Advanced Mathematical Economics

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Contents

Chapter 6

Calculus of variations

Calculus of variations provided the first approach for dealing with optimal variational problems since the early XVIII century (see Goldstine (1980)).

Explicit applications of calculus of variations to economics started, apparently, by a mathematician paper on the optimal pricing of a monopolistic firm Evans (1924). It was the main tools for solving intertemporal optimization problems at least until the interwar period (see Pomini (2018)). However, it is still useful, because it provides a better intuition on the optimization of functionals.

6.1 Calculus of variations: introduction

Calculus of variations problems consist in finding an extreme of a functional over a function $y : \mathrm{X} \rightarrow \mathcal{Y}$, which can be subject to additional requirements. Solving a calculus of variations problem means finding function $y^*(.)$ belonging to an an admissible set of continuous and differentiable functions $\mathcal{Y}$ (not necessarily everywhere).

The **objective functional** takes the form

$$\mathsf{J}[y] \equiv \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \tag{6.1}$$

in which we function $F(.)$ is known.

We assume that $F_{y'}(x, y, y') \equiv \dfrac{\partial F(x, y, y')}{\partial y'} \neq 0$, except maybe on a subset of measure zero (i.e., it may be piecewise-differentiable). This is the characteristic of function $F(.)$ which makes the problem a dynamic optimization problem (if time is the independent variable), in the sense that the optimization involves a trade-off between the current state $y(x)$ and the change in the current state, $y'(x)$. If $F_{y'}(x, y(x), y'(x)) = 0$ globally, for any $x \in \mathrm{X}$ the problem will degenerate to a static functional optimization.

To understand the effect of the derivative on the optimum, consider instead the objective functional

$$\mathsf{J}_0[y] \equiv \int_{x_0}^{x_1} F(x, y(x))\, dx.$$

If there are no other conditions, if $y^*(x)$ is the optimum, a necessary condition is

$$\frac{\delta \mathsf{J}_0[y^*]}{\delta y(x)} = F_y(x, y^*(x)) = 0, \text{ for every } x \in \mathrm{X}$$

where $F_y(x, y) = \dfrac{\partial F(x, y)}{\partial y}$. This condition is a point-wise optimality criterium: the optimum $y^*(x)$ is found by finding an extremum for every point in $x \in \mathrm{X}$ independent of any other point $\in \mathrm{X}$. If the objective function depends on the derivative of function $y(\cdot)$, $y'(\cdot)$, this means that the local interaction influences the value of the problem. This has two consequences: first, the optimum cannot be just determined by a point-wise extremum, and, second, any constraint on the value of y will influence the solution.

This also means that we should look for solutions $y \in C^1(\mathrm{X}; \mathcal{Y})$, where $C^1(\mathrm{X}; \mathcal{Y})$ is the set of continuously differentiable functions mapping X into $\mathcal{Y}$.

Two observations are important referring to the nature of the independent variable, x, and to its domain X.

First, in most economic applications, x is a non-negative real number referring to time , i.e. $x = t$ and $\mathrm{X} = \mathrm{T} \subseteq \mathbb{R}_+$. However in some microeconomic problems or static macroeconomic problems with heterogeneity among agents, in which there are, for example, information or searching frictions, we need to solve optimal control problems in which the independent variable is not time and has a support belonging to a continuum, for instance $\mathrm{X} = [x_0, x_1]$. In time-dependent problems we call $x_0 = t_0$ the initial time and $x_1 = t_1$ the terminal time, or horizon, while for non-time-dependent models the designation depends on the context. For example in models in which x refers to the skill level x_0 refers to the lowest skill in the distribution and x_1 to the highest skill. Therefore, from now on we call x_0 the lower bound and x_1 the upper bound of X.

Second, in time-dependent problems we usually assume that $x_0 = t_0$ and $x_1 = t_1$ may be fixed (v.g., in macroeconomic models) or free (v.g., in microeconomic problems). If x refers to other type of variables x_0 and x_1 may refer to cutoff points which can be free and optimally determined.

At last, another important point to be made, which is particularly important in macroeconomics is related to the boundedness of X. We can consider x_1 to be bounded or unbounded $x_1 = \infty$. In the case in which x refers to time we have to distinguish between **finite or infinite horizon** cases.

6.2 Bounded domains and equality constraints

In this section, we start with the simplest case, in subsection 6.2.1 the case in which the boundary of X and the values of the state variables at that boundary are also known, i.e, x_0, x_1, $y(x_0)$ and $y(x_1)$ are known. In section 6.2.2 we consider the cases in which x_0 are x_1 known but $y(x_0)$ and $y(x_1)$ are free. In section 6.2.3, the cases in which known $y(x_0)$ and $y(x_1)$ are known but x_0 are x_1 are free and the cases in which x_0, x_1, $y(x_0)$ and $y(x_1)$ are all free. At last, in section 6.2.4, we deal with two cases which are common to time-dependent models: the existence of terminal constraints and the infinite horizon problem.

6.2.1 The simplest CV problem

The **simplest problem of calculus of variations** is the following: find a function $y : \mathrm{X} \to \mathrm{Y} \subseteq \mathbb{R}$, that maximizes the **objective functional** (6.1) such that: (i) the set of independent variables is closed and bounded $\mathrm{X} = [x_0, x_1]$, with fixed limits x_0 and x_1 satisfying $x_0 < x_1$, (ii) the functions $y(\cdot)$ is continuous and continuously differentiable (except at a finite number of points) and satisfies $y(x_0) = y_0$ and $y(x_1) = y_1$.

The **admissibility set**,

$$\mathcal{Y} \equiv \left\{ y(x) \in \mathrm{X} : y(x_0) = y_0,\ y(x_1) = y_1 \right\} \subseteq \mathcal{C}^1(\mathbb{R})$$

is the set of all functions continuous and differentiable functions that satisfy the lower and upper boundary data and $\mathrm{X} = [x_0, x_1]$.

Therefore, the problem is to find a function $y^* \in \mathcal{Y}$, which maximizes the functional (6.1).

Formally, the simplest problem is:

$$\begin{aligned} &\max_{y(\cdot)\in\mathcal{Y}} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \\ &\qquad \text{subject to} \\ &\qquad x_0 \text{ and } x_1 \text{ fixed} \\ &\qquad y(x_0) = y_0 \text{ and } y(x_1) = y_1 \text{ fixed} \end{aligned} \tag{P1}$$

We denote by $\varphi = (x_0, x_1, y_0, y_1, .)$ be vector of the data of the problem containing the lower and upper values of the independent variable, the associated values of the state function, and other parameters that might exist in function $F(.)$.

The **value function**, $V(\varphi) = \mathsf{J}[y^*]$, is a real-valued function depending on the data of the problem, that is

$$V(x_0, x_1, y_0, y_1, .) = \mathsf{J}[y^*] \equiv \max_{y\in\mathcal{Y}} \int_{x_0}^{x_1} F(x, y^*(x), y^{*'}(x))\, dx,$$

where $\mathcal{Y}$ is the admissibility set.

