Foundations of Financial Economics
Two period GE: limited participation

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May 21, 2018

Topics

- Types of agents: participation in the risky asset market
- Endogenous market participation
- The equilibrium interest rate with changes in participation
- interest rate response to news

The model

Environment

- Two-period binomial tree with two states $s = L, H$
- Heterogeneity in the priors regarding the states of nature along a continuum $i \in [0, 1]$, but divided into two groups pessimists $i \in [0, \iota]$ giving more weight to $s = L$ and optimists $i \in [\iota, 1]$ giving more weight to $s = H$
- There are marginal agents with $i = \iota$ (very small group).
- Finance economy: there are two assets money and a risky asset with prices and payoffs

$$\mathbf{S} = (1, S_2),\ \mathbf{V} = \begin{pmatrix} 1 & v_{2L} \\ 1 & v_{2H} \end{pmatrix} \Rightarrow \mathbf{R} = (R_1, R_2) = \begin{pmatrix} 1 & R_{2L} \\ 1 & R_{2h} \end{pmatrix}$$

  where $R_{2,s} = v_{2s}/S_2$ for $s = L, H$
- We assume that $R_{2L} < 1 < R_{2H}$: L is a bad state and H is a good state.

The model

The agent i problem

- The problem for household of type $i \in [0,1]$

$$\max_{c_0^i, C_1^i} u(c_0^i) + \beta \mathbb{E}^i[u(C_1^i)]$$

where $\mathbb{E}^i[u(C_1^i)] = \sum_{s \in \{H,L\}} \pi_s^i u(c_{1s}^i)$, subject to

$$\begin{aligned} c_0^i + \theta_1^i + S_2\theta_2^i &= y_0^i + w^i \\ c_{1,s}^i &= \theta_1^i + v_{2s}\theta_2^i, \;\; s = L, H \\ \theta_0^i &\geq 0 \\ \theta_1^i &\geq 0 \end{aligned}$$

- where wealth of agent i

$$w^i = w_1^i + S_2 w_2^i$$

- Simplifying assumption $c_0^i = y_0^i$

Solving the generic agent's problem

- Lagrangean

$$\mathcal{L}^i = u(y_0^i) + \lambda_0^i(w^i - \theta_1^i - S_2\theta_2^i) + \\ + \sum_{s\in\{H,L\}} \beta\pi_s^i u(c_{1s}^i) + \lambda_{1s}^i(\theta_1^i + v_{2s}\theta_2^i - c_{1s}^i) + \\ + \mu_1^i\theta_1^i + \mu_2^i\theta_2^i$$

- Individual arbitrage conditions

$$\lambda_0^i = \beta\left(\sum_{s\in\{H,L\}} \pi_s^i u^{'}(c_{1s}^i)\right) + \mu_1^i$$

$$S_2\lambda_0^i = \beta\left(\sum_{s\in\{H,L\}} \pi_s^i u^{'}(c_{1s}^i)v_{2s}\right) + \mu_2^i$$

- Complementary slackness conditions

$$\mu_1^i\theta_1^i = 0,\ \mu_1^i \geq 0,\ \theta_1^i \geq 0$$

$$\mu_2^i\theta_2^i = 0,\ \mu_2^i \geq 0,\ \theta_2^i \geq 0$$

Behavior of agent of type I:

- Agents of type I sell their initial stock of the risky asset and invests in money: $\theta_1^I > 0$ and $\theta_2^I = 0$
- Then
$$\theta_1^I = w^I = w_1^I + S_2 w_2^I$$
$$c_{1s}^I = w^I$$
is state-independent
- From complementary slackness: $\mu_1^I = 0$ and $\mu_2^I > 0$. Then
$$\lambda_0^I = \beta\left(\sum_{s\in\{H,L\}} \pi_s^I u^{'}(c_{1s}^I)\right) > \beta\left(\sum_{s\in\{H,L\}} \pi_s^I u^{'}(c_{1s}^I) R_{2s}\right)$$
Then $\mathbb{E}^I[u^{'}(C_1^I)] > \mathbb{E}^I[u^{'}(C_1^I)R_2]$
- Equivalently he has a risk-neutral probability distribution such that
$$\boxed{\mathbb{E}^{I_u}[R_2] < 1}$$
agent I invests in the risk-free asset because **he finds** its anticipated return 1 higher than the risky asset.

