Advanced Mathematical Economics

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Part V

First order partial differential equations

Chapter 7

First order quasi-linear partial differential equations

7.1 Introduction

In this chapter we present introductory results on first-order partial differential equations (PDE) and some applications to demography and economics. Those equations can also be called hyperbolic PDEs.

A first-order or hyperbolic PDEs is a known function of one or more unknown functions of more than one independent variable, together with its first-order derivatives. If there is only one unknown function we call the PDE scalar, and if there are two unknown functions we call it planar PDE.

In economics, usually one of the independent variables is time and the other independent variable is the support of some distribution. In physics these equations model advection, travelling or transportation behaviors.

In economics they are used in continuous time overlapping generations models, vintage capital models, interest rate term-structure models in continuous time. They are used in demographics for modelling age-dependent dynamics of population.

The Hamilton-Jacobi equation for deterministic optimal control problems with a finite horizon and a constraint given by a ordinary differential equation is also usually a non-linear first-order PDE.

The field is very large in terms of equations studied and methods involved, and is not generally in the toolbox of economists. We will only present a very brief introduction allowing to study very simple linear models.

There are two benefits from studying these equations: first, they provide a convenient modelling framework for setting up and characterizing the solution for models with heterogeneity, which is becoming topical in economics, second, they provide a better understanding of the implicit assumptions which are introduced when using ODE (or difference equations) models for studying dynamics of heterogeneity.

We assume throughout that there are only two independent variables (x, y) and deal mainly with equations of dimension one, $u(x, y) \in \mathbb{R}$.

Definition A **first-order partial differential equation** in two independent variables $(x, y) \in \Omega \subseteq \mathbb{R}^2$ is a known relation $F : D \to \mathbb{R}$ where $D \subset \mathbb{R}^5$ involving the unknown function $u : \Omega \to \mathbb{R}$ and its gradient

$$F(x, y, u(x, y), \nabla u(x, y)) = 0 \tag{7.1}$$

where $\nabla u(x, y)$ is the gradient of $u(.)$, i.e.

$$\nabla u(x, y) = (u_x(x, y), u_y(x, y))^\top$$

where $u_x(.) = \frac{\partial u(.)}{\partial x}$ and $u_y(.) = \frac{\partial u(.)}{\partial y}$.

Solutions A solution to a first-order PDE is a differentiable function $f(x, y)$ that satisfies the PDE. Existence and uniqueness of solutions for first-order PDE, and for problems involving them, are not guaranteed. Classic solutions are solutions such that $u \in C^1(\Omega)$. Otherwise we call generalised or weak solutions (i.e, non-differentiable or discontinuous solutions).

There are several **methods for obtaining solutions** which can be applied to general or specific problems. The more popular analytical methods are

- method of characteristics
- transformation methods (in particular application of Laplace transforms)

Those methods simplify the first-order PDE into a system of ODE's or a parameterised ODE.

Problems involving PDEs There are two main types of problems involving first-order PDE.

1. the Cauchy problem: there is a single constraint on (x, y) along a surface $\Gamma \in \Omega$:

$$\begin{cases} F(x, y, u(x, y), \nabla u(x, y)) = 0, & (x, y) \in \Omega \\ u \mid_\Gamma = \phi, & (x, y) \in \Gamma \subset \Omega \end{cases}$$

2. problems may involve two constraints, associated with each independent variable, for instance

$$\begin{cases} F(x,y,u(x,y),\nabla u(x,y)) = 0, & (x,y) \in \Omega \\ u\mid_{x=0} = \psi(y), & (0,y) \in \Omega \\ u\mid_{y=0} = \phi(x), & (x,0) \in \Omega \end{cases}$$

Well-posed problems Existence, uniqueness and properties of solutions vary widely. Again we have to distinguish existence properties of the PDE and of the problem involving the PDE (i.e., the PDE and the boundary conditions). A problem is **ill-posed** if, for instance, although the PDE has a solution the problem involving the PDE may not have a solution. A problem is **well-posed** if the general solution to the PDE has a particular solution satisfying the constraints of the problem.

Qualitative theory Linear PDEs, and well-posed problems involving liner PDEs, have explicit solutions. Therefore the distributional dynamics that characterizes their solution can be explicitly discovered.

For non-linear PDEs we are unaware of the existence of a qualitative theory as developed as the qualitative theory for ODE's. In particular, a Grobmann-Hartmann theorem for PDE does not seem to be available. There are phenomena that do not exist in ODE's: traveling waves, front waves, for instance.

Types of first-order PDE First-order PDE are classified into four categories:

- **linear**: it is linear in u_x, u_y and u,

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y) \tag{7.2}$$

- **semi-linear**: it is linear in u_x and u_y and non-linear in u, which only enters into the right-hand side,

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \tag{7.3}$$

- **quasi-linear**: it is linear in u_x and u_y and non-linear in u

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{7.4}$$

- **non-linear**: it is non-linear in u_x and u_y and u

$$F(x,y,u,u_x,u_y) = 0$$

  where F is non-linear in u_x and/or u_y

Linear equations can be classified further as non-autonomous or autonomous, if a, b, c and d are constants, or non-homogeneous or homogeneous, if function $c(x, y, u)$ is homogeneous in u.

In addition we can consider systems of hyperbolic equations

$$\mathbf{F}(\mathbf{x}, \mathbf{u}(\mathbf{x}), D_{\mathbf{x}}\mathbf{u}(\mathbf{x}) = \mathbf{0}$$

where $x \in \mathbb{R}^m$ and $\mathbf{u} : \mathbb{R}^m \to \mathbb{R}^n$.

For, instance a linear planar equation in two independent variables can be

$$\begin{aligned} a_{11}u_x(x, y) + a_{12}u_y(x, y) &= b_{11}u(x, y) + b_{12}v(x, y) \\ a_{21}v_x(x, y) + a_{22}v_y(x, y) &= b_{21}u(x, y) + b_{22}v(x, y) \end{aligned}$$

In the rest of the chapter, in section 7.2 we solve scalar linear equations with an infinite domain, in section 7.3 we deal with semi-linear equations, in section 7.4 we provide brief comments on quasi-linear scalar equations. In section 7.5 we solve some linear equations in the semi-infinite domain by using Laplace transform methods and in section 7.6 we refer to special solutions sometimes used in qualitatitive analysis and in section. Section 7.7 has several applications to economics and demography.

7.2 Scalar equations in the infinite domain

In this section we solve hyperbolic PDE in the infinite domain. We denote the independent variables by (x, y) and assume that the domain of (x, y) is the whole set $\Omega = \mathbb{R}^2$. We consider scalar function $u : \mathbb{R}^2 \Rightarrow \mathbb{R}$ as our dependent variable and consider quasi-linear equations of type [1]

$$a(x, y, u)u_x(x, y) + b(x, y, u)u_y(x, y) = c(x, y, u(x, y)),\ (x, y) \in \Omega$$

and $a(.)$, $b(.)$, and $c(.)$ are known functions.

One useful method to solve the hyperbolic PDE in the infinite domain is the **method of characteristics**.

The following definition is useful

Definition 1. ***Directional derivative*** *Consider a function $f(x, y)$, the derivative of f in the direction given by vector $\mathbf{v} = (v_x, v_y)^\top$ is*

$$\nabla_{\mathbf{v}} f(x, y) = \lim_{h \to 0} \frac{f(x + v_x h, y + v_y h) - f(x, y)}{h}$$

[1] The following notation sometimes is more convenient

$$a(x, y, u)\partial_x u(x, y) + b(x, y, u)\partial_y u(x, y) = c(x, y, u(x, y)),\ (x, y) \in \Omega.$$

if the limit exists.

If function $f(x, y)$ is differentiable, the directional derivative of f in the direction given by vector $\mathbf{v} = (v_x, v_y)^\top$ is equal to the dot product[2]

$$\nabla_{\mathbf{v}} f(x, y) = \nabla f(x, y) \cdot \mathbf{v} = (f_x, f_y) \cdot (v_x, v_y) = f_x(x, y)v_x + f_y(x, y)v_y$$

We start with simple linear PDE to illustrate their solution using the **method of characteristics**. It is very important to remember that we assume, in all this section, that there are no restrictions on the domain of the independent variables, x and y in this section, that is, we assume $(x, y) \in \mathbb{R}^2$.

7.2.1 The two simplest first order PDEs

We start with the two simplest first-order PDE: $u_x(x, y) = 0$ and $u_y(x, y) = 0$.

Proposition 1. *The equation*

$$u_x(x, y) = 0,\ (x, y) \in \Omega = \mathbb{R}^2$$

has the general solution

$$u(x, y) = f(y)$$

where $f \in C^1(\mathbb{R})$ is an arbitrary function.

Proof. First observe that the solution to equation $u_x = 0$ is any function that remains constant along direction $v = (1, 0)^\top$. This can be proved by observing that the directional derivative along that direction is zero,

$$\nabla u(x, y) \cdot (1, 0) = u_x \times 1 + u_y \times 0 = u_x = 0.$$

This is equivalent to any function function, $f(.)$, that remains unchanged along any changes which are parallel to the x-axis, that is $f(y)$. □

In order to have a better intuition on this result, consider an ODE $u_x(x) = 0$, where $u(x)$ is an unknown function of single independent variable $u : \mathbb{R} \to \Omega \subseteq \mathbb{R}$. This equation has the solution $u(x) = k$ where k is an arbitrary **point** in the domain of $u(.)$, $\Omega \subseteq \mathbb{R}$. In the case of the PDE $u_x(x, y) = 0$ the solution is $u(x, y) = f(y)$ where $f(y)$ is an arbitrary differentiable **function** over $\mathbb{R}$.

---

[2] Observe there is a relationship with the total differential. Let $z = f(x, y)$, where $f(.)$ is differentiable. The total differential is $dz = f_x(x, y)dx + f_y(x, y)dy$. If we write $dx = v_x h$ and $dy = v_y h$ then $\nabla f(x, y) \cdot \mathbf{v} = \lim_{h\to 0} \frac{dz}{h}$.

