Advanced Mathematical Economics

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Lecture 2
13.2.2020
(revised 3.3.2020)

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Part II

Linear ordinary differential equations

Chapter 1

Linear ODE: the scalar case

1.1 Introduction

Linear ordinary differential equations are the simplest ordinary differential equations (ODE). They always have one and only one **explicit (or closed form) solution**. This means that the existence and uniqueness of solutions always hold.

There are several reasons to start by studying them. First, characterization of their solution is simple. By this we mean studying their stability, invariant sets, dependence of the solution on parameters. The difference between solving an ODE and a problem involving an ODE can also be clearly stated.

Second, given that existence and uniqueness of solutions for linear ODE is known, the analysis of the solutions for non-linear ODE can be done by comparing them to the linear ODE. In particular, the qualitative theory for ODE is based upon the local approximation of non-linear ODE by linear ODE and by verifying conditions under which a non-linear ODE is (topologically) equivalent to a linear ODE (at least locally).

In this chapter we study **scalar linear** ODE, that is one-dimensional dynamics, which mean that the our unknown function $y(\cdot)$ maps a one-dimensional known variable over the set of real numbers, and the instanteneous variation is a linear function of y.

In particular, a scalar ODE, in the normal form, features a (unknown) function $y(x)$ and an independent variable to be $x \in \mathrm{X} \subseteq \mathbb{R}$, which is a mapping $y : \mathrm{X} \to \mathrm{Y} \subseteq \mathbb{R}$ which is solution of the equation

$$y^{'}(x) \equiv \frac{dy(x)}{dx} = F(y(x), x, \varphi),$$

where $\varphi \in \mathbb{R}^m$ is a vector of parameters.

When the independent variable is tim, we write $t \in \mathrm{T} \subseteq \mathbb{R}_+$ and the (unknown) function is

$y : \mathrm{T} \to \mathrm{Y} \subseteq \mathbb{R}$ the solution of the equation

$$\dot{y} \equiv \frac{dy(t)}{dt} = F(y(t), t, \varphi),$$

where $\varphi \in \mathbb{R}^m$ is a vector of parameters.

In this chapter we assume that F is a **linear** function of y, that is $F(\cdot)$ can take the form $F(y) = a(x)y(x) + b(x)$, where $(a, b) : \mathrm{X} \to \mathbb{R}^2$.

There are two major distinctions among scalar linear ODE, depending on the form of $F(.)$. The first, is related to the functional dependence of F on the independent variable x (or t) and the second refers to homogeneity property of function $F(\cdot)$ regarding y.

We say that the ODE is **autonomous** if $F(\cdot)$ is independent of the independent variable x (or t) and **non-autonomous** if $F(\cdot)$ depends directly on the independent of it. We say that the ODE is **homogeneous** if F is an homogeneous function of y and **non-homogeneous** otherwise.

As in most economic applications the independent variable is time, and the characterization of its solution regarding the time evolution is a major goal of the analysis we address these equation in section 1.2. In section 1.3 we briefly present some results for ODEs as functions of other independent variables.

1.2 Equations having time as the independent variable

A scalar ODE is **autonomous** if the coefficients are constant, i.e, they are independent of the exogenous variable t,

$$\dot{y} = \lambda y + \beta \tag{1.1}$$

where $(\lambda, \beta) \in \mathbb{R}^2$ are known constants. A scalar ODE is **non-autonomous** if the coefficients depend on the the independent variable [1]

$$\dot{y} = \lambda(t)y + \beta(t), \tag{1.2}$$

where $(\lambda, \beta) : \mathrm{T} \to \mathbb{R}^2$ are known functions of t.

A scalar ODE is **homogeneous** if $F(y, .)$ is a homogeneous function of y and it is **non-homogeneous** if $F(y, .)$ is non-homogeneous. Homogeneity of degree n holds if, for a constant ξ we have $F(\xi y) = \xi^n F(y)$. Therefore

$$\dot{y} = \lambda y, \text{ and } \dot{y} = \lambda(t)y$$

are homogeneous and

$$\dot{y} = \lambda y + \beta, \text{ and } \dot{y} = \lambda(t)y + \beta(t)$$

[1] If we redefine the independent variable as $t = \tau$ we can transform the non-autonomous scalar linear ODE into a planar non-linear equation: $\dot{y_1} = \lambda(y_2)y_1 + \beta(y_2)$ $\dot{y}_2 = 1$ where $y_2(t) = t$.

are **non-homogeneous**.

Solving an autonomous ODE means finding a function $y(t) = \phi(t, y; .)$, mapping $\phi : \mathrm{T} \to \mathrm{Y}$, depending on the parameters λ and β and on an arbitrary element $y \in \mathrm{Y}$. **Characterizing** the solution roughly means tracking the behavior of the flow of the elements of Y, denoted as $(y(t))_{t\in\mathrm{T}}$, when the independent variable changes.

This behavior is determined by function $F(y)$. In order to take account of it we introduce the following definitions:

- **equilibrium point** (or steady state): it is a point in range of y such that $F(y) = 0$, that is

$$\bar{y} = \{y \in \mathrm{Y} : F(y) = 0\}$$

- **stability properties** of the equilibrium point: the steady state is asymptotically stable if for any $y \in \mathrm{Y}$ the flow generated by the ODE (1.3), $(y(t))_{t\in\mathrm{T}}$, has the property

$$\lim_{t\to\infty} \phi(t) = \phi(\infty) = \bar{y};$$

  the steady state is **unstable** if for any y in a neighborhood of $\bar{y}$, $y(t)$ does not converge to $\bar{y}$;

- **invariant subsets** are partitions of set Y containing the whole solution path $(y(t))_{t\in\mathrm{T}}$; we call **attractor set** to the subset of points $y \in \mathrm{Y}$ such that the solution will converge to the steady state and **repelling set** to the set of points $y \in \mathrm{Y}$ such that the solution will not converge to the steady state.

At last, we draw a distinction between an ODE and a **problem** involving an ODE. To get an intuition of the difference, observe that equations (1.1) and (1.2) involve a time variation of $y(t)$ independent of the starting or the terminal point. While the solution of the scalar ODE is called **general solution**, the solution to a problem involving an ODE is called **particular solution**.

There are basically two types of problems. We have an **initial-value problem** if we have an ODE together with a known value for y at the initial time, and we have a **terminal-value problem** if we have an ODE together with a known value for y for the terminal time (or there is a functional constraint over the solution). In the first case, we call the ODE a **forward ODE** because the solution will be obtained from future instants (assuming that the present time is $t = 0$) and in the second case we call the ODE a **backward ODE**.

The evolution described by the ODE can be done forward in time (if we know the initial point) or backward in time (if we know a terminal point). With this additional information we can uniquely determine a forward or a backward path.

As we will see, uniqueness of the solution of an ODE is not the same as uniqueness of a problem involving an ODE. And this distinction has important conceptual differences in economic applications.

1.2.1 Autonomous equations

Next we introduce homogeneous and the non-homogeneous autonomous equations and their related problems.

We assume that $\mathrm{Y} = \mathbb{R}$. This means that, with the exception of the ODE and the initial or terminal constraints, there is no other constraint on the evolution of y.

The **scalar homogeneous ODE** is the linear equation

$$\dot{y} = \lambda y, \; y : \mathrm{T} \to \mathrm{Y} \tag{1.3}$$

where $y = y(t)$ is an unknown function with domain T and range Y. Usually $\mathrm{T} = [0, \infty)$ and Y is the set of real numbers ($\mathbb{R}$) or non-negative real numbers ($\mathbb{R}_+$), and λ is a real-valued parameter ($\lambda \in \mathbb{R}$).

As with any equation (or problem) there are three issues related to its solution: existence of solutions, number of solutions and characterisation of the solution

Proposition 1. *The **unique solution of equation** (1.3) is a function $\phi : \mathrm{T} \to \mathrm{Y}$*

$$\phi(t, k; \lambda) = ke^{\lambda t} \tag{1.4}$$

where $k \in \mathrm{Y}$, is an arbitrary member of the range, and $t \in \mathrm{T}$.

Proof. We use the separation of variables approach [2]. It consists in four steps: first, as $\dot{y} \equiv dy/dt$ we can write equation (1.3) in an equivalent way, by separating y from t

$$\frac{dy}{y} = \lambda dt.$$

Second, we integrate both sides of the equation

$$\int \frac{dy}{y} = \int \lambda dt$$

because λ is a constant

$$\int \frac{dy}{y} = \lambda \int dt$$

[2] There are several methods we can employ to find the proof (separation of variables, Laplace transforms, Fourier transforms, transforming into an integral equation, using the concept of generating function, just to name a few)

third, we find the primitives

$$\ln(y) + C_y = \lambda t + C_t$$

where $C_y \in \mathrm{Y}$ and $C_t \in \mathrm{T}$ are two arbitrary constants of integration; at last, if we take exponentials of the two sides we get

$$e^{\ln(y)} = y = e^{\lambda t + (C_t - C_y)}$$

and we write $k = e^{C_t - C_y}$ □

We see that the solution $y(t) = \phi(t, k; \lambda)$ depends on the parameter λ and on an arbitrary point k in the domain Y. This is the **general solution** to equation (1.3).

Characterizing the solution means describing the behavior of the path $(y(t))_{t \in \mathrm{T}}$ from any point k. We readily see that: (1) if $\lambda < 0$ the solution converges to 0 if $t \to \infty$ independently from the value of k; (2) if $\lambda = 0$ the solution becomes $y(t) = k$ for any $t \in \mathrm{T}$, i.e, a constant; and (3) if $\lambda > 0$ the solution depends on the value k; it converges to zero if $k = 0$ and if converges to $+\infty$ if $k > 0$ and to $-\infty$ if $k < 0$.

The **steady states** for equation (1.3) are

$$\bar{y} = \begin{cases} k, & \text{if } \lambda = 0 \\ 0, & \text{if } \lambda \neq 0. \end{cases}$$

In the first case there is an **infinite number** of equilibria, consisting in all points in Y, and in the second there is a **single** equilibria if $0 \in \mathrm{Y}$, or no equilibria if $0 \notin \mathrm{Y}$.

When there is a steady state, that is, when $\lambda \neq 0$ we can characterize its **stability properties**:

- if $\lambda < 0$ then $\lim_{t\to\infty} \phi(t, k, \lambda) = 0 = \bar{y}$ then the equilibrium point is asymptotically stable;
- if $\lambda > 0$ then

$$\lim_{t\to\infty} \phi(t, k, \lambda) = \begin{cases} \pm\infty & \text{, if } k \neq 0 \\ 0 & \text{, if } k = 0 \end{cases}$$

  and the equilibrium point $\bar{y}$ is unstable. In this case we say the solution can be non-stationary.

Therefore, if $\lambda \neq 0$, and $\bar{y} \in \mathrm{Y}$, there are only two kinds of possible **invariant sets**:

- if $\lambda < 0$ the basin of attraction for $\bar{y}$ is the whole set Y and Y **is the attraction set**. Then we say $\bar{y}$ is **globally asymptotically stable**;
- if $\lambda > 0$ then $\bar{y}$ is repelling and unstable and $\mathrm{Y}/\bar{y}$ is the unstable invariant set.

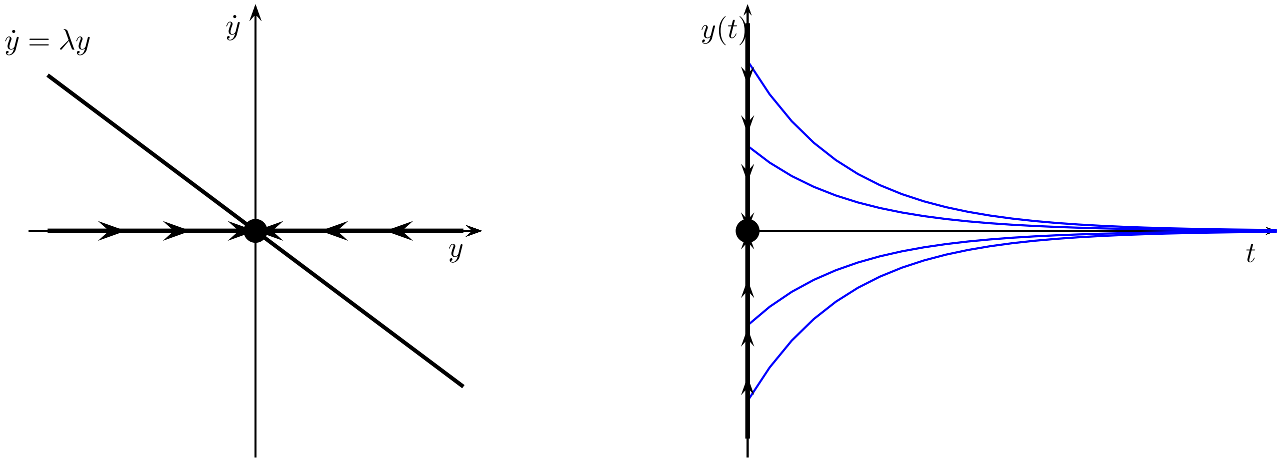

Figure 1.1: Phase diagram and trajectories of equation $\dot{y} = \lambda y$ for $\lambda < 0$

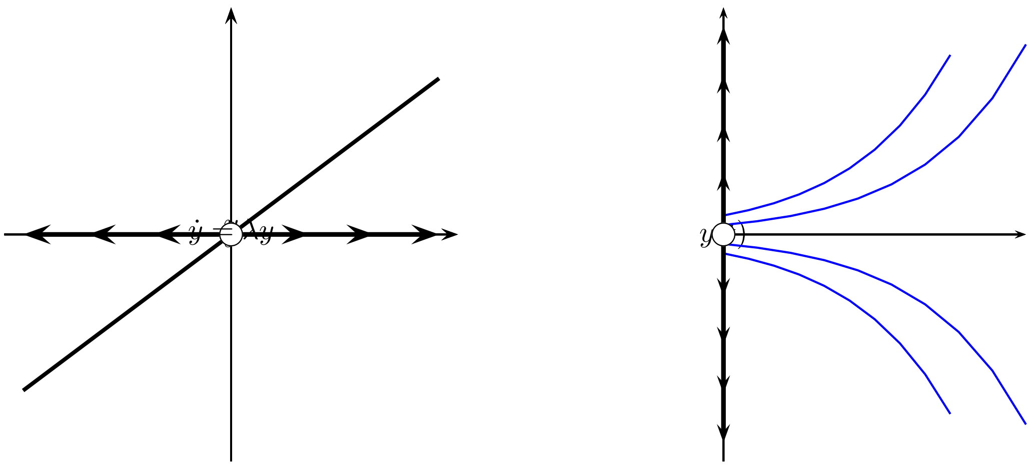

Figure 1.2: Phase diagram and trajectories of equation $\dot{y} = \lambda y$ for $\lambda > 0$

Figures 1.1 and 1.2 contain the **phase diagram** (left hand panel) and representative **orbits** (right-hand panel) for the asymptotically stable and unstable cases of the homogeneous equation, respectively.

The case $\lambda = 0$, where $\dot{y} = 0$, with solution $\phi(t) = k$ a constant is thus a degenerate case in which the solution is **always** time-invariant, i.e., it is independent from the exogenous variable t. Intuitively we can say that there are no dynamics, or that this corresponds to a boundary case between stability and instability. We can also get time-invariant solutions when $\lambda \neq 0$ but only in one case: when $k = \bar{y}$, i.e., if the arbitrary value of Y happens to be the steady state.

The **scalar linear non-homogenous** ODE is

$$\dot{y} = \beta + \lambda y, \ y \in \mathrm{Y} \subseteq \mathbb{R} \tag{1.5}$$

where β and λ are real-valued parameters.

Proposition 2. *The* ***unique solution of equation*** (1.5) *is a function* $\phi : \mathrm{T} \to \mathrm{Y}$

$$\phi(t, k; \beta, \lambda) = \begin{cases} \bar{y} + (k - \bar{y})e^{\lambda t}, & \text{if } \lambda \neq 0 \\ k + \beta t, & \text{if } \lambda = 0 \end{cases} \tag{1.6}$$

where

$$\bar{y} = -\frac{\beta}{\lambda}$$

where $k \in \mathrm{Y}$*, is an arbitrary member of the range, and* $t \in \mathrm{T}$*.*

Proof. First assume that $\lambda \neq 0$. Introduce a change in variables $z(t) = y(t) + \beta/\lambda$. Then $\dot{z} = \lambda z$, because

$$\dot{z} = \dot{y} = \beta + \lambda y = \beta + \lambda \left(z - \frac{\beta}{\lambda} \right) = \lambda z.$$

But we already know that the solution of $\dot{z} = \lambda z$ is $z(t) = k_z e^{\lambda t}$ where k_z belongs to the domain of z. Mapping back to the domain of y we have

$$y(t) + \frac{\beta}{\lambda} = \left(k + \frac{\beta}{\lambda} \right) e^{\lambda t}.$$

As $\bar{y} = -\beta/\lambda$ is the unique equilibrium point of equation (1.5) yields equation (1.6).

Next, let $\lambda = 0$, then from the l'Hôpital's rule.

$$
\begin{aligned}
\phi(t;\lambda=0) = ke^{\lambda t} - \frac{\beta\left(1-e^{\lambda t}\right)}{\lambda}\,|_{\lambda=0} &= \\
= k - \lim_{\lambda\to 0}\frac{\dfrac{d\left(\beta\left(1-e^{\lambda t}\right)\right)}{d\lambda}}{\dfrac{d(\lambda)}{d\lambda}} &= \\
= k - \frac{-\beta t}{1} &= \\
= k + \beta t.
\end{aligned}
$$

□

The solutions are qualitatively similar to the homogeneous case (i.e, when $\beta = 0$) when $\lambda \neq 0$. The only (quantitative) difference are:

- the steady state is also unique, although it is shifted from $\bar{y} = 0$, if $\beta = 0$, to $\bar{y} = -\beta/\lambda$, if $\beta \neq 0$;
- the stability behavior is qualitatively the same but now relative to the equilibrium point $\bar{y} = -\beta/\lambda$: it is asymptotically stable if $\lambda < 0$ and it is unstable if $\lambda > 0$.

