Advanced Mathematical Economics

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Contents

Chapter 5

Scalar non-linear ODE's: the regular case

Time as independent variable. We look for functions $y : \mathrm{T} \times \Phi \rightarrow \mathrm{Y} = \mathbb{R}$, where $\varphi \in \Phi$ is a set of parameters which solve the (autonomous) equation

$$F(\dot{y}, y, \varphi) = 0 \tag{5.1}$$

We say the ODE is **regular** if $F(\dot{y}, y, \cdot)$ is continuous and continuously differentiable and furthermore $\dfrac{d\,F(\dot{y}, y, \varphi)}{d\dot{y}} \neq 0$ for every $y \in \mathrm{Y}$.

This allows us to write equation (5.1) as

$$\dot{y} = f(y, \varphi) \tag{5.2}$$

We assume next that $y : \mathbb{R}_+ \rightarrow \mathrm{Y} \subseteq \mathbb{R}$, and the parameters are real numbers.

In the rest of this chapter we present the normal forms (5.1) and in section we present the qualitative theory for 5.2 for non-linear regular scalar ODE's.

5.1 Normal forms

A **normal form** is the simplest ODE, whose exact solution is usually known and represents a whole family of ODEs by topological equivalence.[1]

We present next some important normal forms for scalar ODEs, which have the generic representation by a polynomial vector field

$$\dot{y} = f(y, \varphi) \equiv a_0 + a_1\, y + a_2\, y^2 + a_3\, y^3, \tag{5.3}$$

[1] In heuristic terms, we say functions $f(y)$ and $g(x)$ are topologically equivalent if there exists a diffeomorphism, i.e., a smooth map h with a smooth inverse h^{-1}, such that if $y = h(x)$ then $h(g(x)) = f(h(x))$. This property may hold globally of locally. The last case is the intuition behind the Grobman-Hartmann theorem.

where $\varphi \equiv (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$.

Linear scalar ODE's, that have been studied in chapter two, are particular cases in which $a_2 = a_3 = a_4 = 0$. In this chapter we address cases in which at least one of coefficients a_2, a_3 or a_4 is different from zero.

It can be shown that (see (Hale and Koçak, 1991, ch. 2)) the following cases are the most relevant: first, ODE's depending on a single parameter (a), as the Ricatti's equation $\dot{y} = a + y^2$, the quadratic Bernoulli equation, $\dot{y} = ay + y^2$, the cubic equations Bernoulli equation, $\dot{y} = ay - y^3$, the Abel equation $\dot{y} = a + y - y^3$; second, ODE's depending on two parameters (a and b) the Abel's equation, $\dot{y} = a + by - y^3$.

We address the solution of those equations from two approaches: First, an **analytical approach**, seeking to find its explicit solution (when it is known); and, second, a **qualitative (or geometric) approach**, which characterizes qualitatively the possible dynamic behavior of the solution, depending on the value of the parameters. The

5.1.1 The linear equation

We already studied the linear scalar ODE

$$\dot{y} = b + a\,y,$$

which is a particular case of equation (5.3) with $a_0 = b$, $a_1 = a$, and $a_2 = a_3 = 0$, where a and b are real numbers. Generically, we have $f_{yy}(\cdot) = f_{yyy}(\cdot) = 0$, and $f_y(\cdot) \neq 0$ can have any sign.

It has the a explicit solutions, whose form depends on the parameters

$$y(t) = \begin{cases} y(0) & \text{if } a = b = 0 \\ y(0)\,e^{a\,t} & \text{if } a \neq 0,\ b = 0 \\ y(0) + b\,t & \text{if } a = 0,\ b \neq 0 \\ y(0)\,e^{a\,t} + \bar{y}\,\left(1 - e^{a\,t}\right) & \text{if } a \neq 0,\ b \neq 0 \end{cases}$$

where $\bar{y} = -b/a$, and $y(0)$Y is arbitrary.

The qualitative properties are: first, it has one unique steady state, $\bar{y} = -b/a$, if $a \neq 0$, which is asymptotically stable if $a < 0$, or unstable if $a > 0$; second, if $a = 0$ it has an infinite number of steady states, if $b = 0$ or no steady state, if $b \neq 0$.

5.1.2 The Ricatti's equation: saddle-node or fold bifurcation

The quadratic equation

$$\dot{y} = f(y, a) \equiv a + y^2 \tag{5.4}$$

is called **Ricatti** equation. This equation is a particular case of equation (5.3) with $a_0 = a$, $a_2 = 1$, and $a_1 = a_3 = a_4 = 0$. Generically, we have $f_{yy}(\cdot) \neq 0$, $f_a(\cdot) \neq 0$ and $f_y(\cdot) = f_{yyy}(\cdot) = 0$.

It has an explicit solution[2] :

$$y(t) = \begin{cases} \frac{y(0)}{1-y(0)\,t} \ , & \text{if } a = 0 \\ \sqrt{a}\left(\tan\left(\sqrt{a}(t - y(0)^{-1})\right)\right) \ , & \text{if } a > 0 \\ -\sqrt{-a}\left(\tanh\left(\sqrt{-a}(t - y(0)^{-1})\right)\right) \ , & \text{if } a < 0 \end{cases}$$

where $y(0) \in \mathrm{Y}$ is an arbitrary constant belonging to the domain of y.

The behavior solution depends again on the value of the parameter a:

- if $a = 0$, the solution takes an infinite value at a finite time $t = \dfrac{1}{y(0)}$ [3], i.e., $\lim_{t \to -y(0)^{-1}} y(t) = \pm\infty$ and tends asymptotically to a steady state $\bar{y} = 0$, that is $\lim_{t\to\infty} y(t) = 0$ independently of the value of k;

- if $a > 0$ the solution takes infinite values for a periodic sequence of times $t \in \{-y(0)^{-1}, \pi - y(0)^{-1}, 2\pi - y(0)^{-1}, \dots, n\pi - y(0)^{-1}, \dots\}$,

$$\lim_{t \to n\pi - y(0)^{-1}} y(t) = \pm\infty, \text{ for } n \in \mathbb{N}$$

  and it has no steady state;

- if $a < 0$, the solution converges to

$$\lim_{t\to\infty} y(t) = \begin{cases} -\sqrt{-a} \ , & \text{if } y(0)^{-1} < \sqrt{-a} \text{ or } -\sqrt{-a} < y(0)^{-1} < \sqrt{-a} \\ +\infty, & \text{if } y(0)^{-1} > \sqrt{-a}. \end{cases}$$

To characterize qualitatively the solution of the differential equation, we have to find the steady state(s), by finding the values of $\bar{y} \in \mathrm{Y}$ such that $f(y, a) = 0$, and characterize its local dynamics, by finding the sign of $f_y(\bar{y}, a) = 2\bar{y}$. Therefore, the qualitative dynamic properties of the ODE depend on the value of a:

- existence and number of steady states: the set of steady states is the set $\bar{y} = \{\, y \in \mathrm{Y} : a + y^2 = 0\}$, i.e, the set of stationary solutions to the ODE. We readily find that: (a) if $a > 0$ there are no steady states, (b) if $a = 0$ there is one steady state $\bar{y} = 0$, and (c) if $a < 0$ there are two steady states $\bar{y} \in \{\bar{y}_1, \bar{y}_2\}$ for $\bar{y}_1 = -\sqrt{-a} < 0$, and $\bar{y}_2 = \sqrt{-a} > 0$;

- local dynamics at a steady state, can only be determined for $a \le 0$. Then: (a) if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable, because $f_y(0) = 0$; and (b) if $a < 0$ steady state $\bar{y}_1$ is asymptotically stable and steady state $\bar{y}_2$ is unstable, then , $f_y(\bar{y}_1) = -2\sqrt{-a} < 0$,

[2]See appendix section 5.A

[3]This is different to the linear case, v.g., $\dot{y} = y$, whose solution $y(t) = y(0)e^t$, if $y(0) \neq 0$, takes an infinite value only in infinite time.

and $f_y(\bar{y}_2) = 2\sqrt{-a} > 0$. Furthermore, we if $a < 0$ the stable manifold associated to (or the basin of attraction of) steady state $\bar{y}_1$, is

$$\mathcal{W}^s_{\bar{y}_1} = \{\ y \in \mathrm{Y}\ : y < \sqrt{-a}\}.$$

Comparing to the linear case, for the case in which the steady state is asymptotically stable, the stable manifold is a subset of Y not the whole Y.

There is a bifurcation point at $(y, a) = (0, 0) \in \mathrm{Y} \times \Phi$, which is called **saddle-node bifurcation**. This bifurcation point is defined by the subset of points in the state space and the space of the parameters $\mathrm{Y} \times \Phi$, given by $\{\ (y, a) :\ f(y, a) = 0,\ f_y(y, a) = 0\}$, such that a steady state changes its dynamic properties, or its phase diagram: .

The bifurcation point, $(0, 0)$, is determined by solving the following system of equations for (y, a):

$$\begin{cases} f(y,a) = 0 \\ f_y(y,a) = 0 \end{cases} \Leftrightarrow \begin{cases} a + y^2 = 0 \\ 2y = 0. \end{cases}$$

Figure 5.1 shows phase diagrams for the $a < 0$ (panel (a)), for the $a = 0$ (panel (b)), and $a > 0$ (panel (c)). Those diagrams provide a geometrical depiction of the results that we have already obtained: there are no steady states in panel (a); there is a steady state in panel (b) but it is neither stable (an initial value higher than 0 will generate an unstable trajectory) nor unstable (an initial value lower than 0 will generate an asymptotically stable trajectory converging to zero); and in panel (c) there are two steady states one which is stable and one unstable. In the last case, we see that the basin of attraction to steady state $\bar{y}_1$ is limited by steady state $\bar{y}_2$, as shown in $\mathcal{W}^s_{\bar{y}_1}$

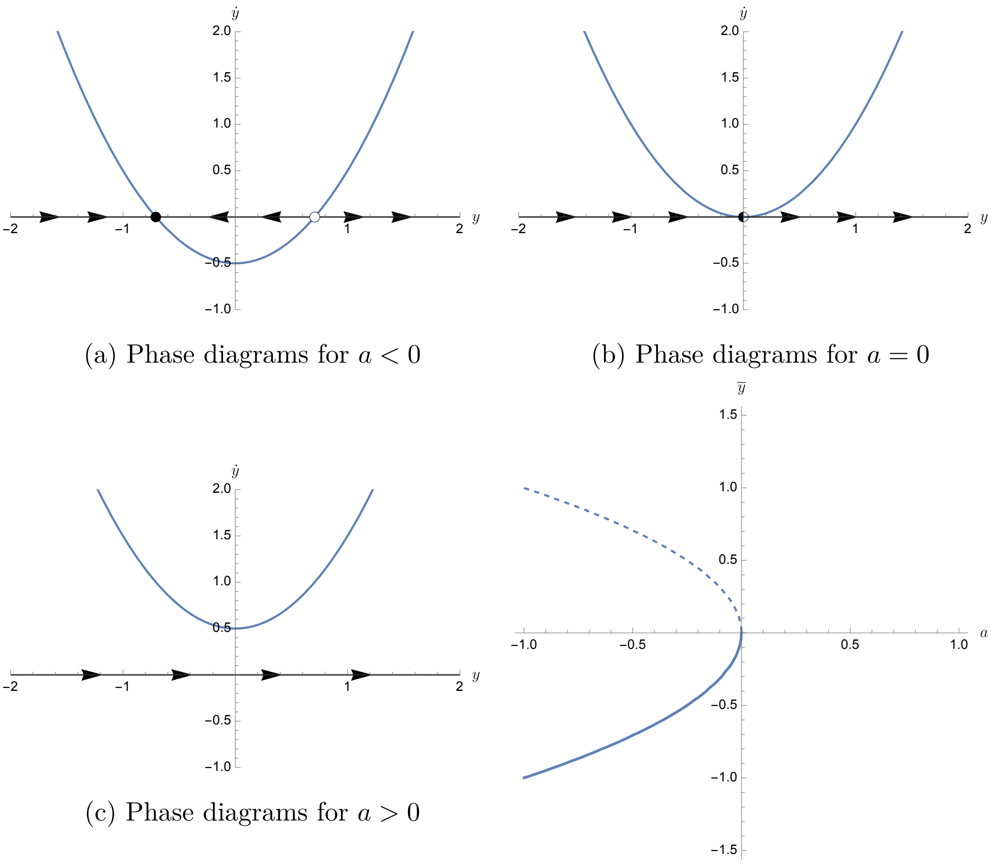
Panel (d) presents the bifurcation diagram for a **saddle-node**. It depict points (a, y) such that $a + y^2 = 0$, say $\bar{y}(a)$, and in solid-line the subset of points such that $f_y(\bar{y}(a)) < 0$ and in dashed-line the subset of points such that $f_y(\bar{y}(a)) > 0$. The first case corresponds to asymptotically stable steady states and the second to unstable steady states. Observe that the curve does not lie in the positive quadrant for a which is the geometrical analogue to the non-existence of steady states. The saddle-node bifurcation point is at the origin $(0, 0)$. This point separates the values for a such that there are two steady states (for $a < 0$) from the points of a such that there is no steady state (for $a > 0$).

