Multipole Fields

Charge distributions can be broken-down into multipole moments, allowing for a simplified description of a complex field. The <u>dipole</u> moment for the electric and magnetic fields are used as an approximation for more complex fields because the properties are well known.

Electric Dipole

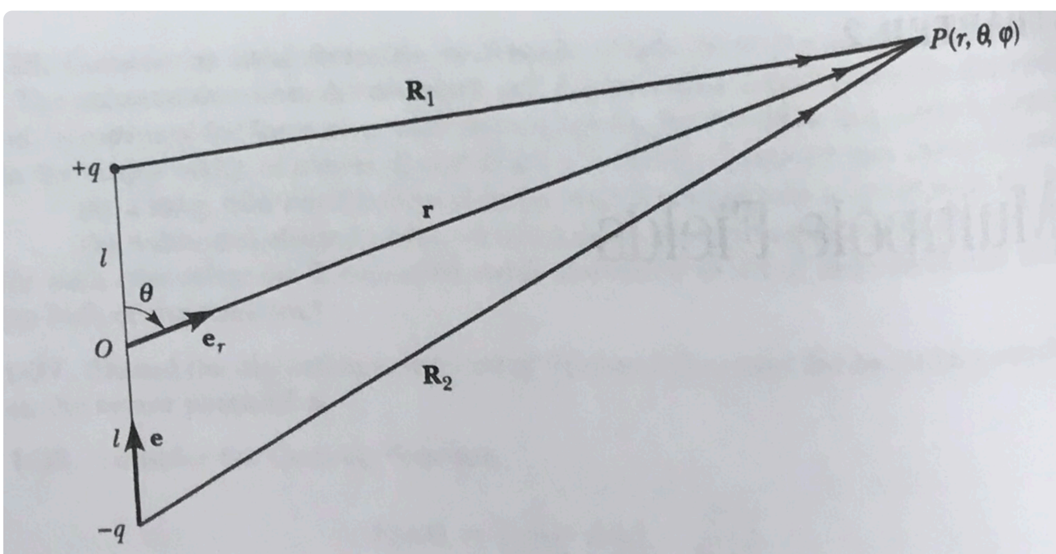


FIGURE 2-1. An elementary charge-pair dipole.

[H+M]

- A positive and negative charge separated by distance $2l$

- The dipole moment $\bar{p}$ points from the negative to the positive charge

$$\bar{p} \equiv 2ql\hat{e}$$

From the diagram, the potential at point $P(r,\theta,\phi)$ is given by

$$\Phi(r,\theta,\phi) = q\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

For $r \gg l$, a Taylor series expansion of $\frac{1}{R_1}$ and $\frac{1}{R_2}$ in terms of $\frac{l}{r}$ provides:

$$\Phi(r,\theta) = 2ql\frac{\cos\theta}{r^2}$$

- azimuthally symmetric $\Rightarrow$ not a function of ϕ

- decreases as $\frac{1}{r^2}$

  ↳ single charge goes as $\frac{1}{r}$

  ↳ as the observation point moves further away, it appears like there is no charge ($q_{total} = q - q = 0$)

For $\hat{e}_r$ as the unit vector to point P, the potential in terms of the dipole moment $\bar{p}$ is

$$\Phi = \frac{\bar{p} \cdot \hat{e}_r}{r^2} = 2ql\,\frac{\cos\theta}{r^2}$$

The electric field is

$$\bar{E} = -\nabla\Phi = -\left[\hat{e}_r \frac{\partial \Phi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\right]$$

$$\Rightarrow \left.\begin{aligned} E_r &= 2p\,\frac{\cos\theta}{r^3} \\ E_\theta &= p\,\frac{\sin\theta}{r^3} \end{aligned}\right\} \frac{1}{r^3} \text{ dependence}$$

