Foundations of Financial Economics
Two period financial markets

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Topics

- Financial assets
- Financial market
- State prices
- Portfolios
- Absence of arbitrage opportunities
- Complete financial markets
- Arbitrage pricing
- Equity premium
- Equity premium puzzle

Information

- Assume again that there is **complete** information regarding time $t = 0$, but **incomplete** information regarding time $t = 1$.
- Until now we have (mostly) assumed that the probability distribution is given by **nature**.
- But next we start assuming that information regarding time $t = 1$ is provided by the **financial market**.
- A financial market is defined by a collection of **contingent claims**.
- The **general idea**: under some conditions, we can extract an **implicit probability distribution** of the states of nature from the structure of the financial market.

Financial assets

Prices and payoffs

A financial asset or contract or contingent claim: is defined by the **price and payoff** pair (S_j, V_j) (for asset j):

- for an **observed** price, S_j, paid at time $t = 0$
- a **contingent** payoff, V_j, will be received at time $t = 1$,

$$V_j = (V_{j1}, \ldots, V_{js}, \ldots, V_{jN})^\top$$

Financial assets

Returns and rates of return

- Return R_j and rate of return r_j of asset j are related as

$$R_j = 1 + r_j,$$

- In the 2-period case: it is the payoff/price ratio

$$R_j = \frac{V_j}{S_j} = \begin{pmatrix} \frac{V_{j,1}}{S_j} \\ \dots \\ \frac{V_{j,s}}{S_j} \\ \dots \\ \frac{V_{j,N}}{S_j} \end{pmatrix} = \begin{pmatrix} 1 + r_{j,1} \\ \dots \\ 1 + r_{j,s} \\ \dots \\ 1 + r_{j,N} \end{pmatrix}$$

Financial assets

Timing, information and flow of funds

Two alternative ways of representing (net) income flows:

- as a price-payoff sequence

$$
\begin{array}{ccc}
-S_j & & \begin{pmatrix} V_{j,1} \\ \dots \\ V_{j,s} \\ \dots \\ V_{j,N} \end{pmatrix} \\
\vdash & \!\!\!\longrightarrow\!\!\! & \dashv \\
0 & & 1
\end{array}
$$

- as an investment-return sequence

$$
\begin{array}{ccc}
-1 & & \begin{pmatrix} R_{j,1} \\ \dots \\ R_{j,s} \\ \dots \\ R_{j,N} \end{pmatrix} \\
\vdash & \!\!\!\longrightarrow\!\!\! & \dashv \\
0 & & 1
\end{array}
$$

Financial assets

Statistics

- From the information, at time $t = 0$, on the probabilities for the states of nature at time $t = 1$ we can compute:
- **Expected return** for asset j at time $t = 1$, from the information at time $t = 0$

$$\mathbb{E}[R_j] = \sum_{s=1}^{N} \pi_s R_{j,s} = \pi_1 R_{j,1} + \ldots \pi_s R_{j,s} + \ldots + \pi_N R_{j,N}$$

- **Variance of the return** for asset j at time $t = 1$, from the information at time $t = 0$

$$\mathbb{V}[R_j] = \sum_{s=1}^{N} \pi_s (R_{js} - \mathbb{E}[R_j])^2 = \pi_1 \left(R_{j,1} - \mathbb{E}[R_j]\right)^2 + \ldots \pi_N \left(R_{j,N} - \mathbb{E}[R_j]\right)^2$$

- Observation: an useful relationship

$$\mathbb{V}[R] = \mathbb{E}[R^2] - \left(\mathbb{E}[R]\right)^2$$

Classification of assets

Risk classification

As regards **risk**:

- risk-less or risk-free asset: $V^\top = (v, \ldots, v)^\top$
- risky asset: $V^\top = (v_1, \ldots, v_N)^\top$ with at least two different elements, i.e. there are at least two elements, v_i and v_j such that $v_i \neq v_j$ for $i \neq j$

Classification of assets

Types of assets

Particular **types** of assets as regards the income flows:

- one period **bonds** with unit facial value:

$$S = \frac{1}{1+i}, \;\; V^{\top} = (1, 1, \ldots, 1)^{\top}$$

- **deposits**:

$$S = 1, \;\; V^{\top} = (1+i, 1+i, \ldots, 1+i)^{\top}$$

- **equity**: the payoff are dividends

$$S_e, \;\; V_e^{\top} = (d_1, \ldots, d_N)^{\top}$$

Classification of assets

Types of assets

Derivatives:

