Mathematical Economics
Discrete time: optimal control problem

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Optimal control problem

We present the optimality conditions for three problems:

- Simplest problem: x_0, x_T and T given
- Constrained terminal state problem: x_0 and T given and x_T constrained
- Discounted infinite horizon problem
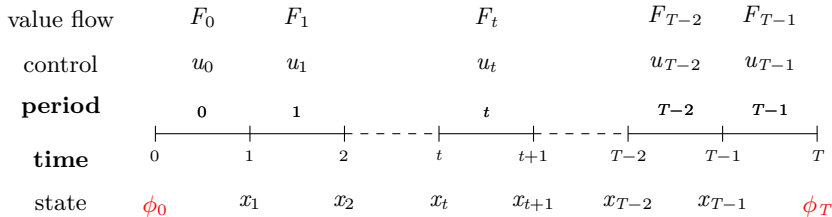
Optimal control problem
Timing and value of the decisions

In discrete time models, it is important to distinguish between moments (time) and periods

- state variables, x, are stock variables and control variables, u, are flow variables
- the stock variable x_t are indexed to **time** t (ex: 31st December 2020)
- a flow variable u_t are indexed to **period** (ex: year 2020)
- F_t is indexed to period t
- **Warning: be careful and consistent as regards the timing you choose**

Optimal control problem

Timing and value of the decisions

value flow		F_0	F_1		F_t		F_{T-2}	F_{T-1}	
control		u_0	u_1		u_t		u_{T-2}	u_{T-1}	
period		0	1		t		$T-2$	$T-1$	
time	0		1	2	t	$t+1$	$T-2$	$T-1$	T
state	ϕ_0		x_1	x_2	x_t	x_{t+1}	x_{T-2}	x_{T-1}	ϕ_T

Optimal control problem

Timing and value of the decisions

- The **action in period** t:
  - in the beginning the state x_t is given,
  - during the period the control u_t is chosen,
  - at the end the state variable will be determined by $x_{t+1} = G(x_t, u_t, t)$
- The **value obtained by** u_t, given the state x_t, is

$$F_t = F(x_t, u_t, t) \text{ period } t = 0, 1, \ldots, T-1$$

Optimal control problem
Timing and value of the decisions

- A strategy :
  - is a **sequence of decisions** $u = \{u_0, u_1, \ldots, u_{T-1}\}$ implying a sequence of states $x = \{x_1, \ldots, x_T\}$
  - with **value**

$$
\begin{aligned}
J[u, x] &= \sum_{t=0}^{T-1} F(u_t, x_t, t) \\
&= F(0, x_0, u_0) + \ldots + F(t, x_t, u_t) + \ldots + F(T-1, x_{T-1}, u_{T-1})
\end{aligned}
$$

  which is a functional: a mapping between a sequence and a number.

- The **optimal sequence** $u^* = \{u_0^*, u_1^*, \ldots, u_{T-1}^*\}$ is the one that maximizes the value of the program

- The **value of the program** is the maximum value attained by choosing an that is

$$
J^* = J[u^*] = \max_u \{J[x, u] : \ (x, u) \in \mathcal{X}\}
$$

Optimal control: simplest problem

- **Problem OCP**: Find $x^* = \{x_0^*, x_1^*, \ldots, x_T^*\}$ and $u^* = \{u_0^*, u_1^*, \ldots, u_T^*\}$ that maximizes

$$\max_{\{u\}} \sum_{t=0}^{T-1} F(u_t, x_t, t)$$

subject to

$$\begin{cases} x_{t+1} = G(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T = \phi_T & t = T \end{cases} \quad \text{(Problem 1)}$$

T, ϕ_0 and ϕ_T given

Pontriyagin's maximum principle

Pontriyagin's maximum principle (PMP)

- This is one method for solving dynamic optimization problems.
- In order to use it, we define the Hamiltonian function

$$H_t = H(\psi_t, x_t, u_t, t) \equiv F(x_t, u_t, t) + \psi_t G(x_t, u_t, t)$$

  where ψ_t is called the co-state variable at time t (obs: it has the same timing as u_t)

- Maximized Hamiltonian is a function

$$H_t^* = H^*(\psi_t, x_t^*, t) = \max_{u_t} H(\psi_t, x_t, u_t, t)$$

  for the optimal control, $u_t^* = u^*(x_t, \psi_t, t)$.

PMP: necessary first order conditions

Proposition

- If $x^* = \{x_t^*\}_{t=0}^{T}$ and $u^* = \{u_t^*\}_{t=0}^{T-1}$ are solutions to the OCP, then there is a sequence of the co-state variable $\psi = \{\psi_t\}_{t=0}^{T-1}$ such that the following conditions are satisfied:

- the optimality condition and the canonical equation

$$\frac{\partial H_t^*}{\partial u_t} = 0, \text{ for every } t = 0, \ldots, T-1$$

$$\psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}, \text{ for every } t = 0, \ldots, T-1$$

- and the admissibility conditions

$$\begin{aligned} x_{t+1}^* &= G(x_t^*, u_t^*, t), \text{ for every } t = 0, \ldots, T-1 \\ x_0^* &= \phi_0, \text{ for } t = 0 \\ x_T^* &= \phi_T, \text{ for } t = T \end{aligned}$$

Proof.

