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On the variational approach to axisymmetric magnetohydrodynamic equilibria

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The variational formulation of the axisymmetric magnetohydrodynamic equilibrium equations with plasma flows is addressed and a more comprehensive method is presented that allows, in particular, for open boundary conditions and discontinuous (shock) solutions. A numerical procedure based on the variational formulation is described and a validation test for an open conical geometry, including also hydrodynamic shocks, is investigated. *© 2008 American Institute of Physics.*
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I. INTRODUCTION

The ideal magnetohydrodynamic description of axisymmetric stationary configurations has found wide application in different fields of plasma physics, from early fusion experiments with no plasma flow (see Refs. 1–4), to astrophysical (see Ref. 5) and present day fusion studies (see Ref. 6). This model provides a convenient tool to describe the large scale features of a wide class of plasma configurations where axisymmetry can be assumed to be a valid approximation. Since the ideal MHD description neglects viscosity and other dissipative terms, Lagrangian and Hamiltonian formulations of this description exist[7,8] which represent a preferred basis for studying the equilibria and the stability of the system.[9] Moreover, calculation methods based on these variational formulations can be used (e.g., the Ritz method or Galerkin methods). In fact, besides the differential approach that is commonly adopted to describe the Grad–Shafranov[2,3] (GS) and the generalized Grad–Shafranov[5,6] (GGS) models, a variational principle for the equilibria was formulated in Refs. 10–13.

A detailed description of axisymmetric MHD equilibria in terms of a Hamiltonian principle is the aim of the present work. The variational approach of Ref. 13 is modified in order to overcome the singularity related to the Alfvén velocity. A more comprehensive method, able to treat open boundary problems (of interest in particular for plasma thrusters and astrophysical jets), is obtained including a natural formulation of open boundary conditions together with an effective description of discontinuous flows. In Sec. II the limitations of the standard formulation are pointed out and the new variational formulation of the problem is discussed. In Sec. III the natural formulation of open boundary conditions is presented and in Sec. IV the case of discontinuous flows is discussed. Finally, in Sec. V some results from a simplified case are presented and the validity of the method is assessed.

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II. VARIATIONAL APPROACH

In the ideal single-fluid limit the dynamics of a plasma is described by the magnetohydrodynamic equations

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}),$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla\left(\frac{v^2}{2}\right) + \mathbf{v} \times (\nabla \times \mathbf{v}) - \rho^{-1} \nabla p - \frac{\rho^{-1}}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{1}$$

$$\frac{\partial S}{\partial t} = -\mathbf{v} \cdot \nabla S, \quad \frac{\partial \mathbf{B}}{\partial t} = -\mathbf{B} \cdot \nabla \mathbf{v} + \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{B},$$

where mass conservation is expressed in terms of the plasma density ρ and the flow velocity $\mathbf{v}$ and the motion of the ideal fluid is subject to pressure gradients and magnetic forces. The pressure p can be expressed in terms of the plasma internal energy per unit mass $U=U(\rho,S)$ as $p=\rho^2 U_\rho$, where the subscript represents partial derivatives. The flow is assumed to be isentropic and, if isothermal conditions are considered, the advection equation of the entropy per unit mass, S, must be substituted by $\partial T/\partial t = -\mathbf{v}\cdot\nabla T$, where T is the fluid temperature. In this latter case the natural thermodynamical variables become (ρ,T) and the pressure $p=\rho^2 A_\rho$ is expressed in terms of the Helmholtz free energy per unit mass $A=U-TS=A(\rho,T)$. The Lorentz term in the momentum equation depends on the magnetic field $\mathbf{B}$, the evolution of which is described by Faraday's law for perfectly conductive fluids.

The above system of equations simplifies in the case of axisymmetric configurations. Introducing cylindrical coordinates (r,ϕ,z), where ϕ is assumed to be the ignorable coordinate, the magnetic field in the r-z plane (the poloidal plane) is determined by the magnetic flux function ψ so that

$$\mathbf{B} = rB_\phi \nabla \phi + \nabla \psi \times \nabla \phi, \tag{2}$$

where B_ϕ is the azimuthal component of the magnetic field. The mass flow can be expressed in a similar way as

$$\rho\mathbf{v} = r\rho v_\phi \nabla \phi + \nabla\chi(\psi) \times \nabla\phi, \qquad (3)$$

where v_ϕ is the azimuthal velocity of the plasma and the stream function χ depends on the magnetic flux function ψ only. Exploiting the definition of ψ, it can be shown that five such flux functions are needed to express the primitive variables in Eq. (1),

$$\begin{aligned}
&F(\psi) = \chi' = 4\pi\rho v_p/B_p, \quad G(\psi) = (v_\phi - v_p B_\phi/B_p)/r, \\
&H(\psi) = rB_\phi - rFv_\phi, \\
&I(\psi) = p\rho^{-\gamma} = \exp[(\gamma-1)S(\psi)/k_B], \\
&J(\psi) = U + \rho U_\rho + v^2/2 - rv_\phi G,
\end{aligned} \qquad (4)$$

where the index p refers to the poloidal plane, and a prime indicates differentiation with respect to ψ. These five flux functions represent the quantities that are conserved during the motion of each fluid particle: the function F is a direct consequence of the mass flow conservation, G represents the electric potential, H is the poloidal vorticity-current density stream function and follows from the azimuthal momentum conservation, I is related to the entropy with γ the adiabatic index and k_B the Boltzmann constant. Finally J is the generalized form of the Bernoulli constant and represents the energy conserved along each streamline. The term $H_E = U + \rho U_\rho = U + p/\rho = \gamma(\gamma-1)^{-1} p/\rho$ represents the enthalpy per unit mass.