- forward contract on an underlying asset with offered price p:

$$S_f,\ V_f^\top = (v_1 - p, \dots, v_N - p)^\top$$

- european call option with exercise price p:

$$S_c,\ V_c^\top = (\max\{v_1 - p, 0\}, \dots, \max\{v_N - p, 0\})^\top$$

- european put option with exercise price p:

$$S_p,\ V_p^\top = (\max\{p - v_1, 0\}, \dots, \max\{p - v_N, 0\})^\top$$

Financial market

Definition: A financial market is a collection of K traded assets. It can be characterized by the structure of prices and payoffs of all K assets : i.e by the pair (S, V):

- vector of observed prices, at time $t = 0$ (we use row vectors for prices)

$$\underbrace{S}_{K\times 1} = (S_1, \ldots, S_K),$$

- and a matrix of contingent (i.e., uncertain) payoffs, at time $t = 1$

$$\underbrace{V}_{N\times K} = \begin{pmatrix} V_{11} & \ldots & V_{K1} \\ \vdots & & \vdots \\ V_{1N} & \ldots & V_{KN} \end{pmatrix}$$

- The participants observe S and know V, but they do not know which payoff they will get for every asset (i.e, which line of V will be realized)

Financial market

The payoff matrix contains the following information:

- each column represents the payoff for**asset** $j = 1, \ldots, K$

$$V = \begin{pmatrix} V_{11} & \ldots & \mathbf{V_{j1}} & \ldots & V_{K1} \\ \vdots & & \vdots & & \vdots \\ V_{1s} & \ldots & \mathbf{V_{js}} & \ldots & V_{Ks} \\ \vdots & & \vdots & & \vdots \\ V_{1N} & \ldots & \mathbf{V_{jN}} & \ldots & V_{KN} \end{pmatrix}$$

- each row represents the outcomes (in terms of payoffs) for **state** of nature $s = 1, \ldots, N$

$$V = \begin{pmatrix} V_{11} & \ldots & V_{j1} & \ldots & V_{K1} \\ \vdots & & \vdots & & \vdots \\ \mathbf{V_{1s}} & \ldots & \mathbf{V_{js}} & \ldots & \mathbf{V_{Ks}} \\ \vdots & & \vdots & & \vdots \\ V_{1N} & \ldots & V_{jN} & \ldots & V_{KN} \end{pmatrix}$$

Financial market

- Equivalently, we can charaterize a financial market by the matrix of returns

$$\underbrace{R}_{N\times K} = \begin{pmatrix} R_{11} & \dots & R_{K1} \\ \vdots & & \vdots \\ R_{1N} & \dots & R_{KN} \end{pmatrix}$$

  where $R_{j,s} = \dfrac{V_{j,s}}{S_j} = 1 + r_{j,s}$

- Therefore

$$\underbrace{R}_{N\times K} = \underbrace{V}_{N\times K} \underbrace{(\text{diag}(S))^{-1}}_{K\times K}$$

  where

$$\underbrace{\text{diag}(S)}_{K\times K} = \begin{pmatrix} S_1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & S_j & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & S_K \end{pmatrix}$$

State prices

Implicit state valuation: the modern approach to financial economics characterizes asset markets by the prices of the states of nature **implicit** in the relationship between present prices S and future payoffs V, i.e.

$$Q := \{x : S = xV\}$$

- if Q is unique it is a vector

$$\underbrace{Q}_{1\times N} = (q_1, \ldots, q_N)$$

satisfying

$$\underbrace{S}_{1\times K} = \underbrace{Q}_{1\times N}\underbrace{V}_{N\times K} \Leftrightarrow \underbrace{S^\top}_{K\times 1} = \underbrace{V^\top}_{K\times N}\underbrace{Q^\top}_{N\times 1}$$

- **Definition**: Q is a state price vector if is it positive , i.e. $q_s > 0$ for **all** $s \in \{1, \ldots, N\}$

Arbitrage opportunities

Definition: if there is at least one $q_s \leq 0$, $s = 1, \ldots, N$ then we say **there are arbitrage opportunities**

Definition: we say **there are no arbitrage opportunities** if $q_s > 0$, for all $s = 1, \ldots, N$

Intuition:

- existence of arbitrage (opportunities) means there are free or negatively valued states of nature
- absence of arbitrage means every state of nature is costly and therefore, positively priced.

Proposition 1

Given (S, V), there are no arbitrage opportunities if and only if Q is a vector of state prices.

