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EVOLUTION OF SOLAR MAGNETIC FIELDS: A NEW APPROACH TO MHD INITIAL-BOUNDARY VALUE PROBLEMS BY THE METHOD OF NEARCHARACTERISTICS

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ABSTRACT

Observations indicate that the magnetic field in the solar atmosphere is subject to continuous agitations by motions and flux changes at the photospheric level. Examinations of such phenomena have been pursued in terms of evolution of force-free magnetic fields, after Tanaka and Nakagawa demonstrated that the excess magnetic energy accumulated in a postulated force-free field by proper (relative) motions of sunspots can account for the energy released in the flares of 1972 August. However, the proper treatment of this subject, namely the evolution of a non–force-free magnetic field including the atmospheric responses, poses a mathematically complex MHD initial-boundary value problem.

A new approach to this MHD initial-boundary problem is presented in this paper. The formulation is based on the method of *nearcharacteristics* developed recently by Werner and Shin and Kot. The main advantage of this method is that the physical causality relationship can be traced from the perturbation to the response as in the method of characteristics, while the mathematical complexity is reduced considerably. The physical validity of the method is demonstrated with examples, and discussion is given of their significance in interpreting observations.

Subject headings: hydromagnetics — Sun: magnetic fields

I. INTRODUCTION

The causes of various solar phenomena have been identified with magnetic activities, such as flares, prominences, enhanced emissions of radio, optical, EUV, and X-rays, and even the modulation of interplanetary solar wind. However, direct observations of the solar magnetic field have been limited to the photospheric level where strong line emissions are available for analyses of the Zeeman effect. Thus, considerable theoretical efforts have been directed to determine the atmospheric magnetic field on the basis of photospheric observations. These are, to name a few, the potential representation (Altschuler *et al.* 1977), the constant α force-free representation (Nakagawa 1973; Nakagawa and Raadu 1972; Raadu and Nakagawa 1971; Nakagawa *et al.* 1971; Nakagawa, Raadu, and Harvey 1973; Nakagawa, Wu, and Tandberg-Hanssen 1978), and the potential plus current sheets representation (Sakurai and Uchida 1977).

Similarly, considerable theoretical effort has been directed to the study of possible evolution of force-free fields (Birn, Goldstein, and Schindler 1978; Heyvaerts *et al.* 1979; Low 1977*a*, *b*, 1979; Low and Nakagawa 1975; Nakagawa and Stenflo 1979), after Tanaka and Nakagawa (1973) successfully demonstrated that the amount of energy released in the flares of 1972 August can be accounted for by the excess magnetic energy accumulated in postulated evolving force-free fields in response to the proper (relative) motions of sunspots. Nevertheless, because of mathematical complexities, a systematic study of a more realistic evolution of non–force-free magnetic fields, including the atmospheric responses, has not been developed. The proper treatment of this subject, namely, the responses of the atmosphere, as well as the atmospheric magnetic field, to finite amplitude perturbations at the photospheric level (i.e., the boundary surface), poses a MHD initial-boundary value problem. A few examples of this type of study have been undertaken; however, they have been either too simple (Bazer and Fleishman 1959) or too limited (Sauerwein 1966) for application to solar or astrophysical problems.

In this paper, therefore, a new method of analysis of the MHD initial-boundary value problem is presented with general application to solar and astrophysical problems in mind. The mathematical formulation is based on the method of *nearcharacteristics* developed by Werner (1968) and modified recently by Shin and Kot (1978). Shin and Kot (1978; hereafter SK) showed for a hydrodynamic problem that the solution obtained with their version is sufficiently accurate in comparison with the results given by the ordinary method of characteristics.

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In the initial-boundary value problem (in order to correlate the consequence of a perturbation on the boundary in a physically consistent manner) the method of characteristics is used to trace the response through the compatibility equations along characteristics. This tractability between the cause and response is retained in the method of nearcharacteristics. In particular, SK's version provides a considerable reduction in mathematical procedures in treating the MHD initial-boundary value problems.

To provide a proper perspective of the method of nearcharacteristics and its relation to the method of characteristics, a general consideration on the subject is presented in § II. In § III, the details of formulation based on the method of nearcharacteristics are described for two-dimensional problems to give a clear physical picture. Then in § IV, the physical validity of formulation is demonstrated with examples of MHD initial-boundary problems. Discussion of the further significance of the formulation is presented in the final section, together with other considerations.

II. GENERAL CONSIDERATIONS

a) Basic Equations and Their Properties

The prevailing physical conditions in the solar atmosphere suggest that appropriate equations for the study are the idealized MHD equations with the inclusion of effects of gravity and radiative energy exchange. Then the basic equations constitute a set of quasi-linear first-order equations of the form

$$A_{klm}\frac{\partial W_m}{\partial \alpha_k} + S_l = 0 \qquad (l = 1, 2, \ldots, 8)\,, \tag{1}$$

where W_m ($m = 1, 2, \ldots, 8$) are the dependent (physical) variables, α_k ($k = 1, 2, 3, 4$) are the independent variables (space and time coordinates), and A_{klm} and S_l are functions of W_m and α_k. (The summational convention of indices is understood.)

It is known (Courant and Hilbert 1962; Jeffrey and Taniuti 1964) that the system of equations (1) is hyperbolic, if this system of equations yields at least one real characteristic surface β_1 = const. in the $\boldsymbol{\alpha}$-coordinates after a suitable transformation of coordinates from the $\boldsymbol{\alpha}$- to $\boldsymbol{\beta}$-coordinates. More precisely, by introducing a system of locally orthogonal $\boldsymbol{\beta}$-coordinates, β_k ($k = 1, 2, 3, 4$), the system of equations (1) can be written in the form

$$A_{klm}\frac{\partial W_m}{\partial \beta_1}\frac{\partial \beta_1}{\partial \alpha_k} + A_{klm}\frac{\partial W_m}{\partial \beta_2}\frac{\partial \beta_2}{\partial \alpha_k} + \cdots + S_l = 0\,. \tag{2}$$

Now, consider the difference of equation (2) between two neighboring points P and P' across the surface β_1 = const. When the points P and P' are brought together, the difference vanishes in the limit (by the orthogonality of $\boldsymbol{\beta}$-coordinates and the continuity of derivatives and physical variables), except for the term containing the derivative $\partial W_m/\partial\beta_1$ normal to the surface β_1 = const. The definition of a characteristic surface is that the normal derivative $\partial W_m/\partial\beta_1$ remains indeterminate, i.e., the determinant consisting of the coefficients of $\partial W_m/\partial\beta_1$ vanishes:

$$\det\left|A_{klm}\frac{\partial \beta_1}{\partial \alpha_k}\right| = 0\,. \tag{3}$$

If equation (3) yields at least one real root $\partial\beta_1/\partial\alpha_k$, the system of equation is called hyperbolic.

Physically, this condition is equivalent to identifying the surface β_1 = const. with the trajectory of an arbitrary discontinuity in physical quantity W_m (i.e., the trajectory of a shock, finite amplitude wave front, and a surface of contact discontinuity) in the $\boldsymbol{\alpha}$ (spacetime)-coordinates.

For a system of hyperbolic equations, the next step in the method of characteristics (see, for example, Sauerwein 1966) is to find a set of linear coefficients λ_l^* from the equation

$$\lambda_l^* A_{klm}\frac{\partial \beta_1}{\partial \alpha_k} = 0\,, \tag{4}$$

and obtain the compatibility equation from equation (2) in the form

$$\lambda_l^* A_{klm}\frac{\partial \beta_2}{\partial \alpha_k}\frac{\partial W_m}{\partial \beta_2} + \lambda_l^* A_{klm}\frac{\partial \beta_3}{\partial \alpha_k}\frac{\partial W_m}{\partial \beta_3} + \cdots + \lambda_l^* S_l = 0\,. \tag{5}$$

After specification of the coordinates β_k as functions of α_k ($k = 1, 2, 3, 4$), this equation can be integrated to yield the variation of W_m along the specified characteristic. Thus, *the compatibility equations provide the physical (causality) relationship between the cause and response along the characteristics.*

It can be proved easily that the basic system of equations for the present problem is hyperbolic (see, for example, Nakagawa, Wu, and Han 1978), with the characteristic surfaces being the trajectories of the fluid element and wave fronts of fast, slow, and transverse MHD waves. In practical application, however, the direct use of the method of characteristics becomes an extremely cumbersome numerical procedure as seen from the analysis by Sauerwein (1966). The complexity arises from the fact that the $\boldsymbol{\beta}$-coordinates become complicated functions of α_k, similarly $\lambda_l{}^*$, as well as other coefficients in equation (5). This complexity can be visualized from the analytical expression of the characteristic surfaces given by Lynn (1962) as a tenth-order algebraic equation in terms of the $\boldsymbol{\alpha}$-coordinates.

The method of nearcharacteristics was proposed to simplify this procedure (Werner 1968) by projecting the real characteristics on the planes parallel to the time coordinates. Under general circumstances, however, the projected characteristics, called the nearcharacteristics, can lie outside of the domain of dependence. Therefore, Werner (1968) suggested to select the direction of nearcharacteristics as close to the real characteristics to secure the accuracy of analysis. Recently, for a two-dimensional hydrodynamic problem, SK obtained a numerical solution of sufficient accuracy by utilizing nearcharacteristics which lie outside the domain of dependence. In SK's version, compatibility equations are derived first along the nearcharacteristics in each projected plane. Then for the final solution these two different directional compatibility equations were combined. This procedure is, in essence, similar to obtaining a properly weighted sum of representative compatibility equations in the method of characteristics (see, for example, Butler 1960). SK's version is found particularly advantageous in MHD problems as the complexity due to nonisotropic propagation of MHD waves in selecting representative characteristics can be replaced by a simple procedure.

b) *Basic Equations of Two-Dimensional Problems*

In order to retain a clear physical picture, let us consider two-dimensional problems (the extension to three-dimensional problems is discussed below). In a system of spherical coordinates (r, θ, ϕ) (where r is the radius, θ the colatitude, and ϕ the longitude) with axial symmetry ($\partial/\partial\phi = 0$), the basic equations can be written in the following matrix form:

$$\left(\boldsymbol{I}\frac{\partial}{\partial t} + \boldsymbol{A}\frac{\partial}{\partial r} + \boldsymbol{B}\frac{\partial}{r\partial\theta}\right)\boldsymbol{W} = \boldsymbol{S}\,, \tag{6}$$

where $\boldsymbol{I}$ is a unit matrix, $\boldsymbol{A}$ and $\boldsymbol{B}$ are 8×8 matrices, and $\boldsymbol{W}$ and $\boldsymbol{S}$ are column vectors. Specifically, they are

$$\boldsymbol{A} = \begin{vmatrix} u & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & \frac{1}{\rho} & 0 & \frac{B_\theta}{\rho} & \frac{B_\phi}{\rho} \\ 0 & 0 & u & 0 & 0 & 0 & -\frac{B_r}{\rho} & 0 \\ 0 & 0 & 0 & u & 0 & 0 & 0 & -\frac{B_r}{\rho} \\ 0 & a^2\rho & 0 & 0 & u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & B_\theta & -B_r & 0 & 0 & 0 & u & 0 \\ 0 & B_\phi & 0 & -B_r & 0 & 0 & 0 & u \end{vmatrix}\,, \tag{7}$$

$$\boldsymbol{B} = \begin{vmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & -\frac{B_\theta}{\rho} & 0 & 0 \\ 0 & 0 & v & 0 & \frac{1}{\rho} & \frac{B_r}{\rho} & 0 & \frac{B_\phi}{\rho} \\ 0 & 0 & 0 & v & 0 & 0 & 0 & -\frac{B_\theta}{\rho} \\ 0 & 0 & a^2\rho & 0 & v & 0 & 0 & 0 \\ 0 & -B_\theta & B_r & 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & B_\phi & -B_\theta & 0 & 0 & 0 & v \end{vmatrix}\,, \tag{8}$$

$$
\boldsymbol{W} = \begin{vmatrix} \rho \\ u \\ v \\ w \\ p \\ B_r \\ B_\theta \\ B_\phi \end{vmatrix}, \qquad \boldsymbol{S} = \begin{vmatrix} -\dfrac{2\rho u}{r} - \dfrac{\rho v}{r}\cot\theta \\ \dfrac{v^2 + w^2}{r} - \dfrac{B_\theta{}^2 + B_\phi{}^2}{\rho r} - g(r) \\ -\dfrac{uv}{r} + \dfrac{w^2}{r}\cot\theta + \dfrac{B_r B_\theta}{\rho r} - \dfrac{B_\phi{}^2}{\rho r}\cot\theta \\ -\dfrac{uw}{r} - \dfrac{vw}{r}\cot\theta + \dfrac{B_r B_\phi}{\rho r} + \dfrac{B_\theta B_\phi}{\rho r}\cot\theta \\ -\dfrac{2a^2\rho\mu}{r} - \dfrac{a^2\rho v}{r}\cot\theta + (\gamma - 1)\Delta Q \\ -\dfrac{2uB_r}{r} - \dfrac{vB_r}{r}\cot\theta \\ -\dfrac{uB_\theta}{r} - \dfrac{vB_r}{r} - \dfrac{vB_\theta}{r}\cot\theta \\ -\dfrac{uB_\phi}{r} - \dfrac{wB_r}{r} - \dfrac{wB_\theta}{r}\cot\theta \end{vmatrix} . \tag{9}
$$

Here the notations are ρ the density, u the radial velocity, v the colatitudinal velocity, w the longitudinal velocity, p the pressure, B_r, B_θ, and B_ϕ the components of magnetic field in the unit of $\sqrt{\mu_0}$, (μ_0 the magnetic permeability), $g(r)$ the gravitational acceleration as a function of r, γ the ratio of specific heats, and ΔQ the net rate of radiative energy exchange per unit volume.

The system of equations (6) is hyperbolic, as equation (3) yields the following real roots:

$$
\frac{D\beta_1}{Dt} = 0 , \quad \left(\frac{D\beta_1}{Dt}\right)^2 - \frac{(\boldsymbol{B}\cdot\nabla\beta_1)^2}{\rho} = 0 , \quad \left(\frac{D\beta_1}{Dt}\right)^2 - c_f{}^2 = 0 , \quad \left(\frac{D\beta_1}{Dt}\right)^2 - c_s{}^2 = 0 , \tag{10a}
$$

where c_f and c_s are the speeds of fast and slow MHD waves in the direction of $\nabla\beta_1$, i.e.,

$$
\begin{aligned}
c_f{}^2 &= \tfrac{1}{2}|\nabla\beta_1|\left\{|\nabla\beta_1|(a^2 + b^2) + \left[(\nabla\beta_1)^2(a^2 + b^2)^2 - 4a^2\frac{(\boldsymbol{B}\cdot\nabla\beta_1)^2}{\rho}\right]^{1/2}\right\} , \\
c_s{}^2 &= \tfrac{1}{2}|\nabla\beta_1|\left\{|\nabla\beta_1|(a^2 + b^2) - \left[(\nabla\beta_1)^2(a^2 + b^2)^2 - 4a^2\frac{(\boldsymbol{B}\cdot\nabla\beta_1)^2}{\rho}\right]^{1/2}\right\} .
\end{aligned} \tag{10b}
$$

Other notations are

$$
\begin{aligned}
\frac{D\beta_1}{Dt} &= \frac{\partial B_1}{\partial t} + u\frac{\partial\beta_1}{\partial r} + v\frac{\partial\beta_1}{r\partial\theta} , \qquad & \nabla\beta_1 &= \frac{\partial\beta_1}{\partial r}\hat{r} + \frac{\partial\beta_1}{r\partial\theta}\hat{\theta} , \\
b^2 &= \frac{B_r{}^2 + B_\theta{}^2 + B_\phi{}^2}{\rho} , \qquad & \boldsymbol{B}\cdot\nabla\beta_1 &= B_r\frac{\partial\beta_1}{\partial r} + B_\theta\frac{\partial\beta_1}{r\partial\theta} ,
\end{aligned} \tag{10c}
$$

a the adiabatic speed of sound, and $\hat{r}$ and $\hat{\theta}$ the unit vectors in their respective directions.

