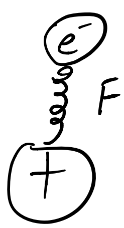
Classical Electron Theory

- Treatment of E+M with matter must properly be completed with quantum theory

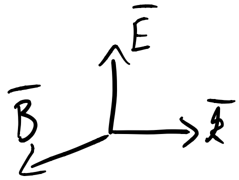
- However, for many purposes, classical treatment can provide insight



$F = -kx \longrightarrow$ Harmonic oscillator $\longrightarrow$ Classical e^- theory

- Why take this approach?
  ↳ It's simple and works well for many applications

Scattering of an E+M wave by Charged Particle



e^- $\Rightarrow \bar{F} = q\left(\bar{E} + \frac{\bar{u}}{c} \times \bar{B}\right)$

$E \sim E_0 e^{-i\omega t} \Rightarrow \bar{F} \sim e^{-i\omega t}$

↳ periodic in time

↳ e^- <u>radiates</u> due to acceleration

⟶ Energy is "absorbed" by the e^- then re-emitted

↳ process is called <u>scattering</u>

Consider a plane wave incident on particle with charge q

$$\bar{E} = \hat{e} E_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

Assuming $\frac{\bar{u}}{c} << 1$, we can ignore the $\bar{B}$ contribution to $\bar{F}$. Also, we select the equilibrium position as $\bar{r} = \emptyset$

$$\bar{F} = q\bar{E} = m\ddot{\bar{r}}$$

↳ mass of particle

The dipole moment is

$$\bar{p}(t) = q\,\bar{r}(t)$$

$$\Rightarrow \ddot{\bar{r}}(t) = \frac{\ddot{\bar{p}}}{q} \Rightarrow q\bar{E} = \frac{m}{q}\ddot{\bar{p}}$$

$$\Rightarrow \ddot{\bar{p}}(t) = \frac{q^2}{m}\bar{E}(t)$$

From the Larmor formula, the time average power radiated per unit solid angle is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\langle [\ddot{p}]^2 \rangle}{4\pi c^3} \sin^2\theta$$

But

$$\langle [\ddot{p}]^2 \rangle = \frac{q^4}{m^2} \langle E^2 \rangle = \frac{q^4 E_0^2}{2m^2}$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \left(\frac{q^2}{mc^2} \right)^2 \frac{c}{8\pi} E_0^2 \sin^2\theta$$

The <u>scattering cross section</u> is the <u>equivalent area</u> at the incident wavefront that delivers the power reradiated by the particle

$$\sigma \equiv \frac{\langle \text{reradiated power} \rangle}{\langle \text{incident power per area} \rangle}$$

cross section

The power/area of the incident wave is the Poynting vector

$$\langle S \rangle = \frac{c}{8\pi} E_0^2$$

↳ free space

The <u>differential</u> scattering cross section (area/solid-angle for scattering in the θ direction)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{polarized}} \equiv \frac{\langle dP/d\Omega \rangle}{\langle \text{incident power/area} \rangle} = \left(\frac{q^2}{mc^2}\right)^2 \sin^2\theta$$

↑

$\bar{p} \parallel \bar{E}$

The <u>total</u> cross section is

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3}\left(\frac{q^2}{mc^2}\right)^2$$

The angle θ in $\sin^2\theta$ is the angle between $\bar{p}$ (or $\bar{E}$) and the outgoing radiation $\hat{n}$