Maximized Hamiltonian dynamic system (MHDS)

- If $\frac{\partial^2 H_t}{\partial u_t^2} \neq 0$ then the optimality condition

$$\frac{\partial H_t^*}{\partial u_t} = \frac{\partial F(x_t^*, u_t^*, t)}{\partial u_t} + \psi_t \frac{\partial G(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

can be solved for the optimal control

$$u_t^* = U(x_t^*, \psi_t, t), \text{ for every } t = 0, \ldots, T-1$$

- Substituting into the f.o.c we get the **MHDS** in (x_t, ψ_t)

$$\begin{cases} x_{t+1}^* = G(x_t^*, \psi_t, t) \\ \psi_t = P(x_{t+1}^*, \psi_{t+1}, t+1) \end{cases}$$

where

$$\begin{array}{rcl} G(x_t^*, \psi_t, t) & = & G(x_t^*, U(x_t^*, \psi_t, t), t) \\ P(x_{t+1}, \psi_{t+1}) & \equiv & \frac{\partial H_{t+1}^*}{\partial x_{t+1}}(x_{t+1}^*, U(x_{t+1}^*, \psi_{t+1}, t+1), \psi_{t+1}, t+1) \end{array}$$

Alternative MHDS

- Alternatively we can solve

$$\frac{\partial H_t^*}{\partial u_t} = \frac{\partial F(x_t^*, u_t^*, t)}{\partial u_t} + \psi_t \frac{\partial G(x_t^*, u_t^*, t)}{\partial u_t} = 0$$

for $\psi_t = q_t(u_t^*, x_t^*, t)$ and we get an alternative **MHDS** in (x_t, u_t)

$$\begin{cases} x_{t+1}^* = G(x_t^*, u_t^*, t) \\ u_{t+1}^* = U(x_t^*, u_t^*, t, t+1) \end{cases}$$

- The system is characterized by a **forward** equation, $x_{t+1} = G(x_t, u_t, t)$, and a **backward** equation, $u_{t+1} = U(x_t, u_t, t, t+1))$

- The solution is the optimal path $\{u_t^*\}_{t=0}^{T-1}$ and $\{x_t^*\}_{t=0}^{T}$ that: (1) solves the the MHDS, (2) and **satisfies the two boundary conditions**

$$\begin{cases} x_0^* = \phi_0 \\ x_T^* = \phi_T \end{cases}$$

Application: resource depletion, or cake eating, problem

The problem

$$\max_{\{C\}} \sum_{t=0}^{T} \beta^t \ln(C_t),$$

$$\text{subject to}$$
$$W_{t+1} = W_t - C_t,$$
$$W_0 = \phi,$$
$$W_T = 0.$$

Application: resource depletion, or cake eating, problem

Applying the PMP:

- The Hamiltonian for this problem is

$$H_t = \beta^t \ln(C_t) + \psi_t(W_t - C_t)$$

- The first order conditions are

$$\begin{cases} \dfrac{\partial H_t^*}{\partial C_t} = \beta^t(C_t^*)^{-1} - \psi_t = 0, & t = 0, \ldots, T-1 \\[2mm] \psi_t = \dfrac{\partial H_{t+1}^*}{\partial W_{t+1}} = \psi_{t+1}, & t = 0, \ldots, T-1 \\[2mm] W_{t+1}^* = W_t^* - C_t^*, & t = 0, \ldots, T-1 \\[2mm] W_T^* = 0, & t = 0 \\[2mm] W_0^* = \phi, & t = T \end{cases}$$

Cake eating problem: MHDS

- As $C_t = \beta^t \psi_t$ and $\psi_{t+1} = \psi_t$ then $C_{t+1} = \beta^{t+1} \psi_{t+1} = \beta\, C_t$.
- Then the f.o.c in (W_t, C_t) are

$$
\begin{align}
W_{t+1}^* &= W_t^* - C_t^*, \ t = 0, \ldots, T-1 \tag{1} \\
C_{t+1}^* &= \beta C_t^* \tag{2} \\
W_T^* &= 0 \tag{3} \\
W_0^* &= \phi \tag{4}
\end{align}
$$

- To find the solution, C^*, W^*, we have to solve this problem.
- I am only interested to show which kind of solution is obtained. Next we will discuss other methods to solve an optimal control problem.

Cake eating problem: recursive solution

1 Solve the "Euler-equation" (2)

$$C_t = C_0 \beta^t. \tag{5}$$

where C_0 is   up to this point;

2 Substitute it in the constraint (1)

$$W_{t+1} = W_t - C_0 \beta^t$$

3 Solve it to find

$$W_t = k - C_0 \sum_{s=0}^{t-1} \beta^s = k - C_0 \frac{1 - \beta^t}{1 - \beta} \tag{6}$$

Cake eating problem: recursive solution (cont.)

