Hochberg-Visser Dynamic Wormholes and curvature invariants

David McNutt

Faculty of Science and Technology,
University of Stavanger, N-4036 Stavanger, Norway

david.d.mcnutt@uis.no

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1 Spherically Symmetric wormholes

Let us consider the imploding spherically symmetric metric in advanced coordinates:

$$ds^2 = -e^{2\beta(v,r)}\left(1 - \frac{2m(v,r)}{r}\right)dv^2 + 2e^{\beta(v,r)}dvdr + r^2 d\Omega^2, \tag{1}$$

where $m(v,r)$ is the mass function and $\beta(v,r)$ is an arbitrary function. In this form all gauge freedom has been used and, in general, further simplification of the Einstein tensor is not possible; for example, for a perfect fluid solution the fluid (or dust) is not, in general, comoving.

We choose the two future pointing radial null geodesic vector fields:

$$\ell = \partial_v + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\partial_r, \quad n = e^{-\beta}\partial_r, \tag{2}$$

where $\ell_a n^a = -1$, and complete the non-coordinate basis using the complex spatial vector and its complex conjugate:

$$m = \frac{1}{\sqrt{2}r}\partial_\theta + \frac{i}{\sqrt{2}r\sin(\theta)}\partial_\phi. \tag{3}$$

Relative to this basis the metric is now diagonalized:

$$g_{ab} = -\ell_{(a}n_{b)} + m_{(a}\bar{m}_{b)}. \tag{4}$$

This implies that relative to the null frame, the future null expansions for these null vectors are:

$$\theta_{(\ell)} = \frac{e^\beta}{r}\left(1 - \frac{2m}{r}\right), \quad \theta_{(n)} = -\frac{2e^{-\beta}}{r}. \tag{5}$$

We will suppose that a wormhole throat exist, and is a spacelike surface determined by the two conditions [1]:

$$\begin{aligned} \theta_{(\ell)} &= 0, \\ \ell^a \nabla_a \theta_{(\ell)} &\geq 0. \end{aligned} \tag{6}$$

Then, the unique spherically symmetric wormhole throat defined by the first condition is described by the equation $r - 2m(v,r) = 0$ and we will denote this hypersurface as $\tilde{\mathcal{H}}$. This hypersurface can be timelike, null or spacelike depending on the sign of the magnitude of the normal vector,

$$n_a = \nabla_a(r - 2m) = (1 - 2m_{,r})dr - 2m_{,v}dv,$$

evaluating the norm on the surface we obtain:

$$|n| = g^{ab}n_a n_b = -4e^{-\beta}m_{,v}(1 - 2m_{,r})|_{\tilde{\mathcal{H}}}. \tag{7}$$

Assuming that $m_{,v} \neq 0$, the surface $\tilde{\mathcal{H}}$ will be spacelike or timelike. We will consider the case when the surface $r = 2m$ is spacelike and hence a wormhole throat. Note that no further constraints will be imposed from the field equations at this stage.

Relative to the coframe given by (2) and (3), the only non-zero component of the Weyl spinor is

$$\begin{aligned} \Psi_2 \;\; = \;\; & \frac{e^{-\beta}\beta_{,v,r}}{6} - \frac{m_{,r}\beta_{,r}}{2r} - \frac{r(r-5m)\beta_{,r}}{6r^3} + \frac{2m_{,r}}{3r^2} - \frac{m_{,r,r}}{6r} \\ & + \frac{r^2(r-2m)\beta_{,r}^2}{6r^3} - \frac{6m - r^2(r-2m)\beta_{,r,r}}{6r^3}, \end{aligned} \tag{8}$$

and so the parameters of the null rotations about ℓ_a and n_a are fixed to identity. The remaining frame freedom consists of boost and spins. However, the boosts can be fixed at zeroth order as well since the non-zero NP curvature scalars for the Ricci spinor and the Ricci scalar $R = \Lambda/24$ are:

$$\Phi_{00} = \frac{e^{2\beta}(r-2m)^2\beta_{,r}}{4r^3} + \frac{e^{\beta}m_{,v}}{r^2}, \tag{9}$$

$$\Phi_{11} = \left(\frac{\beta_{,r}^2}{4r} + \frac{\beta_{,r,r}}{4r} - \frac{3\beta_{,r}}{8r^2}\right)(r-2m) + \frac{\beta_{r,v}e^{-\beta}}{4} - \frac{3\beta_{,r}m_{,r}}{4r} + \frac{3\beta_{,r}}{8r} - \frac{m_{,r,r}}{4r} + \frac{m_{,r}}{2r^2} \tag{10}$$

$$\Phi_{22} = \frac{e^{-2\beta}\beta_{,r}}{r}, \tag{11}$$

$$\Lambda = \left(-\frac{\beta_{,r}^2}{12r} - \frac{\beta_{,r,r}}{12r} - \frac{\beta_{,r}}{24r^2}\right)(r-2m) - \frac{\beta_{,r,v}e^{-\beta}}{12} + \frac{\beta_{,r}m_{,r}}{4r} - \frac{\beta_{,r}}{8r} + \frac{m_{,r,r}}{12r} + \frac{m_{,r}}{6r^2} \tag{12}$$

