

Foundations of Financial Economics

Multi-period finance economies

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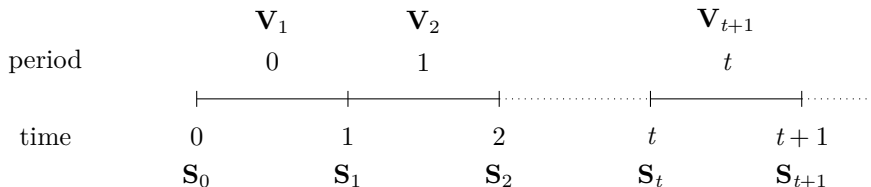
Topics for today

- ▶ Arbitrage asset pricing
- ▶ Equilibrium asset pricing for a homogeneous household economy:
the zero initial wealth case
- ▶ Equity premium puzzle again
- ▶ Non-zero initial wealth case

Arbitrage asset pricing

The structure of the asset market

There are two stochastic processes: $\{\mathbf{V}_t\}_{t=1}^T$ and $\{\mathbf{S}_t\}_{t=0}^{T-1}$



- ▶ at every time $t = 0, \dots, T-1$ (not just at $t = 0$ as before) K assets traded at the vector of price \mathbf{S}_t is set
- ▶ asset deliver payoffs \mathbf{V}_{t+1} in period $t = 0, \dots, T-1$, unknown at the time t of price determination

The price process

- ▶ The price process is $\{\mathbf{S}_t\}_{t=1}^T = \{\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_t, \dots, \mathbf{S}_{T-1}\}$, where

$$\mathbf{S}_t = (S_t^1, \dots, S_t^K), \text{ for } S_t^j = \begin{pmatrix} s_{t,1}^j \\ \dots \\ s_{t,s}^j \\ \dots \\ s_{t,N_t}^j \end{pmatrix}$$

conditional on the information at time $t = 0$

- ▶ or, expanding, the possible realizations for the price at at time $t > 0$ are

$$\mathbf{S}_t = \begin{pmatrix} s_{t,1}^1 & \dots & s_{t,1}^j & \dots & s_{t,1}^K \\ \dots & \dots & \dots & \dots & \dots \\ s_{t,s}^1 & \dots & s_{t,s}^j & \dots & s_{t,s}^K \\ \dots & \dots & \dots & \dots & \dots \\ s_{t,N_t}^1 & \dots & s_{t,N_t}^j & \dots & s_{t,N_t}^K \end{pmatrix}$$

The payoff process

- ▶ The payoff process $\{\mathbf{V}_t\}_{t=1}^T = \{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_t, \dots, \mathbf{V}_T\}$, where

$$\mathbf{V}_t = (V_t^1, \dots, V_t^K), \text{ for } V_t^j = \begin{pmatrix} v_{t,1}^j \\ \dots \\ v_{t,s}^j \\ \dots \\ v_{t,N_t}^j \end{pmatrix}$$

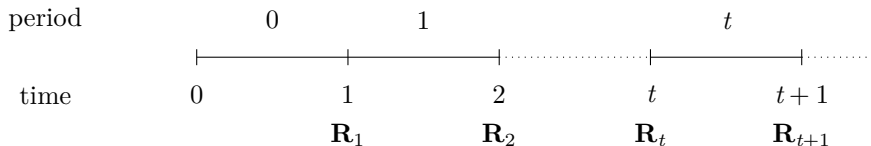
conditional on the information at time $t = 0$

- ▶ or, expanding, the possible realizations for the payoff at time t are

$$\mathbf{V}_t = \begin{pmatrix} v_{t,1}^1 & \dots & v_{t,1}^j & \dots & v_{t,1}^K \\ \dots & \dots & \dots & \dots & \dots \\ v_{t,s}^1 & \dots & v_{t,s}^j & \dots & v_{t,s}^K \\ \dots & \dots & \dots & \dots & \dots \\ v_{t,N_t}^1 & \dots & v_{t,N_t}^j & \dots & v_{t,N_t}^K \end{pmatrix}$$

The structure of the asset market

The return process results: $\{\mathbf{R}_t\}_{t=1}^T = \{\mathbf{R}_1, \dots, \mathbf{R}_t, \dots, \mathbf{R}_T\}$



- ▶ where the returns for every asset $\mathbf{R}_t = (R_t^1 \dots, R_t^K)$,
- ▶ where $R_t^j = (R_{t,1}^j, \dots, R_{t,N_t}^j)^\top$
- ▶ the return of asset j at time t is $R_t^j = \frac{V_t^j + S_t^j}{S_{t-1}^j}$
- ▶ is determined **after** the observation of price S_t^j , i.e., after being sold at the current market price.

Example

- ▶ Two-state binomial tree and $T = 3$ (information conditional at time $t = 0$)
- ▶ Two assets: a and b
- ▶ Prices and payoffs processes
 - ▶ at time $t = 0$ only prices are observed $\mathbf{S}_0 = (S_0^a, S_0^b)$
 - ▶ at time $t = 1$, $\mathbf{V}_1 = \begin{pmatrix} V_{1,1}^a & V_{1,1}^b \\ V_{1,2}^a & V_{1,2}^b \end{pmatrix}$ and $\mathbf{S}_1 = \begin{pmatrix} S_{1,1}^a & S_{1,1}^b \\ S_{1,2}^a & S_{1,2}^b \end{pmatrix}$
 - ▶ at time $t = 2$

$$\mathbf{V}_2 = \begin{pmatrix} V_{2,1}^a & V_{2,1}^b \\ V_{2,2}^a & V_{2,2}^b \\ V_{2,3}^a & V_{2,3}^b \\ V_{2,4}^a & V_{2,4}^b \end{pmatrix}, \quad \mathbf{S}_2 = \begin{pmatrix} S_{2,1}^a & S_{2,1}^b \\ S_{2,2}^a & S_{2,2}^b \\ S_{2,3}^a & S_{2,3}^b \\ S_{2,4}^a & S_{2,4}^b \end{pmatrix}$$

- ▶ at terminal time $t = 3$

$$\mathbf{V}_3 = \begin{pmatrix} V_{3,1}^a & V_{3,1}^b \\ \dots & \dots \\ V_{3,8}^a & V_{3,8}^b \end{pmatrix}$$

Arbitrage asset pricing

Stochastic discount factor: intertemporal form

Definition

stochastic discount factor (SDF) is a process $\{M_t\}_{t=0}^{T-1}$, such that, **for any asset** $j = 1, \dots, K$:

