# ELECTRODYNAMICS, CLASSICAL

J. D. JACKSON, *Department of Physics, University of California at Berkeley, Berkeley, California, U.S.A.*



## INTRODUCTION

Electromagnetism is ultimately at the heart of almost everything we are and do. Indeed, our world and the worlds of other animate and inanimate objects function almost exclusively on electromagnetic interactions. Atoms, the basic building blocks, are bound in their unique structures by electric forces (with dynamics subject to the laws of quantum mechanics). The admittedly nonelectromagnetic energy source, the Sun, transmits its energy to us by electromagnetic radiation.

That radiation is converted by photosynthesis into other forms of energy, stored electromagnetic energy inside complex molecules. More immediately, the process of reading this page occurs through the scattering of light to the reader's eye, where it is absorbed and converted to electrical signals sent to the brain. We human beings and the plants and rocks around us are basically electromagnetic devices, interacting and communicating by means (whether chemical or physical) that are fundamentally electric in nature. Our societies are increasingly governed overtly by electricity and electronics.

The history of the development of our understanding of electricity and magnetism and of light and other electromagnetic radiation is a fascinating subject in itself. Here only the briefest account, with many omissions, can be given. The reader wishing to read a full and authoritative history should consult Whittaker (1951).

The properties of amber ($\eta\lambda\epsilon\kappa\tau\rho o\nu$, Greek) and lodestone were known in ancient times, but the creation of electrodynamics as a quantitative, mathematical subject occurred in about one hundred years (1766–1864). William Gilbert (1544–1603), in his book *De magnete,* published in 1600, clearly differentiated between electrical and magnetic phenomena. By 1733 the distinction had been made between resinous and vitreous electricity (surplus and deficiency of electrons, respectively). Forty years later, electrostatics was developed rapidly and quantitatively in about twenty years. Benjamin Franklin (1706–1790) made a crucial observation that was recognized by Joseph Priestley (1733–1804) as implying the inverse square law for electrical forces (1766). Henry Cavendish (1731–1810) conducted his remarkable experiments in electrostatics from 1771 to 1773. Charles Augustin Coulomb (1736–1806) began publishing his monumental researches in 1785.

These years marked the beginning of quantitative research in electricity and magnetism on a worldwide scale. Technical developments such as the invention of the battery in 1800 by Alessandro Volta (1745–1827) hastened progress in the study of electrical currents and their interactions. By 1820 Hans Christian Oersted (1777–1851) was observing the deflection of a compass needle by the current in a nearby wire, and André Marie Ampère (1775–1836) began publishing his results on the mechanical forces between current-carrying wires. By 1825 he had a mathematical description. In 1831 Michael Faraday (1791–1867) discovered his law of electromagnetic induction; in 1836 he used the famous Faraday cage to establish the law of conservation of electric charge. Faraday's many researches paved the way for the brilliant synthesis by James Clerk Maxwell (1831–1879), a dynamical theory of the electromagnetic field, published in 1864. Along the way, Karl Friedrich Gauss (1777–1855) and Simeon Denis Poisson (1781–1840) contributed importantly to the mathematical development, while practical concepts such as resistance were discovered (1827) by Georg Simon Ohm (1787–1854).

While Maxwell's paper of 1864 provided a complete classical description of electricity, magnetism, and light, its correctness and far-reaching applicability were not understood for some years. Not until 1888, when Heinrich Rudolf Hertz (1857–1894) gave a brilliant demonstration of the generation, transmission, polarization, reflection, and refraction of what we now know as radio waves, were the unity of light, Hertz's waves, and static electricity and magnetism fully appreciated. By the 1890s the basic principles of electromagnetism were thoroughly understood and ripe for the development of the myriad applications in the last hundred years.

