ELSEVIER

Vacuum fluctuation force on a rigid Casimir cavity in a gravitational field

E. Calloni [a,b], L. Di Fiore [b], G. Esposito [b,*], L. Milano [a,b], L. Rosa [a,b]

[a] *Università Federico II di Napoli, Complesso Universitario di Monte Sant' Angelo, Via Cintia, 80126 Naples, Italy*
[b] *INFN, Sezione di Napoli, Complesso Universitario di Monte Sant' Angelo, Via Cintia, 80126 Naples, Italy*

Abstract

We discuss the possibility of verifying the equivalence principle for the zero-point energy of quantum electrodynamics, by evaluating the force, produced by vacuum fluctuations, acting on a rigid Casimir cavity in a weak gravitational field. The resulting force has opposite direction with respect to the gravitational acceleration; the order of magnitude for a multi-layer cavity configuration is derived and experimental feasibility is discussed, taking into account current technological resources. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

Although much progress has been made in the evaluation and experimental verification of effects produced by vacuum energy in Minkowski space–time [1–6], it remains unclear why the observed universe exhibits an energy density much smaller than the one resulting from the application of quantum field theory and the equivalence principle [7,8]. Various hypotheses have also been put forward on the interaction of virtual quanta with the gravitational field. For instance, some arguments seem to suggest that virtual photons do not gravitate [8], while other authors have suggested that Casimir energy contributes to gravitation [9]. So far it seems fair enough to say that no ex-

perimental verification that vacuum fluctuations can be treated according to the equivalence principle has been obtained as yet, even though there are expectations, as we agree, that this should be the case.

Motivated by all these considerations, our Letter computes the effect of a gravitational field on a rigid Casimir cavity, evaluating the net force acting on it. The order of magnitude of the resulting force, although not allowing an immediate experimental verification, turns out to be compatible with the current extremely sensitive force detectors, actually the interferometric detectors of gravitational waves. The Casimir cavity is rigid in that its shape and size remain unchanged under certain particular external conditions such as, for example, the absence of accelerations or impulses. We evaluate the force acting on this non-isolated system at rest in the gravitational field of the earth by studying the regularized energy-momentum tensor. The associated force turns out to have opposite direction with

* Corresponding author.
  E-mail address: giampiero.esposito@na.infn.it (G. Esposito).

respect to gravitational acceleration. Orders of magnitude are discussed bearing in mind the current technological resources as well as experimental problems, and physical relevance of the analysis is stressed.

2. Evaluation of the force

In order to evaluate the force due to the gravitational field let us suppose that the cavity has geometrical configuration of two parallel plates of proper area $A = L^2$ separated by the proper length a. The system of plates is taken to be orthogonal to the direction of gravitational acceleration $\vec{g}$.

In classical general relativity, the force density can be evaluated according to [10]

$$f_\nu = -\frac{1}{\sqrt{-\det g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\det g}\, T_\nu^\mu \right) + \frac{1}{2} \frac{\partial g_{\rho\sigma}}{\partial x^\nu} T^{\rho\sigma}, \tag{1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor of matter, representing the energy densities of all non-gravitational fields, and gravity couples to $T^{\mu\nu}$ via the Einstein equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$. However, we are eventually interested in quantum field theory in curved space–time, where gravitation is still treated classically, while matter fields are quantized. At this stage the key assumption is that one can take the expectation value of the quantum energy-momentum tensor in some state and evaluate the external force as if it were a classical force. This should therefore represent the expectation value of the "quantum force" in the given state. According to this theoretical model, we have to find the regularized energy-momentum tensor $\langle T^{\mu\nu} \rangle$ and insert it into the right-hand side of Eq. (1) to find eventually $\langle f_\nu \rangle$. If we were in Minkowski space–time with metric η, we might exploit the result of Ref. [11], according to which

$$\langle T^{\mu\nu} \rangle = \frac{\pi^2 \hbar c}{180 a^4} \left(\frac{1}{4} \eta^{\mu\nu} - \hat{z}^\mu \hat{z}^\nu \right), \tag{2}$$

