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Magnetic field homogeneity of a conical coaxial coil pair

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An analytical study of the magnetic field created by a double-conical conducting sheet is presented. The analysis is based on the expansion of the magnetic field in terms of Legendre polynomials. It is demonstrated analytically that the angle of the conical surface that produces a nearly homogeneous magnetic field coincides with that of a pair of loops that fulfills the Helmholtz condition. From the results obtained, we propose an electric circuit formed by pairs of isolated conducting loops tightly wound around a pair of conical surfaces, calculating numerically the magnetic field produced by this system and its heterogeneity. An experimental setup of the proposed circuit was constructed and its magnetic field was measured. The results were compared with those obtained by numerical calculation, finding a good agreement. The numerical results demonstrate a significant improvement in homogeneity in the field of the proposed pair of conical coils compared with that achieved with a simple pair of Helmholtz loops or with a double solenoid. Moreover, a new design of a double pair of conical coils based on Braunbek's four loops is also proposed to achieve greater homogeneity. Regarding homogeneity, the rating of the analyzed configurations from best to worst is as follows: (1) double pair of conical coils, (2) pair of conical coils, (3) Braunbek's four loops, (4) Helmholtz pair, and (5) solenoid pair. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.5002572]

I. INTRODUCTION

The importance of achieving a homogeneous magnetic field in a spatial region is well known in physics as in other scientific branches such as biology and medicine.[1] For almost two centuries, it has motivated the study of suitable devices that achieve the greatest homogeneity. In this context, the most analyzed electrical current systems have been those with axial symmetry.

There have been many studies on this kind of system of which Refs. 2–5 are notable among others. In a comprehensive work by Garrett,[2] source systems have included circular filaments, circular current sheets, and thin and thick solenoids. The central uniformity of the symmetrical fields and gradients is analyzed using the zonal harmonic expansion. The theoretical results have been applied to different loop combinations, including eighth-order systems. Franzen[3] used Garrett's theory of axial fields to find the optimum shape of a coil cross section. By choosing optimal dimensions, he showed that relatively thick coils can generate magnetic fields with the same uniformity as that obtained with coils of infinitesimal cross section. In Ref. 5, a multi-pole formulation of the magnetic field, generated by axially symmetric fields, spherical windings, and coaxial coils, is presented with the aim of obtaining maximum homogeneity. Highly uniform magnetic fields are obtained if a large number of the low-order derivatives of the magnetic field with respect to the axial and radial coordinates vanish. Numerical results are provided for an open spherical winding with different opening angles, for systems containing up to ten coils lying on the surface of a sphere (twentieth-order), and for simple eight-order winding combinations of pair of loops with either a solenoid or a spherical winding.

Among other studies of a more restricted type, we mention Ref. 6, in which, using numerical calculations, different current loop systems are rigorously compared, including Helmholtz coils with two loops, the Maxwell system with three loops, and those of Garrett, Barker, and Braunbek with four loops. From the comparison, the eighth-order Braunbek loop system was found to provide an optimal configuration. In other studies,[7,8] a mirror-symmetric system of several pairs of coils with the same radius, each located at a given distance and with a suitable number of turns, are studied. The optimal arrangement with eight loops was calculated from the condition that the magnetic field along the axis has the degree of uniformity desired. The coil system described can be used for calibrating magnetometers. Wang *et al.*[9] studied the homogeneity of the magnetic field created by three loops of the same radius, improving the homogeneity of Maxwell's arrangement by diminishing the value of the sixth derivative of the magnetic field B with respect to z at the center of symmetry.

Other studies differ considerably from the above. In Ref. 10, a system without symmetry of revolution was analyzed, showing that the field profile of square Helmholtz coils had better field homogeneity than that of a circular coil and provided a practical design for the best combination based on field homogeneity. The objective was to perform magnetic measurements of larger high-temperature superconducting tapes. In Ref. 11, a very precise calculation of the magnetic field near the geometric center of a pair of Helmholtz coils is presented. They also considered various detailed effects related to construction such as winding magnetic properties, imperfections, and conductor insulation.

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In all these articles, the magnetic field is generated from free currents in vacuum, whereas in Ref. 12, the study refers to the homogeneity of the magnetic field B generated by Helmholtz coils inside an ideal ferromagnetic shield. The results show that a high uniformity is obtained for B within the space surrounding the central axis.

Recent work has focused on the generation of three-dimensional uniform magnetic fields. As Helmholtz coils are frequently used as uniform magnetic field generators, Beiranvand[13] studied the effects of assembly misalignments and manufacturing mismatches on the uniformity of the generated field in a tri-axial Helmholtz coil set. Chen *et al.*[14] prototyped a three-dimensional field generator with two coil pairs in each direction, obtained by optimizing the design parameters of the generator. A comparison with Helmholtz coils showed a larger uniform region and a smaller volume.

The purpose of this work is to analyze a simple configuration as a source of a homogeneous magnetic field and determine the optimal design. An initial proposal based on a surface conical current has already been made in Ref. 15. In the present work, using analytical calculations, the value of the magnetic field generated by the current in a pair of conical sheets is obtained and the aperture of the cone that provides greater homogeneity is also deduced. Further, an electric circuit formed by N pairs of windings of a conducting wire tightly wound on a pair of insulating cones with an optimum aperture is proposed. The magnetic field created by this configuration is then calculated using numerical methods. This new setup was constructed and both the magnetic field generated and its homogeneity were examined in laboratory experiments. The homogeneities achieved by several known circuits are compared with that of the designed conical system. We also demonstrate that the actual Helmholtz setup commonly used does not produce a magnetic field as homogeneous as that obtained with the system proposed.

II. MAGNETIC FIELD OF A SURFACE CURRENT DENSITY ON A TRUNCATED CONE

Consider a conducting sheet with a truncated conical shape (Fig. 1); its generatrix forms an angle α with the axis of revolution labeled as the OZ axis. The sheet carries a surface current density $\mathbf{j}_s = j_s \mathbf{u}_\phi$.

Taking into account its symmetry, the potential vector of such a system has only a tangential component A_ϕ. Therefore, the expression for this potential in spherical coordinates is

$$A_\phi = \frac{\mu_0 j_s}{4\pi} \int_0^{2\pi} \int_{r'_2}^{r'_2} \frac{r' \sin\alpha \cos\phi' dr' d\phi'}{\sqrt{r'^2 + r^2 - 2rr' \cos\gamma}}, \tag{1}$$

where r and r' are the distances from the origin of the coordinates to the field and source points, respectively, and γ is the angle formed by r and r' (depending on α, ϕ, ϕ', and θ). We are interested in the field near the origin (0, 0, 0), where $r/r' < 1$. Expanding the square root in powers of (r/r'), Eq. (1) gives

$$A_\phi = \frac{\mu_0 j_s \sin\alpha}{4\pi} \int_0^{2\pi} \int_{r'_1}^{r'_2} dr' d\phi' \cos\phi' \sum_0^\infty \left(\frac{r}{r'}\right)^n P_n(\cos\gamma), \tag{2}$$

