# Reflection and Transmission for Normal Incidence

Summarizing the boundary conditions between two media:

$\bar{E}$: tangential component is continuous

$\bar{D}$: normal component is continuous (for case of no surface charge)

$\bar{B}$: normal component is continuous

$\bar{H}$: tangential component is continuous (for case of no surface current)

To simplify our analysis, assume we have a <u>plane E&M wave</u> incident incident upon an infinite boundary. Also, we'll assume

$\sigma = 0 \Rightarrow$ non conducting so no energy loss

$\mu = 1 \Rightarrow$ nonmagnetic

$n_2 > n_1 \Rightarrow$ second medium more optically dense

$\sigma_s = \bar{K}_s = 0 \Rightarrow$ no surface charge or current

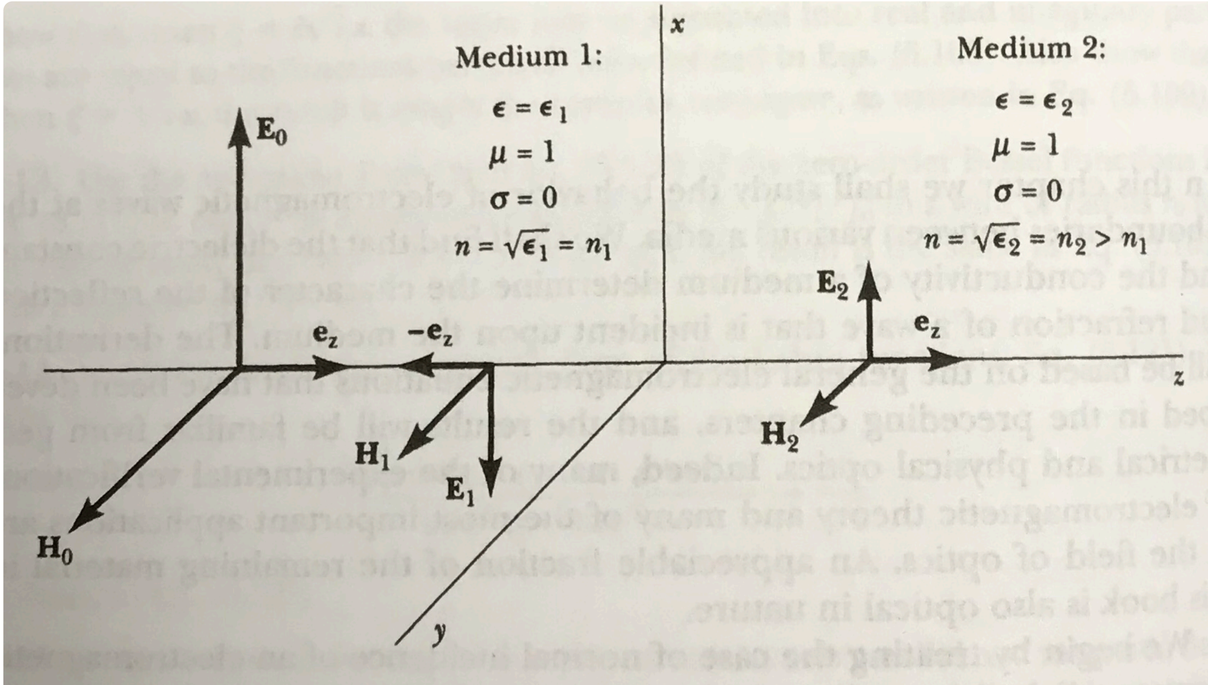



**FIGURE 6-1.** Reflected and transmitted waves for normal incidence.

[H+M]

- The <u>incident electric field $\bar{E}_0$</u> is polarized in the $\hat{e}_x$ direction with $\bar{k}_0$ in the $\hat{e}_z$ direction

- The <u>reflected electric field $\bar{E}_1$</u> suffers a phase change of $\pi$ (we'll see why later) such that we say the polarization is in the $-\hat{e}_x$ direction and $\bar{k}_1$ is in the $-\hat{e}_z$ direction

