Advanced Mathematical Economics

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Part III

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Non-linear differentiable ODE

Non-linear ordinary differential equations in the normal form are defined by

$$y^{'}(x) = f(y, x),\ y : X \subseteq \mathbb{R} \to Y \subseteq \mathbb{R}^n,\ f : X \times Y \to Y$$

where $f(.)$ is a non-linear function of y. Again we have an **autonomous ODE** if function $f(.)$ is independent of the independent variable x, i.e., it is of type $y^{'}(x) = f(y)$. Again, if y is of dimension one, i.e., if $n = 1$ we say it is a **scalar** ODE, if $n = 2$ it is a **planar** ODE, and if $n > 2$ it is a **higher-dimensional** ODE.

As we want to focus on the qualitative properties of the solutions, in particular on qualitative dynamics, we will consider next the case in which the independent variable is time.

If the independent variable is time, the autonomous ODE is written as

$$\dot{y} = f(y),\ y : T \subseteq \mathbb{R}_+ \to Y \subseteq \mathbb{R}^n,\ f : Y \to Y.$$

In general, the non-linear ODE that interests us depends on a parameter or a vector of parameters $\varphi \in \Phi$ is

$$\dot{y} = f(y, \varphi),\ f : Y \times \Phi \to Y. \tag{4.1}$$

In equation (4.1) function $f(.)$ can be **exactly** or just **qualitatively** specified, which is the case in some in economic theory models in which only assumptions on the slope and/or curvature properties are introduced.

We assume in this chapter that $f(.)$ **is continuous and differentiable** (i.e, all its derivatives are finite). In this case it can be proved that *a solution exists, and is unique.* [1] For initial-value problems the solution is also continuous in time.

[1] See Coddington and Levinson (1955), Hartman (1964) and many others.

There are several new aspects introduced by non-linearity when comparing with linear ODE's. First, **most non-linear ODE's do not have a closed form solution**. If an ODE has a closed form solution we can characterise completely its solution. However, if the ODE is not completely specified or a solution is not known, which can be a consequence of the fact that a solution in terms of known functions does not exist, we can still characterise the solution qualitatively. [2] Second, differently from the linear case **there may be a difference between the local and the global properties of the solution** (i.e., the local behavior of the solution may be different at different points of the range set Y). The qualitative (or geometric) study of the solutions of non-linear equations is based upon finding topological equivalence with linear equations or to some non-linear equations with known solutions called **normal forms**. When there is topological equivalence with linear ODE's the local and global properties of a non-linear ODE are the same, but when the topological equivalence is with some non-linear normal form, the local and global properties of the solutions are different.

Qualitatively specified ODE's can only have non-explicit solutions but exactly specified ODE's can have either explicit or non-explicit solutions. In all those cases, we usually need to characterize the qualitative properties of the solutions.

At least for ODEs in which the state space Y is of dimension equal or smaller than two the modern approach to dealing with ODEs emphasises its geometry. A **phase diagram** represents the geometry of the solution on the space Y for a given value of the parameter(s): it is characterized by the number of steady states, their local dynamics and other types of global trajectories (v.g., closed orbits).

The qualitative (or geometrical) theory of ODE's explores that topological equivalence allowing for the characterization of the solution of non-linear ODE's. It consists in the application of three important results:

- the **Grobmann-Hartman theorem**: stating the conditions for the qualitative equivalence between non-linear and linear ODEs;
- the **Poincaré-Bendixon theorem**: associated to the existence of other invariants different from fixed points, that is closed orbits;
- several **bifurcation theorems**, stating the qualitative change of the solutions when some critical parameter or parameters vary.

One important feature of the dynamic analysis is related to the concept of stability of a steady state. We say a steady state $\bar{y}$ is **asymptotically stable** if given an initial value for y, $y(0)$,

[2]There are several handbooks with closed form solutions for non-linear ODE's as Zwillinger (1998), Zaitsev and Polyanin (2003) or Canada et al. (2004). At present most symbolic manipulation software, as Mathematica, Maple, Maxima, the library Sympy of Python have libraries with exact solutions for ODEs

$\lim_{t\to\infty} y(t) = \bar{y}$. The steady state is **locally asymptotically stable** if $y(0)$ is required to be in a small neighborhood of $\bar{y}$ and it is **globally asymptotically stable** if $y(0) \in Y$ can be any value. A fixed point is **stable** if $y(0)$ is in a neighborhood of $\bar{y}$, $||y(0) - \bar{y}|| < \delta$ there is a neighbohood ϵ such that $||y(t) - \bar{y}|| < \epsilon$. That is, if we start close to a steady state we will stay close for any point in time. A steady state is **unstable** if it is not stable: if $y(0)$ is close to a steady state $y(t)$ will not stay close. We say a steady state is **neither stable nor unstable** if depending on the neighborhood of $\bar{y}$ to which $y(0)$ belongs $y(t)$ can converge or not to $\bar{y}$.

For any type of non-linear ODE (explicit or not and with or without explicit solutions) we can characterise the **local dynamics** by the following sequence of steps:

1. first, determine the existence and number of steady states (or time-independent solutions) or of other invariant solutions;

2. second, determine the stability properties for every steady state by linearizing function $f(y)$ in the neighbourhood of every steady state: i.e., by approximating locally a linear equation of type
$$\dot{y} = D_y f(\bar{y}, \varphi)(y - \bar{y})$$
where $\bar{y} \in \{y \in Y : f(y, \varphi) = 0\}$ and $D_y f(\bar{y}, \varphi)$ is the Jacobian of $f(y, \cdot)$ evaluated at the steady state $\bar{y}$. In some cases, some **global dynamics** properties not existing in linear models (heteroclinic and homoclinic trajectories, limit cycles, for instance), can also be identified. We will see that, when $D_y f(\bar{y}, \varphi) = 0$ by Taylor expanding in higher order terms v.g [3],
$$\dot{y} = D_y f(\bar{y}, \varphi)(y - \bar{y}) + \frac{1}{2} D_y^2 f(\bar{y}, \varphi)(y - \bar{y})^2 + o((y - \bar{y})^2),$$
we can find topological equivalente with non-linear normal forms;

3. at last, determining the existence of critical values of the parameters, or bifurcation points. Usually, not only the number and the magnitude of the steady states, but also their dynamic properties, depend on the value of the parameters. We say that the tuple $(\bar{y}(\varphi_0), \varphi_0)$ is a **bifurcation** if introducing a small quantitative change in φ the characteristics of the phase-diagram change qualitatively. There are, again, both **local but also global bifurcations**. A **bifurcation diagram**, plotting $(\bar{y}(\varphi), \varphi)$ for all values $\varphi \in \Phi$, with a reference to the stability properties, is a useful device for conducting bifurcation analysis.

We start by presenting the normal forms for scalar and for some planar non-linear ODE's and next present the main results from the qualitative theory of ODE's.

[3]The rest, $R(y - \bar{y})$, if it is of order $o(y - \bar{y})^2$ in a weak sense, means that $\lim_{y\to\infty} \dfrac{R(y - \bar{y})}{(y - \bar{y})^2} = 0$. See the appendix for the definition of the little-o notation.

4.1 Normal forms

A **normal form** is the simplest ODE, whose exact solution is usually known, which represents a whole family of ODEs by topological equivalence. [4] The simplest case of a normal form is a linear ODE, scalar or planar. It is locally or globally topological equivalent to any ODE which does not have a steady state with a Jacobian with eigenvalues with zero real parts or whose Jacobian, evaluated at any point $y \in Y$ has no singularities.

However, the term normal form is usually reserved to ODE's which are topologically equivalent to ODE's having a polynomial function $f(y)$. We present next the most common normal forms for scalar and planar ODEs.

4.1.1 Scalar ODE's

For the scalar case we have (see (Hale and Koçak, 1991, ch. 2)) two quadratic equations with a single bifurcation parameter (a), the Ricatti's equation $\dot{y} = a + y^2$ and a quadratic Bernoulli equation, $\dot{y} = ay + y^2$, and three cubic equations with one (a) or two bifurcation parameters (a and b): the cubic Bernoulli equation, $\dot{y} = ay - y^3$ and two Abel's equations, $\dot{y} = a + y - y^3$ and $\dot{y} = a + by - y^3$.

For each equation we present: (1) the closed form solution (in most cases) and characterize it; (2) the steady states; and (3) the bifurcation points. Those equations are usually named after the particular bifurcations that they generate. We will also present the relevant bifurcation diagrams.

We assume next that $y : \mathbb{R}_+ \to Y \subseteq \mathbb{R}$, and the parameters are real numbers.

The Ricatti's equation: saddle-node or fold bifurcation The quadratic equation

$$\dot{y} = a + y^2 \tag{4.2}$$

is called Ricatti's equation. It has an explicit solution[5] :

$$y(t) = \begin{cases} -\frac{1}{t+k}, & \text{if } a = 0 \\ \sqrt{a}\left(\tan\left(\sqrt{a}(k+t)\right)\right), & \text{if } a > 0 \\ -\sqrt{-a}\left(\tanh\left(\sqrt{-a}(k+t)\right)\right), & \text{if } a < 0 \end{cases}$$

where k is an arbitrary constant belonging to the domain of y, y.

The behavior of the solution is the following:

[4] In heuristic terms, we say functions $f(y)$ and $g(x)$ are topologically equivalent if there exists a diffeomorphism, i.e., a smooth map h with a smooth inverse h^{-1}, such that if $y = h(x)$ then $h(g(x)) = f(h(x))$. This property may hold globally of locally. The last case is the intuition behind the Grobman-Hartmann theorem.

[5] See appendix section 4.A

- if $a = 0$, the solution takes an infinite value at a finite time $t = -k$ [6], i.e., $\lim_{t\to -k} y(t) = \pm\infty$ and tends asymptotically to a steady state $\bar{y} = 0$, that is $\lim_{t\to\infty} y(t) = 0$ independently of the value of k;

- if $a > 0$ the solution takes infinite values for a periodic sequence of times $t \in \{-k, \pi - k, 2\pi - k, \ldots, n\pi - k, \ldots\}$,
$$\lim_{t\to n\pi - k} y(t) = \pm\infty, \text{ for } n \in \mathbb{N}$$
and it has no steady state;

- if $a < 0$, the solution converges to
$$\lim_{t\to\infty} y(t) = \begin{cases} -\sqrt{-a}, & \text{if } k < \sqrt{-a} \text{ or } -\sqrt{-a} < k < \sqrt{-a} \\ +\infty, & \text{if } k > \sqrt{-a}. \end{cases}$$

Therefore the dynamic properties depend on the value of a:

- existence and number of steady states: $\bar{y} = \{y : \ a + y^2 = 0\}$: if $a > 0$ there are no steady states, if $a = 0$ there is one steady state $\bar{y} = 0$, and if $a < 0$ there are two steady states $\bar{y} \in \{-\sqrt{-a}, \sqrt{-a}\}$;

- local dynamics at a steady state: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable and if $a < 0$ steady state $\bar{y} = -\sqrt{-a}$ is asymptotically stable and steady state $\bar{y} = \sqrt{-a}$ is unstable. This is because $f_y(y) = 2y$ then $f_y(0) = 0$, $f_y(-\sqrt{a}) = -2 - \sqrt{a} < 0$, and $f_y(\sqrt{a}) = 2 - \sqrt{a} > 0$. If $a < 0$ the basin of attraction, or stable manifold associated to steady state $\bar{y} = -\sqrt{-a}$, is
$$\mathcal{W}^s_{-\sqrt{-a}} = \{y \in Y : y < \sqrt{-a}\}.$$
Comparing to the linear case, for the case in which the steady state is asymptotically stable, the stable manifold is a subset of $\mathcal{Y}$ not the whole $\mathcal{Y}$.

