

Electric Fields and Dielectric Materials

The primary vectors used in E+M are

\vec{E} = electric field (statvolt/cm)

\vec{D} = displacement (statvolt/cm)

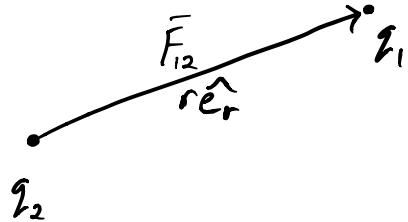
\vec{B} = magnetic field (gauss)

\vec{H} = magnetic intensity (oersted)

* note: some authors use different names
for \vec{B} and \vec{H}

- gauss and oersted are the same
units, but the naming is historical

Coulomb's Law



$$\vec{F}_{12} = \frac{q_1 q_2}{r^2} \hat{e}_r$$

\vec{F}_{12} = force on q_1 due to q_2

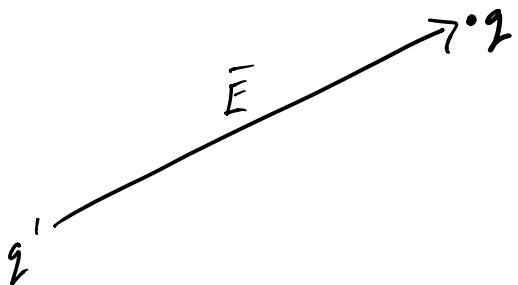
q = point charge in esu

r = distance between charges

\hat{e}_r = unit vector from q_2 to q_1

- If the charges have the same sign, they repel.
- If the signs are opposite, they attract.

The concept of an electric field comes from the force on a test charge:



$$\bar{F} = \frac{q q'}{r^2} \hat{e}_r = q \bar{E}$$

$$\Rightarrow \bar{E} = \frac{q'}{r^2} \hat{e}_r$$

where

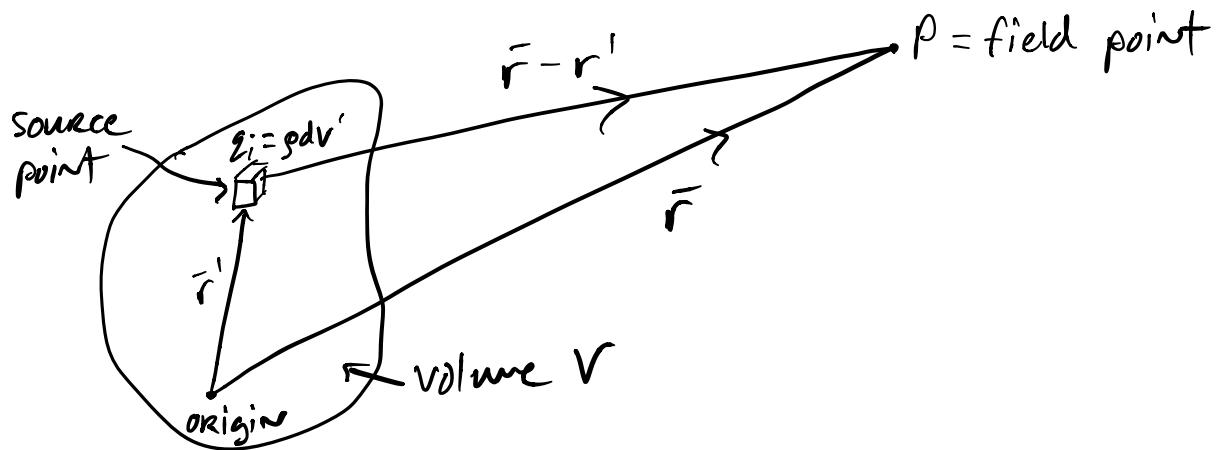
q = test charge

\bar{E} = electric field due to source charge q'

* \bar{E} fields are linear, so they can be added together for different source terms q' to give you a combined electric field (principle of superposition)

$$\bar{E}_{\text{total}} = \bar{E}_1 + \bar{E}_2 + \dots$$

More formally, for a distribution of source charges:



Treating the volume V as a collection of point charges, the electric field at position \vec{r} is:

$$\bar{E}(\vec{r}) = \sum_i \frac{q(\vec{r}_i')}{|\vec{r} - \vec{r}_i'|^2} \hat{e}_{\vec{r} - \vec{r}_i'}$$