- The <u>transmitted electric field $\bar{E}_2$</u> is polarized in the $\hat{e}_x$ direction with $\bar{k}_2$ in the $\hat{e}_z$ direction

$$\bar{E}_0 = \hat{e}_x E_0^0 e^{i(k_1 z - wt)}$$

$$\bar{E}_1 = -\hat{e}_x E_1^0 e^{i(-k_1 z - wt)}$$

$$\bar{E}_2 = \hat{e}_x E_2^0 e^{i(k_2 z - wt)}$$

- $E_2^0, E_1^0, E_0^0$ are <u>complex scalar</u> amplitudes
- $\bar{E}_0$ and $\bar{E}_2$ propagate to the right $(+\hat{e}_z)$ and $\bar{E}_1$ propagates to the left $(-\hat{e}_z)$

The wavenumbers are

$$k_1 = \frac{w}{v_1} = \frac{w}{c} n_1 = \frac{w}{c}\sqrt{\varepsilon_1}$$

$$k_2 = \frac{w}{v_2} = \frac{w}{c} n_2 = \frac{w}{c}\sqrt{\varepsilon_2}$$

Note: $w$ doesn't change between the media but $\lambda$ or $k = \frac{2\pi}{\lambda}$ does... why?

At the boundary ($z=0$) we must have

$$\underbrace{\bar{E}_0 + \bar{E}_1 = \bar{E}_2 \Rightarrow E_0^0 e^{-i\omega_1 t} - E_1^0 e^{-i\omega_1 t} = E_2^0 e^{-i\omega_2 t}}$$

Only way to be true at all times is for $\omega_1 = \omega_2 = \omega$

The magnetic field vectors are given by

$$\underbrace{\bar{H} = \bar{B}}_{\mu=1} = n \hat{e}_k \times \bar{E}$$

direction of $\bar{k}$

$$\Rightarrow \quad \left.\begin{aligned} \bar{H}_0 &= \hat{e}_y\, n_1 E_0^0 e^{i(k_1 z - \omega t)} \\ \bar{H}_1 &= \hat{e}_y\, n_1 E_1^0 e^{i(-k_1 z - \omega t)} \\ \bar{H}_2 &= \hat{e}_y\, n_2 E_2^0 e^{i(k_2 z - \omega t)} \end{aligned}\right\} \text{All pointed in \underline{same direction}}$$

From the tangential BCs:

$$E_0^\circ - E_1^\circ = E_2^\circ$$

$$H_0^\circ + H_1^\circ = H_2^\circ$$

$$\Rightarrow \quad n_1 E_0^\circ + n_1 E_1^\circ = n_2 E_2^\circ$$

Solving for $E_1^\circ$ and $E_2^\circ$ in terms of incident $E_0^\circ$

Reflected $\qquad E_1^\circ = \dfrac{n_2 - n_1}{n_2 + n_1} E_0^\circ$

Transmitted $\qquad E_2^\circ = \dfrac{2n_1}{n_2 + n_1} E_0^\circ$

$\Rightarrow$ $\bar{E}_1$, $\bar{E}_2$, $\bar{H}_0$, $\bar{H}_1$, and $\bar{H}_2$ can all be described by $E_0^\circ$, $n_2$, and $n_1$

Note: We defined $\bar{E}_1$ to point in the $-\hat{e}_x$ direction which gives a positive $E_1^\circ$ from the equation above. If we defined the polarization in the $+\hat{e}_x$ direction we would have a negative $E_1^\circ$

→ When the second medium is more optically dense ($n_2 > n_1$) the $\bar{E}$ is reflected with a <u>$\pi$ phase shift</u>

The average energy flux from the incident wave:

$$\langle \bar{S}_0 \rangle = \frac{c}{8\pi} Re\left(\bar{E}_0 \times \bar{H}_0^*\right)$$

↳ H+M changed notation!