4 Evaluate the solution for W_t at the initial and terminal time

$$\begin{cases} W_0 = k \\ W_T = k - C_0 \frac{1-\beta^T}{1-\beta} \end{cases}$$

5 Remember the initial and terminal constraints (3) and (4): therefore

$$\begin{cases} W_0 = k = \phi \\ W_T = k - C_0 \frac{1-\beta^T}{1-\beta} = 0 \end{cases}$$

6 Solve the system for k and C_0 to get $C_0 = \frac{1-\beta}{1-\beta^T}\phi$ and $k = \phi$

Cake eating problem: recursive solution (cont.)

- Substitute C_0 and k into equations (6) and (5)
- We get the **solution to the optimal control problem**

$$W_t^* = \phi \left(\frac{\beta^t - \beta^T}{1 - \beta^T} \right), \ t = 0, \ldots, T$$

$$C_t^* = \phi \left(\frac{1 - \beta}{1 - \beta^T} \beta^t \right), \ t = 0, \ldots, T - 1.$$

Optimal control problems with free terminal state

Problem OCPTC: find $u = \{u_t\}_{t=0}^{T-1}$ and $x = \{x_t\}_{t=0}^{T}$ that solves

$$\max_u \sum_{t=0}^{T-1} F(u_t, x_t, t)$$

subject to

$$\begin{cases} x_{t+1} = G(x_t, u_t, t) & t = 0, 1, \dots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T \text{ free} & t = T \end{cases} \quad \text{(Problem 2)}$$

T and ϕ_0 known

PMP for the free-terminal state problem

Proposition

- If $x^* = \{x_t^*\}_{t=0}^{T}$ and $u^* = \{u_t^*\}_{t=0}^{T-1}$ are solutions of the OCP, there is a sequence $\psi = \{\psi_t\}_{t=0}^{T-1}$ such that

- the optimality conditions

$$\frac{\partial H_t^*}{\partial u_t} = 0, \; t = 0, 1, \ldots, T-1$$

$$\psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}, \; t = 0, \ldots, T-1$$

- the admissibility conditions

$$x_{t+1}^* = G(x_t^*, u_t^*, t)$$
$$x_0^* = \phi_0$$

- and the transversality conditions

$$\psi_{T-1} = 0$$

Proof.

Optimal control problems for the constrained terminal state

Problem OCPTC: find $u = \{u_t\}_{t=0}^{T-1}$ and $x = \{x_t\}_{t=0}^{T}$ that solves

$$\max_u \sum_{t=0}^{T-1} F(u_t, x_t, t)$$

subject to

$$\begin{cases} x_{t+1} = G(x_t, u_t, t) & t = 0, 1, \ldots, T-1 \\ x_0 = \phi_0 & t = 0 \\ x_T \geq \phi_T & t = T \end{cases} \quad \text{(Problem 3)}$$

T, ϕ_0 and ϕ_T known

PMP for the constrained terminal state problem

Proposition

- If $x^* = \{x_t^*\}_{t=0}^T$ and $u^* = \{u_t^*\}_{t=0}^{T-1}$ are solutions of the OCP, there is a sequence $\psi = \{\psi_t\}_{t=0}^{T-1}$ such that

- the optimality conditions

$$\frac{\partial H_t^*}{\partial u_t} = 0, \ t = 0, 1, \ldots, T-1$$

$$\psi_t = \frac{\partial H_{t+1}^*}{\partial x_{t+1}}, \ t = 0, \ldots, T-1$$

- the admissibility conditions

$$x_{t+1}^* = G(x_t^*, u_t^*, t)$$
$$x_0^* = \phi_0$$

- and the transversality conditions

$$\psi_{T-1}(x_T^* - \phi_T) = 0$$

Proof

Discounted OCP with infinite horizon with a free terminal state

The next problem is very common in macroeconomics and growth theory.

- **Problem OCPIH**: find $(u, x) = \{(u_t, x_t)\}_{t=0}^{\infty}$ that solves

$$\max_u \sum_{t=0}^{\infty} \beta^t f(x_t, u_t), \ 0 < \beta < 1$$

  subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t), & t = 0, 1, \ldots \\ x_0 = \phi_0, \text{ given} & t = 0 \\ \lim_{t \to \infty} x_t \text{ is free} \end{cases} \quad \text{(Problem 4)}$$

  ϕ_0 is known

note this is a free endpoint problem ($T = \infty$ is undetermined)

Discounted OCP with infinite horizon with a constrained terminal state

The next problem is very common in macroeconomics and growth theory.