1. M_t is \mathcal{F}_t -measurable (v.g., has a tree structure) ,
2. $M_0 = m_0 = 1$
3. satisfies

$$M_t S_t^j = \mathbb{E}_t \left[\sum_{\tau=t+1}^T M_\tau V_\tau^j \right], \text{ for } t = 0, \dots, T-1$$

(i.e) the value of any asset j at time t is equal to the (conditional) mathematical expectation of the value of its future payoffs

Arbitrage asset pricing

Stochastic discount factor: intertemporal form

Observations:

1. We say this is SDF definition is in the **intertemporal form**
2. the meaning of the conditional expectation $\mathbb{E}_t[\cdot]$ is

$$M_t S_t^j = \mathbb{E}_t \left[\sum_{\tau=t+1}^T M_\tau V_\tau^j \right] = \mathbb{E} \left[\sum_{\tau=t+1}^T M_\tau V_\tau^{j,t} \mid S^{j,t}, V^t \right], \text{ for any } j = 1, \dots$$

where $S^{j,t} = \{S_0^j, S_1^j, \dots, S_t^j\}$ and $V^{j,t} = \{V_1^j, \dots, V_t^j\}$ are the histories of the asset prices and payoffs of asset t up until time t

Arbitrage asset pricing

Stochastic discount factor: recursive form

Proposition

The stochastic discount factor can be equivalently defined in the recursive form

$$M_t S_t^j = \mathbb{E}_t \left[M_{t+1} (S_{t+1}^j + V_{t+1}^j) \right], \text{ for any } j = 1, \dots, K$$

- **Intuition:** the stochastic discount factor is a stochastic process $\{M_t\}$, that equalizes the **value** of the asset price in period t with the conditional expected value of the **value** of the income in period $t + 1$ (the income is equal to the payoff plus the anticipated market price)

Arbitrage asset pricing

Stochastic discount factor: recursive form

Proof:

- ▶ using the definition of intertemporal form and expanding

$$M_t S_t^j = \mathbb{E}_t \left[M_{t+1} V_{t+1}^j + \sum_{\tau=t+2}^T M_{\tau} V_{\tau}^j \right]$$

- ▶ by the law of iterated expectations

$$\mathbb{E}_t \left[M_{t+1} V_{t+1}^j + \sum_{\tau=t+2}^T M_{\tau} V_{\tau}^j \right] = \mathbb{E}_t \left[M_{t+1} V_{t+1}^j + \mathbb{E}_{t+1} \left[\sum_{\tau=t+2}^T M_{\tau} V_{\tau}^j \right] \right]$$

- ▶ but

$$M_{t+1} S_{t+1}^j = \mathbb{E}_{t+1} \left[\sum_{\tau=t+2}^T M_{\tau} V_{\tau}^j \right],$$

- ▶ then $M_t S_t^j = \mathbb{E}_t \left[M_{t+1} V_{t+1}^j \right] + \mathbb{E}_t \left[M_{t+1} S_{t+1}^j \right]$

Arbitrage asset pricing

Stochastic discount factor and the rate of return

Proposition

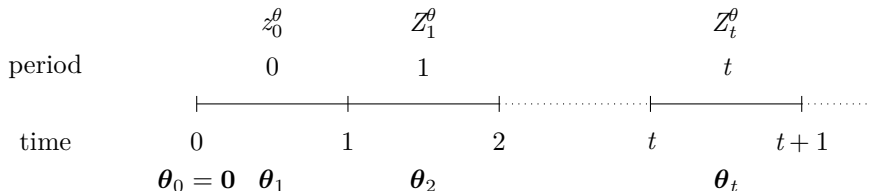
*The stochastic discount factor can be equivalently defined in the **recursive from** using the definition of the return of an asset*

$$M_t = \mathbb{E}_t \left[M_{t+1} R_{t+1}^j \right], \text{ for any } j = 1, \dots, K$$

- **Intuition:** the stochastic discount factor is a stochastic process $\{M_t\}$, that equalizes the **value** of the return of an investment (or loan) to be collected (or paid) at time $t + 1$, to the price of money, conditional on the information at time t

Portfolios and income

Stochastic processes for portfolios and income: $\{\theta_t\}_{t=1}^{T-1}$ and $\{Z_t\}_{t=0}^T$



- ▶ at every time $t = 0, \dots, T-1$ (not just at $t=0$ as before) a portfolio $\theta_{t+1} = (\theta_{t+1}^1, \dots, \theta_{t+1}^K)$ can be determined
- ▶ it generates an income in period $t = 0, \dots, T$, Z_t^θ

Arbitrage asset pricing

Transactions strategy

- The **income** stream $\{Z_t^\theta\}_{t=0}^T$ where (zero initial wealth)

$$z_0^\theta = -\theta_1 S_0 = -\sum_{j=1}^K \theta_1^j S_0^j$$

...

$$Z_t^\theta = \theta_t(S_t + V_t) - \theta_{t+1}S_t = \sum_{j=1}^K \left(\theta_t^j(S_t^j + V_t^j) - \theta_{t+1}^j S_t^j \right),$$

$$Z_T^\theta = \theta_T V_T = \sum_{j=1}^K \theta_T^j V_T^j$$

- where $Z_t^\theta \in \mathbb{R}^{N_t}$ is \mathcal{F}_t -measurable, i.e.

$$Z_t^\theta = (z_{t,1}^\theta, \dots, z_{t,s}^\theta, \dots, z_{t,N_t}^\theta)$$

Arbitrage asset pricing

Transactions strategy

Definition

transactions strategy is a sequence of portfolios $\{\boldsymbol{\theta}_{t+1}\}_{t=0}^{T-1}$, with $\boldsymbol{\theta}_{t+1} = (\theta_{t+1}^1 \dots \theta_{t+1}^K)$, where θ_{t+1}^j is \mathcal{F}_t -measurable, generating an income stream $\{Z_t^\theta\}_{t=0}^T = \{z_0^\theta, Z_1^\theta, \dots, Z_T^\theta\}$.

Definition

If $z_0^\theta = \dots = Z_t^\theta = \dots = Z_T^\theta = \mathbf{0}$ we say the transactions strategy is *self-financed*.

