Universidade de Lisboa
Instituto Superior de Economia e Gestão
Departamento de Economia

Master in Economics
Growth Economics
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Exam **Re-sit exam** (Época de Recurso)
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Solutions: only analytical questions

Part 1 1(a) $\dot{k} = F(k) \equiv (1-c)\, y(k,s) - (\delta+n)\, k$ for $y = k^{\alpha} s^{\beta}$

1(b) Steady state (non-zero): $k^{ss} = \left(\frac{1-c}{\delta+n} s^{\beta}\right)^{\frac{1}{1-\alpha}}$, and as $\frac{\partial F}{\partial k}(k^{ss}) = -\lambda$ for $\lambda \equiv (1-\alpha)(\delta+n) > 0$, k^{ss} is asymptotically stable. Linearized solution $k(t) = k^{ss} + \big(k_0 - k^{ss}\big)e^{-\lambda t}$.

1(c) There is no long-run growth. The long run level of output is $y^{ss} = \left(\left(\frac{1-c}{\delta+n}\right)^{\alpha} s^{\beta}\right)^{\frac{1}{1-\alpha}}$ Level effect of an increase in s : $\frac{\partial y^{ss}}{\partial s} = \left(\frac{\beta}{1-\alpha}\right)\frac{y^{ss}}{s} > 0$.

2(a) Equation for S: in levels, $G = \tau Y(K,L,S) = \dot{S} + \delta S$, in per-capita terms is $\tau y(k,s) = \dot{s} + (\delta+n)s$. Equation for K: the macroeconomic equilibrium is, in levels, $Y = C + I + G$, or, equivalently $(1-c)Y(K,L,S) = \dot{K} + \delta K + \tau Y(K,L,S)$, and, in per-capita terms is the $(1-c-\tau)y(k,s) = \dot{k} + (\delta+n)k$.

2(b) Steady state (for $(k,s) \in \mathbb{R}^2_{++}$) is: $k^{ss} = \left(\frac{\tau^{\beta}(1-c-\tau)^{1-\beta}}{\delta+n}\right)^{\frac{1}{1-\alpha-\beta}}$ and $s^{ss} = \left(\frac{\tau^{\alpha}(1-c-\tau)^{1-\alpha}}{\delta+n}\right)^{\frac{1}{1-\alpha-\beta}}$. Drawing the phase diagram we find that it is asymptotically stable. Therefore there is no long-run growth.

2(c) As $y^{ss} = \left(\left(\frac{\tau}{\delta+n}\right)^{\beta}\left(\frac{1-c-\tau}{\delta+n}\right)^{\alpha}\right)^{\frac{1}{1-\alpha-\beta}}$, then the long-run level effects on output is

$$\frac{\partial y^{ss}}{\partial \tau} = \left(\frac{\beta(1-c) - (\alpha+\beta)\tau}{\tau(1-c-\tau)(1-\alpha-\beta)}\right) y^{ss}.$$

Then the long-run effect on y depends on the level of the tax:

$$\frac{\partial y^{ss}}{\partial \tau} \gtreqless 0 \text{ if } \tau \lesseqgtr \frac{\beta(1-c)}{\alpha+\beta} > 0$$

The effect on the ratio is positive $\frac{\partial}{\partial \tau}\left(\frac{s^{ss}}{k^{ss}}\right) = \frac{1-c}{(1-c-\tau)^2} > 0$.

Part 2 1(a) The detrended system is

$$\dot{n} = \Big(\eta\big(L - L_p(c,n)\big) - (\theta-1)\gamma\Big)n, \text{ for } L_p = c\, n^{\frac{1}{1-\theta}}$$
$$\dot{c} = \Big(r(c,n) - \rho - \gamma\Big)c, \text{ for } r = r_0 + \eta L_p(c,n)$$

and $r_0 = \eta\frac{2-\theta}{\theta-1}$. In order to obtain this we have to set $\gamma_n = (\theta-1)\,\gamma$.

1(b) The long run (endogenous) growth rate is

$$\bar{\gamma} = \frac{1}{\theta}\Big(\frac{\eta}{\theta - 1}L - \rho\Big),$$

and $\bar{L}_p = \bar{c}\,\bar{n}^{\frac{1}{1-\theta}} = \Big(\frac{\theta - 1}{\theta}\Big)\Big(\frac{\rho + \eta L}{\eta}\Big)$. The BGP for N and C is generated by

$$\bar{N}(t) = n_0 e^{(\theta-1)\bar{\gamma}t},\ \bar{C}(t) = \bar{L}_p n_0^{\frac{1}{\theta-1}}\, e^{\bar{\gamma}t},\ t \in [0, \infty)$$

1(c) The BGP for per-capita output is generated by

$$\bar{Y}(t) = \bar{y}\, e^{\bar{\gamma}\, t},\ \text{for}\ \bar{y} = \frac{1}{\theta}\Big(1 + \frac{(\theta - 1)^2\,(\eta L + \rho)}{\theta \eta L}\Big) n_0^{\frac{1}{\theta-1}},\ t \in [0, \infty).$$

2(a) The MHDS is

$$\begin{aligned}\dot{N} &= \eta\, N\,\Big(L - L_p(C, N)\Big),\ \text{for}\ L_p = C\, N^{\frac{1}{1-\theta}}\\ \dot{C} &= C\,\Big(\frac{\eta}{\theta - 1}L - \rho\Big)\end{aligned}$$

2(b) The detrended system is now

$$\begin{aligned}\dot{n} &= \Big(\eta\big(L - L_p(c, n)\big) - (\theta - 1)\gamma\Big)n,\ \text{for}\ L_p = c\, n^{\frac{1}{1-\theta}}\\ \dot{c} &= \Big(\frac{\eta}{\theta - 1}L - \rho - \gamma\Big)c,\ \text{for}\ r = r_0 + \eta L_p(c, n)\end{aligned}$$

Then the long run growth rate is

$$\bar{\gamma}^c = \frac{\eta}{\theta - 1}L - \rho > \bar{\gamma}$$

and the optimal BGP for N and C are generated by

$$\bar{N}(t) = n_0\, e^{(\theta-1)\bar{\gamma}^c\, t}\ ,\ \bar{C}(t) = \rho\Big(\frac{\theta - 1}{\eta}\Big) n_0^{\frac{1}{\theta-1}}\ e^{\bar{\gamma}^c\, t},\ t \in [0, \infty).$$

2(c) BGP for the optimal per capita GDP is generated by

$$\bar{Y}^c(t) = \bar{y}^c\, e^{\bar{\gamma}^c\, t},\ \text{for}\ \bar{y}^c = \frac{1}{\theta}\Big(1 + \frac{(\theta - 1)^2 \rho}{\eta L}\Big) n_0^{\frac{1}{\theta-1}},\ t \in [0, \infty).$$

Because $\theta > 1$ we have: $\bar{\gamma}^c > \bar{\gamma}$ but, if $\bar{\gamma}^c > 0$, $\bar{y}^c < \bar{y}$. This means that initially the labor allocated to production is higher in the decentralized economy, and the higher allocation to research in the optimal economy will generate a higher growth rate.