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Lunar Ejecta

JA1

Orbit equations:

$$\frac{a}{r} = \frac{1}{2\left(1-\frac{v^2}{v_{esc}^2}\right)} \quad (1), \quad e^2 = \left(\frac{2v^2}{v_{esc}^2}-1\right)^2 \sin^2\gamma + \cos^2\gamma \quad (2), \quad r = \frac{a(1-e^2)}{1+e\cos\beta} \quad (3)$$

a = semi major axis of orbit

r = altitude in orbit

v = speed in orbit

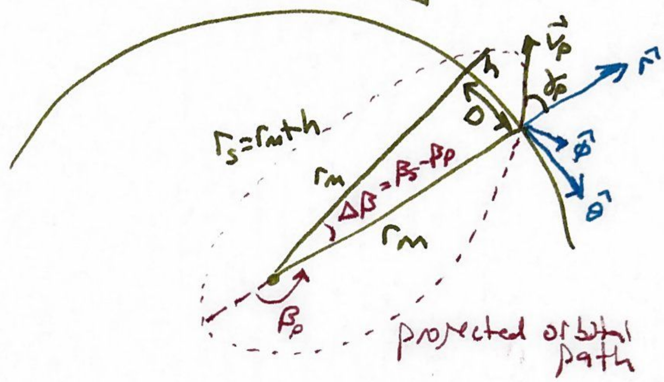
v_{esc} = escape speed of planetary body

γ = angle from local zenith

e = eccentricity of orbit

β = angle from periapsis

$$D = r_m \Delta\beta \Rightarrow \boxed{\frac{D}{r_m} = \beta_s - \beta_p} \quad (4)$$



- Define a and e given v_p and γ_p (projectile speed and zenith angle at initial ejected point)

- Solve eq (3) for $e\cos\beta_p$:

$$e\cos\beta_p = \frac{a}{r_m}(1-e^2) - 1$$

Note: $$1-e^2 = \overbrace{1}^{=\sin^2\gamma_p+\cos^2\gamma_p} - \overbrace{\left(\frac{2v_p^2}{v_{esc}^2}-1\right)^2}^{=4\frac{v_p^4}{v_{esc}^4}-4\frac{v_p^2}{v_{esc}^2}+1}\sin^2\gamma_p - \cos^2\gamma_p$$

$$\boxed{1-e^2 = 4\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p\left(1-\frac{v_p^2}{v_{esc}^2}\right)} \quad (5)$$

- using (1) and (5),

$$\boxed{e\cos\beta_p = 2\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p - 1} \quad (6)$$

- Next, we need $e\sin\beta_p$:

$$e^2\sin^2\beta_p = e^2 - e^2\cos^2\beta_p$$

$$= \left(\frac{2v_p^2}{v_{esc}^2}-1\right)^2\sin^2\gamma_p + \cos^2\gamma_p - \left[2\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p - 1\right]^2$$

$$= 4\frac{v_p^4}{v_{esc}^4}\sin^2\gamma_p - 4\cancel{\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p} + \cancel{\sin^2\gamma_p} + \cancel{\cos^2\gamma_p} - 4\frac{v_p^4}{v_{esc}^4}\sin^4\gamma_p + 4\cancel{\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p} - \cancel{1}$$

$\therefore$

$$\boxed{e\sin\beta_p = 2\frac{v_p^2}{v_{esc}^2}\sin\gamma_p\cos\gamma_p} \quad (7)$$

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- Bringing together eq's (6) and (7):

$$\tan\beta_p = \frac{2\frac{v_p^2}{v_{esc}^2}\sin\gamma_p\cos\gamma_p}{2\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p - 1} \quad (8)$$

- Now, if $h=0$, then we expect the ejecta will land at $-\beta_p$ since the orbit is symmetric about the periapsis.
  $\therefore$ the distance is given by

$$, \Delta\beta = (2\pi - \beta_p) - \beta_p$$
$$= 2\pi - 2\beta_p$$

$$\frac{D}{r_m} = \Delta\beta \Rightarrow \frac{D}{2r_m} = \pi - \beta_p \text{ or}$$

$$\boxed{\beta_p = \pi - \frac{D}{2r_m}} \quad (9)$$

Note $\tan(-\theta) = -\tan\theta$
and $\tan(\pi-\theta) = -\tan\theta$

- and eq (8) and eq (9),

$$\boxed{\tan\left(\frac{D}{2r_m}\right) = \frac{2\frac{v_p^2}{v_{esc}^2}\sin\gamma_p\cos\gamma_p}{1 - 2\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p}} \quad (10)$$

- compare w/ Vickery 1986, eq (1)

