Advanced Mathematical Economics

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Lecture 1
15.9.2021

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Chapter 1

Linear ODE: the scalar case

1.1 Introduction

In this chapter we are interested in finding and characterizing a **distribution** or a **deterministic process** of type $\Big(y(x)\Big)_{\mathrm{X}}$ using a **variational approach**, where $y : \mathrm{X} \to \mathrm{Y} \subseteq \mathbb{R}$, is a mapping $y = y(x)$, where x is a real number belonging to the domain $\mathrm{X} \subseteq \mathbb{R}$, and y is also a real number belonging to the set $\mathrm{Y} \subseteq \mathbb{R}$.

The **variational approach** means that the law governing the process is characterized a functional equation containing the derivative of $y(x)$,

$$y'(x) \equiv \frac{dy(x)}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{x}.$$

Therefore, ODE models phenomena which we describe by the existence of a local interaction.

A **scalar ordinary differential equation** (ODE) is a functional equation, defined over function $y(x)$, of the form

$$F(y'(x), y(x), x) = 0, \; x \in \mathrm{X} \subseteq \mathbb{R} \tag{1.1}$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ is a known continuous and differentiable function.

Solving an ODE means finding function $y(x)$ which solves equation (1.1). That is, while function $F(\cdot)$ is known, function $y(\cdot)$ is unknown. The existence, uniqueness, and the properties of the solutions depend on the nature of function $F(\cdot)$. A well developed theory on the characterization of solutions is provided by the case in which the independent variable is time. In this case, the unknow function is $y : \mathrm{T} \subseteq \mathbb{R} \to \mathbb{R}$ and the derivative is denoted by the Newtonian notation $\dot{y} = \dfrac{dy(t)}{dt}$ and the ODE is

$$F(\dot{y}, y(t), t; \varphi) = 0$$

where $\varphi \in \Phi \subseteq \mathbb{R}^m$ is a vector of parameters. The qualitative theory on the change in behavior of the solution, that is of the dynamic process $\Big(y(t)\Big)_{t \in \mathrm{T}}$ for different values of the parameter φ is called bifurcation theory.

In general a model (or a problem) involving an ODE takes the form

$$\begin{cases} F(y'(x), y(x), x) = 0 \\ \mathsf{F}[y] \ = \text{constant} \end{cases}$$

where $\mathsf{F}[y]$ is a functional over $y \in \mathcal{Y}$, $\mathsf{F} : \mathcal{F} \rightarrow \mathbb{R}$, where $\mathcal{Y}$ is the set of functions $y(\cdot)$.

To distinguish the solution of an ODE from the solution to a model (or a problem) involving an ODE we call **general solution** to the solution of an ODE and **particular solution** to the solution of the latter. Although linear scalar ODE have one unique solution, models (or problems) involving them may not have solutions (if the constraint cannot be satisfied by the solution of the ODE). That is, the fact that a general solution exists and is unique does not imply that the particular solution exists.

The characterization of the solution of a model featuring an ODE has a close relationship to the type of side conditions which are assumed. For instance, in models in which the independent variable is not time the constraint takes the form $\int_{\mathrm{X}} \beta(y(x), x)dx = 0$. In general we have moment conditions, and we are interested in some global characteristics of the solution curve.

In models in which the independent variable is time the constraint sometimes takes to form $\int_{\mathrm{T}} \delta(t - t_0)y(t)dx = y_{t_0}$ where $\delta(\cdot)$ is Dirac's delta generalized function. In this case we fix the value of the function for a particular value of the independent variable time, and want to characterize the evolution across Y across time. This leads to the stability and bifurcation analysis of the model: stability regarding some fixed points of Y, existence of invariant sets, dependence of the solution on parameters.

Because solutions of an ODE depend crucially on function $F(\cdot)$ we classify ODE's according to the its form.

Classification of scalar ODE:

- an ODE is **quasi-linear** when $F(\cdot)$ is a linear function of $y'(x)$. A general form of this equation is
$$A(y(x), x)\, y'(x) + B(y(x), x) = 0;$$
- an ODE is **semi-linear** when $F(\cdot)$ is a linear function of $y'(x)$ and its coefficient is independent of $y(x)$. A general form of this equation is
$$A(x)\, y'(x) + B(y(x), x) = 0;$$
- an ODE is **linear** when $F(\cdot)$ is a linear function of both $y'(x)$ and $y(x)$. A general form of this equation is
$$A(x)\, y'(x) + B\, y(x) + C(x) = 0;$$

An ODE is **non-linear** if the way $y'(x)$ enters equation (1.1) is non-linear: example $(y'(x))^2 = f(y(x), x)$.

Furthermore, we can distinguish between

- autonomous from non-autonomous to semi-linear or linear ODE's when the coefficient functions are constants from the case in which they are dependent on x;
- homogeneous/non-homogeneous when $F(\cdot)$ is a homogeneous function of y.

However, if a linear or a semi-linear equation has a coefficient function $A(\cdot)$ such that $A(x) \neq 0$ for every $x \in \mathrm{X}$, then we can write the ODE in the **normal form**

$$y'(x) = f(y(x), x),\ x \in \mathrm{X} \subseteq \mathbb{R}. \tag{1.2}$$

In most of the dynamic systems literature, in which time is the independent variable, this ODE is called non-linear when $f(y, \cdot)$ is a non-linear function. This is the sense in which we will use the classification of non-linear equation from now on.

Consider a particular point $x_0 \in \mathrm{X} \subseteq \mathbb{R}$, because $\mathbb{R}$ has an order structure. We can classify further the ODE in the integral from as a **forward** ODE if

$$y(x) = y(x_0) + \int_{x_0}^{x} f(y(s), s)\, ds$$

or as a **backward** ODE if

$$y(x^0) = y(x) + \int_{x}^{x^0} f(y(s), s)\, ds.$$

In economics, when the independent variable is time, this distinction is very important as it distinguishes the dynamics generated from past events, usually related to stocks or quantities, from anticipated events, usually related to prices or returns.

The **scalar ODE in normal form** that interests depends on a parameter or a vector of parameters $\varphi \in \Phi \subseteq \mathbb{R}^m$ and takes the form

$$y'(x) \equiv \frac{dy(x)}{dx} = f(y(x), x, \varphi),\ x \in \mathbb{R}.$$

Most of the times we are interested in equations of the form $y'(x) = f(y(x), x, \varphi)$, where $\varphi \in \mathbb{R}^m$ is a vector of parameters and function $f(\cdot)$ is known.

The known function $f(\cdot)$ constrains the family of functions to which function $y(x)$ belongs, which we call $\mathcal{Y}$. If function$f(\cdot)$ is continuous and differentiable, then the solution of solutions of the differential equation are elements of the space of continuous and differentiable functions.

Solving a scalar ODE means finding a function, say $\phi(x)$, such that $\phi : \mathrm{X} \to \mathrm{Y}$ such that $\phi^{'}(x) = f(\phi(x), x, \varphi)$. The question of the existence and uniqueness of solutions is related to number of elements of $\mathcal{Y}$ which satisfy the differential equation.

Most non-linear differential equations do not have **explicit, exact or closed-form** solutions. This is not the case for linear equations of type

$$y'(x) = a(x, \varphi)\, y(x) + b(x, \varphi),$$

which can be solved explicitly, and which will be the object of the rest of this chapter. The qualitative theory of differential equations essentially addresses the solution of non-linear ODEs by using knowledge about the solution of related linear equations.

