Spherical Harmonics

For problems without symmetry about the polar axis, we need to use spherical harmonics for the angular solution.

The ODE to solve is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

$$\text{solutions} = \underbrace{P_\ell^m(\cos\theta)}_{\text{Associated Legendre Polynomials}}$$

m is limited to integer values between $\pm\ell$

$\Rightarrow m = 0, \pm 1, \pm 2, \ldots, \pm\ell$

Orthogonality condition for a given m:

$$\int_{-1}^{1} P_\ell^m(\cos\theta)\,P_{\ell'}^m(\cos\theta)\,dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\ell'}$$

The relationship between P_ℓ^m and P_ℓ^{-m} is:

$$P_\ell^{-m}(\cos\theta) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta)$$

Table 15.3 Associated Legendre Functions

$P_1^1(x) = -(1-x^2)^{1/2} = -\sin\theta$
$P_2^1(x) = -3x(1-x^2)^{1/2} = -3\cos\theta\sin\theta$
$P_2^2(x) = 3(1-x^2) = 3\sin^2\theta$
$P_3^1(x) = -\frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = -\frac{3}{2}(5\cos^2\theta-1)\sin\theta$
$P_3^2(x) = 15x(1-x^2) = 15\cos\theta\sin^2\theta$
$P_3^3(x) = -15(1-x^2)^{3/2} = -15\sin^3\theta$
$P_4^1(x) = -\frac{5}{2}(7x^3-3x)(1-x^2)^{1/2} = -\frac{5}{2}(7\cos^3\theta-3\cos\theta)\sin\theta$
$P_4^2(x) = \frac{15}{2}(7x^2-1)(1-x^2) = \frac{15}{2}(7\cos^2\theta-1)\sin^2\theta$
$P_4^3(x) = -105x(1-x^2)^{3/2} = -105\cos\theta\sin^3\theta$
$P_4^4(x) = 105(1-x^2)^2 = 105\sin^4\theta$

[Arfken + Weber]

Our solution for the ϕ ODE

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0$$

are $$Q_m = C_m e^{im\phi} + D_m e^{-im\phi}$$

Combining the two angular components, we get the <u>Spherical Harmonics</u>

$$\boxed{Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}}$$

The orthogonality relation is

$$\int_{4\pi} Y_\ell^m(\theta,\phi)\, Y_{\ell'}^{m'*}(\theta,\phi)\, d\Omega$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\; Y_\ell^m(\theta,\phi)\, Y_{\ell'}^{m'*}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

$\rightarrow Y_\ell^{m*}$ = complex conjugate of Y_ℓ^m

$$
\left.\begin{aligned}
Y_0^0(\theta,\varphi) &= \sqrt{\frac{1}{4\pi}}
\end{aligned}\right\} \quad l = 0
$$

$$
\left.\begin{aligned}
Y_1^0(\theta,\varphi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_1^{\pm1}(\theta,\varphi) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}
\end{aligned}\right\} \quad l = 1
$$

$$
\left.\begin{aligned}
Y_2^0(\theta,\varphi) &= \sqrt{\frac{5}{16\pi}}\,(2\cos^2\theta - \sin^2\theta) \\
Y_2^{\pm1}(\theta,\varphi) &= \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\, e^{\pm i\varphi} \\
Y_2^{\pm2}(\theta,\varphi) &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}
\end{aligned}\right\} \quad l = 2
$$

