

The Ramsey growth model

Paulo Brito
`pbrito@iseg.ulisboa.pt`

8.3.2018

A short biography

- ▶ Frank Ramsey (see https://en.wikipedia.org/wiki/Frank_P._Ramsey) made several important contributions in his short life (he died at 26) one of them Ramsey (1928)
- ▶ His contribution was only fully recognized in the early 60's (Cass (1965), Koopmans (1965)) as presenting a rigorous alternative to the ad-hoc aspects of the Solow (1956) model (now we call it **exogenous growth theory**)
- ▶ It was rejoined again in the middle of the 1980's which saw the onset of **endogenous growth theory**
- ▶ It is also the founding rock of the DGE (dynamic general equilibrium theory) of macroeconomics

The Ramsey model

The basic idea

- ▶ output is a function of the capital stock and can be used for investment or for consumption
- ▶ consumption is a good thing but too much present consumption can reduce future consumption
- ▶ if people sacrifice present consumption to save and increase the capital stock they improve their prospects for consumption in the future;
- ▶ **savings** is determined by balancing optimally the two effects
- ▶ that is, by an **arbitrage between present and future consumption**
- ▶ for solving the problem we need to solve an **intertemporal optimization problem**

The Ramsey model

Assumptions

- ▶ Production:
 - ▶ closed economy producing a single composite good
 - ▶ production uses two factors: labor and physical capital
 - ▶ production technology: neoclassical (increasing, concave, Inada, CRTS)
- ▶ Reproducible factor:
 - ▶ physical capital (machines)
- ▶ Population:
 - ▶ exogenous and constant

The Ramsey model

Assumptions: cont

- ▶ Households: optimizing behavior
 - ▶ maximize an intertemporal utility functional with consumption as the control variable
 - ▶ subject to a budget constraint
 - ▶ labor is supplied inelastically
 - ▶ they have perfect foresight
- ▶ Equilibrium is Pareto optimal, therefore it is equivalent to a central planner problem

Ramsey model

The model: production technology

- ▶ Technology:

$$Y(t) = F(A, K(t), L(t)) = AK(t)^\alpha N(t)^{1-\alpha}, \quad 0 < \alpha < 1$$

where: A productivity, K stock of capital, N population

- ▶ Preferences: for the representative agent
 - ▶ the intertemporal utility functional is

$$V((c(t))_{t \in [0, \infty)}) = \int_0^\infty u(c(t)) e^{-\rho t} dt$$

- ▶ $\rho > 0$ is the rate of time preference
- ▶ the instantaneous utility function is

$$u(c) = \begin{cases} \frac{c^{1-\theta} - 1}{1-\theta}, & \text{if } \theta \in (0, \infty) \setminus \{1\} \\ \ln(c), & \text{if } \theta = 1 \end{cases}$$

where $1/\theta$ is the elasticity of intertemporal substitution

Ramsey model

The centralized version

- ▶ The central planner solves the problem

$$\max_{(c)_{t \geq 0}} \int_0^{\infty} \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

- ▶ subject to

$$\dot{k} = Ak(t)^\alpha - c(t) - \delta k(t),$$

- ▶ $k(0) = k_0$ given
- ▶ $\lim_{t \rightarrow \infty} h(t)k(t) \geq 0$ physical capital is asymptotically bounded

Ramsey model

Solving by using the Pontryagin's max principle

- The current-value Hamiltonian is

$$H(c, k, q) = \frac{c^{1-\theta} - 1}{1-\theta} + q(Ak^\alpha - c - \delta k)$$

- the optimality conditions are

$$\frac{\partial H}{\partial c} = 0 \Leftrightarrow c^{-\theta}(t) = q(t), \quad t \in [0, \infty)$$

$$\dot{q} = \rho q - \frac{\partial H}{\partial k} \Leftrightarrow \dot{q} = q(t) (\rho + \delta - \alpha Ak(t)^{\alpha-1}), \quad t \in [0, \infty)$$

$$\lim_{t \rightarrow \infty} q(t)k(t)e^{-\rho t} = 0$$

- the admissibility conditions

$$\begin{aligned}\dot{k} &= Ak(t)^\alpha - c(t) - \delta k(t), \quad t \in [0, \infty) \\ k(0) &= k_0, \quad t = 0\end{aligned}$$

Ramsey model

The modified Hamiltonian dynamic system

- ▶ An optimum path $(c^*(t), k^*(t))_{t \in [0, +\infty)}$ is the solution of the (MHDS)

$$\dot{c} = \frac{c}{\theta} (r(k(t)) - \rho - \delta)$$

$$\dot{k} = Ak(t)^\alpha - c(t) - \delta k(t)$$

$$0 = \lim_{t \rightarrow \infty} c(t)^{-\theta} k(t) e^{-\rho t}$$

$$k(0) = k_0 \text{ given}$$

- ▶ with the (gross) rate of return for capital

$$r(k) = \alpha A k^{\alpha-1}$$

The Ramsey model

Steady states

- ▶ they are fixed points of the system

$$\begin{aligned}\frac{c^*}{\theta} (r(k^*) - \rho) &= 0, \\ c^* &= A(k^*)^\alpha - \delta k^*.\end{aligned}$$