The solutions are qualitatively different when $\lambda = 0$. While in the homogenous case (i.e., if $\beta = 0$) the solution is stationary and there is an infinite number of steady states (all the elements of Y) in the non-homogeneous case (i.e, if $\beta \neq 0$) **there are no steady states** and the solution of the ODE is always non-stationary.

Figures 1.3 and 1.4 illustrate the phase diagram (left-hand panel) and orbits (right-hand panel) for the asymptotically stable and unstable cases, respectively, for the non-homogeneous equation.

1.2.2 Problems involving scalar ODE

The previous solutions are usually called **general or fundamental solutions**. In most applications we know (or we set) the value, say $y_\tau \in \mathrm{Y}$, at a particular point in time, say $t = \tau$. So a **problem** involving an ODE takes the form

$$\dot{y} = \lambda y,\ y(\tau) = y_\tau \in \mathrm{Y}. \tag{1.7}$$

Proposition 3. *The unique solution for problem* (1.7)*, usually called* ***particular solution****, is*

$$\phi(t, y_\tau; \lambda) = y_\tau e^{\lambda(t-\tau)}, \textit{for}\ t, \tau \in \mathrm{T},\ \textit{and}\ y_\tau \in \mathrm{Y} \tag{1.8}$$

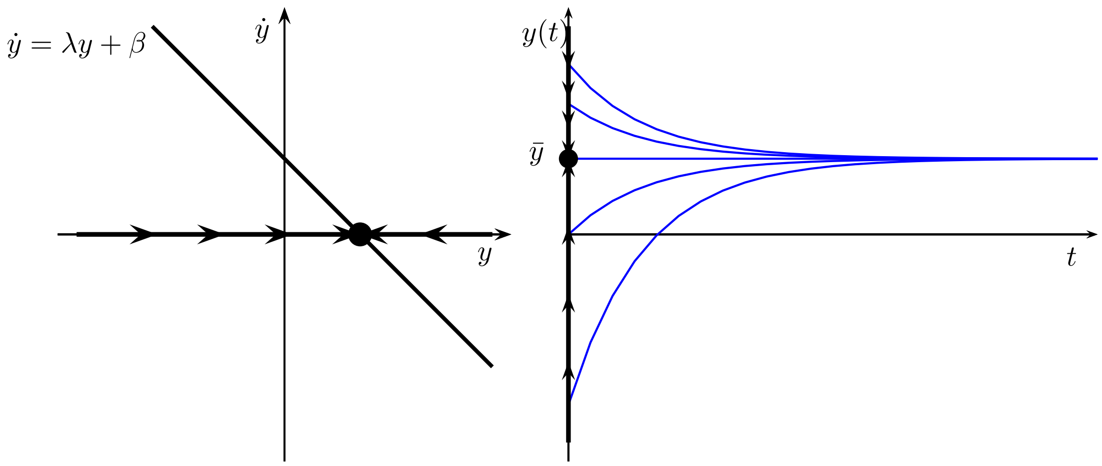

Figure 1.3: Phase diagram and trajectories for $\dot{y} = \lambda y + \beta$ for $\lambda < 0$ and $\beta > 0$

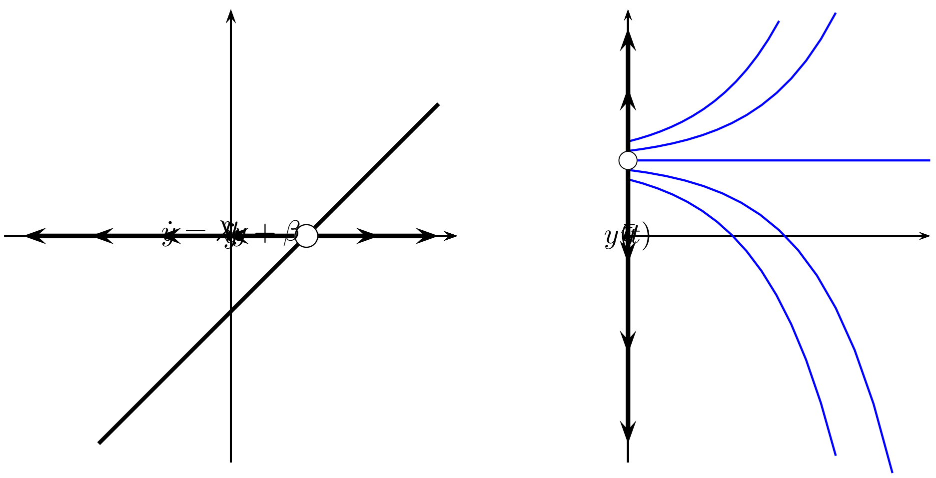

Figure 1.4: Phase diagram and trajectories for $\dot{y} = \lambda y + \beta$ for $\lambda > 0$ and $\beta < 0$

Proof. By using the separation of variables assuming that $t > \tau$ (the method is the same if we consider $t < \tau$) we get

$$\int_{y_\tau}^{y(t)} \frac{dy}{y} = \lambda \int_{\tau}^{t} dt$$

for any $t \geq 0$. Finding the definite integrals we get

$$\ln(y(t)) - \ln(y_\tau) = \ln(y(t)/y_\tau) = \lambda(t - \tau).$$

Taking exponentials to both sides we get again (1.8). □

Proof. (Alternative) Take the general solution for the homogenous equation, (1.4), and evaluate it at point $t = \tau$, to get $\phi(\tau, .) = ke^{\lambda\tau}$. But, to solve the problem we should have $\phi(\tau, .) = y_\tau$, a number. Solving for k we get $k = y_\tau e^{-\lambda\tau}$ and substituting in the general solution, we get function (1.8). □

Exercise 1. Find the solution of the problem $\dot{y} = \beta + \lambda y$, where $y(\tau) = y_\tau \in \mathrm{Y}$.

We have an **initial-value problem** when we have information on the value of the variable y at time $t = 0$, $y(0) = y_0$, and $\mathrm{T} = [0, T]$ for $T > 0$. In this case the solution exists and is unique and is given by

$$\phi(t, y_0; \lambda) = y_0\, e^{\lambda t}, \text{for } t \in [0, T].$$

We call **terminal value problem** (sometimes called boundary value problem) if we know (or want to set) he solution for a terminal time T as $y(T) = y_T$. In this case the solution exists and is unique

$$\phi(t, y_T; \lambda, T) = y_T\, e^{-\lambda(T-t)}, \text{for } t \in [0, T].$$

1.2.3 Non-autonomous equations

The scalar non-autonomous homogeneous equation

$$\dot{y} = a(t)y \tag{1.9}$$

where $a(t)$ is any function defined over T.

Proposition 4. *Equation* (1.16) *has the unique solution*

$$y(t) = ke^{\int a(t)dt} \tag{1.10}$$

where $k \in \mathrm{Y}$.

Proof. Separating variables and integrating we get

$$\int \frac{dy}{y} = \int a(t)dt,$$

which is equivalent to

$$\ln y(t) + C_y = \int a(t)dt$$

where C_y is a constant of integration, thus leading to equation (1.10) if we set $k = e^{C_y}$. □

The **initial value problem**

$$\begin{cases} \dot{y} = a(t)y, \\ y(t_0) = y_0 \end{cases} \tag{1.11}$$

for $\mathrm{T} = [t_0, t_1]$, has the unique solution

$$y(t) = y_0 e^{\int_{t_0}^{t} a(s)ds}, \text{ for } t \in [t_0, t_1]. \tag{1.12}$$

Exercise: prove this.

The initial value problem

$$\begin{cases} \dot{y} = a(t)y + b(t) \\ y(t_0) = y_0 \end{cases} \tag{1.13}$$

for $\mathrm{T} = [t_0, t_1]$.

Proposition 5. *The initial value problem* (1.11) *has the unique (particular) solution*

$$y(t) = y_0 e^{\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} b(s) e^{\int_{s}^{t} a(z)dz} ds, \textit{ for } t \in [t_0, t_1] \tag{1.14}$$

Proof. We apply the variation of constant method[3]. First, we consider the solution for the homogeneous equation, such that $b(t) = 0$ for all $t \in \mathrm{T}$. Its solution is, using equation (1.12)

$$y_h(t, y_C) = y_C e^{\int_{t_0}^{t} a(s)ds}.$$

We expect the solution to problem (1.13) to be

$$y(t) = y_h(t, y_C(t)) = y_C(t) e^{\int_{t_0}^{t} a(s)ds}. \tag{1.15}$$

[3] Due to Lagrange (1811).

Taking time derivatives of the last equation we get

$$\dot{y} = \dot{y}_C e^{\int_{t_0}^{t} a(s)ds} + y_C(t)a(t)e^{\int_{t_0}^{t} a(s)ds}$$

which should be equal to equation (1.13). Equating the right-hand sides of both equations we get the ODE

$$\dot{y}_C = b(t)e^{-\int_{t_0}^{t} a(s)ds}.$$

Applying the separation of variables to solve this equation we find

$$y_C(t) = y_C(t_0) + \int_{t_0}^{t} b(s)e^{-\int_{t_0}^{s} a(z)dz} ds.$$

Substituting in equation (1.15) and because $y_C(t_0) = y(t_0) = y_0$ we finally get solution (1.14) $\square$

In economics the following models are of interest:

Example 1 Consider the initial value problem

$$\dot{y} = ay + b(t), \text{ for } t \in [0, \infty)$$

where $a \neq 0$ and

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^* \\ b_1 & \text{if } t^* \leq t < \infty \end{cases}$$

and $y(0) = y_0$ is given.

The solution is

$$y(t) = \begin{cases} y_0 e^{at} + \frac{b_0}{a}\left(e^{at} - 1\right) & \text{if } 0 \leq t < t^* \\ y_0 e^{at} + \frac{b_0}{a} e^{at} + \left(\frac{b_1 - b_0}{a}\right) e^{a(t-t^*)} - \frac{b_1}{a} & \text{if } t^* \leq t < \infty \end{cases}$$

Observe that the solution, at any point in time, is capitalizing on the past changes of the variable $b(t)$.

Example 2 Consider the terminal value problem

$$\dot{y} = ay + b(t), \text{ for } t \in [0, \infty)$$

where $a > 0$ and

$$b(t) = \begin{cases} b_0 & \text{if } 0 \leq t < t^* \\ b_1 & \text{if } t^* \leq t < \infty \end{cases}$$

and $\lim_{t\to\infty} y(t)e^{-at} = 0$.

The solution for the problem is

$$y(t) = \begin{cases} -\frac{b_0}{a} - \left(\frac{b_1 - b_0}{a}\right) e^{a(t-t^*)} & \text{if } 0 \leq t < t^* \\ -\frac{b_1}{a} & \text{if } t^* \leq t < \infty \end{cases}$$

Comparing to the initial-value problem we see that the solution has an anticipating feature: for $0 < t < t^*$ the solution depends on the value of the variable $b(t)$ **after** its change, b_1, and after the change, for $t \geq t^*$, it is not influenced by the value before the change, b_0.

1.2.4 Economic applications

In economics, the following problem is common

$$\dot{y} = \lambda y, \ \lim_{t\to\infty} y(t)e^{-\mu t} = 0 \tag{1.16}$$

where $\mathrm{T} = [0, \infty)$, $0 \in \mathrm{Y}$ and $\mu > 0$.

To get the solution observe that

$$\lim_{t\to\infty} y(t)e^{-\mu t} = \lim_{t\to\infty} ke^{(\lambda-\mu)t}$$

then we have the following cases: (1) if $\lambda \geq \mu > 0$ then there is only **one** solution if $k = 0$; if $\lambda < \mu$ then there is an **infinity** of solutions, that is the terminal condition is met for any $k \in \mathrm{Y}$.

In economic models we use the following classification of variables and economic equilibrium

- **pre-determined** and **non-pre-determined** variables: the first are observed and the second are anticipated, that is, we have information for $t = 0$ for the first type of variables and we have asymptotic information on the second type of variables;
- **stationary** or **non-stationary** variables if they converge to a constant or are unbounded asymptotically (i.e., when $t \to \infty$);
- **determinacy** or **indeterminacy** if an equilibrium or a state of the economy modelled by a differential equation is unique or not

The relationship between them depends on the existence or not of a steady state and on their stability properties, for states within set Y.

For instance

- if a variable is pre-determined the trajectory described by the solution is always determinate, however, it can be stationary (if $\lambda < 0$) or non-stationary (if $\lambda > 0$). The first case is common in models with adaptative expectations, v.g. $\dot{p} = \lambda(\bar{p} - p)$, for $\lambda > 0$ and p is the log of price. The second case is common in endogenous growth models in which the GDP dynamics is given by $\dot{y} = Ay$, where y is GDP per capita;
- if a variable is non-predetermined the trajectory can be determinate if k is determined uniquely and is indeterminate if k can be any value within set Y. For scalar models the solutions are usually stationary if the terminal condition is of the type $\lim_{t\to\infty} y(t)e^{-\mu t} = 0$ for $\mu > 0$.

Table 1.1 summarizes this concepts, used in dynamic general equilibrium models (DGE).

Table 1.1: Classification of equilibrium paths in DGE models

y	$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
pre-determined	determined and stable	determined and stationary	determined and non-stationary
non- pre-determined	indeterminate	ambiguous	determined

1.3 ODEs depending on other independent variable

Some statistical distributions can be seen as solutions to scalar non-autonomous ODE subject to functional constraints

Pareto

Standard Gaussian

$$\left\{ y'(x) = -x\, y(x),\ x \in \mathbb{R} \int_{-\infty}^{\infty} y(x)dx = 1 \right.$$

Standard log-normal $z(x) = \ln y(x)$

$$\left\{ z'(x) = -(1 + z(x))/x,\ x \in \mathbb{R}_+ \int_0^{\infty} y(x)dx = 1 \right.$$

1.4 References

Mathematics: there is a huge literature on scalar linear ODE, but (Hale and Koçak, 1991, ch 1) is a great modern.

Applications to economics: Gandolfo (1997).

Chapter 2

Linear ODE: planar case

2.1 Introduction

In this chapter we deal with planar linear equations, that is with systems of two coupled ODEs. We will only deal with the autonomous case.

We consider both the homogeneous and the non-homogeneous cases. There are two basic differences as regards the scalar case: first, we can have asymptotic stability and (global) instability as in the scalar case but we can also have saddle point dynamics (or conditional stability); second we can have monotonic dynamics or non-monotonic dynamics.

The saddle-point case is a very common type of dynamics in both macroeconomics and growth theory and charaterizes solutions of most optimal control problems.

The **autonomous linear planar ordinary differential equation** is defined as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B},\ \mathbf{y} : \mathcal{T} \subseteq \mathbb{R}_+ \to Y \subseteq \mathbb{R}^2 \tag{2.1}$$

where matrix $\mathbf{A}$ and vector $\mathbf{B}$ have real elements, that is $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\mathbf{B} \in \mathbb{R}^{2\times 1}$, and can be written as

$$\mathbf{A} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ \mathbf{B} \equiv \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Expanding the matrix operations, we obtain the equivalent system

$$\begin{aligned} \dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + b_1 \\ \dot{y}_2 &= a_{21}y_1 + a_{22}y_2 + b_2. \end{aligned}$$

Equation (2.1) is non-homogeneous (check).

The **homogeneous** linear planar equation is

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y},\ \mathbf{y} : \mathcal{T} \subseteq \mathbb{R}_+ \to Y \subseteq \mathbb{R}^2 \tag{2.2}$$

Next we prove that equation (2.2) has an **unique general solution** $\mathbf{y}(t) = \phi(t, \mathbf{h}; \mathbf{A})$ where

$$\phi(t, \mathbf{h}; \mathbf{A}) \equiv \mathbf{e}^{\mathbf{A}\mathbf{t}}\mathbf{h}$$

where $\mathbf{h} = (h_1, h_2) \in Y$, is an arbitrary element of the range of $\mathbf{y}$, and $\mathbf{e}^{\mathbf{A}\mathbf{t}} : \mathcal{T} \to \mathbb{R}^{2\times 2}$ is a matrix exponential function.

For **characterizing** the solutions we can follow (jointly) two types of approaches:

- an analytical approach studying the existence and uniqueness of the steady states and other particular types of solutions (v.g., periodic solutions) and study their stability properties;
- a geometrical approach by building the **phase diagram**.

In the next section 2.2 we present and characterize the solution to the homogeneous equation (2.2), and the problems associated to it, and in section 2.3 we do the same for the non-homogeneous equation (2.1).

2.2 The homogeneous equation

2.2.1 The matrix exponential function

We saw that the (general) solution to the scalar homogeneous equation $\dot{y} = \lambda y$ was $y(t) = ke^{\lambda t}$ where k is an arbitrary element of $\mathcal{Y} \subseteq \mathbb{R}$.

Recall that the exponential function has the series representation

$$e^{\lambda t} \equiv \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = 1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \frac{1}{6}(\lambda t)^3 + \dots$$

For the planar problem we can also define a **matrix exponential** function

$$\mathbf{e}^{\mathbf{A}\mathbf{t}} \equiv \sum_{n=0}^{+\infty} \frac{1}{n!}\mathbf{A}^n t^n = I + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2 t^2 + \dots \tag{2.3}$$

which is a mapping $\mathbf{e}^{\mathbf{A}\mathbf{t}} : \mathcal{T} \to \mathbb{R}^{2\times 2}$ with the following properties:

Lemma 1. Properties of matrix exponentials $\mathbf{e}^{\mathbf{A}\mathbf{t}}$.

