Waves in Hollow Conductors

- Consider a hollow conducting pipe in a vacuum with perfectly conducting walls
- Instead of treating the problem as a superposition of plane waves, lets solve the <u>wave equation</u> subject to Boundary Conditions

$$\left(\bar{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{bmatrix}\bar{E}\\ \bar{B}\end{bmatrix} = 0$$

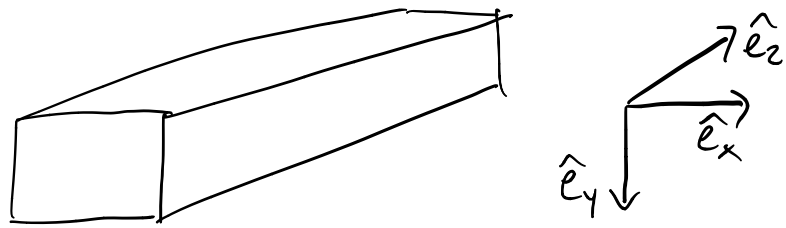
$\hookrightarrow \varepsilon = \mu = n = 1$

Let's assume solutions of the form

$$\begin{bmatrix}\bar{E}\\ \bar{B}\end{bmatrix} = \begin{bmatrix}\bar{E}_0(x,y)\\ \bar{B}_0(x,y)\end{bmatrix} e^{i(k_g z - \omega t)}$$

k_g = guide "effective" propagation constant

→ Assuming harmonic oscillations in $\hat{e}_z$



→ Need to find cross-sectional behavior of $\bar{E}$ and $\bar{B}$ ($\hat{e}_x$ and $\hat{e}_y$ behavior)

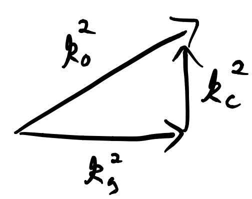
Plugging our "solution" back into the wave eqn

$$\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right.}_{\text{transverse Laplacian}} \left. - k_g^2 + \frac{\omega^2}{c^2}\right)\begin{bmatrix}\bar{E}_0 \\ \bar{B}_0\end{bmatrix} e^{i(k_g z - \omega t)} = \varnothing$$

transverse Laplacian $\bar{\nabla}_t^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Also, $-k_g^2 + \frac{\omega^2}{c^2} = -k_g^2 + k_0^2 = k_c^2$

↳ cutoff wavenumber



$$\Rightarrow \left(\bar{\nabla}_t^2 + k_c^2\right)\begin{bmatrix}\bar{E}\\ \bar{B}\end{bmatrix} = 0$$

$\hookrightarrow$ <u>Helmholtz Equation</u>

Using the form of our solution in Maxwell's eqns

$$\bar{\nabla}\cdot\bar{E} = 0 \qquad\qquad \bar{\nabla}\times\bar{E} = \frac{-1}{c}\frac{\partial\bar{B}}{\partial t} = ik_0\bar{B}$$

$$\bar{\nabla}\cdot\bar{B} = 0 \qquad\qquad \bar{\nabla}\times\bar{B} = \frac{1}{c}\frac{\partial\bar{E}}{\partial t} = -ik_0\bar{E}$$

To simplify our approach, we break the waves into the cross-sectional (transverse) and longitudinal (parallel to axis of pipe)

$$\bar{E} = \bar{E}_z + \bar{E}_t$$

$$\bar{B} = \underbrace{\bar{B}_z}_{\text{longitudinal}} + \underbrace{\bar{B}_t}_{\text{transverse}}$$

$$\Rightarrow \begin{bmatrix} \bar{E}_z \\ \bar{B}_z \end{bmatrix} = \hat{e}_z \begin{bmatrix} E_z^0(x,y) \\ B_z^0(x,y) \end{bmatrix} e^{i(k_g z - \omega t)}$$

$$\begin{bmatrix} \bar{E}_t \\ \bar{B}_t \end{bmatrix} = \begin{bmatrix} \hat{e}_x E_x^0(x,y) + \hat{e}_y E_y^0(x,y) \\ \hat{e}_x \underbrace{B_x^0(x,y)} + \hat{e}_y B_y^0(x,y) \end{bmatrix} e^{i(k_g z - \omega t)}$$

$\hookrightarrow$ why not a fxn of z?

