Electric Dipole Radiation

The $\bar{E}$ due to an accelerating charge
(label charge $= \alpha$)

$$\left(\bar{E}_a\right)_\alpha = \frac{q_\alpha}{c^2}\left[\frac{\bar{R}_\alpha \times (\bar{R}_\alpha \times \bar{a})}{R_\alpha^3}\right]$$

$\bar{R}_\alpha$ = vector from the retarded location of charge to field point P

$[\ ]$ = evaluate at retarded time

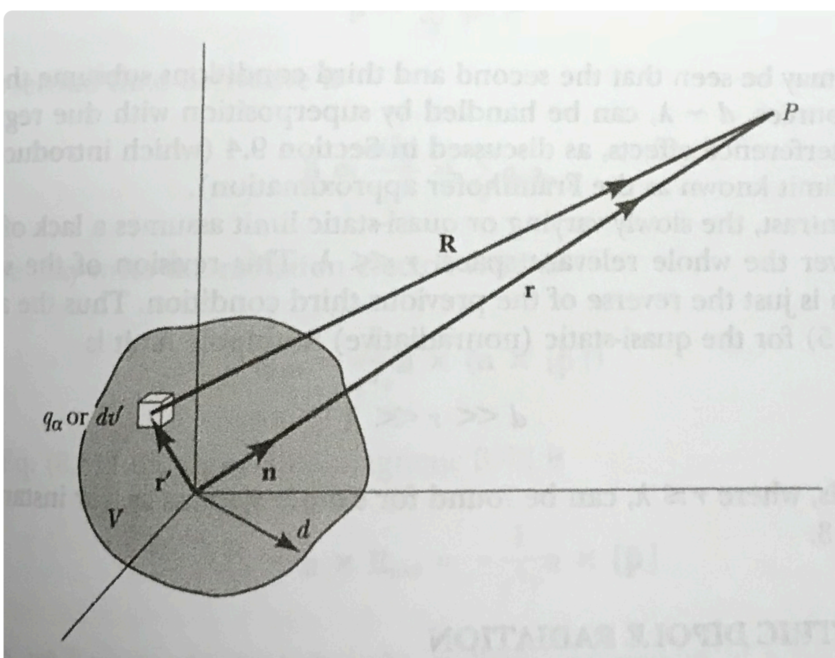

FIGURE 9-1. Source and field coordinates.

[H+M]

For $\quad d \ll \lambda \ll r \quad$ (radiation zone)

$$\Rightarrow \quad \bar{R}_\alpha \approx \bar{r}$$

$$\Rightarrow \bar{E}_{rad} = \sum_\alpha (E_a)_\alpha = \frac{1}{c^2 r} \hat{n} \times \left(\hat{n} \times \left[\sum_\alpha q_\alpha \bar{a}_\alpha\right]\right)$$

$$\hat{n} = \frac{\bar{r}}{r}$$

The vector dipole moment is

$$\bar{p} = \sum_\alpha q_\alpha \bar{r}_\alpha'$$

$$\Rightarrow \quad \ddot{\bar{p}} \equiv \frac{d^2 \bar{p}}{dt^2} = \sum q_\alpha \bar{a}_\alpha$$

$$\bar{E}_{rad} = \frac{1}{c^2 r} \hat{n} \times (\hat{n} \times [\ddot{\bar{p}}])$$

$$\Rightarrow \quad \bar{B}_{rad} = \hat{n} \times \bar{E}_{rad} = -\frac{1}{c^2 r} \hat{n} \times [\ddot{\bar{p}}]$$

In spherical coordinates, if we assume $\bar{p}$ is along the polar axis $(\hat{e}_z = \hat{e}_3)$

