Foundations of Financial Economics
Two period GE: optimality and equilibrium

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Topics

- A central planner economy with heterogeneity
- Implementing the decentralized equilibrium: Negishi-Mantel algorithm
- Optimal taxation

Equilibrium and optimality

Decentralized and centralized allocations

- Consider an economy with heterogenous agents;
- The AD equilibrium we just saw is defined for a **decentralized economy**;
- Question: is there a **centralized economy** which allows for the same allocations of consumption (between agents, time, and the states of nature) ?
- Answer: yes, if the equilibrium is Pareto optimal.

Equilibrium and optimality

Negishi-Mantel algorithm

- the Negishi-Mantel algorithm: allows us to determine the (decentralized) AD equilibrium from a related **centralized** problem
- this can be done for three different reasons:
  - as a way to **compute** the general equilibrium in a simpler way;
  - to determine the Pareto **efficiency** of the GE;
  - to allow for the determination of **optimal redistributive policies** which are implicit in the decentralized equilibrium

Equilibrium in a heterogeneous agent economy

- Assume an economy with **two** agents heterogeneous in the endowments:
- The **General Equilibrium** for this economy is the allocation $((\mathbf{C}^*)^{\mathbf{1}}, (\mathbf{C}^*)^{\mathbf{2}}) \equiv \left(\{c_0^1, C_1^1\}, \{c_0^2, C_1^2\}\right)$ and the discount factor $M^* = (m_1^*, \ldots, m_N^*)$, such that the sequences $\{c_0^i, C_1^i\}$ solve problem for consumer $i = 1, 2$

$$\max_{C^i} \mathbb{E}_0^i \left[u^i(c_0^i) + \beta^i u^i(C_1^i)\right], \text{ s.t. } \mathbb{E}_0^i[c_0^i + MC_i^i] = h^i \equiv \mathbb{E}_0[y_0^i + MY_i^i]$$

and markets clear

$$\begin{aligned} (c_0^*)^1 + (c_0^*)^2 &= y_0^1 + y_0^2 = y_0 \\ (C_1^*)^1 + (C_1^*)^2 &= Y_1^1 + Y_1^2 = Y_1 \end{aligned}$$

where $\{y_0, Y_1\}$ are the sequence of **aggregate** endowments.

Equilibrium in a heterogeneous agent economy

- We found that, if the utility function is logarithmic $(u(c) = \ln(c))$ the equilibrium is

$$
\begin{aligned}
c_0^i &= \frac{h^i}{1+\beta},\ i = 1, 2 \\
c_{1,s}^i &= \frac{\beta h^i}{m_s(1+\beta)},\ i = 1, 2,\ s = 1, \dots, N \\
m_s &= \frac{\beta y_0}{y_{1,s}},\ s = 1, \dots, N
\end{aligned}
$$

- then

$$h^i = y_0^i + \sum_{s=1}^{N} \pi m_s y_{1,s}^i = y_0 \left(\phi_0^i + \beta \sum_{s=1} N \phi_{1,s}^i \right)$$

where $\phi_{t,s}^i = \dfrac{y_{t,s}^i}{y_{t,s}}$ is the weight of agent i on the aggregate endowments at time t and state s.

Pareto efficient equilibria

- An equilibrium allocation $((\mathbf{C}^*)^{\mathbf{1}}, (\mathbf{C}^*)^{\mathbf{2}})$ is **Pareto-efficient**: if there is **no other** feasible allocation, $((C^{'})^1, (C^{'})^2)$ verifying

$$
\begin{aligned}
c_0^{'1} + c_0^{'2} &= y_0^1 + y_0^2 = y_0 \\
c_{1,s}^{'1} + c_{1,s}^{'2} &= y_{1,s}^1 + y_{1,s}^2 = y_{1,s},\ s = 1, \ldots, N.
\end{aligned}
$$

  and *allowing one consumer to be as well of and another better off than with the allocation* $((C^*)^1, (C^*)^2)$.

Welfare theorems

First welfare theorem: A competitive equilibrium with complete markets is Pareto-efficient, under very weak conditions on utility.

Second welfare theorem: Any particular Pareto-efficient allocation can be implemented as an equilibrium allocation if we introduce transfers.

Social planner problem

- Assumptions: two agents, homogeneity in information and preferences, **heterogeneity in endowments**, and log utility.
- The **social welfare function** is

$$\max_{C^1, C^2} \mathbb{E}_0 \left[\alpha \left(\ln(C_0^1) + \beta \ln(C_1^1) \right) + (1-\alpha) \left(\ln(C_0^2) + \beta \ln(C_1^2) \right) \right]$$

  a weighed average of the intertemporal von-Neumann-Morgensten utility functionals for the two households

- where α and $1-\alpha$ are the utility weight for households 1 and 2 $(0 < \alpha < 1)$
- constraints: for every period and every state of nature total consumptions should be equal to total endowments.