In terms of these flux functions, the axisymmetric equilibrium condition reduces to the generalized Grad–Shafranov equation for the function ψ,

$$\begin{aligned}
\nabla\cdot\left[\left(1-\frac{F^2}{4\pi\rho}\right)\frac{\nabla\psi}{r^2}\right]
&= -4\pi\rho r^2(J' + rv_\phi G') - (H + rv_\phi F)(H' + rv_\phi F') \\
&\quad + \frac{4\pi r^2}{\gamma-1}\rho^\gamma I',
\end{aligned} \qquad (5)$$

where $\rho = \rho(r,\psi,\nabla\psi)$ and $v_\phi = v_\phi(r,\psi,\nabla\psi)$ follow from Eqs. (4).

For isothermal flows the functions I and J must be redefined as $I(\psi) = p\rho^{-1} \propto T(\psi)$ and $J(\psi) = A + \rho A_\rho + v^2/2 - rv_\phi G$, where $G_E = A + \rho A_\rho = A + p/\rho = I\ln(\rho/\rho_0)$ represents the Gibbs free energy per unit mass and ρ_0 is a density reference value, and the last term of Eq. (5) must be replaced by $4\pi r^2 \ln(\rho/\rho_0) I'$.

In this differential formulation the plasma density is connected to ψ by the Bernoulli equation which is an algebraic equation that in general cannot be solved analytically so as to set $\rho = \rho(r,\psi,\nabla\psi)$ in explicit form. In addition, from Eqs. (4) the azimuthal velocity can be expressed in terms of ψ and ρ as

$$rv_\phi = (r^2 G + FH/4\pi\rho)(1 - M^2)^{-1}, \qquad (6)$$

where $M^2 = F^2/4\pi\rho = v_p^2/v_A^2$ is the square of the poloidal Alfvén Mach number and $v_A = \sqrt{B_p^2/(4\pi\rho)}$ is the poloidal Alfvén velocity. At the Alfvén surfaces, $M=1$, the expression (6) contains an apparent singularity and, in order to avoid this singularity, the regularity condition $H = -r^2 FG$ must be satisfied. These two mathematical aspects posed limitations for an effective use of the differential approach. In order to overcome the difficulty represented by the implicit Bernoulli equation, a variational approach was proposed in Refs. 10 and 11 and refined in Refs. 12 and 13, in which this implicit equation is automatically obeyed by the extrema of a specific functional of two independent variables: the magnetic flux function ψ and the plasma density ρ. Still, the singularity resulting from Eq. (6) did not find a satisfactory solution in the literature.

A general variational principle for axisymmetric MHD equilibria is described in Refs. 12 and 14, which considers the extrema of the energy functional

$$\mathcal{H} = \frac{1}{2}\int\left\{\rho v^2 + \frac{B^2}{4\pi} + 2\rho U[\rho, S(\psi)]\right\} dV, \qquad (7)$$

constrained by the flow invariants. The entropy is assumed to be a function of ψ and, as described above, for an isothermal flow the internal energy, $U = U(\rho, S)$, must be replaced by the Helmholtz free energy, $A = A(\rho, T)$. The functional

$$\begin{aligned}
\mathcal{F} = \mathcal{H} - \int \rho J(\psi) - \int \frac{1}{4\pi r} B_\phi H(\psi) - \int r\rho v_\phi G(\psi) \\
- \int \frac{1}{4\pi}(\mathbf{v}\cdot\mathbf{B}) F(\psi),
\end{aligned} \qquad (8)$$

can be rewritten in the extended form

$$\begin{aligned}
\mathcal{F} = \int \Bigg[&\frac{1}{2}\rho v_p^2 + \frac{1}{2}\rho v_\phi^2 + \frac{1}{8\pi}\left|\frac{\nabla\psi}{r}\right|^2 + \frac{1}{8\pi}B_\phi^2 + \rho U \\
&- \frac{1}{4\pi}\left(v_\phi F + \frac{H}{r}\right)B_\phi - \rho J - r\rho v_\phi G - \frac{1}{4\pi}(\mathbf{v}_p\cdot\mathbf{B}_p)F\Bigg] dV
\end{aligned} \qquad (9)$$

and depends on the whole set of variables $\zeta = (\mathbf{v}, B_\phi, \rho, \psi)$. Its variation with respect to these variables yields Eqs. (3) and (5), except for the entropy equation in Eqs. (4) that is an assumption of this formulation.