Proposition 1. ***First order necessary conditions for the simplest problem, (P1)*** *: $y^* : [x_0, x_1] \to \mathrm{Y}$ is a solution of the simplest CV problem only if it satisfies the* ***Euler-Lagrange equation*** [1]*:*

$$F_y(x, y^*(x), y^{*'}(x)) = \frac{d}{dx}\Big(F_{y'}(x, y^*(x), y^{*'}(x)) \Big), \ \textit{for}\ x \in (x_0, x_1) \tag{6.2}$$

together with the boundary conditions

$$y^*(x_0) = y_0, \textit{and}\ y^*(x_1) = y_1 \tag{6.3}$$

[1] We use the notation $F_y\ (x, y, y') = \dfrac{\partial F(x, y, y')}{\partial y}$ and $F_{y'}(x, y, y') = \dfrac{\partial F(x, y, y')}{\partial y'}$.

Proof. (Heuristic) Assume we know y^*. Then the maximum value for the functional is

$$\mathsf{J}[y^*] = \int_{x_0}^{x_1} F(x, y^*(x), y^{*'}(x))\, dx.$$

Function y^* is an optimum only if $\mathsf{J}[y^*] \geq \mathsf{J}[y]$ for any other admissible function $y : \mathrm{X} \to \mathrm{Y}$. Take an admissible variation over y^*, $y = y^* + \delta y$ such that the variation is a *parameterized perturbation* of y^*, that is $\delta y(x) = \varepsilon\, \eta(x)$ where $\eta \in \mathcal{Y}$ and ε is a number. A variation to be admissible has to satisfy $y(x_1) = y^*(x_1) = y_1$ and $y(x_0) = y^*(x_0) = y_0$. Therefore, an admissible perturbation has to satisfy $\eta(x_0) = \eta(x_1) = 0$ and it can take arbitrary values $\eta \in \mathrm{Y}$ for x in the interior of the domain X.

The value functional for the perturbed function y is

$$\mathsf{J}[y] = \mathsf{J}[y^* + \varepsilon\eta] \;= \int_{x_0}^{x_1} F(x, y^*(x) + \varepsilon\eta(x), y^{*'}(x) + \varepsilon\eta'(x))dx.$$

The variation of the functional is

$$\begin{aligned}\Delta J(\varepsilon) = &\; \mathsf{J}[y^* + \varepsilon\, \eta] - \; \mathsf{J}[y^*] \\ = &\int_{x_0}^{x_1} \Big(F(x, y^*(x) + \varepsilon\eta(x), y^{*'}(x) + \varepsilon\eta'(x)) - F(x, y^*(x), y^{*'}(x))\Big)\, dx.\end{aligned}$$

Defining the Gâteaux derivative of a functional evaluated at y^* for the perturbation η

$$\delta\mathsf{J}[y^*](\eta) = \lim_{\varepsilon\to 0} \frac{\Delta J(\varepsilon)}{\varepsilon},$$

a first-order expansion of the functional $\mathsf{J}[y]$ in a neighbourhood of y^*,

$$\mathsf{J}[y] \;= \mathsf{J}[y^*] + \delta\mathsf{J}[y^*](\eta)\varepsilon + o(\varepsilon)$$

Then, at the optimum, $\mathsf{J}[y^*] \geq \mathsf{J}[y]$ only if the first integral derivative of J is zero: $\delta\mathsf{J}[y^*](\eta) = 0$. We can find the Gâteaux derivative by using the formula

$$\delta\mathsf{J}[y^*](\eta) = \frac{d}{d\varepsilon}\mathsf{J}[y^* + \varepsilon\eta]\Big|_{\varepsilon=0}.$$

In this case, we have

$$\begin{aligned}\delta\mathsf{J}[y^*](\eta) = &\int_{x_0}^{x_1} \Big(F_y(x, y^*(x), y^{*'}(x))\, \eta(x) + F_{y'}(x, y^*(x), y^{*'}(x)\Big)\, \eta'(x)\Big)\, dx \\ = &\int_{x_0}^{x_1} F_y(x, y^*(x), y^{*'}(x))\, \eta(x)\, dx + \int_{x_0}^{x_1} F_{y'}(x, y^*(x), y^{*'}(x)\Big)\, \eta'(x)\, dx.\end{aligned}$$

Integrating by parts the second integral yields

$$\begin{aligned}\int_{x_0}^{x_1} F_{y'}(x, y^*(x), y^{*'}(x))\, \eta'(x)dx = &\int_{x_0}^{x_1} F_{y'}(x, y^*(x), y^{*'}(x))\, \eta(x) - \int_{x_0}^{x_1} dF_{y'}(x, y^*(x), y^{*'}(x))\, \eta(x) = \\ = &\; F_{y'}(x_1, y^*(x_1), y^{*'}(x_1))\, \eta(x_1) - F_{y'}(x_0, y^*(x_0), y^{*'}(x_0))\, \eta(x_0) \\ &- \int_{x_0}^{x_1} \frac{d}{dx} F_{y'}(x, y^*(x), y^{*'}(x))\, \eta(x)\, dx.\end{aligned}$$

Therefore,

$$
\begin{aligned}
\delta \mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \Big(F_y(x, y^*(x), y^{*'}(x)) - \frac{d}{dx} F_{y'}(x, y^*(x), y^{*'}(x)) \Big)\, \eta(x)\, dx + \\
+ F_{y'}(x_1, y^*(x_1), y^{*'}(x_1))\, \eta(x_1) - F_{y'}(x_0, y^*(x_0), y^{*'}(x_0))\, \eta(x_0).
\end{aligned} \tag{6.4}
$$

As, in this case with fixed boundary values for the variable y, the admissible perturbation satisfies $\eta(x_1) = \eta(x_0) = 0$ equation (6.4) reduces to

$$
\delta \mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \Big(F_y(x, y^*(x), y^{*'}(x)) - \frac{d}{dx} F_{y'}(x, y^*(x), y^{*'}(x)) \Big)\, \eta(x)\, dx
$$

If $F(.)$ is a continuous function we can use the following result (see (Gel'fand and Fomin, 1963, p.9)): if $h := [x_0, x_1] \to \mathbb{R}$ is a continuous function and $\int_{x_0}^{x_1} h(x)\eta(x) dx$ for all C^1 functions η and if $\eta(x_0) = \eta(x_1) = 0$ then $\int_{x_0}^{x_1} h(x)\eta(x)dx = 0$ if and only if $h(x) = 0$ for all $x \in (x_0, x_1)$.

