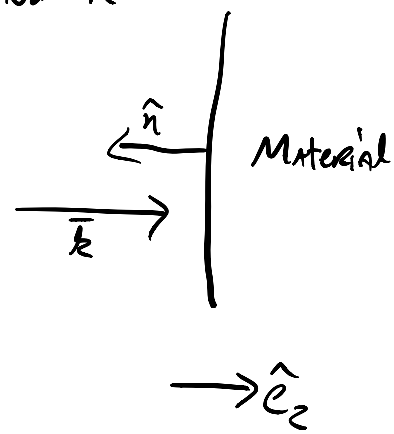
Radiation Pressure

- E+M waves carry energy and momentum
- For a plane wave incident on a perfectly absorbing surface,

$$\underbrace{\frac{\bar{F}}{A}}_{\text{Force per unit area}} = \underbrace{\bar{\bar{T}}}_{\text{Maxwell stress tensor}} \cdot \underbrace{\hat{n}}_{\text{outward normal}}$$

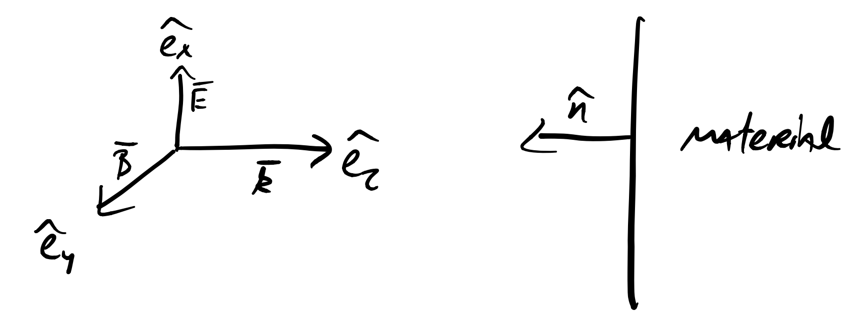


The pressure exerted on the surface is the <u>inward</u> force

$$\underbrace{p}_{\text{pressure}} = \underbrace{-\hat{n}}_{\text{inward component of } \bar{F}/A} \cdot \bar{\bar{T}} \cdot \hat{n}$$

For $\bar{E}$ in $\hat{e}_x$ direction and

$\bar{B}$ in $\hat{e}_y$ direction



$$-\hat{n}\cdot\overline{\overline{T}}\cdot\hat{n} = -\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}T_{xx} & T_{xy} & T_{xz}\\ T_{yx} & T_{yy} & T_{yz}\\ T_{zx} & T_{zy} & T_{zz}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= -\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}T_{xz}\\ T_{yz}\\ T_{zz}\end{pmatrix}$$

$$= -T_{zz}$$

$$T_{ij} = \frac{1}{4\pi}\left[\left(E_iE_j + B_iB_j\right) - \frac{1}{2}\left(E^2+B^2\right)\delta_{ij}\right]$$

$$\Rightarrow T_{zz} = \frac{1}{4\pi}\Big[\underbrace{E_zE_z + B_zB_z}_{\cancel{0}} - \frac{1}{2}(\underset{E_x^2}{E^2} + \underset{B_y^2}{B^2})\Big) = -\frac{1}{8\pi}\left(E_x^2 + B_y^2\right)$$

$$\Rightarrow p = -\hat{n}\cdot\overline{\overline{T}}\cdot\hat{n} = -T_{zz} = \frac{1}{8\pi}\left(E_x^2 + B_y^2\right)$$

For $\quad E_x = E_0 e^{i(kz-\omega t)} \qquad B_y = B_0 e^{i(kz-\omega t)}$

$$p = \frac{1}{8\pi}\left[E_0^2 e^{2i(kz-\omega t)} + B_0^2 e^{2i(kz-\omega t)}\right]$$

The time average is

$$\langle p \rangle = \frac{1}{8\pi}\left[\left\langle E_0^2 e^{2i(kz-\omega t)}\right\rangle + \left\langle B_0^2 e^{2i(kz-\omega t)}\right\rangle\right]$$

But $\quad \langle F\cdot G\rangle = \frac{1}{2}F\cdot G^*$

$$\Rightarrow \langle p \rangle = \frac{1}{8\pi}\left[\frac{E_0^2}{2} + \frac{B_0^2}{2}\right]$$

FROM $\quad n = \sqrt{\varepsilon\mu} = \dfrac{|B_0|}{|E_0|}$

$$\Rightarrow \langle p \rangle = \frac{1}{8\pi} E_0^2$$

↳ equal energy density $\mathcal{E}$ for plane wave!

