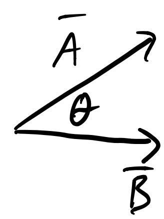
Vector Calculus

Following Appendix A of Heald and Marion:

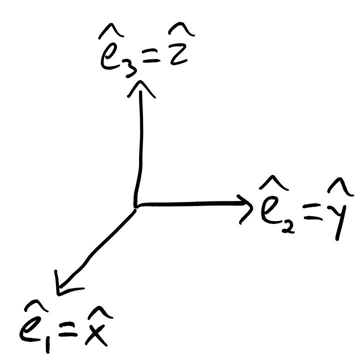
For vectors $\bar{A}$ and $\bar{B}$ in a Cartesian basis, the <u>dot product</u> is:

$$\bar{A} \cdot \bar{B} = \sum_i A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$= |\bar{A}| |\bar{B}| \cos\theta$$



The <u>cross-product</u> is:



$$\bar{A} \times \bar{B} = \sum_{ijk} \varepsilon_{ijk} \hat{e}_i A_j B_k$$

ε_{ijk} = Levi-Civita symbol

$$= \begin{cases} 0 & \text{if any 2 indices are equal} \\ +1 & \text{if } i, j, k \text{ are in cyclic order} \\ -1 & \text{if } i, j, k \text{ are in reversed order} \end{cases}$$

cyclic order

$$(i,j,k) = (1,2,3),\ (3,1,2),\ (2,3,1)$$

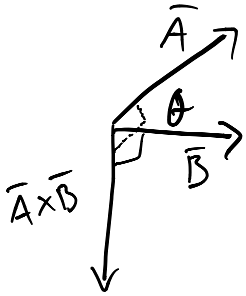
Reversed order

$$(i,j,k) = (3,2,1),\ (1,3,2),\ (2,1,3)$$

Alternatively, a more familiar form of the cross-product is

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \hat{e}_1 (A_2 B_3 - B_2 A_3) - \hat{e}_2 (A_1 B_3 - B_1 A_3) + \hat{e}_3 (A_1 B_2 - B_1 A_2)$$



Right-hand Rule

$$|\bar{A}\times\bar{B}| = |A||B||\sin\theta|$$

Useful Identities:

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \cdots$$
$$= -(\mathbf{B}\times\mathbf{A})\cdot\mathbf{C} = -\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = \cdots$$
$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{A.18}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \quad (\text{"}\textit{BAC-CAB}\text{ rule"}) \tag{A.19}$$

$$\left.\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= \mathbf{A}\cdot[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})] \\ &= \mathbf{A}\cdot[(\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{D}] \\ &= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})\end{aligned}\right\} \tag{A.20}$$

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = [(\mathbf{A}\times\mathbf{B})\cdot\mathbf{D}]\mathbf{C} - [(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}]\mathbf{D} \tag{A.21}$$

$$\mathbf{A}\times[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})] = (\mathbf{B}\cdot\mathbf{D})(\mathbf{A}\times\mathbf{C}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\times\mathbf{D}) \tag{A.22}$$

$$(\mathbf{A}\times\mathbf{B})\cdot[(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A})] = [\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})]^2 \tag{A.23}$$

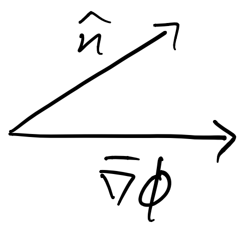
[Heald + Marion]

Vector Differential Operators

The gradient of a scalar function $\phi(x_i)$:

$$\bar{\nabla}\phi = \sum_i \hat{e}_i \frac{\partial \phi}{\partial x_i} = \hat{e}_1 \frac{\partial \phi}{\partial x_1} + \hat{e}_2 \frac{\partial \phi}{\partial x_2} + \hat{e}_3 \frac{\partial \phi}{\partial x_3}$$

The rate of change of ϕ in the direction $\hat{n}$ provides the normal derivative:



$$\hat{n}\cdot\bar{\nabla}\phi \underset{\text{defined}}{\equiv} \frac{\partial \phi}{\partial n}$$

The divergence of a vector $\bar{A}$:

$$\bar{\nabla}\cdot\bar{A} = \sum_i \frac{\partial A_i}{\partial x_i} = \underbrace{\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}}_{\text{SCALAR}}$$