There is a bifurcation point at $(y, a) = (0, 0)$, which is called **saddle-node bifurcation**. We find the bifurcation point by solving, jointly to (y, a) the system
$$\begin{cases} f(y, a) = 0 \\ f_y(y, a) = 0 \end{cases} \Leftrightarrow \begin{cases} a + y^2 = 0 \\ 2y = 0. \end{cases}$$

[6] This is different to the linear case, v.g., $\dot{y} = y$ which, if $y(0) \neq 0$, whose solution $y(t) = y(0)e^t$ takes an infinite value only in infinite time.

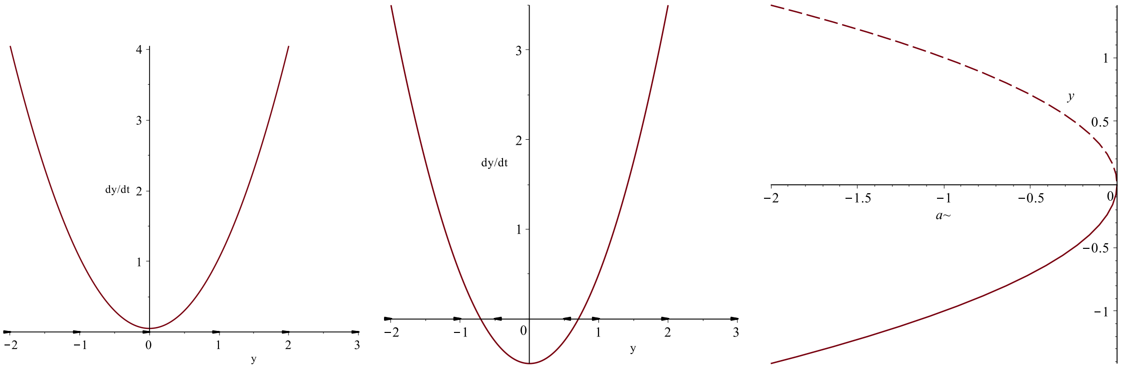


Figure 4.1: Phase diagrams for $a > 0$, and $a < 0$ and bifurcation diagram for equation (4.2)

Figure 4.1 shows phase diagrams for the $a > 0$ (left sub-figure) and for the $a < 0$ (center sub-figure) cases and the ifurcation diagram (right sub-figure). In the bifurcation diagram we depict points (a, y) such that $a + y^2 = 0$, say $\bar{y}(a)$, and in full-line the subset of points such that $f'(\bar{y}(a)) < 0$ and in dashed-line the subset of points such that $f'(\bar{y}(a)) > 0$. The first case corresponds to asymptotically stable steady states and the second to unstable steady states. Observe that the curve does not lie in the positive quadrant for a which is the geometrical analogue to the non-existence of steady states. The saddle-node bifurcation point is at the origin $(0, 0)$.

Quadratic Bernoulli equation: transcritical bifurcation The equation

$$\dot{y} = ay + y^2 \tag{4.3}$$

is a particular case of the Bernoulli's equation $\dot{y} = ay + by^\eta$, for a real number η, and also has an explicit solution [7]:

$$y(t) = \begin{cases} \frac{1}{1/k - t}, & \text{if } a = 0 \\ \frac{a}{(1 + a/k)e^{-at} - 1}, & \text{if } a \neq 0 \end{cases}$$

where k is an arbitrary element of y.

The behavior of the solution is the following:

- if $a > 0$

$$\lim_{t \to \infty} y(t) = \begin{cases} -a, & \text{if } k < 0 \\ +\infty, & \text{if } k > 0 \end{cases}$$

- if $a = 0$, it behaves as the Ricatti's equation when $a = 0$

[7] See appendix section 4.B for the explicit solution for the general Bernoulli ODE.

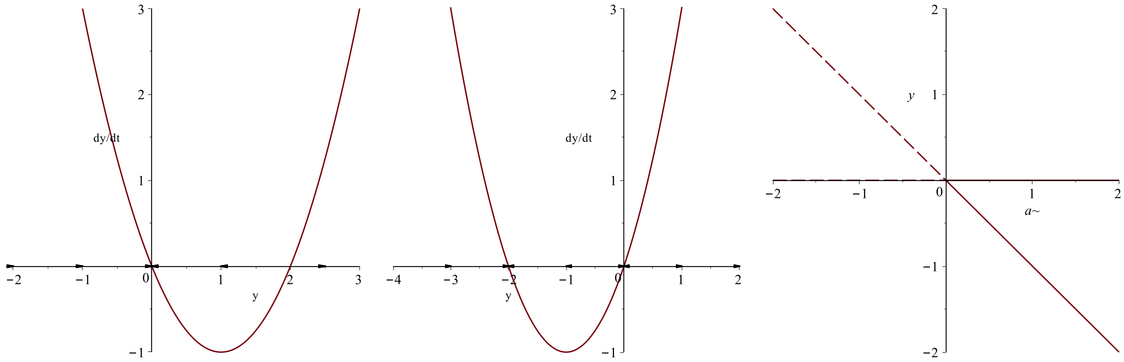


Figure 4.2: Phase diagrams for $a < 0$, and $a > 0$ and bifurcation diagram for equation (4.3)

- if $a < 0$,

$$\lim_{t\to\infty} y(t) = \begin{cases} 0, & \text{if } k < -a \\ +\infty, & \text{if } k > -a \end{cases}$$

The dynamic properties depend on the value of a:

- existence and number of steady states: if $a = 0$ there is one steady state $\bar{y} = 0$, if $a \neq 0$ there are two steady states $\bar{y} = \{0, -a\}$;

- local dynamics at the steady states: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable; if $a < 0$ steady state $\bar{y} = 0$ is asymptotically stable and steady state $\bar{y} = -a$ is unstable; and if $a > 0$ steady state $\bar{y} = 0$ is unstable and steady state $\bar{y} = -a$ is asymptotically stable. The stable manifolds associated to the asymptotically stable equilibrium points are: if $a < 0$

$$\mathcal{W}_0^s = \{y \in Y : y < -a\}.$$

  and, if $a < 0$,

$$\mathcal{W}_{-a}^s = \{y \in Y : y < 0\}.$$

There is a bifurcation point at $(y, a) = (0, 0)$, which is called **transcritical bifurcation**. Figure 4.2 shows two phase diagrams and the bifurcation diagram.

Bernoulli's cubic equation: subcritical pitchfork The equation

$$\dot{y} = ay - y^3 \tag{4.4}$$

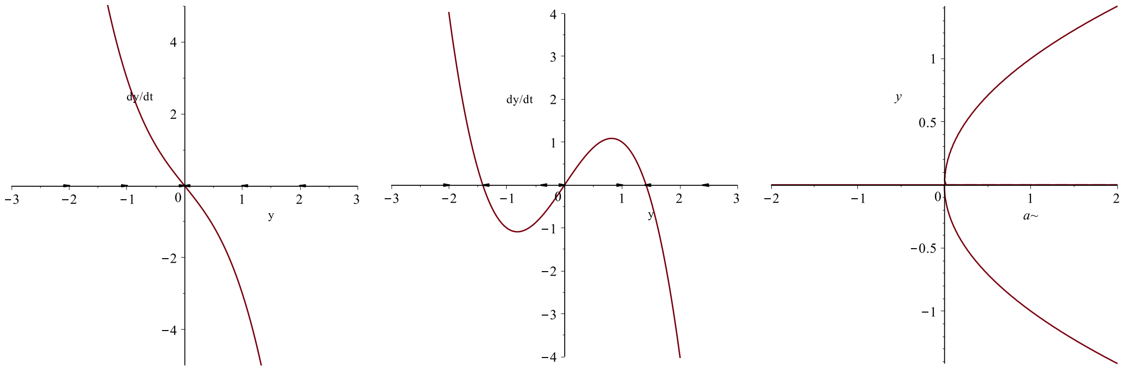


Figure 4.3: Phase diagrams for $a < 0$, and $a > 0$ and bifurcation diagram for equation (4.4)

is also a Bernoulli equation and also has also an explicit solution:

$$y(t) = \pm\sqrt{a}\left[1 - \left(1 - \frac{a}{k^2}\right)e^{-2at}\right]^{-1/2}$$

where k is an arbitrary element of y. The solution trajectories have the following properties for different values of the parameter a:

- if $a \leq 0$, $\lim_{t\to\infty} y(t) = 0$
- if $a > 0$,

$$\lim_{t\to\infty} y(t) = \begin{cases} -\sqrt{a}, & \text{if } k < 0 \\ \sqrt{a}, & \text{if } 0 < k < \sqrt{a} \\ +\infty, & \text{if } k > \sqrt{a} \end{cases}$$

The dynamic properties depend on the value of a:

- existence and number of steady states: there is one steady state $\bar{y} = 0$ and if $a \leq 0$ and there are three steady states $\bar{y} = \{0, -\sqrt{a}, \sqrt{a}\}$ if $a > 0$;
- local dynamics at the steady states: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable; if $a < 0$ steady state $\bar{y} = 0$ is asymptotically stable; and if $a > 0$ steady state $\bar{y} = 0$ is unstable and the other two steady states $\bar{y} = -\sqrt{a}$ and $\bar{y} = \sqrt{a}$ are asymptotically stable.

There is a bifurcation point at $(y, a) = (0, 0)$, which is called **subcritical pitchfork**. Figure 4.3 shows two phase diagrams and the bifurcation diagram.

Exercise: Study the solution for equation $\dot{y} = ay + y^3$. Show that point $(y, a) = (0, 0)$ is also a bifurcation point called **supercritical pitchfork**.

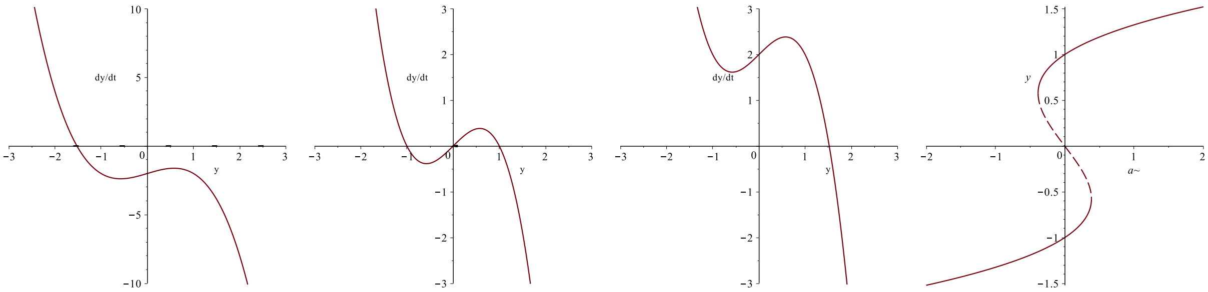


Figure 4.4: Phase diagrams for $a < \frac{2}{3}\sqrt{\frac{1}{3}}$, for $a > \frac{1}{3}$ and for an intermediate value and bifurcation diagram for equation (4.5)

Abel's equation: hysteresis The equation

$$\dot{y} = a + y - y^3 \tag{4.5}$$

is called an Abel equation of the first kind. Although closed form solutions have been found recently [8] they are too cumbersome to report. If $a = 0$ the Abel's equation reduces to a particular Bernoulli's equation (4.4) $\dot{y} = y - y^3$.

Equation (4.5) can have one, two or three equilibrium points, which are the real roots of the polynomial equation $f(y, a) \equiv a + y - y^3 = 0$.

