Foundations of Financial Economics
Two period GE: heterogeneous agents

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Topics for today

- Sources of heterogeneity
- AD equilibrium with heterogeneous agent economies
- Aggregate and idiosyncratic uncertainty

Sources of heterogeneity:
there is heterogeneity if there are at least two agents j and l such that they differ in:

- **information**: their probability spaces may be different $(\Omega_j, P_j) \neq (\Omega_l, P_l)$
- **preferences**: their degree of impatience, and/or attitudes towards risk may differ: $\beta^j \neq \beta_l$, $u^j(.) \neq u^l(.)$
- **endowments**: their wealth may differ: $y^j = \{y_0^j, Y_1^j\} \neq y^l = \{y_0^l, Y_1^l\}$

Types of uncertainty: related with state-dependency

- If $Y_1^j \neq Y_1^l$ we say there is **idiosyncratic uncertainty**
- If $Y_1 = \sum_{i=1}^{I} Y_1^i$ **is state-independent**, i.e., $y_{1,s} = \bar{y}_1$ for all $s = 1, \ldots, N$ then there is **aggregate certainty**,
- If $Y_1 = \sum_{i=1}^{I} Y_1^i$ **is state-dependent**, i.e., there is a pair of components of Y_1 such that $y_{1,s} \neq y_{1,s'}$ for all $s, s^{'} = 1, \ldots, N$ then we say there is **aggregate uncertainty**

Heterogeneity in AD economies

Heterogeneity: sources and types

Then, we can have:

- **idiosyncratic and aggregate certainty**: the GE is deterministic (both consumption at $t = 1$ and asset prices are deterministic) (This was the case studied in chapter 2)
- **idiosyncratic and aggregate uncertainty**: the GE is stochastic (both consumption at $t = 1$ and asset prices are stochastic)
- **idiosyncratic uncertainty and aggregate certainty**: the GE is partially stochastic (consumption at $t = 1$ can be stochastic or deterministic and asset prices are deterministic)

In an **homogeneous** agent economy idiosyncratic and aggregate uncertainty are undistinguishable.

In a **heterogeneous** agent economy they differ.

GE for an AD economy with heterogeneous agents

Definition: General equilibrium (GE):

- is the sequence of **distributions** $\{(c_0^{*1}, \dots c_0^{*I}), (C_1^{*1}, \dots C_1^{*I})\}$ and prices q such that:
  1. every consumer $i = 1, \dots, I$ determines the optimal sequence $\{c_0^i, C_1^i\}$ by solving the problem

$$\max_{\{c_0^i, C_1^i\}} \mathbb{E}^i{}_0 \left[u^i(c_0^i) + \beta^i u^i(C_1^i) \right]$$

$$c_0^i - y^i{}_0 + q(C_1^i - Y_1^i) = 0$$

     given q and $\{y_0^i, Y_1^i\}$;
  2. the good market clears in every period:

$$C_t = Y_t, t = 0, 1$$

where **aggregate** consumption and endowments are

$$C_t = \sum_{i=1}^{I} C_t^i, \;\; Y_t = \sum_{i=1}^{I} Y_t^i, \;\; t = 0, 1$$

Case 1: General equilibrium with heterogeneity in endowments

Assumptions: logarithmic preferences, and **idiosyncratic** uncertainty as regards endowments Y_1^i.

Question: what are the properties of the equilibrium stochastic discount factor ?

Method of determination: we have to solve explicitly the consumers' problems

Case 1: General equilibrium with heterogeneity in endowments

Determination

1. household' $i \in 1, \ldots, I$ problem

$$\max_{c_0^i, c_{11}^i, \ldots, c_{1N}^i} \ln\left(c_0^i\right) + \beta \sum_{s=1}^{N} \pi_s \ln\left(c_{1s}^i\right)$$

   subject to

$$c_0^i + \sum_{s=1}^{N} q_s c_{1s}^i \leq h^i \equiv y_0^i + \sum_{s=1}^{N} q_s y_{1s}^i$$

   where q_s is given to the consumer.