$q(\vec{r}_i')$ = source charge at \vec{r}_i'

$\hat{e}_{\vec{r} - \vec{r}_i'}$ = unit vector from source charge to position of interest

Electric Flux

The integral of the normal component of \vec{E} over a surface Area ΔS is the flux of E :

$$\Phi_E = \int_{\Delta S} \vec{E} \cdot d\vec{a}$$

Φ_E = flux of E through Δs

$$\begin{aligned}\hookrightarrow \text{units} &= (\text{statvolt/cm}) \cdot \text{cm}^2 \\ &= \text{statvolt} \cdot \text{cm}\end{aligned}$$

$d\vec{a} = \hat{n} da$ = differential area at surface
with \hat{n} normal to the
surface

$$\vec{E} \cdot d\vec{a} = E d\vec{a} \cos \theta$$

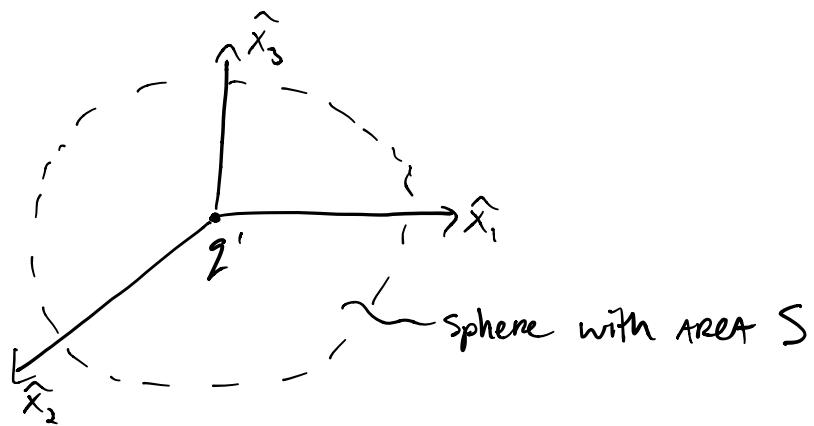
$$\Rightarrow \phi_E = \int_S E d\vec{a} \cos \theta = E \Delta S \cos \theta$$

If $\vec{E} \parallel d\vec{a} \Rightarrow \cos \theta = 1 \Rightarrow \phi_E = \text{maximum}$

$\vec{E} \perp d\vec{a} \Rightarrow \cos \theta = 0 \Rightarrow \phi_E = 0$

↙
no flux

Now, suppose we integrate a point charge over a closed surface S



$$\phi_{E\text{-total}} = \oint_S \bar{E} \cdot d\bar{a} = \oint_S \left(\frac{q'}{r^2} \hat{e}_r \right) \cdot (r^2 \sin\theta d\theta d\phi \hat{e}_r)$$

$$= 4\pi q'$$

This simple example can be generalized
for an arbitrary Gaussian surface S
and charge distribution to give

Gauss' Law

$$\phi_{E\text{-total}} = \oint_S \bar{E} \cdot d\bar{a} = 4\pi q_{\text{enclosed}}$$

ϕ_{total} = total electric flux through surface S

S = closed surface (sphere, cylinder, box, etc)

q_{enclosed} = total charge enclosed

\Rightarrow The total electric flux through a closed surface only depends on the charge enclosed.

This integral equation can be converted to a differential equation by use of the divergence theorem:

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V \nabla \cdot \vec{E} dv$$

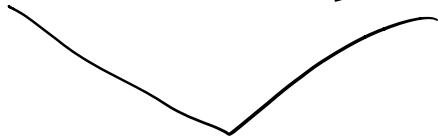
The total charge in the volume V is the integral of the charge density ρ

$$Q_{\text{total}} = \int_V \rho dv$$

$$\Rightarrow \int_V \nabla \cdot \vec{E} dv = 4\pi \int_V \rho dv$$

Because this relation holds for any volume, the integrands must be equal

$$\nabla \cdot \bar{E} = 4\pi\rho$$



Gauss' Law at a point

$$\Rightarrow \nabla \cdot \bar{E} = 0 \text{ if no charge is present}$$

Scalar Potential

The electric field is constant, which means

$$\bar{E}(r) = E(r) \hat{e}_r$$

\Rightarrow electric field points radially out/in and only depends on distance r

Lets suppose that the electric field magnitude can be described as the derivative of some scalar function ϕ

$$\Rightarrow \bar{E} = -\frac{\partial \phi}{\partial r} \hat{e}_r = -\bar{\nabla} \phi$$

The electric field is minus the gradient of the scalar potential

$$\bar{E} = -\bar{\nabla} \phi$$

Because \bar{E} is central and can be described as the gradient of a scalar, it is also conservative:

$$\oint \bar{E} \cdot d\bar{l} = 0$$

Independent of path. Only depends on initial and final positions.