Arbitrage asset pricing

Absence of arbitrage opportunities

Definition

There is **absence of arbitrage opportunities** if there is a positive process $\{M_t\}_{t=0}^{T-1}$ such that the income stream $\{Z_t^\theta\}_{t=0}^T$, generated by the transaction strategy $\{\theta_{t+1}\}_{t=0}^{T-1}$, satisfies

$$\mathbb{E}_0 \left[\sum_{t=0}^T M_t Z_t^\theta \right] = 0$$

Intuition: there are no arbitrage opportunities if, with a **zero initial investment**, the expected value of the present value of any transaction strategy is zero, if the discount factor is positive.

Arbitrage asset pricing

Absence of arbitrage opportunities

Proposition A

necessary condition for the absence of arbitrage opportunities is that:

1. *The terminal price satisfies $M_T S_T = 0$ if T is finite;*
2. ***ruling-out speculative bubbles** condition holds: $\lim_{t \rightarrow \infty} M_t S_t = 0$ if $T = \infty$*

Arbitrage asset pricing

Absence of arbitrage opportunities

Proof (assuming $K = 1$):

- ▶ use the definition of stochastic discount factor (in the recursive form)

$$-M_0 Z_0^\theta = M_0 \theta_1 S_0 = \mathbb{E}_0 [M_1 \theta_1 (S_1 + V_1)]$$

- ▶ use a little trick, introducing $\pm M_1 \theta_2 S_1$;

$$\begin{aligned}\mathbb{E}_0 [M_1 \theta_1 (S_1 + V_1)] &= \mathbb{E}_0 [M_1 \theta_1 (S_1 + V_1) \pm M_1 \theta_2 S_1] = \\ &= \mathbb{E}_0 [M_1 Z_1^\theta + M_1 \theta_2 S_1]\end{aligned}$$

- ▶ use the definition of stochastic discount factor and the law of iterated expectations

$$\begin{aligned}\mathbb{E}_0 [M_1 Z_1^\theta + M_1 \theta_2 S_1] &= \mathbb{E}_0 [M_1 Z_1^\theta + \mathbb{E}_1 [M_2 \theta_2 (S_2 + V_2)]] \\ &= \mathbb{E}_0 [M_1 Z_1^\theta + M_2 \theta_2 (S_2 + V_2)]\end{aligned}$$

Arbitrage asset pricing

Absence of arbitrage opportunities

Proof (assuming $K = 1$ continuation):

► by repeatedly using the previous steps we arrive at

$$-M_0 Z_0^\theta = \mathbb{E}_0 \left[\sum_{t=1}^T M_t Z_t^\theta + M_T \theta_{T+1} S_T \right]$$

► then

$$\mathbb{E}_0 \left[\sum_{t=0}^T M_t Z_t^\theta \right] = 0$$

only if $M_T S_T = 0$

Application 1: zero payoffs

Absence of arbitrage opportunities

- ▶ **Zero payoffs (or no dividends case):** Assume that there are no dividends, i.e., $V_t = \mathbf{0}$ for any $t = 1, \dots, T$.
- ▶ If there are no arbitrage opportunities then

$$M_t S_t^j = \mathbb{E}_t \left[M_{t+1} S_{t+1}^j \right]$$

- ▶ therefore:

Proposition

For a zero dividend process, the process $\{M_t S_t\}_{t=0}^{T-1}$ is a martingale under measure \mathbb{P} ,

Arbitrage asset pricing

Fundamental theorem

Proposition

For a zero-dividend asset, absence of arbitrage opportunities is equivalent to the existence of an equivalent probability measure \mathbb{Q} such that $\{S_t\}_{t=0}^{T-1}$ is a martingale under \mathbb{Q} , that is

$$S_t^j = \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

Arbitrage asset pricing

Fundamental theorem

Sketch of proof

- ▶ Let us define the **conditional** stochastic discount factor

$$M_{t+1|t} \equiv \frac{M_{t+1}}{M_t}$$

- ▶ Then if there are no arbitrage opportunities (because M_t is \mathbb{F}_t -measurable)

$$S_t^j = \mathbb{E}_t[M_{t+1|t} S_{t+1}^j]$$

- ▶ This is valid for the degenerate process $\{\mathbf{1}\}_{t=0}^T = \{\mathbf{1}, \mathbf{1}, \dots, \mathbf{1}\}$, then

$$1 = \mathbb{E}_t[M_{t+1|t}]$$

$$S_t^j = \mathbb{E}_t[M_{t+1|t} S_{t+1}^j] = \frac{\mathbb{E}_t[M_{t+1|t} S_{t+1}^j]}{\mathbb{E}_t[M_{t+1|t}]} = \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

- ▶ from the Radon-Nikodym theorem \mathbb{Q} is an equivalent martingale measure.

Observe that $\{M_t\}$ is also a martingale, under measure \mathbb{Q} because

$$M_t = \mathbb{E}_t^{\mathbb{Q}}[M_{t+1}]$$

Application 2: positive payoffs

- **Positive payoffs (or positive dividends case):** if asset j pays a positive dividend, that is $V_t^j \geq \mathbf{0}$ is a positive vector, then

$$\mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j + V_{t+1}^j] \geq \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

- Then $\{S_t\}$ is a **submartingale** under measure \mathbb{Q}

$$S_t^j \geq \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

Proposition

For a positive-dividend asset, absence of arbitrage opportunities is equivalent to the existence of an equivalent probability measure \mathbb{Q} such that $\{S_t\}_{t=0}^{T-1}$ is a sub-martingale under \mathbb{Q} , that is

$$S_t^j \geq \mathbb{E}_t^{\mathbb{Q}}[S_{t+1}^j]$$

Application 3: Existence of a risk-free asset

- ▶ Consider a bond, issued at every time $t = 0, \dots, T - 1$, with the maturity of one period and paying a (deterministic) payoff with unit face value
- ▶ Then

$$S_t^f = \frac{1}{1 + r_{t+1}}, \quad V_{t+1}^f = \mathbf{1}, \quad V_{t+2}^f = \mathbf{0}, \dots, V_T^f = \mathbf{0}$$

- ▶ If there are no arbitrage opportunities then

$$\frac{1}{1 + r_{t+1}} = \mathbb{E}_t [M_{t+1}|t].$$

Application 3: Existence of a risk-free asset

Proposition

Assume there are no arbitrage opportunities and there is a risk-free asset with the (deterministic) return process $\{R_t^f\}_{t=1}^T$. Then there is a probability process \mathbb{Q} such that the return for asset j satisfies

$$R_{t+1}^f = \mathbb{E}_t^{\mathbb{Q}} \left[R_{t+1}^j \right], \text{ for any } j = 1, \dots, K, \text{ for any } t = 0, \dots, T$$

