Gaussian vs SI units

- Heald and Marion use Gaussian units which may be confusing if you're familiar with SI units from Griffiths

- Jackson's Classical Electrodynamics used to be in Gaussian, but is now in SI

- We'll use Gaussian units following Heald and Marion, but you should be familiar with converting between units (see Appendix D + E)

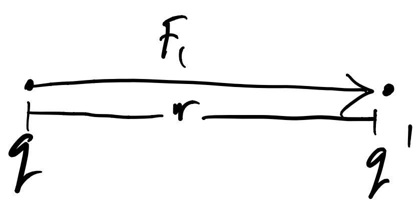
Following Jackson:

Current is defined as the time rate of charge:

$$I = \frac{dq}{dt}, \quad I = \text{current}, \quad q = \text{charge}$$

Coulomb's law provides the force between two point charges q and q' separated by distance r:

$$F_1 = k_1 \frac{q q'}{r^2}$$

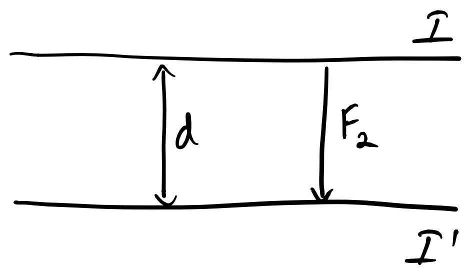


k_1 = proportionality constant which is <u>either</u> determined by pre-specified charge <u>or</u> chosen to define charge

The electric field due to q also depends on k_1:

$$E = k_1 \frac{q}{r^2}$$

From Ampere's observations, the force per unit length between 2 infinitely long parallel wires separated by distance d is:



$$\frac{dF_2}{dl} = 2k_2 \frac{I I'}{d}$$

k_2 = proportionality constant similar to k_1

Combining the equations:

$$\frac{k_1}{k_2} = \frac{F_1 r^2 / qq'}{\frac{dF_2}{dl} d / 2II'} \propto \underbrace{\frac{l^2}{t^2}}_{\text{units}}$$

Comparing the magnitude of the forces, the ratio is found to be:

$$\frac{k_1}{k_2} = c^2$$

c = speed of light in vacuum

The magnetic field, B, due to current I at a distance d follows:

$$B = 2k_2 \alpha \frac{I}{d}$$

α = proportionality constant

From Faraday's law, an additional proportionality constant is required:

$$\bar{\nabla} \times \bar{E} = -k_3 \frac{\partial \bar{B}}{\partial t}$$

k_3 = proportionality constant

These 4 proportionality constants determine the choice of units. Maxwell's equations are:

$$\bar{\nabla} \cdot \bar{E} = 4\pi k_1 \rho$$

$$\bar{\nabla} \times \bar{B} = 4\pi k_2 \alpha \bar{J} + \frac{k_2 \alpha}{k_1} \frac{\partial \bar{E}}{\partial t}$$

$$\bar{\nabla} \times \bar{E} = -k_3 \frac{\partial \bar{B}}{\partial t}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

For source free regions, the two curl equations can be combined to give the wave equation:

$$\nabla^2 \bar{B} - k_3 \frac{k_2 \alpha}{k_1} \frac{\partial^2 \bar{B}}{\partial t^2} = 0$$

The velocity of an electromagnetic wave in free space is known

$$\Rightarrow \quad \frac{k_3 k_2 \alpha}{k_1} = \frac{1}{c^2}$$

$$\Rightarrow \quad k_3 = \frac{1}{\alpha} \qquad \text{because} \qquad \frac{k_1}{k_2} = c^2$$

✻ Gaussian units rely on c for the proportionality constants while SI relies on ε_0 and μ_0

Then for source-free regions the two curl equations can be combined into the wave equation,

$$\nabla^2 \mathbf{B} - k_3 \frac{k_2 \alpha}{k_1} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \tag{A.9}$$

The velocity of propagation of the waves described by (A.9) is related to the combination of constants appearing there. Since this velocity is known to be that of light, we may write

$$\frac{k_1}{k_3 k_2 \alpha} = c^2 \tag{A.10}$$

Combining (A.5) with (A.10), we find

$$k_3 = \frac{1}{\alpha} \tag{A.11}$$

an equality holding for both magnitude and dimensions.

3 Various Systems of Electromagnetic Units

The various systems of electromagnetic units differ in their choices of the magnitudes and dimensions of the various constants above. Because of relations (A.5) and (A.11) there are only two constants (e.g., k_2, k_3) that can (and must) be chosen arbitrarily. It is convenient, however, to tabulate all four constants (k_1, k_2, α, k_3) for the commoner systems of units. These are given in Table 1. We note that, apart from dimensions, the em units and SI units are very similar, differing only in various powers of 10 in their mechanical and electromagnetic units. The Gaussian and Heaviside–Lorentz systems differ only by factors of 4π.

