Plane Waves in Nonconducting Media

For no free charges or current

$$\bar{\nabla}\cdot\bar{E} = 0$$

$$\bar{\nabla}\cdot\bar{B} = 0$$

$$\bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial\bar{B}}{\partial t} = 0$$

$$\bar{\nabla}\times\bar{B} - \frac{\varepsilon\mu}{c}\frac{\partial E}{\partial t} = 0$$

Where we have assumed that ε and μ are constant over all space and

$$\bar{D} = \varepsilon\bar{E} \qquad \bar{B} = \mu\bar{H}$$

Taking the curl of $\bar{\nabla}\times\bar{E}$:

$$\bar{\nabla}\times\bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial}{\partial t}\bar{\nabla}\times\bar{B} = 0$$

From the vector identity

$$\bar{\nabla}\times\bar{\nabla}\times\bar{E} = \bar{\nabla}(\bar{\nabla}\cdot\bar{E}) - \bar{\nabla}^2 E$$

$$\Rightarrow \quad \underbrace{\bar{\nabla}(\bar{\nabla}\cdot\bar{E})}_{=0,\ \text{no } \rho} - \bar{\nabla}^2\bar{E} + \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\bar{\nabla}\times\bar{B}}_{\frac{\varepsilon\mu}{c}\frac{\partial\bar{E}}{\partial t}} = 0$$

$$\Rightarrow \quad \boxed{\bar{\nabla}^2\bar{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\bar{E}}{\partial t^2} = 0}$$

Wave equation with $\rho = \bar{J} = 0$

Following a similar procedure for $\bar{\nabla}\times\bar{B}$:

$$\boxed{\bar{\nabla}^2\bar{B} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\bar{B}}{\partial t^2} = 0}$$

The velocity of propagation is

$$V = \frac{c}{\sqrt{\varepsilon\mu}}$$

In free space, $\mu = \varepsilon = 1$

$$\Rightarrow \quad V = c \simeq 3\times 10^{10}\ \text{cm/s}$$

For typical <u>media</u>, $\mu \simeq 1$ and $\varepsilon > 1$

$$\Rightarrow v < c$$

In a plasma, $\varepsilon < 1$. What does this mean?

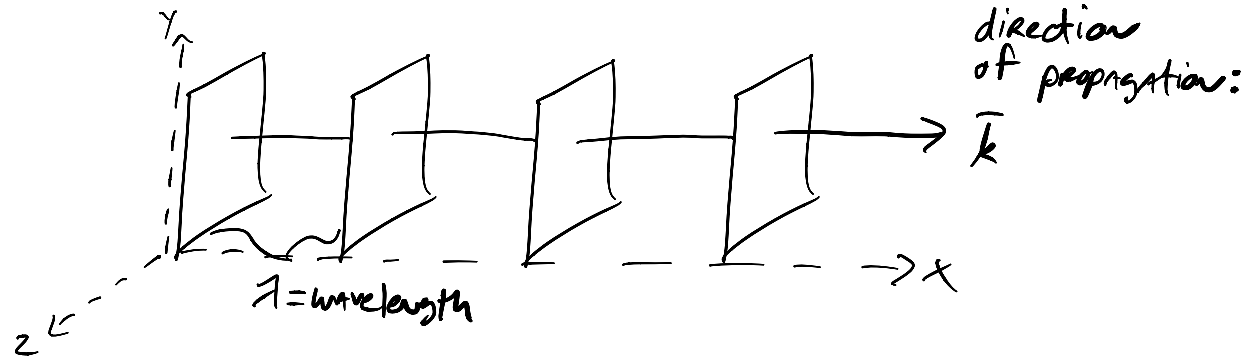
The vector form of the wave equations can be broken down to each rectangular coordinate and treated as a scalar:

$$\nabla^2 \psi - \frac{\varepsilon \mu}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

$$\psi = E_x, E_y, E_z, B_x, B_y, B_z$$

Plane wave solutions to the wave equation:

- Amplitutude is a function of $\perp$ distance to origin (and time)
- planes are infinite in extent and travel in one direction



Suppose the waves are traveling in the $\hat{e}_x$ direction

$$\left.\begin{aligned} &\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \\ &v = \frac{c}{\sqrt{\epsilon\mu}} \end{aligned}\right\} \text{ no } x, y \text{ dependence}$$