Application 3: Existence of a risk-free asset

- Proof: for any other risky asset, j , we can write

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}} \left[S_{t+1}^j + V_{t+1}^j \right] &= \frac{\mathbb{E}_t \left[M_{t+1|t} \left(S_{t+1}^j + V_{t+1}^j \right) \right]}{\mathbb{E}_t \left[M_{t+1|t} \right]} = \\ &= (1 + r_{t+1}) \mathbb{E}_t \left[M_{t+1|t} \left(S_{t+1}^j + V_{t+1}^j \right) \right] = \\ &= (1 + r_{t+1}) S_t^j\end{aligned}$$

- Then

$$S_t^j = \frac{1}{1 + r_{t+1}} \mathbb{E}_t^{\mathbb{Q}} \left[S_{t+1}^j + V_{t+1}^j \right]$$

- Dividing by S_t^j we find

$$R_{t+1}^f = \mathbb{E}_t^{\mathbb{Q}} \left[R_{t+1}^j \right]$$

- It can also be proved that

$$S_t^j = \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{\tau=t+1}^T D_{t+1,\tau} V_{\tau}^j \right]$$

the asset price at time t is the conditional expected value of the present value of the future payoffs;

- where the discount factor is

$$D_{t+1,\tau} = \prod_{h=t+1}^{\tau} \frac{1}{1 + r_h}, \quad \tau \geq t + 1.$$

- Exercise: prove this.

Equilibrium asset pricing

Equilibrium asset pricing

Real part of the economy: resources

- ▶ There is a **given** sequence of endowments

$$Y^T \equiv \{Y_t\}_{t=0}^T = \{Y_0, Y_1, \dots, Y_t, \dots, Y_T\}$$

- ▶ where Y_t is \mathcal{F}_t -measurable, such that

$$Y_t = \begin{pmatrix} y_{t,1} \\ \dots \\ y_{t,N_t} \end{pmatrix}$$

Equilibrium asset pricing

Real part of the economy: preferences and distribution

- ▶ households choose a contingent-consumption sequence belonging to the set

$$C^T \equiv \{C_t\}_{t=0}^T = \{C_0, C_1, \dots, C_t, \dots, C_T\}$$

where C_t is \mathcal{F}_t -measurable,

- ▶ through an intertemporal von-Neumann-Morgenstern functional

$$\mathbb{E}_0 \left[\sum_{t=0}^T \beta^t u(C_t) \right]$$

- ▶ expansion of the utility functional

$$\begin{aligned} \mathbb{E}_0 \left[\sum_{t=0}^T \beta^t u(C_t) \right] &= \sum_{t=0}^T \beta^t \mathbf{P}_t u(C_t) = \\ &= u(C_0) + \beta \mathbf{P}_1 u(C_1) + \dots + \beta^t \mathbf{P}_t u(C_t) + \dots + \beta^T \mathbf{P}_T u(C_T) \end{aligned}$$

where

$$\mathbf{P}_t u(C_t) = \sum_{s=1}^{N_t} \pi_{t,s} u(c_{t,s})$$

Equilibrium asset pricing

Market structure

There are assets markets with the structure we have just presented, opening at every time $t \in \{0, \dots, T\}$

Equilibrium asset pricing: zero initial wealth

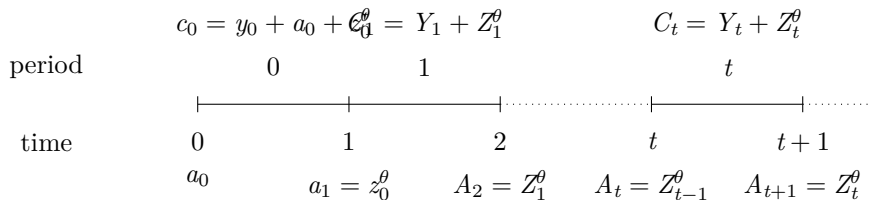
Assumption: the level of initial net wealth is zero $a_0 = 0$

Consequence: we can transform the household problem in a finance economy into the household problem in an equivalent AD economy

Non-zero initial wealth: we have to apply other methods for solving the household-investor problem (v.g, dynamic programming or optimal control)

Flow and stock accounting

Adaptoing the timing



Equilibrium asset pricing: Zero initial wealth

Radner or sequential general equilibrium

Definition

The **Radner or sequential general equilibrium** is defined by the processes $\{C_t\}_{t=0}^T$, $\{\theta_t\}_{t=1}^T$ and $\{S_t\}_{t=0}^{T-1}$ such that, **given** the processes of endowments $\{Y_t\}_{t=0}^T$ and payoffs $\{V_t\}_{t=1}^T$:

- (1) the household solves his **consumption-portfolio problem**, with rational expectations regarding future asset prices, and
- (2) the **markets clear**,

$$C_t = Y_t, \quad t = 0, \dots, T$$

$$\theta_t = 0, \quad t = 1, \dots, T.$$

Equilibrium asset pricing: Zero initial wealth

The (sequential) household-investor problem

Find the process for consumption $\{C_t\}_{t=0}^T$ and a transactions' strategy $\{\theta_t\}_{t=1}^T$

- ▶ that maximizes the value functional

$$V_0(\{C_t\}, \{\theta_t\}) \equiv \mathbb{E}_0 \left[\sum_{t=0}^T \beta^t u(C_t) \right]$$

- ▶ subject to the **sequential** budget constraints

$$c_0 + \sum_{j=1}^K \theta_1^j S_0^j \leq y_0$$

...

$$C_t + \sum_{j=1}^K \theta_{t+1}^j S_t^j \leq Y_t + \sum_{j=1}^K \theta_t^j (S_t^j + V_t^j), \quad t = 1, \dots, T-1 \quad (\mathcal{F}_t - \text{adapted})$$

...

$$C_T \leq Y_T + \sum_{j=1}^K \theta_T^j V_T^j \quad (\mathcal{F}_T - \text{adapted})$$

Equilibrium asset pricing: Zero initial wealth

The (sequential) household problem

- We can write the sequence of budget constraints equivalently as

$$C_0 \leq Y_0 + Z_0^\theta$$

...

$$C_t \leq Y_t + Z_t^\theta, \quad t = 1, \dots, T-1 \quad (\mathcal{F}_t - \text{adapted})$$

...

$$C_T \leq Y_T + Z_T^\theta \quad (\mathcal{F}_T - \text{adapted})$$

where Z_t^θ is the income generated at time t by the transaction strategy $\{\theta_t\}_{t=1}^T$.