5.1.3 Quadratic Bernoulli equation: transcritical bifurcation

The quadratic equation

$$\dot{y} = ay + y^2 \tag{5.5}$$

is called **quadratic Bernoulli** equation. This equation is a particular case of equation (5.3) with $a_1 = a$, $a_2 = 1$, and $a_0 = a_3 = 0$. Generically, we have $f_{yy}(\cdot) \neq 0$ and $f_{ya}(\cdot) \neq 0$. It has the



(a) Phase diagrams for $a < 0$ (b) Phase diagrams for $a = 0$

(c) Phase diagrams for $a > 0$

(d) Bifurcation diagram

Figure 5.1: Phase diagrams and bifurcation diagram for equation (5.4)

explicit solution [4]:

$$y(t) = \begin{cases} \frac{y(0)}{1-y(0)\,t}\ , & \text{if } a = 0 \\ \frac{a}{(1+a/y(0))e^{-at}-1}, & \text{if } a \neq 0 \end{cases}$$

where $y(0)$ is an arbitrary element of Y.

The behavior of the solution is the following (see Figure 5.3):

- if $a < 0$, as we can see in panel (a),

$$\lim_{t\to\infty} y(t) = \begin{cases} 0\ , & \text{if } y(0) < -a \\ +\infty, & \text{if } y(0) > -a \end{cases}$$

- if $a = 0$, as we can see in panel (a), it behaves as the Ricatti's equation when $a = 0$

$$\lim_{t\to\infty} y(t) = \begin{cases} 0\ , & \text{if } y(0) < 0 \\ +\infty, & \text{if } y(0) > 0 \end{cases}$$

[4]See appendix section 5.B for the explicit solution for the general Bernoulli ODE.

- if $a > 0$, as we can see in panel (c),

$$\lim_{t \to \infty} y(t) = \begin{cases} -a\ , & \text{if } y(0) < 0 \\ +\infty, & \text{if } y(0) > 0 \end{cases}$$

The dynamic properties depend on the value of a:

- existence and number of steady states: if $a = 0$ there is one steady state $\bar{y} = 0$, if $a \neq 0$ there are two steady states $\bar{y} = \{\ \bar{y}_1, \bar{y}_2\}\ = \{0, -a\}$;
- local dynamics at the steady states: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable, that is, $(y, a) = (0, 0)$ is again a bifurcation point; if $a < 0$ the steady state $\bar{y}_1 = 0$ is asymptotically stable and steady state $\bar{y}_2 = -a$ is unstable; and if $a > 0$ the two steady states change stability with steady state $\bar{y}_1 = 0$ being unstable and steady state $\bar{y}_2 = -a$ being asymptotically stable. Then, a non-empty basin of attraction always exists, because the stable manifolds associated to the asymptotically stable equilibrium points are: if $a < 0$ the stable manifold is

$$\mathcal{W}^s_{\bar{y}_1} = \{\ y \in Y : y < -a\}.$$

and, if $a > 0$, the stable manifold is

$$\mathcal{W}^s_{\bar{y}_2} = \{\ y \in Y : y < 0\}.$$

The bifurcation point, $(y, a) = (0, 0)$, is determined by solving the following system of equations for (y, a):

$$\begin{cases} f(y, a) = 0, \\ f_y(y, a) = 0, \end{cases} \Leftrightarrow \begin{cases} a\,y + y^2 = 0, \\ a + 2\,y = 0, \end{cases} \Leftrightarrow \begin{cases} 2\,a\,y + 2y^2 = 0, \\ a\,y + 2\,y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} a\,y = 0, \\ a + 2\,y = 0. \end{cases}$$

That bifurcation point is called **transcritical bifurcation**, as Figure 5.3, panel (d), shows that after crossing the bifurcation point the number of steady states is the same (two) but their stability properties change.

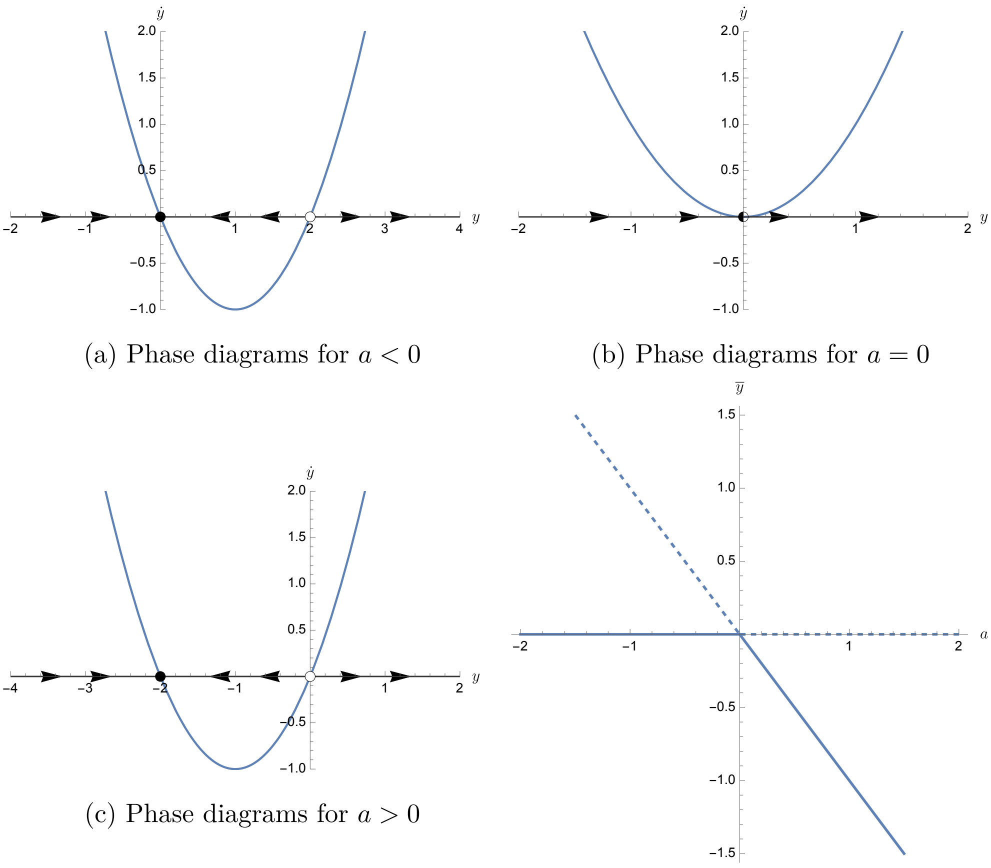
Exercise Show that $\dot{y} = a\,y - y^2$ also has a transcritical bifurcation.

5.1.4 Bernoulli's cubic equation: subcritical pitchfork

The equation

$$\dot{y} = ay - y^3 \tag{5.6}$$

is a **cubic Bernoulli** equation. This equation is a particular case of equation (5.3) with $a_1 = a$, $a_3 = -1$, and $a_0 = a_2 = 0$. Generically, we have $f_{yyy}(\cdot) \neq 0$ and $f_{ya}(\cdot) \neq 0$.



(a) Phase diagrams for $a < 0$

(b) Phase diagrams for $a = 0$

(c) Phase diagrams for $a > 0$

(d) Bifurcation diagram

Figure 5.2: Phase diagrams and bifurcation diagram for equation (5.5)

Being a Bernoulli equation, it also has an explicit solution:

$$y(t) = \begin{cases} \left(y(0)^{-2} \ + 2\,t\right)^{-\frac{1}{2}} & \text{if } \ a = 0 \\ \pm\sqrt{a} \ \left[1 - \left(1 - \frac{a}{y(0)^{-2}}\right) e^{-2at} \ \right]^{-1/2} & \text{if } \ a \neq 0 \end{cases}$$

where $y(0)$ is an arbitrary element of Y. The solution trajectories have the following properties for different values of the parameter a (see Figure 5.3):

- if $a \leq 0$, panels (a) and (b) show that, for any $y(0) \in \mathrm{Y}$, we have $\lim_{t\to\infty} y(t) = 0$;
- if $a > 0$, panel (c) shows that

$$\lim_{t\to\infty} y(t) = \begin{cases} -\sqrt{a}\ , & \text{if } y(0) < 0 \\ \sqrt{a}\ , & \text{if } 0 < y(0) < \sqrt{a} \\ +\infty, & \text{if } y(0) > \sqrt{a} \end{cases}$$

The dynamic properties depend on the value of a:

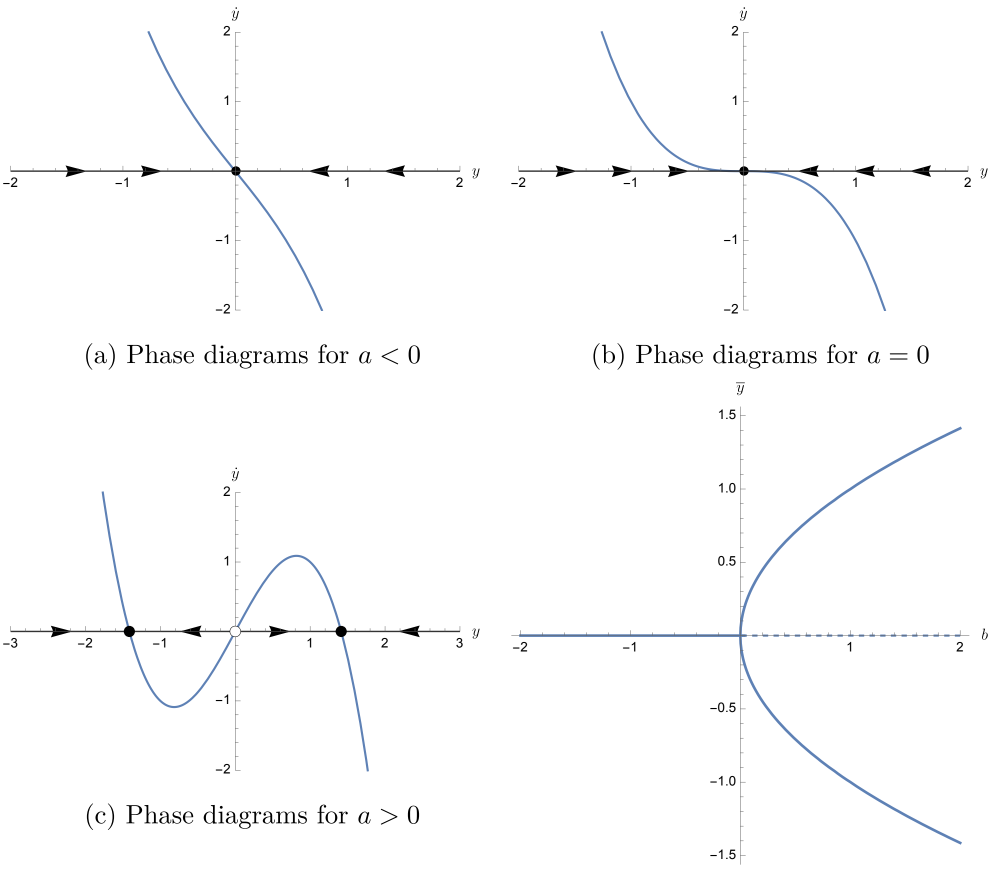
- existence and number of steady states: there is **one** steady state $\bar{y} = 0$ and if $a \le 0$ and there are **three** steady states $\bar{y} = \{0, -\sqrt{a}, \sqrt{a}\}$ if $a > 0$;
- local dynamics at the steady states: if $a = 0$ the steady state $\bar{y} = 0$ is neither stable nor unstable; if $a < 0$ steady state $\bar{y} = 0$ is asymptotically stable; and if $a > 0$ steady state $\bar{y} = 0$ is unstable and the other two steady states $\bar{y} = -\sqrt{a}$ and $\bar{y} = \sqrt{a}$ are asymptotically stable.

To find the bifurcation point, we solve jointly $f(y, a) = 0$ and $f_y(y, a) = 0$ for (y, a), yielding

$$\begin{cases} a\,y - y^3 = 0 \\ a - 3\,y^2 = 0 \end{cases} \iff \begin{cases} 3\,a\,y - 3\,y^3 = 0 \\ a\,y - 3\,y^3 = 0 \end{cases} \iff \begin{cases} 2\,a\,y = 0 \\ a - 3\,y^2 = 0 \end{cases}$$

then the there is a **subcritical pitchfork** at $(y, a) = (0, 0)$.

Figure **??** shows two phase diagrams and the bifurcation diagram.



(a) Phase diagrams for $a < 0$

(b) Phase diagrams for $a = 0$

(c) Phase diagrams for $a > 0$

(d) Bifurcation diagram

Figure 5.3: Phase diagrams and bifurcation diagram for equation (5.6).

Exercise: Study the solution for equation $\dot{y} = ay + y^3$. Show that point $(y, a) = (0, 0)$ is also a bifurcation point called **supercritical pitchfork**.

5.1.5 Abel's equation: hysteresis

The following ODE

$$\dot{y} = a + y - y^3 \tag{5.7}$$

is called an Abel equation of the first kind, in which the following properties hold: $f_{yyy}(\cdot) \neq 0$, $f_y(\cdot) \neq 0$ and $f_a\ (\cdot) \neq 0$.

Although closed form solutions have been found recently [5] they are too cumbersome to report. If $a = 0$ the Abel's equation reduces to a particular Bernoulli's equation (5.6) $\dot{y} = y - y^3$.

Equation (5.7) can have one, two or three equilibrium points, which are the real roots of the polynomial equation $f(y, a) \equiv a + y - y^3 = 0$.

