Mathematical Economics
Discrete time: optimal control problem

Paulo Brito

[1] pbrito@iseg.ulisboa.pt
University of Lisbon

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Optimal control problem

We present the optimality conditions for three problems:

- Simplest problem: x_0, x_T and T given
- Constrained terminal state problem: x_0 and T given and x_T constrained
- Discounted infinite horizon problem

Optimal control problem

Timing and value of the decisions

value flow		F_0		F_1				F_t				F_{T-2}		F_{T-1}	
control		u_0		u_1				u_t				u_{T-2}		u_{T-1}	
period		0		1				t				$T-2$		$T-1$	
time	0		1		2		t		$t+1$		$T-2$		$T-1$		T
state	ϕ_0		x_1		x_2		x_t		x_{t+1}		x_{T-2}		x_{T-1}		ϕ_T

Optimal control problem

Timing and value of the decisions

- The **action in period** t: in the beginning the state x_t is given, during the period the control u_t is chosen, at the end the state variable will be $x_{t+1} = G(x_t, u_t, t)$
- The **value obtained by** u_t, given the state x_t, is

$$F_t = F(x_t, u_t, t) \text{ period } t = 0, 1, \ldots, T-1$$

- The **value of choosing** the sequence $u = \{u_0, u_1, \ldots, u_{T-1}\}$ is

$$J(u, x) = F(0, x_0, u_0) + \ldots + F(t, x_t, u_t) + \ldots + F(T-1, x_{T-1}, u_{T-1})$$

- The **optimal sequence** u^* is such that

$$J^* = J[u^*] = \max_u \{J[x, u] : (x, u) \in \mathcal{D}\}$$

Optimal control: simplest problem

- **Problem OCP**: Find $x^* = \{x_0^*, x_1^*, \ldots, x_T^*\}$ and $u^* = \{u_0^*, u_1^*, \ldots, u_T^*\}$ that maximizes

$$J[x, u] = \sum_{t=0}^{T-1} F(u_t, x_t, t)$$

subject to

$$\begin{cases} x_{t+1} = G(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T = \phi_T & t = T \end{cases}$$

T, ϕ_0 and ϕ_T given

Pontryagin's maximum principle (PMP)

- This is a second method for solving dynamic optimization problems.
- In order to use it, we define the Hamiltonian function

$$H_t = H(\psi_t, x_t, u_t, t) = f(x_t, u_t, t) + \psi_t G(x_t, u_t, t)$$

  where ψ_t is called the co-state variable at time t (obs: it has the same timing as x_t)
- Maximized Hamiltonian is a function

$$H_t^*(\psi_t, x_t^*) = \max_u H_t(\psi_t, x_t, u_t)$$

  for the optimal control, $u_t^* = u^*(x_t, \psi_t)$.

PMP: necessary first order conditions from

Proposition

- *If* $x^* = \{x^*_t\}_{t=0}^{T}$ *and* $u^* = \{u^*_t\}_{t=0}^{T-1}$ *are solutions to the OCP, there is a sequence of the co-state variable* $\psi = \{\psi_t\}_{t=0}^{T-1}$ *such that the following conditions hold:*
- *the optimality conditions*

$$
\begin{aligned}
\frac{\partial H^*_t}{\partial u_t} &= 0,\ t = 0, \ldots, T-1 \\
\psi_t &= \frac{\partial H^*_{t+1}}{\partial x_{t+1}},\ t = 0, \ldots, T-1
\end{aligned}
$$

- *and the admissibility conditions*

$$
\begin{aligned}
x^*_{t+1} &= G(x^*_t, u^*_t, t)\ t = 0, \ldots, T-1 \\
x^*_0 &= \phi_0,\ t = 0 \\
x^*_T &= \phi_T,\ t = T
\end{aligned}
$$

Proof.