We can determine bifurcation points in the space $\mathcal{Y} \times \Phi$ by solving for (y, a)

$$\begin{cases} f(y, a) = 0, \\ f_y(y, a) = 0. \end{cases}$$

Because

$$\begin{cases} a + y - y^3 = 0, \\ 1 - 3y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3(a + y) - 3y^3 = 0, \\ y - 3y^3 = 0, \end{cases} \Leftrightarrow \begin{cases} 3a + 2y = 0 \\ y = \pm\sqrt{1/3}, \end{cases}$$

we readily find that the ODE (4.5) has two critical points, called **hysteresis** points:

$$(y, a) = \left\{ \left(-\sqrt{\frac{1}{3}}, \frac{2}{3}\sqrt{\frac{1}{3}} \right), \left(\sqrt{\frac{1}{3}}, -\frac{2}{3}\sqrt{\frac{1}{3}} \right) \right\}.$$

By looking at figure 4.4 (to the right sub-figure) we see that:

- for $a > \frac{2}{3}\sqrt{\frac{1}{3}}$ or for $a < -\frac{2}{3}\sqrt{\frac{1}{3}}$ there is one asymptotically stable steady state

[8]For known closed form solutions of ODEs see, Zaitsev and Polyanin (2003) or Zwillinger (1998).

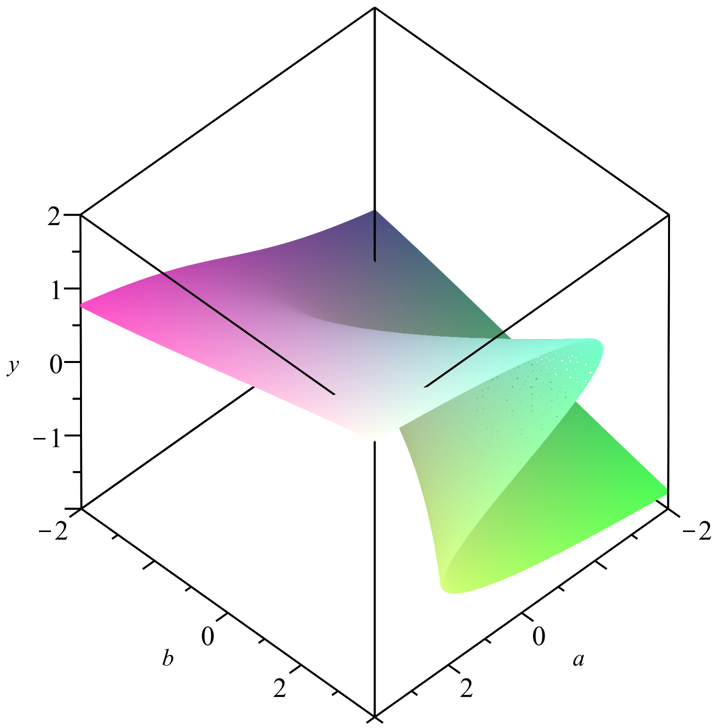


Figure 4.5: Bifurcation diagram for equation $\dot{y} = a + by - y^3$

- for $a = \frac{2}{3}\sqrt{\frac{1}{3}}$ there are two steady states: one asymptotically stable equilibrium and a bifurcation point for $\bar{y} = \sqrt{\frac{1}{3}}$, for $a = -\frac{2}{3}\sqrt{\frac{1}{3}}$ there are two steady states: one asymptotically stable equilibrium and a bifurcation point for $\bar{y} = -\sqrt{\frac{1}{3}}$
- for $-\frac{2}{3}\sqrt{\frac{1}{3}} < a < \frac{2}{3}\sqrt{\frac{1}{3}}$ there are three steady states, two asymptotically stable (the extreme ones) and one unstable (the middle one)

Cubic equation: cusp The equation

$$\dot{y} = f(y, a, b) \equiv a + by - y^3 \tag{4.6}$$

is also an Abel equation of the first kind. Observe that we have now two parameters, a and b, that also allow for critical changes of its solution.

This ODE can have one, two or three equilibrium points, depending on the values of the parameters a and b. We can determine them by solving the cubic polynomial equation $a+by-y^3 = 0$

(see appendix section 4.C). Calling discriminant to

$$\Delta \equiv \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3; \tag{4.7}$$

it can be proven that: if $\Delta < 0$ there are three steady states, if $\Delta = 0$ there are two steady states, and if $\Delta > 0$ there is one steady state.

We can determine critical points (co-dimension one bifurcation points) by solving the system:

$$\begin{cases} f(y, a, b) = 0 \\ f_y(y, a, b) = 0. \end{cases} \tag{4.8}$$

Applying to equation (4.6) we have

$$\begin{cases} a + by - y^3 = 0, \\ b - 3y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3a + 2by = 0, \\ b - 3y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3a + 2by = 0 \\ 2b^2 + 9ay = 0, \end{cases} \Leftrightarrow \begin{cases} 27a^2 + 18aby = 0 \\ 4b^3 + 18aby = 0. \end{cases}$$

The solutions to the system must verify

$$18aby = -12a^2 = -4b^3 \Leftrightarrow \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3 = 0$$

that is $\Delta = 0$. Function $f(y, a, b) = 0$ defines a surface in the surface in the three-dimensional space for (a, b, y) called **cusp** which is depicted in Figure 4.9 [9]. Because we have two parameters, the bifurcation loci, obtained from system (4.8) defines a line in the three-dimensional space (a, b, y). We can see how it changes by imagining horizontal slices in Figure 4.9 and project them in the (a, b)-plane. This would convince us that if $a = 0$ we would get the bifurcation diagram for the pitchfork, for equation (4.4), and if $a \neq 0$ and $b = 1$ we obtain the hysteresis diagram, for equation (4.5). This result would be natural because those two equations are a particular case of the cusp equation.

By solving the system

$$\begin{cases} f(y, a, b) = 0 \\ f_y(y, a, b) = 0 \\ f_b(y, a, b) = 0 \end{cases}$$

we find a bifurcation point $(y, a, b) = (0, 0, 0)$ corresponding to a bifurcation for a higher level of degeneracy (co-dimension two bifurcation points).

[9] This was one of the famous cases of catastrophe theory very popular in the 1980's see `https://en.wikipedia.org/wiki/Catastrophe_theory`.

4.1.2 Planar ODE's

Next we consider the planar ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi)$, in vector notation, $\mathbf{y} : T \to Y \subseteq \mathbb{R}^2$, depending on a vector of parameters, $\varphi \in \mathbb{R}^n$ for $n \geq 1$. Expanding, we have,

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \varphi) \\ \dot{y}_2 &= f_2(y_1, y_2, \varphi) \end{aligned} \tag{4.9}$$

There are a large number of normal forms that have been studied for planar ODEs (see Kuznetsov (2005).

In principle, we could consider combining all the previous scalar normal forms to have an idea of the number of possible cases, and extend the previous method to study the dynamics. That method consisted in finding critical points, corresponding to steady states and values of the parameters such that the derivatives of the steady variables would be equal to zero. However, for planar equation, to fully characterise the dynamics, we may have to study local dynamics in invariant orbits other than steady states. In general there are, at least, three types of **invariant orbits** that do not exist in planar linear models: homoclinic and heteroclinic orbits and limit cycles.

In the next section we present a general method to finding bifurcation points associated to steady states. In the rest of this section we presents ODE's in which those invariant curves exist and are generic (in the sense that they hold for any values of a parameter, except for some particular values) and non-generic. The non-generic cases consist in one-parameter bifurcations for non-linear planar equations associated to heteroclinic and homoclinic orbits and limit cycles.

Heteroclinic orbits We say there is an **heteroclinic orbit** if, in a planar ODE in which there are at least two steady states, say $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$, and there are solutions $\mathbf{y}(t)$ that entirely lie in a curve joining $\bar{\mathbf{y}}^1$ to $\bar{\mathbf{y}}^2$ say $\mathrm{Het}(\mathbf{y})$. Therefore, if $\mathbf{y}(0) \in \mathrm{Het}(\mathbf{y})$ then $\mathbf{y}(t) \in \mathrm{Het}(\mathbf{y})$ for $t > 0$ and either $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^1$ and $\lim_{t\to-\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^2$ or $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^2$ and $\lim_{t\to-\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^1$. Heteroclinics can exist if the stability type of the steady states are different or equal. In the first case, they connect stable or unstable node and a saddle point or a stable and one unstable node. In the last case, the only possibility is if the two steady states are saddle points and we say we have a **saddle connection**.

Heteroclinic networks can also exist when there are more than two steady states which are connected.

Generic heteroclinic orbits Although there are several normal forms generating generic heteroclinic orbits, we focus next in the following case:

$$\begin{aligned} \dot{y}_1 &= a y_1 y_2 \\ \dot{y}_2 &= 1 + y_1^2 - y_2^2 \end{aligned} \tag{4.10}$$

where $a \neq 0$. This equation has two steady states: $\bar{\mathbf{y}}^1 = (0, -1)$ and $\bar{\mathbf{y}}^2 = (0, 1)$. Calling,

$$\mathbf{f}(\mathbf{y}) = \begin{pmatrix} a y_1 y_2 \\ 1 + y_1^2 - y_2^2 \end{pmatrix}$$

we have the Jacobian

$$J(\mathbf{y}) \equiv D_{\mathbf{y}} \mathbf{f}(\mathbf{y}) = \begin{pmatrix} a y_2 & a y_1 \\ 2 y_1 & -2 y_2 \end{pmatrix},$$

which has trace and determinant depending on the parameter a

$$\begin{aligned} \text{trace}\,(J(\mathbf{y})) &= (a-2) y_2 \\ \det\,(J(\mathbf{y})) &= -2a(y_1^2 + y_2^2). \end{aligned}$$

Then, remembering again that we assumed $a \neq 0$ and because, for any steady state $y_1^2 + y_2^2 > 0$ then $\det\,(J(\mathbf{y})) > 0$ if $a < 0$ and $\det\,(J(\mathbf{y})) < 0$ if $a > 0$.

Therefore, if $a < 0$, steady state $\bar{\mathbf{y}}^1$ is an unstable node, because $\text{trace}\left(J(\bar{\mathbf{y}}^1)\right) > 0$ and $\det\left(J(\bar{\mathbf{y}}^1)\right) > 0$, and steady state $\bar{\mathbf{y}}^2$ is a saddle point, because $\det\left(J(\bar{\mathbf{y}}^2)\right) < 0$. Then for any trajectory starting from any element of $y/\bar{\mathbf{y}}^2$ there is convergence to steady state $\bar{\mathbf{y}}^1$ (see the left subfigure in figure 4.6). If we denote $\text{Het}(\mathbf{y})$ the set points connecting $\bar{\mathbf{y}}^2$ to $\bar{\mathbf{y}}^1$ we readily see that $\text{Het}(\mathbf{y}) = y$, which means there are an infinite number of heteroclinic orbits, and that this set is coincident with the stable manifold $\mathcal{W}^s_{\bar{\mathbf{y}}^1}$ (see the left subfigure in Figure 4.6).

However, if $a > 0$ both steady states, $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$, are saddle points, because $\det\left(J(\bar{\mathbf{y}}^1)\right) = \det\left(J(\bar{\mathbf{y}}^2)\right) < 0$. In this case, there is one heteroclinic surface

$$\text{Het}(\mathbf{y}) = \{(y_1, y_2) : y_1 = 0,\ -1 \leq y_2 \leq 1\}$$

which is the locus of points connecting $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$ such that for any initial value $\mathbf{y}(0) \in \text{Het}(\mathbf{y})$ the solution will converge to $\bar{\mathbf{y}}^2$ (see the right subfigure in figure 4.6). In this case $\text{Het}(\mathbf{y})$ is the set of points belonging to the intersection of the unstable manifold of $\bar{\mathbf{y}}^1$ and to the stable manifold of $\bar{\mathbf{y}}^2$: $\text{Het}(\mathbf{y}) = \mathcal{W}^u_{\bar{\mathbf{y}}^1} \cap \mathcal{W}^s_{\bar{\mathbf{y}}^2}$.