In section 1.2 we find solutions for linear scalar equations, and related problems, for a generic independent variable. In **??** we present the characterization of solutions when the independent variable is time. This is an elementary introduction to dynamic systems's theory with a view to economic applications. We follow an applied, heuristic, approach.

1.2 General linear ODE

A **linear scalar ODE in normal form** is an ODE in which $f(\cdot)$ is a linear function of y. The most general form is

$$y'(x) = f(y(x), x) \equiv a(x)\, y + b(x).$$

1.2.1 Autonomous equation

We start with the linear scalar ODE autonomous equation

$$y'(x) = a\, y(x) + b,\ x \in \mathbb{R}.$$

If $b = 0$ the equation is homogeneous and if $b \neq 0$ the equation is non-homogeneous. The reason for this simple: while $f(y) = a\, y$ is an homogeneous function, $f(y) = a\, y + b$ is non-homogenous.[1]

Let us start with the homogeneous equation,

$$y'(x) = a\, y(x),\ x \in \mathbb{R} \tag{1.3}$$

with an arbitrary real coefficient, $a \in \mathbb{R}$.

There are several methods for solving this equation[2]. We present two methods that will be useful in subsequent chapters: the methods of separation of variables and recursive integration method.

Proposition 1. *The unique solution of ODE* (1.3) *is a function* y

$$y(x) = \begin{cases} y(x_0)\, e^{a\,(x-x_0)} & \text{if } a \neq 0 \\ y(x_0) & \text{if } a = 0 \end{cases} \tag{1.4}$$

where $x_0 \leq x$ *is an element of* X *and* $y(x_0)$ *is the arbitrary element of* Y *associated to it.*

[1] Recall that function $f(x)$ is homogeneous of degree n if multiplying the independent variable by an arbitrary real number λ then the value of the function multiplied by λ^n, that is $f(\lambda\, x) = \lambda^n\, f(x)$.

[2] There are several methods we can employ to find the proof (separation of variables, Laplace transforms, Fourier transforms, transforming into an integral equation, using the concept of generating function, just to name a few). See Zwillinger (1998)

Proof. We use the separation of variables approach. It consists in four steps: first, as $y'(x) \equiv dy(x)/dx$ we can write equation (1.3) in an equivalent way, by separating y from x

$$\frac{dy}{y} = a\,dx.$$

Second, we integrate both sides of the equation by quadrature,

$$\int_{y(x_0)}^{y(x)} \frac{dy}{y} = \int_{x_0}^{x} a\,ds.$$

Third, we simplify both sides of the equation by computing the elementary integrals

$$\int_{y(x_0)}^{y(x)} d\ln(y) = a \int_{x_0}^{x} ds \iff \ln(y(x)) - \ln(y(x_0)) = a\,(x - x_0).$$

Taking exponentials of the two sides, we find equation (1.4). In the special case in which $a = 0$ we have the general solution

$$y(x) = y(x_0), \text{ for any } x, x_0 \in \mathrm{X}.$$

□

Proof. It is instructive[3] to use another method of proof, by observing that the equation (1.3) can be written as

$$y(x) = y(x_0) + \int_{x_0}^{x} a\,y(s)ds$$

Substituting $y(s)$ inside the integral yields

$$\begin{aligned} y(x) &= y(x_0) + \int_{x_0}^{x} a\left(y(x_0) + \int_{x_0}^{s} a\,y(s')\,ds'\right) ds \\ &= y(x_0) + a\,y(x_0)\int_{x_0}^{x} ds + \int_{x_0}^{x}\int_{x_0}^{s} a^2\,y(s')\,ds'\Big)\,ds \\ &= y(x_0) + a\,y(x_0)\,(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{s} a^2\,y(s')\,ds'\,ds \end{aligned}$$

Substituting again the solution $y(s')$ inside the integral yields

$$\begin{aligned} y(x) &= y(x_0)\,(1 + a\,(x - x_0)) + \int_{x_0}^{x}\int_{x_0}^{s} a^2\left(y(x_0) + \int_{x_0}^{s'} a\,y(s'')\,ds''\right) ds'\,ds \\ &= y(x_0)\left(1 + a\,(x - x_0) + a^2\int_{x_0}^{x}\int_{x_0}^{s} ds'\,ds\right) + \int_{x_0}^{x}\int_{x_0}^{s}\int_{x_0}^{s'} a^3\,y(s'')\,ds''\,ds'\,ds \\ &= y(x_0)\left(1 + a\,(x - x_0) + \frac{a^2\,(x - x_0)^2}{2}\right) + \int_{x_0}^{x}\int_{x_0}^{s}\int_{x_0}^{s'} a^3\,y(s'')\,ds''\,ds'\,ds. \end{aligned}$$

If we continue we find

$$y(x) = y(x_0)\sum_{n=0}^{\infty}\frac{a^n\,(x - x_0)^n}{n}$$

[3]When we deal with planar ODE's or stochastic differential equations.

which is the series representation of the solution we already found by other methods (separation of variables, for instance). □

Equation (1.4) is called a general solution. As it can be seen it depends on an arbitrary point $(x_0, y(x_0))$ in the space $\mathrm{X} \times \mathrm{Y}$, that is on an arbitrary value for the independent variable and the associated value for the dependent variable. This should be intuitive from the fact that an ODE results from a variational approach for uncovering the economic phenomenon that the we want to study, meaning that we describe it by the change in the dependent variable from a marginal change in the independent variable. We will see next how the complete specification of the behavior or y is provided by a side-condition.

Now consider a scalar linear autonomous non-homogeneous equation

$$y'(x) = a\,y(x) + b,\ x \in \mathbb{R} \tag{1.5}$$

with an arbitrary real coefficient, $a \in \mathbb{R}$ and $b \neq 0$.

Proposition 2. *The unique solution of ODE* (1.5) *is a function* y

$$y(x) = \begin{cases} \bar{y} + (y(x_0) - \bar{y})\, e^{a\,(x-x_0)} & \text{if } a \neq 0 \\ y(x_0) + b\,(x - x_0) & \text{if } a = 0 \end{cases} \tag{1.6}$$

where

$$\bar{y} \ = -\frac{b}{a}$$

for any $x \geq x_0 \in \mathrm{X}$.

Proof. First assume that $a \neq 0$. Then, there is a number $\bar{y} \ = -b/a$ such that if $y(x) = \bar{y}$ then $y'(x) = 0$. Introduce a change in variables $z(x) = y(x) - \bar{y}$. Then $z'(x) = y'(x) = a\,y(x) + b = a\,(z(x)+\bar{y})+b = a\,z(x)$ form the definition of $\bar{y}$. We already know that the solution of $z'(x) = a\,z(x)$ is $z(x) = z(x_0)\,e^{a\,(x-x_0)}$. Mapping back to y we have

$$y(x) - \bar{y} \ = \ z(x_0)\,e^{a\,(x-x_0)} - \bar{y} \ = (y(x_0) - \bar{y})\,e^{a\,(x-x_0)} - \bar{y}.$$

Then $y(x) = \bar{y} + (y(x_0) - \bar{y})\,e^{a\,(x-x_0)}$ if $a \neq 0$. Now assume that $a = 0$.

$$\begin{aligned} y(x) &= \lim_{a\to 0} \left(\ y(x_0)e^{a\,(x-x_0)} - \frac{b(1 - e^{a\,(x-x_0)}}{a} \ \right) \\ &= y(x_0) - \lim_{a\to 0} \frac{\frac{d}{da}\left(1 - e^{a\,(x-x_0)t}\right)}{\frac{d}{da}\ a} = \\ &= y(x_0) - \frac{-b(x - x_0)}{1} \end{aligned}$$

using the l'Hôpital's rule. □

1.2.2 Non-autonomous equation

Let us start with the **non-autonomous homogeneous** equation,

$$y'(x) = a(x)\, y(x),\ y : \mathrm{X} \to\ \mathrm{Y} \tag{1.7}$$

where both X are Y are subsets of $\mathbb{R}$. The coefficient function $a(x)$ is arbitrary. It can be a constant, piecewise constant, or be an arbitrary function of x.