- If the utility function $u(\cdot)$ displays no-satiation the constraints hold with equality in the optimum.

Equilibrium asset pricing: Zero initial wealth

Equivalent simultaneous household problem

- **If there are no arbitrage opportunities**, then there is stochastic discount factor process $\{M_t\}_{t=0}^{T-1}$, such that

$$-\mathbb{E}_0 \left[\sum_{t=0}^T M_t Z_t^\theta \right] = \mathbb{E}_0 \left[\sum_{t=0}^T M_t (Y_t - C_t) \right] = 0.$$

- Then, the household's problem is (the same as in the AD economy)

$$\max_{\{C_t\}_{t=0}^T} \mathbb{E}_0 \left[\sum_{t=0}^T \beta^t u(C_t) \right] \text{ s.t. } \mathbb{E}_0 \left[\sum_{t=0}^T M_t (Y_t - C_t) \right] = 0$$

- We already found the f.o.c

$$u'(c_0)M_t = \beta^t u'(C_t), \text{ } (\mathcal{F}_t - \text{adapted})$$

Equilibrium asset pricing: Zero initial wealth

Equilibrium stochastic discount factor

- ▶ The household arbitrage condition and the market equilibrium conditions

$$\begin{cases} u'(c_0)M_t = \beta^t u'(C_t) & t = 1, \dots, T \\ C_t = Y_t & t = 0, \dots, T \end{cases}$$

- ▶ imply that, at equilibrium, as in the AD economy

$$M_t = \beta^t \frac{u'(Y_t)}{u'(Y_0)} \quad (\mathcal{F}_t - \text{adapted})$$

- ▶ In terms of the possible realizations

$$M_t = \begin{pmatrix} m_{t,1} \\ \dots \\ m_{t,N_t} \end{pmatrix}, \quad t = 0, \dots, T-1$$

where

$$m_{ts} = \beta^t \frac{u'(y_{ts})}{u'(y_0)}, \quad \text{for } s = 1, \dots, N_t, \text{ and, } t = 0, \dots, T-1.$$

Equilibrium asset pricing: Zero initial wealth

Equilibrium asset pricing

- ▶ If there are no arbitrage opportunities, we proved that, for any asset j

$$M_t S_t^j = \mathbb{E}_t \left[M_{t+1} (V_{t+1}^j + S_{t+1}^j) \right], \quad j = 1, \dots, K$$

- ▶ Then the **GE equilibrium** asset pricing is

$$u'(Y_t) S_t^j = \beta \mathbb{E}_t \left[u'(Y_{t+1}) (V_{t+1}^j + S_{t+1}^j) \right], \quad j = 1, \dots, K$$

- ▶ determines asset price process $\{S_t^j\}$ given the processes $\{V_t^j\}$ and $\{Y_t\}$.

Equilibrium asset pricing: Zero initial wealth

Equilibrium asset pricing

Equivalent representations:

1. The **equilibrium rate of return** for asset j is determined from

$$\mathbb{E}_t \left[M_{t+1|t} R_{t+1}^j \right] = 1$$

where the **equilibrium recursive stochastic discount factor** is

$$M_{t+1|t} \equiv \beta \frac{u'(Y_{t+1})}{u'(Y_t)}$$

and the return is

$$R_{t+1}^j = \frac{V_{t+1}^j + S_{t+1}^j}{S_t^j}$$

2. or, equivalently

$$u'(Y_0) S_0^j = \mathbb{E}_0 \left[\sum_{t=1}^T \beta^t u'(Y_t) V_t^j \right], \quad j = 1, \dots, K.$$

Infinite horizon case, $T = \infty$

- The arbitrage condition is, of course, still valid.

Proposition

Fundamental equilibrium arbitrage condition: if we rule out speculative bubbles, then the price for asset j satisfies

$$S_t^j = \mathbb{E}_t \left[\sum_{\tau=1}^{\infty} \beta^{\tau} \frac{u'(C_{t+\tau})}{u'(C_t)} V_{t+\tau}^j \right], \quad j = 1, \dots, K, \quad t \in [0, \infty) \quad (1)$$



Infinite horizon case, $T = \infty$

Proof:

$$\begin{aligned}u'(C_t)S_t^j &= \beta \mathbb{E}_t \left[u'(C_{t+1})(S_{t+1}^j + V_{t+1}^j) \right] = \\&= \lim_{k \rightarrow \infty} \beta^k \mathbb{E}_t \left[u'(C_{t+k})S_{t+k}^j \right] + \\&\quad + \mathbb{E}_t \left[\sum_{\tau=1}^{\infty} \beta^\tau u'(C_{t+\tau}) V_{t+\tau}^j \right]\end{aligned}$$

If we rule out speculative bubbles, that is

$$\lim_{k \rightarrow \infty} \beta^k \mathbb{E}_t \left[u'(C_{t+k})S_{t+k}^j \right] = 0$$

we get equation (1)

Risky and risk-free assets

- For a risky asset

$$\mathbb{E}_t \left[M_{t+1|t} R_{t+1}^j \right] = 1$$

- For a riskless asset with return $R_t^f = 1 + r_t^f$ we have

$$\mathbb{E}_t \left[M_{t+1|t} \right] R_{t+1}^f = 1$$

- Then, for any asset j

$$\mathbb{E}_t \left[M_{t+1|t} R_{t+1}^j \right] = \mathbb{E}_t \left[M_{t+1|t} \right] R_{t+1}^f$$

Equilibrium equity premium: example

Equilibrium risk premium for a Markovian case

► Assumptions:

1. Homogeneous agent finance economy
2. CRRA Bernoulli utility function
3. growth factor for the return is Markovian following an iid log-normal distribution
4. there is one riskless and one risky asset such that the return is Markovian following an iid log-normal distribution

► Problem: **Derive the distribution for the multiplicative risk premium for the risky asset R^j/R^f**

► Solution: the risk premium for asset j , satisfies

$$\ln \mathbb{E}_t[R_{t+1}^j] = \ln R_{t+1}^f + \zeta \text{Cov}_t \left[\ln(1 + \gamma_{t+1}), \ln R_{t+1}^j \right], \quad j = 1, \dots, K$$

