Laplace's Equation

From Gauss' Law, we derived Poisson's Equation for the scalar potential

$$\bar{\nabla}^2 \Phi = -4\pi\rho$$

In regions with no charge (neutral) such that $\rho = 0$

$$\underbrace{\bar{\nabla}^2 \Phi = 0}_{\text{Laplace's Equation}}$$

- This equation is well studied in both physics and math and provides analytic solutions depending on the geometry.

- The solutions to Laplace's equation are known as <u>Harmonic Functions</u> and have special properties

Properties of Harmonic Functions

1) Superposition: If Φ_1 and Φ_2 are solutions, so is $a\Phi_1 + b\Phi_2$

$a, b =$ constants

2) Uniqueness: If $\Phi(r)$ satisfies Laplace's Equation and the boundary conditions of the enclosing surface, then $\Phi(r) =$ the only solution

$\Rightarrow$ Doesn't matter how you get the solution, if it works then it's the only solution.

3) Smoothing: If Φ is a solution to Laplace's equation, then the <u>maximum</u> or <u>minimum</u> of Φ occurs on the <u>boundaries</u> (not in the interior). Harmonic functions <u>smooth</u> or <u>average</u> in the interior of the region of interest.

Laplace's Equation in Rectangular Coordinates

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Using separation of variables to solve the partial differential equation (PDE), we assume that $\Phi(x,y,z)$ can be written as 3 separate functions, which are each a function of only a single variable:

$$\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\Rightarrow \nabla^2 \Phi = YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2}$$

$$= 0$$

Divide by $\Phi = XYZ$

$$\Rightarrow \underbrace{\frac{1}{X} \frac{\partial^2 X}{\partial x^2}}_{\alpha'^2} + \underbrace{\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}}_{\beta'^2} + \underbrace{\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2}}_{\gamma'^2} = 0$$

$$\alpha'^2 + \beta'^2 + \gamma'^2 = 0$$

constants

Because each term depends on different independent variables, the only way for this to be true in general is if each term is a constant.

Can all of the constants α, β, and γ be real? Can they all be imaginary?

$\rightarrow$ No, they must be a mix, or all equal to zero (trivial)

$$\underbrace{\alpha'^2 + \beta'^2 + \gamma'^2 = 0}_{\text{auxiliary condition}} \Rightarrow \gamma'^2 = -\alpha'^2 - \beta'^2$$

Each variable/constant gives us an independent equation:

$$\frac{\partial^2 X}{\partial x^2} - \alpha'^2 X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} - \beta'^2 Y = 0$$

$$\frac{\partial^2 Z}{\partial z^2} - \gamma'^2 Z = 0$$

Solutions are exponentials

$$\frac{\partial^2 X}{\partial x^2} - \alpha'^2 X = 0$$

$$X(x) = Ae^{\alpha' x} + Be^{-\alpha' x}$$

Substitute to verify

$$\frac{\partial^2 X}{\partial x^2} = A\alpha'^2 e^{\alpha' x} + B\alpha'^2 e^{-\alpha' x} = \alpha'^2 X$$

Solutions for Y and Z are also exponentials

$$X(x) = Ae^{\alpha' x} + Be^{-\alpha' x}$$

$$Y(y) = Ce^{\beta' x} + De^{-\beta' x}$$

$$Z(z) = Ee^{\gamma' z} + Fe^{-\gamma' z}$$

Depending on the boundary conditions we can pre-select the α', β', γ' constants to be real or imaginary.

A general guideline is:

- <u>imaginary constant</u> for symmetric boundary conditions and finite dimensions

  ↳ oscillatory ($e^{\pm i\alpha x}$, $\sin \alpha x$, $\cos \alpha x$)

- <u>real constant</u> for asymmetric boundary conditions of infinite dimensions

  ↳ exponential ($e^{\pm \alpha x}$, $\sinh \alpha x$, $\cosh \alpha x$)