This article is concerned with the fundamental principles of classical electrodynamics. Some derivations are sketched, but space limitations prevent lengthy proofs. Many results are just stated. References for details or elaboration are given in the text and also in the Further Reading section at the end. Applications are discussed elsewhere, under Electromagnetic Technology or Optics, for instance, as are more detailed treatments of special aspects, for example, ELECTROSTATICS, MAGNETOSTATICS, and ELECTROMAGNETIC RADIATION. The motion of charged particles under the action of external electric and magnetic fields is an important topic in its own right. The reader interested in these aspects should consult the articles on CHARGED-PARTICLE OPTICS; ACCELERATORS, LINEAR; CYCLOTRONS; SYNCHROTRONS; BETATRONS; PLASMA PHYSICS; INTENSE PARTICLE BEAMS; and FUSION, MAGNETIC CONFINEMENT.

The four laws that comprise the Maxwell equations are described physically and then mathematically in differential and integral form, with brief comments on the limitations imposed by the discreteness of physical charges and of quantum-mechanical modifications. The equations governing macroscopic fields inside material media, constitutive relations, and boundary conditions on fields at media interfaces precede a treatment of the frequency dependence of simple dielectric constants. The concept of causality is used to develop relations between the absorptive and dispersive parts of the dielectric constant (dispersion relations). The energies associated with electric and magnetic fields are exploited to define coefficients of capacitance and inductance, quantities useful in electric circuit analysis, and some examples of both are given. The scalar and vector potentials are introduced and their arbitrariness discussed (gauge transformations). Solutions of the wave equations in terms of the potentials are derived and the appropriateness of advanced and retarded solutions discussed. A general formula for radiation by a harmonically varying source is given. Poynting's theorem on the conservation of energy for fields and sources is derived, as is the conservation of momentum. The lumped-circuit concepts of impedance and admittance are discussed in terms of the fields by means of Poynting's theorem for harmonic fields. Because of its close connection to electrodynamics, the main ideas and results of special relativity are presented, along with a relativistic formulation of electrodynamics and a brief treatment of classical Lagrangian field theory.

## 1. UNITS, SYMBOLS, AND CONVENTIONS

The dimensions and units of quantities in electricity and magnetism are often viewed as contentious, with much heat and little light generated in polemics on the subject. The truth is that convenience of use is the only sensible criterion. Provided the schema is consistent, the choices of dimensions and units of magnitude of electromagnetic quantities are largely arbitrary, with no set more fundamental than any other. The interrelations among the various physical quantities through the laws of electrodynamics means, of course, that one does not have complete freedom of choice. The interested reader may consult the appendix of Jackson (1975) for a nonpartisan discussion. Traditionally (because mechanics preceded electromagnetism as a coherent subject), mass, length, and time are viewed as distinct in their dimensions. Systems of units differ in the amounts defined as unit magnitudes of each.

In this article the Système International (SI) of units is used, except when explicitly stated otherwise. In electromagnetism, the units of mass (the kilogram, abbreviated as kg), length (the meter, abbreviated as m), and time (the second, abbreviated as s) are augmented by a fourth, the unit of charge (the coulomb, abbreviated as C). The second is defined in terms of the frequency of a specific atomic transition, while the meter is defined as the distance light travels in vacuum in 1/299 792 458 seconds, or equivalently such that the speed of light in vacuum is $c = 299\,792\,458$ m/s. The electromagnetic parameters, the permittivity ($\epsilon_0$) and the permeability ($\mu_0$) of free space, are therefore both without experimental uncertainty. The permeability of free space is defined to be $\mu_0 = 4\pi \times 10^{-7}$ H/m, while the permittivity is $\epsilon_0 = 10^7/(4\pi c^2)$ F/m, where the henry (H) and the farad (F) are units of inductance and capacitance, respectively (see below). With the speed of light as a nine-digit number, $(4\pi\epsilon_0)^{-1} = 10^{-7}c^2$ is an exact 17-digit number (times a power of 10), the first six of which are $8.987\,55 \times 10^9$. To the same accuracy, $\epsilon_0 = 8.854\,19 \times 10^{-12}$ F/m. The quantity $\sqrt{\mu_0/\epsilon_0} = 376.730\ \Omega$ is called the impedance of free space. (See METROLOGY; ELECTROMAGNETIC QUANTITIES, MEASUREMENT OF BASIC.)