- **Problem OCPIH**: find $(u, x) = \{(u_t, x_t)\}_{t=0}^{\infty}$ that solves

$$\max_u \sum_{t=0}^{\infty} \beta^t f(x_t, u_t), \ 0 < \beta < 1$$

subject to

$$\begin{cases} x_{t+1} = g(x_t, u_t), & t = 0, 1, \dots \\ x_0 = \phi_0, \text{ given} & t = 0 \\ \lim_{t \to \infty} x_t \geq 0 \end{cases} \quad \text{(Problem 5)}$$

ϕ_0 is known

note this is constrained endpoint problem ($T = \infty$ is undetermined)

Discounted OCP with infinite horizon (cont.)

- The discounted-value Hamiltonian is

$$H_t = \beta^t f(u_t, x_t) + \psi_t g(u_t, x_t)$$
$$= \beta^t \left(f(u_t, x_t) + \beta^{-t} \psi_t g(u_t, x_t) \right)$$
$$= \beta^t h_t$$

- We define the current-value Hamiltonian

$$h_t \equiv h(x_t, \eta_t, u_t) = f(u_t, x_t) + \eta_t g(u_t, x_t)$$

where the current-value co-state variable is

$$\eta_t = \beta^{-t} \psi_t$$

PMP for the infinite horizon problems

Proposition

- *The solution of problem OCPIH verifies the following conditions:*

$$\frac{\partial h_t^*}{\partial u_t} = 0, \ t = 0, \ldots, \infty \tag{7}$$

$$\eta_t = \beta \frac{\partial h_{t+1}^*}{\partial x_{t+1}}, \ t = 0, \ldots, \infty \tag{8}$$

$$x_{t+1}^* = g(x_t^*, u_t^*), \ t = 0, \ldots, \infty \tag{9}$$

$$x_0^* = \phi_0, \ t = 0 \tag{10}$$

- *plus: terminal values and transversality conditions*

$$\lim_{t \to \infty} x_t \ free, \ \lim_{t \to \infty} \beta^t \eta_t = 0 \tag{11}$$

or

$$\lim_{t \to \infty} x_t \geq 0 \ \lim_{t \to \infty} \beta^t \eta_t x_t^* = 0 \tag{12}$$

Solving the MHDS: methods

- In OPCIH problems the MHDS can be written as a system of autonomous difference equations

$$x_{t+1}^* = g(u_t^*, x_t^*)$$
$$u_{t+1}^* = k(u_t^*, x_t^*)$$

- **Main difficulty in solving the system**: is that we have only an initial condition for x (x_0). Another boundary condition should be obtained before we can find an explicit solution to the problem.

Solving the MHDS: methods

If the system is linear we can use the following rule of thumb to solve the system:

1. try to **reduce the dimensionality** of the system: this is the case, v.g.

   1. if the system is recursive (**method 1**): solve the scalar equation and substitute the solution in the other equation;
   2. find other type of reduction: if we can find a single variable like $z_t = \eta_t x_t$ and use the terminal constraint (**method 2**)

2. if we **cannot reduce the dimensionality** of the system: use the solution of planar linear difference equation (**method 3**)

Application: consumption-investment problem

- Find the optimal consumption-investment strategy that solves the problem: find $C = \{C_t\}_{t=0}^{\infty}$ that

$$\max_C \sum_{t=0}^{\infty} \beta^t \ln(C_t) \text{ (inter-temporal utility)}$$

- subject to the constraints:

$$\begin{cases} W_{t+1} = (1+r)W_t - C_t, \text{ (intra-temporal budget constraint)} \\ W_0 = \phi, \text{ (initial wealth given)} \\ \lim_{t \to \infty}(1+r)^{-t}W_t \geq 0, \text{ (Non-Ponzi game condition)} \end{cases}$$

where $r > 0$ is the (given and constant) interest rate.

Solvingby using the PMP

- Discounted Hamiltonian

$$h_t = \ln(C_t) + \eta_t((1+r)W_t - C_t)$$

- PMP optimality conditions

$$\begin{cases} \dfrac{1}{C_t} = \eta_t \\ \eta_t = \beta(1+r)\eta_{t+1} \\ W_{t+1} = (1+r)W_t - C_t \\ W_0 = \phi_0 \\ \lim_{t \to \infty} \beta^t \eta_t W_t = 0 \end{cases}$$

Optimality conditions

- Eliminating η we get:
- the maximized Hamiltonian dynamic system (MHDS)

$$
\begin{aligned}
C_{t+1} &= \beta(1+r)C_t & (13) \\
W_{t+1} &= (1+r)W_t - C_t & (14)
\end{aligned}
$$

- and the initial and transversality conditions

$$
\begin{aligned}
W_0 &= \phi_0 & (15) \\
\lim_{t \to \infty} \beta^t \frac{W_t}{C_t} &= 0 & (16)
\end{aligned}
$$

Solving the MHDS: method 1

1 Solve equation (13):

$$C_t = C_0 \beta^t (1+r)^t, \ t \in \{0, 1, \dots, \infty\}$$

where C_0 is unknown

2 Substitute in equation (14) to get

$$W_{t+1} = (1+r) W_t - C_0 \beta^t (1+r)^t, \ t \in \{0, 1, \dots, \infty\} \qquad (17)$$

3 Solve equation (17)

$$
\begin{aligned}
W_t &= W_0 (1+r)^t - C_0 \sum_{s=0}^{t-1} (1+r)^{t-s-1} (1+r)^s \beta^s = \\
&= W_0 (1+r)^t - C_0 (1+r)^{t-1} \sum_{s=0}^{t-1} \beta^s = \\
&= (1+r)^t \left(W_0 - \frac{C_0}{1+r} \left(\frac{1-\beta^t}{1-\beta} \right) \right)
\end{aligned}
$$

Use the terminal conditions: cont.