With three components of magnetic fields, the loci of characteristic surfaces given by equations (10a) are the ϕ-directional projection (with $\partial/\partial\phi = 0$) of the three-dimensional characteristic surfaces to the (r, θ)-plane. Figure 1 illustrates these loci in a stationary medium (i.e., $u = v = w = 0$); the deformed ellipse is the loci of fast wave fronts resulting from a point source at $0'$. For slow waves, the loci are a pair of cusped triangles, and for the incompressible transverse waves, the loci are two points A' and C'.

The characteristic surfaces in the (r, θ, t)-space with a fluid motion is shown in Figure 2. In contrast to simple linear trajectories of the fluid element (PQ) and transverse waves (AQ and BQ), the trajectory of fast waves forms a conical surface, while the trajectories of slow waves become a pair of triangular cones. It can be seen from Figure 2 that even for two-dimensional problems the determination of $\boldsymbol{\beta}$-coordinates in terms of α_k is a considerable mathematical task.

It is important to note that the roots $\partial\beta_1/\partial\alpha_k = 0$ obtained in equations (10a) are mutually independent. Thus, the solution of nonlinear equation (6) can be obtained as a sum of arbitrary discontinuities propagating along these

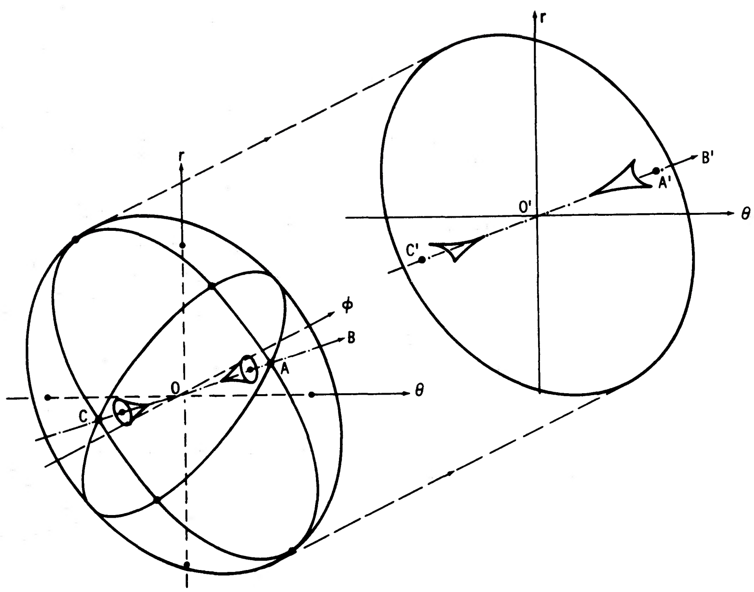

FIG. 1.—The relationship between three- and two-dimensional loci of characteristic surfaces. The direction of magnetic field in the (r, θ, ϕ)-coordinates is represented by OB and the projection in the (r, θ)-plane by $O'B'$. A' and C' are the projection of A and C.

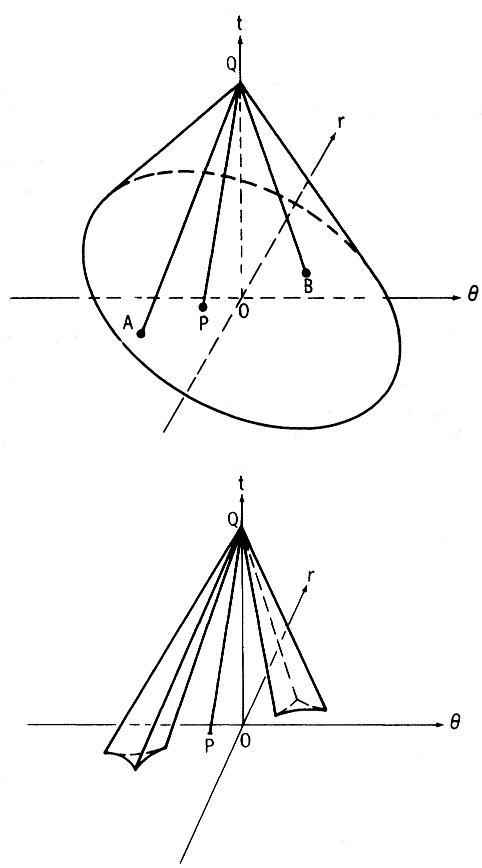

FIG. 2.—The characteristic surfaces in the (r, θ, t)-space, where linearity of trajectories is assumed for simplicity. PQ is the trajectory of a fluid element, and AQ and BQ are the trajectories of transverse waves.

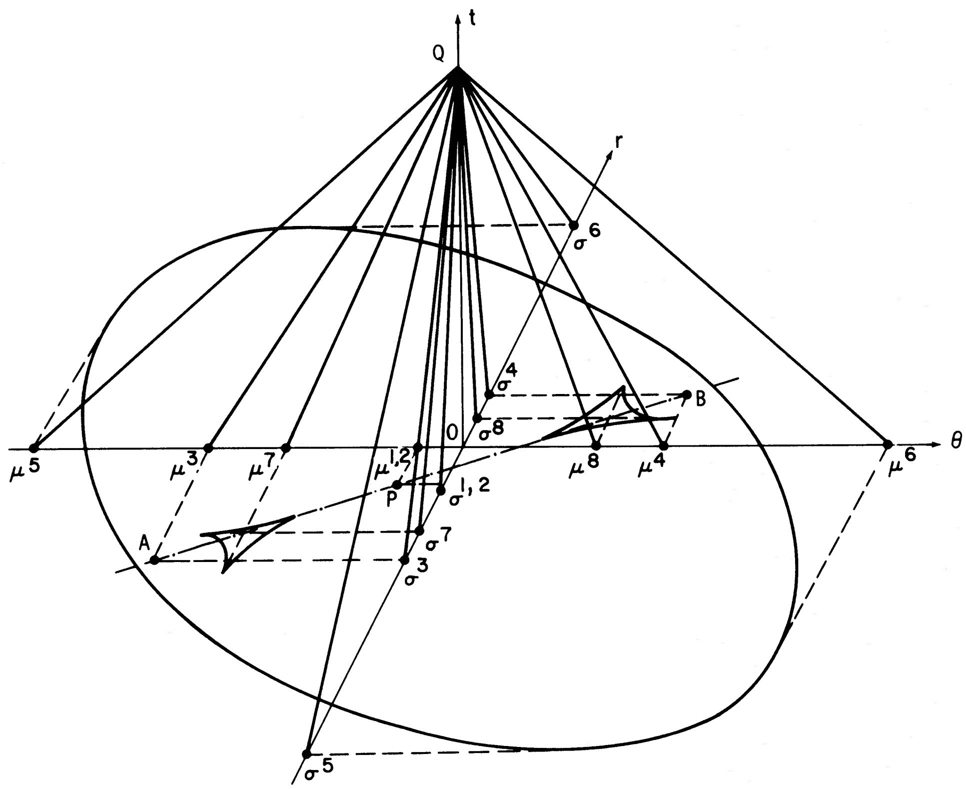

FIG. 3.—The loci of characteristics and the foot points of nearcharacteristics. The variables σ^l and μ^l ($l = 1, 2, \ldots, 8$) denote the foot points of near characteristics, and APB is the direction of magnetic field. The foot points are determined by the interactions of planes tangential to the loci with the r- or θ-axis.

characteristic directions. The arbitrariness is then removed by satisfying all compatibility equations of the form of equation (5).

III. FORMULATION OF THE METHOD OF NEARCHARACTERISTICS

a) *Determination of Nearcharacteristics*

In the present problem, two sets of nearcharacteristics are obtained: one in the (r, t)-plane and the other in (θ, t)-plane (see Fig. 3). In each plane the direction of nearcharacteristic is given by a specific value of either dr/dt or $rd\theta/dt$. Let $\sigma^{(l)}$ ($l = 1, 2, \ldots, 8$) be the direction of nearcharacteristic in the (r, t)-plane and $\mu^{(l)}$ be the direction of nearcharacteristic in the (θ, t)-plane. Then these directions $\sigma^{(l)}$ and $\mu^{(l)}$ are given by the eigenvalues of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ from the equations

$$\det|\boldsymbol{A} - \sigma^{(l)}\mathbf{1}| = 0\,, \qquad \det|\boldsymbol{B} - \mu^{(l)}\mathbf{1}| = 0\,. \tag{11}$$

Explicitly, the eigenvalues are

$$\begin{aligned}
\sigma^{(1)} &= u\,, & \mu^{(1)} &= v\,,\\
\sigma^{(2)} &= u\,, & \mu^{(2)} &= v\,,\\
\sigma^{(3)} &= u + U_A\,, & \mu^{(3)} &= v + V_A\,,\\
\sigma^{(4)} &= u - U_A\,, & \mu^{(4)} &= v - V_A\,,\\
\sigma^{(5)} &= u + U_f\,, & \mu^{(5)} &= v + V_f\,,\\
\sigma^{(6)} &= u - U_f\,, & \mu^{(6)} &= v - V_f\,,\\
\sigma^{(7)} &= u + U_s\,, & \mu^{(7)} &= v + V_s\,,\\
\sigma^{(8)} &= u - U_s\,, & \mu^{(8)} &= v - V_s\,,
\end{aligned} \tag{12a}$$

where

$$U_A = |b_r|\,, \qquad V_A = |b_\theta|\,,$$
$$U_f^2 = \tfrac{1}{2}\{a^2 + b^2 + [(a^2+b^2)^2 - 4a^2b_r^2]^{1/2}\}\,, \qquad V_f^2 = \tfrac{1}{2}\{a^2 + b^2 + [(a^2+b^2)^2 - 4a^2b_\theta^2]^{1/2}\}\,,$$
$$U_s^2 = \tfrac{1}{2}\{a^2 + b^2 - [(a^2+b^2)^2 - 4a^2b_r^2]^{1/2}\}\,, \qquad V_s^2 = \tfrac{1}{2}\{a^2 + b^2 - [(a^2+b^2)^2 - 4a^2b_\theta^2]^{1/2}\}\,, \tag{12b}$$

and, apart from the quantities a and b defined previously,

$$b_r = \frac{B_r}{\sqrt{\rho}}\,, \quad b_\theta = \frac{B_\theta}{\sqrt{\rho}}\,, \quad b_\phi = \frac{B_\phi}{\sqrt{\rho}} \qquad (b^2 = b_r^2 + b_\theta^2 + b_\phi^2)\,. \tag{12c}$$

In Figure 3, the nearcharacteristics with the values of $\sigma^{(l)}$ and $\mu^{(l)}$ given in equation (12a) are shown with the relative locations of their foot points to the loci of characteristics. Note that only the foot points of nearcharacteristics $\sigma^{(5)}$, $\sigma^{(6)}$, $\mu^{(5)}$, and $\mu^{(6)}$ lie outside of the domain of dependence (i.e., outside of the fast-wave cone in Fig. 2).

b) Determination of Left-Eigenvectors

The procedure of determining a set of linear coefficients λ_l^* in the method of characteristics, i.e., equation (4), is replaced in the method of nearcharacteristics by finding the left-eigenvectors of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ associated with the eigenvalues. Denoting the left-eigenvectors associated with $\sigma^{(l)}$ by $\boldsymbol{\lambda}^{(l)}$ and for $\mu^{(l)}$ by $\boldsymbol{\eta}^{(l)}$, these eigenvectors are determined from the equations

$$\boldsymbol{\lambda}^{(l)}\boldsymbol{A} = \sigma^{(l)}\boldsymbol{\lambda}^{(l)}\,, \qquad \boldsymbol{\eta}^{(l)}\boldsymbol{B} = \mu^{(l)}\boldsymbol{\eta}^{(l)} \qquad (l = 1, 2, \ldots, 8)\,. \tag{13}$$

In this determination, one component of left-eigenvectors remains arbitrary; however, the choice of this arbitrary component does not affect the final results.

Solving equation (13), we find

for $\sigma^{(1)} = u$,
$$\boldsymbol{\lambda}^{(1)} = \{a^2, 0, 0, 0, -1, 0, 0, 0];\tag{14a}$$

for $\sigma^{(2)} = u$,
$$\boldsymbol{\lambda}^{(2)} = \left[0, 0, 0, 0, 0, \frac{1}{\sqrt{\rho}}, 0, 0\right];\tag{14b}$$

for $\sigma^{(3)} = u + U_A$,
$$\boldsymbol{\lambda}^{(3)} = \left[0, 0, -b_r b_\phi, b_r b_\theta, 0, 0, \frac{b_\phi U_A}{\sqrt{\rho}}, -\frac{b_\theta U_A}{\sqrt{\rho}}\right];\tag{14c}$$

for $\sigma^{(4)} = u - U_A$,
$$\boldsymbol{\lambda}^{(4)} = \left[0, 0, b_r b_\phi, -b_r b_\theta, 0, 0, \frac{b_\phi U_A}{\sqrt{\rho}}, -\frac{b_\theta U_A}{\sqrt{\rho}}\right];\tag{14d}$$

for $\sigma^{(5)} = u + U_f$,
$$\boldsymbol{\lambda}^{(5)} = \left[0, (U_f^2 - b_r^2)U_f, -b_r b_\theta U_f, -b_r b_\phi U_f, \frac{U_f^2 - b_r^2}{\rho}, 0, \frac{b_\theta U_f^2}{\sqrt{\rho}}, \frac{b_\phi U_f^2}{\sqrt{\rho}}\right];\tag{14e}$$

for $\sigma^{(6)} = u - U_f$,
$$\boldsymbol{\lambda}^{(6)} = \left[0, -(U_f^2 - b_r^2)U_f, b_r b_\theta U_f, b_r b_\phi U_f, \frac{U_f^2 - b_r^2}{\rho}, 0, \frac{b_\theta U_f^2}{\sqrt{\rho}}, \frac{b_\phi U_f^2}{\sqrt{\rho}}\right];\tag{14f}$$

for $\sigma^{(7)} = u + U_s$,
$$\boldsymbol{\lambda}^{(7)} = \left[0, (b_r^2 - U_s^2)U_s, b_r b_\theta U_s, b_r b_\phi U_s, \frac{b_r^2 - U_s^2}{\rho}, 0, -\frac{b_\theta U_s^2}{\sqrt{\rho}}, -\frac{b_\phi U_s^2}{\sqrt{\rho}}\right]\tag{14g}$$

for $\sigma^{(8)} = u - U_s$,
$$\boldsymbol{\lambda}^{(8)} = \left[0, -(b_r^2 - U_s^2)U_s, -b_r b_\theta U_s, -b_r b_\phi U_s, \frac{b_r^2 - U_s^2}{\rho}, 0, -\frac{b_\theta U_s^2}{\sqrt{\rho}}, -\frac{b_\phi U_s^2}{\sqrt{\rho}}\right];\tag{14h}$$

