Foundations of Financial Economics
Two period GE: limited participation

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Topics

- Types of agents: participation in the risky asset market
- Endogenous market participation (related to priors on the likelihood of the good and bad state: pessimists and optimists)
- The equilibrium interest rate depends on the participation is risky asset markets
- Interest rate response to news in an asymetric way
- **Welcome to financial economics post 2008 !**

Differences with the benchmark model

- The participation in the risky asset market is not proportional to household wealth across households: U.S data similar shape for different countries
- We introduce a **friction**: agents cannot have short positions in assets
- Wealth takes the form of financial wealth only
- There is positive net zero wealth (external finance: external money and a another risky asset in positive net supply)

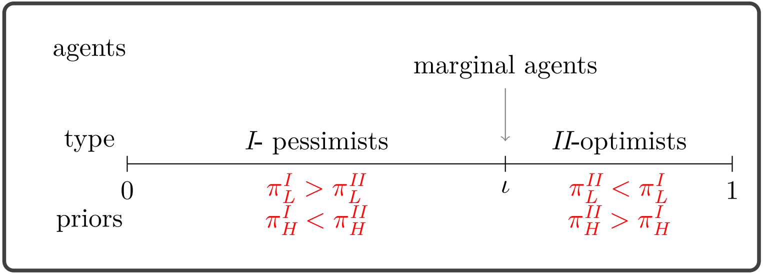
Possible extensions and applications

- The model is in the other extreme of the benchmark model we studied until this point
- A half-way model would consider that internal finance (short positions) is possible, but it is constrained by **collateral constraints**: short positions are limited by the existence of a long position in another asset that should be offered as collateral (for instance money)
- This partly explains
  - the demand for liquidity, for instance for firms,
  - the characteristics of the 2008 and the Euro crises (a crisis may be brewing without signs in the behavior of interest rates)
  - and the increasing consideration of the so-called balance effects in macro-finance models.

The model

Environment: fundamentals

- Information: two-period binomial tree with two states $s = L, H$
- **Heterogeneity in priors regarding the states of nature**.
  - there is a continuum of agents $i \in [0, 1]$
  - divided into two main groups: pessimists $i \in [0, \iota]$ giving more weight to $s = L$ and optimists $i \in [\iota, 1]$ giving more weight to $s = H$
  - There are marginal agents with $i = \iota$ (very small group).



- off course: $\pi^I_L + \pi^I_H = \pi^{II}_L + \pi^{II}_H = 1$

The model

Markets

- I assume a finance economy in which there are two assets: **money** and a **risky asset**
- The asset prices and payoffs are

$$\mathbf{S} = (1, S_2), \ \mathbf{V} = \begin{pmatrix} 1 & v_{2L} \\ 1 & v_{2H} \end{pmatrix}$$

  L is a bad state and H is a good state: $v_{2L} < v_{2H}$

- Therefore, the return matrix is

$$\mathbf{R} = (R_1, R_2) = \begin{pmatrix} 1 & R_{2L} \\ 1 & R_{2h} \end{pmatrix}$$

  where $R_{2,s} = v_{2s}/S_2$ for $s = L, H$

- Assume there are no arbitrage opportunities:

$$R_{2L} < 1 < R_{2H}$$

The model

The agent i problem

- The problem for household of type $i \in [0,1]$

$$\max_{c_0^i, C_1^i, \theta^i} u(c_0^i) + \beta \mathbb{E}^i[u(C_1^i)]$$

where $\mathbb{E}^i[u(C_1^i)] = \sum_{s \in \{H,L\}} \pi_s^i u(c_{1s}^i)$, subject to

$$\begin{aligned} c_0^i + \theta_1^i + S_2 \theta_2^i &= y_0^i + w^i \\ c_{1,s}^i &= \theta_1^i + v_{2s}\theta_2^i, \;\; s = L, H \\ \theta_1^i &\geq 0 \text{ (friction: no short position allowed)} \\ \theta_2^i &\geq 0 \text{ (friction: no short position allowed)} \end{aligned}$$

- Assume that the initial wealth composition may contain the two types of assets for any type of agent

$$w^i = w_1^i + S_2 w_2^i$$

with $w_1^i > 0$ and $w_2^i > 0$ for any i given.