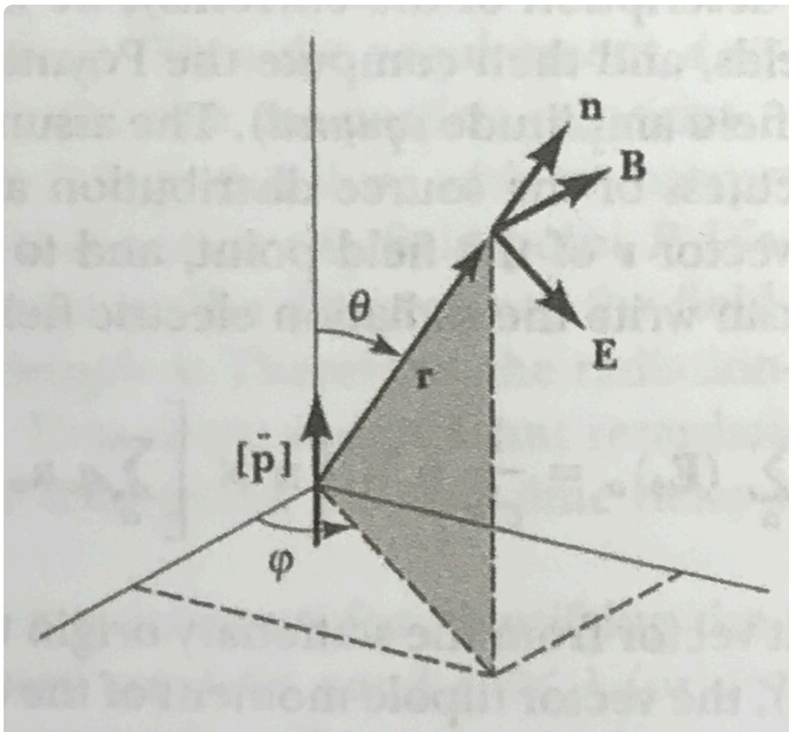

FIGURE 9-2. Fields of a radiating dipole.

[H+M]

$$\Rightarrow \quad \bar{E}_{rad} = \frac{[\ddot{\bar{p}}]}{c^2 r} \sin\theta \, \hat{e}_\theta$$

$$\bar{B}_{rad} = \frac{[\ddot{\bar{p}}]}{c^2 r} \sin\theta \, \hat{e}_\phi$$

$$\Rightarrow \quad \boxed{\frac{dP}{d\Omega} = \frac{e^2[a^2]}{4\pi c^3} \sin^2\theta = \frac{[\ddot{\bar{p}}^2]}{4\pi c^3} \sin^2\theta}$$

