Boundary Conditions

What happens to the field vectors at a boundary?

Starting with the Electric/Displacement Field:

From Gauss' Law

$$\oint_S \bar{D} \cdot \hat{n}\, da = 4\pi q_f$$

⟶ The total flux of $\bar{D}$ through a closed surface S depends on the amount of free charge q_f

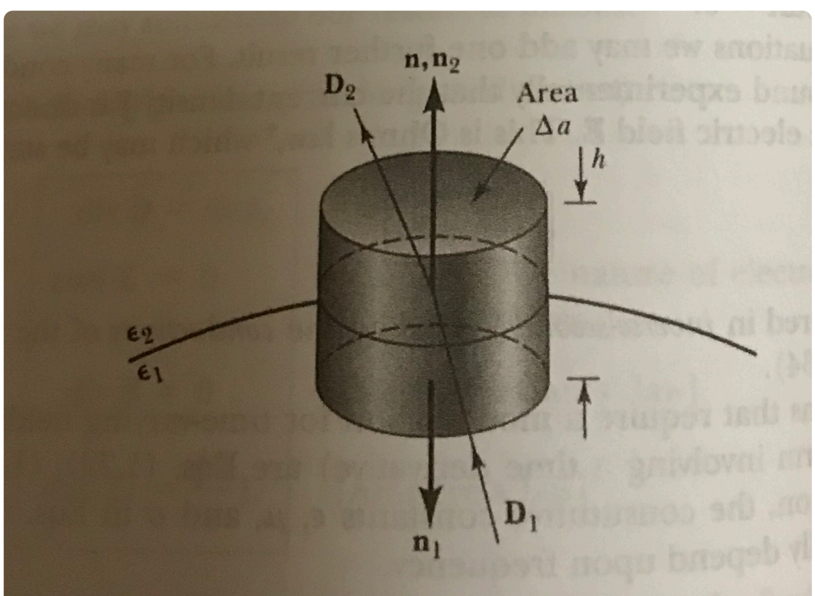


FIGURE 1-11. Gaussian pillbox at interface.

For a $\bar{D}$ directed from a material with a dielectric constant of ε_1 to a material with ε_2 (figure above), we construct a <u>Gaussian pillbox</u> with area Δa at the top/bottom.

$h \to 0$ so the side flux is ignored.

$$\Rightarrow \oint_S \bar{D} \cdot \bar{n}\, da = \left[\bar{D}_2 \cdot \hat{n} + \bar{D}_1 \cdot (-\hat{n})\right] \Delta a$$

$$= 4\pi \underbrace{(\rho_s)_f}_{\text{surface charge density at interface (charge/area)}} \Delta a$$

$$\Rightarrow \boxed{(\bar{D}_2 - \bar{D}_1) \cdot \hat{n} = 4\pi (\rho_s)_f}$$

Boundary condition on <u>normal</u> component of $\bar{D}$

In terms of the electric field, this becomes:

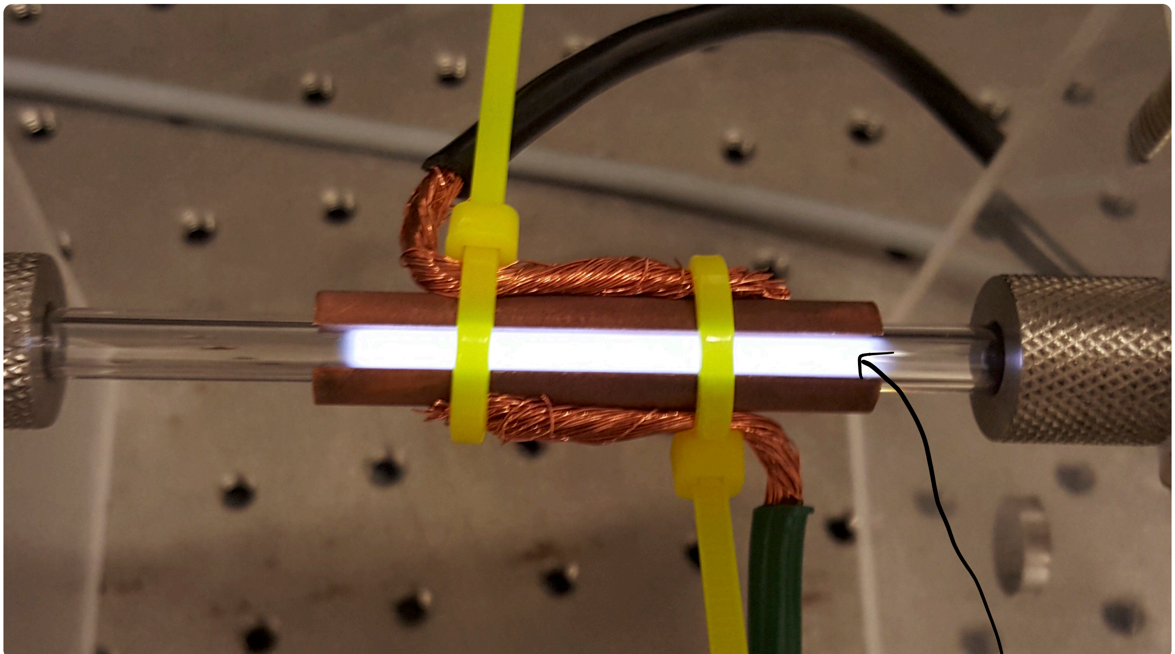
$$\left(\varepsilon_2 \bar{E}_2 - \varepsilon_1 \bar{E}_1\right) \cdot \hat{n} = 4\pi (\rho_s)_f$$

If $(\rho_s)_f = 0$ so there is no free charge on the boundary, what does that say about $\bar{D}_1$ and $\bar{D}_2$? What about $\bar{E}_1$ and $\bar{E}_2$?

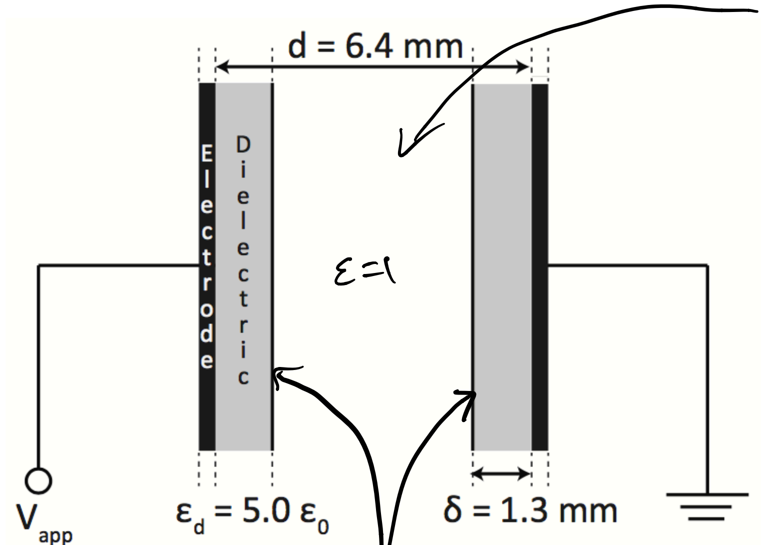
$$\longrightarrow \underbrace{\bar{D}_1 = \bar{D}_2}_{\text{continuous across boundary}} \quad \text{or} \quad \varepsilon_2 \bar{E}_2 = \varepsilon_1 \bar{E}_1 \quad \text{if } (\rho_s)_f = 0$$

Example:

Radio Frequency Dielectric Barrier Discharge



PLASMA



What is the electric field at the boundaries?

As the plasma forms, current flows to the dielectric boundaries, creating a surface charge:

$$\hat{n}\cdot\big(\underbrace{\bar{E}_p}_{\substack{\text{field in}\\ \text{plasma}\\ \text{(vacuum)}}} - \underbrace{\varepsilon_d \bar{E}_d}_{\text{dielectric}}\big) = \underbrace{4\pi(\rho_s)_f}_{\substack{\text{surface charge}\\ \text{density}}}$$

$$\frac{d(\rho_s)_f}{dt} = \hat{n}\cdot\bar{J}_e + \hat{n}\cdot\bar{J}_i$$

→ surface charge density changes over time depending on the electron and ion current densities, $\bar{J}_e$ and $\bar{J}_i$

☆ How does current flow to/from the electrodes?