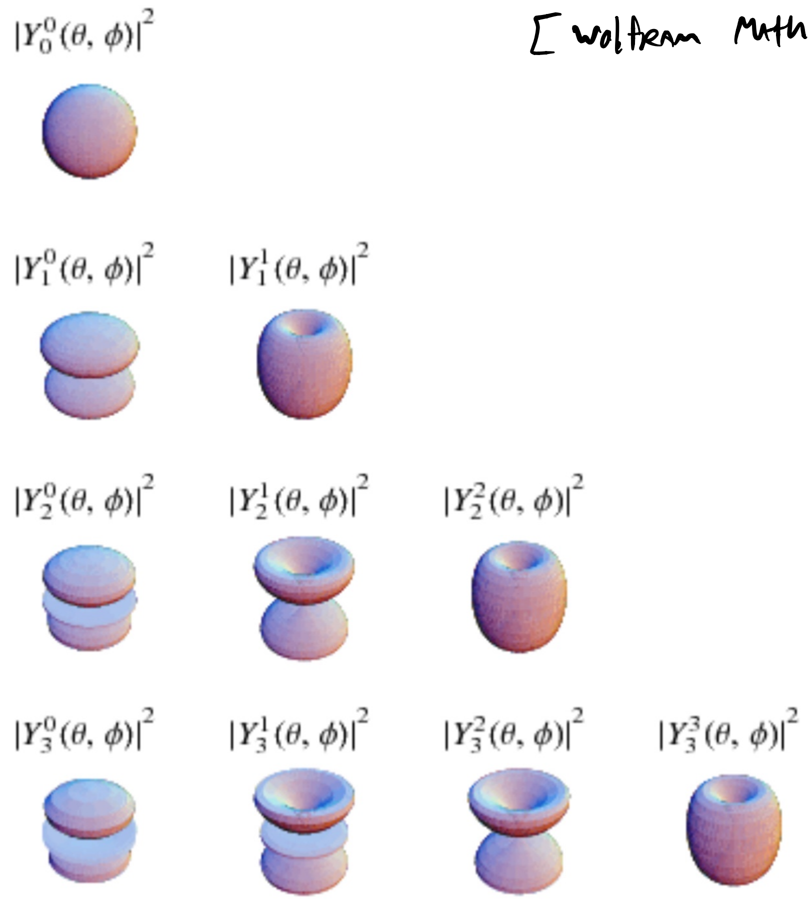
$$
\left.\begin{aligned}
Y_3^0(\theta,\varphi) &= \sqrt{\frac{7}{16\pi}}\,(2\cos^3\theta - 3\cos\theta\sin^2\theta) \\
Y_3^{\pm1}(\theta,\varphi) &= \mp\sqrt{\frac{21}{64\pi}}\,(4\cos^2\theta\sin\theta - \sin^3\theta)e^{\pm i\varphi} \\
Y_3^{\pm2}(\theta,\varphi) &= \sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta\, e^{\pm 2i\varphi} \\
Y_3^{\pm3}(\theta,\varphi) &= \mp\sqrt{\frac{35}{64\pi}}\sin^3\theta\, e^{\pm 3i\varphi}
\end{aligned}\right\} \quad l = 3
$$

[H+M]

The general solution for Φ becomes

$$
\boxed{\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_l^m r^l + \frac{B_l^m}{r^{l+1}}\right]Y_l^m(\theta,\phi)}
$$

[wolfram Mathworld]



Spherical Harmonics can also be used for the multipole expansion:

$$\Phi(\bar{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_l^m \frac{Y_l^m(\theta, \phi)}{r^{l+1}}$$

$$q_l^m = \underbrace{\int_V Y_l^{m*}(\theta', \phi') r'^l \rho(\bar{r}') dv'}_{\text{integrate over source } dv'} = \text{multipole moment}$$

Cylindrical Coordinates

$$\bar{\nabla}^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Separation of variables:

$$\Phi(r, \theta, z) = R(r)\, Q(\theta)\, Z(z)$$

$$\Rightarrow \underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{r^2}{Z}\frac{d^2 Z}{dz^2}}_{\text{function of } r \text{ and } z} = \underbrace{\frac{-1}{Q}\frac{d^2 Q}{d\theta^2}}_{\text{function of } \theta}$$