(i) $\mathbf{e}^{\mathbf{A}(\mathbf{t}+\mathbf{s})} = \mathbf{e}^{\mathbf{A}\mathbf{t}}\mathbf{e}^{\mathbf{A}\mathbf{s}}$

(ii) $\left(\mathbf{e}^{\mathbf{A}\mathbf{t}}\right)^{-1} = \mathbf{e}^{-\mathbf{A}\mathbf{t}}$

(iii) $\frac{d}{dt}\mathbf{e}^{\mathbf{A}\mathbf{t}} = \mathbf{A}\mathbf{e}^{\mathbf{A}\mathbf{t}} = \mathbf{e}^{\mathbf{A}\mathbf{t}}\mathbf{A}$

(iv) If matrices $\mathbf{A}$ *and* $\mathbf{B}$ *commute, (i.e., if* $\mathbf{A}\,\mathbf{B} = \mathbf{B}\,\mathbf{A}$*) then* $\mathbf{e^{(A+B)t}} = \mathbf{e^{At}e^{Bt}}$

(v) Let $\mathbf{P}$ *be a non-singular and square matrix. Then* $\mathbf{e^{P^{-1}APt}} = \mathbf{P}^{-1}\mathbf{e^{At}P}$.

In the appendix 2.A.1 we gather some useful results from matrix algebra. In particular we prove that any matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \mathbf{P\Lambda P}^{-1}$$

where $\mathbf{P}$ is the (always non-singular) eigenvector matrix and $\mathbf{\Lambda}$ is the Jordan canonical form which is similar to $\mathbf{A}$ (i.e., has the same eigenvalues). Recalling that the **eigenvalues** of $\mathbf{A}$ are

$$\lambda_1 = \frac{\text{trace}(\mathbf{A})}{2} + \sqrt{\Delta(\mathbf{A})},\ \lambda_2 = \frac{\text{trace}(\mathbf{A})}{2} - \sqrt{\Delta(\mathbf{A})}$$

where the discriminant is $\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A})$, there are three Jordan canonical forms

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},\ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},\ \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{2.4}$$

In the appendix we present the fundamental result that $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ if $\Delta(\mathbf{A}) > 0$, $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ if $\Delta(\mathbf{A}) = 0$ and $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ if $\Delta(\mathbf{A}) < 0$.

From Lemma 1 *(v)* as $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{\Lambda}$ then $\mathbf{e^{\Lambda t}} = \mathbf{e^{P^{-1}APt}} = \mathbf{P}^{-1}\mathbf{e^{At}P}$ or, equivalently

$$\mathbf{e^{At}} = \mathbf{Pe^{\Lambda t}} \mp^{-1}.$$

Therefore, for any matrix $\mathbf{A}$ the time dependency of $\mathbf{e^{At}}$ is determined from the matrix exponential of the similar Jordan canonical form $\mathbf{e^{\Lambda t}}$. This is an important result which means that the types of solutions, and the associated phase diagrams, can be enumerated.

Now, we need to determine the exponential matrices for the Jordan canonical forms:

Lemma 2. Matrix exponentials for the Jordan canonical forms, $\mathbf{e^{\Lambda t}}$

Let $\mathbf{\Lambda}$ *be a matrix in a arbitrary Jordan canonical form as in equation* (2.4) *and let* λ_1, λ_2, λ, α *and* β. *be real numbers. Then,*

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ *then*

$$\mathbf{e^{\Lambda t}} = \mathbf{e^{\Lambda_1 t}} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}. \tag{2.5}$$

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ *then*

$$\mathbf{e^{\Lambda t}} = \mathbf{e^{\Lambda_2 t}} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{2.6}$$

- *If* $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ *then*

$$\mathbf{e^{\Lambda t}} = \mathbf{e^{\Lambda_3 t}} = e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}. \tag{2.7}$$

Proof. Consider the definition of matrix exponential, equation (2.3) and the Jordan canonical form matrices in equation (2.4). In the first case, we have

$$\mathbf{e^{\Lambda_1 t}} = \mathbf{I}_2 + \mathbf{\Lambda}_1 t + \frac{1}{2}(\mathbf{\Lambda}_1)^2 t^2 + \ldots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda_1^2 t^2 & 0 \\ 0 & \lambda_2^2 t^2 \end{pmatrix} + \ldots$$

then, performing the matrix additions,

$$\mathbf{e^{\Lambda_1 t}} = \begin{pmatrix} 1 + \lambda_1 t + \frac{1}{2}\lambda_1^2 t^2 + \ldots & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2}\lambda_2^2 t^2 + \ldots \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

because $e^y = \sum_{n=0}^{+\infty} \frac{1}{n!} y^n$. That result is straightforward because the Jordan matrix is diagonal. This is not the case for Jordan matrix $\mathbf{\Lambda}_2$, though. But if we decompose $\mathbf{\Lambda}_2$ as

$$\mathbf{\Lambda}_2 = \mathbf{\Lambda}_{2,1} + \mathbf{\Lambda}_{2,2} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and because the two matrices commute, i.e. $\mathbf{\Lambda}_{2,1}\mathbf{\Lambda}_{2,2} = \mathbf{\Lambda}_{2,2}\mathbf{\Lambda}_{2,1}$, then applying property (iv) of Lemma 1 we obtain

$$\mathbf{e^{\Lambda_2 t}} = \mathbf{e^{(\Lambda_{2,1}+\Lambda_{2,2})t}} = \mathbf{e^{\Lambda_{2,1} t}}\mathbf{e^{\Lambda_{2,2} t}}$$

where

$$\mathbf{e^{\Lambda_{2,1} t}} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} = e^{\lambda t}\mathbf{I}_2.$$

Using again formula (2.3) for matrix $\mathbf{\Lambda}_{2,2}$ we get

$$\mathbf{e^{\Lambda_{2,2} t}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \ldots = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

that multiplying to matrix $\mathbf{e^{\Lambda_{2,1} t}}$ yields (2.6).

In the third case, $\mathbf{\Lambda}_3$ is again non-diagonal, but it can also be decomposed into the sum of two matrices that commute

$$\mathbf{\Lambda}_3 = \mathbf{\Lambda}_{3,1} + \mathbf{\Lambda}_{3,2} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Applying again property (iv) of Lemma 1 we get

$$\mathbf{e^{\Lambda_3 t}} = \mathbf{e^{\Lambda_{3,1} t}}\,\mathbf{e^{\Lambda_{3,2} t}},$$

where

$$\mathbf{e^{\Lambda_{3,1}t}} = e^{\alpha t}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Using again formula (2.3) for matrix $\mathbf{\Lambda}_{3,2}$ we get

$$\mathbf{e^{\Lambda_{3,2}t}} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & \beta t\\ -\beta t & 0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}\beta^2 t^2 & 0\\ 0 & -\beta^2 t^2\end{pmatrix} + \ldots = \begin{pmatrix}\cos\beta t & \sin\beta t\\ -\sin\beta t & \cos\beta t\end{pmatrix},$$

because $\sin y = \sum_{n=0}^{+\infty}\frac{y^{2n+1}}{(2n+1)!}$ and $\cos y = \sum_{n=0}^{+\infty}\frac{y^{2n}}{(2n)!}$, we get (2.7). □

Now we have the general solution for a homogeneous ODE:

Proposition 1. *Consider the planar homogeneous ODE* (2.2)*. The general solution for this equation is*

$$\mathbf{y}(t) = \mathbf{e^{At}h} = \mathbf{Pe^{\Lambda t}h} \tag{2.8}$$

where $\mathbf{e^{\Lambda t}}$ *is the matrix exponential for the Jordan canonical form which is similar to* $\mathbf{A}$*,* $\mathbf{P}$ *is the eigenvector matrix and* $\mathbf{h} = \mathbf{P}^{-1}\mathbf{k}$*, where* $\mathbf{k}$ *is an arbitrary element of* Y*.*

Proof. Let the solution to equation (2.2) be the vectorial function $\phi(t) = \mathbf{e^{Bt}k}$ for an arbitrary 2×2 matrix $\mathbf{B}$ and for an arbitrary $\mathbf{k}\in Y$. Introducing this function into equation (2.2), and using property (*iii*) from Lemma 1 we get

$$\frac{d\phi(t)}{dt} = \mathbf{Be^{Bt}k} = \mathbf{B}\,\phi(t)$$

Then $\mathbf{y}(t) = \phi(t)$ if and only if $\mathbf{B} = \mathbf{A}$ and the solution to equation (2.2) is $\mathbf{e^{At}k}$. But $\mathbf{A} = \mathbf{P\Lambda P}^{-1}$, where $\mathbf{\Lambda}$ is the Jordan form of $\mathbf{A}$, Then $\mathbf{e^{At}} = \mathbf{e^{P\Lambda P^{-1}t}} = \mathbf{Pe^{\Lambda t}P^{-1}k}$ by property (v) of Lemma 1. □

The solution of an arbitrary homogeneous ODE, with coefficient matrix $\mathbf{A}$, can be written as a linear transformation of the solution of a planar homogeneous ODE with a coefficient vector in the Jordan canonical form. To be the case, the linear operator should be the eigenvector associated to $\mathbf{A}$ and the coefficient of the second ODE should be the Jordan canonical form which is similar to matrix $\mathbf{A}$:

Corollary 1. *Consider the coefficient matrix* $\mathbf{A}$ *and let* $\mathbf{P}$ *and* $\mathbf{\Lambda}$ *be its associated eigenvector matrix and Jordan canonical form. Then, the solution of the ODE* (2.2) *is*

$$\mathbf{y}(t) = \mathbf{Pw}(t)$$

where $\mathbf{w}(t)$ *is the solution of the ODE* $\dot{\mathbf{w}} = \Lambda\mathbf{w}$*,*

$$\mathbf{w}(t) = \mathbf{e^{\Lambda t}}\,\mathbf{k} \tag{2.9}$$

for $\mathbf{k} = \mathbf{Ph} \in \mathcal{Y}$ *.*

Proof. Let $\mathbf{w}(t) = \mathbf{P}^{-1}\mathbf{y}(t)$ then $\dot{\mathbf{w}} = \mathbf{P}^{-1}\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{y} = \mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} = \mathbf{\Lambda}\mathbf{w}$ which has the general solution (2.9). □

The relationship between the arbitrary elements of Y in equations (2.8) and (2.9) is $\mathbf{h} = \mathbf{P}^{-1}\mathbf{k}$.

2.2.2 Characterizing solutions when A is a Jordan normal form

Characterizing the possible dynamics for coefficient matrix in one of the canonical forms is necessary if matrix $\mathbf{A}$ is in a Jordan canonical form and it is useful if the coefficient matrix is not in a Jordan canonical form. This is because an implication of Corollary 1 is that the time dependency of the solution results from the dynamics of the associated canonical form. Furthermore, this means that the number of cases to analyse can be explicitly enumerated.

The solution of equation

$$\dot{\mathbf{w}} = \mathbf{\Lambda}\mathbf{w} \tag{2.10}$$

is $\mathbf{w}(t) = \mathbf{e}^{\mathbf{\Lambda t}}\mathbf{k}$ and can take one and only one of the following forms:

1. if $\Delta(\mathbf{A}) > 0$ then $\Lambda = \mathbf{\Lambda}_1$ and
$$\mathbf{w}(t) = \begin{pmatrix} k_1 e^{\lambda_1 t} \\ k_2 e^{\lambda_2 t} \end{pmatrix} \tag{2.11}$$
2. if $\Delta(\mathbf{A}) = 0$ then $\Lambda = \mathbf{\Lambda}_2$ and
$$\mathbf{w}(t) = e^{\lambda t}\begin{pmatrix} k_1 + k_2 t \\ k_2 \end{pmatrix} \tag{2.12}$$
3. if $\Delta(\mathbf{A}) < 0$ then $\Lambda = \mathbf{\Lambda}_3$ and
$$\mathbf{w}(t) = e^{\alpha t}\begin{pmatrix} k_1 \cos\beta t + k_2 \sin\beta t \\ k_2 \cos\beta t - k_1 \sin\beta t \end{pmatrix} \tag{2.13}$$

The previous solutions are functions of parameters, and of an arbitrary element of the state space Y. We will see that their specific values determine the time behavior of the solution.

We can enumerate the types of solutions along several criteria. We will focus on two criteria: first, the time dependency of the solution, and, second, the asymptotic behavior of the solution, i.e, the path of $\mathbf{w}(t)$ when t tends to infinity.

Time dependency of solutions

From the first perspective we can have the following type of solutions: stationary, monotonic, oscillatory, periodic solutions and hump-shaped.

Stationary solutions We say the solution is stationary if $\mathbf{w}(t)$ is a constant for all $t \in T$. In this case $\dot{\mathbf{w}}(t) = \mathbf{0}$ for all t.

Monotonic solutions We say the solution is monotonic if $\text{sign}(\dot{\mathbf{w}}(t))$ is the same for all $t \in T$. This means that the solution is monotonically increasing if $\dot{\mathbf{w}}(t) > \mathbf{0}$ for all t, it is monotonically decreasing if $\dot{\mathbf{w}}(t) < \mathbf{0}$ for all t. A stationary solution can be seen as a particular type of monotonic solution.

Oscillatory solutions A solution is oscillatory if $\mathbf{w}(t) = \mathbf{w}(t+p(t))$ for $t \in T$ and time-dependent period $p(t) \in T$: the solution is repeated in increasing intervals if $p^{'}(t) > 0$ or in decreasing intervals if $p^{'}(t) < 0$. For these solutions, there is a sequence of points, increasing or decreasing in time $\tau \in \{t_0, t_1, \ldots, t_s, \ldots\}$ such that $\dot{\mathbf{w}}(\tau) = 0$. In our case if there are two complex eigenvalues with non-zero real part, that is $\alpha \neq 0$, then the solution is oscillatory

$$\mathbf{w}(t) = e^{\alpha t} \begin{pmatrix} k_1 \cos \beta t + k_2 \sin \beta t \\ k_2 \cos \beta t - k_1 \sin \beta t \end{pmatrix}.$$

Periodic solutions If a solution satisfies $\mathbf{w}(t) = \mathbf{w}(t+p)$ for $t \in T$ and $p \in \mathcal{T}$ it is a periodic solution period p. This is a particular case of an oscillatory solution in which the period is constant. In our case if there are two complex eigenvalues with zero real part then the solution is periodic

$$\mathbf{w}(t) = \begin{pmatrix} k_1 \cos \beta t + k_2 \sin \beta t \\ k_2 \cos \beta t - k_1 \sin \beta t \end{pmatrix}.$$

This case occurs if and only if $\text{trace}(\mathbf{A}) = 2\alpha = 0$. Observe that in this case and if we transform the system into polar coordinates (see appendix 2.A.2 we have $r(t) = r_0$ constant and $\theta(t) = \theta_0 - \beta t$.

Hump-shaped solutions If the solution of a planar equation is such that only one variable satisfies $\dot{w}_i(t) = 0$ for a finite $t \in T$ and the other variable w_{-i} is monotonic, then we say the solution is hump-shaped. This case only occurs for the general homogeneous equation when there are eigenvalues with real parts.

Steady states and stability analysis

The second perspective on equations deals with their convergence as regards steady states.

Steady states

Definition 1. ***Steady states*** *A steady state is a fixed point to equation 2.2 such that* $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} = \mathbf{0}$.

If $\mathbf{A} = \mathbf{\Lambda}$ is in the Jordan form we define the set of steady states

$$\overline{\mathbf{w}} = \{\mathbf{w} \in Y : \mathbf{\Lambda}\mathbf{w} = 0\}.$$

An important distinction should be made: while a stationary solution is a function of t such that $\mathbf{w}(t)$ is constant, for all $t \in T$, a steady state is a fixed point of the vector field generated by the differential equation. However, for a planar ODE the solution of the differential equation is stationary if and only if it is a steady state. A stationary solution can only exist for particular values of $\mathbf{k} \in Y$. We will see that both the number of steady states and the convergence to or divergence from a steady state are determined by the parameters (in vector $\mathbf{\Lambda}$).

Let $\mathbf{0} \in Y$. Then steady states always exist but need not be unique. We have again three main cases: First if $\mathbf{\Lambda} = \mathbf{\Lambda}_1$ the two eigenvalues are real and distinct and we have four possible cases. i.e.,

$$\overline{\mathbf{w}} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ k_2 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\}$$

where $\mathbf{k} = (k_1, k_2)^\top$ is an arbitrary element of Y. The **steady state is an unique point** $\bar{\mathbf{w}} = \mathbf{0} \in Y$ If the two eigenvalues are non-zero, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. The steady state is a **one-dimensional manifold** in Y If $\lambda_1 = 0$, $\lambda_2 \neq 0$ and $k_1 \neq 0$ or if $\lambda_1 \neq 0$, $\lambda_2 = 0$ and $k_2 \neq 0$. Every point in Y is a steady state if $\lambda_1 = \lambda_2 = 0$. If the steady state is unique we call it a **node** and when it is not unique we call it a **degenerate node**.

Second, if $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ then the two eigenvalues are real and equal and we have two possible cases

$$\overline{\mathbf{w}} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k_1 \\ 0 \end{pmatrix} \right\}$$

The steady state is unique if $\overline{\mathbf{w}} = \mathbf{0} \in Y$ and the eigenvalue if different from zero, and the steady state is one dimensional manifold in Y if the eigenvalue is equal to zero and $k_1 \neq 0$. If the steady state is unique we call it a **node with multiplicity** and if it is not unique it is a **degenerate node with multiplicity**.

Third, if $\mathbf{\Lambda} = \mathbf{\Lambda}_3$ then the steady state is unique

$$\overline{\mathbf{w}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In this case, the steady state is called a **focus**.