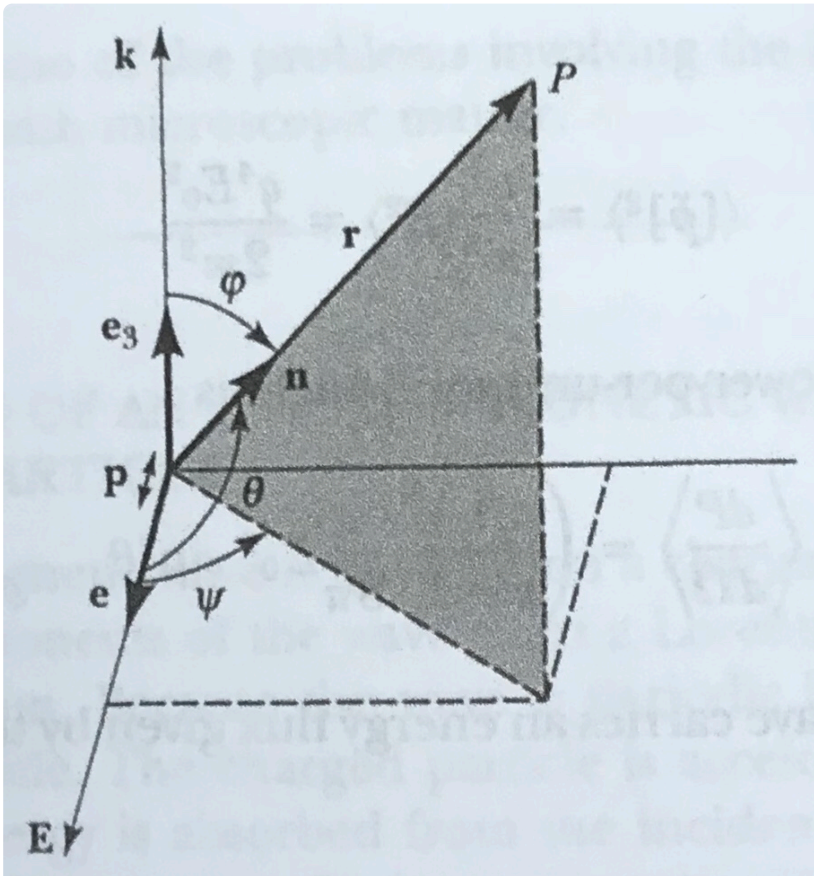


FIGURE 10-1. Scattering geometry.

[H+M]

To calculate the cross section of a randomly polarized wave, we now define

$\hat{e}_3$ = direction of incident wave ($\hat{k}$)

$\hat{n}$ = direction of outgoing wave

ϕ = angle between $\hat{e}_3$ and $\hat{n}$

We must average over all possible azimuthal orientations of $\vec{E}$ (angle ψ)

$$\cos\theta = \cos\psi \sin\phi$$

$$\Rightarrow \sin^2\theta = 1 - \cos^2\psi \sin^2\phi$$

Averaging over ψ:

$$\overline{\sin^2\theta} = 1 - \overline{\cos^2\psi} \sin^2\phi = 1 - \frac{1}{2}\sin^2\phi$$

$$= \frac{1}{2}(1 + \cos^2\phi)$$

Substituting $\overline{\sin^2\theta}$ for $\sin^2\theta$ in $\left\langle \frac{d\sigma}{d\Omega} \right\rangle$

$$\Rightarrow \left\langle \frac{d\sigma}{d\Omega} \right\rangle_{\text{unpolarized}} = \left(\frac{q^2}{mc^2}\right)^2 \frac{1 + \cos\phi}{2}$$

$$\Rightarrow \sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{8\pi}{3}\left(\frac{q^2}{mc^2}\right)^2$$

↳ same as before

For an e^- as the particle

$$\boxed{\sigma_{thomson} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \approx 0.665 \times 10^{-24} \text{ cm}^2}$$

Thomson scattering

Note: $\frac{e^2}{m_e c^2} \approx 3 \times 10^{-13}$ cm = classical e^- radius

↳ matches results of Thomson

- classical σ is not a fxn of frequency

- if $\hbar\omega \approx m_e c^2$ (~ 0.5 MeV for e^-) then quantum theory must be used (compton scattering)

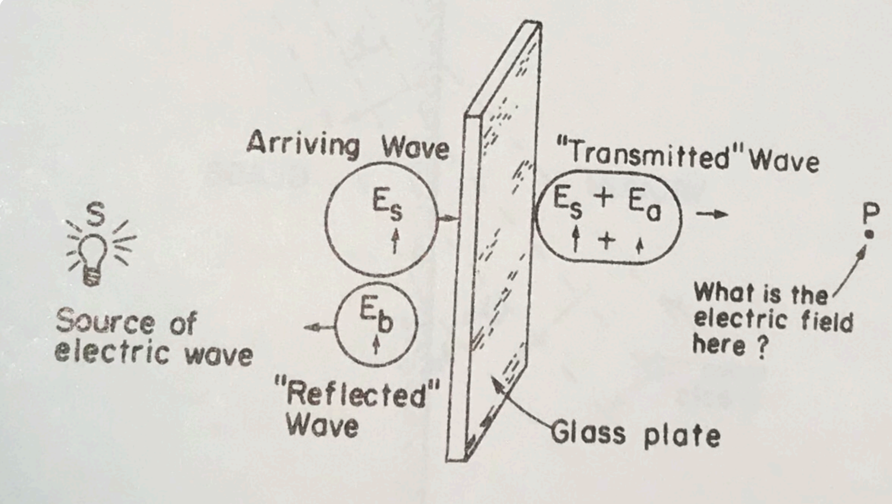


Fig. 31–1. Electric waves passing through a layer of transparent material. [Feynman]