- ▶ there are three steady states

$$(c^*, k^*) = \{(0, 0), (0, (A/\delta)^{1/(1-\alpha)}), (\bar{c}, \bar{k})\}$$

for

$$\bar{k} = \left(\frac{\alpha A}{\delta + \rho} \right)^{1/(1-\alpha)}, \quad \bar{c} = \frac{\rho + \delta(1 - \alpha)}{\alpha} \bar{k}$$

- ▶ the last one verifies the transversality condition (the second not: check)
- ▶ then steady state GDP levels

$$\bar{y} = A\bar{k}^\alpha = \left[A \left(\frac{\alpha}{\delta + \rho} \right)^\alpha \right]^{1/(1-\alpha)}.$$

The Ramsey model

Solving the Ramsey model

- ▶ In general the Ramsey **does not have an explicit solution** (also called exact or closed form)
- ▶ We can only find an exact solution for the case $\theta = \alpha$ (which is counterfactual)
- ▶ Analytical methods for finding the solution:
 - ▶ get a **linear approximate** system and force the solution to converge to the steady state;
 - ▶ use exact methods by **transforming** the MHDS into a known differential equation (only for that very special case)
- ▶ In all cases, **it is always a good idea to build the phase diagram**

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- ▶ there is no explicit solution
- ▶ we study dynamics of the approximate system in a neighbourhood of (\bar{c}, \bar{k})
- ▶ the linearised MHDS is

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} 0 & \bar{c}r'(\bar{k})/\theta \\ -1 & \rho \end{pmatrix} \begin{pmatrix} c(t) - \bar{c} \\ k(t) - \bar{k} \end{pmatrix}$$

- ▶ where $r' = (\alpha - 1)\alpha Ak^{\alpha-2}|_{k=\bar{k}} = -\frac{(1-\alpha)\rho}{\bar{k}} < 0$
- ▶ and $\bar{c}r'(\bar{k})/\theta = -(1-\alpha)\frac{(\rho+\delta(1-\alpha))(\rho+\delta)}{\alpha\theta} = -d < 0$

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- ▶ the system is of type $\dot{x} = Jx$
- ▶ where the Jacobian matrix is

$$\mathbf{J} = \begin{pmatrix} 0 & -d \\ -1 & \rho \end{pmatrix}$$

- ▶ the solution is of type

$$x(t) = h_s \mathbf{V}^s e^{\lambda_s t} + h_u \mathbf{V}^u e^{\lambda_u t}$$

- ▶ where λ_j are the eigenvalues and \mathbf{V}^j are the associated eigenvectors of J and h_s are arbitrary constants

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- ▶ the eigenvalues of \mathbf{J} are

$$\lambda_u = \frac{\rho}{2} + \left[\left(\frac{\rho}{2} \right)^2 + d \right]^{1/2} > \rho > 0$$

$$\lambda_s = \frac{\rho}{2} - \left[\left(\frac{\rho}{2} \right)^2 + d \right]^{1/2} < 0$$

- ▶ satisfying $\lambda_s + \lambda_u = \rho > 0$, $\lambda_s \lambda_u = -d$
- ▶ then (\bar{c}, \bar{k}) is a **saddle-point**

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- ▶ the eigenvectors are determined as follows
- ▶ \mathbf{V}^s solves the homogeneous system

$$(\mathbf{J} - \lambda_s \mathbf{I}_2) \mathbf{V}^s = \mathbf{0}$$

- ▶ that is

$$\begin{pmatrix} -\lambda_s & -d \\ -1 & \rho - \lambda_s \end{pmatrix} \begin{pmatrix} \mathbf{V}_1^s \\ \mathbf{V}_2^s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- ▶ the members of vector \mathbf{V}^s should satisfy

$$\frac{\mathbf{V}_1^s}{\mathbf{V}_2^s} = -\frac{d}{\lambda_s} = \lambda_u$$

(because $\rho - \lambda_s = \lambda_u$)

- ▶ for \mathbf{V}^u we find (prove this)

$$\frac{\mathbf{V}_1^u}{\mathbf{V}_2^u} = -\frac{d}{\lambda_u} = \lambda_s$$

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- ▶ the general solution becomes

$$\begin{pmatrix} c(t) - \bar{c} \\ k(t) - \bar{k} \end{pmatrix} = h_u \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix} e^{\lambda_s t} + h_s \begin{pmatrix} \lambda_s \\ 1 \end{pmatrix} e^{\lambda_u t}$$