For any boundary condition, we still need

$$\gamma'^2 = -\alpha'^2 - \beta'^2$$

As an example, lets assume

$$\left.\begin{aligned} \alpha' &= i\alpha \\ \beta' &= i\beta \end{aligned}\right\rangle \text{ imaginary}$$

$$\gamma' = \gamma \;\rangle\; \text{real}$$

$$\Rightarrow \quad \left.\begin{array}{l} X(x) = Ae^{i\alpha x} + Be^{-i\alpha x} \\ Y(y) = Ce^{i\beta x} + De^{-i\beta x} \end{array}\right\} \text{oscillatory}$$

$$\left. Z(z) = Ee^{\gamma z} + Fe^{-\gamma z} \right\} \text{exponential}$$

The constants from the auxiliary condition α, β, γ and the corresponding constants A, B, C, D, E, F can have many (infinite) values, and must be determined from the <u>boundary conditions</u>

$$\alpha_r = \alpha_1, \alpha_2, \alpha_3 \ldots$$

$$\beta_s = \beta_1, \beta_2, \beta_3 \ldots$$

$$\gamma_{rs} = \sqrt{\alpha_r^2 + \beta_s^2}$$

The general solution follows

$$\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$= \sum_{r,s=1}^{\infty} \left(A_r e^{i\alpha_r x} + B_r e^{-i\alpha_r x}\right)\left(C_s e^{i\beta_s y} + D_s e^{-i\beta_s y}\right)$$

$$\cdot \left(E_{rs} e^{\gamma_{rs} z} + F_{rs} e^{-\gamma_{rs} z}\right)$$

In shorthand, we can write

$$\Phi(x,y,z) = \sum_{r,s} \underbrace{H_{rs}}_{\text{constant dependent on } r,s} e^{\pm i\alpha_r x} e^{\pm i\beta_s y} e^{\pm \gamma_{rs} z}$$

Example

To show how we determine the constants based on the boundary conditions, lets work through an example.

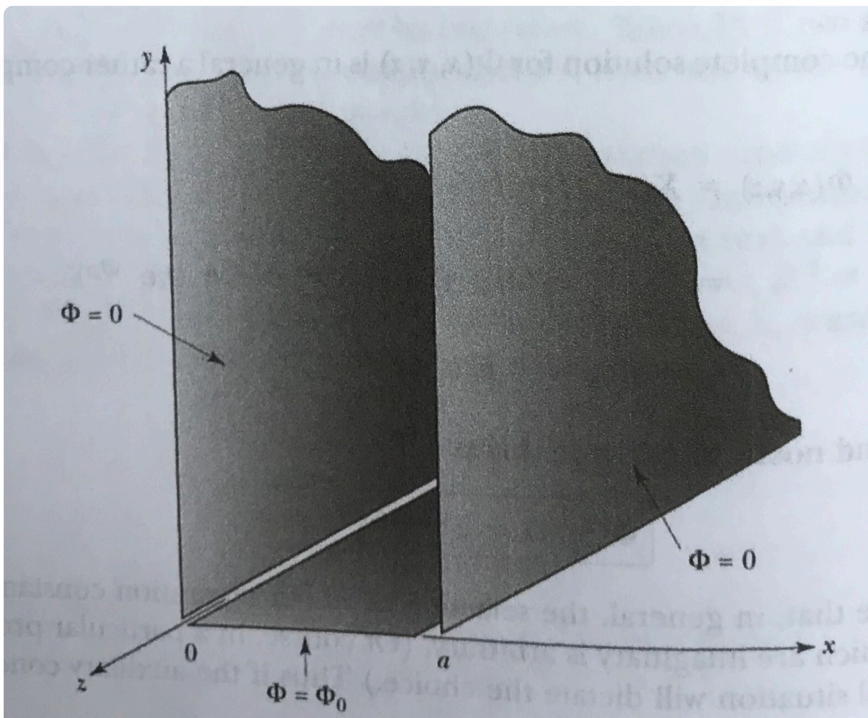


FIGURE 3-1. Semi-infinite channel.