The <u>curl</u> of vector $\bar{A}$:

$$\bar{\nabla} \times \bar{A} = \sum_{ijk} \varepsilon_{ijk} \hat{e}_i \frac{\partial A_k}{\partial x_j} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{e}_1 \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) - \hat{e}_2 \left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} \right)$$

$$+ \hat{e}_3 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)$$

The <u>Laplacian</u> of a scalar function ϕ:

$$\bar{\nabla}^2 \phi = \bar{\nabla} \cdot \bar{\nabla} \phi = \sum_i \frac{\partial^2 \phi}{\partial x_i^2} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}$$

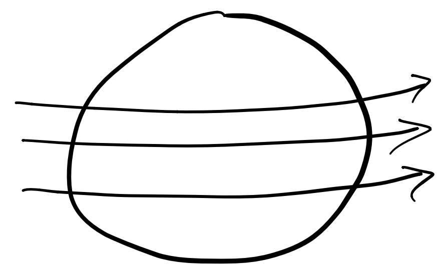
For vector $\bar{A}$ represented in Cartesian coordinates, the gradient of $\bar{A}$ is:

$$\bar{\nabla}\bar{A} = \begin{pmatrix} \frac{\partial A_1}{\partial x_1} & \frac{\partial A_2}{\partial x_1} & \frac{\partial A_3}{\partial x_1} \\ \frac{\partial A_1}{\partial x_2} & \frac{\partial A_2}{\partial x_2} & \frac{\partial A_3}{\partial x_2} \\ \frac{\partial A_1}{\partial x_3} & \frac{\partial A_2}{\partial x_3} & \frac{\partial A_3}{\partial x_3} \end{pmatrix}$$

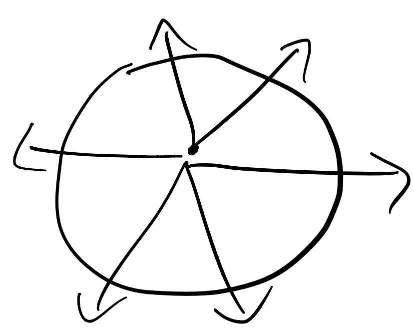
Physically, what do these differential operators mean?

