Electromagnetic Induction

- Experiments by Faraday in 1831 found that moving a magnet, or a current carrying wire, <u>induces</u> a current in a nearby circuit. (Joseph Henry found this induction first but didn't publish the results right away!)

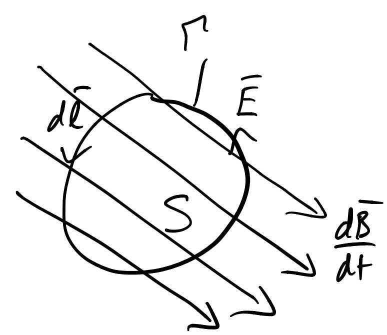
- <u>Faraday's Law</u> = the electromotive force (EMF) produced in a circuit is proportional to the time rate of change of the magnetic flux, Φ_m:

$$\text{EMF} = -\frac{1}{c}\frac{d\Phi_m}{dt}$$

artifact of Gaussian units

$$\text{EMF} \equiv \oint_{\Gamma} \bar{E} \cdot d\bar{l}$$

$$\Phi_m \equiv \int_S \bar{B} \cdot d\bar{a}$$



Note the minus sign: the EMF produces a current that goes against the change in flux: <u>Lenz's Law</u>

$$\boxed{\text{Nature abhors a change in flux}}$$

– Griffiths

Another way of writing Faraday's Law

$$\oint_{\Gamma} \bar{E} \cdot d\bar{l} = -\frac{1}{c} \int_{S} \frac{\partial \bar{B}}{\partial t} \cdot d\bar{a}$$

$$\Big\downarrow \text{Stoke's}$$

$$\int_{S} \bar{\nabla} \times \bar{E} \cdot d\bar{a} = -\frac{1}{c} \int_{S} \frac{\partial \bar{B}}{\partial t} \cdot d\bar{a}$$

$$\Rightarrow \boxed{\bar{\nabla} \times \bar{E} = -\frac{1}{c} \frac{\partial \bar{B}}{\partial t}}$$

Faraday's Law in differential form

- Faraday's Law holds anywhere, whether or not a circuit wire or other material is present.
- The $\bar{E}$ produced by $\frac{\partial \bar{B}}{\partial t}$ is called the <u>Faraday</u> electric field, as opposed to the **<u>Coulomb</u>** electric field due to charges.

<u>Maxwell's Modification of Ampère's Law:</u>

Steady-state version $\Rightarrow$ $\bar{\nabla} \times \bar{H} = \frac{4\pi}{c} \bar{J}_{free}$

$$\Rightarrow \bar{\nabla} \cdot (\bar{\nabla} \times \bar{H}) = 0 = \frac{4\pi}{c} \bar{\nabla} \cdot \bar{J}_{free}$$

$$\Rightarrow \bar{\nabla} \cdot \bar{J} = 0$$

But, for non-static conditions, the continuity equation holds

$$\bar{\nabla} \cdot \bar{J} = -\frac{\partial \rho}{\partial t}$$

FROM GAUSS' LAW:

$$\bar{\nabla} \cdot \bar{D} = 4\pi \rho$$

$$\Rightarrow \bar{\nabla} \cdot \bar{J} = -\frac{\partial}{\partial t}\left(\bar{\nabla} \cdot \frac{\bar{D}}{4\pi}\right) = -\bar{\nabla} \cdot \frac{\partial}{\partial t}\left(\frac{\bar{D}}{4\pi}\right)$$

$$\Rightarrow \bar{\nabla} \cdot \underbrace{\left[\bar{J} + \frac{1}{4\pi}\frac{\partial \bar{D}}{\partial t}\right]}_{\text{CALL this } \bar{J}'} = 0$$

Going back to Ampère's LAW and allowing for a time changing ρ $\left(\frac{d\rho}{dt} \neq 0\right)$