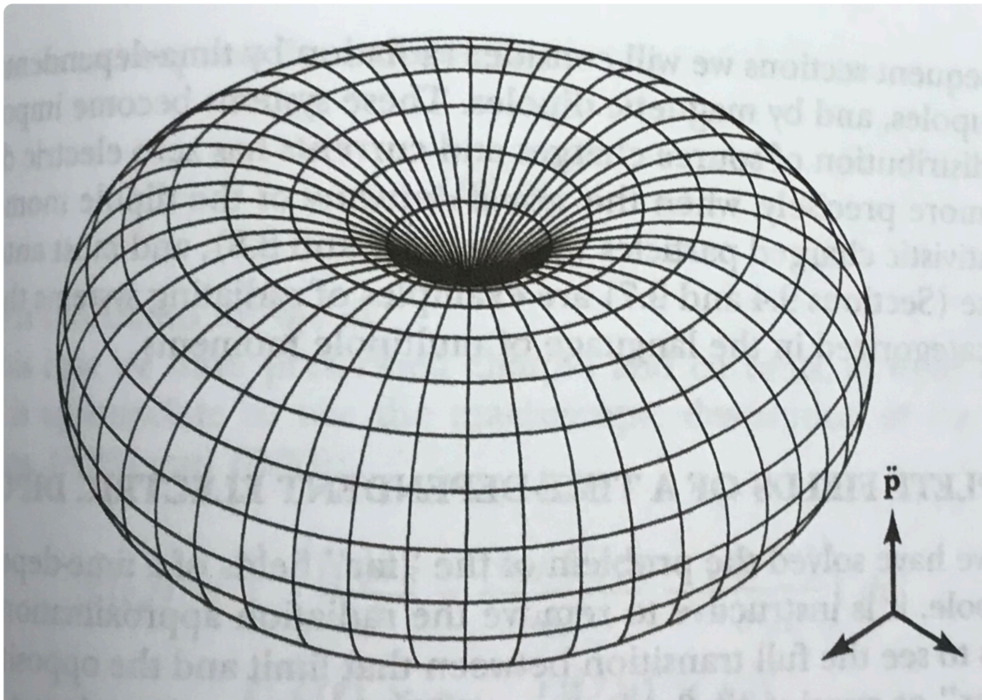

FIGURE 9-3. Radiation pattern of an accelerated electric dipole. [H+M]

↳ strongest radiation around its "waist"

The total radiated power is

$$\boxed{P = \frac{2e^2[a^2]}{3c^3} = \frac{2[\ddot{p}^2]}{3c^3}}$$

→ Larmor formulas apply to electric-dipole radiation

For a dipole oscillating at a fixed angular frequency

$$p(t) = p_0 \, e^{-i\omega t}$$

$$\Rightarrow \quad \langle \ddot{p}^2 \rangle = p_0^2 \omega^4 \langle \cos^2 \omega t \rangle = \frac{1}{2} p_0^2 \omega^4$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{p_0^2 \omega^4}{8\pi c^3} \sin^2\theta$$

$$\langle P \rangle = \frac{p_0^2 \omega^4}{3c^3}$$

$\hookrightarrow$ INCREASES AS ω^4

$\hookrightarrow$ MORE ACCELERATION AS $\omega \uparrow$

Linear Antennas

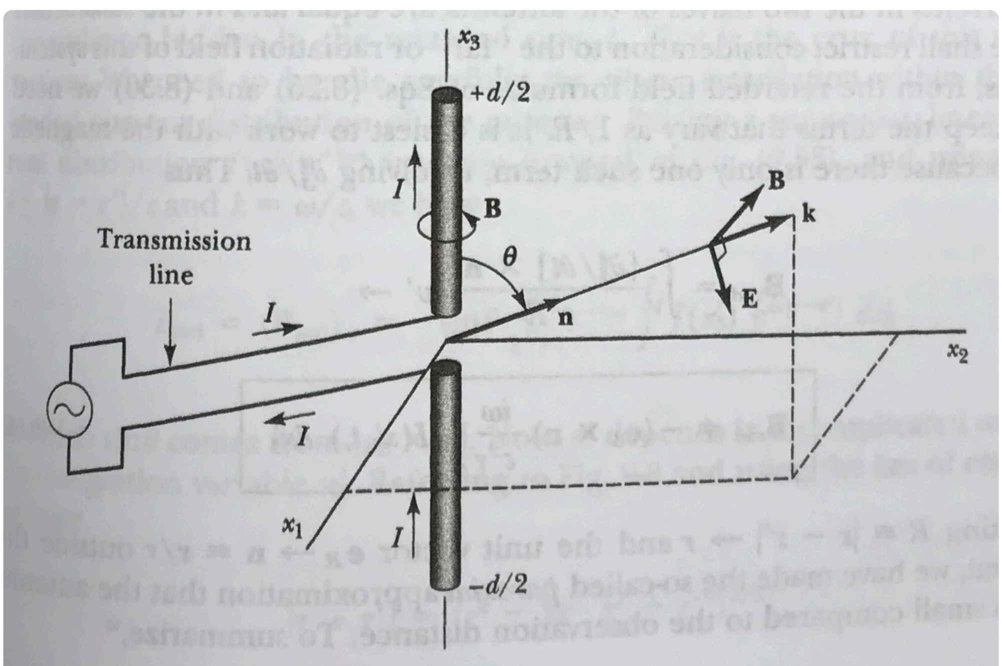

FIGURE 9-7. Center-driven linear antenna.

