Complex dielectric constant

Instead of using a complex $\hat{k}$ to derive our wave behavior in a conductor, we can assume a complex ε and get the same result:

Ampère's Law $\longrightarrow$ $\bar{\nabla} \times \bar{B} - \dfrac{\varepsilon \mu}{c} \dfrac{\partial \bar{E}}{\partial t} = \dfrac{4\pi \sigma \mu}{c} \bar{E}$

For a periodic variation with $E \sim e^{-i\omega t}$

$$\Rightarrow \bar{\nabla} \times \bar{B} - \underbrace{\left(\varepsilon + i \frac{4\pi\sigma}{\omega} \right)} \frac{\mu}{c}(-i\omega) \bar{E} = 0$$

$$\boxed{\hat{\varepsilon} \equiv \varepsilon + i \frac{4\pi\sigma}{\omega}}$$

Complex dielectric constant

$$\Rightarrow \bar{\nabla} \times \bar{B} - \hat{\varepsilon} \frac{\mu}{c}(-i\omega) \bar{E} = 0$$

Our equation for $\hat{k}$ becomes

$$\hat{k}^2 = \frac{\varepsilon \mu \omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\varepsilon\omega}\right) = \frac{\hat{\varepsilon}\mu\omega^2}{c^2}$$

$$\Rightarrow \boxed{\hat{k} = \frac{\omega}{c}\sqrt{\hat{\varepsilon}\mu}}$$

Similarly, from $\hat{k} = \hat{n}\frac{\omega}{c}$

$$\Rightarrow \boxed{\hat{n} = \sqrt{\hat{\varepsilon}\mu}}$$

This form of $\hat{\varepsilon}$, $\hat{k}$, and $\hat{n}$ <u>looks the same</u> as in the <u>nonconducting</u> scenario. However, the calculations will still be difficult because $\hat{n}$ is of the form

$$\hat{n} = (\alpha + i\beta)\frac{c}{\omega}$$

Also, we have $\sqrt{i}$ from $\sqrt{\hat{\varepsilon}}$ and

$$i = e^{i(\pi/2 + 2\pi n)} \Rightarrow \sqrt{i} = e^{i\left(\frac{\pi}{4} + \pi n\right)} = \pm \frac{1}{\sqrt{2}}(1 + i)$$

integer

Using a complex $\hat{n}$, the $\bar{E}_0$ and $\bar{B}_0$ become

$$\bar{B}_0 = \hat{n}\, \hat{e}_k \times \bar{E}_0$$

$$\bar{E}_0 = \frac{1}{\hat{n}} \hat{e}_k \times \bar{B}_0$$

Rewriting $\hat{n}$ and $\hat{k}$ in polar form:

$$\hat{k} = \hat{n} \frac{\omega}{c} = \alpha + i\beta = \sqrt{\alpha^2 + \beta^2}\, e^{i\phi}$$

For

$$\alpha = \frac{\omega}{c} \sqrt{\frac{\mu\varepsilon}{2}} \left[\sqrt{1 + \left(\frac{4\pi\sigma}{\omega\varepsilon}\right)^2} + 1 \right]^{1/2}$$

$$\beta = \frac{\omega}{c} \sqrt{\frac{\mu\varepsilon}{2}} \left[\sqrt{1 + \left(\frac{4\pi\sigma}{\omega\varepsilon}\right)^2} - 1 \right]^{1/2}$$

We get

$$\Rightarrow \quad |\hat{k}| = |\hat{n}| \frac{\omega}{c} = \sqrt{\alpha^2 + \beta^2} = \frac{\omega\sqrt{\varepsilon\mu}}{c}\left[1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2\right]^{1/4}$$