Completeness of a financial market

Definition: if Q **is unique**, then we say markets are **complete**

Definition: if Q **is not unique** then we say markets are **incomplete**

Intuition:

- completeness: there is an unique valuation of the states of nature.
- incompleteness: there is not an unique valuation of the states of nature.

Completeness of a financial market

Conditions for completeness

We can classify completeness of a market by looking at the number of assets, number of states of nature and the independence of payoffs:

1. if $K = N$ and $\det(V) \neq 0$ then markets are complete and all assets are independent;
2. if $K > N$ and $\det(V) \neq 0$ then markets are complete and there are N independent and $K - N$ **redundant** assets;
3. if $K < N$ or $K \geq N$ and $\det(V) = 0$ then markets are incomplete.

Proposition 2

Given (S, V), markets are complete if and only if $dim(V) = N$

$\dim(V)$ = number of linearly independent columns (i.e., assets)

Characterization of a financial market

No arbitrage opportunities and completeness

- Consider a financial market (S, V), such that $K = N$ and $\det(V) \neq 0$
- We defined state prices from $S = QV$
- As $K = N$ and $\det(V) \neq 0$ then V^{-1} exists and is unique
- Therefore, we obtain uniquely

$$\underbrace{Q}_{1\times N} = \underbrace{S}_{1\times K}\underbrace{V^{-1}}_{K\times N}$$

that is

$$(q_1, \ldots, q_N) = (S_1, \ldots, S_K)\begin{pmatrix} V_{1,1} & \ldots & V_{1,K} \\ \ldots & \ldots & \ldots \\ V_{N,1} & \ldots & V_{N,K} \end{pmatrix}^{-1}$$

- If $Q \gg 0$ we say there are no arbitrage opportunities
- And because Q is unique we say markets are complete

Characterization of a financial market

No arbitrage opportunities and completeness: alternative 1

Alternatively, because

$$\underbrace{V^\top}_{K\times N}\underbrace{Q^\top}_{N\times 1} = \underbrace{S^\top}_{K\times 1}$$

If $\det(V) \neq 0$ then

$$Q^\top = \left(V^\top\right)^{-1} S^\top$$

that is

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} V_{1,1} & \dots & V_{N,1} \\ \dots & \dots & \dots \\ V_{1,K} & \dots & V_{N,K} \end{pmatrix}^{-1} \begin{pmatrix} S_1 \\ \vdots \\ S_K \end{pmatrix}$$

Characterization of a financial market

No arbitrage opportunities and completeness: alternative 2

As $R_{js} = V_{js}/S_j$ equivalently, we can write

$$\underbrace{Q}_{1\times N}\underbrace{R}_{N\times K} = \underbrace{\mathbf{1}^\top}_{1\times K}$$

$$(q_1, \ldots, q_N)\begin{pmatrix} R_{1,1} & \ldots & R_{1,K} \\ \ldots & \ldots & \ldots \\ R_{N,1} & \ldots & R_{N,K} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

then

$$\boxed{Q = \mathbf{1}^\top R^{-1}}$$

Characterization of a financial market

No arbitrage opportunities and completeness: alternative 3

Or, alternatively,

$$R^{\top} Q^{\top} = \mathbf{1}$$

then

$$\boxed{\underbrace{Q^{\top}}_{N\times 1} = \underbrace{(R^{\top})^{-1}}_{N\times K} \underbrace{\mathbf{1}}_{K\times 1}}$$

matricially

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} R_{1,1} & \dots & R_{1,K} \\ \dots & \dots & \dots \\ R_{N,1} & \dots & R_{N,K} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Characterization of a financial market

Market incompleteness

- Consider a financial market (S, V), such that $K < N$, and in the partition

$$\underbrace{V^\top}_{K\times N} = \big(\underbrace{V_1^\top}_{(K\times K)} \quad \underbrace{V_2^\top}_{(N-K\times K)} \big)$$

  assume that $\det(V_1) \neq 0$

- We defined state prices from $V^\top Q^\top = S^\top$
- But now we have

$$\begin{pmatrix} V_1^\top & V_2^\top \end{pmatrix} \begin{pmatrix} Q_1^\top \\ Q_2^\top \end{pmatrix} = S^\top$$

  where Q_1 is $(1 \times K)$ and Q_2 is $(1 \times N - K)$.

