PHYS 775

Ionosphere II
Winter 2017
Conductivity
Schunk 5.10, 5.11

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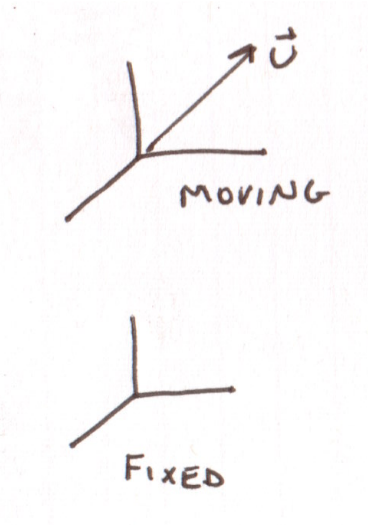
E & B Transformations



- In classical mechanics used Gallilean transformation to go from one reference frame to another

E & B Transformations

- Special relativity tells us that this is not valid as $v \to C$
- In E & M, Gallilean transformation not valid even if $v \ll C$
- Will need Lorentz transformation
- Trying to find out what $\vec{E}$ and $\vec{B}$ are in a moving reference frame

E & B Transformations

- i.e. What does the electric field look like if we are moving along with the neutral wind?

$$
\begin{aligned}
\vec{v}' &= \vec{u} + \vec{v} \\
\vec{E}' &= \vec{E} + ? \\
\vec{B}' &= \vec{B} + ?
\end{aligned}
$$

- For the good students, the derivation is carried out in Jackson, Chapter 11

E & B Transformations

- For the rest of us, the transformed fields are

$$
\begin{aligned}
\vec{E}' &= \vec{E} + \gamma \vec{v} \times \vec{B} + \left(\frac{\gamma - 1}{v^2}\right) \vec{v} \times \left(\vec{E} \times \vec{v}\right) \\
\vec{B}' &= \vec{B} + \gamma \vec{E} \times \frac{\vec{v}}{C^2} + \left(\frac{\gamma - 1}{v^2}\right) \vec{v} \times \left(\vec{E} \times \vec{v}\right) \\
\gamma &= \left(1 - \frac{v^2}{C^2}\right)^{-1/2}
\end{aligned}
$$

E & B Transformations

- For $v \ll C$,

$$\begin{aligned} \gamma &\sim 1 \\ \vec{E}' &= \vec{E} + \vec{v} \times \vec{B} \\ \vec{B}' &= \vec{B} \end{aligned}$$

- Why do these transformations?
- Makes the math easier, after a while

E & B Transformations

- Transform to a reference frame moving with the neutral wind:

$$\begin{aligned}
\vec{u}_s{}' &= \vec{u}_s - \vec{u}_n \\
\vec{u}_s &= \vec{u}_s{}' + \vec{u}_n \\
\vec{E}' &= \vec{E} + \vec{u}_n \times \vec{B} \\
\vec{E} &= \vec{E}' - \vec{u}_n \times \vec{B} \\
\vec{B}' &= \vec{B}
\end{aligned}$$

- $\vec{E}$ and $\vec{B}$ have no independent existence
- A purely magnetic or electric field in one coordinate system will appear as a mixture in another coordinate system

E & B Transformations

- Once again, we go back to the momentum equation with the diffusion approximation and no stress nor heat transport

$$\vec{\nabla} p_s - \rho_s \vec{G} - n_s q(\vec{E} + \vec{u}_s \times \vec{B}) \;=\; \sum_t \rho_s \nu_{st}(\vec{u}_t - \vec{u}_s)$$

- Consider only collisions with neutrals
- $\sum_t \rightarrow \sum_n$

E & B Transformations

$$
\begin{aligned}
\vec{\nabla} p_s - \rho_s \vec{G} - n_s q_s(\vec{E} + \vec{u}_s \times \vec{B}) &= \sum_n \rho_n \nu_{sn}(\vec{u}_n - \vec{u}_s) \\
\vec{\nabla} p_s - \rho_s \vec{G} & \\
-n_s q_s \left((\vec{E}\,' - \vec{u}_n \times \vec{B}\,') + (\vec{u}_s\,' + \vec{u}_n) \times \vec{B} \right) &= \rho_s \nu_{nt} \left(\vec{u}_n - (\vec{u}_s\,' + \vec{u}_n) \right) \\
\vec{\nabla} p_s - \rho_s \vec{G} - n_s q_s \left(\vec{E}\,' + \vec{u}_s\,' \times \vec{B}'\,' \right) &= -\rho_s \nu_{sn} \vec{u}_s\,' \\
\rho_s \nu_{sn} \vec{u}_s\,' - n_s q_s\,' \vec{u}_s\,' \times \vec{B}\,' &= -\vec{\nabla} p_s + \rho_s \vec{G} + n_s q_s \vec{E}\,' \\
n_s m_s \nu_{sn} \vec{u}_s\,' - n_s q_s\,' \vec{u}_s\,' \times \vec{B}\,' &= -\vec{\nabla} p_s + \rho_s \vec{G} + n_s q_s \vec{E}\,' \\
\vec{u}_s\,' - \frac{q_s \vec{u}_s\,' \times \vec{B}\,'}{m_s \nu_{sn}} &= \frac{1}{m_s \nu_{sn}} \left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right)
\end{aligned}
$$