Substituting the explicit dependence of $(\mathbf{v}, B_\phi)$ in terms of (ρ, ψ) into Eq. (9), the variational principle used in Ref. 13 is recovered

$$\mathcal{F}^* = -\int\left[\frac{M^2-1}{8\pi}\left(\frac{\nabla\psi}{r}\right)^2 + \frac{\mathbf{\Pi}_1}{M^2} + \frac{\mathbf{\Pi}_2}{\gamma M^{2\gamma}} + \frac{\mathbf{\Pi}_3}{M^2-1}\right] dV, \qquad (10)$$

where the dependence on the plasma density is hidden in the Alfvén Mach number and the three arbitrary functions $\mathbf{\Pi}_1 = 4\pi F^2(J + r^2 G/2)$, $\mathbf{\Pi}_2 = \gamma(\gamma-1)^{-1}(4\pi F^2)^\gamma I$, $\mathbf{\Pi}_3 = 2\pi F^2 (r^{-1}H - rG)^2$ are a convenient combination of the five flux functions, Eqs. (4).

In order to avoid the apparent singularity in Eq. (10) for $M=1$, a less general reduction is possible which represents the first result of the present paper. We consider the explicit dependence of $(\mathbf{v}_p, B_\phi)$ in terms of (v_ϕ, ρ, ψ) and substitute these into Eq. (9), thus considering v_ϕ as an additional independent function in the variational formulation. The solutions of Eq. (5) with the two implicit conditions given by the Bernoulli equation and by Eq. (6) are obtained by finding the extrema of the functional

$$\mathcal{L}(v_\phi,\rho,\psi) = -\int_\Omega l(\mathbf{x},v_\phi,\rho,\psi,\nabla\psi)dV, \tag{11}$$

where

$$l = \left(\frac{F^2}{4\pi\rho}-1\right)\frac{1}{8\pi}\left(\frac{\nabla\psi}{r}\right)^2 + \frac{1}{8\pi}\left(\frac{H+rv_\phi F}{r}\right)^2 - \frac{1}{2}\rho v_\phi^2 + \rho(J+rv_\phi G) - \frac{\rho^\gamma I}{\gamma-1}. \tag{12}$$

In this variational approach, where v_ϕ is kept as an independent variable, the additional condition $H=-r^2FG$ need not be imposed at the Alfvén surface. In fact, when the extremum is found for a regular function v_ϕ, this condition is automatically satisfied since the Euler–Lagrange equation obtained by varying the functional Eq. (12) with respect to v_ϕ is Eq. (6).

III. BOUNDARY CONDITIONS

In order to complete the description of axisymmetric MHD equilibria, boundary conditions for the magnetic flux function are needed. A main distinction can be made between systems in which the boundary consists of a fixed magnetic flux surface, ψ=const on $\partial\Omega$ (as is the case for equilibria in fusion experiments inside a closed domain Ω with boundary $\partial\Omega$) and systems in which the magnetic field lines are free to intersect a part of the boundary (as is the case for plasma thrusters and astrophysical jets). In the latter case a further distinction is possible. Let us call the region of $\partial\Omega$ where the magnetic field $\mathbf{B}$ intersects the boundary $\partial_1\Omega$. In this region we can assume that the magnetic field lines are either tied or free to move; for tied magnetic field lines the condition $\psi=\psi(s)$ holds on $\partial_1\Omega(s)$, where $\partial_1\Omega(s)$ is a curvilinear parameterization of the boundary $\partial_1\Omega$ and $\psi(s)$ is a given function of the boundary arc length s. On the contrary, in the case of free magnetic lines the physical problem leads to *natural* boundary conditions. These conditions follow directly from the variational approach. The first variation of the functional Eq. (11) yields

$$\delta\mathcal{L} = \int_\Omega [l]_\psi \delta\psi dV + \int_\Omega [l]_\rho \delta\rho dV + \int_\Omega [l]_{v_\varphi}\delta v_\varphi dV + \int_{\partial_1\Omega}\left(l_{\psi_z}\frac{dr}{ds} - l_{\psi_r}\frac{dz}{ds}\right)\delta\psi d\Sigma, \tag{13}$$

where $[l]_\psi$, $[l]_\rho$, $[l]_{v_\varphi}$ represent the Euler–Lagrange equations [Eqs. (5) and (6) and the expression for $J(\psi)$] associated with the three variables of our formulation. In order to find an extremum also, the surface integral in Eq. (13), where $d\Sigma$ is the surface element, must vanish. If the class of admissible functions is chosen such that the variations of ψ are constrained to vanish at the boundaries this term is identically zero. If not, for this term to vanish, the natural boundary condition must hold

$$\frac{\partial l(\mathbf{x},v_\varphi,\rho,\psi,\nabla\psi)}{\partial(\partial\psi/\partial r)}\frac{dz}{ds} - \frac{\partial l(\mathbf{x},v_\varphi,\rho,\psi,\nabla\psi)}{\partial(\partial\psi/\partial z)}\frac{dr}{ds} = 0, \tag{14}$$

which depends on the functional to be extremized and thus on the specific physical problem. The natural boundary conditions do not restrict the class of admissible functions so that a direct variational solution will satisfy these conditions automatically and thus there is no need to impose them, e.g., in a numerical procedure.