Characterization of a financial market

Market incompleteness

- Then

$$V_1^\top Q_1^\top + V_2^\top Q_2^\top = S^\top$$

- Because $\det(V_1) \neq 0$ we can make

$$Q_1^\top = (V_1^\top)^{-1}\left(S^\top - V_2^\top Q_2^\top\right)$$

- There are only K independent prices: i.e, Q is indeterminate and the degree of indeterminacy is $N - K$,
- As Q is not uniquely determined (i.e, we can fix arbitrarilty $N - K$ state prices) the market is **incomplete**

Characterization of a financial market

Market completeness with redundant assets

- Consider a financial market (S, V), such that $K > N$, partition

$$\underbrace{V}_{N\times K} = \Big(\underbrace{V_1}_{(N\times N)} \ \underbrace{V_2}_{(K-N\times N)} \ \Big)$$

and assume that $\det(V_1) \neq 0$

- We defined state prices from $V^\top Q^\top = S^\top$
- But now we have

$$\begin{pmatrix} V_1^\top \\ V_2^\top \end{pmatrix} \left(Q^\top\right) = \begin{pmatrix} S_1^\top \\ S_2^\top \end{pmatrix}$$

where S^1 is $(1 \times N)$ and S^2 is $(1 \times K - N)$.

Characterization of a financial market

Market completeness with redundant assets

- Then
$$\begin{cases} V_1^\top Q^\top = S_1^\top \\ V_2^\top Q^\top = S_2^\top \end{cases}$$
- Because $\det(V_1) \neq 0$ we can determine Q uniquely
$$Q^\top = (V_1^\top)^{-1} S_1^\top$$
- Which implies that the prices of the remaining $K - N$ assets can be obtained from the prices S_1
$$S_2^\top = V_2^\top Q^\top = V_2^\top \left((V_1^\top)^{-1} S_1^\top \right)$$
- There are $K - N$ **redundant assets** (i.e, they do not add new information on Q)
- As Q is uniquely determined the market is **complete**

Characterization of a financial market

Suggestion on how to apply this theory

- First: obtain R^T which is a $(K \times N)$ matrix
- Second: check the number of assets, K and the number of states of nature N
  - if $N < K$ then markets are incomplete
  - if $N \geq K$ check the number of independent rowsn of R
    - if $\dim(V) < N$ then markets are incomplete
    - if $\dim(V) \geq N$ then markets are complete
- Third: obtaining the state prices Q
  - if $K = N$ and $\det R \neq 0$ compute $Q^\top = (R^\top)^{-1}\mathbf{1}$
  - if $K > N$ and $\tilde{R}$ a $N \times N$ partition of R has $\det \tilde{R} \neq 0$ then compute $Q^\top = (\tilde{R}^\top)^{-1}\mathbf{1}$
  - if $K = N$ but $\det R = 0$, then, solve the independent equations in $R^\top Q^\top = \mathbf{1}$
  - if $K < N$ solve the independent equations in $R^\top Q^\top = \mathbf{1}$.

Example 1

- Let $S = (1, 1)$ and $V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- then
$$R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},\ R^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
- Therefore
$$Q^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

  then markets are complete but there are arbitrage opportunities.

Example 2

- Let $S = (1, 1)$ and $V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- then

$$R = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = R^\top$$

- Therefore

$$Q^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

  markets are complete and there are no arbitrage opportunities

Example 3

- Let $S = (1, 1, 2)$ and $V = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
- then $R = \begin{pmatrix} 2 & 1 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} \end{pmatrix}$ and $R^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}$
- we pick all the combinations of two assets

$$Q^\top = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$Q^\top = \begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} \\ -1 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

and

$$Q^\top = \begin{pmatrix} 1 & 2 \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{4}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

then markets are complete, there are no arbitrage opportunities and there is one redundant asset

Example 4

- Let $S = (1, 2)$ and $V = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$
- then $R = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ and $R^\top = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ two assets have the same returns
- then $R^\top Q^\top = \mathbf{1}^\top$ becomes

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has an infinite number of solutions

$$Q^\top = \begin{pmatrix} (1-k)/2 \\ k \end{pmatrix}$$

for an arbitrary k, then markets are incomplete and if $0 < k < 1$ there are no arbitrage opportunities;

Example 5

- Let $S = (1)$ and $V = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- then
$$R = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
- then we also have
$$(q_1, q_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1$$
or
$$2q_1 + q_2 = 1$$
has an infinite number of solutions $Q = ((1-k)/2, k)$, for any k, then markets are incomplete and if $0 < k < 1$ there are no arbitrage opportunities