Table 1 Magnitudes and Dimensions of the Electromagnetic Constants for Various Systems of Units

The dimensions are given after the numerical values. The symbol c stands for the velocity of light in vacuum ($c \simeq 2.998 \times 10^{10}$ cm/s $\simeq 2.998 \times 10^8$ m/s). The first four systems of units use the centimeter, gram, and second as their fundamental units of length, mass, and time (l, m, t). The SI system uses the meter, kilogram, and second, plus current (I) as a fourth dimension, with the ampere as unit.

System	k_1	k_2	α	k_3
Electrostatic (esu)	1	$c^{-2}(t^2 l^{-2})$	1	1
Electromagnetic (emu)	$c^2(l^2 t^{-2})$	1	1	1
Gaussian	1	$c^{-2}(t^2 l^{-2})$	$c(lt^{-1})$	$c^{-1}(tl^{-1})$
Heaviside–Lorentz	$\frac{1}{4\pi}$	$\frac{1}{4\pi c^2}(t^2 l^{-2})$	$c(lt^{-1})$	$c^{-1}(tl^{-1})$
SI	$\frac{1}{4\pi\epsilon_0} = 10^{-7} c^2$ $(ml^3 t^{-4} I^{-2})$	$\frac{\mu_0}{4\pi} \equiv 10^{-7}$ $(mlt^{-2} I^{-2})$	1	1

Only in the Gaussian (and Heaviside–Lorentz) system does k_3 have dimensions. It is evident from (A.7) that, with k_3 having dimensions of a reciprocal velocity, **E** and **B** have the same dimensions. Furthermore, with $k_3 = c^{-1}$, (A.7) shows that for electromagnetic waves in free space **E** and **B** are equal in magnitude as well.

For SI units, (A.10) reads $1/(\mu_0\epsilon_0) = c^2$. With c now defined as a nine-digit number and $k_2 \equiv \mu_0/4\pi = 10^{-7}$ H/m, also by definition, 10^7 times the constant k_1 in Coulomb's law is

$$\frac{10^7}{4\pi\epsilon_0} = c^2 = 89\ 875\ 517\ 873\ 681\ 764$$

an exact 17-digit number (approximately 8.9876×10^{16}). Use of the speed of light without error to define the meter in terms of the second removes the anomaly in SI units of having one of the fundamental proportionality constants ϵ_0 with experimental errors. Note that, although the right-hand side above is the square of the speed of light, the *dimensions* of ϵ_0 (as distinct from its magnitude) are not seconds squared per meter squared because the numerical factor on the left has the dimensions of μ_0^{-1}. The dimensions of $1/\epsilon_0$ and μ_0 are given in Table 1. It is conventional to express the dimensions of ϵ_0 as farads per meter and those of μ_0 as henrys per meter. With $k_3 = 1$ and dimensionless, **E** and c**B** have the same dimensions in SI units; for a plane wave in vacuum they are equal in magnitude.

Only electromagnetic fields in free space have been discussed so far. Consequently only the two fundamental fields **E** and **B** have appeared. There remains the task of defining the macroscopic field variables **D** and **H**. If the averaged electromagnetic properties of a material medium are described by a macroscopic polarization **P** and a magnetization **M**, the general form of the definitions of **D** and **H** are

$$\left.\begin{aligned} \mathbf{D} &= \epsilon_0\mathbf{E} + \lambda\mathbf{P} \\ \mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \lambda'\mathbf{M} \end{aligned}\right\} \tag{A.12}$$

where $\epsilon_0, \mu_0, \lambda, \lambda'$ are proportionality constants. Nothing is gained by making **D** and **P** or **H** and **M** have different dimensions. Consequently λ and λ' are chosen as pure numbers ($\lambda = \lambda' = 1$ in rationalized systems, $\lambda = \lambda' = 4\pi$ in unrationalized systems). But there is the choice as to whether **D** and **P** will differ in dimensions from **E**, and **H** and **M** differ from **B**. This choice is made for convenience and simplicity, usually to make the macroscopic Maxwell equations have a relatively simple, neat form. Before tabulating the choices made for different systems, we note that for linear, isotropic media the constitutive relations are always written

$$\left.\begin{aligned} \mathbf{D} &= \epsilon\mathbf{E} \\ \mathbf{B} &= \mu\mathbf{H} \end{aligned}\right\} \tag{A.13}$$

Thus in (A.12) the constants ϵ_0 and μ_0 are the vacuum values of ϵ and μ. The relative permittivity of a substance (often called the *dielectric constant*) is defined as the dimensionless ratio (ϵ/ϵ_0), while the relative permeability (often called the *permeability*) is defined as (μ/μ_0).