Plasma motion due to applied forces

- Let:

$$\omega_c \equiv \frac{qB}{m}$$
$$K \equiv \frac{\omega_c}{\nu_{sn}}$$
$$\vec{u}_s{}' - K\,\vec{u}_s{}' \times \hat{b} = \frac{1}{m_s \nu_{sn}} \left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right)$$

- Let $\vec{W}$ be the fluid velocity

$$\vec{W} = \vec{u}_s{}' - K\,\vec{u}_s{}' \times \hat{b} = \frac{1}{m_s \nu_{sn}} \left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right)$$

Plasma motion due to applied forces

- When $K \ll 1$, "unmagnetized" case

$$
\begin{aligned}
\vec{u}_s{}' &= \frac{1}{m_s \nu_{sn}} \left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right) \\
\vec{u}_s{}' &= \frac{1}{m_s \nu_{sn}} \left(-n_s\, k \frac{\vec{\nabla} T_s}{n_s} - T_s\, k \frac{\vec{\nabla} n_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right) \\
\vec{u}_s{}' &= \frac{-kT}{m_s \nu_{sn}} \left(\frac{\vec{\nabla} T_s}{T_s} + \frac{\vec{\nabla} n_s}{n_s} + \frac{m_s g \hat{z}}{kT} - \frac{q_s \vec{E}\,'}{kT} \right) \\
\Gamma_s = n_s \vec{u}_s{}' &= -D \left(\frac{n_s}{T_s} \vec{\nabla} T_s + \vec{\nabla} n_s + \frac{n_s}{H} \hat{z} - \frac{n_s\, q_s \vec{E}\,'}{kT} \right)
\end{aligned}
$$

Plasma motion due to applied forces

- If $\vec{E}$ dominates forces

$$\Gamma_s = n_s \vec{u}_s{}' = D \left(\frac{n_s\, q_s \vec{E}\,'}{kT} \right)$$

$$\Gamma_s = n_s \vec{u}_s{}' = \frac{kT}{m_s \nu_{sn}} \left(\frac{n_s\, q_s \vec{E}\,'}{kT} \right)$$

$$\Gamma_s = n_s \vec{u}_s{}' = \frac{n_s\, q_s \vec{E}\,'}{m_s \nu_{sn}}$$

Plasma motion due to applied forces

- From your favorite E & M or plasma class:

$$
\begin{aligned}
\vec{J} &\equiv \sum_s n_s\, q_s\, \vec{u}_s = \sum_s \Gamma_s\, q_s \\
&= \sum_s \frac{n_s\, q_s^2}{m_s \nu_{sn}} \vec{E}\,' \\
&= \sum_s \sigma_s \vec{E}\,' \\
\sigma_s &\equiv \frac{n_s\, q_s^2}{m_s\, \nu_{sn}}
\end{aligned}
$$

- σ is the conductivity
- When K is small ("unmagnetized") there will be a current in the direction of $\vec{E}$

Plasma motion due to applied forces

- When $K \gg 1$, "magnetized" case

$$
\begin{aligned}
-K\,\vec{u}_s{}' \times \hat{b} &= \frac{1}{m_s \nu_{sn}} \left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right) \\
-\frac{q_s\,\vec{u}_s{}' \times \vec{B}}{m_s\,\nu_{sn}} &= \frac{1}{m_s \nu_{sn}} \left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right) \\
-\frac{q_s\,\vec{u}_s{}' \times \vec{B}}{m_s\,\nu_{sn}} \times \vec{B} &= \frac{1}{m_s \nu_{sn}} \left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right) \times \vec{B} \\
u_\perp &= \frac{1}{q_s B^2} \left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,' \right) \times \vec{B} \\
&= -\frac{\vec{\nabla} p_s \times \vec{B}}{n_s q_s B^2} + m_s \frac{\vec{G} \times \vec{B}}{q_s B^2} + \frac{\vec{E}\,' \times \vec{B}}{B^2}
\end{aligned}
$$

Conductivity

- What happens in between?
- In between regime, $K \sim 1$

$$
\begin{aligned}
\vec{u}_s{}' - K\,\vec{u}_s{}' \times \hat{b} &= \frac{1}{m_s \nu_{sn}}\left(-\frac{\vec{\nabla} p_s}{n_s} + m_s \vec{G} + q_s \vec{E}\,'\right) \\
\vec{u}_s{}' - K\,\vec{u}_s{}' \times \hat{b} &= \vec{W} \\
\vec{u}_s{}' + K\,\hat{b} \times \vec{u}_s{}' &= \vec{W}
\end{aligned}
$$

