Vector Calculus

Two of the most important vector calculus theorems in E+M are the Divergence Theorem and Stoke's Theorem

Divergence Theorem

normal to surface

$$\underbrace{\int_V \bar{\nabla} \cdot \bar{A} \, dv}_{\text{Volume integral of divergence}} = \underbrace{\oint_S \bar{A} \cdot \hat{n} \, da}_{\text{Surface integral over surface covering volume } V}$$

The integral of the divergence of a vector function, $\bar{\nabla} \cdot \bar{A}$, over a volume V is equal to the integral of the dot product of $\bar{A}$ with the surface normal $\hat{n}$ over the surface covering V.

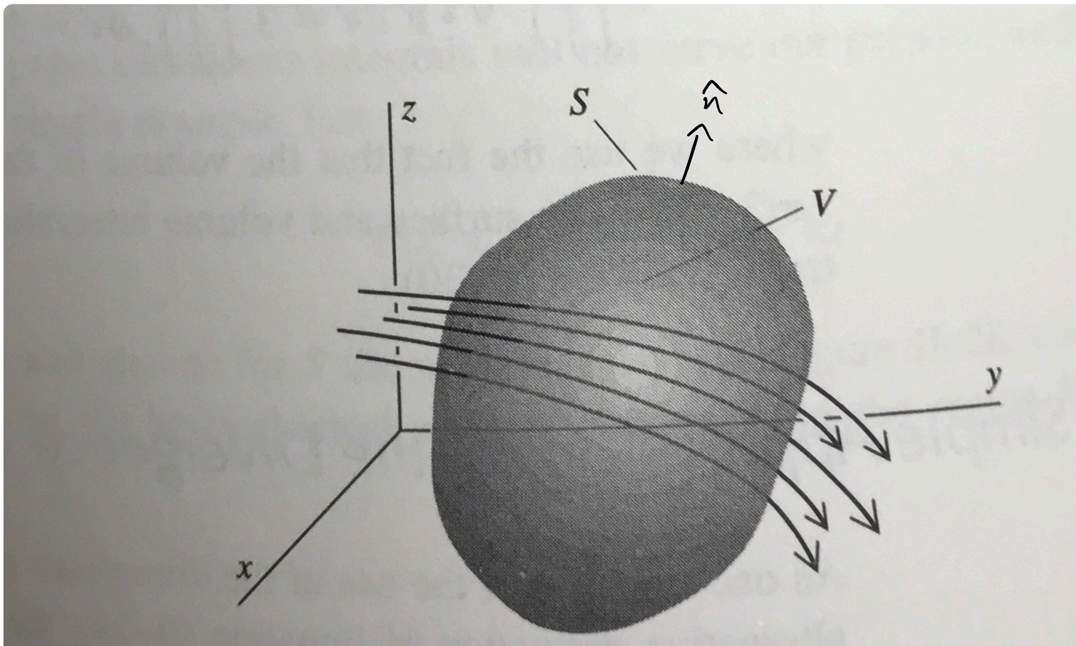


Figure II–29

[Schey]

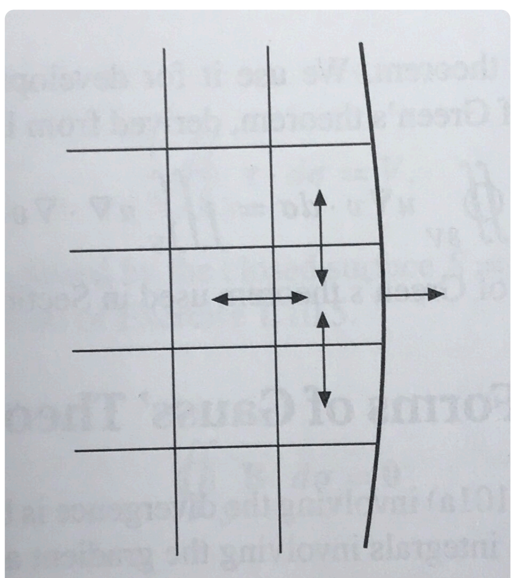


FIGURE 1.28 Exact cancellation of $d\boldsymbol{\sigma}$'s on **interior** surfaces. No cancellation on the **exterior** surface.

Break the volume into cubes and the interior surfaces cancel each other

$\Rightarrow$ only the exterior surfaces matter!