Therefore $\delta \mathsf{J}[y^*](\eta) = 0$ if an only if $F_y(x, y^*(x), y^{*'}(x)) - \frac{d}{dx} F_{y'}(x, y^*(x), y^{*'}(x)) = 0$ for every $x \in \mathrm{X} = [x_0, x_1]$. $\square$

Writing $F^*\ (x) \equiv F(x, y^*(x), y^{*'}(x))$ and $F_j^*\ (x) \equiv F_j(x, y^*(x), y^{*'}(x))$ for $j \in y, y'$, its derivatives evaluated at the optimum, the Euler-Lagrange equation is a 2nd order ODE (ordinary differential equation) if $F_{y'y'} \neq 0$: if we expand the right-hand-side we find

$$
F^*_{y'y'}(x)\, y''(x) + F^*_{y'y}(x)\, y'\ (x) + F^*_{y'x}(x) - F^*_y(x) = 0, \text{ for each } \ x \in [x_0, x_1].
$$

We can transform it into a system of first order ODE's if we define $y_1 = y$ and $y_2 = y'$ then

$$
\begin{aligned}
y_1' &= y_2 \\
F_{y_2 y_2}(x, y_1, y_2)\, y_2' &= F_{y_1}(x, y_1, y_2) - F_x(x, y_1, y_2) - F_{y_2 y_1}(x, y_1, y_2) y_2.
\end{aligned}
$$

The first order necessary condition only allows for the determination of an extremum. In order to get the a necessary condition for a maximand we need a second order condition:

Proposition 2. ***Second order necessary conditions****: the solution to the CV problem $y^* : \mathrm{X} \to \mathrm{Y}$ is a maximand only if it satisfies the Legendre-Clebsch condition*

$$
F_{y'y'}(x, y^*(x), y^{*'}(x)) \leq 0 \tag{6.5}
$$

Proof. (Heuristic but more complicated). Performing a second -order expansion of the functional $\mathsf{J}[x]$ in a neighbourhood of y^*, we obtain

$$
\mathsf{J}[y]\ = \mathsf{J}[y^*] + \delta \mathsf{J}[y^*](\eta)\varepsilon + \frac{1}{2}\ \delta^2 \mathsf{J}[y^*](\eta)\varepsilon^2 + o(\varepsilon^2),
$$

where

$$
\delta^2 \mathsf{J}[y^*](\eta) = \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} \mathsf{J}[y^*\ + \varepsilon\eta].
$$

Because at the optimum for any admissible perturbation η we have $\delta \mathsf{J}[y^*](\eta) = 0$, and at a have a maximum $\mathsf{J}[y] \ \leq \mathsf{J}[y^*]$, a necessary condition is $\delta^2 \mathsf{J}[y^*](\eta) \leq 0$.

The second-order functional derivative is

$$\delta^2 \mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \Big(F_{yy}(x, y^*(x), y^{*'}(x))\, \eta(x)^2 + \\ + 2F_{yy'}(x, y^*(x), y^{*'}(x))\, \eta(x)\, \eta'(x) + F_{y'y'}(x, y^*(x), y^{*'}(x))\, (\eta'(x))^2 \Big)\, dx.$$

As

$$\begin{aligned}\int_{x_0}^{x_1} 2F^*_{yy'}(x)\, \eta(x)\, \eta'(x) dx &= \int_{x_0}^{x_1} F^*_{yy'}(x) \frac{d}{dx}\ (\eta(x)^2) dx = \\ &= F^*_{yy'}(x)\ \eta(x)^2 \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\ \Big(F^*_{yy'}\Big) (\eta(x)^2) dx \\ &= - \int_{x_0}^{x_1} \frac{d}{dx}\ \Big(F^*_{yy'}\Big) (\eta(x)^2) dx\end{aligned}$$

because of the admissibility conditions $\eta(x_0) = \eta(x_1) = 0$. Then

$$\delta^2 \mathsf{J}[y^*](\eta) = \int_{x_0}^{x_1} \left(\Big(F^*_{yy}(x) - \frac{d}{dx} F^*_{yy'}(x) \Big) \eta(x)^2 + F^*_{y'y'}(x) (\eta'(x))^2 \right) dx.$$

Following (Liberzon, 2012, p.59-60)), it can be shown that $\delta^2 \mathsf{J}[y^*](\eta) \leq 0$ only if condition (6.5) holds. □

Proposition 3. ***Sufficient conditions****: let $y^* \in \mathcal{Y}$ verify*

$$F_y\ (t, y^*, y^{*'}) = \frac{d}{dx}\ F_{y'}(t, y^*, y^{*'})\ \text{ and } F_{y'y'}(t, y^*, y^{*'}) \leq 0$$

then (under some additional conditions on the trajectory of y) y^* is an optimiser to $\mathsf{J}[y]$.

Proof. See (Liberzon, 2012, p.62-68) □

Proposition 4. ***Necessary and sufficient conditions****: consider the simplest calculus of variations problem and assume that $F_{y'y'}(x, y(x), y'(x)) \leq 0$ for every $x \in [x_0, x_1]$ then equations* (6.2) *and* (6.3) *are necessary and sufficient conditions.*

6.2.2 Free boundary values for the state variable

Now we consider the problem: find function y^* among admissible functions $y \in \mathcal{Y}$ having the following properties: $y : \mathrm{X} \to Y \subseteq \mathbb{R}$, where $\mathrm{X} = [x_0, x_1]$ has known boundaries, x_0 and x_1, and such that $y(x_0)$ and/or $y(x_1)$ are free. The objective functional is again (6.1).

Formally, the problem are:

$$
\begin{aligned}
&\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \\
&\quad \text{subject to} \\
&\quad x_0 \text{ and } x_1 \text{ fixed} \\
&\quad y(x_0) = y_0 \text{ and } y(x_1) = y_1 \text{ free} && \text{(P2a)} \\
&\quad y(x_0) = y_0 \text{ fixed, } \; y(x_1) = y_1 \text{ free} && \text{(P2b)} \\
&\quad y(x_0) = y_0 \text{ free } y(x_1) = y_1 \text{ fixed} && \text{(P2c)}
\end{aligned}
$$

We have the following data, value functions and admissibility sets:

- in the case of the problem (P2a), the parameter set is $\varphi = (x_0, x_1, .)$, the value function is

  $$V(x_0, x_1, .) = \max_{y \in \mathcal{Y}} \int_{x_0}^{x_1} F(x, y(x), y'(x))dx.$$

  and the admissible set is

  $$\mathcal{Y} \equiv \Big\{ y(x) \in \mathrm{Y} : \; x \in \mathrm{X} \Big\} \; \subseteq \mathcal{C}^1(\mathbb{R});$$

- in the case of the problem (P2b), the parameter set is $\varphi = (x_0, x_1, y_0, .)$, the value function is

  $$V(x_0, x_1, y_0, .) = \max_{y \in \mathcal{Y}} \int_{x_0}^{x_1} F(x, y(x), y'(x))dx.$$

  and the admissible set is

  $$\mathcal{Y} \equiv \Big\{ y(x) \in \mathrm{Y} : \; y(x_0) = y_0, x \in \mathrm{X} \Big\} \; \subseteq \mathcal{C}^1(\mathbb{R});$$

- in the case of the problem (P2c), the parameter set is $\varphi = (x_0, x_1, y_1, .)$, the value function is

  $$V(x_0, x_1, y_1 .) = \max_{y \in \mathcal{Y}} \int_{x_0}^{x_1} F(x, y(x), y'(x))dx.$$

  and the admissible set is

  $$\mathcal{Y} \equiv \Big\{ y(x) \mathrm{Y} : \; y(x_1) = y_1, \; x \in \mathrm{X} \Big\} \; \subseteq \mathcal{C}^1(\mathbb{R}).$$

