Poisson's Equation

Laplace's equation is a homogeneous PDE

$$\bar{\nabla}^2 \Phi = \emptyset$$

equal to zero = no source term

= homogeneous

Poisson's Equation is the nonhomogeneous version of Laplace's equation

$$\bar{\nabla}^2 \Phi = -4\pi \rho(\bar{r}) \quad \Big\} \text{Poisson's Equation}$$

↳ source term = nonhomogeneous

Poisson's Equation must be used when there is a charge density present, which acts as a source term for the PDE.

- How do we solve Poisson's equation in general?

- Can we use separation of variables?

  ↳ No, not for an arbitrary $\rho(\vec{r})$

$$\vec{\nabla}^2 \Phi(x_1, x_2, x_3) = -4\pi \rho(x_1, x_2, x_3)$$

$$\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} = \underbrace{-4\pi \rho(x_1, x_2, x_3)}_{\text{function of } x_1, x_2, x_3 \text{ so how do we separate?}}$$

A common approach to solving Poisson's equation is to use a <u>Green's Function</u>.

Green's Functions

- Green's functions provide a method of calculating the particular solution (nonhomogeneous part) of a differential equation by converting the differential equation to an integral of the source term times the Green's function.

- Following Wolfram MathWorld, a differential operator (take $\bar{\nabla}^2$ for example) $\mathcal{L}$ with:

$$\mathcal{L}\, u(x) = f(x)$$

differential operator   solution   source term has a Green's function (particular to $\mathcal{L}$) that follows:

$$\mathcal{L}\, G(x, x') = \delta(x - x')$$

Green's fxn   delta fxn

Because of the property above, we can multiply both sides of the equation by $f(x')$ and integrate:

$$\underbrace{\int \mathcal{L}\, G(x, x')\, f(x')\, dx'}_{} = \underbrace{\int \delta(x - x')\, f(x')\, dx'}_{}$$

$$\Rightarrow \mathcal{L}\left(\int G(x, x')\, f(x')\, dx' \right) = f(x)$$

operates on x, so we can take it outside of the integral

But, we know

$$\mathcal{L}\, u(x) = f(x)$$

$$\Longrightarrow \quad u(x) = \underbrace{\int G(x,x')\, f(x')\, dx'}$$

solution is integral
of Green's function with
the source term $f(x')$

<u>Example, Poisson's Equation:</u>

$$\bar{\nabla}^2 \Phi(\bar{r}) = -4\pi \rho(\bar{r})$$

The Green's function satisfies

$$\bar{\nabla}^2 G(\bar{r},\bar{r}') = \delta(\bar{r}-\bar{r}')$$

From the Divergence Theorem, we found

$$\bar{\nabla}^2 \frac{1}{|\bar{r}-\bar{r}'|} = -4\pi\,\delta(\bar{r}-\bar{r}')$$

$$\Rightarrow \quad G(\bar{r},\bar{r}') = -\frac{1}{4\pi}\frac{1}{|\bar{r}-\bar{r}'|} + F(\bar{r},\bar{r}')$$

where $F(\bar{r},\bar{r}')$ is the homogeneous solution satisfying

$$\bar{\nabla}^2 F(\bar{r},\bar{r}') = 0$$

$F(\bar{r},\bar{r}')$ is selected to help satisfy the BCs.

The general solution is

$$\Phi = \int_V \rho(\bar{r}')\, G(\bar{r},\bar{r}')\, dv'$$

$$+ \frac{1}{4\pi} \oint_S \left[G(\bar{r},\bar{r}') \frac{\partial \Phi}{\partial n'} - \Phi(\bar{r}') \frac{\partial G(\bar{r},\bar{r}')}{\partial n'} \right] da'$$

Boundary conditions

Neumann

$$\frac{\partial \Phi_{BC}}{\partial n'} = \Phi_0$$

Dirichlet

$$\Phi_{BC} = \Phi_0$$

Once you know $G(\bar{r},\bar{r}')$, you can solve for Φ using integration

↳ powerful for arbitrary $\rho(\bar{r})$

$\boxed{\text{Green's Function Homework Problem}}$

In this problem, we'll use SI units.

In 1-D, Poisson's equation is

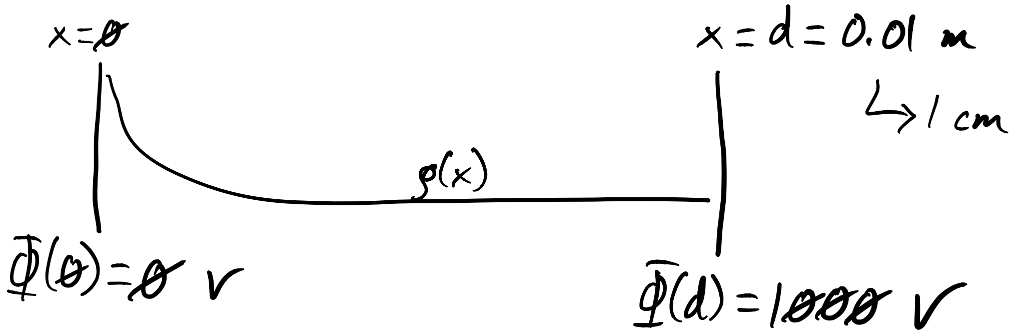
$$\frac{\partial^2 \Phi}{\partial x^2} = -\frac{\rho(x)}{\varepsilon_0}$$