Tangential Boundary Condition

From $\bar{\nabla} \times \bar{E} = 0$ we can integrate around a <u>Stokesian loop</u> Γ to obtain:

$$\underbrace{\int_S \bar{\nabla} \times \bar{E} \cdot \hat{n}_0 \, da = \oint_\Gamma \bar{E} \cdot d\bar{l}}_{\text{Stoke's theorem}} = 0$$

normal to surface S

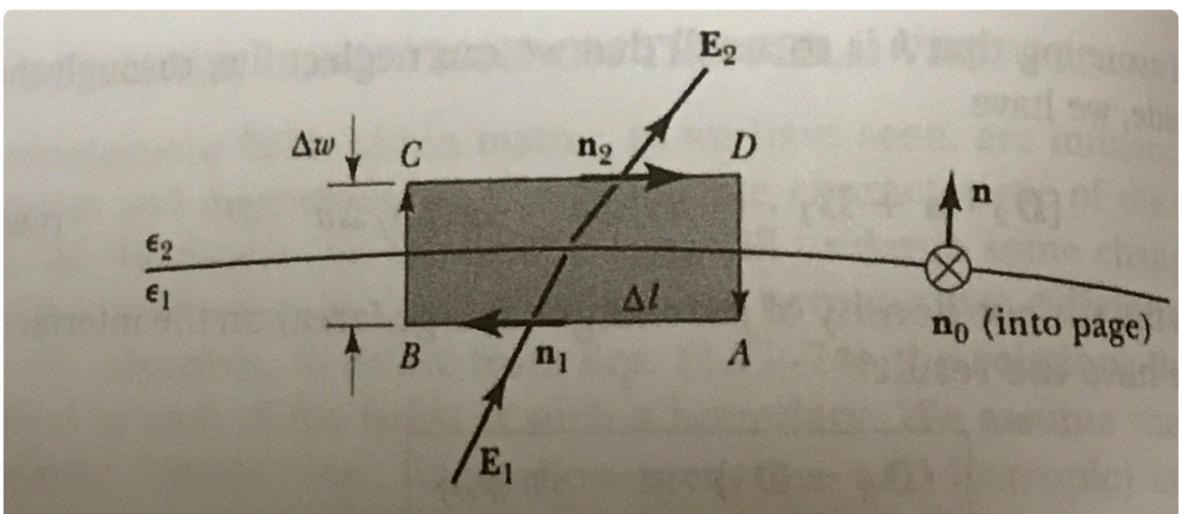


FIGURE 1-12. Stokesian rectangle at interface.

For $\Delta w \to 0$, we neglect the contribution from the sides

$$\Rightarrow \left[\bar{E}_1 \cdot \hat{n}_1 + \bar{E}_2 \cdot \hat{n}_2\right] \Delta l = 0$$

The unit vectors follow

$$\hat{n}_2 = -\hat{n}_1 = \underbrace{\hat{n}_0}_{\text{normal to surface}} \times \underbrace{\hat{n}}_{\text{normal to boundary}}$$

$$\Rightarrow -\bar{E}_1 \cdot \hat{n}_2 + \bar{E}_2 \cdot \hat{n}_2 = (\bar{E}_2 - \bar{E}_1) \cdot \hat{n}_2 = 0$$

$$\Rightarrow (\bar{E}_2 - \bar{E}_1) \cdot (\hat{n}_0 \times \hat{n}) = 0$$

From the vector identity

$$\bar{C} \cdot (\bar{A} \times \bar{B}) = \bar{A} \cdot (\bar{B} \times \bar{C})$$