Auxiliary: log-normal distributions

Some properties

Assume two random variables X and Y following log-normal distributions. Then $\ln X$ and $\ln Y$ are normally distributed. Then:

$$\ln \mathbb{E}[X] = \mathbb{E}[\ln X] + \frac{1}{2}\mathbb{V}[\ln X]$$

$$\ln \mathbb{E}[\alpha X] = \ln \alpha + \mathbb{E}[\ln X] + \frac{1}{2}\mathbb{V}[\ln X], \alpha \text{ constant}$$

$$\ln \mathbb{E}[\alpha X^\beta] = \ln \alpha + \beta \mathbb{E}[\ln X] + \frac{\beta^2}{2}\mathbb{V}[\ln X], \alpha, \beta, \text{ constants}$$

$$\ln \mathbb{E}[XY] = \mathbb{E}[\ln X] + \mathbb{E}[\ln Y] + \frac{1}{2} \{ \mathbb{V}[\ln X] + \mathbb{V}[\ln Y] - 2 \text{Cov}(\ln X, \ln Y) \}$$

$$\begin{aligned} \ln \mathbb{E}[X^\beta Y] &= \beta \mathbb{E}[\ln X] + \mathbb{E}[\ln Y] + \\ &\quad + \frac{1}{2} \{ \beta^2 \mathbb{V}[\ln X] + \mathbb{V}[\ln Y] - 2\beta \text{Cov}(\ln X, \ln Y) \} \end{aligned}$$

because $\text{Cov}[\beta X, Y] = \beta \text{Cov}[XY]$.

Equilibrium equity premium example: proof

solution

- ▶ The risky asset j follows a iid log-normal distribution: then

$$\ln \mathbb{E}_t[R_{t+1}^j] = \mathbb{E}_t[\ln R_{t+1}^j] + \frac{1}{2} \mathbb{V}_t[\ln R_{t+1}^j]$$

- ▶ the endowment process satisfies $Y_{t+1} = (1 + \gamma_{t+1}) Y_t$, where the growth factor follows also a iid log-normal distribution: then

$$\ln \mathbb{E}_t[1 + \gamma_{t+1}] = \mathbb{E}_t[\ln(1 + \gamma_{t+1})] + \frac{1}{2} \mathbb{V}_t[\ln(1 + \gamma_{t+1})]$$

- ▶ the utility function is CRRA $u(C) = (1 - \zeta)^{-1} C^{1-\zeta}$ then the stochastic discount factor is

$$M_{t+1|t} = \beta(1 + \gamma_{t+1})^{-\zeta}$$

- ▶ then, for any asset, the arbitrage condition holds as

$$1 = \beta \mathbb{E}_t \left[(1 + \gamma_{t+1})^{-\zeta} R_{t+1}^j \right]$$

Equilibrium equity premium example: proof

solution (cont.)

- for the riskless asset, after taking logs to the arbitrage condition, we have

$$\begin{aligned} 0 &= \ln \beta + \ln \mathbb{E}_t \left[(1 + \gamma_{t+1})^{-\zeta} R_{t+1}^f \right] \\ &= \ln \beta + \mathbb{E}_t \left[\ln \left((1 + \gamma_{t+1})^{-\zeta} R_{t+1}^f \right) \right] + \\ &\quad + \frac{1}{2} \mathbb{V}_t \left[\ln \left((1 + \gamma_{t+1})^{-\zeta} R_{t+1}^f \right) \right] \\ &= \ln \beta - \zeta \mathbb{E}_t [\ln(1 + \gamma_{t+1})] + \ln R_{t+1}^f + \frac{\zeta^2}{2} \mathbb{V}_t [\ln(1 + \gamma_{t+1})] \end{aligned}$$

Equilibrium equity premium example: proof

solution (cont.)

- for the risky asset j , after taking logs to the arbitrage condition, we have

$$\begin{aligned} 0 &= \ln \beta + \ln \mathbb{E}_t \left[(1 + \gamma_{t+1})^{-\zeta} R_{t+1}^j \right] = \\ &= \ln \beta - \zeta \mathbb{E}_t [\ln(1 + \gamma_{t+1})] + \mathbb{E}_t [\ln R_{t+1}^j] + \\ &\quad + \frac{1}{2} \left\{ \zeta^2 \mathbb{V}_t [\ln(1 + \gamma_{t+1})] + \mathbb{V}_t [\ln R_{t+1}^j] - \right. \\ &\quad \left. - 2\zeta \text{Cov}_t [\ln(1 + \gamma_{t+1}), \ln R_{t+1}^j] \right\} = \\ &= -\ln R_{t+1}^f + \mathbb{E}_t [\ln R_{t+1}^j] + \frac{1}{2} \mathbb{V}_t [\ln R_{t+1}^j] - \\ &\quad - \zeta \text{Cov}_t [\ln(1 + \gamma_{t+1}), \ln R_{t+1}^j] = \\ &= -\ln R_{t+1}^f + \ln \mathbb{E}_t [R_{t+1}^j] - \zeta \text{Cov}_t [\ln(1 + \gamma_{t+1}), \ln R_{t+1}^j]. \end{aligned}$$

(end of proof)

Equilibrium equity premium

Hansen-Jaganathan bounds

- ▶ Let us write the **Equity premium** for asset risky j as:
 $R_{t+1}^j - R_{t+1}^f$

- ▶ Expected premium and standard deviation

$$\mathbb{E}_t \left[R_{t+1}^j - R_{t+1}^f \right], \sigma_t \left[R_{t+1}^j - R_{t+1}^f \right]$$

- ▶ **Equilibrium equity premium** for risky asset j satisfies, under the assumptions of the model:

$$\mathbb{E}_t \left[M_{t+1|t} \left(R_{t+1}^j - R_{t+1}^f \right) \right] = 0$$

- ▶ Then,

$$\left| \frac{\mathbb{E}_t \left[R_{t+1}^j - R_{t+1}^f \right]}{\sigma_t \left[R_{t+1}^j \right]} \right| \leq \frac{\sigma_t[M_{t+1|t}]}{\mathbb{E}_t[M_{t+1|t}]} \quad (2)$$

the l.h.s is called the Sharpe ratio and r.h.s. the Hansen-Jaganathan bounds

Equilibrium equity premium

Hansen-Jaganathan bounds

- Proof:
- From a standard result on the covariance between two random variables

$$\begin{aligned} & \mathbb{E}_t \left[M_{t+1|t} \left(R_{t+1}^j - R_{t+1}^f \right) \right] = \\ &= \mathbb{E}_t \left[M_{t+1|t} \right] \mathbb{E}_t \left[R_{t+1}^j - R_{t+1}^f \right] + \text{Cov}_t \left[M_{t+1|t}, \left(R_{t+1}^j - R_{t+1}^f \right) \right] = \\ &= 0 \end{aligned}$$

- But

$$\begin{aligned} & \text{Cov}_t \left[M_{t+1|t}, \left(R_{t+1}^j - R_{t+1}^f \right) \right] \\ &= \rho_{M_{t+1|t}, R_{t+1}^j - R_{t+1}^f} \sigma_t(M_{t+1|t}) \sigma_t \left(R_{t+1}^j - R_{t+1}^f \right) \quad (3) \end{aligned}$$

where $\rho_{M_{t+1|t}, R_{t+1}^j - R_{t+1}^f}$ is the correlation coefficient between $M_{t+1|t}$ and $R_{t+1}^j - R_{t+1}^f$

Equilibrium equity premium

Hansen-Jaganathan bounds (cont.)

