TE and TM modes boundary conditions

TE $\longrightarrow$ $B_z^0(x,y)$ completely determines the field
↳ H-waves

TM $\longrightarrow$ $E_z^0(x,y)$ completely determines the field
↳ E-waves

To find $B_z^0(x,y)$ and $E_z^0(x,y)$, we solve the $\hat{e}_z$ component of the Helmholtz Equation

$$TE \longrightarrow \left(\bar{\nabla}_t^2 + k_c^2\right) B_z^0 = 0$$

$$TM \longrightarrow \left(\bar{\nabla}_t^2 + k_c^2\right) E_z^0 = 0$$

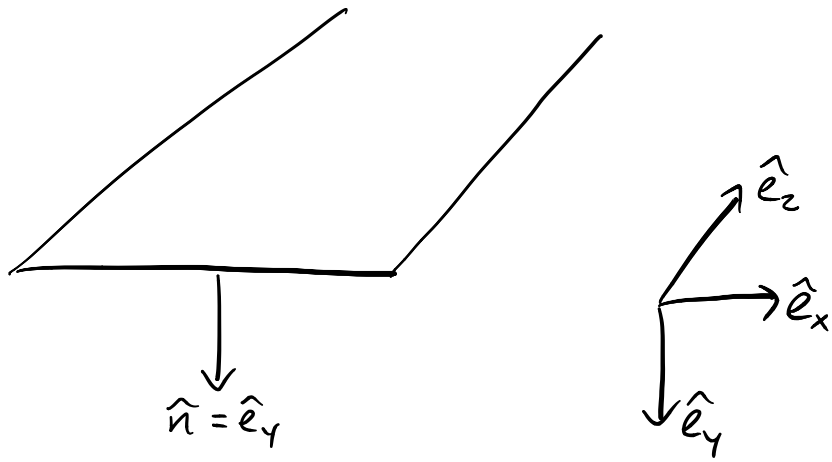
with the BCs

$$\left.\begin{aligned} \bar{E}_{\text{tangential}}\big|_S &= \hat{n} \times \bar{E}\big|_S = 0 \\ \bar{B}_{\text{normal}}\big|_S &= \hat{n} \cdot \bar{B}\big|_S = 0 \end{aligned}\right\} \text{At conductor boundary}$$

Where

$\hat{n}$ = normal to surface S of conducting wall

To visualize the BCs, let's analyze a single wall with



For a <u>TE mode</u> at this boundary:

$$E_{\text{tangential}}\Big|_s = \underbrace{\hat{e}_y \times \bar{E}}_{} = 0 \Rightarrow E_x = 0$$

$$\hookrightarrow E_z = 0, \ \hat{e}_y \times E_y \hat{e}_y = 0$$

$$B_{\text{normal}}\Big|_s = \hat{e}_y \cdot \bar{B} = B_y = 0$$

But, from

$$\bar{B}_{t0} = \frac{k_g}{k_0}(\hat{e}_z \times \bar{E}_{t0})$$

$$\Rightarrow B_y = \frac{k_g}{k_0} E_x$$

→ B_y and E_x are related

→ can collapse the two BCs to a single BC

We also note that

$$E_x^0 = \frac{i}{k_c^2} k_0 \frac{\partial B_z^0}{\partial y} \qquad B_y^0 = \frac{i}{k_c^2} k_g \frac{\partial B_z^0}{\partial y}$$

↓

Both depend on $\frac{\partial B_z^0}{\partial y}$

Therefore,

$$E_x^0 = B_y^0 = 0 \Rightarrow \frac{\partial B_z^0}{\partial y} = 0$$

Making this more general, we extend to other boundaries using the <u>normal derivative</u> $\frac{\partial}{\partial n}$

$$\boxed{\left.\frac{\partial B_z^0}{\partial n}\right|_s = 0}$$ BC for TE mode

For $\underline{\text{TM modes}}$, $B_z = \emptyset$

$$E_{tangential}\Big|_S = \hat{e}_y \times \bar{E} = \emptyset$$

$$\Rightarrow E_x = E_z = \emptyset$$

$$B_{normal}\Big|_S = \hat{e}_y \cdot \bar{B} = B_y = \emptyset$$

From

$$\bar{E}_{t0} = -\frac{k_g}{k_0}\left(\hat{e}_z \times \bar{B}_{t0}\right)$$

$$\Rightarrow E_x^0 = \frac{k_g}{k_0} B_y^0 \longrightarrow E_x^0, B_y^0 \text{ are related}$$