Behavior of agent of type II:

- Agents of type *II* sell their initial stock of money and invest in risky asset: $\theta_1^{II} = 0$ and $\theta_2^{II} > 0$
- Then

$$\theta_2^{II} = \frac{w^{II}}{S_2} = \frac{w_1^{II} + S_2 w_2^{II}}{S_2}$$
$$c_{1s}^{II} = v_{2s} \frac{w^{II}}{S_2} = \frac{w^{II}}{R_{2s}}$$

  is state-dependent

- From complementary slackness: $\mu_1^{II} > 0$ and $\mu_2^{II} = 0$. Then

$$\lambda_0^{II} = \beta \left(\sum_{s \in \{H,L\}} \pi_s^{II} u'(c_{1s}^{II}) R_{2s} \right) > \beta \left(\sum_{s \in \{H,L\}} \pi_s^{II} u'(c_{1s}^{II}) \right)$$

  Then $\mathbb{E}^{II}[u'(C_1^{II})] < \mathbb{E}^{II}[u'(C_1^{II}) R_2]$.

- Then

$$\boxed{\mathbb{E}^{II_u}[R_2] > 1}$$

  he finds the risky asset better.

Marginal agent

- Agents of type I prefer holding money to holding the risky asset because
$$\mathbb{E}^{I_u}[R_2] < 1$$

- Agents of type II prefer holding the risky asset rather than money because
$$\mathbb{E}^{II_u}[R_2] > 1$$

- There should be a marginal agent (with wealth weight of zero) from whom

$$\mathbb{E}^{\iota}[R_2] = 1 \Leftrightarrow \boxed{S_2 = \pi^{\iota} v_{2L} + (1 - \pi^{\iota}) v_{2H}} \tag{1}$$

Equilibrium in the asset markets

- Generic equilibrium conditions

$$\iota\theta_1^I + (1-\iota)\theta_1^{II} = w_1$$
$$\iota\theta_2^I + (1-\iota)\theta_2^{II} = S_2 w_2$$

where $w_j = w_j^I + w_j^{II}$ is the aggregate stock of total of asset $j = 1, 2$ and ι is the proportion in the population (of size equal to 1) of agents of type I: **non-investors in the risky asset**

- Using the previous demand conditions

$$\iota\theta_1^I = \iota w^I = w_1$$
$$(1-\iota)\theta_2^{II} = (1-\iota) w^{II} = S_2 w_2$$

(remember that $\theta_1^{II} = \theta_2^I = 0$ and $\theta_1^I = w^I$ and $\theta_2^{II} = w^{II}$)

Equilibrium in the asset markets

- Assumption: homogeneity in the distribution of wealth, that is $w^I = w^{II} = \bar{w}$
- Then the **equilibrium price for the risky asset** is

$$\boxed{S_2^* = S_2(\iota) = \left(\frac{1-\iota}{\iota}\right)\frac{w_1}{w_2}} \tag{2}$$

- The asset price decreases with ι because

$$\frac{\partial S_2}{\partial \iota} = -\frac{w_1}{\iota^2 w_2} < 0$$

Equilibrium distribution of agents

- Assumption: the probability distribution of the marginal investor, π^{ι}, is a function of their weight in the total population ι. For simplicity let $\pi^{\iota} = \iota$.
- Then, from equations (1) and (2), the equilibrium value $\iota^* = \{\iota \in (0,1) : \mathcal{I}(\iota) = 0\}$ where

$$\mathcal{I}(\iota) \equiv (1-\iota)w_1 - (\iota v_{2L} + (1-\iota)v_{2H})\,\iota w_2$$