$$E_\phi = 0 \quad \Big\} \text{ no azimuth dependence}$$

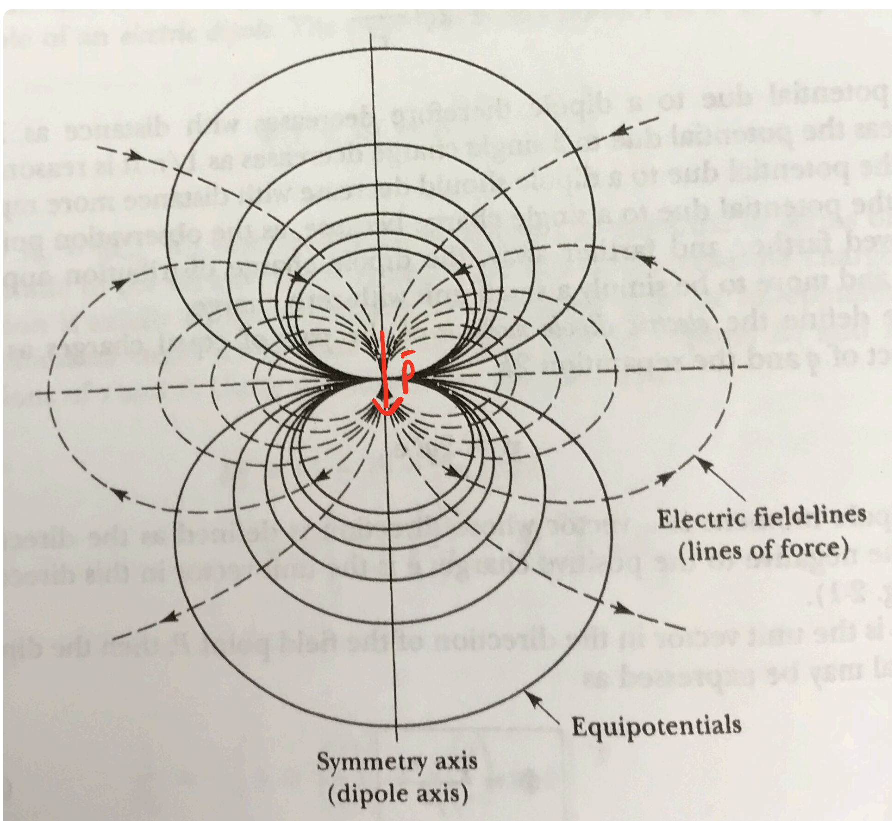


FIGURE 2-2. Dipole equipotentials and field-lines.

[H+M]

Multipole Expansion

For an arbitrary collection of charges q_α located around the origin

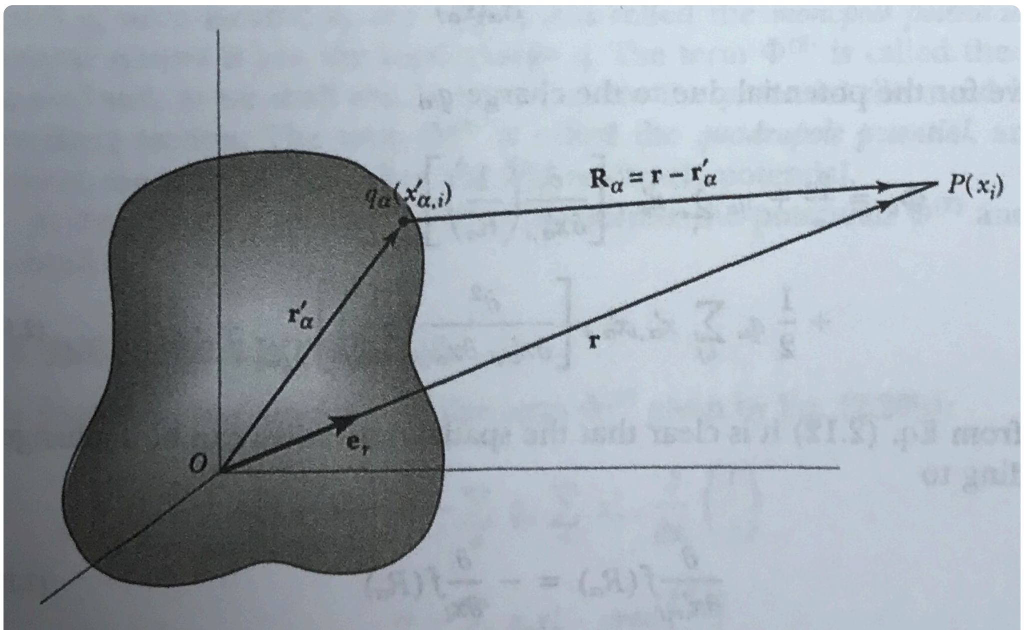


FIGURE 2-3. Source and field coordinates.