Maximized Hamiltonian dynamic system (MHDS)

- If $\partial^2 H_t / \partial u_t^2 \neq 0$ then the optimality condition

$$\frac{\partial H_t^*}{\partial u_t} = \frac{\partial F(x_t^*, u_t^*, t)}{\partial u_t} + \psi_t \frac{\partial G(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

  can be solved for the optimal control

$$u_t^* = U(x_t^*, \psi_t, t)$$

- Substituting into the f.o.c we get the **MHDS** in (x_t, ψ_t)

$$\begin{cases} \psi_t = p(x_{t+1}^*, \psi_{t+1}, t+1) \\ x_{t+1}^* = G(x_t^*, \psi_t, t) \\ x_0^* = \phi_0 \\ x_T^* = \phi_T \end{cases}$$

  where

$$\begin{aligned} p(x_{t+1}, \psi_{t+1}) &\equiv \frac{\partial H_{t+1}^*}{\partial x_{t+1}}(x_{t+1}^*, U(x_{t+1}^*, \psi_{t+1}, t+1), \psi_{t+1}, t+1) \\ G(x_t^*, \psi_t, t) &= G(x_t^*, U(x_t^*, \psi_t, t), t) \end{aligned}$$

Alternative MHDS

- Alternatively we can solve

$$\frac{\partial H_t^*}{\partial u_t} = \frac{\partial F(x_t^*, u_t^*, t)}{\partial u_t} + \psi_t \frac{\partial G(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

for $\psi_t = q_t(u_t^*, x_t^*, t)$ and we get an alternative **MHDS** in (x_t, u_t)

$$\begin{cases} x_{t+1}^* = G(x_t^*, u_t^*, t) \\ u_{t+1}^* = U(x_t^*, u_t^*, t, t+1) \\ x_0^* = \phi_0 \\ x_T^* = \phi_T \end{cases}$$

- If we compare with the f.o.c for an optimum for the calculus of variations problem, instead of a scalar second order difference equation we have a planar first-order equation.
- The system is characterized by a **forward** equation, $x_{t+1} = G(x_t, u_t, t)$, and a **backward** equation, $u_{t+1} = U(x_t, u_t, t, t+1))$

Application: cake eating problem

- The problem

$$\max_{\{C\}} \sum_{t=0}^{T} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = W_t - C_t,\ W_0 = \phi,\ W_T = 0.$$

- The Hamiltonian for this problem is

$$H_t = \beta^t \ln(C_t) + \psi_t(W_t - C_t)$$

- The first order conditions are

$$\begin{cases} \dfrac{\partial H_t^*}{\partial C_t} = \beta^t (C_t^*)^{-1} - \psi_t = 0, & t = 0, \ldots, T-1 \\ \psi_t = \dfrac{\partial H_{t+1}^*}{\partial W_{t+1}} = \psi_{t+1}, & t = 0, \ldots, T-1 \\ W_{t+1}^* = W_t^* - C_t^*, & t = 0, \ldots, T-1 \\ W_T^* = 0, & t = 0 \\ W_0^* = \phi, & t = T \end{cases}$$

Cake eating problem: MHDS

- The MHDS in (W_t, C_t) is

$$
\begin{aligned}
C^*_{t+1} &= \beta C^*_t && (1)\\
W^*_{t+1} &= W^*_t - C^*_t, \ t = 0, \dots, T-1 && (2)\\
W^*_T &= 0 && (3)\\
W^*_0 &= \phi && (4)
\end{aligned}
$$

- To find the solution we have to solve it.
- There are several methods to do it. Examples: (1) solve recursively; (2) as a system of linear DE

Cake eating problem: recursive solution

1 Solve the "Euler-equation" (1)

$$C_t = C_0 \beta^t. \tag{5}$$

where C_0 is an arbitrary constraint;

2 Substitute it in the constraint (2)

$$W_{t+1} = W_t - C_0 \beta^t$$

3 Solve it to find

$$W_t = k - C_0 \sum_{s=0}^{t-1} \beta^s = k - C_0 \frac{1-\beta^t}{1-\beta} \tag{6}$$

Cake eating problem: recursive solution (cont.)

