_s$ is given to the consumer.

2. optimal consumption of household $i \in 1, \ldots, I$ (without satiation)

$$\begin{aligned} c_0^i &= \frac{1}{1+\beta} h^i \\ c_{1s}^i &= \frac{\pi_s \beta}{q_s(1+\beta)} h^i \end{aligned}$$

# Case 1: General equilibrium with heterogeneity in endowments

## Determination: continuation

1. Aggregate supply

$$y_0 = \sum_{i=1}^{I} y_0^i$$

$$y_{1,s} = \sum_{i=1}^{I} y_{1,s}^i, \; s = 1, \ldots, N$$

2. Aggregate demand

$$c_0 = \sum_{i=1}^{I} c_0^i = \frac{1}{1+\beta} h$$

$$c_{1,s} = \sum_{i=1}^{I} c_{1,s}^i = \frac{\beta \pi_s}{q_s(1+\beta)} h, \; s = 1, \ldots, N$$

# Case 1: General equilibrium with heterogeneity in endowments

## Determination: continuation

1. Aggregate wealth

$$h = \sum_{i=1}^{I} h^i = y_0 + \sum_{s=1}^{N} q_s y_{1,s}$$

2. Market clearing conditions

$$\begin{aligned} c_0 &= y_0 \Leftrightarrow \frac{1}{1+\beta} h = y_0 \\ c_{1,s} &= y_{1,s} \Leftrightarrow \frac{\beta \pi_s}{q_s(1+\beta)} h = y_{1,s}, \ s = 1, \ldots, N \end{aligned}$$

3. Then

$$\frac{\beta \pi_s y_0}{q_s} = y_{1,s}$$

# Case 1: General equilibrium with heterogeneity in endowments

## Characterization

### Proposition 1

*Consider a AD economy in which there is heterogeneity in endowments and homogeneity in preferences and information. Then the equilibrium stochastic discount factor is independent of the distribution of income.*

Let $y_{1,s} = (1+\gamma_s)y_0$ and assume a logarithmic utility function. Then he **equilibrium discount factor** is

$$m_s = \frac{q_s}{\pi_s} = \beta \frac{y_0}{y_{1,s}} = \frac{\beta}{1+\gamma_s},\ s = 1, \ldots, N$$

# Case 1: General equilibrium with heterogeneity in endowments

Interpretation: the equilibrium discount factor $M = (m_1, \dots m_N)$ where

$$m_s = \frac{\beta}{1+\gamma_s}, \text{ for } s = 1, \dots, N$$

- is independent of the distribution of endowments among agents (only depends on the growth factor of the **aggregate endowments**
- if there is **aggregate uncertainty** then it is **state-dependent** (stochastic)
- if there is **aggregate certainty** (even if there is idiosyncratic uncertainty) then it is **state-independent** (i.e, deterministic):

$$m_s = m = \frac{\beta}{1+\gamma}, \text{ for all } s = 1, \dots, N.$$

# Case 1: General equilibrium with heterogeneity in endowments

## Characterization

### Proposition 2

*Consider the previous economy, in which there is idiosyncratic uncertainty but aggregate certainty (i.e, $Y_1 = y_1$ for all states $s = 1, \ldots, N$). Then there is* ***perfect insurance*** *consumption at time $t = 1$ is state independent.*

Next we prove that

$$c_{1s}^{*i} = c_1^{*i} = \frac{1+\gamma}{1+\beta} h^{*i}, \ \forall s = 1, \ldots, N$$

is state-independent if $Y_1 = y_1 = (1+\gamma) y_0$

# Case 1: General equilibrium with heterogeneity in endowments

## Proof of Proposition 2

- In equilibrium

$$c_{1s}^{i} = \frac{\beta}{m_s^*(1+\beta)} h^i = \frac{1+\gamma_s}{1+\beta} h^i$$

- The **equilibrium distribution** of human wealth is (if we substitute $m_s$)

$$h^{*i} = y_0^i + \beta \sum_s \frac{\pi_s y_{1,s}^i}{1+\gamma_s} = y_0^i \left( 1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma_s^i}{1+\gamma_s} \right) \; i = 1, \ldots, I$$