At last, we should notice that in both cases the heteroclinic orbits are generic, in the sense that they persist for a wide range of values for parameter a. This is not the case for the next example.

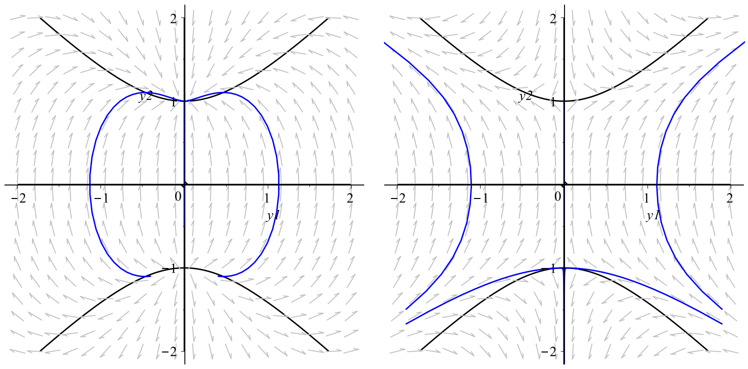


Figure 4.6: Phase diagrams for equation 4.10 for $a < 0$, and $a > 0$

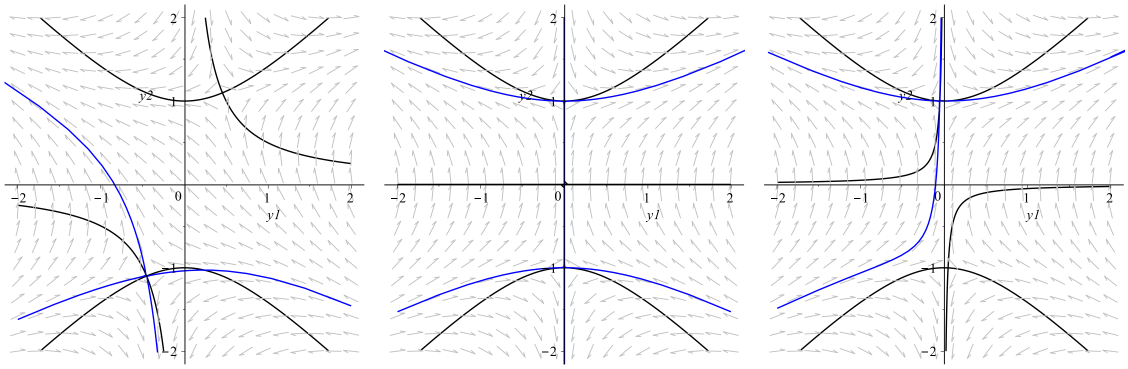


Figure 4.7: Phase diagrams for equation 4.11 for $\lambda < 0$, $\lambda = 0$, $\lambda > 0$

Heteroclinic saddle connection bifurcation Assuming a related but slightly different normal form generates an heteroclinic bifurcation meaning we may have a bifurcation parameter that when it crosses a specific value heteroclinic orbits cease to exist. The following model is studied, for instance, in (Hale and Koçak, 1991, p.210).

$$\begin{aligned} \dot{y}_1 &= \lambda + 2y_1y_2 \\ \dot{y}_2 &= 1 + y_1^2 - y_2^2 \end{aligned} \tag{4.11}$$

In this case we have, for $\lambda = 0$, an heteroclinic orbit, connecting the two steady states exists and we have the second case in the previous model. When λ is perturbed away from zero we will have only one steady state which is a saddle point. See Figure .

Homoclinic orbits We say there is an **homoclinic orbit** if, in a planar ODE, there is a subset of points Hom($\mathbf{y}$) connecting the steady state with itself. This is only possible if the steady state

$\bar{\mathbf{y}}$ is a saddle point in which the stable manifold contains a closed curve, that we call homoclinic curve. Because of this fact, homoclinic orbits exist jointly with periodic trajectories.

Again, homoclinic orbits can be generic or non-generic. Next we illustrate both cases.

Generic homoclinic orbits Consider the non-linear planar ODE depending on one parameter, a, of type

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 - a y_1^2. \end{aligned} \tag{4.12}$$

It has two steady states $\bar{\mathbf{y}}^1 = (0,0)$ and $\bar{\mathbf{y}}^2 = (1/a, 0)$. The Jacobian

$$J(\mathbf{y}) \equiv D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ 1 - 2ay_1 & 0 \end{pmatrix}$$

has following the trace and the determinant

$$\begin{aligned} \operatorname{trace}(J(\mathbf{y})) &= 0 \\ \det(J(\mathbf{y})) &= 2ay_1 - 1. \end{aligned}$$

It is easy to see that steady state $\bar{\mathbf{y}}^1$ is always a saddle point, because $\det\big(J(\bar{\mathbf{y}}^1)\big) = -1 < 0$, and the steady state $\bar{\mathbf{y}}^2$ is always locally a center, because $\det\big(J(\bar{\mathbf{y}}^2)\big) = 1 > 0$ and $\operatorname{trace}\big(J(\bar{\mathbf{y}}^2)\big) = 0$, for any value of a.

Furthermore, we can prove that there is an invariant curve, such that solutions follow a potential or first integral curve which is constant.

In order to see this we introduce a **Lyapunov function** which is a differentiable function $H(\mathbf{y})$ such that the time derivative is $\dot{H} = D_{\mathbf{y}} H \dot{\mathbf{y}}$, that is $\dot{H} = H_{y_1}\dot{y}_1 + H_{y_2}\dot{y}_2$. A **first integral** is a set of points (y_1, y_2) such that $\dot{H} = 0$. In this case the orbits passing through those points allow for a conservation of energy in some sense and $H(\mathbf{y}(t)) =$ constant. For values such that $H(\mathbf{y}(t)) = 0$ that curve passes through a steady state.

For this case consider the function

$$H(y_1, y_2) = -\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{a}{3}y_1^3.$$

If we time-differentiate this Lyapunov function and substitute equations (4.13) we have

$$\begin{aligned} \dot{H} &= (ay_1 - 1)y_1\dot{y}_1 + y_2\dot{y}_2 = \\ &= (ay_1 - 1)y_1 y_2 + y_2 y_1(1 - y_1) = \\ &= 0. \end{aligned}$$

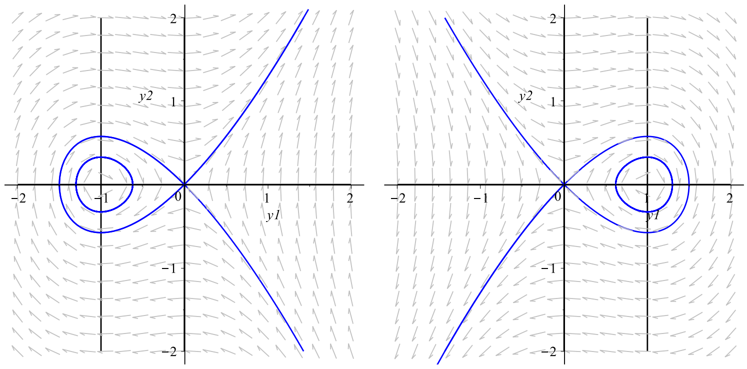


Figure 4.8: Phase diagrams for equation 4.12 for $a < 0$, and $a > 0$

Then $\dot{H} = 0$, for any values of $\mathbf{y}$ and a. We call homoclinic surface to the set of points such that there are homoclinic orbits. In our case, homoclinic orbits converge both for $t \to \infty$ and $t \to -\infty$ to point $\bar{\mathbf{y}}^1$. Therefore the homoclinic surface is the set of points

$$\text{Hom}(\bar{\mathbf{y}}^1) = \{(y_1, y_2) : H(y_1, y_2) = 0,\ \text{sign}(\bar{y}_1) = \text{sign}(a)\}$$

Figure 4.8 depicts phase diagrams for the case in which $a < 0$ (left sub-figure) and $a > 0$ (right sub-figure).

We see that the homoclinic trajectories are generic, i.e, they exist for different values of the parameters. This is not always the case as we show next.

Homoclinic or saddle-loop bifurcation This model is studied, for instance, in (Hale and Koçak, 1991, p.210) and (Kuznetsov, 2005, ch. 6.2). It is a non-linear ODE depending on one parameter, a, of type

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 + a\, y_2 - y_1^2. \end{aligned} \tag{4.13}$$

In this case, we have

$$\mathbf{f}(\mathbf{y}, a) = \begin{pmatrix} y_2 \\ y_1 + a\, y_2 - y_1^2. \end{pmatrix}.$$

The set of equilibrium point is $\bar{\mathbf{y}} = \{\mathbf{y} : \mathbf{f}(\mathbf{y}, a) = \mathbf{0}\}$. For equation (4.13) we have two equilibrium points,

$$\bar{\mathbf{y}}^1 = (\bar{y}_1^1, \bar{y}_2^1) = (0, 0),\ \bar{\mathbf{y}}^2 = (\bar{y}_1^2, \bar{y}_2^2) = (1, 0)$$

In order to determine the local dynamics we evaluate the Jacobian for any point $\mathbf{y} = (y_1, y_2)$,

$$D_{\mathbf{y}}\mathbf{f}(\mathbf{y}, a) = \begin{pmatrix} 0 & 1 \\ 1 - 2y_1 & a \end{pmatrix}.$$

The eigenvalues of the Jacobian are functions of the variables and of the parameter a,

$$\lambda_{\pm}(\mathbf{y}, a) = \frac{a}{2} \pm \left[\left(\frac{a}{2}\right)^2 + 1 - 2y_1 \right]^{\frac{1}{2}}.$$

If we evaluate the eigenvalues at the steady state $\bar{\mathbf{y}}^1 = (0, 0)$, we find it is a saddle point, because the eigenvalues of the Jacobian $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^1)$ are

$$\lambda^1_{\pm} \equiv \lambda_{\pm}(\bar{\mathbf{y}}^1, a) = \frac{a}{2} \pm \left[\left(\frac{a}{2}\right)^2 + 1 \right]^{\frac{1}{2}}$$

yielding $\lambda^1_- < 0 < \lambda^1_+$. At the steady state $\bar{\mathbf{y}}^2 = (1, 0)$ the eigenvalues of the Jacobian $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^2)$ are

$$\lambda^2_{\pm} = \lambda_{\pm}(\bar{\mathbf{y}}^2, a) = \frac{a}{2} \pm \left[\left(\frac{a}{2}\right)^2 - 1 \right]^{\frac{1}{2}}$$

yielding $\text{sign}(\text{Re}(\lambda_{\pm}(\bar{\mathbf{y}}^2, a)) = \text{sign}(a)$.

Therefore steady state $\bar{\mathbf{y}}^1$ is always a saddle point, and steady state $\bar{\mathbf{y}}^2$ is a stable node or a stable focus if $a < 0$, it is an unstable node or an unstable focus if $a > 0$, or it is a centre if $a = 0$.

When $a = 0$ another type of dynamics occurs. We introduce the following Lyapunov function

$$H(y_1, y_2) = -\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{3}y_1^3.$$

and prove that it can only be a first integral if $a = 0$. To show this, if we time-differentiate this Lyapunov function and substitute equations (4.13) we have

$$\begin{aligned} \dot{H} &= (y_1 - 1)y_1\dot{y}_1 + y_2\dot{y}_2 = \\ &= (y_1 - 1)y_1 y_2 + y_2 y_1(1 - y_1) + a y_2^2 = \\ &= a y_2^2. \end{aligned}$$

Then $\dot{H} = 0$, for any values of $\mathbf{y}$, if and only if and only if $a = 0$.

In our case this generates an **homoclinic orbit** which is a trajectory that exits a steady state and returns to the same steady state. In this case, a homoclinic orbit exists if $a = 0$ and it does not exist if $a \neq 0$.