We can determine bifurcation points in the space $\mathrm{Y} \times \Phi$ by solving for (y, a)

$$\begin{cases} f(y, a) = 0, \\ f_y(y, a) = 0. \end{cases}$$

Because

$$\begin{cases} a + y - y^3 = 0, \\ 1 - 3y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3(a + y) - 3y^3 = 0, \\ y - 3y^3 = 0, \end{cases} \Leftrightarrow \begin{cases} 3a + 2y = 0 \\ y = \pm\sqrt{1/3}, \end{cases}$$

we readily find that the ODE (5.7) has two critical points, called **hysteresis** points:

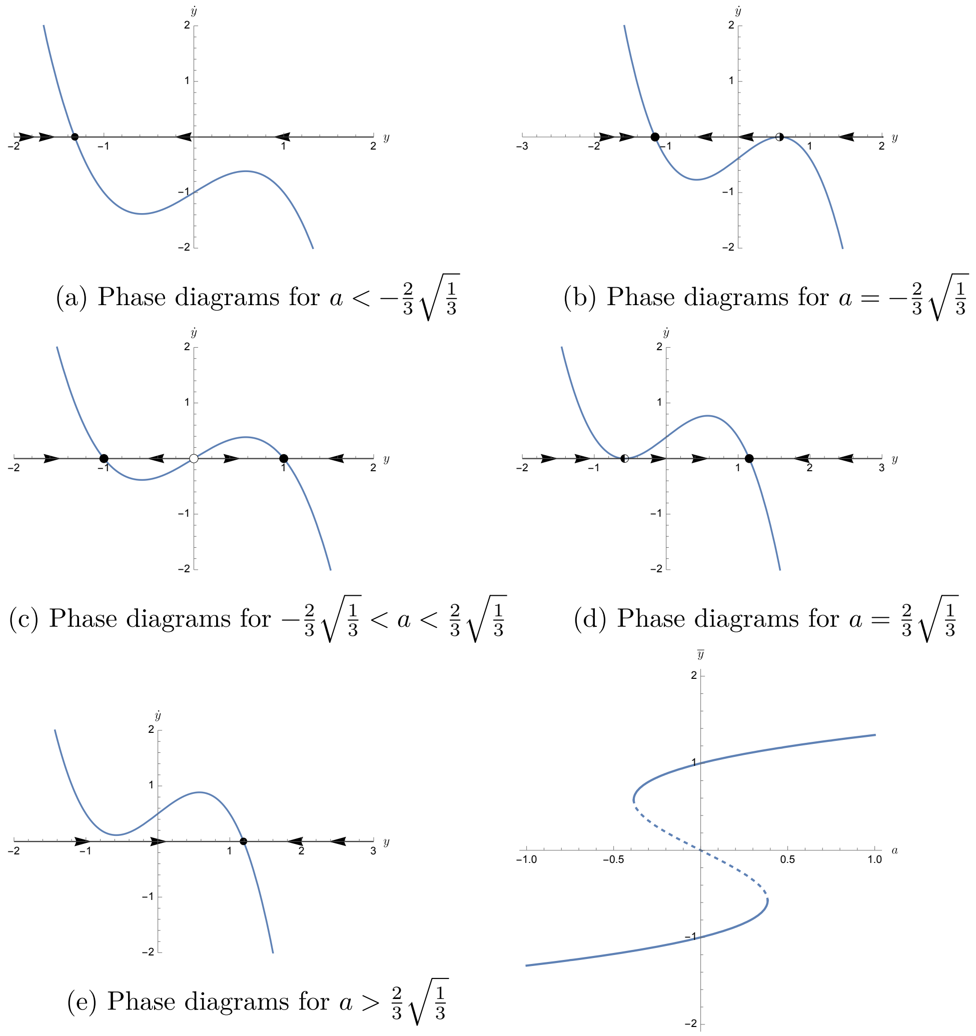
$$(y, a) = \left\{ \left(-\sqrt{\frac{1}{3}}, \frac{2}{3}\sqrt{\frac{1}{3}}\right), \left(\sqrt{\frac{1}{3}}, -\frac{2}{3}\sqrt{\frac{1}{3}}\right) \right\}.$$

Figure 5.4 shows that:

- for $a > \frac{2}{3}\sqrt{\frac{1}{3}}$ or for $a < -\frac{2}{3}\sqrt{\frac{1}{3}}$ there is one asymptotically stable steady state, see panels (a) and (e), respectively;
- for $a = \frac{2}{3}\sqrt{\frac{1}{3}}$ there are two steady states: one asymptotically stable equilibrium and a bifurcation point for $\bar{y} = \sqrt{\frac{1}{3}}$, see panel (b);
- for $a = -\frac{2}{3}\sqrt{\frac{1}{3}}$ there are two steady states: one asymptotically stable equilibrium and a bifurcation point for $\bar{y} = -\sqrt{\frac{1}{3}}$, see panel (d);
- for $-\frac{2}{3}\sqrt{\frac{1}{3}} < a < \frac{2}{3}\sqrt{\frac{1}{3}}$ there are three steady states, two asymptotically stable (the extreme ones) and one unstable, see panel (c). (the middle one)

Panel (f) in Figure 5.4, which illustrates the bifurcation diagram, displays the hysteresis curve. If we start will small values of a there will be a unique asymptotically steady state. If a a bifurcation is reached such that the number of steady states will increase to three, with an emergence of a

[5] For known closed form solutions of ODEs see, Zaitsev and Polyanin (2003) or Zwillinger (1998).



(a) Phase diagrams for $a < -\frac{2}{3}\sqrt{\frac{1}{3}}$

(b) Phase diagrams for $a = -\frac{2}{3}\sqrt{\frac{1}{3}}$

(c) Phase diagrams for $-\frac{2}{3}\sqrt{\frac{1}{3}} < a < \frac{2}{3}\sqrt{\frac{1}{3}}$

(d) Phase diagrams for $a = \frac{2}{3}\sqrt{\frac{1}{3}}$

(e) Phase diagrams for $a > \frac{2}{3}\sqrt{\frac{1}{3}}$

(f) Bifurcation diagram

Figure 5.4: Phase diagrams and bifurcation diagram for equation (5.7).

new intermediate steady state and a another asymptotically steady state, corresponding to greater values for y. This scenario will be maintained for greater values of a until a new bifurcation is reached. Further increases of a will imply that we will have again one unique asymptotically stable steady state greater than in the start of the exercise.

For intermediate values of a, specifically for $-\frac{2}{3}\sqrt{\frac{1}{3}} < a < \frac{2}{3}\sqrt{\frac{1}{3}}$, we have three steady states such that $\bar{y}_1 < \bar{y}_2 = 0 < \bar{y}_3$, where $\bar{y}_1$ and $\bar{y}_3$ are asymptotically stable and $\bar{y}_2$ is unstable. This type of phase diagram will not occur in linear models. In particular, the state space Y is partitioned as $\mathrm{Y} = \mathcal{W}^s_{\bar{y}_1} \cup \{\bar{y}_2\} \cup \mathcal{W}\ s_{\bar{y}_3}$ where $\mathcal{W}^s_{\bar{y}_1}$ and $\mathcal{W}^s_{\bar{y}_3}$ are the stable manifolds associated to steady states $\bar{y}_1$ and $\bar{y}_3$, respectively, which are

$$\mathcal{W}^s_{\bar{y}_1} = \Big\{ y \in \mathrm{Y} : y < \bar{y}_2 \Big\} \text{ and } \mathcal{W}^s_{\bar{y}_3} = \Big\{ y \in \mathrm{Y} : y > \bar{y}_2 \Big\}.$$

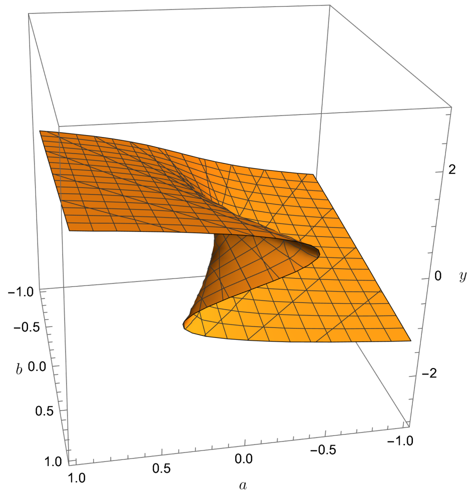


Figure 5.5: Bifurcation diagram for equation $\dot{y} = a + by - y^3$

5.1.6 Cubic equation: cusp

The next ODE

$$\dot{y} = f(y, a, b) \equiv a + by - y^3 \tag{5.8}$$

is also an Abel equation of the first kind. It has two parameters, a and b, and has the following properties: $f_{yyy}(\cdot) \neq 0$ and $f_a\ (\cdot) \neq 0$ and $f_y(\cdot)$ can have any sign depending on the parameter b. This last property allow for critical changes of its solution.

This ODE can have one, two or three equilibrium points, depending on the values of the parameters a and b. We can determine them by solving the cubic polynomial equation $a+b\,y-y^3 = 0$ (see appendix section 5.C). The number of real roots of this polynomial depends on the value of

$$\Delta \equiv \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3; \tag{5.9}$$

which is called the discriminant. It is known that: if $\Delta < 0$ there are three real roots, if $\Delta = 0$ there are two real roots (one is multiple), and if $\Delta > 0$ there is one real root and a pair of complex conjugate roots.

Therefore, regarding our ODE: if $\Delta < 0$ there are three steady states, if $\Delta = 0$ there are two steady states, and if $\Delta > 0$ there is one steady state.

We already know we can determine critical points by solving the system:

$$\begin{cases} f(y, a, b) = 0 \\ f_y(y, a, b) = 0. \end{cases} \tag{5.10}$$

Applying to equation (5.8) we have

$$\begin{cases} a + b\,y - y^3 = 0, \\ b - 3\,y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3\,a + 2\,b\,y = 0, \\ b - 3\,y^2 = 0, \end{cases} \Leftrightarrow \begin{cases} 3\,a + 2\,b\,y = 0 \\ 2\,b^2 + 9\,a\,y = 0, \end{cases} \Leftrightarrow \begin{cases} 27\,a^2 + 18\,a\,b\,y = 0 \\ 4\,b^3 + 18\,a\,b\,y = 0. \end{cases}$$

The solutions to the system must verify

$$18\,a\,b\,y = -12\,a^2 = -4\,b^3 \Leftrightarrow \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3 = 0$$

that is $\Delta = 0$. Function $f(y, a, b) = 0$ traces out a surface in the three-dimensional space for (a, b, y) called **cusp** which is depicted in Figure 6.4 [6]. Because we have two parameters, the bifurcation loci, obtained from system (5.10) defines a line in the three-dimensional space (a, b, y). We can see how it changes by imagining horizontal slices in Figure 6.4 and project them in the (a, b)-plane. This would convince us that if $a = 0$ we would get the bifurcation diagram for the pitchfork, for equation (5.6), and if $a \neq 0$ and $b = 1$ we obtain the hysteresis diagram, for equation (5.7). This result would be natural because those two equations are a particular case of the cusp equation.

Those points refer to just one level of degeneracy, and are given by a particular relationship between two parameters, and not by specific values for them. This is the reason that we call them **co-dimension one bifurcation points**. As we have two parameters, we can find a higher level of degeneracy by finding a particular values of (y, a, b) that solves the system

$$\begin{cases} f(y, a, b) = 0 \\ f_y(y, a, b) = 0 \\ f_b(y, a, b) = 0, \end{cases}$$

and call them **co-dimension two bifurcation points**. We find that this point is unique $(y, a, b) = (0, 0, 0)$ and is called a **cusp point**. Looking at Figure 6.4 we observe that this point separates two types of bifurcation diagrams we have already seen: first, if we project the cusp surface into a vertical plane passing through this point we obtain a bifurcation diagram for the pitchfork, and, second, if we project the cusp surface into a horizontal plane passing through this point we obtain a bifurcation diagram for the hysteresis.

5.2 Qualitative theory

A **normal form** is the simplest ODE, whose exact solution is usually known and represents a whole family of ODEs by topological equivalence.[7] The simplest case of a normal form is a linear ODE, scalar or planar. It is locally or globally topological equivalent to any ODE with one steady state and whose Jacobian, evaluated at that steady state, does not eigenvalues with zero real parts or whose Jacobian, evaluated at any point $y \in Y$ has no singularities (i.e., infinitely valued eigenvalues).

[6] This was one of the famous cases of catastrophe theory very popular in the 1980's see `https://en.wikipedia.org/wiki/Catastrophe_theory`.

[7] In heuristic terms, we say functions $f(y)$ and $g(x)$ are topologically equivalent if there exists a diffeomorphism, i.e., a smooth map h with a smooth inverse h^{-1}, such that if $y = h(x)$ then $h(g(x)) = f(h(x))$. This property may hold globally of locally. The last case is the intuition behind the Grobman-Hartmann theorem.

However, the term normal form is usually reserved to ODE's which are topologically equivalent to ODE's in which $f(y)$ is a polynomial in y.

If a small variation of the parameter changes the phase diagram we say we have a bifurcation. As you saw, there are local (fixed points) and global bifurcations (heteroclinic connection, etc). Those bifurcations were associated to particular normal forms of both scalar and planar ODEs. This fact allows us to find classes of ODE's which are topologically equivalent to those we have already presented.

5.2.1 Bifurcations for scalar ODE's

Consider the scalar ODE

$$\dot{y} = f(y, \varphi),\ Y, \varphi \in \mathbb{R}.$$

Fold bifurcation (see (Kuznetsov, 2005, ch. 3.3)): Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yy}(0, 0) \neq 0,\ f_{\varphi}(0, 0) \neq 0.$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi \pm y^2,$$

that is to the Ricatti's model (5.4).

Transcritical bifurcation: Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yy}(0, 0) \neq 0,\ f_{\varphi y}(0, 0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^2$$

that is to the Bernoulli model (5.5).

Pitchfork bifurcation: Let $f \in C^2(\mathbb{R})$ and consider $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yyy}(0, 0) \neq 0,\ f_{\varphi y}(0, 0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^3$$

that is to the Bernoulli model (5.6).

5.3 References

- (Hale and Koçak, 1991, Part I , III): very good introduction.
- Brock and Malliaris (1989), (Grass et al., 2008, ch.2) has a compact presentation of all the important results with some examples in economics and management science.