[H+M]

Assuming <u>driving current</u> is sinusoidal with frequency ω, and produces <u>standing waves</u> with nodes at the ends

$$\bar{J}(\bar{r}', t_r)\, dv' \rightarrow I(x_3', t_r)\, d\bar{x}_3'$$

$$= \hat{e}_3 I_0 e^{-i\omega t_r} \sin k\left(\frac{d}{2} - |x_3'|\right) dx_3'$$

The input signal is

$$I_{gap}(t_r) = I_0 \sin \frac{kd}{2} e^{-i\omega t_r}$$

From the generalized Biot-Savart law

$$\bar{B}(\bar{r}, t) = \int_V \left(\frac{[J] \times \hat{e}_R}{cR^2} + \frac{[\partial \bar{J}/\partial t] \times \hat{e}_R}{c^2 R} \right) dv'$$

In the "far" radiation zone, we only need to keep the $1/R$ term (why we use $\bar{B}$ instead of $\bar{E}$)

$$\Rightarrow \bar{B}_{RAD} = \int_V \frac{[\partial \bar{J}/\partial t] \times \hat{n}}{c^2 R}\, dv'$$

$$= -\left(\hat{e}_3 \times \hat{n}\right) \frac{i\omega}{c^2 r} \int I(x_3', t_r)\, dx_3'$$

where $R = |\bar{r} - \bar{r}'| \approx r$

$$\hat{e}_R = \hat{n} = \frac{\bar{r}}{r}$$

We could take r and $\hat{n}$ out of the integral by using the paraxial approximation:

$$r \gg \begin{cases} \lambda & \text{(radiation limit)} \\ d & \text{(paraxial limit)} \end{cases}$$

$\hookrightarrow$ antenna size

From $\bar{B}_{RAD}$ we get

$$\bar{E}_{RAD} = -\hat{n} \times \bar{B}_{RAD}$$

$$\Rightarrow \quad \bar{S}_{RAD} = \frac{c}{4\pi} \bar{E}_{RAD} \times \bar{H}_{RAD} = \frac{c}{4\pi} B_{RAD}^2 \hat{n}$$

goes as the magnitude of $\bar{B}_{RAD}$

Because $t_r = t - |\bar{r} - \bar{r}'|$ we can't remove it from the integral over dx_3'

$$B_{RAD} = (\bar{B}_{RAD})_\phi = -\sin\theta \frac{i\omega}{c^2 r} e^{-i\omega t} \int I(x_3') e^{ik|\bar{r}-\bar{r}'|} dx_3'$$

$\sin\theta$ from $|\hat{e}_3 \times \hat{n}|$ $\qquad k = \frac{\omega}{c}$

Also, $\bar{r}'$ depends on x_3'

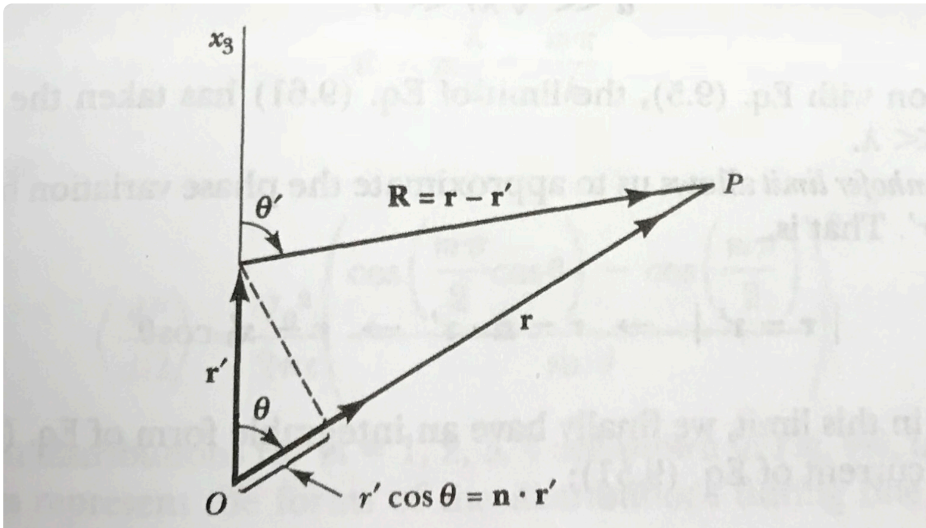

FIGURE 9-8. Geometry determining phase of antenna elements.

