Retarded Potentials and Fields

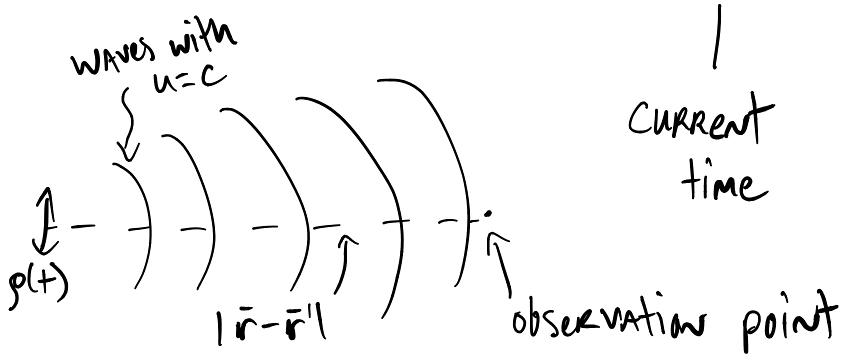
For <u>static</u> fields, our potentials are

$$\Phi(\bar{r}) = \int_v \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

$$\bar{A}(\bar{r}) = \frac{1}{c}\int_v \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

If $\bar{J}$ and ρ are functions of time, then we have to account for the finite propagation speed, c

$$\text{Retarded time} = \underbrace{t}_{\text{current time}} - \underbrace{\frac{|\bar{r}-\bar{r}'|}{c}}_{\text{propagation (delay) time}}$$



The retarded potentials are

$$\Phi(\bar{r},t) = \int_V \frac{\rho(\bar{r}', t - |\bar{r}-\bar{r}'|/c)}{|\bar{r}-\bar{r}'|}\, dv'$$

$$\bar{A}(\bar{r},t) = \frac{1}{c}\int_V \frac{\bar{J}(\bar{r}', t - |\bar{r}-\bar{r}'|/c)}{|\bar{r}-\bar{r}'|}\, dv'$$

Substituting $R \equiv |\bar{r}-\bar{r}'|$

$$\rho(\bar{r}', t - R/c) \equiv [\rho(\bar{r}')]$$

brackets mean evaluated at retarded time

$$\Rightarrow \quad \Phi(\bar{r},t) = \int_V \frac{[\rho(\bar{r}')]}{R}\, dv'$$

$$\bar{A}(\bar{r},t) = \frac{1}{c}\int_V \frac{[\bar{J}(\bar{r}')]}{R}\, dv'$$

Retarded fields

From $\quad \bar{E} = -\bar{\nabla}\Phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$

we can calculate $\bar{E}$. However,

$$\bar{\nabla}\Phi = \bar{\nabla}\int_V \frac{[\rho(\bar{r}')]}{R}\,dv' = \int_V \bar{\nabla}\,\frac{\rho(\bar{r}', t - \frac{R}{c})}{R}\,dv'$$

Acts on $\bar{r}$ (unprimed, position of interest)

$R = |\bar{r} - \bar{r}'| \quad$ is a function of $\bar{r}$

$$\bar{\nabla}\frac{1}{R} = -\frac{\hat{e}_R}{R^2}$$

$$\bar{\nabla}[\rho] = \left[\frac{\partial\rho}{\partial t}\right]\bar{\nabla}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right) = -\frac{1}{c}\left[\frac{\partial\rho}{\partial t}\right]\hat{e}_R$$

$$\Rightarrow \boxed{\bar{E}(\bar{r},t) = \int_V \left(\frac{[\rho]\hat{e}_R}{R^2} + \frac{[\partial\rho/\partial t]\,\hat{e}_R}{cR} - \frac{[\partial\bar{J}/\partial t]}{c^2 R} \right) dv'}$$

Generalized Coulomb-Faraday Law

Similarly, from $\bar{B} = \bar{\nabla} \times \bar{A}$

$$\boxed{\bar{B}(\bar{r},t) = \int_V \left(\frac{[\bar{J}] \times \hat{e}_R}{cR^2} + \frac{[\partial\bar{J}/\partial t] \times \hat{e}_R}{c^2 R} \right) dv'}$$