where $\hat{z}^\mu = (0, 0, 0, 1)$ is the unit spacelike 4-vector orthogonal to the plates' surface. If such an analysis is performed for an accelerated but locally orthonormal reference frame in curved space–time, corresponding to our system at rest in the earth's gravitational field, the Minkowski line element should be replaced by the line element near the observer's world line. As shown in detail in Ref. [12] this reads (neglecting possible rotation effects)

$$ds^2 = -\left(1 + 2A_j x^j\right) \left(dx^0\right)^2 + \delta_{jk}\, dx^j\, dx^k + O_{\alpha\beta}\left(\left|x^j\right|^2\right) dx^\alpha\, dx^\beta. \tag{3}$$

Here $c^2 \vec{A}$ (with components $(0, 0, |\vec{g}|)$), the observer's acceleration with respect to the local freely falling frame, shows up in the correction term $-2A_j x^j$ to g_{00}, which is proportional to distance along the acceleration direction. Note that first-order corrections to the line element are unaffected by space–time curvature. Only at second order, which is beyond our aims, does curvature begin to show up. With this understanding we find, on setting

$$K \equiv \frac{\pi^2 \hbar c}{180 a^4}, \tag{4}$$

that the non-vanishing components of the regularized energy-momentum tensor which contribute to $\langle f_z \rangle$ are

$$\langle T_3^3 \rangle = \langle T^{33} \rangle = -\frac{3}{4} K, \tag{5}$$

$$\langle T^{00} \rangle = -\frac{K}{4} \frac{1}{1 + 2Az}. \tag{6}$$

Since (of course $x^3 \equiv z$ and hence $\langle f_3 \rangle$ denotes $\langle f_z \rangle$)

$$\langle f_z \rangle = -\frac{1}{\sqrt{1 + 2Az}} \frac{\partial}{\partial x^\mu} \left(\sqrt{1 + 2Az}\, T_3^\mu \right) + \frac{1}{2} \frac{\partial g_{\eta\rho}}{\partial z} T^{\eta\rho} = \left(\frac{3}{4} K + \frac{1}{4} K \right) \frac{A}{1 + 2Az}, \tag{7}$$

we find for the component of the full force along the z-axis the formula

$$\langle F_z \rangle \cong a L^2 \langle f_z \rangle \cong a L^2 g K, \tag{8}$$

where integration over the volume $V = aL^2$ has reduced to a simple multiplication by aL^2 to our order of approximation.

Interestingly, the resulting force has opposite direction with respect to the gravitational acceleration. Note also that the force density in Eq. (7), and hence the force itself in Eq. (8), is the sum of two terms: the latter, proportional to $g \frac{K}{4}$, arises from the Casimir energy encoded into T^{00}, and can be interpreted as the Newtonian repulsive force (i.e., "push") on an object

with negative energy. The former term, proportional to $\frac{3}{4}gK$, results from the pressure along the acceleration axis and can be interpreted as the mass contribution of the spatial part of the energy-momentum tensor. With this understanding, our result agrees with the equivalence principle, according to which for every pointlike event of space–time, there exists a sufficiently small neighbourhood such that in every local, freely falling frame in that neighbourhood, all laws of physics obey the laws of special relativity (as a corollary, the gravitational binding energy of a body contributes equally to the inertial mass and to the passive gravitational mass). In particular, we agree with the statement that a system with a given rest energy momentum tensor $T^{\overline{00}}$ has the inertial mass tensor $m^{ij} = T^{\overline{00}}\delta^{ij} + T^{\overline{ij}}$ [12].