Set both sides equal to n^2:

$$\frac{d^2 Q}{d\theta^2} + n^2 Q = 0$$

$$\Rightarrow Q_n(\theta) = C_n e^{in\phi} + D_n e^{-in\phi}$$

$n =$ integer

The equation for r and z becomes

$$\underbrace{\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r^2}}_{=-k^2} = \underbrace{\frac{-1}{Z}\frac{d^2Z}{dz^2}}_{=-k^2}$$

$$\Rightarrow \quad \frac{d^2Z}{dz^2} - k^2 Z = 0$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(k^2r^2 - n^2\right)R = 0$$

The solutions for Z are exponential

$$Z_k(z) = E_k e^{kz} + F_k e^{-kz}$$

For the R equation, we make the substitutions

$$u = kr$$

$$\frac{d}{dr} = k\frac{d}{du}$$

$$\Rightarrow \underbrace{u^2 \frac{d^2R}{du^2} + u\frac{dR}{du} + (u^2 - n^2)R = 0}_{\text{Bessel's Equation}}$$

Solutions are <u>Bessel Functions</u>

$$J_n(u) = \frac{u^n}{2^n n!} \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma}{2^{2\gamma}\gamma!(n+1)(n+2)\dots(n+\gamma)} u^{2\gamma}$$

$$N_n(u) = \frac{J_n(u)\cos n\pi - J_{-n}(u)}{\sin n\pi}$$

$\rightarrow J_n$ = Bessel function of order n

N_n = Neumann function (Bessel function of the second kind)

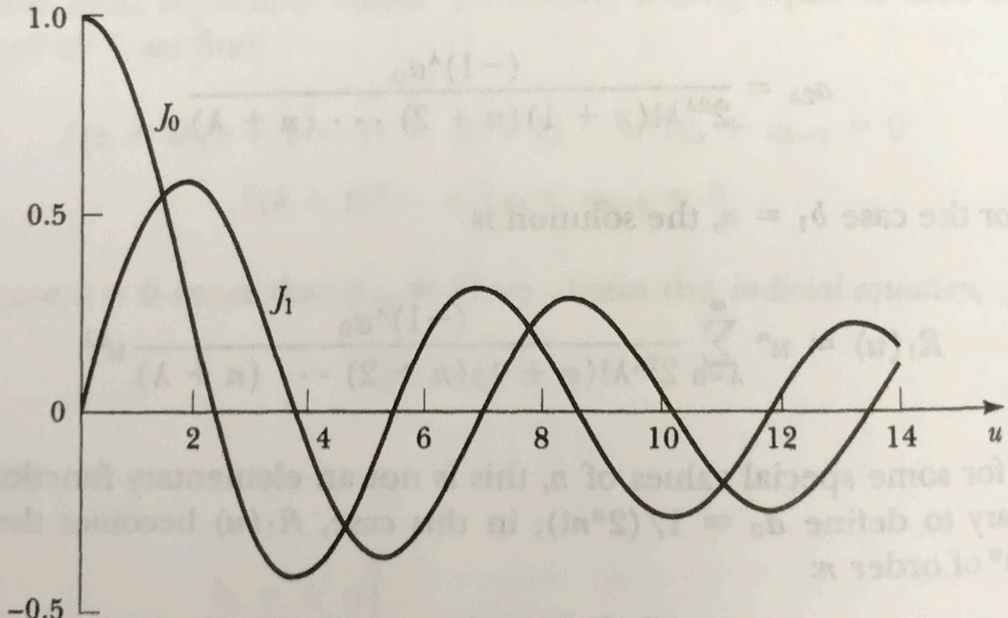


FIGURE 3-6. Bessel functions.