$$\Rightarrow \hat{n}_0 \cdot [\hat{n} \times (\bar{E}_2 - \bar{E}_1)] = 0$$

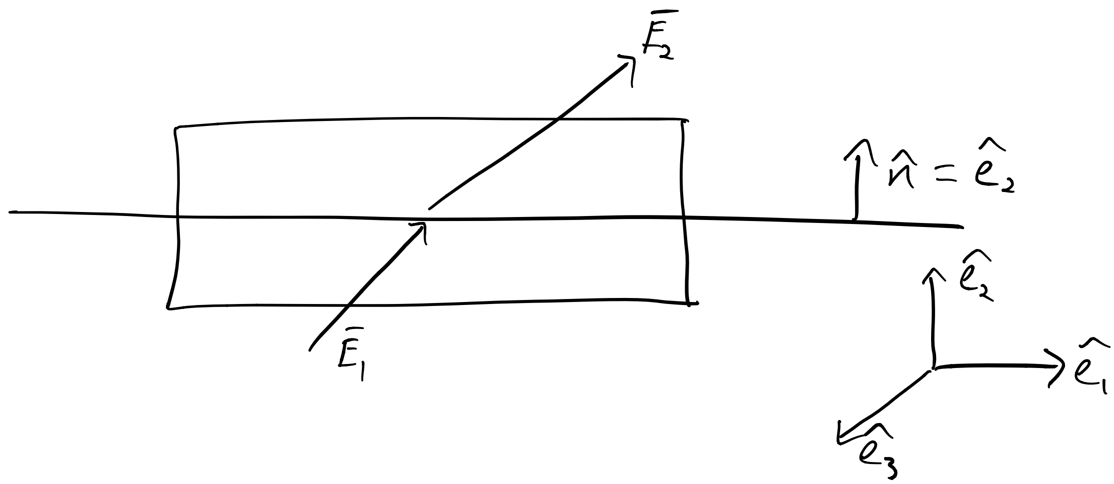
Also, $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$

$$\Rightarrow -\hat{n}_0 \cdot [(\bar{E}_2 - \bar{E}_1) \times \hat{n}] = 0$$

Because $\hat{n}_0$ is non-zero with arbitrary orientation, we must have

$$\boxed{(\bar{E}_2 - \bar{E}_1) \times \hat{n} = 0}$$

☆ Tangential components of $\bar{E}$ are continuous across a boundary



Suppose $\bar{E}_2 - \bar{E}_1$ has a component in the $\hat{e}_1$ tangential direction with magnitude ΔE

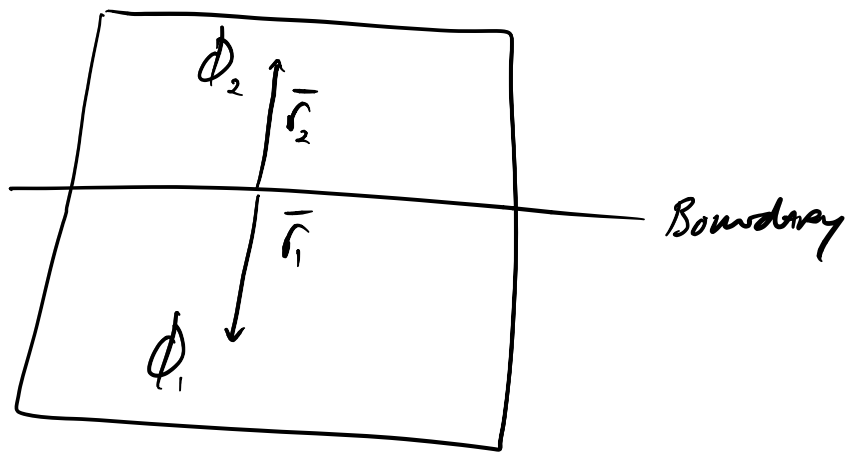
$$\bar{E}_2 - \bar{E}_1 = \Delta E \, \hat{e}_1$$

$$\Rightarrow (\bar{E}_2 - \bar{E}_1) \times \hat{n} = \Delta E \, \hat{e}_1 \times \hat{e}_2 = \Delta E \, \hat{e}_3$$

