$$\boxed{\text{Laplacian Lecture \#2}}$$

We could spend weeks on this chapter alone, but I'll introduce the spherical and cylindrical forms of the Laplacian, then we'll move forward.

Spherical Coordinates

$$\bar{\nabla}^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Separation of variables:

$$\Phi(r, \theta, \phi) = R(r)P(\theta)Q(\phi)$$

Substituting and dividing by $\Phi = RPQ$

$$\frac{1}{r^2 R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{r^2 P \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \frac{1}{r^2 Q \sin^2\theta}\frac{d^2 Q}{d\phi^2} = 0$$

Multiply by $r^2 \sin^2\theta$

$$\underbrace{\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)}_{\text{function of } r \text{ and } \theta} = \underbrace{-\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{\text{function of } \phi}$$

Because RHS is only a function of ϕ, it must be constant

$$\Rightarrow -\frac{1}{Q}\frac{d^2Q}{d\phi^2} = m^2 \Rightarrow \frac{d^2Q}{d\phi^2} + m^2 Q = 0$$

Solutions follow

$$Q_m = C_m e^{im\phi} + D_m e^{-im\phi}$$

m is an integer so that $im(\phi + 2\pi n) = im\phi$

Going back to the r, θ dependence with the RHS $= m^2$, and dividing by $\sin^2\theta$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{function of } r} = \underbrace{-\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{m^2}{\sin^2\theta}}_{\text{function of } \theta}$$

Let's set the RHS to $\ell(\ell+1)$

reason becomes apparent later

$$\Rightarrow \quad \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \ell(\ell+1)R = 0$$

$$\Rightarrow \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

The radial solutions are obtained from a power series:

$$R_\ell(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}$$

To solve the ODE for the polar angle, we make the following substitutions:

$$\cos\theta \to x$$

$$-\frac{1}{\sin\theta}\frac{d}{d\theta} \to \frac{d}{dx}$$

So the equation for θ becomes

$$\underbrace{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0}_{\text{Associated Legendre Equation}}$$

If $m=0$ so that we have azimuthal symmetry and Φ is not a function of ϕ

$$\underbrace{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0}_{\text{Legendre's Equation}}$$

Solutions are Legendre Polynomials of order ℓ

Legendre Polynomials
- complete
- orthogonal

$$\left\{ \begin{aligned} &P_\ell(x=+1) = 1 \\ &P_0(x) = 1 \\ &P_1(x) = x \\ &P_2(x) = \frac{1}{2}(3x^2-1) \\ &P_3(x) = \frac{1}{2}(5x^3-3x) \\ &\vdots \end{aligned} \right.$$

Rodrigues' formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

Orthogonality

$$\int_{-1}^{+1} P_{\ell'}(x) P_\ell(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

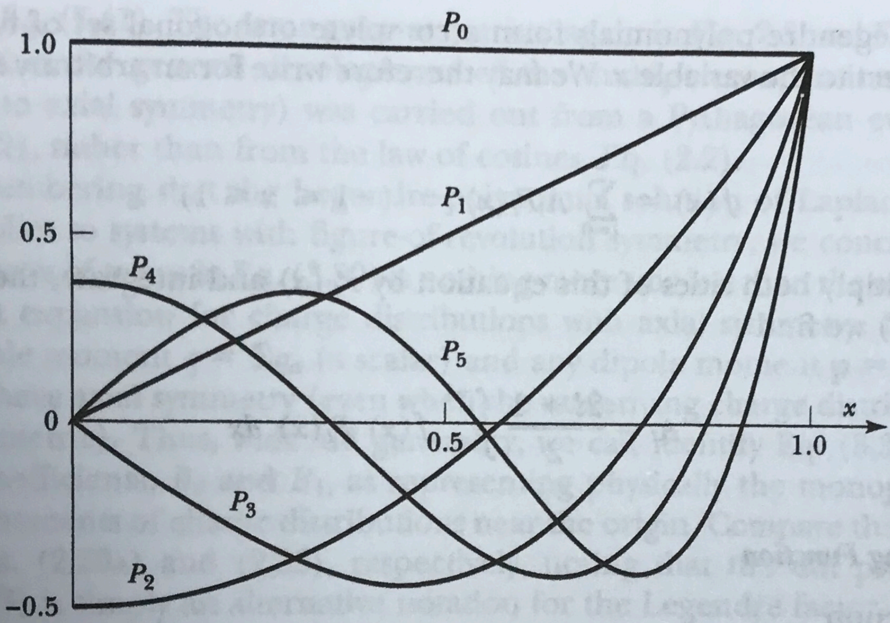

FIGURE 3-3a. Legendre polynomials as functions of $x = \cos\theta$.