$$\frac{\text{Force}}{\text{Area}} = \frac{\text{energy}}{\text{volume}}$$

Alternatively, we can analyze the time rate of change in momentum transferred to the surface:

$$\bar{g}_{\text{field}} = \frac{1}{c^2}\bar{S}$$

momentum/volume $\qquad$ Poynting vector

The momentum is carried at a speed c

$$\Rightarrow \quad \langle p \rangle = \underbrace{\langle c\, g_{\text{field}} \rangle}_{\text{units} = \frac{\text{distance}}{\text{time}} \times \frac{\text{momentum}}{\text{volume}} = \frac{\text{Force}}{\text{Area}}} = \frac{1}{c}\langle S \rangle = \frac{1}{8\pi} E_0^2$$

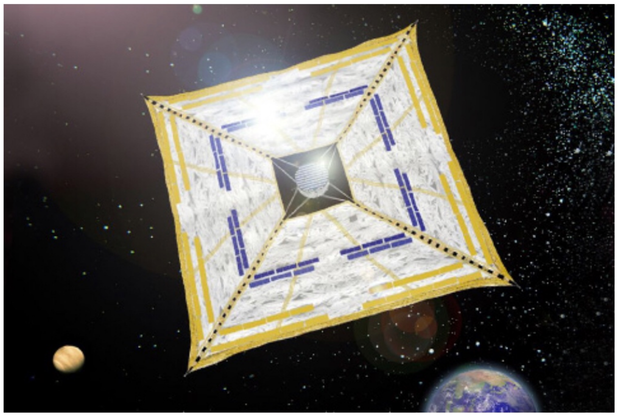
- If the surface is a good <u>conductor</u>, there will be a <u>reflected</u> wave with an <u>equal amplitude</u> to the incident wave

$$\hookrightarrow \langle p \rangle_{\text{conductor}} = 2\langle p \rangle$$

why?

Example

About Small Solar Power Sail Demonstrator "IKAROS"



Space yacht accelerated by radiation of the Sun

A Solar Sail gathers sunlight as propulsion by means of a large membrane while a Solar "Power" Sail gets electricity from thin film solar cells on the membrane in addition to acceleration by solar radiation. What's more, if the ion-propulsion engines with high specific impulse are driven by such solar cells, it can become a "hybrid" engine that is combined with photon acceleration to realize fuel-effective and flexible missions.
JAXA is studying two missions to evaluate the performance of the solar power sails. The project name for the first mission is IKAROS (Interplanetary Kite-craft Accelerated by Radiation Of the Sun). This craft was launched with the Venus Climate Orbiter "AKATSUKI", using an H-IIA launch vehicle. This will be the world's first solar powered sail craft employing both photon propulsion and thin film solar power generation during its interplanetary cruise.

Launch date: May 21, 2010

[global.jaxa.jp/projects/sat/ikaros]

Plane Waves in Conducting Media

In a conductor with conductivity σ the $\bar{E}$ drives a $\bar{J}$ (Ohm's law)

$$\bar{J} = \sigma \bar{E}$$

Assuming a linear, homogeneous, isotropic medium

$$\bar{D} = \varepsilon \bar{E} \quad \text{and} \quad \bar{B} = \mu \bar{H}$$

$$\Rightarrow \left.\begin{aligned} \bar{\nabla}\cdot\bar{E} &= 0 \\ \bar{\nabla}\cdot\bar{B} &= 0 \\ \bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial\bar{B}}{\partial t} &= 0 \\ \bar{\nabla}\times\bar{B} - \frac{\varepsilon\mu}{c}\frac{\partial\bar{E}}{\partial t} &= \frac{4\pi\sigma\mu}{c}\bar{E} \end{aligned}\right\}$$

Maxwell's Equations for $\bar{J} = \sigma\bar{E}$ with no other free charges or currents

The wave equations are (you derive something similar in your HW):

$$\boxed{\begin{aligned} \bar{\nabla}^2 \bar{E} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \bar{E}}{\partial t} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} = 0 \\ \bar{\nabla}^2 \bar{B} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \bar{B}}{\partial t} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \bar{B}}{\partial t^2} = 0 \end{aligned}}$$

Limits for behavior with respect to σ:

σ = small

$\hookrightarrow$ wave attenuates as it propagates

σ = large

$\hookrightarrow$ $\frac{\varepsilon\mu}{c^2}\frac{\partial^2 \bar{B}}{\partial t^2}$ term is negligible $\rightarrow$ reduced to <u>diffusion</u> equation

Considering plane wave solutions of the form

$$\bar{E} = \bar{E}_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)} = \bar{E}_0 e^{i(kr_5 - \omega t)}$$

$$\bar{B} = \bar{B}_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)} = \bar{B}_0 e^{i(kr_5 - \omega t)}$$

where ζ is the $\perp$ distance to the origin.

$$\Rightarrow \underbrace{\left(k^2 - i\,\frac{4\pi\sigma\mu\omega}{c^2} - \frac{\varepsilon\mu\omega^2}{c^2}\right)}_{=0 \text{ because } \longrightarrow} \underbrace{\bar{E}_0\, e^{i(k\zeta - \omega t)}}_{\neq 0} = 0$$