For $(\bar{E}_2 - \bar{E}_1) \times \hat{n} = 0$ to hold, ΔE must be zero

$\Rightarrow$ tangential component of $\bar{E}$ is continuous

From $\Delta \phi = \phi_2 - \phi_1 = -\int_{r_1}^{r_2} \bar{E} \cdot d\bar{l}$



As $\bar{r}_2 - \bar{r}_1 \to 0$ and the two points became continuous, $\Delta \phi \to 0$

$$\lim_{\bar{r}_2 \to \bar{r}_1} -\int_{r_1}^{r_2} \bar{E} \cdot d\bar{l} = 0 = \Delta \phi$$

$$\Rightarrow \boxed{\phi_2 = \phi_1}$$

Potential is continuous across a boundary

Magnetic Field Boundary Conditions

Following the examples for the electric field:

From $\bar{\nabla} \cdot \bar{B} = 0$ we have

$$\oint_S \bar{B} \cdot \hat{n} \, da = 0$$

Using the same Gaussian pillbox as before, the boundary condition on the normal component is

$$\boxed{(\bar{B}_2 - \bar{B}_1) \cdot \hat{n} = 0}$$

* normal components of magnetic field are continuous across a boundary

For the tangential component we start from Ampère's law:

$$\bar{\nabla} \times \bar{H} = \frac{4\pi}{c} \bar{J}_f$$

$$\Rightarrow \int_S \underbrace{\bar{\nabla} \times \bar{H} \cdot \hat{n}_0 \, da = \oint_\Gamma}_{\text{Stoke's theorem}} \bar{H} \cdot d\bar{l} = \int \frac{4\pi \bar{J}_f}{c} \cdot \hat{n}_0 \, da$$

$$= \frac{4\pi}{c} (I_f)_{link}$$

free current

Using the same Stokesian Loop as before, and assuming there's a surface current density $\bar{K}_f$ (current/width) in the direction of $\hat{n}_0$ (normal to surface S enclosed by Stokesian Loop Γ)

$$\boxed{\left(\bar{H}_2 - \bar{H}_1\right) \times \hat{n} = -\frac{4\pi}{c} \bar{K}_f}$$

☆ The tangential component of the magnetic intensity $\bar{H}$ is discontinuous if a surface current is present

If $\bar{K}_f = 0 \longrightarrow \underbrace{(\bar{H}_2)_t = (\bar{H}_1)_t}_{\text{tangential to boundary}}$

Using the magnetic field,

$$\left(\frac{\bar{B}_2}{\mu_2} - \frac{\bar{B}_1}{\mu_1}\right) \times \hat{n} = -\frac{4\pi}{c} \bar{K}_f$$

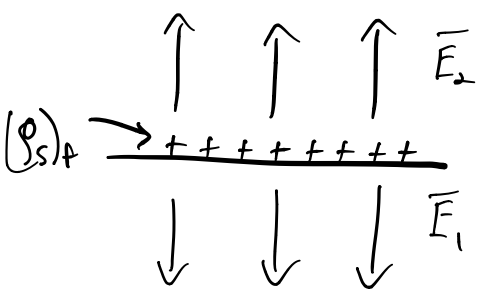
If $\quad \bar{K}_f = 0 \quad \rightarrow \quad \frac{(\bar{B}_2)_t}{\mu_2} = \frac{(\bar{B}_1)_t}{\mu_1}$

$\Rightarrow (\bar{B}_2)_t \neq (\bar{B}_1)_t \quad$ unless $\quad \mu_1 = \mu_2$

To remember the BCs, think

Electric

If charge is on the surface:

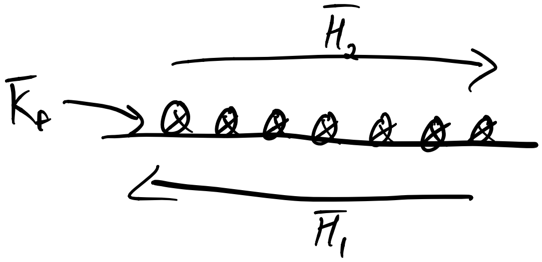


$(E_2)_n \neq (E_1)_n$

normal

Magnetic

If surface current is present:



$(\bar{H}_2)_t \neq (\bar{H}_1)_t$

tangential