Proposition 3. *The unique solution of ODE* (1.7) *is a function* y

$$y(x) = y(x_0)\, e^{\int_{x_0}^{x}\ a(s)\, ds},\ \textit{for any}\ \ x, x_0 \in \mathrm{X} \tag{1.8}$$

where x_0 *is an element of* X *and* $y(x_0)$ *is the arbitrary element of* Y *associated to it.*

Proof. Next we use the method of separation of variables to determine the solution for ODE (1.7). Recalling that we denoted the derivative as $\dfrac{dy}{dx} = y'(x)$ we can write the ODE (1.7) as $\dfrac{dy}{dx} = a(x)\, y$. Using integration by quadratures we find

$$\int_{y(x_0)}^{y(x)} \frac{dy}{y} = \int_{x_0}^{x} a(s)\, ds$$

as the anti-derivative of $\dfrac{1}{y}$ is $\ln y$ then $\ln\left(\frac{y(x)}{y(x_0)}\right) = \int_{x_0}^{x} a(s)\, ds$ Taking exponentials on both sides yields the general solution (1.8). □

The **non-autonomous and non-homogeneous** scalar linear ODE is

$$y'(x) = a(x)y + b(x),\ y : \mathrm{X} \to\ \mathrm{Y} \tag{1.9}$$

where, again, both X are Y are subsets of $\mathbb{R}$.

Proposition 4. *The unique solution of ODE* (1.9) *is a function* y

$$y(x) = y(x_0)\, e^{\int_{x_0}^{x}\ a(s)\, ds} + \int_{x_0}^{x} e^{\int_{s}^{x} a(z)\, dz}\, b(s) ds \tag{1.10}$$

where x_0 *is an element of* X *and* $y(x_0)$ *is the arbitrary element of* Y *associated to it.*

Proof. We apply the variation of constant method[4]. First, we consider the solution for the homogeneous equation, such that $b(x) = 0$ for all $x \in \mathrm{X}$. From equation (1.8) its solution for the fixed interval (x_0, x), such that $x_0 < x$ is

$$y_h(x, y_0) = y_0\, e^{\int_{x_0}^{x} a(s) ds}.$$

But we expect the solution to equation (1.9) to be, for an arbitrary $x > x_0$,

$$y(x) = y_h(x, y_0(x)) = y_0(x)\, e^{\int_{x_0}^{x} a(s) ds}. \tag{1.11}$$

[4]Due to Lagrange (1811).

Taking derivatives for x, in the last equation, we obtain

$$y'(x) = y_0'(x)\, e^{\int_{x_0}^{x} a(s)ds} + y_0(x)\, a(x)\, e^{\int_{x_0}^{x} a(s)ds} = y_0'(x)\, e^{\int_{x_0}^{x} a(s)ds} + a(x)\, y(x)$$

which should be equal to equation (1.9). By equating the right-hand sides of both equations we get the ODE

$$y_0'(x) = b(x)e^{-\int_{x_0}^{x} a(s)ds}.$$

As function $y_0(.)$ is continuous, from the fundamental theorem of calculus $_{x_0}^{x} y_0'(s)\, ds = y_0(x) - y_0(x_0)$, and

$$y_0(x) = y_0(x_0) + \int_{x_0}^{x} b(s)e^{-\int_{x_0}^{s} a(z)dz} ds.$$

Substituting in equation (1.11) and because $y_0(x_0) = y(x_0)$ we finally get solution (1.10) $\square$

1.2.3 ODE problems

The choice of pair $(x_0, y(x_0))$, in any of the equations (1.8) and (1.10), and therefore the exact determination of their solution, depends on the side conditions we impose. We can generically saw that they take the form of a functional. Assuming that $\underline{x}$ and $\bar{x}$ denote the infimum and the supremum of X, the side conditions take generically the form

$$\int_{\underline{x}}^{\bar{x}} C(y(x), x)\, dx = \text{constant}$$

where $C(\cdot)$ is a knwon function. In this case we consider (1.8) in the form

$$y(x) = y(\underline{x})\, e^{\int_{\underline{x}}^{x} a(s)\, ds}$$

and analogously for equation (**??**), and solve

$$\int_{\underline{x}}^{\bar{x}} C(y(\underline{x})\, e^{\int_{\underline{x}}^{x} a(s)\, ds}, x)\, dx = \text{constant}$$

for k. This would allows us to obtain a **particular solution** which is an exact solution for $y(\cdot)$.

This approach is sufficiently general to encompass cases in which we fix a value for y, as $y(x_0) = y_0$, for a particular point x_0 in the domain (or its closure) X. For instance, for the side condition

$$\int_{\underline{x}}^{\bar{x}} \delta(x - x_0)\, y(x)\; dx = y_0$$

where y_0 is a know number. Substituting the general solution

$$\int_{\underline{x}}^{\bar{x}} \delta(x - x_0)\, y(\underline{x})\, e^{\int_{\underline{x}}^{x} a(s)\, ds} dx = y_0$$

we find $y(\underline{x}) = y_0\, e^{-\int_{\underline{x}}^{x_0}\ a(s)\, ds}$ which yields the particular solution

$$y(x) = y_0\, e^{\int_{\underline{x}}^{x}\ a(s)\, ds}.$$

Next we see some applications of models involving non-autonomous ODE's, which include particular side constraimnts

1.2.4 Some applications

Utility theory

The generalized logarithmic function[5]

$$u(x) = \ln_\sigma(x) \equiv \frac{x^{1-\sigma} - 1}{1-\sigma} \text{ for } \ \sigma > 0$$

has many uses, not only in economics. In economics, in deterministic models it is called the iso-elastic utility function, or, in stochastic models, it is called constant relative risk aversion (CRRA) Bernoulli utility function CRRA.

Using the analysis in Pratt (1964) it can be showed that it is a solution of the problem

$$\begin{cases} -\dfrac{u''(x)\, x}{u'(x)} = \sigma,\ x \in \mathrm{X} = (0, \infty) & (1.12a) \\ \sigma \displaystyle\int_1^\infty \frac{u'(x)}{x}\ dx = 1 & (1.12b) \\ u(1) = 0 & (1.12c) \end{cases}$$

The first equation is a definition of the relative risk aversion, as the symmetric of the elasticity of $u(\cdot)$ being a constant equal to σ. The first constraint conditions the relative slope of $u(\cdot)$ on all its domain and the last constraint fixes the value of utility of consumption at one (this condition makes transparent that a logarithm is hidden behind the utility function).