This gives us the <u>dispersion relation</u>

$$\hat{k}^2 = \frac{\varepsilon\mu\omega^2}{c^2}\left(1 + i\,\frac{4\pi\sigma}{\varepsilon\omega}\right)$$

hat on k means <u>complex</u>

Defining complex $\hat{k}$ as

$$\hat{k} \equiv \alpha + i\beta$$

The spatial part of our solution follows

$$\Rightarrow e^{i\hat{k}\zeta} = \underbrace{e^{i\alpha\zeta}}_{\text{periodic}}\ \underbrace{e^{-\beta\zeta}}_{\text{exponential decay (damping)}}$$

The wave decays by $\frac{1}{e}$ over a distance

$$\zeta = \frac{1}{\beta}$$

Squaring $\hat{k}$ we get

$$\hat{k}^2 = (\alpha + i\beta)(\alpha + i\beta) = \underbrace{(\alpha^2 - \beta^2)}_{\text{real}} + \underbrace{2i\alpha\beta}_{\text{imaginary}}$$

From before, we know that

$$\hat{k}^2 = \frac{\varepsilon \mu \omega^2}{c^2}\left(1 + i\,\frac{4\pi\sigma}{\varepsilon\omega}\right)$$

$$\Longrightarrow \alpha^2 - \beta^2 = \frac{\mu\varepsilon\omega^2}{c^2}$$

$$2\alpha\beta = \frac{4\pi\omega\sigma\mu}{c^2}$$

Solving for α and β:

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{4\pi\sigma}{\omega\varepsilon}\right)^2}+1\right]^{1/2}$$

$$\beta = \frac{\omega}{c}\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{4\pi\sigma}{\omega\varepsilon}\right)^2}-1\right]^{1/2}$$

where the signs have been selected to describe a wave progressing in the $+\zeta$ direction

From $\hat{k} = \alpha + i\beta$

$\alpha = \frac{2\pi}{\lambda}$ = real, periodic part of $\hat{k}$

β = damping (exponential decay) coefficient

$\hookrightarrow$ imaginary part of $\hat{k}$

$\boxed{\text{In class problem}}$

Find the <u>dispersion relation</u> for a plane wave in a vacuum:

$$\bar{\nabla}^2 \bar{E} - \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} = 0$$

What are α and β in $\hat{k} = \alpha + i\beta$?

Hints:

$$\bar{\nabla}\bar{E} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \begin{pmatrix} E_x & E_y & E_z \end{pmatrix} = \begin{pmatrix} \frac{\partial E_x}{\partial x} & \frac{\partial E_y}{\partial x} & \frac{\partial E_z}{\partial x} \\ \frac{\partial E_x}{\partial y} & \frac{\partial E_y}{\partial y} & \frac{\partial E_z}{\partial z} \\ \frac{\partial E_x}{\partial z} & \frac{\partial E_y}{\partial z} & \frac{\partial E_z}{\partial z} \end{pmatrix}$$

Assume $\bar{k}$ is in $\hat{e}_z$ direction and $\bar{E}$ is in $\hat{e}_x$ direction

$$\Rightarrow \bar{E} = E_0 \hat{e}_x e^{i(kz - \omega t)}$$

$\boxed{\text{Solution}}$

Time part:

$$-\frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} = -\frac{1}{c^2}\left(-\omega^2 \bar{E}\right) = \frac{\omega^2}{c^2}\bar{E}$$

For the Laplacian:

$$\bar{\nabla}^2 \bar{E} = \bar{\nabla}\cdot(\bar{\nabla}\bar{E})$$

$$\bar{\nabla}\bar{E} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}\begin{pmatrix} E_x & E_y & E_z \end{pmatrix} = \begin{pmatrix} \frac{\partial E_x}{\partial x} & \frac{\partial E_y}{\partial x} & \frac{\partial E_z}{\partial x} \\ \frac{\partial E_x}{\partial y} & \frac{\partial E_y}{\partial y} & \frac{\partial E_z}{\partial z} \\ \frac{\partial E_x}{\partial z} & \frac{\partial E_y}{\partial z} & \frac{\partial E_z}{\partial z} \end{pmatrix}$$

$$\Rightarrow \bar{\nabla} E_0 \hat{e}_x e^{i(kz-\omega t)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ikE & 0 & 0 \end{pmatrix}$$

only $\frac{\partial E_x}{\partial z} \neq 0$

$$\Rightarrow \bar{\nabla} \cdot (\bar{\nabla} \bar{E}) = \begin{pmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ikE & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -k^2 E & 0 & 0 \end{pmatrix}$$

$$= -k^2 \bar{E}$$

$$\Rightarrow \bar{\nabla}^2 \bar{E} = -k^2 \bar{E}$$

Combining

$$-k^2 \bar{E} + \frac{\omega^2}{c^2} \bar{E} = 0$$

$$\Rightarrow \boxed{k = \frac{\omega}{c}}$$