Proposition 5. ***First order necessary conditions for the free terminal state problem:*** *$y^* \in \mathcal{Y}$ is the solution to one of the CV problem with free boundary values for the state variable and known terminal values for the independent variable, x_0 and x_1, problems P2a, P2b, or P2c, only if it satisfies the Euler equation* (6.2) *and the boundary conditions:*

1. *if both boundary values are free (problem* (P2a)*)*

$$F_{y'}(x_0, y^*(x_0), y^{*'}(x_0)) = 0, \; \textit{and} \; F_{y'}(x_1, y^*(x_1), y^{*'}(x_1)) = 0 \tag{6.7}$$

2. *if the lower boundary value is given by* $y(x_0) = y_0$*, and the upper boundary value is free (problem* (P2b)*)*

$$y^*(x_0) = y_0, \text{ and } F_{y'}(x_1, y^*(x_1), y^{*'}(x_1)) = 0 \tag{6.8}$$

3. *if the upper boundary value is given by* $y(x_1) = y_1$*, and the lower boundary value is free (problem* (P2c)*)*

$$F_{y'}(x_0, y^*(x_0), y^{*'}(x_0)) = 0, \text{ and } y^*(x_1) = y_1. \tag{6.9}$$

Proof. (Heuristic) Now the boundary values for perturbation are $\eta(x_0)$ and $\eta(x_1)$ can take any value, including zero if the associated boundary value $y(x)$, for $x \in \{x_0, x_1\}$ is fixed. The proof follows the same steps as in the proof of Proposition 1. However, in equation (6.4), in order to get $\delta \mathsf{J}[y^*](\eta) = 0$, and after introducing the Euler-Lagrange condition, we should have

$$F_{y'}(x_j, y^*(x_j), y^{*'}(x_j))\, \eta(x_j) = 0, \text{ for } j = 0, 1. \tag{6.10}$$

Thus we have two cases, concerning the adjoint conditions at boundary x_j, for $j = 0, 1$, for an optimum. First, if the value of the state variable for the boundary x_j is known, i.e., $y(x_j) = y_j$, an admissible perturbation should verify $\eta(x_j) = 0$, implying that condition (6.10) holds automatically. This is the case in Proposition 1. Second, if the value of the state variable for the boundary x_j is free, then the related perturbation value is arbitrary and $\eta(x_j) \neq 0$ in general. The optimally condition (6.10) holds if and only if $F_{y'}(x_j, y^*(x_j), y^{*'}(x_j)) = 0$ which provides one adjoint condition allowing for the determination of the optimal boundary value for the state variable $y^*(x_j)$. This is how we adjoint (6.7) to (6.9) depending on which boundary value for the state variable is free. □

In time-varying models in which the value of the state variable is known at time $t = 0$ and the terminal value of the state variable is endogenous we supplement the Euler-Lagrange with condition (6.8).

However, there are models in which the initial value of the state variable is unknown. This is the case, for instance, in optimal taxation models of the Mirrlees (1971) type in which the independent variable are skill values and the initial condition is related to the cutoff level of skill bellow which taxes should be zero. In this case condition (6.7) can be used.

Observation: as the Euler-Lagrange is a second-order differential equation, in order to fully solve a model we need to have information on the value of y at the two boundaries for $x = x_0$ and $x = x_1$.

6.2.3 Free boundary values for the independent variable

Now we consider the problem: find function $y^* \in \mathcal{Y}$ which is the set of functions $y : \mathrm{X}^* \to \mathbb{R}$, where X^* has at least one unknown boundary, x_0^* and/or x_1^*, but such that the terminal values for the state variable are known. That is $\mathrm{X}^* = [x_0^*, x_1]$ or $\mathrm{X}^* = [x_0, x_1^*]$ or $\mathrm{X}^* = [x_0^*, x_1^*]$ where x_j is known and x_j^* is free. If a boundary value for the independent variable is free the related boundary value for the state variable is known, that is $y(x_j^*) = y_j$. The objective functional is again (6.1).

Formally, the problem are :

$$\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx$$

subject to

$y(x_0) = y_0$ and $y(x_1) = y_1$ fixed

x_0 and x_1 free (P3a)

x_0 fixed, x_1 free (P3b)

x_0 free x_1 fixed (P3c)

In this case the data of the problem is $\varphi = (y_0, y_1, .)$. and the value functional is

$$V(y_0, y_1, .) = \max_{y\in\mathcal{Y}, x_0, x_1} \int_{x_0}^{x_1} F(x, y(x), y'(x))dx = \int_{x_0^*}^{x_1^*} F(x, y^*(x), y^{*'}(x))dx$$

where we x_0^* and/or x_1^* are determined endogenously. Now, we have $\mathrm{X}^* = [x_0^*, x_1^*]$ and the admissible set for problem (P3a) is

$$\mathcal{Y} = \{\ y(x) \in \mathrm{Y} : y(x_0) = y_0,\ y(x_1) = y_1,\ x \in \mathrm{X}^*\}.$$

The other admissible sets are defined accordingly.

Proposition 6. ***First order necessary conditions for the free boundaries value problem:*** *$y^* \in \mathcal{Y}$ is the solution to the CV problem with known boundary values for the state variable, y_0 and y_1, and free terminal values for the independent variable, problems P3a, P3b, or P3c, only if it satisfies the Euler equation* (6.2) *and the boundary conditions:*

1. *if both boundary values for the indepdendent variable are free (problem P3a)*

$$F(x_0^*, y_0, y^{*'}(x_0^*)) - F_{y'}(x_0^*, y_0, y^{*'}(x_0^*))\, y^{*'}(x_0^*) = 0$$
$$\text{and}\ \ F(x_1^*, y_1, y^{*'}(x_1^*)) - F_{y'}(x_1^*, y_1, y^{*'}(x_1^*))\, y^{*'}(x_1^*) = 0 \quad (6.12)$$

2. *if the lower boundary value for the independent variable is known, $x_0^* = x_0$, and the upper boundary for the independent variable is free (problem P3b)*

$$x_0^* = x_0,\ \text{and}\ F(x_1^*, y_1, y^{*'}(x_1^*)) - F_{y'}(x_1^*, y_1, y^{*'}(x_1^*))y^{*'}(x_1^*) = 0 \quad (6.13)$$

3. *if the upper boundary value for the independent variable is known, $x_1^* = x_1$, and the lower boundary for the independent variable is free (problem P3c)*

$$F(x_0^*, y_0, y^{*'}(x_0^*)) - F_{y'}(x_0^*, y_0, y^{*'}(x_0^*))y^{*'}(x_0^*) = 0,\ \text{and}\ x_1^* = x_1. \quad (6.14)$$

Proof. (Heuristic) Let us assume that we know the solution $y^*(x)$ for $x \in [x_0^*, x_1^*]$, that is for all values of the independent variable contained between the two optimally chosen boundary values.