General solutions to the 1-D wave equation is a combination of arbitrary functions of $x + vt$ and $x - vt$:

$$\psi(x,t) = \underbrace{f(x + vt)}_{\text{propagate } -\hat{e}_x} + \underbrace{g(x - vt)}_{\text{propagate } +\hat{e}_x}$$

Test $f(x+vt)$ solution: say $u = x + vt$ and use the chain rule

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}\underbrace{\left(\frac{\partial u}{\partial x}\right)^2}_{1} + \frac{\partial f}{\partial u}\underbrace{\frac{\partial^2 u}{\partial x^2}}_{0} = \frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial u^2}\underbrace{\left(\frac{\partial u}{\partial t}\right)^2}_{v^2} + \frac{\partial f}{\partial u}\underbrace{\frac{\partial^2 u}{\partial t^2}}_{0} = v^2\frac{\partial^2 f}{\partial u^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial u^2} - \frac{v^2}{v^2}\frac{\partial^2 f}{\partial u^2} = 0$$

$\hookrightarrow$ f = arbitrary function

If we consider sinusoidal waveforms and limit the solution to $+\hat{e}_x$ propagation:

Monochromatic wave $\rangle$ $$\Psi(x,t) = \Psi_0 \, e^{ik(x-vt)}$$

Ψ_0 = Amplitude of oscillation

k = wavenumber

The wavenumber can be described by

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{\omega}{v} = \frac{\omega\sqrt{\varepsilon\mu}}{c} = n\,\frac{\omega}{c}}$$

λ = wavelength

$\omega = 2\pi f$ = Angular frequency

n = index of refraction

The index of refraction gives the ratio of c to v:

$$\boxed{n = \frac{c}{v} = \sqrt{\varepsilon\mu}}$$

The monochromatic wave solution can be rewritten as:

$$\psi(x,t) = \psi_0 \, e^{i(kx - \omega t)}$$

Extending to multiple dimensions, we define a wave vector (propagation vector) $\bar{k}$ that points in the direction of propagation and has a magnitude k. The wave solution becomes:

$$\boxed{\psi(\bar{r},t) = \psi_0 \, e^{i(\bar{k}\cdot\bar{r} - \omega t)}}$$

If a plane wavefront has a $\perp$ distance ℓ_2 to the origin:

$$\bar{k}\cdot\bar{r} = k\ell_2 \Rightarrow \psi(\bar{r},t) = \psi_0 \, e^{i(k\ell_2 - \omega t)}$$