[Arfken + Weber Mathematical Methods for Physicists]

Stokes' Theorem

$$\int_S \bar{\nabla} \times \bar{A} \cdot \hat{n}\, da = \oint_\Gamma \bar{A} \cdot d\bar{s}$$

Integral of curl over open surface

Integral over the closed line Γ that bounds the surface S

Ampere's law

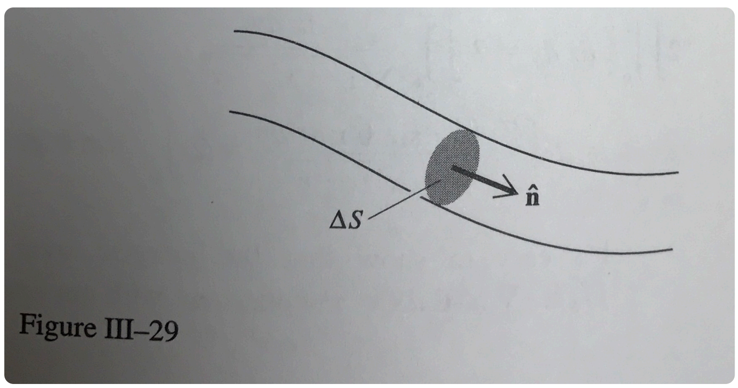


Figure III–29

[Schay]

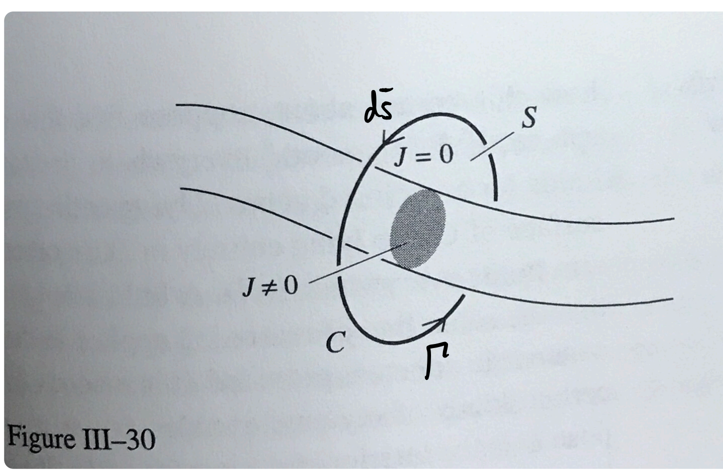


Figure III–30

[Schay]

The integral of the curl of a vector function, $\bar{\nabla} \times \bar{A}$, over an open surface S is equal to the line integral over the closed path Γ bounding surface S.

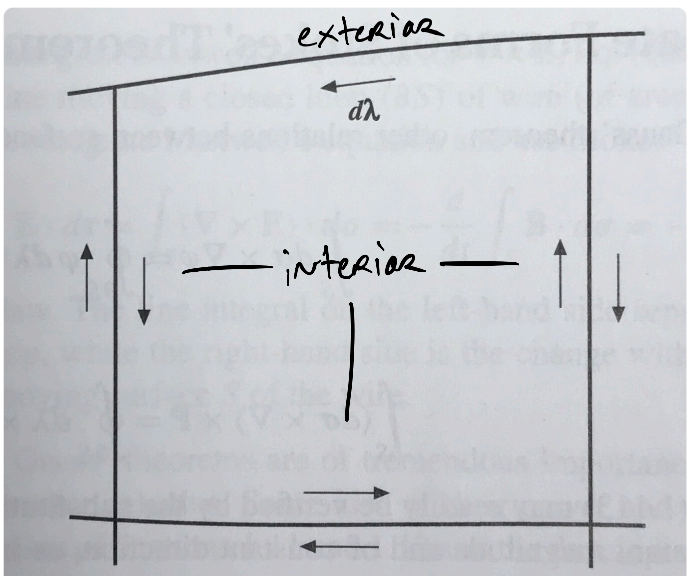


FIGURE 1.30 Exact cancellation on **interior** paths. No cancellation on the **exterior** path.

Interior line integrals cancel leaving only the outer line integral that makes a contribution

Helpful identities with $\vec{\nabla}$

IDENTITIES INVOLVING THE OPERATOR ∇*

$$\nabla(fg) = f\,\nabla g + g\,\nabla f$$
$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$$
$$\nabla \cdot (f\mathbf{F}) = f\,\nabla \cdot + \mathbf{F} \cdot \nabla f$$
$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$
$$\nabla \cdot \nabla \times \mathbf{F} = 0$$
$$\nabla \times (f\mathbf{F}) = f\,\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F}$$
$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F})$$
$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$
$$\nabla \times \nabla f = 0$$

[Schey]