4 Evaluate the solution for W_t at the initial and terminal time

$$\begin{cases} W_0 = k \\ W_T = k - C_0 \frac{1-\beta^T}{1-\beta} \end{cases}$$

5 Remember the the initial and terminal constraints (3) and (4)

$$\begin{cases} W_0 = k = \phi \\ W_T = k - C_0 \frac{1-\beta^T}{1-\beta} = 0 \end{cases}$$

6 Solve the system for k and C_0 to get $C_0 = \frac{1-\beta}{1-\beta^T}\phi$ and $k = \phi$

Cake eating problem: recursive solution (cont.)

- Substitute C_0 and k into equations (6) and (5)
- We get the **solution to the optimal control problem**

$$W_t^* = \phi \left(\frac{\beta^t - \beta^T}{1 - \beta^T} \right), \ t = 0, \ldots, T$$

$$C_t^* = \phi \left(\frac{1 - \beta}{1 - \beta^T} \beta^t \right), \ t = 0, \ldots, T - 1.$$

Optimal control problems with terminal constraints

Problem OCPTC: find $u = \{u_t\}_{t=0}^{T-1}$ and $x = \{x_t\}_{t=0}^{T}$ that solves

- $$\max_u \sum_{t=0}^{T-1} F(u_t, x_t, t)$$

  with T finite and known
- subject to the constraints

$$\begin{cases} x_{t+1} = G(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \end{cases}$$

- and, the alternative terminal constrains

$$x_T \text{ free}, \text{ or } x_T \geq \phi_T$$

PMP for the free-endpoint problem

Proposition

- *If* $x^* = \{x_t^*\}_{t=0}^{T}$ *and* $u^* = \{u_t^*\}_{t=0}^{T-1}$ *are solutions of the OCP, there is a sequence* $\psi = \{\psi_t\}_{t=0}^{T-1}$ *such that*
- *the optimality conditions*

$$\begin{aligned} \frac{\partial H_t^*}{\partial u_t} &= 0,\ t = 0, 1, \dots, T-1 \\ \psi_t &= \frac{\partial H_{t+1}^*}{\partial x_{t+1}},\ t = 0, \dots, T-1 \end{aligned}$$

- *the admissibility conditions*

$$\begin{aligned} x_{t+1}^* &= G(x_t^*, u_t^*, t) \\ x_0^* &= \phi_0 \end{aligned}$$

- *and the transversality conditions*

$$\psi_{T-1} = 0, \text{ or } \psi_{T-1}(x_T^* - \phi_T) = 0$$

Discounted OCP with infinite horizon

- **Problem OCPIH**: find $(u, x) = \{(u_t, x_t)\}_{t=0}^{\infty}$ such that

$$\max_{u} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t),\ 0 < \beta < 1$$

- subject to

$$\begin{aligned} x_{t+1} &= g(x_t, u_t),\ t = 0, 1, \ldots \\ x_0 &= \phi_0,\ \text{given} \end{aligned}$$

note this is a free endpoint problem ($T = \infty$ is undetermined)

Discounted OCP with infinite horizon (cont.)

- The discounted-value Hamiltonian is

$$\begin{aligned} H_t &= \beta^t f(u_t, x_t) + \psi_t g(u_t, x_t) \\ &= \beta^t \left(f(u_t, x_t) + \beta^{-t} \psi_t g(u_t, x_t) \right) \\ &= \beta^t h_t \end{aligned}$$

- We define the current-value Hamiltonian

$$h_t \equiv h(x_t, \eta_t, u_t) = f(u_t, x_t) + \eta_t g(u_t, x_t)$$

where the current-value co-state variable is

$$\eta_t = \beta^{-t} \psi_t$$

PMP for the infinite horizon problem

Proposition

- *The solution of problem OCPIH verifies the following conditions:*

$$
\begin{aligned}
\frac{\partial h_t^*}{\partial u_t} &= 0, \; t = 0, \ldots, \infty && (7)\\
\eta_t &= \beta \frac{\partial h_{t+1}^*}{\partial x_{t+1}}, \; t = 0, \ldots, \infty && (8)\\
x_{t+1}^* &= g(x_t^*, u_t^*), \; t = 0, \ldots, \infty && (9)\\
x_0^* &= \phi_0, \; t = 0 && (10)
\end{aligned}
$$