- If there is no aggregate uncertainty $1+\gamma_s = 1+\gamma$ for every $s = 1, \ldots, N$

# Case 1: General equilibrium with heterogeneity in endowments

## Consumption distribution

### Proposition 3

*In equilibrium, the weight of agents' i consumption relative to aggregate consumption is stationary (i.e, time-independent), state independent and is equal to its share of aggregate wealth.*

# Case 1: General equilibrium with heterogeneity in endowments

## Consumption distribution

- The equilibrium aggregate human wealth is

$$h^* = y_0 + \beta \sum_s \frac{\pi_s y_{1,s}}{1+\gamma_s} = y_0 \left( 1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma_s}{1+\gamma_s} \right) = y_0(1+\beta)$$

- The distribution of consumption at $t=0$ is

$$\frac{c_0^{*i}}{c_0} = \frac{1}{1+\beta} \frac{h^{*i}}{y_0} = \frac{h^{*i}}{h} = \frac{y_0^i}{y_0} \left( \frac{1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma_s^i}{1+\gamma_s}}{1+\beta} \right)$$

- and at $t=1$ is

$$\frac{c_{1s}^{*i}}{c_{1s}} = \frac{1+\gamma_s}{1+\beta} \frac{h^{*i}}{y_{1s}} = \frac{1}{1+\beta} \frac{h^{*i}}{y_0} = \frac{h^{*i}}{h}, \text{for all } s = 1, \ldots, N$$

# Case 1: General equilibrium with heterogeneity in endowments

- Observation: the fact that every consumer can perfectly insure (i.e, the distribution of consumption for every consumer is state independent) does not mean that the distribution of consumption among households is symmetric
- The consumption for every household is dependent of their specific (idiosyncratic wealth)

$$c_1^i = \frac{1+\gamma}{1+\beta} h^i$$

- **There is perfect insurance but not perfect equality in consumption.**

# Example 1: homogeneous agent economy

| | $t = 0$ | $t = 1$ | |
|---|---|---|---|
| | | $s = 1$ | $s = 2$ |
| $y^a$ | 50 | 45 | 55 |
| $y^b$ | 50 | 45 | 55 |
| $y = y^a + y^b$ | 100 | 90 | 110 |
| $\mathbf{m}$ | | **1.089** | **0.891** |
| $c^a$ | 50 | 45 | 55 |
| $c^b$ | 50 | 45 | 55 |

Table: Two homogeneous agents ($a$ and $b$). Common parameter: $\beta = 1/1.02$. Idiosyncratic and aggregate uncertainty

# Example 2: heterogenous agents and aggregate uncertainty

| | $t=0$ | $t=1$ | |
|---|---|---|---|
| | | $s=1$ | $s=2$ |
| $y^a$ | 30 | 27 | 33 |
| $y^b$ | 70 | 63 | 77 |
| $y=y^a+y^b$ | 100 | 90 | 110 |
| **m** | | **1.089** | **0.891** |
| $c^a$ | 30 | 27 | 33 |
| $c^b$ | 70 | 63 | 77 |

Table: Two heterogeneous agents ($a$ and $b$). Common parameter: $\beta = 1/1.02$. Idiosyncratic and aggregate uncertainty. In this case there is no insurance

# Example 3: idiosyncratic uncertainty and aggregate certainty

| | $t=0$ | $t=1$ | |
|---|---|---|---|
| | | $s=1$ | $s=2$ |
| $y^a$ | 50 | 45 | 55 |
| $y^b$ | 50 | 55 | 45 |
| $y$ | 100 | **100** | **100** |
| **m** | | **0.98** | **0.98** |
| $c^a$ | 50 | 50 | 50 |
| $c^b$ | 50 | 50 | 50 |

Table: Two heterogeneous agents ($a$ and $b$). Common parameter: $\beta = 1/1.02$. Idiosyncratic uncertainty and aggregate certainty: **perfect insurance**