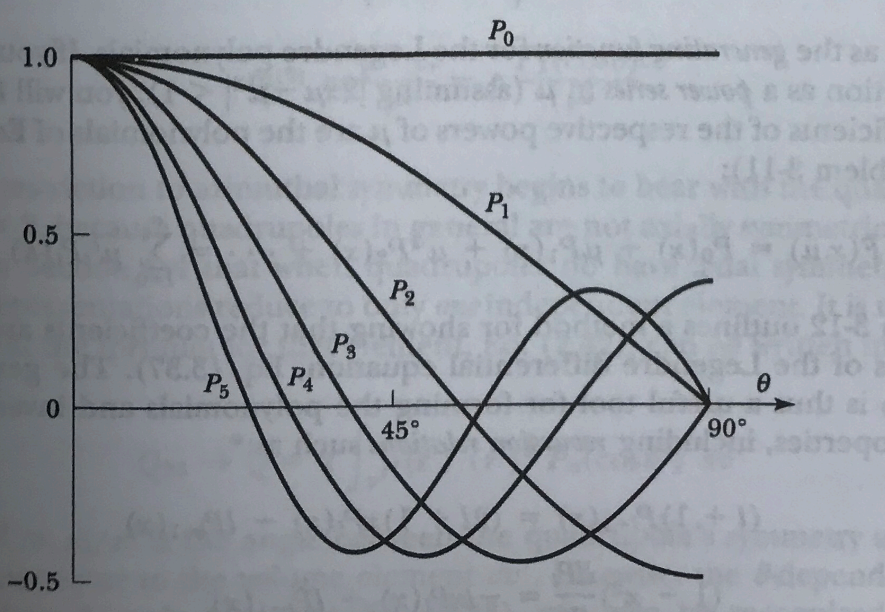

FIGURE 3-3b. Legendre polynomials as functions of θ.

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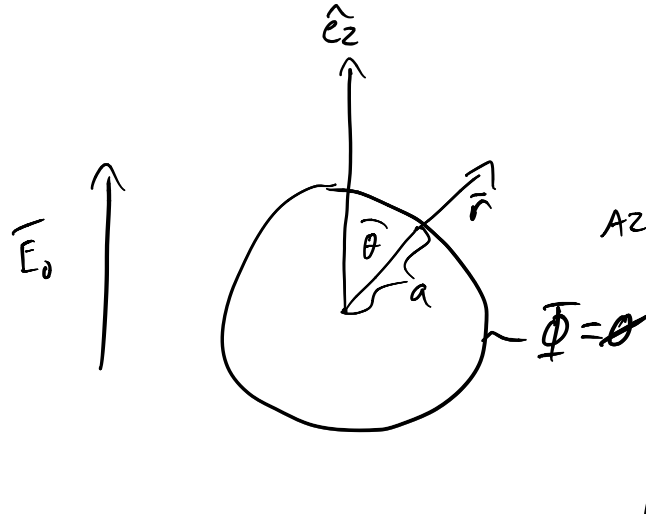
Combining the r and θ solutions for an azimuthally symmetric scenario:

$$\boxed{\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l \frac{1}{r^{l+1}}\right] P_l(\cos\theta)}$$

<u>Example</u>

Compute Φ for all points in space exterior to a grounded conducting sphere of radius a placed in a uniform electric field $\bar{E}_0$



Azimuthally symmetric

background potential

$$\bar{E}_0 = E_0 \hat{e}_z = -\bar{\nabla}\Phi_0 \implies \Phi_0 = -E_0 z$$

Converting to spherical coordinates

$$z = r\cos\theta \implies \Phi_0 = -E_0 r\cos\theta$$

But $P_1(\cos\theta) = \cos\theta$

$$\implies \Phi_0 = -E_0 r P_1(\cos\theta)$$

Note $\Phi_0(r \to \infty) \neq 0$ because we assumed a uniform background field of infinite extent

The general solution is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l \frac{1}{r^{l+1}}\right] P_l(\cos\theta)$$

For large r, we can ignore $\frac{1}{r^{l+1}}$ and the solution should match the background

$$\Phi(\text{large } r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r P_1(\cos\theta)$$

Because P_ℓ are orthogonal and we only need the $P_{\ell=1}$ term, we have

$$A_1 r P_1(\cos\theta) = -E_0 r P_1(\cos\theta)$$

$$\Longrightarrow A_1 = -E_0 \quad , \text{All other } A_\ell = 0$$

$$\Rightarrow \Phi(r,\theta) = -E_0 r P_1(\cos\theta) + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

At $\Phi(r=a)$ we need $\Phi = 0$

$$\Rightarrow -E_0 a P_1(\cos\theta) + \sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta) = 0$$

Multiply by $P_{\ell'}$ and integrate to take advantage of orthogonality

$$E_0 a \int_{-1}^{1} P_1(\cos\theta) P_{l'}(\cos\theta)\, d\cos\theta = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} \int_{-1}^{1} P_l(\cos\theta) P_{l'}(\cos\theta)\, d\cos\theta$$