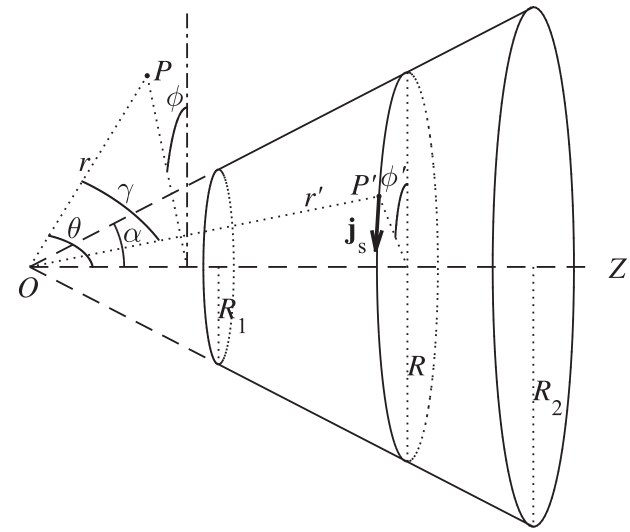

FIG. 1. Conducting conical sheet showing the geometric quantities involved to obtain the expression for the magnetic field generated by current density $\mathbf{j}_s$.

where P_n are the Legendre polynomials. Applying the addition theorem to $P_n(\cos\gamma)$,[16] integrating over ϕ', setting $\phi = 0$, and taking into account that the integral of $\cos\phi'$ from 0 to 2π is zero, we have

$$A_\phi = \frac{\mu_0 j_s \sin\alpha}{2} \int_{r'_1}^{r'_2} dr' \sum_{n=1}^\infty \left(\frac{r}{r'}\right)^n \frac{(n-1)!}{(n+1)!} P_n^1(\cos\alpha) P_n^1(\cos\theta), \tag{3}$$

where P_n^1 are the associate Legendre polynomials. By taking the curl of the potential vector A_ϕ, we obtain the magnetic field components

$$B_{1r} = \frac{\mu_0 j_s}{2} \sin^2\alpha \cos\theta \ln\left(\frac{R_2}{R_1}\right), \tag{4}$$

$$B_{2r} = \frac{3\mu_0 j_s}{4} \cos\alpha \sin^3\alpha \times \left(2\cos^2\theta - \sin^2\theta\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) r, \tag{5}$$

$$B_{3r} = \frac{3\mu_0 j_s}{32} \sin^4\alpha \left(5\cos^2\alpha - 1\right)\cos\theta \times \left[5\left(\cos^2\theta - \sin^2\theta\right) - 1\right]\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right) r^2, \tag{6}$$

$$B_{4r} = \frac{5\mu_0 j_s}{96} \sin^5\alpha \cos\alpha \left[(7\cos^2\theta - 3)(2\cos^2\theta - \sin^2\theta) - 14\cos^2\theta \sin^2\theta\right](7\cos^2\alpha - 3)\left(\frac{1}{R_1^3} - \frac{1}{R_2^3}\right) r^3, \tag{7}$$

$$B_{5r} = \frac{15\mu_0 j_s}{512} \sin^6\alpha (21\cos^4\alpha - 14\cos^2\alpha + 1)\cos\theta \times \left[21\cos^4\theta - 14\cos^2\theta + 14\sin^2\theta \times (1 - 3\cos^2\theta) + 1\right]\left(\frac{1}{R_1^4} - \frac{1}{R_2^4}\right) r^4, \tag{8}$$

where B_{ir} represents the radial component of the magnetic field corresponding to the i-term of the expansion of the magnetic field in powers of r.

By again using A_ϕ, the terms of the development corresponding to the component θ of the magnetic field are

$$B_{1\theta} = -\frac{\mu_0 j_s}{2}\sin^2\alpha\sin\theta\ln\left(\frac{R_2}{R_1}\right), \tag{9}$$

$$B_{2\theta} = -\frac{9\mu_0 j_s}{4}\cos\alpha\sin^3\alpha\cos\theta\sin\theta\left(\frac{1}{R_1}-\frac{1}{R_2}\right)r, \tag{10}$$

$$B_{3\theta} = -\frac{3\mu_0 j_s}{16}\sin^4\alpha\sin\theta\left(5\cos^2\alpha-1\right) \times(5\cos^2\theta-1)\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)r^2, \tag{11}$$

$$B_{4\theta} = -\frac{25\mu_0 j_s}{96}\sin^5\alpha\cos\alpha\sin\theta\cos\theta \times(7\cos^2\alpha-3)(7\cos^2\theta-3)\left(\frac{1}{R_1^3}-\frac{1}{R_2^3}\right)r^3, \tag{12}$$

$$B_{5\theta} = -\frac{45\mu_0 j_s}{512}\sin^6\alpha\sin\theta(21\cos^4\alpha-14\cos^2\alpha+1) \times(21\cos^4\theta-14\cos^2\theta+1)\left(\frac{1}{R_1^4}-\frac{1}{R_2^4}\right)r^4. \tag{13}$$

Once we have obtained the magnetic field **B** produced by the conducting conical sheet, the calculation of the field produced by another current, mirror-symmetric with respect to the $z = 0$ plane, is straightforward. Indeed, as the angle formed by its generatrix with axis OZ is $\pi - \alpha$, we obtain the following radial components of the field:

$$B_{1r}^T = \mu_0 j_s\sin^2\alpha\cos\theta\ln\left(\frac{R_2}{R_1}\right), \tag{14}$$

$$B_{2r}^T = 0, \tag{15}$$

$$B_{3r}^T = \frac{3\mu_0 j_s}{16}\sin^4\alpha\left(5\cos^2\alpha-1\right)\cos\theta \times(5\cos 2\theta-1)\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)r^2, \tag{16}$$

$$B_{4r}^T = 0, \tag{17}$$

$$B_{5r}^T = \frac{15\mu_0 j_s}{256}\sin^6\alpha(21\cos^4\alpha-14\cos^2\alpha+1)\cos\theta \times\left[21\cos^4\theta-14\cos^2\theta+14\sin^2\theta \times(1-3\cos^2\theta)+1\right]\left(\frac{1}{R_1^4}-\frac{1}{R_2^4}\right)r^4, \tag{18}$$

where B_{ir}^T is the total field corresponding to the i-term of the expansion. Examining this result, we see that the second and fourth terms vanish, as expected by symmetry considerations. The third term however vanishes only if $5\cos^2\alpha - 1 = 0$, i.e., $\alpha = \tan^{-1} 2 \approx 63.43°$.

Inspecting Eqs. (9)–(13), $\cos\alpha$ appears as a factor in Eqs. (10) and (12). Therefore the components $B_{i\theta}^T$ are zero for $i = 2$ and $i = 4$. Furthermore, the condition $\alpha = \tan^{-1} 2$ makes $B_{3\theta}^T = 0$. Hence, by constructing conical sheets in such a way that the generatrix forms an inclination of $\alpha = \tan^{-1} 2$, a strong homogeneity in the magnetic field is achieved over a significant volume around the center of symmetry, i.e., the origin.

Another important advantage of this system is that the homogeneity and intensity of the magnetic field increases with cone length.

III. MAGNETIC FIELD GENERATED BY A PAIR OF CIRCULAR LOOPS

The system proposed above has the advantage of not only being able to generate strong homogeneous magnetic fields but also being very simple in terms of mechanical properties. However, in practice, the system has a drawback due to the difficulty in obtaining a uniform surface current density perpendicular to the generatrix of the cone. Rather than generating tangential currents in this way, wires are commonly used through which electric current flows. Therefore, the proposal here involves using a conical sheet that has the above-mentioned advantages but without its major drawback. The system proposed consists of a conical coil that is made by tightly winding an enameled conducting wire around a cone, the resulting coil having its loops close together in the shape of a helix with increasing radius. In the numerical calculations, the coil is considered to be made of a series of circular loops of wire that are wrapped around the cone; if the diameter of the wire is very small, the turns are independent of each other but are very close. We begin the study of the magnetic field generated by a pair of conical coils with that produced by a single pair of circular loops.