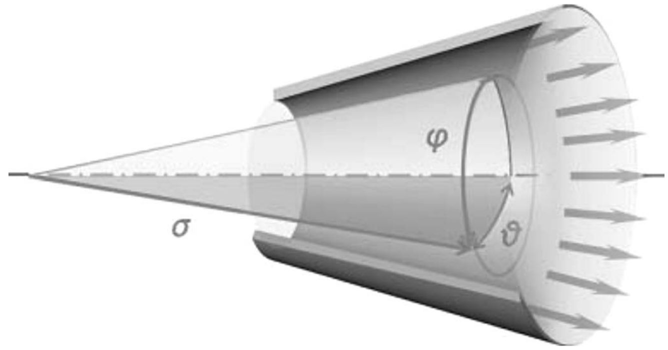

FIG. 1. A conic nozzle and the spherical coordinate system used to describe its geometry.

A simple representation of an open geometry configuration is the conic magnetic nozzle shown in Fig. 1. Configurations closed to conic geometry have been adopted in aerospace propulsion systems based on magneto plasma dynamics to obtain an efficient plasma acceleration (a case description can be found in Ref. 15). Due to its technological interest and to the simplicity of conic geometry, the conic nozzle will be used as a test case in order to illustrate the next steps in the present investigation.

In particular, conic geometry can be easily described using spherical coordinates (σ,θ,ϕ), as illustrated in Fig. 1, where the domain is defined as $\sigma\in[\sigma_{\text{inlet}},\sigma_{\text{outlet}}]$ and $\theta\in[0,\theta_{\text{wall}}]$.

As a consequence of the definition of the flux function ψ, we can now assume the Dirichlet condition on the nozzle's wall and on the symmetry axis whereas in the open parts of the boundary we shall prescribe the natural conditions. By inserting into Eq. (14) the function l given in Eq. (12) we obtain the expression of the natural condition for the MHD problem,

$$\frac{1}{r^2}\left(\frac{F^2}{4\pi\rho}-1\right)\nabla\psi\cdot\mathbf{n}=0. \tag{15}$$

In the open parts of the boundary, where we assume the natural condition to hold, this equation yields, for $M\neq 1$, $\nabla\psi\cdot\mathbf{n}=0$. Note that while the boundary conditions of the form ψ=const on $\partial\Omega$ or $\psi=\psi(s)$ on $\partial_1\Omega(s)$ are related to the normal component of $\mathbf{B}$, the natural conditions depend on its tangential component $\nabla\psi\cdot\mathbf{n}$.

If required by physical conditions, more general boundary conditions can be obtained adding appropriate surface terms to the variational functional Eq. (12).

IV. DISCONTINUOUS SOLUTIONS

The application of the variational principle (11) and of the boundary conditions discussed in Sec. III needs be reexamined when Eq. (5) is hyperbolic. The coefficient $\nabla\cdot[(1-M^2)\nabla\psi/r^2]$ of the highest derivative in Eq. (5) depends on the square of the Mach number M^2, where M^2 is the solution of the Bernoulli equation and, hence, a function of $\nabla\psi$. As detailed in Refs. 5 and 13, Eq. (5) is hyperbolic when

$$(M^2-M_s^2)(M^2-M_f^2)/(M^2-M_c^2)>0, \tag{16}$$

where $M_{f,s}^2\equiv[(4\pi\gamma p+B^2)/2B_p^2]\{1\pm[1-16\pi\gamma pB_p^2/(4\pi\gamma p+B^2)^2]^{1/2}\}$ correspond to the fast and slow magnetosonic velocities and $M_c^2\equiv\gamma p/(\gamma p+B^2/4\pi)$ to the cusp front velocity. Hyperbolicity occurs for $M_c^2<M^2<M_s^2$ (slow regime) and $M^2>M_f^2$ (fast regime).

As described in the previous section, boundary conditions for the variational approach are assigned in each part of the boundary and, in general, they are not compatible with hyperbolic regimes. However, if we consider adiabatic instead of isentropic plasma flows, discontinuous solutions that admit hyperbolic regimes are possible. In order to satisfy the prescribed boundary condition the solution must pass from hyperbolic to elliptic and, since an entropy jump is admissible, this transition occurs as a "shock," i.e., the fluid properties must change discontinuously through a shock surface. These solutions are characterized by the value of the entropy jump ΔS and by the position of the shock surface (a detailed description of MHD shocks can be found in Ref. 16).

Let us assume that there exists a discontinuity surface λ (a curve in the poloidal plane) which divides the domain into two parts: Ω_1 and Ω_2. Thus, given a generic function $\chi(\mathbf{x})$, we define as $[\![\chi]\!]$ the value of its jump through the surface λ at a generic point of λ. The mass conservation through the discontinuity can be rewritten as

$$[\![\rho v_n]\!]=0, \tag{17}$$

where the subscript n denotes the component normal to the surface. Similar results follow from the remaining Eqs. (1), aside from the entropy equation that must be substituted by the equation for energy conservation

$$\int\rho v_n(U+v^2/2)d\Sigma+\int pv_n d\Sigma+(c/4\pi)\int\mathbf{E}\times\mathbf{B}\cdot\mathbf{n}d\Sigma=0, \tag{18}$$

where $\mathbf{n}$ is the unit vector normal to the surface λ and $d\Sigma$ is its surface element. In this way we obtain, together with Eq. (17), a number of jump conditions equal to the number of conservation equations,

