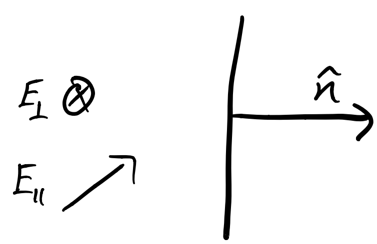
Analyzing the tangential components of $\bar{E}$ at a boundary (we could get the same results from the normal components and Snell's law)

$$\underbrace{\left(\bar{E}_0 + \bar{E}_1\right) \times \hat{n}}_{E_{\text{tangetial-left}}} = \underbrace{\hat{E}_2 \times \hat{n}}_{E_{\text{tangential-right}}}$$

$$\left(\bar{H}_0 + \bar{H}_1\right) \times \hat{n} = \bar{H}_2 \times \hat{n}$$

We can break down any plane wave as the superposition of two waves:

1) $\bar{E}_{\parallel}$ polarized parallel to the plane of incidence
2) $\bar{E}_{\perp}$ polarized $\perp$ to the $\underbrace{\text{plane of incidence}}_{\text{plane of the page}}$



Analyzing the two cases separately (because they give different results)

$\bar{E} \perp$ to plane of incidence (page)

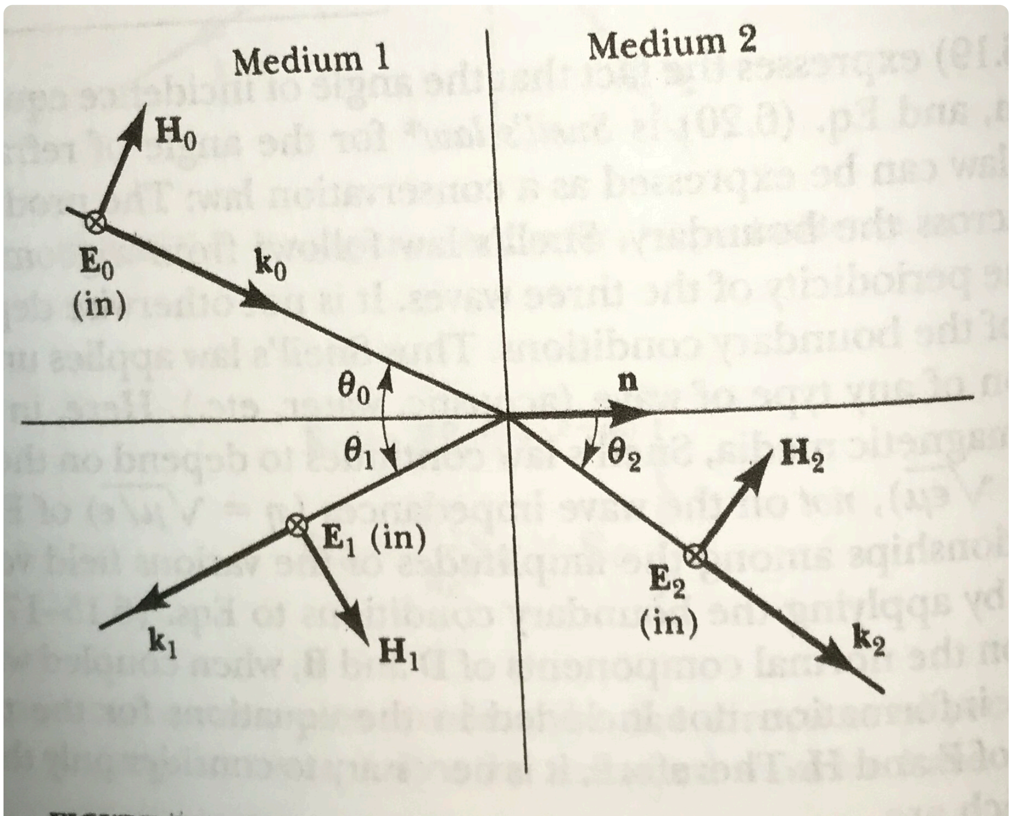


FIGURE 6-3. Electric field perpendicular to plane of incidence.