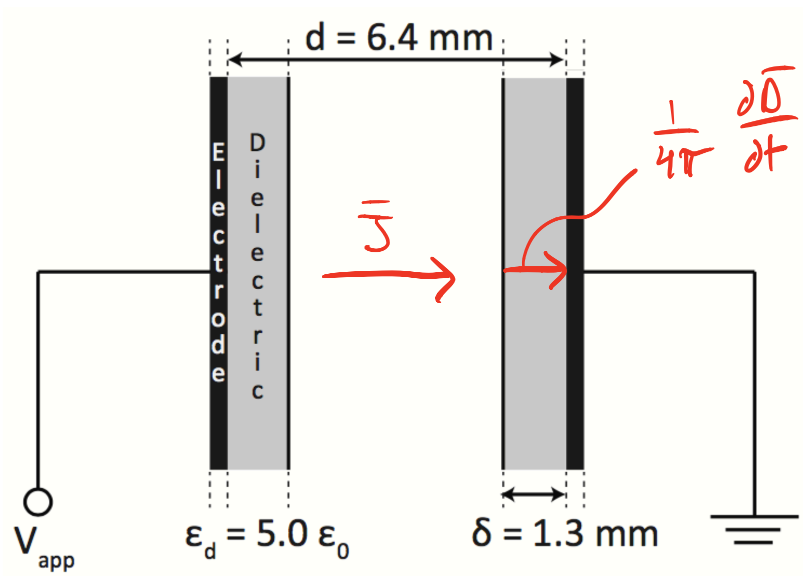
$$\boxed{\bar{\nabla} \times \bar{H} = \frac{4\pi}{c}\bar{J}' = \frac{4\pi}{c}\bar{J} + \frac{1}{c}\frac{\partial \bar{D}}{\partial t}}$$

Ampère-MAXWELL LAW

The new current term is known as the <u>displacement current</u>

$$\bar{J}_d = \frac{1}{4\pi} \frac{\partial \bar{D}}{\partial t}$$

We saw an example of the displacement current that takes place in a radio frequency dielectric barrier discharge



Without the displacement current, Ampere's law would not hold.

To see this, consider Ampère's law for the following circuit

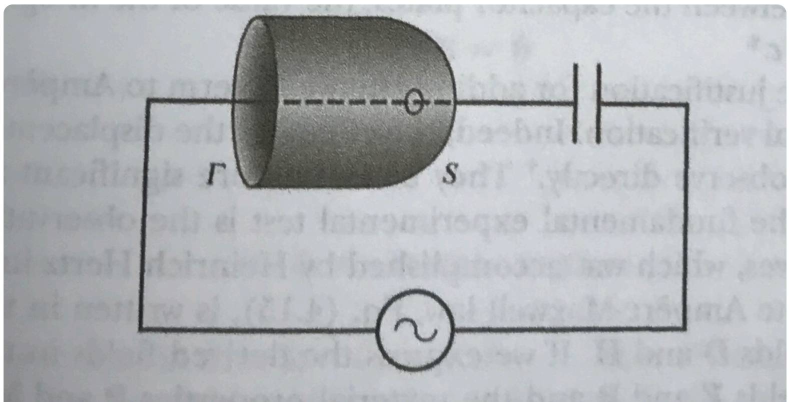


FIGURE 4-1. Surface intercepting conduction current.

[H+M]

Using the steady-state Ampère's Law:

$$\oint_{\Gamma} \bar{H} \cdot d\bar{l} = \frac{4\pi}{c} \int_{S} \bar{J} \cdot d\bar{a} = \frac{4\pi}{c} I$$

This result needs to be independent of S, but if we use the following S:

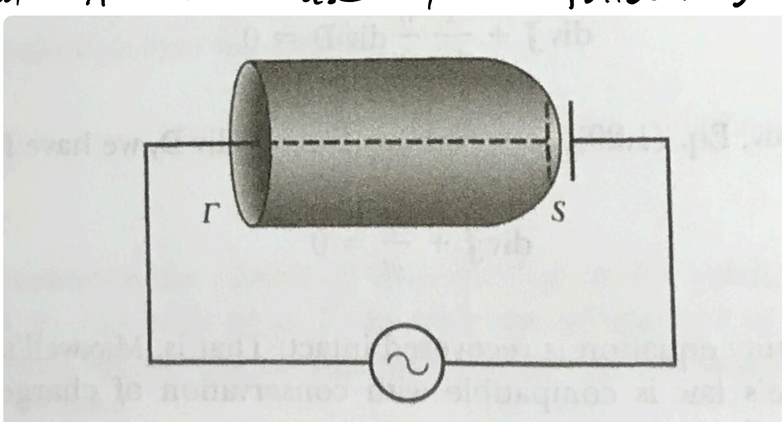


FIGURE 4-2. Surface intercepting displacement current.