5.A Solution of Ricatti's equation (5.4)

Start with the case: $a = 0$. Separating variables, we have

$$\frac{dy}{y^2} = dt$$

integrating both sides

$$\int_{y(0)}^{y(t)} \frac{dy}{y^2} = \int_0^t ds \Leftrightarrow -\frac{1}{y(t)} = t - \frac{1}{y(0)}.$$

Then the solution is

$$y(t) = \frac{y(0)}{1 - y(0)\,t}.$$

Now let $a \neq 0$. By using the same method we have

$$\frac{dy}{a + y^2} = dt. \tag{5.11}$$

At this point it is convenient to note that

$$\frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}\ ,\ \frac{d\tanh^{-1}(x)}{dx} = \frac{1}{1-x^2},$$

where

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}, \quad \tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}.$$

Then we should deal separately with the cases $a > 0$ and $a < 0$. If $a > 0$ integrating equation (5.11)

$$\int \frac{dy}{a + y^2} = dt \Leftrightarrow \frac{1}{\sqrt{a}} \int \frac{1}{1+x^2} dx = t + k \Leftrightarrow \frac{1}{\sqrt{a}} \tan^{-1}(x) = t + k$$

where we defined $x = y/\sqrt{a}$. Solving the last equation for x and mapping back to y we get

$$y(t) = \sqrt{a}\left(\tan\left(\sqrt{a}(t + k)\right)\right).$$

If $a < 0$ we integrate equation (5.11) by using a similar transformation, but instead with $x = y/\sqrt{-a}$ to get

$$\int \frac{dy}{a + y^2} = dt \Leftrightarrow -\frac{1}{\sqrt{-a}} \int \frac{1}{1-x^2} dx = t + k \Leftrightarrow -\frac{1}{\sqrt{-a}} \tanh^{-1}(x) = t + k.$$

Then

$$y(t) = -\sqrt{-a}\left(\tanh\left(\sqrt{-a}(t + k)\right)\right).$$

5.B Solution for a general Bernoulli equation

Consider the Bernoulli equation

$$\dot{y} = ay + by^{\eta},\ a \neq 0,\ b \neq 0 \tag{5.12}$$

where $y : T \to \mathbb{R}$. We intoduce a first transformation $z(t) = y(t)^{1-\eta}$, which leads to a linear ODE

$$\dot{z} = (1-\eta)(az + b) \tag{5.13}$$

beause

$$\begin{aligned}
\dot{z} &= (1-\eta)y^{-\eta}\dot{y} = \\
&= (1-\eta)(ay^{1-\eta} + b) = \\
&= (1-\eta)(az + b).
\end{aligned}$$

To solve equation (5.13) we introduce a second transformation $w(t) = z(t) + \frac{b}{a}$. Observing that $\dot{w} = \dot{z}$ we obtain a homogeneous ODE $\dot{w} = a(1-\eta)w$ which has solution

$$w(t) = w(0)\, e^{a(1-\eta)t}.$$

Then the solution to equation (5.13) is

$$z(t) = -\frac{b}{a} + \Big(w(0) + \frac{b}{a}\Big)e^{a(1-\eta)t}$$

because $w(0) = y(0) + \frac{b}{a}$.

We finally get the solution for the Bernoulli equation (5.12)

$$y(t) = \Big[-\frac{b}{a} + \Big(y(0)^{1-\eta} + \frac{b}{a}\Big)\, e^{a(1-\eta)t}\Big]^{\frac{1}{1-\eta}} \tag{5.14}$$

because $z(0) = y(0)^{1-\eta}$.

If $a = 0$ the solution is

$$y(t) = \Big(y(0)^{1-\eta} \ + b\,(1-\eta)\,t\Big)^{\frac{1}{1-\eta}}.$$

Æxercise: Prove this.

5.C Solution to the cubic polynomial equation

In this section we present the solutions to the cubic polynomial equation [8]

$$x^3 + \alpha_2\, x^2 + \alpha_1\, x + \alpha_0 = 0.$$

[8]See `cubicpolynomialhistory` for the history of this equation.

By performing a Tschirnhausen transformation (see King (1996)) we can transform into a simpler, irreducible, equation. Defining $y = x - \dfrac{\alpha_2}{3}$ we obtain the polynomial

$$y^3 + \left(\alpha_1 - \frac{\alpha_2^2}{3}\right) y + \alpha_0 - \frac{\alpha_1\,\alpha_2}{3} + \frac{2\,\alpha_2^3}{27} \;= 0$$

This equation is a (monic) cubic polynomial equation, of type

$$y^3 - by - a = 0. \tag{5.15}$$

If $a = 0$ we have $y\,(y^2 - b\,y) = 0$ and the solutions are $y = 0$ and the solutions of the quadratic equation are $y = \pm\sqrt{b}$.

If $a \neq 0$ we prove that the solutions of the monic cubic equation are

$$y_j = w^{j-1}\theta^{\frac{1}{3}} + \frac{b}{3}\left(\omega^{j-1}\theta^{\frac{1}{3}}\right)^{-1},\; j = 1, 2, 3. \tag{5.16}$$

where ω and θ are presented next.

Write $y = u + v$. Then we get the equivalent representation

$$u^3 + v^3 + 3\left(uv - \frac{b}{3}\right)(u + v) - a = 0.$$

As $y = u + v \neq 0$, u and v solve simultaneously

$$\begin{cases} u^3 + v^3 = a \\ uv = \dfrac{b}{3} \end{cases} \quad\Leftrightarrow\quad \begin{cases} u^3u^3 + u^3v^3 - u^3a = 0 \\ u^3v^3 = \left(\dfrac{b}{3}\right)^3 \end{cases} \quad\Leftrightarrow\quad \begin{cases} u^6 - au^3 + \left(\dfrac{b}{3}\right)^3 = 0 \\ uv = \frac{b}{3}. \end{cases}$$

The first equation is a quadratic polynomial in u^3 which has roots

$$u^3 = \frac{a}{2} \pm \sqrt{\Delta},\text{ where } \Delta \equiv \left(\frac{a}{2}\right)^2 - \left(\frac{b}{3}\right)^3$$

where Δ is the discriminant in equation (5.9). We can take any solution of the previous equation and set $\theta \equiv \frac{a}{2} + \sqrt{\Delta}$.

At this stage it is useful to observe that the solutions of equation $x^3 = 1$ are

$$x_1 = 1,\; x_2 = \omega,\; x_3 = \omega^2.$$

where $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2}(1 - \sqrt{3}i)$ and $\omega^2 = e^{\frac{4\pi i}{3}} = -\frac{1}{2}(1 + \sqrt{3}i)$. Therefore $u^3 = \theta$ has also three solutions

$$u_1 = \theta^{\frac{1}{3}},\; u_2 = \omega\theta^{\frac{1}{3}},\; u_3 = \omega^2\theta^{\frac{1}{3}}$$

and because $v = \frac{b}{3u}$, we finally obtain equation (5.16), which are the solutions to equation (5.15).

5.D Approximating functions

Assume we have a function $f(x)$ for $x \in \mathrm{X} \subseteq \mathbb{R}$. If function $f(.)$ is non-linear, or non explicitly specified, we need sometimes to compare it with another function whose behaviour can be simpler, or we need to know only about how it grows or decays, or its concavity or symmetry, without worrying about the details. The notations $O(.)$ and $o(.)$ (called big-O and littlele-o notations) provide a usefull approach.

5.D.1 Big O notation

A function is of **constant order** if there is a non-zero constant c such that we can write, equivalently

$$\lim_{x \to \infty} \frac{f(x)}{c} = 1 \,\text{or}\, \lim_{x \to \infty} f(x) = c$$

We say in this case that

$$f(x) \in O(1)$$

More generally:

Definition 1. *A function $f(x)$ **is big-O of function** $g(x)$, written as $f(x) \in O(g(x))$ (sometimes written $f(x) = O(g(x))$ if $f(x)$ is of the same order than $g(x)$ (they grow or decay at the same rate)*

$$f(x) \in O(g(x)) \text{ or } \lim_{x \to \infty} \frac{f(x)}{g(x)} = c$$

or $\frac{f(x)}{g(x)} \in O(1)$.

Examples:

- $ax + b \in O(x)$ if $a \neq 0$
- $ax^2 + bx + c \in O(x^2)$ if $a \neq 0$
- $ax^{-2} + bx^{-1} \in O(x^{-1})$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \in O(x^2)$

5.D.2 Little o notation

There is an associate function, the *little-o* notation that means that a functions is asymptotically smaller than another function. Function $f(x)$ is asymptotically smaller than a non-zero constant c if

$$\lim_{x \to \infty} \frac{f(x)}{c} = 0, \,\text{or} f(x) \in o(1)$$

In general, function $f(x)$ *is little-o of function* $g(x)$, written as of $f(x) \in o(g(x))$ if for $n \to \infty$ there is a number N such that

$$|f(x)| < \epsilon |x| \,\text{for}\, n > N$$

If $g(x) \neq 0$ in all its domain them $f(x)$ is a small-o of function $g(x)$ if and only if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

Examples:

- $2x = o(x^2)$

Notation	Limit definition (a)	Limit definition (b)
$f \in o(g)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$\limsup \frac{f(x)}{g(x)} = 0$
$f \in O(g)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$\limsup \frac{f(x)}{g(x)} < \infty$

- $e^x = 1 + x + \frac{1}{2}x^2 + o(x^3)$,
- $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^4)$
- $\ln(x) \in o(x)$

For a frequent function used in economics $f(x) = x^\alpha$ where $\alpha > 0$; we have

- if $0 \leq \alpha < 1$ then $f(x)$ if $o(x)$
- if $1 \leq \alpha < 2$ then $f(x)$ if $o(x^2)$
- if $2 \leq \alpha < 3$ then $f(x)$ if $o(x^3)$
- if $n - 1 \leq \alpha < n$, for n an integer, then $f(x)$ if $o(x^n)$.

If $f(x) \in o(g(x))$ then $f(x) \in O(g(x))$ but the converse may not be true.
There are several **properties** for $O(.)$ and $o(.)$ functions, this is a small sample:

- If $f(x) \in O(g(x))$ then $c\, f(x) \in O(g(x))$ for any constant c,
- If $f_1(x)$ and $f_2(x)$ are both $O(g(x))$ then $f_1(x) + f_2(x) \in O(g(x))$,
- If $f_1(x) \in O(g(x))$ and $f_2(x) \in o(g(x))$ then $f_1(x) + f_2(x) \in O(g(x))$,
- If $f_1(x) \in O(f_2(x))$ and $f_2(x) \in o(g(x))$ then $f_1(x) \in o(g(x))$,
- $x^n o(x^m) = o\left(x^{n+m}\right)$
- $o\left(x^n\right) o\left(x^m\right) = o\left(x^{n+m}\right)$
- $x^m = o\left(x^n\right)$ if $n < m$

5.D.3 Calculus with big-O and little-o

The function f has a strong derivative at x if

$$f(x + \epsilon) = f(x) + f^{'}(x)\epsilon + O(\epsilon)$$

for ϵ sufficiently small.

If f is n-times differentiable at x the Taylor's theorem states

$$f(x+\epsilon) = f(x) + f'(x)\,\epsilon + \frac{f''(x)}{2}\ \epsilon^2 + ... + \frac{f^{(n)}}{n!}\epsilon^n + O(\epsilon^n).$$

Weak derivative

$$f(x+\epsilon) = f(x) + f^{'}(x)\epsilon + o(\epsilon^2)$$

The Lagrange formula can be written as

$$f(x+\epsilon) = f(x) + f^{'}(x)\epsilon + \frac{f^{''}(x)}{2!}\ \epsilon^2 + ... + \frac{f^{(n)}(x)}{n!}\ \epsilon^n + o(\epsilon^{n+1}).$$

Chapter 6

Planar non-linear ODE's: the regular case

Time as independent variable. We look for functions $y : \mathrm{T} \times \Phi \rightarrow \mathrm{Y} = \mathbb{R}$, where $\varphi \in \Phi$ is a set of parameters which solve the (autonomous) equation

$$F(\dot{y}, y, \varphi) = 0 \tag{6.1}$$

We say the ODE is **regular** if $F(\dot{y}, y, \cdot)$ is continuous and continuously differentiable and furthermore $\dfrac{d\,F(\dot{y}, y, \varphi)}{d\dot{y}} \neq 0$ for every $y \in \mathrm{Y}$.

This allows us to write equation (5.1) as

$$\dot{y} = f(y, \varphi) \tag{6.2}$$

We assume next that $y : \mathbb{R}_+ \rightarrow \mathrm{Y} \subseteq \mathbb{R}$, and the parameters are real numbers.

In the rest of this chapter we present the normal forms (5.1) and in section we present the qualitative theory for 5.2 for non-linear regular scalar ODE's.

6.1 Normal forms

A **normal form** is the simplest ODE, whose exact solution is usually known and represents a whole family of ODEs by topological equivalence.[1]

We present next some important normal forms for scalar ODEs, which have the generic representation by a polynomial vector field

$$\dot{\mathbf{y}} = \mathbf{f}\,(\mathbf{y}, \varphi) \equiv \mathbf{A}_0 + \mathbf{A}_1\,\mathbf{y} + \mathbf{A}_2\,<\mathbf{y}, \mathbf{y}> + \mathbf{A}_3\,<\mathbf{y}, \mathbf{y}, \mathbf{y}>, \tag{6.3}$$

[1] In heuristic terms, we say functions $f(y)$ and $g(x)$ are topologically equivalent if there exists a diffeomorphism, i.e., a smooth map h with a smooth inverse h^{-1}, such that if $y = h(x)$ then $h(g(x)) = f(h(x))$. This property may hold globally of locally. The last case is the intuition behind the Grobman-Hartmann theorem.

where $\boldsymbol{\varphi} \equiv (\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) \in \mathbb{R}^4$.

