Universidade de Lisboa
Instituto Superior de Economia e Gestão

PhD in Economics
Advanced Mathematical Economics
2018-2019

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Exam: **Época Normal**
23.1.2019 (18.00h-21.00h)

1. (a) The solution to the TVP is $y(t) = \bar{y} + (y(0) - \bar{y})e^{gt}$, where $\bar{y} = -b/g$. The solution is indeterminate because $y(0)$ is arbitrary.

   (b) The solution to the TVP is $y(t) = \bar{y}$ for any $t \geq 0$. The solution is unique.

2. (a) The solution to the ODE is

$$y(t) = h_- \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + h_+ \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t}$$

   (c) The solution to the IVP is

$$y(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

3. (a) The solution to the ODE is $y(t) = \left(1 + \left(\frac{1}{y(0)} - 1\right) e^{-\lambda t}\right)^{-1}$

   (b) There are two steady states $\bar{y}_1 = 0$ and $\bar{y}_2 = 1$ if $\lambda \neq 0$ and there are an infinite number of steady states if $\lambda = 0$. If $\lambda < 0$, $\bar{y}_1$ is stable and $\bar{y}_2$ is unstable. If $\lambda > 0$, $\bar{y}_1$ is unstable and $\bar{y}_2$ is stable.

4. (a) The current value Hamiltonian is $h(c, z, q) = \ln\left(cz^{-\varphi}\right) + \mu q(c - z)$ and the optimality conditions according to the PMP are

$$\begin{cases} \mu q c = -1 \\ \dot{q} = (\rho + \mu) q + \varphi z^{-1} \\ \dot{z} = \mu (c - z) \\ z(0) = z_0 \\ \lim_{t\to\infty} q(t) z(t) e^{-\rho t} = 0 \end{cases}$$

   (b) The MHDS in (z, c) is

$$\begin{cases} \dot{z} = \mu (c - z) \\ \dot{c} = \left(\mu \varphi c z^{-1} - (\rho + \mu)\right) c \\ z(0) = z_0 \\ \lim_{t\to\infty} \dfrac{z(t)}{c(t)} e^{-\rho t} = 0 \end{cases}$$

(c) Setting $x(t) = \dfrac{z(t)}{c(t)}$ the MHDS is equivalent to $\dot{x} = \mu(1-\varphi) + \rho x$ and $\lim_{t\to\infty} x(t)e^{-\rho t} = 0$. The solution is $x(t) = \dfrac{\mu(\varphi-1)}{\rho}$. Therefore

$$c^*(t) = \frac{\rho}{\mu(\varphi-1)}, \text{ and } z^*(t),\ z^*(t) = z_0 e^{\xi t}$$

where $\xi = \dfrac{\rho - \mu(\varphi-1)}{\varphi - 1}$

(d) If $\xi > 0$ then $\lim_{t\to\infty} c^*(t) = \lim_{t\to\infty} z^*(t) = +\infty$. If $\xi < 0$ then $\lim_{t\to\infty} c^*(t) = \lim_{t\to\infty} z^*(t) = 0$

5. (a) The solution to the first-order PDE is

$$y(t,x) = f\left(xe^{-at}\right)$$

(b) The solution to the IVP is

$$y(t,x) = e^{-(xe^{-at})^2}$$

6. (a) The solution to the parabolic PDE is

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} u(0,\xi) e^{-\dfrac{(x-\xi)^2}{4t}} d\xi, \text{ for } t > 0$$

(b) The solution to the IVP is

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\dfrac{(x-x_0)^2}{4t}}, \text{ for } t > 0.$$

7. (a) The solution to the IVP, for $X(0) = x_0$ is

$$X(t) = e^{\gamma t}\left(x_0 + \sigma \int_0^t e^{-\gamma s} dW(s)\right)$$

(b) The expected value and variance are

$$\mathbb{E}[X(t)] = e^{\gamma t} x_0,\ \mathbb{V}[X(t)] = \frac{\sigma^2}{2\gamma}\left(e^{2\gamma t} - 1\right)$$

because $\mathbb{E}[X(t)^2] = x_0^2 e^{2\gamma t} + \dfrac{\sigma^2}{2\gamma}\left(e^{2\gamma t} - 1\right)$.

(c) The forward Kolmogorov equation is a parabolic PDE on $p = p(t,x)$

$$p_t = -\left(\gamma p + \gamma x p_x\right) + \frac{\sigma^2}{2} p_{xx}$$