$$\oint_{\Gamma} \bar{H} \cdot d\bar{l} = \cancel{0} \Longrightarrow$$ WRONG! The current is there in the form of the displacement current

$$\oint_{\Gamma} \bar{H} \cdot d\bar{l} = \underbrace{\frac{4\pi}{c} \int_S \bar{J} \cdot d\bar{a}}_{\cancel{0}} + \underbrace{\frac{1}{c} \frac{d}{dt} \int_S \bar{D} \cdot d\bar{a}}_{\frac{4\pi}{c} I} = \frac{4\pi}{c} I$$

Expanding $\bar{H}$ and $\bar{D}$, we get the microscopic form of Ampère–Maxwell

$$\bar{\nabla} \times \bar{B} = \frac{4\pi}{c} \underbrace{\left(\bar{J}_{free} + \underbrace{\frac{\partial \bar{P}}{\partial t} + c \bar{\nabla} \times \bar{M}}_{J_{bound}} \right)}_{\bar{J}_{total}} + \underbrace{\frac{1}{c} \frac{\partial \bar{E}}{\partial t}}_{\text{Maxwell Induction}}$$

Maxwell's Equations

Macroscopic

$$\bar{\nabla} \cdot \bar{D} = 4\pi \rho_{free}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} + \frac{1}{c} \frac{\partial \bar{B}}{\partial t} = 0$$

$$\bar{\nabla} \times \bar{H} - \frac{1}{c} \frac{\partial \bar{D}}{\partial t} = \frac{4\pi}{c} \bar{J}_{free}$$

Microscopic

$$\bar{\nabla} \cdot \bar{E} = 4\pi \rho_{total}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} + \frac{1}{c} \frac{\partial \bar{B}}{\partial t} = 0$$

$$\bar{\nabla} \times \bar{B} - \frac{1}{c} \frac{\partial \bar{E}}{\partial t} = \frac{4\pi}{c} \bar{J}_{total}$$

Potential Functions of the Electromagnetic Field

For static fields, $\bar{\nabla} \times \bar{E} = 0$

$$\Longrightarrow \bar{E} = -\bar{\nabla} \bar{\Phi}$$

From $\bar{\nabla} \cdot \bar{B} = 0$ we always have (static or not)

$$\Longrightarrow \bar{B} = \bar{\nabla} \times \bar{A}$$

For the non-static scenario, from Faraday's Law

$$\bar{\nabla} \times \bar{E} = -\frac{1}{c} \frac{\partial \bar{B}}{\partial t} = -\frac{1}{c} \frac{\partial}{\partial t} \bar{\nabla} \times \bar{A}$$

$$= \bar{\nabla} \times \left(-\frac{1}{c} \frac{\partial \bar{A}}{\partial t} \right)$$

The $\bar{\nabla} \bar{\Phi}$ term vanishes from the curl

$$\Longrightarrow \boxed{\bar{E} = -\bar{\nabla} \bar{\Phi} - \frac{1}{c} \frac{\partial \bar{A}}{\partial t}}$$

From $\bar{B} = \bar{\nabla} \times \bar{A}$, the choice of $\bar{A}$ is unchanged by the addition of the gradient of an arbitrary scalar function

$$\bar{A}' = \bar{A} + \bar{\nabla}\varepsilon$$

$$\Rightarrow \bar{\nabla} \times \bar{A}' = \bar{\nabla} \times \bar{A} + \underbrace{\bar{\nabla} \times \bar{\nabla}\varepsilon}_{=0} = \bar{B}$$

If we use A' in the equation for E

$$\bar{E} = -\bar{\nabla}\bar{\Phi} - \frac{1}{c}\frac{\partial}{\partial t}\left(\bar{A} + \bar{\nabla}\varepsilon\right)$$

$$= -\bar{\nabla}\left(\bar{\Phi} + \frac{1}{c}\frac{\partial \varepsilon}{\partial t}\right) - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$

To make this equal to our original E, we need to replace $\bar{\Phi}$ with $\bar{\Phi}'$

$$\bar{\Phi}' = \bar{\Phi} - \frac{1}{c}\frac{\partial \varepsilon}{\partial t}$$

$\longrightarrow$ By adding $\bar{\nabla}\varepsilon$ to $\bar{A}$

$$\bar{A}' = \bar{A} + \bar{\nabla}\varepsilon$$

And subtracting $\frac{1}{c}\frac{\partial \varepsilon}{\partial t}$ from Φ

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial \varepsilon}{\partial t}$$

the equation for $\bar{E}$ is <u>unchanged!</u>

$\Rightarrow$ <u>Gauge Transformation</u>

We are free to pick the choice of gauge (ε) to impose additional conditions on $\bar{A}$. A vector field is not completely specified by its <u>curl</u>, but it is uniquely determined if both the <u>curl</u> and <u>divergence</u> are specified.