We also note that

$$E_x^0 = \frac{i k_g}{k_c^2} \frac{\partial E_z^0}{\partial x} \qquad B_y^0 = \frac{i k_0}{k_c^2} \frac{\partial E_z^0}{\partial x}$$

Both depend on $\dfrac{\partial E_z^0}{\partial x}$

$$\Rightarrow \quad E_x^0 = B_y^0 = 0 \quad \Rightarrow \quad \frac{\partial E_z^0}{\partial x} = 0$$

Also, because we need $E_z = 0$ on the boundary and $E_z = 0$ in the conductor

$$E_z^0 = 0 \Rightarrow \frac{\partial E_z^0}{\partial x} = 0 \quad \text{on boundary}$$

$$\Rightarrow \quad \boxed{E_z^0 \Big|_S = 0} \quad \text{BC for TM mode}$$

For both TE and TM modes

$$\underbrace{\underset{\text{normal}}{E_y} \propto \underset{\text{tangential}}{B_x}}_{\text{satisfy BCs for surface charge + current}} \neq 0 \quad \text{in general}$$

Rectangular Waveguides

- Ordinary circuits fail at high frequencies because of distributed capacitance + inductance

- Microwaves ($\nu \sim 3-100$ GHz or $\lambda \sim 10-0.3$ cm) can use hollow waveguides to transmit energy

Consider a TE mode in a rectangular waveguide:

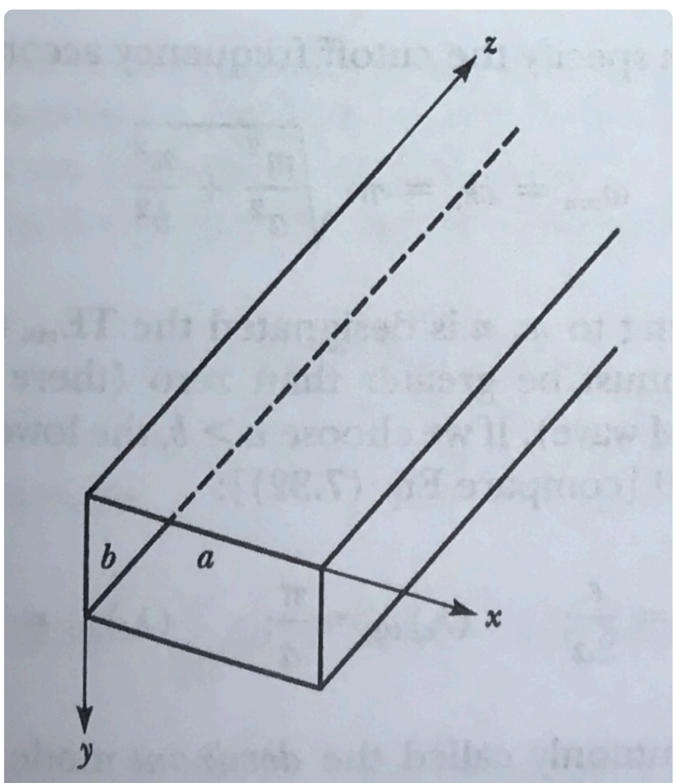


FIGURE 7-5. Rectangular waveguide.

[H+M]

→ If we can find B_z^0, then we'll have all of the other field components

→ B_z^0 is determined by the Helmholtz Eqn

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) B_z^0 = 0$$

with the BC

$$\left.\frac{\partial B_z^0}{\partial n}\right|_S = 0$$

For this geometry, the BC becomes:

$$\left.\frac{\partial B_z^0}{\partial x}\right|_{x=0,a} = 0 \qquad \left.\frac{\partial B_z^0}{\partial y}\right|_{y=0,b} = 0$$

Which gives us a solution

$$B_z^0 = B^0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

↳ Amplitude

Check solution

$$\frac{\partial B_z^0}{\partial x} \sim \sin\left(\frac{m\pi x}{a}\right) \rightarrow 0 \quad \text{at} \quad x = 0, a$$

$$\frac{\partial B_z^0}{\partial y} \sim \sin\left(\frac{n\pi y}{b}\right) \rightarrow 0 \quad \text{at} \quad y = 0, b$$

Plug into Helmholtz:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) B^0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) = 0$$

$$\Rightarrow -\underbrace{\left(\frac{m\pi}{a}\right)^2}_{\text{from } x} - \underbrace{\left(\frac{n\pi}{b}\right)^2}_{\text{from } y} + k_c^2 = 0$$

$$\Rightarrow k_c = \pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

↳ cutoff wavenumber

$$\Rightarrow \underbrace{\omega_c = \omega_{mn}} = ck_c = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$\hookrightarrow$ cutoff frequency