The next figure shows the phase diagrams for the cases $a < 0$, $a = 0$ and $a > 0$. If $a < 0$ there is a saddle point and a stable focus, if $a = 0$ there is a saddle point, an infinite number of centres surrounded by an homoclinic orbit. If $a > 0$ there is a saddle point and an unstable focus.

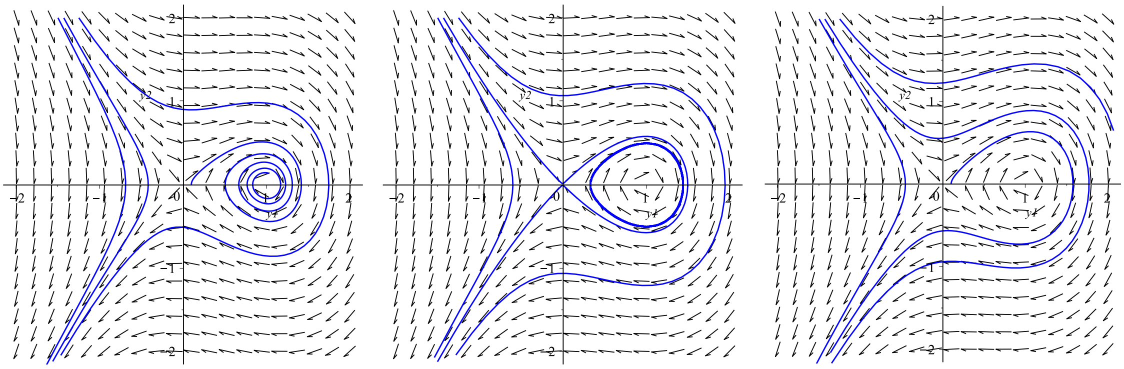


Figure 4.9: Phase diagrams for equation 4.13 for $a < 0$, $a = 0$, $a > 0$

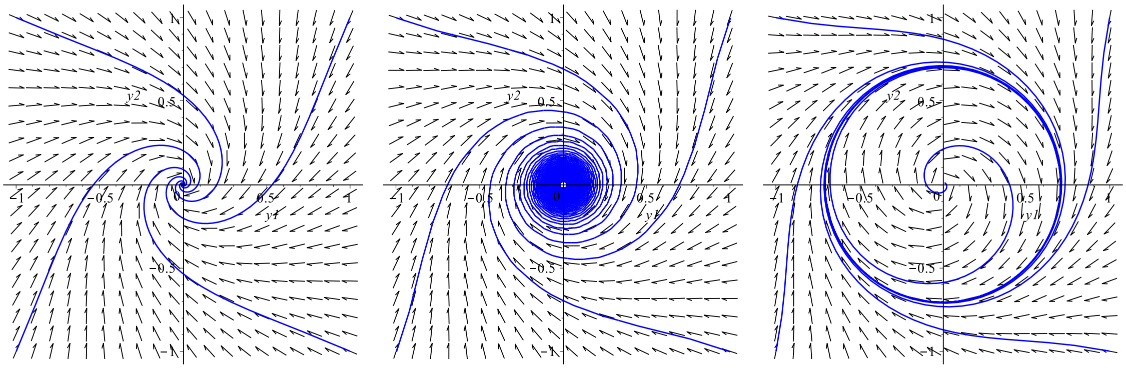


Figure 4.10: Phase diagrams for equation 4.14 for $\lambda < 0$, $\lambda = 0$, $\lambda > 0$

Planar equation: Andronov-Hopf bifurcation This model is studied, for instance, in (Hale and Koçak, 1991, p.212).

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2) \equiv y_2 + y_1(\lambda - y_1^2 - y_2^2) \\ \dot{y}_2 &= f_2(y_1, y_2) \equiv -y_1 + y_2(\lambda - y_1^2 - y_2^2) \end{aligned} \tag{4.14}$$

It has a single steady state $\bar{\mathbf{y}} = (0, 0)$. However, it has another invariant curve. In order to see this, we compute the Jacobian

$$J(\mathbf{y}) = \begin{pmatrix} \lambda - 3y_1^2 - y_2^2 & 1 - 2y_1y_2 \\ -1 - 2y_1y_2 & \lambda - y_1^2 - 3y_2^2 \end{pmatrix}$$

which has eigenvalues

$$\lambda_{\pm} = \lambda - 2(y_1^2 + y_2^2) \pm (y_1^2 + y_2^2)$$

In figure 4.10 we see the following: if $\lambda < 0$ there will be only one steady state which is a stable node with multiplicity, although the speed of convergence to the steady state increases very much when λ converges to zero, if $\lambda > 0$ a **limit circle** appears and the steady state becomes a unstable focus. According to the Bendixson-Dulac criterium (see Theorem 3) as

$$\frac{\partial f_1(y_1, y_2)}{\partial y_1} + \frac{\partial f_2(y_1, y_2)}{\partial y_2} = 2\lambda - (2y_1)^2 - (2y_2)^2$$

changes sign for $\lambda > 0$, in a subset of $\mathcal{Y}$, then a closed curve can exist. This closed curve is a limit cycle which is a curve such that $y^1 + y^2 = \lambda$. To prove this, we transform the system in polar coordinates (see Appendix to chapter 1) and get [10]

$$\dot{r} = r(\lambda - r^2)$$
$$\dot{\theta} = -1$$

there is thus a periodic orbit with radius $\bar{r} = \sqrt{\lambda}$.

4.2 Qualitative theory of ODE

Next we present a short introduction to the qualitative (or geometrical) theory of ODE's.

We consider a generic ODE

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}),\ \mathbf{f} : \mathcal{Y} \to \mathcal{Y},\ \mathbf{y} : \mathcal{T} \to \mathcal{Y} \tag{4.15}$$

where $\mathbf{f} \in C^1(\mathcal{Y})$, i.e., $f(.)$ is continuously differentiable up to the first order.

The qualitative theory of ODEs consists in finding a (topological) equivalence between a non-linear (or even incompletely defined) function $\mathbf{f}(.)$ and a linear or a normal form ODE. This allows us to characterize the dynamics in the neighborhood of a steady state or of a periodic orbit or other invariant sets (homoclinic and heteroclinic orbits or limit cycles). If there are more than one invariant orbit or steady state we distinguish between local dynamics (in the neighborhood of a steady state or invariant orbit) from global dynamics (in all set y). If there is only one invariant set then local dynamics is qualitatively equivalent to global dynamics.

One important component of qualitative theory is **bifurcation analysis**, which consists in describing the change in the dynamics (that is, in the phase diagram) when one or more parameters take different values within its domain.

[10] We define $r^2 = y_1^2 + y_2^2$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y_2}{y_1}$, and take time derivatives, obtaining

$$\dot{r} = \frac{y_1\dot{y}_1 + y_2\dot{y}_2}{r}$$
$$\dot{\theta} = \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{r^2}$$

.

4.2.1 Local analysis

We study local dynamics of equation (4.15) by performing a local analysis close to an equilibrium point or a periodic orbit. There are three important results that form the basis of the local analysis: the Grobman-Hartmann, the manifold and the Poincaré-Bendixson theorems. The first two are related to using the knowledge on the solutions of an equivalent linearized ODE to study the local properties close to the a steady-state for a non-linear ODE and the third introduces a criterium for finding periodic orbits.

Equivalence with linear ODE's

Assume there is (at least) one equilibrium point $\bar{\mathbf{y}} \in \{\mathbf{y} \in Y \subseteq \mathbb{R}^n : \mathbf{f}(\mathbf{y}) = \mathbf{0}\}$, for $n \geq 1$, and consider the Jacobian of $\mathbf{f}(.)$ evaluated at that equilibrium point

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \frac{\partial f_1(\bar{y})}{\partial y_1} & \dots & \frac{\partial f_1(\bar{y})}{\partial y_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n(\bar{y})}{\partial y_1} & \dots & \frac{\partial f_n(\bar{y})}{\partial y_n} \end{pmatrix}.$$

An equilibrium point is **hyperbolic** if the Jacobian J has no eigenvalues with zero real parts. An equilibrium point is **non-hyperbolic** if the Jacobian has at least one eigenvalue with zero real part.

Theorem 1 (**Grobman-Hartmann theorem**). *Let $\bar{\mathbf{y}}$ be a hyperbolic equilibrium point. Then there is a neighbourhood U of $\bar{\mathbf{y}}$ and a neighborhood U_0 of $\mathbf{y}(0)$ such that the ODE restricted to U is topologically equivalent to the variational equation*

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}}),\ \mathbf{y} - \bar{\mathbf{y}} \in U_0$$

The original paper are Grobman (1959) and Hartman (1964).

Stability properties of $\bar{\mathbf{y}}$ are characterized from the eigenvalues of Jacobian matrix $J(\bar{\mathbf{y}}) = D_y\mathbf{f}(\bar{\mathbf{y}})$.

If all eigenvalues λ of the Jacobian matrix have negative real parts, $\mathrm{Re}(\lambda) < 0$, then $\bar{y}$ is asymptotically stable. If there is at least one eigenvalue λ such that $\mathrm{Re}(\lambda) > 0$ then $\bar{y}$ is unstable.

Example 1 Consider the scalar ODE

$$\dot{y} = f(y) \equiv y^{\alpha} - a \tag{4.16}$$

where a and α are two constants, with $a > 0$, and $y \in \mathbb{R}_+$. Then there is an unique steady state $\bar{y} = a^{\frac{1}{\alpha}}$. As

$$f_y(y) = \alpha y^{\alpha-1}$$

then

$$f_y(\bar{y}) = \alpha a^{\frac{\alpha-1}{\alpha}}.$$

Set $\lambda \equiv f_y(\bar{y})$. Therefore the steady state is hyperbolic if $\alpha \neq 0$ and it is non-hyperbolic if $\alpha = 0$. In addition, if $\alpha < 0$ the hyperbolic steady state $\bar{y}$ is asymptotically stable and if $\alpha > 0$ it is unstable.

If $\alpha \neq 0$ we can perform a first-order Taylor expansion of the ODE (4.16) in the neighborhood of the steady state

$$\dot{y} = \lambda(y - \bar{y}) + o((y - \bar{y}))$$

which means that the solution to (4.16) can be locally approximated by

$$y(t) = \bar{y} + (k - \bar{y})e^{\lambda t}$$

for any $k \in \mathbb{R}_+$. In particular, if we fix $y(0) = y_0$ then $k = y_0$.

Example 2 Consider the non-linear planar ODE

$$\begin{aligned} \dot{y}_1 &= y_1^\alpha - a, \ 0 < \alpha < 1, \ a \geq 0, \\ \dot{y}_2 &= y_1 - y_2 \end{aligned} \tag{4.17}$$

It has an unique steady state $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2) = (a^{\frac{1}{\alpha}}, a^{\frac{1}{\alpha}})$. The Jacobian evaluated at any point $\mathbf{y}$ is

$$J(\mathbf{y}) = D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} \alpha y_1^{\alpha-1} & 0 \\ 1 & -1 \end{pmatrix}.$$

If we approximate the system in a neighborhood of the steady state, $\bar{\mathbf{y}}$, we have the linear planar ODE

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})\,(\mathbf{y} - \bar{\mathbf{y}})$$

where $J(\bar{\mathbf{y}})$ is the Jacobian evaluated at the steady state,

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \alpha a^{\frac{\alpha-1}{\alpha}} & 0 \\ 1 & -1 \end{pmatrix}.$$

We already saw that the solution to this equation is

$$\mathbf{y}(t) = \mathbf{y} + \mathbf{P}\mathbf{e}^{J(\bar{\mathbf{y}})t}\mathbf{h}.$$

Because

$$\begin{aligned} \text{trace}(J(\bar{\mathbf{y}})) &= \alpha a^{\frac{\alpha-1}{\alpha}} - 1 \\ \det\left(J(\bar{\mathbf{y}})\right) &= -\alpha a^{\frac{\alpha-1}{\alpha}} \\ \Delta(J(\bar{\mathbf{y}})) &= \left(\frac{\alpha a^{\frac{\alpha-1}{\alpha}} + 1}{2}\right)^2 \end{aligned}$$

which implies that the eigenvalues of the Jacobian $J(\bar{\mathbf{y}})$ are

$$\lambda_{\pm} = \frac{\alpha a^{\frac{\alpha-1}{\alpha}} - 1}{2} \pm \frac{\alpha a^{\frac{\alpha-1}{\alpha}} + 1}{2}.$$

that is $\lambda_{+} = \alpha a^{\frac{\alpha-1}{\alpha}}$ and $\lambda_{-} = -1$. Therefore, the steady state is hyperbolic if $\alpha \neq 0$ and non-hyperbolic if $\alpha = 0$.