# Case 1: General equilibrium with heterogeneity in endowments

## Characterization

- **Summing up**:
  - if there is **aggregate certainty** then: the stochastic discount factor is **deterministic** and there is **perfect insurance** $c_1^i$ is state-independent (because $\gamma$ is state-independent);
  - if there is **aggregate uncertainty** then: the stochastic discount factor is **stochastic** and there is **not** perfect insurance $c_1^i$ is state-dependent (because $\gamma$ is state-dependent);
- Then:
  - only aggregate variables determine the stochastic discount factor;
  - the **distribution of income is irrelevant** for the determination of the stochastic discount factors
  - **Those results extend to a finance economy with complete asset markets.** (see next)

# Case 1: General equilibrium with heterogeneity in endowments

## Comparing a representative agent with a heterogeneous agent economy

- In a representative agent economy we can only have two cases
  - Aggregate and individual (idiosyncratic) certainty
  - Both aggregate and individual (idiosyncratic) uncertainty. In this case there is not insurance
- In a heterogeneous agent economy we have three cases
  - Aggregate and individual (idiosyncratic) certainty
  - Both aggregate and individual (idiosyncratic) uncertainty. In this case there is some insurance
  - Aggregate certainty and individual (idiosyncratic) uncertainty. In this case there can be **perfect insurance** and redistribution.

# Case 2: General equilibrium with heterogeneity in endowments and preferences

## Assumptions

- homogeneous utility function: logarithmic
- heterogeneity in **impatience** ($\beta^i$). Let the distribution of psychological discount factors be represented by

$$B = (\beta^1, \ldots, \beta^i, \ldots \beta^I)$$

- **idiosyncratic uncertainty as regards endowments** $Y_1^i$

# Case 2: General equilibrium with heterogeneity in endowments and preferences

The consumption problem is now

$$\max_{c_0^i, c_{11}^i, \ldots, c_{1N}^i} \ln\left(c_0^i\right) + \beta^i \sum_{s=1}^{N} \pi_s \ln\left(c_{1s}^i\right)$$

subject to

$$c_0^i + \sum_{s=1}^{N} \pi_s m_s c_{1s}^i \leq h^i \equiv y_0^i + \sum_{s=1}^{N} \pi_s m_s y_{1s}^i$$

# Case 2: General equilibrium with heterogeneity in endowments and preferences

## Solution to the household $i$ problem

- The optimal consumption process for household $i$ is

$$\begin{aligned} c_0^i &= \frac{1}{1+\beta^i} h^i, \ i = 1, \ldots, I \\ c_{1s}^i &= \frac{\beta^i}{m_s(1+\beta^i)} h^i, \ i = 1, \ldots, I \end{aligned}$$

# Case 2: General equilibrium with heterogeneity in endowments and preferences

## Endowment distribution

- Define the process for the shares of household $i$ in the aggregate endowments, $\{\phi_0^i, \Phi_1^i\}$,
- At time $t = 0$ we have

$$\phi_0^i = \frac{y_0^i}{y_0} = \frac{y_0^i}{\sum_{i=1}^I y_0^i} \text{ for } i = 1, \ldots, I$$

  where $\sum_{i=1}^I \phi_0^i = 1$ and
- At time $t = 1$ we have

$$\phi_{1,s}^i = \frac{y_{1,s}^i}{y_{1,s}} = \frac{y_{1,s}^i}{\sum_{i=1}^I y_{1,s}^i} \text{ for } s = 1, \ldots, N, \quad i = 1, \ldots, I$$

  where $\sum_{i=1}^I \phi_{1,s}^i = 1$ for all $s = 1, \ldots, N$

# Case 2: General equilibrium with heterogeneity in endowments and preferences

## Wealth distribution

- Then the human wealth of consumer $i$ can be written as

$$h^i = \left( \phi_0^i + \sum_{s=1}^{N} m_s \pi_s (1+\gamma_s) \phi_{1,s}^i \right) y_0, \;\; i = 1, \ldots, I$$

because $y_0^i = \phi_0^i y_0$ and $y_{1s}^i = \phi_{1s}^i y_{1s} = \phi_{1s}^i (1+\gamma_s) y_0$