- Rewrite in matrix form
- To do this it is easier to define $\vec{B}$ in the $\hat{z}$ direction

Conductivity

- If

$$\vec{u}_s{}' = \begin{pmatrix} u_s' & 0 & 0 \end{pmatrix}$$

- Then,

$$\begin{aligned} \hat{b} \times \vec{u}_s{}' &= \begin{pmatrix} i & j & k \\ 0 & 0 & 1 \\ u_s' & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & u_s' & 0 \end{pmatrix} \end{aligned}$$

- If

$$\vec{u}_s{}' = (0 \quad u_s \quad 0) \rightarrow \hat{b} \times \vec{u}_s{}' = \begin{pmatrix} -u_s' & 0 & 0 \end{pmatrix}$$

- If

$$\vec{u}_s{}' = (0 \quad 0 \quad u_s) \rightarrow \hat{b} \times \vec{u}_s{}' = (0 \quad 0 \quad 0)$$

Conductivity

$$\hat{b} \times \vec{u}_s{}' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \vec{u}_s{}'$$

$$\begin{aligned} \vec{u}_s{}' &= \underline{\underline{I}} \cdot \vec{u}_s{}' \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{u}_s{}' \end{aligned}$$

Conductivity

$$\vec{W} = \vec{u}_s{}' + K\,\hat{b} \times \vec{u}_s{}'$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{u}_s{}' + K \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \vec{u}_s{}'$$

$$\vec{W} = \begin{pmatrix} 1 & -K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{u}_s{}'$$

Conductivity

- Use this to rewrite the momentum equation

$$\begin{aligned}
\underline{\underline{A}} \cdot \vec{x} &= \vec{b} \\
\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} \cdot \vec{x} &= \underline{\underline{A}}^{-1} \cdot \vec{b} \\
\vec{x} &= \underline{\underline{A}}^{-1} \cdot \vec{b}
\end{aligned}$$

Conductivity

- In this case

$$\underline{\underline{A}}^{-1} = \frac{1}{1+K^2}\begin{pmatrix} 1 & K & 0 \\ -K & 1 & 0 \\ 0 & 0 & 1+K^2 \end{pmatrix}$$

- So

$$\vec{u}_s{}' = \frac{1}{1+K^2}\begin{pmatrix} 1 & K & 0 \\ -K & 1 & 0 \\ 0 & 0 & 1+K^2 \end{pmatrix} \cdot \vec{W}$$

Conductivity ***UPDATED***

- Look at the results in vector components

$$
\begin{aligned}
\vec{u}_s{}' &= u_x{}' + u_y{}' + u_z{}' \\
&= \frac{1}{1+K^2}\begin{pmatrix} 1 & K & 0 \\ -K & 1 & 0 \\ 0 & 0 & 1+K^2 \end{pmatrix} \cdot \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix} \\
&= \frac{1}{1+K^2}\begin{pmatrix} W_x + KW_y \\ -KW_x + W_y \\ (1+K^2)W_z \end{pmatrix} \\
&= \frac{1}{1+K^2}\left((W_x + KW_y)\hat{i} + (-KW_x + W_y)\hat{j} + (1+K^2)W_z\hat{k}\right)
\end{aligned}
$$

Conductivity

$$
\begin{aligned}
\vec{u}_s{}' &= u_{\parallel}{}' + u_{\perp}{}' \\
&= \frac{1}{1+K^2}\begin{pmatrix} 1 & K & 0 \\ -K & 1 & 0 \\ 0 & 0 & 1+K^2 \end{pmatrix} \cdot (W_{\neq z} + W_z) \\
&= \frac{1}{1+K^2}\begin{pmatrix} 1 & K & 0 \\ -K & 1 & 0 \\ 0 & 0 & 1+K^2 \end{pmatrix} \cdot \left(W_{\perp} + W_{\parallel}\right)
\end{aligned}
$$

Conductivity

- This results in:

$$
\begin{aligned}
u_{\parallel}{}' &= \frac{1}{1+K^2}\left(1+K^2\right) W_{\parallel} \\
&= W_{\parallel} \\
&= \frac{1}{m_s \nu_{sn}} \left(-\frac{\vec{\nabla}_{\parallel} p_s}{n_s} + m_s G_{\parallel} + q_s E_{\parallel}{}' \right) \\
&= \frac{-kT}{m_s \nu_{sn}} \left(\frac{1}{T} \vec{\nabla}_{\parallel} T + \frac{1}{n_s} \vec{\nabla}_{\parallel} n_s - \frac{m_s G_{\parallel}}{kT} - \frac{q_s E_{\parallel}{}'}{kT} \right) \\
&= -D \left(\frac{1}{T} \vec{\nabla}_{\parallel} T + \frac{1}{n_s} \vec{\nabla}_{\parallel} n_s - \frac{m_s G_{\parallel}}{kT} - \frac{q_s E_{\parallel}{}'}{kT} \right)
\end{aligned}
$$