Proposition 2. *The equation*

$$u_y(x,y) = 0,\ (x,y) \in \Omega = \mathbb{R}^2$$

has the general solution

$$u(x,y) = f(x)$$

where $f \in C^1(\mathbb{R})$ is an arbitrary function.

Proof. Not the PDE solution is constant along the direction $v = (0,1)^\top$, because it is equivalent to the directional derivative along that direction being equal to zero,

$$\nabla u(x,y) \cdot (0,1) = u_y = 0.$$

In this case, the slution is any function function, $f(.)$, that remains unchanged along any changes which are parallel to the y-axis, that is $f(x)$. □

From those two previous results we can understand more general linear first order scalar PDE's as being constant along particular directions, which are called **characteristics**.

7.2.2 Linear equation with constant coefficients

Next we consider linear equations without side constrains and Cauchy problems for linear hyperbolic equations defined in the infinite domain.

Free boundary problems

Consider the first order linear autonomous PDE

$$u_x(x,y) + a u_y(x,y) = 0,\ (x,y) \in \Omega = \mathbb{R}^2 \tag{7.5}$$

where $a \neq 0$ is an arbitrary constant.

Proposition 3. *The general solution of PDE (7.5) is*

$$u(x,t) = f(y - ax),$$

where $f \in C^1(\mathbb{R})$ is an arbitrary function.

Proof. First, observe that the PDE (7.5) determines a function $u(x,y)$ which is constant along the direction $v = (1,a)^\top$, because

$$\nabla u \cdot (1,a) = u_x + a u_y = 0.$$

To interpret this geometrically consider the three-dimensional surface

$$S \equiv \{(x, y, u(x, y))\}.$$

A particular solution $(x_0, y_0, u(x_0, y_0))$ belongs to the surface S, and the PDE traces out a curve C over the surface, in which u remains constant.

In order to determine curve C we parametrize the two independent variables as $x = X(s)$, $y = Y(s)$, where $s \in \mathbb{R}$. Then, we get a parameterized value for u, as $u = U(s) = u(X(s), Y(s))$. Therefore C can be represented by

$$C = \{(X(s), Y(s), U(s))\}.$$

Taking derivatives to $u = U(s) = u(X(s), Y(s))$ we find

$$\frac{dU}{ds} = \frac{du(X(s), Y(s))}{ds} = u_x \frac{dX}{ds} + u_y \frac{dY}{ds}$$

The PDE will hold if and only if the following conditions hold:

- the characteristic system

$$\begin{aligned} \frac{dX}{ds} &= 1 \\ \frac{dY}{ds} &= a \end{aligned}$$

- the compatibility condition

$$\frac{dU}{ds} = 0$$

Solving the characteristic system and the compatibility equation we get

$$\begin{aligned} x &= X(s) = s + c_1 \\ y &= Y(s) = as + c_2 \\ u &= U(s) = f(k) \end{aligned}$$

where c_1 and c_2 are arbitrary constants, and $f(k)$ is an arbitrary function evaluated at an arbitrary point. If we eliminate s, from the solution of the characteristic system, we find

$$y - ax = c_2 - ac_1 = k$$

where k is a constant. Then we find the general solution for (7.5) to be constant along the direction $(1, a)^\top$,

$$u(x, y) = U(s) = f(k) = f(y - ax),$$

where f is an arbitrary C^1 function. □

In order to check that this is a solution, assume that $u(x,y) = f(y-ax)$. Then

$$u_x(x,y) + au_y(x,y) = -af^{'}(y-ax) + af^{'}(y-ax) = 0$$

which is equation (7.5).

We call **projected characteristic** to the line $y = k + ax$, where $k \in \mathbb{R}$ is arbitrary, and we call $f(y-ax)$ the **first integral** of the PDE.

Figure 7.1 depicts projected characteristic lines for cases $a > 0$ and $a < 0$. These curves correspond to the projection in the space (x,y) of the solution curves of the PDE (7.5) over which $u(x,y)$ is constant.

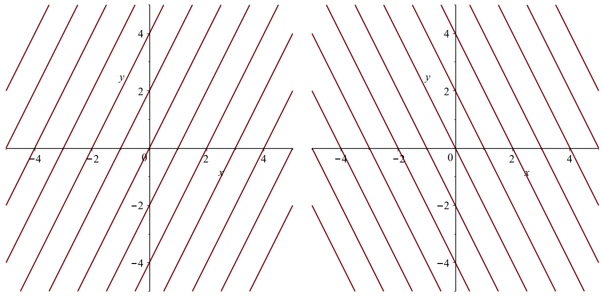


Figure 7.1: Characteristic lines for (7.5) for $a > 0$ (left figure) and $a < 0$ (right figure)

Next we introduce the linear first-order PDE with an homogeneous right-hand side

$$u_x + au_y = bu, (x,y) \in \mathbb{R}^2 \tag{7.6}$$

where $a \neq 0$ and $b \neq 0$ are constants.

Proposition 4. *The general solution of PDE* (7.6) *is*

$$u(x,y) = f(y-ax)e^{bx}$$

where $f(.)$ is an arbitrary C^1 function.

Proof. To solve it by using the method of characteristics we parameterize again both the independent variables, $x = X(s)$ and $y = Y(s)$, and the unknown function $u = u(X(s), Y(s)) = U(s)$ and solve the system

$$\frac{dX}{ds} = 1, \ \frac{dY}{ds} = a, \ \frac{dU}{ds} = bU$$

which have solutions

$$x = X(s) = s + c_1, \ y = Y(s) = as + c_2, \ u = U(s) = g(k)e^{bs}.$$

where c_1 and c_2 are arbitrary constants and $g(k)$ is an arbitrary function. Then $s = x - c_1$ and the projected characteristic if again $y - ax = c_2 - ac_1 = k$ and $u = g(k)e^{bc_1}e^{bx} = f(k)e^{bx}$. $\square$

This equation has the same projected characteristics as shown in figure 7.1 but now, the value of $u(.)$ will not remain constant along the characteristics, as in the case of equation (7.5): it will grow or decay along the characteristic at the rate b, respectively, if $b > 0$ or if $b < 0$.

Cauchy problems

Consider again equation (7.5) and assume that we know the distribution for y for a particular value of x, say $x = 0$. If x is interpreted at time, and y as another independent variable, we call the problem an **initial-value problem** (which is a particular case of the Cauchy problem)

$$\begin{cases} u_x + a u_y = 0, & (x, y) \in \mathbb{R}_+ \times \mathbb{R} \\ u = \phi(y), & (x, y) \in \{x = 0\} \times \mathbb{R} \end{cases} \tag{7.7}$$

where ϕ is a **known** C^1 function. We can write the initial condition as $u(0, y) = \phi(y)$ where $\phi(.)$ is known.

Proposition 5. *The general solution to the Cauchy problem* (7.7) *is*

$$u(x, y) = \phi(y - ax),\ (x, y) \in \Omega = \mathbb{R}^2$$

Proof. In the three-dimensional surface S, previously presented, the **constraint defines a curve** $(0, y, \phi(y))$ that has a **projection** in the (x, y) space characterized by a curve passing through point $\{(0, y)\}$. Using the same method that we used to determine the characteristic curve C, we parameterize the constraint Γ by a new variable r, such that it defines a direction $\Gamma = \{(0, r)\}$.

Introducing the two parameterizations (associated to the characteristic curve and the initial condition) we define

$$x = X(s, r),\ y = Y(s, r),\ u = U(s, r) = u(X(s, r), Y(s, r)).$$

The characteristic system and the compatibility condition become the system of parameterized (by r) ODE's over the independent variable s

$$\begin{aligned} \frac{\partial X(s, r)}{\partial s} &= 1 \\ \frac{\partial Y(s, r)}{\partial s} &= a \\ \frac{\partial U(s, r)}{\partial s} &= 0 \end{aligned}$$

that we can solve, together with the (given) initial conditions

$$\begin{aligned} X(0,r) &= 0 \\ Y(0,r) &= r \\ U(0,r) &= \phi(r). \end{aligned}$$

The solution to the three ODE initial value problems allows us to obtain a relationship between the initial independent variables and the parameters related to the characteristic and the initial condition

$$x = X(s,r) = s \tag{7.8}$$

$$y = Y(s,r) = as + r \tag{7.9}$$

and

$$u = U(s,r) = \phi(r).$$

To get the solution in the original independent variables, we have to obtain the reversed relationships, say $s = S(x,y)$ and $r = R(x,y)$. In order to get it, observe that the solution for the characteristic system can be written as $(x,y) = G(s,r)$. If this system is invertible then $(s,r) = G^{-1}(x,y)$. The system (7.8)-(7.9) can provide this solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} s \\ r \end{pmatrix} \Leftrightarrow \begin{pmatrix} s \\ r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y - ax \end{pmatrix}$$

Therefore, $s = x$ and $r = y - ax$. Then $u(x,y) = U(s,r) = \phi(r) = \phi(y - ax)$ □

Example If the initial distribution is $u(0,y) = \phi(y) = e^{-y^2}$, then the solution to the Cauchy problem (7.7) is

$$u(x,y) = e^{-(x-y)^2}.$$

Figure 7.2 illustrates this case. The projected characteristics are again as those depicted in figure **??**.

Now, consider an equation (7.6) and the associated Cauchy problem

$$\begin{cases} u_x + u_y = bu, & (x,y) \in \mathbb{R}^2 \\ u = \phi(y), & (x,y) \in \{x = 0\} \times \mathbb{R} \end{cases} \tag{7.10}$$

where $c \neq 0$ is a constant.

