Energy in the Electromagnetic Field

From the two Maxwell curl equations:

$$\frac{1}{c}\frac{\partial \bar{B}}{\partial t} = -\bar{\nabla}\times\bar{E}$$

$$\frac{1}{c}\frac{\partial \bar{D}}{\partial t} = \bar{\nabla}\times\bar{H} - \frac{4\pi}{c}\bar{J}_f$$

Take the dot product with respect to $\bar{H}$ and $\bar{E}$ and combine the equations:

$$\frac{1}{c}\left(\bar{H}\cdot\frac{\partial \bar{B}}{\partial t} + \bar{E}\cdot\frac{\partial \bar{D}}{\partial t}\right) = -\frac{4\pi}{c}\bar{E}\cdot\bar{J}_f - \left(\bar{H}\cdot\bar{\nabla}\times\bar{E} - \bar{E}\cdot\bar{\nabla}\times\bar{H}\right)$$

From the vector identity

$$\bar{B}\cdot\bar{\nabla}\times\bar{A} - \bar{A}\cdot\bar{\nabla}\times\bar{B} = \bar{\nabla}\cdot(\bar{A}\times\bar{B})$$

The final term becomes

$$\bar{\nabla}\cdot(\bar{E}\times\bar{H}) = \bar{H}\cdot\bar{\nabla}\times\bar{E} - \bar{E}\cdot\bar{\nabla}\times\bar{H}$$

From this, we define the <u>Poynting vector</u> $\bar{S}$ as

$$\boxed{\bar{S} \equiv \frac{c}{4\pi} \bar{E} \times \bar{H}}$$

For simplicity, let's assume the medium is linear so that ε and μ can be used.

$$\bar{D} = \varepsilon \bar{E} \quad , \quad \bar{B} = \mu \bar{H}$$

Then, we get <u>Poynting's Theorem</u>:

$$\boxed{\frac{\partial}{\partial t}\left[\frac{1}{8\pi}\left(\varepsilon E^2 + \frac{B^2}{\mu} \right) \right] + \bar{\nabla} \cdot \bar{S} + \bar{E} \cdot \bar{J}_f = 0}$$

Let's integrate over a volume V and apply the divergence theorem

$$\frac{d}{dt} \int_V \frac{1}{8\pi}\left(\varepsilon E^2 + \frac{B^2}{\mu} \right) dv + \int_V \bar{E} \cdot \bar{J}_f \, dv + \oint_S \bar{S} \cdot d\bar{a} = 0$$

Poynting's theorem is an <u>energy conservation</u> equation. The different terms are

$$\mathcal{E} = \frac{1}{8\pi}\left(\bar{E}\cdot\bar{D} + \bar{H}\cdot\bar{B}\right) \rightarrow \frac{1}{8\pi}\left(\varepsilon E^2 + \frac{B^2}{\mu}\right)$$

$\mathcal{E}$ = energy density of the E+M fields (erg/cm^3)

$$\Rightarrow \frac{d}{dt}\int_V \mathcal{E}\, dv = \text{time rate of change of E+M energy in volume } V$$

The $\bar{E}\cdot\bar{J}_f$ term gives the work done by the E+M field per unit time on the charges making $\bar{J}_f$:

$$\int_V \bar{E}\cdot\bar{J}_f\, dv \rightarrow \sum_\alpha \underbrace{q_\alpha}_{\text{charge}} \overbrace{\bar{u}_\alpha}^{\text{velocity}} \cdot \underbrace{\bar{E}_\alpha}_{\bar{E} \text{ at } q_\alpha}$$

The Force on the αth particle is

$$\bar{F}_\alpha = q_\alpha \left(\bar{E}_\alpha + \frac{1}{c} \bar{u}_\alpha \times \bar{B}_\alpha \right)$$

The work per unit time is

$$\frac{dW_\alpha}{dt} = \bar{F}_\alpha \cdot \bar{u}_\alpha = q_\alpha \bar{u}_\alpha \cdot \bar{E}_\alpha$$

Note: the magnetic field does not attribute to the work $\rightarrow \left(\bar{u}_\alpha \times \bar{B}_\alpha \right) \cdot \bar{u}_\alpha = 0$

- In a vacuum, the $\bar{E}$ field increases the kinetic energy of the particles

- In a conductor, the work is shared with the medium through collisions, producing the <u>I^2R Joule Heating term</u>

- In both cases, the $\bar{E} \cdot \bar{J}$ term is a <u>loss of energy</u> from the E+M field