for $\mu^{(1)} = v$,

$$\boldsymbol{\eta}^{(1)} = [a^2, 0, 0, 0, -1, 0, 0, 0]\,; \tag{15a}$$

for $\mu^{(2)} = v$,

$$\boldsymbol{\eta}^{(2)} = \left[0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{\rho}}, 0\right]; \tag{15b}$$

for $\mu^{(3)} = v + V_A$,

$$\boldsymbol{\eta}^{(3)} = \left[0, -b_\theta b_\phi, 0, b_r b_\theta, 0, \frac{b_\phi V_A}{\sqrt{\rho}}, 0, -\frac{b_r V_A}{\sqrt{\rho}}\right]; \tag{15c}$$

for $\mu^{(4)} = v - V_A$,

$$\boldsymbol{\eta}^{(4)} = \left[0, b_\theta b_\phi, 0, -b_r b_\theta, 0, \frac{b_\phi V_A}{\sqrt{\rho}}, 0, -\frac{b_r V_A}{\sqrt{\rho}}\right]; \tag{15d}$$

for $\mu^{(5)} = v + V_f$,

$$\boldsymbol{\eta}^{(5)} = \left[0, -b_r b_\theta V_f, (V_f^2 - b_\theta^2)V_f, -b_\theta b_\phi V_f, \frac{V_f^2 - b_\theta^2}{\rho}, \frac{b_r V_f^2}{\sqrt{\rho}}, 0, \frac{b_\phi V_f^2}{\sqrt{\rho}}\right]; \tag{15e}$$

for $\mu^{(6)} = v - V_f$,

$$\boldsymbol{\eta}^{(6)} = \left[0, b_r b_\theta V_f, -(V_f^2 - b_\theta^2)V_f, b_\theta b_\phi V_f, \frac{V_f^2 - b_\theta^2}{\rho}, \frac{b_r V_f^2}{\sqrt{\rho}}, 0, \frac{b_\phi V_f^2}{\sqrt{\rho}}\right]; \tag{15f}$$

for $\mu^{(7)} = v + V_s$,

$$\boldsymbol{\eta}^{(7)} = \left[0, b_r b_\theta V_s, (b_\theta^2 - V_s^2)V_s, b_\theta b_\phi V_s, \frac{b_\theta^2 - V_s^2}{\rho}, -\frac{b_r V_s^2}{\sqrt{\rho}}, 0, -\frac{b_\phi V_s^2}{\sqrt{\rho}}\right]; \tag{15g}$$

for $\mu^{(8)} = v - V_s$,

$$\boldsymbol{\eta}^{(8)} = \left[0, -b_r b_\theta V_s, -(b_\theta^2 - V_s^2)V_s, -b_\theta b_\phi V_s, \frac{b_\theta^2 - V_s^2}{\rho}, -\frac{b_r V_s^2}{\sqrt{\rho}}, 0, -\frac{b_\phi V_s^2}{\sqrt{\rho}}\right]. \tag{15h}$$

c) Derivation of Compatibility Equations

After the left-eigenvectors are determined, the compatibility equation along each nearcharacteristic can be obtained by a scalar multiplication of eigenvector to equation (6). Formally, this procedure yields, along $dr/dt = \sigma^{(l)}$,

$$\boldsymbol{\lambda}^{(l)}\left(\mathbf{1}\frac{\partial}{\partial t} + \boldsymbol{A}\frac{\partial}{\partial r}\right)\boldsymbol{W} = \boldsymbol{\lambda}^{(l)}\cdot\boldsymbol{S} - \boldsymbol{\lambda}^{(l)}\left(\boldsymbol{B}\frac{\partial}{r\partial\theta}\right)\boldsymbol{W}, \tag{16a}$$

and along $rd\theta/dt = \mu^{(l)}$,

$$\boldsymbol{\eta}^{(l)}\left(\boldsymbol{I}\frac{\partial}{\partial t} + \boldsymbol{B}\frac{\partial}{r\partial\theta}\right)\boldsymbol{W} = \boldsymbol{\eta}^{(l)}\cdot\boldsymbol{S} - \boldsymbol{\eta}^{(l)}\left(\boldsymbol{A}\frac{\partial}{\partial r}\right)\boldsymbol{W}, \tag{16b}$$

where to emphasize the directionality of these compatibility equations the terms not contained in the planes of projections are transferred to the right-hand side. The exact forms of these compatibility equations are given in Appendix A.

It is possible to express (Jeffrey and Taniuti 1964; Werner 1968) equations (16a) and (16b) in terms of the temporal variation of $\boldsymbol{W}$ at a given location and time. Accordingly, with the use of equation (13), we hereafter write the compatibility equations in the form, for $dr/dt = \sigma^{(l)}$,

$$\boldsymbol{\lambda}^{(l)}\frac{\partial \boldsymbol{W}}{\partial t} = -\boldsymbol{\sigma}^{(l)}\left(\frac{\partial}{\partial r} + \boldsymbol{\lambda}^{(l)}\boldsymbol{B}\frac{\partial}{r\partial\theta}\right)\boldsymbol{W} + \boldsymbol{\lambda}^{(l)}\boldsymbol{S} = \lambda_l\,, \tag{17a}$$

and for $rd\theta/dt = \mu^{(l)}$,

$$\eta^{(l)}\frac{\partial \boldsymbol{W}}{\partial t} = -\boldsymbol{\eta}^{(l)}\left(\frac{\partial}{\partial r} + {}^{(l)}\frac{\partial}{r\partial\theta}\right)\boldsymbol{W} + \boldsymbol{\eta}^{(l)}\boldsymbol{S} = \eta_l\,, \tag{17b}$$

where new notations λ_l and η_l denote the terms on the right-hand sides (R.H.S.) with the direction of nearcharacteristics referred through the index l ($l = 1, 2, \ldots, 8$).

In terms of these notations, the compatibility equations constitute two sets of eight simultaneous algebraic equations in the (r, t)- and (θ, t)-planes, relating the temporal variations of eight physical quantities $\partial W_m/\partial t$ ($m = 1, 2, \ldots, 8$) as summarized in Table 1. It is then possible to determine the eight quantities $\partial W_m/\partial t$ uniquely by solving eight equations in each projected plane. However, as $l = 1$ and 2 (i.e., those with $dr/dt = u$ and $rd\theta/dt = v$) can be decoupled from the rest of equations, it is convenient to determine six temporal variations, i.e., $\partial u/\partial t$, $\partial v/\partial t$, $\partial w/\partial t$, $\partial p/\partial t$, $\partial B_\theta/\partial t$, and $\partial B_\phi/\partial t$, apart from six compatibility equations in the (r, t)-plane, and similarly determine another set of six variations, i.e., $\partial u/\partial t$, $\partial v/\partial t$, $\partial w/\partial t$, $\partial p/\partial t$, $\partial B_r/\partial t$, and $\partial B_r/\partial t$, from six compatibility equations in the (θ, t)-plane. Then in the final determination, the directional effect is compensated as described in Appendix B in detail.

In actual application, numerical computations are necessary. In the difference form, the final equation for the determination of temporal variation of the physical variables at a given spatial location, say, at a grid point (i, j) (where i refers to the radius, j the colatitude) and the time step $n\Delta t$ (where Δt is the unit time step) becomes, after some algebra and rearrangement described in Appendix C,

$$\left(\frac{\partial \boldsymbol{W}}{\partial t}\right)_{i,j}^{n} = (\boldsymbol{C})_{i,j}^{n}(\Delta_r \boldsymbol{W})_{i,j}^{n} + (\boldsymbol{D})_{i,j}^{n}(\Delta_\theta \boldsymbol{W})_{i,j}^{n} + (\boldsymbol{E})_{i,j}^{n}(\delta_r \boldsymbol{W})_{i,j}^{n} + (\boldsymbol{F})_{i,j}^{n}(\delta_\theta \boldsymbol{W})_{i,j}^{n} + (\boldsymbol{S}^*)_{i,j}^{n}\,, \tag{18}$$

where (suppressing indices) the quantities $\partial \boldsymbol{W}/\partial t$, $\Delta_r \boldsymbol{W}$, $\delta_r \boldsymbol{W}$, $\Delta_\theta \boldsymbol{W}$, $\delta_\theta \boldsymbol{W}$, are column vectors, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{F}$ are 8×8 matrices, and $\boldsymbol{S}^*$ is the averaged column vector $\boldsymbol{S}$, i.e.,

$$\boldsymbol{S}^* = \tfrac{1}{4}(\boldsymbol{S}_{i+1,j} + \boldsymbol{S}_{i-1,j} + \boldsymbol{S}_{i,j+1} + \boldsymbol{S}_{i,j-1})\,. \tag{19}$$

Apart from the matrices $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{F}$ given in Appendix C, the column vectors denote the following quantities:

$$(\Delta_r \boldsymbol{W})_{i,j}^{n} = \frac{(\boldsymbol{W})_{i+1,j}^{n} - (\boldsymbol{W})_{i-1,j}^{n}}{2\Delta r}\,, \qquad (\Delta_\theta \boldsymbol{W})_{i,j}^{n} = \frac{(\boldsymbol{W})_{i,j+1}^{n} - (\boldsymbol{W})_{i,j-1}^{n}}{2r_i\Delta\theta}\,,$$

$$(\delta_r \boldsymbol{W})_{i,j}^{n} = \frac{(\boldsymbol{W})_{i+1,j}^{n} - 2(\boldsymbol{W})_{i,j}^{n} + (\boldsymbol{W})_{i-1,j}^{n}}{2\Delta r}\,, \qquad (\delta_\theta \boldsymbol{W})_{i,j}^{n} = \frac{(\boldsymbol{W})_{i,j+1}^{n} - 2(\boldsymbol{W})_{i,j}^{n} + (\boldsymbol{W})_{i,j-1}^{n}}{2r_i\Delta\theta}\,, \tag{20}$$

where r_i is the radial distance of the grid point (i, j), and the spatial distances Δr, $r_i\Delta\theta$, and the time step Δt must satisfy the Courant-Friedrichs-Lewy criterion of stability modified by SK, i.e.,

$$2\Delta r > (|u| + |U_f|)\Delta t\,, \qquad 2r_i\Delta\theta > (|u| + |V_f|)\Delta t\,. \tag{21}$$

d) Physical Interpretation and Transformation in Other Systems of Coordinates

By comparing equation (18) with the original equation (6), it is found that equation (18) represents the centered-difference form of the original equations supplemented by the weighted effects of perturbations traveling along the nearcharacteristics. The terms associated with $\Delta_r \boldsymbol{W}$ and $\Delta_\theta \boldsymbol{W}$ represent the original equations together with $\boldsymbol{S}^*$. Hence, the net weighted effects due to perturbations through the compatibility equations can be identified with the terms containing $\delta_r \boldsymbol{W}$ and $\delta_\theta \boldsymbol{W}$. This identification facilitates the tracing of the physical consequence of a perturbation in the initial-boundary value problem.

The present formulation derived in the system of spherical coordinates is applicable to other two-dimensional problems. For problems in the system of cylindrical coordinates (R, ϕ, x) (where R is the radius, ϕ the aximuthal angle, x the vertical coordinate) with the axial symmetry ($\partial/\partial\phi = 0$), all the equations are valid with the following change of identities for the dependent and independent variables:

$$u \to v_x\,, \qquad v \to v_R\,, \qquad w \to v_\phi\,, \qquad B_r \to B_x\,, \qquad B_\theta \to B_R\,, \qquad B_\phi \to B_\phi\,,$$

$$\frac{\partial}{\partial r} \to \frac{\partial}{\partial x}\,, \qquad \frac{\partial}{r\partial\theta} \to \frac{\partial}{\partial R}\,, \qquad \frac{dr}{dt} \to \frac{dx}{dt}\,, \qquad r\frac{d\theta}{dt} \to \frac{dR}{dt}\,, \tag{22a}$$

including the following changes for the quantities in $\boldsymbol{S}$:

$$\frac{1}{r} \to 0\,, \qquad \frac{1}{r}\cot\theta \to \frac{1}{R}\cdot \tag{22b}$$

For problems in a system of Cartesian coordinates (x, y, z) (where x is the vertical, y and z the horizontal coordinates) with the condition of two-dimensionality represented by $\partial/\partial z = 0$, all the equations remain valid with

TABLE 1

SUMMARY OF COMPATIBILITY EQUATIONS

Eigenvalue	$\frac{\partial \rho}{\partial t}$	$\frac{\partial u}{\partial t}$	$\frac{\partial v}{\partial t}$	$\frac{\partial w}{\partial t}$	$\frac{1}{\rho}\frac{\partial \rho}{\partial t}$	$\frac{1}{\sqrt{\rho}}\frac{\partial B_r}{\partial t}$	$\frac{1}{\sqrt{\rho}}\frac{\partial B_\theta}{\partial t}$	$1\sqrt{\rho}\, \partial B_\phi \partial t$	R.H.S.
$\frac{dr}{dt} = u$	a^2	...	...	...	$-\rho$	...	...	...	λ_1
$\frac{dr}{dt} = u$	...	...	...	...	...	1	...	...	λ_2
$\frac{dr}{dt} = u + U_A$	...	...	$-b_r b_\phi$	$b_r{}^{b}{}_{\theta}$	...	...	$b_\phi U_A$	$-b_\theta U_A$	λ_3
$\frac{dr}{dt} = u - U_A$	...	...	$b_r b_\phi$	$-b_r b_\theta$	...	...	$b_\phi U_A$	$-b_\theta U_A$	λ_4
$\frac{dr}{dt} = u + U_f$	...	$U_f(U_f^2 - b_r^2)$	$-b_r b_\theta U_f$	$-b_r b_\phi U_f$	$U_f^2 - b_r^2$	...	$b_\theta U_f^2$	$b_\phi U_f^2$	λ_5
$\frac{dr}{dt} = u - U_f$	...	$-U_f(U_f^2 - b_r^2)$	$b_r b_\theta U_f$	$b_r b_\phi U_f$	$U_f^2 - b_r^2$	...	$b_\theta U_f^2$	$b_\phi U_f^2$	λ_6
$\frac{dr}{dt} = u + U_s$	...	$U_s(b_r^2 - U_s^2)$	$b_r b_\theta U_s$	$b_r b_\phi U_s$	$b_r^2 - U_s^2$	...	$-b_\theta U_s^2$	$-b_\phi U_s^2$	λ_7
$\frac{dr}{dt} = u - U_s$	...	$-U_s(b_r^2 - U_s^2)$	$-b_r b_\theta U_s$	$-b_r b_\phi U_s$	$b_r^2 - U_s^2$	...	$-b_\theta U_s^2$	$-b_\phi U_s^2$	λ_8
$r\frac{d\theta}{dt} = v$	a^2	...	...	...	$-\rho$	...	...	...	η_1
$r\frac{d\theta}{dt} = v$	...	...	...	...	...	...	1	...	η_2
$r\frac{d\theta}{dt} = v + V_A$	...	$-b_\theta b_\phi$	...	$b_r b_\theta$	...	$b_\phi V_A$	...	$-b_r V_A$	η_3
$r\frac{d\theta}{dt} = v - V_A$	...	$b_\theta b_\phi$	...	$-b_r b_\theta$	...	$b_\phi V_A$	...	$-b_r V_A$	η_4
$r\frac{d\theta}{dt} = v + V_f$	...	$-b_r b_\theta V_f$	$V_f(V_f^2 - b_\theta^2)$	$-b_\theta b_\phi V_f$	$V_f^2 - b_\theta^2$	$b_r V_f^2$	...	$b_\phi V_f^2$	η_5
$r\frac{d\theta}{dt} = v - V_f$	...	$b_r b_\theta V_f$	$-V_f(V_f^2 - b_\theta^2)$	$b_\theta b_\phi V_f$	$V_f^2 - b_\theta^2$	$b_r V_f^2$	...	$b_\phi V_f^2$	η_6
$r\frac{d\theta}{dt} = v + V_s$	...	$b_r b_\theta V_s$	$V_s(b_0^2 - V_s^2)$	$b_\theta b_\phi V_s$	$b_\theta^2 - V_s^2$	$-b_r V_s^2$	...	$-b_\phi V_s^2$	η_7
$r\frac{d\theta}{dt} = v - V_s$	...	$-b_r b_\theta V_s$	$-V_s(b_\theta^2 - V_s^2)$	$-b_\theta b_\phi V_s$	$b_\theta^2 - V_s^2$	$-b_r V_s^2$	...	$-b_\phi V_s^2$	η_8

the following change of identities:

$$u \to v_x\,, \quad v \to v_y\,, \quad w \to v_z\,, \quad B_r \to B_x\,, \quad B_\theta \to B_y\,, \quad B_\phi \to B_z\,,$$

$$\frac{\partial}{\partial r} \to \frac{\partial}{\partial x}\,, \quad \frac{\partial}{r\partial\theta} \to \frac{\partial}{\partial y}\,, \quad \frac{dr}{dt} \to \frac{dx}{dt}\,, \quad r\frac{d\theta}{dt} \to \frac{dy}{dt}\,; \tag{23a}$$

also

$$\frac{1}{r} \to 0\,, \quad \frac{1}{r}\cot\theta \to 0\,. \tag{23b}$$

IV. EXAMPLES OF APPLICATION TO INITIAL-BOUNDARY VALUE PROBLEMS

In order to illustrate the validity of the present formulation, let us consider practical examples of MHD initial-boundary problems related to evolution of the solar atmospheric magnetic fields.