Unfortunately, at present, there is not yet a direct calculation of inertial mass of a Casimir cavity in 4 dimensions; nevertheless direct calculations of inertial mass on a two-dimensional space–time [13] are in agreement with our calculation if "extended" to four dimensions. Furthermore, it is very important to note that, when considering the total force acting on the real cavity, which is an isolated system, the contribution to the force resulting from the spatial part of the energy-momentum tensor is balanced by the contribution of the mechanical energy-momentum tensor, and hence should not be considered for experimental evaluation. The resulting force is then the Newtonian force on the sum of the rest Casimir energy and rest mechanical mass: the contribution of vacuum fluctuations leads to a gravitational push on the Casimir apparatus expressed by the formula

$$\vec{F} \cong \frac{\pi^2 L^2 \hbar c}{720 a^3} \frac{g}{c^2} \vec{e}_r, \tag{9}$$

which should be tested against observation. As far as we can see, our calculation suggests that the electromagnetic vacuum state in a weak gravitational field is red-shifted, possibly adding evidence in favour of virtual quanta being able to gravitate, a nontrivial property on which no universal agreement has been reached in the literature [8,9], as we already noticed in the introduction. To further appreciate this point, we now find it appropriate and helpful to show that the same result can be derived from a mode-by-mode analysis. Recall indeed that in

Minkowski space–time the zero-point energy of the system can be evaluated, in the case of perfect conductors, as

$$U = \frac{\hbar c L^2}{2} \sum_{n=-\infty}^{\infty} \int \frac{d^2 k}{(2\pi)^2} \sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2}, \tag{10}$$

where we have considered the normal modes labelled by the integer n and the transverse momentum k. On performing the integral by dimensional regularization, the energy takes the well-known Casimir expression

$$U_{\text{reg}} = -\frac{\pi^2 L^2 \hbar c}{720 a^3}, \tag{11}$$

where the final result is independent of the particular regularization method.

Consider next the cavity at rest in a Schwarzschild geometry. On assuming that virtual quanta do gravitate, and bearing in mind that the modes remain unchanged, the energy of each mode is red-shifted by the factor $\sqrt{g_{00}} = \sqrt{1 - (\alpha/r)}$, with $\alpha \equiv \frac{2GM}{c^2}$. Hence the total energy can be written as (the suffix S referring to Schwarzschild)

$$U_{\text{S}} = \frac{\hbar c L^2}{2}$$
$$\times \sum_{n=-\infty}^{\infty} \int \frac{d^2 k}{(2\pi)^2} \sqrt{\left(k^2 + \left(\frac{n\pi}{a}\right)^2\right)\left(\sqrt{g_{00}}\right)^2}. \tag{12}$$

On performing dimensional regularization one finds therefore

$$(U_{\text{S}})_{\text{reg}} = (g_{00})^{1/2} U_{\text{reg}}$$
$$= -\left(1 - \frac{\alpha}{r}\right)^{1/2} \frac{\pi^2 L^2 \hbar c}{720 a^3}. \tag{13}$$

Now we assume that minus the gradient of $(U_{\text{S}})_{\text{reg}}$ with respect to r yields the force exerted by the gravitational field on the rigid Casimir cavity. If $r \gg \alpha$, we find the result in Eq. (9) by working at the same level of approximation. Higher-order contributions are of order $\alpha^2 r^{-3}$.

3. Problems of experimental verification

In considering the possibility of experimental verification of the extremely small forces linked to this

effect we point out that such measurements cannot be performed statically; this would make it necessary to compare the weight of the assembled cavity with the sum of the weights of its individual parts, a measure impossible to perform. On the contrary, the measurements we are interested in should be performed dynamically, by modulating the force in a known way; the effect will be detected if the modulation signal will be higher than the sensitivity of the detector. In this spirit we focus on the sensitivity reached by the present technology in detection of very small forces on a macroscopic body, on earth, paying particular attention to detectors of the extremely small forces induced by a gravitational wave. As an example, gravitational wave signals h of order $\approx 10^{-25}$, corresponding to forces of magnitude $\approx 5 \times 10^{-17}$ N at frequency of few tens of Hz, are expected to be detected with the Virgo gravitational wave detector presently under construction [14,15] after a month of integration time.