$\rightarrow$ z dependence is in $e^{i(k_g z - \omega t)}$

Plugging into Maxwell's Equations:

$$\bar{\nabla} \cdot \bar{E} = 0 \Rightarrow \frac{\partial E_x^0}{\partial x} + \frac{\partial E_y^0}{\partial y} + i k_g E_z^0 = 0$$

$$\bar{\nabla} \cdot \bar{B} = 0 \Rightarrow \frac{\partial B_x^0}{\partial x} + \frac{\partial B_y^0}{\partial y} + i k_g B_z^0 = 0$$

$$\bar{\nabla} \times \bar{E} = i k_0 \bar{B} \Rightarrow \frac{\partial E_z^0}{\partial y} - i k_g E_y^0 = i k_0 B_x^0$$

$$i k_g E_x^0 - \frac{\partial E_z^0}{\partial x} = i k_0 B_y^0 \qquad \star$$

$$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = i k_0 B_z^0$$

$$\bar{\nabla} \times \bar{B} = -ik_0 \bar{E} \Rightarrow \quad \frac{\partial B_z^0}{\partial y} - ik_g B_y^0 = -ik_0 E_x^0 \qquad \star$$

$$ik_g B_x^0 - \frac{\partial B_z^0}{\partial x} = -ik_0 E_y^0$$

$$\frac{\partial B_y^0}{\partial x} - \frac{\partial B_x^0}{\partial y} = -ik_0 E_z^0$$

Solving $\star$ for E_x^0:

$$E_x^0 = \frac{i}{k_c^2}\left(k_0 \frac{\partial B_z^0}{\partial y} + k_g \frac{\partial E_z^0}{\partial x}\right)$$

$$k_c^2 = k_0^2 - k_g^2$$

Similarly, we find

$$E_y^0 = \frac{-i}{k_c^2}\left(k_0 \frac{\partial B_z^0}{\partial x} - k_g \frac{\partial E_z^0}{\partial y}\right)$$

$$B_x^0 = \frac{-i}{k_c^2}\left(k_0 \frac{\partial E_z^0}{\partial y} - k_g \frac{\partial B_z}{\partial x}\right)$$

$$B_y^0 = \frac{i}{k_c^2}\left(k_0 \frac{\partial E_z^0}{\partial x} + k_g \frac{\partial B_z^0}{\partial y}\right)$$

$\rightarrow$ Transverse components ($E^0_{x,y}$ and $B^0_{x,y}$) are specified entirely by the longitudinal components (E^0_z and B^0_z)

$\rightarrow$ Solutions depend on the mode (TE, TM, or TEM)

First, we analyze TEM modes in hollow waveguides.

$$\text{TEM} \rightarrow E^0_z = B^0_z = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\begin{bmatrix} E^0_x \\ B^0_x \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} E^0_y \\ B^0_y \end{bmatrix} = 0$$