Additionally, because $\bar{E} = -\bar{\nabla} \phi$

$$\bar{\nabla} \times \bar{E} = \bar{\nabla} \times (-\bar{\nabla} \phi) = 0$$

CURL of gradient = 0

Note: this is true in the static scenario when $\frac{\partial \bar{B}}{\partial t} = 0$ because \bar{E} is conservative

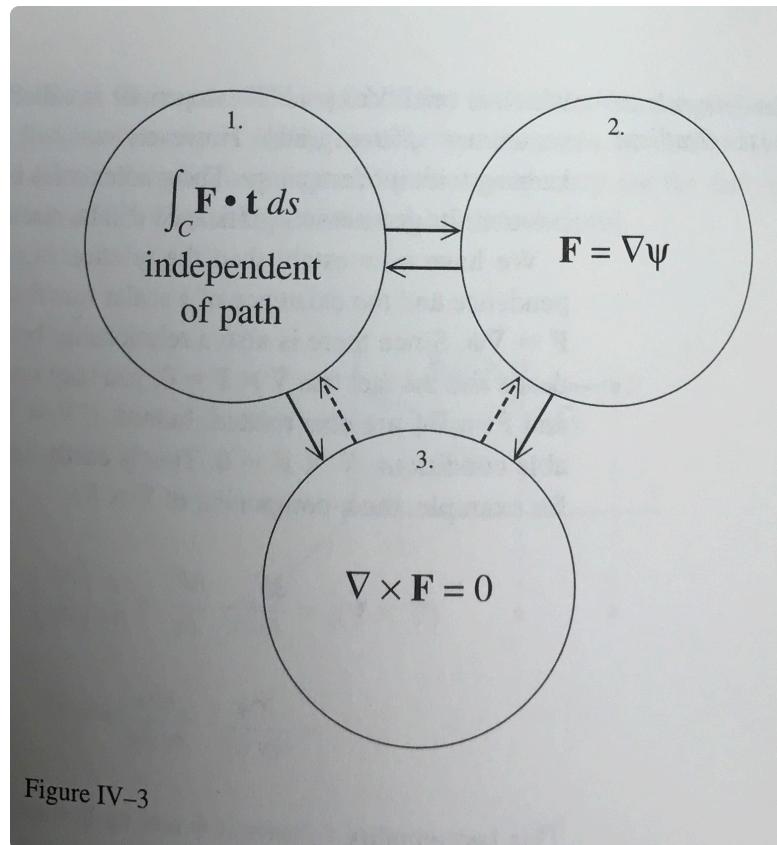


Figure IV-3

The scalar potential (voltage) can be defined by:

$$\phi(\vec{r}) = - \int_{C}^{r_0} \vec{E} \cdot d\vec{l}$$

r_0 = arbitrary reference point

C = any path from r_0 to r

From Gauss' law, we have

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla} \phi) = -\vec{\nabla}^2 \phi = 4\pi\rho$$

Poisson's Equation

If no charge is present

$$\vec{\nabla}^2 \phi = 0$$

LAPLACE'S EQUATION

For a point charge q :

$$\nabla \phi = -\vec{E} = -\frac{q}{r^2} \hat{e}_r$$

$$\Rightarrow \frac{\delta \phi}{\delta r} = -\frac{q}{r^2}$$

$$\Rightarrow \int_{\phi_0}^{\phi_i} d\phi = -q \int_{r_0}^{r_i} \frac{dr}{r^2}$$

$$\Rightarrow \phi_i - \phi_0 = q \left[\frac{1}{r} \right] \Big|_{r_0}^{r_i} = q \left(\frac{1}{r_i} - \frac{1}{r_0} \right)$$

If we choose $\phi_0 = 0$ at $r_0 = \infty$

$$\Rightarrow \phi_i = \frac{q}{r_i}$$

 potential at distance r from q