- Let's relax h such that $h \neq 0$.
  → Goal: Find an equation for v_p in terms of γ_p, D, and r_s

- First, let's solve eq (10) for $\frac{v_p}{v_{esc}}$, so we have something to compare to

↳ From (10),

$$\cot\left(\frac{D}{2r_m}\right) = \frac{1 - \frac{v_p^2}{v_{esc}^2}\left[1 - \cos(2\gamma_p)\right]}{\frac{v_p^2}{v_{esc}^2}\sin(2\gamma_p)}$$

- Collect v^2's

Note! $\sin(2\theta) = 2\sin\theta\cos\theta$
$\cos(2\theta) = 1 - 2\sin^2\theta$
or
$2\sin^2\theta = 1 - \cos(2\theta)$

$$\frac{v_p^2}{v_{esc}^2}\left[\sin(2\gamma_p)\cot\left(\frac{D}{2r_m}\right) + 1 - \cos(2\gamma_p)\right] = 1$$

$\therefore$

$$\boxed{\frac{v_p}{v_{esc}} = \frac{1}{\sqrt{1 - \cos(2\gamma_p) + \sin(2\gamma_p)\cot\left(\frac{D}{2r_m}\right)}}} \quad (11)$$

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Start w/ $\cos\beta_s = \cos\left(\frac{D}{r_m}+\beta_p\right)$, using eq(4)

$= \cos\left(\frac{D}{r_m}\right)\cos\beta_p - \sin\left(\frac{D}{r_m}\right)\sin\beta_p$ - multiply by e, and sub in eq(6), (7), ~~and~~ and (12)

Note: using eq(3)

$$r_s = \frac{a(1-e^2)}{1+e\cos\beta_s} \Rightarrow e\cos\beta_s = \frac{a}{r_s}(1-e^2)-1$$

- using eq (1) and (5),

$$\boxed{e\cos\beta_s = 2\frac{r_m}{r_s}\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p - 1} \quad (12)$$

$$2\frac{r_m}{r_s}\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p - 1 = \left(2\frac{v_p^2}{v_{esc}^2}\sin^2\gamma_p - 1\right)\cos\left(\frac{D}{r_m}\right) - \left(2\frac{v_p^2}{v_{esc}^2}\sin\gamma_p\cos\gamma_p\right)\sin\left(\frac{D}{r_m}\right)$$

$$\frac{v_p^2}{v_{esc}^2}\left[2\frac{r_m}{r_s}\cancel{\frac{v_p^2}{v_{esc}^2}}\sin^2\gamma_p - 2\cancel{\frac{v_p^2}{v_{esc}^2}}\sin^2\gamma_p\cos\left(\frac{D}{r_m}\right) + 2\cancel{\frac{v_p^2}{v_{esc}^2}}\sin\gamma_p\cos\gamma_p\sin\left(\frac{D}{r_m}\right)\right] = 1 - \cos\left(\frac{D}{r_m}\right)$$

$$= \overbrace{2\cancel{\frac{v_p^2}{v_{esc}^2}}\sin^2\gamma_p}^{=1-\cos(2\gamma_p)}\left[\frac{r_m}{r_s} - \cos\left(\frac{D}{r_m}\right)\right] + \cancel{\frac{v_p^2}{v_{esc}^2}}\sin(2\gamma_p)\sin\left(\frac{D}{r_m}\right)$$

Note:

$\sin\left(\frac{D}{r_m}\right) = 2\sin\left(\frac{D}{2r_m}\right)\cos\left(\frac{D}{2r_m}\right)$

$1-\cos\left(\frac{D}{r_m}\right) = 2\sin^2\left(\frac{D}{2r_m}\right)$

$\therefore$

$$\frac{v_p^2}{v_{esc}^2}\left[\left[1-\cos(2\gamma_p)\right]\left[\frac{r_m}{r_s} - \cos\left(\frac{D}{r_m}\right)\right] + 2\sin(2\gamma_p)\sin\left(\frac{D}{2r_m}\right)\cos\left(\frac{D}{2r_m}\right)\right] = 1 - \cos\left(\frac{D}{r_m}\right)$$

- divide by RHS, keep on 1st term, use $2\sin^2\left(\frac{D}{2r_m}\right)$ on 2nd term,

$$\boxed{\frac{v_p}{v_{esc}} = \frac{1}{\sqrt{\left[\frac{\frac{r_m}{r_s} - \cos\left(\frac{D}{r_m}\right)}{1-\cos\left(\frac{D}{r_m}\right)}\right]\left[1-\cos(2\gamma_p)\right] + \sin(2\gamma_p)\cot\left(\frac{D}{2r_m}\right)}}} \quad (13)$$

- compare w/ eq(11)