In this case we have to introduce two types of perturbations: a perturbation to the state variable $y(x) = y^*(x) + \varepsilon\eta(x)$, by function $\eta(\cdot)$, and to the independent variable $x = x^* + \varepsilon\chi$ by a constant χ. If we denote $y^*_j = y^*(x^*_j)$, for $j = 0, 1$, the two boundary values for the independent and dependent variables are $P^*_j \equiv (x^*_j, y^*_j)$ for $j = 0, 1$ at the optimum. The related terminal points for the perturbed solution are written as $P_j = (x^*_j + \varepsilon\chi_j, y^*_j + \varepsilon\eta_j)$ for $j = 0, 1$.

At the optimum the objective functional is

$$\mathsf{J}[y^*; x^*] = \int_{x^*_0}^{x^*_1} F(x, y^*(x), y^{*'}(x))dx$$

and

$$\mathsf{J}[y^* + \varepsilon\eta; x^* + \varepsilon\chi] = \int_{x^*_0+\varepsilon\chi_0}^{x^*_1+\varepsilon\chi_1} F(x, y^*(x) + \varepsilon\eta(x), y^{*'}(x) + \varepsilon\eta'\ (x))dx.$$

Then, denoting $\Delta\mathsf{J}(\varepsilon) = \mathsf{J}[y^* + \varepsilon\eta; x^* + \varepsilon\chi] - \mathsf{J}[y^*; x^*]$ we have

$$\begin{aligned}
\Delta\mathsf{J}(\varepsilon) &= \int_{x^*_0+\varepsilon\chi_0}^{x^*_1+\varepsilon\chi_1} F\big(x, y^*(x) + \varepsilon\eta(x), y^{*'}(x) + \varepsilon\,\eta'(x)\big)\,dx - \int_{x^*_0}^{x^*_1} F(x, y^*(x), y^{*'}(x))\,dx \\
&= \int_{x^*_0}^{x^*_1} \Big(F(x, y^*(x) + \varepsilon\eta(x), y^{*'}(x) + \varepsilon\eta'\ (x)) - F(x, y^*(x), y^{*'}(x))\Big)dx + \\
&\quad + \int_{x^*_1}^{x^*_1+\varepsilon\chi_1} F(x, y^*(x) + \varepsilon\eta(x), y^{*'}(x) + \varepsilon\eta'\ (x))dx - \\
&\quad - \int_{x^*_0+\varepsilon\chi_0}^{x^*_0} F(x, y^*(x) + \varepsilon\eta(x), y^{*'}(x) + \varepsilon\eta'\ (x))dx
\end{aligned}$$

Denoting $F^*(x) = F(t, y^*(x), \dot{y}^*(x))$ and using the mean-value theorem,

$$\Delta\mathsf{J}(\varepsilon) = \varepsilon \int_{x^*_0}^{x^*_1} \Big(F^*_y(x)\eta(x) + F^*_{y'}(x)\eta'(x)\Big)\,dx + F(\tilde{x}_1)\,\varepsilon\,\chi_1 - F(\tilde{x}_0)\,\varepsilon\,\chi_0$$

where $\tilde{x}_1 \in (x^*_1, x^*_1 + \varepsilon\,\chi_1)$ and $\tilde{x}_0 \in (x^*_0, x^*_0 + \varepsilon\,\chi_0)$. Taking $\delta\mathsf{J}[y^*; x^*]\ (\eta, \chi) = \lim_{\varepsilon\to 0} \dfrac{\Delta\mathsf{J}(\varepsilon)}{\varepsilon}$, the functional derivative becomes

$$\delta\mathsf{J}[y^*; x^*]\ (\eta, \chi) = \int_{x^*_0}^{x^*_1} \Big(F^*_y(x)\eta(x) + F^*_{y'}(x)\,\eta'(x)\Big)dx + F^*(x)\Big|_{x=x^*_1}\chi_1 - F^*(x)\Big|_{x=x^*_0}\chi_0,$$

where $\chi = (\chi_0, \chi_1)$. Integration by parts yields

$$\begin{aligned}
\delta\mathsf{J}[y^*; x^*]\ (\eta, \chi) &= \int_{x^*_0}^{x^*_1} \left(F^*_y(x) - \frac{d}{dx}F^*_{y'}(x)\right)\eta(x)dx + \\
&\quad + F^*_{y'}(x)\eta(x)\Big|_{x=x^*_1} - F^*_{y'}(x)\eta(x)\Big|_{x=x^*_0} + F^*\,(x)\big|_{x=x^*_1}\,\chi_1 - F^*(t)\big|_{x=x^*_0}\,\chi_0.
\end{aligned}$$

We only know the perturbations for the state variables at the perturbed boundaries x_0 and x_1 and not at x^*_0 and x^*_1, which inhibits the computation of the integral in the last equation. In order to find $\eta(x^*_j)$, using the approximation $\eta(x_j) \approx \eta(x^*_j) + y'(x^*_j)\,\chi_j$, we introduce

$$\eta(x^*_j) \approx \eta_j - y'(x^*_j)\chi_j,\ \text{for } j = 0, 1.$$

Therefore,

$$\delta \mathsf{J}[y^*; x^*]\ (\eta, \chi) = \int_{x_0^*}^{x_1^*} \left(F_y^*(x) - \frac{d}{dx} F_{y'}^*(x) \right) \eta(x) dx + F_{y'}^*(x_1^*)\eta_1 - F_{y'}^*(x_0^*)\eta_0 + \\ + \left(F^*(x) - F_{y'}^*(x) y'(x) \right)\Big|_{x=x_1^*} \chi_1 - \left(F^*(x) - F_{y'}^*(x) y'(x) \right)\Big|_{x=_0^*} \chi_0$$

As the terminal values of the state variables, $y(x_0^*) = y_0$ and $y(x_1^*) = y_1$, are known then the terminal perturbation for the independent variable should satisfy $\eta_0 = \eta_1 = 0$. Therefore, $\delta \mathsf{J}[y^*; x^*]\ (\eta, \chi) = 0$ if and only if the Euler-Lagrange equation holds and $\left(F^*(x) - F_{y'}^*(x) y'(x) \right)\Big|_{x=x_1^*} \chi_1 = 0$ and/or $\left(F^*(x) - F_{y'}^*(x) y'(x) \right)\Big|_{x=_0^*} \chi_0 = 0$. This encompasses the three cases in equations (P5), (6.13) and (6.14). □

6.2.4 Free boundaries for both independent and dependent variables

The most general problem is: find function $y^* \in \mathcal{Y}$ among functions $y : \mathrm{X}^* \to \mathbb{R}$, where X^* has at least one unknown boundary, x_0^* and/or x_1^*, as in the previous subsection, and the terminal values for the state variables, $y(x_0^*)$ and/or $y(x_1^*)$ are also free. The objective functional is again (6.1).