- Same as before

Conductivity

- Now in the perpendicular direction

$$
\begin{aligned}
{u_\perp}' &= \frac{1}{1+K^2}\begin{pmatrix} 1 & K \\ -K & 1 \end{pmatrix}\cdot(W_\perp) \\
&= \frac{1}{1+K^2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\vec{W}_\perp + K\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\cdot W_\perp\right) \\
&= \frac{1}{1+K^2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\vec{W}_\perp - K\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot W_\perp\right)
\end{aligned}
$$

Conductivity

- Last term in equation can be rewritten

$$
\begin{aligned}
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \vec{W}_\perp &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} W_x \\ W_y \end{pmatrix} \\
&= \left(-W_y \hat{i} + W_x \hat{j}\right) \\
&= \begin{pmatrix} i & j & k \\ 0 & 0 & 1 \\ W_x & W_y & W_z \end{pmatrix} \\
&= \hat{b} \times \vec{W}
\end{aligned}
$$

- So,

$$
\begin{aligned}
u_\perp{}' &= \frac{1}{1+K^2} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{W}_\perp - K \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot W_\perp \right) \\
u_\perp{}' &= \frac{1}{1+K^2} \left(\vec{W}_\perp + K \vec{W}_\perp \times \hat{b} \right)
\end{aligned}
$$

Conductivity

$$
\begin{aligned}
u_\perp{}' &= \frac{1}{1+K^2}\left(\vec{W}_\perp + K\vec{W}_\perp \times \hat{b}\right) \\
\vec{W}_\perp &= \frac{1}{m_s\nu_{sn}}\left(-\frac{\vec{\nabla}_\perp p_s}{n_s} + m_s\vec{G}_\perp + q_s\vec{E_\perp}{}'\right) \\
&= \frac{-k_B T}{m_s\nu_{sn}}\left(\frac{\vec{\nabla}_\perp T_s}{T_s} + \frac{\vec{\nabla}_\perp n_s}{n_s} - \frac{m_s\vec{G}_\perp}{k_B T_s} - \frac{q_s}{k_B T_s}\vec{E_\perp}{}'\right) \\
u_\perp{}' &= \frac{-k_B T_s}{m_s\nu_{sn}}\frac{1}{1+K^2}\left(\frac{\vec{\nabla}_\perp T_s}{T_s} + \frac{\vec{\nabla}_\perp n_s}{n_s} - \frac{m_s G_\perp}{k_B T_s} - \frac{q_s}{k_B T_s}E_\perp{}'\right) \\
&\quad + \frac{K}{1+K^2}\frac{1}{m_s\nu_{sn}}\left(-\frac{\vec{\nabla}_\perp p_s}{n_s} + m_s G_\perp + q_s E_\perp{}'\right) \times \hat{b}
\end{aligned}
$$

Conductivity

- Multiply and divide the last term by K

$$
\begin{aligned}
u_\perp{}' \quad = \quad & \frac{-kT}{m_s\nu_{sn}}\frac{1}{1+K^2}\left(\frac{1}{T}\vec{\nabla}_\perp T + \frac{1}{n_s}\vec{\nabla}_\perp n_s - \frac{m_s G_\perp}{kT} - \frac{q_s}{kT}E_\perp{}'\right) \\
& + \frac{K^2}{1+K^2}\frac{1}{K\; m_s\nu_{sn}}\left(-\frac{\vec{\nabla}_\perp p_s}{n_s} + m_s G_\perp + q_s E_\perp{}'\right) \times \hat{b}
\end{aligned}
$$

- Note:

$$
\frac{1}{K}\frac{1}{m_s\nu_{sn}} \quad = \quad \frac{\nu_{sn}}{\omega_c}\frac{1}{m_s\nu_{sn}} = \frac{m_s}{q_s B}\frac{1}{m_s} = \frac{1}{q_s\, B}
$$

Conductivity

Last term becomes
$$\frac{K^2}{1+K^2}\left(-\frac{\vec{\nabla}_\perp p_s}{q_s B n_s}+\frac{m_s G_\perp}{q_s B}+\frac{E_\perp{}'}{B}\right)\times\hat{b}$$
$$\frac{K^2}{1+K^2}\left(\vec{u}_{\vec{\nabla}p}+\vec{u}_G+\vec{u}_{E\times B}\right)$$

- Combining all terms

$$u_\perp{}' = \underbrace{\frac{-D}{1+K_s^2}\left(\frac{\vec{\nabla}_\perp T_s}{T_s}+\frac{\vec{\nabla}_\perp n_s}{n_s}-\frac{m_s G_\perp}{k_B T_s}-\frac{q_s}{k_B T_s}E_\perp{}'\right)}_{\text{dominates when } K\ll 1}$$
$$+\underbrace{\frac{K_s^2}{1+K_s^2}\left(\vec{u}_{\vec{\nabla}p}+\vec{u}_G+\vec{u}_{E\times B}\right)}_{\text{dominates when } K\gg 1} \quad \text{Schunk 5.103}$$