► Then

$$\frac{\mathbb{E}_t \left[R_{t+1}^j - R_{t+1}^f \right]}{\sigma_t \left[R_{t+1}^j \right]} = \rho_{M_{t+1}|t, R_{t+1}^j - R_{t+1}^f} \frac{\sigma_t [M_{t+1}|t]}{\mathbb{E}_t [M_{t+1}|t]}$$

► We use the fact $|\rho_{M_{t+1}|t, R_{t+1}^j - R_{t+1}^f}| \in [0, 1]$

Equilibrium equity premium

Example

- If we assume that $Y_{t+1} = (1 + \gamma_{t+1}) Y_t$, the utility function is homogeneous, and $R_{t+1}^f \approx 1/\beta$ then

$$\left| \frac{\mathbb{E}_t \left[R_{t+1}^j - R_{t+1}^f \right]}{\sigma_t \left[R_{t+1}^j \right]} \right| \leq \sigma_t [u'(1 + \gamma_{t+1})]$$

- if the utility function is homogeneous, from the equilibrium arbitrage condition

$$\beta \mathbb{E}_t [u'(1 + \gamma_{t+1})] R_{t+1}^f = 1$$

- if $R_{t+1}^f \approx 1/\beta$ then

$$\mathbb{E}_t [u'(1 + \gamma_{t+1})] = 1$$

Equilibrium equity premium: example

- If we assume a CRRA utility function

$$u(C) = \frac{C^{1-\zeta}}{1-\zeta}$$

Then

$$M_{t+1|t} = \beta(1 + \gamma_{t+1})^{-\zeta}$$

$$\sigma_t[u'(1 + \gamma_{t+1})] = \sigma_t[(1 + \gamma_{t+1})^{-\zeta}]$$

The higher η the lower $\sigma_t[M_{t+1|t}]$ is.

Equilibrium equity premium puzzle

- **Equity premium puzzle:** if we set $\zeta \approx 2$, we find excessive risk premium in the data:

$$\text{Sharpe ratio} = 0.37 > \frac{\sigma_t[M_{t+1}|t)]}{\mathbb{E}_t[M_{t+1}|t]} \approx \frac{0.002}{0.96}$$

- This means that the data displays a higher risk premium than the model would predict (or consumption displays a lower relative volatility than the model predicts)

Equilibrium equity premium puzzle

- ▶ This has led to a whole research program (still going on) for macro finance: see <http://academicwebpages.com/preview/mehra/pdf/FIN200201.pdf> for a survey, by introducing in the model:
 - ▶ changes in preferences: habit formation, non-additive preferences concerning risk
 - ▶ transactions costs, taxes, etc
 - ▶ distributions
 - ▶ imperfectly competitive environments
- ▶ The basic change we have to introduce should do the following: consumption (and investment) should have a smoother behaviour than the model predicts, which means that the reaction of portfolios to changes in asset prices is more rigid, which implies a higher variation in prices to unpredicted shocks.

Non-zero initial wealth

Equilibrium asset pricing: non-zero initial wealth

Assumption assume that the level of initial net wealth is different from zero

Implications:

- ▶ we cannot transform the household problem into a household problem for an AD economy
- ▶ the household problem becomes a stochastic optimal control problem where financial wealth, A_t , is the state variable and consumption, C_t is the control variable

Equilibrium asset pricing: non-zero initial wealth

Timing and information sequence

- ▶ When we introduce the stock of financial wealth A_t , we have to be careful as regards the stock-flow accounting and the exact timing of information should be specified. From now on, we assume the information timing for the end period t , is taken **after** observing C_t and Y_t and **before** observing S_t and \mathbb{V}_t .
- ▶ That is, in period t :
 1. at the start of period t , between times t and $t + 1$ we know the portfolio θ_t and we observe S_{t-1}^j (set at the end of period $t - 1$);
 2. along period t we observe the flow Y_t and we decide over the flow C_t ;
 3. close to the time $t + 1$, we observe the payoffs V_t^j for $j = 1, \dots, K$ and the asset markets open and draw S_t^j ;
 4. we buy a new portfolio θ_{t+1}^j at the market prices S_t^j
 5. And so on.

Equilibrium asset pricing: non-zero initial wealth

Sequential budget constraints

- ▶ The stock of asset j at time t is

$$A_t^j = \theta_t^j S_{t-1}^j$$

- ▶ and the return of asset j computed at the end of period t is

$$R_t^j = \frac{S_t^j + V_t^j}{S_{t-1}^j} = 1 + r_t^j$$

- ▶ Then the **period budget constraint** for period t is

$$\sum_j R_t^j A_t^j + Y_t = C_t + \sum_j A_{t+1}^j$$

- ▶ This is also equivalent to

$$C_t = Y_t + Z_t^\theta$$

Equilibrium asset pricing: non-zero initial wealth

Sequential budget constraints

- The **total** wealth at the beginning of period t is

$$A_t = \sum_{j=1}^K A_t^j$$

- If we define the weight of asset j in total wealth as

$$w_t^j \equiv \frac{A_t^j}{A_t}, \quad \sum_{j=1}^K w_t^j = 1$$

then

$$R_t^A A_t - A_{t+1} = (1 + r_t^A) A_t - A_{t+1}$$

where the average return is the rate of return for the whole portfolio

$$R_t^A = \sum_j w_t^j R_t^j, \quad r_t^A = \sum_j w_t^j r_t^j,$$

Equilibrium asset pricing: non-zero initial wealth

Sequential budget constraints

- ▶ Then the period budget constraint for period t is

$$A_{t+1} = Y_t - C_t + R_t^A A_t, \quad t = 0, \dots, T,$$

- ▶ Of course, given $A_t = a_t$ at the beginning of period t (starting a time t and ending at time $t + 1$) A_{t+1} is a distribution

$$A_{t+1} = (a_{t+1,1}, \dots, a_{t+1,N_{t+1}|t})^\top$$

where $N_{t+1|t}$ is the number of nodes at $t + 1$ subsequent to the node s_t at time t ;