One choice of gauge is the <u>Lorenz gauge</u>

$$\bar{\nabla}\cdot\bar{A} = -\frac{1}{c}\frac{\partial\Phi}{\partial t}$$

$$\Rightarrow \bar{\nabla}\cdot\bar{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$$

$$= \bar{\nabla}\cdot\bar{A}' - \bar{\nabla}^2\varepsilon + \frac{1}{c}\frac{\partial\Phi'}{\partial t} + \frac{1}{c^2}\frac{\partial^2\varepsilon}{\partial t^2}$$

$$= \bar{\nabla}\cdot A' + \frac{1}{c}\frac{\partial\Phi'}{\partial t} - \underline{\left(\bar{\nabla}^2\varepsilon - \frac{1}{c^2}\frac{\partial^2\varepsilon}{\partial t^2}\right)}$$

For A' and Φ' to also satisfy the Lorenz condition, we must have

$$\underline{\bar{\nabla}^2\varepsilon - \frac{1}{c^2}\frac{\partial^2\varepsilon}{\partial t^2} = 0}$$

↳ wave equation

Using the potentials from the Lorenz guage in Maxwell's equations:

$$\bar{\nabla} \cdot \bar{E} = 4\pi\rho$$

$$\bar{E} = -\bar{\nabla}\Phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$$

$$\Rightarrow \quad \bar{\nabla}^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}\bar{\nabla}\cdot\bar{A} = -4\pi\rho$$

From the Lorenz condition:

$$\bar{\nabla}\cdot\bar{A} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

$$\Rightarrow \bar{\nabla}^2\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho$$

Also,

$$\bar{\nabla}\times\bar{B} - \frac{1}{c}\frac{\partial \bar{E}}{\partial t} = \frac{4\pi}{c}\bar{J}$$

$$\bar{B} = \bar{\nabla}\times\bar{A}$$

$$\Rightarrow \underbrace{\bar{\nabla}\times\bar{\nabla}\times\bar{A}} + \frac{1}{c}\bar{\nabla}\frac{\partial\Phi}{\partial t} + \frac{1}{c^2}\frac{\partial^2\bar{A}}{\partial t^2} = \frac{4\pi}{c}\bar{J}$$

$$\bar{\nabla}\times\bar{\nabla}\times\bar{A} = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \bar{\nabla}^2\bar{A}$$

$$\Rightarrow \bar{\nabla}^2\bar{A} - \frac{1}{c^2}\frac{\partial^2\bar{A}}{\partial t^2} - \bar{\nabla}\underbrace{\left(\bar{\nabla}\cdot\bar{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right)}_{=\emptyset \text{ from Lorenz condition}} = -\frac{4\pi}{c}\bar{J}$$

$$\Rightarrow \boxed{\begin{aligned} \bar{\nabla}^2\bar{A} - \frac{1}{c^2}\frac{\partial^2\bar{A}}{\partial t^2} &= -\frac{4\pi}{c}\bar{J} \\ \bar{\nabla}^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -4\pi\rho \end{aligned}}$$

uncoupled wave equations

- Now, the equations for $\bar{\Phi}$ and $\bar{A}$ are uncoupled and can be solved separately.

- If no charge or current is present

$$\left.\begin{aligned} \bar{\nabla}^2 \bar{\Phi} - \frac{1}{c^2}\frac{\partial^2 \bar{\Phi}}{\partial t^2} &= 0 \\ \bar{\nabla}^2 \bar{A} - \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} &= 0 \end{aligned}\right\} \text{no local sources}$$

If the time derivatives are zero:

$$\left.\begin{aligned} \bar{\nabla}^2 \bar{\Phi} &= -4\pi\rho \\ \bar{\nabla}^2 \bar{A} &= -\frac{4\pi}{c}\bar{J} \end{aligned}\right\} \text{static}$$

with solutions

$$\left.\begin{aligned} \bar{\Phi}(\bar{r}) &= \int_V \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv' \\ \bar{A}(\bar{r}) &= \frac{1}{c}\int_V \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv' \end{aligned}\right\} \text{static}$$