The conductivity tensor

- Back to general solution

$$\vec{u}_s{}' = \frac{1}{1+K_s^2}\begin{pmatrix} 1 & K_s & 0 \\ -K_s & 1 & 0 \\ 0 & 0 & 1+K_s^2 \end{pmatrix} \cdot \vec{W}_s$$

$$\vec{W}_s = \frac{1}{m_s \nu_{sn}}\left(-\frac{1}{n_s}\vec{\nabla} p_s + m_s \vec{G} + q_s \vec{E}\,'\right)$$

The conductivity tensor

- We want to define a relationship between $\vec{J}\,'$ and $\vec{E}\,'$
- In the simple case of $K_s \ll 1$

$$\begin{aligned} \vec{J}' &= \sum_s \sigma_s \vec{E}\,' \\ \sigma_s &= \frac{n_s q_s^2}{m_s \nu_{sn}} \end{aligned}$$

- Current will flow in the direction of the electric field

The conductivity tensor

- Now do the general case
- Since we are only interested in $\vec{E}\,'$,

$$
\begin{aligned}
\vec{W} &\sim \frac{q_s}{m_s \nu_{sn}} \vec{E}\,' \\
\vec{u}_s\,' &= \frac{1}{1+K_s^2} \begin{pmatrix} 1 & K_s & 0 \\ -K_s & 1 & 0 \\ 0 & 0 & 1+K_s^2 \end{pmatrix} \cdot \frac{q_s}{m_s \nu_{sn}} \vec{E}\,' \\
\vec{J}' &\equiv \sum_s n_s \vec{u}_s\,' q_s \\
&= \sum_s n_s \frac{1}{1+K_s^2} \begin{pmatrix} 1 & K_s & 0 \\ -K_s & 1 & 0 \\ 0 & 0 & 1+K_s^2 \end{pmatrix} \cdot \frac{q_s}{m_s \nu_{sn}} \vec{E}\,' q_s \\
&= \sum_s \frac{1}{1+K_s^2} \begin{pmatrix} 1 & K_s & 0 \\ -K_s & 1 & 0 \\ 0 & 0 & 1+K_s^2 \end{pmatrix} \cdot \frac{n_s q_s^2}{m_s \nu_{sn}} \vec{E}\,'
\end{aligned}
$$

The conductivity tensor

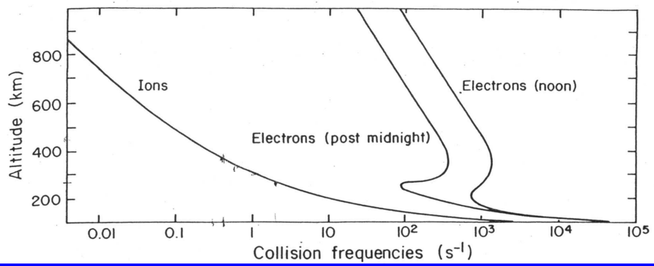
$$\vec{j}' = \sum_s \begin{pmatrix} \frac{1}{1+K_s^2} & \frac{K_s}{1+K_s^2} & 0 \\ \frac{-K_s}{1+K_s^2} & \frac{1}{1+K_s^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \sigma_s \vec{E}\,'$$

Parallel or specific conductivity

- Look at the conductivity tensor component parallel to the magnetic field

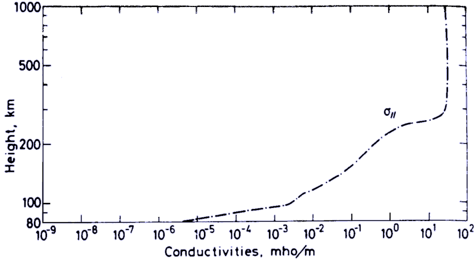
$$\begin{aligned}\sigma_0 &= \sum_s \sigma_s = \sum_s \frac{n_s q_s^2}{m_s \nu_{sn}} \\ &= \frac{n_e q_e^2}{m_e \nu_{en}} + \frac{n_i q_i^2}{m_i \nu_{in}} \simeq \frac{n_e q_e^2}{m_e \nu_{en}}\end{aligned}$$

- Dominated by electron motion since it goes as $1/m_s$



Parallel or specific conductivity

- What happens with altitude?



$$\sigma_0 \simeq \frac{n_e q_e^2}{m_e \nu_{en}}$$

- At low altitude, n is small, ν_{en} is large, $\sigma_0 \to 0$
- Going higher, $n \uparrow$, $\nu_{en} \downarrow$, σ_0 grows rapidly
- What happens at high altitude? $n \downarrow$, $\nu_{en} \downarrow$, σ_0 ?