Linear scalar ODE's, that have been studied in chapter two, are particular cases in which $a_2 = a_3 = a_4 = 0$. In this chapter we address cases in which at least one of coefficients a_2, a_3 or a_4 is different from zero.

It can be shown that (see (Hale and Koçak, 1991, ch. 2)) the following cases are the most relevant: first, ODE's depending on a single parameter (a), as the Ricatti's equation $\dot{y} = a + y^2$, the quadratic Bernoulli equation, $\dot{y} = ay + y^2$, the cubic equations Bernoulli equation, $\dot{y} = ay - y^3$, the Abel equation $\dot{y} = a + y - y^3$; second, ODE's depending on two parameters (a and b) the Abel's equation, $\dot{y} = a + by - y^3$.

We address the solution of those equations from two approaches: First, an **analytical approach**, seeking to find its explicit solution (when it is known); and, second, a **qualitative (or geometric) approach**, which characterizes qualitatively the possible dynamic behavior of the solution, depending on the value of the parameters. The

6.1.1 Planar ODE's

Next we consider the planar ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi)$, in vector notation, $\mathbf{y} : T \to Y \subseteq \mathbb{R}^2$, depending on a vector of parameters, $\varphi \in \mathbb{R}^n$ for $n \geq 1$. Expanding, we have,

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \varphi) \\ \dot{y}_2 &= f_2(y_1, y_2, \varphi) \end{aligned} \tag{6.4}$$

There are a large number of normal forms that have been studied for planar ODEs (see Kuznetsov (2005)).

In principle, we could consider combining all the previous scalar normal forms to have an idea of the number of possible cases, and extend the previous method to study the dynamics. That method consisted in finding critical points, corresponding to steady states and values of the parameters such that the derivatives of the steady variables would be equal to zero. However, for planar equation, to fully characterise the dynamics, we may have to study local dynamics in invariant orbits other than steady states. In general there are, at least, three types of **invariant orbits** that do not exist in planar linear models: homoclinic and heteroclinic orbits and limit cycles.

In the next section we present a general method to finding bifurcation points associated to steady states. In the rest of this section we presents ODE's in which those invariant curves exist and are generic (in the sense that they are verified for any values of a parameter, except for some particular values) and non-generic. The non-generic cases consist in one-parameter bifurcations for non-linear planar equations associated to heteroclinic and homoclinic orbits and limit cycles.

Heteroclinic orbits We say there is an **heteroclinic orbit** if, in a planar ODE in which there are at least two steady states, say $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$, and there are solutions $\mathbf{y}(t)$ that entirely lie in a curve joining $\bar{\mathbf{y}}^1$ to $\bar{\mathbf{y}}^2$ say $\text{Het}(\mathbf{y})$. Therefore, if $\mathbf{y}(0) \in \text{Het}(\mathbf{y})$ then $\mathbf{y}(t) \in \text{Het}(\mathbf{y})$ for $t > 0$ and either $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^1$ and $\lim_{t\to-\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^2$ or $\lim_{t\to\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^2$ and $\lim_{t\to-\infty} \mathbf{y}(t) = \bar{\mathbf{y}}^1$.

Heteroclinics can exist if the stability type of the steady states are different or equal. In the first case, they connect stable or unstable node and a saddle point or a stable and one unstable node. In the last case, the only possibility is if the two steady states are saddle points and we say we have a **saddle connection**.

Heteroclinic networks can also exist when there are more than two steady states which are connected.

Generic heteroclinic orbits Although there are several normal forms generating generic heteroclinic orbits, we focus next in the following case:

$$\begin{aligned} \dot{y}_1 &= a y_1 y_2 \\ \dot{y}_2 &= 1 + y_1^2 - y_2^2 \end{aligned} \tag{6.5}$$

where $a \neq 0$. This equation has two steady states: $\bar{\mathbf{y}}^1 = (0, -1)$ and $\bar{\mathbf{y}}^2 = (0, 1)$. Calling,

$$\mathbf{f}(\mathbf{y}) = \begin{pmatrix} a y_1 y_2 \\ 1 + y_1^2 - y_2^2 \end{pmatrix}$$

we have the Jacobian

$$J(\mathbf{y}) \equiv D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} a y_2 & a y_1 \\ 2 y_1 & -2 y_2 \end{pmatrix},$$

which has trace and determinant depending on the parameter a

$$\begin{aligned} \operatorname{trace}(J(\mathbf{y})) &= (a-2) y_2 \\ \det(J(\mathbf{y})) &= -2a(y_1^2 + y_2^2). \end{aligned}$$

Then, remembering again that we assumed $a \neq 0$ and because, for any steady state $y_1^2 + y_2^2 > 0$ then $\det(J(\mathbf{y})) > 0$ if $a < 0$ and $\det(J(\mathbf{y})) < 0$ if $a > 0$.

Therefore, if $a < 0$, steady state $\bar{\mathbf{y}}^1$ is an unstable node, because $\operatorname{trace}(J(\bar{\mathbf{y}}^1)) > 0$ and steady state $\bar{\mathbf{y}}^2$ is a stable node, because $\operatorname{trace}(J(\bar{\mathbf{y}}^2)) < 0$. Then for any trajectory starting from any element of $\mathbf{y} \neq \bar{\mathbf{y}}^2$ there is convergence to steady state $\bar{\mathbf{y}}^2$ (see the left subfigure in figure 6.1).

If we denote $\text{Het}(\mathbf{y})$ the set points connecting $\bar{\mathbf{y}}^2$ to $\bar{\mathbf{y}}^1$ we readily see that $\text{Het}(\mathbf{y}) = \mathrm{Y}/\{\mathbf{y}_1\}$, which means there is an infinite number of heteroclinic orbits, and that this set is coincident with the stable manifold $\mathcal{W}^s_{\bar{\mathbf{y}}^1}$ (see the left subfigure in Figure 6.1).

However, if $a > 0$ both steady states, $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$, are saddle points, because $\det(J(\bar{\mathbf{y}}^1)) = \det(J(\bar{\mathbf{y}}^2)) < 0$. In this case, there is one heteroclinic surface

$$\text{Het}(\mathbf{y}) = \{\ (y_1, y_2) : y_1 = 0,\ -1 \leq y_2 \leq 1\}$$

which is the locus of points connecting $\bar{\mathbf{y}}^1$ and $\bar{\mathbf{y}}^2$ such that for any initial value $\mathbf{y}(0) \in \text{Het}(\mathbf{y})$ the solution will converge to $\bar{\mathbf{y}}^2$ (see the right subfigure in figure 6.1). In this case $\text{Het}(\mathbf{y})$ is the set of points belonging to the intersection of the unstable manifold of $\bar{\mathbf{y}}^1$ and to the stable manifold of $\bar{\mathbf{y}}^2$: $\text{Het}(\mathbf{y}) = \mathcal{W}^u_{\bar{\mathbf{y}}^1} \cap \mathcal{W}^s_{\bar{\mathbf{y}}^2}$.

At last, we should notice that in both cases the heteroclinic orbits are generic, in the sense that they persist for a wide range of values for parameter a. This is not the case for the next example.

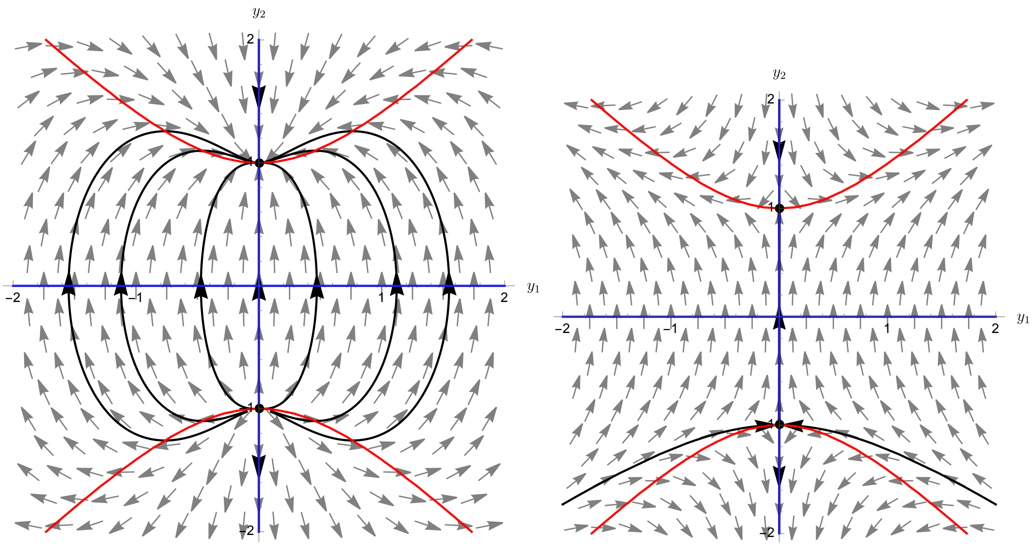


Figure 6.1: Phase diagrams for equation (6.5) for $a < 0$, and $a > 0$

Heteroclinic saddle connection bifurcation Assuming a related but slightly different normal form generates an heteroclinic bifurcation meaning we may have a bifurcation parameter that when it crosses a specific value heteroclinic orbits cease to exist. The following model is studied, for instance, in (Hale and Koçak, 1991, p.210).

$$\begin{aligned} \dot{y}_1 &= \lambda + 2y_1y_2 \\ \dot{y}_2 &= 1 + y_1^2 - y_2^2 \end{aligned} \tag{6.6}$$

In this case we have, for $\lambda = 0$, an heteroclinic orbit, connecting the two steady states exists and we have the second case in the previous model. When λ is perturbed away from zero we will have only one steady state which is a saddle point. See Figure 6.2.

Homoclinic orbits We say there is an **homoclinic orbit** if, in a planar ODE, there is a subset of points $\text{Hom}(\mathbf{y})$ connecting the steady state with itself. This is only possible if the steady state $\bar{\mathbf{y}}$ is a saddle point in which the stable manifold contains a closed curve, that we call homoclinic curve. Because of this fact, homoclinic orbits exist jointly with periodic trajectories.

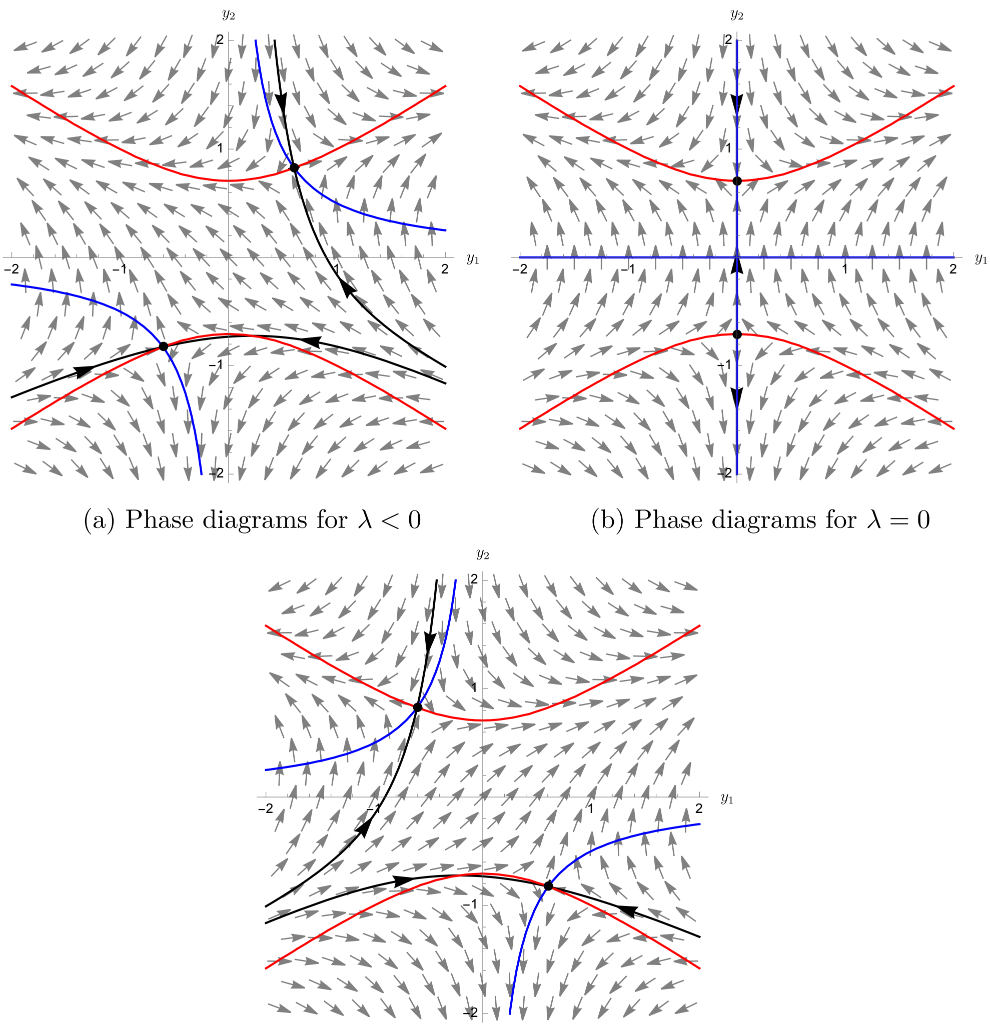
Again, homoclinic orbits can be generic or non-generic. Next we illustrate both cases.