[H+M]

$$\Rightarrow |\bar{r} - \bar{r}'| = \sqrt{r^2 - 2\bar{r}\cdot\bar{r}' + r'^2}$$

For $r' << r$

$$|\bar{r} - \bar{r}'| \approx r\left[1 - \frac{\hat{n}\cdot\bar{r}'}{r} + \frac{r'^2}{2r^2} - \frac{1}{8}\left(\frac{2\hat{n}\cdot\bar{r}'}{r}\right)^2 + \dots\right]$$

$$\approx r\left[1 - \frac{r'}{r}\cos\theta + \frac{r'^2}{2r^2}\sin^2\theta + \dots\right]$$

$\hookrightarrow$ determines the <u>phase</u>

In the <u>Fraunhofer limit</u>, the quadratic terms and higher can be ignored

$$k\left(\frac{d}{2}\right)^2 / 2r << 2\pi \implies r >> \frac{d^2}{8\lambda}$$

$\hookrightarrow$ little contribution to phase

In this limit, the phase varies linearly with r'

$$|\bar{r}-\bar{r}'| \rightarrow r - \hat{n}\cdot\bar{r}' \rightarrow r - x_3'\cos\theta$$

$$\Rightarrow B_{RAD}(r,\theta,t) = -\sin\theta \, \frac{\omega I_0}{c^2 r} \, i e^{i(kr-\omega t)} \int_{-d/2}^{d/2} \sin k\left(\frac{d}{2} - |x_3'|\right) e^{-ikx_3'\cos\theta} \, dx_3'$$

$$= -\frac{2I_0}{cr} \, i e^{i(kr-\omega t)} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta} \right)$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = r^2 \langle \bar{S}_{rad} \rangle \cdot \hat{n} = \frac{c r^2}{4\pi} \langle B_{rad}^2 \rangle$$

$$= \frac{I_0^2}{2\pi c} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta} \right)^2$$

$\hookrightarrow$ fxn of $\frac{kd}{2}$

For situations where

$$d = m\frac{\lambda}{2} = m\frac{\pi}{k}$$

$\hookrightarrow$ integer multiple of $\lambda/2$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{I_0^2}{2\pi c} \left(\frac{\cos\left(\frac{m\pi}{2}\cos\theta\right) - \cos\frac{m\pi}{2}}{\sin\theta} \right)^2$$

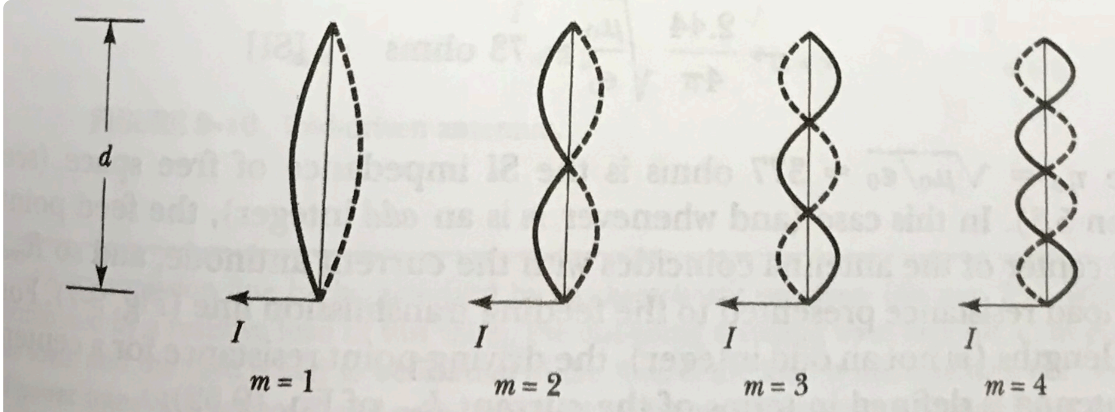

FIGURE 9-9. Center-driven antennas. current distributions [H+M]