The model

The agent i problem

- Simplifying assumptions: $c_0^i = y_0^i$ and $Y_1^i = \mathbf{0}$: consumption in period 1 is only financed by financial returns)
- Then the constraint in period zero simplifies to:

$$w^i = \theta_1^i + S_2\theta_2^i$$

- The problem for household of type $i \in [0,1]$ is simplified to

$$\max_{\theta_1^i, \theta_2^i} u(y_0^i) + \beta \mathbb{E}^i[u(\theta_1^i + V_2\theta_2^i)]$$

subject to

$$\begin{aligned} \theta_1^i + S_2\theta_2^i &= w^i \\ \theta_1^i &\geq 0 \\ \theta_2^i &\geq 0 \end{aligned}$$

Solving the generic agent's problem

- Lagrangean

$$\mathcal{L}^i = u(y_0^i) + \sum_{s\in\{H,L\}} \beta\pi_s^i u\big(\theta_1^i + v_{2s}\theta_2^i\big) + \lambda^i(w^i - \theta_1^i - S_2\theta_2^i) +$$
$$+ \mu_1^i\theta_1^i + \mu_2^i\theta_2^i$$

- Individual arbitrage conditions

$$\frac{\partial\mathcal{L}^i}{\partial\theta_1^i} = 0 \Leftrightarrow \lambda^i = \beta\left(\sum_{s\in\{H,L\}} \pi_s^i u^{'}(c_{1s}^i)\right) + \mu_1^i$$
$$\frac{\partial\mathcal{L}^i}{\partial\theta_2^i} = 0 \Leftrightarrow S_2\lambda^i = \beta\left(\sum_{s\in\{H,L\}} \pi_s^i u^{'}(c_{1s}^i) v_{2s}\right) + \mu_2^i$$

- Complementary slackness conditions

$$\mu_1^i\theta_1^i = 0,\ \mu_1^i \geq 0,\ \theta_1^i \geq 0$$
$$\mu_2^i\theta_2^i = 0,\ \mu_2^i \geq 0,\ \theta_2^i \geq 0$$

Behavior of agent of type I (pessimist):

- Agents of type I sell their initial stock of the risky asset and invests in money: $\theta_1^I > 0$ and $\theta_2^I = 0$
- Then

$$\theta_1^I = w^I = w_1^I + S_2 w_2^I$$
$$c_{1s}^I = w^I$$

is state-independent

- From complementary slackness: $\mu_1^I = 0$ and $\mu_2^I > 0$. Then

$$\lambda^I = \beta\left(\sum_{s\in\{H,L\}} \pi_s^I u^{'}(c_{1s}^I)\right) > \beta\left(\sum_{s\in\{H,L\}} \pi_s^I u^{'}(c_{1s}^I) R_{2s}\right)$$

- Then $\mathbb{E}^I[u^{'}(C_1^I)] > \mathbb{E}^I[u^{'}(C_1^I)R_2]$

Behavior of agent of type I (pessimist):

- This is equivalent to

$$\beta\, u^{'}(w^I) > \beta\, u^{'}(w^I) \sum_{s \in \{H,L\}} \pi_s^I R_{2s}$$

- Then pessimists have a prior, i.e., a risk- probability distribution $(\mathbb{P}^{I_u})$ such that

$$\boxed{R_1 = 1 > \mathbb{E}^{I_u}[R_2]}$$

agent I invests in the risk-free asset because **he finds** its anticipated return on money (i.e., 1) higher than that of the risky asset.