[H+M]

Because $\bar{E}$ is parallel to the boundary surface, the tangential BC is

$$E_0 e^{i(\bar{k}_0 \cdot \bar{x} - \omega t)} + E_1 e^{i(\bar{k}_1 \cdot \bar{x} - \omega t)} = E_2 e^{i(\bar{k}_2 \cdot \bar{x} - \omega t)}$$

As we found before, for this to be true at all positions at the boundary for all times, we must have

$$\bar{k}_0 \cdot \bar{x} - \omega t = \bar{k}_1 \cdot \bar{x} - \omega t = \bar{k}_2 \cdot \bar{x} - \omega t$$

$$\Rightarrow E_0 + E_1 = E_2$$

Additionally, from expanding

$$(\bar{H}_0 + \bar{H}_1) \times \hat{n} = \bar{H}_2 \times \hat{n}$$

in terms of $\bar{k}$, we get the following (see Heald + Marion for details)

$$(E_0^0 - E_1^0) \cos\theta_0 = \frac{n_2}{n_1} E_2^0 \cos\theta_2$$

Solving for E_1^0 and E_2^0 in terms of E_0^0

$$E_1^0 = \frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)} E_0^0 \qquad E_2^0 = \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_2 + \theta_0)} E_0^0$$

$E \parallel$ to plane of incidence (page)

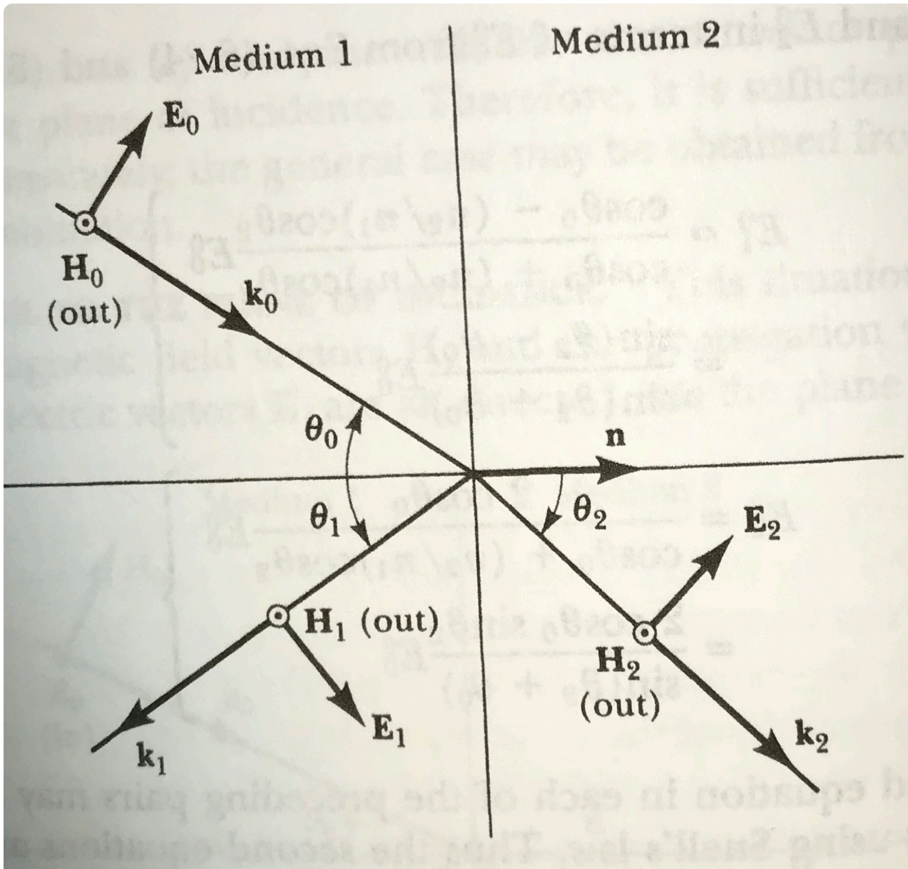
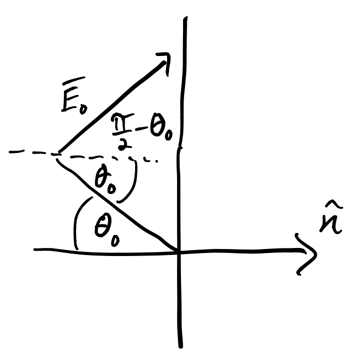


FIGURE 6-4. Electric field parallel to plane of incidence.