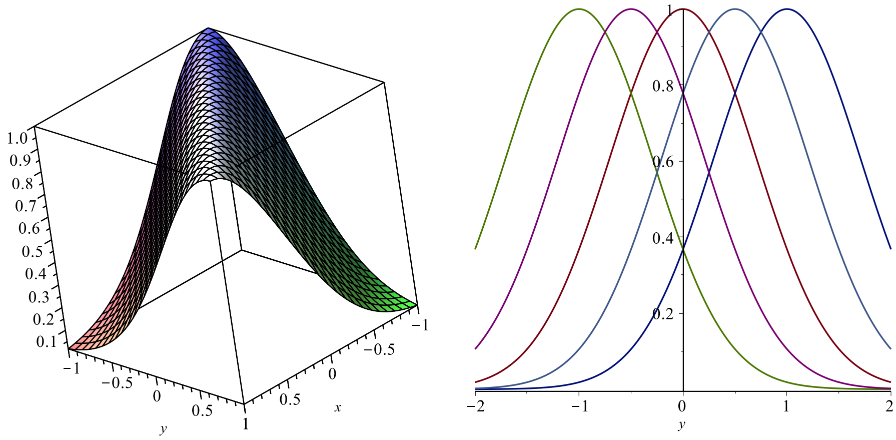


Figure 7.2: Solution for the problem $u_x + u_y = 0$ and $u(0, y) = e^{-y^2}$, 3d plot and 2d plot for $x \in \{-1, -0.5, 0, 0.5, 1\}$

Proposition 6. *The solution do problem* (7.10) *is*

$$u(x, y) = \phi(y - x)e^{bx}$$

Exercise: prove this.

In Figure 7.3 we present an illustration. Observe that for $b > 0$ the solution has both a advection (i.e., transport) and a growing behavior.

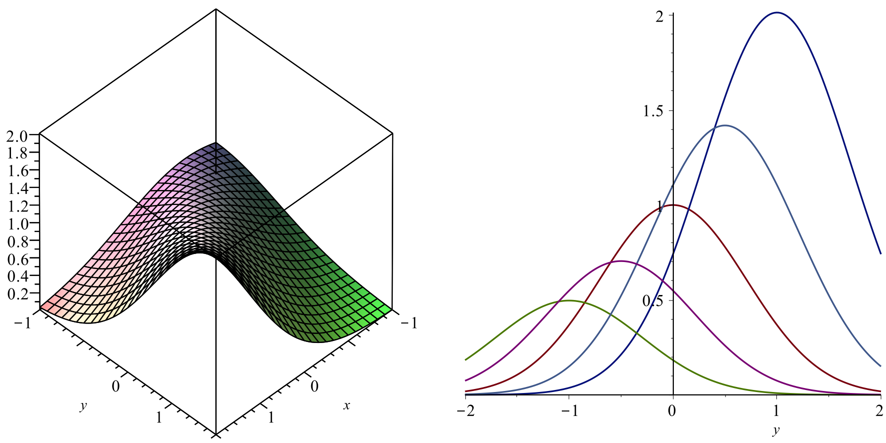


Figure 7.3: Solution for the problem $u_x + u_y = 0.7u$ and $u(0, y) = e^{-y^2}$, 3d plot and 2d plot for $x \in \{-1, -0.5, 0, 0.5, 1\}$

Next, we will see what we can learn from the application of the method of characteristics to solving the semi-linear and the quasi-linear equations.

7.3 Semi-linear equation in the infinite domain

We consider first one simple semi-linear equation that can be solved by transformation to a linear equation. Next we present conditions for the existence of solutions to more general semi-linear equations, with or without a zero right-hand side, i.e, with $c(x, y, u) = 0$ or $c(x, y, u) \neq 0$.

7.3.1 The transport equation

We consider a simple example called the **transport equation**. To simplify assume that the independent variables are (t, x) and that their domain is unbounded, i.e., $(t, x) \in \mathbb{R}^2$ [3] :

$$\begin{cases} \partial_t u(t,x) + \partial_x \left(\mu x u(t,x)\right) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0,x) = \phi(x), & (t,x) \in \{t=0\} \times \mathbb{R} \end{cases} \tag{7.11}$$

Proposition 7. *The solution to the transport equation Cauchy problem* (7.11) *is*

$$u(t,x) = e^{-\mu t} \phi\left(x e^{-\mu t}\right).$$

Proof. The PDE can be equivalently written as

$$u_t(t,x) + \mu x u_x(t,x) + \mu u(t,x) = 0.$$

Let us consider a change in variables: $x = X(y) = e^{\mu y}$ and

$$v(t,y) = e^{\mu t} u(t, X(y)).$$

Taking derivatives for t and y and applying the relationship in equation (7.11) we find that

$$v_t(t,y) + v_y(t,y) = 0$$

if and only if $u_t(t,x) + \mu x u_x(t,x) + \mu u(t,x) = 0$. This equation has the form of (7.5), with $a = 1$. As $v(0,y) = u(0, X(y)) = \phi(X(y))$ we can use the solution to Cauchy problem (7.7) to obtain

$$v(t,y) = \phi(X(y-t)) = \phi\left(e^{\mu(y-t)}\right) = \phi\left(X(y) e^{-\mu t}\right).$$

To obtain the solution we just need to transform back to the original function $u(.)$ and substitute $X(y) = x$. ☐

[3] We use the notation $\partial_{x_i} f(\mathbf{x}) = \dfrac{\partial f(\mathbf{x})}{\partial x_i}$, where $\mathbf{x} = (x, \dots, x_i, \dots x_n) \in \mathbb{R}^n$.

The projected characteristics are as in Figure 7.4 for $\mu > 0$. Differently from the previous cases we see that the characteristics are non-linear, and, in this case exponentially growing.

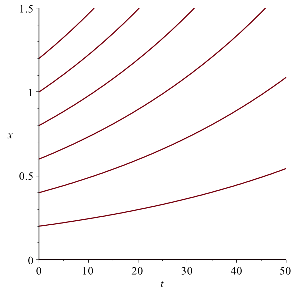


Figure 7.4: Characteristic lines for (7.11) for $a > 0$

7.3.2 Semi-linear equation with zero right-hand-side

We consider the problem for a more general case, in which the coefficient functions are not specified

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = 0, & (x,y) \in \mathbb{R}^2 \\ u\mid_\Gamma = \phi, & (x,y) \in \Gamma \subset \mathbb{R}^2 \end{cases}$$

where $a(.)$, $b(.)$ and $c(.)$ are C^1 functions in $\mathbb{R}^2$, and there is a constraint given by curve Γ. The solution to this problem depends on the form of the constraint surface Γ.

Let us consider the points in the constrained set parameterised by r, write $\Gamma = \{(x,y) = (\gamma_1(r), \gamma_2(r))\}$ and define

$$A(r) \equiv a(\gamma_1(r), \gamma_2(r))$$
$$B(r) \equiv b(\gamma_1(r), \gamma_2(r))$$

We say that the constraint Γ is **characteristic** if it is tangent to the projected characteristic and Γ is **non-characteristic** if it is not tangent to the projected characteristic.

Therefore, Γ is characteristic if

$$\frac{A(r)}{B(r)} = \frac{\gamma_1'(r)}{\gamma_2'(r)}$$

and Γ is non-characteristic if

$$\frac{A(r)}{B(r)} \neq \frac{\gamma_1'(r)}{\gamma_2'(r)} \tag{7.12}$$

Proposition 8. *Consider the Cauchy problem* (7.4). *A unique solution exists if* Γ *is non-characteristic in all its domain. The local solution to the problem exist and is unique, and can be written as*

$$u(x,y) = \phi(G^{-1}(x,y)).$$

where $\det G(x,y) \neq 0$.

Proof. In order to see this we proceed by two phases.

- In the first phase we apply the same method as before. We introduce the change in coordinates $x = X(s,r)$, $y = Y(s,r)$, implying $u = U(s,r) = u(x(s,r), y(s,r))$. The characteristic system and the compatibility condition become

$$\begin{aligned} \frac{\partial X(s,r)}{\partial s} &= a(X(s,r), Y(s,r)) \\ \frac{\partial Y(s,r)}{\partial s} &= b(X(s,r), Y(s,r)) \\ \frac{\partial U(s,r)}{\partial s} &= 0 \end{aligned}$$

  and the constraints on their values introduced by Γ that we associate with $s = 0$ are

$$\begin{aligned} X(0,r) &= \gamma_1(r) \\ Y(0,r) &= \gamma_2(r) \\ U(0,r) &= \phi(r). \end{aligned}$$

  If we solve the ODE characteristic system together with the initial conditions we obtain the transformation $(x,y) = G(s,r)$, where

$$x = X(s,r) \tag{7.13}$$
$$y = Y(s,r). \tag{7.14}$$

  In order to obtain the solution satisfying $u = U(0,r) = \phi(r)$ we need to solve system (7.13)-(7.14), that is, we need to find $(s,r) = G^{-1}(x,y)$.

- Second phase: The system is locally invertible to $s = S(x,y)$ and $r = R(x,y)$ if we can apply the inverse function theorem $(s,r) = G^{-1}(x,y)$. This is possible if the Jacobian of G has a non-zero determinant evaluated at points $(0,r)$.

  The Jacobian of system (7.13)-(7.14) evaluated at point $(s,r) = (0,r)$ is

$$D(G)|_\Gamma = \begin{pmatrix} X_s(0,r) & X_r(0,r) \\ Y_s(0,r) & Y_r(0,r) \end{pmatrix} = \begin{pmatrix} a(\gamma_1(r), \gamma_2(r)) & \gamma_1'(r) \\ b(\gamma_1(r), \gamma_2(r)) & \gamma_2'(r) \end{pmatrix} = \begin{pmatrix} A(r) & \gamma_1'(r) \\ B(r) & \gamma_2'(r) \end{pmatrix}$$

Then $\det\left(D(G|_\Gamma)\right) \neq 0$ if condition (7.12) holds, and, using the inverse function theorem, we can (at least locally) determine $(s,r)|_{s=0} = G^{-1}(x,y)$, and the solution will have the generic form $u(x,y) = \phi(G^{-1}(x,y))$ □

This means that, geometrically, the solution will propagate not along parallel characteristic lines but along lines which can change slope depending on the values of x and y.