Stability properties A solution $\mathbf{w}(t)$ is **asymptotically stable** if $\lim_{t\to\infty}\mathbf{w}(t) = \overline{\mathbf{w}} = \mathbf{0}$ for any $\mathbf{k} \neq \mathbf{0}$, i.e., the solution converges to the steady state.

A solution is **unstable** if for any $\mathbf{k} \neq \overline{\mathbf{w}} = \mathbf{0}$ then $\lim_{t\to\infty}\mathbf{w}(t) = \pm\infty$, i.e., the solution diverges. A solution is **semi-stable** (or conditionally stable) if there is a a subset of values $\mathcal{E}^s \in Y$ such that if $\mathbf{k} \in \mathcal{E}^s$ then $\lim_{t\to\infty}\mathbf{w}(t) = \overline{\mathbf{w}} = \mathbf{0}$ but if $\mathbf{k} \notin \mathcal{E}^s$ then $\lim_{t\to\infty}\mathbf{w}(t) = \pm\infty$, i.e, the solution is asymptotically stable for some values $\mathbf{k}$ but is unstable for others.

The eigenvalues of Λ not only determine the number of steady states but also their stability properties:

Proposition 2. *The asymptotic dynamic characteristics of the solution of equation* (2.10) *is determined by the real part of the eigenvalues,* $Re(\lambda_i)$, $i = 1, 2$ *of matrix* $\mathbf{\Lambda}$*:*

1. *if all the eigenvalues have negative real parts then all solutions of ODE* (2.10) *are asymptotically stable;*
2. *if all eigenvalues have positive real parts then all solutions are unstable;*
3. *if there is one negative and one positive eigenvalue* ($\lambda_1 > 0 > \lambda_2$) *then the solution to ODE* $\dot{\mathbf{w}} = \mathbf{\Lambda}\mathbf{w}$ *is semi-stable: it is unstable if* $k_1 \neq 0$ *and it is asymptotically stable if* $k_1 = 0$*;*
4. *if there is one zero eigenvalue the fixed point is a one-dimensional manifold (a center manifold), the solution will converge to it if the other eigenvalue is negative (i.e., in case* $\lambda_1 = 0$ *and* $\lambda_2 < 0$*) and will not converge to it if the other eigenvalue is positive (i.e., in case* $\lambda_1 > 0$ *and* $\lambda_2 = 0$*). In the first case there is a **degenerate stable node** and in the second case a **degenerate unstable node***

Proof. (1) If we consider the solutions (2.11)-(2.13) such that the real parts of the eigenvalues are negative (i.e, $\lambda_1 < 0$ and $\lambda_2 < 0$, or $\lambda < 0$ or $\alpha < 0$) then we see that the solutions tend to the fixed point $\overline{\mathbf{w}} = 0$ for any k_1 and k_2. (2) If there is an eigenvector with a positive real part (i.e, $\lambda_1 > 0$ or $\lambda_2 > 0$, or $\lambda > 0$ or $\alpha > 0$) then given any point $\mathbf{k} \neq \overline{\mathbf{w}}$ then the solution will be unbounded. All the other cases can be characterized in an analogous way. □

Eigenspaces The solutions of equation $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ is a weighted average of two elementary functions weighted by $\mathbf{h} = (h_1, h_2)$. For example, if $\Lambda = \mathbf{\Lambda}_1$ we could write its solution (2.8) as

$$\mathbf{y}(t) = h_1\mathbf{P}^1 e^{\lambda_1 t} + h_2\mathbf{P}^2 e^{\lambda_2 t}. \tag{2.14}$$

That is, the solution of the ODE is a superposition of two elementary function $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, acting on the directions given by $\mathbf{P}^1$ and $\mathbf{P}^2$, respctively, and weighted by the arbitrary constants h_1 and h_2. In other words, the elementary components of the time behavior of the solutions, $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, are linearly transformed by the eigenvectors $\mathbf{P}^1$ and $\mathbf{P}^2$.

We define the **eigenspaces** as the subsets of space Y which are followed by those two elementary solutions:

$$\begin{aligned}\mathcal{E}^1 &= \{\mathbf{w} \in Y : \text{ spanned by } \mathbf{P}^1\}\\ \mathcal{E}^2 &= \{\mathbf{w} \in Y : \text{ spanned by } \mathbf{P}^2\}\end{aligned}$$

Clearly the range of $\mathbf{y}$ satisfies $Y = \mathcal{E}^1 \oplus \mathcal{E}^2$.

In the case of equation $\dot{\mathbf{w}} = \mathbf{\Lambda w}$ because $\mathbf{P} = \mathbf{I}$ we have

$$\begin{aligned}\mathcal{E}^1 &= \{\mathbf{w} \in Y : \ w_2 = 0\}\\ \mathcal{E}^2 &= \{\mathbf{w} \in Y : \ w_1 = 0\}\end{aligned}$$

which is equivalent to setting $k_2 = 0$ in the first case and $k_1 = 0$ in the second.

We call stable, unstable and center eigenspaces to the subsets of Y which are spanned by the eigenspaces associated to the eigenvalues with negative, positive and zero real parts. Formally the **stable eigenspace** is

$$\mathcal{E}^s \equiv \oplus\{\mathcal{E}^j : \text{Re}(\lambda_j) < 0\},$$

the **unstable eigenspace** is

$$\mathcal{E}^u \equiv \oplus\{\mathcal{E}^j : \text{Re}(\lambda_j) > 0\},$$

and the **center eigenspace** is

$$\mathcal{E}^c \equiv \oplus\{\mathcal{E}^j : \text{Re}(\lambda_j) = 0\}.$$

Again we have

$$\mathcal{E}^s \oplus \mathcal{E}^u \oplus \mathcal{E}^c = Y.$$

Let n_-, n_+ and n_c be respectively the number of eigenvalues with negative, positive and zero real parts. Another way to see the relationship between the eigenspaces and the range of the dynamical system is based on the observation that

$$n_- + n_+ + n_c = 2.$$

and that the dimension of the there eigenspaces are therefore

$$\dim(\mathcal{E}^s) = n_-, \ \dim(\mathcal{E}^u) = n_+, \ \dim(\mathcal{E}^c) = n_c,$$

implying

$$\dim(\mathcal{E}^s) + \dim(\mathcal{E}^u) + \dim(\mathcal{E}^c) = \dim(Y) = 2.$$

Therefore, for a planar ODE we have:

1. if all eigenvalues have negative real parts, i.e., if $n_- = 2$, then $\mathcal{E}^s = \mathcal{E}^1 \oplus \mathcal{E}^2 = Y$, and $\mathcal{E}^u$ and $\mathcal{E}^c$ are empty, which means that $\mathcal{E}^s$ is spanned by $\mathcal{E}^1$ and $\mathcal{E}^2$ (i.e, the elements in $\mathcal{E}^s$ are a weighted sum of elements of $\mathcal{E}^1$ and $\mathcal{E}^2$). Then $\mathcal{Y}$ is the **attracting set**;
2. if all eigenvalues have positive real parts, i.e., if $n_+ = 2$, then $\mathcal{E}^u = \mathcal{E}^1 \oplus \mathcal{E}^2 = Y$, and $\mathcal{E}^s$ and $\mathcal{E}^c$ are empty. Then $\mathcal{Y}/\bar{y}$ is the **repelling set**
3. if there is a saddle point, i.e., if $n_- = n_+ = 1$, then $\mathcal{E}^s = \mathcal{E}^2$, $\mathcal{E}^u = Y/E^s$ and , and $\mathcal{E}^c$ is empty. Then $\mathcal{E}^s$ is the **attracting set** and $\mathcal{E}^u$ is the **repelling set**
4. if there is at least one eigenvalue with zero real part, i.e., if $n^c \in \{1, 2\}$, then $\mathcal{E}^c$ is non-empty.

Phase diagrams

The **geometrical** approach for solving ODE consists in drawing a **phase diagram**.

Phase diagrams for planar autonomous ODE are drawn in the space (w_1, w_2) and contain the following elements:

1. **isoclines (or nullclines)** are lines in space (w_1, w_2) such that w_1 or w_2 are constant, that is
$$\mathbb{I}_{w_1} = \{(w_1, w_2) \in Y : \dot{w}_1 = 0\}, \text{ and } \mathbb{I}_{w_2} = \{(w_1, w_2) \in Y : \dot{w}_2 = 0\}.$$
The steady states are the locus or loci where isoclines intersect;
2. the **eigenspaces** $\mathcal{E}^1$ and $\mathcal{E}^2$ are lines in Y whose slopes are given by those of the eigenvectors $\mathbf{P}^1$ and $\mathbf{P}^2$. They span the stable, unstable and center manifolds, $\mathcal{E}^s$, $\mathcal{E}^u$, and $\mathcal{E}^c$, which are lines or two-dimensional subsets of Y;
3. some representative trajectories, also called **integral curves**, that is parametric curves of the solution to the ODE within space Y. They are usually represented with direction arrows showing the direction of the solution with time.
4. the **vector field** indicating the direction of time evolution for a grid of points in Y.

There are four main types of phase diagrams: **nodes**, if all eigenvalues are real and have the same sign, **saddles** if there is one positive and one negative eigenvalue, **foci** if the two eigenvalues are complex conjugate with non-zero real parts, and **centers** if the two eigenvalues are complex conjugate with zero real parts.

Next we present a complete list **phase diagrams**:

Stable nodes A stable node exists if there is at least one real negative eigenvalue and there are no positive eigenvalues. There are three cases: the non-degenerate stable nodes (figure 2.1), the degenerate stable node (see figure 2.2) and the stable node with multiplicity (see figure 2.3).

In the case of figure 2.1 the phase diagram contains the following elements

- there are two isoclines: the abcissa, associated $\dot{w}_2 = 0$ which is the loci where w_2 is constant, and the ordinate, associated $\dot{w}_1 = 0$ which is the loci where w_1 is constant
- a fixed point where the two isoclines cross at $(w_1, w_2) = (0, 0)$
- the eigenspace $\mathcal{E}^1$ which is coincident with $\dot{w}_2 = 0$ associated to the eigenvector λ_1 and eigenspace $\mathcal{E}^2$ which is coincident with $\dot{w}_1 = 0$ associated to the eigenvector λ_2. This coincidence occurs for decoupled systems where $\mathbf{A}$ has the Jordan form $\mathbf{\Lambda}_1$. The whole space Y (with the exception of the fixed point) corresponds to the stable eigenspace $\mathcal{E}^s$. Both the unstable eigenspace and the center eigenspace are empty.
- four representative trajectories. Observe that the slope of the trajectories is parallel to $\mathcal{E}^2$ for initial points far away from the fixed point and they tend asymptotically to $\mathcal{E}^1$. To prove this we write their slope in the phase diagram, for any t, $s(t)$

  $$\frac{w_2(t)}{w_1(t)} = s(t) \equiv \frac{k_2}{k_1} e^{(\lambda_2 - \lambda_1)t}.$$

  We see that $s(0) = \frac{k_2}{k_1}$, $\lim_{t \to -\infty} s(t) = \infty$ and $\lim_{t \to \infty} s(t) = 0$ because $\lambda_2 - \lambda_1 < 0$. This means that all trajectories converge to the steady state along trajectories which are tangent to line $w_2 = 0$, that is, to the eigenspace $\mathcal{E}^1$, or to the direction defined by the eigenvector $\mathbf{P}^1$ which is associated with the eigenvalue with **smaller** absolute value.

In the case of the degenerate stable node, such that $0 = \lambda_1 > \lambda_2$, in figure 2.2, we have $\mathcal{E}^c = \mathcal{E}^2$ and $\mathcal{E}^s = Y/\mathcal{E}^2$. $\mathcal{E}^c$ is also loci of fixed points which are in infinite number.

In the case of multiplicity (see figure 2.3) the trajectories approach P^1 whose slope is given by the simple eigenvector $\mathbf{P}^1 = (1, 0)^\top$.

Saddle point A saddle points exists if the two eigenvalues are real and $\lambda_2 < 0 < \lambda_1$. Figure 2.4 presents the phase diagram containing the following elements

- there are two isoclines: the abcissa, associated $\dot{w}_2 = 0$ which is the loci where w_2 is constant, and the ordinate, associated $\dot{w}_1 = 0$ which is the loci where w_1 is constant
- a fixed point where the two isoclines cross at $(w_1, w_2) = (0, 0)$

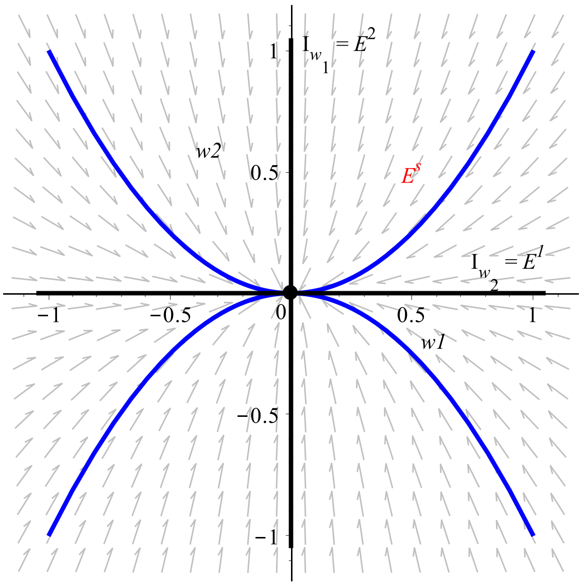

Figure 2.1: Stable node: phase diagram and representative trajectories for the ODE $\dot{w}_1 = -0.5w_1$, $\dot{w}_2 = -w_2$.

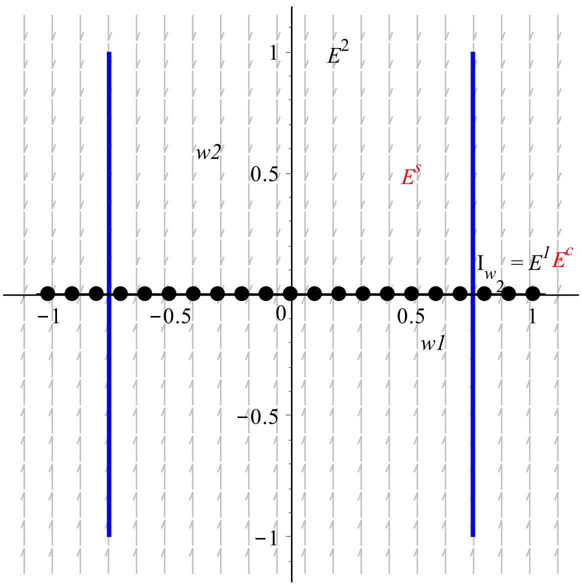

Figure 2.2: Degenerate stable node: $\dot{w}_1 = 0$, $\dot{w}_2 = -w_2$.

- the unstable eigenspace $\mathcal{E}^1$, which is coincident with $\dot{w}_2 = 0$, associated to the eigenvector $\lambda_1 > 0$ and the stable eigenspace $\mathcal{E}^s = \mathcal{E}^2$, which is coincident with $\dot{w}_1 = 0$, associated to the eigenvector $\lambda_2 < 0$. The unstable eigenspace $\mathcal{E}^u$ is almost coincident with all set Y, as $\mathcal{E}^u = Y/\mathcal{E}^s$This coincidence occurs again for decoupled systems where $\mathbf{A}$ has the Jordan form $\mathbf{\Lambda}_1$

Unstable nodes A unstable node exists if there is at least one real positive eigenvalue and there are no negative eigenvalues. There are three cases: the non-degenerate unstable nodes (figure 2.5), the degenerate unstable node (see figure 2.6).

The interpretation is analogous to the stable nodes, if we introduce a time reversal, and if we substitute the stable eigenspace with unstable eigenspace.

Stable foci

A stable focus exists if there are two complex conjugate eigenvalues with negative real parts (see figures 2.8 for $\beta > 0$ and 2.9 for $\beta < 0$).

In this case we see that there is asymptotic stability, as for the case of the stable node in figure 2.1, but the trajectories are oscillatory. We also see that the stable node with multiplicity 2.3 is a boundary case between stable node and foci. The stable eigenspace is coincident with the whole space Y and the unstable and center eigenspaces are empty.

Unstable foci

An unstable focus exists if there are two complex conjugate eigenvalues with positive real parts (see figures 2.10 for $\beta > 0$ and 2.11 for $\beta < 0$). The unstable eigenspace is coincident with the whole space Y and the stable and center eigenspaces are empty.

Center A center (see figure 2.12) exists , if eigenvalues are complex conjugate and have zero real parts. The center eigenspace, $\mathcal{E}^c$, is coincident with the whole space Y and the stable and the unstable eigenspaces are empty. If $w \neq 0$ then all the trajectories are periodic.

2.2.3 Characterizing solutions when A is not in the Jordan form

Now we address the general planar linear homogeneous equation $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ already derived in equation (2.8),

$$\mathbf{y}(t) = \mathbf{P}\mathbf{e}^{\mathbf{\Lambda t}}\mathbf{h}$$

where $\mathbf{h} = \mathbf{P}^{-1}\mathbf{k}$. By observing that $\mathbf{y}(t) = \mathbf{P}\mathbf{w}(t)$ we see that the solution in this case is a linear transformation of the solution for the case in which the coefficient matrix is in the Jordan form.