4 Use the initial condition (15),

$$W_t = (1+r)^t \left(\phi - \frac{C_0}{1+r} \left(\frac{1-\beta^t}{1-\beta} \right) \right)$$

5 Use the transversality condition (16) to determine C_0

$$\lim_{t \to \infty} \beta^t \frac{W_t}{C_t} = \frac{1}{C_0} \left(\phi - \frac{C_0}{(1+r)(1-\beta)} + \lim_{t \to \infty} \frac{C_0 \beta^t}{(1+r)(1-\beta)} \right) =$$

$$= \frac{1}{C_0} \left(\phi - \frac{C_0}{(1+r)(1-\beta)} \right) = 0$$

if and only if

$$C_0 = \phi(1+r)(1-\beta)$$

❶ Then the solution is

$$W_t^* = \phi \beta^t (1+r)^t, \ t = 0, 1, \ldots, \infty$$
$$C_t^* = (1+r)(1-\beta) W_t^* \ t = 0, 1, \ldots, \infty$$

(18)

Solving the MHDS: method 2

1. Introduce a transformation of variables $z_t \equiv W_t/C_t$
   (suggestion: use the transversality condition)

2. We get a scalar linear difference equation equivalent to equations (13) and (14)

$$z_{t+1} = \frac{W_{t+1}}{C_{t+1}} = \frac{(1+r)W_t - C_t}{\beta(1+r)C_t} = \frac{1}{\beta}\left(z_t - \frac{1}{1+r}\right)$$

3. Jointly with condition (16) we have a simpler boundary value problem for z_t

$$\begin{cases} z_{t+1} = \frac{1}{\beta}\left(z_t - \frac{1}{1+r}\right) \\ \lim_{t\to\infty} \beta^t z_t = 0. \end{cases}$$

4. The general solution for z_t is

$$z_t = \bar{z} + (k - \bar{z})\beta^{-t}k.$$

where

$$\bar{z} = \frac{1}{(1-\beta)(1+r)}$$

Solving the MHDS: method 2, cont

4 We use equation (16) to determine k

$$\lim_{t \to \infty} \beta^t z_t = \lim_{t \to \infty} \beta^t \bar{z} + k - \bar{z} = k - \bar{z} = 0$$

if and only if $k = \bar{z}$. Then z_t is time-independent

$$z_t = \bar{z} = \frac{1}{(1+r)(1-\beta)}, \quad t = 0, 1, \ldots, \infty$$

5 Because $C_t^* = (1-\beta)(1+r) W_t^*$, substituting in equation (14) and using condition (15) we can solve the initial value problem

$$\begin{cases} W_{t+1}^* = (1+r) W_t^* - C_t^* = \beta(1+r) W_t^*, \quad t = 0, 1, \ldots \\ W_0^* = \phi \end{cases}$$

6 Which, after solving, yields the same solution (18)

Solving the MHDS: method 3

1 We write equations (13) and (14) in matrix notation

$$\begin{pmatrix} C_{t+1} \\ W_{t+1} \end{pmatrix} = \begin{pmatrix} \beta(1+r) & 0 \\ -1 & 1+r \end{pmatrix} \begin{pmatrix} C_t \\ W_t \end{pmatrix}$$

2 The general solution of this planar equation has the form (Planar).

$$\begin{pmatrix} C_t \\ W_t \end{pmatrix} = h_+ \mathbf{P}^+ \lambda_+^t + h_- \mathbf{P}^- \lambda_-^t \qquad (19)$$

3 Compute the eigenvalues $\lambda_\pm$. The characteristic polynomial is

$$\begin{aligned} c(\lambda) &= \lambda^2 - (1+r)(1+\beta)\lambda + \beta(1+r)^2 = \\ &= (\lambda - (1+r))(\lambda - \beta(1+r)) \end{aligned}$$

it happens to factorize (if not use the general formula). Then

$$\lambda_+ = 1+r, \ \lambda_- = \beta(1+r)$$

Solving the MHDS: method 3, cont

4 Compute the eigenvectors $\mathbf{P}^+$ and $\mathbf{P}^-$

$$\begin{pmatrix} (1+r)(\beta-1) & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p_1^+ \\ p_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{P}^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ -1 & (1+r)(1-\beta) \end{pmatrix} \begin{pmatrix} p_1^- \\ p_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{P}^- = \begin{pmatrix} (1+r)(1-\beta) \\ 1 \end{pmatrix}$$