Behavior of agent of type II:

- Agents of type II sell their initial stock of money and invest in risky asset: $\theta_1^{II} = 0$ and $\theta_2^{II} > 0$
- Then

$$\theta_2^{II} = \frac{w^{II}}{S_2} = \frac{w_1^{II} + S_2 w_2^{II}}{S_2}$$

$$c_{1s}^{II} = \frac{v_{2s}}{S_2} w^{II} = \frac{w^{II}}{R_{2s}}$$

  is state-dependent (i.e., risky)

- From complementary slackness: $\mu_1^{II} > 0$ and $\mu_2^{II} = 0$. Then

$$\lambda_0^{II} = \beta \left(\sum_{s \in \{H,L\}} \pi_s^{II} u^{'}(c_{1s}^{II}) R_{2s} \right) > \beta \left(\sum_{s \in \{H,L\}} \pi_s^{II} u^{'}(c_{1s}^{II}) \right)$$

  Then $\mathbb{E}^{II}[u^{'}(C_1^{II}) R_1] = \mathbb{E}^{II}[u^{'}(C_1^{II})] < \mathbb{E}^{II}[u^{'}(C_1^{II}) R_2]$.

- Then optimists have a prior, i.e., an equivalent probability distribution $(\mathbb{P}^{II_u})$ such that

$$\boxed{\mathbb{E}^{II_u}[R_2] > 1}$$

Marginal agent

- Agents of type I prefer holding money to holding the risky asset because
$$\mathbb{E}^{I_u}[R_2] < 1$$

- Agents of type II prefer holding the risky asset rather than money because
$$\mathbb{E}^{II_u}[R_2] > 1$$

- By continuity, **there should exist a marginal agent (with wealth weight of zero)** having a probability distribution such that
$$\mathbb{E}^{\iota}[R_2] = 1 \Leftrightarrow \boxed{S_2 = \pi^{\iota} v_{2L} + (1 - \pi^{\iota}) v_{2H}} \tag{1}$$

Equilibrium in the asset markets

- Generic equilibrium conditions (total de mand = total supply)

$$\iota\theta_1^I + (1-\iota)\theta_1^{II} = w_1 = w_1^I + w_1^{II}$$
$$\iota\theta_2^I + (1-\iota)\theta_2^{II} = S_2 w_2 = S_2\left(w_2^I + w_2^{II}\right)$$

  where w_j is the aggregate stock of total of asset $j = 1, 2$ and ι is the proportion in the population (of size equal to 1) of agents of type I: **non-investors in the risky asset**

- Using the previous demand agent-level results

$$\iota\theta_1^I = \iota w^I = w_1$$
$$(1-\iota)\theta_2^{II} = (1-\iota)w^{II} = S_2 w_2$$

  (remember that $\theta_1^{II} = \theta_2^I = 0$ and $\theta_1^I = w^I$ and $\theta_2^{II} = w^{II}$)

- **The equilibrium values for S_2 and ι are jointly determined**: the asset price depends on the rate of participation.

Equilibrium in the asset markets

- Assumption: homogeneity in the distribution of wealth, that is $w^I = w^{II} = \bar{w}$
- Then the **equilibrium price for the risky asset** is

$$\boxed{S_2^* = S_2(\iota) = \left(\frac{1-\iota}{\iota}\right)\frac{w_1}{w_2}} \tag{2}$$

- The asset price decreases with ι because

$$\frac{\partial S_2}{\partial \iota} = -\frac{w_1}{\iota^2 w_2} < 0$$

- The asset price increases with the stock of money w_1

Equilibrium distribution of agents

- Assumption: the probability distribution of the marginal investor, π^ι, is a function of their weight in the total population ι. For simplicity let $\pi^\iota = \iota$.
- Then, from equations (1) and (2), the equilibrium value $\iota^* = \{\iota \in (0,1) : \mathcal{I}(\iota) = 0\}$ where

$$\mathcal{I}(\iota) \equiv (1-\iota) w_1 - (\iota v_{2L} + (1-\iota) v_{2H})\, \iota w_2$$

Proposition

There is one unique value $\iota^* \in (0,1)$,

$$\iota^* = \frac{v_{2H} w_2 + w_1}{2(v_{2H} - v_{2L}) w_2} - \left[\left(\frac{v_{2H} w_2 - w_1}{2(v_{2H} - v_{2L}) w_2} \right)^2 + \frac{4 v_{2L} w_1 w_2}{4 (v_{2H} - v_{2L})^2 w_2^2} \right]^{\frac{1}{2}}$$