The $\oint_S \bar{S}\cdot d\bar{a}$ term is the rate at which energy is transported by the E+M field outward through a surface encompassing volume V

$\Longrightarrow$ $\bar{S}$ = power per cross-sectional area transported by E+M field

$\hookrightarrow$ direction gives direction of energy flow

Combining all terms

$$\underbrace{\frac{d}{dt}\int_V \frac{1}{8\pi}\left(\varepsilon E^2 + \frac{B^2}{\mu}\right)dv}_{\text{energy stored in fields}} + \underbrace{\int_V \bar{E}\cdot\bar{J}_f\, dv}_{\text{energy loss due to matter}} + \underbrace{\oint_S \bar{S}\cdot d\bar{a}}_{\text{energy flow transported by fields}} = 0$$

$\rightarrow$ E+M fields possess and <u>transport energy</u>

If no mobile charges are present:

$$\bar{\nabla} \cdot \bar{S} = -\frac{\partial \mathcal{E}}{\partial t}$$

Compare to the charge conservation equation

$$\bar{\nabla} \cdot \bar{J} = -\frac{\partial \rho}{\partial t}$$

E+M fields carry energy... do they also have momentum?

Electrostatic Energy

Consider charged particles at rest in volume V

$$\bar{J} = \emptyset \Rightarrow \bar{B} = \emptyset$$

The electric potential energy is

$$U_{es} = \frac{1}{8\pi} \int_V \bar{D} \cdot \bar{E} \, dV$$

From $\bar{E} = -\bar{\nabla} \Phi$ we get

$$U_{es} = \frac{-1}{8\pi} \int_V \bar{D} \cdot \bar{\nabla} \Phi \, dV$$

Using the vector identity

$$\bar{D} \cdot \bar{\nabla} \Phi = \bar{\nabla} \cdot (\Phi \bar{D}) - \Phi \bar{\nabla} \cdot \bar{D}$$

$$\Rightarrow U_{es} = \frac{1}{8\pi} \int_V \Phi \bar{\nabla} \cdot \bar{D} \, dV - \frac{1}{8\pi} \int_V \bar{\nabla} \cdot (\Phi \bar{D}) \, dV$$

From the divergence theorem:

$$\int_V \bar{\nabla} \cdot (\Phi \bar{D}) dV = \oint_S \Phi \bar{D} \cdot d\bar{a} \sim \oint_S \frac{q}{r} \frac{q}{r^2} r^2 \sin\theta d\theta d\phi$$

$$\sim \oint_S \frac{q^2}{r} \sin\theta d\theta d\phi$$

$\hookrightarrow \frac{1}{r} \to 0$ as $r \to \infty$

For V as All-space, the $\frac{1}{r}$ goes to zero:

$$U_{es} = \frac{1}{8\pi} \int_{\text{All-space}} \Phi \bar{\nabla} \cdot \bar{D} \, dV$$

But $\bar{\nabla} \cdot \bar{D} = 4\pi \rho_f$

$$\Rightarrow \boxed{U_{es} = \frac{1}{2} \int_{\text{All-space}} \Phi \rho_f \, dV}$$

When conductors are present, the surface charge will contribute to U_{es}:

$$U_{es} = \frac{1}{2}\underbrace{\int \Phi \rho_f \, dV}_{\substack{\text{All space} \\ \text{outside of} \\ \text{conductors}}} + \frac{1}{2}\underbrace{\int \Phi (\rho_s)_f \, da}_{\substack{\text{surface area} \\ \text{of all conductors}}}$$

This equation is for free charges, ρ_f. What happens when dielectrics are present?

$\rightarrow$ Dielectrics modify $\Phi(r) = -\int \bar{E} \cdot d\bar{l}$ in material

Our two forms of U_{es} <u>localize</u> energy differently

$$U_{es} = \frac{1}{8\pi}\int_V \bar{D} \cdot \bar{E} \, dV \quad \Big\rangle \text{ energy spread throughout } V$$

$$= \frac{1}{2}\underbrace{\int \Phi \rho_f \, dV}_{\text{All space}} \quad \Big\rangle \text{ energy localized at } \rho$$

Which form of energy localization is correct?

- While both forms give the correct total energy, the $\bar{E}$ field carries energy, so the $\bar{D}\cdot\bar{E}$ term is the correct <u>localization</u>

For <u>point charges</u>

$$U_{es} = \frac{1}{2}\sum_{\alpha} q_\alpha \bar{\Phi}_\alpha$$

This potential energy is the energy required to bring a collection of point charges from infinity ($\bar{\Phi}_\alpha = 0$) to their final position. The $\frac{1}{2}$ makes sure that we don't double count each pair of charges contributing to $q_\alpha \bar{\Phi}_\alpha$.