$$\left[\!\!\left[v_n\mathbf{v}+p\mathbf{n}+\frac{B^2}{8\pi}\mathbf{n}-\frac{B_n}{4\pi}\mathbf{B}\right]\!\!\right]=0, \tag{19}$$

which shows the conservation of momentum and

$$\left[\!\!\left[H_E+\frac{v^2}{2}+\frac{B^2}{4\pi\rho}-(\mathbf{v}\cdot\mathbf{B})\frac{B_n}{4\pi\rho v_n}\right]\!\!\right]=0, \tag{20}$$

which shows the conservation of energy and follows from Eq. (18) where the enthalpy per unit mass H_E has been used together with the continuity of the mass flow, while the Maxwell equations for the magnetic field yield

$$[\![\mathbf{B}\cdot\mathbf{n}]\!]=0,\quad \mathbf{n}\times[\![\mathbf{v}\times\mathbf{B}]\!]=\mathbf{0}. \tag{21}$$

The system of Eqs. (17) and (19)–(21) defines the jump conditions for a plasma flow through an entropy discontinuity. This system can also be derived from the variational formulation of the problem. The same jump conditions are indeed obtained by looking for the extrema of the functional Eq. (11) with ψ continuous and v_ϕ, ρ, and $\nabla\psi$ piecewise continuous. The continuity of the stream function ψ yields the continuity of the normal component of the magnetic field, $[\![B_n]\!]=0$ and also implies the continuity of the flux functions $F(\psi)$, $G(\psi)$, $H(\psi)$, $J(\psi)$ in Eqs. (4), i.e., with the exception of $I(\psi)$ which depends on the entropy per unit mass S.

By considering an arbitrary discontinuity surface λ, Eq. (12) can be rewritten as

$$\mathcal{L}(v_\phi,\rho,\psi)=\int_{\Omega_1}l_1(\mathbf{x},v_\phi,\rho,\psi,\nabla\psi)dV+\int_{\Omega_2}l_2(\mathbf{x},v_\phi,\rho,\psi,\nabla\psi)dV, \tag{22}$$

where the only difference between l_1 and l_2 is in the entropy term. In the two subdomains v_ϕ, ρ, and $\nabla\psi$ are continuous, hence the solution satisfies the classical Euler–Lagrange equations associated with the variational principle. However, variations of the flux function on the shock surface yield

$$[\![l_{\psi_r}]\!]n_r+[\![l_{\psi_z}]\!]n_z=0, \tag{23}$$

for each point on λ, while the variation induced by the arbitrariness of the shock surface gives one more condition in the form

$$[\![l]\!]=l_{\psi_r}|_1[\![\psi_r]\!]+l_{\psi_z}|_1[\![\psi_z]\!], \tag{24}$$

where the index 1 means that the expression is evaluated on the Ω_1 side. In fact, substituting the expression (12) in Eq. (23) we obtain the tangential component of Eq. (19). Due to expression (12), Eq. (24) reduces to

$$[\![l]\!]=\left(\frac{F^2}{4\pi\rho}-1\right)_1\left[\frac{1}{r^2}\frac{\partial\psi}{\partial z}\bigg|_1\left[\!\!\left[\frac{\partial\psi}{\partial z}\right]\!\!\right]+\frac{1}{r^2}\frac{\partial\psi}{\partial r}\bigg|_1\left[\!\!\left[\frac{\partial\psi}{\partial r}\right]\!\!\right]\right] \tag{25}$$

that yields the remaining equation, i.e., the component of Eq. (19) normal to the shock surface.

So, the problem can be equivalently solved using the differential approach, with the boundary conditions and the shock conditions like Eq. (17), or finding an extremum of the variational principle amid piecewise-continuous solutions for v_ϕ, ρ, and $\nabla\psi$. From a numerical point of view it is thus

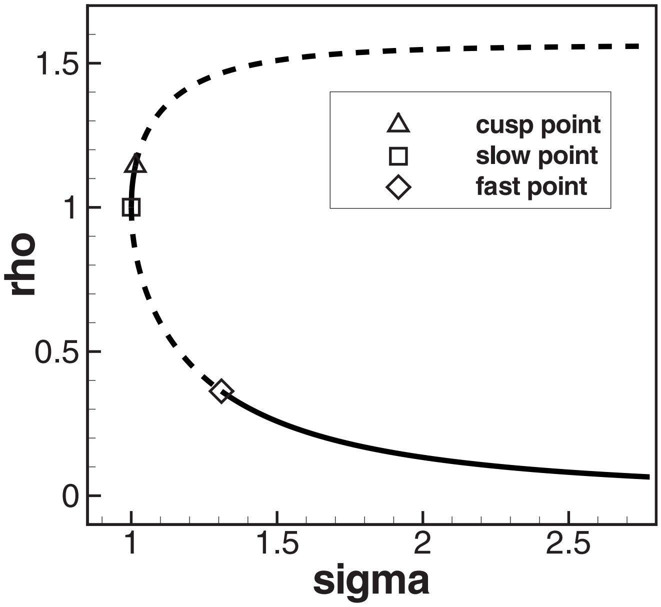

FIG. 2. Density solution and transition points. Solid lines indicate the hyperbolic regions while dotted lines are used for the elliptic ones. The curve is plotted in dimensionless units.