2. optimal consumption of household $i \in 1, \ldots, I$ (without satiation)

$$\begin{aligned} c_0^i &= \frac{1}{1+\beta} h^i \\ c_{1s}^i &= \frac{\pi_s \beta}{q_s(1+\beta)} h^i \end{aligned}$$

Case 1: General equilibrium with heterogeneity in endowments

Determination: continuation

1. Aggregate supply

$$
\begin{aligned}
y_0 &= \sum_{i=1}^{I} y_0^i \\
y_{1,s} &= \sum_{i=1}^{I} y_{1,s}^i, \ s = 1, \ldots, N
\end{aligned}
$$

2. Aggregate demand

$$
\begin{aligned}
c_0 &= \sum_{i=1}^{I} c_0^i = \frac{1}{1+\beta} h \\
c_{1,s} &= \sum_{i=1}^{I} c_{1,s}^i = \frac{\beta \pi_s}{q_s (1+\beta)} h, \ s = 1, \ldots, N
\end{aligned}
$$

Case 1: General equilibrium with heterogeneity in endowments

Determination: continuation

1. Aggregate wealth

$$h = \sum_{i=1}^{I} h^i = y_0 + \sum_{s=1}^{N} q_s y_{1,s}$$

2. Market clearing conditions

$$\begin{aligned} c_0 &= y_0 \Leftrightarrow \frac{1}{1+\beta} h = y_0 \\ c_{1,s} &= y_{1,s} \Leftrightarrow \frac{\beta \pi_s}{q_s(1+\beta)} h = y_{1,s}, \ s = 1, \ldots, N \end{aligned}$$

3. Then

$$\frac{\beta \pi_s y_0}{q_s} = y_{1,s}$$

Case 1: General equilibrium with heterogeneity in endowments

Characterization

Proposition 1

Consider a AD economy in which there is heterogeneity in endowments and homogeneity in preferences and information. Then the equilibrium stochastic discount factor is independent of the distribution of income.

Let $y_{1,s} = (1+\gamma_s)y_0$ and assume a logarithmic utility function. Then he **equilibrium discount factor** is

$$m_s = \frac{q_s}{\pi_s} = \beta \frac{y_0}{y_{1,s}} = \frac{\beta}{1+\gamma_s}, \ s = 1, \ldots, N$$

Case 1: General equilibrium with heterogeneity in endowments

Interpretation: the equilibrium discount factor $M = (m_1, \dots m_N)$ where

$$m_s = \frac{\beta}{1+\gamma_s}, \text{ for } s = 1, \dots, N$$

- is independent of the distribution of endowments among agents (only depends on the growth factor of the **aggregate endowments**
- if there is **aggregate uncertainty** then it is **state-dependent** (stochastic)
- if there is **aggregate certainty** (even if there is idiosyncratic uncertainty) then it is **state-independent** (i.e, deterministic):

$$m_s = m = \frac{\beta}{1+\gamma}, \text{ for all } s = 1, \dots, N.$$

Case 1: General equilibrium with heterogeneity in endowments

Characterization

Proposition 2

Consider the previous economy, in which there is idiosyncratic uncertainty but aggregate certainty (i.e, $Y_1 = y_1$ for all states $s = 1, \ldots, N$). Then there is ***perfect insurance*** *consumption at time $t = 1$ is state independent.*

Next we prove that

$$c_{1s}^{*i} = c_1^{*i} = \frac{1+\gamma}{1+\beta} h^{*i},\ \forall s = 1, \ldots, N$$

is state-independent if $Y_1 = y_1 = (1+\gamma)y_0$

Case 1: General equilibrium with heterogeneity in endowments

Proof of Proposition 2

- In equilibrium

$$c^i_{1s} = \frac{\beta}{m^*_s(1+\beta)} h^i = \frac{1+\gamma_s}{1+\beta} h^i$$

- The **equilibrium distribution** of human wealth is (if we substitute m_s)

$$h^{*i} = y^i_0 + \beta \sum_s \frac{\pi_s y^i_{1,s}}{1+\gamma_s} = y^i_0 \left(1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma^i_s}{1+\gamma_s}\right) \; i = 1, \ldots, I$$

- If there is no aggregate uncertainty $1 + \gamma_s = 1 + \gamma$ for every $s = 1, \ldots, N$

Case 1: General equilibrium with heterogeneity in endowments

Consumption distribution

Proposition 3

In equilibrium, the weight of agents' i consumption relative to aggregate consumption is stationary (i.e, time-independent), state independent and is equal to its equal to its share of aggregate wealth.

