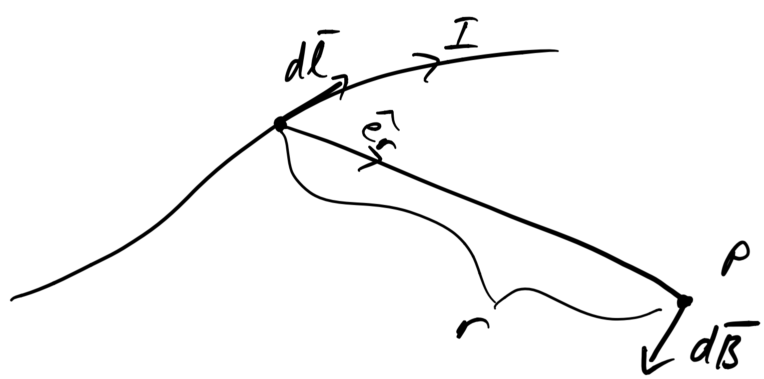
Magnetostatics

- The observation that currents produce magnetic fields was first discovered by Oersted in 1820.

- Additional experiments by Ampere, Biot, and Savart provide the basis for the magnetic field

Biot - Savart Law

$$\bar{B} = \frac{1}{c} \oint \frac{I\, d\bar{l} \times \hat{e}_r}{r^2}$$



$\bar{B}$ = magnetic field due to I

$\hat{e}_r$ = points to some source P

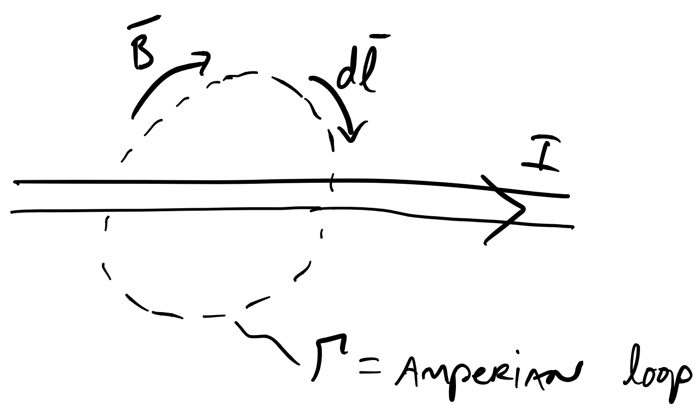
$I d\bar{l}$ = source element

$\oint$ = integral over current loop

r = distance to point P

Ampère's Law

$$\oint_\Gamma \bar{B} \cdot d\bar{l} = \frac{4\pi}{c} I_{link}$$



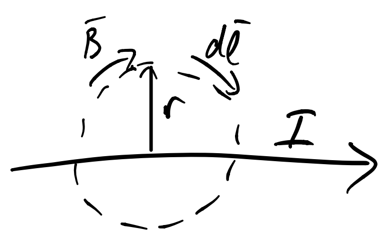
$\bar{B}$ = magnetic field $\Rightarrow$ direction given by right-hand rule with thumb pointed towards I

$d\bar{l}$ = differential along path Γ

Γ = Amperian loop

I_{link} = Sum of all currents through Γ

<u>Example of Ampère's Law</u>

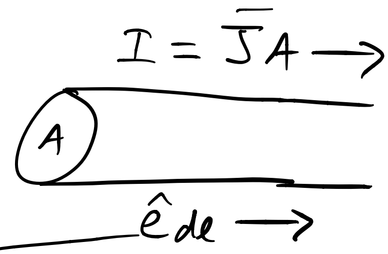


$$\oint \underbrace{\bar{B} \cdot d\bar{l}}_{\text{same direction } \hat{e}_\theta} = B\, 2\pi r = \frac{4\pi}{c} I \Rightarrow B = \frac{2I}{cr}$$

$$\Rightarrow \bar{B}(r) = \frac{2I}{cr} \hat{e}_\theta$$

Current density

$$\bar{J} = \frac{I}{A}\,\hat{e}_{dl}\ (\text{statamp}/\text{cm}^2)$$

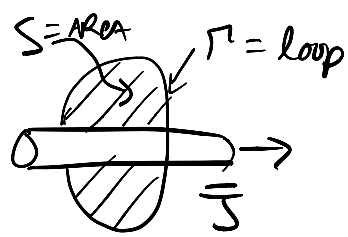


unit vector along $I\,d\bar{l}$

$\bar{J}$ is a vector, providing a "density" to I where I is just a total amount.