- ▶ We are assuming that all the possible realizations of the budget constraint are of the form:

$$a_{t+1,s_{t+1}|s_t} = y_{t,s_t} - c_{t,s_t} + R_{t,s_{t+1}|s_t}^A a_{t,s_t}, \quad \text{for } s_t = 1, \dots, N_t, \quad s_{t+1}|s_t = 1, \dots,$$

Equilibrium asset pricing: non-zero initial wealth

- ▶ at time t the household observes A_t
- ▶ along period t he gets Y_t and decides on consumption C_t
- ▶ at the the end of period he receives a signal R_t^A and decides on the portfolio composition θ_{t+1} such that $A_{t+1} = \sum_{j=1}^K \theta_{t+1}^j S_t^j$
- ▶ in our case only the savings decision (not the financial decision over $\{\theta_t\}_{t=1}^T$) matters for the determination of the equilibrium stochastic discount factor

Conditional evolution of the asset position

$$\begin{array}{ccc} & \begin{pmatrix} Y_{t,1} - C_{t,1} + R_{t,1}^A \\ \vdots \\ Y_{t,N_{t+1}|t} - C_{t,N_{t+1}|t} + R_{t,N_{t+1}|t}^A \end{pmatrix} & A_t \\ \hline t & & t+1 \\ A_t & & \begin{pmatrix} A_{t+1,1} \\ \vdots \\ A_{t+1,N_{t+1}|t} \end{pmatrix} \end{array}$$

Equilibrium asset pricing: non-zero initial wealth

The representative agent problem

$$\max_{\{C_t\}_{t=0}^T} \mathbb{E}_0 \left[\sum_{t=0}^T \beta^t u(C_t) \right]$$

subject to the **sequence** of random constraints

$$A_{t+1} = Y_t - C_t + R_t^A A_t, \quad t = 0, \dots, T-1$$

given A_0 .

Equilibrium asset pricing: non-zero initial wealth

Solving the problem using Dynamic Programming

- We define the **value function** at time t

$$V(A_t) = \max_{\{C_\tau\}_{\tau=t}^T} \mathbb{E}_t \left[\sum_{\tau=t}^T \beta^{(\tau-t)} u(C_\tau) \right]$$

- If there is an optimal solution, $\{C_t^*\}_{t=0}^T$ for the agent's problem, it satisfies the **Hamilton-Jacobi-Bellman** (HJB) equation

$$V(A_t) = \max_{C_t} \{ u(C_t) + \beta \mathbb{E}_t [V(A_{t+1})] \}$$

Solving the household-investor problem using DP

- ▶ Assume there is only one asset: $K = 1$.
- ▶ In order to solve the problem:
 1. we derive the first order conditions, for an optimal consumption,

$$u'(C_t) = \beta \mathbb{E}_t[V'(A_{t+1})]$$

because $A_{t+1} = Y_t - C_t + R_t A_t$ and $\partial A_{t+1} / \partial C_t = -1$

2. and we use the envelope condition (taking the derivative to A_t)

$$V'(A_t) = \beta \mathbb{E}_t[V'(A_{t+1})R_t]$$

Equilibrium asset pricing: non-zero initial wealth

- **Optimality condition for the household:** again

$$u'(C_t) = \mathbb{E}_t[u'(C_{t+1})R_{t+1}]$$

- Proof: using the law of iterated expectations, the envelopment theorem and the measurability properties of the variables, we have

$$\begin{aligned}u'(C_t) &= \beta \mathbb{E}_t[V'(A_{t+1})] \\&= \beta \mathbb{E}_t[\beta \mathbb{E}_{t+1}[V'(A_{t+2})R_{t+1}]] \\&= \beta \mathbb{E}_t[\beta \mathbb{E}_{t+1}[V'(A_{t+2})] R_{t+1}] \\&= \beta \mathbb{E}_t[u'(C_{t+1}) R_{t+1}]\end{aligned}$$

- ▶ Now, for any number of assets. $K > 1$
- ▶ The first order conditions for an optimal consumption is the same

$$u'(C_t) = \beta \mathbb{E}_t [V'(A_{t+1})]$$

- ▶ but we have $A_t = A_t^1 + \dots + A_t^K$ and $R_t^A A_t = R_t^1 A_t^1 + \dots + R_t^K A_t^K$
- ▶ applying the envelope condition for any asset j

$$V'(A_t) \frac{\partial A_t}{\partial A_t^j} = \beta \mathbb{E}_t \left[V'(A_{t+1}) \frac{\partial A_{t+1}}{\partial A_t^j} \right], \quad j = 1, \dots, K$$

- ▶ we get

$$V'(A_t) = \beta \mathbb{E}_t \left[V'(A_{t+1}) R_t^j \right], \quad j = 1, \dots, K$$

- Then the **arbitrage condition** for the household choice of asset j is

$$u'(C_t) = \beta \mathbb{E}_t \left[u'(C_{t+1}) R_{t+1}^j \right], \quad j = 1, \dots, K$$

- or, equivalently

$$u'(C_t) S_t^j = \beta \mathbb{E}_t \left[u'(C_{t+1}) (V_{t+1}^j + S_{t+1}^j) \right], \quad j = 1, \dots, K, \quad t = 0, \dots, T$$

Using stochastic optimal control

- ▶ We obtain a similar result using the stochastic Pontryagin's principle
- ▶ Define the hamiltonian, for period f

$$H_t = u(C_t) + \Lambda_t (Y_t - C_t + R_t A_t)$$

where $\{\Lambda_t\}_{t=0}^{\infty}$ is an adapted \mathcal{F}_t process, but λ_t is conditional on the information at period t

- ▶ The optimality condition is, conditional on the information at t

$$\frac{\partial h_t}{\partial c_t} = 0 \iff u'(c_t) = \lambda_t$$

- ▶ The Euler equation is

$$\lambda_t = \beta \mathbb{E}_t \left[\frac{\partial H_{t+1}}{\partial C_{t+1}} \right] = \beta \mathbb{E}_t \left[\Lambda_{t+1} R_{t+1} \right]$$

- ▶ Then

$$u'(C_t) = \beta \mathbb{E}_t \left[u'(C_{t+1}) R_{t+1} \right]$$