——— = distributions during ½ cycle

- - - - - = distributions during other ½ cycle

Important cases:

$$m=1 \Rightarrow \text{half-wave} \Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{I_0^2}{2\pi c} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

$$m=2 \Rightarrow \text{full-wave} \Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{2I_0^2}{\pi c} \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

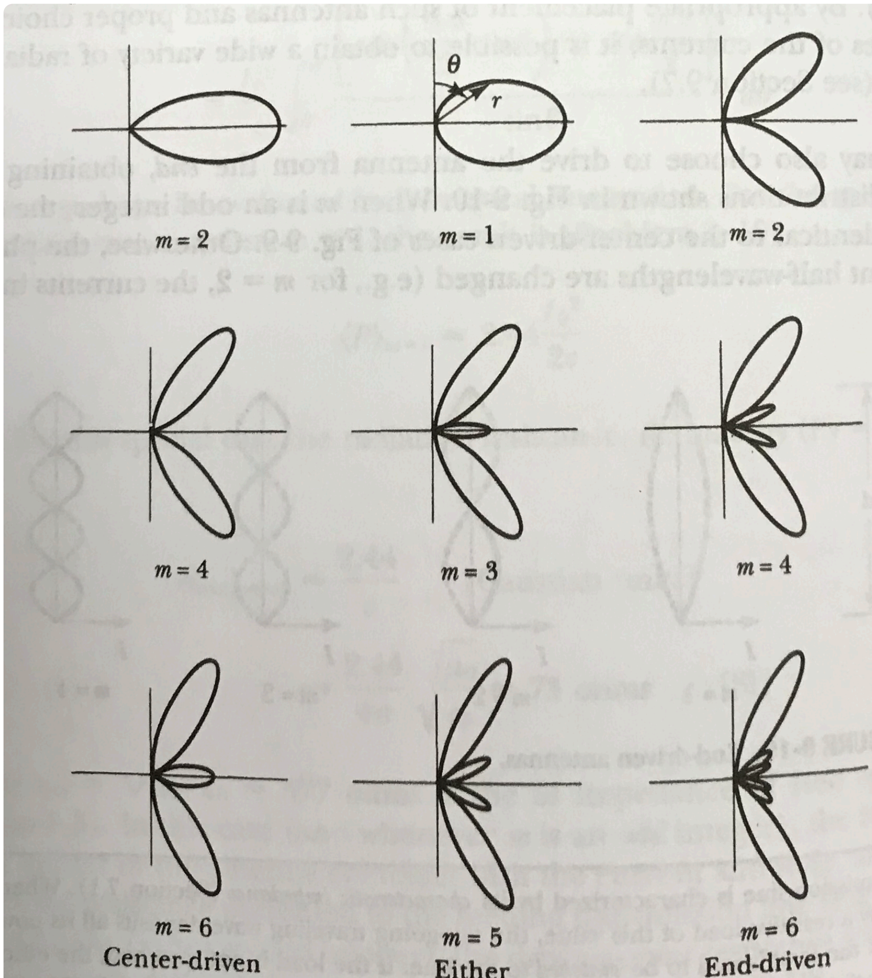

FIGURE 9-11. Radiation patterns of linear antennas.

[H+M]

Symmetric about vertical axis

- Center driven m=2 and m=1 looks like familiar $\sin^2\theta$ pattern

- End-driven m=2 and center driven m=4 look like quadrupole radiators