Implicit market probabilities: risk-free probabilities

- Let $Q = (q_1, \ldots, q_N)$ be a state-price vector
- If there are no arbitrage opportunities then $q_s > 0$, i.e $Q \gg 0$
- A **Radon-Nikodyn derivative** is defined as

$$\boxed{\pi_s^Q = q_s/\bar{q}}$$

  where $\bar{q} = \sum_{s=1}^N q_s$.
- Therefore, if there are no arbitrage opportunities then

$$0 < \pi_s^Q < 1, \text{ and} \sum_{s=1}^N \pi_s^Q = 1$$

  that is π_s^Q **is a probability measure**
- This is a probability measure implicit in the financial market (S, V)
- Therefore:
  - If markets are complete the probability measure is unique
  - If markets are incomplete the probability measure is not unique

And asset pricing

- Using the definition for q_s

$$S_j = \sum_{s=1}^{N} q_s \; V_{sj} = \sum_{s=1}^{N} \bar{q} \frac{q_s}{\bar{q}} \; V_{sj}$$

  then, for any asset

$$\boxed{S_j = \bar{q} \, \mathbb{E}^Q \Big[V_j \Big]}$$

- This means prices are proportional to the expected value of the payoff, using an implicit market probability.

Definition

- Let $Q = (q_1, \ldots, q_N)$
- Assume we have a (objective or subjective) probability distribution for the states of nature

$$\mathbb{P} = (\pi_1, \ldots, \pi_N)$$

  such that $0 < \pi_s < 1$ and $\sum_{s=1}^{N} \pi_s = 1$

- The stochastic discount factor for state s is

$$\boxed{m_s = \frac{q_s}{\pi_s},\ s = 1, \ldots, N}$$

- The stochastic discount factor is the random variable

$$\underbrace{M}_{1 \times N} = (m_1, \ldots, m_s, \ldots, m_N)$$

And asset pricing

- Using the definition for q_s

$$S_j = \sum_{s=1}^{N} q_s \, V_{sj} = \sum_{s=1}^{N} \pi_s \frac{q_s}{\pi_s} \, V_{sj}$$

  then, for any asset

$$\boxed{S_j = \mathbb{E}\Big[M \, V_j\Big]}$$

- This means that the stochastic discount factor combines both market and fundamental information.

The state price approach to asset markets

Summing up:

- The state price translates the information structure implicit in financial transactions
- The relevant information is related to:
  - how costly is the insurance against the states of nature
  - the uniqueness of that insurance cost
  - suggests a method for pricing new assets: pricing by redundancy

Portfolios

- **Definition**: A **portfolio** is a vector specifying the positions, θ, in all the assets in the market

$$\theta^\top = (\theta_1, \ldots, \theta_K)^\top \in \mathbb{R}^K$$

If $\theta_j > 0$ there is a **long** position in asset j
if $\theta_j < 0$ there is a **short** position in asset j

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Portfolios

- The stream of income generated by position θ_j is

$$
\begin{array}{ccc}
 & & \begin{pmatrix} \theta_j V_{j,1} \\ \cdots \\ \theta_j V_{j,s} \\ \cdots \\ \theta_j V_{j,N} \end{pmatrix} \\
-S_j\theta_j & & \\
\vdash & \text{———————} & \dashv \\
0 & & 1
\end{array}
$$

  - long position ($\theta_j > 0$): pay $S_j\theta_j$ at time $t = 0$ and receive the contingent payoff $\theta_j V_j$ at time $t = 1$
  - short position ($\theta_j < 0$): receive $S_j\theta_j$ at time $t = 0$ and pay the contingent payoff $\theta_j V_j$ at time $t = 1$

Portfolios

A portfolio generates a (stochastic) stream of income $\{z_0^\theta, Z_1^\theta\}$

$$\begin{aligned} z_0^\theta &= -C(\theta, S) = \underbrace{-S\theta}_{1\times 1} = \sum_{j=1}^{K} S_j\theta_j \\ Z_1^\theta &= \underbrace{V\theta}_{N\times 1} = \sum_{j=1}^{K} V_j\theta_j \end{aligned}$$

where

$$Z_1^\theta = \begin{pmatrix} z_{1,1}^\theta \\ \dots \\ z_{1,s}^\theta \\ \dots \\ z_{1,N}^\theta \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{K} V_{j,1}\theta_j \\ \dots \\ \sum_{j=1}^{K} V_{j,s}\theta_j \\ \\ \sum_{j=1}^{K} V_{j,N}\theta_j \end{pmatrix}$$