5 Substituting in equation (19) we get

$$\begin{pmatrix} C_t \\ W_t \end{pmatrix} = h_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+r)^t + h_- \begin{pmatrix} (1+r)(1-\beta) \\ 1 \end{pmatrix} \beta^t (1+r)^t$$

Then the general solution for C_t and W_t is

$$C_t = (1-\beta)(1+r)^{1+t}\beta^t, \ W_t = (1+r)^t \left(h_+ + h_- \beta^t \right)$$

5 To obtain the particular solution, we use the transversality condition

$$\lim_{t \to \infty} \beta^t \frac{W_t}{C_t} = \lim_{t \to \infty} \frac{h_+ + h_- \beta^t}{(1+r)(1-\beta)h_-} = \frac{h_+}{(1+r)(1-\beta)h_-} = 0.$$

Condition (16) holds if and only if $h_+ = 0$.

6 Therefore

$$W_t = h_-(1+r)^t \beta^t$$

and condition (15) holds if and only if $h_- = \phi$.

7 We get the same solution (18)

Optimal consumption-saving: characterization of the solution

- As

$$C_t^* = (1 + r)(1 - \beta) W_t^*$$

the dynamics of consumption is monotonously related to financial wealth

- The optimal stock of financial wealth is

$$W_t^* = \phi \left(\beta(1 + r)\right)^t = \phi \left(\frac{1 + r}{1 + \rho}\right)^t, \ t = 0, 1, \ldots, \infty$$

where

$$\beta = \frac{1}{1 + \rho}$$

$\rho =$ rate of time preference

- Characterisation of the solution
  - if $r > \rho$ then $\lim_{t \to \infty} W_t^* = \infty$ and $\lim_{t \to \infty} C_t^* = \infty$
  - if $r = \rho$ then $\lim_{t \to \infty} W_t^* = \phi$ and $\lim_{t \to \infty} C_t^* = \rho\phi$
  - if $r < \rho$ then $\lim_{t \to \infty} W_t^* = 0$ and $\lim_{t \to \infty} C_t^* = 0$

- Even though wealth and consumption may be unbounded (if $\rho < r$) the value functional is bounded

- The value of the intertemporal utility for the optimal consumption path is

$$
\begin{aligned}
J^* &= \sum_{t=0}^{\infty} \beta^t \ln\left(C_t^*\right) = \sum_{t=0}^{\infty} \beta^t \ln\left((1+r)(1-\beta)\,W_t^*\right) = \\
&= \sum_{t=0}^{\infty} \beta^t \ln\left((1+r)(1-\beta)\phi\left(\beta(1+r)\right)^t\right) = \\
&= \ldots \\
&= \frac{1}{1-\beta} \ln\left(\left[(1+r)(1-\beta)^{1-\beta}\beta^\beta\right]^{1/(1-\beta)}\phi\right)
\end{aligned}
$$

- is bounded if ϕ is bounded for any r and ρ

- This is a consequence of the transversality condition: what matters is boundedness in present value terms not at the asymptotic levels of the variables.

Dynamic programming

Infinite-horizon discounted problem

- We present an alternative approach to solving Problems 4 and 5
  Problem OCPIH Consider the infinite-horizon discounted optimal
  control problem: find $(x^*, u^*)_{t=0}^{\infty}$ that solves the problem

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t)$$

$$\text{subject to}$$

$$x_{t+1} = g(x_t, u_t),$$

$$x_0 = \phi_0$$

  where $0 < \beta < 1$ and ϕ_0 is given.

- The approach:
  - find the optimal policy function $u^* = U(x)$
  - substitute in the constraint and solve

$$\begin{cases} x_{t+1} = g\big(x_t, U(x_t)\big) & t \in \{0, \dots, \infty\} \\ x_0 = .\phi_0 & t = 0 \end{cases}$$

Infinite-horizon discounted problem

Proposition

Let (x^*, u^*) be a solution to problem OCPIH: then it verifies the Hamilton-Jacobi-Bellman condition

$$V(x) = \max_u \{f(x, u) + \beta\, V(g(x, u))\}$$

for any admissible $x_t^* = x$ for $t \in \{0, \dots \infty\}$. Proof.

HJB equation: properties

- The properties of V are hard to determine: in general it is continuous, but differentiability is not assured

- if $H(x, u) \equiv f(x, u) + \beta\, V(g(x, u))$ has second order derivatives as regards u then we can determine the optimal control through the **optimality condition**

$$\frac{\partial H(x, u)}{\partial u} = 0$$

and we get the **policy function**

$$u^* = U(x)$$

- HJB equation becomes a non-linear functional equation

$$V(x) = f(x, h(x)) + \beta\, V[g(x, U(x))].$$

- both U and V have closed form solution only in very rare cases.