The <u>power reflection coefficient R</u> is the relative energy flux reflected at the boundary

$$R \equiv \frac{\langle \bar{S}_1 \rangle \cdot (-\hat{e}_z)}{\langle \bar{S}_0 \rangle \cdot \hat{e}_z} = \frac{Re\left(\bar{E}_1 \times \bar{H}_1^*\right) \cdot (-\hat{e}_z)}{Re\left(\bar{E}_0 \times \bar{H}_0^*\right) \cdot \hat{e}_z}$$

$$\bar{H}_0 = \hat{e}_y\, n_1 E_0^0 e^{i(k_1 z - \omega t)}$$

$$\bar{H}_1 = \hat{e}_y\, n_1 \left(\frac{n_2 - n_1}{n_2 + n_1}\right) E_0^0 e^{i(-k_1 z - \omega t)}$$

$$\bar{E}_0 = \hat{e}_x E_0^0 e^{i(k_1 z - \omega t)}$$

$$\bar{E}_1 = -\hat{e}_x \left(\frac{n_2 - n_1}{n_2 + n_1}\right) E_0^0 e^{i(-k_1 z - \omega t)}$$

$$\Rightarrow \boxed{R = \frac{n_1 \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 \left(E_0^0\right)^2}{n_1 \left(E_0^0\right)^2} = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2}$$

Similarly, the <u>power transmission coefficient $T$</u> is defined by

$$T \equiv \frac{\langle \bar{S}_2 \rangle \cdot \hat{e}_z}{\langle \bar{S}_0 \rangle \cdot \hat{e}_z}$$

<u>In class problem</u>

Show that

$$\boxed{T = \frac{4 n_1 n_2}{(n_2 + n_1)^2}}$$

Because no energy is stored at the interface

$$\boxed{R + T = 1}$$

As an example, for a wave incident on glass ($n \simeq 1.5$) from air ($n \simeq 1$)

$\rightarrow R = 0.04 \qquad T = 0.96$

$\hookrightarrow$ very little reflection

# Oblique Incidence - the Fresnel Equations

Extending our previous example to a case with oblique incidence:

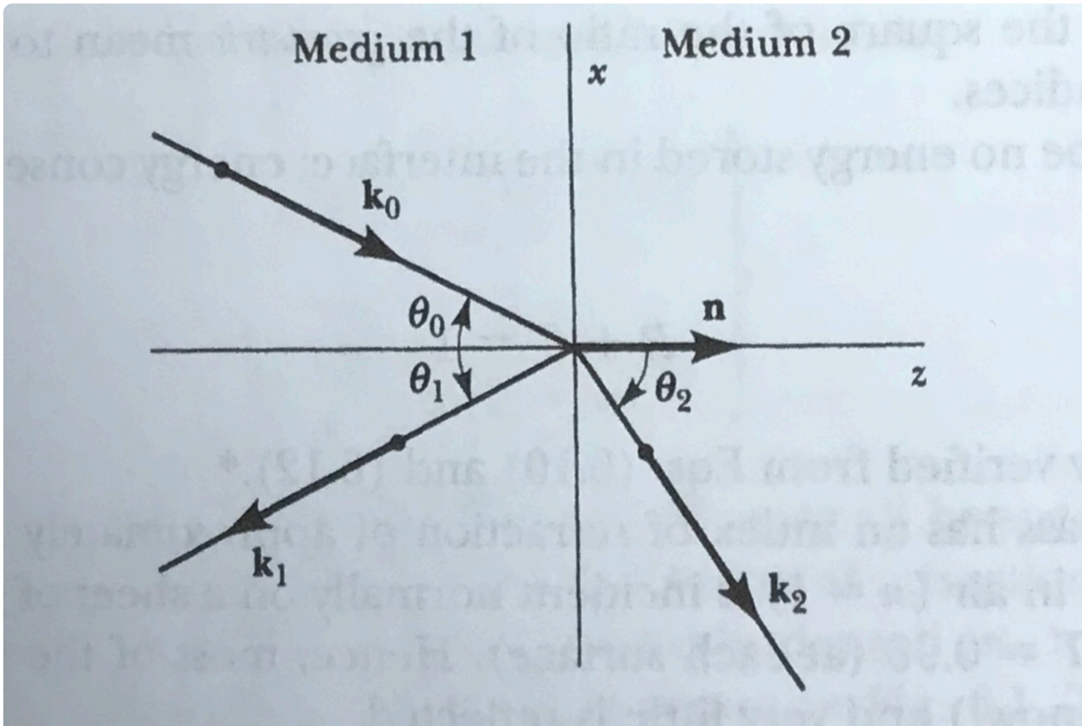



**FIGURE 6-2.** Propagation vectors for oblique incidence.

[H+M]

