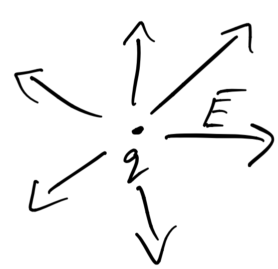
Point charges and the Delta-Function

Divergence theorem assumes that the vector field $\bar{A}$ and its first derivative are continuous (Arfken + Weber)

$$\int_V \nabla \cdot \bar{A} \, dv = \oint_S \bar{A} \cdot \hat{n} \, da$$

For a point charge, the electric field is

$$\bar{E} = \frac{q}{r^2} \hat{e}_r$$



What happens as $r \to 0$?

Is $\frac{\partial \bar{E}}{\partial r}$ continuous at $r = 0$?

what is $\bar{\nabla}\cdot\bar{E}$ for a point charge?

$$\bar{\nabla}\cdot\bar{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{q}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(q) = \frac{0}{r^2}$$

For $r \neq 0$, $\bar{\nabla}\cdot\bar{E} = 0$.

what happens at $r=0$?

From the divergence theorem, we get

$$\underbrace{\int_V}_{\text{Includes origin}} \bar{\nabla}\cdot\bar{E}\, dv = \oint_S \bar{E}\cdot d\bar{a}$$

Gauss' Law tells us

$$\oint \bar{E}\cdot d\bar{a} = 4\pi q_{enc}$$

$$\Rightarrow \int_V \bar{\nabla} \cdot \bar{E} \, dV = \oint \bar{E} \cdot d\bar{a} = 4\pi q_{enc}$$

For a point charge, $q_{enc} = q$ (non-zero)
but $\bar{\nabla} \cdot \bar{E} = 0$ for $r \neq 0$

→ How does this work?

$\bar{\nabla} \cdot \bar{E}$ is singular at the origin and zero everywhere else, and when integrated it gives a finite value

→ Answer = <u>Dirac delta function</u>

Dirac delta-function

$$\delta(x - x') = 0 \quad \text{for} \quad x \neq x'$$

$$\int_{-\infty}^{\infty} \delta(x-x')\,dx = 1$$

$\Rightarrow$ δ is zero everywhere except $x = x'$ where it is infinite

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x')\,dx = f(x')$$

In 3-D:

$$\int_{V} \delta(\bar{r}-\bar{r}')\,dv = 1$$

$$\int_{V} f(\bar{r})\,\delta(\bar{r}-\bar{r}')\,dv = f(\bar{r}')$$

Treating our point charge as a delta-function

$$\underbrace{\rho}_{\text{charge density}} = q\,\underbrace{\delta(\bar{r})}_{\text{located at } \bar{r}=0}$$

From Gauss' Law

$$\bar{\nabla}\cdot\bar{E} = 4\pi\rho$$

$$\Rightarrow \int_V \bar{\nabla}\cdot\bar{E}\,dv = \int_V 4\pi\rho\,dv$$

$$\Rightarrow \int_V \bar{\nabla}\cdot\frac{q\hat{e}_r}{r^2}\,dv = \int_V 4\pi q\,\delta(\bar{r})\,dv$$

divide by q

$$\Rightarrow \int_V \bar{\nabla}\cdot\frac{\hat{e}_r}{r^2}\,dv = \int_V 4\pi\,\delta(\bar{r})\,dv$$

This holds over an arbitrary volume V, so we can say

$$\bar{\nabla} \cdot \frac{\hat{e_r}}{r^2} = 4\pi\,\delta(\bar{r})$$

Alternatively, in terms of the Laplacian (which comes from the same derivation but starting with $\bar{E} = -\bar{\nabla}\Phi$)

$$\bar{\nabla}^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\bar{r})$$

- So, even though $\frac{\partial E}{\partial r}$ is discontinuous, we're able to use the Divergence theorem and integrate over the singularity because we get a finite result from the Dirac delta-function.

- There are methods for integrating over discontinuities (see Lebesgue Integration for example) but they are outside the scope of this course.

- For this class, the divergence theorem holds for point charges even though there's a discontinuity