Similar to $\quad \frac{q}{V} = \rho$ (total: q; density: ρ)

The total current is the integral of $\bar{J}$ over a surface



$$\int_S \bar{J}\cdot d\bar{a} = I$$

normal to surface

From Stoke's theorem,

$$\oint_{\Gamma} \bar{B}\cdot d\bar{l} = \int_S \bar{\nabla}\times\bar{B}\cdot d\bar{a} = \frac{4\pi}{c} I = \frac{4\pi}{c}\int_S \bar{J}\cdot d\bar{a}$$

$$\Longrightarrow \underbrace{\bar{\nabla}\times\bar{B} = \frac{4\pi}{c}\bar{J}}$$

Differential form of Ampère's Law

Note: steady-state $\rightarrow$ Additional term comes from $\frac{\partial E}{\partial t}$

Because no magnetic monopoles have been observed in nature, the <u>magnetic Gauss' Law</u> is:

$$\bar{\nabla}\cdot\bar{B} = 0$$

$\uparrow$ $\rho_{\text{magnetic}} = 0$

$\Rightarrow \bar{B}$ are composed of <u>closed curves</u>

$\bar{E}$ originates and terminates at charges

Because $\bar{\nabla} \cdot \bar{B} = 0$, we can use the vector identity

$$\bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) = 0$$

to introduce the idea of a vector potential

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

vector potential

Without going through the derivation, $\bar{A}$ can be described by:

$$\bar{A} = \frac{1}{c}\underbrace{\oint \frac{I\,d\bar{l}}{r}}_{\text{integral over loop}} = \frac{1}{c}\underbrace{\int_V \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}\,dv'}_{\text{integral over loop and area for } \bar{J}}$$

$\bar{J}\,dA\,dl = \bar{J}\,dv$

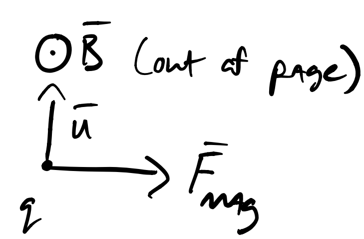
$\bar{r}'$ = source coordinate

$\bar{r}$ = position of interest

Lorentz Force

The magnetic force on a moving charge:

$$\bar{F}_{mag} = \frac{q\bar{u}}{c} \times \bar{B}$$



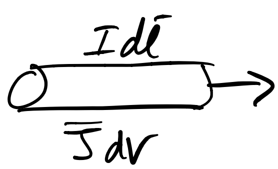
Taking the electric force into account

$$\bar{F}_{el} = q\bar{E}$$

gives the <u>Lorentz force</u>

$$\boxed{\bar{F} = q\left(\bar{E} + \frac{\bar{u}}{c} \times \bar{B}\right)}$$

To convert the magnetic force to a force on element $d\bar{l}$ of a wire:



drift velocity

$$I d\bar{l} = \bar{J} dv \quad \text{and} \quad \bar{J} = n q \bar{u}$$

density of charge carriers

$$n = \frac{N}{V}$$

$$\Rightarrow \bar{J} dv = n q \bar{u}\, dv$$

$$\bar{F} = \frac{q\bar{u}}{c} \times \bar{B} \implies d\bar{F} = \frac{nq\bar{u}\,dv}{c} \times B$$

single q

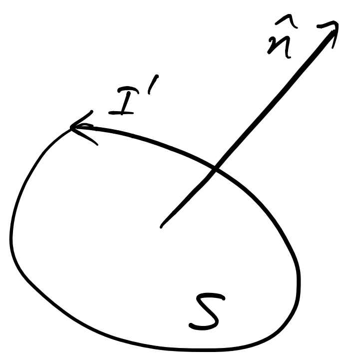
force on volume element dv for charge density $\rho = nq$

$$\implies d\bar{F} = \underbrace{\frac{I\,d\bar{l}}{c} \times \bar{B}}_{\text{Force on element } d\bar{l} \text{ of wire}}$$

Magnetic Materials

Similar to the idea of $\bar{p}$ for electric dipoles, we introduce the idea of the magnetic dipole moment

$$\bar{m} \equiv \frac{I'}{c}\bar{S} = \frac{I'}{c} S\,\hat{n}$$



$\bar{m}$ = magnetic dipole moment

I' = magnitude of Amperian loop current

S = area enclosed by loop

$\hat{n}$ = normal to area S

$\bar{S}$ = directed area

The idea of a magnetic dipole follows the idea of an electric dipole, with the torques defined as:

$$\bar{\tau}_e = \bar{p} \times \bar{E} \longrightarrow \text{electric}$$

$$\bar{\tau}_m = \bar{m} \times \bar{B} \longrightarrow \text{magnetic}$$

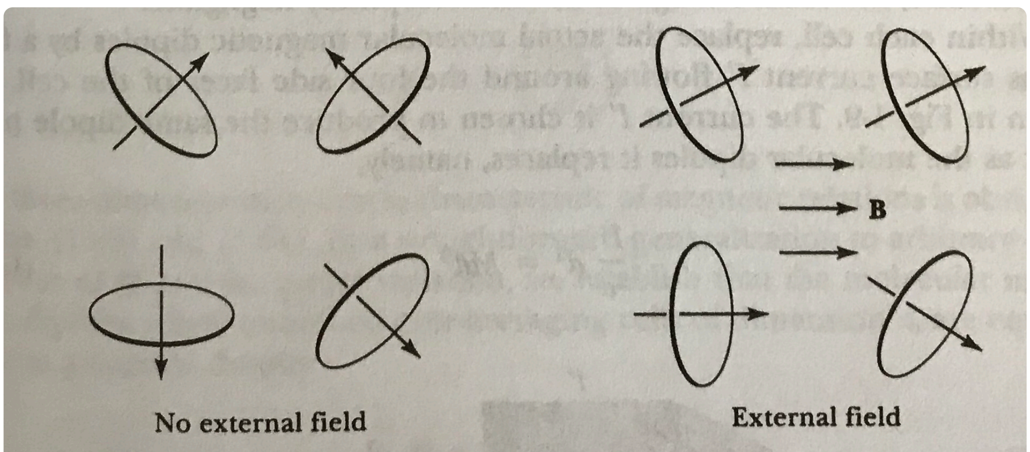


FIGURE 1-8. Molecular dipoles aligned by magnetic field.

[H+M]

Magnetic dipoles align with an applied $\bar{B}$

- diamagnetism: $\bar{m}$ antiparallel to $\bar{B}$
- paramagnetism: $\bar{m}$ parallel to $\bar{B}$
- ferromagnetism: self-aligned (strong effect)

Macroscopic Description:

Similar to $\bar{P}$, the magnetization $\bar{M}$ is the net magnetic dipole moment per volume:

$$M(\bar{r}) = \bar{m}_0(\bar{r})\, N(\bar{r})$$

$N(\bar{r})$ → molecular density

$\bar{m}_0(\bar{r})$ → moment for each molecule

Similar to the derivation for

$$-\bar{\nabla} \cdot \bar{P} = \rho'$$

the magnetic analog is

$$\bar{\nabla} \times \bar{M} = \frac{J'}{c}$$

J' ← bound current density

Separating the current density to free and bound

$$\bar{\nabla} \times \bar{B} = \frac{4\pi}{c} \bar{J} = \frac{4\pi}{c} (\bar{J}_f + \bar{J}_b)$$

$$\Rightarrow \bar{\nabla} \times \underbrace{(\bar{B} - 4\pi \bar{M})} = \frac{4\pi}{c} \bar{J}_f$$

$\bar{H} = \bar{B} - 4\pi \bar{M}$ = magnetic intensity

Which gives us the macroscopic version of Ampére's law in a magnetic medium:

$$\bar{\nabla} \times \bar{H} = \frac{4\pi}{c} \bar{J}_f$$

Again, following the electric analog for ε, the magnetization is experimentally found to be proportional to $\bar{H}$ for para and diamagnetic materials:

$$\bar{M} = \chi_m \bar{H}$$

↳ magnetic susceptibility

$$\Rightarrow \bar{B} = (1 + 4\pi \chi_m)\bar{H} = \mu \bar{H}$$

$\hookrightarrow$ permeability

$\mu = 1$ in free space

$\mu > 1$ in paramagnetic

$\mu < 1$ in diamagnetic

$\mu \gg 1$ in ferromagnetic (strong!)

As it turns out, **H** is a more useful quantity than **D**. In the laboratory you will frequently hear people talking about **H** (more often even than **B**), but you will never hear anyone speak of **D** (only **E**). The reason is this: To build an electromagnet you run a certain (free) current through a coil. The *current* is the thing you read on the dial, and this determines **H** (or at any rate, the line integral of **H**); **B** depends on the specific materials you used and even, if iron is present, on the history of your magnet. On the other hand, if you want to set up an *electric* field, you do *not* plaster a known free charge on the plates of a parallel plate capacitor; rather, you connect them to a battery of known *voltage*. It's the *potential difference* you read on your dial, and that determines **E** (or at any rate, the line integral of **E**); **D** depends on the details of the dielectric you're using. If it were easy to measure charge, and hard to measure potential, then you'd find experimentalists talking about **D** instead of **E**. So the relative familiarity of **H**, as contrasted with **D**, derives from purely practical considerations; theoretically, they're all on equal footing.

Many authors call **H**, not **B**, the "magnetic field." Then they have to invent a new word for **B**: the "flux density," or magnetic "induction" (an absurd choice, since that term already has at least two other meanings in electrodynamics). Anyway, **B** is indisputably the fundamental quantity, so I shall continue to call it the "magnetic field," as everyone does in the spoken language. **H** has no sensible name: just call it "**H**".[4]

[4]For those who disagree, I quote A. Sommerfeld's *Electrodynamics* (New York: Academic Press, 1952), p. 45: "The unhappy term 'magnetic field' for **H** should be avoided as far as possible. It seems to us that this term has led into error none less than Maxwell himself . . ."

[Griffiths]