Social planner problem

- The problem:

$$\max_{C^1, C^2} \mathbb{E}_0 \left[\alpha \left(\ln\left(c_0^1\right) + \beta \ln\left(C_1^1\right) \right) + (1-\alpha) \left(\ln\left(c_0^2\right) + \beta \ln\left(C_1^2\right) \right) \right]$$

subject to

$$\begin{aligned} c_0^1 + c_0^2 &= y_0^1 + y_0^2 = y_0 \\ c_{1,s}^1 + c_{1,s}^2 &= y_{1,s}^1 + y_{1,s}^2 = y_{1,s}, \ s = 1, \ldots, N \end{aligned}$$

Social planner problem

Solving the SPP

- The Lagrangean is

$$\mathcal{L} = E_0 \left[\alpha \left(\ln(c_0^1) + \beta \ln(c_1^1)\right) + (1-\alpha)\left(\ln(c_0^2) + \beta \ln(c_1^2)\right) + \right.$$
$$\left. \mu_0 \left(y_0 - c_0^1 - c_0^2\right) + \mu_1 \cdot \left(y_1 - c_1^1 - c_1^2\right)\right] =$$
$$= \alpha \ln(c_0^1) + \alpha\beta \sum_{s=1}^{N} \pi_s \ln(c_{1,s}^1) +$$
$$+ (1-\alpha)\ln(c_0^2) + (1-\alpha)\beta \sum_{s=1}^{N} \pi_s \ln(c_{1,s}^2)$$
$$+ \mu_0 \left(y_0 - c_0^1 - c_0^2\right) +$$
$$+ \sum_{s=1}^{N} \mu_{1,s} \left(y_{1,s} - c_{1,s}^1 - c_{1,s}^2\right)$$

Solving the SPP (cont.)

For $(c_0^1, c_0^2, (c_{1,s}^1)_{s=1}^N, (c_{1,s}^2)_{s=1}^N, \mu_0, (\mu_{1,s})_{s=1}^N)$, the first order conditions are:

$$\begin{aligned}
c_0^1 &= \frac{\alpha}{\mu_0} \\
c_0^2 &= \frac{1-\alpha}{\mu_0} \\
c_{1,s}^1 &= \beta\pi_s \frac{\alpha}{\mu_{1,s}}, \ s = 1, \ldots, N \\
c_{1,s}^2 &= \beta\pi_s \frac{(1-\alpha)}{\mu_{1,s}}, \ s = 1, \ldots, N \\
c_0^1 + c_0^2 &= y_0 \\
c_{1,s}^1 + c_{1,s}^2 &= y_{1,s} = (1+\gamma_s)y_0, \ s = 1, \ldots, N
\end{aligned}$$

Solving the SPP (cont.)

- Then the Lagrange multipliers become

$$\mu_0 = 1/y_0$$
$$\mu_{1,s} = \beta\pi_s/y_{1,s}$$

- Observe that the ratio of the Lagrange multipliers and the stochastic discount factor for a AD (decentralized) economy $q_s = \pi_s m_s$ or

$$\frac{\mu_{1,s}}{\mu_0} = \frac{\beta\pi_s}{1+\gamma_s} \Leftrightarrow m_s = \frac{\mu_{1,s}}{\pi_s\mu_0}$$

- Then the **optimal allocation of consumption** is

$$c_0^1 = \alpha y_0,\ c_0^2 = (1-\alpha)y_0$$
$$c_{1,s}^1 = \alpha y_{1,s},\ c_{1,s}^2 = (1-\alpha)y_{1,s} s = 1, \dots, N$$

consumption is a proportion of total endowments equal to the social welfare weights.

Relationship between equilibrium and optimality

We can rely on the welfare theorems to relate equilibrium and optimality

- Negishi-Mantel algorithm: assuming that the central planner wants to implement the equilibrium outcome which weights should it use ?
- Optimal tax-transfer policy: if the central planer wants to implement the optimal distribution of consumptions (given its weights) what should be the tax-transfer policy ?

1. Negishi-Mantel algorithm

The **Negishi-Mantel algorithm** uses the welfare theorems as a method to determine competitive equilibria from a centralized planner problem.