and

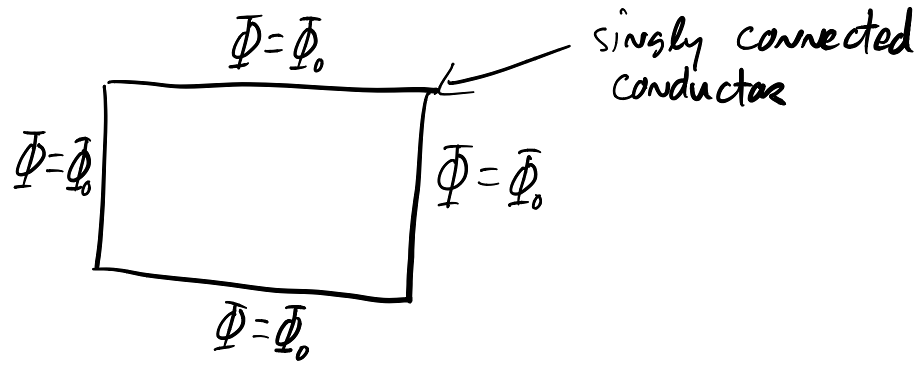
$$\frac{\partial}{\partial x}\begin{bmatrix} E^0_y \\ B^0_y \end{bmatrix} - \frac{\partial}{\partial y}\begin{bmatrix} E^0_x \\ B^0_x \end{bmatrix} = 0$$

Solutions are (plug in to verify)

$$\begin{bmatrix} E_x^0 \\ B_x^0 \end{bmatrix} = \frac{\partial \Phi}{\partial x} \qquad \begin{bmatrix} E_y^0 \\ B_y^0 \end{bmatrix} = \frac{\partial \Phi}{\partial y}$$

$$\bar{\nabla}_t^2 \Phi = 0$$

↳ Laplace's Equation



$\Rightarrow \Phi(x,y) = \Phi_0 \longrightarrow$ equipotential

$\Rightarrow \bar{E}_t = 0$

If $E_z = 0$ and $\bar{E}_t = 0 \Rightarrow \bar{B} = 0$

$\boxed{\text{☆ TEM WAVES CANNOT PROPOGATE IN A HOLLOW PIPE}}$

Note: If the bounding surfaces are not connected, as in a coaxial cable, TEM modes can propagate.

TE and TM waves

TE mode $\Rightarrow E_z^0 \equiv 0,\ B_z^0 \neq 0$

$$\Rightarrow E_x^0 = \frac{ik_0}{k_c^2}\frac{\partial B_z^0}{\partial y} \qquad E_y^0 = \frac{-ik_0}{k_c^2}\frac{\partial B_z^0}{\partial x}$$

$$B_x^0 = \frac{ik_g}{k_c^2}\frac{\partial B_z^0}{\partial x} \qquad B_y^0 = \frac{ik_g}{k_c^2}\frac{\partial B_z^0}{\partial y}$$

$$\Rightarrow \bar{\nabla} B_z^0 = \hat{e}_x \frac{\partial B_z^0}{\partial x} + \hat{e}_y \frac{\partial B_z^0}{\partial y} = \frac{k_c^2}{ik_g}\left(\hat{e}_x B_x^0 + \hat{e}_y B_y^0\right)$$

or

$$\boxed{\bar{\nabla} B_z^0 = -\frac{ik_c^2}{k_g}\bar{B}_{+0}} \qquad \text{TE mode}$$

Also, from $E_z^0 = \emptyset$ we have:

$$B_x^0 = -\frac{k_g}{k_0} E_y^0 \qquad\qquad B_y^0 = \frac{k_g}{k_0} E_x^0$$

$$\Rightarrow \bar{B}_{t0} = \hat{e}_x B_x^0 + \hat{e}_y B_y^0 = \frac{k_g}{k_0}\left(-\hat{e}_x E_y^0 + \hat{e}_y E_x^0\right)$$

OR

$$\boxed{\bar{B}_{t0} = \frac{k_g}{k_0}\left(\hat{e}_z \times \bar{E}_{t0}\right)}$$ TE mode

For TM modes $\Rightarrow B_z^0 \equiv \emptyset,\ E_z^0 \neq \emptyset$

$$\Rightarrow \boxed{\begin{aligned} \bar{E}_{t0} &= -\frac{k_g}{k_0}\left(\hat{e}_z \times \bar{B}_{t0}\right) \\ \bar{\nabla} E_z^0 &= \frac{-ik_c^2}{k_g} \bar{E}_{t0} \end{aligned}}$$ TM modes