Equation (1.12a) is a second order ODE. We can transform it into a first order ODE by defining $y(x) = \ln u'(x) \iff u'(x) = e^{z(x)}$. Then we obtain an the linear ODE

$$z'(x) = b(x) \equiv -\frac{\sigma}{x}.$$

This equation has solution $z(x) = z(x_0) + \int_{x_0}^{x} b(s)ds$, which we can prove simplifies to $z(x) = z(x_0) - \sigma\, \ln\left(\frac{x}{x_0}\right)$, for an arbitrary $x_0 > 0$. Therefore

$$u'(x) = e^{z(x)}\ = u'(x_0)\left(\frac{x}{x_0}\right)^{-\sigma}$$

[5] If $\sigma = 1$ it can be shown that it is $u(x) = \ln(x)$.

which is again a linear differential equation. Solving it, and observing that $u'(x_0)$ is an arbitrary constant, yields

$$\begin{aligned} u(x) &= u(x_0) + u'(x_0) \int_{x_0}^{x} \left(\frac{s}{x_0}\right)^{-\sigma} ds \\ &= u(x_0) + u'(x_0) x_0^{\sigma} \left(\frac{x^{1-\sigma}}{1-\sigma} - \frac{x_0^{1-\sigma}}{1-\sigma}\right). \end{aligned} \tag{1.13}$$

The two side conditions (1.12b) and (1.12c) allow us, in principle, to determine the arbitrary constants $u(x_0)$ and $u'(x_0)$. First, using the expression obtained for $u'(x)$ we have

$$\int_{x_0}^{\infty} \frac{u'(x)}{x} dx = u'(x_0)\, x_0^{\sigma} \int_{x_0}^{\infty} x^{-\sigma-1} \, dx = \frac{u'(x_0)}{\sigma},$$

which, considering constraint (1.12b) we for $x_0 = 1$, we have $\dfrac{u'(1)}{\sigma}\dfrac{1}{\sigma}$ that is, $u'(1) = 1$. Setting again $x_0 = 1$ in equation (1.13) we have

$$u(x) = u(1) + \frac{x^{1-\sigma} - 1}{1-\sigma}$$

which, upon introducing side-condition (1.12c) yields the generalized logarithm.

The Gaussian distribution

We can derive the standard Gaussian probability density function from the ODE problem,

$$\begin{cases} y'(x) = -x\, y(x), \text{ for } x \in \mathrm{X} = (-\infty, \infty) & (1.14a) \\ \displaystyle\int_{-\infty}^{\infty} y(x) \, dx = 1 & (1.14b) \end{cases}$$

While equation (1.14a) means that the rate of decay between to adjacent points in X is equal to the value of x in which we measure it, equation (1.14b) constraints $(y(x))_{\mathrm{X}}$ to be a distribution.

The solution to the problem is

$$y(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\,\pi}} \tag{1.15}$$

To prove this, we already know that the solution to equation (1.14a) is

$$y(x) = y(x_0)\, e^{\int_{x_0}^{x} s\, ds} = y(x_0)\, e^{-\frac{x^2}{2} + \frac{x_0^2}{2}} = y(x_0)\, e^{\frac{x_0^2}{2}} \; e^{-\frac{x^2}{2}}$$

for an arbitrary point $(x_0, y(x_0))$. As[6]

$$y(x_0)\, e^{\frac{x_0^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = y(x_0)\, e^{\frac{x_0^2}{2}} \sqrt{2\pi}$$

if we substitute this general solution in the constraint (1.14b) we have $y(x_0)\, e^{\frac{x_0^2}{2}} \sqrt{2\pi} = 1$, which yields teh standard Gaussian probability density function (1.15).

[6]The Gaussian integral is $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi}$.

1.3 Time as the independent variable and stability properties

Most applications of differential equations have time, t, as the independent variable. In this case, the convention is to use Newton's notation for the derivative, i.e., $\dot{y} \equiv \frac{dy(t)}{dt}$ and write the general scalar linear ODE as

$$\dot{y} = a(t)\, y + b(t),\ t \in \mathrm{T} \subseteq \mathbb{R}_+ \tag{1.16}$$

where $y : \mathrm{T} \to \mathbb{R}$. This is a non-autonomous and non-homogeneous equation.[7]

Again, an ODE problem is defined by finding the solution to equation (1.16) such that it satisfies a side-condition. In the case of the time-dependent case, the most common problems in economics are **initial-value** problems and **terminal-value problems**. In the first case we add information on a pair $(t_0, y_0) \in \mathrm{T} \times \mathrm{Y}$, and in the second we fix asymptotic values for $t = \infty$ and $y(\infty)$.

The existence and uniqueness of solutions for linear ODE has already been established in the previous section (becaue we are able to find explicit solutions) to the ODE and to ODE problems. However, in the case of time dependent ODEs there is a rich geometrical theory for the characterization of the solutions. That is, we can describe the process $(y(t))_{t\in\mathrm{T}}$ without the nedd to explicitly solving the ODE. As most non-linear ODEs do not have explicit solutions, the characterization of their solutions may be possible by comparing them, at least locally, to the linear ODEs. In particular, the qualitative theory for ODE is based upon the local approximation of non-linear ODE by linear ODE and by verifying conditions under which a non-linear ODE is (topologically) equivalent to a linear ODE (at least locally).

Next we present solutions to autonomous equations, present the qualitative theory of the solutions of linear present solutions to non-autonomous equations, and describe their main applications to economics.

1.3.1 Solution to the autonomous equation

A scalar ODE is **autonomous** if the coefficients are constant, i.e, they are independent of the exogenous variable t,

$$\dot{y} = a\, y + b \tag{1.17}$$

where $(a, b) \in \Phi \subseteq \mathbb{R}^2$ are known constants. From now on we let initial value of the independent variable be equal to zero, that is $t_0 = 0$.

Using the results of the previous section, we can state (without proof) that

Proposition 5 (Solutions to equation (1.16))**.** *For a given pair of parameters $(a, b) \in \Phi$ the solution*

[7] If we redefine the independent variable as $t = \tau$ we can transform the non-autonomous scalar linear ODE into a planar non-linear equation: $\dot{y_1} = \lambda(y_2)y_1 + \beta(y_2)\ \dot{y}_2 = 1$ where $y_2(t) = t$. This means that the behavior of the solution when the coefficients are functions of time can be quite different.