Furthermore, the steady state is a saddle point if $\alpha > 0$ and it is a stable node if $\alpha < 0$. We can also find the eigenvector matrix of $J(\bar{\mathbf{y}})$,

$$\mathbf{P} = \left(\mathbf{P}^{+}\mathbf{P}^{-}\right) = \begin{pmatrix} 1+\lambda_{+} & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore, the approximate solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = h_{+} \begin{pmatrix} 1+\lambda_{+} \\ 1 \end{pmatrix} e^{\lambda_{+}t} + h_{-} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_{-}t}.$$

If $\alpha < 0$ the stable eigenspace is $\mathcal{E}^s = \{(y_1, y_2) : y_1 = \bar{y}_1\}$, and, if $\alpha > 0$ the stable eigenspace is the whole space, $\mathcal{E}^s = \mathcal{Y}$.

Local manifolds

Consider a neighbourhood $U \subset \mathcal{Y} \subseteq \mathbb{R}^n$ of $\bar{\mathbf{y}}$: the local stable manifold is the set

$$\mathcal{W}^s_{loc}(\bar{\mathbf{y}}) = \{\mathbf{k} \in U : \lim_{t\to\infty} \mathbf{y}(t,k) = \bar{y},\ y(t,\mathbf{k}) \in U, t \geq 0\}$$

the local unstable manifold is the set

$$\mathcal{W}^u_{loc}(\bar{\mathbf{y}}) = \{\mathbf{k} \in U : \lim_{t\to\infty} \mathbf{y}(-t,\mathbf{k}) = \bar{\mathbf{y}},\ \mathbf{y}(-t,\mathbf{k}) \in U, t \geq 0\}$$

The center manifold is denoted $\mathcal{W}^c_{loc}(\bar{\mathbf{y}})$. Let n_{-}, n_{+} and n_0 denote the number of eigenvalues of the Jacobian evaluated at steady state $\bar{\mathbf{y}}$ with negative, positive and zero real parts.

Theorem 2 (Manifold Theorem). *: suppose there is a steady state $\bar{\mathbf{y}}$ and $J(\bar{\mathbf{y}})$ is the Jacobian of the ODE* (4.15) *. Then there are local stable, unstable and center manifolds, $\mathcal{W}^s_{loc}(\bar{\mathbf{y}})$, $\mathcal{W}^u_{loc}(\bar{\mathbf{y}})$ and $\mathcal{W}^c_{loc}(\bar{\mathbf{y}})$, of dimensions n_{-}, n_{+} and n_0, respectively, such that $n = n_{-} + n_{+} + n_0$. The local manifolds are tangent to the local eigenspaces $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$ of the (topologically) equivalent linearized ODE*

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}}).$$

The first two, eigenspaces $\mathcal{E}^s$ and $\mathcal{E}^u$, are unique, and $\mathcal{E}^c$need not be unique (see (Grass et al., 2008, ch.2)).

The eigenspaces are spanned by the eigenvectors of the Jacobian matrix $J(\bar{y})$ which are associated to the eigenvalues with negative, positive and zero real parts.

Example 2 Consider example 2 and let $\alpha > 0$ which implies that the steady state $\bar{\mathbf{y}}$ is a saddle point. Because the eigenvector associated to eigenvalue λ_- is $\mathbf{P}^- = (0, 1)^\top$, then the stable eigenspace is

$$\mathcal{E}^s = \{(y_1, y_2) \in \mathbb{R}_+ : y_1 = \bar{y}_1 = a^{\frac{1}{\alpha}}\}.$$

The local stable manifold $\mathcal{W}^s_{loc}(\bar{\mathbf{y}})$ is tangent to $\mathcal{E}^s$ in a neighborhood of the steady state.

Periodic orbits

We saw that solution trajectories can converge or diverge not only as regards equilibrium points but also to periodic trajectories (see the Andronov-Hopf model).

The **Poincaré-Bendixson** theorem ((Hale and Koçak, 1991, p.367)) states that if the limit set is bounded and it is not an equilibrium point it should be a periodic orbit.

In order to determine if there is a periodic orbit in a compact subset of y the Bendixson criterium provides a method ((Hale and Koçak, 1991, p.373)):

Theorem 3 (**Bendixson-Dulac criterium**). *Let D be a compact region of $y \subseteq \mathbb{R}^n$ for $n \geq 2$. If,*

$$div(\mathbf{f}) = f_{1,y_1}(y_1, y_2) + f_{2,y_2}(y_1, y_2)$$

has constant sign, for $(y_1, y_2) \in D$, then $\dot{y} = f(y)$ has not a constant orbit lying entirely in D.

4.2.2 Global analysis

While local analysis consists in studying local dynamics in the neighbourhood of steady states or periodic orbits, this may not be enough to characterise the dynamics.

We already saw that there are orbits that are invariant and that cannot be determined by local methods, for instance heteroclinic and homoclinic orbits.

Homoclinic and heteroclinic orbits

There are methods to determine if there are homoclinic or heteroclinic orbits. They essentially consist in building a trapping area for the trajectories and proving there should exist trajectories that do not exit the ”trap”.

Global manifolds

There are global extensions of the local manifolds by continuation in time (in the opposite direction) of the local manifolds: $\mathcal{W}^s(\bar{y})$, $\mathcal{W}^u(\bar{y})$, $\mathcal{W}^c(\bar{y})$.

A trajectory $y(.)$ of the ODE is called a **stable path** of $\bar{y}$ if the orbit $\text{Or}(y_0)$ is contained in the stable manifold $\text{Or}(y_0) \subset \mathcal{W}^s(\bar{y})$ and $\lim_{t\to\infty} y(t, y_0) = \bar{y}$.

A trajectory $y(.)$ of the ODE is called a **unstable path** of $\bar{y}$ if the orbit $\text{Or}(y_0)$ is contained in the stable manifold $\text{Or}(y_0) \subset \mathcal{W}^u(\bar{y})$ and $\lim_{t\to\infty} y(-t, y_0) = \bar{y}$.

4.3 Dependence on parameters

We already saw that the solution of linear ODE's, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, may depend on the values for the parameters in the coefficient matrix $\mathbf{A}$.

We can extend this idea to non-linear ODE's of type

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi),\ \varphi \in \Phi \subseteq R^q$$

where φ is a vector of parameters of dimension $q \geq 1$

We can distinguish two types of parameter change:

- **bifurcations** when a parameter change induces a qualitative change in the dynamics, i.e, the phase diagram. By qualitative change we mean change the number or the stability properties of steady states or other invariants. Close to a bifurcation point, a change in a parameter changes the qualitative characteristics of the dynamics;
- **perturbations** when parameter changes do not change the qualitative dynamics, i.e., they do not change the phase diagram. This is typically the case in economics when one performs comparative dynamics exercises.

4.3.1 Bifurcations

If a small variation of the parameter changes the phase diagram we say we have a bifurcation. As you saw, there are local (fixed points) and global bifurcations (heteroclinic connection, etc). Those bifurcations were associated to particular normal forms of both scalar and planar ODEs. This fact allows us to find classes of ODE's which are topologically equivalent to those we have already presented.

Bifurcations for scalar ODE's

Consider the scalar ODE

$$\dot{y} = f(y, \varphi),\ Y, \varphi \in \mathbb{R}.$$

Fold bifurcation (see (Kuznetsov, 2005, ch. 3.3)): Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yy}(0, 0) \neq 0,\ f_{\varphi}(0, 0) \neq 0.$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi \pm y^2,$$

that is to the Ricatti's model (4.2).

Transcritical bifurcation: Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yy}(0, 0) \neq 0,\ f_{\varphi y}(0, 0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^2$$

that is to the Bernoulli model (4.3).

Pitchfork bifurcation: Let $f \in C^2(\mathbb{R})$ and consider $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yyy}(0, 0) \neq 0,\ f_{\varphi y}(0, 0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^3$$

that is to the Bernoulli model (4.4).

Bifurcations for planar ODE's

Consider the planar ODE

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi),\ \mathbf{y} \in \mathbb{R}^2, \varphi \in \mathbb{R}$$

Andronov-Hopf bifurcation (see (Kuznetsov, 2005, ch. 3.4)): Let $\mathbf{f} \in C^2(\mathbb{R})$ and consider $(\bar{\mathbf{y}}, \varphi_0) = (\mathbf{0}, 0)$ the Jacobian at $(\mathbf{0}, 0)$ has eigenvalues

$$\lambda_{\pm} = \eta(\varphi) \pm i\omega(\varphi)$$

where $\eta(0) = 0$ and $\omega(0) > 0$. If some additional conditions are satisfied then the ODE is locally topologically equivalent to

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

4.3.2 Comparative dynamics in economics

As mentioned, **comparative dynamics** exercises consist in introducing perturbation in a dynamic system: i.e., a small variation of the parameter that does not change the phase diagram. This kind of analysis only makes sense if the steady state is hyperbolic, that is if $\det(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)) \neq 0$ or $\mathrm{trace}(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)) \neq 0$ if $\det(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)) > 0$.

In this case let the steady state be for a given value of the parameter $\varphi = \varphi_0$

$$\bar{\mathbf{y}}_0 = \{y \in Y : \mathbf{f}(\mathbf{y}, \varphi_0) = \mathbf{0}\}.$$

If $\bar{\mathbf{y}}_0$ is a hyperbolic steady state, then we can expand the ODE into a linear ODE

$$\dot{\mathbf{y}} = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)(\mathbf{y} - \bar{\mathbf{y}}_0) + D_{\varphi}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)(\varphi - \varphi_0). \tag{4.18}$$

This equation can be solved as a linear ODE. Setting $\varphi = \varphi_0 + \delta_\varphi$ and because $\bar{\mathbf{y}} = \bar{\mathbf{y}}(\varphi)$ and $\bar{\mathbf{y}}_0 = \bar{\mathbf{y}}(\varphi_0)$ we have

$$D_{\varphi}\bar{\mathbf{y}}(\varphi_0) = \lim_{\delta_\varphi \to 0} \frac{\bar{\mathbf{y}}(\varphi_0 + \delta_\varphi) - \bar{\mathbf{y}}(\varphi_0)}{\delta_\varphi} = -D_{\mathbf{y}}^{-1}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0) D_{\varphi}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)$$

which are called the **long-run multipliers** associated to a permanent change in φ. Solving the linearized system allows us to have a general solution to the problem of finding the **short-run** or **transition multipliers**, $d\mathbf{y}(t) \equiv \mathbf{y}(t) - \bar{\mathbf{y}}_0$ for a change in the parameter φ.

4.4 Application to economics

4.4.1 The optimality conditions for the Ramsey model

The Ramsey (1928) model (see also Cass (1965) and Koopmans (1965)) is the workhorse of modern macroeconomics and growth theory. It is a normative model (but can also be seen as a positive model if its behavior fits the data) on the optimal choice of consumption and where savings leads to the accumulation of capital, and therefore to future consumption. Therefore, the optimal trade-off between present and future consumption guides the accumulation of capital.