Harmonically varying quantities, or superpositions of them, play an important role in electromagnetism. Time-averaged entities such as power and energy are conveniently handled by means of complex numbers. The conventions are as follows.

*Complex numbers:*

$$z = x + iy, \quad \text{where} \quad i = \sqrt{-1}$$

$$\mathrm{Re}(z) = x, \quad \mathrm{Im}(z) = y,$$

$$z = \rho e^{i\theta}, \text{ where } \rho = |z| = \sqrt{x^2 + y^2} \text{ and}$$

$$\theta = \tan^{-1} y/x$$

$$z^* = x - iy = \rho e^{-i\theta}.$$

*Harmonic quantities:*

$$A(t)=A_0\cos(\omega t-\alpha)$$
$$=\mathrm{Re}(A_0 e^{i\alpha-i\omega t})$$
$$\rightarrow A_0 e^{i\alpha-i\omega t}.$$

*Time averages:*

$$\langle A(t)B(t)\rangle=\tfrac{1}{2}\mathrm{Re}[A(t)B^*(t)],$$

where $A(t)$ and $B(t)$ are understood to be complex amplitudes whose real parts correspond to the real harmonically varying physical quantities. It is often useful to discuss phenomena in either the time (space) or the frequency (wave-number) domain. An important tool is the Fourier integral, written here for time–frequency conversion, but easily adapted to space–wave-number domains in more than one dimension.

*Fourier integrals:*

$$A(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\ A(t)e^{i\omega t}, \qquad (1)$$

$$A(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\ A(\omega)e^{-i\omega t}.$$

If $A(t)$ is real, then $A(-\omega)=A^*(\omega)$. Subject to appropriate existence and convergence properties, the following relation holds between $A(t)$ and $A(\omega)$:

$$\int_{-\infty}^{\infty} |A(\omega)|^2 d\omega=\int_{-\infty}^{\infty} |A(t)|^2\, dt. \qquad (2)$$

If $A(t)$ is real, the frequency integral can be written as an integral over only positive frequencies of $2|A(\omega)|^2$.

## 2. THE MAXWELL EQUATIONS IN VACUUM IN DIFFERENTIAL AND INTEGRAL FORM

The basic equations governing electromagnetic phenomena are the Maxwell equations, supplemented by constitutive relations describing the responses of material media. We first summarize the Maxwell equations in vacuum in terms of the electric and magnetic fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, and electric charge and current densities, $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$.

### 2.1 Coulomb's Law

The first of the Maxwell equations stems from the 18th-century observations of Cavendish, Coulomb, Franklin, and others on the forces between electrically charged objects. The conclusion that the force between small charged bodies ("point charges") varies as the product of the charges, inversely as the square of the distance between them, and along the line joining them is expressed mathematically by the divergence equation,

$$\nabla\cdot\mathbf{E}=\rho/\epsilon_0, \qquad (3)$$

where $\mathbf{E}(\mathbf{r},t)$ is the electric field strength, defined as the force per unit charge, with units of volts per meter (V/m), while $\rho(\mathbf{r},t)$ is the electric charge density in units of coulombs per cubic meter (C/m$^3$). By means of the divergence theorem of vector calculus, Eq. (3) can be cast into integral form,

$$\int_S \mathbf{n}\cdot\mathbf{E}\, da=\frac{1}{\epsilon_0}\int_V \rho\, d^3x. \qquad (4)$$

The volume $V$ is bounded by the closed surface(s) $S$, with $\mathbf{n}$ being the outwardly directed normal to its surface. The left-hand integral is thus the total or integral flux of electric field emerging from the volume; the right-hand integral is the total charge inside, divided by $\epsilon_0$. Equation (4) is the integral electric flux theorem, or integral form of Coulomb's law. The simplest application of Eq. (4) is a point charge $Q$ at the center of a sphere of radius $r$. The electric field points radially outward and has magnitude $|\mathbf{E}(\mathbf{r})|=Q/4\pi\epsilon_0 r^2$. (See ELECTROSTATICS.)