Generalized Biot-Savart Law

Instead of analyzing distributions ρ and $\bar{J}$, we can focus on <u>a point charge</u>

$$\Phi(\bar{r},t) = e \int_{-\infty}^{\infty} \frac{\delta(t' - t + |\bar{r} - \bar{r}'|/c)}{|\bar{r} - \bar{r}_e|} dt'$$

charge of e^-

location is a delta fxn in time

Defining $\bar{R} \equiv R\,\hat{e}_R$

$$\bar{\beta} \equiv \frac{\bar{u}}{c}$$ — $\bar{u}$ = particle's velocity

$$\bar{K} \equiv 1 - \frac{\bar{\beta}\cdot\bar{R}}{R}$$

We can rewrite Φ as (see book for details)

$$\boxed{\Phi(\bar{r},t) = \frac{e}{[KR]}}$$

from $\bar{J} = \rho_e \bar{u}$ we obtain

$$\boxed{\bar{A}(\bar{r},t) = e\left[\frac{\bar{\beta}}{KR}\right]}$$

Liénard-Wiechert potentials

From the potentials, "after some heroic algebra" we obtain the Liénard-Wiechert fields

$$\boxed{\begin{aligned} \bar{E} &= e\left[\frac{(\hat{n}-\bar{\beta})(1-\beta^2)}{K^3R^2} + \frac{\hat{n}\times((\hat{n}-\bar{\beta})\times\bar{a})}{c^2K^3R}\right] \\ \bar{B} &= [\hat{n}]\times\bar{E} \end{aligned}}$$

where $\hat{n} = \hat{e}_R = \dfrac{\bar{R}}{R}$

$\bar{a}$ = particle's acceleration

Radiation From Accelerated Charged Particle at low velocities

For $u \ll c \Rightarrow \beta \ll 1, \quad \kappa \approx 1$

the acceleration fields become

$$\bar{E}_a = \frac{e}{c^2 R^3}\left\{(\bar{R}\cdot\bar{a})\bar{R} - R^2\bar{a}\right\}$$

$$\bar{B}_a = \hat{n}\times\bar{E}_a$$

We have dropped the [retarded time] because we're now focused on radiation leaving rather than approaching.

The Poynting Vector is

$$\bar{S}_a = \frac{c}{4\pi}\left(\bar{E}_a\times\bar{B}_a\right) = \frac{c}{4\pi}\left\{\frac{\bar{E}_a\times(\bar{R}\times\bar{E}_a)}{R}\right\}$$

$$= \frac{c}{4\pi}\left\{\frac{E_a^2\bar{R} - (\bar{E}_a\cdot\bar{R})\bar{E}_a}{R}\right\}$$

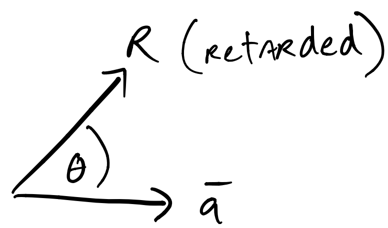
But, $\qquad \bar{E}_a \perp \bar{R} \implies \bar{E}_a \cdot \bar{R} = 0$

$$\implies \quad \bar{S}_a = \frac{c}{4\pi} E_a^2 \frac{\bar{R}}{R} = \frac{c}{4\pi} E_a^2 \hat{n}$$

And

$$E_a^2 = \bar{E}_a \cdot \bar{E}_a = \frac{e^2}{c^4 R^4} \left\{ R^2 a^2 - (\bar{R} \cdot \bar{a})^2 \right\}$$

Defining



$$\implies E_a^2 = \frac{e^2 a^2}{c^4 R^2} \left(1 - \cos^2\theta \right) = \frac{e^2 a^2}{c^4 R^2} \sin^2\theta$$

$$\implies \quad \bar{S}_a = \frac{e^2 a^2 \sin^2\theta}{4\pi c^3 R^2} \hat{n}$$

$$S_a = \frac{\text{power}}{\Delta A} = \frac{\text{power}}{R^2 \Delta \Omega}$$

↳ unit area ↳ solid angle

$$\Rightarrow \boxed{(\bar{S}_a \cdot \hat{n}) R^2 = \frac{dP}{d\Omega} = \frac{e^2 a^2}{4\pi c^3} \sin^2\theta}$$

Radiated power per unit solid angle

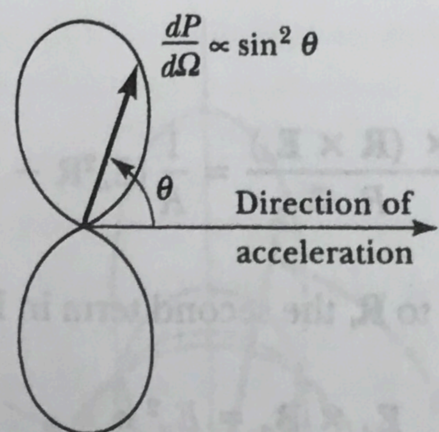


FIGURE 8-5. Angular dependence of radiation from a slow accelerated charge.