Generic homoclinic orbits Consider the non-linear planar ODE depending on one parameter, a, of type

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 - ay_1^2. \end{aligned} \tag{6.7}$$

It has two steady states $\bar{\mathbf{y}}^1 = (0,0)$ and $\bar{\mathbf{y}}^2 = (1/a, 0)$. The Jacobian

$$J(\mathbf{y}) \equiv D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ 1 - 2ay_1 & 0 \end{pmatrix}$$



(a) Phase diagrams for $\lambda < 0$

(b) Phase diagrams for $\lambda = 0$

(c) Phase diagrams for $\lambda > 0$

Figure 6.2: Phase diagrams for equation (6.6).

has following the trace and the determinant

$$\begin{aligned} \operatorname{trace}\left(J(\mathbf{y})\right) &= 0 \\ \det\left(J(\mathbf{y})\right) &= 2ay_1 - 1. \end{aligned}$$

It is easy to see that steady state $\bar{\mathbf{y}}^1$ is always a saddle point, because $\det\left(J(\bar{\mathbf{y}}^1)\right) = -1 < 0$, and the steady state $\bar{\mathbf{y}}^2$ is always locally a center, because $\det\left(J(\bar{\mathbf{y}}^2)\right) = 1 > 0$ and $\operatorname{trace}\left(J(\bar{\mathbf{y}}^2)\right) = 0$, for any value of a.

Furthermore, we can prove that there is an invariant curve, such that solutions follow a potential or first integral curve which is constant.

In order to see this we introduce a **Lyapunov function** which is a differentiable function $H(\mathbf{y})$ such that the time derivative is $\dot{H} = D_{\mathbf{y}}H \cdot \dot{\mathbf{y}}$, that is $\dot{H} = H_{y_1}\dot{y}_1 + H_{y_2}\dot{y}_2$. A **first integral** is a set of points (y_1, y_2) such that $\dot{H} = 0$. In this case the orbits passing through those points allow for

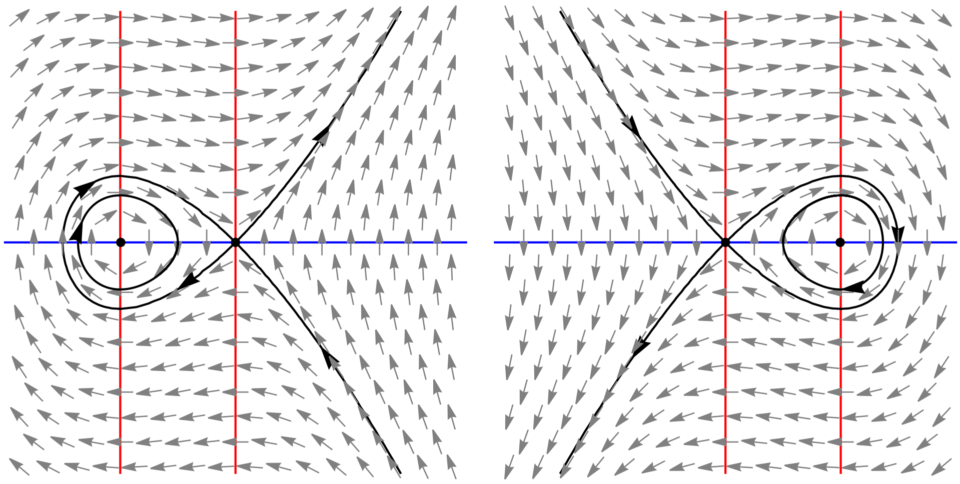


Figure 6.3: Phase diagrams for equation (6.7) for $a < 0$, and $a > 0$

a conservation of energy in some sense and $H(\mathbf{y}(t)) =$ constant. For values such that $H(\mathbf{y}(t)) = 0$ that curve passes through a steady state.

For this case consider the function

$$H(y_1, y_2) = -\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{a}{3}y_1^3.$$

If we time-differentiate this Lyapunov function and substitute equations (6.8) we have

$$\begin{aligned}\dot{H} &= (ay_1 - 1)y_1\dot{y}_1 + y_2\dot{y}_2 = \\ &= (ay_1 - 1)y_1 y_2 + y_2 y_1(1 - ay_1) = \\ &= 0.\end{aligned}$$

Then $\dot{H} = 0$, for any values of $\mathbf{y}$ and a. We call homoclinic surface to the set of points such that there are homoclinic orbits. In our case, homoclinic orbits converge both for $t \to \infty$ and $t \to -\infty$ to point $\bar{\mathbf{y}}^1$. Therefore the homoclinic surface is the set of points

$$\mathrm{Hom}(\bar{\mathbf{y}}^1) = \{\ (y_1, y_2) : H(y_1, y_2) = 0,\ \mathrm{sign}(\bar{y}_1) = \mathrm{sign}(a)\}$$

Figure 6.3 depicts phase diagrams for the case in which $a < 0$ (left sub-figure) and $a > 0$ (right sub-figure).

We see that the homoclinic trajectories are generic, i.e, they exist for different values of the parameters. This is not always the case as we show next.

Homoclinic or saddle-loop bifurcation This model is studied, for instance, in (Hale and Koçak, 1991, p.210) and (Kuznetsov, 2005, ch. 6.2). It is a non-linear ODE depending on one parameter, a, of type

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 + a\,y_2 - y_1^2.\end{aligned} \tag{6.8}$$

In this case, we have

$$\mathbf{f}(\mathbf{y}, a) = \begin{pmatrix} y_2 \\ y_1 + a\, y_2 - y_1^2. \end{pmatrix}.$$

The set of equilibrium point is $\bar{\mathbf{y}} = \{\ \mathbf{y} : \mathbf{f}(\mathbf{y}, a) = \mathbf{0}\}$. For equation (6.8) we have two equilibrium points,

$$\bar{\mathbf{y}}^1 = \begin{pmatrix} \bar{y}_1^1 \\ \bar{y}_2^1 \end{pmatrix} = \mathbf{0},\ \bar{\mathbf{y}}^2 = \begin{pmatrix} \bar{y}_1^2 \\ \bar{y}_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In order to determine the local dynamics we evaluate the Jacobian for any point $\mathbf{y} = (y_1, y_2)$,

$$D_{\mathbf{y}}\mathbf{f}(\mathbf{y}, a) = \begin{pmatrix} 0 & 1 \\ 1 - 2y_1 & a \end{pmatrix}.$$

The eigenvalues of the Jacobian are functions of the variables and of the parameter a,

$$\lambda_{\pm}(\mathbf{y}, a) = \frac{a}{2} \pm \left[\ \left(\frac{a}{2}\right)^2 + 1 - 2y_1\ \right]^{\frac{1}{2}}.$$

If we evaluate the eigenvalues at the steady state $\bar{\mathbf{y}}^1 = (0, 0)$, we find it is a saddle point, because the eigenvalues of the Jacobian $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^1)$ are

$$\lambda_{\pm}^1 \equiv \lambda_{\pm}(\bar{\mathbf{y}}^1, a) = \frac{a}{2} \pm \left[\ \left(\frac{a}{2}\right)^2 + 1\ \right]^{\frac{1}{2}}$$

yielding $\lambda_-^1 < 0 < \lambda_+^1$. At the steady state $\bar{\mathbf{y}}^2 = (1, 0)$ the eigenvalues of the Jacobian $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^2)$ are

$$\lambda_{\pm}^2 = \lambda_{\pm}(\bar{\mathbf{y}}^2, a) = \frac{a}{2} \pm \left[\ \left(\frac{a}{2}\right)^2 - 1\ \right]^{\frac{1}{2}}$$

yielding $\text{sign}(\text{Re}(\lambda_{\pm}(\bar{\mathbf{y}}^2, a)) = \text{sign}(a)$.

Therefore steady state $\bar{\mathbf{y}}^1$ is always a saddle point, and steady state $\bar{\mathbf{y}}^2$ is a stable node or a stable focus if $a < 0$, it is an unstable node or an unstable focus if $a > 0$, or it is a centre if $a = 0$.

When $a = 0$ another type of dynamics occurs. We introduce the following Lyapunov function

$$H(y_1, y_2) = -\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{3}y_1^3.$$

and prove that it can only be a first integral if $a = 0$. To show this, if we time-differentiate this Lyapunov function and substitute equations (6.8) we have

$$\begin{aligned} \dot{H} &= (y_1 - 1)y_1\dot{y}_1 + y_2\dot{y}_2 = \\ &= (y_1 - 1)y_1y_2 + y_2y_1(1 - y_1) + ay_2^2 = \\ &= ay_2^2. \end{aligned}$$

Then $\dot{H} = 0$, for any values of $\mathbf{y}$, if and only if and only if $a = 0$.

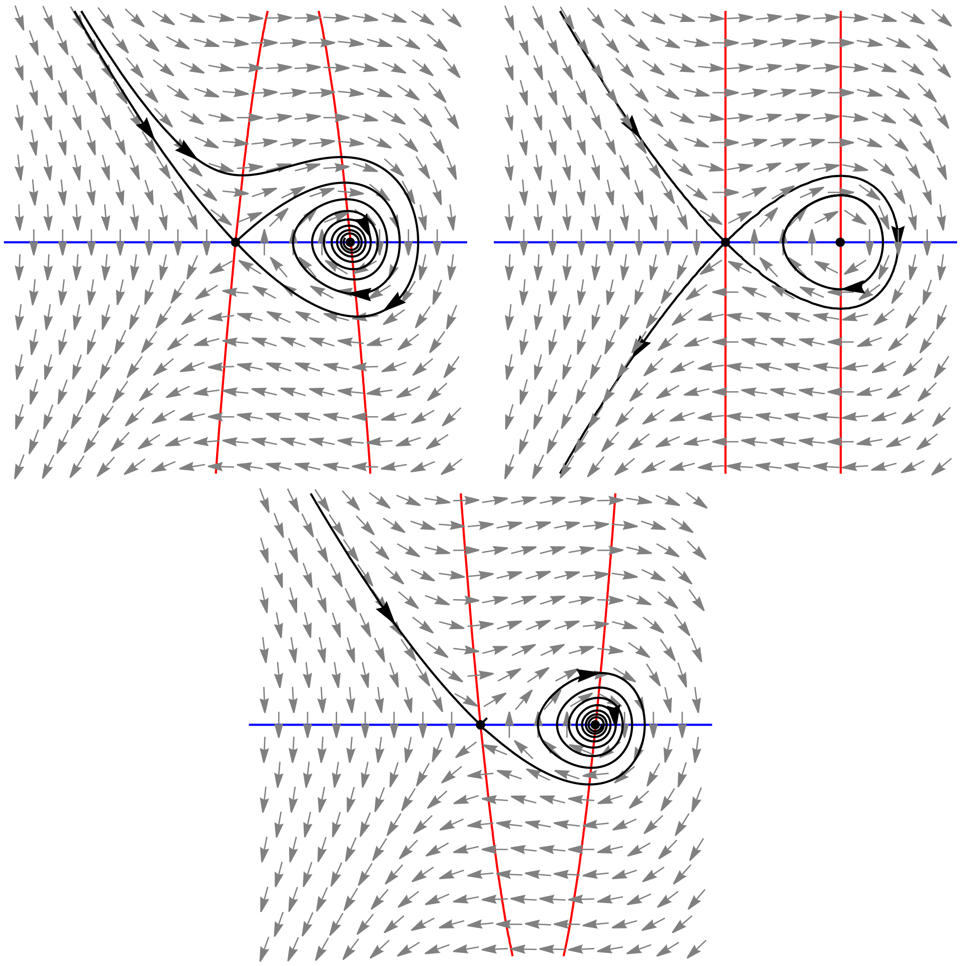


Figure 6.4: Phase diagrams for equation (6.8) for $a < 0$, $a = 0$, $a > 0$

In our case this generates an **homoclinic orbit** which is a trajectory that exits a steady state and returns to the same steady state. In this case, a homoclinic orbit exists if $a = 0$ and it does not exist if $a \neq 0$.

The next figure shows the phase diagrams for the cases $a < 0$, $a = 0$ and $a > 0$. If $a < 0$ there is a saddle point and a stable focus, if $a = 0$ there is a saddle point, an infinite number of centres surrounded by an homoclinic orbit. If $a > 0$ there is a saddle point and an unstable focus.