$\hookrightarrow$ reoriented

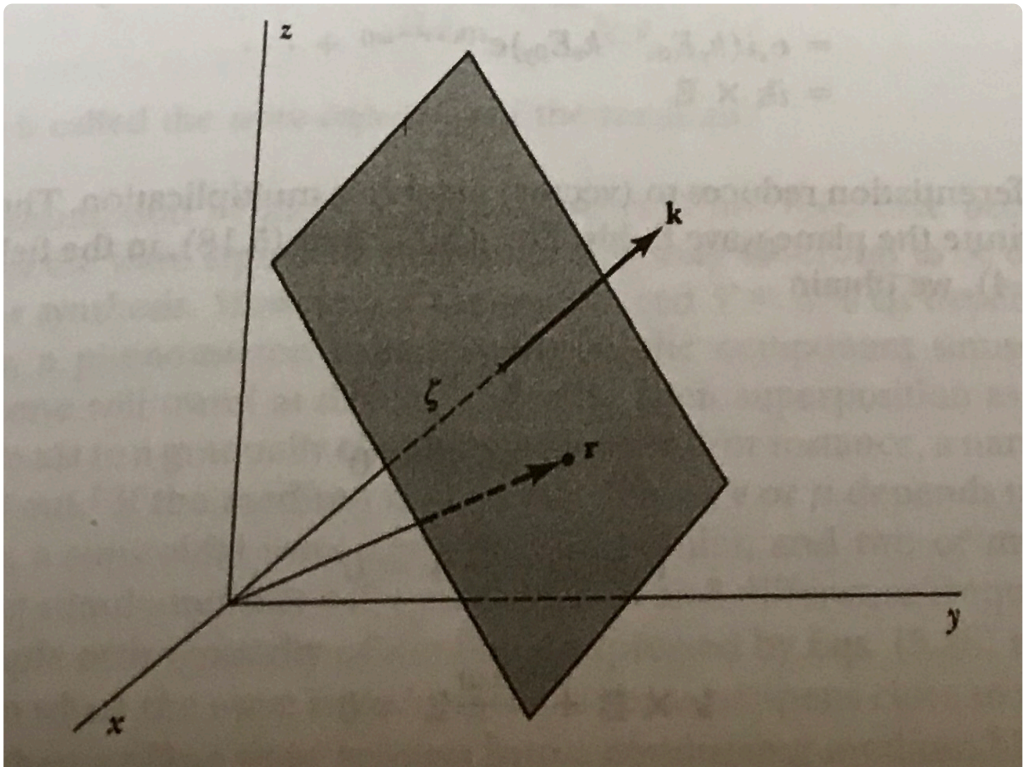


FIGURE 5-1. Plane wavefront in arbitrary direction.

Plane, monochromatic, electromagnetic waves are described by

$$\bar{E}(\bar{r},t) = \bar{E}_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

$$\bar{B}(\bar{r},t) = \bar{B}_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

$\bar{E}_0$ and $\bar{B}_0$ are constant amplitude vectors.

Analyzing the solution, we can simplify certain vector operations:

<u>Work through in class</u>

Show

$$\frac{\partial \bar{E}}{\partial t} = -i\omega \bar{E}$$

$$\bar{\nabla} \cdot \bar{E} = i\bar{k} \cdot \bar{E}$$

$$\bar{\nabla} \times \bar{E} = i\bar{k} \times \bar{E}$$

From the relationships above, Maxwell's equations become

$$\left.\begin{aligned} \bar{k} \cdot \bar{E} &= 0 \\ \bar{k} \cdot \bar{B} &= 0 \end{aligned}\right\} \bar{E}, \bar{B} \text{ are } \perp \text{ to } \bar{k}$$

(<u>transverse</u>)

$$\left.\begin{aligned} \bar{k} \times \bar{E} - \frac{\omega}{c}\bar{B} &= 0 \\ \bar{k} \times \bar{B} + \frac{\epsilon\mu\omega}{c}\bar{E} &= 0 \end{aligned}\right\} \bar{k}, \bar{E}, \bar{B} \text{ are all } \perp$$

↳ TEM wave

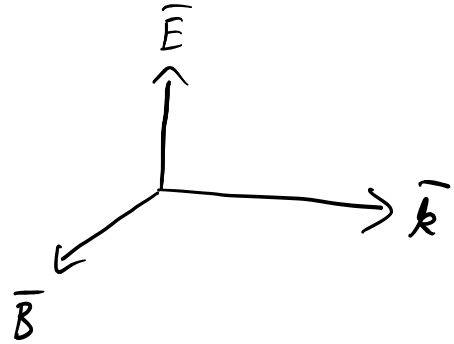
(transverse electromagnetic)

In terms of the index of refraction

$$\bar{B} = n\,\hat{e}_K \times \bar{E}$$

$$\bar{E} = -\frac{1}{n}\,\hat{e}_K \times \bar{B}$$

$\hat{e}_K$ → direction of $\bar{k}$



- The Poynting vector $\bar{S} = \frac{c}{4\pi}\left(\bar{E} \times \frac{\bar{B}}{\mu}\right)$ is parallel to $\bar{k}$.

- For a traveling wave, $\bar{E}$ and $\bar{B}$ are in phase

The ratio of field amplitudes in a linear medium is

$$\frac{|B_0|}{|E_0|} = n = \sqrt{\varepsilon\mu}$$

In free space, $n=1 \Rightarrow |B_0| = |E_0|$. Note, that this is true in Gaussian, but not SI

(In SI $\frac{|B_0|}{|E_0|} = \frac{n}{c}$)

The ratio in terms of the magnetic intensity (which is in the Poynting vector):

$$\frac{|E_0|}{|H_0|} = \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

η = wave impedence

- Waveforms do not have to be monochromatic because we can use Fourier synthesis (combination of plane waves)

- If ε or μ is a function of frequency

$$\underbrace{n(\omega) = \sqrt{\varepsilon(\omega)\mu(\omega)} \Rightarrow V(\omega) = \frac{c}{n(\omega)}}_{\text{dispersive media}}$$

$\hookrightarrow$ waveform will change

- Orthogonality of $\bar{E}, \bar{B}$, and $\bar{k}$ breaks down when
  - → wave is close to localized source (not plane)
  - → wave refracts into conducting medium
  - → wave is in anisotropic and inhomogeneous media