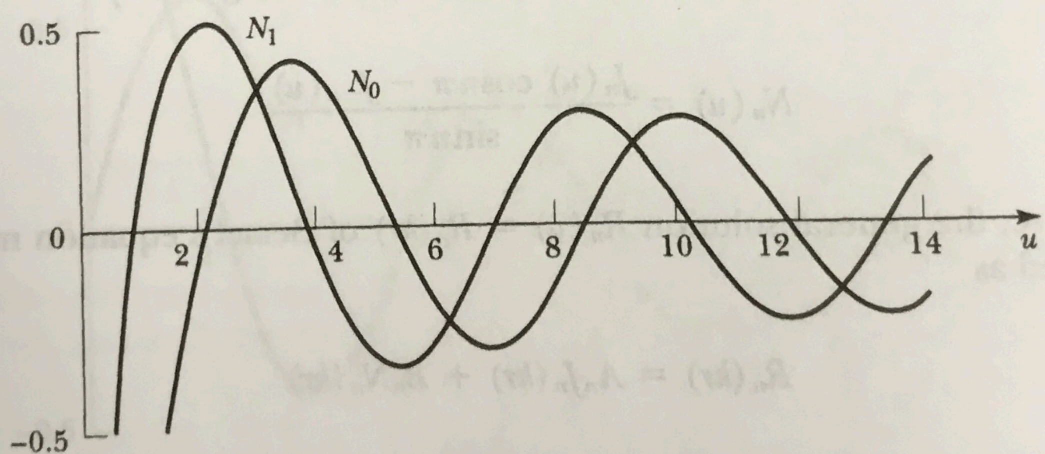


FIGURE 3-7. Neumann functions.

[H+M]

J_n is used for problems that involve the origin because $N_n(kr) \rightarrow -\infty$ as $kr \rightarrow 0$.

Orthogonality:

For $k_m \rho$ as the mth root of $J_n(kr)$

$$\int_0^\rho J_n(k_m r) J_n(k_{m'} r)\, r\,dr = \frac{\rho^2}{2} J_{n+1}^2(k_{m'}\rho)\,\delta_{mm'}$$

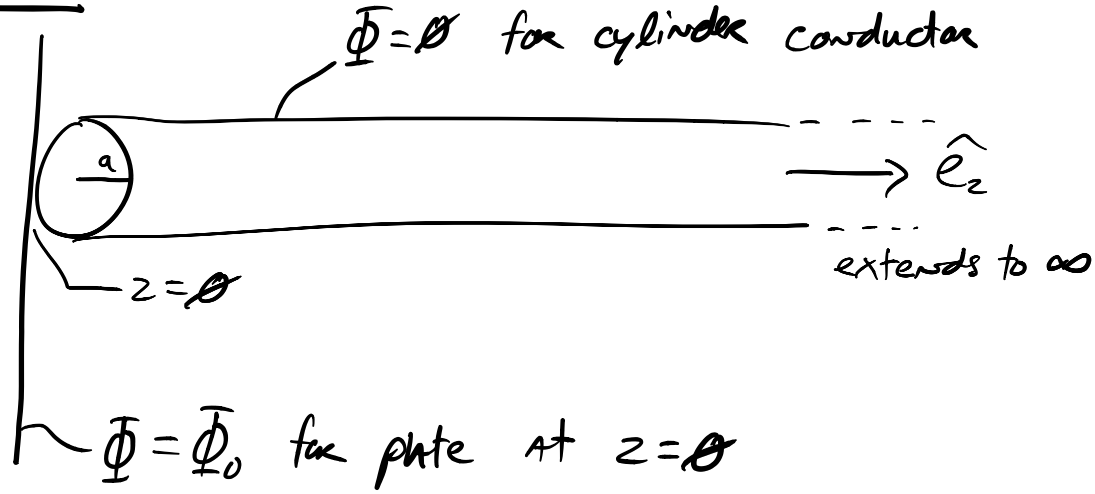
The general solution for R becomes:

$$R_n(kr) = A_n J_n(kr) + B_n N_n(kr)$$

Taking into account the fact that various k values can lead to acceptable solutions, we'll label the k solutions as k_m and the general solution for Φ is

$$\boxed{\Phi(r,\theta,z) = \sum_{m,n}\left[A_{mn} J_n(k_m r) + B_{mn} N_n(k_m r)\right] e^{\pm i n\theta} e^{\pm k_m z}}$$

<u>Example:</u>



☆ Find Φ in cylinder

- We have azimuthal symmetry, so $n = 0$ and we can ignore $Q(\theta)$.