In the course of studying experimental possibilities of verification of the force on a rigid Casimir cavity, we evaluate this force on a macroscopic body, having essentially the same dimensions of mirrors for gravitational wave detection and obtained through a multi-layer sedimentation by a series of rigid cavities. Each rigid cavity consists of two thin metallic disks, of thickness of order 100 nm [16] separated by a dielectric material, inserted to maintain the cavity sufficiently rigid: introduction of a dielectric is equivalent to enlarging the optical path length by the refractive index n $(a \mapsto na)$. By virtue of presently low costs and facility of sedimentation, and low absorption in a wide range of frequencies [17], SiO_2 can be an efficient dielectric material.

From an experimental point of view we point out that the Casimir force has so far been tested down to a distance a of about 60 nm, corresponding to a frequency ν_{min} of the fundamental mode equal to 2.5×10^{15} Hz. This limit results from the difficulty to control the distance between two separate bodies, as in the case of measurements of the Casimir pressure. As stated before, in our rigid case, present technologies allow for cavities with much thinner separations between the metallic plates, of the order of few nanometers. At distances of order 10 nm, finite conductivity and dielectric absorption are expected to play an important role in decreasing the effective Casimir pressure, with respect to the case of perfect

mirrors [18,19]. In this Letter we discuss experimental problems by relying on present technological resources, considering cavities with plates' separation of 5 nm and estimating the effect of finite conductivity by considering the numerical results of Ref. [18]; for a separation of 6.5 nm this corresponds to a decreasing factor η of about 7×10^{-2} for Al. Moreover, to increase the total force and obtain macroscopic dimensions, $N_l = 10^6$ layers can be used, each having a diameter of 35 cm, and thickness of 100 nm, for a total thickness of about 10 cm. Last, one should also consider corrections resulting from finite temperature and roughness of the surfaces, although one might hope to minimize at least the former by working at low temperatures.

With these figures, the total force $\vec{F}^T$ acting on the body can be calculated with the help of Eq. (9), modified to take into account the refractive index n for SiO_2, the decreasing factor η, the area A of disk-shape plates, and the N_l layers:

$$\vec{F}^T \approx \eta N_l \frac{A\pi^2\hbar c}{720(na)^3} \frac{g}{c^2} \vec{e}_r. \qquad (14)$$

This formula describes a static effect, while the need for a feasible experiment makes it mandatory to modulate the force, and various experimental possibilities [20–22] are currently under study. In particular, we are investigating the possibility of modulating η, by varying the temperature, so as to achieve a periodic transition from conductor to superconductor regime. By doing so one can obtain η_{max} of order 5×10^{-1} [23], and the magnitude of the force at the modulation frequency can reach 10^{-14} N. Even though such a force is apparently more than two orders of magnitude larger than the force which the Virgo gravitational antenna is expected to detect, we should consider that the signal there is at reasonable high frequency (some tens of Hz), while our calculated signal remains at lower frequencies, i.e., tens of mHz. Moreover, the technical problems resulting from stress induced in changing temperature require careful consideration before saying that modulation is feasible.

4. Conclusions

The relation of the Casimir energy with the geometry of bounding surfaces has been properly under-

stood only very recently, thanks to the outstanding work in Ref. [24], and another open problem of modern physics, i.e., the cosmological constant problem, results from calculations which rely on the application of the equivalence principle to vacuum energy [7], and this adds interest to our calculations, that we have performed by focusing on a Casimir apparatus.

Our original contributions are given by the evaluation of the force acting on a macroscopic body which mimics the rigid Casimir cavity, and by a detailed estimate of the expected order of magnitude of such a force.

As far as we can see, there is room left for an assessment of our investigation, i.e.:

  (i) How to make sure that the cavity is sufficiently rigid;
 (ii) The task of verifying that corrugations or yet other defects do not affect substantially our estimates;
(iii) How to perform signal modulation, which is still beyond our reach.