The solution using a Green's function approach for Dirichlet BCs is:

$$\Phi(x) = \int_0^x \frac{x'}{d}(d-x)\frac{\rho(x')}{\varepsilon_0}\,dx'$$

$$+\int_x^d x\left(1-\frac{x'}{d}\right)\frac{\rho(x')}{\varepsilon_0}\,dx'$$

$$-\Phi(0)\left(\frac{x}{d}-1\right) + \Phi(d)\frac{x}{d}$$



$$\rho(x) = \rho_0 \left[1 - \tanh\left(\frac{x}{d/10} \right) \right]$$

$$\rho_0 = q_e \cdot 10^{17} \quad (\text{Coulomb}/m^3)$$

charge of electron

⟶ Plot $\Phi(x)$ and $E(x)$

Charge Conservation

The current through a closed surface is

$$I = \oint_S \bar{J} \cdot d\bar{a}$$

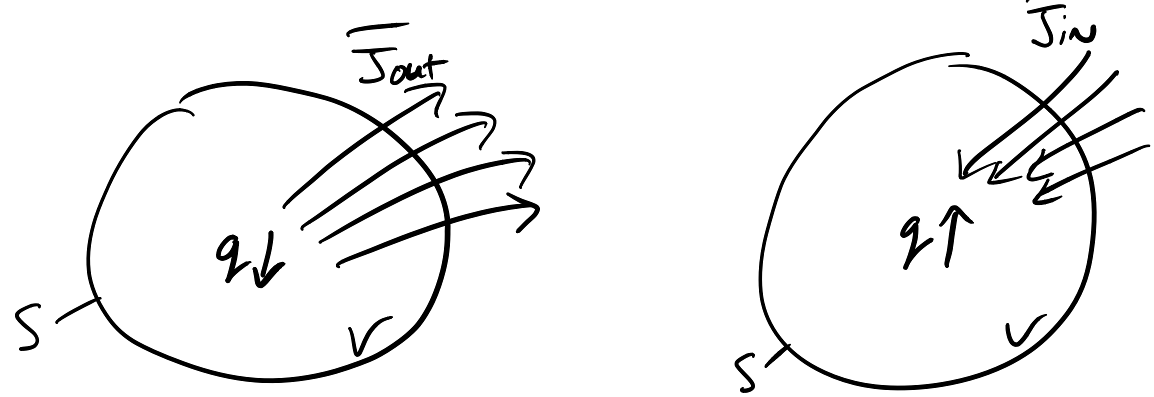
I is positive if there's an <u>outward</u> flow of <u>positive</u> charge.

If no charge is created or lost inside the volume V bounded by S (no source term), then an <u>outward</u> flow of current <u>decreases</u> the total charge in the volume:

total charge

$$\oint_S \bar{J} \cdot d\bar{a} = -\frac{dq}{dt} = -\frac{d}{dt} \int_V \rho \, dv$$

charge density



Total charge decrease

Total charge increase

For a fixed S and V, the derivative must be applied strictly to ρ:

$$\oint_S \bar{J} \cdot d\bar{a} = -\int_V \frac{\partial \rho}{\partial t} \, dv$$

Divergence Theorem

$$\int_V \bar{\nabla} \cdot \bar{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} \, dv$$

Because V is arbitrary

$$\boxed{\bar{\nabla} \cdot \bar{J} = -\frac{\partial \rho}{\partial t}}$$

Continuity Equation

This is a very important equation that can be found in many branches of physics.

Example: Consider a volume with a homogeneous material that obeys Ohm's Law

$$\bar{J} = \sigma \bar{E}$$

$\hookrightarrow$ conductivity (units $= \frac{1}{S}$)

and has a conductivity σ and dielectric constant ε. At time $t = 0$, there is free charge density $\rho_0(\bar{r})$. How does ρ change with time for an applied E?

$$\longrightarrow \quad -\frac{\partial \rho}{\partial t} = \bar{\nabla}\cdot\bar{J} = \bar{\nabla}\cdot\sigma\bar{E} = \sigma\bar{\nabla}\cdot\bar{E}$$

But $\bar{\nabla}\cdot\bar{D} = 4\pi\rho$ and $\bar{D} = \varepsilon\bar{E}$

$$\Rightarrow \quad -\frac{\partial \rho}{\partial t} = \frac{4\pi\sigma}{\varepsilon}\rho$$

$$\Rightarrow \quad \rho(t) = \rho_0 \exp\left[-\frac{4\pi\sigma}{\varepsilon}t\right]$$

$$= \rho_0 \exp\left[-t/\tau\right]$$

$$\tau = \frac{\varepsilon}{4\pi\sigma} = \text{relaxation time}$$

$\longrightarrow$ charge density decreases exponentially over time

$\longrightarrow$ $\bar{J} = \sigma\bar{E}$ moves charge out of the volume