Planar equation: Andronov-Hopf bifurcation This model is studied, for instance, in (Hale and Koçak, 1991, p.212).

$$
\begin{aligned}
\dot{y}_1 &= f_1(y_1, y_2) \equiv y_2 + y_1(\lambda - y_1^2 - y_2^2) \\
\dot{y}_2 &= f_2(y_1, y_2) \equiv -y_1 + y_2(\lambda - y_1^2 - y_2^2)
\end{aligned} \tag{6.9}
$$

It has a single steady state $\bar{\mathbf{y}} = (0, 0)$. However, it has another invariant curve. In order to see

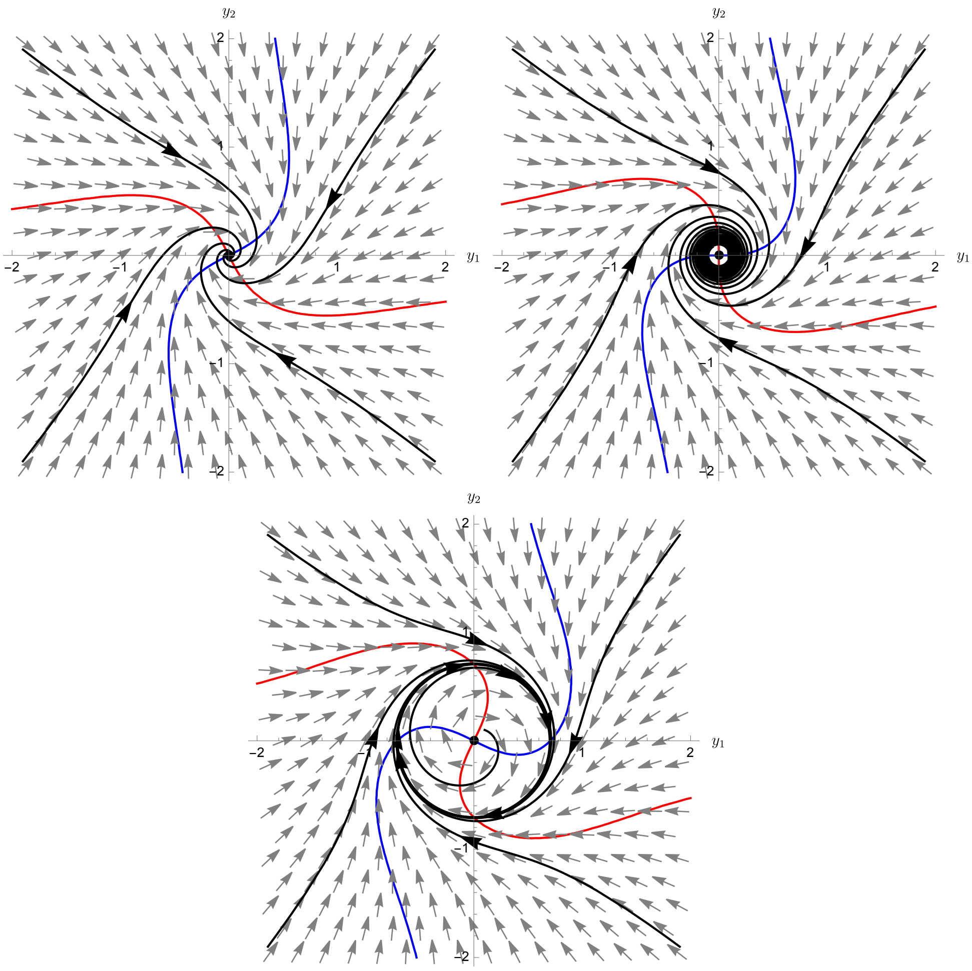


Figure 6.5: Phase diagrams for equation (6.9) for $\lambda < 0$, $\lambda = 0$, $\lambda > 0$

this, we compute the Jacobian

$$J(\mathbf{y}) = \begin{pmatrix} \lambda - 3y_1^2 - y_2^2 & 1 - 2y_1y_2 \\ -1 - 2y_1y_2 & \lambda - y_1^2 - 3y_2^2 \end{pmatrix}$$

which has eigenvalues

$$\lambda_{\pm} = \lambda - 2(y_1^2 + y_2^2) \pm (y_1^2 + y_2^2)$$

In figure 6.5 we see the following: if $\lambda < 0$ there will be only one steady state which is a stable node with multiplicity, although the speed of convergence to the steady state increases very much when λ converges to zero, if $\lambda > 0$ a **limit circle** appears and the steady state becomes a unstable focus. According to the Bendixson-Dulac criterium (see Theorem 3) as

$$\frac{\partial f_1(y_1, y_2)}{\partial y_1} + \frac{\partial f_2(y_1, y_2)}{\partial y_2} = 2\lambda - (2y_1)^2 - (2y_2)^2$$

changes sign for $\lambda > 0$, in a subset of $\mathbf{y}$, then a closed curve can exist. This closed curve is a limit cycle which is a curve such that $y^1 + y^2 = \lambda$. To prove this, we transform the system in polar coordinates (see Appendix to chapter 1) and get [2]

$$\begin{aligned} \dot{r} &= r(\lambda - r^2) \\ \dot{\theta} &= -1 \end{aligned}$$

there is thus a periodic orbit with radius $\bar{r} = \sqrt{\lambda}$.

Planar equation: Bogdanov-Takens bifurcation The normal form of the Bogdanov-Takens bifurcation is

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2) \equiv y_2 \\ \dot{y}_2 &= f_2(y_1, y_2) \equiv \lambda_1 + \lambda_2\, y_1 + y_1^2 \pm y_1\, y_2 \end{aligned} \tag{6.10}$$

Has co-dimension one local bifurcations: saddle-node and Andronov-Hopf and a global bifurcation: homoclinic bifurcation.

6.2 Qualitative theory of ODE

Next we present a short introduction to the qualitative (or geometrical) theory of ODE's.

We consider a generic ODE

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}),\ \mathbf{f} : \mathrm{Y} \to \mathrm{Y},\ \mathbf{y} : \mathrm{T} \to \mathrm{Y} \tag{6.11}$$

where $\mathbf{f} \in C^1(\mathrm{Y})$, i.e., $f(.)$ is continuously differentiable up to the first order.

The qualitative theory of ODEs consists in finding a (topological) equivalence between a non-linear (or even incompletely defined) function $\mathbf{f}(.)$ and a linear or a normal form ODE. This allows us to characterize the dynamics in the neighborhood of a steady state or of a periodic orbit or other invariant sets (homoclinic and heteroclinic orbits or limit cycles). If there are more than one invariant orbit or steady state we distinguish between local dynamics (in the neighborhood of a steady state or invariant orbit) from global dynamics (in all set y). If there is only one invariant set then local dynamics is qualitatively equivalent to global dynamics.

One important component of qualitative theory is **bifurcation analysis**, which consists in describing the change in the dynamics (that is, in the phase diagram) when one or more parameters take different values within its domain.

---

[2]We define $r^2 = y_1^2 + y_2^2$ and $\theta = \arctan \dfrac{y_2}{y_1}$, and take time derivatives, obtaining

$$\begin{aligned} \dot{r} &= \frac{y_1 \dot{y}_1 + y_2 \dot{y}_2}{r} \\ \dot{\theta} &= \frac{y_1 \dot{y}_2 - y_2 \dot{y}_1}{r^2} \end{aligned}$$

.

6.2.1 Local analysis

We study local dynamics of equation (6.11) by performing a local analysis close to an equilibrium point or a periodic orbit. There are three important results that form the basis of the local analysis: the Grobman-Hartmann, the manifold and the Poincaré-Bendixson theorems. The first two are related to using the knowledge on the solutions of an equivalent linearized ODE to study the local properties close to the a steady-state for a non-linear ODE and the third introduces a criterium for finding periodic orbits.

Equivalence with linear ODE's

Assume there is (at least) one equilibrium point $\bar{\mathbf{y}} \in \{\, \mathbf{y} \in Y \subseteq \mathbb{R}^n : \mathbf{f}(\mathbf{y}) = \mathbf{0}\}$, for $n \geq 1$, and consider the Jacobian of $\mathbf{f}(.)$ evaluated at that equilibrium point

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \frac{\partial f_1(\bar{y})}{\partial y_1} & \dots & \frac{\partial f_1(\bar{y})}{\partial y_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n(\bar{y})}{\partial y_1} & \dots & \frac{\partial f_n(\bar{y})}{\partial y_n} \end{pmatrix}.$$

An equilibrium point is **hyperbolic** if the Jacobian J has no eigenvalues with zero real parts. An equilibrium point is **non-hyperbolic** if the Jacobian has at least one eigenvalue with zero real part.

Theorem 1 (**Grobman-Hartmann theorem).** *Let $\bar{\mathbf{y}}$ be a hyperbolic equilibrium point. Then there is a neighbourhood U of $\bar{\mathbf{y}}$ and a neighborhood U_0 of $\mathbf{y}(0)$ such that the ODE restricted to U is topologically equivalent to the variational equation*

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}}),\ \mathbf{y} - \bar{\mathbf{y}} \in U_0$$

The original paper are **?** and **?**.

Stability properties of $\bar{\mathbf{y}}$ are characterized from the eigenvalues of Jacobian matrix $J(\bar{\mathbf{y}}) = D_y\mathbf{f}(\bar{\mathbf{y}})$.

If all eigenvalues λ of the Jacobian matrix have negative real parts, $\mathrm{Re}(\lambda) < 0$, then $\bar{y}$ is asymptotically stable. If there is at least one eigenvalue λ such that $\mathrm{Re}(\lambda) > 0$ then $\bar{y}$ is unstable.

Example 1 Consider the scalar ODE

$$\dot{y} = f(y) \equiv y^{\alpha} - a \tag{6.12}$$

where a and α are two constants, with $a > 0$, and $y \in \mathbb{R}_+$. Then there is an unique steady state $\bar{y} = a^{\frac{1}{\alpha}}$. As

$$f_y(y) = \alpha y^{\alpha-1}$$

then

$$f_y(\bar{y}) = \alpha a^{\frac{\alpha-1}{\alpha}}.$$

Set $\lambda \equiv f_y(\bar{y})$. Therefore the steady state is hyperbolic if $\alpha \neq 0$ and it is non-hyperbolic if $\alpha = 0$. In addition, if $\alpha < 0$ the hyperbolic steady state $\bar{y}$ is asymptotically stable and if $\alpha > 0$ it is unstable.

If $\alpha \neq 0$ we can perform a first-order Taylor expansion of the ODE (6.12) in the neighborhood of the steady state

$$\dot{y} = \lambda(y - \bar{y}) + o((y - \bar{y}))$$

which means that the solution to (6.12) can be locally approximated by

$$y(t) = \bar{y} + (k - \bar{y})e^{\lambda t}$$

for any $k \in \mathbb{R}_+$. In particular, if we fix $y(0) = y_0$ then $k = y_0$.

Example 2 Consider the non-linear planar ODE

$$\begin{aligned} \dot{y}_1 &= y_1^\alpha - a,\ 0 < \alpha < 1,\ a \geq 0, \\ \dot{y}_2 &= y_1 - y_2 \end{aligned} \tag{6.13}$$

It has an unique steady state $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2) = (a^{\frac{1}{\alpha}}, a^{\frac{1}{\alpha}})$. The Jacobian evaluated at any point $\mathbf{y}$ is

$$J(\mathbf{y}) = D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \begin{pmatrix} \alpha y_1^{\alpha-1} & 0 \\ 1 & -1 \end{pmatrix}.$$

If we approximate the system in a neighborhood of the steady state, $\bar{\mathbf{y}}$, we have the linear planar ODE

$$\dot{\mathbf{y}} \ = J(\bar{\mathbf{y}})\,(\mathbf{y} - \bar{\mathbf{y}})$$

where $J(\bar{\mathbf{y}})$ is the Jacobian evaluated at the steady state,

$$J(\bar{\mathbf{y}}) = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}) = \begin{pmatrix} \alpha a^{\frac{\alpha-1}{\alpha}} & 0 \\ 1 & -1 \end{pmatrix}.$$

We already saw that the solution to this equation is

$$\mathbf{y}(t) = \mathbf{y} + \mathbf{P}\mathbf{e}^{J(\bar{\mathbf{y}})t}\mathbf{h}.$$

Because

$$\begin{aligned} \operatorname{trace}(J(\bar{\mathbf{y}})) &= \alpha a^{\frac{\alpha-1}{\alpha}} - 1 \\ \det\left(J(\bar{\mathbf{y}})\right) &= -\alpha a^{\frac{\alpha-1}{\alpha}} \\ \Delta(J(\bar{\mathbf{y}})) &= \left(\frac{\alpha a^{\frac{\alpha-1}{\alpha}} + 1}{2}\right)^2 \end{aligned}$$

which implies that the eigenvalues of the Jacobian $J(\bar{\mathbf{y}})$ are

$$\lambda_\pm = \frac{\alpha a^{\frac{\alpha-1}{\alpha}} - 1}{2} \pm \frac{\alpha a^{\frac{\alpha-1}{\alpha}} + 1}{2}.$$

that is $\lambda_+ = \alpha a^{\frac{\alpha-1}{\alpha}}$ and $\lambda_- = -1$. Therefore, the steady state is hyperbolic if $\alpha \neq 0$ and non-hyperbolic if $\alpha = 0$.

Furthermore, the steady state is a saddle point if $\alpha > 0$ and it is a stable node if $\alpha < 0$. We can also find the eigenvector matrix of $J(\bar{\mathbf{y}})$,

$$\mathbf{P} = (\mathbf{P}^+ \mathbf{P}^-) = \begin{pmatrix} 1+\lambda_+ & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore, the approximate solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = h_+ \begin{pmatrix} 1+\lambda_+ \\ 1 \end{pmatrix} e^{\lambda_+ t} + h_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_- t}.$$

If $\alpha < 0$ the stable eigenspace is $\mathcal{E}^s = \{\ (y_1, y_2) : y_1 = \bar{y}_1\}$, and, if $\alpha > 0$ the stable eigenspace is the whole space, $\mathcal{E}^s = \mathrm{Y}$.

Local manifolds

Consider a neighbourhood $U \subset \mathrm{Y} \subseteq \mathbb{R}^n$ of $\bar{\mathbf{y}}$: the local stable manifold is the set

$$\mathcal{W}^s_{loc}(\bar{\mathbf{y}}) = \{\ \mathbf{k} \in U :\ \lim_{t\to\infty} \mathbf{y}(t,k) = \bar{y},\ y(t,\mathbf{k}) \in U, t \geq 0\}$$

the local unstable manifold is the set

$$\mathcal{W}^u_{loc}(\bar{\mathbf{y}}) = \{\ \mathbf{k} \in U :\ \lim_{t\to\infty} \mathbf{y}(-t,\mathbf{k}) = \bar{\mathbf{y}},\ \mathbf{y}(-t,\mathbf{k}) \in U, t \geq 0\}$$

The center manifold is denoted $\mathcal{W}^c_{loc}(\bar{\mathbf{y}})$. Let n_-, n_+ and n_0 denote the number of eigenvalues of the Jacobian evaluated at steady state $\bar{\mathbf{y}}$ with negative, positive and zero real parts.