Our result expressed by Eq. (9), according to which the gravitational field on a rigid Casimir cavity gives rise to a net force, whose direction is opposite to the one of gravitational acceleration, is simple but non-trivial, because it is a possible first step towards testing the equivalence principle in configurations involving the regularized zero-point energy of the electromagnetic field. The effect can be understood in terms of the change in rest mass because the system is more tightly bound as the cavity walls are brought closer together. Interestingly, we therefore end up by suggesting that a very tiny force produced by variations of rest mass is detectable under suitable conditions, whereas standard estimates of the change in rest mass for chemical reactions show that this is unlikely to be the case [25].

The experimental verification of the calculated force, which was our main concern, might be performed if the problem of signal modulation could be solved. On the other hand, in the authors' opinion, the order of magnitude of the calculated force can be already of interest to demonstrate that experiments involving effects of gravitation and vacuum fluctuations are not far from what can be obtained with the help of present technological resources.

References

[1] B.S. DeWitt, Phys. Rep. 19 (1975) 295;
    M. Bordag, U. Mohideen, V.M. Mostepanenko, Phys. Rep. 353 (2001) 1.
[2] M. Bordag, B. Geyer, G.L. Klimchitskaya, V.M. Mostepanenko, Phys. Rev. D 60 (1999) 055004.
[3] K. Milton, hep-th/0009173.
[4] S.A. Fulling, Aspects of Quantum Field Theory in Curved Space-Time, Cambridge University Press, Cambridge, 1989.
[5] S.K. Lamoreaux, Phys. Rev. Lett. 83 (1999) 3340;
    U. Mohideen, A. Roy, Phys. Rev. Lett. 81 (1998) 4549.
[6] H.B. Chan, V.A. Aksyuk, R.N. Kleiman, D.J. Bishop, F. Capasso, Science 291 (2001) 1941.
[7] S. Weinberg, Rev. Mod. Phys. 61 (1989) 1.
[8] R.P. Feynman, A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York, 1965, pp. 244–246;
    For an up-to-date presentation, with references, see, P.S. Wesson, Astrophys. J. 378 (1991) 466.
[9] D.W. Sciama, in: S. Saunders, H.R. Brown (Eds.), The Philosophy of Vacuum, Clarendon, Oxford, p. 137.
[10] C. Moller, The Theory of Relativity, Oxford University Press, Oxford, 1972.
[11] L.S. Brown, G.J. Maclay, Phys. Rev. 184 (1969) 1272.
[12] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, Freeman, New York, 1973.
[13] M.T. Jaekel, S. Reynaud, J. Phys. I 3 (1993) 1093.
[14] D. Blair, The Detection of Gravitational Waves, Cambridge University Press, Cambridge, 1991.
[15] VIRGO Collaboration, Nucl. Instrum. Methods A 360 (1995) 258;
    VIRGO Collaboration, in: Cosmology and Particle Physics, Verbier, 2000, p. 138.
[16] M. Bordag, B. Geyer, G.L. Klimchitskaya, V.M. Mostepanenko, Phys. Rev. D 62 (2000) 011701.
[17] R.A. Street, Hydrogenated Amorphous Silicon, Cambridge University Press, Cambridge, 1991;
    B. Chapman, Discharge Processes, John Wiley, New York, 1980.
[18] A. Lambrecht, S. Reynaud, Eur. Phys. J. D 8 (2000) 309.
[19] G.J. Maclay, Phys. Rev. A 61 (2000) 052110.
[20] S. Hunklinger, H. Geisselmann, W. Arnold, Rev. Sci. Instrum. 43 (1972) 584.

[21] P.R. Saulson, Interferometric Gravitational Wave Detectors, World Scientific, Singapore, 1994.

[22] R. Esquivel-Sirvent, C. Villareal, G.H. Cocoletzi, Phys. Rev. A 64 (2001) 052108.

[23] E. Calloni, L. Di Fiore, G. Esposito, L. Milano, L. Rosa, quant-ph/0109142, Int. J. Mod. Phys. A 17 (2002) 804.

[24] G. Barton, J. Phys. A 34 (2001) 4083.

[25] E.F. Taylor, J.A. Wheeler, Spacetime Physics: Introduction to Special Relativity, 2nd edn., Freeman, New York, 1992.