$\bar{k}_0$ = incident wavevector

$\bar{k}_1$ = reflected wavevector

$\bar{k}_2$ = transmitted wavevector

The $\bar{E}$ and $\bar{H}$ are:

$$\left.\begin{aligned} \bar{E}_0 &= E_0^0 e^{i(\bar{k}_0 \cdot \bar{r} - \omega t)} \\ \bar{H}_0 &= \frac{n_1}{k_0} \bar{k}_0 \times \bar{E}_0 \end{aligned}\right\} \text{incident}$$

$$\left.\begin{aligned} \bar{E}_1 &= E_1^0 e^{i(k_1 \cdot \bar{r} - \omega t)} \\ \bar{H}_1 &= \frac{n_1}{k_1} \bar{k}_1 \times \bar{E}_1 \end{aligned}\right\} \text{reflected}$$

$$\left.\begin{aligned} \bar{E}_2 &= E_2^0 e^{i(\bar{k}_2 \cdot \bar{r} - \omega t)} \\ \bar{H}_2 &= \frac{n_2}{k_2} \bar{k}_2 \times \bar{E}_2 \end{aligned}\right\} \text{transmitted}$$

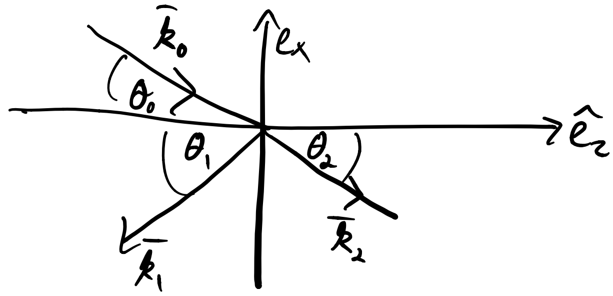
The phase components need to be equal at the boundary so that the spatial and temporal variations at $z=0$ are the same for all fields:

$$i(k_0 \cdot \bar{r} - \omega t) = i(\bar{k}_1 \cdot \bar{r} - \omega t) = i(\bar{k}_2 \cdot \bar{r} - \omega t)$$

$$\Rightarrow \bar{k}_0 \cdot (x\hat{e}_x + \underbrace{z\hat{e}_z}_{z=0 \text{ at boundary}}) = \bar{k}_0 \cdot (x\hat{e}_x) = \bar{k}_1 \cdot (x\hat{e}_x) = \bar{k}_2 \cdot (x\hat{e}_x)$$

This must hold true for arbitrary $x$:

$$\Rightarrow \quad \bar{k}_0 \cdot \hat{e}_x = \bar{k}_1 \cdot \bar{e}_x = \bar{k}_2 \cdot \hat{e}_x$$




$$\Rightarrow k_0 \sin\theta_0 = k_1 \sin\theta_1 = k_2 \sin\theta_2$$

But, $\quad k_0 = k_1 \quad$ from $\quad k = \frac{\omega}{c} n$

(same $n \Rightarrow$ same $\lambda \Rightarrow$ same $k$)

$$\Rightarrow k_0 \sin\theta_0 = k_0 \sin\theta_1 \Rightarrow \boxed{\theta_0 = \theta_1}$$

<u>Angle of incidence equals angle of reflection</u>

Also

$$\left.\frac{k_1}{n_1} = \frac{k_2}{n_2} = \frac{\omega}{c}\right\} \text{constant}$$

And
$$\frac{k_0 \sin\theta_0}{k_2 \sin\theta_2} = 1$$

But $k_0 = \frac{n_1}{n_2} k_2$

$$\Rightarrow \frac{n_1 \sin\theta_0}{n_2 \sin\theta_2} = 1 \Rightarrow n_1 \sin\theta_0 = n_2 \sin\theta_2$$

or, because $\theta_1 = \theta_0$

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

Snell's Law