a) *Evolution of Magnetic Field Due to Photospheric Shearing Motions*

The evolution of magnetic fields by photospheric shearing motions has been investigated by various authors in terms of topologically continuous force-free fields (Birn, Goldstein, and Schindler 1978; Heyvaerts *et al.* 1979; Low 1977*a*, *b*, 1979; Low and Nakagawa 1975). The basic geometry of the problem is illustrated in Figure 4*a*, where the

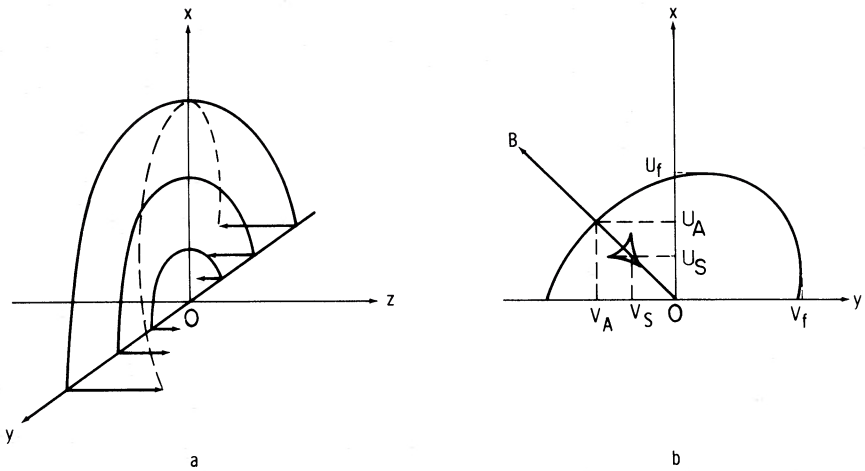

FIG. 4.—(*a*) A schematic evolution of magnetic field by horizontal shearing motions (*indicated by arrows*). Solid curves refer to the initial state, dashed curve the later stage. (*b*) Typical loci of characteristics at $t = 0$ and the foot points of nearcharacteristics at $x = 0$ surface: OB denotes the direction of magnetic field.

notations defined in equation (23a), with the x-axis along the vertical and all physical variables being as functions of x, y, and t.

Suppose that at $t = 0$, the atmosphere is in hydrostatic equilibrium permeated by a potential field. Then the initial magnetic field lines are contained in the (x, y)-plane. Now let us consider the consequence of the following shearing motion at the boundary surface $x = 0$:

$$w = w(y, t)\,, \qquad v = 0\,, \qquad \text{for } t > 0\,, \tag{24}$$

where the condition $v = 0$ imposes the foot points of field lines to remain at the same distance from the (x, z)-plane.

Under such circumstances, to determine the physical consequence at the photospheric level (i.e., $x = 0$), six physically meaningful compatibility equations can be obtained along the nearcharacteristics with their foot points in the (x, y)-plane as shown in Figure 4*b*. These nearcharacteristics are the projections of the loci of real characteristics in the domain $x > 0$. As summarized in Table 2 the compatibility equations can be divided into two sets of three compatibility equations in the (x, t)- and (y, t)-planes. Note that in the last column of Table 2 the quantities λ_l and η_l are represented by the primed quantities, i.e., by λ_4', λ_6', λ_8', η_3', η_6', and η_7', and they are the quantities λ_l and η_l defined in equations (18a) and (18b) and Appendix A with the $(\partial v/\partial t)$-component of the original equations and the terms containing v and $\partial v/\partial y$ set equal to zero.

At first glance, the six relations among five temporal variations in Table 2 appear to be overspecification.

TABLE 2

AVAILABLE COMPATIBILITY EQUATIONS AT $x = 0$ SURFACE WITH $\dfrac{\partial w}{\partial t}$ GIVEN AND $\dfrac{\partial v}{\partial t} = \dfrac{\partial v}{\partial y} = v = 0$

Eigenvalue	$\dfrac{\partial u}{\partial t}$	$\dfrac{1}{\rho}\dfrac{\partial \rho}{\partial t}$	$\dfrac{1}{\sqrt{\rho}}\dfrac{\partial B_x}{\partial t}$	$\dfrac{1}{\sqrt{\rho}}\dfrac{\partial B_y}{\partial t}$	$\dfrac{1}{\sqrt{\rho}}\dfrac{\partial B_z}{\partial t}$	R.H.S.
$\dfrac{dx}{dt} = u - U_A$................	...	...	...	$b_z U_A$	$-b_y U_A$	$b_x b_y \dfrac{\partial w}{\partial t} + \lambda_4'$
$\dfrac{dx}{dt} = u - U_f$................	$-U_f(U_f^2 - b_x^2)$	$U_f^2 - b_x^2$	...	$b_y U_f^2$	$b_z U_f^2$	$-b_x b_z U_f \dfrac{\partial w}{\partial t} + \lambda_6'$
$\dfrac{dx}{dt} = u - U_s$................	$-U_s(b_x^2 - U_s^2)$	$b_x^2 - U_s^2$	...	$-b_y U_s^2$	$-b_z U_s^2$	$+b_x b_z U_s \dfrac{\partial w}{\partial t} + \lambda_8'$
$\dfrac{dy}{dt} = v + V_A$................	$-b_y b_z$	...	$b_z V_A$	...	$-b_x V_A$	$-b_x b_y \dfrac{\partial w}{\partial t} + \eta_3'$
$\dfrac{dy}{dt} = v - V_f$................	$b_x b_y V_f$	$V_f^2 - b_y^2$	$b_x V_f^2$	...	$b_z V_f^2$	$-b_y b_z V_f \dfrac{\partial w}{\partial t} + \eta_6'$
$\dfrac{dy}{dt} = v + V_s$................	$b_x b_y V_s$	$b_y^2 - V_s^2$	$-b_x V_s^2$	...	$-b_z V_s^2$	$-b_y b_z V_s \dfrac{\partial w}{\partial t} + \eta_7'$

However, recalling that the temporal variation can be determined independently in each different direction, we can utilize these six compatibility equations to determine the five temporal variations in the following manner. First, we solve the three compatibility equations in the (x, t)-plane for $\partial p/\partial t$, $\partial B_y/\partial t$, $\partial B_z/\partial t$ as functions of $\partial u/\partial t$, $\partial w/\partial t$, λ_4', λ_6', and λ_8'. Then solve the second set of three compatibility equations in the (y, t)-plane for $\partial u/\partial t$, $\partial B_x/\partial t$ and $\partial B_z/\partial t$ as a function of $\partial w/\partial t$, $\partial p/\partial t$, η_3', η_6', and η_7'. After combining these independently obtained results and eliminating the similar temporal variations, we obtain

$$\frac{\partial u}{\partial t} = L_1 \frac{\partial w}{\partial t} + M_1(\lambda_6', \lambda_8', \eta_6', \eta_7') , \qquad \frac{1}{\rho}\frac{\partial p}{\partial t} = L_2 \frac{\partial w}{\partial t} + M_2(\lambda_6', \lambda_8', \eta_6', \eta_7') ,$$

$$\frac{1}{\sqrt{\rho}}\frac{\partial B_x}{\partial t} = L_3 \frac{\partial w}{\partial t} + M_3(\lambda_6', \lambda_8', \eta_3', \eta_6', \eta_7') ,$$

$$\frac{1}{\sqrt{\rho}}\frac{\partial B_y}{\partial t} = L_4 \frac{\partial w}{\partial t} + M_4(\lambda_4', \lambda_6', \lambda_8', \eta_6', \eta_7') , \qquad \frac{1}{\sqrt{\rho}}\frac{\partial B_z}{\partial t} = L_5 \frac{\partial w}{\partial t} + M_5(\lambda_6', \lambda_6', \eta_3', \eta_6', \eta_7') , \tag{25}$$

where L_i $(i = 1, 2, \ldots, 5)$ are somewhat lengthy expressions given in Appendix D, and M_i $(i = 1, 2, \ldots, 5)$ are complex functions of their argument λ_l' and η_l'.

Equations (25) are the proper and physically self-consistent boundary conditions determining the physical consequence of shearing motion at $x = 0$ in the present initial-boundary value problem. It should be noted that equations (24) and (25) give seven boundary conditions for eight physical variables, leaving the boundary condition on ρ unspecified. However, this ambiguity can be removed with the specification of the equation of state.

Returning to the problem of evolution with the initial condition of a hydrostatic atomosphere permeated by a potential magnetic field, we find at $x = 0$ and $t = 0$

$$\lambda_4' = \lambda_6' = \lambda_8' = \eta_3' = \eta_6' = \eta_7' = 0 , \qquad u = v = w = B_z = b_z = 0 . \tag{26}$$

Then with the introduction of $w(x, y, t)$ for $t >$, the only nonvanishing initial response (denoted, hereafter, by dW) given by equation (26) (see Appendix D) becomes

$$bB_z = -\sqrt{\rho}\, dw \qquad \text{or} \qquad db_z = -dw . \tag{27}$$

In other words, dw produces dB_z of the magnitude equal to the Alfvén speed in the z-direction. For subsequent changes of physical quantities at $x = 0$, as well as $x > 0$, we must pursue numerical solutions because the quantities λ_l' and η_l' no longer remain null.

The above problem resembles the concentration of axisymmetric magnetic flux by rotational shearing motion considered by Nakagawa and Stenflo (1979). The basic geometry of the problem is shown in Figure 5. By following topologically continuous constant α force-free fields, Nakagawa and Stenflo (1979) concluded that the shearing motions lead to the uplift of material from the photospheric level, i.e., $\partial u/\partial t > 0$, $\partial p/\partial t < 0$ for $t > 0$ at $x = 0$. As noted in Appendix D, it is found that such a result could follow from equations (25) by pursuing the responses as a function of $\partial w/\partial t$, while neglecting contributions due to functions M_i $(i = 1, 2, \ldots, 5)$.

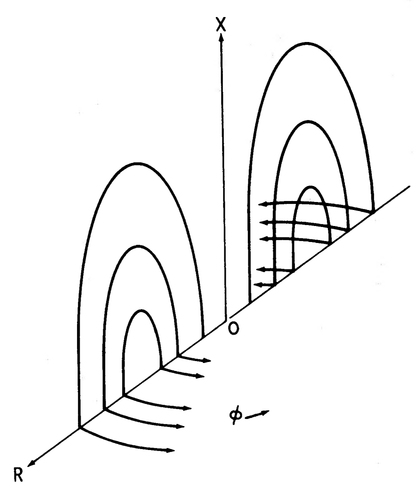

FIG. 5.—A schematic evolution of axisymmetric magnetic field by rotational shearing motions (*indicated by arrows*)

b) *Growth of Axisymmetric Magnetic Fields*

In contrast to the subtle responses found in the evolution of a magnetic field due to fluid motions, more active changes are found with direct growth of a magnetic field (often called the flux emergence). Although various speculations have been made, physically self-consistent treatment of this problem has not yet been known. With the present formulation, it is possible to examine this problem in a physically self-consistent manner. To be more specific, let us consider the growth of an axisymmetric sunspot with no rotational motion. The boundary conditions at $x = 0$ are, then,

$$B_x = B_x(R, t)\,, \qquad w = 0\,, \qquad \text{for } t > 0\,, \tag{28}$$

where the notations refer to those defined in equation (23a), i.e., x the vertical, R the radial, ϕ the azimuthal coordinates, and w the azimuthal velocity. With equation (28), we find six compatibility equations, as in the previous case, available for the determination of the time-dependent boundary conditions at $x = 0$, as summarized in Table 3.

Proceeding as in the previous case, we obtain the following equations, for the determination of temporal variations of physical variables at $x = 0$ and $t \geq 0$:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= L_1'\left(\frac{1}{\sqrt{\rho}}\frac{\partial B_x}{\partial t}\right) + M_1'(\lambda_6', \lambda_8', \eta_4', \eta_5', \eta_8')\,,\\
\frac{\partial v}{\partial t} &= L_2'\left(\frac{1}{\sqrt{\rho}}\frac{\partial B_x}{\partial t}\right) + M_2'(\lambda_6', \lambda_8', \eta_4', \eta_5', \eta_8')\,,\\
\frac{1}{\rho}\frac{\partial p}{\partial t} &= L_3'\left(\frac{1}{\sqrt{\rho}}\frac{\partial B_x}{\partial t}\right) + M_3'(\lambda_6', \lambda_8', \eta_4', \eta_5', \eta_8')\,,\\
\frac{1}{\sqrt{\rho}}\frac{\partial B_R}{\partial t} &= L_4'\left(\frac{1}{\sqrt{\rho}}\frac{\partial B_x}{\partial t}\right) + M_4'(\lambda_4', \lambda_6', \lambda_8', \eta_4', \eta_5', \eta_8')\,,\\
\frac{1}{\sqrt{\rho}}\frac{\partial B_\phi}{\partial t} &= L_5'\left(\frac{1}{\sqrt{\rho}}\frac{\partial B_x}{\partial t}\right) + M_5'(\lambda_6', \lambda_8', \eta_4', \eta_5', \eta_8')\,,
\end{aligned} \tag{29}$$

where L_i' $(i = 1, 2, 5)$ denote somewhat complex functions given in Appendix E and M_i' $(i = 1, \ldots, 5)$ are complex functions of their argument.