Perpendicular components

- The perpendicular component of the conductivity tensor is a little bit more complicated
- Use a right hand coordinate system with the magnetic field in the $\hat{k}$ and the electric field in the $\hat{i}$ direction
- Go back to the equation we solved earlier

$$\vec{u}_s{}' - K_s \left(\vec{u}_s{}' \times \hat{b}\right) = \frac{1}{m_s \nu_{sn}} q_s \vec{E}\,'$$

and solve for the individual components

Perpendicular components

$$\vec{u}_s{}' - K_s\left(\vec{u}_s{}' \times \hat{b}\right) = \frac{1}{m_s \nu_{sn}} q_s \vec{E}\,'$$

$$\left(u_i\hat{i} + u_j\hat{j}\right) - K_s\left(\left(u_i\hat{i} + u_j\hat{j}\right) \times \hat{k}\right) = \frac{qE_i'}{m\nu_{sn}}\hat{i}$$

$$u_i\hat{i} \times \hat{k} = -u_i\hat{j}$$

$$u_j\hat{j} \times \hat{k} = u_j\hat{i}$$

$$\left(u_i\hat{i} + u_j\hat{j}\right) - K_s\left(u_j\hat{i} - u_i\hat{j}\right) = \frac{qE_i'}{m\nu_{sn}}\hat{i}$$

Perpendicular components

- Separate into components

$$\left(u_i\hat{i} + u_j\hat{j}\right) - K_s\left(u_j\hat{i} - u_i\hat{j}\right) = \frac{qE_i'}{m\nu_{sn}}\hat{i}$$

$$u_i\hat{i} - K_s\,u_j\hat{i} = \frac{qE_i'}{m\nu_{sn}}\hat{i}$$

$$u_j\hat{j} + K_s\,u_i\hat{j} = 0$$

- Solve

$$u_j = -K_s\,u_i$$

$$u_i + K_s^2 u_i = \frac{qE_i'}{m\nu_{sn}}$$

$$u_i = \frac{1}{1+K_s^2}\frac{qE_i'}{m\nu_{sn}}$$

$$u_j = \frac{-K_s}{1+K_s^2}\frac{qE_i'}{m\nu_{sn}}$$

Pedersen conductivity

- Now look at the conductivity in the $\hat{i}$ direction

$$
\begin{aligned}
J &= \sum_r n_r q_r u_r \\
&= \sum_r n_r q_r \frac{1}{1+K_r^2} \frac{q_r E_i'}{m_r \nu_{rn}} \\
&= \sum_r \sigma_r E_i'
\end{aligned}
$$

Pedersen conductivity

$$\sum_r \sigma_r = \sum_r \frac{1}{1+K_r^2} \frac{q_r^2\, n_r}{m_r \nu_{rn}}$$

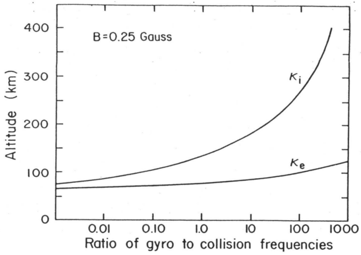
$$\sigma_p = \sum_r \sigma_r = \sum_r \frac{1}{1+K_r^2} \frac{q_r^2\, n_r}{m_r \nu_{rn}}$$

$$\sigma_p \equiv \sum_s \frac{1}{1+K_s^2} \frac{q_s^2\, n_s}{m_s \nu_{sn}}$$

$$\sigma_p \equiv \sum_s \frac{\sigma_s}{1+K_s^2}$$

- This is the Pedersen conductivity
- $\perp$ to $\vec{B}$, $\parallel$ to $\vec{E}$

Pedersen conductivity



- Pedersen conductivity

$$\sigma_P \equiv \sum_s \frac{\sigma_s}{1 + K_s^2}$$

- Typically $K_e \gg K_i$

$$\sigma_P \simeq \sum_i \frac{\sigma_i}{1 + K_i^2}$$

Hall conductivity

- Now look at the conductivity in the $\hat{j}$ direction

$$
\begin{aligned}
J &= \sum_r n_r q_r u_r \\
&= \sum_r n_r q_r \frac{-K_r}{1+K_r^2} \frac{q_r E_i'}{m_r \nu_{rn}} \\
&= \sum_r \sigma_r E' \\
\sum_r \sigma_r &= \sum_r \frac{-K_r}{1+K_r^2} \frac{q_r^2 n_r}{m_r \nu_{rn}} \\
\sigma_H = \sum_r \sigma_r &= \sum_s \frac{-K_s}{1+K_s^2} \frac{q_s^2 n_s}{m_s \nu_{sn}} \\
\sigma_H &\equiv \sum_s \frac{-K_s}{1+K_s^2} \sigma_s
\end{aligned}
$$

- This is the Hall conductivity; Perpendicular to $\vec{E}$ and to $\vec{B}$
- What's with the negative sign?