$\hookrightarrow$ fxn of mode integers $m+n$

The mode corresponding to m, n is

TE_{mn} mode

At least one integer of m or n

must be NON-ZERO or

$$B_z^0 = B^0 \Rightarrow E_x^0, B_y^0 \sim \frac{\partial B_z^0}{\partial y} = 0$$

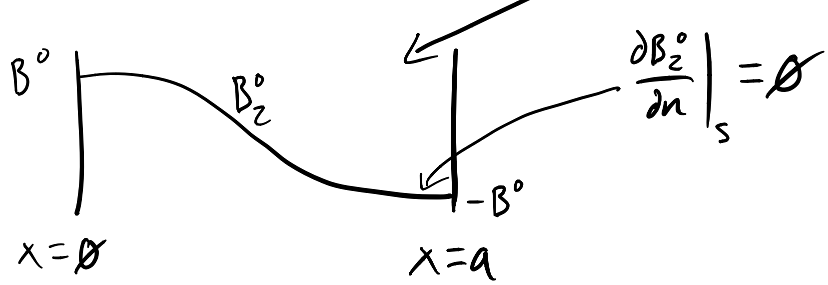
$$E_y^0, B_x^0 \sim \frac{\partial B_z^0}{\partial x} = 0$$

$\hookrightarrow E = B = 0$

For $a > b$ the lowest cutoff frequency is

$$m = 1,\ n = 0 \Rightarrow TE_{10}$$

$$\Rightarrow k_{10} = \frac{\pi}{a},\quad \omega_{10} = \frac{\pi c}{a},\quad \lambda_{10} = 2a$$



- TE_{10} is called the <u>dominant mode</u>

- No frequency below ω_{10} can $\underbrace{\text{propagate}}_{\text{transport energy}}$

- Waveguide dimensions are selected so that <u>only TE_{10}</u> propagates $\Rightarrow$ don't need to control several modes

- Ratio $a/b \approx 2$ maximizes single mode bandwidth
  ↳ HW problem

- Rectangular cross-section is used instead of circular because theory is easier and polarization is fixed

- The field components for TE_{10} are

$E_y^0 = i\dfrac{k_0 a}{\pi} B^0 \sin\left(\dfrac{\pi x}{a}\right)$		$E_x^0 = E_z^0 = \emptyset$
$B_y^0 = \emptyset$	$B_x^0 = -i\dfrac{k_g a}{\pi} B^0 \sin\left(\dfrac{\pi x}{a}\right)$	$B_z^0 = B^0 \cos\left(\dfrac{\pi x}{a}\right)$

Multiply by $\underline{\exp[i(k_g z - \omega t)]}$ to get $z + t$ dependence

The energy flow is calculated from

$$\langle \bar{S} \rangle = \frac{c}{8\pi} \mathrm{Re}(\bar{E} \times \bar{B}^*)$$

For the TE_{10} mode

$$\langle \bar{S} \rangle_{10} = \frac{c}{8\pi} \mathrm{Re}\left(E_y^0 B_z^{0*} \hat{e}_x - E_y^0 B_x^{0*} \hat{e}_z \right)$$

But, we found that

$$E_y^0 = i \frac{k_0 a}{\pi} B^0 \sin\left(\frac{\pi x}{a}\right) \qquad B_z^0 = B^0 \cos\left(\frac{\pi x}{a}\right)$$

$$B_x^0 = \frac{-i k_g a}{\pi} B^0 \sin\left(\frac{\pi x}{a}\right)$$

$$\Rightarrow \mathrm{Re}\left(\underbrace{E_y^0 B_z^{0*}}_{\text{IMAGINARY}} \right) = 0$$

$$\Rightarrow \langle S \rangle_{10} = \frac{c}{8\pi} \mathrm{Re}\left(-E_y^0 B_x^{0*} \hat{e}_z \right)$$

$$= \hat{e}_z \frac{c}{8\pi} \left(\frac{a}{\pi} B^0 \right) k_0 k_g \sin^2\left(\frac{\pi x}{a}\right)$$

$\hookrightarrow$ transmitted down the waveguide

The total transmitted power can be found by integrating over the cross section

$$P_{10} = \int_0^a \langle S \rangle_{10} \, b \, dx = \frac{c}{16\pi} \left(\frac{a}{\pi} B^0 \right)^2 k_0 k_g \, ab$$

