Propagation of Waves Between Conducting Planes

We start by analyzing propagation between 2 parallel and perfectly conducting planes

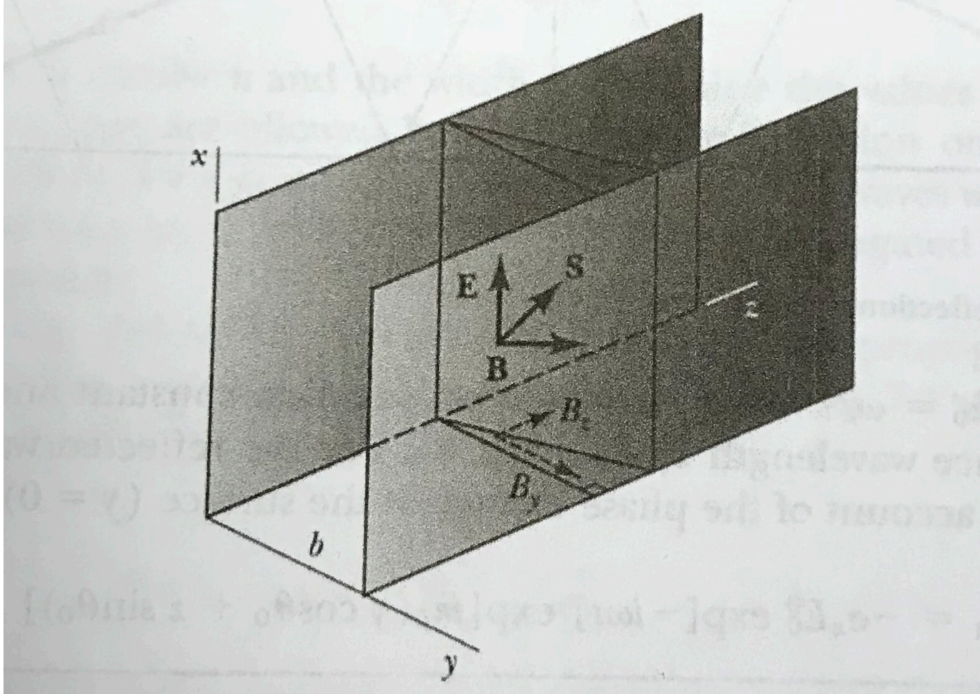

FIGURE 7-3. Plane wave propagating between parallel planes. [H+M]

Inside perfect conductor $\Rightarrow E = 0$

$B = 0$

From the BCs

$$(\bar{E}_2 - \bar{E}_c) \times \hat{n} = 0$$

$$(\bar{B}_2 - \bar{B}_c) \cdot \hat{n} = 0$$

where $\bar{E}_c = \bar{B}_c = \emptyset$ are the fields in a conductor

$$\Rightarrow \underbrace{E_{\text{tangential}} = B_{\text{normal}} = \emptyset}_{\text{at the surface}}$$

Assuming the planes are in a vacuum ($\mu = \varepsilon = 1$, $\sigma = \rho = \emptyset$), we introduce an E+M wave that propagates parallel to the planes by reflecting off of the conductors

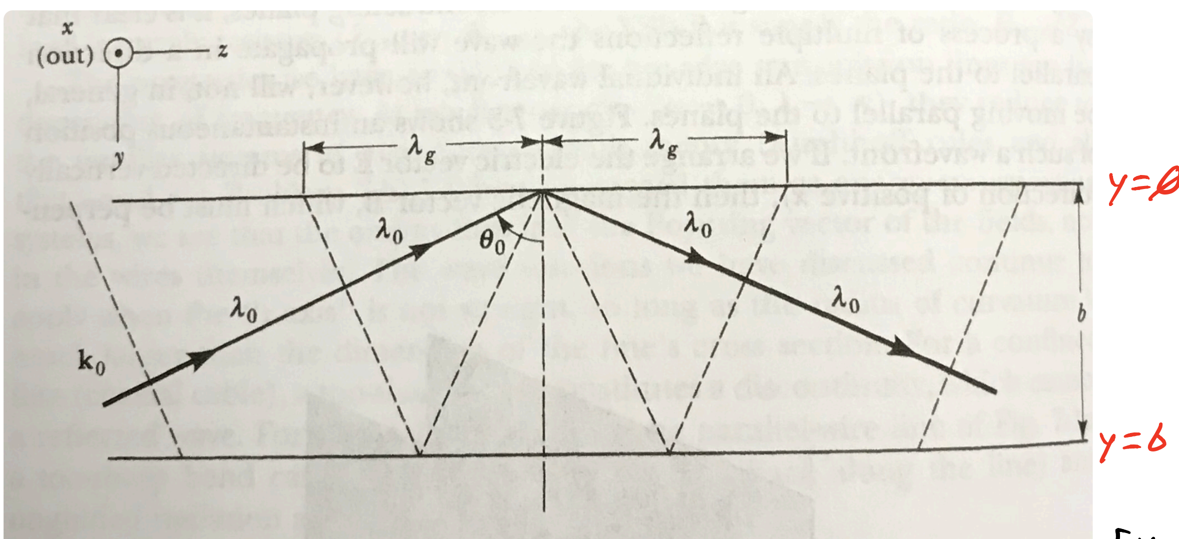

FIGURE 7-4. Reflection between conducting planes.