- ▶ but as $\lim_{t \rightarrow \infty} e^{\lambda_u t} = \infty$ we set $h_u = 0$
- ▶ and determine h_s such that $k(0) = k_0$

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- the approximate solution is, for $t \in [0, \infty)$

$$\begin{aligned}c(t) &= \bar{c} + \lambda_u(k_0 - \bar{k})e^{\lambda_s t}, \\k(t) &= \bar{k} + (k_0 - \bar{k})e^{\lambda_s t}.\end{aligned}$$

Ramsey model

Case $\theta \neq \alpha$: approximate solution

- ▶ at $t = 0$ we have

$$\begin{pmatrix} c(0) \\ k(0) \end{pmatrix} = \begin{pmatrix} \bar{c} + \lambda_u(k_0 - \bar{k}) \\ k_0 \end{pmatrix}$$

observe that λ_u gives the variation of consumption as $c(0) - \bar{c} = \lambda_u(k_0 - \bar{k})$ and the initial consumption is determined from **future data** (\bar{c} and \bar{k})

- ▶ asymptotically (i.e., in the long run)

$$\lim_{t \rightarrow \infty} \begin{pmatrix} c(t) \\ k(t) \end{pmatrix} = \begin{pmatrix} \bar{c} \\ \bar{k} \end{pmatrix} = \begin{pmatrix} \frac{\rho + \delta(1-\alpha)}{\alpha} \bar{k} \\ \bar{k} \end{pmatrix}$$

the solution converges to the steady state (this means that the transversality condition is satisfied)

- ▶ the saddle path dynamics implies that the solution is unique

Ramsey model

Case $\theta = \alpha$: exact solution

- ▶ there is an explicit solution:

$$c(t) = \frac{\delta + \rho(1 - \alpha)}{\alpha} k(t),$$
$$r(t) = \frac{r(0)(\delta + \rho)}{r(0) + (\delta + \rho - r(0))e^{-[(1-\alpha)(\delta+\rho)/\alpha]t}},$$

with $k(t) = (\alpha A / r(t))^{1/(1-\alpha)}$

- ▶ given $k(0)$ we get explicitly

$$c(0) = \frac{\delta + \rho(1 - \alpha)}{\alpha} k(0)$$

- ▶ convergences asymptotically to the steady state,

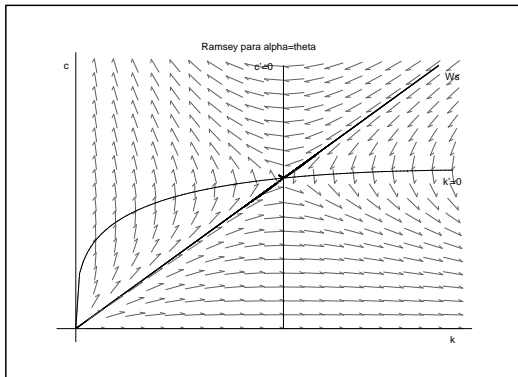
$$\lim_{t \rightarrow \infty} c(t) = \bar{c}$$

$$\lim_{t \rightarrow \infty} r(t) = \bar{r} = \delta + \rho$$

$$\lim_{t \rightarrow \infty} k(t) = \bar{k}$$

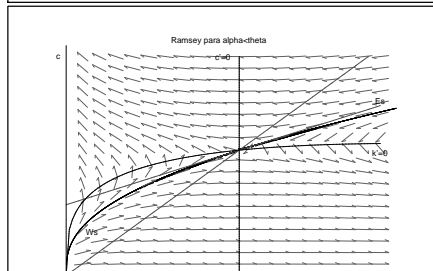
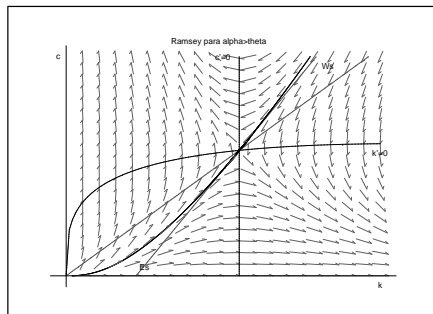
Ramsey model

Case $\theta = \alpha$: phase diagram



Ramsey model

Case $\theta \neq \alpha$: phase diagrams for $\theta < \alpha$ and $\theta > \alpha$



Ramsey model

Properties of the solution paths

1. if $k(0) \neq \bar{k}$ then $\lim_{t \rightarrow \infty} k(t) = \bar{k}$,
2. given any initial value for k , $k(0)$, there is only a value for c , $c(0)$ which is determined endogenously such that $\lim_{t \rightarrow \infty} c(t) = \bar{c}$;
3. **the solution is determinate, i.e., unique**: this is the only solution for the ode system such that the transversality condition holds;
4. the saddle path is asymptotically tangent to the straight line

$$c(t) - \bar{c} = \lambda_u(k(t) - \bar{k})$$

5. the **approximate** per-capita output path is

$$y(t) = \left[\bar{y}^{1/\alpha} + (y(0)^{1/\alpha} - \bar{y}^{1/\alpha}) e^{\lambda_s t} \right]^\alpha$$

the model only displays transitional dynamics as $\lambda_s < 0$.