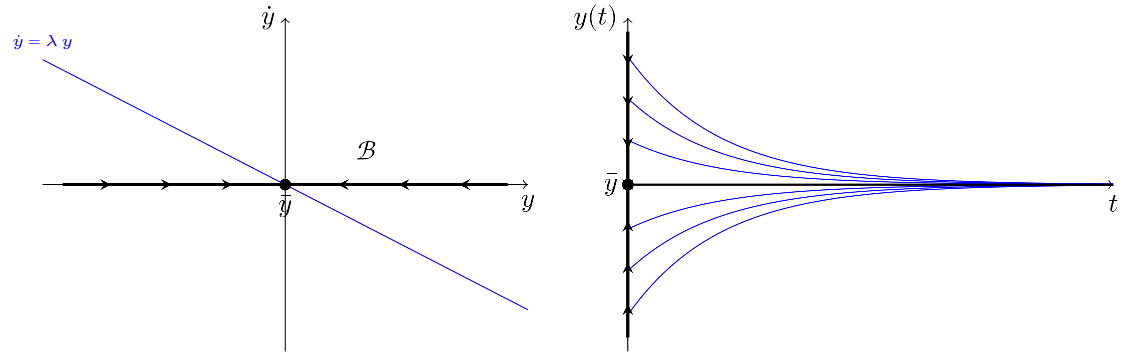

Figure 1.1: Trajectories (left) and phase diagram (right) of ODE $\dot{y} = ay$ for $a < 0$

is a unique mapping $\phi : \mathrm{T} \times \mathrm{Y} \times \Phi \to \mathrm{Y}$*. In particular,*

$$y(t) = \phi(t, y(0); a, b) = \begin{cases} y(0)\, e^{a\,t} - \dfrac{b}{a}\left(1 - e^{a\,t}\right) & \textit{if } \ a \neq 0, \ \textit{and} \ \ b \neq 0 \\ y(0)\, e^{a\,t} & \textit{if } \ a \neq 0, \ \textit{and} \ \ b = 0 \\ y(0) + b\,t & \textit{if } \ a = 0, \ \textit{and} \ \ b \neq 0 \\ y(0) & \textit{if } \ a = b = 0 \end{cases} \ \textit{for any } \ t \in \mathrm{T}, \tag{1.18}$$

where $y(t\,|\,t = 0) = y(0)$ *is an arbitrary element of* Y*.*

The solution of a scalar linear a autonomous ODE is a function $y(t) = \phi(t, y(0); a, b)$ of time and on an arbitrary element of Y, whose behavior depends on the parameters a and b. **Characterizing the dynamics** inscribed in the ODE the solution means tracking the behavior of the path $\Big(y(t)\Big)_{t\in\mathrm{T}}$ traveled within Y when time independent variable changes from $t = 0$ to $t = \infty$.

The trajectories for the autonomous and homogeneous ODE $\dot{y} = a\,y$ for different values initial values $y(0)$ are represented in the left panels of figures 1.1, for the case in which $a < 0$, and in 1.2, for the case in which $a > 0$. We observe that when $a < 0$, independently of $y(0)$, all the trajectories converge to $y = 0$, asymptotically, and that when $a > 0$, all the trajectories diverge to $+\infty$ if $y(0) > 0$ and to $-\infty$ if $y(0) < 0$. However, in both cases if $y(0) = 0$ the trajectories are stationary, that is $y(t) = 0$ for any $t \in [0, \infty)$.

The trajectories for the autonomous and non-homogeneous ODE $\dot{y} = a\,y + b$ for different values initial values $y(0)$ are represented in the left panels of figures 1.3, for the case in which $a < 0$ and $b > 0$, and in 1.4, for the case in which $a > 0$ and $b < 0$. The trajectories qualitative behavior of the trajectories, in the sense of being convergent or divergent in time, to the homogeneous ODE, however it can be quantitatively different. That is, we observe that when $a < 0$, independently of $y(0)$, all the trajectories converge to a point $y = -b/a$, asymptotically, and that when $a > 0$, all the trajectories again diverge to $+\infty$ if $y(0) > -b/a$ and to $-\infty$ if $y(0) < -b/a$. However, in both cases if $y(0) = -b/a$ the trajectories are stationary, that is $y(t) = -b/a$ for any $t \in [0, \infty)$.

The case $a = b = 0$, where $\dot{y} = 0$, with solution $y(t) = y(0)$ a constant is thus a degenerate case in which the solution is **always** time-invariant, i.e., it is independent from the exogenous variable

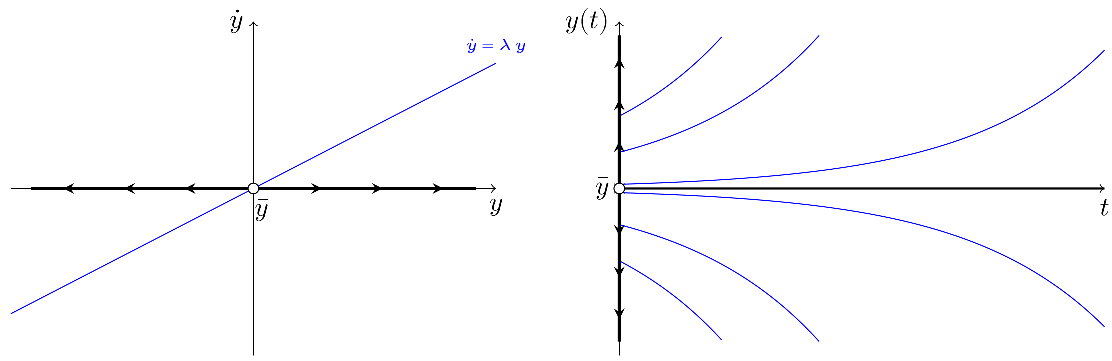

Figure 1.2: Trajectories (left) and phase diagram (right) of ODE $\dot{y} = ay$ for $a > 0$

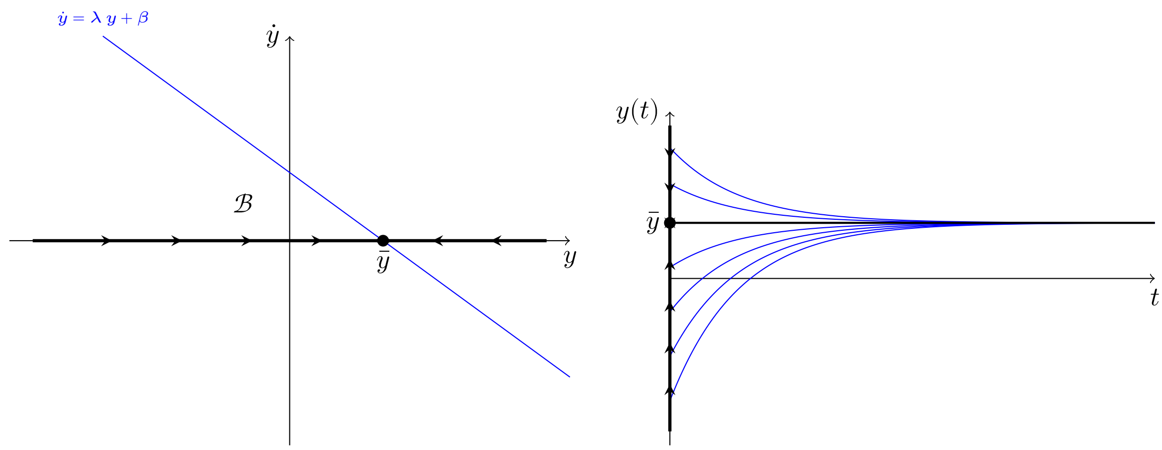

Figure 1.3: Trajectories (left) and phase diagram (right) of ODE $\dot{y} = a\,y + b$ for $a < 0$ and $b > 0$