And

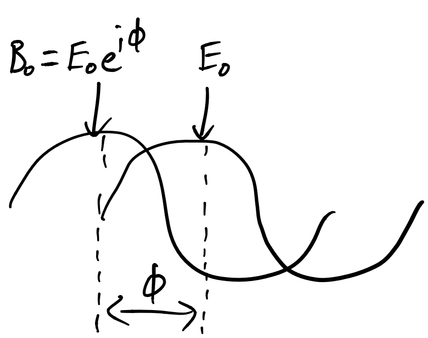
$$\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4\pi\sigma}{\varepsilon\omega}\right)$$

polar angle
(phase)

Now, from $\quad \hat{n} \sim e^{i\phi}$

$$\Rightarrow \qquad \bar{B}_0 = \hat{n}\,\hat{e}_k \times \bar{E}_0 \sim e^{i\phi}\,\bar{E}_0$$

$$\bar{E}_0 = \frac{1}{\hat{n}}\hat{e}_k \times \bar{B}_0 \sim \underbrace{e^{-i\phi}}_{\text{out of phase by } \phi}\bar{B}_0$$



out of phase by ϕ

B_0 lags behind E_0

Analyzing the two <u>limits</u> of $\hat{k}$ with respect to the <u>conductivity</u>

$$\hat{k} = \alpha + i\beta = \frac{\omega}{c}\sqrt{\hat{\varepsilon}\mu} = \frac{\omega}{c}\sqrt{\mu}\left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right)^{1/2}$$

<u>Low conductivity, $4\pi\sigma << \varepsilon\omega$</u>

Applies to moderately good conductors at very high frequencies

Taylor series $(1+x)^{1/2} \simeq 1 + \frac{1}{2}x + \ldots$

$$\Rightarrow \hat{k} \simeq \frac{\omega}{c}\sqrt{\mu\varepsilon}\left(1 + i\frac{2\pi\sigma}{\varepsilon\omega} + \ldots\right)$$

$$\Rightarrow \hat{k} = \alpha + i\beta$$

$$\alpha \simeq \frac{\omega}{c}\sqrt{\mu\varepsilon} = \frac{\omega n}{c} = \frac{2\pi}{\lambda} \longrightarrow$$ Same as nonconducting scenario

$$\beta \simeq \frac{2\pi\sigma}{c}\sqrt{\frac{\mu}{\varepsilon}} \longrightarrow$$ decays in conductor

High conductivity, $4\pi\sigma \gg \varepsilon\omega$

Applies to most metals for radio, microwave, infrared and visible frequencies.

$\rightarrow$ conduction current $\gg$ displacement current

$$\Rightarrow \hat{k} = \frac{\omega}{c}\sqrt{\mu}\left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right)^{1/2} \simeq \frac{\omega}{c}\sqrt{i\frac{4\pi\sigma\mu}{\omega}}$$

$$= \frac{\omega}{c}\sqrt{\frac{2\pi\sigma\mu}{\omega}}\,(1+i)$$

$$= \frac{1}{c}\sqrt{2\pi\sigma\mu\omega}\,(1+i)$$

$$\Rightarrow \underbrace{\alpha \simeq \beta \simeq \frac{1}{c}\sqrt{2\pi\sigma\mu\omega}}_{\text{periodic and damping coefs are equal}}$$

e-folding distance

$$\frac{1}{\beta} \simeq \frac{1}{\alpha} \simeq \frac{\lambda}{2\pi}$$

$\Rightarrow$ Amplitude decreases by $e^{2\pi} \simeq 535$ in one wavelength!

$\hookrightarrow$ Heavily damped

The wavelength in the conductor is much less than the wavelength in a non conducting medium

$$\frac{2\pi/\alpha}{2\pi c/\omega\sqrt{\epsilon\mu}} = \frac{2\pi c/\sqrt{2\pi\sigma\mu\omega}}{2\pi c/\omega\sqrt{\epsilon\mu}} = \sqrt{\frac{\epsilon\omega}{2\pi\sigma}} \ll 1$$

in conductor $\rightarrow$ (numerator)

non conductor $\rightarrow$ (denominator)

The e-folding distance is used so much that it's given a name and symbol

$$\boxed{\text{skin depth} \equiv \delta \equiv \frac{c}{\sqrt{2\pi\mu\sigma\omega}}}$$