Again with the assumption that initially the atmosphere is in the state of hydrostatic equilibrium permeated by a potential magnetic field, the response immediately following the introduction of $B_x(R, t)$ is deduced from equations (29). In the present case, the response becomes functions of the radial distance according to the situations shown in Figure 6.

TABLE 3

AVAILABLE COMPATIBILITY EQUATIONS AT $x = 0$ SURFACE WITH $\dfrac{\partial B_x}{\partial t}$ GIVEN AND $\dfrac{\partial w}{\partial t} = 0$

Eigenvalue	$\dfrac{\partial u}{\partial t}$	$\partial v \partial t$	$\dfrac{1}{\rho}\dfrac{\partial p}{\partial t}$	$\dfrac{1}{\sqrt{\rho}}\dfrac{\partial B_R}{\partial t}$	$\dfrac{1}{\sqrt{\rho}}\dfrac{\partial B_\phi}{\partial t}$	R.H.S.
$\dfrac{dz}{dt} = u - U_A$	...	$b_z b_\phi$	...	$b_\phi U_A$	$-b_R U_A$	λ_4'
$\dfrac{dz}{dt} = u - U_f$	$-U_f(U_f^2 - b_z^2)$	$b_R b_z U_f$	$U_f^2 - b_z^2$	$b_R U_f^2$	$b_\phi U_f^2$	λ_6'
$\dfrac{dz}{dt} = u - U_s$	$-U_s(b_z^2 - U_s^2)$	$-b_R b_z U_s$	$b_z^2 - U_s^2$	$-b_R U_s^2$	$-b_\phi U_s^2$	λ_8'
$\dfrac{dR}{dt} = v - V_A$	$b_R b_\phi$	...	...	...	$-b_z V_A$	$-b_\phi V_A \dfrac{1}{\sqrt{\rho}}\dfrac{\partial B_z}{\partial t} + \eta_4'$
$\dfrac{dR}{dt} = v + V_f$	$-b_R b_z V_f$	$V_f(V_f^2 - b_R^2)$	$V_f^2 - b_R^2$	...	$b_\phi V_f^2$	$-b_z V_f^2 \dfrac{1}{\sqrt{\rho}}\dfrac{\partial B_z}{\partial t} + \eta_5'$
$\dfrac{dR}{dt} = v - V_s$	$-b_R b_z V_s$	$-V_s(b_R^2 - V_s^2)$	$b_R^2 - V_s^2$	...	$-b_\phi V_s^2$	$b_z V_s^2 \dfrac{1}{\sqrt{\rho}}\dfrac{\partial B_z}{\partial t} + \eta_8'$

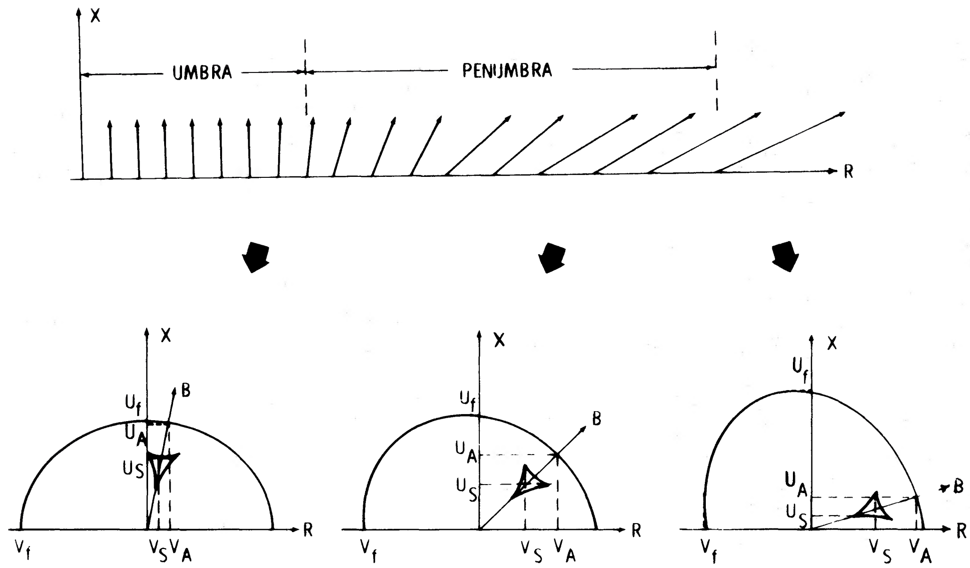

FIG. 6.—A schematic configuration of axisymmetric sunspot magnetic field and variation of parameters with radial distance

In the umbra, with $b_x = U_A \gg b_R = V_A$, and $U_A > a$, after evaluating contributions from various terms in L^i $(i = 1, \ldots, 5)$, we find

$$du \approx -\left(\frac{U_A}{a}\right)\frac{1}{\sqrt{\rho}}\,dB_x\,, \qquad dv \approx -\left(\frac{a}{U_A}\right)\frac{1}{\sqrt{\rho}}\,dB_x\,, \qquad dp \approx -\sqrt{\rho}\,U_A dB_x\,, \qquad dB_R = dB_\phi \approx 0\,. \tag{30}$$

In the penumbra where the field lines are roughly 45° from the vertical, with $U_A \approx V_A \approx (b/\sqrt{2}) > a$, $b = (b_x^2 + b_R^2)^{1/2}$, the responses become

$$du \approx dv \approx \left(\frac{b}{a}\right)\frac{1}{\sqrt{\rho}}\,dB_x\,, \qquad dp \approx \sqrt{\rho}\,\rho\, b dB_x\,, \qquad dB_R \approx dB_\phi \approx 0\,. \tag{31}$$

Then near the penumbra boundary with the field lines close to the horizontal plane and $U_A \ll V_A^4$, and $V_A > a$, equations (29) yield

$$du \approx \frac{1}{\sqrt{\rho}}\,dB_x\,, \qquad dv \approx \frac{aU_A^2}{V_A^3}\frac{1}{\sqrt{\rho}}\,dB_x\,, \qquad dp \approx \sqrt{\rho}\left(\frac{aU_A}{V_A}\right)dB_x\,, \qquad dB_R \approx dB_x\,, \qquad dB_\phi \approx 0\,. \tag{32}$$

Assuming for simplicity that the density, pressure, and temperature increase or decrease simultaneously, the results obtained in equations (30)–(32) can be summarized as follows. The growth of a sunspot is accompanied in the umbra by the appearance of a large downward motion with a slight lateral convergence and large decrese of pressure, temperature, and density. In the penumbra a large outward material motion approximately 45° from the vertical appears together with substantial increases of pressure, temperature, and density, and at the penumbra boundary an outflow of materials ensues with an upward motion and increases of pressure, temperature, and density.

Some of these results are suggestive in interpreting known observations, such as the formation of a bright ring around the penumbra of growing sunpots and the ejection of materials before flares. Also, the downward motions over the umbra have been observed (Zwaan 1978). Applying this result to small-scale phenomena, such as the growth of a magnetic field at the corner of supergranules, the ejection of material can be identified with the spicules having their geometrical properties slightly modified by the peculiar physical conditions at the boundaries of supergranules as discussed by Nakagawa (1977). It is, therefore, evident that the present formulation provides a powerful and physically valid new tool to investigate the physics of solar activities.

V. CONCLUDING REMARKS

The new formulation described in this paper clearly provides a method of systematic analysis of the MHD initial-boundary value problem while removing some of the mathematical complexity which has prevented realistic studies of solar activity. In addition, the validity and physical significance of the present formulations are demonstrated by the examples considered. It is, however, necessary to pursue numerical computations for a complete analysis (which is currently in progress).

In previous theoretical studies the evolution of magnetic fields is considered only in terms of topologically continuous force-free fields, ignoring the response of the atmosphere. In contrast the present method offers the means of examining not only the evolution of non–force-free fields, but also the changes of physical conditions in the atmosphere accompanying the evolution. This is important as the results obtained by the present method can be incorporated into observational verification such as evolutional changes of the optical and radio emissions. This change of physical conditions can provide also the basis for examining different types of plasma, as well as MHD instabilities, namely, physical processes which can induce such dissipative effects in evolution. However, the inclusion of such dissipative effects is clearly the subject of future investigations.

It is evident that the extension of the present formulation to three-dimensional problems poses no basic difficulty, apart from the fact that the inclusion of a third spatial dimension will add another set of eight compatibility equations and an increased complexity in numerical procedures. In this connection, it may be noted that in the difference form, the present formulation can include shock discontinuities which are considered the limit of applicability of the method of characteristics. In the present formulation the shock is replaced by a somewhat diffused transition; however, this fault can be easily remedied by using some sophisticated numerical formulations, for example, the scheme of flux-corrected transport by Boris and Book (1976).

Other improvements in the accuracy of computation are possible, for example, with the choice of nearcharacteristics as close as possible to the real characteristics suggested by Werner (1968) and discussed in Appendix C. Similarly, improvements are possible by replacing the present *explicit* scheme by an *implicit* scheme (see, Kneer and Nakagawa 1976). However, these subjects are clearly beyond the scope of the present paper, and such problems and the inclusion of additional physical processes will be considered in future papers together with the comparisons of numerical results with observations.

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APPENDIX A

Along each specific nearcharacteristic the following compatibility equations result from equations (16a) and (16b). Along $dr/dt = u$,

$$a^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right) - \left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r}\right) = -a^2 v\frac{\partial \rho}{r\partial \theta} + v\frac{\partial p}{r\partial \theta} - (\gamma - 1)\Delta Q\,. \tag{A1}$$

Along $dr/dt = u$,

$$\frac{1}{\sqrt{\rho}}\left(\frac{\partial B_r}{\partial t} + u\frac{\partial B_r}{\partial t}\right) = b_\theta\frac{\partial u}{r\partial \theta} - b_r\frac{\partial v}{r\partial \theta} - v\frac{\partial B_r}{\sqrt{\rho}\,r\partial \theta} - \frac{b_r}{r}(2u + v\cot\theta)\,. \tag{A2}$$

Along $dr/dt = u \pm U_A$,

$$\begin{aligned}
&\mp b_r b_\phi\left[\frac{\partial v}{\partial t} + (u \pm U_A)\frac{\partial v}{\partial r}\right] \pm b_r b_\theta\left[\frac{\partial w}{\partial t} + (u \pm U_A)\frac{\partial w}{\partial r}\right] \\
&\quad + \frac{b_\phi U_A}{\sqrt{\rho}}\left[\frac{\partial B_\theta}{\partial t} + (u \pm U_A)\frac{\partial B_\theta}{\partial r}\right] - \frac{b_\theta U_A}{\sqrt{\rho}}\left[\frac{\partial B_\phi}{\partial t} + (u \pm U_A)\frac{\partial B_\phi}{\partial r}\right] \\
&\quad = (b_\theta U_A \pm b_r v)\left[b_\phi\frac{\partial v}{r\partial \theta} - b_\theta\frac{\partial w}{r\partial \theta}\right] \pm b_r b_\phi\frac{\partial p}{\rho r\partial \theta} \pm b_r^2 b_\phi\frac{\partial B_r}{\sqrt{\rho}\,r\partial \theta} - b_\phi U_A v\frac{\partial B_\theta}{\sqrt{\rho}\,r\partial \theta} \\
&\quad + [b_\theta U_A v \pm b_r(b_\theta^2 + b_\phi^2)]\frac{\partial B_\phi}{\sqrt{\rho}\,r\partial \theta} - \frac{U_A}{r}(b_\phi v - b_\theta w)(b_r + b_\theta\cot\theta) \\
&\quad \pm \frac{b_r u}{r}(b_\phi v - b_\theta w) \mp \frac{b_r w}{r}(b_\theta v + b_\phi w)\cot\theta \pm \frac{b_r b_\phi}{r}(b_\theta^2 + b_\phi^2)\cot\theta\,,
\end{aligned} \tag{A3}$$

Here the upper and lower signs correspond to the upper and lower signs of the eigenvalues, i.e., $u \pm U_A$. This convention is used hereafter in all applicable cases.

Along $dr/dt = u \pm U_f$,

$$
\begin{aligned}
&\pm(U_f{}^2 - b_r{}^2)U_f\left[\frac{\partial u}{\partial t} + (u \pm U_f)\frac{\partial u}{\partial r}\right] \mp b_r b_\theta U_f\left[\frac{\partial v}{\partial t} + (u \pm U_f)\frac{\partial v}{\partial r}\right] \\
&\quad \mp b_r b_\phi U_f\left[\frac{\partial w}{\partial t} + (u \pm U_f)\frac{\partial w}{\partial r}\right] + \frac{(U_f{}^2 - b_r{}^2)}{\rho}\left[\frac{\partial p}{\partial t} + (u \pm U_f)\frac{\partial p}{\partial r}\right] \\
&\quad + \frac{b_\theta U_f{}^2}{\sqrt{\rho}}\left[\frac{\partial B_\theta}{\partial t} + (u \pm U_f)\frac{\partial B_\theta}{\partial r}\right] + \frac{b_\phi U_f{}^2}{\sqrt{\rho}}\left[\frac{\partial B_\phi}{\partial t} + (u \pm U_f)\frac{\partial B_\phi}{\partial r}\right] \\
&\quad = \mp(U_f{}^2 - b_r{}^2)U_f v\frac{\partial u}{r\partial\theta} - U_f[U_f(U_f{}^2 - b_r{}^2 - b_\theta{}^2) \mp b_r b_\theta v]\frac{\partial v}{r\partial\theta} \\
&\quad + b_\phi U_f(b_\theta U_f \pm b_r v)\frac{\partial w}{r\partial\theta} - [(U_f{}^2 - b_r{}^2)v \mp b_r b_\theta U_f]\frac{\partial p}{\rho r\partial\theta} \\
&\quad \pm b_\theta U_f{}^3\frac{\partial B_r}{\sqrt{\rho}\, r\partial\theta} - b_\theta U_f{}^2 v\frac{1}{\sqrt{\rho}}\frac{1}{r}\frac{\partial B_\theta}{\partial\theta} - b_\phi U_f{}^2 v\frac{\partial B_\phi}{\sqrt{\rho}\, r\partial\theta} \\
&\quad - \frac{u}{r}U_f{}^2(2U_f{}^2 - b^2 - b_r{}^2) - \frac{v}{r}U_f{}^2(U_f{}^2 - b^2)\cot\theta - \frac{U_f{}^2}{r}(b_\theta v + b_\phi w)(b_r + b_\theta\cot\theta) \\
&\quad \pm\frac{u}{r}(b_\theta v + b_\phi w)b_r U_f \pm\frac{w}{r}(b_\phi v - b_\theta w)b_r U_f\cot\theta \pm\frac{U_f}{r}(v^2 + w^2)(U_f{}^2 - b_r{}^2) \\
&\quad \mp\frac{U_f{}^3}{r}(b_\theta{}^2 + b_\phi{}^2) \mp (U_f{}^2 - b_r{}^2)U_f g + \frac{1}{\rho}(U_f{}^2 - b_r{}^2)(\gamma - 1)\Delta Q\,.
\end{aligned}
\tag{A4}
$$