Application: the cake strikes again

- The HJB equation

$$V(W) = \max_C \left\{ \ln(C) + \beta V(W - C) \right\},$$

- Finding the optimal control: the optimality condition

$$\frac{\partial \left\{ \ln(C) + \beta V(W - C) \right\}}{\partial C} = 0.$$

- The best we can do is to say that optimal consumption is a function of the size of the cake

$$\frac{1}{C} - \beta V'(W - C) = 0 \iff C^* = C(W)$$

- and that the HJB has the form

$$V(W) = \ln(C(W)) + \beta V[W - C(W)].$$

The cake problem: solution

- Step 1: solve the HJB equation explicitly
  1. we use a *trial function of W* depending upon some undetermined coefficients;
  2. if the form of the function is right, then we use the *method of the undetermined coefficients* (try to get the unknown coefficients by substituting in the HJB equation)
  3. we get an *explicit* solution for C as a function of W

- Step 2: substitute $C(W)$ in the constraints of the problem to get

$$W_{t+1} = W_t - C(W_t)$$

- Step 3: solve the difference equation with $W_0 = \phi$

The cake problem: solving the HJB equation

- Trial solution:

$$V(W) = a + b \ln(W)$$

  where a and b are unknown constants;

- Policy function:

$$\frac{1}{C} = \frac{\beta\, b}{W - C} \;\Rightarrow\; C = \frac{1}{1 + b\,\beta}\, W$$

- Substituting in the HJB equation

$$a + b \ln(W) = \ln(W) - \ln(1 + b\,\beta) + \beta \left(a + b \ln \left(\left(1 - \frac{1}{1 + b\,\beta} \right) W \right) \right),$$

- collecting terms

$$(b(1 - \beta) - 1) \ln(W) = a(\beta - 1) - \ln(1 + b\beta) + \beta b \ln \left(\frac{b\beta}{1 + b\beta} \right).$$

Step 1: solving the HJB equation

F

- then we determine

$$b = \frac{1}{1-\beta}, \ a = \ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta) = \left(\beta^\beta(1-\beta)^{1-\beta}\right)^{1/(1-\beta)}$$

- and

$$V(W) = \frac{1}{1-\beta}\ln\left(\left(\beta^\beta(1-\beta)^{1-\beta}\right)^{1/(1-\beta)} W\right).$$

- Then because

$$C^* = (1-\beta)\,W^*$$

is the optimal policy function and holds at all times

Step 2: optimal budget constraint

- Substituting the policy function in the intratemporal budget constraint we get

$$W_{t+1}^* = W_t^* - (1 - \beta) W_t^*, \ t = 0, 1, \ldots, \infty$$

- given

$$W_0 = \phi, \ \text{given}$$

Step 3: solution for the cake-eating problem

The infinite horizon cake eating problem has the solution:

- the optimal sequence of cake size $W^* = \{W_t^*\}_{t=0}^{\infty}$ is generated by

$$W_t^* = \phi\beta^t, \ t = 0, 1, \ldots, \infty$$

- the optimal sequence of cake consumption $C^* = \{C_t^*\}_{t=0}^{\infty}$ is generated by

$$C_t^* = \phi(1 - \beta)\beta^t, \ t = 0, 1, \ldots, \infty$$

Proofs

Proof of proposition 1

- Assume we know the solution $(u^*, x^*) = \{x_t^*, u_t^*\}_{t=0}^{T}$ for the problem.
- The optimal value is

$$J^* = \sum_{t=0}^{T-1} F(x_t^*, u_t^*, t)$$

- We write the Lagrangean

$$
\begin{aligned}
L &= \sum_{t=0}^{T-1} F(x_t, u_t, t) + \psi_t(G(x_t, u_t, t) - x_{t+1}) \\
&= \sum_{t=0}^{T-1} H_t(\psi_t, x_t, u_t, t) - \psi_t x_{t+1} = \\
&= \sum_{t=0}^{T-1} \ell(\psi_t, x_t, u_t, x_{t+1}, t)
\end{aligned}
$$

Proof of proposition 1

- Consider an arbitrary perturbation away from the solution to the problem, such that $x_t = x_t^* + \epsilon_t^x$. The perturbation is admissible if $\epsilon_0^x = \epsilon_T^x = 0$, and $u_t = u_t^* + \epsilon_t^u$. It induces the variation in value

$$L - J^* = \frac{\partial H_0}{\partial x_0}\epsilon_0^x + \sum_{t=1}^{T-1}\left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t}\right)\epsilon_t^x - \psi_{T-1}\epsilon_T^x +$$

$$\sum_{t=0}^{T-1}\frac{\partial H_t}{\partial u_t}\epsilon_t^u + \sum_{t=0}^{T-1}\left(\frac{\partial H_t}{\partial \psi_t} - x_{t+1}\right)\epsilon_t^\psi =$$

$$= \sum_{t=1}^{T-1}\left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t}\right)\epsilon_t^x + \sum_{t=0}^{T-1}\frac{\partial H_t}{\partial u_t}\epsilon_t^u + \sum_{t=0}^{T-1}\left(G(x_t, u_t, t) - x_{t+1}\right)\epsilon_t^\psi$$

- Then $L = J^*$ only if $\psi_{t-1} - \frac{\partial H_t}{\partial x_t} = \frac{\partial H_t}{\partial u_t} = x_{t+1} - G(x_t, u_t, t) = 0$.