Portfolios

The flow of income generated by a portfolio θ is:

$$\begin{array}{ccc} z_0^{\theta} & & \begin{pmatrix} z^{\theta}_{1,1} \\ \dots \\ z^{\theta}_{1,s} \\ \dots \\ z^{\theta}_{1,N} \end{pmatrix} \\ 0 & & 1 \end{array}$$

Portfolios and arbitrage opportunities

Proposition 3

Assume there are ***arbitrage opportunities****. Then there exists at least one portfolio ϑ such that*

$$z_0^{\vartheta} = 0 \text{ and } Z_1^{\vartheta} > 0$$

or

$$z_0^{\vartheta} > 0 \text{ and } Z_1^{\vartheta} = 0.$$

(Obs. A positive vector $X > 0$ has non-negative elements, and has at least one equal to zero. A strictly positive vector $X \gg 0$ has only positive elements)

Intuition

with a **zero cost** we can get a **positive income** in at least one state of nature.

If we have an **initial income**, we will pay a **zero cost** in the future, in every state of nature.

Example 1

- We consider Example 1 again: $S = (1, 1)$ and $V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- We can build a portfolio $\vartheta = (\vartheta_1, \vartheta_2)^\top$ such that $z_0^\vartheta = -S\vartheta = -(\vartheta_1 + \vartheta_2) = 0$, that is

$$\vartheta_1 = -x, \vartheta_2 = x$$

- The return at time $t = 1$ is

$$Z_1^\vartheta = V\vartheta = \begin{pmatrix} -x + x \\ -x + 2x \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

then if we take a long position in the risky asset ($x > 0$) we have a positive income at $t = 1$ with a zero investment.

Portfolios and arbitrage opportunities

Proposition 4

Assume there are ***no arbitrage opportunities****. A portfolio with cost, $z_0^{\vartheta} = 0$, generates a* ***nonpositive*** *income at time 1, Z_1^{ϑ}.*

- Non-positive Z_1^{ϑ} means that there is at least one state of nature, s such that $z_{1,s}^{\theta} < 0$.

  Intuition: with a zero cost although we can get a positive income in several states of nature there is **at least one state of nature** in which there is a negative income.

Example 2

- Consider the previous Example 2: $S = (1, 1)$ and $V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- We can build a portfolio $(\vartheta_1, \vartheta_2)^\top$ such that $z_0^\vartheta = -S\vartheta = 0$:

$$\vartheta_1 = -x, \vartheta_2 = x$$

- The return at time $t = 1$ is

$$Z_1^\vartheta = V\vartheta = \begin{pmatrix} -2x + x \\ -x + 2x \end{pmatrix} = \begin{pmatrix} -x \\ x \end{pmatrix}$$

then whatever the position, long ($x > 0$) or short ($x < 0$), in the risky asset we will not have positive income at $t = 1$ (in some states of the nature we win, but we will lose in others).

Replicating portfolios

Definition: **Replicating porfolio**: Assume we have a contingent income $W = (w_1, \ldots, w_N)^\top$. To the portfolio, ϑ, build using the assets in the market, generating the same (contingent) income

$$\vartheta \equiv \{\theta : \; Z_1 = V\theta = W\}$$

we call **replicating portfolio**. The **cost** of the replicating portfolio is

$$C(\vartheta, S) = S\vartheta$$

Arbitrage asset pricing theory (APT)

- Deals with the determination of asset prices **through replication**.
- APT vs GEAP (general equilibrium asset pricing):
  - in APT we take (S, V) as given (S is exogenous)
  - in GEAP we take V as given and want to determine S from the fundamentals (S is endogenous)
- Both theories have three equivalent formulations:
  - using state prices Q
  - using the stochastic discount factor M
  - using market or risk-neutral probabilities π^Q

Arbitrage asset pricing

Using state prices

Proposition 5

Assume there is a financial market (S, V) such that there are no arbitrage opportunities. Then, given a new asset with payoff V_k its price can be determined by redundancy by using the expression

$$S_k = QV_k$$

where $Q = SV^{-1} \gg 0$ *is determined from the market pair* (S, V).

That is

$$S_k = \sum_{s=1}^{N} q_s V_{ks}$$

Arbitrage asset pricing

Using state prices

From what we have previously seen there are two important results

Proposition 6

*If markets are **complete** then the value, S_k, is **unique**. If markets are **incomplete** then S_k is **not unique**.*

Proposition 7

*Assume a financial market (S, V) is complete and there are $K - N$ redundant assets. Then the **price of any asset is equal to the cost of their replicating portfolios build with N independent assets**.*

Arbitrage asset pricing

Using stochastic discount factors

- If Q is a state price vector and π is the vector or probabilities, we define the stochastic discount factor (SDF) for state s by

$$m_s \equiv \frac{q_s}{\pi_s}$$

Proposition 8

Assume there is a financial market (S, V) such that there are no arbitrage opportunities. Then, given a new asset with payoff V_k its price can be determined by redundancy by using the expression

$$S_k = \mathbb{E}[MV_k]$$

where $M = (m_1, \ldots, m_N)^\top$ is the stochastic discount factor.