Current Distribution in Conductors

In a good conductor, we can ignore the displacement current term

$$\bar{\nabla} \times \bar{B} - \underbrace{\frac{\varepsilon\mu}{c}\frac{\partial \bar{E}}{\partial t}}_{\text{ignore}} = \frac{4\pi\sigma\mu}{c}\bar{E}$$

$$\Rightarrow \bar{\nabla} \times \bar{B} = \frac{4\pi\sigma\mu}{c}\bar{E}$$

From

$$\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}$$

$$\bar{\nabla} \times \bar{\nabla} \times \bar{E} = \bar{\nabla}(\underbrace{\bar{\nabla} \cdot \bar{E}}_{\emptyset}) - \bar{\nabla}^2 \bar{E} = -\bar{\nabla}^2 \bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{B})$$

$$\Rightarrow \bar{\nabla}^2 \bar{E} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \bar{E}}{\partial t} = \emptyset$$

Or, using Ohm's law $\bar{J} = \sigma \bar{E}$ we get

$$\underbrace{\bar{\nabla}^2 \bar{J} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \bar{J}}{\partial t} = 0}_{\text{Diffusion equation}}$$

For $\bar{J} = \bar{J}_0 e^{-i\omega t}$

$$\Rightarrow \bar{\nabla}^2 \bar{J}_0 + \tau^2 \bar{J}_0 = 0$$

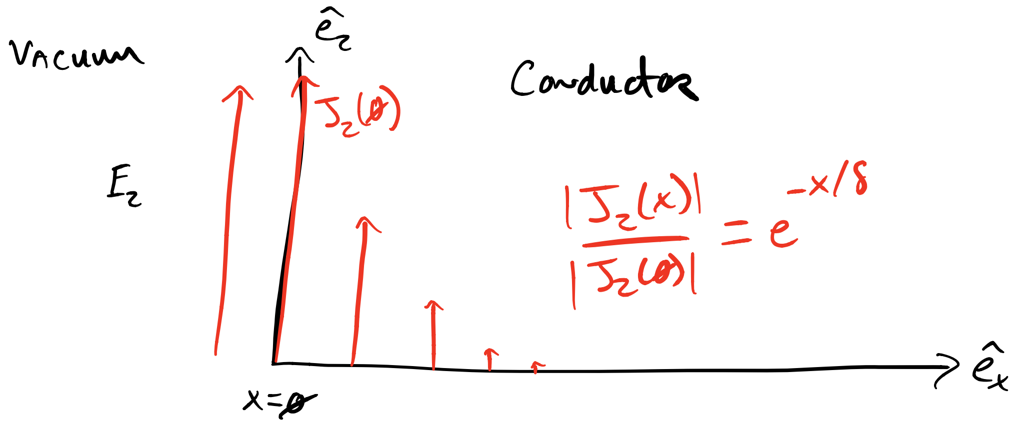
$$\hookrightarrow \tau^2 \equiv i\,\frac{4\pi\sigma\mu\omega}{c^2}$$

$$\Rightarrow \text{positive root} = \tau = \frac{\sqrt{2\pi\sigma\mu\omega}}{c}(1+i)$$

$$= \frac{1+i}{\delta}$$

For $\delta = \dfrac{c}{\sqrt{2\pi\sigma\mu\omega}}$ } same skin depth from before

Consider a 2-D conductor and a plane wave polarized in the $\hat{e}_z$ direction



The solution is

$$J_z(x) = J_z(0) e^{i\gamma x} = J_z(0) e^{(i-1)x/\delta}$$

$\Rightarrow$ The <u>current density magnitude decreases exponentially</u>

$$\frac{|J_z(x)|}{|J_z(0)|} = e^{-x/\delta}$$

- Skin depth δ gives the depth at which $|J|$ decreases by a factor of $1/e$ in a conductor
- In copper for a microwave frequency of 5000 MHz $\longrightarrow$ $\delta \simeq 10^{-4}$ cm