7.3.3 General semi-linear equation

The Cauchy problem for a semi-linear equation and an associated boundary in a surface Γ is

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = c(x,y,u), & (x,y) \in \mathbb{R}^2 \\ u\mid_\Gamma = \phi, & (x,y) \in \Gamma \subset \mathbb{R}^2 \end{cases}$$

where $a(.)$ and $b(.)$ are C^1 functions in $\mathbb{R}^2$ and $c(.)$ is a C^1 function in $\mathbb{R}^3$. Observe that the function u enters, possibly in a non-linear from, in the right-hand side.

Again we introduce a parameterisation associated with the characteristic surface and the boundary surface by a pair (s,r) and set $x = X(s,r)$ and $y = Y(s,r)$ and $u = U(s,r) = u(X(s,r), Y(s,r))$

In this case the characterstic equation system and the compatibility condition become

$$\begin{aligned} \frac{\partial X(s,r)}{\partial s} &= a\left(X(s,r), Y(s,r)\right) \\ \frac{\partial Y(s,r)}{\partial s} &= b\left(X(s,r), Y(s,r)\right) \\ \frac{\partial U(s,r)}{\partial s} &= c\left(X(s,r), Y(s,r), U(s,r)\right) \end{aligned}$$

and the constrains on their values introduced by Γ that we associate with $s = 0$

$$\begin{aligned} X(0,r) &= \gamma_1(r) \\ Y(0,r) &= \gamma_2(r) \\ U(0,r) &= \phi(r) \end{aligned}$$

We observe again that from the solution of the two first ODE's we get a relationship $(x,y) = G(s,r)$ and if Γ is non-characteristic we get, at least locally $(s,r) = G^{-1}(x,y)$, which allows for uniqueness and existence of solutions for the PDE problem. The only difference is related to the fact that now the right hand side of the compatibility condition for U depends on U.

7.4 Quasi-linear equations

Let us consider the semi-linear equation and an associated boundary in a surface Γ

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), & (x,y) \in \mathbb{R}^2 \\ u \mid_\Gamma = \phi, & (x,y) \in \Gamma \subset \mathbb{R}^2 \end{cases}$$

where $a(.)$, $b(.)$ and $c(.)$ is a C^1 functions in $\mathbb{R}^3$.

Again we introduce a parameterisation associated with the characteristic surface and the boundary surface by a pair (s,r) and set $x = X(s,r)$ and $y = Y(s,r)$ and $u = U(s,r) = u(X(s,r), Y(s,r))$

In this case the characterstic equation system and the compatibility condition become

$$\begin{aligned} \frac{\partial X(s,r)}{\partial s} &= a\left(X(s,r), Y(s,r), U(s,r)\right) \\ \frac{\partial Y(s,r)}{\partial s} &= b\left(X(s,r), Y(s,r), U(s,r)\right) \\ \frac{\partial U(s,r)}{\partial s} &= c\left(X(s,r), Y(s,r), U(s,r)\right). \end{aligned}$$

This system, differently from the previous cases, lost their recursive structure, in the sense that we cannot separate the determination of the solutions for $X(.)$ and $Y(.)$ from $U(.)$: the two independent variables, X and Y, and the dependent variable, U, are jointly determined. In order to solve the system, Γ provides the boundary conditions for $s = 0$:

$$\begin{aligned} X(0,r) &= \gamma_1(r) \\ Y(0,r) &= \gamma_2(r) \\ U(0,r) &= \phi(r) \end{aligned}$$

Now the non-characteristic conditions for (Γ, ϕ) are more involved because all three differential equations depend on (X, Y, U) and the conditions for the application of the non-characteristic condition may not hold.

The geometric meaning is the following: while for linear and semi-linear PDE the characteristic lines are parallel and do not cross, for the quasi-linear case this may not be the case. At singularity points the uniqueness and even the existence of solutions may break down.

A well known quasi-linear first-order PDE is the inviscid Burger's equation (see `https://en.wikipedia.org/wiki/Burgers%27_equation`)

$$\begin{cases} u_t + uu_x = 0, & (t,x) \in \mathbb{R}^2 \\ u(0,x) = \phi(x) & (t,y) \in \{t = 0\} \times \mathbb{R} \end{cases}$$

It can be proved that the characteristic equations can intersect which implies that the solutions cannot be unique at those singular points. Introducing some solvability conditions, gives birth to shock waves, which is a type of behavior not presented in linear hyperbolic PDE's.

7.5 The linear equation in the semi-infinite domain

In the previous cases we assumed that the independent variables were defined in the space $\Omega = \mathbb{R}^2$. The solution of the first-order PDE and/or of the associated problems varies both in terms of the existence and of the methods of determination if the domain is different, that is $\Omega \subset \mathbb{R}^2$. In this case we may have as solutions not functions (single-valued continuous mappings) but generalized functions (also called weak solutions).

7.5.1 Linear equation with zero right-hand side

To illustrate this, assume that $\Omega = \mathbb{R}^2_{++}$, that is $x > 0$ and $y > 0$ and consider the problem

$$\begin{cases} u_x + a u_y = 0, & (x, y) \in R^2_{++} \\ u(x, 0) = \psi(x), & (x, y) \in R_{++} \times \{y = 0\} \\ u(0, y) = \phi(y), & (x, y) \in \{x = 0\} \times R_{++} \times \end{cases} \tag{7.15}$$

A convenient way to solve this equation is to use **Laplace transforms** instead of the method of characteristics (see the Appendix). In order to do this we pick one of the independent variables as a parameter (for instance x) and keep one variable as an independent variable (for instance y)[4]. Laplace transforms are convenient because the domain of transformation is the semi-infinite interval $[0, \infty)$.

The method of solution follows the steps:

1. First, we apply Laplace transforms to go from the PDE into a parameterized ODE
2. Second, we solve the ODE and apply the transforms of the boundary conditions
3. Finally, we apply inverse Laplace transforms to obtain the solution

Proposition 9. *The solution to Cauchy problem* (7.15) *is we get*

$$u(x, y) = \phi(y - ax) H(y - ax) + \mathcal{L}^{-1} \left[\int_0^x \psi(s) e^{-a\xi(x-s)} ds \right] (y) \tag{7.16}$$

where

$$H(z) = \begin{cases} 0, & \text{if } z \leq 0 \\ 1, & \text{if } z > 0 \end{cases}$$

is the Heaviside "function" and $\mathcal{L}^{-1}[f(x)](y)$ *is the inversa Laplace transform.*

[4] The choice can be done in a way to simplify the solution of the problem, given the constraints.

Proof. Let $U(x, \xi)$ be the Laplace transform of $u(x, y)$ taking variable x as a parameter, that is

$$\mathcal{L}[u(x,y)](\xi) = \int_0^\infty e^{-\xi y} u(x,y) dy = U(x,\xi),$$

where $\xi > 0$.

Equation $u_x(x,y) + a u_y(x,y) = 0$ holds if and only if

$$\int_0^\infty e^{-\xi y} \left(u_x(x,y) + a u_y(x,y) \right) dy = 0$$

But

$$\begin{aligned}\int_0^\infty e^{-\xi y} \left(u_x(x,y) + a u_y(x,y) \right) dy &= \int_0^\infty e^{-\xi y} \frac{\partial u}{\partial x}(x,y) dy + a \int_0^\infty e^{-\xi y} \frac{\partial u}{\partial y}(x,y) dy = \\ &= U_x(x,\xi) + a \left(\int_0^\infty e^{-\xi y} u(x,y) - \int_0^\infty u(x,y) \frac{d}{dy} \left(e^{-\xi y} \right) dy \right) = \\ &= U_x(x,\xi) - a u(x,0) + a \xi U(x,\xi) = 0\end{aligned}$$

using integration by parts and

$$U_x(x,\xi) = \mathcal{L}[u_x(x,y)](\xi) = \int_0^\infty e^{-\xi y} u_x(x,y) dy.$$

We found that

$$\mathcal{L}[u_y(x,y)](\xi) = \int_0^\infty e^{-\xi y} u_y(x,y) dy = \xi U_y(x,\xi) - u(x,0) = \xi U_y(x,\xi) - \psi(x).$$

Then the PDE, in Laplace transforms, is equivalent to the linear ODE in the transformed variable ξ and parameterized by x,

$$U_x(x,\xi) + a(\xi U(x,\xi) - \psi(x)) = 0.$$

In order to solve the Cauchy problem, we also need to introduce the Laplace transform of $\phi(y)$, that is

$$\mathcal{L}[u(0,y)](\xi) = \int_0^\infty e^{-\xi y} \phi(y) dy = \Phi(\xi).$$

Then we get an initial-value problem for the parameterized (by ξ) ODE

$$\begin{cases} U_x(x,\xi) = -a\xi U(x,\xi) - a\psi(x), & x > 0 \\ U(0,\xi) = \Phi(\xi), & x = 0. \end{cases}$$

The solution is

$$U(x,\xi) = \Phi(\xi) e^{-a\xi x} + \int_0^x \psi(s) e^{-a\xi(x-s)} ds$$

Then, applying an inverse Laplace transform

$$u(x,y) = \mathcal{L}^{-1}\left[U(x,\xi) \right](y) = \frac{1}{2\pi i} \lim_{Y \to \infty} \int_{\gamma - iY}^{\gamma + iY} e^{\xi y} F(z) dz$$

we get the solution (7.15). $\square$

Example 1. In order to have an intuition on the solution consider the case: $\phi(y) = 0$ and $\psi(x) = \psi$, a constant. In this case

$$U(x,\xi) = \psi \int_0^x e^{-a\xi(x-s)} ds = \frac{\psi}{a}\left(\frac{1-e^{-a\xi x}}{\xi}\right).$$

Then

$$u(x,y) = \frac{\psi}{a}\mathcal{L}^{-1}\left[\frac{1-e^{-a\xi x}}{\xi}\right](y) = \frac{\psi}{a}H(ax-y)$$

That is the solution is

$$u(x,y) = \begin{cases} 0 & \text{for } y \geq ax \\ \dfrac{\psi}{a} & \text{for } 0 < y < ax \end{cases}$$

In this case the solution takes a constant value for $\{(x,y) : 0 < y < ax\}$ where $a > 0$ and $x > 0$, and it is equal to zero elsewhere.