### 2.2 Ampère–Maxwell Law

The second Maxwell equation concerns magnetic fields caused by currents and is known as the Ampère–Maxwell equation. Ampère's researches in the early 19th century were the study of forces between current loops carrying time-independent currents; Maxwell's contribution in mid-19th century was to recognize that Ampère's equation needed modification for time-varying sources and fields. The abstraction of a magnetic flux density $\mathbf{B}$ from the forces between currents is inherently more complex than that for the electric field $\mathbf{E}$ because there are no "point magnetic charges" [see the fourth Maxwell equation, Eq. (18)]. The magnetic force at a point (on a moving point electric charge, for example) is caused by a distribution of current; furthermore, the force is not central. The increment of magnetic flux density $d\mathbf{B}$ caused by an infinitesimal line element of current $I\ d\mathbf{l}$ is

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{I\ d\mathbf{l}\times\mathbf{r}}{r^3}, \qquad (5)$$

where $\mathbf{r}$ is the coordinate vector from the current element to the observation point and $I$ is the current (units of amperes=coulombs per second, abbreviated as A). The units of the magnetic induction are tesla (abbreviated as T). In reality, one must sum over all the current elements in a closed circuit, but the infinitesimal expression shows the parallels and differences with the electric field of a point charge $q$, $\mathbf{E}=q\mathbf{r}/4\pi\epsilon_0 r^3$. The magnetic flux density $\mathbf{B}$ of a long straight wire of circular cross section carrying a current $I$ points in the azimuthal direction (circles centered on the wire, with sense given by the right-hand rule) and has a magnitude $|\mathbf{B}|=\mu_0 I/2\pi d$, where $d$ is the perpendicular distance from the wire (Biot–Savart law).

If an infinitesimal current element $I\ d\mathbf{l}$ is in a magnetic flux density $\mathbf{B}$, the infinitesimal force on it is $d\mathbf{F}=I\ d\mathbf{l}\times\mathbf{B}$, a relation that may be viewed as an empirical result, but that has a more fundamental basis in the Lorentz force on a charge $q$ in motion with a velocity $\mathbf{v}$, $\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. The combination of Eq. (5) with the expression for the force on a current in a magnetic flux density permits the definition of the unit of current (1 A) in terms of the mechanical force per unit length between two very long, straight, and parallel current-carrying wires with separation $d$, namely,

$$\frac{dF}{dl}=\frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}. \qquad (6)$$

The force is attractive if the currents flow in the same direction and repulsive for opposite flow. Since $\mu_0$ is defined to have a definite numerical value, measurements of the mechanical force per unit length and the separation of the wires permit the current to be defined without reference to the magnetic flux density $\mathbf{B}$. Note, however, that electrodynamics is involved, because the meter is defined in terms of the speed of light. It is amusing that one of Ampère's basic observations in the 1820s is the basis for the accepted definition of the SI unit of current.

The general expression for the force on a closed circuit $C_1$, fixed in position and carrying a steady current $I_1$, caused by a steady current $I_2$ flowing in a fixed closed circuit $C_2$ is

$$\mathbf{F}_{12}=-\frac{\mu_0 I_1 I_2}{4\pi}\oint\oint\frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\mathbf{r}_{12}}{r_{12}^3}, \qquad (7)$$

where the integration is over both of the closed circuits.