We will derive the optimality conditions when we study optimal control. In this section we assume that there are two primitives for the model related with technology and preferences: (1) the production function, $f(k)$ and (2) the elasticity of intertemporal substitution $\eta(c)$ and the rate of time preference ρ.

The first order conditions for an optimum take the form of two non-linear differential equations. Let k and c denote per-capita physical capital and consumption, respectively, and let the two variables be non-negative. That is $(k, c) \in \mathbb{R}^2_+$. The Ramsey model is the planar ODE

$$\begin{aligned} \dot{k} &= f(k) - c \qquad (4.19) \\ \dot{c} &= \eta(c)\, c \left(f^{'}(k) - \rho \right), \qquad (4.20) \end{aligned}$$

supplemented with an initial value for k, $k(0) = k_0$ and the transversality condition $\lim_{t\to\infty} u^{'}(c)k(t)e^{-\rho t} = 0$, where $u(c)$ is the utility function from which we determine the elasticity of intertemporal substitution. For this section we will be concerned with trajectories that are bounded asymptotically, that is converging to a steady state.

The ODE system (also called modified Hamiltonian dynamic system MHDS) is non-linear when the two primitive functions are not completely specified, as is the case with system (4.19)-(4.20). Next we assume a smooth case and the following assumptions

1. preferences are specified by a constant elasticity of intertemporal substitution, $\eta(c) = \eta > 0$ is constant;

2. the rate of time preference is positive $\rho > 0$;

3. the production function is of the Inada type: it is positive for positive levels of capital, it is monotonously increasing and globally concave. Formally: $f(0) = 0$, $f(k) > 0$ for $k > 0$, $f^{'}(k) > 0$, $\lim_{k\to 0} f^{'}(k) = +\infty$, $\lim_{k\to +\infty} f^{'}(k) = 0$, and $f^{''}(k) < 0$ for all $k \in \mathbb{R}_+$. [11]

Given the smoothness of the vector field, i.e, of functions $f_1(k, c) \equiv f(k) - c$ and $f_2(k, c) \equiv \eta c(f^{'}(k) - \rho)$, we know that a solution exists and it is unique. Therefore, in order to characterize the dynamics we can use the qualitative theory of ODE's presented previously in this section.

In particular we will

1. determine the existence and number of steady states

[11] Observe that $f(k)$ is locally but not globally Lipschitz, i.e, a small change in k close to zero induces a large change in $f(k)$.

2. characterize them regarding hyperbolicity and local dynamics, performing, if necessary, a local bifurcation analysis
3. try to find other invariant trajectories of a global nature
4. conduct comparative dynamics analysis in the neigborhood of relevant hyperbolic steady states.

Steady states Any steady-state, $(\bar{k}, \bar{c})$, belongs to the set

$$(\bar{k}, \bar{c}) = \{(k, c) \in \mathbb{R}^2_+ : \ \dot{k} = \dot{c} = 0\} = \{(0,0), (k^*, c^*)\}$$

where $k^* = g(\rho)$, where $g(.) = (f^{'})^{-1}(.)$ and $c^* = f(k^*) = f(g(\rho))$.

To prove the existence and uniqueness of a positive steady state level for k we use the Inada and global concavity properties of the production function: first, $\dot{c} = 0$ if there is a value k that solves the equation $f^{'}(k) = \rho$; second, because $\rho > 0$ is finite and $f^{'}(k) \in (0, \infty)$ then there is at least one value for k that solves that equation; at last, because the function $f(.)$ is globally strictly concave then $f^{'}(k)$ is monotonously decreasing which implies that the solution is unique.

Characterizing the steady states In order to characterize the steady states, we find the Jacobian of system (4.19)-(4.20), is

$$D_{(k,c)}\mathbf{F}(k, c) = \begin{pmatrix} f^{'}(k) & -1 \\ \eta c f^{''}(k) & \eta\left(f^{'}(k) - \rho\right). \end{pmatrix} \tag{4.21}$$

The eigenvalues of $D_{(k,c)}\mathbf{F}(k, c)$ evaluated at steady state $(\bar{k}, \bar{c}) = (0, 0)$ are

$$\lambda^0_s = \eta(f^{'}(0) - \rho) = +\infty, \ \lambda^0_u = f^{'}(0) = +\infty,$$

which means that this steady state is singular (see chapter 8). This is a consequence of the fact that $f(k)$ is not locally Lipschitz close to $k = 0$.

For steady state $(\bar{k}, \bar{c}) = (k^*, c^*)$, the trace and the determinant of the Jacobian are

$$\mathrm{trace}(D_{(k,c)}\mathbf{F}(k^*, c^*)) = \rho > 0, \ \det(D_{(k,c)}F(k^*, c^*)) = \eta c^* f^{''}(k^*) < 0$$

and the eigenvalues are

$$\lambda^*_s = \frac{\rho}{2} - \left(\left(\frac{\rho}{2}\right)^2 - \eta c^* f^{''}(k^*)\right)^{\frac{1}{2}} < 0, \ \lambda^*_u = \frac{\rho}{2} + \left(\left(\frac{\rho}{2}\right)^2 - \eta c^* f^{''}(k^*)\right)^{\frac{1}{2}} < 0$$

satisfy the relationships

$$\lambda^*_s + \lambda^*_u = \rho, \ \lambda^*_s\lambda^*_u = \eta c^* f^{''}(k^*) < 0.$$

The steady state (k^*, c^*) is also hyperbolic and it is a saddle-point. The intuition behind this property is transparent when we look at the expression for the determinant: the mechanism generating stability is related to the existence of decreasing marginal returns in producion. Because capital accumulation is equal to savings, and savings sustains future increases in consumption by increasing production, the existence of decreasing marginal returns implies that the marginal increase in production will tend to zero thus stopping the incentives for future capital accumulation.

As the Jacobians of system (4.19)-(4.20), evaluated at every steady state, does not have eigenvalues with zero real parts both steady states are hyperbolic and there are no local bifurcation points.

In addition, from the Grobman-Hartmann theorem the system (4.19)-(4.20) can be approximated by a (topologically equivalent) linear system in the neighborhood of every steady state.

Let us consider the steady state (k^*, c^*). As the Jacobian in this case is

$$D_{(k,c)}\mathbf{F}(k^*, c^*) = \begin{pmatrix} \rho & -1 \\ \eta c^* f''(k^*) & 0 \end{pmatrix}$$

we can consider the **variational system**

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \eta c^* f''(k^*) & 0 \end{pmatrix} \begin{pmatrix} k - k^* \\ c - c^* \end{pmatrix}$$

as giving the approximated dynamics in the neighborhood of the steady state (k^*, c^*).

Because

$$D_{(k,c)}\mathbf{F}(k^*, c^*) - \lambda_s^* \mathbf{I}_2 = \begin{pmatrix} \rho - \lambda_s^* & -1 \\ \eta c^* f''(k^*) & -\lambda_s^* \end{pmatrix} = \begin{pmatrix} \lambda_u^* & -1 \\ \lambda_s^* \lambda_u^* & -\lambda_s^* \end{pmatrix}$$

we get the eigenvector associated to λ_s^*

$$\mathbf{P}_s^* = (1, \lambda_u^*)^\top.$$

This implies that the stable eigenspace of the linearized ODE,

$$\mathcal{E}^s = \{(k, c) \in N^* : c = \lambda_u^* k\}$$

gives the locus of points in the domain, wich are tangent to the local stable manifold for the original ODE (4.19)-(4.20)

$$\mathcal{W}_{loc}^s = \{(k, c) \in N^* : \lim_{t \to \infty} (k(t), c(t)) = (k^*, c^*)\}$$

where $N^* = \{(k, c) \in \mathbb{R}_+^2 : ||(k, c) - (k^*, c^*)|| < \delta\}$ for a small δ.

Global invariants We can prove that there is an heteroclinic orbit connecting steady states $(0,0)$ and (k^*, c^*). Furthermore, the points in that orbit belong to the stable manifold

$$\mathcal{W}^s = \{(k,c) \in \mathbb{R}^2_+ : \lim_{t\to\infty}(k(t), c(t)) = (k^*, c^*)\},$$

and take the form $c = h(k)$. Although we cannot determine explicitly the function $h(.)$ we can prove that it exists (see Figure 4.11.

We already know that the steady state $(0,0)$ is an unstable node, which means that any small deviation will set a diverging path, and, because steady state (k^*, c^*) is a saddle point there is one unique path converging to it. There is an heteroclinic orbit if this path starts from from $(0,0)$. In order to prove this is the case we can consider a "trapping area" $T = \{(k,c) : c \le f(k),\ 0 \le k \le k^*\}$, where the isoclines $\dot{k} = 0$ and $\dot{c} = 0$ define the boundaries $S_1 = \{(k,c) : c = f(k), 0 \le k \le k^*\}$ and $S_2 = \{(k,c) : 0 \le c \le c^*,\ k = k^*\}$. We can see that all the trajectories coming from inside will exit T: first, the trajectories that cross S_1 will exit T because $\dot{k}|_{S_1} = 0$ and $\dot{c}|_{S_1} = \eta c(f^{'}(k) - \rho) = \eta f(k)(f^{'}(k) - f^{'}(k^*)) > 0$ because $f^{'}(k) > f^{'}(k^*)$ for $k < k^*$, second all trajectories that cross S_2 will exit T because $\dot{k}|_{S_2} = f(k^*) - c = c^* - c > 0$ and $\dot{c}|_{S_2} = 0$.

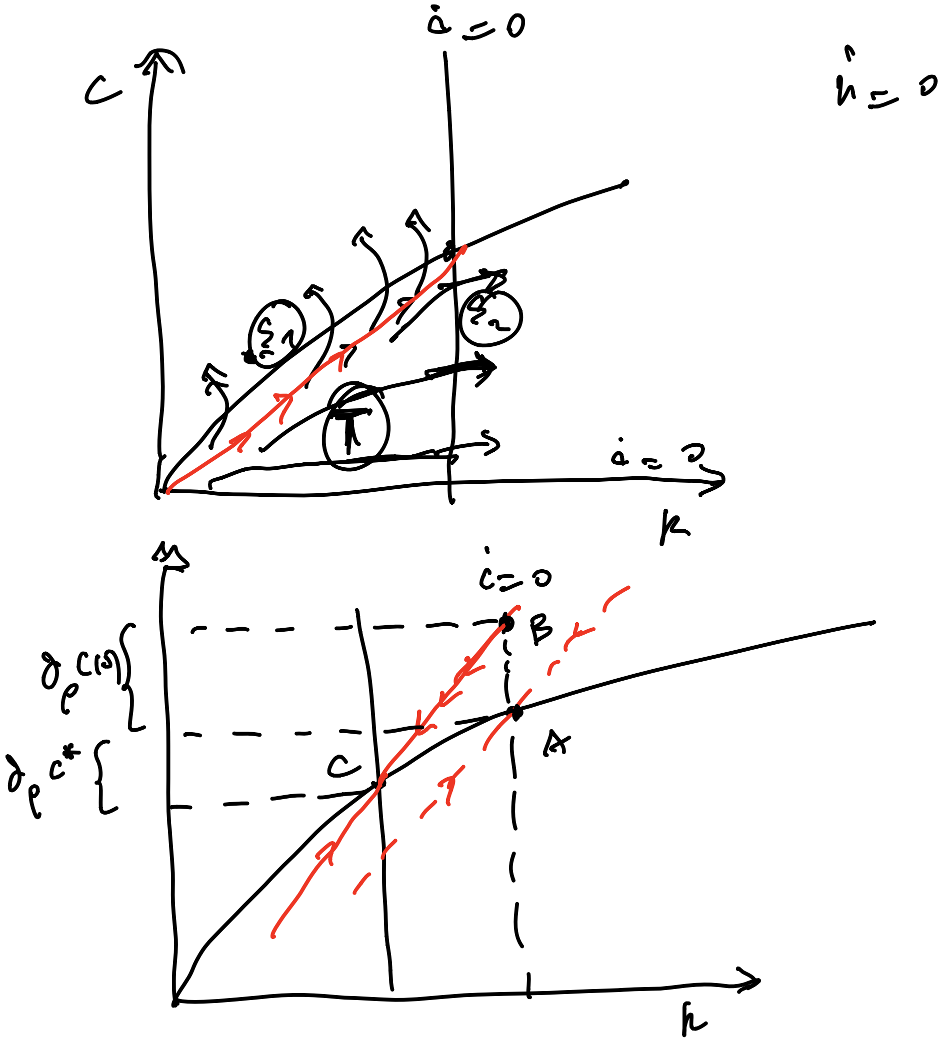
Comparative dynamics Let us consider the steady state (k^*, c^*). As we saw that it is an hyperbolic point, small perturbations by a parameter will not change the local dynamic properties of the steady state, only its quantitative level. Therefore, we can perform a comparative dynamics exercise in its neighborhood.