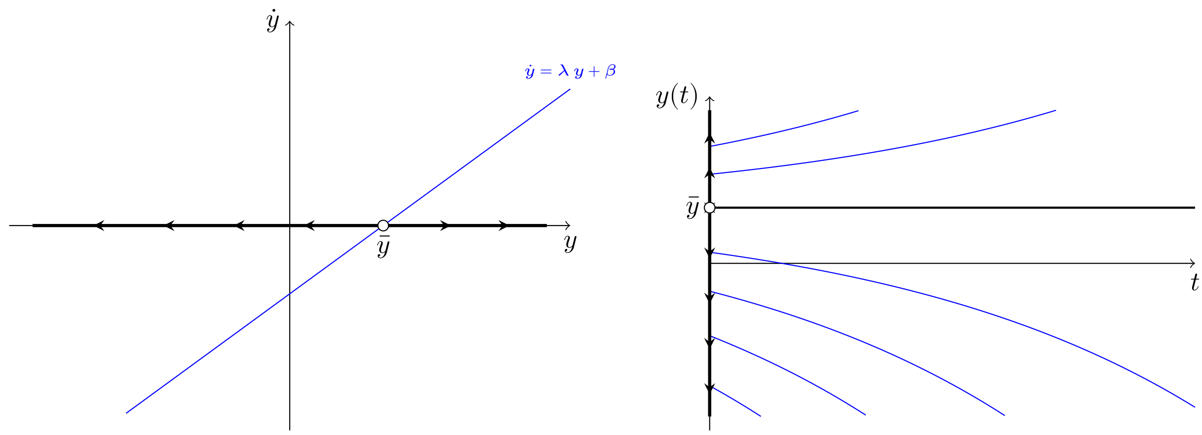

Figure 1.4: Trajectories (left) and phase diagram (right) of ODE $\dot{y} = a\,y + b$ for $a > 0$ and $b < 0$

t and from the initial point $y(0)$. Intuitively we can say that there are no dynamics, or that this corresponds to a boundary case between stability and instability.

1.3.2 Qualitative theory

This dynamics of the equation is determined by function $f(y) = a\,y + b$.

To introduce qualitative dynamics analysis we introduce some definitions.

- **steady state** (or equilibrium point): it is an element in range of y, Y, such that $f(y) = 0$, that is
$$\bar{y} = \{y \in \mathrm{Y} : f(y) = 0\}.$$

- a steady state be of one of the following types according to its **stability properties**: First, the steady state is **asymptotically stable** if for any $y = y(x_0) \in \mathrm{Y}$ the flow generated by the ODE $\dot{y} = f(y)$, that we can denote by $(y(t))_{t\in\mathrm{T}}$, has the property
$$\lim_{t\to\infty} y(t) = y(\infty) = \bar{y}.$$
Second, the steady state is **unstable** if for any $y = y(x_0)$ in a neighborhood of $\bar{y}$, $y(t)$ does not converge to $\bar{y}$.

- **Invariant subsets** are partitions of set Y containing the whole solution path $(y(t))_{t\in\mathrm{T}}$; we call **attractor set** (or basin of attraction) to the subset of points $y \in \mathrm{Y}$ such that the solution will converge to the steady state and **repelling set** to the set of points $y \in \mathrm{Y}$ such that the solution will not converge to the steady state.

- A **phase diagram** is a graphical representation of the set Y in which we represent the steady states, and the invariant sets. The invariant sets representation includes the representation of the variation of the solution with increasing time.

We will see that for planar and nonlinear ODEs there are other types of dynamics.

We start by applying those definitions to the homogeneous equation $\dot{y} = a\,y$. The existence and number of **steady states** depend on a
$$\bar{y} = \begin{cases} y(0), & \text{if } a = 0 \\ 0, & \text{if } a \neq 0. \end{cases}$$

In the first case there is an **infinite number** of equilibria, consisting in all points in Y, and in the second there is a **single** equilibria if $0 \in \mathrm{Y}$, or no equilibria if $0 \notin \mathrm{Y}$.

When there is a steady state, that is, when $a \neq 0$ we can characterize its **stability properties**:

- if $a < 0$ then $\lim_{t\to\infty} y(t) = 0 = \bar{y}$, for any $y(0)$, then the equilibrium point is asymptotically stable;

- if $a > 0$ then
$$\lim_{t\to\infty} y(t) = \begin{cases} \pm\infty & \text{, if } y(0) \neq 0 \\ 0 & \text{, if } y(0) = 0 \end{cases}$$
and the equilibrium point $\bar{y}$ is unstable. In this case we say the solution can be non-stationary if $y(0) \neq 0$

Therefore, if $a \neq 0$, and $\bar{y} = 0 \in \mathrm{Y}$, there are only two kinds of possible **invariant sets**:

- if $a < 0$ the basin of attraction for $\bar{y}$ is the whole set Y and Y **is the attraction set**. Then we say $\bar{y}$ is **globally asymptotically stable**;

- if $a > 0$ then $\bar{y}$ is repelling and unstable and $\mathrm{Y}/\bar{y}$ is the unstable invariant set.

The right-hand panel in Figures 1.1 and 1.2 illustrate the phase diagram for the asymptotically stable and unstable cases, respectively. In the first case we label the basin of attraction of $\bar{y}$ by $\mathcal{B}$.

For the non-homogeneous equation $\dot{y} \; = a\,y + b$ the qualitative dynamics is similar, except for the case in which $a = 0$. For $a \neq 0$ there are only some quantitative differences:

- the steady state is also unique, although it is shifted from $\bar{y} = 0$, if $b = 0$, to $\bar{y} = -b/a$, if $b \neq 0$;

- the stability behavior is qualitatively the same but now relative to the equilibrium point $\bar{y} = -b/a$: it is asymptotically stable if $a < 0$ and it is unstable if $a > 0$.

The right-hand panel in Figures 1.3 and 1.4 illustrate the phase diagram for the asymptotically stable and unstable cases, respectively. In the first case we label the basin of attraction of $\bar{y}$ by $\mathcal{B}$.

The dynamics are qualitatively different when $a = 0$. While in the homogenous case (i.e., if $b = 0$) the solution is stationary and there is an infinite number of steady states (all the elements of Y) in the non-homogeneous case (i.e, if $b \neq 0$) **there are no steady states** and the solution of the ODE is always non-stationary.

Table 1.1, which we can call a **bifurcation table**, summarizes the main types of dynamics for the scalar linear autonomous ODE:

Table 1.1: Types of dynamics for the linear scalar ODE

<table>
<tr><th></th><th>$a < 0$</th><th>$a = 0$</th><th>$a > 0$</th></tr>
<tr><td>$b = 0$</td><td rowspan="2">one steady state
asymptotically stable</td><td>infinite number of steady states</td><td rowspan="2">one steady state
unstable</td></tr>
<tr><td>$b \neq 0$</td><td>no steady states</td></tr>
</table>

1.3.3 Problems involving scalar ODE

At last, we draw a distinction between an ODE and a **problem** involving an ODE. To get an intuition of the difference, observe that equations (1.18) involve a dependence on an arbitrary point $y(0)$, this is the reason why these solution, for specific scalar ODEs, are called **general solutions**. An **ODE problem** involving side-conditions and the solution it is called **particular solution**[8] In the first case, we call the ODE a **forward ODE** because the solution will be obtained from future instants (assuming that the present time is $t = 0$) and in the second case we call the ODE a **backward ODE**. The evolution described by the ODE can be done forward in time (if we know the initial point) or backward in time (if we know a terminal point). With this additional information we can sometimes uniquely determine a forward or a backward path.