[H+M]

$\bar{r}'_\alpha$ = location of q_α

$\bar{r}$ = vector to point P from the origin

$\bar{R}_\alpha = \bar{r} - \bar{r}'_\alpha$ = vector from q_α to point P

The potential due to charge q_α is

$$\Phi_\alpha = \frac{q_\alpha}{R_\alpha}$$

$$R_\alpha = |\bar{r} - \bar{r}_\alpha'| = \sqrt{(x_1 - x_{\alpha 1}')^2 + (x_2 - x_{\alpha 2}')^2 + (x_3 - x_{\alpha 3}')^2}$$

$$= \sqrt{\sum_i (x_i - x_{\alpha i}')^2}$$

As a reminder, a 1-D Taylor (MacLaurin) series expansion about $x=0$ is

$$f(x) = f(0) + x f'(0) + \frac{1}{2!} x^2 f''(0) + \ldots$$

In 2-D, this becomes

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0)$$
$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$
$$+ \ldots$$

Performing a Taylor (MacLaurin) series expansion around $\bar{r}'_\alpha = 0$

$$\Phi_\alpha = \frac{q_\alpha}{r} + q_\alpha \sum_i x'_{\alpha i} \left[\frac{\partial}{\partial x'_{\alpha i}} \left(\frac{1}{R_\alpha} \right) \right]_{R_\alpha = r}$$

$$+ \frac{1}{2} q_\alpha \sum_{ij} x'_{\alpha i} x'_{\alpha j} \left[\frac{\partial^2}{\partial x'_{\alpha i} \partial x'_{\alpha j}} \left(\frac{1}{R_\alpha} \right) \right]_{R_\alpha = r}$$

$$+ \dots$$

From $R_\alpha = \sqrt{\sum_i (x_i - x'_{\alpha i})^2}$

$$\Rightarrow \left[\frac{\partial}{\partial x'_{\alpha i}} \left(\frac{1}{R_\alpha} \right) \right]_{R_\alpha = r} = - \left[\frac{\partial}{\partial x_i} \left(\frac{1}{R_\alpha} \right) \right]_{R_\alpha = r}$$

$$= - \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right)$$

$$\Rightarrow \Phi_\alpha = \frac{q_\alpha}{r} - q_\alpha \sum_i x'_{\alpha i} \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) + \frac{1}{2} q_\alpha \sum_{ij} x'_{\alpha i} x'_{\alpha j} \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) - \dots$$

Combining the collection of charges q_α:

$$\Phi = \sum_\alpha \Phi_\alpha = \Phi^{(1)} + \Phi^{(2)} + \Phi^{(4)} + \dots + \Phi^{(2^\ell)} + \dots$$

where

$$\Phi^{(1)} \equiv \sum_\alpha \frac{q_\alpha}{r} = \frac{q_{total}}{r} = \text{monopole potential}$$

$$\Phi^{(2)} \equiv -\sum_\alpha q_\alpha \sum_i x'_{\alpha i} \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = \text{dipole potential}$$

$$\Phi^{(4)} \equiv \frac{1}{2} \sum_\alpha q_\alpha \sum_{ij} x'_{\alpha i} x'_{\alpha j} \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \text{quadrupole potential}$$

Dipole Potential

$$\Phi^{(2)} = -\sum_\alpha q_\alpha \sum_i x'_{\alpha i} \frac{d}{dx_i}\left(\frac{1}{r}\right)$$

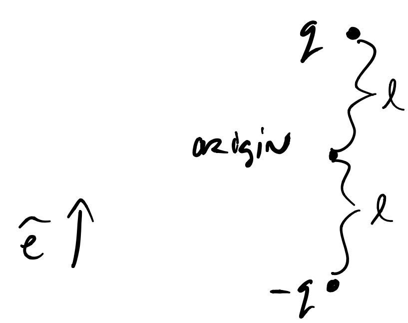
$$= -\sum_\alpha q_\alpha \bar{r}'_\alpha \cdot \bar{\nabla}\left(\frac{1}{r}\right)$$

with respect to x_i

Following our simple dipole for two charges, $\bar{p} = 2q\ell\hat{e}$, we can generalize this to an arbitrary collection of charges

$$\boxed{\bar{p} = \sum_\alpha q_\alpha \bar{r}'_\alpha}$$

Example



$$\bar{p} = q\ell\hat{e} + (-q)(-\ell)\hat{e}$$

$$= 2q\ell\hat{e}$$

The potential becomes

$$\bar{\Phi}^{(2)} = -\bar{p} \cdot \bar{\nabla}\left(\frac{1}{r}\right) = -\bar{p} \cdot \frac{-\hat{e}_r}{r^2}$$

$$\Rightarrow \boxed{\bar{\Phi}^{(2)} = \frac{\bar{p} \cdot \hat{e}_r}{r^2}}$$

The electric field for the dipole is

$$E^{(2)} = -\bar{\nabla}\bar{\Phi}^{(2)} = -\bar{\nabla}\left(\frac{\bar{p} \cdot \hat{e}_r}{r^2}\right) = -\bar{\nabla}\left(\frac{\bar{p} \cdot \bar{r}}{r^3}\right)$$

Work through in-class

$$\Rightarrow E^{(2)} = \frac{1}{r^5}\left[3(\bar{p}\cdot\bar{r})\bar{r} - \bar{p}r^2\right]$$

It's tough to visualize this, but in your homework, you'll derive:

$$E^{(2)}(r,\theta) = \frac{p}{r^3}\left(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta\right)$$

for $\bar{p}$ in the $\hat{e}_3$ direction.