Table 2 displays the values of ϵ_0 and μ_0, the defining equations for **D** and **H**, the macroscopic forms of the Maxwell equations, and the Lorentz force equation

Table 2 Definitions of ϵ_0, μ_0, **D**, **H**, Macroscopic Maxwell Equations, and Lorentz Force Equation in Various Systems of Units

Where necessary the dimensions of quantities are given in parentheses. The symbol c stands for the velocity of light in vacuum with dimensions (lt^{-1}).

System	ϵ_0	μ_0	D, H	Macroscopic Maxwell Equations				Lorentz Force per Unit Charge
Electrostatic (esu)	1	c^{-2} (t^2l^{-2})	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ $\mathbf{H} = c^2\mathbf{B} - 4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D} = 4\pi\rho$	$\nabla\times\mathbf{H} = 4\pi\mathbf{J} + \dfrac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E} + \dfrac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \mathbf{v}\times\mathbf{B}$
Electromagnetic (emu)	c^{-2} (t^2l^{-2})	1	$\mathbf{D} = \dfrac{1}{c^2}\mathbf{E} + 4\pi\mathbf{P}$ $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D} = 4\pi\rho$	$\nabla\times\mathbf{H} = 4\pi\mathbf{J} + \dfrac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E} + \dfrac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \mathbf{v}\times\mathbf{B}$
Gaussian	1	1	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$	$\nabla\cdot\mathbf{D} = 4\pi\rho$	$\nabla\times\mathbf{H} = \dfrac{4\pi}{c}\mathbf{J} + \dfrac{1}{c}\dfrac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E} + \dfrac{1}{c}\dfrac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \dfrac{\mathbf{v}}{c}\times\mathbf{B}$
Heaviside–Lorentz	1	1	$\mathbf{D} = \mathbf{E} + \mathbf{P}$ $\mathbf{H} = \mathbf{B} - \mathbf{M}$	$\nabla\cdot\mathbf{D} = \rho$	$\nabla\times\mathbf{H} = \dfrac{1}{c}\left(\mathbf{J} + \dfrac{\partial\mathbf{D}}{\partial t}\right)$	$\nabla\times\mathbf{E} + \dfrac{1}{c}\dfrac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \dfrac{\mathbf{v}}{c}\times\mathbf{B}$
SI	$\dfrac{10^7}{4\pi c^2}$ $(I^2t^4m^{-1}l^{-3})$	$4\pi\times10^{-7}$ $(mlI^{-2}t^{-2})$	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ $\mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M}$	$\nabla\cdot\mathbf{D} = \rho$	$\nabla\times\mathbf{H} = \mathbf{J} + \dfrac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{E} + \dfrac{\partial\mathbf{B}}{\partial t} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\mathbf{E} + \mathbf{v}\times\mathbf{B}$

in the five common systems of units of Table 1. For each system of units the continuity equation for charge and current is given by (A.1), as can be verified from the first pair of the Maxwell equations in the table in each case.* Similarly, in all systems the statement of Ohm's law is $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity.

4 *Conversion of Equations and Amounts Between SI Units and Gaussian Units*

The two systems of electromagnetic units in most common use today are the SI and Gaussian systems. The SI system has the virtue of overall convenience in

Table 3 Conversion Table for Symbols and Formulas

The symbols for mass, length, time, force, and other not specifically electromagnetic quantities are unchanged. To convert any equation in SI variables to the corresponding equation in Gaussian quantities, on both sides of the equation replace the relevant symbols listed below under "SI" by the corresponding "Gaussian" symbols listed on the left. The reverse transformation is also allowed. Residual powers of $\mu_0\epsilon_0$ should be eliminated in favor of the speed of light ($c^2\mu_0\epsilon_0 = 1$). Since the length and time symbols are unchanged, quantities that differ dimensionally from one another only by powers of length and/or time are grouped together where possible.

Quantity	**Gaussian**	**SI**
Velocity of light	c	$(\mu_0\epsilon_0)^{-1/2}$
Electric field (potential, voltage)	$\mathbf{E}(\Phi, V)/\sqrt{4\pi\epsilon_0}$	$\mathbf{E}(\Phi, V)$
Displacement	$\sqrt{\epsilon_0/4\pi}\,\mathbf{D}$	$\mathbf{D}$
Charge density (charge, current density, current, polarization)	$\sqrt{4\pi\epsilon_0}\,\rho(q, \mathbf{J}, I, \mathbf{P})$	$\rho(q, \mathbf{J}, I, \mathbf{P})$
Magnetic induction	$\sqrt{\mu_0/4\pi}\,\mathbf{B}$	$\mathbf{B}$
Magnetic field	$\mathbf{H}/\sqrt{4\pi\mu_0}$	$\mathbf{H}$
Magnetization	$\sqrt{4\pi/\mu_0}\,\mathbf{M}$	$\mathbf{M}$
Conductivity	$4\pi\epsilon_0\sigma$	σ
Dielectric constant	$\epsilon_0\epsilon$	ϵ
Magnetic permeability	$\mu_0\mu$	μ
Resistance (impedance)	$R(Z)/4\pi\epsilon_0$	$R(Z)$
Inductance	$L/4\pi\epsilon_0$	L
Capacitance	$4\pi\epsilon_0 C$	C