But

$$\int_{-1}^{1} P_l(\cos\theta) P_{l'}(\cos\theta)\, d\cos\theta = \frac{2}{2l+1}\delta_{ll'}$$

$$\Rightarrow E_0 a \frac{2}{2l'+1}\delta_{1l'} = \frac{B_{l'}}{a^{l'+1}} \frac{2}{2l'+1}$$

$$\Rightarrow B_l = E_0 a^{l+2} \delta_{1l} = E_0 a^3$$

The final solution is

$$\Phi(r,\theta) = -E_0 r P_1(\cos\theta) + \frac{B_1}{r^2} P_1(\cos\theta)$$

$$= -E_0 \left(1 - \frac{a^3}{r^3}\right) r\cos\theta$$

The electric field is

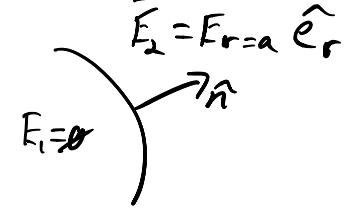
$$E_r = -\frac{\partial \Phi}{\partial r} = E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta$$

$$E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta$$

From our boundary equation for $\bar{E}$, we have

$$\left(\bar{E}_2 - \bar{E}_1\right)\cdot\hat{n} = 4\pi\sigma_s$$

surface charge density



$$\Rightarrow \frac{E_{r=a}}{4\pi} = \frac{E_0}{4\pi}3\cos\theta = \sigma_s$$

If we integrate ρ_s over the sphere, it vanishes $\Rightarrow$ charge on top and bottom of sphere are equal and opposite in charge, like a <u>dipole</u>

We can rewrite Φ as

$$\Phi(r,\theta) = -E_0\left(1 - \frac{a^3}{r^3}\right) r\cos\theta = \Phi_0 + E_0 a^3 \frac{\cos\theta}{r^2}$$

from $\Phi_0 = -E_0 r\cos\theta$

Which, if we define

$$\bar{p} = E_0 a^3 \hat{e}_z$$

from $\bar{p} = \oint_S \rho_s \bar{r}\, da$

$\uparrow$
integrate over surface of sphere

Then

$$\Phi(r,\theta) = \Phi_0 + \underbrace{\frac{\bar{p}\cdot\hat{e}_r}{r^2}}$$

Conducting sphere acts like a dipole with a dipole moment of $p = E_0 a^3$

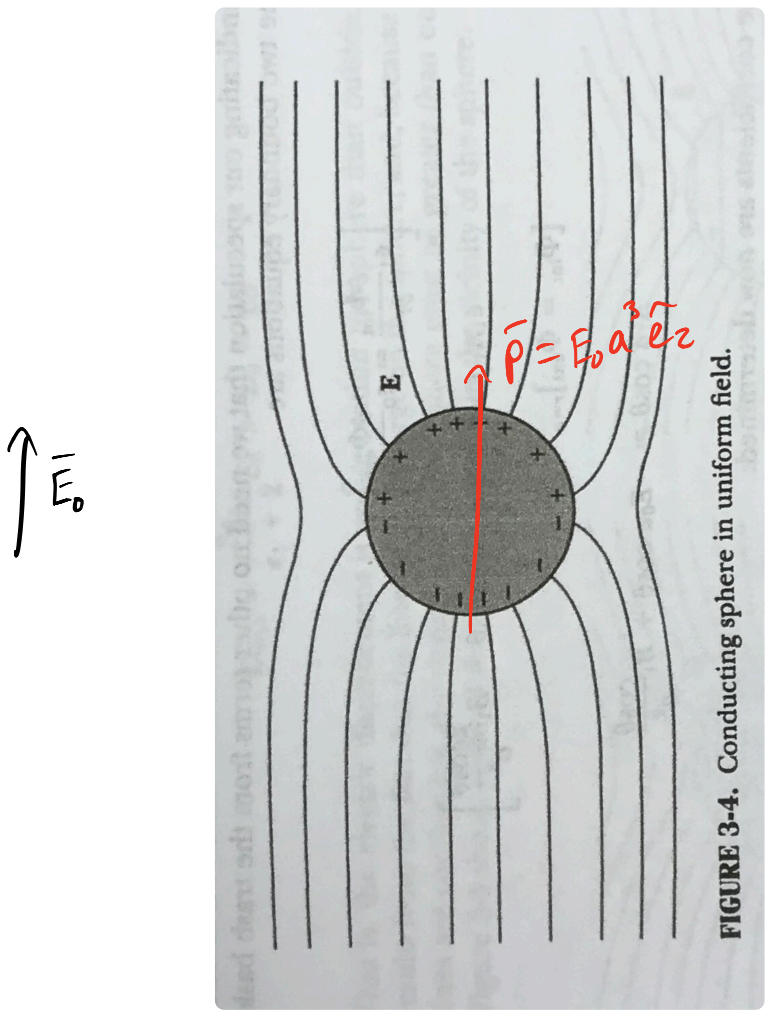


FIGURE 3-4. Conducting sphere in uniform field.

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