In a more useful form

Gaussian $$P_{10} = \left(\frac{c E_0^2}{16\pi} \right) ab \sqrt{1 - \left(\frac{\lambda_0}{2a} \right)^2}$$

SI $$P_{10} = \left(\frac{E_0^2}{4\eta_0} \right) ab \sqrt{1 - \left(\frac{\lambda_0}{2a} \right)^2}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377 \ \Omega \ \text{(ohms)}$$

↳ Impedence of free space

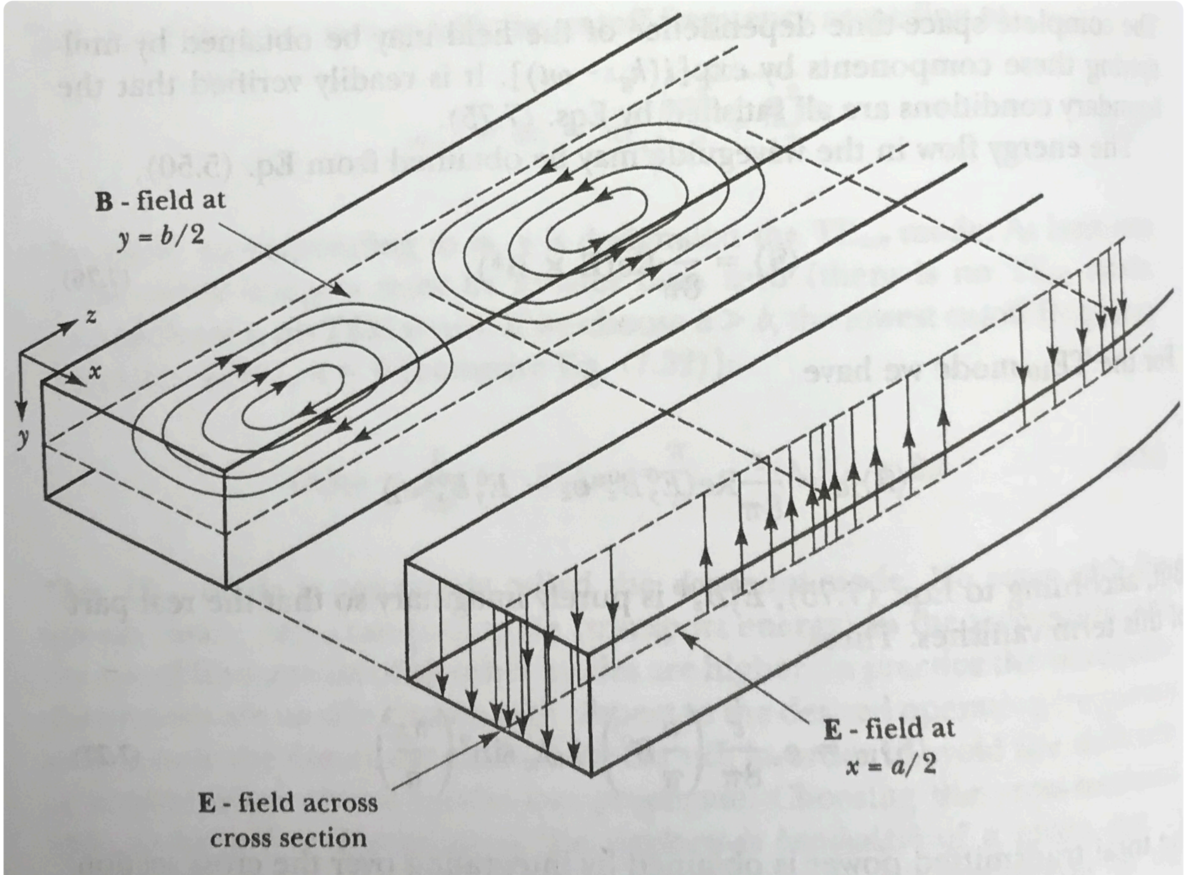


FIGURE 7-6. Fields of the TE_{10} dominant mode.

[H+M]

- $\bar{E}$ and $\bar{B}$ are in-phase
- Peak E_y occurs when $\bar{B} = B\hat{e}_x$
- $\bar{S}$ is mainly in the $\hat{e}_z$ direction
- If an endplate is added → resonant cavity
  - ↳ produces standing waves
  - ↳ $\bar{E}$ and $\bar{B}$ are $\frac{\pi}{2}$ out of phase
  - ↳ no net energy flow