Thus the Weyl tensor is of algebraic type **D**, and the Ricci tensor is generally of algebraic type **I** ($\Phi_{00} \neq 0$) relative to the alignment classification [2]. At zeroth order the isotropy group of the Riemann tensor consists of spins [1]. Modulo a choice of boost, the components of the curvature tensor and its covariant derivatives can be treated as Cartan invariants.

[1] In fact, the spins belong to the isotropy group of the metric, and so all higher covariant derivatives of the Riemann tensor are invariant under spins.

We will use the covariant derivative of the Weyl tensor to detect the wormhole throat. Applying the differential Bianchi identities, the components the covariant derivative of the Weyl tensor can be expressed in terms of Ψ_2, $\Phi_{00} = \Phi_{22}$,Φ_{11}, $\Delta\Phi_{11}$ and the spin coefficients:

$$\epsilon = \tfrac{r(r-2m)e^{\beta}\beta_{,r}}{2r^2} + \tfrac{\beta_{,v}}{2} - \tfrac{e^{\beta}m_{,r}}{r} + \tfrac{me^{\beta}}{2r^2}, \tag{13}$$

$$\tfrac{1}{2}\theta_{(\ell)} = \rho = -\tfrac{e^{\beta}(r-2m)}{2r^2}, \tag{14}$$

$$\tfrac{1}{2}\theta_{(n)} = \mu = -\tfrac{e^{-\beta}}{r}. \tag{15}$$

The existence of the wormhole throat affects the structure of this tensor and this is reflected in the vanishing of a particular scalar polynomial curvature invariant (SPI) constructed from higher order SPIs [**?** 3].

Theorem 1.1. *For any spherically symmetric wormhole metric, the wormhole throat $r = 2m(v, r)$ is detected by the vanishing of the first order SPI:*

$$J = 4I_1I_3 - I_5, \tag{16}$$

where $I_1 = C_{abcd}C^{abcd}$, $\;I_3 = C_{abcd;e}C^{abcd;e}$ *and* $I_5 = I_{1,a}{I_1}^a$.

Proof. To prove this we may compute the explicit forms of the invariants I_1, I_3 and I_5 and combine them to show that

$$J = 4I_1I_3 - I_5 = (2^{12})(3^3)\rho\mu\Psi_2^4 = \frac{2^{11}3^3(r-2m)\Psi_2^4}{r^3}. \tag{17}$$

As $r - 2m = 0$ on $\tilde{\mathcal{H}}$, the invariant vanishes on $\tilde{\mathcal{H}}$. □

Since $I_1 = 48\Psi_2^2$ for spherically symmetric metrics[4], we may normalize the above SPI to produce an invariant whose vanishing is necessary and sufficient to detect the wormhole throat,

$$\tilde{J} = \frac{J}{2^4 3 I_1^2} = \rho\mu. \tag{18}$$

This invariant will be proportional to the Cartan invariant $\rho = 2\theta_{(\ell)}$ which vanishes on the wormhole throat.

Remark 1.2. The authors in [5] argue that the vanishing of the SPI J determines the *geometric horizon* (GH) for spherically symmetric blackhole metrics. The detection of the throat for spherically symmetric wormhole metrics presents an opportunity to revise the definition of a GH. In principle, a wormhole throat and GH could be distinguished by the flare-out condition.

Computing the norm of the gradient of $\tilde{J}$ we have a new curvature invariant constructed from the ratio of first order and second order SPIs. This curvature invariant will partially determine the flare-out condition on the throat:

$$|\nabla\tilde{J}|^2\big|_{\mathcal{H}} = \frac{m_{,v}(2m_{,r}-1)e^{-\beta}}{r^6}\big|_{\mathcal{H}}. \tag{19}$$

This invariant almost specifies the sign of $m_{,v}$ on the wormhole throat. To determine the sign of $m_{,r}$ on the throat invariantly and hence fully determine the flare-out condition, we require one more SPI.

References

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