Proof of proposition 1 b

- From the previous proof, because the terminal state is free, an admissible is such that $\epsilon_0^x = 0$ but ϵ_T^x is free. Therefore,

$$L - J^* = \sum_{t=1}^{T-1} \left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t} \right) \epsilon_t^x +$$

$$+ \sum_{t=0}^{T-1} \frac{\partial H_t}{\partial u_t} \epsilon_t^u + \sum_{t=0}^{T-1} \left(G(x_t, u_t, t) - x_{t+1} \right) \epsilon_t^\psi -$$

$$- \psi_{T-1} \epsilon_T^x$$

- Then $L = J^*$ only if $\psi_{t-1} - \frac{\partial H_t}{\partial x_t} = \frac{\partial H_t}{\partial u_t} = x_{t+1} - G(x_t, u_t, t) = 0$ and $\psi_{T-1} = 0$

.

Proof of proposition 1c

- Because of the terminal constraint the Lagrangean is

$$L = \sum_{t=0}^{T-1} H_t(\psi_t, x_t, u_t, t) - \psi_t x_{t+1} + \mu_T(\phi_T - x_T)$$

- where $\mu_T(\phi_T - x_T) = 0$.
- Taking the previous proof, because the terminal state is free, an admissible is such that $\epsilon_0^x = 0$ but ϵ_T^x is free. Therefore,

$$L - J^* = \sum_{t=1}^{T-1} \left(-\psi_{t-1} + \frac{\partial H_t}{\partial x_t} \right) \epsilon_t^x +$$

$$+ \sum_{t=0}^{T-1} \frac{\partial H_t}{\partial u_t} \epsilon_t^u + \sum_{t=0}^{T-1} \left(G(x_t, u_t, t) - x_{t+1} \right) \epsilon_t^\psi -$$

$$- (\psi_{T-1} + \mu_T)\epsilon_T^x$$

- Then $L = J^*$ only if $\psi_{t-1} - \frac{\partial H_t}{\partial x_t} = \frac{\partial H_t}{\partial u_t} = x_{t+1} - G(x_t, u_t, t) = 0$ and $\psi_{T-1} = -\mu_T$. Then $\mu_T(\phi_T - x_T) = \psi_{T-1}(x_T - \phi_T) = 0$.

Return.

Proof of proposition 2

The value functional for x_t^* is

$$V(x_t^*) = \sum_{s=t}^{\infty} \beta^{s-t} f(x_s^*, u_s^*) =$$

$$= \max_{\{u_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} f(x_s, u_s) =$$

$$= \max_{\{u_s\}_{s=t}^{\infty}} \left\{ F(x_t, u_t) + \beta \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} F(x_s, u_s) \right\}$$

(by the principle of dynamic programming)

$$= \max_{u_t} \left\{ F(x_t, u_t) + \beta \left(\max_{\{u_s\}_{s=t+1}^{\infty}} \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} F(x_s, u_s) \right) \right\}$$

$$= \max_{u_t} \left\{ f(x_t, u_t) + \beta V(x_{t+1}^*) \right\}$$

Proof of proposition 2, cont.

- But to be admissible $x_{t+1}^* = g(x_t^*, u_t^*)$, and the previous equation should hold for any $t \in \{0, \ldots, \infty\}$ and for any admissible value for $x_t^* = x$,

$$V(x) = \max_u \{f(x, u) + \beta V(g(x, u))\}$$

.

Solution of planar linear difference equations

- Consider a linear difference equation $\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}$ and assume that $\det(\mathbf{I} - \mathbf{A}) \neq 0$

$$\begin{pmatrix} y_{1,t+1} \\ y_{2,t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- The solution of this equation can be written as

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + h_+ \mathbf{P}^+ \lambda_+^t + h_- \mathbf{P}^- \lambda_-^t$$

  where:

  - $\bar{\mathbf{y}}$ is the steady state of the planar equation
  - $\lambda_{\pm}$ are the eigenvalues of matrix $\mathbf{A}$
  - vectors $\mathbf{P}^+$ and $\mathbf{P}^-$ are the eigenvectors associated to λ_+ and λ_-,
  - the arbitrary constants h_+ and h_- are determined by using the initial and the terminal or tranversality conditions

The components of the solution:

- The steady state is

$$\bar{\mathbf{y}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

- the eigenvalues $\lambda_{\pm}$ of matrix $\mathbf{A}$ which are the roots of the characteristic equation

$$C(\lambda) = \lambda^2 - \text{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

- the eigenvectors vectors $\mathbf{P}^+$ and $\mathbf{P}^-$, associated to λ_+ and λ_-, are determined from the homogeneous equation

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{P}^i = \mathbf{0}, \text{ for, } i = +, -$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{0} = (0, 0)^{\top}$

.