Formally, the problem are :

$$\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx$$

subject to

$$y(x_0) = y_0 \text{ and } x_0 \text{ free, } y(x_1) = y_1 \text{ and } x_1 \text{ fixed} \tag{P4a}$$

$$y(x_0) = y_0 \text{ and } x_0 \text{ fixed, } y(x_1) = y_1 \text{ and } x_1 \text{ free} \tag{P4b}$$

$$y(x_0) = y_0,\ x_0,\ y(x_1) = y_1 \text{ and } x_1 \text{ free} \tag{P4c}$$

In this case the data of the problem, $\varphi = (.)$, only involves parameters that may be present in function $F(.)$. The value functional is

$$V(.) = \max_{y \in \mathcal{Y}, x_0, x_1} \int_{x_0}^{x_1} F(x, y(x), y'(x)) dx = \int_{x_0^*}^{x_1^*} F(x, y^*(x), y^{*'}(x)) dx$$

where we x_0^* and/or x_1^* and $y^*(x_0^*)$ and/or $y^*(x_1^*)$ are determined endogenously.

Proposition 7. ***First order necessary conditions for the free terminal boundary problem:*** *$y^* \in \mathcal{Y}$ is the solution to the CV problem with free boundary values for the state variable and for the independent variable, only if it satisfies the Euler equation* (6.2) *and the boundary conditions:*

1. *if both values for lower boundary are free (problem P4a)*

$$F_y(x_0^*, y^*(x_0^*), y^{*'}(x_0^*)) = F_{y'}(x_0^*, y^*(x_0^*), y^{*'}(x_0^*)) = 0 \tag{6.16}$$

2. *if both values for upper boundary are free (problem P4b)*

$$F_y(x_1^*, y^*(x_1^*), y^{*'}(x_1^*)) = F_{y'}(x_1^*, y^*(x_1^*), y^{*'}(x_1^*)) = 0 \tag{6.17}$$

3. *if all terminal values for x and $y(x)$ are free (problem P4c)*

$$F_y(x_0^*, y^*(x_0^*), y^{*'}(x_0^*)) = F_{y'}(x_0^*, y^*(x_0^*), y^{*'}(x_0^*)) = 0 \tag{6.18a}$$

$$F_y(x_1^*, y^*(x_1^*), y^{*'}(x_1^*)) = F_{y'}(x_1^*, y^*(x_1^*), y^{*'}(x_1^*)) = 0 \tag{6.18b}$$

Proof. We use the previous proof and, in equation (6.15), we consider $\eta_0 \neq 0$, $\eta_1 \neq 0$, $\chi_0 \neq 0$ and $\chi_1 \neq 0$. □

Table 6.1 assembles all the previous results. Observe that if we consider all the possible combinations of the information on both boundaries we have **16 possible cases**.

Table 6.1: Adjoint conditions for bounded domain CV problems

data		optimum	
x_j	$y(x_j)$	x_j^*	$y^*(x_j^*)$
fixed	fixed	x_j	y_j
fixed	free	x_j	$F_{y'}(x_j, y^*(x_j), y^{*'}(x_j)) = 0$
free	fixed	$F(x_j^*, y_j, y^{*'}(x_j^*)) - y^{*'}(x_j^*)F_{y'}(x_j^*, y_j, y^{*'}(x_j^*)) = 0$	y_j
free	free	$F\left(x_j^*, y^*(x_j^*), y^{*'}(x_j^*)\right) = 0$	$F_{y'}(x_j^*, y^*(x_j^*), y^{*'}(x_j^*)) = 0$

The index refers to the lower boundary when $j = 0$ and to the upper boundary when $j = 1$

6.2.5 Inequality terminal constraint

As for the static optimization problem we can consider inequality constraints, for instance inequality constraints on the value of the variable y for some value of the independent variable.

We consider a problem in which the two limits for independent variable are known, i.e, x_0 and x_1 are known, $y(x_0) = y_0$ is known, but we $y(x_1)$ is constrained by the condition $R(x_1, y(x_1)) \geq 0$.

Formally, the problem are :

$$\begin{aligned} &\max_{y(\cdot)} \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \\ &\text{subject to} \\ &x_0 \text{ and } x_1 \text{ fixed} \\ &y(x_0) = y_0 \text{ fixed} \\ &R(x_1, y(x_1)) \geq 0. \end{aligned} \tag{P5}$$

Proposition 8. ***First order necessary conditions for the constrained terminal state problem:*** *$y^* \in \mathcal{Y}$ is the solution to the CV problem (P5) only if it satisfies the Euler-Lagrange equation* (6.2)*, the initial condition $y^*(x_0) = y_0$ and the boundary condition*

$$F_{y'}(x_1, y^*(x_1), y^{*'}(x_1))R(x_1, y^*(x_1)) = 0 \tag{6.19}$$

Proof. In this case we consider the functional we introduce a Lagrange multiplier (a real number) associated to the terminal condition, yielding the Lagrange functional

$$L[y] = \int_{x_0}^{x_1} F(x, y(x), y'(x))dx + \mu R(x_1, y(x_1)).$$

We proceed as previously to get the optimality conditions for a perturbation $\eta \in \mathcal{Y}$ over the optimal function y^*. The first order necessary condition is

$$\delta L[y^*](\eta) = \int_{x_0}^{x_1} \left(F_y^*(x) - \frac{d}{dx}\, F_{y'}^*(x) \right) \eta(x)dx + \left(F_{y'}^*(x_1) + \mu R_y^*(x_1) \right) \eta(x_1) = 0$$

where $F^*(x) = F(x, y^*(x), {y^*}'(x))$ and $R_y^*(x_1) = \partial_y R(x_1, y^*(x_1))$. Because of the free terminal state, admissible perturbations are such that $\eta(x_1) \neq 0$. Therefore $\delta L[y^*](\eta) = 0$ requires that the adjoint condition $F_{y'}^*(x_1) + \mu R_y^*(x_1) = 0$ holds.

Due to the existence of a static inequality constraint at the boundary x_1, the Karush-Kuhn-Tucker (KKT) complementarity slackness conditions are also necessary:

$$\mu R^*(x_1) = 0,\ \mu \geq 0 \text{ and } \ R^*(x_1) \geq 0$$

where $R^*(x_1) = R(x_1, y^*(x_1))$. Multiplying the adjoint condition by $R^*(x_1)$ we obtain an equivalent condition

$$R^*(x_1)\, F_{y'}^*(x_1) + \mu\, R^*(x_1)\, R_y^*(t_y) = 0,$$

which is equivalent to $F_{y'}^*(x_1) R^*(x_1) = 0$, after considering the KKT condition. Therefore $\delta L[y^*](\eta) = 0$ if the Euler-Lagrange equation (6.2) and adjoint boundary condition (6.19) hold. □

6.3 Calculus of variations in time

We can directly apply the previous results for problems in which time is the independent variable. When time is the independent variable the domain of the independent variable it $\mathrm{T} \subseteq \mathbb{R}_+$, if we have a finite interval $\mathrm{T} = [t_0, t_1]$, the dependent variable is $y(t)$, which is a mapping $y : \mathrm{T} \ \to \mathcal{Y} \subseteq \mathbb{R}$, and we denote the time derivative by $\dot{y} = \dfrac{dy(t)}{dt}$.