[H+M]

3-D extension = doughnut

→ This radiation pattern is characteristic of <u>dipole radiation</u> and simple antennas

The total radiated power is

$$\boxed{P = \int_{4\pi} \frac{dP}{d\Omega}\, d\Omega = \frac{2e^2a^2}{3c^3}}$$

Larmor Formulas

Collinear Velocity and Acceleration Radiation

For $\bar{u} \parallel \bar{a} \Rightarrow \bar{\beta} \times \bar{a} = 0$

$$\Rightarrow \bar{E}_a = \frac{e}{c^2K^3R^3}\left\{(\bar{R}\cdot\bar{a})\bar{R} - R^2\bar{a}\right\}$$

↳ now has retardation factor

$K = 1 - \hat{n}\cdot\bar{\beta}$

$$\Rightarrow \bar{S}_a = \frac{c}{4\pi}E_a^2\,\hat{n} = \frac{e^2a^2\sin^2\theta}{4\pi c^3K^6R^2}\,\hat{n}$$

The incremental amount of energy lost by the particle and radiated to solid angle θ <u>measured</u> during time dt:

$$-dW(\theta) = (\bar{S}_a \cdot \hat{n}) R^2 \, dt$$

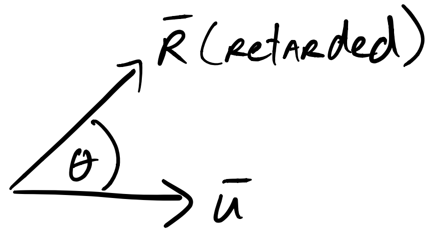
$$\Rightarrow \quad \frac{dP}{d\Omega} = -\frac{dW(\theta)}{dt'} = (\bar{S}_a \cdot \hat{n}) R^2 \frac{dt}{dt'}$$

Radiated during t' to be measured at t

From $t' = t - R(t')/c$

$$\Rightarrow \quad \frac{dt}{dt'} = 1 - \frac{\bar{\beta} \cdot \bar{R}}{R} = K = 1 - \beta \cos\theta$$

$$\Rightarrow \quad \boxed{\frac{dP}{d\Omega} = \frac{e^2 a^2 \sin^2\theta}{4\pi c^3 (1 - \beta\cos\theta)^5}}$$



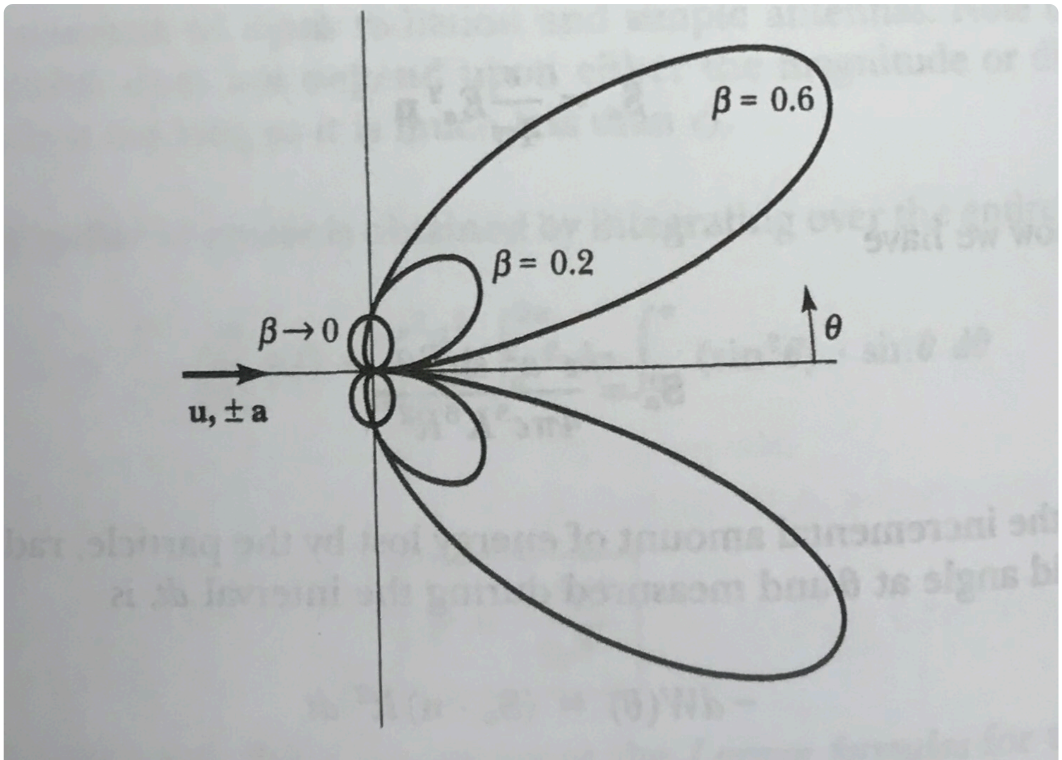


FIGURE 8-6. Radiation from accelerated charge with **u** parallel to **a**.

[H+M]

- For $\beta \ll 1$, we get the Larmor formula
- As $\beta \to 1$, the radiation intensity increases in the forward direction
  - ↳ bremsstrahlung (braking radiation)
  - ↳ produce x-rays by "braking" fast e^-