Let us consider a system of two coaxial loops carrying the same current I, with their centers on the symmetry axis OZ, equidistant from the coordinate origin, and located at $z = -d$ and $z = d$, respectively (Fig. 2). The subtending angle α of the coils from the origin is given by $\tan\alpha = R/d$.

In Sec. II, a spherical coordinate system was used because the expansion of the components of the magnetic field in terms of Legendre polynomials proved very suitable. However, given the axial symmetry of the coil system, a cylindrical coordinate system is used hereafter. Axial symmetry implies that the tangential component of the magnetic field **B** created by the loops vanishes everywhere. The magnetic field at point $P(\rho, z)$ on a plane that contains the symmetry axis has, in general, non-zero components B_ρ and B_z.

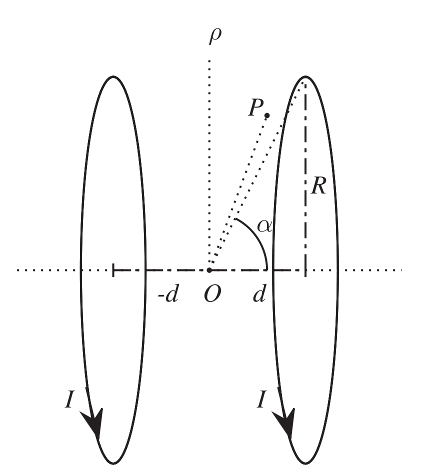

FIG. 2. Pair of circular coaxial loops, symmetrical with respect to the origin of coordinates, showing its geometric characteristics and current intensities.

The magnetic field generated by the current in the circular loop can be calculated using numerical procedures. Among other formulations suitable for the application of the numerical calculation, we have those of Schill[1] and Smythe.[17] From the latter, a pair of loops produces radial and axial field components as follows:

$$B_\rho = \frac{1}{2}\frac{\mu_0 I(z+d)\left[-K(k_+) + \dfrac{(R^2+\rho^2+(z+d)^2)E(k_+)}{(R-\rho)^2+(z+d)^2}\right]}{\pi\rho\sqrt{R^2+2R\rho+\rho^2+(z+d)^2}} + \frac{1}{2}\frac{\mu_0 I(z-d)\left[-K(k_-) + \dfrac{(R^2+\rho^2+(z-d)^2)E(k_-)}{(R-\rho)^2+(z-d)^2}\right]}{\pi\rho\sqrt{R^2+2R\rho+\rho^2+(z-d)^2}}, \tag{19}$$

$$B_z = \frac{1}{2}\frac{\mu_0 I\left[K(k_+) + \dfrac{(R^2-\rho^2-(z+d)^2)E(k_+)}{(R-\rho)^2+(z+d)^2}\right]}{\pi\sqrt{R^2+2R\rho+\rho^2+(z+d)^2}} + \frac{1}{2}\frac{\mu_0 I\left[K(k_-) + \dfrac{(R^2-\rho^2-(z-d)^2)E(k_-)}{(R-\rho)^2+(z-d)^2}\right]}{\pi\sqrt{R^2+2R\rho+\rho^2+(z-d)^2}}, \tag{20}$$

where K and E are the complete elliptic integrals of the first and second kinds, respectively, and

$$k_+ = 2\sqrt{\frac{R\rho}{(R+\rho)^2+(z+d)^2}} \text{ and } k_- = 2\sqrt{\frac{R\rho}{(R+\rho)^2+(z-d)^2}}. \tag{21}$$

The field in the neighborhood of the point O, half-way between the loops, is very nearly uniform, provided that the loops are separated by a distance, $2d$, equal to their common radius, R. The condition $d = R/2$ yields an almost uniform magnetic field. Referred to as the Helmholtz coil configuration, it has been widely used in practice. The subtended angle of the loops from origin O is $\alpha = \tan^{-1} 2 \approx 63.43°$, which coincides with that calculated in Sec. II for the conducting conical sheet.

We calculated the magnetic field at various points $P(\rho, z)$ using Eqs. (19) and (20) in either Maple® or Matlab® code. With $\mu_0 = 4\pi \times 10^{-7}$ m kg s^{-2} A^{-2} and $d = R/2$, then by setting the values of R and I, the only two remaining variables are the coordinates (ρ, z) for the points on a plane, permitting the field components $B_\rho(\rho, z)$ and $B_z(\rho, z)$ to be evaluated given the current in the pair of loops.

Applying the procedure described above to $R = 1$ m and $I = 1$ A yields the set of vectors $\mathbf{B}$, represented by arrows in Fig. 3. At the center, the field components are $B_{z0} = 8.9918 \times 10^{-7}$ T and $B_{\rho 0} = 0$ T. Four circles mark the cross sections of the loops in the plane of the drawing. By simple inspection, a high homogeneity is observed in the magnetic field near the center but is heterogeneous in the region close to the wires. The accessible region, which is of the greatest interest here, is bounded by a cylinder with its two circular ends confined by both loops of the Helmholtz coils and with a length equal to its radius. Its volume is πR^3 and its axial cross section in the plane ρOZ is a rectangle of area $2R^2$.

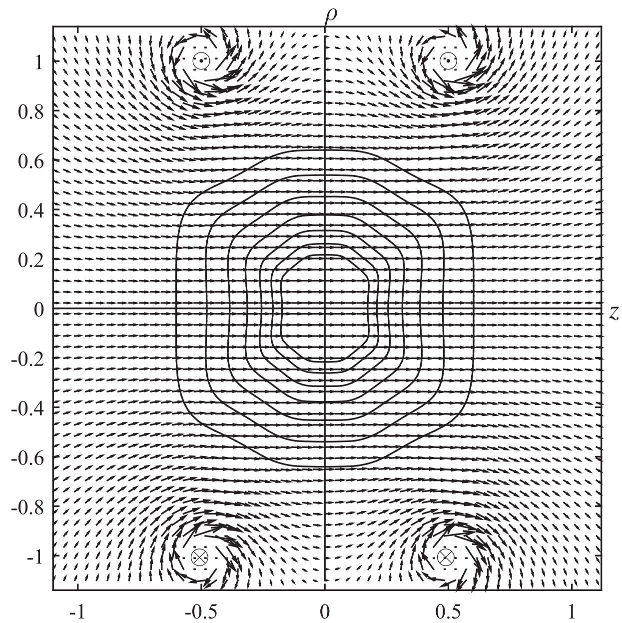

FIG. 3. Magnetic field generated by a Helmholtz pair in the plane ρOZ, showing good homogeneity near the center. From innermost to outermost, the contours indicate equal heterogeneity, for h = 0.100%, 0.215%, 0.464%, 1.000%, 2.154%, 4.642%, and 10.000%, respectively. Distances are expressed in m.