Ramsey model

Case $\theta = \alpha$: GDP exact dynamics

- ▶ the **exact** per-capita output path is

$$y(t) = A \left[\frac{\alpha A k(0)^{\alpha-1} (\delta + \rho)}{\alpha A k(0)^{\alpha-1} + (\delta + \rho - \alpha A k(0)^{\alpha-1}) e^{-[(1-\alpha)(\delta+\rho)/\alpha]t}} \right]^{\alpha},$$

- ▶ the solution converges asymptotically to the steady state

$$\lim_{t \rightarrow \infty} y(t) = \bar{y} = \left[A \left(\frac{\alpha}{\delta + \rho} \right)^{\alpha} \right]^{1/(1-\alpha)}$$

The Ramsey model

Growth implications

- ▶ there is no long-run growth $\bar{g} = 0$
- ▶ the long-run level \bar{y} depends on $(A, \delta, \rho, \alpha)$: productivity, the rate of depreciation, the rate of time preference (impatience) and on the income shares;
- ▶ there is transitional dynamics: the speed and the pattern of convergence depends on the relationship between the capital share, α , in income and the intertemporal elasticity of substitution θ

The Neoclassical DGE model

Assumption

- ▶ Representative household: has initial financial wealth b and gets financial income (rb), and decides on consumption (c) and savings (\dot{b}) ;
- ▶ Households own firms with physical capital (k) which is only financed by bonds: thus $b = k$. Firms transform capital and labor into output (y)
- ▶ There are accounting restrictions.
- ▶ All markets are competitive
- ▶ Other assumptions: infinite-lived households with isoelastic utility and Cobb-Douglas production, function and no frictions.

The Neoclassical DGE model

- Household's problem

$$\max_{c(.)} \int_0^{\infty} \frac{c(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{b} = r(t)b(t) + w(t) - c(t), \quad t \geq 0$$

$$b(0) = b_0$$

$$\lim_{t \rightarrow \infty} e^{-\int_t^{\infty} r(s) ds} \geq 0$$

where b =bonds, w =wage

- Optimality conditions

$$\dot{c} = c(t) \frac{(r(t) - \rho)}{\theta}$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} c(t)^{-\theta} b(t) = 0$$

The Neoclassical DGE model

- Firm's problem (price taker in all the markets)

$$\max_i \int_0^{\infty} (Ak(t)^{\alpha} - w(t) - i(t)) e^{-\int_t^{\infty} r(s) ds} dt$$

subject to

$$\dot{k} = i - \delta k$$

$$k(0) = k_0$$

- F.o.c

$$r(t) = \alpha Ak(t)^{\alpha-1} - \delta$$

The Neoclassical DGE model

- ▶ Micro-macro constraints:
 - ▶ Accounting identity $b(t) = k(t)$,
 - ▶ Then $\dot{b}(t) = \dot{k}(t)$,
 - ▶ Wage determination $w = y - rk = (1 - \alpha)Ak^\alpha$,
- ▶ Then get the same dynamic system as in the Ramsey model

$$\begin{aligned}\dot{c} &= c(t) \frac{(r(t) - \rho)}{\theta} \\ \dot{k} &= Ak(t)^\alpha - c(t) - \delta k(t)\end{aligned}$$

- ▶ We say that the **equilibrium is Pareto efficient**)

References

- ▶ Ramsey (1928), Cass (1965) Koopmans (1965)
- ▶ (Acemoglu, 2009, ch. 8) , (Aghion and Howitt, 2009, ch. 1), (Aghion and Howitt, 2009, ch. 1), (Barro and Sala-i-Martin, 2004, ch. 2)

Daron Acemoglu. *Introduction to Modern Economic Growth*. Princeton University Press, 2009.

Philippe Aghion and Peter Howitt. *The Economics of Growth*. MIT Press, 2009.

Robert J. Barro and Xavier Sala-i-Martin. *Economic Growth*. MIT Press, 2nd edition, 2004.

D. Cass. Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies*, 32:233–40, 1965.

T. Koopmans. On the concept of optimal economic growth. In *The Econometric Approach to Development Planning*. Pontificiae Acad. Sci., North-Holland, 1965.

Frank P. Ramsey. A mathematical theory of saving. *Economic Journal*, 38(152):543–559, December 1928.