What is the self-energy for a point charge?

$$U_{self} = q\,\Phi_{self} = q\,\frac{q}{r_{self}}$$

But, $r_{self} = \emptyset \implies U_{self} = \infty$

↳ Takes an infinite amount of work to pack charge in a $\emptyset$ volume

To account for this, we exclude the self-energy.

A similar approach can be taken to find the magnetic energy for constant (slowly varying) currents:

$$U_{ms} = \frac{1}{8\pi}\int_V \bar{H}\cdot\bar{B}\,dV$$

Maxwell Stress Tensor

The force on a distribution can be written as:

$$\bar{F} = \int_V \left(\rho \bar{E} + \frac{1}{c} \bar{J} \times \bar{B} \right) dV$$

Using Gauss' law and the Ampère-Maxwell law, we can rewrite the force strictly in terms of the fields:

$$\bar{F} = \int_V \frac{1}{4\pi} \Big[\underbrace{(\bar{\nabla} \cdot \bar{E})}_{4\pi\rho} \bar{E} + \underbrace{\left(\bar{\nabla} \times \bar{B} - \frac{1}{c} \frac{\partial \bar{E}}{\partial t} \right)}_{\frac{4\pi}{c} \bar{J}} \times \bar{B} \Big] dV$$

With some "now obvious foresight" in combination with the other Maxwell equations, we obtain:

$$\bar{F} + \frac{d}{dt} \int_V \frac{1}{4\pi c} (\bar{E} \times \bar{B}) dV$$

$$= \int_V \underbrace{\frac{1}{4\pi} \Big[(\bar{\nabla} \cdot \bar{E}) \bar{E} - \bar{E} \times (\bar{\nabla} \times \bar{E}) + (\bar{\nabla} \cdot \bar{B}) \bar{B} - \bar{B} \times (\bar{\nabla} \times \bar{B}) \Big]}_{= \bar{\nabla} \cdot \bar{\bar{T}}} dV$$

$\bigg\}\ \bar{\bar{T}} =$ Maxwell Stress Tensor

Using an extension of the divergence theorem applied to symmetric tensors:

$$\Rightarrow \quad \bar{F} + \frac{d}{dt}\int_V \frac{1}{4\pi c}(\bar{E}\times\bar{B})\,dV = \oint_S \bar{\bar{T}}\cdot\hat{n}\,da$$

$\rightarrow$ $\bar{\bar{T}}$ gives a <u>force per unit area</u> (stress, pressure)

$\rightarrow$ $\bar{\bar{T}}$ is a second-rank tensor (matrix)

because it operates on $\hat{n}\,da$

give vector $d\bar{F}$

$$d\bar{F} = \bar{\bar{T}}\cdot\hat{n}\,da$$

$$\Rightarrow \quad dF_i = \sum_{j=1}^{3} T_{ij}\, n_j\, da$$

Diagonal elements = <u>normal</u> forces

Off-diagonal elements = <u>shear</u> forces

In terms of momentum, $\bar{p}$:

$$\underbrace{\bar{F}}_{\frac{d}{dt}\bar{p}_{\text{matter}}} + \underbrace{\frac{d}{dt}\int_V \frac{1}{4\pi c}(\bar{E}\times\bar{B})\,dV}_{\bar{p}_{\text{EM field}}} = \underbrace{\oint_S \bar{\bar{T}}\cdot\hat{n}\,da}_{\text{stress forces}}$$

$$\Rightarrow \frac{d}{dt}\left(\bar{p}_{\text{matter}} + \bar{p}_{\text{EM-field}}\right) = \oint_S \bar{\bar{T}}\cdot\hat{n}\,da$$

↳ The E+M stresses integrated over a surface equals the time rate of change of the <u>total momentum</u>.

The momentum of the EM field is related to the Poynting vector:

$$\bar{p}_{\text{EM-field}} = \int_V \bar{g}_{\text{field}}\,dV = \int_V \frac{1}{4\pi c}(\bar{E}\times\bar{B})\,dV$$

↳ momentum density of field

$$\Rightarrow \quad \bar{g}_{field} = \frac{1}{4\pi c}(\bar{E} \times \bar{B}) = \frac{\bar{S}}{c^2}$$