- We prove next that there is one unique value $\iota^* \in (0,1)$ is

$$\iota^* = \frac{v_{2H}w_2 + w_1}{2(v_{2H} - v_{2L})w_2} - \left[\left(\frac{v_{2H}w_2 - w_1}{2(v_{2H} - v_{2L})w_2}\right)^2 + \frac{4v_{2L}w_1w_2}{4(v_{2H} - v_{2L})^2 w_2^2}\right]^{\frac{1}{2}}$$

- Observe that $\iota^* = \iota^*(v_{2L}, v_{2H}, w_1, w_2)$ is a function of the payoffs and of the aggregate stocks of the two assets

Equilibrium distribution of agents

Proof

- **Proof that $\iota^* \in (0,1)$ exists and is unique.**
  Function $\mathcal{I}(\iota)$ is convex in ι (U-shaped) and therefore there can be zero, one or two values of ι such that $\mathcal{I}(\iota) = 0$ for $-\infty < \iota < \infty$. However, the domain of ι is $(0,1)$. It is easy to see that $\mathcal{I}(0) = w_1 > 0$, $\mathcal{I}^{'}(0) = -(w_1 + v_{2H}w_2) < 0$ and $\mathcal{I}(1) = -(w_1 + v_{2L}w_2) < 0$: therefore, in the interval $(0,1)$ there is one and only one value of ι, ι^* such that $\mathcal{I}(\iota) = 0$. Because the function is convex it has two points $0 < \iota_- < 1 < \iota_+$ such that $\mathcal{I}(\iota) = 0$ and the first one is the solution we are looking for.

Equilibrium distribution of agents

Properties

- We know that ι^* depends on the payoffs in the good and bad states, v_{2H} and v_{2L} (and on the aggregate relative endowment of the two assets)
- It can be proved that the proportion of non-investors in the risky asset market (and therefore of risky investors) responds

$$\boxed{\frac{\partial \iota^*}{\partial v_{2s}} < 0, \text{ for } s = H, L}$$

- Interpretation: an increase in the payoff of the risky asset, for any state of nature, will decrease the proportion of agents that invest only in the risk-free asset. That is, it increases the participation in the risky asset market.
- We also have:

$$\frac{\partial \iota^*}{\partial w_1} > 0, \ \frac{\partial \iota^*}{\partial w_2} < 0$$

Equilibrium distribution of agents

Proof

- **Proof of the sign relationships for $\dfrac{\partial \iota^*}{\partial v_{2s}}$**

  We know that $\mathcal{I}(\iota, v_{2H}, v_{2L}) = 0$. Therefore, the response of ι to the payoffs is

$$\frac{\partial \iota^*}{\partial v_{2s}} = - \left. \frac{\mathcal{I}_{v_{2s}}}{\mathcal{I}_\iota} \right|_{\iota = \iota^*}, \; s = L, H$$

- Where $\mathcal{I}_{v_{2H}} = -\iota^*(1-\iota^*)w_2 < 0$ and $\mathcal{I}_{v_{2L}} = -(\iota^*)^2 w_2 < 0$ and

$$\mathcal{I}_\iota = 2(v_{2H} - v_{2L})w_2 \left(\iota^* - \frac{w_1 + v_{2H}w_2}{2(v_{2H} - v_{2L})w_2} \right) < 0$$

  because $0 < \iota^* < \dfrac{w_1 + v_{2H}w_2}{2(v_{2H} - v_{2L})w_2}$

Equilibrium rate of return for the risky asset

- Equilibrium rate of return of the risky asset is

$$R^*_{2,s} = \frac{v_{2s}}{S_2(v_{2L}, v_{2H}, .)}, s = L, H \tag{3}$$

- Because: $S_2 = S_2(\iota^*(v_{2L}, v_{2H}, .))$. Because

$$\frac{\partial S_2}{\partial \iota} < 0, \ \frac{\partial \iota}{\partial v_{2s}} < 0, \ s = H, L$$

  then S_2 increases with both v_{2H} and v_{2L}.
- This means that if there is an increase in v_{2s} generates two effects on R_{2s}:
  - a direct positive effect
  - a negative indirect effect, because the prices increases as a result of the change in the participation in the risky asset market
  - The final effect is ambiguous.