## Case 2: General equilibrium with heterogeneity in endowments and preferences

### Market clearing conditions

- The market clearing conditions are

$$
\begin{aligned}
c_0 &= y_0 \Leftrightarrow \sum_{i=1}^{I} \frac{h^i}{1+\beta^i} = y_0 \\
c_{1,s} &= y_{1,s} \Leftrightarrow \frac{1}{m_s}\left(\sum_{i=1}^{I} \frac{\beta^i h^i}{1+\beta^i}\right) = (1+\gamma_s) y_0,\ s = 1, \ldots, N
\end{aligned}
$$

- Observation: now we are summing not only over wealth $h^i$ but also over the distribution of the discount factors $\beta^i$ $(B)$

# Case 2: General equilibrium with heterogeneity in endowments and preferences

## Market clearing conditions

- Define

$$z_0 = z_0(B) \equiv \sum_{i=1}^{I} \frac{\beta^i \phi_0^i}{1+\beta^i},$$

$$z_{1,s} = z_{1,s}(B) \equiv \sum_{i=1}^{I} \frac{\beta^i \phi_{1,s}^i}{1+\beta^i}$$

- Then, the equilibrium conditions for $t = 1$ can be written as (check !)

$$z_0(B) + \sum_{s=1}^{N} \pi_s m_s (1+\gamma_s) z_{1,s}(B) = m_s(1+\gamma_s), \ s = 1, \dots, N$$

- This implies $m_1(1+\gamma_1) = \ldots = m_s(1+\gamma_s) = \ldots = m_N(1+\gamma_N)$.

# Case 2: General equilibrium with heterogeneity in endowments and preferences

- Then
$$\sum_{s=1}^{N} \pi_s m_s (1+\gamma_s) z_{1,s}(B) = m_s(1+\gamma_s)\mathbb{E}[z_1(B)]$$
  for any $s$
- Then we determine the **equilibrium discount factor**

$$\boxed{m_s = \tilde{\beta}(B)\frac{1}{1+\gamma_s},\ \tilde{\beta}(B) \equiv \left(\frac{z_0(B)}{1-\mathbb{E}[z_1(B)]}\right)}$$

- where $B$ is the distribution of the psychological discount factors
$$B = (\beta^1, \ldots \beta^i, \ldots, \beta^I)$$

# Case 2: General equilibrium with heterogeneity in endowments and preferences

**Conclusions**:

- if there is heterogeneity in the psychological discount factor and there is idiosyncratic uncertainty then the **equilibrium stochastic discount factor is formally similar to the homogeneous case**: it multiplies a weighted psychological discount factor with the inverse of the endowment growth factor;
- the weighted psychological discount factor, $\tilde{\beta}$ **depends upon the distribution of income** but is state-independent and constant;
- If there is **no** aggregate uncertainty then the stochastic discount factor $m$ is **state-independent**.

# Example 2 bis: heterogenous agents and aggregate uncertainty

| | $t=0$ | $t=1$ | |
|---|---|---|---|
| | | $s=1$ | $s=2$ |
| $y^a$ | 30 | 27 | 33 |
| $y^b$ | 70 | 63 | 77 |
| $y$ | 100 | 90 | 110 |
| $\mathbf{m}$ | | **1.094** | **0.895** |
| $c^a$ | 30.2 | 26.8 | 32.8 |
| $c^b$ | 69.8 | 63.2 | 77.2 |

Table: Two heterogeneous agents ($a$ and $b$). Heterogenous preferences: $\beta^a = 1/1.025$ $\beta^b = 1/1.015$ . Idiosyncratic and aggregate uncertainty

# Example 3 bis: idiosyncratic uncertainty and aggregate certainty

| | $t=0$ | $t=1$ | |
|---|---|---|---|
| | | $s=1$ | $s=2$ |
| $y^a$ | 50 | 45 | 55 |
| $y^b$ | 50 | 55 | 45 |
| $y$ | 100 | **100** | **100** |
| **m** | | **0.9804** | **0.9804** |
| $c^a$ | 50.2 | 49.8 | 49.8 |
| $c^b$ | 49.8 | 50.2 | 50.2 |

Table: Two heterogeneous agents ($a$ and $b$) where $b$ is more patient than $a$: $\beta^a = 1/1.025$ $\beta^b = 1/1.015$. There is both idiosyncratic uncertainty and aggregate certainty: **perfect insurance**. But as $b$ is more patient the time profile of consumption is different from $a$ which is less patient.