Arbitrage asset pricing

Using stochastic discount factors

- Proof: as $S_k = \sum_{s=1}^{N} q_s V_{ks}$ and using the definition of SDF we get
$$S_k = \sum_{s=1}^{N} q_s V_{ks} = \sum_{s=1}^{N} \pi_s m_s V_{ks}$$
- **Intuition**: if there are no arbitrage opportunities the price of an asset is the expected value of the future payoffs discounted by the stochastic discount factor (which is the market discount for uncertain payoffs)

Arbitrage asset pricing

Using risk-neutral probabilities

- Therefore,

$$S_k = \sum_{s=1}^{N} q_s V_{ks} = \bar{q} \sum_{s=1}^{N} \pi_s^Q V_{ks} = \bar{q}\mathbb{E}^Q[V_k]$$

or compactly

$$\boxed{S_k = \bar{q}\mathbb{E}^Q[V_k]}$$

- **Intuition**: if there are no arbitrage opportunities there is an equivalent probability measure such that the price of an asset is proportional of the expected value of the future payoffs
- Although $\bar{q}$ is mysterious we can calculate it a intuitive way.

Arbitrage asset pricing

Existence of risk-free asset

Proposition 9

Assume there are no arbitrage opportunities and there is a risk-free bond with price $1/(1+i)$ and face value 1. Then the price of any asset verifies the relationship

$$S_j = \frac{1}{1+i}\mathbb{E}^Q[V_j]$$

Exercise: prove this.

Arbitrage asset pricing

Existence of risk-free asset

Proposition 10

Assume there are ***no arbitrage opportunities*** *and there is a risk-free asset. Then* ***there is a (market) probability measure*** *such that the expected returns of every asset is equal to the return of the risk-free asset.*

- Proof. From Proposition 8 we have

$$1 + i = \mathbb{E}^Q[R_j], \text{ for any, } j = 1, \ldots, K$$

As this holds for any asset, therefore π^Q has the property

$$1 + i = \mathbb{E}^Q[R_1] = \ldots = \mathbb{E}^Q[R_K]$$

Application

Arbitrage and completeness with two assets

- Assume that $N = 2$ and there is a risky and a risk-free asset

$$S = \left(\frac{1}{1+i}, p \right), \quad V = \begin{pmatrix} 1 & d_1 \\ 1 & d_2 \end{pmatrix}$$

- and that $r_1 < i < r_2$.

  Then markets are complete and there are no arbitrage opportunities.

  The return matrix is

$$R = \begin{pmatrix} \frac{1}{\frac{1}{1+i}} & \frac{d_1}{p} \\ \frac{1}{\frac{1}{1+i}} & \frac{d_2}{p} \end{pmatrix} = \begin{pmatrix} 1+i & 1+r_1 \\ 1+i & 1+r_2 \end{pmatrix}$$

Application

Arbitrage asset pricing

- If there are risk-free assets and absence of arbitrage opportunities, then any asset can be priced by redundancy as

$$S_k = q_1 V_{k,1} + q_2 V_{k,2}$$

where

$$q_1 = \frac{i - r_2}{(1+i)(r_1 - r_2)},\ q_2 = \frac{r_1 - i}{(1+i)(r_1 - r_2)}.$$

- Proof: assume there is a risk-free and a risky asset such that $QR = \mathbf{1}^\top$ becomes

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1+i & 1+i \\ 1+r_1 & 1+r_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and if there are no arbitrage opportunities $r_1 < i < r_2$ implying $q_1 > 0$ and $q_2 > 0$.