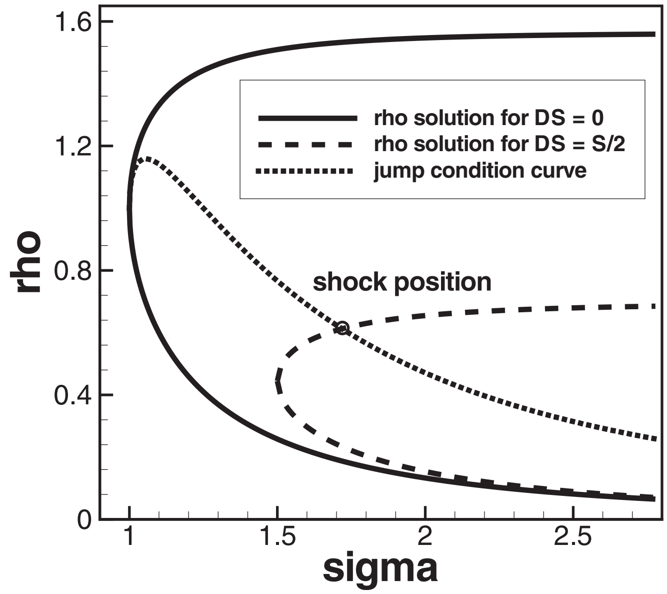

FIG. 3. Behavior of two density solutions with different entropy (the jump conditions are also shown that lead to a discontinuous solution).

possible to model magnetohydrodynamic flows with shock surfaces within the variational theory by including the position of the shock among the unknowns.

V. NUMERICAL PROCEDURE AND VALIDATION RESULTS

After the development of the theoretical model, we apply a numerical procedure that permits a complete description of plasma flow and the achievement of a reasonable understanding of the acceleration processes. Exploiting the variational formulation of the problem, the numerical approach is based on Ritz's method, i.e., the extremum is confined in a finite-dimensional functions subspace and the solution is obtained through a system of nonlinear algebraic equations. By rescaling the variational functional it is first possible to make all the equations dimensionless. Next, we search for an approximation of ψ, ρ, and v_ϕ within a finite dimensional function subspace and express ψ and ρ as finite sums of base functions. Thus we write

$$\psi(x,y)=\sum_{n=0}^{n_T}\psi_n\mathcal{F}_n(x,y),\quad \rho(x,y)=\sum_{m=0}^{m_T}\rho_m\mathcal{G}_m(x,y),$$
$$v_\phi(x,y)=\sum_{l=0}^{l_T}v_{\phi l}\mathcal{H}_l(x,y), \tag{26}$$

where $\mathcal{F}_n$, $\mathcal{G}_m$, and $\mathcal{H}_l$ are three families of base functions and n_T, m_T, and l_T are the number at which each series expansion has been truncated. By substituting these expressions into Eq. (12), the extremization process can be performed by differentiating the variational functional L with respect to the three sets of coefficients $\{\psi_n,\rho_m,v_{\phi l}\}$,

$$\frac{\partial L}{\partial\psi_n}=\int_\Omega\frac{\partial\mathcal{L}}{\partial\psi_n}dV,\quad \frac{\partial L}{\partial\rho_m}=\int_\Omega\frac{\partial\mathcal{L}}{\partial\rho_m}dV,$$
$$\frac{\partial L}{\partial v_{\phi l}}=\int_\Omega\frac{\partial\mathcal{L}}{\partial v_{\phi l}}dV. \tag{27}$$

The coefficients of the approximate solution are obtained by solving the nonlinear algebraic equation system

$$\frac{\partial L}{\partial\psi_0}=\cdots=\frac{\partial L}{\partial\psi_{n_T}}=\frac{\partial L}{\partial\rho_0}=\cdots=\frac{\partial L}{\partial\rho_{m_T}}=\frac{\partial L}{\partial v_{\phi 0}}=\cdots$$
$$=\frac{\partial L}{\partial v_{\phi l_T}}=0. \tag{28}$$

Since each derivative (27) is a smooth function of the unknown coefficients and can be easily differentiated with respect to these coefficients, the equation system (28) is solved by using the Newton–Raphson algorithm.

A simple solution is used to validate the procedure. For uniform inlet conditions ($\mathbf{B}\cdot\mathbf{n}|_i=B_0$, $\mathbf{v}\cdot\mathbf{n}|_i=V_0$, $\rho|_i=\rho_0$, and $p|_i=p_0$), zero azimuthal velocity ($v_\phi=0$), and zero azimuthal magnetic field ($B_\phi=0$), it is possible to obtain an analytic solution of the conical nozzle problem. First we deduce the five stream functions $[F,G,H,I,J]$ from the inlet conditions: the ratio between the mass flow rate and the magnetic field in the poloidal plane gives $F(\psi)=4\pi\rho_0V_0/B_0=F_0$; the zero azimuthal velocity and azimuthal magnetic field imply $G(\psi)=0$ and $H(\psi)=0$; the Bernoulli equation written at the inlet surface leads to $J(\psi)=V_0^2/2+\gamma(\gamma-1)^{-1}p_0/\rho_0=J_0$; the flux entropy is uniform and the related function is $I(\psi)=p_0/\rho_0^\gamma$. By substituting these functional dependencies on the right-hand side of Eq. (5), we obtain in spherical coordinates,