[H+M]

Our BC on $\bar{E}$ gives



points ↓

$$\Rightarrow \sin\left(\tfrac{\pi}{2}-\theta_0\right)E_0 - \sin\left(\tfrac{\pi}{2}-\theta_1\right)E_1 = \sin\left(\tfrac{\pi}{2}-\theta_2\right)E_2$$

But $\sin\left(\frac{\pi}{2}-\theta_0\right) = \cos(\theta_0)$

$$\Rightarrow \quad \cos(\theta_0)E_0 - \cos(\theta_1)E_1 = \cos(\theta_2)E_2$$

Because $\bar{H}$ is tangential to the surface and

$$\bar{H} = n\,\hat{e}_k \times \bar{E} \quad \Rightarrow \quad |\bar{H}| = n|\bar{E}|$$

$$\Rightarrow \quad n_1 E_0^0 + n_1 E_1^0 = n_2 E_2^0$$

Combining, we obtain

$$E_1^0 = \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)} E_0^0$$

$$E_2^0 = \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_0+\theta_2)\cos(\theta_0-\theta_2)} E_0^0$$

The equations for $E_\perp$ and $E_\parallel$ are known as the <u>Fresnel equations</u>

Brewster's Angle

For the case of $E \parallel$ to the plane of incidence, the reflected E is

$$E_1^0 = \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)} = \frac{\sin(\theta_0 - \theta_2)}{\cos(\theta_0 - \theta_2)} \frac{\cos(\theta_0 + \theta_2)}{\sin(\theta_0 + \theta_2)}$$

$= 0$ if $\theta_0 = \theta_2$ } trivial $(n_1 = n_2)$

or

$\theta_0 + \theta_2 = \frac{\pi}{2}$ } $\theta_0 = \theta_B =$ Brewster's Angle

$\Rightarrow$ When the reflected and refracted rays are $\perp$ $\left(\theta_0 + \theta_2 = \theta_1 + \theta_2 = \frac{\pi}{2}\right)$ no energy is carried by the reflected ray polarized $\parallel$ to the plane of incidence

$\Rightarrow$ The reflected wave is polarized $\perp$ to the plane of incidence

From Snell's law:

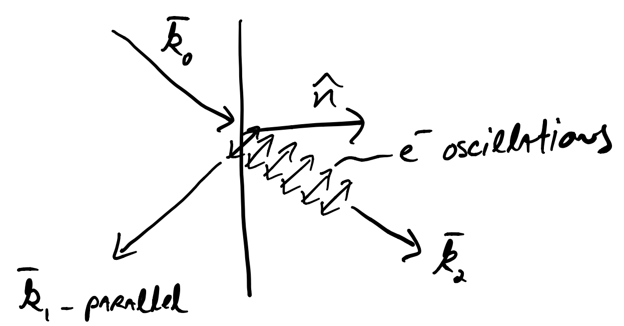
Brewster's Angle

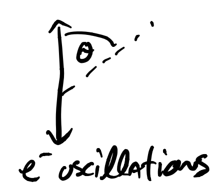
$$\frac{n_2}{n_1} = \frac{\sin \theta_B}{\sin[\frac{\pi}{2} - \theta_B]} = \tan \theta_B$$

For light incident upon glass ($n_2 = 1.5$) from air ($n_1 = 1$), $\theta_B \approx 56°$

Why does this happen?

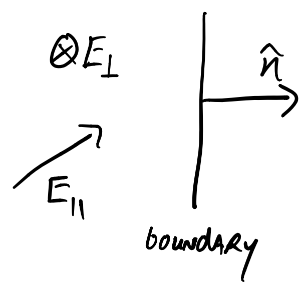
- At the plane surface, e^- are linearly accelerated by $\bar{E}$
- Linear oscillations of e^- radiate energy with a $\sin^2\theta$ dependence (linear antenna)
- If the reflected ray is $\perp$ to oscillations from the transmitted wave, $\sin^2\theta = \sin^2\pi = 0$





Reflection Coefficients

For the cases of $\bar{E}$ perpendicular ($\perp$) and $\bar{E}$ parallel ($\parallel$) to the plane of incidence



The reflection and transmission coefficients are:

$$R_\perp = \frac{\langle \bar{S}_1 \rangle_\perp \cdot (-\hat{n})}{\langle \bar{S}_0 \rangle_\perp \cdot \hat{n}} = \frac{|E_1^0|^2}{|E_0^0|^2} = \frac{\sin^2(\theta_2 - \theta_0)}{\sin^2(\theta_2 + \theta_0)}$$

$\hookrightarrow \cos\theta_0 = \cos\theta_1$

$$T_\perp = \frac{\langle \bar{S}_2 \rangle_\perp \cdot \hat{n}}{\langle \bar{S}_0 \rangle_\perp \cdot \hat{n}} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_0} \frac{|E_2^0|^2}{|E_0^0|^2} = \frac{\sin 2\theta_0 \sin 2\theta_2}{\sin^2(\theta_2 + \theta_0)}$$