# Perfect insurance in a finance economy

## Proposition 4

*Assume a finance economy in which there is idiosyncratic uncertainty regarding endowments. If asset markets are complete and the aggregate endowment is state independent (i.e., there is no aggregate uncertainty) then there is perfect insurance.*

# Perfect insurance in a finance economy

## Proof

- Assume again that there is **heterogeneity in endowments**
- Remember the problem for agent $i$ in a **finance economy**

$$\max_{i_0, C_1^i)\}, \theta^i} u(c_0^i) + \beta \mathbb{E}[u(C_1^i)]$$

subject to

$$\begin{aligned} c_0^i &= y_0^i - S\,\theta^i \\ C_1^i &= Y_1^i + V\theta^i \end{aligned}$$

for $C_1 = (c_{1,s})_{s=1}^N$ and $\theta = (\theta_j)_{j=1}^K$

- There are two sources of uncertainty: endowments $Y_1 = (\dots, y_{1,s}, \dots)^\top$ and financial $V = \left(v_{j,s}\right)_{j=1,s=1}^{N,K}$
- There is **idiosyncratic uncertainty** if the endowment is uncertain and agent-specific

# Perfect insurance in a finance economy

## Proof, cont.

- If markets are complete, then:
  - $\pi_s \hat{m}_s = \hat{q}_s$ and $\hat{Q} = Q = S\,V^{-1}$: the state price is equal to the implicit market state price.
  - and the f.o.c.for household $i$ are

$$\beta u^{'}(c^i_{1,s}) = m_s u^{'}(c^i_0),\ s = 1, \ldots, N$$

$$c^i_0 + \mathbb{E}[\hat{M}\, C^i_1] = H^i_0 \equiv y^i_0 + \mathbb{E}[\hat{M}\, Y^i_1]$$

- There is **perfect insurance**, if consumption is the same for every state of nature

$$c_{1,s} = c_1, \text{ for every } s = 1, \ldots, N$$

- Using the f.o.c, we see that **there is perfect insurance if and only if the equilibrium stochastic discount factor is state-independent**, i.e,

$$m_s = m \text{ for every } s = 1, \ldots, M$$

# Perfect insurance in a finance economy

## Proof, cont.

- The equilibrium stochastic discount factor is

$$m_s = \beta \frac{u^{'}(y_{1,s})}{u^{'}(y_0)}, \; s = 1, \dots, N$$

where the **aggregate endowments** are

$$y_0 = \sum_{i=1}^{I} y_0^i$$

$$Y_1 = \sum_{i=1}^{I} Y_{1,s}^i = \begin{pmatrix} \dots \\ \sum_{i=1}^{I} y_{1,s}^i \\ \dots \end{pmatrix} = \begin{pmatrix} \dots \\ y_{1,s} \\ \dots \end{pmatrix}$$

- Therefore $m_s = m$ if and only if the **aggregate endowment is state-independent**, that is

$$y_{1,s} = y_1, \text{for every } s = 1, \dots, N.$$

if there is **no aggregate uncertainty**.

# Perfect insurance in a finance economy

- We say there is **perfect insurance** if any consumer although having an uncertain endowment, by trading in the financial markets, he/she can finance a state-independent consumption at time $t = 1$.
- In the previous case $\mathbb{V}[Y_1^i] > \mathbb{V}[C_1^i] = 0$: the variance of consumption is zero while the variance of endowments is positive
- There is **imperfect insurance** if $\mathbb{V}[Y_1^i] > \mathbb{V}[C_1^i] > 0$: although the variance of consumption is positive, it is smaller than the one of endowments. This may be possible even if markets are incomplete.