Assume we start at a steady state and introduce a small change in ρ. As the steady state is a function of ϕ, this means that, after the change, the steady state will move and the initial point is not a steady state. That is we can see it as an arbitrary initial point out of the (new) steady state. From hyperbolicity, the new steady state is still a saddle point, which means that the small perturbation will generate unbounded orbits unless there is a "jump" to the new stable manifold associated to the new steady state. This is the intuition behind the comparative dynamics exercise in most perfect foresight macro models (see Blanchard and Khan (1980) and Buiter (1984)) that we illustrate next. We basically assume that variable k is continuous in time (it is pre-determined) and that c is piecewise continuous in time (it is non-predetermined).

Formally, as we also saw that it is a function of the rate of time preference, let us introduce a permanent change in its value from ρ to $\rho + d\rho$. This will introduce a time-dependent change in the two variables, from (k^*, c^*) to $(k(t), c(t))$ where $k(t) = k^* + dk(t)$ and $c(t) = c^* + dc(t)$. In order to find $(dk(t), dc(t))$ we make a first-order Taylor expansion on (k,c) generated by $d\rho$ to get

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = D_{(k,c)}\mathbf{F}(k^*, c^*) \begin{pmatrix} dk(t) \\ dc(t) \end{pmatrix} + D_\rho \mathbf{F}(k^*, c^*) d\rho \tag{4.22}$$

where $D_\rho \mathbf{F}(k^*, c^*) = (0, -\eta c^*)^\top$. This is a linear planar non-homogeneous ODE.


ċ = 0
c
ḣ = 0
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B
A
C
k

From $\dot{k} = \dot{c} = 0$ we can find the long-run multipliers

$$\begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} = \begin{pmatrix} \frac{dk^*}{d\rho} \\ \frac{dc^*}{d\rho} \end{pmatrix} = -\Big(D_{(k,c)}\mathbf{F}(k^*, c^*)\Big)^{-1} D_\rho \mathbf{F}(k^*, c^*),$$

that is

$$\begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} = \begin{pmatrix} \rho & -1 \\ \eta c^* f''(k^*) & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \eta c^* \end{pmatrix}.$$

Then $\partial_\rho k^* = \dfrac{1}{f''(k^*)} < 0$ and $\partial_\rho c^* = \rho \partial_\rho k^* < 0$. A permanent unanticipated change in ρ will reduce the long run capital stock and consumption level.

We are only interested in the trajectories that converge to the new steady state after a perturbation, $k^* + \partial_\rho k^* d\rho$ and $c^* + \partial_\rho c^* d\rho$, that is a saddle point. In order to make sure this is the case, we solve the variational system for the saddle path to get

$$\begin{pmatrix} \partial_\rho k(t) \\ \partial_\rho c(t) \end{pmatrix} = \begin{pmatrix} \partial_\rho k^* \\ \partial_\rho c^* \end{pmatrix} + x \begin{pmatrix} 1 \\ \lambda_u^* \end{pmatrix} e^{\lambda_s^* t},$$

where x is a positive arbitrary element. If we assume that the variable k is pre-determined, that is it can only be changed in a continuous way from the initial steady state value k^*, we set $\partial_\rho k(0) = 0$. Then, from

$$\partial_\rho k(0) = \partial_\rho k^* + x = 0 \Rightarrow x = -\partial_\rho k^*$$

At last we obtain the **short-run multipliers**

$$\begin{aligned} \partial_\rho k(t) &= \frac{1}{f''(k^*)} \left(1 - e^{\lambda_s^* t}\right) \\ \partial_\rho c(t) &= \frac{1}{f''(k^*)} \left(\rho - \lambda_u \, e^{\lambda_s^* t}\right) \end{aligned}$$

for $t \in [0, \infty)$. In particular we get the impact multipliers, for $t = 0$

$$\begin{aligned} \partial_\rho k(0) &= 0 \\ \partial_\rho c(0) &= \partial_\rho k^* \lambda_s > 0 \end{aligned}$$

which quantify the "jump" to the new stable eigenspace, and the long-run multipliers

$$\begin{aligned} \lim_{t\to\infty} \partial_\rho k(t) &= \partial_\rho k^* < 0 \\ \lim_{t\to\infty} \partial_\rho c(t) &= \rho \partial_\rho k^* = \partial_\rho c^* < 0. \end{aligned}$$

Therefore, on impact consumption increases, which reduces capital accumulation, which reduces again consumption through time. The process stops because the reduction in the per-capita stock will increase marginal productivity which reduces the incentives for further reduction in consumption.

Add figure

Observe also that we should have a ”jump” to the stable manifold to have convergence towards the new steady state. As we have determined convergence to the steady state within the stable eigenspace of the variational system, the trajectory we have determined is qualitatively but not quantitatively exact.

4.5 References

- (Hale and Koçak, 1991, Part I , III): very good introduction.
- (Guckenheimer and Holmes, 1990, ch. 1, 3, 6) Is a classic reference on the field.
- Kuznetsov (2005) Very complete presentation of bifurcations for planar systems.
- (Grass et al., 2008, ch.2) has a compact presentation of all the important results with some examples in economics and management science.

4.A Solution of the Ricatti's equation (4.2)

Start with the case: $a = 0$. Separating variables, we have

$$\frac{dy}{y^2} = dt$$

integrating both sides

$$\int \frac{dy}{y^2} = \int dt \Leftrightarrow -\frac{1}{y} = t - k$$

where k is an arbitrary constant. Then we get the solution

$$y(t) = -\frac{1}{t+k}$$

Now let $a \neq 0$. By using the same method we have

$$\frac{dy}{a+y^2} = dt. \tag{4.23}$$

At this point it is convenient to note that

$$\frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}, \quad \frac{d\tanh^{-1}(x)}{dx} = \frac{1}{1-x^2},$$

where

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}, \quad \tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}.$$

Then we should deal separately with the cases $a > 0$ and $a < 0$. If $a > 0$ integrating equation (4.23)

$$\int \frac{dy}{a+y^2} = dt \Leftrightarrow \frac{1}{\sqrt{a}} \int \frac{1}{1+x^2} dx = t + k \Leftrightarrow \frac{1}{\sqrt{a}} \tan^{-1}(x) = t + k$$

where we defined $x = y/\sqrt{a}$. Solving the last equation for x and mapping back to y we get

$$y(t) = \sqrt{a}\left(\tan\left(\sqrt{a}(t+k)\right)\right).$$

If $a < 0$ we integrate equation (4.23) by using a similar transformation, but instead with $x = y/\sqrt{-a}$ to get

$$\int \frac{dy}{a+y^2} = dt \Leftrightarrow -\frac{1}{\sqrt{-a}} \int \frac{1}{1-x^2} dx = t + k \Leftrightarrow -\frac{1}{\sqrt{-a}} \tanh^{-1}(x) = t + k.$$

Then

$$y(t) = -\sqrt{-a}\left(\tanh\left(\sqrt{-a}(t+k)\right)\right).$$

4.B Solution for a general Bernoulli equation

Consider the Bernoulli equation

$$\dot{y} = ay + by^{\eta},\ a \neq 0,\ b \neq 0 \tag{4.24}$$

where $y : T \to \mathbb{R}$. We intoduce a first transformation $z(t) = y(t)^{1-\eta}$, which leads to a linear ODE

$$\dot{z} = (1-\eta)(az + b) \tag{4.25}$$

beause

$$\begin{aligned} \dot{z} &= (1-\eta)y^{-\eta}\dot{y} = \\ &= (1-\eta)(ay^{1-\eta} + b) = \\ &= (1-\eta)(az + b). \end{aligned}$$

To solve equation (4.25) we introduce a second transformation $w(t) = z(t) + \frac{b}{a}$. Observing that $\dot{w} = \dot{z}$ we obtain a homogeneous ODE $\dot{w} = a(1-\eta)w$ which has solution

$$w(t) = k_w e^{a(1-\eta)t}.$$

Then the solution to equation (4.25) is

$$z(t) = -\frac{b}{a} + (k_z + \frac{b}{a})e^{a(1-\eta)t}$$

because $k_w = k_z + \frac{b}{a}$.

We finally get the solution for the Bernoulli equation (4.24)

$$y(t) = \left(-\frac{b}{a} + (k^{1-\eta} + \frac{b}{a})e^{a(1-\eta)t}\right)^{\frac{1}{1-\eta}} \tag{4.26}$$

4.C Solution to the cubic polynomial equation

Consider the (monic) cubic polynomial equation

$$y^3 - by - a = 0 \tag{4.27}$$

Write $y = u + v$. Then we get the equivalent representation

$$u^3 + v^3 + 3\left(uv - \frac{b}{3}\right) - a = 0$$

which holds if u and v solve simultaneously

$$\begin{cases} u^3+v^3=a \\ uv=\dfrac{b}{3} \end{cases} \Leftrightarrow \begin{cases} u^3u^3+u^3v^3-u^3a=0 \\ u^3v^3=\left(\dfrac{b}{3}\right)^3 \end{cases} \Leftrightarrow \begin{cases} u^6-au^3+\left(\dfrac{b}{3}\right)^3=0 \\ uv=\frac{b}{3}. \end{cases}$$

The first equation is a quadratic polynomial in u^3 which has roots

$$u^3=\frac{a}{2}\pm\sqrt{\Delta},\ \text{where } \Delta\equiv\left(\frac{a}{2}\right)^2-\left(\frac{b}{3}\right)^3$$

where Δ is the discriminant in equation (4.7). We can take any solution of the previous equation and set $\theta\equiv\frac{a}{2}+\sqrt{\Delta}$.

At this stage it is useful to observe that the solutions of equation $x^3=1$ are

$$x_1=1,\ x_2=\omega,\ x_3=\omega^2.$$

where $\omega=e^{\frac{2\pi i}{3}}=-\frac{1}{2}(1-\sqrt{3}i)$ and $\omega^2=e^{\frac{4\pi i}{3}}=-\frac{1}{2}(1+\sqrt{3}i)$. Therefore $u^3=\theta$ has also three solutions

$$u_1=\theta^{\frac{1}{3}},\ u_2=\omega\theta^{\frac{1}{3}},\ u_3=\omega^2\theta^{\frac{1}{3}}$$

and because $v=\frac{b}{3u}$, we finally get the solutions to equation (4.27) are

$$y_j=w^{j-1}\theta^{\frac{1}{3}}+\frac{b}{3}\left(\omega^{j-1}\theta^{\frac{1}{3}}\right)^{-1},\ j=1,2,3.$$

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