- *plus: terminal values and transversality conditions*

$$\lim_{t\to\infty} x_t \, \textit{free}, \; \lim_{t\to\infty} \beta^t \eta_t = 0 \tag{11}$$

or

$$\lim_{t\to\infty} x_t \geq 0 \; \lim_{t\to\infty} \beta^t \eta_t x_t^* = 0 \tag{12}$$

Solving the MHDS: methods

- In OPCIH problems the MHDS can be written as a system of autonomous difference equations

$$\begin{aligned} u^*_{t+1} &= k(u^*_t, x^*_t) \\ x^*_{t+1} &= g(u^*_t, x^*_t) \end{aligned}$$

- **If the system is linear** we can use the following rule of thumb to solve the system:
  1. try to **reduce the dimensionality** of the system: this is the case, v.g.
     1. if the system is recursive (**method 1**): solve the scalar equation and substitute the solution in the other equation;
     2. find other type of reduction: if we can express the system into a variable like $z_t = \eta_t x_t$ (**method 2**)
  2. if we **cannot reduce the dimensionality** of the system: use the solution of planar linear difference equation (**method 3**)

Application: consumption-investment problem

- Find the optimal consumption-investment strategy that solves the problem: find $C = \{C_t\}_{t=0}^{\infty}$ that

$$\max_{C} \sum_{t=0}^{\infty} \beta^t \ln (C_t) \text{ (inter-temporal utility)}$$

- subject to the constraints:

$$\begin{cases} W_{t+1} = (1+r) W_t - C_t, \text{ (intra-temporal budget constraint)} \\ W_0 = \phi, \text{ (initial wealth given)} \\ \lim_{t\to\infty} (1+r)^{-t} W_t \geq 0, \text{ (Non-Ponzi game condition)} \end{cases}$$

where $r > 0$ is the (given and constant) interest rate.

Solving through the PMP

- Discounted Hamiltonian

$$h_t = \ln\left(C_t\right) + \eta_t((1+r)\,W_t - C_t)$$

- PMP optimality conditions

$$\begin{cases} \dfrac{1}{C_t} = \eta_t \\ \eta_t = \beta(1+r)\eta_{t+1} \\ W_{t+1} = (1+r)\,W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t\to\infty} \beta^t \eta_t W_t = 0 \end{cases}$$

MHDS

- Eliminating η we get:
- the maximized Hamiltonian dynamic system (MHDS)

$$
\begin{aligned}
C_{t+1} &= \beta(1+r)C_t && (13)\\
W_{t+1} &= (1+r)W_t - C_t && (14)
\end{aligned}
$$

- and the initial and transversality conditions

$$
\begin{aligned}
W_0 &= \phi_0 && (15)\\
\lim_{t\to\infty} \beta^t \frac{W_t}{C_t} &= 0 && (16)
\end{aligned}
$$

Solving the MHDS: method 1

1 Solve equation (13):

$$C_t = C_0 \beta^t (1+r)^t,\ t \in \{0, 1, \ldots, \infty\}$$

where C_0 is unknown

2 Substitute in equation (14) to get

$$W_{t+1} = (1+r)\, W_t - C_0 \beta^t (1+r)^t,\ t \in \{0, 1, \ldots, \infty\} \tag{17}$$

3 Solve equation (17)

$$\begin{aligned} W_t &= W_0 (1+r)^t - C_0 \sum_{s=0}^{t-1} (1+r)^{t-s-1} (1+r)^s \beta^s = \\ &= W_0 (1+r)^t - C_0 (1+r)^{t-1} \sum_{s=0}^{t-1} \beta^s = \\ &= (1+r)^t \left(W_0 - \frac{C_0}{1+r} \left(\frac{1-\beta^t}{1-\beta} \right) \right) \end{aligned}$$

Solving the MHDS: method 1, cont.

