Foundations of Financial Economics
DSGE: two-period Arrow-Debreu economy

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Topics

- Two period Arrow-Debreu exchange economy

AD exchange economy: contracts

AD contract: is a **real forward contract** such that

- for a price associated to state $s = i$, $\tilde{q}_i$ paid at time $t = 0$
- there is delivery of a contingent good at time $t = 1$ at state $s = i$

$$x_{1,i} = \begin{cases} 1, & \text{if } s = i \\ 0, & \text{if } s \neq i \end{cases}$$

$$\begin{array}{ccc} -\tilde{q}_i & & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{c} s = 1 \\ \vdots \\ s = i-1 \\ s = i \\ s = i+1 \\ \vdots \\ s = N \end{array} \\ \vdash & \text{———————} & \dashv \\ 0 & & 1 \end{array}$$

This allows to extend the static GE theory to the present intertemporal and stochastic economy context

AD exchange economy: markets

Existing markets:

- 1 spot market operating at time $t = 0$, where the price p_0 is set
- N markets for AD contracts operating at time $t = 0$, where the price vector $\tilde{Q}$ clears the market.

We can characterize the AD markets by the payoff sequence $(\tilde{Q}, X_1)$ where

- prices are
$$\tilde{Q} = (\tilde{q}_1, \ldots, \tilde{q}_s, \ldots, \tilde{q}_N)$$
- and the deliveries are
$$X_1 = (x_{1,s})_{s=1}^N = \begin{pmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & 1 \end{pmatrix}$$

AD exchange economy: spot market

Transactions in the spot market:

the net demand: z_0.

then total expenditure: $p_0 z_0$

AD exchange economy: Arrow-Debreu markets

Transactions in every AD market:

- The number of contracts is

$$Z_1 = (z_{1,1}, \dots, z_{1,s}, \dots, z_{1,N})^\top$$

  where
  - if the agent is a buyer of the k-contract, then $z_{1,k} > 0$, and
    - **pays** $\tilde{q}_k z_k$ at time $t = 0$
    - **receives** z_k units of the good at time $t = 1$ if the state k occurs and 0 otherwise
  - if the agent is a seller of the l-contract, then $z_{1,l} < 0$, and
    - **receives** $\tilde{q}_l z_l$ at time $t = 0$ and
    - **delivers** z_l units of the good at time $t = 1$ if the state l occurs and 0 otherwise
- Then total net expenditure in all AD markets is

$$\tilde{Q}.Z_1 = \sum_{s=1}^{N} \tilde{q}_s z_{1,s}$$

- **The budget constraint is** $p_0 z_0 + \tilde{Q}.Z_1 = 0$

AD exchange economy: consumption financing

- The consumer has a sequence of endowments

$$Y = \{y_0, Y_1\}$$

- allowing financing the (random) sequence of consumption, $c = \{c_0, C_1\}$, out of his endowment, such that
  - in the period $t = 0$

$$c_0 = z_0 + y_0$$

  - at time $t = 1$, contingent on the information and contracts performed at time $t = 0$

$$C_1 = Z_1 + Y_1 = \begin{pmatrix} c_{1,1} \\ \dots \\ c_{1,s} \\ \dots \\ c_{1,N} \end{pmatrix} = \begin{pmatrix} z_1 \\ \dots \\ z_s \\ \dots \\ z_N \end{pmatrix} + \begin{pmatrix} y_{1,1} \\ \dots \\ y_{1,s} \\ \dots \\ y_{1,N} \end{pmatrix}$$

AD exchange economy: consumer's budget constraint

As

$$\begin{cases} c_0 - y_0 = z_0, & \text{at } t = 0 \\ c_{1,s} - y_{1,s} = z_{1,s}, & \text{at } t = 1, \text{ for every } s = 1, \ldots, N \end{cases}$$

i.e. for every period and for any state of nature total income is equal to total expenditure

then the budget constraint at time $t = 0$ is

$$p_0(c_0 - y_0) + \tilde{Q}.(C_1 - Y_1) = p_0(c_0 - y_0) + \sum_{s=1}^{N} \tilde{q}_s(c_{1,s} - y_{1,s}) = 0$$