The parallel coefficients are

$$R_{\parallel} = \frac{\tan^2(\theta_2 - \theta_0)}{\tan^2(\theta_2 + \theta_0)}$$

$$T_{\parallel} = \frac{\sin 2\theta_0 \sin 2\theta_2}{\sin^2(\theta_0 + \theta_2)\cos^2(\theta_0 - \theta_2)}$$

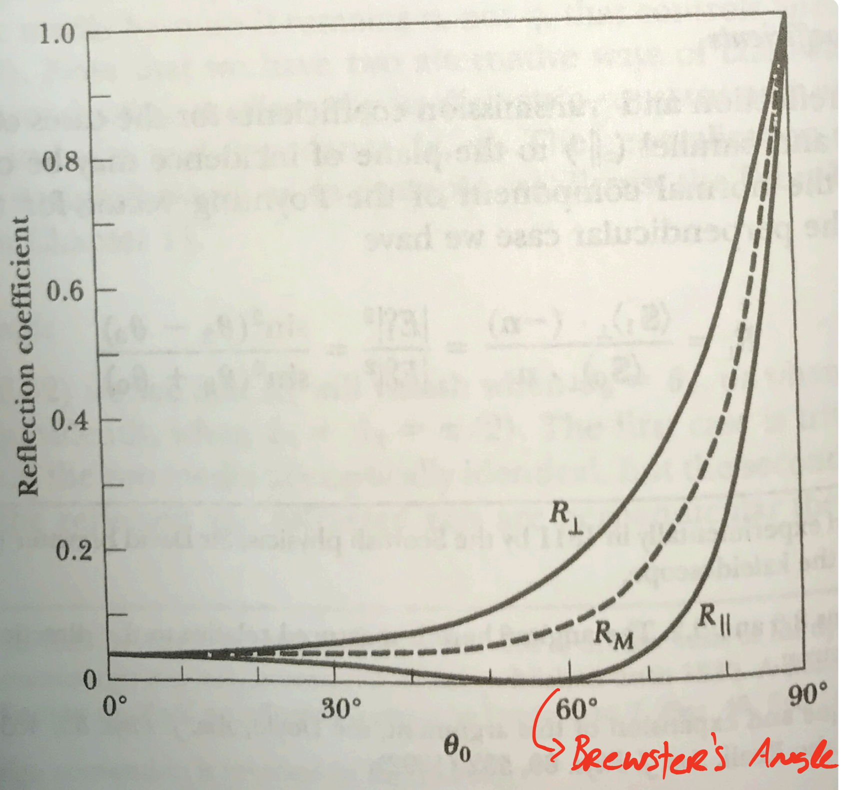


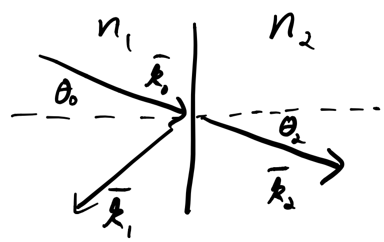
FIGURE 6-5. Reflection coefficients for air-glass interface.

[H+M]

The reflected light is mostly polarized $\perp$ to the plane of incidence

Total Internal Reflection

For the case of $n_1 > n_2$



From Snell's Law

$$n_1 \sin\theta_0 = n_2 \sin\theta_2$$

$$\Rightarrow \underbrace{\frac{n_1}{n_2}}_{>1} \underbrace{\sin\theta_0}_{<1} = \sin\theta_2$$

which will produce $\theta_2 = \frac{\pi}{2}$ at

$$\sin\theta_c = \frac{n_2}{n_1}$$

$\hookrightarrow$ critical angle

The refraction angle can be written as

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - \frac{\sin^2\theta_0}{\sin^2\theta_c}}$$

As θ_0 is increased beyond θ_c,

$$1 - \frac{\sin^2\theta_0}{\sin^2\theta_c} < 1 \implies \cos\theta_2 = iQ$$

$\hookrightarrow$ imaginary

where

$$Q = \sqrt{\frac{\sin^2\theta_0}{\sin^2\theta_c} - 1} \qquad \text{for} \quad \theta_0 > \theta_c$$

From this, we find
(see Heald + Marion for details)

$$|(E_1^0)_\perp| = |(E_0^0)_\perp|$$

$$|(E_1^0)_\parallel| = |(E_0^0)_\parallel|$$

Reflected = incident

$\Rightarrow$ total internal reflection