Equilibrium distribution of agents

- **Proof that $\iota^* \in (0,1)$ exists and is unique.**
  Function $\mathcal{I}(\iota)$ is convex in ι (U-shaped) and therefore there can be zero, one or two values of ι such that $\mathcal{I}(\iota) = 0$ for $-\infty < \iota < \infty$. However, the domain of ι is $(0,1)$. It is easy to see that $\mathcal{I}(0) = w_1 > 0$, $\mathcal{I}^{'}(0) = -(w_1 + v_{2H}w_2) < 0$ and $\mathcal{I}(1) = -(w_1 + v_{2L}w_2) < 0$: therefore, in the interval $(0,1)$ there is one and only one value of ι, ι^* such that $\mathcal{I}(\iota) = 0$. Because the function is convex it has two points $0 < \iota_- < 1 < \iota_+$ such that $\mathcal{I}(\iota) = 0$ and the first one is the solution we are looking for.

Equilibrium distribution of agents

Properties

Proposition

The participation rate $(1-\iota^*)$ increases with the payoff (for any state of nature) and the aggregate stock of the risky asset and reduces with the aggregate stock of money.

- We showed that $\iota^* = \iota(v_{2H}, v_{2L}, w_1, w_2)$, and next we prove that

$$\boxed{\frac{\partial \iota^*}{\partial v_{2s}} < 0, \text{ for } s = H, L, \ \frac{\partial \iota^*}{\partial w_1} > 0, \ \frac{\partial \iota^*}{\partial w_2} < 0}$$

Equilibrium distribution of agents

Proof

- **Proof of the sign relationships for $\dfrac{\partial \iota^*}{\partial v_{2s}}$**

  We know that $\mathcal{I}(\iota, v_{2H}, v_{2L}) = 0$. Therefore, the response of ι to the payoffs is

  $$\frac{\partial \iota^*}{\partial v_{2s}} = -\left.\frac{\mathcal{I}_{v_{2s}}}{\mathcal{I}_{\iota}}\right|_{\iota=\iota^*}, \ s = L, H$$

- Where $\mathcal{I}_{v_{2H}} = -\iota^*(1-\iota^*)w_2 < 0$ and $\mathcal{I}_{v_{2L}} = -(\iota^*)^2 w_2 < 0$ and

  $$\mathcal{I}_{\iota} = 2(v_{2H} - v_{2L})w_2 \left(\iota^* - \frac{w_1 + v_{2H}w_2}{2(v_{2H} - v_{2L})w_2}\right) < 0$$

  because $0 < \iota^* < \dfrac{w_1 + v_{2H}w_2}{2(v_{2H} - v_{2L})w_2}$

Equilibrium rate of return for the risky asset

- Equilibrium rate of return of the risky asset is

$$\boxed{R^*_{2,s} = \frac{v_{2s}}{S_2(v_{2L}, v_{2H}, .)}, s = L, H} \tag{3}$$

- As

$$\frac{\partial S_2}{\partial \iota} < 0, \ \frac{\partial \iota}{\partial v_{2s}} < 0, \ s = H, L$$

  then

$$\frac{\partial S_2}{\partial v_{2s}} > 0 \text{ for any } s = H, L$$

- This means that **if there is an increase in** v_{2s} **generates two effects on** R_{2s}:
  - a direct positive effect (of the payoff in the "own" state)
  - a negative indirect effect, because the prices increases as a result of the change in the participation in the risky asset market
  - The **final effect is ambiguous**.

Equilibrium rate of return for the risky asset

- For the case in which there is **no change in participation** we have

$$\frac{d\bar{R}_{2s}}{dv_{2s}} = \frac{1}{\bar{S}_2} > 0, \frac{d\bar{R}_{2s'}}{dv_{2s}} = 0, \; s \neq s^{'} = H, L$$

- The rate of return outcome for a particular state of nature **only changes when the payoff outcome for the same state of nature changes**.