We can extend the multipoles to an arbitrary 2^ℓth-moment but we'll stop at dipoles. Be aware of the other multipoles.

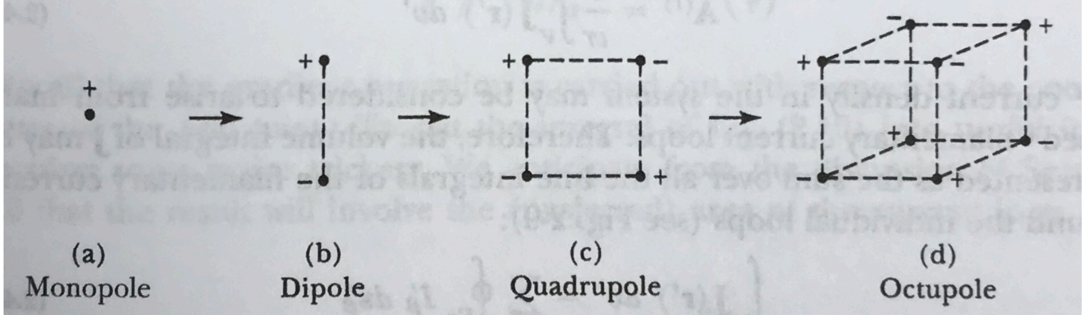


FIGURE 2-8. A sequence of multipoles.

[H+M]

Magnetic Multipoles

Following the same outline as for the electric multipoles, except now we focus on the vector potential $\bar{A}$

$$\bar{A}(\bar{r}) = \frac{1}{c} \int_V \frac{\bar{J}(\bar{r}')}{R} \, dv'$$

Taylor series expansion of $\frac{1}{R}$:

$$\bar{A}(\bar{r}) = \underbrace{\frac{1}{cr} \int_V \bar{J}(\bar{r}') \, dv'}_{\text{monopole}} - \underbrace{\frac{1}{c} \int_V \bar{J}(\bar{r}') \left[\bar{r}' \cdot \bar{\nabla}\left(\frac{1}{r}\right) \right] dv'}_{\text{dipole}} + \ldots$$

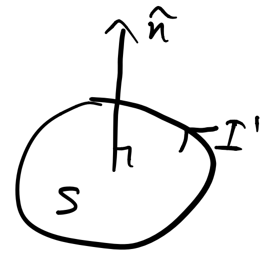
$$= A^{(1)} \quad + \quad A^{(2)} \quad + \ldots$$

The monopole term is zero
(no magnetic monopoles)

$$A^{(1)} = 0$$

With some tricky math due to the integrals around current loops, and the definition of a magnetic dipole moment $\bar{m}$

$$\bar{m} = \frac{I'\bar{S}}{c} = \frac{I' S \hat{n}}{c}$$



we find

$$A^{(2)} = \frac{\bar{m}_{total} \times \hat{e}_r}{r^2}$$

$$\bar{m}_{total} = \sum_{B} \bar{m}_B \quad \Big\} \text{ sum over all current loops}$$