Along $dr/dt = u \pm U_s$,

$$
\begin{aligned}
&\pm(b_r{}^2 - U_s{}^2)U_s\left[\frac{\partial u}{\partial t} + (u \pm U_s)\frac{\partial u}{\partial r}\right] \pm b_r b_\theta U_s\left[\frac{\partial v}{\partial t} + (u \pm U_s)\frac{\partial v}{\partial r}\right] \\
&\quad \pm b_r b_\phi U_s\left[\frac{\partial w}{\partial t} + (u \pm U_s)\frac{\partial w}{\partial r}\right] + \frac{(b_r{}^2 - U_s{}^2)}{\rho}\left[\frac{\partial p}{\partial t} + (u \pm U_s)\frac{\partial p}{\partial r}\right] \\
&\quad - \frac{b_\theta U_s{}^2}{\sqrt{\rho}}\left[\frac{\partial B_\theta}{\partial t} + (u \pm U_s)\frac{\partial B_\theta}{\partial r}\right] - \frac{b_\phi U_s{}^2}{\sqrt{\rho}}\left[\frac{\partial B_\phi}{\partial t} + (u \pm U_s)\frac{\partial B_\phi}{\partial r}\right] \\
&\quad = \mp(b_r{}^2 - U_s{}^2)U_s v\frac{\partial u}{r\partial\theta} - U_s[U_s(b_r{}^2 + b_\theta{}^2 - U_s{}^2) \pm b_r b_\theta v]\frac{\partial v}{r\partial\theta} \\
&\quad - b_\phi U_s(b_\theta U_s \pm b_r v)\frac{\partial w}{r\partial\theta} - [(b_r{}^2 - U_s{}^2)v \pm b_r b_\theta U_s]\frac{\partial p}{\rho r\partial\theta} \\
&\quad \mp b_\theta U_s{}^3\frac{\partial B_r}{\sqrt{\rho}\, r\partial\theta} + b_\theta U_s{}^2 v\frac{\partial B_\theta}{\sqrt{\rho}\, r\partial\theta} + b_\phi U_s{}^2 v\frac{\partial B_\phi}{\sqrt{\rho}\, r\partial\theta} \\
&\quad - \frac{u}{r}U_s{}^2(b^2 + b_R{}^2 - 2U_s{}^2) - \frac{v}{r}U_s{}^2(b^2 - U_s{}^2)\cot\theta + \frac{U_s{}^2}{r}(b_\theta v + b_\phi w)(b_r + b_\theta\cot\theta) \\
&\quad \mp\frac{u}{r}(b_\theta v + b_\phi w)b_r U_s \mp\frac{w}{r}b_r U_s(b_\phi v - b_\theta w)\cot\theta \pm\frac{U_s}{r}(v^2 + w^2)(b_r{}^2 - U_s{}^2) \\
&\quad \pm\frac{U_s{}^3}{r}(b_\theta{}^2 + b_\phi{}^2) \mp (b_r{}^2 - U_s{}^2)U_s g + \frac{1}{\rho}(b_r{}^2 - U_s{}^2)(\gamma - 1)\Delta Q\,.
\end{aligned}
\tag{A5}
$$

Along $r(d\theta/dt) = v$,

$$
a^2\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{r\partial\theta}\right) - \left(\frac{\partial p}{\partial t} + v\frac{\partial p}{r\partial\theta}\right) = -a^2 u\frac{\partial \rho}{\partial r} + u\frac{\partial p}{\partial r} - (\gamma - 1)\Delta Q\,. \tag{A6}
$$

Along $r(d\theta/dt) = v$,

$$\frac{1}{\sqrt{\rho}}\left(\frac{\partial B_\theta}{\partial t} + v\frac{\partial B_\theta}{r\partial\theta}\right) = -b_\theta\frac{\partial u}{\partial r} + b_r\frac{\partial v}{\partial r} - u\frac{\partial B_\theta}{\sqrt{\rho}\,\partial r} - \frac{ub_\theta}{r} - \frac{v}{r}(b_r + b_\theta\cot\theta)\,. \tag{A7}$$

Along $r(d\theta/dt) = v \pm V_A$,

$$\begin{aligned}
&\mp b_\theta b_\phi\left[\frac{\partial u}{\partial t} + (v \pm V_A)\frac{\partial u}{r\partial\theta}\right] \pm b_r b_\theta\left[\frac{\partial w}{\partial t} + (v \pm V_A)\frac{\partial w}{r\partial\theta}\right] \\
&\quad + \frac{b_\phi V_A}{\sqrt{\rho}}\left[\frac{\partial B_r}{\partial t} + (v \pm V_A)\frac{\partial B_r}{r\partial\theta}\right] - \frac{b_r V_A}{\sqrt{\rho}}\left[\frac{\partial B_\phi}{\partial t} + (v \pm V_A)\frac{\partial B_\phi}{r\partial\theta}\right] \\
&= (b_r V_A \pm b_\theta u)\left[b_\phi\frac{\partial u}{\partial r} - b_r\frac{\partial w}{\partial r}\right] \pm b_\theta b_\phi\frac{\partial p}{\rho\partial r} - b_\phi V_A u\frac{\partial B_r}{\sqrt{\rho}\,\partial r} \pm b_\theta^2 b_\phi\frac{\partial B_\theta}{\sqrt{\rho}\,\partial r} \\
&\quad + [b_r V_A u \pm b_\theta(b_r^2 + b_\phi^2)]\frac{\partial B_\phi}{\sqrt{\rho}\,\partial r} - \frac{b_r V_A}{r}(b_\phi v - b_\theta w)\cot\theta \\
&\quad - \frac{b_r V_A}{r}(b_\phi u - b_r w) \mp \frac{b_\theta b_\phi}{r}[(v^2 + w^2) - (b_\theta^2 + b_\phi^2)] \\
&\quad \mp \frac{b_r b_\theta}{r}[(u + v\cot\theta)w - b_\phi(b_r + b_\theta\cot\theta)] \pm b_\theta b_\phi g\,.
\end{aligned} \tag{A8}$$

Along $r(d\theta/dt) = v \pm V_f$,

$$\begin{aligned}
&\mp b_r b_\theta V_f\left[\frac{\partial u}{\partial t} + (v \pm V_f)\frac{\partial u}{r\partial\theta}\right] \pm (V_f^2 - b_\theta^2)V_f\left[\frac{\partial v}{\partial t} + (v \pm V_f)\frac{\partial v}{r\partial\theta}\right] \\
&\quad \mp b_\theta b_\phi V_f\left[\frac{\partial w}{\partial t} + (v \pm V_f)\frac{\partial w}{r\partial\theta}\right] + \frac{(V_f^2 - b_\theta^2)}{\rho}\left[\frac{\partial p}{\partial t} + (v \pm V_f)\frac{\partial p}{r\partial\theta}\right] \\
&\quad + \frac{b_r V_f^2}{\sqrt{\rho}}\left[\frac{\partial B_r}{\partial t} + (v \pm V_f)\frac{\partial B_r}{r\partial\theta}\right] + \frac{b_\phi V_f^2}{\sqrt{\rho}}\left[\frac{\partial B_\phi}{\partial t} + (v \pm V_f)\frac{\partial B_\phi}{r\partial\theta}\right] \\
&= -V_f[V_f(V_f^2 - b_r^2 - b_\theta^2) \mp b_r b_\theta u]\frac{\partial u}{\partial r} \mp (V_f^2 - b_\theta^2)V_f u\frac{\partial v}{\partial r} \\
&\quad + b_\phi V_f(b_r V_f \pm b_\theta u)\frac{\partial w}{\partial r} - [(V_f^2 - b_\theta^2)u \mp b_r b_\theta V_f]\frac{\partial p}{\rho\partial r} \\
&\quad - b_r V_f^2 u\frac{\partial B_r}{\sqrt{\rho}\,\partial r} \pm b_r V_f^3\frac{\partial B_\theta}{\sqrt{\rho}\,\partial r} - b_\phi V_f^2 u\frac{\partial B_\phi}{\sqrt{\rho}\,\partial r} \\
&\quad - \frac{u}{r}V_f^2(2V_f^2 - 2b_\theta^2 - b_\phi^2) - \frac{v}{r}V_f^2(V_f^2 - b_\theta^2 - b_\phi^2)\cot\theta - \frac{w}{r}b_\phi V_f^2(b_r + b_\theta\cot\theta) \\
&\quad \mp \frac{u}{r}V_f[(V_f^2 - b_\theta^2)v - b_\theta b_\phi w] \pm \frac{w}{r}V_f[b_\theta b_\phi v + (V_f^2 - b_\theta^2)w]\cot\theta \mp \frac{V_f}{r}b_r b_\theta(v^2 + w^2) \\
&\quad \pm \frac{V_f^3}{r}(b_r b_\theta - b_\phi^2\cot\theta) \pm b_r b_\theta V_f g + \frac{1}{\rho}(V_f^2 - b_\theta^2)(\gamma - 1)\Delta Q\,.
\end{aligned} \tag{A9}$$

Along $r(d\theta/dt) = v \pm V_s$,

$$\begin{aligned}
&\pm b_r b_\theta V_s\left[\frac{\partial u}{\partial t} + (v \pm V_s)\frac{\partial u}{r\partial\theta}\right] \pm (b_\theta^2 - V_s^2)V_s\left[\frac{\partial v}{\partial t} + (v \pm V_s)\frac{\partial v}{r\partial\theta}\right] \\
&\quad \pm b_\theta b_\phi V_s\left[\frac{\partial w}{\partial t} + (v \pm V_s)\frac{\partial w}{r\partial\theta}\right] + \frac{(b_\theta^2 - V_s^2)}{\rho}\left[\frac{\partial p}{\partial t} + (v \pm V_s)\frac{\partial p}{r\partial\theta}\right]
\end{aligned}$$

$$-\frac{b_r V_s^2}{\sqrt{\rho}}\left[\frac{\partial B_r}{\partial t} + (v \pm V_s)\frac{\partial B_r}{r\partial\theta}\right] - \frac{b_\phi V_s^2}{\sqrt{\rho}}\left[\frac{\partial B_\phi}{\partial t} + (v \pm V_s)\frac{\partial B_\phi}{r\partial\theta}\right]$$

$$= -V_s[V_s(b_r^2 + b_\theta^2 - V_s^2) \pm b_r b_\theta u]\frac{\partial u}{\partial r} \mp (b_\theta^2 - V_s^2)V_s u\frac{\partial v}{\partial r}$$

$$- b_\phi V_s(b_r V_s \pm b_\theta u)\frac{\partial w}{\partial r} - [(b_\theta^2 - V_s^2)u \pm b_r b_\theta V_s]\frac{\partial p}{\rho\partial r}$$

$$+ b_r V_s^2 u\frac{\partial B_r}{\sqrt{\rho}\,\partial r} \mp b_r V_s^3\frac{\partial B_\theta}{\sqrt{\rho}\,\partial r} + b_\phi V_s^2 u\frac{\partial B_\phi}{\sqrt{\rho}\,\partial r}$$

$$- \frac{u}{r}V_s^2(2b_\theta^2 + b_\phi^2 - 2V_s^2) - \frac{v}{r}V_s^2(b_\theta^2 + b_\phi^2 - V_s^2)\cot\theta + \frac{w}{r}b_\phi V_s^2(b_r + b_\theta\cot\theta)$$

$$\mp \frac{u}{r}V_s[(b_\theta^2 - V_s^2)v + b_\theta b_\phi w] \mp \frac{w}{r}V_s[b_\theta b_\phi v - (b_\theta^2 - V_s^2)w]\cot\theta \pm \frac{V_s}{r}b_r b_\theta(v^2 + w^2)$$

$$\mp \frac{V_s^3}{r}(b_r b_\theta - b_\phi^2\cot\theta) \mp b_r b_\theta V_s g + \frac{1}{\rho}(b_\theta^2 - V_s^2)(\gamma - 1)\Delta Q\,. \qquad \text{(A10)}$$

It should be noted that the quantities λ_l and η_l defined in equations (17a) and (17b) can be obtained from the above equations by simply transfering appropriate terms from the left- to right-hand side.

APPENDIX B

After solving various temporal variations of physical quantities from the compatibility equations in the (r, t)- and (θ, t)-planes separately, the final compensations of the directional effects is achieved by the following combinations:

$$2b_\theta(b_r^2 + b_\phi^2)[U_f U_s(U_f^2 - U_s^2) + V_f V_s(V_f^2 - V_s^2)]\frac{\partial u}{\partial t}$$

$$= -b_r[V_s(b_\theta^2 - V_s^2)(\eta_5 - \eta_6) - V_f(V_f^2 - b_\theta^2)(\eta_7 - \eta_8)]$$

$$+ b_\theta(b_r^2 + b_\phi^2)[U_s(\lambda_5 - \lambda_6) + U_f(\lambda_7 - \lambda_8)] - b_\phi V_f V_s(V_f^2 - V_s^2)(\eta_3 - \eta_4)\,, \qquad \text{(B1)}$$

$$2b_r(b_\theta^2 + b_\phi^2)[U_f U_s(U_f^2 - U_s^2) + V_f V_s(V_f^2 - V_s^2)]\frac{\partial v}{\partial t}$$

$$= b_r(b_\theta^2 + b_\phi^2)[V_s(\eta_5 - \eta_6) + V_f(\eta_7 - \eta_8)]$$

$$- b_\theta[U_s(b_r^2 - U_s^2)(\lambda_5 - \lambda_6) - U_f(U_f^2 - b_r^2)(\lambda_7 - \lambda_8)] - b_\phi U_f U_s(U_f^2 - U_s^2)(\lambda_3 - \lambda_4)\,, \qquad \text{(B2)}$$

$$2[b_r(b_\theta^2 + b_\phi^2)U_f U_s(U_f^2 - U_s^2) + b_\theta(b_r^2 + b_\phi^2)V_f V_s(V_f^2 - V_s^2)]\frac{\partial w}{\partial t}$$

$$= b_r V_f V_s(V_f^2 - V_s^2)(\eta_3 - \eta_4) + b_\theta U_f U_s(U_f^2 - U_s^2)(\lambda_3 - \lambda_4)$$

$$- b_\phi[U_s(b_r^2 - U_s^2)(\lambda_5 - \lambda_6) - U_f(U_f^2 - b_r^2)(\lambda_7 - \lambda_8)]$$

$$- b_\phi[V_s(b_\theta^2 - V_s^2)(\eta_5 - \eta_6) - V_f(V_f^2 - b_\theta^2)(\eta_7 - \eta_8)]\,, \qquad \text{(B3)}$$

$$2[b_r^2(U_f^2 - U_s^2) + b_\theta^2(V_f^2 - V_s^2)]\frac{1}{\rho}\frac{\partial p}{\partial t} = U_s^2(\lambda_5 + \lambda_6) + U_f^2(\lambda_7 + \lambda_8) + V_s^2(\eta_5 + \eta_6) + V_f^2(\eta_7 + \eta_8)\,, \qquad \text{(B4)}$$

$$4b_\theta^2(b_r^2 + b_\phi^2)V_A(V_f^2 - V_s^2)\frac{1}{\sqrt{\rho}}\frac{\partial B_r}{\partial t} = b_r V_A[(b_\theta^2 - V_s^2)(\eta_5 + \eta_6) - (V_f^2 - b_\theta^2)(\eta_7 + \eta_8)]$$

$$+ b_\theta^2 b_\phi(V_f^2 - V_s^2)(\eta_3 + \eta_4) + 2b_\theta^2(b_r^2 + b_\phi^2)V_A(V_f^2 - V_s^2)\lambda_2\,, \qquad \text{(B5)}$$

$$4b_r^2(b_\theta^2 + b_\phi^2)U_A(U_f^2 - U_s^2)\frac{1}{\sqrt{\rho}}\frac{\partial B_\theta}{\partial t} = b_\theta U_A[(b_r^2 - U_s^2)(\lambda_5 + \lambda_6) - (U_f^2 - b_r^2)(\lambda_7 + \lambda_8)]$$
$$+ b_r^2 b_\phi(U_f^2 - U_s^2)(\lambda_3 + \lambda_4) + 2b_r^2(b_\theta^2 + b_\phi^2)U_A(U_f^2 - U_s^2)\eta_2\,, \quad \text{(B6)}$$

$$2[b_r^2(b_\theta^2 + b_\phi^2)U_A(U_f^2 - U_s^2) + b_\theta^2(b_r^2 + b_\phi^2)V_A(V_f^2 - V_s^2)]\frac{1}{\sqrt{\rho}}\frac{\partial B_\phi}{\partial t}$$
$$= -b_r^2 b_\theta(U_f^2 - U_s^2)(\lambda_3 + \lambda_4) - b_r b_\theta^2(V_f^2 - V_s^2)(\eta_3 + \eta_4)$$
$$+ b_\phi U_A[(b_r^2 - U_s^2)(\lambda_5 + \lambda_6) - (U_f^2 - b_r^2)(\lambda_7 + \lambda_8)] + b_\phi V_A[(b_\theta^2 - V_s^2)(\eta_5 + \eta_6) - (V_f^2 - b_\theta^2)(\eta_7 + \eta_8)]\,. \quad \text{(B7)}$$