Divergence

$\bar{\nabla}\cdot\bar{A} = 0 \Rightarrow$



$\bar{\nabla}\cdot\bar{A} \neq 0 \Rightarrow$



vectors diverge from a point

Curl

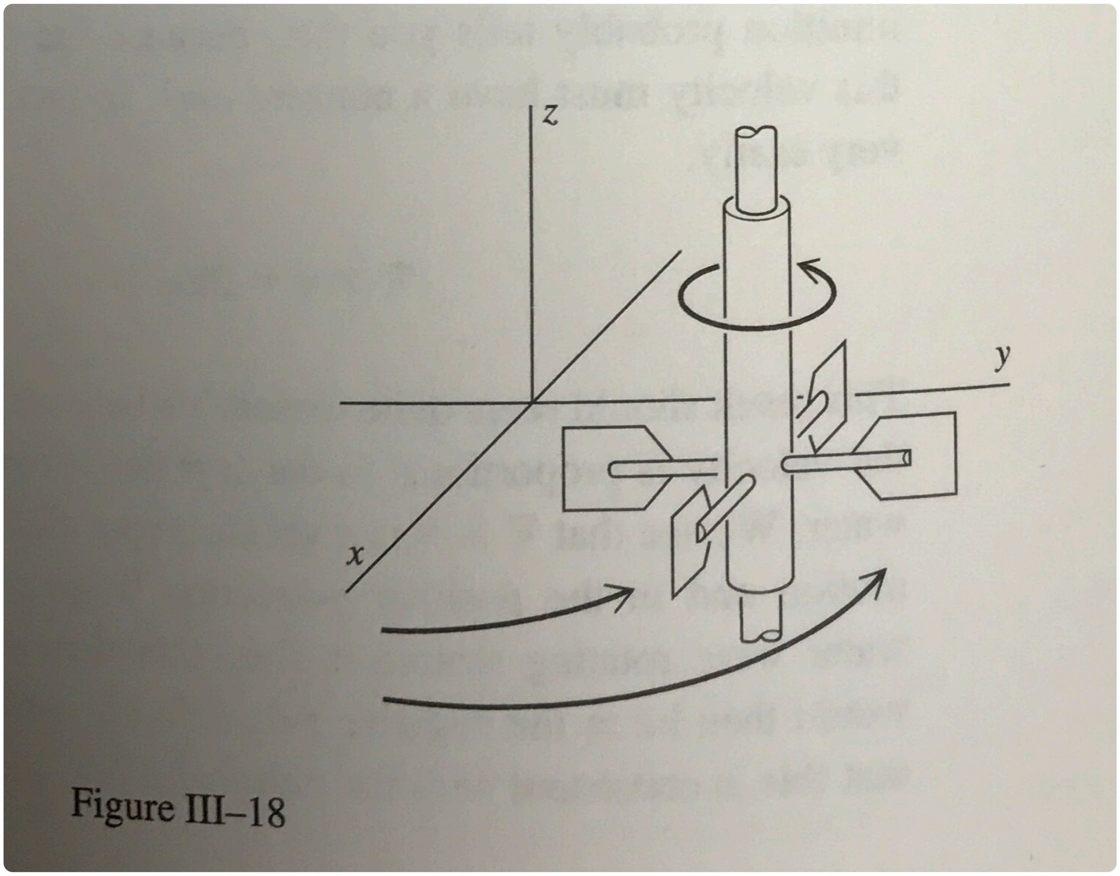


Figure III–18

[From <u>div grad curl and all that</u>, Schey]

$\bar{\nabla} \times \bar{A} = 0 \Rightarrow$ paddle <u>does not</u> rotate

$\bar{\nabla} \times \bar{A} \neq 0 \Rightarrow$ paddle rotates

Gradient

The gradient can be used in two separate manners:

normal derivative $\quad \frac{\partial \phi}{\partial n} = \hat{n} \cdot \bar{\nabla}\phi$

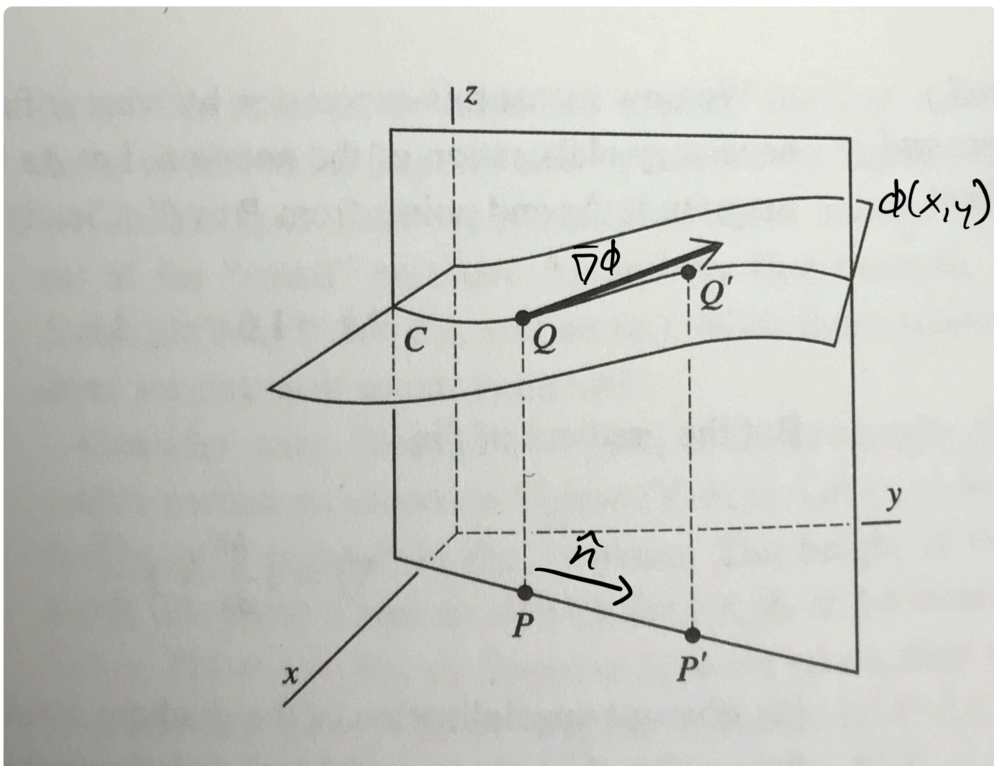


Figure IV–8(b)

[Schey]

$\hat{n} \cdot \bar{\nabla}\phi$ is a maximum when $\hat{n}$ points in the direction of $\bar{\nabla}\phi$

$\Rightarrow$ $\bar{\nabla}\phi$ points in the direction of the greatest rate of change and provides the magnitude of the change

(think derivative and tangent vector)