AD exchange economy: stochastic discount factor

We define:

- the relative price of AD contracts also called the price of the state of nature
$$Q^\top = \Big(q_1, \dots, q_s, \dots, q_N\Big)$$
where
$$q_s \equiv \frac{\tilde{q}_s}{p_0},\ s = 1, \dots, N.$$

- the stochastic discount factor is
$$M^\top = \Big(m_1, \dots, m_s, \dots, m_N\Big)$$
where
$$m_s \equiv \frac{q_s}{\pi_s},\ s = 1, \dots, N.$$

AD exchange economy: consumer's problem

- Objective: choose a **contingent plan** $C = \{c_0, C_1\}$ that maximizes the **intertemporal utility** functional

$$U(c_0, C_1) = \mathbb{E}_0 \left[u(c_0) + \beta u(C_1) \right]$$

- subject to the **intertemporal (instantaneous) budget constraint**

$$\mathbb{E}_0 \left[c_0 + m C_1 \right] \leq \mathbb{E}_0 \left[y_0 + m Y_1 \right] = h_0$$

- given: stochastic discount factor and endowments (M, Y),

  where we define the **wealth of the consumer**

$$h_0 \equiv \mathbb{E}_0 \left[y_0 + m Y_1 \right]$$

AD exchange economy: general equilibrium

Definition: GE in an AD exchange economy: The general equilibrium for an AD economy is **defined** by the sequence of distribution of consumptions $(C^{i*})_{i=1}^{I}$ and by the stochastic discount factor M^*, such that $(C^{i*})_{i=1}^{I} = \left(\{c_0^{i*}, C_1^{i*}\}\right)_{i=1}^{I}$, for a given distribution of endowments

$$Y = \left(Y^1, \dots Y^I\right), \text{ for } Y^i = \{y_0^i, Y_1^i\}, \; i = 1, \dots, I$$

such that:

- every consumer $i \in \mathcal{I}$ determines the optimal sequence of consumption

$$C^{i*} = \arg\max \left\{\mathbb{E}_0^i \left[u^i(c_0^i) + \beta^i u^i(C_1^i)\right] s.t. \mathbb{E}_0^i \left[c_0^i + mC_1^i\right] \leq h_0^i\right\}$$

  given Y^i and M,

- and markets clear:

$$\sum_{i=1}^{I} c_0^i = \sum_{i=1}^{I} y_0^i, \; \dots \sum_{i=1}^{I} c_{1,s}^i = \sum_{i=1}^{I} y_{1,s}^i, \; s = 1, \dots, N$$

AD exchange and homogeneous economy: general equilibrium

Assume agents are homogeneous: same preferences, same information, same endowments

Definition: GE in an AD exchange homogeneous economy: The general equilibrium for an AD economy is **defined** by the sequence of consumption and prices $(\{c_0^*, C_1^*\}, M^*)$ such that:

- the representative consumer determines the optimal sequence

$$C^* = \arg\max \{\mathbb{E}_0 [u(C_0) + \beta u(C_1)] \; s.t. \mathbb{E}_0 [C_0 + mC_1] \leq h_0\}$$

  given $Y = \{Y_0, Y_1\}$ and M,

- markets clear

$$c_0^* = y_0, \ldots, C_1^* = Y_1$$

  or, equivalently

$$c_{t,s}^* = y_{t,s}, \; t = 0, 1, \; s = 1, \ldots, N$$

AD general equilibria: intuition

Allows for the determination of the equilibrium value for the stochastic discount factor M: market price for transactions across time and the states of nature:

- Homogeneous agent economy: dependent upon the preferences, information and the endowments of the economy
- Heterogeneous agent economy: dependent upon the preferences, information and the endowments of the economy **and their distribution among agents** (i.e, when there are differences in information, attitudes towards risk and wealth)

AD exchange and homogeneous economy

Determination of equilibrium prices

We determine the equilibrium in two steps:

1. first, determine the optimality conditions

$$u^{'}(c_0^*)m_s = \beta u^{'}(c_{1,s}^*),\ s = 1, \dots, N$$

   if we assume there is no satiation;