This means that

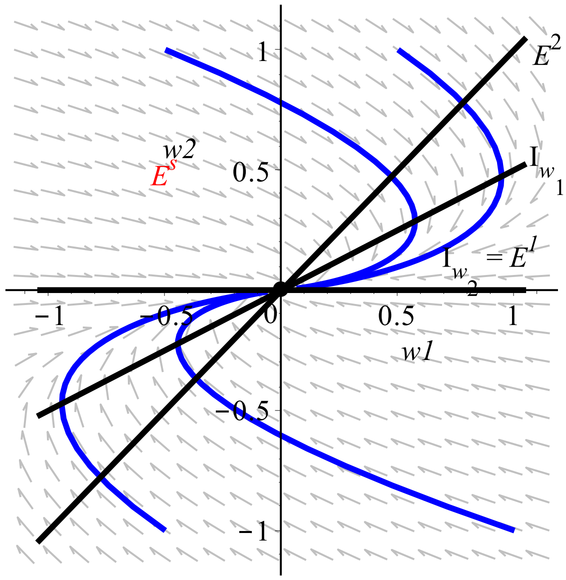

Figure 2.3: Stable node with multiplicity: $\dot{w}_1 = -0.5w_1 + w_2$, $\dot{w}_2 = -0.5w_2$.

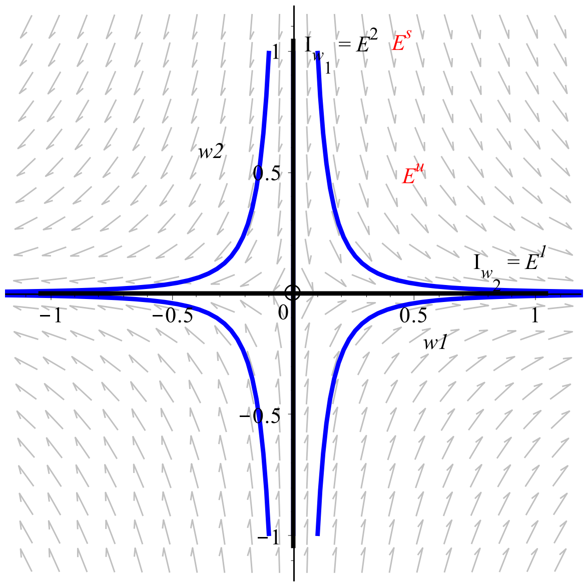

Figure 2.4: Saddle point: $\dot{w}_1 = 0.5w_1$, $\dot{w}_2 = -w_2$.

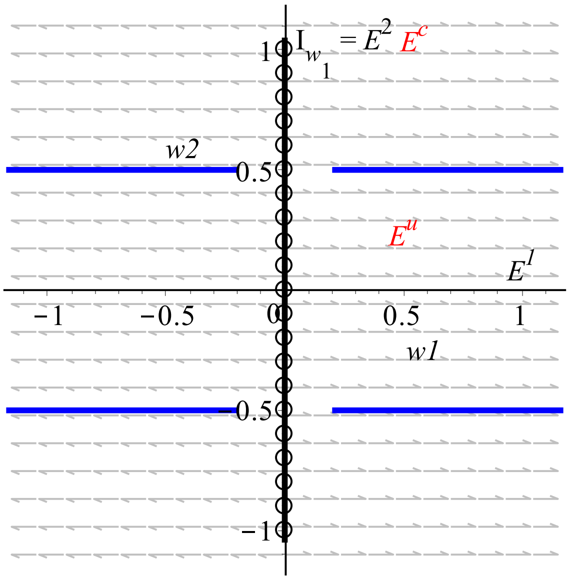

Figure 2.5: Unstable degenerate node: $\dot{w}_1 = 0.5w_1$, $\dot{w}_2 = 0$.

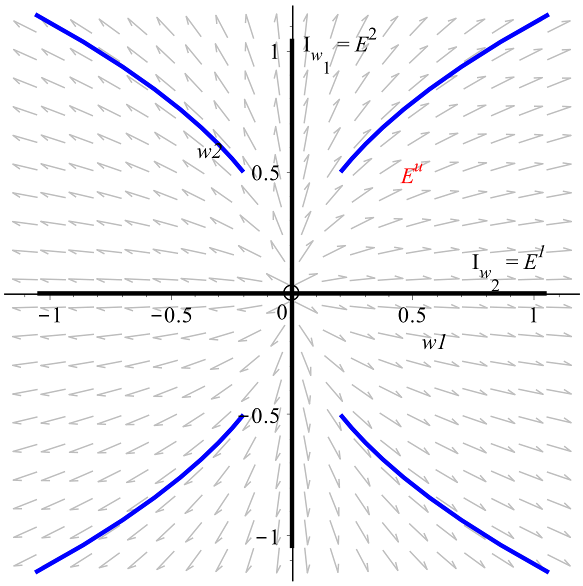

Figure 2.6: Unstable node: $\dot{w}_1 = w_1$, $\dot{w}_2 = 0.5w_2$.

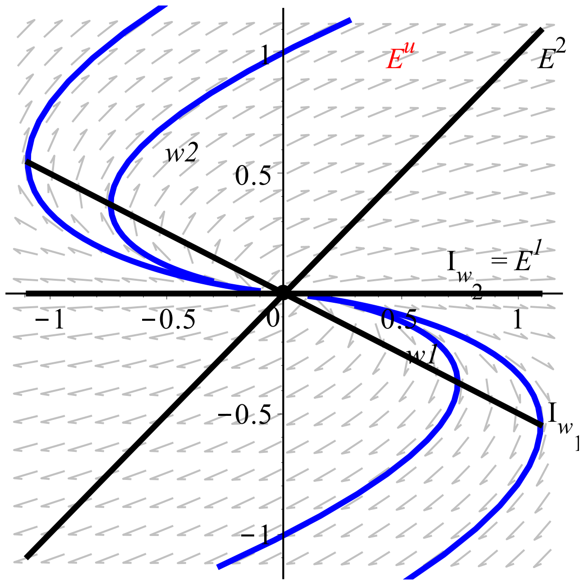

Figure 2.7: Unstable node with multiplicity: $\dot{w}_1 = 0.5w_1 + w_2$, $\dot{w}_2 = 0.5w_2$.

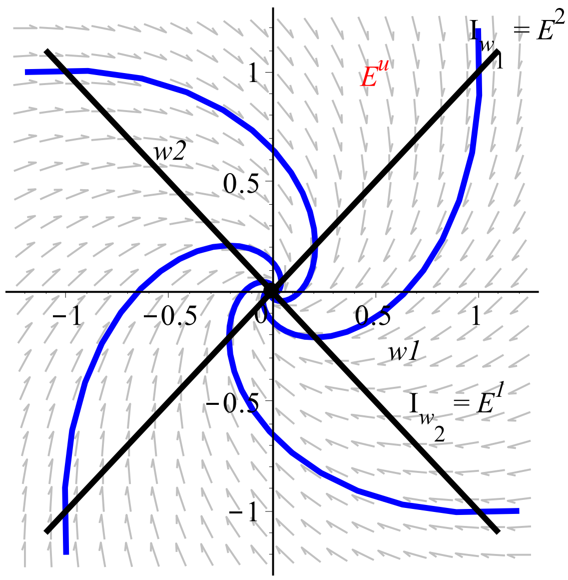

Figure 2.8: Stable focus: $\dot{w}_1 = -0.5w_1 + 0.5w_2$, $\dot{w}_2 = -0.5w_1 - 0.5w_2$ (case $\alpha < 0$ and $\beta > 0$).

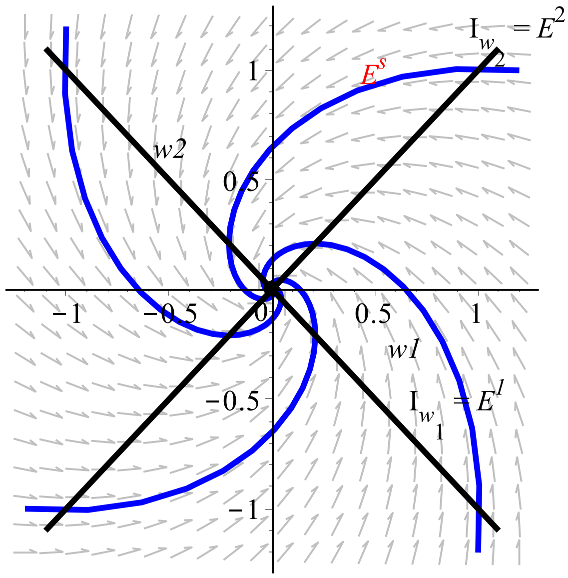

Figure 2.9: Stable focus:$\dot{w}_1 = -0.5w_1 - 0.5w_2$, $\dot{w}_2 = 0.5w_1 - 0.5w_2$ (case $\alpha < 0$ and $\beta < 0$).

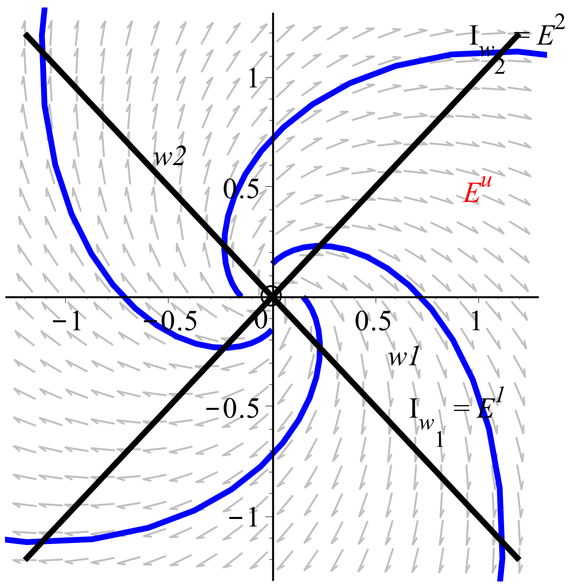

Figure 2.10: Unstable focus: $\dot{w}_1 = 0.5w_1 + 0.5w_2$, $\dot{w}_2 = -0.5w_1 + 0.5w_2$ (case $\alpha > 0$ and $\beta > 0$).

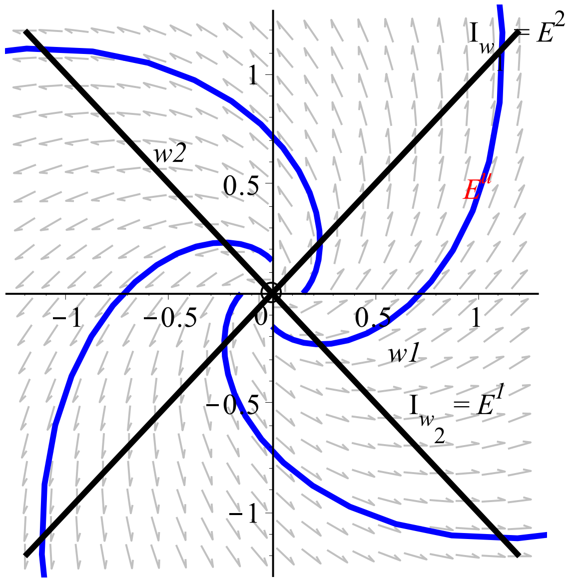

Figure 2.11: Unstable focus: $\dot{w}_1 = 0.5w_1 - 0.5w_2$, $\dot{w}_2 = 0.5w_1 + 0.5w_2$ (case $\alpha < 0$ and $\beta < 0$).

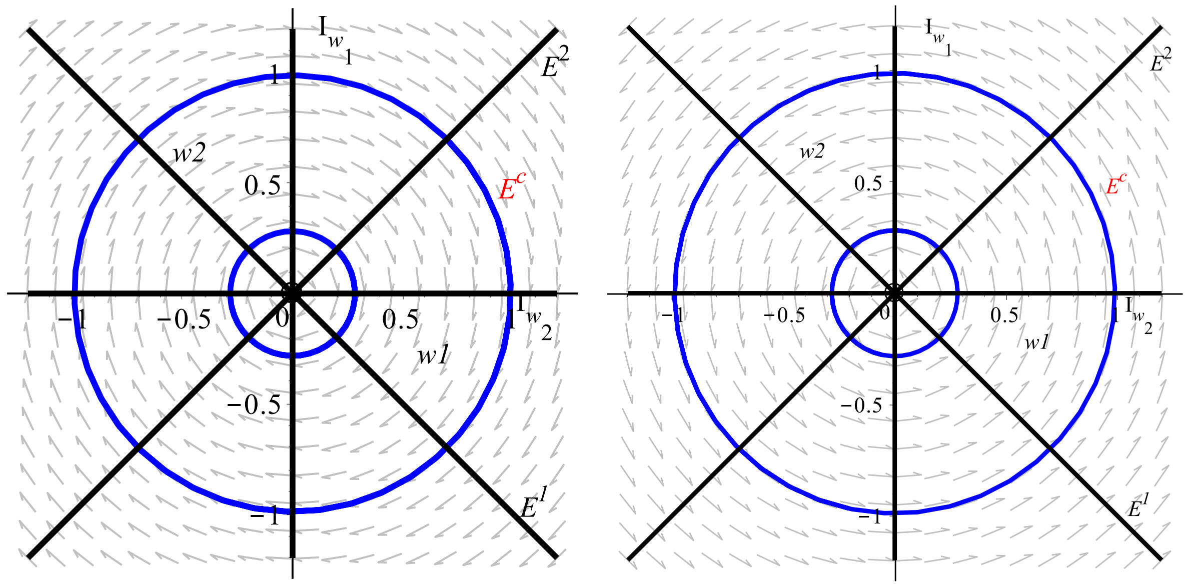

Figure 2.12: Center: $\dot{w}_1 = -0.5w_2$, $\dot{w}_2 = 0.5w_1$ and $\dot{w}_1 = 0.5w_2$, $\dot{w}_2 = -0.5w_1$ (cases $\alpha = 0$ and $\beta > 0$, and $\alpha = 0$ and $\beta < 0$).

1. the qualitative properties of the dynamics are the same, in particular, the number and stability type of the steady state(s)

2. the dimensions of the stable, unstable and center eigenspaces, partitioning the range Y, is the same

3. the only difference is related to the slopes of the eigenspaces and therefore of the solution trajectories, because the eigenvector matrix $\mathbf{P}$ is different from the identity matrix.

In particular we can have one of the following (general) solutions

1. if $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, the general solution is

$$\mathbf{y}(t) = h_1 e^{\lambda_1 t}\mathbf{P}^1 + h_2 e^{\lambda_2 t}\mathbf{P}^2$$

or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = h_1 e^{\lambda_1 t} \begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} + h_2 e^{\lambda_2 t} \begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix}$$

2. if $\mathbf{\Lambda} = \mathbf{\Lambda}_2$, the general solution is

$$\mathbf{y}(t) = e^{\lambda t}\left(\mathbf{P}^1(h_1 + h_2 t) + h_2\mathbf{P}^2\right)$$

or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{\lambda t}\left((h_1 + h_2 t)\begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} + h_2 \begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix}\right)$$

3. if $\mathbf{\Lambda} = \mathbf{\Lambda}_3$, the general solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{\alpha t}\left((h_1 \cos\beta t + h_2 \sin\beta t)\mathbf{P}^1 + (h_2\cos\beta t - h_1 \sin\beta t)\mathbf{P}^2\right) = \\ &= e^{\alpha t}\left(h_1(\cos\beta t\mathbf{P}^1 - \sin\beta t\mathbf{P}^2) + h_2(\sin\beta t\mathbf{P}^1 + \cos\beta t\mathbf{P}^2)\right). \end{aligned}$$

or, equivalently,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{\alpha t}\left(h_1 \begin{pmatrix} P_1^1\cos\beta t - P_1^2 \sin\beta t \\ P_2^1 \cos\beta t - P_2^2 \sin\beta t \end{pmatrix} + h_2 \begin{pmatrix} P_1^1 \sin\beta t + P_1^2 \cos\beta t \\ P_2^1 \sin\beta t + P_2^2\cos\beta t \end{pmatrix}\right).$$

The next example allows for a comparison between the solutions of an homogeneous problem when the coefficient matrix is a Jordan form with the case in which it is a similar matrix but not in the Jordan normal form

Example 1 Solve the planar ODE assuming that $\mathbf{w} \in \mathbb{R}^2$.

$$\dot{w}_1 = 3w_1$$
$$\dot{w}_2 = -3w_2$$

We readily see that the coefficient matrix in in the Jordan form Λ_1

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}.$$

The (general) solution of the ODE is

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} h_1 e^{3t} \\ h_2 e^{-3t} \end{pmatrix} = h_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + h_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

Therefore: (1) there is a unique steady state $\overline{\mathbf{w}} = (\overline{w}_1, \overline{w}_2) = (0, 0)$; (2) the steady state is a saddle point; (3) the eigenvalues of the coefficient matrix are $\lambda_1 = 3$ and $\lambda_2 = -3$ and the associated eigenvectors are $\mathbf{P}^1 = (1, 0)^\top$ and $\mathbf{P}^2 = (0, 1)^\top$; (4) the eigenspaces associated to the eigenvalues λ_1 and λ_2 are

$$\mathcal{E}^1 = \{\mathbf{w} \in \mathbb{R}^2 : w_2 = 0\}, \mathcal{E}^2 = \{\mathbf{w} \in \mathbb{R}^2 : w_1 = 0\};$$

(5) then the center eigenspace $\mathcal{E}^c$ is empty and the stable and unstable eigenspaces are both of dimension 1 and the unstable and stable eigenspaces are

$$\mathcal{E}^s = \mathcal{E}^2,\ \mathcal{E}^u = \mathbb{R}^2 / \mathcal{E}^s$$

meaning that for any $\mathbf{k} \neq (0, k_2)$ the solution is unstable.

That is, trajectories belonging to the stable subspace, that is converging to the steady state, should have $k_1 = 0$, that is they are

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ h_2 e^{\lambda_2 t} \end{pmatrix}$$

The phase diagram for this equation is very similar to the one depicted in Figure 2.4.