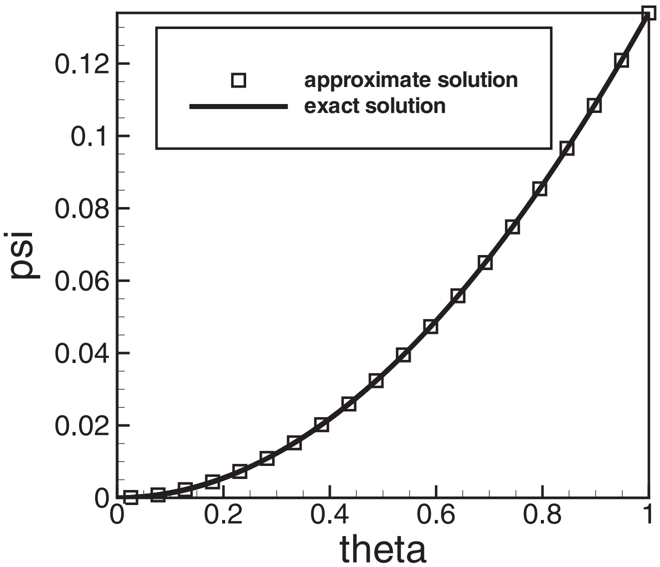

FIG. 4. Exact and approximate stream function solutions for the test case, plotted in dimensionless units.

$$\frac{\partial}{\partial\sigma}\left[\left(\frac{F_0^2}{4\pi\rho}-1\right)\frac{\partial\psi}{\partial\sigma}\right]+\frac{\sin\theta}{\sigma^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\left(\frac{F_0^2}{4\pi\rho}-1\right)\frac{\partial\psi}{\partial\theta}\right]$$
$$=0,$$

where the density ρ is given by the algebraic Bernoulli equation

$$\frac{1}{2}\left(\frac{F_0}{\sigma\sin\theta}\frac{\nabla\psi}{4\pi\rho}\right)^2+\frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0^\gamma}\rho^{\gamma-1}=J_0. \tag{29}$$

Considering a stream function of the form

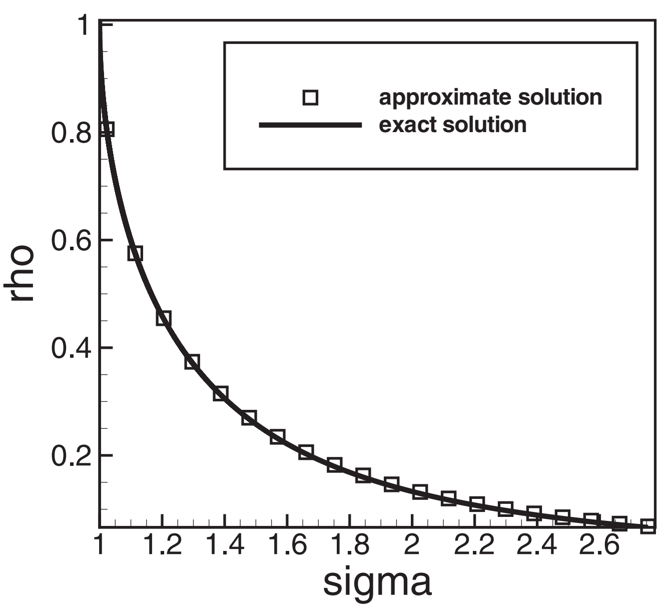

FIG. 5. Exact and approximate density solutions for the test case, plotted in the dimensionless unit.

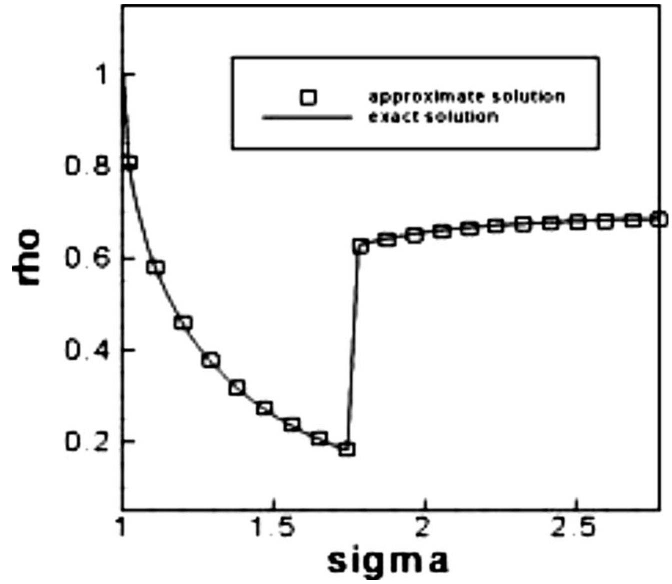

FIG. 6. ρ solution with shock, sonic inlet conditions, and $\Delta S=S/2$.