2. second, use the market equilibrium conditions

$$c_{t,s}^* = y_{t,s},\ t = 0, 1,\ s = 1, \dots, N$$

Then, the **equilibrium** stochastic discount factor is

$$\boxed{m_s^* = \beta \left(\frac{u^{'}(y_{1,s})}{u^{'}(y_0)} \right),\ s = 1, \dots, N}$$

or equivalently the **equilibrium AD price** is

$$q_s^* = \beta \pi_s \left(\frac{u^{'}(y_{1,s})}{u^{'}(y_0)} \right),\ s = 1, \dots, N$$

AD exchange and homogeneous economy

Equilibrium values

Then the general equilibrium when consumers are homogeneous and there is no satiation :

- consumption is similar to the case in an autarkic economy

$$\{C_t^*\}_{t=0}^1 = \{Y_t\}_{t=0}^1$$

  but, in this case, we say there is **aggregate uncertainty**;
- equilibrium stochastic discount factor (SDF)

$$M^* = \left(\beta\left(\frac{u'(y_{1,1})}{u'(y_0)}\right), \ldots, \beta\left(\frac{u'(y_{1,N})}{u'(y_0)}\right)\right)^\top$$

  is functions of the **fundamentals** (resources and preferences)
- equilibrium relative price for AD contracts

$$Q^* = \left(\beta\pi_1\left(\frac{u'(y_{1,1})}{u'(y_0)}\right), \ldots, \beta\pi_N\left(\frac{u'(y_{1,N})}{u'(y_0)}\right)\right)^\top$$

  is a function of the **fundamentals** (resources, preferences and information)

An example with log utility

SDF for state s

Assuming:

- logarithmic Bernoully utility function

$$u(c) = \ln(c)$$

- stochastic endowment's growth factor

$$y_{1,s} = (1 + \gamma_s) y_0, \; s = 1, \ldots, N$$

- How does uncertainty affects the stochastic discount factor and the utility of the consumer ?

An example with log utility

Distribution of the SDF

- the stochastic discount factor is $m^*_s = \frac{\beta}{1+\gamma_s}$

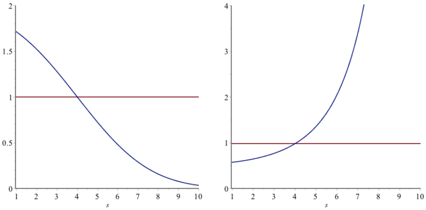

Figure: Growth factor $(1+\Gamma)$ and stochastic the associated discount factor M

- Conclusions:
  1. there is **aggregate uncertainty**
  2. stochastic discount factor is **negatively correlated** with rate of growth

An example with log utility

Sampling the SDF

- the stochastic discount factor is

$$m_s^* = \frac{\beta}{1+\gamma_s}$$

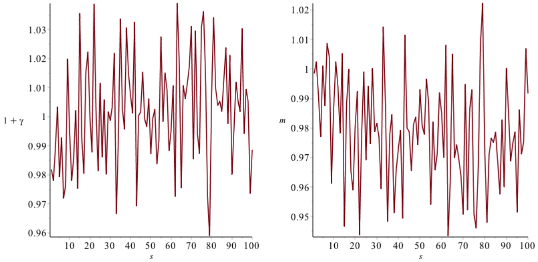

Figure: Sampling from $\gamma \sim N(0, 0.02)$ and the stochastic discount factor

An example with log utility

Aggregate uncertainty and lack of insurance

- The utility for the consumer is (prove it)

$$\begin{aligned} U(C^*) &= \ln(c_0^*) + \beta \mathbb{E}_0[\ln(C_1^*)] = \\ &= \ln(y_0) + \beta \mathbb{E}_0[\ln(Y_1)] = \\ &= \ln\left(y_0^{1+\beta}(G\mathbb{E}_0[1+\Gamma])^\beta\right) \end{aligned}$$

  increases with y_0 and with the geometric mean of the growth rate.

- Question: why this looks like the utility in a Robinson-Crusoe economy ?
- Question: what are the consequences of more volatility, to the stochastic discount factor and to consumer's utility ?

References

- (LeRoy and Werner, 2014, Part III), (Lengwiler, 2004, ch. 2), (Altug and Labadie, 2008, ch. 3)

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