4 Use the initial condition (15),

$$W_t = (1+r)^t \left(\phi - \frac{C_0}{1+r} \left(\frac{1-\beta^t}{1-\beta} \right) \right)$$

5 Use the transversality condition (16) to determine C_0

$$\begin{aligned} \lim_{t\to\infty} \beta^t \frac{W_t}{C_t} &= \frac{1}{C_0} \left(\phi - \frac{C_0}{(1+r)(1-\beta)} + \lim_{t\to\infty} \frac{C_0 \beta^t}{(1+r)(1-\beta)} \right) = \\ &= \frac{1}{C_0} \left(\phi - \frac{C_0}{(1+r)(1-\beta)} \right) = 0 \end{aligned}$$

if and only if

$$C_0 = \phi(1+r)(1-\beta)$$

1. Then the solution is

$$\begin{aligned} W_t^* &= \phi \beta^t (1+r)^t, \ t = 0, 1, \ldots, \infty \\ C_t^* &= (1+r)(1-\beta)\, W_t^* \ t = 0, 1, \ldots, \infty \end{aligned} \tag{18}$$

Solving the MHDS: method 2

1 Introduce a transformation of variables $z_t \equiv W_t / C_t$ (suggestion: use the transversality condition)

2 We get a scalar linear difference equation equivalent to equations (13) and (14)

$$z_{t+1} = \frac{W_{t+1}}{C_{t+1}} = \frac{(1+r)\,W_t - C_t}{\beta(1+r)\,C_t} = \frac{1}{\beta}\left(z_t - \frac{1}{1+r}\right)$$

3 Jointly with condition (16) we have a simpler boundary value problem for z_t

$$\begin{cases} z_{t+1} = \frac{1}{\beta}\left(z_t - \frac{1}{1+r}\right) \\ \lim_{t\to\infty} \beta^t z_t = 0. \end{cases}$$

4 The general solution for z_t is

$$z_t = \bar{z} + (k - \bar{z})\beta^{-t}k.$$

where

$$\bar{z} = \frac{1}{(1-\beta)(1+r)}$$

Solving the MHDS: method 2, cont

4 We use equation (16) to determine k

$$\lim_{t\to\infty} \beta^t z_t = \lim_{t\to\infty} \beta^t \bar{z} + k - \bar{z} = k - \bar{z} = 0$$

if and only if $k = \bar{z}$. Then z_t is time-independent

$$z_t = \bar{z} = \frac{1}{(1+r)(1-\beta)},\ t = 0, 1, \ldots, \infty$$

5 Because $C_t^* = (1-\beta)(1+r)\, W_t^*$, substituting in equation (14) and using condition (15) we can solve the initial value problem

$$\begin{cases} W_{t+1}^* = (1+r)\, W_t^* - C_t^* = \beta(1+r)\, W_t^*,\ t = 0, 1, \ldots \\ W_0^* = \phi \end{cases}$$

6 Which, after solving, yields the same solution (18)

Solving the MHDS: method 3

1 We write equations (13) and (14) in matrix notation

$$\begin{pmatrix} C_{t+1} \\ W_{t+1} \end{pmatrix} = \begin{pmatrix} \beta(1+r) & 0 \\ -1 & 1+r \end{pmatrix} \begin{pmatrix} C_t \\ W_t \end{pmatrix}$$

2 The general solution of this planar equation has the form Planar.

$$\begin{pmatrix} C_t \\ W_t \end{pmatrix} = h_+ \mathbf{P}^+ \lambda_+^t + h_- \mathbf{P}^- \lambda_-^t \tag{19}$$

3 Compute the eigenvalues $\lambda_\pm$. The characteristic polynomial is

$$\begin{aligned} c(\lambda) &= \lambda^2 - (1+r)(1+\beta)\lambda + \beta(1+r)^2 = \\ &= (\lambda - (1+r))\,(\lambda - \beta(1+r)) \end{aligned}$$

it happens to factorize (if not use the general formula). Then

$$\lambda_+ = 1+r,\ \lambda_- = \beta(1+r)$$

Solving the MHDS: method 3, cont

4 Compute the eigenvectors $\mathbf{P}^+$ and $\mathbf{P}^-$

$$\begin{pmatrix} (1+r)(\beta-1) & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{P}^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ -1 & (1+r)(1-\beta) \end{pmatrix} \begin{pmatrix} p_1^- \\ p_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{P}^- = \begin{pmatrix} (1+r)(1-\beta) \\ 1 \end{pmatrix}$$