Application

Simple Black and Scholes (1973) equation

- Consider an european call option with exercise price p on an underlying asset with payoff

$$V^{\top}_{underlying} = (d_1, d_2)$$

- Then the contingent payoff of the option is

$$V^{\top}_{option} = (\max\{d_1 - p, 0\}, \max\{d_2 - p, 0\})$$

- Question: assuming there are no arbitrage opportunities and there is a risk-free asset what is the market price of the option ?
- Answer: the price for a call option with exercise price p is

$$S_{option} = \frac{(r_2 - i)\max\{d_1 - p, 0\} + (r_1 - i)\max\{d_2 - p, 0\}}{(1+i)(r_1 - r_2)}$$

(prove this)

Equity premium

- We call equity premium to the difference in the rates of return between the risky and a risk-free asset

$$r - i = \begin{pmatrix} r_1 - i \\ r_2 - i \end{pmatrix}$$

- The Sharpe index is defined as

$$\frac{\mathbb{E}[r-i]}{\sigma[r]} = \frac{\sum_{s=1}^{2} \pi_s (r_s - i)}{\sqrt{\sum_{s=1}^{2} \pi_s ((r_s - i) - \mathbb{E}[r-i])^2}}$$

where $\sigma[r-i] = \sqrt{\mathbb{V}[r-i]}$ is the standard deviation of the equity premium (prove that $\sigma[r-i] = \sigma[r]$).

Risk neutral probabilities

Proposition 11

If there are no arbitrage opportunities then there is a (market) risk neutral probability distribution such that the expected value of the equity risk premium is zero,

$$E^Q[r - i] = 0$$

This is the reason why π_s^Q are called risk neutral probabilities: $\mathbb{P}^Q = (\pi_1^Q, \ldots \pi_N^Q)$ is a probability measure such that the expected value of the equity premium is zero.

Risk neutral probabilities

- Proof: Let
$$R^\top = \begin{pmatrix} 1+i & 1+i \\ 1+r_1 & 1+r_2 \end{pmatrix}$$
- Then, because $R^\top Q^\top = \mathbf{1}$
$$\begin{cases} (1+i)q_1 + (1+i)q_2 = 1 \\ (1+r_1)q_1 + (1+r_2)q_2 = 1 \end{cases}$$
- Then $(1+r_1)q_1 + (1+r_2)q_2 = (1+i)q_1 + (1+i)q_2$.
- Therefore
$$q_1(r_1 - i) + q_2(r_2 - i) = 0$$

Risk neutral probabilities

- Proof (cont)Define $\bar{q} = q_1 + q_2$

$$\frac{q_1}{\bar{q}}(r_1 - i) + \frac{q_1}{\bar{q}}(r_2 - i) = 0$$

- Define again $\pi_s^Q = \frac{q_s}{\bar{q}}$. If there are no arbitrage opportunities, then $q_s > 0$ and $\pi_s^Q > 0$. Because $\sum_{s=1}^{2} \pi_s^Q = 1$ then π_s^Q are probabilities.
- At last

$$\pi_1^Q(r_1 - i) + \pi_2^Q(r_2 - i) = 0$$

The Sharpe index and the SDF

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Proposition 12

Assume there are no arbitrage opportunities and there is a risk-free asset. Then the Sharpe index verifies the relationship

$$\frac{\mathbb{E}[r-i]}{\sigma[r]} = -\rho_{r-i,m}\left(\frac{\mathbb{E}[M]}{\sigma[M]}\right)^{-1}$$

where $\rho_{r-i,m}$ is the coefficient of correlation between the equity premium and the stochastic discount factor.

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The Sharpe index and the SDF

- Proof: Using our previous definition of the stochastic discount factor $m_s = q_s/\pi_s$ then

$$\mathbb{E}[M(r-i)] = 0$$

- But $\mathbb{E}[M(r-i)] = Cov[M(r-i)] + \mathbb{E}[M]\,\mathbb{E}[r-i]$
- Using the definition of correlation

$$\rho_{r-i,m} = \frac{Cov[M(r-i)]}{\sigma[M]\,\sigma[r-i]} \in (-1,1)$$

- Then $\rho_{r-i,m}\,\sigma[M]\,\sigma[r-i] + \mathbb{E}[M]\,\mathbb{E}[r-i] = 0$

The equity premium and DGE models

- **Intuition**: the Sharpe index is equal symmetric to the product between coefficient of variation of the stochastic discount factor and correlation coefficient between the equity premium and the stochastic discount factor, $\rho_{r-i,m}$
- The coefficient of variation of the stochastic discount factor contains the aggregate market value of risk.
- $\mathbb{E}[M]/\sigma[M]$ can be derived from simple DSGE models (as we will see next)
- Observe that the Sharpe index satisfies

$$\frac{\mathbb{E}[R - R^f]}{\sigma[R]}$$

where $R^f = 1 + i$ is the risk-free return and R can be taken as a return on a market index. (see `http://web.stanford.edu/~wfsharpe/art/sr/SR.htm`