Example 2 Solve the homogeneous ODE over the domain $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$:

$$\begin{aligned} \dot{y}_1 &= -2y_1 + 5y_2, \\ \dot{y}_2 &= y_1 + 2y_2. \end{aligned} \tag{2.15}$$

where

$$\mathbf{A} = \begin{pmatrix} -2 & 5 \\ 1 & 2 \end{pmatrix}.$$

As $\mathrm{trace}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) = -9$ the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$, which means that the coefficient matrix is similar to the previous example. The eigenvector matrix is

$$\mathbf{P} = \left(\mathbf{P}^1, \mathbf{P}^2\right) = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}.$$

This means that the eigenspaces are

$$\mathcal{E}^1 = \{\mathbf{y} \in \mathbb{R}^2 : y_1 - y_2 = 0\},\ \mathcal{E}^2 = \{(\mathbf{y} \in \mathbb{R}^2 : y_1 + 5\,y_2 = 0\}$$

As $\det(\mathbf{A}) \neq 0$ then the fixed point exists and is unique and is $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2)^\top = (0,0)^\top$.

The (general) solution of the equation, $\mathbf{y}(t) = \mathbf{P}\mathbf{e}^{\mathbf{\Lambda t}}\mathbf{h}$, is

$$y(t) = h_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + h_2 \begin{pmatrix} -5 \\ 1 \end{pmatrix} e^{-3t}. \tag{2.16}$$

The stable and the unstable eigenspaces (the center eigenspace is empty. Why ?) are

$$\mathcal{E}^s = \{(y_1, y_2) :\ y_1 + 5y_2 = 0\},\ \mathcal{E}^u = \mathbb{R}^2/(\mathcal{E}^s \cup \{(0,0)\}.$$

and the stable subspace is equal to the eigenspace $\mathcal{E}^2$. Then, if the initial point is such that $(y_1(0), y_2(0)) = (-5y_2(0), y_2(0))$ for any choice of $y_2(0)$ the solution converges to the steady state $\bar{y} = (0,0)^\top$. For any other initial point the solution is asymptotically unbounded.

We can prove this in two different but equivalent ways: First, we can consider the general solution in equation of (2.19) and set $h_1 = 0$. Comparing to the case in which the coefficient matrix is in the similar Jordan form we have

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} k_1 + 5k_2 \\ -k_1 + k_2 \end{pmatrix}.$$

we see that this holds if and only if $y_1(0)+5y_2(0) = 0$, because the constant k_1 and k_2 are arbitrary. The second way (which we can used without having to determine h_1) consider the general solution and the observation, again, that we can only eliminate the unbounded part of the solution if we have h_1. This means that the solution along the stable subspace is

$$y_1(t) = -5h_2 e^{-3t},\ y_2(t) = h_2 e^{-3t}$$

By eliminating $h_2 e^{-3t}$ in the two equations we have $y_1(t) = -5y_2(t)$.

To study the equation geometry we draw the **phase diagram** (see figure 2.13). Given the fact that we have a positive and a negative eigenvalue we know that it is a saddle. However, to determine their configuration in this case, we draw the following elements:

1. the isoclines, that is, the loci for $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$

$$\mathbb{I}_{y_1} = \{(y_1, y_2) : \ -2y_1 + 5y_2 = 0\}, \ \mathbb{I}_{y_2} = \{(y_1, y_2) : \ y_1 + 2y_2 = 0\}$$

2. the eigenspaces $\mathcal{E}^1$ and $\mathcal{E}^2$;

3. the vector field;

4. as the model is linear and the vector field should show us that the stable eigenspace is coincident with the eigenspace associated to the eigenvector $\mathbf{P}^2$;

5. all the isoclines and the eigenvectors cross at the steady state $(0, 0)$;

6. if the initial point is not at the origin, then two types of paths are possible: first, if they start at $\mathcal{E}^s$ they will converge to the origin; second, if they do not start at the origin they will be parallel to $\mathcal{E}^2$ at the beginning and will converge to $\mathcal{E}^1$ asymptotically. Observe that when they cross any isocline they should change direction as regards the variable associated to the isocline. For instance, if they cross $\mathbb{I}_{y_1}$ ($\mathbb{I}_{y_2}$) they should be tangent to a vertical (horizontal) line.

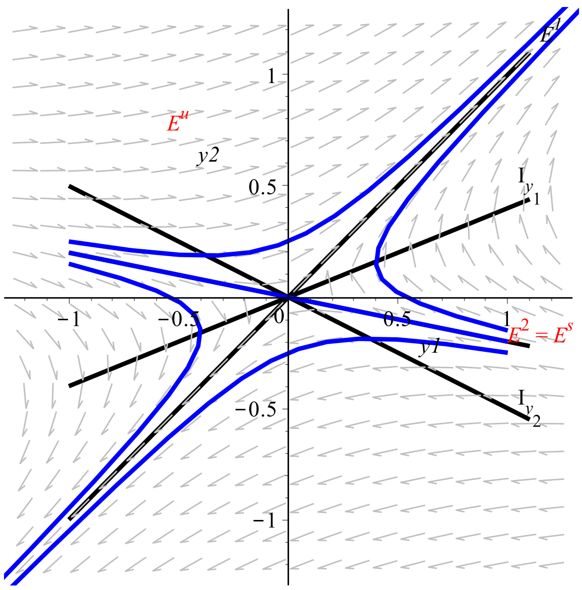

Figure 2.13: Saddle: $\dot{y}_1 = -2y_1 + 5y_2$, $\dot{y}_2 = y_1 + 2y_2$.

2.3 The non-homogeneous equation

In this section we solve the non-homogeneous equation (2.1), $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, where $\mathbf{A}$ is similar to one of the Jordan forms already presented or is equal to a new matrix

$$\mathbf{\Lambda}_4 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

It is convenient to start by addressing the existence and number of steady states, or stationary solutions.

Steady states are defined as the elements of the set

$$\overline{\mathbf{y}} = \{\mathbf{y} \in Y : \ \mathbf{A}\mathbf{y} + \mathbf{B} = 0\}$$

Again we write the eigenvector matrix associated to coefficient matrix $\mathbf{A}$

$$\mathbf{P} = \begin{pmatrix} P_1^1 & P_1^2 \\ P_2^1 & P_2^2 \end{pmatrix}.$$

Proposition 3. *(Existence and number of fixed points)*

1. *If $\mathbf{A}$ has no zero eigenvalues then a steady state exists and is unique and it is*

$$\overline{\mathbf{y}} = -\mathbf{A}^{-1}\mathbf{B}.$$

2. *If $\Delta(\mathbf{A}) > 0$, and $\lambda_1 = 0$, $\lambda_2 < 0$, and $P_2^2 b_2 = P_1^2 b_1$ then there is an infinite number of equilibrium points belonging to a one-dimensional manifold (a line)*

$$\overline{y} \in \{(y_1, y_2) \in Y : \ P_1^1(\lambda_2 y_2 - b_2) = P_2^1(\lambda_1 y_1 - b_1)\}.$$

3. *If $\Delta(\mathbf{A}) > 0$, and $\lambda_1 > 0$, $\lambda_2 = 0$, and $P_1^1 b_2 = P_2^1 b_1$ then there is an infinite number of equilibrium points belonging to a one-dimensional manifold*

$$\overline{y} \in \{(y_1, y_2) \in Y : \ P_2^2(\lambda_1 y_1 - b_1) = P_1^2(\lambda_2 y_2 - b_2)\}.$$

4. *If $\Delta(\mathbf{A}) = 0$, $\lambda = 0$, and $P_2^2 b_1 = P_1^2 b_2$ then there is an infinite number of equilibrium points belonging to a one-dimensional manifold*

$$\overline{y} \in \{(y_1, y_2) \in Y : \ P_2^1(y_1 - b_1) = P_1^1(y_2 - b_2)\}.$$

5. *if* $\mathbf{A} = 0$ *and* $P_2^2 b_2 - P_1^2 b_1 = P_1^1 b_2 - P_2^1 b_1 = 0$ *then we have an infinity of equilibrium points belonging to a two-dimensional manifold (i.e., the whole space* Y*).*

6. *If none of the former conditions hold there are no steady states.*

Proof. A steady state is a point $\mathbf{y}$ such that $\mathbf{A}\mathbf{y} = -\mathbf{B}$. If $\det(\mathbf{A}) \neq 0$ then a there is a unique inverse matrix $\mathbf{A}^{-1}$ and therefore a unique fixed point exits $\overline{\mathbf{y}} = -\mathbf{A}^{-1}\mathbf{B}$. If matrix $\mathbf{A}$ is singular, that is $\det(\mathbf{A}) = 0$, then a classical inverse does not exist. In this case, observe that $\mathbf{A}\mathbf{y} = -\mathbf{B}$ is equivalent to $\mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} = -\mathbf{B}$ and also $\mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{y} = -\mathbf{P}^{-1}\mathbf{B}$. Because in this case there only real eigenvalues, there are two forms for expanding this equation. The first form is

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} P_2^2 & -P_1^2 \\ -P_2^1 & P_1^1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_2^2 & -P_1^2 \\ -P_2^1 & P_1^1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for $\lambda_1 = 0$ and $\lambda_2 \neq 0$, $\lambda_1 \neq 0$ and $\lambda_2 = 0$ or $\lambda_1 = \lambda_2 = 0$. Then, in the first case,

$$P_2^2 b_2 = P_1^2 b_1, \text{ and } P_1^1(\lambda_2 y_2 - b_2) = P_2^1(\lambda_2 y_1 - b_1)$$

in the second case

$$P_1^1 b_2 = P_2^1 b_1, \text{ and } P_2^2(\lambda_1 y_1 - b_1) = P_1^2(\lambda_1 y_2 - b_2)$$

in the third case, we have $P_2^2 b_2 - P_1^2 b_1 = P_1^1 b_2 - P_2^1 b_1 = 0$ which is a condition for existence.

The second form is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_2^2 & -P_1^2 \\ -P_2^1 & P_1^1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_2^2 & -P_1^2 \\ -P_2^1 & P_1^1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

which is equivalent to

$$P_2^2 b_1 = P_1^2 b_2, \text{ and } P_2^1(y_1 - b_1) = P_1^1(y_2 - b_2)$$

In all other cases, fixed points will not exist. □

Next we derive the general solution for the case in which there is a steady state

Proposition 4. *Consider the planar ode* (2.1)*, and assume that an equilibrium point* $\overline{\mathbf{y}} \in Y$ *exists. Then, the unique solution is*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}\mathbf{t}}\mathbf{P}^{-1}(\mathbf{h} - \overline{\mathbf{y}}) \tag{2.17}$$

where $\mathbf{h} \in Y$ *is an arbitrary element of the range of* $\mathbf{y}$.

Proof. Assume that a fixed point $\overline{\mathbf{y}}$ exists. Let $\mathbf{y}(t) - \overline{\mathbf{y}} = \mathbf{P}\mathbf{w}(t)$. Then $\mathbf{w}(t) = \mathbf{P}^{-1}(\mathbf{y}(t) - \overline{\mathbf{y}})$ and $\dot{\mathbf{w}} = \mathbf{P}^{-1}\dot{\mathbf{y}} = \mathbf{P}^{-1}(\mathbf{A}\mathbf{y} + \mathbf{B}) = \mathbf{P}^{-1}\mathbf{A}\left((\mathbf{P}\mathbf{w} + \overline{\mathbf{y}}) + \mathbf{B}\right) = \mathbf{\Lambda}\mathbf{w} + \mathbf{\Lambda}\mathbf{P}^{-1}\mathbf{A}\overline{\mathbf{y}} + \mathbf{P}^{-1}\mathbf{B} = \mathbf{\Lambda}\mathbf{w} - \mathbf{P}^{-1}\mathbf{B} + \mathbf{P}^{-1}\mathbf{B} = \mathbf{\Lambda}\mathbf{w}$ for any matrix $\mathbf{\Lambda}$. Then, we get equivalently $\dot{\mathbf{w}} = \mathbf{\Lambda}\mathbf{w}$, which has solution $\mathbf{w}(t) = \mathbf{e}^{\mathbf{\Lambda}\mathbf{t}}\mathbf{k}$, where $\mathbf{k}$ is a vector of arbitrary constants. Therefore, the solution for $\mathbf{y}$ is $\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{w}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}\mathbf{t}}\mathbf{P}^{-1}(\mathbf{h} - \overline{\mathbf{y}})$ where $\mathbf{h}$ is a vector of arbitrary constants, in the units of $\mathbf{y}$. □

The solution (2.17) can be written as

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda}\mathbf{t}}\mathbf{k}$$

where

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{\det(\mathbf{P})} \begin{pmatrix} P_2^2 & -P_1^2 \\ -P_2^1 & P_1^1 \end{pmatrix} \begin{pmatrix} h_1 - \overline{y}_1 \\ h_2 - \overline{y}_2 \end{pmatrix}.$$

Then, recalling what we have learned from solving equation (2.2), it can take one of the following three forms

1. if $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, the general solution is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + k_1 e^{\lambda_1 t}\mathbf{P}^1 + k_2 e^{\lambda_2 t}\mathbf{P}^2$$

   or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + k_1 e^{\lambda_1 t} \begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} + k_2 e^{\lambda_2 t} \begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix}$$

2. if $\mathbf{\Lambda} = \mathbf{\Lambda}_2$, the general solution is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + e^{\lambda t}\left(\mathbf{P}^1(k_1 + k_2 t) + k_2\mathbf{P}^2\right)$$

   or, equivalently

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\lambda t}\left((k_1 + k_2 t) \begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} + k_2 \begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix}\right)$$

3. if $\Lambda = \Lambda_3$, the general solution is

$$\begin{aligned} \mathbf{y}(t) &= \overline{\mathbf{y}} + e^{\alpha t}\left((k_1 \cos\beta t + k_2 \sin\beta t)\mathbf{P}^1 + (k_2 \cos\beta t - k_1 \sin\beta t)\mathbf{P}^2\right) = \\ &= \overline{y} + e^{\alpha t}\left(k_1(\cos\beta t\mathbf{P}^1 - \sin\beta t\mathbf{P}^2) + k_2(\sin\beta t\mathbf{P}^1 + \cos\beta t\mathbf{P}^2)\right). \end{aligned}$$

   or, equivalently,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix} + e^{\alpha t}\left(k_1 \begin{pmatrix} P_1^1 \cos\beta t - P_1^2 \sin\beta t \\ P_2^1 \cos\beta t - P_2^2 \sin\beta t \end{pmatrix} + k_2 \begin{pmatrix} P_1^1 \sin\beta t + P_1^2 \cos\beta t \\ P_2^1 \sin\beta t + P_2^2 \cos\beta t \end{pmatrix}\right).$$

Eigenspaces and stability analysis Let $\mathbf{\Lambda} = \mathbf{\Lambda}_1$. We can determine again the eigenspaces, spanned by eigenvectors $\mathbf{P}^1$ and $\mathbf{P}^2$, by making $k_2 = 0$ and $k_1 = 0$, respectively. Then [1]

$$\mathcal{E}^1 = \{\mathbf{y} \in Y : \ P_1^1(y_2 - \overline{y}_2) = P_2^1(y_1 - \overline{y}_1)\}$$

and

$$\mathcal{E}^2 = \{\mathbf{y} \in Y : \ P_1^2(y_2 - \overline{y}_2) = P_2^2(y_1 - \overline{y}_1)\}$$

We can also partition the state space according to the stability properties of the solutions belonging to them. We define the **stable eigenspace** as

$$\mathcal{E}^s = \{\mathbf{h} \neq \overline{\mathbf{y}} \in Y : \ \lim_{t\to\infty} \mathbf{y}(t, \mathbf{h}) = \overline{\mathbf{y}}\}$$

we define the **unstable eigenspace** as

$$\mathcal{E}^u = \{\mathbf{h} \neq \overline{\mathbf{y}} \in Y : \ \lim_{t\to-\infty} \mathbf{y}(t, \mathbf{h}) = \overline{\mathbf{y}}\}$$

and the **center eigenspace** as

$$\mathcal{E}^c = \{\mathbf{h} \in Y : \ \mathbf{y}(t, \mathbf{h}) = \text{const}\}$$

if there is at least one eigenvalue with zero real part.

If all eigenvalues have negative real parts then we have **asymptotic stability**, and $\mathcal{E}^s = \mathcal{E}^1 \oplus \mathcal{E}^2 = Y$, and $\mathcal{E}^u$ and $\mathcal{E}^c$ are empty. If all eigenvalues have positive real parts then we have **instability** and $\mathcal{E}^u = \mathcal{E}^1 \oplus \mathcal{E}^2 = Y$, and $\mathcal{E}^s$ and $\mathcal{E}^c$ are empty. If there is one negative and one positive eigenvalue then we have a **saddle point** and $\mathcal{E}^s = \mathcal{E}^2$ and $\mathcal{E}^u = Y/\mathcal{E}^2$.

Changes in the phase diagrams Next we extend the case in Example 2 to show that adding a vector $\mathbf{B}$ only changes the value of the steady state but not its stability properties, as regards the associated homogeneous case.

Example 3 Consider the ODE, where $y \in \mathbb{R}^2$, which is slightly modification of equation (2.15):

$$\begin{aligned} \dot{y}_1 &= -2y_1 + 5y_2 - 1/5, \\ \dot{y}_2 &= y_1 + 2y_2 - 4/5. \end{aligned} \tag{2.18}$$

This is an non-homogenous equation of type $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, where matrix $\mathbf{A}$ is as in example (2.15). As trace$(\mathbf{A}) = 0$ and $\det(A) = -9$ then the eigenvalues are $\lambda \in \{-3, 3\}$. The steady state is

$$\bar{y} = -A^{-1}B = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix}.$$

[1] If we set $k_2 = 0$ we have $h_1 e^{\lambda_1 t} P_1^1 = y_1(t) - \overline{y}_1$ and $h_1 e^{\lambda_1 t} P_2^1 = y_2(t) - \overline{y}_2$. Thus $h_1 e^{\lambda_1 t} = \dfrac{y_1(t) - \overline{y}_1}{P_1^1} = \dfrac{y_2(t) - \overline{y}_2}{P_2^1}$. We proceed in an analogous way for $\mathcal{E}^2$.