5 Substituting in equation (19) we get

$$\begin{pmatrix} C_t \\ W_t \end{pmatrix} = h_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+r)^t + h_- \begin{pmatrix} (1+r)(1-\beta) \\ 1 \end{pmatrix} \beta^t (1+r)^t$$

then

$$C_t = (1-\beta)(1+r)^{1+t}\beta^t, \;\; W_t = (1+r)^t \left(h_+ + h_- \beta^t \right)$$

Solving the MHDS: method 3, cont

5 Then

$$\lim_{t\to\infty} \beta^t \frac{W_t}{C_t} = \lim_{t\to\infty} \frac{h_+ + h_- \beta^t}{(1+r)(1+\beta)h_-} = \frac{h_+}{(1+r)(1+\beta)h_-} = 0.$$

Condition (16) holds if and only if $h_+ = 0$.

6 Then

$$W_t = h_-(1+r)^t \beta^t$$

and condition (15) holds if and only if $h_- = \phi$.

7 We get the same solution (18)

Optimal consumption-saving: characterisation of the solution

- As
$$C_t^* = (1+r)(1-\beta)\,W_t^*$$
the dynamics of consumption is monotonously related to financial wealth
- The optimal stock of financial wealth is
$$W_t^* = \phi\left(\beta(1+r)\right)^t = \phi\left(\frac{1+r}{1+\rho}\right)^t,\ t = 0, 1, \ldots, \infty$$
where
$$\beta = \frac{1}{1+\rho}$$
$\rho =$ rate of time preference
- Characterisation of the solution
  - if $r > \rho$ then $\lim_{t\to\infty} W_t^* = \infty$ and $\lim_{t\to\infty} C_t^* = \infty$
  - if $r = \rho$ then $\lim_{t\to\infty} W_t^* = \phi$ and $\lim_{t\to\infty} C_t^* = \rho\phi$
  - if $r < \rho$ then $\lim_{t\to\infty} W_t^* = 0$ and $\lim_{t\to\infty} C_t^* = 0$

- Even though wealth and consumption may be unbounded (if $\rho < r$) the value functional is bounded
- The value of the intertemporal utility for the optimal consumption path is

$$
\begin{aligned}
J^* &= \sum_{t=0}^{\infty} \beta^t \ln\left(C_t^*\right) = \\
&= \sum_{t=0}^{\infty} \beta^t \ln\left((1+r)(1-\beta)\phi\left(\beta(1+r)\right)^t\right) = \\
&= \ldots \\
&= \frac{1}{1-\beta} \ln\left(\left[(1+r)(1-\beta)^{1-\beta}\beta^{\beta}\right]^{1/(1-\beta)} \phi\right)
\end{aligned}
$$

- is bounded if ϕ is bounded for any r and ρ
- This is a consequence of the transversality condition: what matters is boundedness in present value terms not at the asymptotic levels of the variables.

Infinite-horizon discounted problem: dynamic programming approach

Problem OCPIH Consider the infinite-horizon discounted optimal control problem: find $(x^*, u^*)_{t=0}^{\infty}$ that solves the problem

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \text{ s.t. } x_{t+1} = g(x_t, u_t),\ x_0 = \phi_0$$

where $0 < \beta < 1$ and ϕ_0 is given

Proposition

Let (x^*, u^*) be a solution to problem OCPIH: then it verifies the Hamilton-Jacobi-Bellman condition

$$V(x) = \max_u \{f(x, u) + \beta V(g(x, u))\}$$

for any admissible $x^*_t = x$ for $t \in \{0, \dots \infty\}$. Proof.

HJB equation: properties

- the properties of V are hard to determine: in general it is continuous, but differentiability is not assured
- if H is $\mathcal{C}^2$ then we can determine the optimal control through the **optimality condition**

$$\frac{\partial H(x,u)}{\partial u} = 0$$

  and we get the **policy function**

$$u^* = h(x)$$

- HJB equation becomes a non-linear functional equation

$$V(x) = f(x, h(x)) + \beta V[g(x, h(x))].$$

- both h and V have closed form solution only in very rare cases.