- The origin is involved, so we'll use J_n instead of N_n

- $\Phi \to 0$ as $z \to \infty$, so we can't use the e^{+kz} solution

$$\Rightarrow \quad \Phi(r,\theta,z) = \sum_m A_{m0} J_0(k_m r)\, e^{-k_m z}$$

for $r < a$ and $z > 0$

We need

$$\Phi(a,\theta,z) = 0$$

$$\Rightarrow \quad \sum_m A_{m0} J_0(k_m a)\, e^{-k_m z} = 0$$

So $k_m a$ must be the zeroes of $J_0(kr)$

$$\Rightarrow \quad k_m = \frac{k_{B0}}{a} \longleftarrow J_0 \text{ Roots}$$

TABLE 3.5
Some Roots of J_0, J_1, and J_2

Bessel Function	*1st Root*	*2nd Root*	*3rd Root*	*4th Root*
J_0	2.405	5.520	8.654	11.792
J_1	3.832	7.016	10.173	13.324
J_2	5.136	8.417	11.620	14.796

Also, we need

$$\Phi(r, \theta, 0) = \Phi_0$$

$$\Rightarrow \sum_m A_{m0} J_0(k_m r) = \Phi_0$$

Using the orthogonality of k_m, we can multiply both sides by $r J_0(k_{m'} r)$ and integrate from 0 to a:

$$\int_0^a \left[\sum_m A_{m0} J_0(k_m r)\right] J_0(k_{m'} r)\, r\, dr = A_{m0} \frac{a^2}{2} J_1^2(k_m a)$$

$$= \int_0^a \Phi_0 J_0(k_m r)\, r\, dr$$

$$\Rightarrow A_{m0} = \frac{2\Phi_0}{a^2 J_1^2(k_m a)} \int_0^a J_0(k_m r)\, r\, dr$$

But,
$$\int_0^a kr\, J_0(kr)\, d(kr) = ka\, J_1(ka)$$

$$\Rightarrow \int_0^a r\, J_0(kr)\, dr = \frac{a}{k} J_1(ka)$$

$$\Rightarrow A_{m0} = \frac{2\,\Phi_0}{a^2 J_1^2(k_n a)} \frac{a}{k_n} J_1(k_m a)$$

$$= \frac{2\,\Phi_0}{a\, k_n\, J_1(k_n a)}$$

The complete solution is

$$\Phi(r,\theta,z) = 2\Phi_0 \sum_m \frac{J_0(k_m r)}{a k_n J_1(k_n a)} e^{-k_m z}$$