In this case the general solution is

$$y(t) = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + k_2 \begin{pmatrix} -5 \\ 1 \end{pmatrix} e^{-3t}. \tag{2.19}$$

where

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} h_1 - \bar{y}_1 \\ h_2 - \bar{y}_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} h_1 + 5h_2 - 7/5 \\ -h_1 + h_2 + 1/5 \end{pmatrix}.$$

Therefore, the eigenspaces are

$$\mathcal{E}^1 = \{(y_1, y_2) : y_1 + 5y_2 - 7/5 = 0\},\ \mathcal{E}^2 = \{(y_1, y_2) : -y_1 + y_2 + 1/5 = 0\}$$

The fixed point is again a saddle point and the stable eigenspace is again $\mathcal{E}^s = \mathcal{E}^1$

The phase diagram is in figure 2.14. If we compare with figure 2.13 we see that they have the same shape (i.e, the isoclines and the eigenspaces have the same slopes) with the fixed point shifted from the origin to the new steady state $\bar{y} = (2/5, 1/5)^\top$.

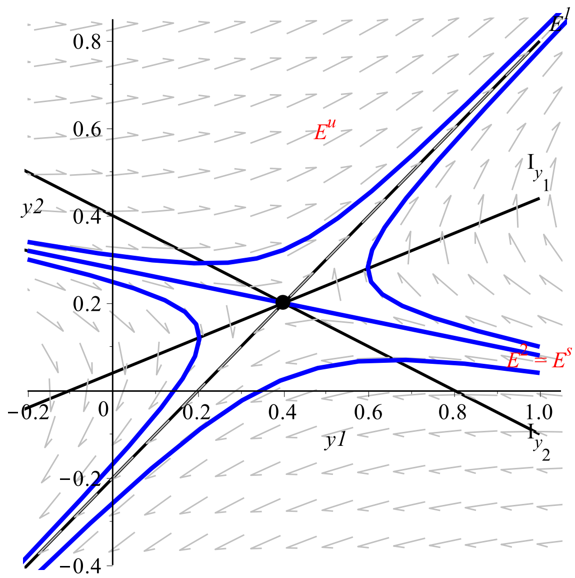

Figure 2.14: Saddle: $\dot{y}_1 = -2y_1 + 5y_2 - 0.2$, $\dot{y}_2 = y_1 + 2y_2 - 0.8$.

Comparing examples 1, 2, and 3, with phase diagrams in 2.4, 2.13 and 2.14 lead to the following observations:

1. as we already saw, when the equation is homogeneous, i.e., $\mathbf{B} = \mathbf{0}$, but matrix $\mathbf{A}$ is not in a Jordan normal form, the steady state is still in the origin but the isoclines and the eigenspaces are rotated (compare figures 2.4 and 2.13);

2. when the equation is non-homogeneous, i.e., vector $\mathbf{B} \neq \mathbf{0}$, the steady state is shifted out of the origin but the isoclines and the eigenspaces are the same as for the similar homogeneous equation (compare figures 2.13 and 2.14).

Next we present a case in which we have a **stable node** and show that **hump-shaped** trajectories can occur for a non-homogeneous ODE.

Example 4 Consider the ODE, where $y \in \mathbb{R}^2$:

$$\begin{aligned} \dot{y}_1 &= -2y_1 + y_2 + 1/5, \\ \dot{y}_2 &= y_1 - 2y_2 + 4/5. \end{aligned} \tag{2.20}$$

Prove that the solution for the initial value problem, for any $\mathbf{y}(0) = (y_1(0), y_2(0))$ is

$$y(t) = \begin{pmatrix} \frac{4}{15} \\ \frac{1}{15} \end{pmatrix} + \frac{1}{2}\left(-y_1(0) + y_2(0) - \frac{1}{15}\right)\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3\,t} + \frac{1}{2}\left(y_1(0) + y_2(0) - \frac{9}{15}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\,t}$$

The phase diagram is in figure 2.15. We see that it is a stable node. In addition, the unstable and the centre subspaces are empty and the stable subspace is the whole set minus the fixed point $\bar{y} = (\frac{4}{15}, \frac{1}{15})^\top$. In addition observe that the trajectories at $t = 0$ tend to be parallel to the eigenspace associated to the negative eigenvalue larger in absolute value $\mathcal{E}^1 = \{(y_1, y_2) : \ y_1 + y_2 = 0\}$ and they become asymptoticaly tangent to the eigenspace associated to the negative eigenvalue smaller in absolute value $\mathcal{E}^2 = \{(y_1, y_2) : \ y_1 - y_2 = 0\}$. This means that are trajectories that cross the isoclines and which are, therefore, non-monotonous.

Non-monotonous trajectories The previous example displays another difference between homogeneous and similar non-homogeneous equations (see phase diagram in Figure 2.14). Consider a planar ODE (homogeneous or not) in which the coefficient matrix $\mathbf{A}$ is such that there is a unique steady state which is a stable nodes (i.e., there are two real eigenvalues with negative and distinct real parts). Now compare the cases with similar matrices for the case in $\mathbf{A}$ is a Jordan form, as in figure 2.1, and it is not, as in figure 2.15). We observe that in the last case we see that there are **hump-shaped trajectories**, that is, trajectories that converge to the steady state after crossing an isocline, but affecting only one variable which.

This also allows for a partition of space Y. The isoclines $\mathbb{I}_{y_1}$ and $\mathbb{I}_{y_2}$ allow for a partition of Y into for subsets, say

$$\begin{aligned} Y^{++} &= \{\mathbf{y} \in Y : \dot{y}_1 > 0,\ \dot{y}_2 > 0\} \\ Y^{-+} &= \{\mathbf{y} \in Y : \dot{y}_1 < 0,\ \dot{y}_2 > 0\} \\ Y^{+-} &= \{\mathbf{y} \in Y : \dot{y}_1 > 0,\ \dot{y}_2 < 0\} \\ Y^{--} &= \{\mathbf{y} \in Y : \dot{y}_1 < 0,\ \dot{y}_2 < 0\} \end{aligned}$$

As we saw, for stable nodes, the solution path will be asymptotically attracted to the direction defined by the eigenspace associated to the smaller eigenvalue in absolute value. In our case it is $\mathcal{E}^1$. This eigenspace will be contained in the union of two of the subsets defined by the isoclines. It can be proved that if $\mathbf{h}$ does not belong to union of those subsets then one of the variables will be hump-shaped.

For example, let $\mathcal{E}^1 \subset Y^{++} \cup Y^{--}$. If $\mathbf{h} \in Y^{-+} \cup Y^{+-}$ then one of the solution paths will be hump-shaped, depending which isocline is crossed: if it crosses $\mathbb{I}_{y_1}$ then $y_1(t)$ will be hump-shaped and $y_2(t)$ will be monotonous and if it crosses $\mathbb{I}_{y_2}$ then $y_2(t)$ will be hump-shaped and $y_1(t)$ will be monotonous

Summing up, we may have three types of trajectories, independently from the uniqueness and stability properties of steady states:

1. **monotonous trajectories** both stable and unstable: if the steady state is a saddle point, or a node and the coefficient matrix $\mathbf{A}$ is in the Jordan form or if it is not the arbitrary constant does not involve trajectories crossing isoclines;

2. **oscillatory trajectories** both stable and unstable: when there is a focus

3. **hump-shaped trajectories**: when there is a node, the coefficient matrix is not in the Jordan form and trajectories cross isoclines.

2.4 Main result on stability theory

The dynamic behavior of the solution for equation (2.1) is similar to that of equation (**??**), but relative to a fixed point which is not necessarily coincident with the origin .

Theorem 1. *Consider the planar ODE* (2.1). *Assume that a fixed point* $\overline{\mathbf{y}} \in \mathcal{Y}$ *exists if* $\det(\mathbf{A}) \neq 0$ *or that an infinite number of fixed points exist if* $\det(\mathbf{A}) = 0$. *The asymptotic properties of the solution as a function of the trace and determinant of* $\mathbf{A}$ *are:*

1. *asymptotic stability if and only if trace*$(\mathbf{A}) < 0$ *and* $\det(\mathbf{A}) \geq 0$*;*

2. *saddle path (or conditional) stability if and only if* $\det(\mathbf{A}) < 0$*;*

3. *instability if and only if trace*$(\mathbf{A}) > 0$ *and* $\det(\mathbf{A}) \geq 0$*;*

4. *stability but not asymptotic stability if trace*$(\mathbf{A}) = 0$ *and* $\det(\mathbf{A}) \geq 0$.

In figure 2.16 we represent the phase diagrams associated to the different values of the trace and determinant of $\mathbf{A}$.

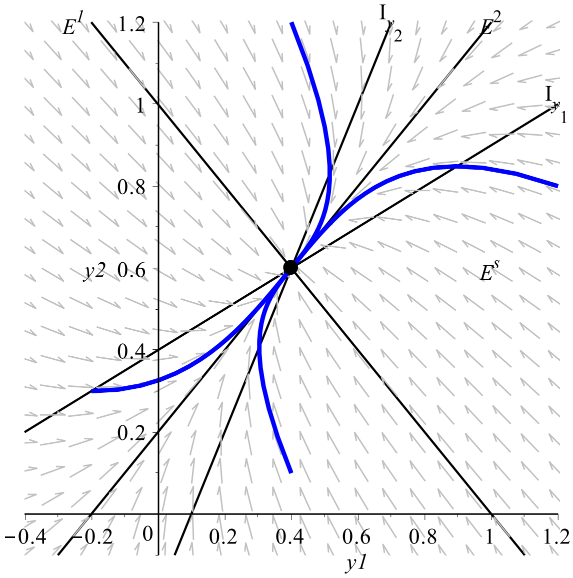

Figure 2.15: A sink or stable node: $\dot{y}_1 = -2y_1 + y_2 + 0.2$, $\dot{y}_2 = y_1 - 2y_2 + 0.8$.

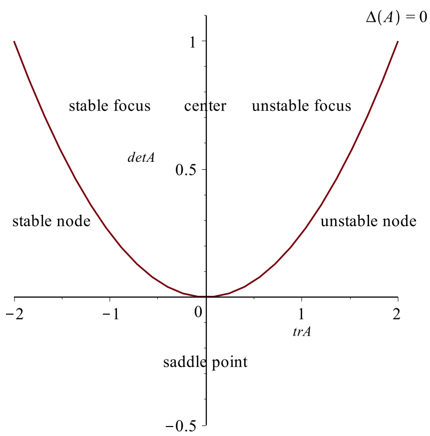

Figure 2.16: Types of phase diagrams in the trace-determinant space

2.5 Problems involving planar ODE's

As we saw all the solutions involve a vector of arbitrary elements of Y, $\mathbf{k}$ or $\mathbf{h}$. This means that we have existence but not uniqueness for **general** solutions.

In applications we introduce further information on the system. The type of **problem** involving planar ODE's depends on this additional information. We can define the following types of problems:

- if we know the initial point $\mathbf{y}(0) = \mathbf{y}_0 = (y_{1,0}, y_{2,0})$ and want to solve the problem forward in time, we say we have an **initial-value problem**;
- if we know the value of at least one variable at a point in time $T > 0$, $\mathbf{y}(T) = \mathbf{y}_T$, or $y_1(T) = y_{1,T}$, $y_2(T) = y_{2,T}$, we say we have a **boundary-value problem**;
- in economics a common problem is a mixes initial-terminal value problem, where we know the initial value for one variable and a boundary condition for the asymptotic value of another. Example: $y_1(0) = y_{1,0}$ and $\lim_{t\to\infty} e^{-\mu t} y_2(t) = 0$, where μ is a non-negative constant.

When the initial, boundary or terminal conditions are imposed we say we have **particular** solutions. Off course, the issues of existence, uniqueness and characterization still hold.

In economics it has been standard to refer to problems having an unique solution as **determinate** and to problems having multiple solutions as **indeterminate**.

2.5.1 Initial-value problems

Proposition 5. *Let $\mathbf{y}(0) = \mathbf{y}_0$ then the solution for the initial-value problem is unique*

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda t}}\mathbf{P}^{-1}(\mathbf{y}_0 - \overline{\mathbf{y}})$$

Proof. The general solution for a planar non-homogeneous equation is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda t}}\mathbf{k}.$$

As $\mathbf{e}^{\mathbf{\Lambda t}}|_{t=0} = \mathbf{I}$ then evaluating the solution at time $t = 0$, we have

$$\mathbf{y}(0) - \overline{\mathbf{y}} = \mathbf{P}\mathbf{k}$$

then, because $\mathbf{P}$ is non-singular

$$\mathbf{k} = \mathbf{P}^{-1}(\mathbf{y}(0) - \overline{\mathbf{y}})$$

Plugging the initial condition we have a particular value for $\mathbf{k}$

$$\mathbf{h} = \mathbf{P}^{-1}(\mathbf{y}_0 - \overline{\mathbf{y}}).$$

□

2.5.2 Terminal value problems

Proposition 6. *Consider the problem defined by planar non-homogeneous equation and the limiting constraint*

$$\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}} \in Y.$$

Then:

(1) if $\overline{\mathbf{y}}$ is a stable node or a stable focus then the solution is indeterminate

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \mathbf{P}\mathbf{e}^{\mathbf{\Lambda t}}\mathbf{k}$$

for any $\mathbf{k} = \mathbf{P}^{-1}(\mathbf{h} - \overline{\mathbf{y}}$ with $\mathbf{h} \in Y$;

(2) if $\overline{\mathbf{y}}$ is an unstable node or an unstable focus then the solution is determinate

$$\mathbf{y}(t) = \overline{\mathbf{y}},\ \textit{for all}\, t \in T$$

(3) if $\overline{\mathbf{y}}$ is a saddle-point then the solution is indeterminate

$$\mathbf{y}(t) = \overline{\mathbf{y}} + k_2\mathbf{P}^2 e^{\lambda_2 t}.$$

Proof. (1) If all the eigenvalues of $\mathbf{A}$ have negative real parts then

$$\lim_{t\to\infty} \mathbf{e}^{\mathbf{\Lambda t}} = \mathbf{I}_{2\times 2}$$

which implies $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{y}$ independently of the value of $\mathbf{h}$. (2) if all the eigenvalues of $\mathbf{A}$ have positive real parts then all the exponential functions $e^{\lambda_1 t}$, $e^{\lambda_2 t}$, $e^{\lambda t}$ or $e^{\alpha t}$ become unbounded, which means that we can only have $\lim_{t\to\infty} \mathbf{P}\mathbf{e}^{\mathbf{\Lambda t}}\mathbf{k} = \mathbf{0}$ if and only if $\mathbf{k} = \mathbf{0}$. Then as $\mathbf{k}$ is uniquely determined, the solution is unique. (3) If the steady state is a saddle point we know that the Jacobian form of $\mathbf{A}$ is $\mathbf{\Lambda}_1$, the solution takes the form

$$\mathbf{y}(t) = \overline{\mathbf{y}} + k_1\mathbf{P}^1 e^{\lambda_1 t} + k_2\mathbf{P}^2 e^{\lambda_2 t}$$

and $\lim_{t\to\infty} e^{\lambda_1 t} = +\infty$ and $\lim_{t\to\infty} e^{\lambda_2 t} = 0$. Therefore $\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}}$ if and only if $k_1 = 0$, and the solution is

$$\mathbf{y}(t) = \overline{\mathbf{y}} + k_2\mathbf{P}^2 e^{\lambda_2 t}.$$

□

2.5.3 Initial-terminal value problems

Proposition 7. *Consider the problem defined by planar non-homogeneous equation in which the steady state is a saddle point, the limiting constraint*

$$\lim_{t\to\infty} \mathbf{y}(t) = \overline{\mathbf{y}}$$

and the initial value $y_1(0) = y_{10}$ hold. Then the solution exists and is unique

$$\mathbf{y}(t) = \overline{\mathbf{y}} + \frac{(y_{1,0} - \overline{y}_1)}{P_1^2}\mathbf{P}^2 e^{\lambda_2 t}.$$

Proof. We can take the solution of case (3) of the terminal-valure problem and evaluate it at time $t = 0$ to get

$$\mathbf{y}(0) = \overline{\mathbf{y}} + k_2\mathbf{P}^2 \Leftrightarrow k_2\mathbf{P}^2 + \overline{\mathbf{y}} - \mathbf{y}(0) = \mathbf{0},$$

or, expanding and substituting the initial condition

$$\begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix} k_2 + \begin{pmatrix} \overline{y}_1 - y_{1,0} \\ \overline{y}_2 - y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As we want to solve this system for for $y_2(0) - \overline{y}_2$ and k_2 it is convenient to re-arrange it as

$$\begin{pmatrix} P_1^2 & 0 \\ P_2^2 & 1 \end{pmatrix} \begin{pmatrix} k_2 \\ \overline{y}_2 - y_2(0) \end{pmatrix} = \begin{pmatrix} y_{1,0} - \overline{y}_1 \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} k_2 \\ \overline{y}_2 - y_2(0) \end{pmatrix} &= \begin{pmatrix} P_1^2 & 0 \\ P_2^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1(0) - \overline{y}_1 \\ 0 \end{pmatrix} = \\ &= \frac{1}{P_1^2} \begin{pmatrix} 1 & 0 \\ -P_2^2 & P_1^2 \end{pmatrix} \begin{pmatrix} y_{1,0} - \overline{y}_1 \\ 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 \\ -P_2^2 \end{pmatrix} \frac{(y_{1,0} - \overline{y}_1)}{P_1^2}. \end{aligned}$$

□

In this case the initial value for $y_2(0)$ is determined

$$y_2(0) = \overline{y}_2 + \frac{P_2^2}{P_1^2}(y_{1,0} - \overline{y}_1)$$

where $\dfrac{P_2^2}{P_1^2}$ is the slope of $\mathcal{E}^2$ which is co-incident with the stable eigenspace $\mathcal{E}^s$.