Application: the cake strikes again

- The HJB equation

$$V(W) = \max_{C} \left\{ \ln(C) + \beta V(W - C) \right\},$$

- Finding the optimal control: the optimality condition

$$\frac{\partial \left\{ \ln(C) + \beta V(W - C) \right\}}{\partial C} = 0.$$

- The best we can do is to say that optimal consumption is a function of the size of the cake

$$\frac{1}{C} - \beta V'(W - C) = 0 \Leftrightarrow C^{*} = C(W)$$

- and that the HJB has the form

$$V(W) = \ln(C(W)) + \beta V[W - C(W)].$$

The cake problem: solution

- Step 1: solve the HJB equation explicitly
  1. we use a trial function of W depending upon some undetermined coefficients;
  2. if the form of the function is right, then we use the method of the undetermined coefficients (try to get the unknown coefficients by substituting in the HJB equation)
  3. we get an explicit solution for C as a function of W
- Step 2: substitute $C(W)$ in the constraints of the problem to get

$$W_{t+1} = W_t - C(W_t)$$

- Step 3: solve the difference equation with $W_0 = \phi$

The cake problem: solving the HJB equation

- Trial solution:
$$V(W) = a + b\ln(W)$$
where a and b are unknown constants;
- Policy function:
$$\frac{1}{C} = \frac{\beta b}{W - C} \;\Rightarrow\; C = \frac{1}{1 + b\beta} W$$
- Substituting in the HJB equation
$$a + b\ln(W) = \ln(W) - \ln(1 + b\beta) + \beta\left(a + b\ln\left(\left(1 - \frac{1}{1 + b\beta}\right) W\right)\right),$$
- collecting terms
$$(b(1 - \beta) - 1)\ln(W) = a(\beta - 1) - \ln(1 + b\beta) + \beta b\ln\left(\frac{b\beta}{1 + b\beta}\right).$$

Step 1: solving the HJB equation

- then we determine

$$b = \frac{1}{1-\beta},\ a = \ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta)$$

- and

$$V(W) = \frac{1}{1-\beta}\ln(\chi W),\ \text{where } \chi \equiv \left(\beta^{\beta}(1-\beta)^{1-\beta}\right)^{1/(1-\beta)}.$$

- Then because

$$C^* = (1-\beta)\,W^*$$

is the optimal policy function and holds at all times

Step 2: optimal budget constraint

- Substituting the policy function in the intratemporal budget constraint we get

$$W^*_{t+1} = W^*_t - (1-\beta)\, W^*_t,\ t = 0, 1, \ldots, \infty$$

- given

$$W_0 = \phi, \text{ given}$$

Step 3: solution for the cake-eating problem

The infinite horizon cake eating problem has the solution:

- the optimal sequence of cake size $W^* = \{W_t^*\}_{t=0}^{\infty}$ is generated by

$$W_t^* = \phi\beta^t,\ t = 0, 1, \ldots, \infty$$

- the optimal sequence of cake consumption $C^* = \{C_t^*\}_{t=0}^{\infty}$ is generated by

$$C_t^* = \phi(1-\beta)\beta^t,\ t = 0, 1, \ldots, \infty$$

Proofs

Proof of proposition 1

- Assume we know the solution $(u^*, x^*) = \{x_t^*, u_t^*\}_{t=0}^{T}$ for the problem.
- The optimal value

$$J^* = \sum_{t=0}^{T-1} F(x_t^*, u_t^*, t)$$

- We write the Lagrangean

$$\begin{aligned} L &= \sum_{t=0}^{T-1} F(x_t, u_t, t) + \psi_t(G(x_t, u_t, t) - x_{t+1}) \\ &= \sum_{t=0}^{T-1} H_t(\psi_t, x_t, u_t, t) - \psi_t x_{t+1} = \\ &= \sum_{t=0}^{T-1} \ell(\psi_t, x_t, u_t, x_{t+1}, t) \end{aligned}$$