[H+M]

- Two planes parallel to y-axis at $x = 0, a$
- Planes extend to infinity in $+y$ direction starting from $y = 0$
- Two y-planes are grounded ($\Phi = 0$)
- Third plane along x-axis held at potential Φ_0

To start, we note that there is no z-dependence

$$\Rightarrow \frac{\partial^2 Z}{\partial z^2} = \gamma'^2 Z = 0 \Rightarrow \gamma' = 0$$

$$\Rightarrow \gamma'^2 = 0 = -\alpha'^2 - \beta'^2$$

$$\Rightarrow \alpha'^2 = -\beta'^2$$

Because x extent is finite and the BCs are symmetric, we say

$\alpha' = i\alpha$ > oscillatory solution

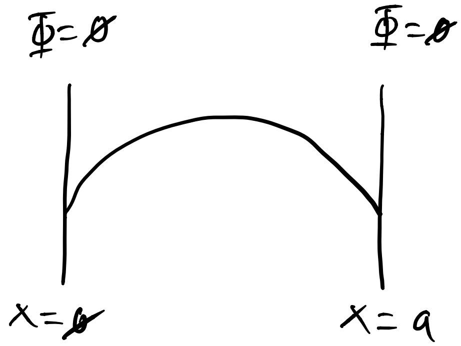
Also, the y extent is infinite, so

$$\beta' = \beta$$

$$\Rightarrow -\alpha^2 = -\beta^2 \Rightarrow \alpha = \beta$$

↳ keep positive root

For the X solution, we have



$$X_r(x) = A_r e^{i\alpha_r x} + B_r e^{-i\alpha_r x}$$

equivalent

Or, in our problem it's easier to deal with sin and cos:

$$e^{i\alpha x} = \cos(\alpha x) + i\sin(\alpha x)$$

$$X_r(x) = A_r \sin(\alpha_r x) + B_r \cos(\alpha_r x)$$

Because we need $\Phi(x=0) = 0$ we'll use the sin component

$$\implies B_r = 0$$

$$\Rightarrow \quad X_r(x) = A_r \sin(\alpha_r x)$$

To satisfy $\Phi(x=a) = 0$ we must have

$$\underbrace{\alpha_r = \frac{r\pi}{a}}_{\sin r\pi = 0 \text{ for integer } r} \qquad \text{for} \qquad r = 1, 2, 3, \ldots$$

$$\Rightarrow \quad X_r(x) = A_r \sin\left(\frac{r\pi}{a} x\right)$$

For the Y component, we have

$$Y_r(y) = C_r e^{\alpha_r y} + D_r e^{-\alpha_r y}$$

But, for positive α_r, $e^{\alpha_r y}$ blows up as $y \to \infty$

$$\Rightarrow Y_r(y) = D_r e^{-\alpha_r y} = D_r e^{-\frac{r\pi}{a} y}$$

Combining, and allowing for all possible solutions

$$\Phi(x,y,z) = \sum_{r=1}^{\infty} \Phi_r(x,y,z) = \sum_{r=1}^{\infty} X_r(x) Y_r(y)$$

$$= \sum_{r=1}^{\infty} A_r \sin\left(\frac{r\pi}{a}x\right) D_r e^{-\frac{r\pi}{a}y}$$

combine

$$= \sum_{r=1}^{\infty} H_r \sin\left(\frac{r\pi}{a}x\right) e^{-\frac{r\pi}{a}y}$$

Also, we need $\Phi(y=0) = \Phi_0$. Evaluating at $y=0$

$$\Phi(y=0) = \sum_{r=1}^{\infty} H_r \sin\left(\frac{r\pi}{a}x\right) = \Phi_0$$

To solve for H_r, we use the orthogonality relation for sin

$$\int_0^a \sin\frac{r\pi x}{a}\sin\frac{s\pi x}{a}\,dx = \frac{a}{2}\delta_{rs}$$

Integrating $\Phi(y=0)$, we get

$$\int_0^a\left[\sum_{r=1}^{\infty} H_r \sin\left(\frac{r\pi}{a}x\right)\right]\sin\left(\frac{r\pi}{a}x\right)dx = \int_0^a \Phi_0 \sin\left(\frac{r\pi}{a}x\right)dx$$

$$\Rightarrow \quad \frac{a}{2}H_r = \Phi_0\frac{2a}{r\pi} \qquad \text{for } \underline{\text{odd}}\ r$$

$$\Rightarrow H_r = \frac{4\Phi_0}{r\pi} \qquad \text{for odd } r$$

$$\Rightarrow \quad \Phi(x,y,z) = \frac{4\Phi_0}{\pi}\sum_{r\,\text{odd}}\frac{1}{r}\sin\left(\frac{r\pi x}{a}\right)e^{-\frac{r\pi y}{a}}$$