Equilibrium rate of return for the risky asset

- For the case in which there is no change in participation we have

$$\frac{d\bar{R}_{2s}}{dv_{2s}} = \frac{1}{\bar{S}_2} > 0, \frac{d\bar{R}_{2s'}}{dv_{2s}} = 0, \ s \neq s^{'} = H, L$$

- The rate of return outcome for a particular state of nature only changes when the payoff outcome for the same state of nature changes.

Equilibrium rate of return for the risky asset

- When there is a change in participation we have

$$\frac{\partial R_{2s}}{\partial v_{2s}} = \frac{1 - \epsilon_{\iota}^{S_2}\epsilon_{v_{2s}}^{\iota}}{S_2(\iota^*)}, \ \frac{\partial R_{2s'}}{\partial v_{2s}} = -\frac{v_{2s'}}{v_{2s}}\frac{\epsilon_{\iota}^{S_2}\epsilon_{v_{2s}}^{\iota}}{S_2(\iota^*)}, \ s \neq s^{'} = L, H$$

  where

  - the elasticity of S_2 to ι is

$$\epsilon_{\iota}^{S_2} = \frac{\partial S_2}{\partial \iota}\frac{\iota}{S_2} - \frac{1}{1-\iota^*} < -1$$

  - the elasticity of ι to v_{2s} is

$$\epsilon_{v_{2s}}^{\iota} = \frac{\partial \iota^*}{\partial v_{2s}}\frac{v_{2s}}{\iota}, \ s = H, L$$

- The rate of return outcome for a particular state of nature changes with with payoff changes for any state of nature due to the change in participation.

Equilibrium rate of return for the risky asset

- For a change in v_{2H} we have a change in the distribution of R_2
  - if the good state occurs
$$\frac{\partial \bar{R}_{2H}}{\partial v_{2H}} = -\frac{1}{S_2(\iota^*)}\left(\frac{2(v_{2H}-v_{2L})w_2\iota^*(1-\iota_+)}{\mathcal{I}_\iota}\right) > 0$$
  - if the bad state state occurs
$$\frac{\partial \bar{R}_{2L}}{\partial v_{2H}} = -\frac{1}{S_2(\iota^*)}\frac{v_{2L}}{v_{2H}}\epsilon_\iota^{S_2}\epsilon_{v_{2H}}^\iota < 0$$
- For a change in v_{2L} we have a change in the distribution of R_2
  - if the good state occurs
$$\frac{\partial \bar{R}_{2L}}{\partial v_{2L}} = -\frac{w_2}{\mathcal{I}_\iota} > 0$$
  - if the bad state state occurs
$$\frac{\partial \bar{R}_{2H}}{\partial v_{2L}} = -\frac{1}{S_2(\iota^*)}\frac{v_{2H}}{v_{2L}}\epsilon_\iota^{S_2}\epsilon_{v_{2L}}^\iota < 0$$

Equilibrium rate of return for the risky asset

Conclusion

- If there is a change in the participation, then a change in any of the anticipated outcomes in the payoff distribution will change the rate of return whatever the state of nature that occurs at time $t = 1$, but will do it in a state-dependent way:
  - a positive news regarding the good state v_{2H}, $\Delta v_{2H} > 0$, generates an increase in the rate of return if the good state occurs and a decrease in the rate of return if the bad state occurs:

$$\Delta v_{2H} > 0 \Rightarrow \Delta R_{2L} < 0 < \Delta R_{2H}$$

  - a negative news regarding the bad state, v.g., $\Delta v_{2L} < 0$, there is an increase in the rate of return if the good state occurs and a reduction if the bad state occurs

$$\Delta v_{2L} < 0 \Rightarrow \Delta R_{2L} < 0 < \Delta R_{2H}$$

References

This lecture is adapted from Geanakoplos (2010) and Fostel and Geanakoplos (2014).

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