Hall conductivity

$$
\begin{aligned}
\sigma_H &= -\sum_s \sigma_s \frac{K_s}{1+K_s^2} \\
&= -\sum_s \frac{\sigma_s}{K_s} \frac{K_s^2}{1+K_s^2} \\
&= -\sum_s \frac{n_s q_s^2}{m_s \nu_{sn}} \frac{m_s \nu_{sn}}{q_s B} \frac{K_s^2}{1+K_s^2} \\
&= -\sum_s \frac{n_s q_s}{B} \frac{K_s^2}{1+K_s^2} \\
&= \frac{n_e\, e}{B} \frac{K_e^2}{1+K_e^2} - \sum_i \frac{n_i\, e}{B} \frac{K_i^2}{1+K_i^2}
\end{aligned}
$$

- Since typically $K_e \gg K_i$, σ_H is positive

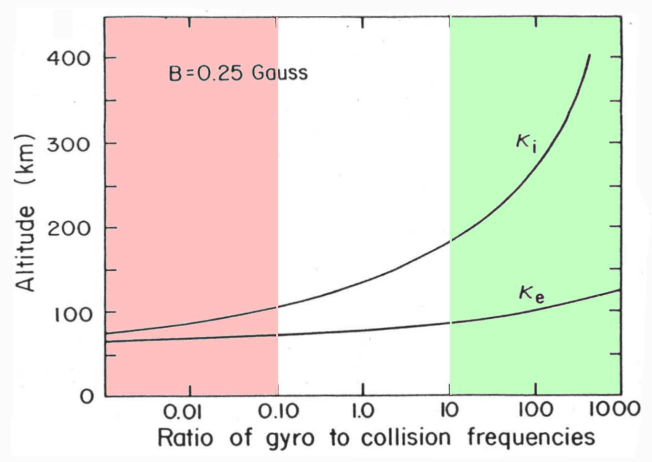
The conductivity tensor

$$\vec{J} = \sum_s \begin{pmatrix} \frac{1}{1+K^2} & \frac{K}{1+K^2} & 0 \\ \frac{-K}{1+K^2} & \frac{1}{1+K^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \sigma_s \vec{E}\,'$$

$$\vec{J} = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix} \cdot \vec{E}\,'$$

Currents

- Now revisit the three cases we looked at earlier
  - Unmagnetized ($K \ll 1$)
  - Magnetized ($K \gg 1$)
  - Intermediate case ($K \sim 1$)



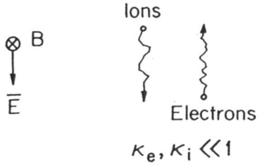
Unmagnetized case: $K \ll 1$

$$\sigma_0 = \frac{n\,e^2}{m_e \nu_{en}}$$

$$\sigma_P = \sum_s \frac{\sigma_s}{1+K_s^2} \to \sum \sigma_s = \sigma_0$$

$$\sigma_H = -\sum_s \sigma_s \frac{K_s}{1+K_s^2} \to 0$$

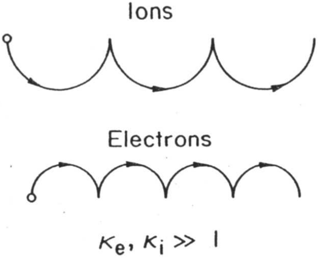
$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix}$$



- In an unmagnetized plasma, we can drive a current equally in all directions that there is an $\vec{E}$

Magnetized case: $K \gg 1$

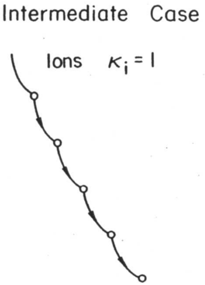
$$
\begin{aligned}
\sigma_0 &= \frac{n\, e^2}{m_e \nu_{en}} \\
\sigma_P &= \sum_s \frac{\sigma_s}{1+K_s^2} \to 0 \\
\sigma_H &= -\sum_s \sigma_s \frac{K_s}{1+K_s^2} \\
&= \frac{n_e\, e}{B} \frac{K_e^2}{1+K_e^2} - \sum_i \frac{n_i\, e}{B} \frac{K_i^2}{1+K_i^2} \\
&= \frac{n_e\, e}{B}(1-1) = 0 \\
\underline{\underline{\sigma}} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix}
\end{aligned}
$$



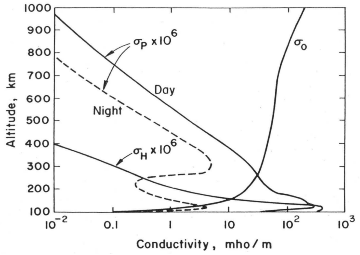
- In an magnetized plasma, $\vec{E} \times \vec{B}$ drift dominates and we can only drive a current parallel to $\vec{B}$