Equilibrium R distribution and news

Proposition

If there is a change in the participation, then a change in any of the anticipated outcomes in the payoff distribution will change the rate of return whatever the state of nature that occurs at time $t = 1$, but will do it in a state-dependent way

Equilibrium rate of return for the risky asset

- Proof: When there is a change in participation we have

$$\frac{\partial R_{2s}}{\partial v_{2s}} = \frac{1-\epsilon_{\iota}^{S_2}\epsilon_{v_{2s}}^{\iota}}{S_2(\iota^*)}, \ \frac{\partial R_{2s'}}{\partial v_{2s}} = -\frac{v_{2s'}}{v_{2s}}\frac{\epsilon_{\iota}^{S_2}\epsilon_{v_{2s}}^{\iota}}{S_2(\iota^*)}, \ s \neq s^{'} = L, H$$

where

- the elasticity of S_2 to ι is

$$\epsilon_{\iota}^{S_2} = \frac{\partial S_2}{\partial \iota}\frac{\iota}{S_2} - \frac{1}{1-\iota^*} < -1$$

- the elasticity of ι to v_{2s} is

$$\epsilon_{v_{2s}}^{\iota} = \frac{\partial \iota^*}{\partial v_{2s}}\frac{v_{2s}}{\iota}, \ s = H, L$$

- The rate of return outcome for a particular state of nature changes with **changes in the payoff of any state of nature** due to the change in participation.

Equilibrium R distribution and news

- Proof (cont): For a change in v_{2H} we have a change in the distribution of R_2
  - if the good state occurs
$$\frac{\partial \bar{R}_{2H}}{\partial v_{2H}} = -\frac{1}{S_2(\iota^*)}\left(\frac{2(v_{2H}-v_{2L})w_2\iota^*(1-\iota_+)}{\mathcal{I}_\iota}\right) > 0$$
  - if the bad state state occurs
$$\frac{\partial \bar{R}_{2L}}{\partial v_{2H}} = -\frac{1}{S_2(\iota^*)}\frac{v_{2L}}{v_{2H}}\epsilon_\iota^{S_2}\epsilon_{v_{2H}}^\iota < 0$$
- For a change in v_{2L} we have a change in the distribution of R_2
  - if the good state occurs
$$\frac{\partial \bar{R}_{2L}}{\partial v_{2L}} = -\frac{w_2}{\mathcal{I}_\iota} > 0$$
  - if the bad state state occurs
$$\frac{\partial \bar{R}_{2H}}{\partial v_{2L}} = -\frac{1}{S_2(\iota^*)}\frac{v_{2H}}{v_{2L}}\epsilon_\iota^{S_2}\epsilon_{v_{2L}}^\iota < 0$$

Equilibrium R distribution and news

- A positive news regarding the good state v_{2H}, $\Delta v_{2H} > 0$, generates an increase in the rate of return if the good state occurs and a decrease in the rate of return if the bad state occurs:

$$\Delta v_{2H} > 0 \ \Rightarrow \Delta R_{2L} < 0 < \Delta R_{2H}$$

  This is because

$$v_{2H} \uparrow \rightarrow \iota \downarrow \rightarrow S_2 \uparrow \rightarrow \begin{cases} R_{2L} = v_{2L}/S_2 & \downarrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

- a negative news regarding the bad state, v.g., $\Delta v_{2L} < 0$, there is an increase in the rate of return if the good state occurs and a reduction if the bad state occurs

$$\Delta v_{2L} < 0 \ \Rightarrow \Delta R_{2L} < 0 < \Delta R_{2H}$$

  this is because

$$v_{2L} \downarrow \rightarrow \iota \uparrow \rightarrow S_2 \downarrow \rightarrow \begin{cases} R_{2L} = v_{2L}/S_2 & \downarrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

Conclusions

- We showed that when priors differ, and there are participation frictions in the asset market asymmetric expected changes in payoffs have an effect on the whole distribution of the rate of return for risky assets
- Good news regarding the good state or bad news regarding the bad state lead to a kind of an **amplification** response of the rate of return: a higher realized rate of return if the good state realizes and a lower rate of return if the bad state realizes.
- Other results: an expansion in the money supply $M = w_1$ will increase the rate of return for all states of nature

$$M \uparrow \to \iota \uparrow \to S_2 \downarrow \to \begin{cases} R_{2L} = v_{2L}/S_2 & \uparrow \\ R_{2H} = v_{2H}/S_2 & \uparrow \end{cases}$$

References

This lecture is adapted from Geanakoplos (2010) and Fostel and Geanakoplos (2014).

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