Example 2. If, instead, we had the case $\phi(y) = e^{by}$ and $\psi(x) = 0$ we would have

$$u(x,y) = \phi(y-ax)H(y-ax) = e^{b(y-ax)}H(y-ax)$$

$$u(x,y) = \begin{cases} 0 & \text{for } 0 < y < ax \\ e^{b(y-ax)} & \text{for } y > ax. \end{cases}$$

In this case the projected characteristics are as in figure :

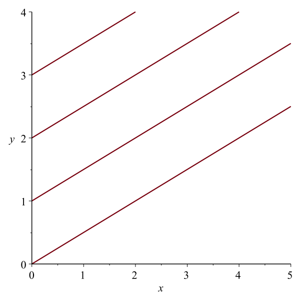


Figure 7.5: Characteristic lines for (7.15) for $a > 0$ $\psi(x) = 0$ and $\phi(y) = e^{by}$

7.5.2 Linear equation with homogeneous right-hand side

Now consider the problem

$$\begin{cases} u_x + u_y = au, & x > 0, y > 0 \\ u(x,0) = e^{bx}, & x > 0 \\ u(0,y) = 0, & y > 0 \end{cases}$$

Using the same method we find the solution

$$u(x,y) = H(y-x)e^{(a-b)y}\left(1-e^{bx}\right)$$

or, equivalently,

$$u(x,y) = \begin{cases} e^{(a-b)y}\left(1-e^{bx}\right), & \text{if } y \le x \\ 0, & \text{if } y > x \end{cases}$$

A graphical depiction of the solution for $a = -0.01$ and $b = 0.1$ presented in Figure 7.6. The projected characteristics are as in Figure 7.5 but, differently from that case where u is constant, now the solution growth at the rate a along the characteristic lines.

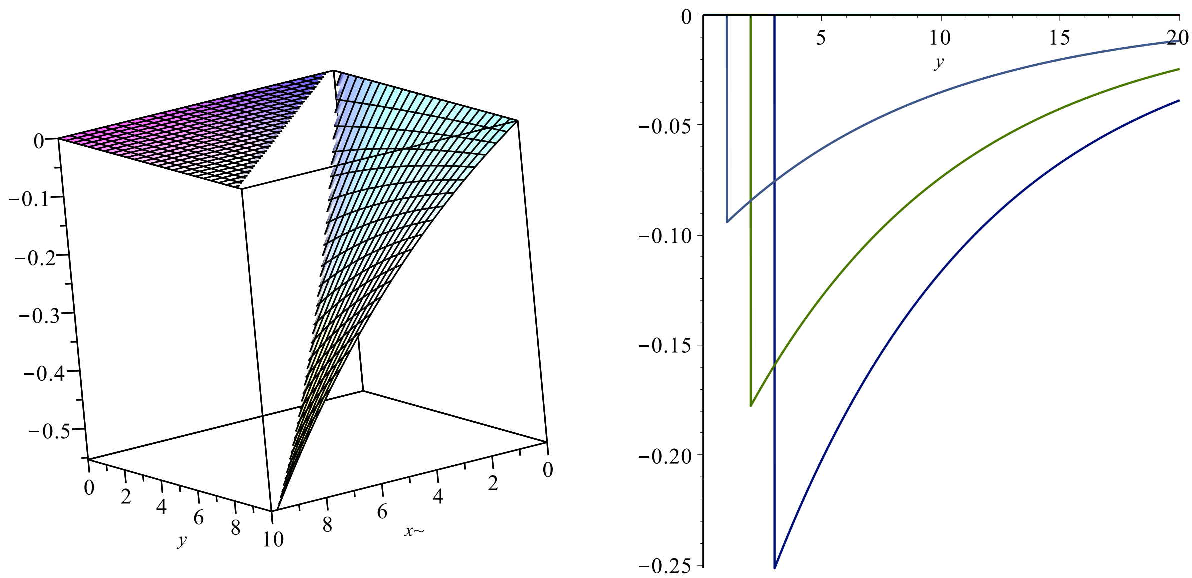


Figure 7.6: Solution for the problem $u_x + u_y = -0.01u$, $u(0,y) = 0$, $u(x,0) = e^{0.1x}$ and defined for $x > 0$ and $y > 0$ 3d plot and 2d plot for $x \in \{0, 0.5, 1\}$

7.6 Particular solutions

Some particular solutions are sometimes useful for characterising more general cases:

7.6.1 Steady-states

For the autonomous semi-linear equation, in which $u = u(t, x)$ where we assume the independent variable t is time

$$u_t + u_x = F(x, u).$$

This equation describes an distribution moving along time where both advection and growth dynamics are considered.

A steady-state is a **function** $\bar{u}(x)$ such that $u_t(t, x) = 0$, formally

$$\bar{u}(x) = \{h(x) : h^{'}(x) = F(x, h(x))\}$$

If there is asymptotic stability in a sense similar to asymptotic stability for ODE's (ordinary differential equation) then $\bar{u}(x)$ would be a limiting distribution for the first-order PDE

$$\lim_{t\to\infty} u(t, x) = \bar{u}(x)$$

Observe that $\bar{u}(x)$ is the solution of an ODE in which the independent variable is x. Therefore, while an ODE solution converges to a point a first-order PDE solution converges to a function of x.

7.6.2 Traveling-wave solutions

Consider the problem

$$u_t + cu_x = F(u)$$

Let $\xi = x - \lambda t$ and look for solutions of the from $u(x - \lambda t) = U(\xi)$. Then we can write the PDE as an ODE over the independent variable ξ, because $u_x = U^{'}(\xi)\xi_x = U^{'}(\xi)$ and $u_t = U^{'}(\xi)\xi_t = -U^{'}(\xi)\lambda$ then the travelling-wave solution of the PDE verify

$$U^{'}(\xi)(-\lambda + c) = F(U(\xi))$$

which is a non-linear ODE. Off course, along a characteristic $\lambda = c$ and $U(\xi)$ is constant.

7.7 Applications

7.7.1 Age-structured population dynamics

The exponential model for population dynamics, $\dot{n} = \mu n$, where μ is the difference between the fertility rate and the mortality rate, has the solution $n(t) = n(0)e^{\mu t}$, Although this model may be a good approximation asymptotically, in the shorter run there is a large deviation. One of the reasons for the deviation is related to the fact that both fertility and mortality rates are age-dependent. If we introduce age-dependent mortality and fertility rates the dynamics of the population is governed by a first-order PDE.

Let $n(a,t)$ be the number of females of age a at time t in a population. The number of females between ages a and $a+\Delta a$ at time t is $n(a,t)da$. The rate of change of the number of females in the age interval Δa is equal to the rate of entry at a minus the rate of exit at $a+\Delta a$ minus the number of deaths,

$$\frac{\partial n}{\partial t}\Delta a = J(a,t) - J(a+\Delta a,t) - \mu(a.t)n(a,t)\Delta a$$

where $J(.)$ is the flow of entry and $\mu(.)$ is the mortality rate. Dividing by Δa and letting it go to zero we find

$$\frac{\partial n}{\partial t} = -\frac{\partial J}{\partial a} - \mu(a,t)n$$

But

$$J(a,t) = n(a,t)\frac{da}{dt}$$

where $da/dt = 1$ is the flow of individual in age per unit of time. Therefore we have the equation for an age-dependent population

$$n_t + n_a = \mu(a,t)n.$$

The McKendry model further assumes an initial population distribution and an age-dependent fertility

$$\begin{cases} n_t + n_a = -\mu(a,t)n, & (a,t) \in (0,\omega) \times (0,\infty) \\ n(a,0) = n_0(a), & (a,t) \in (0,\omega) \times \{t=0\} \\ n(0,t) = b(t), & (a,t) \in \{a=0\} \times (0,\infty) \end{cases} \tag{7.17}$$

where the newborns are determined as

$$b(t) = \int_0^\omega \beta(a,t)n(a,t)da$$

The total population is

$$N(t) = \int_0^\omega n(a,t)da$$

If we compared to the PDE already presented, the McKendrick model has two new features:

1. first, it has two boundary conditions: an initial distribution for the population (at $t=0$) and for the population at age $a=0$;

2. second, the boundary condition referring to the newborns is non-local, that is, it depends on the distribution of the total population. This last feature implies that it is hard to solve, requiring the solution of an integral equation.

Assuming away that global nature of fertility, the Mc-Kendrick equation features a different type of dynamics depending in the difference between a and t: for $a < t$ the dynamics depends on the newborns, i.e., population with age $a = 0$, while for $a > t$ the dynamics is governed by the initial age-distribution of the population. Of course, asymptotically the first type of behavior prevails.

Consider the case

$$\begin{cases} n_t + n_a = -\mu n, & (a,t) \in (0,\omega) \times (0,\infty) \\ n(a,0) = n_0(a), & (a,t) \in (0,\omega) \times \{t = 0\} \\ n(0,t) = \phi(t), & (a,t) \in \{a = 0\} \times (0,\infty) \end{cases}$$

where $n_0(a)$ is the initial distribution of population and $\phi(t)$ is the number of offspring here assumed as exogenous, i.e., independent of the distribution of population.