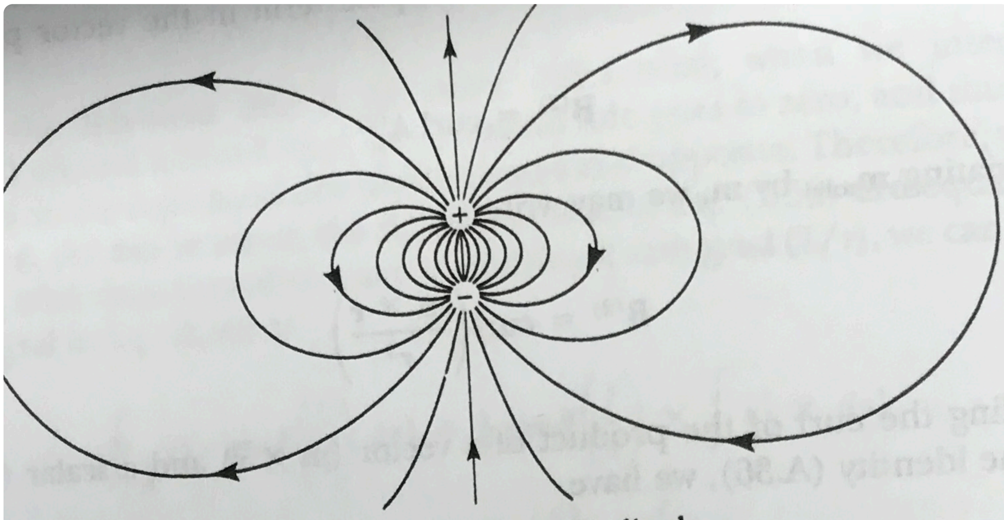
The magnetic field from the dipole is

$$\bar{B}^{(2)} = \bar{\nabla} \times \bar{A}^{(2)} = \underbrace{\frac{1}{r^5}\left(3[\bar{m}\cdot\bar{r}]\,\bar{r} - \bar{m}\,r^2\right)}_{\text{SAME AS } \bar{E} \text{ due to electric dipole!}}$$

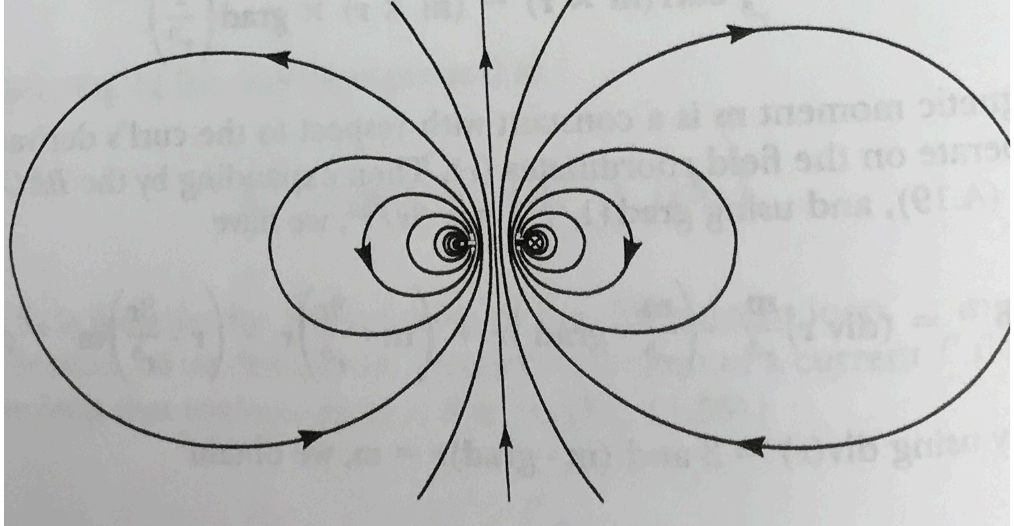
For $\bar{m} = m\hat{e}_3$

$$\bar{B}^{(2)}(r,\theta) = \frac{m}{r^3}\left(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta\right)$$

$\Rightarrow$ $B^{(2)}$ and $E^{(2)}$ are the same <u>outside</u> the dipole, but much different <u>inside</u> the dipole



(a) Electric (charge-pair) dipole



(b) Magnetic (current-loop) dipole

[H+M]

Supplemental (Following Jackson)

The electric potential can be expanded in terms of spherical harmonics, $Y_{\ell m}(\theta, \phi)$, which provides:

$$\Phi(\bar{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}}$$

$$q_{\ell m} = \int Y^{*}_{\ell m}(\theta', \phi') \, r'^{\ell} \rho(\bar{r}') \, dv'$$

multipole moments

This expansion is very useful if $\rho(\bar{r})$ is easily described by spherical harmonics because of the orthogonality of $Y_{\ell m}$:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, Y^{*}_{\ell' m'}(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell' \ell} \delta_{m' m}$$

$\delta_{l'l}$ = Kronecker delta

$$= \begin{cases} 0 & \text{if } l' \neq l \\ 1 & \text{if } l' = l \end{cases}$$

SPHERICAL HARMONICS $Y_{lm}(\theta, \phi)$

$$l = 0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1 \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\dfrac{3}{4\pi}} \cos\theta \end{cases}$$

$$l = 2 \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}} \sin^2\theta e^{2i\phi} \\ Y_{21} = -\sqrt{\dfrac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \\ Y_{20} = \sqrt{\dfrac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \end{cases}$$

$$l = 3 \begin{cases} Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}} \sin^3\theta e^{3i\phi} \\ Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}} \sin^2\theta \cos\theta e^{2i\phi} \\ Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}} \sin\theta \left(5\cos^2\theta - 1\right) e^{i\phi} \\ Y_{30} = \sqrt{\dfrac{7}{4\pi}} \left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \end{cases}$$

[Jackson]