where $k_m a$ = zeroes of J_0

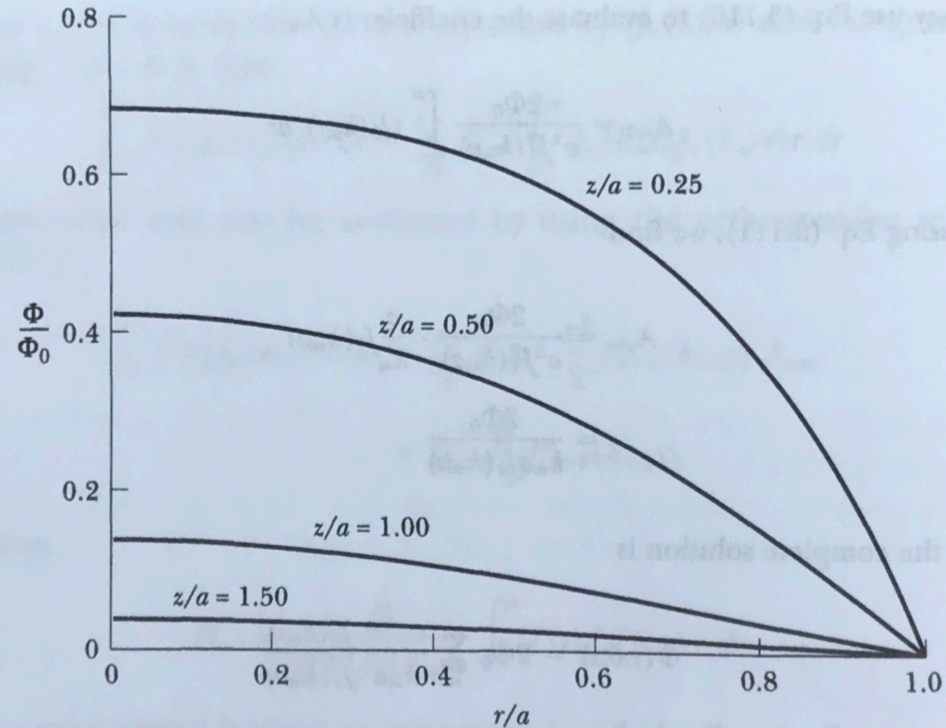
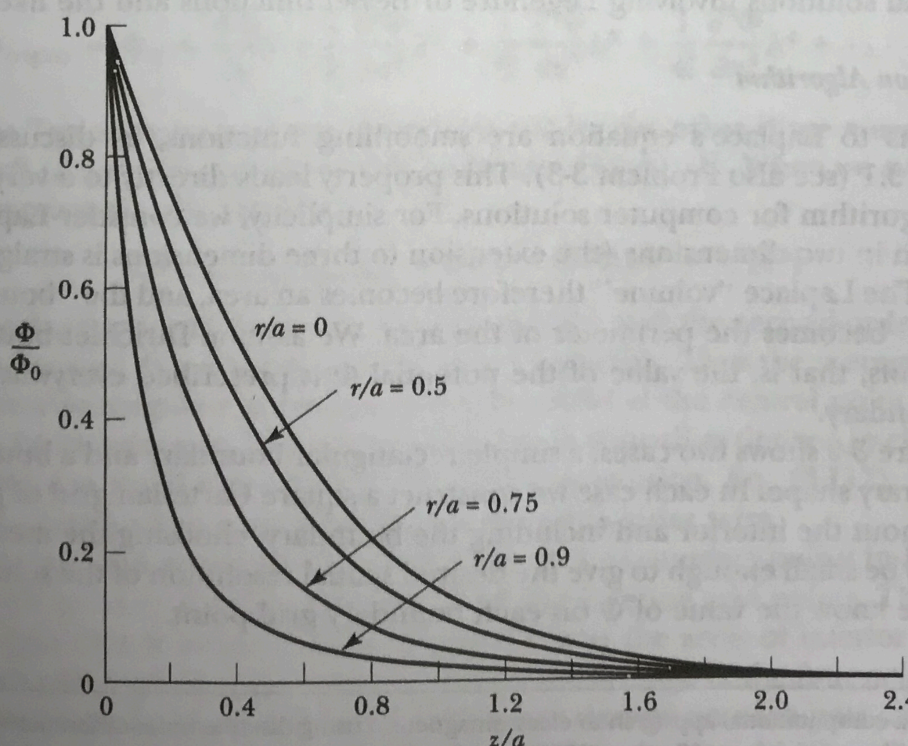


FIGURE 3-8b. Solutions of Example 3.5, continued.



[H+M]

FIGURE 3-8a. Solutions of Example 3.5.

In general:

Spherical Problems $\longrightarrow$ think <u>spherical harmonics</u>

Cylindrical Problems $\longrightarrow$ think <u>Bessel functions</u>