E_s = source E

E_a = field produced by charges in plate

→ In the forward direction, $E_a + E_s$ are coherent and constructive

↳ effectively modify wave speed due to phase shift

↳ index of refraction!

Dispersion in Gases

- Dilute gas $\Rightarrow$ no mutual interaction
- Treat as simple harmonic oscillators
- Calculate frequency-dependence of speed

Starting with the wave equation:

$$\bar{\nabla}^2 \bar{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} = \emptyset$$

$$n \equiv \frac{c}{v} = \sqrt{\varepsilon\mu}$$

$\hookrightarrow$ phase velocity

Assuming $\mu = 1$

$$\Rightarrow n = \sqrt{\varepsilon}$$

- If ε is a fxn of ω, n and v will be fxns of $\omega \Rightarrow$ dispersive medium

- In a dispersive medium, wave form changes (narrow pulse spreads out) as it travels

Allowing for conductivity (complex n + ε)

$$\hat{n}(\omega) = \sqrt{\hat{\varepsilon}(\omega)}$$

The <u>Lorentz classical model</u> of a molecule

- α^{th} e^- attached to molecule core with spring constant K_α
- spring has zero equilibrium length (exclude molecules with permanent dipole)
- Assume the e^- experiences <u>damping force</u> proportional to its velocity

The equation of motion is

$$m\ddot{\bar{r}}_\alpha + l_\alpha \dot{\bar{r}}_\alpha + K_\alpha \bar{r}_\alpha = \underbrace{-e\bar{E}}_{\text{Force on } e^- \text{ from E+M wave}}$$

$$\Rightarrow \quad \ddot{\bar{r}}_\alpha + 2\beta_\alpha \dot{\bar{r}}_\alpha + \omega_\alpha^2 \bar{r}_\alpha = \frac{-e}{m} E_0 e^{-i\omega t}$$

$$\omega_\alpha \equiv \sqrt{\frac{K_\alpha}{m}} = \text{characteristic frequency}$$

$$\beta_\alpha \equiv \frac{l_\alpha}{2m} = \text{normalized damping coef}$$

This is a damped, driven harmonic oscillator with solution:

$$\bar{r}_\alpha(t) = \frac{-(e/m)\bar{E}_0}{(\omega_\alpha^2 - \omega^2) - 2i\beta_\alpha \omega} e^{-i\omega t}$$

↳ physical solution is real part

The induced dipole moment is

$$\bar{P}_\alpha = -e\bar{r}_\alpha(t) = \frac{(e^2/m)\bar{E}}{(\omega_\alpha^2 - \omega^2) - 2i\beta_\alpha \omega}$$

For an e^- density N $\left(\frac{\#}{\Delta V}\right)$ and a fraction f_α with characteristic frequency ω_α the dipole moment per unit volume is

$$\bar{P} = \sum_\alpha N f_\alpha \bar{p}_\alpha = \bar{E} \sum_\alpha \frac{N f_\alpha e^2/m}{(\omega_\alpha^2 - \omega^2) - 2i\beta_\alpha \omega} \equiv \bar{E} \hat{\chi}_e$$

electric susceptibility

$$\Rightarrow \bar{D} = \bar{E} + 4\pi \bar{P} = \underbrace{\left(1 + 4\pi \hat{\chi}_e\right)}_{\equiv \hat{\varepsilon}} \bar{E} = \hat{\varepsilon} \bar{E}$$

$$\Rightarrow \hat{\varepsilon} = 1 + 4\pi \sum_\alpha \frac{N f_\alpha e^2/m}{(\omega_\alpha^2 - \omega^2) - 2i\beta_\alpha \omega}$$

$$\Rightarrow \hat{n} = \sqrt{1 + 4\pi \sum_\alpha \frac{N f_\alpha e^2/m}{(\omega_\alpha^2 - \omega^2) - 2i\beta_\alpha \omega}}$$