Theorem 2 (**Manifold Theorem**). *: suppose there is a steady state $\bar{\mathbf{y}}$ and $J(\bar{\mathbf{y}})$ is the Jacobian of the ODE* (6.11) *. Then there are local stable, unstable and center manifolds, $\mathcal{W}^s_{loc}(\bar{\mathbf{y}})$, $\mathcal{W}^u_{loc}(\bar{\mathbf{y}})$ and $\mathcal{W}^c_{loc}(\bar{\mathbf{y}})$, of dimensions n_-, n_+ and n_0, respectively, such that $n = n_- + n_+ + n_0$. The local manifolds are tangent to the local eigenspaces $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$ of the (topologically) equivalent linearized ODE*

$$\dot{\mathbf{y}} = J(\bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}}).$$

The first two, eigenspaces $\mathcal{E}^s$ and $\mathcal{E}^u$, are unique, and $\mathcal{E}^c$need not be unique (see (Grass et al., 2008, ch.2)).

The eigenspaces are spanned by the eigenvectors of the Jacobian matrix $J(\bar{y})$ which are associated to the eigenvalues with negative, positive and zero real parts.

Example 2 Consider example 2 and let $\alpha > 0$ which implies that the steady state $\bar{\mathbf{y}}$ is a saddle point. Because the eigenvector associated to eigenvalue λ_ is $\mathbf{P}^- = (0,1)^\top$, then the stable eigenspace is

$$\mathcal{E}^s = \{(y_1, y_2) \in \mathbb{R}_+ : y_1 = \bar{y}_1 = a^{\frac{1}{\alpha}}\}.$$

The local stable manifold $\mathcal{W}^s_{loc}(\bar{\mathbf{y}})$ is tangent to $\mathcal{E}^s$ in a neighborhood of the steady state.

6.2.2 Periodic orbits

We saw that solution trajectories can converge or diverge not only as regards equilibrium points but also to periodic trajectories (see the Andronov-Hopf model).

The **Poincaré-Bendixson** theorem ((Hale and Koçak, 1991, p.367)) states that if the limit set is bounded and it is not an equilibrium point it should be a periodic orbit.

In order to determine if there is a periodic orbit in a compact subset of y the Bendixson criterium provides a method ((Hale and Koçak, 1991, p.373)):

Theorem 3 (**Bendixson-Dulac criterium**). *Let D be a compact region of $y \subseteq \mathbb{R}^n$ for $n \geq 2$. If,*

$$div(\mathbf{f}) = f_{1,y_1}(y_1, y_2) + f_{2,y_2}(y_1, y_2)$$

has constant sign, for $(y_1, y_2) \in D$, then $\dot{y} = f(y)$ has not a constant orbit lying entirely in D.

6.2.3 Global analysis

While local analysis consists in studying local dynamics in the neighbourhood of steady states or periodic orbits, this may not be enough to characterise the dynamics.

We already saw that there are orbits that are invariant and that cannot be determined by local methods, for instance heteroclinic and homoclinic orbits.

Homoclinic and heteroclinic orbits

There are methods to determine if there are homoclinic or heteroclinic orbits. They essentially consist in building a trapping area for the trajectories and proving there should exist trajectories that do not exit the "trap".

Global manifolds

There are global extensions of the local manifolds by continuation in time (in the opposite direction) of the local manifolds: $\mathcal{W}^s(\bar{y})$, $\mathcal{W}^u(\bar{y})$, $\mathcal{W}^c(\bar{y})$.

A trajectory $y(.)$ of the ODE is called a **stable path** of $\bar{y}$ if the orbit $\text{Or}(y_0)$ is contained in the stable manifold $\text{Or}(y_0) \subset \mathcal{W}^s(\bar{y})$ and $\lim_{t\to\infty} y(t, y_0) = \bar{y}$.

A trajectory $y(.)$ of the ODE is called a **unstable path** of $\bar{y}$ if the orbit $\text{Or}(y_0)$ is contained in the stable manifold $\text{Or}(y_0) \subset \mathcal{W}^u(\bar{y})$ and $\lim_{t\to\infty} y(-t, y_0) = \bar{y}$.

6.2.4 Dependence on parameters

We already saw that the solution of linear ODE's, $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}$, may depend on the values for the parameters in the coefficient matrix $\mathbf{A}$.

We can extend this idea to non-linear ODE's of type

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi), \ \varphi \in \Phi \subseteq R^q$$

where φ is a vector of parameters of dimension $q \geq 1$

We can distinguish two types of parameter change:

- **bifurcations** when a parameter change induces a qualitative change in the dynamics, i.e, the phase diagram. By qualitative change we mean change the number or the stability properties of steady states or other invariants. Close to a bifurcation point, a change in a parameter changes the qualitative characteristics of the dynamics;

- **perturbations** when parameter changes do not change the qualitative dynamics, i.e., they do not change the phase diagram. This is typically the case in economics when one performs comparative dynamics exercises.

Bifurcations

If a small variation of the parameter changes the phase diagram we say we have a bifurcation. As you saw, there are local (fixed points) and global bifurcations (heteroclinic connection, etc). Those bifurcations were associated to particular normal forms of both scalar and planar ODEs. This fact allows us to find classes of ODE's which are topologically equivalent to those we have already presented.

Bifurcations for scalar ODE's Consider the scalar ODE

$$\dot{y} = f(y, \varphi),\ Y, \varphi \in \mathbb{R}.$$

Fold bifurcation (see (Kuznetsov, 2005, ch. 3.3)): Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yy}(0, 0) \neq 0,\ f_{\varphi}(0, 0) \neq 0.$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi \pm y^2,$$

that is to the Ricatti's model (5.4).

Transcritical bifurcation: Let $f \in C^2(\mathbb{R})$ and consider the point $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yy}(0, 0) \neq 0,\ f_{\varphi y}(0, 0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^2$$

that is to the Bernoulli model (5.5).

Pitchfork bifurcation: Let $f \in C^2(\mathbb{R})$ and consider $(\bar{y}, \varphi_0) = (0, 0)$, such that $f(0, 0) = 0$, with $f_y(0, 0) = 0$ and

$$f_{yyy}(0, 0) \neq 0, \; f_{\varphi y}(0, 0) \neq 0$$

then the ODE is topologically equivalent to

$$\dot{y} = \varphi y \pm y^3$$

that is to the Bernoulli model (5.6).

Bifurcations for planar ODE's Consider the planar ODE

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \varphi), \; \mathbf{y} \in \mathbb{R}^2, \varphi \in \mathbb{R}$$

Andronov-Hopf bifurcation (see (Kuznetsov, 2005, ch. 3.4)): Let $\mathbf{f} \in C^2(\mathbb{R})$ and consider $(\bar{\mathbf{y}}, \varphi_0) = (\mathbf{0}, 0)$ the Jacobian at $(\mathbf{0}, 0)$ has eigenvalues

$$\lambda_{\pm} = \eta(\varphi) \pm i\omega(\varphi)$$

where $\eta(0) = 0$ and $\omega(0) > 0$. If some additional conditions are satisfied then the ODE is locally topologically equivalent to

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

6.2.5 Comparative dynamics in economics

As mentioned, **comparative dynamics** exercises consist in introducing perturbation in a dynamic system: i.e., a small variation of the parameter that does not change the phase diagram. This kind of analysis only makes sense if the steady state is hyperbolic, that is if $\det\left(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)\right) \neq 0$ or $\text{trace}(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)) \neq 0$ if $\det\left(D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}, \varphi_0)\right) > 0$.

In this case let the steady state be for a given value of the parameter $\varphi = \varphi_0$

$$\bar{\mathbf{y}}_0 = \{\, y \in Y : \mathbf{f}(\mathbf{y}, \varphi_0) = \mathbf{0}\}.$$

If $\bar{\mathbf{y}}_0$ is a hyperbolic steady state, then we can expand the ODE into a linear ODE

$$\dot{\mathbf{y}} = D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)(\mathbf{y} - \bar{\mathbf{y}}_0) + D_{\varphi}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)(\varphi - \varphi_0). \tag{6.14}$$

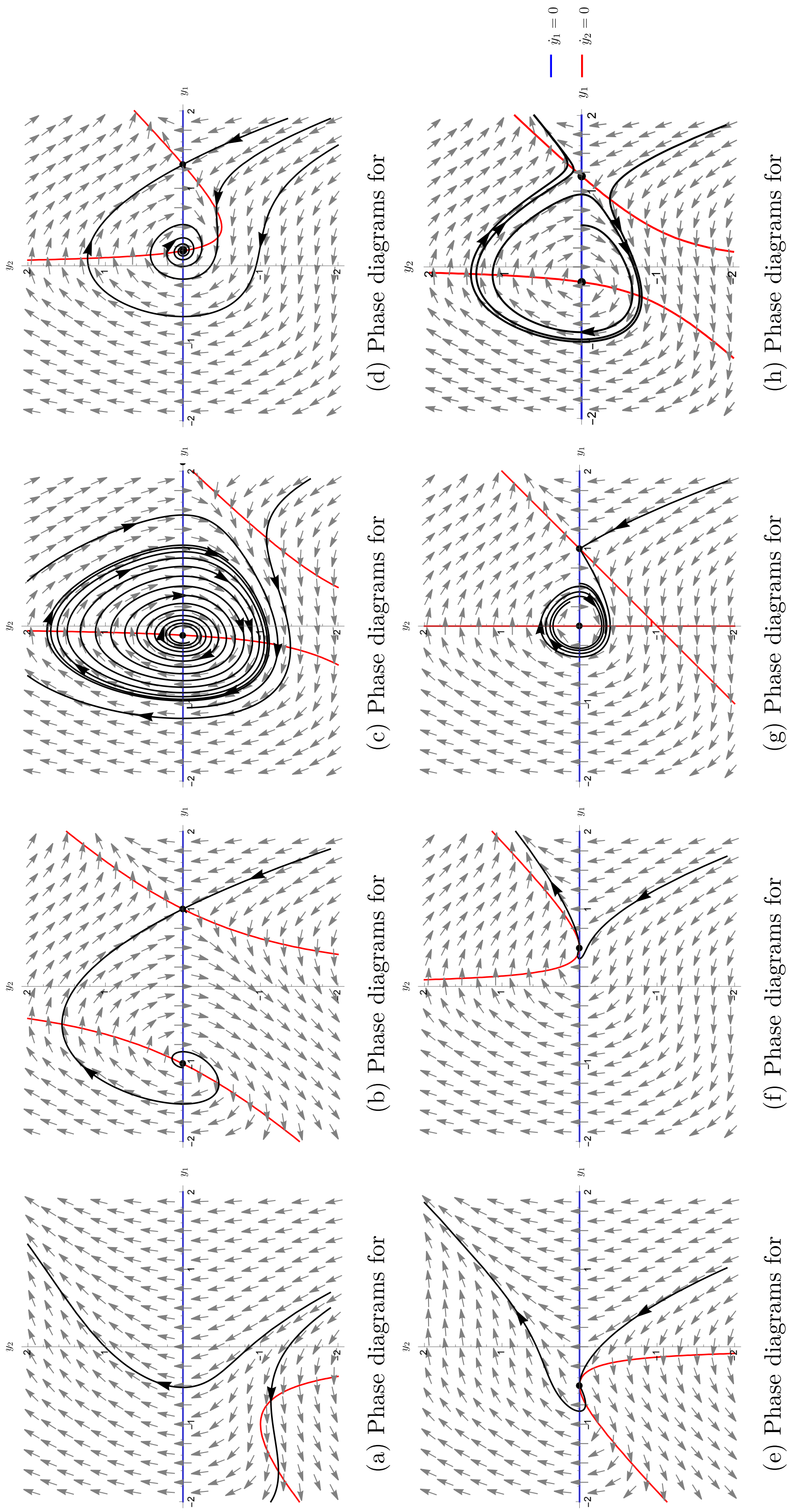
This equation can be solved as a linear ODE. Setting $\varphi = \varphi_0 + \delta_\varphi$ and because $\bar{\mathbf{y}} = \bar{\mathbf{y}}(\varphi)$ and $\bar{\mathbf{y}}_0 = \bar{\mathbf{y}}(\varphi_0)$ we have

$$D_{\varphi}\bar{\mathbf{y}}(\varphi_0) = \lim_{\delta_\varphi \to 0} \frac{\bar{\mathbf{y}}(\varphi_0 + \delta_\varphi) - \bar{\mathbf{y}}(\varphi_0)}{\delta_\varphi} = -D_{\mathbf{y}}^{-1}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0) D_{\varphi}\mathbf{f}(\bar{\mathbf{y}}_0, \varphi_0)$$

which are called the **long-run multipliers** associated to a permanent change in φ. Solving the linearized system allows us to have a general solution to the problem of finding the **short-run** or **transition multipliers**, $d\mathbf{y}(t) \equiv \mathbf{y}(t) - \bar{\mathbf{y}}_0$ for a change in the parameter φ.

6.3 References

- (Hale and Koçak, 1991, Part I , III): very good introduction.
- (Guckenheimer and Holmes, 1990, ch. 1, 3, 6) Is a classic reference on the field.
- Kuznetsov (2005) Very complete presentation of bifurcations for planar systems.
- Brock and Malliaris (1989), (Grass et al., 2008, ch.2) has a compact presentation of all the important results with some examples in economics and management science.



(a) Phase diagrams for

(b) Phase diagrams for

(c) Phase diagrams for

(d) Phase diagrams for

(e) Phase diagrams for

(f) Phase diagrams for

(g) Phase diagrams for

(h) Phase diagrams for

Figure 6.6: Phase diagrams for equation (6.10).

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