In between case: $K \sim 1$

$$
\begin{aligned}
\sigma_0 &= \frac{n\,e^2}{m_e \nu_{en}} \\
\sigma_P &= \sum_i \frac{\sigma_i}{1+K_i^2} \\
\sigma_H &= \frac{n_e\,e}{B}\frac{K_e^2}{1+K_e^2} - \sum_i \frac{n_i\,e}{B}\frac{K_i^2}{1+K_i^2}
\end{aligned}
$$



In between case: $K \sim 1$



- Since $K_e > K_i$ there exists a region where σ_H is non-negligible
- Roughly $z = 115km$

The conductivity tensor

- How do we measure the neutral wind and $\vec{E}$?
- Usually from an Earth fixed reference frame
- Finally, transform back to Earth fixed coordinate system

$$
\begin{aligned}
\vec{E}\,' &= \vec{E} + \vec{u}_n \times \vec{B} \\
\vec{J}\,' &= \underline{\underline{\sigma}} \cdot \vec{E}\,' \\
\vec{J} &= \underline{\underline{\sigma}} \cdot (\vec{E} + \vec{u}_n \times \vec{B})
\end{aligned}
$$

- Don't need an electric field to drive a current, just a neutral wind

Vector tensor dot product

$$
\begin{aligned}
\vec{a} &= a_1\,\hat{e}_1 + a_2\,\hat{e}_2 + a_3\,\hat{e}_3 \\
\vec{b} &= b_1\,\hat{e}_1 + b_2\,\hat{e}_2 + b_3\,\hat{e}_3 \\
\vec{a}\cdot\vec{b} &= a_1\,b_1 + a_2\,b_2 + a_3\,b_3 \\
\vec{b}\cdot\vec{a} &= b_1\,a_1 + b_2\,a_2 + b_3\,a_3 \\
\vec{a}\cdot\vec{b} &= \vec{b}\cdot\vec{a}
\end{aligned}
$$

Vector tensor dot product

$$
\begin{aligned}
\vec{a} &= a_1\,\hat{e}_1 + a_2\,\hat{e}_2 + a_3\,\hat{e}_3 \\
&= a_i \cdot e_i \\
&= a_i e_i \delta ii \\
&= a_i e_i
\end{aligned}
$$

Vector tensor dot product

$$
\begin{aligned}
\vec{a} &= a_i e_i \\
\vec{b} &= b_j e_j \\
\vec{a} \cdot \vec{b} &= a_i e_i \cdot b_j e_j \\
&= a_i b_j \, e_i \cdot e_j \\
&= a_i b_j \, \delta ij \\
&= a_i b_i \\
&= a_1 \, b_1 + a_2 \, b_2 + a_3 \, b_3 \\
\vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{a}
\end{aligned}
$$

Vector tensor dot product

$$\begin{aligned}
\underline{\underline{W}} &= \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix} \\
&= \quad W_{11}\hat{e}_1\hat{e}_1 + W_{12}\hat{e}_1\hat{e}_2 + W_{13}\hat{e}_1\hat{e}_3 \\
&\quad + W_{21}\hat{e}_2\hat{e}_1 + W_{22}\hat{e}_2\hat{e}_2 + W_{23}\hat{e}_2\hat{e}_3 \\
&\quad + W_{31}\hat{e}_3\hat{e}_1 + W_{32}\hat{e}_3\hat{e}_2 + W_{33}\hat{e}_3\hat{e}_3 \\
&= W_{jk}\, e_j\, e_k
\end{aligned}$$

Vector tensor dot product

$$
\begin{aligned}
\underline{\underline{W}} \cdot \vec{a} &= W_{jk}\, e_j\, e_k \cdot a_i\, e_i \\
&= W_{jk}\, a_i\, e_j\, e_k \cdot e_i \\
&= W_{jk}\, a_i\, e_j\, \delta_{ki} \\
&= W_{jk}\, a_k\, e_j
\end{aligned}
$$

Vector tensor dot product

$$
\begin{aligned}
\underline{\underline{W}} \cdot \vec{a} &= \quad W_{11}\, a_1 \hat{e}_1 + W_{12}\, a_2 \hat{e}_1 + W_{13}\, a_3 \hat{e}_1 \\
&\quad + W_{21}\, a_1 \hat{e}_2 + W_{22}\, a_2 \hat{e}_2 + W_{23}\, a_3 \hat{e}_2 \\
&\quad + W_{31}\, a_1 \hat{e}_3 + W_{32}\, a_2 \hat{e}_3 + W_{33}\, a_3 \hat{e}_3 \\
&= \quad \left(W_{31}\, a_1 + W_{12}\, a_2 + W_{13}\, a_3 \right) \hat{e}_1 \\
&\quad + \left(W_{21}\, a_1 + W_{22}\, a_2 + W_{23}\, a_3 \right) \hat{e}_2 \\
&\quad + \left(W_{31}\, a_1 + W_{32}\, a_2 + W_{33}\, a_3 \right) \hat{e}_3
\end{aligned}
$$