Prove that the solution is

$$n(a,t) = \begin{cases} \phi(t-a)\pi(a), & \text{if } a \leq t \\ n(a-t,0)\frac{\pi(a)}{\pi(a-t)}, & \text{if } a \geq t \end{cases}$$

where

$$\pi(a) = e^{-\mu a}$$

is the probability of survival until age a.

Reference: McKendrick (1926) and for a recent textbook presentation Kot (2001)

7.7.2 Cohort's budget constraint

Let $w(a,t)$ be the financial wealth of an agent with age a at time t. The budget constraint is

$$w_t + w_a = s(a,t) + rw(a,t) \tag{7.18}$$

where $s(a,t)$ is the savings at age a at time t and r is the interest rate. If we assume that the initial stock of wealth is unbounded then $w : (0,A) \times (0,T) \to \mathbb{R}$ and the initial wealth distribution is $w(0,t) = 0$.

The general solution of equation (7.18) is

$$w(a,t) = \left(\int_0^a s(z, z-a+t)e^{-rz}dz + f(t-a) \right) e^{ra}$$

for an arbitrary $f(.)$.

If we assume that there are no bequests, that is no wealth at birth, $w(a,t)=0$ and $s(a,t)=e^{ba(K-a)+gt}-c$ the solution becomes

$$w(a,t)=\frac{\sqrt{\pi}}{2\sqrt{b}}\left(\Phi\left(\frac{Kb+g-r}{2\sqrt{b}}\right)-\Phi\left(\frac{(K-2a)b+g-r}{2\sqrt{b}}\right)\right)e^{\frac{K^2b^2((2K+4(t-a))g-2(K-2a)r)b+(g-r)^2}{4b}}-\\ -\frac{c}{r}\left(1-e^{ra}\right) \quad (7.19)$$

where $\Phi(x)=\text{erf}(x)=(2/\sqrt{\pi})\int_0^x e^{-z^2}dz$. Figure **??** illustrates equation (7.19). It displays a life-cycle behavior of savings: the agent tends to be a net borrower at young age and lender at older ages, although it dissaves later in life.

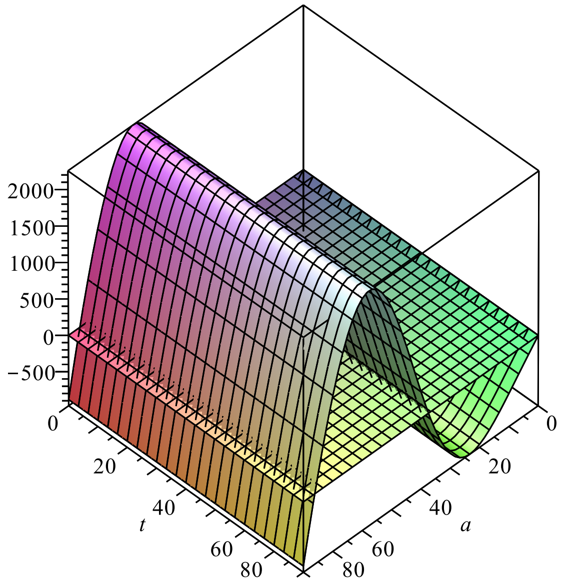


Figure 7.7: Illustration of equation (7.19) for $K=88$, $b=0.0029$, $r=0.02$, $g=0.02$ and $c=50$.

7.7.3 Interest rate term-structure

We consider an economy in which there is perfect foresight and in which there are two types of assets: a bank account and a continuum of bonds with maturity dates ranging from zero to infinity, $x\in[0,\infty)$.

The bank account's balance, for an initial deposit of $B(0)$, follows the process

$$B(t)=B(0)e^{\int_0^t r(\tau)d\tau}$$

where $r(t)$ is the spot interest rate. The price, at time t, for a bond that matures at time $t+x$

$$P(t,t+x)=e^{-\int_0^x f(t,y)dy}$$

where $f(t,x)$ is the forward interest rate for maturity time x.

If there are no arbitrage opportunities, the instantaneous rate of return for any two investments should be equal. In particular, the rate of return for the bank account should be equal to teh rate of return for a bond of any maturity

$$\frac{dB(t)}{B(t)} = \frac{dP(t,t+x)}{P(t,t+x)} \; \forall (t,x).$$

The rate of return for a bank account is

$$\frac{dB(t)}{B(t)} = r(t)dt$$

and the rate of return for a bond with maturity x

$$\frac{dP(t,t+x)}{P(t,t+x)} = f(t,x)dt - \int_0^x \frac{\partial f(t,x)}{\partial t} dydt$$

(where $dx + dt = 0$). If we assume that the forward interest rate follows a one-factor model,

$$\frac{\partial f(t,x)}{\partial t} = \mu(t,x)$$

then the arbitrage condition is equivalent to

$$f(t,x) - \int_0^x \mu(t,y)dy = r(t), \; \forall t \in [0,\infty)$$

If we take the derivative relative to x we get

$$\frac{\partial f(t,x)}{\partial x} = \mu(t,x)$$

Then the forward rate follows, in a deterministic setting, follows a first order partial differential equation

$$f_x(t,x) - f_t(t,x) = 0$$

that has the general solution

$$f(t,x) = h(t+x)$$

where $h(.)$ is an arbitrary function. As the following condition should be true: the instantaneous forward rate should be equal to the spot rate

$$f(t,0) = r(t).$$

then the particular solution is

$$f(t,x) = r(t+x)$$

the forward rate, at time t, for a bond maturing at time $t+x$ should be equal to the spot rate at time $t+x$.

7.7.4 Optimality condition for a consumer choice problem

Consider the following general consumer problem $\max_{c_1,c_2} u(c_1, c_2)$ subjec to the following constraints

$$\begin{cases} E(c_1, c_2) = p_1 c_1 + p_2 c_2 \leq W \\ 0 \leq c_1 \leq \bar{c}_1 \qquad\qquad\qquad 0 \leq c_2 \leq \bar{c}_2 \end{cases}$$

Assume that $u(.)$ is continuous, differentiable, increasing and concave in both arguments. Forming The Lagrangean

$$\begin{aligned} \mathcal{L} = u(c_1, c_2) + \lambda(W - E(c_1, c_2)) - \eta_1 c_1 - \eta_2 c_2 + \\ + \zeta_1(\bar{c}_1 - c_1) + \zeta_2(\bar{c}_2 - c_2) \end{aligned}$$

The solution (which always exists) (c_1^*, c_2^*) verifies the Karush-Kuhn-Tucker conditions

$$\begin{aligned} & u_{c_i}(c_1, c_2) - \lambda p_j - \eta_j - \zeta_j = 0,\ j = 1, 2 \\ & \eta_j c_j = 0,\ \eta_j \geq 0,\ c_j \geq 0,\ j = 1, 2 \\ & \zeta_j(\bar{c}_j - c_j) = 0,\ \zeta_j \geq 0,\ c_j \leq \bar{c}_j,\ j = 1, 2 \\ & \lambda(W - E(c_1, c_2)) = 0,\ \lambda \geq 0,\ E(c_1, c_2) \leq W \end{aligned}$$

For an interior solution, we have

- Let $c_1^* \in (0, \bar{c}_1)$ and $c_2^* \in (0, \bar{c}_2)$
- It verifies the conditions

$$p_2 u_{c_1}(c_1^*, c_2^*) = p_1 u_{c_2}(c_1^*, c_2^*) \tag{7.20}$$

$$E(c_1^*, c_2^*) = W \tag{7.21}$$

- Equation (7.20) is a first-order partial differential equation with solution

$$u(c_1^*, c_2^*) = v\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

- if we use equation (7.21) and define $w \equiv W/p_1$ in the optimum we have

$$u(c_1^*, c_2^*) = v(w)$$

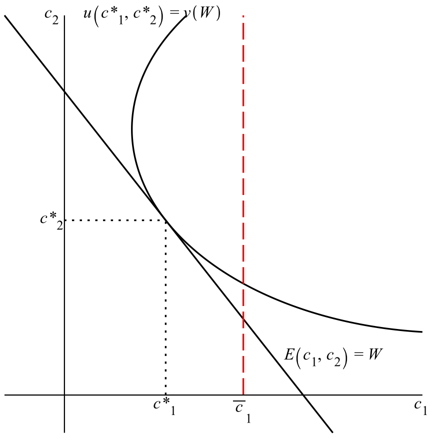


Figure 7.8: Interior optimum

- if the utility function is strictly concave then with very weak conditions (differentiability) we have an unitque interior optimum. It is clear that the budget set, in real terms, is a projected characteristic.

Corner solutions for c_1 verify

- Let $c_1^* = 0$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- It verifies the conditions

$$\begin{aligned} p_2 u_{c_1}(c_1^*, c_2^*) &= p_1 u_{c_2}(c_1^*, c_2^*) - p_2 \eta_1 && (7.22) \\ E(c_1^*, c_2^*) &= W && (7.23) \end{aligned}$$

- Equation (7.22) is a first-order partial differential equation with solution

$$u(c_1^*, c_2^*) = \frac{\eta_1 c_2^*}{p_1} + v\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

- if we use equation (7.25) in the optimum we have

$$u(c_1^*, c_2^*) = -\frac{\eta_1 p_2 c_2^*}{p_1} + v(w) < v(w)$$

- then the indirect utility level is smaller than for the unconstrained case

The second corner solution for c_1

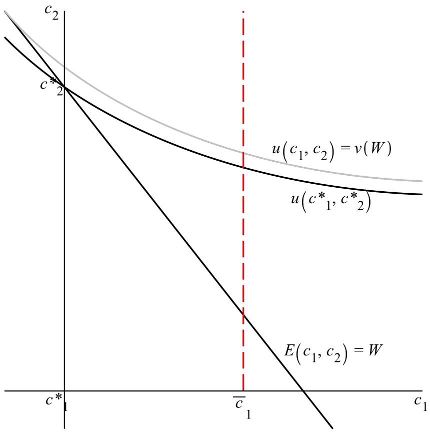


Figure 7.9: Corner solution: zero consumption

- Let $c_1^* = \bar{c}_1$ and $c_2^* \in (0, \bar{c}_2)$ and let the budget constraint be saturated;
- It verifies the conditions

$$\begin{aligned} p_2 u_{c_1}(c_1^*, c_2^*) &= p_1 u_{c_2}(c_1^*, c_2^*) + p_2 \zeta_1 && (7.24) \\ E(c_1^*, c_2^*) &= W && (7.25) \end{aligned}$$

- Equation (7.24) is a first-order partial differential equation with solution

$$u(c_1^*, c_2^*) = -\frac{\zeta_1 c_2^*}{p_1} + v\left(\frac{p_1 c_1^* + p_2 c_2^*}{p_1}\right)$$

- if we use equation (7.25) in the optimum we have

$$u(c_1^*, c_2^*) = \frac{\zeta_1 p_2 c_2^*}{p_1} + v(w) < v(w)$$

- then the indirect utility level is smaller than for the unconstrained case

We verify that the interior optimum has characteristics given by $c_1 + \frac{p_1}{p_2} c_2$, and in general they coincide with the budget constraint.