Sometimes if we assume we know the initial value for variable y_2, $y_2(0) = y_{2,0}$ the difference $y_{2,0} - \left(\overline{y}_2 + \dfrac{P_2^2}{P_1^2}(y_{1,0} - \overline{y}_1)\right)$ is interpreted as the initial "jump" to the saddle path.

2.6 Applications in Economics

Types of variables and types of problems

1. macroeconomics pre rational expectations models: are usually initial-value problems in which the dynamic system is a for stable node or stable focus;

2. post rational expectations and DGE (dynamic general equilibrium) models: are usually initial-terminal value problems in which the dynamic system is a saddle point. This structure allows for the both forward (pre-determined) and backward (non-predetermined or expected) dynamics and for existence and uniqueness of DGE paths;

3. neo-Keynesian DGE models: are interested in cases in which for initial-terminal value problem in which the dynamic system can be a stable node or stable focus. This structure allows for the both forward (pre-determined) and backward (non-predetermined or expected) dynamics, for the existence of DGE paths but non necessarily for their uniqueness. If DGE paths are not unique the dynamics is said to be indeterminate, meaning that self-fulfilling prophecies are possible, and these are related with the existence of imperfections in the markets (externalities, incompleteness of contracts, policy rules, etc);

4. growth theory models: are usually initial or initial-terminal value problems in which there are no positively valued steady states or steady states are a degenerate node (with a zero and a positive eigenvalue). Two-dimensional endogenous growth models usually feature dynamic systems with a zero and a positive real eigenvalue which is associated with the existence of a balanced-growth path.

2.7 Bibliographic references

Mathematical textbooks: Hirsch and Smale (1974), (Hale and Koçak, 1991, ch 8) and Perko (1996)

Economics textbooks: on dynamical systems applied to economics (Gandolfo (1997), Tu (1994)), general mathematical economics textbooks with chapters on dynamic systems (Simon and Blume, 1994, ch. 24,25), de la Fuente (2000).

2.A Appendix

2.A.1 Review of matrix algebra

Consider matrix $\mathbf{A}$ of order 2 with real entries

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

that is $\mathbf{A} \in \mathbb{R}^{2\times 2}$. The **trace** and the **determinant** of $\mathbf{A}$ are, respectively,

$$\text{trace}(\mathbf{A}) = a_{11} + a_{22},\ \det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

The kernel (or null space) of matrix $\mathbf{A}$ is a vector $\mathbf{v}$ defined as

$$\text{kern}(\mathbf{A}) = \{\mathbf{v} : \mathbf{A}\mathbf{v} = \mathbf{0}\}$$

The dimension of the kernel gives a measure of the linear independence between the rows of $\mathbf{A}$.

The characteristic polynomial of matrix $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda \mathbf{I}_2) = \lambda^2 - \text{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) \tag{2.21}$$

where $\lambda \in \mathbb{C}$ is an eigenvalue, which is complex valued.

The **spectrum** of $\mathbf{A}$ is the **set of eigenvalues**

$$\sigma(\mathbf{A}) \equiv \{\lambda \in \mathbb{C} : \det(\mathbf{A} - \lambda \mathbf{I}_2) = 0\}$$

The **eigenvalues** of any 2×2 matrix $\mathbf{A}$ are

$$\lambda_1 = \frac{\text{trace}(\mathbf{A})}{2} + \Delta(\mathbf{A})^{\frac{1}{2}},\ \lambda_2 = \frac{\text{trace}(\mathbf{A})}{2} - \Delta(\mathbf{A})^{\frac{1}{2}} \tag{2.22}$$

where the discriminant is

$$\Delta(\mathbf{A}) \equiv \left(\frac{\text{trace}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A}).$$

A useful result on the relationship between the eigenvalues and the trace and the determinant of $\mathbf{A}$:

Lemma 3. *Let λ_1 and λ_2 be the eigenvalues of a 2×2 matrix $\mathbf{A}$. Then they are verify:*

$$\begin{aligned} \lambda_1 + \lambda_2 &= \textit{trace}(\mathbf{A}) \\ \lambda_1 \lambda_2 &= \det(\mathbf{A}). \end{aligned}$$

Three cases can occur:

1. if $\Delta(\mathbf{A}) > 0$ then λ_1 and λ_2 are real and distinct and $\lambda_1 > \lambda_2$
2. if $\Delta(\mathbf{A}) = 0$ then $\lambda_1 = \lambda_2 = \lambda = \text{trace}(\mathbf{A})/2$ are real and multiple,
3. if $\Delta(\mathbf{A}) < 0$ then λ_1 and λ_2 are complex conjugate $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ where $\alpha = \frac{\text{tr}(A)}{2}$ and $\beta = \sqrt{|\Delta(\mathbf{A})|}$ and $i = \sqrt{-1}$.

In the last case, we can write the eigenvalues in polar coordinates as

$$\lambda_1 = r(\cos\theta + \sin\theta i),\ \lambda_2 = r(\cos\theta - \sin\theta i)$$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\tan\theta = \beta/\alpha$, or

$$\alpha = r\cos\theta,\ \beta = r\sin\theta$$

Jordan canonical forms Two matrices $\mathbf{A}$ and $\mathbf{A}'$ with the equal eigenvalues are called **similar**. This allows for classifying matrices according to their eigenvalues.

The Jordan canonical forms for 2×2 matrices are

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},\ \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},\ \mathbf{\Lambda}_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{2.23}$$

Lemma 4 (Jordan canonical from of matrix $\mathbf{A}$)**.** *Consider any 2×2 matrix with real entries and its discriminant $\Delta(\mathbf{A})$. Then*

1. *If $\Delta(\mathbf{A}) > 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_1$.*
2. *If $\Delta(\mathbf{A}) = 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_2$.*
3. *If $\Delta(\mathbf{A}) < 0$ then the Jordan canonical form associated to $\mathbf{A}$ is $\mathbf{\Lambda}_3$.*

The Jordan canonical form $\mathbf{\Lambda}_3$ can also be represented by a diagonal matrix with complex entries

$$\mathbf{\Lambda}_3 = \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix}.$$

In this sense, if $\Delta(\mathbf{A}) \neq 0$ then matrix $\mathbf{A}$ is diagonalizable and it is not diagonalizable if $\Delta(\mathbf{A}) = 0$.

Figure 2.17 presents the different cases in a $(\text{trace}(\mathbf{A}), \det(\mathbf{A}))$ diagram. It has the following information:

- Jordan canonical forms are associated to the following areas: $\mathbf{\Lambda}_1$ is outside the parabola; $\mathbf{\Lambda}_3$ is inside the parabola, and $\mathbf{\Lambda}_2$ is represented by the parabola;

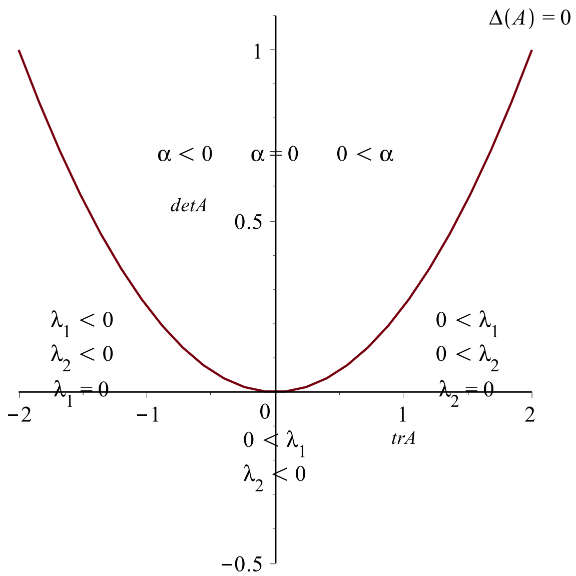

Figure 2.17: Eigenvalues of $\mathbf{A}$ in the trace-determinant space:

- in the positive orthant the two eigenvalues have positive real parts, in the negative orthant they have negative real parts and bellow the abcissa there are two real eigenvalues with opposite signs;
- the abcissa corresponds to the locus of points in which there is at least one zero-valued eigenvalue, the upper part of the ordinate corresponds to complex eigenvalues with zero real part, and the origin to the case in which there are two eigenvalues equal to zero.

Eigenvectors of A

Lemma 5. *Let* $\mathbf{A}$ *be a* 2×2 *matrix with real entries. Then, there exists a non-singular matrix* $\mathbf{P}$ *such that*

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$$

where $\mathbf{\Lambda}$ *is the Jordan canonical form of* $\mathbf{A}$*, and matrix* $\mathbf{P}$ *is a* 2×2 ***eigenvector*** *matrix associated to* $\mathbf{A}$.

There are two types of eigenvectors:

1. **simple eigenvectors** if $\Delta(\mathbf{A}) \neq 0$. In this case the eigenvector is $\mathbf{P} = (\mathbf{P}^1, \mathbf{P}^2)$ concatenating the eigenvectors $\mathbf{P}^1$ and $\mathbf{P}^2$ associated to the eigenvalues λ_1 and λ_2, which are obtained from solving the homogeneous system

$$(\mathbf{A} - \lambda_j \mathbf{I}_2)\mathbf{P}^j = 0, \ j = 1, 2$$

where $\mathbf{I}_2$ is the identity matrix of order 2. Observe that $\mathbf{P}^j = \text{kern}(\mathbf{A} - \lambda_j \mathbf{I}_2)$, i.e, it is the null space of matrix $(\mathbf{A} - \lambda_j \mathbf{I}_2)$;

2. **generalized eigenvectors** if $\Delta(\mathbf{A}) = 0$, that is, when we have multiple eigenvalues $\lambda_1 = \lambda_2 = \lambda$. In this case we determine $\mathbf{P} = (\mathbf{P}^1, \mathbf{P}^2)$ where $\mathbf{P}^1$ is a simple eigenvalue and $\mathbf{P}^2$ is a generalized eigenvalue. They are obtained in the following way: first, $\mathbf{P}^1$ solves $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P}^1 = 0$, where $\mathbf{I} = \mathbf{I}_2$; second, (a) if $(\mathbf{A} - \lambda \mathbf{I})^2 \neq \mathbf{0}$ we determine $\mathbf{P}^2$ from $(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{P}^2 = 0$; however, (b) if $(\mathbf{A} - \lambda \mathbf{I})^2 = \mathbf{0}$ then we determine $\mathbf{P}^2$ from $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P}^2 = \mathbf{P}^1$.

When $\Delta(\mathbf{A}) < 0$ we can use one of the following two approaches:

1. either we write the Jordan matrix as a complex-valued matrix

$$\Lambda_3 = \begin{pmatrix} \alpha + \beta i & 0 \\ 0 & \alpha - \beta i \end{pmatrix}$$

   and compute $\mathbf{P}^j$ as a complex-valued vector from

$$(\mathbf{A} - \lambda_j \mathbf{I}_2)\mathbf{P}^j = 0,$$

2. or we write the Jordan matrix as a real-valued matrix as in equation (2.23) and compute $\mathbf{P}$ as a real-valued matrix by setting $\mathbf{P} = (\mathbf{u}, \mathbf{v})$ where $\mathbf{Q} = \mathbf{u} + \mathbf{v}i$ is the solution of the homogeneous system

$$(\mathbf{A} - (\alpha + \beta i)\mathbf{I}_2)\mathbf{Q} = 0$$

Conclusion: given a matrix $\mathbf{A}$, we can find matrices $\mathbf{\Lambda}$ and $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\Lambda\mathbf{P}^{-1}$ where $\mathbf{P}$ is invertible. Equivalently $\Lambda = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Proposition 8. *The eigenvector matrices associated to the Jordan canonical forms are:*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{2.24}$$

for $\Lambda = \mathbf{\Lambda}_1$, $\mathbf{\Lambda} = \mathbf{\Lambda}_2$ *and* $\mathbf{\Lambda} = \mathbf{\Lambda}_3$, *respectively*

Proof. For $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, because $(\mathbf{\Lambda}_1 - \lambda_1 \mathbf{I})\mathbf{P}^1 = 0$ and $(\mathbf{\Lambda}_1 - \lambda_2 \mathbf{I})\mathbf{P}^2 = 0$ are

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 - \lambda_1 \end{pmatrix} \begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

then we get $\mathbf{P} = (\mathbf{P}^1\mathbf{P}^2) = \mathbf{I}$, because $\lambda_1 \neq \lambda_2$. For $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_2$ we determine the simple eigenvector from $(\boldsymbol{\Lambda}_2 - \lambda\mathbf{I})\mathbf{P}^1 = 0$. To determine the second eigenvector as $(\Lambda_2 - \lambda\mathbf{I})^2 = \mathbf{0}$, because

$$(\Lambda_2 - \lambda\mathbf{I})^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then we use $(\boldsymbol{\Lambda}_2 - \lambda\mathbf{I})\mathbf{P}^2 = \mathbf{P}^1$,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1^2 \\ P_2^2 \end{pmatrix} = \begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix},$$

to get $\mathbf{P}^1 = (1, 0)$ and $\mathbf{P}^2 = (1, 1)$.

For $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_3$ consider eigenvalue $\lambda_1 = \alpha + \beta i$ and assume that there is a complex vector

$$\mathbf{z} = \begin{pmatrix} u_1 + v_1 i \\ u_2 + v_2 i \end{pmatrix}$$

that solves $(\Lambda_3 - (\alpha + \beta i)I)\mathbf{z} = 0$, that is [2]

$$\begin{cases} \beta\,(u_2 + v_1 + (v_2 - u_1)i) & = 0 \\ \beta\,((v_2 - u_1) - (u_2 + v_1)i) & = 0 \end{cases}$$

then we should have $u_1 = v_2$ and $u_2 = -v_1$. We can arbitrarily set $u_1 = 1$ and $v_1 = 1$, in $\mathbf{P}^1 = (u_1, u_2)^\top$ and $\mathbf{P}_2 = (v_1, v_2)^\top$, to get the third eigenvector matrix. $\square$

Eigenspaces As matrix $\mathbf{P}$ is non singular it forms a basis for vector space $\mathbf{A}$. Then vector space $\mathbf{A}$ can be seen as a direct sum $\mathbf{A} = \mathcal{E}^1 \oplus \mathcal{E}^2$ where

$$\begin{aligned} \mathcal{E}^1 &= \{\text{eigenspace associated with } \lambda_1\} \\ \mathcal{E}^2 &= \{\text{eigenspace associated with } \lambda_2\}. \end{aligned}$$

2.A.2 Polar coordinates

When the eigenvalues are complex (or the model is non-linear) sometimes we can simplify the solution and get a better geometrical intuition of it, if we transform the ODE from cartesian coordinates $(y_1, y_2) \in \mathbb{R}$ into polar coordinates (r, θ) by using the transformation:

$$y_1(t) = r(t)\cos(\theta(t)),\ y_2(t) = r(t)\sin(\theta(t)).$$

[2] We use the rules for sums and multiplications of complex numbers: if $x_1 = a_1 + b_1 i$ and $x_2 = a_2 + b_2 i$, then $x_1 + x_2 = (a_1 + a_2) + (b_1 + b_2)i$ and $x_1 x_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$ because $i^2 = -1$.

where r measures the distance from a reference point (the radius) and θ the angular coordinate.

The following relationships hold $r^2 = y_1^2 + y_2^2$, because $\cos(\theta)^2 + \sin(\theta)^2 = 1$ and $\tan(\theta) = \sin(\theta)/\cos(\theta) = y_2/y_1$. If we take time derivatives of this two relationships we find

$$r' = \frac{y_1\dot{y}_1 + y_2\dot{y}_2}{r}$$
$$\theta' = \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{r^2}$$

Exercise: provide a proof (hint $d(\tan(\theta(t))/dt = (1 + \tan(\theta)^2)\theta' = (1 + (y_2/y_1)^2)\theta'$.

In order to apply this transformation, consider the ODE

$$\dot{y}_1 = \alpha y_1 + \beta y_2$$
$$\dot{y}_2 = -\beta y_1 + \alpha y_2$$

The ODE in polar coordinates becomes

$$r' = \alpha r$$
$$\theta' = -\beta$$

which has the general solution

$$r(t) = r_0 e^{\alpha t}$$
$$\theta(t) = \theta_0 - \beta t$$

If $\alpha < 0$ the radius converges to zero (meaning that the the dynamics is stable) and if $\theta > 0$ the movement is clockwise.

2.A.3 Second order equations

Consider a general second order equation.

$$\ddot{y} - a_1\dot{y} + a_0 y = 0$$

If we define $y_1 = y$ and $y_2 = \dot{y} = \dot{y}_1$, then, we can transform the equation into the system

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = a_0 y_1 + a_1 y_2.$$

Bibliography

de la Fuente, A. (2000). *Mathematical Methods and Models for Economists.* Cambridge University Press.

Gandolfo, G. (1997). *Economic Dynamics.* Springer-Verlag.

Hale, J. and Koçak, H. (1991). *Dynamics and Bifurcations.* Springer-Verlag.

Hirsch, M. and Smale, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press.

Perko, L. (1996). *Differential Equations and Dynamical Systems, 2nd Ed.* Springer-Verlag.

Simon, C. P. and Blume, L. (1994). *Mathematics for Economists.* W. W. Norton & Co., New York.

Tu, P. (1994). *Dynamical Systems, 2nd ed.* Springer-Verlag.