Finally, the directional compensation of the temporal variation of density is achieved by

$$\frac{\partial \rho}{\partial t} = \frac{1}{a^2}\left[\frac{\partial p}{\partial t} + \tfrac{1}{2}(\lambda_1 + \eta_1)\right]. \quad \text{(B8)}$$

APPENDIX C

The difference equation (19) is obtained by expressing derivatives contained in λ_l and η_l ($l = 3, \ldots, 8$) according to the rules described below. Depending upon the sign of $\sigma^{(l)}$ and $\mu^{(l)}$ as seen in Figure 3, the foot points of nearcharacteristics can be classified into the following four groups. In reference to a grid point (i, j) for $\sigma^{(l)} < 0$, the foot points are located between the radial grid points i and $i + 1$, and for $\sigma^{(l)} > 0$ between i and $i - 1$. Similarly for $\mu^{(l)} < 0$, the foot points are located between j and $j + 1$ and for $\mu^{(l)} > 0$ between j and $j - 1$. Accordingly, the derivatives contained in λ_l and η_l ($l = 3, \ldots, 8$) in equations (17a) and (17b) are given to the lowest order of approximation by the following formulae. Suppressing unnecessary indices,

$$\left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\sigma^{(l)}<0} = \frac{(\boldsymbol{W})_{i+1,j} - (\boldsymbol{W})_{i,j}}{\Delta r}\,, \qquad \left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\sigma^{(l)}>0} = \frac{(\boldsymbol{W})_{i,j} - (\boldsymbol{W})_{i-1,j}}{\Delta r}\,,$$
$$\left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\sigma^{(l)}<0} = \left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\sigma^{(l)}>0} = \frac{1}{r_i}\frac{(\boldsymbol{W})_{i,j+1} - (\boldsymbol{W})_{i,j-1}}{2\Delta\theta}\,; \quad \text{(C1)}$$

$$\left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\mu^{(l)}<0} = \frac{1}{r_i}\frac{(\boldsymbol{W})_{i,j+1} - (\boldsymbol{W})_{i,j}}{\Delta\theta}\,, \qquad \left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\mu^{(l)}>0} = \frac{1}{r_i}\frac{(\boldsymbol{W})_{i,j} - (\boldsymbol{W})_{i,j-1}}{\Delta\theta}\,,$$
$$\left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\mu^{(l)}<0} = \left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\mu^{(l)}>0} = \frac{(\boldsymbol{W})_{i+1,j} - (\boldsymbol{W})_{i-1,j}}{2\Delta r}\,. \quad \text{(C2)}$$

From these definitions it follows that

$$\Delta_r \boldsymbol{W} = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\sigma^{(l)}<0} + \left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\sigma^{(l)}>0}\right] = \left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\mu^{(l)}<0} = \left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\mu^{(l)}>0}\,,$$
$$\Delta_\theta \boldsymbol{W} = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\mu^{(l)}<0} + \left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\mu^{(l)}>0}\right] = \left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\sigma^{(l)}<0} = \left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\sigma^{(l)}>0}\,,$$
$$\delta_r \boldsymbol{W} = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\sigma^{(l)}<0} - \left(\frac{\partial \boldsymbol{W}}{\partial r}\right)_{\sigma^{(l)}>0}\right], \qquad d_\theta \boldsymbol{W} = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\mu^{(l)}<0} - \left(\frac{\partial \boldsymbol{W}}{r\partial \theta}\right)_{\mu^{(l)}>0}\right]. \quad \text{(C3)}$$

Then by assembling these derivatives the final form of equation (18) is obtained.

As noted in the text, the terms associated with $(\Delta_r \boldsymbol{W})$ and $(\Delta_\theta \boldsymbol{W})$ reproduce the original equations. Therefore, the components of matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ are identical with those of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, except for the change of signs due to transfer of the terms from the left- to right-hand side. Explicitly, they are

$$C_{11} = -u\,, \quad C_{12} = -\rho\,, \quad C_{22} = -u\,, \quad C_{25} = -\frac{1}{\rho}\,, \quad C_{27} = -\frac{B_\theta}{\rho}\,,$$
$$C_{28} = -\frac{B_\phi}{\rho}\,, \quad C_{33} = -u\,, \quad C_{37} = \frac{B_r}{\rho}\,, \quad C_{44} = -u\,, \quad C_{48} = \frac{B_r}{\rho}\,,$$
$$C_{52} = -a^2\rho\,, \quad C_{55} = -u\,, \quad C_{66} = -u\,, \quad C_{72} = -B_\theta\,, \quad C_{73} = B_r\,,$$
$$C_{77} = -u\,, \quad C_{82} = -B_\phi\,, \quad C_{84} = B_r\,, \quad C_{88} = -u\,; \quad \text{(C4)}$$

$$D_{11} = -v\,, \quad D_{13} = -\rho\,, \quad D_{22} = -v\,, \quad D_{26} = \frac{B_\theta}{\rho}\,, \quad D_{33} = -v\,,$$
$$D_{35} = -\frac{1}{\rho}\,, \quad D_{36} = -\frac{B_r}{\rho}\,, \quad D_{38} = -\frac{B_\phi}{\rho}\,, \quad D_{44} = -v\,, \quad D_{48} = \frac{B_\theta}{\rho}\,,$$
$$D_{53} = -a^2\rho\,, \quad D_{55} = -v\,, \quad D_{62} = B_\theta\,, \quad D_{63} = -B_r\,, \quad D_{66} = -v\,,$$
$$D_{77} = -v\,, \quad D_{83} = -B_\phi\,, \quad D_{84} = B_\theta\,, \quad D_{88} = -v\,, \tag{C5}$$

and all other components are null.

The determination of the components of matrices $\boldsymbol{E}$ and $\boldsymbol{F}$ requires considerable algebraic manipulation. Excluding the terms E_{1m} and F_{1m} $(m = 1, 2 \ldots, 8)$ which can be obtained from E_{5m} and F_{5m} by dividing by a^2, other nonvanishing components are:

$$\begin{aligned}
G_1E_{22} &= ab_\theta(b_r^2 + b_\phi^2)(a^2 + b_\theta^2 + b_\phi^2 + aU_A)U_A(U_f - U_s)\,,\\
G_1E_{23} &= -ab_rb_\theta^2(b_r^2 + b_\phi^2)U_A(U_f - U_s)\,,\\
G_1E_{24} &= -ab_rb_\theta b_\phi(b_r^2 + b_\phi^2)U_A(U_f - U_s)\,,\\
G_1E_{25} &= \frac{1}{\rho}\,b_\theta(b_r^2 + b_\phi^2)(a + U_A)U_A(U_f - U_s)u\,,\\
G_1E_{27} &= \frac{1}{\sqrt{\rho}}\,ab_\theta^2(b_r^2 + b_\phi^2)U_A(U_f - U_s)u\,,\\
G_1E_{28} &= \frac{1}{\sqrt{\rho}}\,ab_\theta b_\phi(b_r^2 + b_\phi^2)U_A(U_f - U_s)u\,;
\end{aligned} \tag{C6}$$

$$\begin{aligned}
G_2E_{32} &= -ab_r^2b_\theta(b_\theta^2 + b_\phi^2)U_A(U_f - U_s)\,,\\
G_2E_{33} &= ab_r^3[b_\phi^2(U_f + U_s) + b_\theta^2(a + U_A)](U_f - U_s)\,,\\
G_2E_{34} &= -ab_r^3b_\theta b_\phi[U_f + U_s - (a + U_A)](U_f - U_s)\,,\\
G_2E_{35} &= \frac{1}{\rho}\,b_r^2b_\theta(b_\theta^2 + b_\phi^2)(U_f - U_s)u\,,\\
G_2E_{37} &= -\frac{1}{\sqrt{\rho}}\,ab_r^2[b_\phi^2(U_f + U_s) + b_\theta^2(a + U_A)](U_f - U_s)u\,,\\
G_2E_{38} &= \frac{1}{\sqrt{\rho}}\,ab_r^2b_\theta b_\phi[U_f + U_s - (a + U_A)](U_f - U_s)u\,;
\end{aligned} \tag{C7}$$

$$\begin{aligned}
G_3E_{42} &= -ab_r^2b_\phi(b_\theta^2 + b_\phi^2)U_A(U_f - U_s)\,,\\
G_3E_{43} &= -ab_r^3b_\theta b_\phi[U_f + U_s - (a + U_A)](U_f - U_s)\,,\\
G_3E_{44} &= ab_r^3[b_\theta^2(U_f + U_s) + b_\phi^2(a + U_A)](U_f - U_s)\,,\\
G_3E_{45} &= \frac{1}{\rho}\,b_r^2b_\phi(b_\theta^2 + b_\phi^2)(U_f - U_s)u\,,\\
G_3E_{47} &= \frac{1}{\sqrt{\rho}}\,ab_r^2b_\theta b_\phi[U_f + U_s - (a + U_A)](U_f - U_s)u\,,\\
G_3E_{48} &= -\frac{1}{\sqrt{\rho}}\,ab_r^2[b_\theta^2(U_f + U_s) + b_\phi^2(a + U_A)](U_f - U_s)u\,;
\end{aligned} \tag{C8}$$

$$\begin{aligned}
G_4E_{52} &= \rho ab_r^2(a + U_A)(U_f - U_s)u\,, &\quad G_4E_{53} &= \rho ab_rb_\theta U_A(U_f - U_s)u\,,\\
G_4E_{54} &= \rho ab_rb_\phi U_A(U_f - U_s)u\,, &\quad G_4E_{55} &= ab_r^2(a + U_A)(U_f - U_s)\,,\\
G_4E_{57} &= \sqrt{\rho}\,a^2b_r^2b_\theta(U_f - U_s)\,, &\quad G_4E_{58} &= \sqrt{\rho}\,a^2b_r^2b_\phi(U_f - U_s)\,;
\end{aligned} \tag{C9}$$

$$
\begin{aligned}
2G_6E_{72} &= \sqrt{\rho}\, b_r^2 b_\theta (b_\theta^2 + b_\phi^2) U_A (U_f - U_s) u\,, \\
2G_6E_{73} &= -\sqrt{\rho}\, b_r^3 [b_\phi^2 (U_f + U_s) + b_\theta^2 (a + U_A)](U_f - U_s) u\,, \\
2G_6E_{74} &= \sqrt{\rho}\, b_r^3 b_\theta b_\phi [U_f + U_s - (a + U_A)](U_f - U_s) u\,, \\
2G_6E_{75} &= \frac{1}{\sqrt{\rho}} b_r^2 b_\theta (b_\theta^2 + b_\phi^2) U_A (U_f - U_s)\,, \\
2G_6E_{77} &= b_r^2 [b_r^2 b_\phi^2 (U_f + U_s) + b_\theta^2 (b^2 + aU_A) U_A](U_f - U_s)\,, \\
2G_6E_{78} &= -b_r^2 b_\theta b_\phi [b_r^2 (U_f + U_s) - (b^2 + aU_A) U_A](U_f - U_s)\,;
\end{aligned}
\tag{C10}
$$

$$
\begin{aligned}
G_7E_{82} &= \sqrt{\rho}\, b_r^2 b_\phi (b_\theta^2 + b_\phi^2) U_A (U_f - U_s) u\,, \\
G_7E_{83} &= \sqrt{\rho}\, b_r^3 b_\theta b_\phi [U_f + U_s - (a + U_A)](U_f - U_s) u\,, \\
G_7E_{84} &= -\sqrt{\rho}\, b_r^3 [b_\theta^2 (U_f + U_s) + b_\phi^2 (a + U_A)](U_f - U_s) u\,, \\
G_7E_{85} &= \frac{1}{\sqrt{\rho}} b_r^2 b_\phi (b_\theta^2 + b_\phi^2) U_A (U_f - U_s)\,, \\
G_7E_{87} &= -b_r^2 [b_\theta b_\phi b_r^2 (U_f + U_s) - (aU_A + b^2) U_A](U_f - U_s)\,, \\
G_7E_{88} &= b_r^2 [b_r^2 b_\theta^2 (U_f + U_s) + b_\phi^2 (aU_A + b^2) U_A](U_f - U_s)\,;
\end{aligned}
\tag{C11}
$$

$$
\begin{aligned}
G_1F_{22} &= ab_\theta^3 [b_\phi^2 (V_f + V_s) + b_r^2 (a + V_A)](V_f - V_s)\,, \\
G_1F_{23} &= -ab_r b_\theta^2 (b_r^2 + b_\phi^2) V_A (V_f - V_s)\,, \\
G_1F_{24} &= -ab_r b_\theta^3 b_\phi [V_f + V_s - (a + V_A)](V_f - V_s)\,, \\
G_1F_{25} &= \frac{1}{\rho} b_r b_\theta^2 (b_r^2 + b_\phi^2)(V_f - V_s) v\,, \\
G_1F_{26} &= -\frac{1}{\sqrt{\rho}} ab_\theta^2 [b_\phi^2 (V_f + V_s) + b_r^2 (a + V_A)](V_f - V_s) v\,, \\
G_1F_{28} &= \frac{1}{\sqrt{\rho}} ab_r b_\theta^2 b_\phi [V_f + V_s - (a + V_A)](V_f - V_s) v\,;
\end{aligned}
\tag{C12}
$$

$$
\begin{aligned}
G_2F_{32} &= -ab_r^2 b_\theta (b_\theta^2 + b_\phi^2) V_A (V_f - V_s)\,, \\
G_2F_{33} &= ab_r (b_\theta^2 + b_\phi^2)(a^2 + b_r^2 + b_\phi^2 + aV_A) V_A (V_f - V_s)\,, \\
G_2F_{34} &= -ab_r b_\theta b_\phi (b_\theta^2 + b_\phi^2) V_A (V_f - V_s)\,, \\
G_2F_{35} &= \frac{1}{\rho} b_r (b_\theta^2 + b_\phi^2)(a + V_A) V_A (V_f - V_s) v\,, \\
G_2F_{36} &= \frac{1}{\sqrt{\rho}} ab_r^2 (b_\theta^2 + b_\phi^2) V_A (V_f - V_s) v\,, \\
G_2F_{38} &= \frac{1}{\sqrt{\rho}} ab_r b_\phi (b_\theta^2 + b_\phi^2) V_A (V_f - V_s) v\,;
\end{aligned}
\tag{C13}
$$