IV. FIELD HETEROGENEITY

The heterogeneity or discrepancy h of the magnetic field $\mathbf{B}$ at an arbitrary point $P(\rho, z)$ can be quantified from the magnitude of the difference between the field vector at $P(\rho, z)$ and that at a point taken as reference. If the reference is the magnetic field $\mathbf{B}_0$ at the center of symmetry, the relative value of the difference $\mathbf{B} - \mathbf{B}_0$ is

$$h \equiv \frac{|\mathbf{B} - \mathbf{B}_0|}{|\mathbf{B}_0|}, \tag{22}$$

which can be used as a definition of heterogeneity;[6] h may also be expressed as a percentage. Figure 3 shows various contour lines of equal heterogeneity h (for the two loops of Fig. 2); these are isotesla contours given for seven values of h up to 10%. For instance, the innermost contour ($h = 0.001$) encloses the set of points whose heterogeneity is less than one per thousand; that is, the magnetic field varies by one per thousand of that obtained at the origin. There is a gradual decrease in field homogeneity with distance from the center. The positions of the pair of symmetrical current loops at $\rho = 1$ m and $z = 0.5$ m (marked by circles) yield a ratio of 2 and hence coincide with the Helmholtz coil condition, $\tan\alpha = 2$.

Figure 3 obtained here can be compared with Fig. 2 by Caprari,[6] both figures being in strong agreement, confirming the accuracy of both calculations. The volume of the spatial region within which homogeneity is smaller than a given value can be calculated along with its fractional value relative to πR^3.

The homogeneity of the magnetic field is independent of the current I, as can be seen by substituting Eqs. (19) and (20) into (22), but is related to the size of the pair of circular loops. This is easily shown if the points of the field are on the axis. Using the expression for the magnetic field along the axis of a circular loop,[18] it follows that for the Helmholtz pair, $h_{\rho=0} \approx 1.1z/R$. Hence, a pair of circular loops with large radii produces a larger area with greater homogeneity, although the magnitude of the magnetic field at the center is smaller.

V. PROPOSED CONICAL CURRENT LOOP SYSTEM TO GENERATE HOMOGENEOUS MAGNETIC FIELDS

From the analytical calculations presented in Sec. II, the conical electric currents produce a nearly homogeneous magnetic field, as the resulting magnetic field is zero for terms with subscripts 2, 3, and 4. With the Legendre polynomials defined as coefficients in the Taylor series expansion, the proposed conical system can be considered as fourth-order because our fifth term corresponds to the fourth derivative. The conical configuration is to a degree an improved version of the Helmholtz coils as this new configuration does not have the same geometrical difficulties inherent in the usual two short straight solenoids in the Helmholtz configuration. Helmholtz coils are actually made with $2N$ turns to increase the magnetic field, but the ratio of the distance between each pair of loops to their radius is not equal to one for all loop pairs, producing higher fields but more heterogeneous.

Prototyping a conical arrangement as proposed in Sec. II presents some difficulties. Therefore certain appropriate modifications need to be introduced. Our proposal is to replace the conical conductor with a set of loops wrapped tightly around an insulating cone (Fig. 4). The cone also has the property that the ratio of the radius R of the circle to the distance d of the circle to the apex is a constant. Therefore, if N coaxial loops are tightly wound on a conical surface with a semi-angle α, the quotient R/d takes the same value for all loops. That is, if a pair of loops on the symmetrical cones (Fig. 4) satisfies the Helmholtz condition, then all pairs fulfill the same condition. For the pair of loops with index i, their common radius can be expressed as $R_i = 2d_i$, with $i = 1, 2, \ldots, N$; if the loops of the wire are tightly wound, the distance of loop i to the center of symmetry is $d_i = d_1 + (i - 1)D\cos\alpha = d_1 + 5^{-0.5}(i - 1)D$, where D is the diameter of the conducting wire.

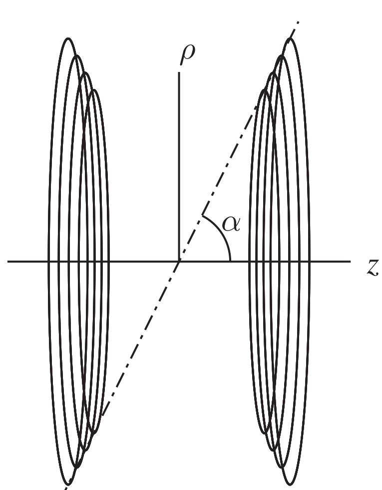

FIG. 4. Electric circuit formed only by four turns on each side and wound on a cone of angle $\alpha \approx 63.43°$.

We next calculate the components B_ρ and B_z of the magnetic field $\mathbf{B}$ generated by the currents in the circular coaxial loops on the conical surfaces. Note that there are $2N$ turns. To this purpose, we begin with the smallest pair of loops using Eqs. (19) and (20) at points in the useful plane, i.e., the region between the first pair of loops. Current I, flowing through the loops in the same direction, is assumed to be known, and the experiment is performed in vacuum. In this way, field values for $\mathbf{B}_{\rho,1}$ and $\mathbf{B}_{z,1}$, corresponding to the smallest pair of loops, are obtained, and the values of h are then calculated at the points in the ρOZ plane.

The process is repeated for the second pair of loops to produce $\mathbf{B}_{\rho,2}$ and $\mathbf{B}_{z,2}$, and so on up to the Nth pair of loops. The addition of the field components of $\mathbf{B}$ for each pair of loops gives the resulting magnetic field $\mathbf{B}$ generated by the conical coaxial circular current loops. The set of N radial and tangential components $\mathbf{B}_{\rho i}$ and $\mathbf{B}_{zi}$ can be placed in a matrix array, which helps in calculating the total magnetic field of the $2N$ turns.

The square root of the square of the difference $B_z - B_{z0}$, element by element, plus the square of B_ρ, divided by B_{z0}, element by element, give the heterogeneity of $\mathbf{B}$ at points in the region of interest, and the result is represented by matrix $\mathbf{h}$.

VI. EXPERIMENTAL LAYOUT

To perform the experiment, two truncated cones were constructed from aluminum sheets with aperture given by the Helmholtz condition, i.e., $\alpha \approx 63.43°$. The cones were mounted symmetrically on a structure also made of aluminum and supported on a granite anti-vibration table to avoid perturbations of the magnetic field generated by ferromagnetic materials, as shown in Fig. 5. Double-sided thin adhesive tape was adhered to the aluminum cones. Then, the enameled copper wire of diameter $D = 0.73$ mm was wound around each cone, the number of turns being $N = 179$. Other geometric features include the smallest and largest radii of $R_1 = 27.78$ mm and $R_N = 147$ mm and distances from the origin to the planes of the smallest and largest loops of $d_1 = 13.89$ mm and $d_N = 73.5$ mm; their separation is $d_N - d_1 = 59.61$ mm. The number of turns that

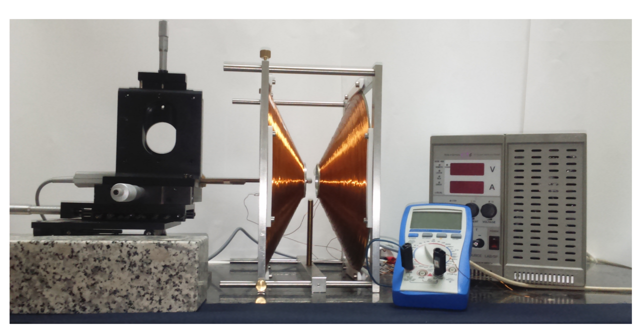

FIG. 5. Experimental layout showing from left to right: micrometric platform on which the magnetic field sensor is placed, conical coils with their supports, ampère-meter, and current source.

can theoretically be wound in a single layer is $N = 1 + (d_N - d_1)/(D\cos\alpha) = 1 + 5^{0.5}(d_N - d_1)/D = 184$ loops, agreeing well with the counted turns. The length of each loop is $l_i = 2\pi R_i = 4\pi d_i$; hence, the total length of the wire is $L = 2 \times 4\pi\Sigma d_i = 8\pi N(d_1 + d_N)/2 = 4\pi(1 + 5^{0.5}(d_N - d_1)/D)(d_1 + d_N)$, which in the current setup gives a length of $L \approx 170$ m.