The second application comes from analyzing iso-surfaces. An iso-surface is a surface with a constant value of a scalar function:

$$\phi(x_1, x_2, x_3) = \text{constant}$$

For $\hat{n}$ <u>tangent</u> to the iso-surface

$$\hat{n} \cdot \bar{\nabla}\phi = 0$$

because $\phi = \text{constant}$ along the surface. Since

$$\hat{n} \neq \bar{0} \qquad \text{and} \qquad \bar{\nabla}\phi \neq \bar{0}$$

$$\Rightarrow \quad \bar{\nabla}\phi \perp \hat{n}$$

perpendicular

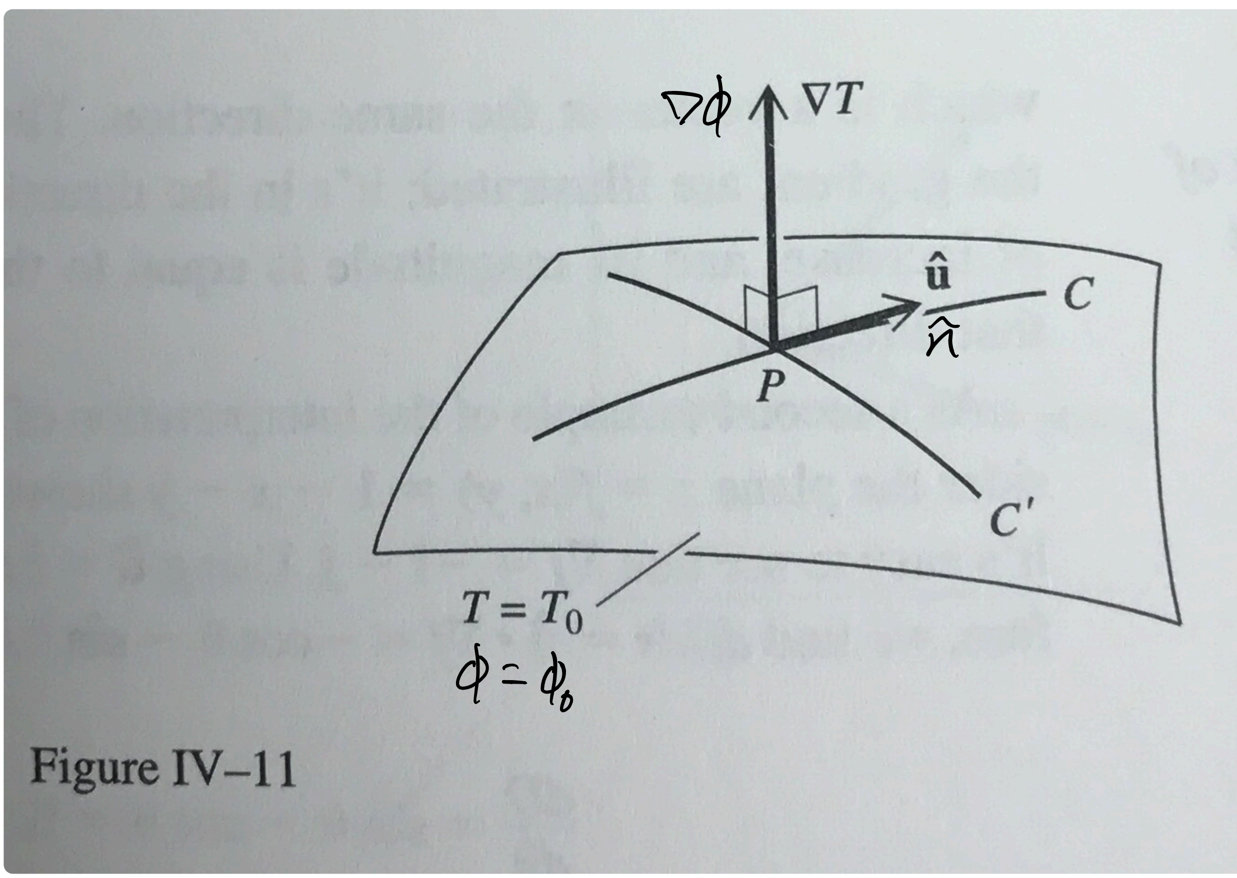


Figure IV–11

[Schey]

Curvilinear Coordinates

Cylindrical:

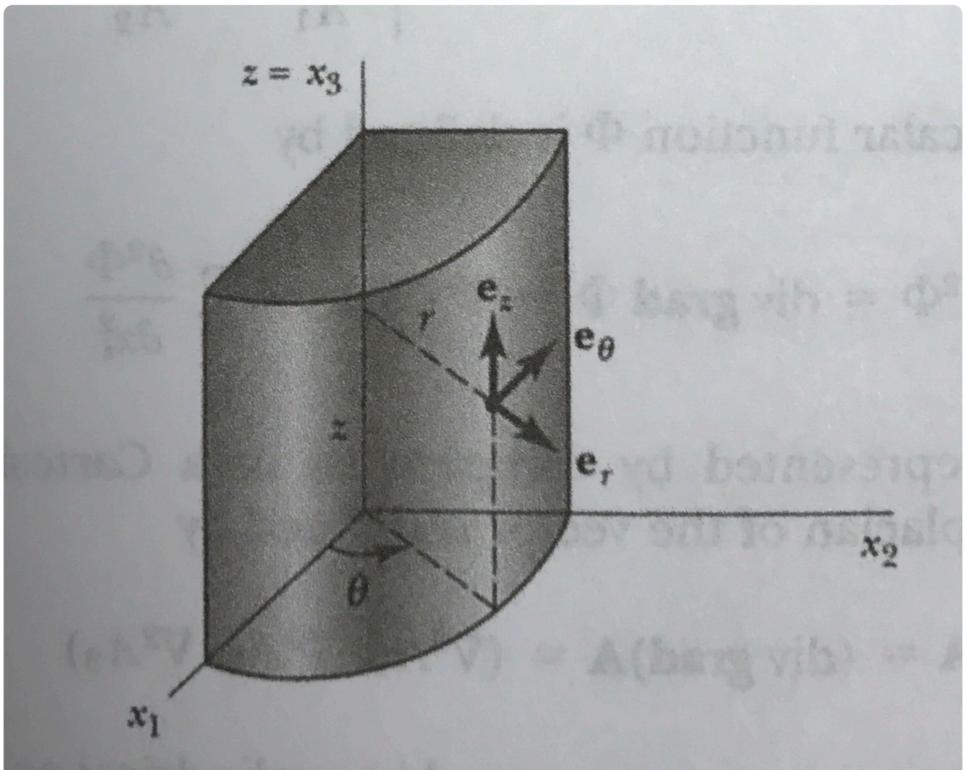


FIGURE A-2. Cylindrical coordinates.

[Heald + Marion]

$$\bar{\nabla}\psi = \hat{e}_r \frac{\partial \psi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{e}_z \frac{\partial \psi}{\partial z}$$

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\bar{\nabla} \times \bar{A} = \begin{vmatrix} \hat{e}_r/r & \hat{e}_\theta & \hat{e}_z/r \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$\bar{\nabla}^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical:

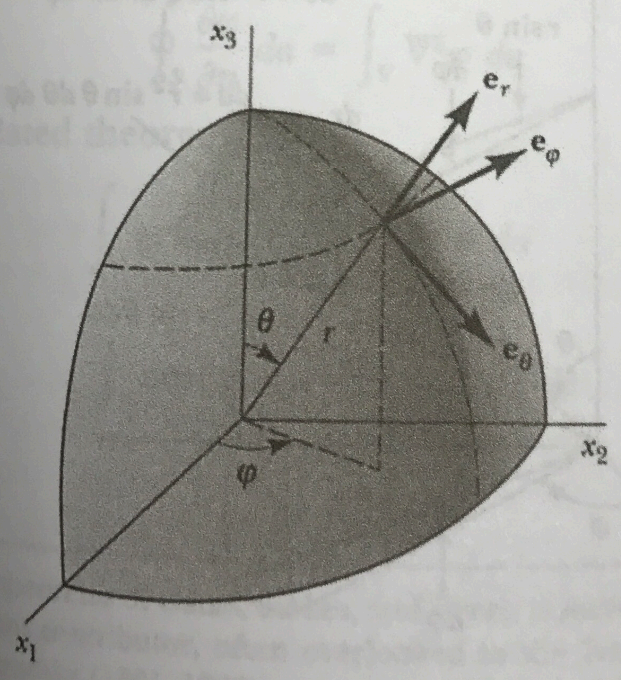


FIGURE A-4. Spherical coordinates.

[Hedd + Marion]

$$\bar{\nabla}\psi = \hat{e}_r \frac{\partial \psi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{e}_r/r^2\sin\theta & \hat{e}_\theta/r\sin\theta & \hat{e}_\phi/r \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\bar{\nabla}^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$