$$
\begin{aligned}
G_3F_{42} &= -ab_r b_\theta^3 b_\phi [V_f + V_s - (a + V_A)](V_f - V_s)\,, \\
G_3F_{43} &= -ab_\theta^2 b_\phi (b_r^2 + b_\phi^2) V_A (V_f - V_s)\,, \\
G_3F_{44} &= ab_\theta^3 [b_r^2 (V_f + V_s) + b_\phi^2 (a + V_A)](V_f - V_s)\,,
\end{aligned}
$$

$$G_3F_{45} = \frac{1}{\rho} b_\theta^2 b_\phi (b_r^2 + b_\phi^2)(V_f - V_s)v\,,$$

$$G_3F_{46} = \frac{1}{\sqrt{\rho}} ab_r b_\theta^2 b_\phi [V_f + V_s - (a + V_A)](V_f - V_s)v\,,$$

$$G_3F_{48} = -\frac{1}{\sqrt{\rho}} ab_\theta^2 [b_r^2(V_f + V_s) + b_\phi^2(a + V_A)](V_f - V_s)v\,; \tag{C14}$$

$$\begin{aligned} G_4F_{52} &= \rho ab_r b_\theta V_A(V_f - V_s)v\,, & G_4F_{53} &= \rho ab_\theta^2(a + V_A)(V_f - V_s)v\,, \\ G_4F_{54} &= \rho ab_\theta b_\phi V_A(V_f - V_s)v\,, & G_4F_{55} &= ab_\theta^2(a + V_A)(V_f - V_s)\,, \\ G_4F_{56} &= \sqrt{\rho}\, a^2 b_r b_\theta^2 (V_f - V_s)\,, & G_4F_{58} &= \sqrt{\rho}\, a^2 b_\theta^2 b_\phi (V_f - V_s)\,; \end{aligned} \tag{C15}$$

$$\begin{aligned} 2G_5F_{62} &= -\sqrt{\rho}\, b_\theta^3[b_\phi^2(V_f + V_s) + b_r^2(a + V_A)](V_f - V_s)v\,, \\ 2G_5F_{63} &= \sqrt{\rho}\, b_r b_\theta^2 (b_r^2 + b_\phi^2) V_A (V_f - V_s)v\,, \\ 2G_5F_{64} &= \sqrt{\rho}\, b_r b_\theta^3 b_\phi [V_f + V_s - (a + V_A)](V_f - V_s)v\,, \\ 2G_5F_{65} &= \frac{1}{\sqrt{\rho}} b_r b_\theta^2 (b_r^2 + b_\phi^2) V_A (V_f - V_s)\,, \\ 2G_5F_{66} &= b_\theta^2 [b_\theta^2 b_\phi^2 (V_f + V_s) + b_r^2(aV_A + b^2)V_A](V_f - V_s)\,, \\ 2G_5F_{68} &= -b_r b_\theta^2 b_\phi [b_\theta^2(V_f + V_s) - (aV_A + b^2)V_A](V_f - V_s)\,; \end{aligned} \tag{C16}$$

$$\begin{aligned} G_7F_{82} &= \sqrt{\rho}\, b_r b_\theta^3 b_\phi [V_f + V_s - (a + V_A)](V_f - V_s)v\,, \\ G_7F_{83} &= \sqrt{\rho}\, b_\theta^2 b_\phi (b_r^2 + b_\phi^2) V_A (V_f - V_s)v\,, \\ G_7F_{84} &= -\sqrt{\rho}\, b_\theta^3 [b_r^2(V_f + V_s) + b_\phi^2(a + V_A)](V_f - V_s)v\,, \\ G_7F_{85} &= \frac{1}{\sqrt{\rho}} b_\theta^2 b_\phi (b_r^2 + b_\phi^2) V_A (V_f - V_s)\,, \\ G_7F_{86} &= -b_r b_\theta^2 b_\phi [b_\theta^2 (V_f + V_s) - (aV_A + b^2)V_A](V_f - V_s)\,, \\ G_7F_{88} &= b_\theta^2 [b_r^2 b_\theta^2 (V_f + V_s) + b_\phi^2 (aV_A + b^2) V_A](V_f - V_s)\,; \end{aligned} \tag{C17}$$

where the common factors G_i ($i = 1, 2, \ldots, 7$) are

$$\begin{aligned} G_1 &= ab_\theta(b_r^2 + b_\phi^2)[U_A(U_f^2 - U_s^2) + V_A(V_f^2 - V_s^2)]\,, \\ G_2 &= ab_r(b_\theta^2 + b_\phi^2)[U_A(U_f^2 - U_s^2) + V_A(V_f^2 - V_s^2)]\,, \\ G_3 &= a[b_r(b_\theta^2 + b_\phi^2)U_A(U_f^2 - U_s^2) + b_\theta(b_r^2 + b_\phi^2)V_A(V_f^2 - V_s^2)]\,, \\ G_4 &= b_r^2(U_f^2 - U_s^2) + b_\theta^2(V_f^2 - V_s^2)\,, \\ G_5 &= b_\theta^2(b_r^2 + b_\phi^2)V_A(V_f^2 - V_s^2)\,, \\ G_6 &= b_r^2(b_\theta^2 + b_\phi^2)U_A(U_f^2 - U_s^2)\,, \\ G_7 &= G_5 + G_6\,. \end{aligned} \tag{C18}$$

In obtaining these results, the following relations, which follow from the defining equation (13), are used:

$$\begin{gathered} U_f U_s = aU_A\,, \qquad V_f V_s = aV_A\,, \qquad U_f^2 + U_s^2 = V_f^2 + V_s^2 = a^2 + b^2\,, \\ U_f^2(U_f^2 - b^2) = a^2(U_f^2 - b_r^2)\,, \qquad U_s^2(b^2 - U_s^2) = a^2(b_r^2 - U_s^2)\,, \\ V_f^2(V_f^2 - b^2) = a^2(V_f^2 - b_\theta^2)\,, \qquad V_s^2(b^2 - V_s^2) = a^2(b_\theta^2 - V_s^2)\,, \\ (U_f^2 - b_r^2)(b_r^2 - U_s^2) = b_r^2(b_\theta^2 + b_\phi^2)\,, \qquad (V_f^2 - b_\theta^2)(b_\theta^2 - V_s^2) = b_\theta^2(b_r^2 + b_\phi^2). \end{gathered} \tag{C19}$$

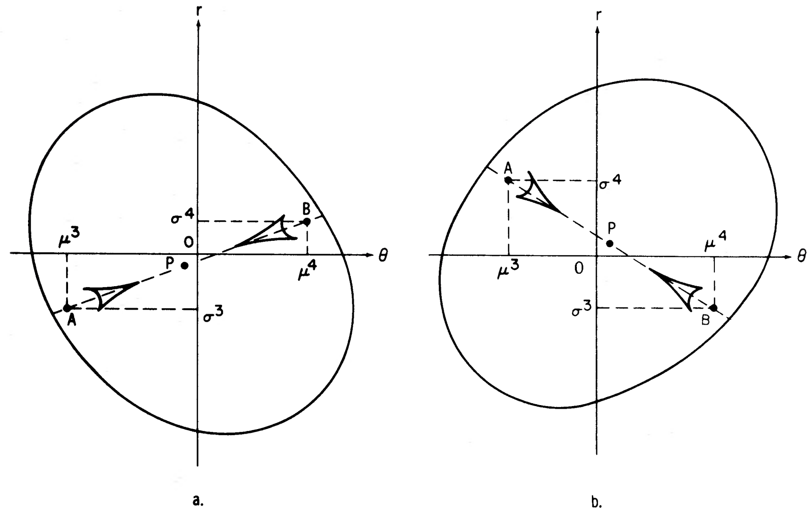

FIG. 7.—The dependence of the foot point of nearcharacteristics on the fluid motion and the direction of magnetic field. (*a*) B_r and B_θ are same signs. (*b*) B_r and B_θ are opposite signs.

It should be noted that each component listed above has its counterpart, as the consequence of directional compensation. In other words, when b_r is replaced by b_θ, U_A by V_A, U_f by V_f, U_s by V_s, and u by v, each component finds its counterpart, as can be seen in the common factors $G_1, G_2, \ldots, G_6$, and G_7.

The components given in the above equations appear lengthy. However, this formulation is a considerable simplification compared with the original formulation suggested by Werner (1968), because the nonisotropic nature of the original loci is removed. If the nearcharacteristics are selected as close as real characteristics, additional terms result in every component, and the additional terms change signs depending upon the direction of the magnetic field and the fluid velocity. For example, including such effects, the component E_{25} becomes

$$\rho G_1 E_{25} = b_\theta(b_r^2 + b_\phi^2)(a + U_A)U_A(U_f - U_s)u + ab_r^2 b_\theta V_A(b_\theta^2 - V_s^2)\left[\operatorname{sgn}\left(\frac{u}{|u|}\right)\right] + b_r b_\theta^2(b_r^2 + b_\phi^2)V_f u[\operatorname{sgn}(B_r, B_\theta)]\,, \tag{C20a}$$

where

$$\left[\operatorname{sgn}\left(\frac{u}{|u|}\right)\right] = \begin{cases} +1 & \text{if } u > 0 \\ -1 & \text{if } u < 0\,, \end{cases} \qquad [\operatorname{sgn}(B_r, B_\theta)] = \begin{cases} +1 & \text{if } B_r \text{ and } B_\theta \text{ are same signs} \\ -1 & \text{if } B_r \text{ and } B_\theta \text{ are opposite signs}\,. \end{cases} \tag{C20b}$$

The cause of this sign change can be seen from the locations of the foot points of nearcharacteristics relative to the origin O in the (r, θ)-plane as shown in Figure 7. When B_r and B_θ are same signs, the foot point A of the nearcharacteristic designed by $dr/dt = u + U_A$ and $r(d\theta/dt) = v + V_A$ is located in the third quadrant. Thus, their derivatives can be evaluated in the manner similar to the points $\sigma^{(l)} > 0$ and $\mu^{(l)} > 0$. However, with the opposite signs of B_r and B_θ, the foot point of the nearcharacteristic $dr/dt = u + U_A$ shifts to the fourth quadrant (i.e., to B), while the foot point of the nearcharacteristic $rd\theta/dt = v + V_A$ moves into the second quadrant (i.e., A) reflecting the nonisotropic manner of propagation of MHD waves. This shift of foot point from the negative to positive domains of the coordinates also depends on the sign of velocity. Therefore, these shifts lead to modification of the difference evaluation of the derivatives and subsequently effect the component of matrices $\boldsymbol{E}$ and $\boldsymbol{F}$. However, with the use of the SK's version, these changes drop out because the location of projected foot points (σ^3 and μ^3 or σ^4 and μ^4) is unaffected by the directions of B_r and B_θ as shown in Figure 7. In addition, the directional compensation removes the dependence on the signs of u and v.

APPENDIX D

The coefficients L_i ($i = 1, 2, \ldots, 5$) in equations (26a)–(26e) are

$$L_1 = \frac{b_z}{b_x}\left[\frac{V_A(V_f - V_s)}{b_y}\frac{N}{R} - 1\right], \tag{D1}$$

$$L_2 = -ab_z\frac{N}{R}, \tag{D2}$$

$$L_3 = -\frac{b_z}{b_x}\left[\frac{V_A}{b_y} + \frac{b_x^2(V_A - a) - b_z^2(V_f - V_s)}{b_x^2 + b_z^2}\frac{N}{R}\right], \tag{D3}$$

$$L_4 = b_z\left[\frac{b_x b_y(a - U_A)}{(b_y^2 + b_z^2)U_A(U_f - U_s)} - \frac{b_y}{b_x(U_f - U_s)} + \frac{V_A(V_f - V_s)}{b_x(U_f + U_s)}\frac{N}{R}\right], \tag{D4}$$

$$L_5 = \frac{V_A}{b_y} - \frac{b_z^2(V_f - V_s + V_A - a)}{b_x^2 + b_z^2}\frac{N}{R}, \tag{D5}$$

with

$$N = 2U_A U_f + a(U_f - U_s) > 0\,, \qquad R = b_x b_y(U_f + U_s) + (a + U_A)V_A(V_f - V_s) > 0\,. \tag{D6}$$

Note that with $U_A \gg a$, $b_x = U_A$, $b_y = -V_A$, $b_z = 0$, $U_f \approx V_f$, and $U_s \approx V_s \approx a$, we obtain

$$L_1 = L_2 = L_3 = L_4 = 0\,, \qquad L_5 = -1\,, \tag{D7}$$

which leads to the result given in equation (27). In cylindrical coordinates with the sign change, i.e., $b_y = V_A$, we find after $b_z > 0$,

$$L_1 > 0\,, \qquad L_2 < 0\,, \tag{D8}$$

which yields the result quoted in the text.

APPENDIX E

The coefficients L_i' $(i = 1, 2, \ldots, 5)$ in equation (29) are

$$L_1' = \frac{(b_x^2 + b_\phi^2)}{b_R T} V_A\left[V_f - V_s + b_x(b_x^2 + b_\phi^2)U_A\frac{N'}{R'}\right], \tag{E1}$$

$$L_2' = (b_x^2 + b_\phi^2)U_A\frac{N'}{R'}, \tag{E2}$$

$$L_3' = \frac{a(b_x^2 + b_\phi^2)}{T}\left\{b_x + [b_x^2(V_f - V_s) - b_\phi^2(V_A - a)]U_A\frac{N'}{R'}\right\}, \tag{E3}$$

$$L_4' = \frac{1}{b_R(b_R^2 + b_\phi^2)}\left\{L_1'[b_R^2(U_f + U_s) + b_\phi^2(a + U_A)] - \frac{L_3'}{a}[b_R^2(a + U_A) + b_\phi^2(U_f + U_s)]\right\}, \tag{E4}$$

$$L_5' = \frac{b_\phi}{T}\left[b_x(V_f - V_s + V_A - a) + (b_x^2 + b_\phi^2)U_A\frac{N'}{R'}\right], \tag{E5}$$

with

$$\begin{aligned} R' &= b_R(V_A - a)[b_x^3 b_R + b_\phi^2 U_A(U_f - U_s)] - b_R^2(V_f - V_s)[b_x b_\phi^2 + b_R U_A(U_f + U_s)] \\ &\quad + b_x(b_x^2 + b_\phi^2)(a + U_A)U_A V_A\,, \\ N' &= b_x b_R(U_f + U_s) - (a + U_A)V_A(V_f - V_s)\,, \\ T &= b_x^2(V_A - a) - b_\phi^2(V_f - V_s)\,. \end{aligned} \tag{E6}$$

For the initial responses, with $b_x = U_A$, $b_r = V_A$, and $b_\phi = 0$, we have

$$R' \approx U_A^5 V_A\,, \qquad N' \approx U_A V_A[U_f + U_s - (V_f - V_s)]\,, \qquad T \approx U_A^2(V_A - a)\,. \tag{E7}$$

The sign and magnitude of quantities N' and T are functions of the radial distance, and the results given in equations (30)–(32) follow with the change of the radial location.

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