The maximum permissible current is calculated by taking into account that, in the steady state, the electric power supplied $\rho_r L I^2/(\pi D^2/4)$ is equal to the heat power $h_c \delta T \pi D L$ transferred from the wire to air, where ρ_r is the electric resistivity, h_c is the heat convection coefficient, and δT is the temperature difference between the wire and the air. Setting $\rho_r = 1.7 \times 10^{-8}$ Ωm, $h_c = 9$ W/(m^2 K), and $\delta T = 60$ K gives $I \approx 6$ A.

Field **B** was measured using a magnetometer (Phywe Teslameter) with a resolution of 0.01 mT. The probe of the device was rigidly attached to a 3D movable stage platform with a translation precision of 0.01 mm. The terrestrial magnetic field and other possible fields generated through other causes have been partially compensated in each reading by resetting the magnetometer prior to each measurement of the desired field.

Table I shows the matrix $\mathbf{B}_z$ constructed with the experimentally measured values of the magnetic field for a 1-A coil current. In this table, the absence of values in certain points of the useful space is because the magnetometer probe was physically unable to reach the region. Its finite size prevents probing near the cone, which supports the loops. Similarly, Tables II and III show the matrix $\mathbf{B}_\rho$ and heterogeneity formed from the experimentally measured values at the accessible experimental points.

TABLE I. Experimental values of B_z (in mT) at accessible points in the useful region.

B_z (mT)	$z = 0$	$z = 2$	$z = 4$	$z = 6$	$z = 8$	$z = 10$	$z = 12$	$z = 14$ mm
$\rho = 0$	2.26	2.26	2.27	2.26	2.26	2.25	2.26	2.24
$\rho = 2$	2.26	2.26	2.25	2.26	2.26	2.26	2.26	2.25
$\rho = 4$	2.25	2.26	2.27	2.27	2.26	2.26	2.25	2.25
$\rho = 6$	2.26	2.26	2.28	2.26	2.27	2.27	2.27	2.26
$\rho = 8$	2.26	2.25	2.27	2.27	2.27	2.27	2.27	2.26
$\rho = 10$	2.26	2.25	2.26	2.27	2.27	2.28	2.28	2.27
$\rho = 12$	2.26	2.25	2.26	2.26	2.27	2.29	2.29	2.28
$\rho = 14$	2.26	2.24	2.27	2.26	2.28	2.31	2.31	2.30
$\rho = 16$	2.25	2.24	2.26	2.27	2.29	2.32	2.32	2.35
$\rho = 18$ mm	2.23	2.24	2.24	2.26	2.29	2.32	2.35	2.37

TABLE II. Matrix B_ρ (in mT) of experimentally measured values in the useful region.

B_ρ (mT)	$z = 0$	$z = 2$	$z = 4$	$z = 6$ mm
$\rho = 0$	0.02	0.02	0.02	0.02
$\rho = 2$	0.02	0.02	0.02	0.02
$\rho = 4$	0.02	0.01	0.03	0.03
$\rho = 6$	0.02	0.02	0.02	0.03
$\rho = 8$	0.02	0.02	0.03	0.02
$\rho = 10$	0.02	0.02	0.02	0.02
$\rho = 12$	0.02	0.01	0.01	0.01
$\rho = 14$	0.02	0.00	0.00	0.00
$\rho = 16$	0.02	0.00	0.00	−0.02
$\rho = 18$	0.01	0.00	−0.02	−0.04
$\rho = 20$	0.02	−0.01	−0.04	−0.08
$\rho = 22$	0.02	−0.02	−0.07	−0.11
$\rho = 24$	0.02	−0.03	−0.09	−0.15
$\rho = 26$ mm	0.02	−0.04	−0.11	−0.18

TABLE III. Values of h (in percentages) from the experimentally measured field values.

h	$z = 0$	$z = 2$	$z = 4$	$z = 6$ mm
$\rho = 0$	0.0000	0.0000	0.4425	0.0000
$\rho = 2$	0.0000	0.0000	0.4425	0.0000
$\rho = 4$	0.4425	0.4425	0.6257	0.6257
$\rho = 6$	0.0000	0.0000	0.8849	0.4425
$\rho = 8$	0.0000	0.4425	0.6257	0.4425
$\rho = 10$	0.0000	0.4425	0.0000	0.4425
$\rho = 12$	0.0000	0.6257	0.4425	0.4425
$\rho = 14$	0.0000	1.2515	0.9894	0.8849
$\rho = 16$	0.4425	1.2515	0.8849	1.8243
$\rho = 18$ mm	1.3992	1.2515	1.9787	2.6548

VII. NUMERICAL CALCULATION OF B GENERATED BY THE CONICAL COAXIAL CIRCULAR CURRENTS OF *N* LOOP PAIRS

Taking into account the characteristics of the experimental setup, the magnetic field components B_ρ and B_z for the smallest pair of turns have been numerically calculated. The points chosen fall within the useful half-plane, inside of a rectangle delineated by the coordinate axes, lines $z = 13.89$ mm, and $\rho = 27.78$ mm and meshed with a grid of size 2 mm × 2

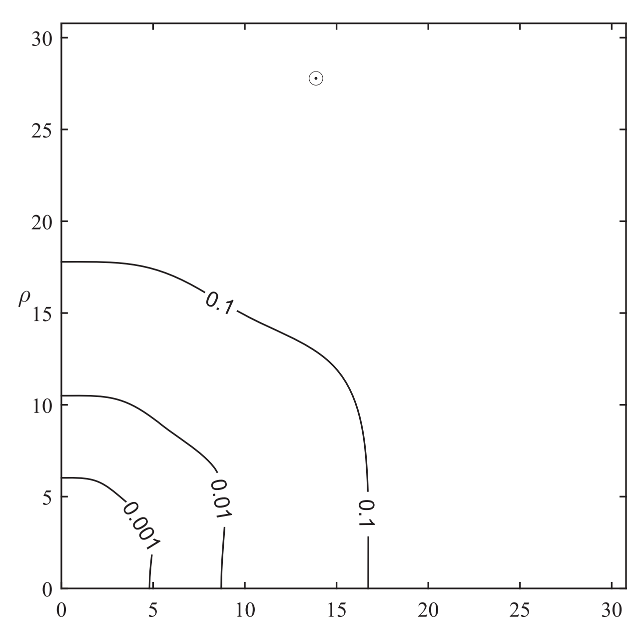

FIG. 6. Calculated contours with equal heterogeneity for $h = 0.001$, 0.01, and 0.1, respectively, corresponding to the smallest loop pair of the experimental layout.

TABLE IV. Calculated values of the component B_z (in mT) for the conical coils.