$$c = 2.997\,924\,58 \times 10^{8}\ \text{m/s}$$

$$\epsilon_0 = 8.854\,187\,8\ldots \times 10^{-12}\ \text{F/m}$$

$$\mu_0 = 1.256\,637\,0\ldots \times 10^{-6}\ \text{H/m}$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 376.730\,3\ldots\ \Omega$$

*Some workers employ a modified Gaussian system of units in which current is defined by $I = (1/c)(dq/dt)$. Then the current density $\mathbf{J}$ in Table 2 must be replaced by $c\mathbf{J}$, and the continuity equation is $\nabla \cdot \mathbf{J} + (1/c)(\partial\rho/\partial t) = 0$. See also the footnote to Table 4.

Table 4 Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many SI or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.997 924 58), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.997\,924\,58 \times 4\pi \times 10^5$) and "9" is actually $10^{-16}\,c^2 = 8.987\,55\ldots$. Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or SI units.

Physical Quantity	Symbol	SI		Gaussian
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (g)
Time	t	1 second (s)	1	second (s)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton (N)	10^5	dynes
Work Energy	W U	1 joule (J)	10^7	ergs
Power	P	1 watt (W)	10^7	ergs s^{-1}
Charge	q	1 coulomb (C)	3×10^9	statcoulombs
Charge density	ρ	1 C m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (A)	3×10^9	statamperes
Current density	J	1 A m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1} (Vm^{-1})	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt (V)	$\frac{1}{300}$	statvolt
Polarization	P	1 C m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 C m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 siemens m^{-1}	9×10^9	s^{-1}
Resistance	R	1 ohm (Ω)	$\frac{1}{9} \times 10^{-11}$	s cm^{-1}
Capacitance	C	1 farad (F)	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber (Wb)	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla (T)	10^4	gauss (G)
Magnetic field	H	1 A m^{-1}	$4\pi \times 10^{-3}$	oersted (Oe)
Magnetization	M	1 A m^{-1}	10^{-3}	magnetic moment cm^{-3}
Inductance*	L	1 henry (H)	$\frac{1}{9} \times 10^{-11}$	

*There is some confusion about the unit of inductance in Gaussian units. This stems from the use by some authors of a modified system of Gaussian units in which current is measured in electromagnetic units, so that the connection between charge and current is $I_m = (1/c)(dq/dt)$. Since inductance is defined through the induced voltage $V = L(dI/dt)$ or the energy $U = \frac{1}{2}LI^2$, the choice of current defined in Section 2 means that our Gaussian unit of inductance is equal in magnitude and dimensions (t^2l^{-1}) to the electrostatic unit of inductance. The electromagnetic current I_m is related to our Gaussian current I by the relation $I_m = (1/c)I$. From the energy definition of inductance, we see that the electromagnetic inductance L_m is related to our Gaussian inductance L through $L_m = c^2L$. Thus L_m has the dimensions of length. The modified Gaussian system generally uses the electromagnetic unit of inductance, as well as current. Then the voltage relation reads $V = (L_m/c)(dI_m/dt)$. The numerical connection between units of inductance is

$$1 \text{ henry} = \tfrac{1}{9} \times 10^{-11} \text{ Gaussian (es) unit} = 10^9 \text{ emu}$$

Example of conversion <u>from</u> Gaussian <u>to</u> SI units:

Gauss' law $\quad \bar{\nabla} \cdot \bar{E} = 4\pi \rho$

Following Table 3, we replace

$$\frac{\bar{E}}{\sqrt{4\pi\varepsilon_0}} \rightarrow \bar{E} \quad \text{and} \quad \sqrt{4\pi\varepsilon_0}\,\rho \rightarrow \rho$$

Gaussian

$$\Rightarrow \bar{\nabla} \cdot \left(\frac{E}{\sqrt{4\pi\varepsilon_0}} \right) \sqrt{4\pi\varepsilon_0} = 4\pi \frac{\left(\sqrt{4\pi\varepsilon_0}\,\rho\right)}{\sqrt{4\pi\varepsilon_0}}$$

Replace $\rightarrow$ SI

$$\bar{\nabla} \cdot \bar{E} \sqrt{4\pi\varepsilon_0} = 4\pi \frac{\rho}{\sqrt{4\pi\varepsilon_0}}$$

$$\Rightarrow \bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0}$$