It consists in two steps:

1. First step: determine the optimal allocation of consumption for a centralized social planner problem, by defining a **social welfare function**, in which there are transfers between consumers, **parameterized by the consumers' weights**;
2. Second step: use the second welfare theorem, to **determine the transfers**which would be verified in a competitive equilibrium.

By doing this we can determine the **weights** of the centralized problem which would support a competitive equilibrium.

Step 1

- We already solved the social planner problem

$$c_0^1 = \alpha y_0,\ c_0^2 = (1-\alpha)y_0$$
$$c_{1,s}^1 = \alpha y_{1,s},\ c_{1,s}^2 = (1-\alpha)y_{1,s} s = 1, \ldots, N$$

- This solution implies there are implicit transfers between agents:
  - for agent 1

$$c_0^1 - y_0^1 = \alpha y_0 - y_0^1,\ \ c_{1,s}^1 - y_{1,s}^1 = \alpha y_{1,s} - y_{1,s}^1$$

  - for agent 2

$$c_0^2 - y_0^2 = (1-\alpha)y_0 - y_0^2,\ \ c_{1,s}^2 - y_{1,s}^2 = (1-\alpha)y_{1,s} - y_{1,s}^2$$

Step 1

- The implicit present value of the expected transfers that agent i, using the relative Lagrange multipliers as relative prices $\frac{\mu_{1,s}}{\mu_0} = \beta \frac{\pi_s}{1+\gamma_s}$

$$\begin{aligned} \tau^i(\alpha) &= c_0^i(\alpha) - y_0^i + \beta \sum_{s=1}^{N} \pi_s \frac{c_{1,s}^i(\alpha) - y_{1,s}^i}{1+\gamma_s} \\ &= \left(\omega^i - \phi_0^i + \beta \sum_{s=1}^{N} \pi_s (\omega^i - \phi_{1,s}^i) \right) y_0 \end{aligned}$$

where $\omega^1 = \alpha$ and $\omega^2 = 1 - \alpha$.

- The adding-up constraint holds

$$\tau^1(\alpha) + \tau^2(\alpha) = 0.$$

because $\sum_{i=1}^2 \omega^i = \sum_{i=1}^2 \phi_0^i = \sum_{i=1}^2 \phi_{1,s}^i = 1$.

Step 2

- That is the transfers satisfy

$$c_0^i(\alpha) + \frac{\beta}{1+\gamma}\mathbb{E}\left[C_1^i(\alpha)\right] = \tau^i(\alpha) + y_0^i + \mathbb{E}\left[\frac{\beta}{1+\gamma}Y_1^i\right]$$

- Comparing to the definition of wealth in equilibrium for the decentralized economy we have

$$c_0^i(\alpha) + \frac{\beta}{1+\gamma}\mathbb{E}\left[C_1^i(\alpha)\right] = \tau^i(\alpha) + h^i$$

- But, substituting the optimal consumption we have

$$c_0^i(\alpha) + \frac{\beta}{1+\gamma}\mathbb{E}\left[C_1^i(\alpha)\right] = \alpha^i\left(y_0 + \frac{\beta}{1+\gamma}\mathbb{E}\left[Y_1\right]\right) = \alpha^i h$$

  where $h = h^1 + h^2 = \mathbb{E}_0\left[y_0^1 + y_0^2 + \frac{\beta}{1+\gamma}(y_1^1 + y_1^2)\right]$

- Then $\tau^i(\alpha) + h^i = \alpha^i h$

Implicit Pareto weights in (decentralized) equilibrium

- *Determination of the weights by using the second welfare theorem*: in a market economy the intertemporal budget constraint of every agent should hold. Therefore $\tau^i(\alpha) = 0$.

$$
\begin{aligned}
\tau^1(\alpha) &= \alpha h - h^1 = 0 \\
\tau^2(\alpha) &= (1-\alpha)h - h^2 = 0
\end{aligned}
$$

- Then we get the Pareto-weights

$$\boxed{\alpha = \frac{h^1}{h},\ 1-\alpha = \frac{h^2}{h}}$$

- **Conclusion**: the **Pareto weights** in a centralized problem which implements the allocations of consumption for a competitive equilibrium, are **equal to the share of each consumer in the aggregate wealth**.

2. Optimal taxation

- If the government wants to introduce a tax policy implement such that it implements the optimal allocation, by using a non-distortionary, it can only do it (in an AD economy) by introducing a tax at time $t = 0$
- If we assume that it introduces a non-distortionary tax we can invert the previous relationship to find

$$\begin{aligned}
\tau^1 + h^1 &= \alpha h \\
\tau^2 + h^2 &= (1-\alpha) h
\end{aligned}$$

References

Brito (2014, chapter 7)