B_z (mT)	$z = 0$	$z = 2$	$z = 4$	$z = 6$	$z = 8$	$z = 10$	$z = 12$	$z = 14$ mm
$\rho = 0$	2.2561	2.2561	2.2559	2.2552	2.2535	2.2499	2.2438	2.2345
$\rho = 2$	2.2561	2.2561	2.2560	2.2555	2.2539	2.2505	2.2446	2.2355
$\rho = 4$	2.2560	2.2561	2.2564	2.2563	2.2552	2.2524	2.2471	2.2384
$\rho = 6$	2.2557	2.2560	2.2568	2.2575	2.2574	2.2557	2.2513	2.2434
$\rho = 8$	2.2550	2.2555	2.2571	2.2590	2.2604	2.2602	2.2573	2.2506
$\rho = 10$	2.2533	2.2543	2.2569	2.2605	2.2641	2.2662	2.2655	2.2604
$\rho = 12$	2.2502	2.2516	2.2558	2.2618	2.2684	2.2738	2.2761	2.2735
$\rho = 14$	2.2447	2.2468	2.2530	2.2623	2.2731	2.2831	2.2897	2.2904
$\rho = 16$	2.2358	2.2388	2.2476	2.2612	2.2777	2.2943	2.3072	2.3127
$\rho = 18$ mm	2.2221	2.2262	2.2383	2.2574	2.2816	2.3075	2.3298	2.3424

mm. A current $I = 1$ A is used in the calculation. The matrices $\mathbf{B}_{\rho,1}$ and $\mathbf{B}_{z,1}$ corresponding to the smallest coil pair are then obtained along with the h values at points in the useful half-plane. Figure 6 shows the calculated iso-curves for $h = 0.001$, 0.01, and 0.1, respectively.

Following the procedure outlined in Sec. V, the radial and axial components of the magnetic field generated by the current in all turns were numerically computed. The resulting magnetic field **B** is then obtained by adding the field components. Finally, the heterogeneity h is calculated from the field **B**. Because the size of the probe limits measurements, only the fields near the center of the coil system were measured and compared with the numerical results.

Table IV gives the calculated values of the B_z components to four decimal places and the grid of 10 × 8 points corresponding to one quadrant of the area of interest. Note that in this table, the numerical values listed are those at points where the magnetic field has been measured. From the symmetry of the device, similar results are expected for the other quadrants. Similarly, Tables V and VI give the calculated values of the B_ρ component and the heterogeneity h. The h values are very small being near the origin. In Fig. 6, corresponding only to the first coil pair, a greater area than that between the loops is shown and values as large as h = 10% are obtained at points far away from the origin.

Taking all loops in the double cone into consideration, we plotted contours of constant h (Fig. 7), corresponding to values

TABLE V. Calculated values of the component B_ρ (in mT) for the conical layout.

B_ρ (mT)	$z = 0$	$z = 2$	$z = 4$	$z = 6$ mm
$\rho = 0$	0.0000	0.0000	0.0000	0.0000
$\rho = 2$	0.0000	0.0000	0.0001	0.0005
$\rho = 4$	0.0000	−0.0001	0.0001	0.0008
$\rho = 6$	0.0000	−0.0004	−0.0003	0.0005
$\rho = 8$	0.0000	−0.0010	−0.0014	−0.0006
$\rho = 10$	0.0000	−0.0021	−0.0034	−0.0030
$\rho = 12$	0.0000	−0.0040	−0.0067	−0.0072
$\rho = 14$	0.0000	−0.0067	−0.0119	−0.0140
$\rho = 16$	0.0000	−0.0107	−0.0194	−0.0243
$\rho = 18$	0.0000	−0.0162	−0.0301	−0.0392
$\rho = 20$	0.0000	−0.0235	−0.0445	−0.0602
$\rho = 22$	0.0000	−0.0325	−0.0629	−0.0883
$\rho = 24$	0.0000	−0.0428	−0.0846	−0.1234
$\rho = 26$ mm	0.0000	−0.0536	−0.1076	−0.1624

TABLE VI. Calculated values of h (as percentages) for the conical layout.

h	$z = 0$	$z = 2$	$z = 4$	$z = 6$ mm
$\rho = 0$	0.0000	0.0005	0.0076	0.0377
$\rho = 2$	0.0002	0.0008	0.0066	0.0346
$\rho = 4$	0.0029	0.0046	0.0131	0.0357
$\rho = 6$	0.0151	0.0172	0.0343	0.0660
$\rho = 8$	0.0487	0.0510	0.0761	0.1306
$\rho = 10$	0.1224	0.1245	0.1554	0.2374
$\rho = 12$	0.2625	0.2641	0.2988	0.4081
$\rho = 14$	0.5054	0.5063	0.5435	0.6784
$\rho = 16$	0.8991	0.8994	0.9391	1.0990
$\rho = 18$ mm	1.5045	1.5052	1.5501	1.7390

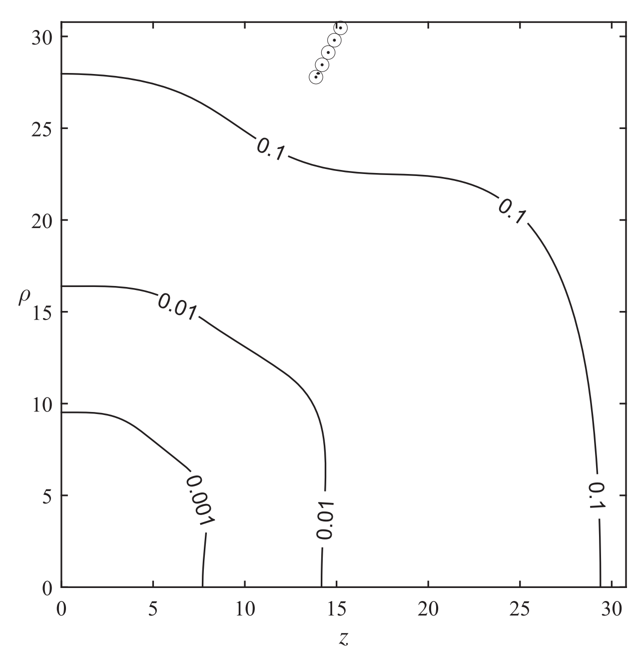

FIG. 7. Contours with equal heterogeneity for $h = 0.001$, 0.01, and 0.1, respectively, corresponding to the conical coil pair. The area enclosed by $h = 0.001$ is estimated to be 2.5 times greater than that in Fig. 6, the corresponding volume being 4 times greater.

$h = 0.001$, 0.01, and 0.1. From the innermost to the outermost contour, the curves expand smoothly.

VIII. COMPARISON OF THE NUMERICALLY CALCULATED VALUES OF THE MAGNETIC FIELD WITH THE EXPERIMENTAL RESULTS

Comparing the calculated values with the experimental results, we find that the differences in the B_z components (Tables I and IV) are smaller than 2%, the differences in B_ρ components (Tables II and V) are smaller than or equal to 0.01 mT, and the differences in h (Tables III and VI) are equal to or less than 0.9%.

To understand these differences, one must keep in mind that the numerical and experimental results in the table have uncertainties associated with their values. Thus, Table I of experimental data shows some measured values of the magnetic field component B_z in terms of the point positions (coordinates ρ and z) for which there are uncertainties. Because the sensitivity of the probe is limited to 0.01 mT, there is an inaccuracy of at least 0.01 mT. Measurements of the position also have an intrinsic difficulty because the sensitivity of the platform is limited to 0.01 mm. In addition, the position of the Hall sensor in the probe as well as its finite size also contributes to the uncertainty of about 1 mm for both. Another problem is the position of the origin, which can increase the uncertainty by 1 mm. Moreover, the terrestrial magnetic field varies over time, and there are possible parasitic small signals in the environment, being observable oscillations of order 0.01 mT in a duration smaller than it takes between resetting the teslameter and taking measurements.