Proof of proposition 1

- Consider an arbitrary perturbation away from the solution to the problem, such that $x_t = x_t^* + \epsilon_t^x$. The perturbation is admissible if $\epsilon_0^x = \epsilon_T^x = 0$, and $u_t = u_t^* + \epsilon_t^u$. It induces the variation in value

$$L - J^* = \frac{\partial H_0}{\partial x_0}\epsilon_0^x + \sum_{t=1}^{T-1}\left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t}\right)\epsilon_t^x - \psi_{T-1}\epsilon_T^x + \sum_{t=0}^{T-1}\frac{\partial H_t}{\partial u_t}\epsilon_t^u +$$

$$+ \sum_{t=0}^{T-1}\left(\frac{\partial H_t}{\partial \psi_t} - x_{t+1}\right)\epsilon_t^\psi =$$

$$= \sum_{t=1}^{T-1}\left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t}\right)\epsilon_t^x + \sum_{t=0}^{T-1}\frac{\partial H_t}{\partial u_t}\epsilon_t^u + \sum_{t=0}^{T-1}\left(G(x_t, u_t, t) - x_{t+1}\right)\epsilon_t^\psi$$

- Then $L \leq J^*$ only if $\psi_{t-1} - \frac{\partial H_t}{\partial x_t} = \frac{\partial H_t}{\partial u_t} = x_{t+1} - G(x_t, u_t, t) = 0$.

Return.

Proof of proposition 2

- The value functional for x_t^* is

$$
\begin{aligned}
V(x_t^*) &= \sum_{s=t}^{\infty} \beta^{s-t} F(x_s^*, u_s^*) = \\
&= \max_{\{u_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} F(x_s, u_s) = \\
&= \max_{\{u_s\}_{s=t}^{\infty}} \left\{ F(x_t, u_t) + \beta \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} F(x_s, u_s) \right\} \\
&= \max_{u_t} \left\{ F(x_t, u_t) + \beta \left(\max_{\{u_s\}_{s=t+1}^{\infty}} \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} F(x_s, u_s) \right) \right\}
\end{aligned}
$$

- applying the principle of dynamic programming

Proof of proposition 2, cont.

- Then
$$V(x_t^*) = \max_{u_t} \{ f(x_t, u_t) + \beta\, V(x_{t+1}^*) \}$$

- But to be admissible $x_{t+1}^* = g(x_t^*, u_t^*)$, and the previous equation should hold for any $t \in \{0, \dots, \infty\}$ and for any admissible value for $x_t^* = x$,
$$V(x) = \max_{u} \{ f(x, u) + \beta\, V(g(x, u)) \}$$

Return.

Solution of planar linear difference equations

- Consider a linear difference equation $\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}$ and assume that $\det(\mathbf{I} - \mathbf{A}) \neq 0$

$$\begin{pmatrix} y_{1,t+1} \\ y_{2,t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- The solution of this equation can be written as

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + h_+ \mathbf{P}^+ \lambda_+^t + h_- \mathbf{P}^- \lambda_-^t$$

  where:

  - $\bar{\mathbf{y}}$ is the steady state of the planar equation
  - $\lambda_\pm$ are the eigenvalues of matrix $\mathbf{A}$
  - vectors $\mathbf{P}^+$ and $\mathbf{P}^-$ are the eigenvectors associated to λ_+ and λ_-,
  - the arbitrary constants h_+ and h_- are determined by using the initial and the terminal or tranversality conditions

The components of the solution:

- The steady state is

$$\bar{\mathbf{y}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

- the eigenvalues $\lambda_{\pm}$ of matrix $\mathbf{A}$ which are the roots of the characteristic equation

$$C(\lambda) = \lambda^2 - \mathrm{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

- the eigenvectors vectors $\mathbf{P}^{+}$ and $\mathbf{P}^{-}$, associated to λ_{+} and λ_{-}, are determined from the homogeneous equation

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{P}^i = \mathbf{0}, \text{ for, } i = +, -$$

  where $\mathbf{I}$ is the identity matrix and $\mathbf{0} = (0, 0)^{\top}$

Return.