We can have several types of side-conditions but in economics the two most common conditions are initial conditions, if the point $(t, y(t)) = (0, y_0)$ is know, or terminal conditions, if the point $(t, y(t)) = (T, y_T)$ for finite-time problems where $\mathrm{T} = [0, T]$, or if $\lim_{t\to\infty} \ y(t) = y_\infty$ for infinite-time problems where $\mathrm{T} = [0, \infty)$,

In the first case, we have an **initial-value problem**

$$\begin{cases} \dot{y} \ = a\,y + b \text{ for } \ t \in \mathrm{T} & (1.19a) \\ \quad y(0) = y_0 \text{ for } \ t = 0, \ y_0 \in \mathrm{Y} & (1.19b) \end{cases}$$

Observe that while $y(0)$ represents function $y(t)$ evaluated at $t = 0$, y_0 is a number belonging to the range of y. If $a \neq 0$ the solution is unique. If furthermore, $b \neq 0$ the solution to the previous problem is (prove this)

$$y(t) = \bar{y} \ + (y_0 - \bar{y})\, e^{a\,t}, \ t \in \mathrm{T}$$

for $\bar{y} \ = -b/a$.

A common **terminal-value problem** is the following

$$\begin{cases} \dot{y} \ = a\,y + b \text{ for } \ t \in \mathrm{T} & (1.20a) \\ \quad y(T) = y_T \text{ for } \ t = T, \ y_0 \in \mathrm{Y} & (1.20b) \end{cases}$$

Observe that while $y(T)$ represents function $y(t)$ evaluated at $t = T$, y_T is a number belonging to the range of y. If $a \neq 0$ and $b \neq 0$ the solution is unique

$$y(t) = \bar{y} \ + (y_T - \bar{y})\, e^{-a\,(T-t)}, \ t \in \mathrm{T} = [0, T].$$

In this case, we observe that $y(0) = \bar{y} \ + (y_T - \bar{y})\, e^{-a\,T}$ becomes endogenous.

A common **infinite-horizon terminal-value problem** in economics is the following

$$\begin{cases} \qquad \dot{y} \ = a\,y + b \text{ for } \ t \in \mathrm{T} \ = [0, \infty) & (1.21a) \\ \lim\limits_{t\to\infty} \ e^{-\mu\,t} \ y(t) = 0 \text{ where } \ \mu > 0 & (1.21b) \end{cases}$$

[8] Although the solution to a linear ODE always exists, the solution to an ODE problem may not exist if the side conditions are incompatible with the general solution of the ODE.

We can prove that solutions always exist, but are not necessarily unique. Specifically: (1) if $a < \mu$ there is an infinite number of solutions

$$y(t) = \bar{y} \ + (y(0) - \bar{y})\, e^{a\,t}, \ t \in [0, \infty)$$

where $y(0)$ is an arbitrary value for y. In this case we say that the solution to the problem is **indeterminate**; (2) if $a \geq \mu$ then the solution is unique and stationary

$$y(t) = \bar{y} \text{ for all } \ t \in [0, \infty).$$

In this case we say that the solution to the problem is **determinate**.

To prove this: first, take the appropriate solution to the ODE (1.21a) from equation (1.18),

$$y(t) = \bar{y} \ + (y(0) - \bar{y})\, e^{a\,t}, \ t \in \mathrm{T} = [0, \infty)$$

where $y(0)$ is an arbitrary number from Y; second, write

$$\begin{aligned} \lim_{t\to\infty} \ e^{-\mu\,t} \ y(t) &= \lim_{t\to\infty} \ e^{-\mu\,t}\Big(\bar{y} \ + (y(0) - \bar{y})\, e^{a\,t}\Big) \\ &= 0 + \lim_{t\to\infty} \,(y(0) - \bar{y})\, e^{(a-\mu)\,t}, \end{aligned}$$

at last, from equation (1.21b) we should have

$$\lim_{t\to\infty} (y(0) - \bar{y})\, e^{(a-\mu)\,t} = 0,$$

which is verified for any $y(0) - \bar{y}$ if $a - \mu < 0$ and only for $y(0) - \bar{y} = 0$ if $a - \mu \geq 0$.

Therefore, uniqueness (and existence) of the solution of an ODE is not the same as uniqueness of a problem involving an ODE. And this distinction has important conceptual differences in economic applications

In economic models we use the following classification of variables and economic equilibrium

- **pre-determined** and **non-pre-determined** variables: the first are observed and the second are anticipated, that is, we have information for $t = 0$ for the first type of variables and we have asymptotic information on the second type of variables;
- **stationary** or **non-stationary** variables if they converge to a constant or are unbounded asymptotically (i.e., when $t \to \infty$);
- **determinacy** or **indeterminacy** if an equilibrium or a state of the economy modelled by a differential equation is unique or not

The relationship between them depends on the existence or not of a steady state and on their stability properties, for states within set Y.

For instance

- if a variable is pre-determined the trajectory described by the solution is always determinate, however, it can be stationary (if $a < 0$) or non-stationary (if $a > 0$). The first case is common in models with adaptative expectations, v.g. $\dot{p} = \lambda(\bar{p} - p)$, for $\lambda > 0$ and p is the log of price. The second case is common in endogenous growth models in which the GDP dynamics is given by $\dot{y} = Ay$, where y is GDP per capita;

- if a variable is non-predetermined the trajectory can be determinate if $y(0)$ is determined uniquely and is indeterminate if k can be any value within set Y. For scalar models the solutions are usually stationary if the terminal condition is of the type $\lim_{t\to\infty} y(t)e^{-\mu t} = 0$ for $\mu > 0$.

Table 1.2 summarizes this concepts, used in dynamic general equilibrium models (DGE).

Table 1.2: Classification of equilibrium paths in DGE models

y	$a < 0$	$a = 0$	$a > 0$
pre-determined	determined and stable	determined and stationary	determined and non-stationary
non- pre-determined	indeterminate	ambiguous	determined

Example: budget constraint dynamics

A fundamental differential equation in economics is the budget constraint equation. Let $a(t) \in R$ be the asset position of an economic entity[9] at time t, which is a stock variable which can be read in its balance. If $a > 0$ we say the agent is a net creditor and if $a < 0$ it is a net debtor. Assume that the the asset has an instantaneous return $r(t)$ and that the entity has a flow of non-financial income denoted by $y(t)$ and a flow of expenditures denoted by $e(t)$. One of the iron "laws" of economics is that the change in the asset position, or investment, is equal to savings. Savings, denoted by $s(t)$, is equal to total income minus expenditure. Therefore

$$\dot{a}(t) = s(t) = r(t)\, a(t) + y(t) - e(t), \text{for every } \; t \in \mathrm{T}. \tag{1.22}$$

Let us assume that all the exogenous variables are constant and parametrically given.

$$\dot{a}(t) = s(t) = r\, a(t) + y - e, \text{for every } \; t \in \mathrm{T}. \tag{1.23}$$

We can answer the question: given an initial asset position $a(0) = a_0$ what will be the asset positions in the future ? How will they change constant, permanent changes in any of the parameters r, y or e or in the initial level a_0 ? We see (1.23) as a forward equation and answer those questions by solving the initial-value problem

$$\begin{cases} \dot{a}(t) = r\, a(t) + y - e \text{ for } \; t \in \mathrm{T} & (1.24\text{a}) \\ a(0) = a_0 \text{ for } \; t = 0 & (1.24\text{b}) \end{cases}$$

[9]It can be a household, a the government, or an economy. In the first case, n represents the net asset position, in the second it is usually the government debt, and in the third the net asset position of a country regarding the rest of the world.