In-class / Homework problems from Schey

$\boxed{\text{II} - 8}$

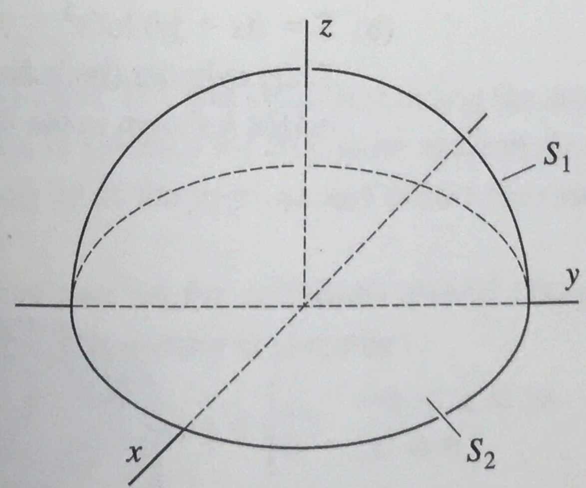
II–8 An electrostatic field is given by

$$\mathbf{E} = \lambda(\mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy),$$

where λ is a constant. Use Gauss' law to find the total charge enclosed by the surface shown in the figure consisting of S_1, the hemisphere

$$z = (R^2 - x^2 - y^2)^{1/2},$$

and S_2, its circular base in the xy-plane.



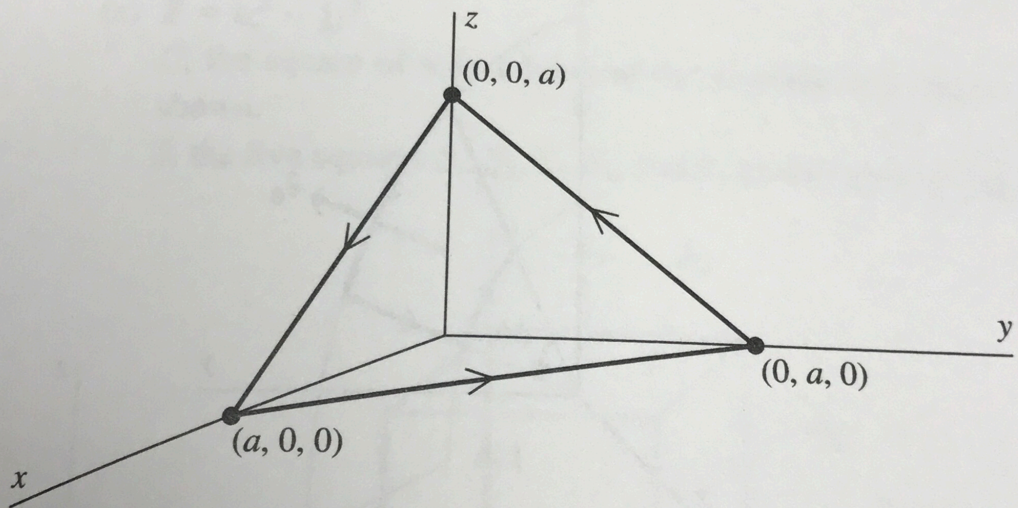
Gauss' Law $\longrightarrow$ $\displaystyle\oint_S \vec{E} \cdot \hat{n}\, da = 4\pi q_{encl}$

$\boxed{\text{III - 5 (a)}}$

III–5 (a) Calculate $\oint \mathbf{F} \cdot \hat{\mathbf{t}}\, ds$ where

$$\mathbf{F} = \mathbf{k}(y + y^2)$$

over the perimeter of the triangle shown in the figure (integrate in the direction indicated by the arrows).



$$\hat{t}(s) = \hat{\imath}\,\frac{dx}{ds} + \hat{\jmath}\,\frac{dy}{ds} + \hat{k}\,\frac{dz}{ds}$$

= unit vector tangent to path s

S = path of integration

$\Rightarrow$ ds = differential distance along s

The example below should help

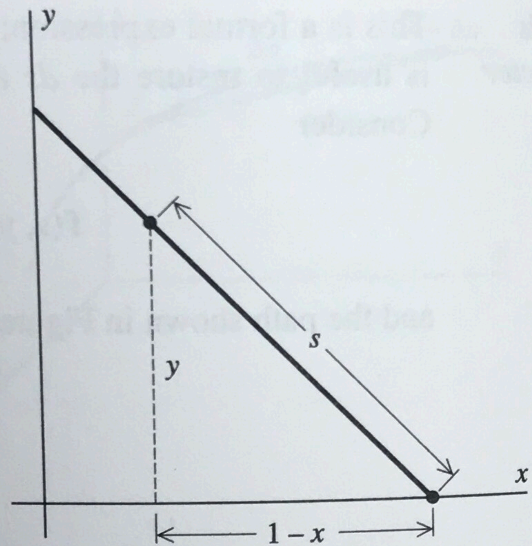


Figure III–6(b)

But $(1 - x)/s = \cos 45° = 1/\sqrt{2}$ and $y/s = \sin 45° = 1/\sqrt{2}$ [Figure III–6(b)]. Thus,

$$\left.\begin{aligned} x &= 1 - \frac{s}{\sqrt{2}} \Rightarrow \frac{dx}{ds} = -\frac{1}{\sqrt{2}} \\ y &= \frac{s}{\sqrt{2}} \Rightarrow \frac{dy}{ds} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \quad 0 \le s \le \sqrt{2}.$$