A particular important problem is the discounted infinite-horizon problem

6.3.1 Discounted infinite horizon

The most common problem in macroeconomics and growth theory has three main common features. First, time is the independent variable, and assumes that the initial time and values are known, usually $x_0 = 0$ and $y(0) = y_0$, and an unbounded value for the terminal time, $x_1 \to \infty$. Second, the objective function is of type $F(t, y, \dot{y}) = f(y, \dot{y})e^{-\rho t}$, where $e^{-\rho t}$ is a discount factor with a time-independent rate of discount $\rho \geq 0$, and the current value objective function $f(y, \dot{y})$ is time -independent. Third, there are two main versions to the problem depending on the terminal value of the state variable, that can be free or constrained.

Free asymptotic state

Find function $y^* \in \mathcal{Y}$ which is the set of functions $y : [0, \infty) \to \mathbb{R}$ such that $y(0) = y_0$ where y_0 are given that maximizes

$$\mathsf{J}[y] \equiv \int_0^{\infty} f(y(t), \dot{y}(t))e^{-\rho t}dt, \ \rho \geq 0. \tag{6.20}$$

This can be treated as a problem with a fixed initial time and value for the state variable, a fixed terminal time but a free terminal value for the state variable.

Proposition 9. ***First order necessary conditions for the discounted infinite horizon problem with free terminal state:*** *$y^* \in \mathcal{Y}$ is the solution to the discounted infinite horizon CV problem with a known initial data, $\varphi = (y_0, \rho, .)$, and with a free terminal state only if it satisfies the Euler-Lagrange equation*

$$\frac{d}{dt}\left(f_{\dot{y}}(y^*(t), \dot{y}^*(t))\right) = f_y(y^*(t), \dot{y}^*(t)) + \rho f_{\dot{y}}(y^*(t), \dot{y}^*(t)), \ \textit{for} \ \ t \in [0, \infty), \tag{6.21}$$

the so-called transversality condition

$$\lim_{t\to\infty} f_{\dot{y}}(y^*(t), \dot{y}^*(t))e^{-\rho t} = 0 \tag{6.22}$$

and the initial condition $y^*(0) = y_0$

Proof. In the proof for the free boundaries value problem we extend $x_1 \to \infty$ and take it as fixed but let $\lim_{t\to\infty} y^*(t)$ be free. In this discounted problem the Euler-Lagrange equation (6.2) $F^*_y(t) = \frac{d}{dt}F^*_{\dot{y}}(t)$ is equivalent to

$$e^{-\rho t}f_y(y^*, \dot{y}^*) = \frac{d}{dt}\left(e^{-\rho t}f_{\dot{y}}(y^*, \dot{y}^*)\right),$$

and the terminal condition (6.22) is obtained fro the boundary condition $\lim_{t\to\infty} F^*_{\dot{y}}(t) = 0$. □

Observe that the Euler-Lagrange is again a 2nd order non-linear autonomous ODE

$$f_y(y^*, \dot{y}^*) + \rho f_{\dot{y}}(y^*, \dot{y}^*) - f_{\dot{y}y}(y^*, \dot{y}^*)\dot{y} - f_{\dot{y}\dot{y}}(y^*, \dot{y}^*)\ddot{y} = 0.$$

The constrained terminal state problem

In several problems in economics the former condition can lead to an asymptotic state which does not make economic sense (v.g, a negative level for a capital stock).

The most common discounted infinite horizon model is usually the following: find function $y^* \in \mathcal{Y}$ which is the set of functions $y : [0, \infty) \to \mathbb{R}$ such that $y(0) = y_0$ and $\lim_{t\to\infty} R(t, y(t)) \geq 0$, where $x_0 = 0$ and $y(x_0) = y_0$ are given, that maximizes the objective functional (6.20)

Proposition 10. ***First order necessary conditions for the discounted infinite horizon problem with constrained terminal state:*** *$y^* \in \mathcal{Y}$ is the solution to the discounted infinite horizon CV problem with a known initial data, $(x_0, y(x_0)) = (0, y_0)$, and with a terminal state constrained by $\lim_{t\to\infty} R(t, y(t)) \geq 0$ only if it satisfies the Euler-Lagrange equation (6.21), the initial condition $y^*(0) = y_0$, and the (so-called) transversality condition*

$$\lim_{t\to\infty} f_{\dot{y}}(y^*(t), \dot{y}^*(t)) R(t, y^*(t)) e^{-\rho t} = 0 \tag{6.23}$$

Exercise: prove this. Observe that as the terminal constraint is $\lim_{t\to\infty} y(t) \geq 0$ we have to introduce a Lagrange multiplier associated to the terminal time.

6.3.2 Applications

The resource depletion problem

Assume we have a resource W (v.g., a cake) with initial size W_0 and we want to consume it along period $[0, \bar{t}]$. If $C(t)$ denotes the consumption at time x we evaluate the consumption of the resource by the functional $\int_0^{\bar{t}} \ln(C(t)) e^{-\rho t} dt$. Several properties: (1) we are impatient (we discount time at a rate $\rho > 0$); (2) the felicity at every point in time is only a function of the instantaneous consumption (preferences are inter temporally additive); (3) more consumption means more felicity but at a decreasing rate (the increase in utility for big bites is smaller than for small bites); and (4) there is no satiation (there is not a bite with a zero or negative marginal utility): consumption is always good.

Cake eating problem with the terminal state given CE problem: find $C^* = (C^*(t))_{0\leq t\leq \bar{t}}$ that

$$\max_C \int_0^{\bar{t}} \ln(C(t)) e^{-\rho t} dt$$

subject to

$$\dot{W}(t) = -C(t), \text{ for } t \in [0, T]$$

given $W(0) = W_0$ and $W(\bar{t}) = 0$.

Formulated as a CV problem: find $W^* = (W^*(t))_{0\leq t\leq \bar{t}}$ such that

$$V(W_0, \bar{t}, \rho) = \max_W \mathsf{J}[W] = \max_W \int_0^{\bar{t}} \ln(-\dot{W}(t)) e^{-\rho t} dt$$

given $W(0) = W_0$ and $W(\bar{t}) = 0$. The data of the problem is the vector of constants $\varphi = (0, \bar{t}, W_0, 0, \rho)$

The solution of the problem, $(W^*(t))_{t=0}^{\bar{t}}$, is obtained from

$$\begin{cases} \ddot{W}^* + \rho \dot{W}^* = 0, & 0 < t < \bar{t} \\ W^*(0) = W_0, & t = 0 \\ W^*(T) = 0, & t = T \end{cases}$$

The solution of the Euler equation is [2]

$$W(t) = W(0) - \frac{k}{\rho}\left(1 - e^{-\rho t}\right)$$

where k is an arbitrary constant. Using the adjoint conditions $W^*(\bar{t}) = 0$ and $W^*(0) = W_0$ we find the solution

$$W^*(t) = \frac{e^{-\rho t} - e^{-\rho \bar{t}}}{1 - e^{-\rho \bar{t}}} \, W_0, \text{ for } t \in [0, \bar{t}].$$

The value of the cake is

$$\begin{aligned} V(\varphi) &= \int_0^{\bar{t}} \ln\left(-\dot{W}^*(t)\right) e^{-\rho t} dt = \\ &= \frac{1}{\rho}\left[\left(1 + \ln\left(\frac{1 - e^{-\rho \bar{t}}}{\rho W_0}\left(e^{-\rho \bar{t}} - 1\right)\right)\right)\right] + \bar{t} e^{-\rho \bar{t}} \end{aligned}$$

if the consumer is rational this should be equal its reservation price for the cake. If $\rho = 0.01$ and the cake lasts for one week and the calorie content is $W_0 = 1000$ then the reservation price for should be $V(10, 0.01, 1/52) \approx 0.12$ per 100 calories.