Furthermore, numerical calculations require input data. These data are provided by the measurements in the laboratory of α, R_1, d_1, d_N, and I. All the data bring their respective uncertainties. Even supposing that the numerical calculation is accurate, the final result has a certain uncertainty.

Within the ranges given by these uncertainties, the results for the measured magnetic field intensities and those from numerical calculation lead to the conclusion that both sets of values are acceptable and a good agreement has been achieved.

IX. MAGNETIC FIELD GENERATED BY A PAIR OF CYLINDRICAL SOLENOIDS OF 2*N* LOOPS

Devices formed using two cylindrical coils in an approximate Helmholtz pair arrangement are widely used in laboratories to obtain homogeneous magnetic fields. For this reason, we now study a similar setup, two solenoids with a constant radius R and pairs of loops at distance $d_i = R/2 + (i - 1)D$ from the origin, with the nearest at $d_1 = R/2$ (Fig. 8 shows this configuration for $N = 4$). To compare results with the conical setup, we had set $R = R_1 = 27.78$ mm and the length of each solenoid to $d_N - d_1 = 59.61$ mm (similar to the cone length); hence, the number of turns is $N = 82$ loops. Recall that only the coil pair closest to the coordinate origin satisfies the Helmholtz condition.

A numerical calculation was performed in a similar manner as for the cone. The contours of $h = 0.1\%$, 1%, and 10% are presented in Fig. 9.

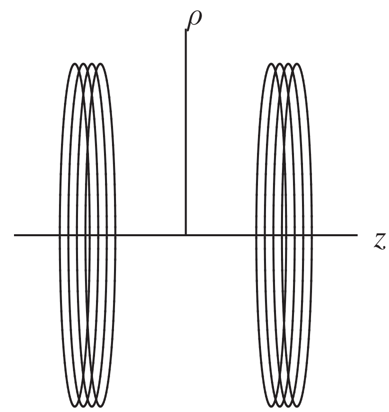

FIG. 8. Circuit formed by a double solenoid of radius equal to that of the smallest loop of the experimental layout.

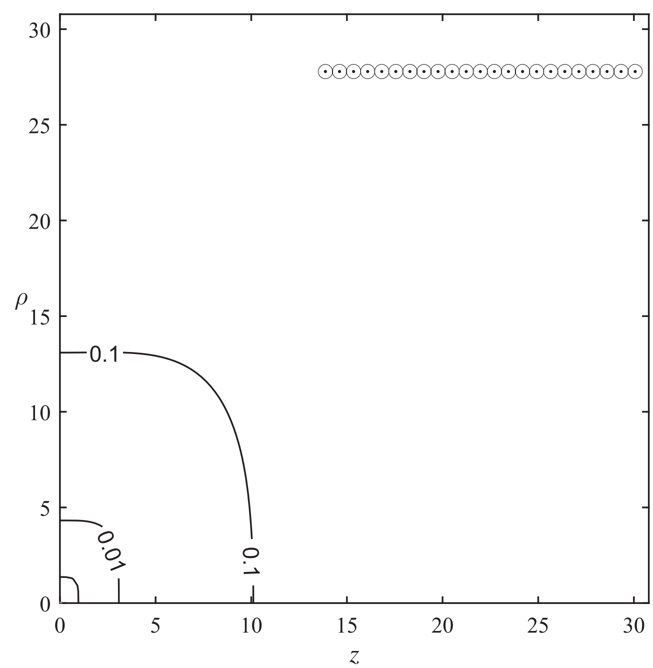

FIG. 9. Contours of constant heterogeneity calculated for the double solenoid with $h = 0.001$, 0.01, and 0.1, respectively. The surface area within the smallest contour has area 13 times smaller than that corresponding to a Helmholtz pair.

X. MAGNETIC FIELD DUE TO THE CURRENT IN TWO PAIRS OF OVERLAPPING CONES

As a single pair of turns, Helmholtz coils form a 4th-order current loop system. The other loop systems with two pairs of loops, such as those proposed by Garrett, Barker, and Braunbek, are 8th-order (the first nonzero derivative is the 8th). Consequently, these assemblies generate magnetic fields that are more homogeneous than those of a single pair.

Taking into consideration the layout of the double cone analyzed in Sec. V, we propose an assembly with the advantages of two pairs of loops instead of one and the replacement of each loop by a conical coil. Figure 10 presents a schematic of the proposed system comprising two cone pairs, with each quartet of coils placed in the optimal layout given by Braunbek.[19] The current in both conical windings are set

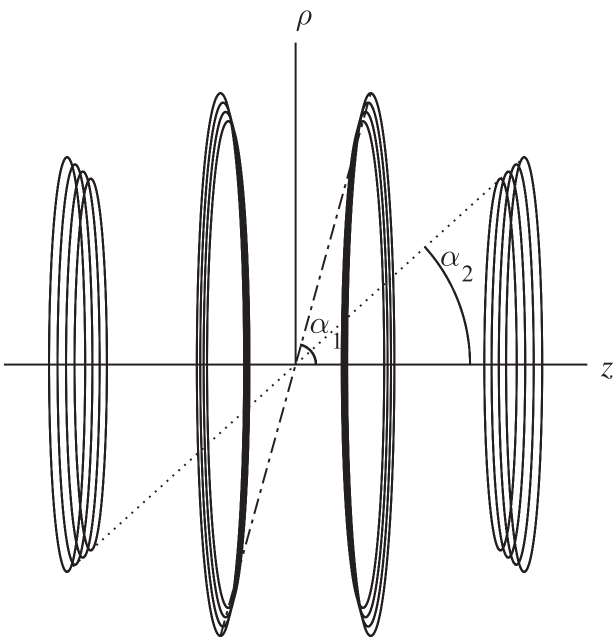

FIG. 10. Circuit formed by four coils wound on two pairs of conical surfaces of angles $\alpha_1 \approx 74.46°$ and $\alpha_2 \approx 42.09°$.

equal, $I_1 = I_2$ and the approximate angle of the first cone is $\alpha_1 = 74.46°$, whereas for the second cone, the angle is $\alpha_2 = 42.09°$.

To calculate the magnetic field generated by the two pairs of conical coils, the smallest radius of the truncated cone closest to the center (angle at the apex is α_1) is assumed to be the same as that for the single double cone, i.e., $R_{1,1} = R_1 = 27.78$ mm. From this value, the distance of the plane of the closest loop to the center[6] is $d_{1,1} = d_1 = 7.72$ mm. For the second winding (angle at the apex is α_2), the radius and distance to the center are $R_{1,2} = 21.22$ mm and $d_{1,2} = 23.49$ mm, respectively. The number of turns in each coil equals that for the double cone, i.e., $N = 179$; the diameter of the wire is also the same. Once the coils have been tightly wound on the outer cones, fulfilling the Braunbek condition for the quartet of loops requires calculating the required distance between the inner two loops. The numerical calculation for this setup provides the components of the magnetic field of each pair of conical coils and their sum. This enables the contours corresponding to $h = 0.1\%$, 1%, and 10% to be calculated (Fig. 11) along with the area of the region in the plane delimited by each contour.

The number of cones can be increased to improve field homogeneity as well as the magnetic field strength.