[H+M]

For $\bar{E} = E\hat{e}_x$ and $\bar{B} = \hat{e}_k \times \bar{E}$

(plane wave in vacuum, $n=1$)

with the wave propagation ultimately in $\hat{e}_z$ after reflections

$\Rightarrow$ $\bar{B}$ does have a $\hat{e}_z$ component

$\bar{E}$ does not have a $\hat{e}_z$ component

$\hookrightarrow$ Transverse Electric (TE) wave

Alternatively, for $\bar{B} = B\hat{e}_x$ and $\bar{E} = -\hat{e}_k \times \bar{B}$

$\Rightarrow$ $\bar{B}$ does not have a $\hat{e}_z$ component

$\bar{E}$ does have a $\hat{e}_z$ component

$\hookrightarrow$ Transverse Magnetic (TM) wave

The field that does not have a component in the direction of propagation is transverse.

What is a plane wave in free space?

- Transverse electric and magnetic (TEM)

In a hollow conductor, we can only have TE or TM (no TEM... we'll see why later)

Assuming we have a TE wave in the geometry of Figure 7-4, $\bar{E}$ incident is

$$\bar{E}_0 = \hat{e}_x E_0^0 \exp[i(\bar{k}_0 \cdot \bar{r} - \omega t)]$$

$$= \hat{e}_x E_0^0 \exp[i\omega t] \exp[ik_0(-y\cos\theta_0 + z\sin\theta_0)]$$

Where

$$k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0} \longleftarrow \text{free space wavelength}$$

After the wave is reflected at $y=0$, a π phase shift provides

$$\bar{E}_1 = -\hat{e}_x E_0^0 \exp[-i\omega t] \exp[ik_0(y\cos\theta_0 + z\sin\theta_0)]$$

Combining the two waves:

$$\bar{E}_{total} = \bar{E}_0 + \bar{E}_1$$

$$= \hat{e}_x E_0^0 \exp[-i\omega t] \exp[ik_0 z \sin\theta_0] \left(\exp[-ik_0 y \cos\theta_0] - \exp[ik_0 y \cos\theta_0] \right)$$

$$= -2i \hat{e}_x E_0^0 \sin(k_0 y \cos\theta_0) \exp[ik_0 z \sin\theta_0 - \omega t]$$

$\hookrightarrow$ <u>nonplane wave</u> traveling in $\hat{e}_z$ direction with $k = k_0 \sin\theta_0$

From an energy transport view, this can be treated as <u>2 plane waves</u> zigzagging back and forth

Because $\bar{E} = E\hat{e}_x$, the BCs are

$$\bar{E}(y=0) = \bar{E}(y=b) = 0$$

This BC is already satisfied at $y=0$

At $y=b$, we need

$$\sin(k_0 b\cos\theta_0) = 0 \Rightarrow k_0 b\cos\theta_0 = n\pi \quad \text{for } n=1,2,3\ldots$$

In the transverese $\hat{e}_y$ direction, E has the form of a standing wave

$$\frac{2\pi}{\lambda_c} = \frac{2\pi}{\lambda_0}\cos\theta_0$$

↳ effective (cutoff) wavelength

$$\Rightarrow \frac{2\pi}{\lambda_0}\cos\theta_0\, b = \frac{2\pi}{\lambda_c} b = n\pi$$

$$\Rightarrow \lambda_c = \frac{2b}{n} \longrightarrow \lambda_c \text{ is "quantized"}$$

↳ $n=$ mode number

The frequencies are also discrete

$$\omega_n = c k_0 = \frac{c \, n\pi}{b\cos\theta_0}$$

The effective wavelength in the propagation direction $\hat{e}_z$ is

$$\lambda_g = \frac{\lambda_0}{\sin\theta_0} \implies k_g = \frac{2\pi}{\lambda_g}$$

guide wavelength

guide propagation constant

Combining the equations for λ:

$$\boxed{\begin{gathered} \frac{1}{\lambda_0^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} \\ k_0^2 = k_c^2 + k_g^2 \end{gathered}}$$

Dispersion relation

Expanding k_g in terms of λ_c

$$k_g = \frac{2\pi}{\lambda_g} = \frac{2\pi\sqrt{\lambda_c^2 - \lambda_0^2}}{\lambda_c \lambda_0} = \frac{n\pi\sqrt{\frac{4b^2}{n^2} - \lambda_0^2}}{\lambda_0 b}$$

↳ fxn of width and mode number

k_g is: <u>real</u> if $\lambda_c > \lambda_0$

↳ propagates

<u>imaginary</u> if $\lambda_c < \lambda_0$

↳ exponentially attenuated

↳ <u>evanescent wave</u>

↳ no phase change (infinite λ)

↳ no transport of energy

$\Rightarrow$ waveguide is high pass filter with a <u>cutoff frequency</u> $\nu_c = \frac{\omega_c}{2\pi} = \frac{c}{\lambda_c} = \frac{nc}{2b}$

Phase and Group Velocities

From the superposition of 2 plane waves, we can extract 2 characteristic velocities:

1) The point of intersection of of the phase fronts with the conducting plane

→ **Phase Velocity:** $u_{ph} = \dfrac{c}{\sin\theta_0} = \dfrac{c\lambda_g}{\lambda_0}$

↳ greater than c (intersection of scissor blades travels faster than blades themselves)

2) $\hat{e}_z$ component of diagonal velocity of wave front

→ **group velocity:** $u_{gr} = c\sin\theta_0$

↳ less than c

↳ rate at which energy is transported down the waveguide (no information travels faster than c!)

The phase and group velocities are related by

$$u_{gr} u_{ph} = c^2$$

Waveguides are <u>dispersive</u>: phase velocity varies with frequency

In dispersive media

$$\boxed{u_{ph} = \frac{\omega}{k_g} \quad \text{and} \quad u_{gr} = \frac{d\omega}{dk_g}}$$

In class problem:

Calculate u_{gr} using $k_0 = \frac{\omega}{c}$ from

$$k_0^2 = k_c^2 + k_g^2$$