Cake eating problem: infinite horizon If we assume an infinite horizon and the terminal condition $\lim_{t \to \infty} W(t) \geq 0$, meaning that we cannot have a negative level of resource asymptotically. The first order conditions are:

$$\begin{cases} \rho \dot{W}^*(t) + \ddot{W}^*(t) = 0 \\ W^*(0) = W_0 \\ -\lim_{t \to \infty} e^{-\rho t} \frac{W^*(t)}{\dot{W}^*(t)} = 0 \end{cases}$$

We already found

$$W(t) = W_0 - \frac{k}{\rho}\left(1 - e^{-\rho t}\right)$$

then

$$\dot{W}(t) = -k e^{-\rho t}$$

Solution (as $k = \rho W_0$)

$$W^*(t) = W_0 e^{-\rho t}, \ t \in \mathbb{R}_+$$

Again $\lim_{t \to \infty} W^*(t) = 0$.

The benchmark representative problem

The benchmark representative consumer problem in macroeconomics is to find optimal consumption and asset holdings (C, A) such that $C : \mathbb{R}_+ \to \mathbb{R}_+$ and $A : \mathbb{R}_+ \to \mathbb{R}$ that maximize the value functional

$$U[C] = \int_0^{\infty} u(C(t)) e^{-\rho t} dt$$

[2]Hint: setting $z = \dot{W}$ we get a first-order ODE $\dot{z} = -\rho z$ with solution $\dot{z} = k e^{-\rho t}$. As $dW(t) = z(t)dt$, if we integrate we have $\int_{W(0)}^{W(t)} dW = \int_0^t z(s)ds = \int_0^t k e^{-\rho s} ds$.

subject to the instantaneous budget constraint

$$\dot{A} = Y - C + rA$$

given $A(0) = A_0$ and the non-Ponzi game condition $\lim_{t\to} A(t)e^{-rt} \geq 0$. In the above equations Y and r denote, respectively the non-financial income and the interest rate, and are both positive. The following assumptions on utility are standard: $u(0) = 0$, $u^{'}(C) > 0$ and $u^{''}(C) < 0$.

The inverse of the elasticity of intertemporal substitution can be proved to be

$$\theta(C) = -\frac{u^{''}(C)C}{u^{'}(C)} > 0.$$

Assumption: the elasticity of intertemporal substitution $\theta(C) = \theta$ is constant and

$$\theta > \frac{r-\rho}{r} > 0.$$

We can transform it into a CV problem by observing that consumption is a function of the both wealth and savings, $\dot{A}$,

$$C = C(A, \dot{A}) \equiv Y + rA - \dot{A}.$$

Therefore, the problem becomes a CV problem with value functional

$$\mathsf{J}[A] = \int_0^\infty u\left(Y + rA(t) - \dot{A}(t)\right) e^{-\rho t} dt$$

where $f(A(t), \dot{A}(t)) = u\left(Y + rA(t) - \dot{A}(t)\right)$. The optimality conditions (which are necessary and sufficient in this case) are

$$\begin{cases} (r-\rho)u^{'}(C(A,\dot{A})) + (r\dot{A} - \ddot{A})u^{''}(C(A,\dot{A})) = 0 \\ A(0) = A_0 \\ -\lim_{t\to\infty} e^{-\rho t}u^{'}(C(A,\dot{A}))A(t) \ = 0 \end{cases}$$

Observing that $\dot{C} = r\dot{A} - \ddot{A}$ and using the definition of the inverse intertempopral elasticity of substitution we can transform the Euler equation into

$$\dot{C} = \gamma C, \text{ for } \gamma \equiv \frac{r-\rho}{\theta} > 0.$$

This allows us to find a general solution for optimal consumption

$$C(t) = C(0)\, e^{\gamma t},$$

where $C(0)$ is an arbitrary unknown admissible level for consumption, i.e., it should be non-negative. In order to find that value we use the transversality condition. But for this we need to determine admissible values for A. The asset dynamics is then governed by

$$\dot{A} = Y + rA - ke^{\gamma t}, \text{for } t > 0,\ A(0) = A_0, \text{ for } t = 0$$

The solution to this initial value problem is

$$A(t) = -\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right) e^{rt} + \frac{C(0)}{r-\gamma} \left(e^{rt} - e^{\gamma t}\right).$$

With the previous assumption we have $r > \gamma$. As $u^{'}(C) = C^{-\theta}$ with an isoelastic utility function we find

$$\begin{aligned}
\lim_{t\to\infty} u^{'}(C(t))A(t)e^{-\rho t} &= \lim_{t\to\infty} \left(C(0)e^{\gamma t}\right)^{-\theta} A(t)e^{-\rho t} = \\
&= \lim_{t\to\infty} C(0)^{-\theta} e^{-rt} \left[-\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right) e^{rt} + \frac{C(0)}{r-\gamma} \left(e^{rt} - e^{\gamma t}\right)\right] = \\
&= \lim_{t\to\infty} C(0)^{-\theta} \left[A_0 + \frac{Y}{r} + \frac{C(0)}{r-\gamma} - \frac{C(0)}{r-\gamma} e^{(\gamma - r)t}\right] = \\
&= C(0)^{-\theta} \left[A_0 + \frac{Y}{r} + \frac{C(0)}{r-\gamma}\right] = 0
\end{aligned}$$

if and only if $C(0) = (r-\gamma)\left(A_0 + \dfrac{Y}{r}\right)$. Therefore the optimal consumption and asset holdings are

$$C^*(t) = (r-\gamma)\left(A_0 + \frac{Y}{r}\right) e^{\gamma t},\ t \in [0,\infty) \tag{6.24}$$

$$A^*(t) = -\frac{Y}{r} + \left(A_0 + \frac{Y}{r}\right) e^{\gamma t},\ t \in [0,\infty). \tag{6.25}$$

Observations: First, if we define human capital as the present value, at rate r, of the non-financial income

$$H(t) = \int_t^\infty Y e^{r(t-s)} ds$$

we find $H(0) = \dfrac{Y}{r}$. Therefore the solution is a linear function of the total capital, financial and non-financial

$$C^*(t) = (r-\gamma)(A_0 + H(0))e^{\gamma t},\ A^*(t) = -H(0) + (A_0 + H(0))e^{\gamma t}$$

Second, because $\gamma > 0$ then the asymptotic value of the optimal A becomes unbounded. However, it still satisfies that boundary condition $\lim_{t\to\infty} A^*(t)e^{-rt} = 0$ because, by assumption, $r > \gamma$. What matters is not the absolute level of A but its level in present-value terms.

6.4 Bibliography

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