7.7.5 Growth and inequality dynamics

Let us assume that $N(t,k)$ is the number of people having an asset endowment k at time t and let $k \in [\underline{k}(t), \overline{k}(t)] \subset \mathbb{R}_+$. Then,

$$N(t) = \int_{\underline{k}(t)}^{\overline{k}(t)} N(t,k)dk.$$

We can denote the population density by $n(t,k) = N(t,k)/N(t)$. In this case

$$\int_{\underline{k}(t)}^{\overline{k}(t)} n(t,k)dk = 1.$$

Assume that the capital accumulates in linearly as

$$\frac{dk}{dt} = \gamma k(t).$$

Then, using Leibnitz rule

$$\begin{aligned}
\frac{d}{dt}\int_{\underline{k}(t)}^{\overline{k}(t)} n(t,k)dk &= \int_{\underline{k}(t)}^{\overline{k}(t)} n_t(t,k)dk + n(t,\overline{k}(t))\frac{d\overline{k}(t)}{dt} - n(t,\underline{k}(t))\frac{d\underline{k}(t)}{dt} \\
&= \int_{\underline{k}(t)}^{\overline{k}(t)} n_t(t,k)dk + n(t,\overline{k}(t))\gamma\overline{k}(t) - n(t,\underline{k}(t))\gamma\underline{k}(t) = \\
&= \int_{\underline{k}(t)}^{\overline{k}(t)} n_t(t,k) + \frac{\partial\left(\gamma k n(t,k)\right)}{\partial k}dk
\end{aligned}$$

by the mean-value theorem. Then the density satisfies the PDE

$$n_t(t,k) + \gamma k n_k(t,k) + \gamma n(t,k) = 0$$

which has the form of the transport equation (7.11).

Given an initial density $n_0(k)$ the solution is then

$$n(t,k) = e^{-\gamma t} n_0\left(k e^{-\gamma t}\right),\ (t,k) \in \mathbb{R}_+.$$

Assuming a log-normal distribution

$$\phi(k) = \left(2\pi k^2\sigma^2\right)^{-\frac{1}{2}} \exp\left(-\frac{(\log(k) - \mu)^2}{2\sigma^2}\right)$$

Figure **??** presents the dynamics of capital distribution, that is, the dynamics of the density of population for several levels of capital

Growth can be seen as travelling of the density of population for higher levels of capital wealth. We can compute several statistics to characterize the growth and distributional facts from this simple model (see Figure 7.11:

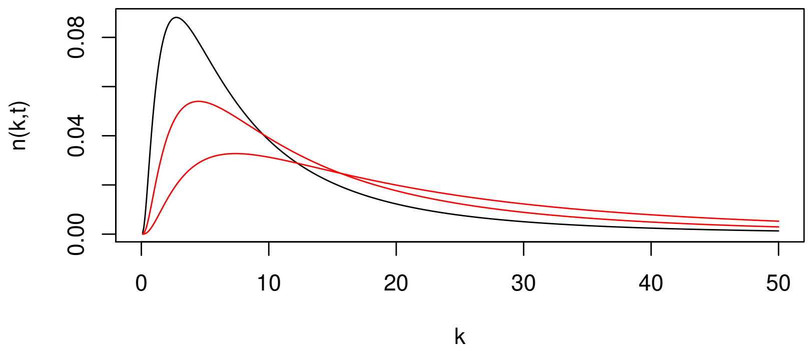
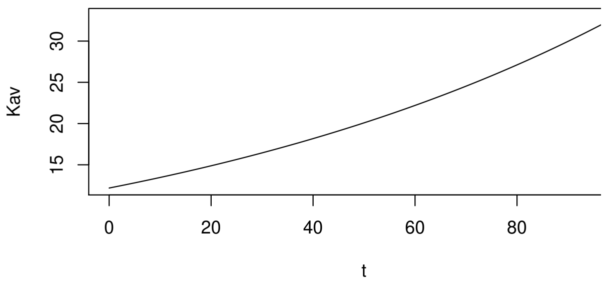


Figure 7.10: Density for several increasing dates

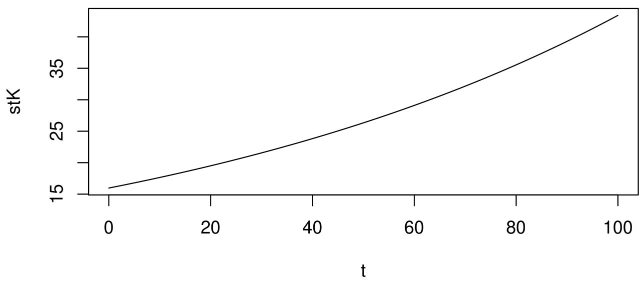
1. There is unbounded growth, with a constant growth rate $\gamma(t) = \gamma$
2. The distribution is non-ergodic (the average and variance tends to infinity)
$$\bar{k}(t) = e^{\gamma t + \mu + \frac{\sigma^2}{2}},\ \sigma_k(t) = (\bar{k}(t))^2 \left(e^{\sigma^2} - 1\right)$$
3. But the inequality measures are constant: Gini and Theil indices
$$G(t) = \mathrm{erf}\left(\frac{\sigma}{2}\right),\ \mathrm{Th}(t) = \frac{\sigma^2}{2}$$
4. Ratio of the quantiles is also constant
$$\frac{k_{90}}{k_{10}} = -\sigma\sqrt{2}\left[\mathrm{erf}^{-1}\left(1 - 2\frac{9}{10}\right) - \mathrm{erf}^{-1}\left(1 - 2\frac{1}{10}\right)\right]$$

7.8 References

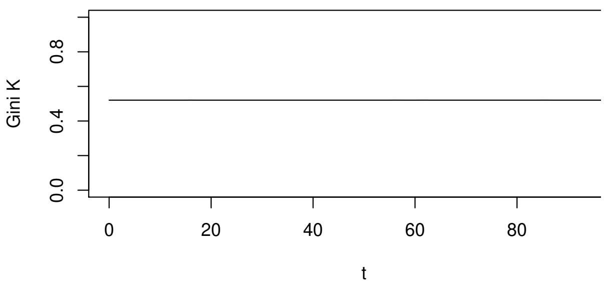
- Mathematics of PDE: introductory Olver (2014) more advanced (Evans, 1998, ch 3)
- Applications to economics: Hritonenko and Yatsenko (2013)
- Application to mathematical demography: (Kot, 2001, ch. 23)
- A useful site: `http://eqworld.ipmnet.ru/en/solutions/fpde/fpdetoc1.htm`



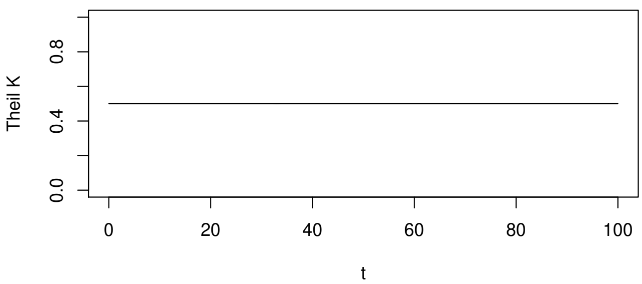
(a) Increasing average



(b) Increasing st. dev.



(c) Constant Gini coefficient



(d) Constant Theil index

Figure 7.11: Linear accumulation function for $\gamma > 0$ and an initial log-normal distribution

7.1 Laplace transforms and inverse Laplace transforms

Consider function $f(x)$ where $x > 0$. The Laplace transform of $f(x)$ is

$$\mathcal{L}[f(x)](s) = \int_0^\infty e^{-sx} f(x)dx = F(s).$$

The application of Laplace transforms to the solution of differential equations is convenient because it allows for the transformation of a ODE into an non-differential equation and the transformation of a PDE into an ODE.

The Laplace transform of $f^{'}(x) = df(x)/dx$ is

$$\mathcal{L}[f^{'}(x)](s) = \frac{d}{dx}\left(\int_0^\infty e^{-sx} f^{'}(x)dx\right) = s\int_0^\infty e^{-sx} f(x)dx + e^{-sx}f(x)\mid_{x=0}^{\infty} = sF(s) - f(0)$$

if the function $f(.)$ is bounded.