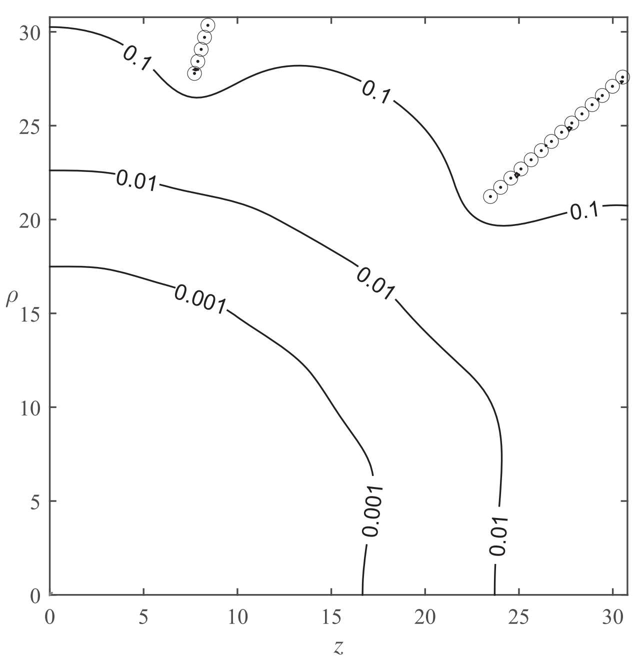

FIG. 11. Contours of homogeneity for $h = 0.001$, 0.01, and 0.1 calculated for two pairs of conical coils wound on pairs of truncated cones. The area enclosed by the smallest contour has an area 3.4 times larger than that of a pair of conical coils (Fig. 7).

XI. COMPARISON OF RESULTS

A. Comparison of calculated values for the double-conical conducting sheet with experimental results

Taking into account the geometric dimensions of the cones (Sec. VI), we find that the number of turns, per unit of generatrix length, is $n = 1343$ m^{-1}. For a current of 1 A, the current per length unit is then $nI = 1343$ Am^{-1}. Hence, the equivalent surface current density in the tangential direction to the cone is $j_s = 1343$ Am^{-1}. From Eqs. (14)–(18) and the values $\alpha = 63.43$, $R_1 = 27.78$ mm, and $R_2 = 147$ mm, the value obtained for the magnetic field at the center is 2.249 mT, which differs from the experimental value by 0.49% (see Table I).

B. Comparison of numerical results for a pair of cones and a Helmholtz pair

To compare numerically the homogeneity in the magnetic field generated by a pair of conical coils with that of the Helmholtz pair, we compared the areas under the corresponding heterogeneity contours. A simple visual inspection of Figs. 6 and 7 suggests that the areas bounded by these contours of equal heterogeneity are larger in the latter. This infers that the homogeneity achieved with a conical coil pair is better than that obtained with the Helmholtz pair.

The area A enclosed by a h-contour can be quantified using geometric and numerical methods or by counting the number of elements of the matrix h with values less than a certain threshold. Table VII shows the results for areas and volumes calculated by numerical integration, corresponding to the Helmholtz pair and three different values of h. The area values presented correspond to those of Fig. 6. From the symmetry, the total area enclosed by a contour is four times that of a quadrant. The set of volumes that appear in Table VII correspond to the entire space enclosed by the surfaces of constant h.

For the pair of conical coils, Table VIII lists the results for the areas of regions confined by the three h-contours given in Fig. 7. In this table, the volumes correspond to the entire space enclosed by the h-contour. Each value was calculated by numerical integration. A comparison of Tables VII and VIII

TABLE VII. Areas and volumes of the regions within various h-contours for the Helmholtz pair.

h	A (mm^2)	V (mm^3)
0.001	25.69	869.80
0.01	80.86	4 767.52
0.1	255.24	25 135.57

TABLE VIII. Areas and volumes of the regions within various h-contours for the conical coil pair.

h	A (mm^2)	V (mm^3)
0.001	64.98	3 477.38
0.01	204.17	18 747.72
0.1	678.05	101 451.41

indicates that, for the three h-contours, both the areas and the volumes are much larger for the double-conical coil than for the Helmholtz pair.

In the following calculations, we limit our analysis to the smallest h-contour. For the pair of conical coils, the area is A (h = 0.001) = 64.98 mm^2, which represents approximately 17% of the quadrant considered useful ($R_1 d_1$ = 386 mm^2). For the Helmholtz pair (Table VII), the area is approximately 25.69 mm^2, i.e., the area for the double cone is 2.5 times that of the loop pair. From the volumes (Tables VII and VIII), the double cone has four times the volume of a pair of loops. Therefore, as far as homogeneity is concerned, a pair of cones is better than a simple Helmholtz pair.

C. Comparison of results for the cylindrical solenoid and the Helmholtz pair

The area under the contour for h = 0.001 for the solenoid was estimated to be approximately 2 mm^2 (see Fig. 9), whereas for the Helmholtz pair, it is 25.69 mm^2. A comparison shows that the Helmholtz pair has an area 13 times greater than that obtained with solenoids. The volume of the homogeneous region can be estimated as proportional to the area raised to the power of 3/2, thereby giving a factor approximately 47 times greater for the Helmholtz pair than that of the solenoid pair. This establishes the Helmholtz pair as the better setup of the two. The proposed two-cylinder system is poorly suited to produce a homogeneous magnetic field because only the first loop pair satisfies the Helmholtz condition.

D. Comparison of a pair of conical coils with a double pair of conical coils

From Figs. 11 and 7, for a double pair of conical coils, the area delimited by the smallest contour is approximately 3.4 times greater than that for a pair of conical coils. Hence, the ratio of the respective volumes is approximately 6. Consequently, the double pair of conical coils is much better than the simple pair of conical coils, if the objective is to produce a homogeneous magnetic field.

E. Comparison of the Helmholtz pair with a double pair of loops

Caprari[6] calculated and compared the homogeneity of the Braunbek double pair (four loops) with that of the Helmholtz pair (two loops); he found that the quotient of the volume enclosed by the surface h = 0.01 and the volume enclosed by the loop system is equal to 0.227 for the Braunbek double pair, whereas it is 0.071 for the Helmholtz pair, i.e., such a quotient for the Braunbek layout is 3.2 times that for the Helmholtz pair. Similar results are obtained for other values of h; the quotient of the volumes in all cases is higher for the Braunbek coils than for the Helmholtz configuration.

XII. CONCLUSIONS

In this work, different electrical configurations are investigated, both theoretically and experimentally, in order to obtain a large volume of homogeneous magnetic field. A new conical current system to generate homogeneous magnetic fields is designed. From the analytical calculations, it is found that the optimal angle between the generatrix of the cone and its axis is 63.43°. A copper wire is wound on a pair of truncated cones so that each pair of loops satisfies the Helmholtz condition. Then, a high homogeneity in the magnetic field is achieved around the center of symmetry.

The numerical results show that for a field heterogeneity of 0.1%, the proposed pair of conical coils provides a greater homogeneity than the Helmholtz pair; the corresponding volume is 4 times that of the Helmholtz configuration. In addition, the numerical results for the conical coils are in good agreement with the experimental ones. To improve the homogeneity, we also propose a new layout with two pairs of conical coils, which is based on the optimal configuration with four loops presented by Braunbek. For the contour with a heterogeneity of 0.1%, the four conical coils yield a volume that is 6 times greater than that obtained with a single conical pair.

According to the results obtained, in regard to rating the homogeneity of the magnetic field produced, the configurations analyzed are listed from best to worst as follows:

1. Double pair of conical coils.
2. Pair of conical coils.
3. Braunbek four loops.
4. Helmholtz pair.
5. Cylindrical solenoid pair.

All the proposed layouts are cheap and easy to assemble. Moreover, the conical setup proposed could represent the starting point of future designs with practical applications.

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