The solution is

$$a(t) = \bar{a} \ + (a_0 - \bar{a})e^{r\,t},\ \bar{a} \ = -\frac{y-e}{r}\ t \in [0, \infty).$$

If $r > 0$ we can see that the process $\big(a(t)\big)_{t\in[0,\infty)}$ is unstable: if $a_0 > \bar{a}$, then $\lim_{t\to\infty}\ a(t) = +\infty$ and the agent will become a very large (indeed unboundly large) creditor; if $a_0 < \bar{a}$, then $\lim_{t\to\infty}\ a(t) = -\infty$ and the agent will become a very large (indeed unboundly large) debtor; or if $a_0 = \bar{a}$ its asset position will be stationary $a(t) = \bar{a}$ for any t.

Some times, a represents a ratio of debt over another indexing variable (as population, prices, GDP, etc). In this case, r represent the interest rate net of rate of growth of the indexing, which makes possible that $r \leq 0$. In this case, particularly when $r < 0$ the dynamics change radically: the process for $\big(a(t)\big)_{t\in[0,\infty)}$ becomes asymptotically stable and converges to $\bar{a}$, for any initial asset position a_0.

We can see how the solution of the equation changes with variations in the parameters. For instance, we call finding $\dfrac{\partial a(t)}{\partial r}$ an exercise of comparative dynamics. This should not be confused with finding $\dfrac{\partial \bar{a}}{\partial r}$ which is a comparative statics exercise.

A different question is: what is the sustainable level of the asset position $a(0)$? The question posed like this is close to meaningless, before we translate mathematically "sustainability" by some criterium. One commonly used is: we say that the asset position is sustainable if the asymptotic present value of a is equal to zero, that is

$$\begin{cases} \dot{a}(t) = r\,a(t) + y - e \text{ for } \ t \in \mathrm{T} = [0, \infty) & (1.25a) \\ \lim\limits_{t\to\infty}\ a(t)e^{-\rho\,t} \ = 0\rho > 0 & (1.25b) \end{cases}$$

Using our previous example, the answer depends on the relationship between r and ρ: if $r < \rho$ then the solution is

$$a(t) = \bar{a} \ + (a(0) - \bar{a})e^{r\,t},\ \bar{a} \ = -\frac{y-e}{r}\ t \in [0, \infty).$$

any initial asset position, $a(0)$ is sustainable; however if $r \geq \rho$ then the solution is

$$a(t) = \bar{a},\ \text{for all}\ \ t \in [0, \infty),$$

which means that $a(0) = \bar{a}$. If the entity is an initial debtor, say $a_0 < 0$ then this level of debt is sustainable only if it satisfies, for every point in time, $y - e = -ra_0 > 0$, i.e, its income is permanently higher than its expenditure.

1.3.4 Non-autonomous equations

The scalar linear non-autonomous having time as an independent variable, has been already written in equation (1.16).

From Proposition 1.3.4 its general solution is, adapting equation (1.10)

$$y(t) = y(0)\,e^{\int_0^t\ a(s)\,ds} + \int_0^t\ e^{\int_s^t a(z)\,dz}\,b(s)ds,\ t \in \mathrm{T} \tag{1.26}$$

where $y(0)$ is an arbitrary element of Y associated to it.

A common **initial value problem** is

$$\begin{cases} \dot{y} &= a(t)\, y + b(t) \text{ for } t \in \mathrm{T} & (1.27a) \\ & y(0) = y_0 \text{ for } t = 0,\ y_0 \in \mathrm{Y} & (1.27b) \end{cases}$$

has the solution

$$y(t) = y_0\, e^{\int_0^t a(s)\, ds} + \int_0^t e^{\int_s^t a(z)\, dz}\, b(s) ds,\ t \in \mathrm{T}$$

Exercise: prove this.

A common terminal value problem is

$$\begin{cases} \dot{y} &= a(t)\, y + b(t) \text{ for } t \in \mathrm{T} & (1.28a) \\ \lim_{t\to\infty} e^{-\int_0^t a(s) ds} & y(t) = 0 \text{ for } t = 0,\ y_0 \in \mathrm{Y}\ . & (1.28b) \end{cases}$$

Has the solution

$$y(t) = -\int_t^\infty e^{-\int_t^s a(z) dz} b(s) ds$$

Exercise: Take the budget constraint (1.22) and solve it. Solve the associated initial and terminal value problems.

In economics the following models are featuring anticipated shocks in exogenous variables, are of interest.

Example: future shock in an exogenous variable for pre-determined endogenous variable Let us assume that y is a pre-determined variable, in which we know the value at time $t = 0$, and assume there will be an additive shock in a prescribed future data in an exogenous variable that affects the dynamics of y.

We can address this case through the initial value problem

$$\dot{y} = ay + b(t),\ \text{for } t \in [0, \infty)$$

where $a \neq 0$ by assumption and

$$b(t) = \begin{cases} b_0 & \text{if } 0 \le t < t^* \\ b_1 & \text{if } t^* \le t < \infty \end{cases}$$

and $y(0) = y_0$ is given.

The solution is

$$y(t) = \begin{cases} y_0 e^{at} + \dfrac{b_0}{a} \left(e^{at} - 1\right) & \text{if } 0 \le t < t^* \\ y_0 e^{at} + \dfrac{b_0}{a} e^{at} + \left(\dfrac{b_1 - b_0}{a}\right) e^{a(t-t^*)} - \dfrac{b_1}{a} & \text{if } t^* \le t < \infty \end{cases}$$

Observe that the solution, at any point in time, is capitalizing on the past changes of the variable $b(t)$. It only responds to the shock **after** it is observed.

Example: anticipated shock in an exogenous variable Let us assume that y is a npn-pre-determined variable and assume there will be an additive shock in a prescribed future data in an exogenous variable that affects the dynamics of y. Assume that a rulling-out bubble, a transversality or a sustainability condition should be satisfied.

We can address this case through the terminal value problem

$$\dot{y} = ay + b(t), \text{ for } t \in [0, \infty)$$

where $a > 0$ and

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^* \\ b_1 & \text{if } t^* \leq t < \infty \end{cases}$$

and $\lim_{t\to\infty} y(t)e^{-at} = 0$.

The solution for the problem is

$$y(t) = \begin{cases} -\frac{b_0}{a} - \left(\frac{b_1 - b_0}{a}\right) e^{a(t-t^*)} & \text{if } 0 \leq t < t^* \\ -\frac{b_1}{a} & \text{if } t^* \leq t < \infty \end{cases}$$

Comparing to the initial-value problem we see that the solution has an anticipating feature: for $0 < t < t^*$ the solution depends on the expect value of the variable $b(t)$ **after** its change, b_1, and after the change, for $t \geq t^*$, it is not influenced by the value before the change, b_0.

These two cases illustrate two fundamental types of dynamics in macro-economics. Dynamic general equilibrium models have usualy both dynamics coupled.

1.4 References

Mathematics: there is a huge literature on scalar linear ODE, but (Hale and Koçak, 1991, ch 1) is a great modern.

Applications to economics: Gandolfo (1997).

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