Example: consider the differential equation

$$f^{'}(x) = af(x).$$

Applying the Laplace transform to both sides, yields

$$\mathcal{L}[f^{'}(x)](s) = a\mathcal{L}[f(x)](s),$$

which is equivalent to the the algebraic equation in $F(s)$

$$sF(s) - f(0) = aF(s).$$

Therefore

$$F(s) = \frac{f(0)}{s-a}.$$

To go back to the solution as a function of the independent variable x, we apply the inverse Laplace transform

$$\mathcal{L}^{-1}[F(s)](x) = f(0)\mathcal{L}^{-1}[\frac{1}{s-a}](x).$$

But

$$f(x) = \mathcal{L}^{-1}[F(s)](x) = \int_0^\infty e^{-xs}F(s)ds$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)(x) = e^{ax}$$

Therefore,

$$f(x) = f(0)\, e^{ax}.$$

Table 7.1: Laplace transforms and inverse Laplace transforms

$f(x)$	$F(s)$	
a	$\frac{1}{s}$	
x	$\frac{1}{s^2}$	
e^{ax}	$\frac{1}{s-a}$	
$H(a-x)$	$\frac{e^{-as}}{s}$	$a > 0$

Some Laplace transforms used in the main text are presented in Table 7.1

The Laplace transform and the inverse Laplace transforms are tabulated in many textbooks on calculus or in the web, see `http://tutorial.math.lamar.edu/pdf/Laplace_Table.pdf`. We can compute them using Mathematica `http://reference.wolfram.com/language/ref/LaplaceTransform.html` and `http://reference.wolfram.com/language/ref/InverseLaplaceTransform.html`.

Chapter 8

Optimal control of first order PDE's

In this section we assume that one of the independent variables is time. Optimization or optimal control problems where the system we want to control are represented by first-order PDE's consist in finding one solution which maximises a functional on the two independent variables. There should be an intrinsic heterogeneity along a support represented by variable x. Roughly, it involves choosing, if there is any, an optimal distribution involving along time.

This is also a huge area, although there are not many application of optimal control of first-order PDE's to economics.

We present next the (necessary) first order conditions for calculus of variations problems and for the optimal control problem using the Pontryagin's principle. We also show a simple application to economics.

8.1 The calculus of variations problem

Given boundaries Consider the calculus of variations problem: among the functions $u(t,x)$: $[0,T] \times [0,X] \to \mathbb{R}$ verifying $u(0,x) = \phi_0(x)$, $u(T,x) = \phi_T(x)$, $u(t,0) = \varphi_0(t)$ and $u(t,X) = \varphi_X(t)$, choose the function (or functions) that maximizes the functional

$$J(u) = \int_0^T \int_0^X F\left(x, t, u(t,x), u_t(t,x), u_x(t,x)\right) dt dx.$$

Let $u^*(t,x)$ by a function which solves the CV problem, then it verifies the Euler-Lagrange equation

$$F^*_u\left(x,t\right) = \frac{\partial}{\partial t}\left(F^*_{u_t}\left(x,t\right)\right) + \frac{\partial}{\partial x}\left(F^*_{u_x}\left(x,t\right)\right), \; (t,x) \in (0,T) \times (0,X) \tag{8.1}$$

where $F^*(t,x) = F\left(t, x, u^*(t,x), u_t^*(t,x), u_x^*(t,x)\right)$ together with the conditions

$$\begin{cases} u^*(0,x) = \phi_0(x), \ (t,x) \in \{t=0\} \times (0,X) \\ u^*(T,x) = \phi_T(x), \ (t,x) \in \{t=T\} \times (0,X) \\ u^*(t,0) = \varphi_0(t), \ (t,x) \in (0,T) \times \{x=0\} \\ u^*(t,X) = \varphi_X(t), \ (t,x) \in (0,T) \times \{x=X\} \end{cases}$$

The Euler-Lagrange condition is a second-order PDE.

Free boundaries If we assume there are only the constraints $u(0,x) = \phi_0(x)$, and $u(t,0) = \varphi_0(t)$ the necessary conditions are the Euler-Lagrange equation (8.1) together with the conditions

$$\begin{cases} u^*(0,x) = \phi_0(x), \ (t,x) \in \{t=0\} \times (0,X) \\ F^*_{u_t}(T,x) = 0, \ (t,x) \in \{t=T\} \times (0,X) \\ u^*(t,0) = \varphi_0(t), \ (t,x) \in (0,T) \times \{x=0\} \\ F^*_{u_x}(t,X) = 0, \ (t,x) \in (0,T) \times \{x=X\} \end{cases}$$

Infinite horizon Consider the calculus of variations problem: among the functions $u(t,x) : [0,\infty) \times [0,X] \to \mathbb{R}$ verifying $u(0,x) = \phi_0(x)$ and $u(t,0) = \varphi_0(t)$ choose the function (or functions) that maximizes the functional

$$J(u) = \int_0^\infty \int_0^\infty n(x) g(t) f\left(u(t,x), u_t(t,x), u_x(t,x)\right) dt dx.$$

The Euler-Lagrange condition is

$$\begin{aligned} f_u^*(t,x) = \frac{g'(t)}{g(t)} f^*_{u_t}(t,x) + \frac{n'(x)}{n(x)} f^*_{u_x}(t,x) + f^*_{u_t u}(t,x) u_t^*(t,x) + f^*_{u_x u}(t,x) u_x^*(t,x) + \\ + f^*_{u_t u_t}(t,x) u^*_{tt}(t,x) + f^*_{u_t u_x}(t,x) u^*_{tx}(t,x) + f^*_{u_x u_t}(t,x) u^*_{xt}(t,x) + f^*_{u_x u_x}(t,x) u^*_{xx}(t,x) \end{aligned} \tag{8.2}$$

where $f^*(t,x) = f(u^*(t,x), u_t^*(t,x), u_x^*(t,x))$ and analogously for the derivatives.

8.2 Optimal control: Pontriyagin's principle

We consider the optimal control problem: among the functions $u(t,x) : [0,T] \times [0,X] \to \mathbb{R}$ and $k(t,x) : [0,T] \times [0,X] \to \mathbb{R}$, called respectively control and state functions, verifying $k(0,x) = \phi_0(x)$, $k(t,0) = \varphi_0(t)$ and the first-order PDE

$$k_t + k_x = G(t,x,u,k), \ (t,x) \in (0,T) \times (0,X) \tag{8.3}$$

find $u^*(t,x) = \text{argmax}_{u(.)} J(u)$ that maximises the functional

$$J(u) = \int_0^T \int_0^X F(t,x,u,k)dxdt$$

Necessary conditions according to the principle of Pontryiagin.

Assume there is a solution $(u^*(.), k^*(.))$ for the optimal control problem. Then there is a piecewise continuous function $\lambda : (0,T) \times (0,X) \to \mathbb{R}$ and a Hamiltonian

$$H(t,x,u,k,\lambda) \equiv F(t,x,u,k) + \lambda G(t,x,u,k)$$

such that the any admissible solution verifies the following conditions

$$\lambda_t + \lambda_x = -H_k^*(t,x) \tag{8.4}$$

where

$$H^*(t,x) = H(t,x,u^*,k^*,\lambda) = \max_u H(t,x,u,k,\lambda)$$

and the following conditions hold

$$k^*(T,x)\text{given or } \lambda(T,x) = 0 \text{ for } (t,x) \in \{t = T\} \times (0,X)$$

and

$$k^*(t,X)\text{given or } \lambda(t,X) = 0 \text{ for } (t,x) \in (0,T) \times \{x = X\}.$$

Differently from the calculus of variations case, a necessary condition for an optimum is that the solution verifies the **system of first-order PDE's**

$$\begin{cases} \lambda_t + \lambda_x = -\left(F_k(t,x,u^*,k^*) + \lambda G_k(t,x,u^*,k^*)\right) \\ k_t^* + k_x^* = G(t,x,u^*,k^*) \end{cases} \tag{8.5}$$

together with the static condition

$$H_u^*(t,x,u^*,k^*) = F_u(t,x,u^*,k^*) + \lambda G_u(t,x,u^*,k^*) = 0.$$

If $\det(H_{uu}^*) \neq 0$ then we can apply the inverse function theorem to get $u^* = h(t,x,\lambda,k)$ and the system (8.5) can be written as a system of first-order PDE's over $k(t,x)$ and $\lambda(t,x)$

$$\begin{cases} \lambda_t + \lambda_x = -\left(F_k(t,x,h(t,x,\lambda,,k^*),k^*) + \lambda G_k(t,x,h(t,x,\lambda,,k^*),k^*)\right) = -\ell(t,x,\lambda,k^*) \\ k_t^* + k_x^* = G(t,x,h(t,x,\lambda,k^*)k^*) = g(t,x,\lambda,k^*) \end{cases} \tag{8.6}$$

This is an infinite-dimensional extension of the modified Hamiltonian dynamic system for ODE control. Intuitively, observe that as the objective of the planner is to find the optimal dynamics, through time, of the distribution of k given an initial distribution $\phi_0(x)$, the shadow price, represented by λ should also be a distribution varying in time.

The solution of a system (8.6) is beyond the scope of this course (there are some results ifor instance in (Evans, 1998, ch 11).

8.3 Application: a generalised cake-eating problem

Problem: given a the wealth process by

$$w_t + w_a = -c(a,t)$$

given $w(a,0) = \phi_0(a)$ and $w(a,T) = 0$, find the consumption function $c(a,t)$ which maximises the functional

$$J(c) = \int_0^A \int_0^T \ln\left(c(a,t)\right) n(a) g(t) dt da.$$

We can transform the problem into a calculus of variations problem: find $w(a,t)$ that maximises the functional

$$J(w) = \int_0^A \int_0^T \ln\left(-(w_t(a,t) + w_a(a,t))\right) n(a) g(t) dt da.$$

given $w(a,0) = \phi_0(a)$ and $w(a,T) = 0$. We assume that $g(t) = e^{-\rho t}$ and $n(a) = e^{-\mu a}$ where $\rho > 0$ and $\mu > 0$.

The Euler-Lagrange equation is

$$w_{tt} + 2w_{at} + w_{aa} - (\rho + \mu)(w_t + w_a) = 0.$$

This equation has the general solution

$$w(a,t) = f_1(a-t) + f_2(t-a)e^{(\rho+\mu)t}$$

where $f_1(.)$ and $f_2(.)$ are arbitrary functions defined over the domain $\mathbb{R}^2$.

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