Case 1: General equilibrium with heterogeneity in endowments

Consumption distribution

- The equilibrium aggregate human wealth is

$$h^* = y_0 + \beta \sum_s \frac{\pi_s y_{1,s}}{1+\gamma_s} = y_0 \left(1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma_s}{1+\gamma_s} \right) = y_0 (1+\beta)$$

- The distribution of consumption at $t = 0$ is

$$\frac{c_0^{*i}}{c_0} = \frac{1}{1+\beta} \frac{h^{*i}}{y_0} = \frac{h^{*i}}{h} = \frac{y_0^i}{y_0} \left(\frac{1 + \beta \sum_{s=1}^{N} \pi_s \frac{1+\gamma_s^i}{1+\gamma_s}}{1+\beta} \right)$$

- and at $t = 1$ is

$$\frac{c_{1s}^{*i}}{c_{1s}} = \frac{1+\gamma_s}{1+\beta} \frac{h^{*i}}{y_{1s}} = \frac{1}{1+\beta} \frac{h^{*i}}{y_0} = \frac{h^{*i}}{h}, \text{for all } s = 1, \ldots, N$$

Example 1: homogeneous agent economy

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	50	45	55
y^b	50	45	55
y	100	90	110
m		**1.089**	**0.891**
c^a	50	45	55
c^b	50	45	55

Table: Two homogeneous agents (a and b). Common parameter: $\beta = 1/1.02$. Idiosyncratic and aggregate uncertainty

Example 2: heterogenous agents and aggregate uncertainty

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	30	27	33
y^b	70	63	77
y	100	90	110
$\mathbf{m}$		**1.089**	**0.891**
c^a	30	27	33
c^b	70	63	77

Table: Two heterogeneous agents (a and b). Common parameter: $\beta = 1/1.02$. Idiosyncratic and aggregate uncertainty

Example 3: idiosyncratic uncertainty and aggregate certainty

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	50	45	55
y^b	50	55	45
y	100	**100**	**100**
m		**0.98**	**0.98**
c^a	50	50	50
c^b	50	50	50

Table: Two heterogeneous agents (a and b). Common parameter: $\beta = 1/1.02$. Idiosyncratic uncertainty and aggregate certainty: **perfect insurance**

Case 1: General equilibrium with heterogeneity in endowments

Characterization

- **Summing up**:
  - if there is **aggregate certainty** then:
    the stochastic discount factor is **deterministic** and there is **perfect insurance** c_1^i is state-independent (because γ is state-independent);
  - if there is **aggregate uncertainty** then:
    the stochastic discount factor is **stochastic** and there is **not** perfect insurance c_1^i is state-dependent (because γ is state-dependent);
- Then:
  - only aggregate variables determine the stochastic discount factor;
  - the **distribution of income is irrelevant** for the determination of the stochastic discount factors
- **Those results extend to a finance economy with complete asset markets.**

Case 1: General equilibrium with heterogeneity in endowments

Comparing a representative agent with a heterogeneous agent economy

- In a representative agent economy we can only have two cases
  - Aggregate and individual (idiosyncratic) certainty
  - Both aggregate and individual (idiosyncratic) uncertainty. In this case there is not insurance
- In a heterogeneous agent economy we have three cases
  - Aggregate and individual (idiosyncratic) certainty
  - Both aggregate and individual (idiosyncratic) uncertainty. In this case there is some insurance
  - Aggregate certainty and individual (idiosyncratic) uncertainty. In this case there can be **perfect insurance** and redistribution.

Case 2: General equilibrium with heterogeneity in endowments and preferences

Assumptions

- homogeneous utility function: logarithmic
- heterogeneity in **impatience** (β^i). Let the distribution of psychological discount factors be represented by

$$B = (\beta^1, \ldots, \beta^i, \ldots \beta^I)$$

- **idiosyncratic uncertainty as regards endowments** Y_1^i

Case 2: General equilibrium with heterogeneity in endowments and preferences

The consumption problem is now

$$\max_{c_0^i, c_{11}^i, \ldots, c_{1N}^i} \ln\left(c_0^i\right) + \beta^i \sum_{s=1}^{N} \pi_s \ln\left(c_{1s}^i\right)$$

subject to

$$c_0^i + \sum_{s=1}^{N} \pi_s m_s c_{1s}^i \leq h^i \equiv y_0^i + \sum_{s=1}^{N} \pi_s m_s y_{1s}^i$$

Case 2: General equilibrium with heterogeneity in endowments and preferences

Solution to the household i problem

- The optimal consumption process for household i is

$$
\begin{aligned}
c_0^i &= \frac{1}{1+\beta^i} h^i, \ i = 1, \ldots, I \\
c_{1s}^i &= \frac{\beta^i}{m_s(1+\beta^i)} h^i, \ i = 1, \ldots, I
\end{aligned}
$$