$$\psi=\sigma_0^2B_0[1-\cos\theta] \tag{30}$$

and inserting it into Eq. (29), we find that the density depends only on the spherical radius, $\rho=\rho(\sigma)$, and obeys the polynomial equation

$$(\rho/\rho_0)^{\gamma+1}-(K+1)(\rho/\rho_0)^2+K(\sigma/\sigma_0)^{-4}=0, \tag{31}$$

with K defined by $K=(\gamma-1)\rho_0V_0^2/(2\gamma p_0)$. Equation (31) can be interpolated numerically and the result can be compared with that obtained from the variational algorithm. The flow velocity in the inlet region is taken to be equal to the speed of sound, c_s, and the flow after it is assumed to be supersonic. Figure 2 shows the solution of Eq. (31): both the supersonic branch (where the density decreases with the radius) and the subsonic branch (where the density increases with the radius) of the solution are represented, the dimensionless

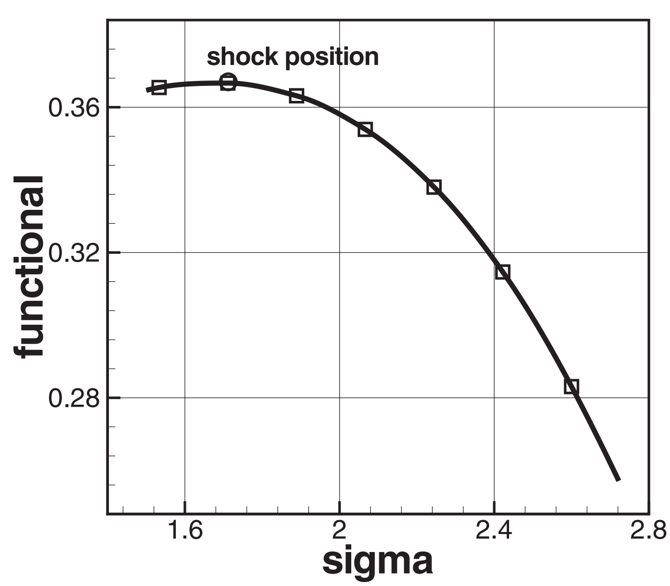

FIG. 7. Functional Eq. (12) values evaluated at different shock positions for $\Delta S=S/2$.

density ρ/ρ_0 is plotted against the dimensionless radius σ/σ_0 and the three transition points described in Sec. IV are marked. In fact, by substituting the expressions for ρ and ψ in the definition of the Alfvén–Mach numbers, the hyperbolicity condition (16) can be solved: for $M=M_c$ we obtain the *cusp* transition point, σ_c; since $M_s=c_s/v_A$, in this case the *slow* transition point corresponds to the inlet radius, $\sigma_s=\sigma_0$; for $M=M_f=1$ we obtain the *fast* transition radius, σ_f. The different regions defined by the transition points are also shown in Fig. 2.

Considering an entropy jump $\Delta S=S/2$ inside the nozzle domain, the flux function ψ defined by Eq. (30) is still a solution of the problem while Fig. 3 shows the density profiles associated with S and $S+\Delta S$. A hydrodynamic shock, i.e., a shock in which the magnetic field is normal to the shock surface on both sides of the discontinuity (see Ref. 16), between the two density solutions with different entropies is allowed. In Fig. 3 the hydrodynamic jump condition curve is also plotted, which represents the value of the density after an hydrodynamic shock depending on the shock position itself. The crossing point of the jump condition curve with the density curve at higher entropy gives the shock position for the assigned entropy jump ΔS. As follows from the entropy condition, an admissible jump ($\Delta S>0$) is possible only on the lower (supersonic) branch of the density solution. However, it is still possible to distinguish between fast and slow shocks depending whether the shock radius is greater than σ_f or not.

The numerical results for the simple problem described above appear satisfactory. The fields are the two unknown functions (ψ,ρ). The comparison between the exact solution and our approximated results is good, as can be seen in Figs. 4–6. Figure 4 shows the flux function solution, Fig. 5 shows the density solution with no entropy discontinuity, while Fig. 6 shows the density solution for the case in which an entropy jump $\Delta S=S/2$ is allowed. For this latter case, the behavior of the variational functional (12) with respect to the shock position is shown in Fig. 7, which allows us to locate the position of the shock.

VI. CONCLUSIONS

A magnetohydrodynamical model of an axially symmetric, stationary plasma flow is discussed, its features are presented, and a variational principle for solving this model is formulated. In particular, it is shown that open boundary conditions and discontinuous solutions can be easily included in the variational treatment, so that this formulation can be used to investigate a wide class of physical phenomena and technological problems. From this variational formulation a numerical procedure is derived and tests have been successfully performed on simple cases.

The formulation and the procedure described above can provide substantial help in the description of plasma acceleration processes in the magnetohydrodynamic regime and will allow us to investigate different plasma flows beyond the simple cases discussed in the present work. At the same time, the Hamiltonian structure of the model, exploited in the present formulation, deserves an in-depth study in order to characterize the stability of the solution. Both the development of this theoretical framework and the investigation of different flow patterns and of their stability are currently the object of our ongoing research.

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