Case 2: General equilibrium with heterogeneity in endowments and preferences

Endowment distribution

- Define the process for the shares of household i in the aggregate endowments, $\{\phi_0^i, \Phi_1^i\}$,
- At time $t = 0$ we have

$$\phi_0^i = \frac{y_0^i}{y_0} = \frac{y_0^i}{\sum_{i=1}^I y_0^i} \text{ for } i = 1, \ldots, I$$

  where $\sum_{i=1}^I \phi_0^i = 1$ and

- At time $t = 1$ we have

$$\phi_{1,s}^i = \frac{y_{1,s}^i}{y_{1,s}} = \frac{y_{1,s}^i}{\sum_{i=1}^I y_{1,s}^i} \text{ for } s = 1, \ldots, N, \quad i = 1, \ldots, I$$

  where $\sum_{i=1}^I \phi_{1,s}^i = 1$ for all $s = 1, \ldots, N$

Case 2: General equilibrium with heterogeneity in endowments and preferences

Wealth distribution

- Then the human wealth of consumer i can be written as

$$h^i = \left(\phi_0^i + \sum_{s=1}^{N} m_s \pi_s (1+\gamma_s) \phi_{1,s}^i \right) y_0, \ i = 1, \ldots, I$$

because $y_0^i = \phi_0^i y_0$ and $y_{1s}^i = \phi_{1s}^i y_{1s} = \phi_{1s}^i (1+\gamma_s) y_0$

Case 2: General equilibrium with heterogeneity in endowments and preferences

Market clearing conditions

- The market clearing conditions are

$$
\begin{aligned}
c_0 &= y_0 \Leftrightarrow \sum_{i=1}^{I} \frac{h^i}{1+\beta^i} = y_0 \\
c_{1,s} &= y_{1,s} \Leftrightarrow \frac{1}{m_s} \left(\sum_{i=1}^{I} \frac{\beta^i h^i}{1+\beta^i} \right) = (1+\gamma_s) y_0, \; s = 1, \ldots, N
\end{aligned}
$$

- Observation: now we are summing not only over wealth h^i but also over the distribution of the discount factors β^i (B)

Case 2: General equilibrium with heterogeneity in endowments and preferences

Market clearing conditions

- Define

$$z_0 = z_0(B) \equiv \sum_{i=1}^{I} \frac{\beta^i \phi_0^i}{1+\beta^i},$$

$$z_{1,s} = z_{1,s}(B) \equiv \sum_{i=1}^{I} \frac{\beta^i \phi_{1,s}^i}{1+\beta^i}$$

- Then, the equilibrium conditions for $t = 1$ can be written as (check !)

$$z_0(B) + \sum_{s=1}^{N} \pi_s m_s(1+\gamma_s) z_{1,s}(B) = m_s(1+\gamma_s), \; s = 1, \ldots, N$$

- This implies $m_1(1+\gamma_1) = m_2(1+\gamma_2) = \ldots = m_N(1+\gamma_N)$.

Case 2: General equilibrium with heterogeneity in endowments and preferences

Then we determine the **equilibrium discount factor**

$$\boxed{m_s = \tilde{\beta}(B)\frac{1}{1+\gamma_s},\ \tilde{\beta}(B) \equiv \left(\frac{z_0(B)}{1-\mathbb{E}[z_1(B)]}\right)}$$

Conclusions:

- if there is heterogeneity in the psychological discount factor and there is idiosyncratic uncertainty then the **equilibrium stochastic discount factor is formally similar to the homogeneous case**: it multiplies a weighted psychological discount factor with the inverse of the endowment growth factor;
- the weighted psychological discount factor, $\tilde{\beta}$ **depends upon the distribution of income** but is state-independent and constant;
- If there is **no** aggregate uncertainty then the stochastic discount factor m is **state-independent**.

Example 2 bis: heterogenous agents and aggregate uncertainty

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	30	27	33
y^b	70	63	77
y	100	90	110
m		**1.094**	**0.895**
c^a	30.2	26.8	32.8
c^b	69.8	63.2	77.2

Table: Two heterogeneous agents (a and b). Heterogenous preferences: $\beta^a = 1/1.025$ $\beta^b = 1/1.015$. Idiosyncratic and aggregate uncertainty

Example 3 bis: idiosyncratic uncertainty and aggregate certainty

	$t=0$	$t=1$	
		$s=1$	$s=2$
y^a	50	45	55
y^b	50	55	45
y	100	**100**	**100**
$\mathbf{m}$		**0.9804**	**0.9804**
c^a	50.2	49.8	49.8
c^b	49.8	50.2	50.2

Table: Two heterogeneous agents (a and b) where b is more patient than a: $\beta^a = 1/1.025$ $\beta^b = 1/1.015$. There is both idiosyncratic uncertainty and aggregate certainty: **perfect insurance**. But as b is more patient the time profile of consumption is different from a which is less patient.