In differential form, the Ampère–Maxwell equation relating the magnetic flux density to the current density is

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \qquad (8)$$

where the right-hand side is the sum of the conduction current density ($\mathrm{A/m^2}$) times $\mu_0$ and the Maxwell "displacement current" density. The displacement-current term is necessary to make Eq. (8) compatible with Eq. (3), Coulomb's law, and the conservation of charge and current densities,

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0. \qquad (9)$$

Use of Stokes's theorem from vector calculus converts Eq. (8) into the integral statement of Ampère's law:

$$\int_C \mathbf{B}\cdot d\mathbf{l}=\int_S \mathbf{n}\cdot\left(\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)da. \qquad (10)$$

The left-hand side is the line integral of $\mathbf{B}$ along a closed path $C$, while the right-hand side is $\mu_0$ times the total conduction and displacement current flowing through the open area bounded by $C$. (For more on Ampère's law and its use, see MAGNETOSTATICS.)

### 2.3 Faraday's Law

The third of the Maxwell equations is Faraday's law, codifying the observations of Faraday in the first half of the 19th century that time-varying magnetic fields produce electromotive forces. In differential form, Faraday's law reads

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0}. \tag{11}$$

Its integral form, obtained by application of Stokes's theorem for an open surface $S$ bounded by a closed curve $C$, is

$$\int_C \mathbf{E}\cdot d\mathbf{l}=-\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{n}\ da. \tag{12a}$$

If the circuit $C$ is stationary, the time derivative can be taken outside the integral sign to give

$$\int_C \mathbf{E}\cdot d\mathbf{l}=-\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}\ da. \tag{12b}$$

In words, the electromotive force (emf) or line integral of the electric field around the closed circuit $C$ (left-hand side) is equal to the negative of the time rate of change of the magnetic flux through the surface $S$ (right-hand side). The negative sign reflects the empirical observation called Lenz's law, whereby the current induced by the emf tends to counteract the change in magnetic field.

Application of Faraday's law often involves nonstationary circuits. Care must be taken in using Eqs. (12a) or (12b). To understand the exact meaning of Faraday's law, we go back to Faraday's observations of 1831. He discovered that transient current flow occurred in a circuit when conditions were changed in adjacent current-carrying circuits. Specifically, he found a transient current induced when (a) the steady current flowing in a nearby circuit was turned on or off, (b) a nearby circuit with a steady current was moved relative to the first, or (c) a permanent magnet was thrust into or out of the circuit. No transient occurred unless the adjacent circuit's current changed or there was relative motion of the two circuits. Other observations showed that alteration of the shape of the first circuit in the presence of a circuit with a steady current caused a transient current in the first while its shape was being altered. Faraday interpreted the transient current in every situation as being caused by a changing magnetic flux linking the observed circuit. The current in the circuit is driven, according to Ohm's law, by the force acting on the charge carriers in the wire. We are thus led to consider the current to be proportional to the electromotive force $\mathscr{E}$, defined for a stationary circuit as the line integral of the electric field around the path of the circuit, as given by the left-hand sides of Eqs. (12a) and (12b). With the flux through the circuit defined as

$$\Phi_M=\int_S \mathbf{B}\cdot\mathbf{n}\ da, \tag{13}$$

where $S$ is any open surface spanning the circuit $C$, Faraday's law in integral form is

$$\mathscr{E}=-\frac{d\Phi_M}{dt}. \tag{14}$$

To accommodate the possibility of movement of the circuit in time, we must generalize the definition of the electromotive force $\mathscr{E}$. Motion of the first circuit in the presence of a stationary second circuit or the same relative motion with the first circuit stationary produces the same transient current. But if an element $d\mathbf{l}$ of the first circuit is in motion with velocity $\mathbf{v}$, the charge carriers comoving with that element experience a force per unit charge equal to $\mathbf{E}+\mathbf{v}\times\mathbf{B}$. This means that the total electromotive force $\mathscr{E}$ causing the current flow in a circuit $C$ whose path elements have velocity $\mathbf{v}(\mathbf{l})$ must be

$$\mathscr{E}=\oint_C (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}. \tag{15}$$

With Eq. (15) for the electromotive force around a circuit in motion (translating and perhaps changing shape if $\mathbf{v}$ is different for different parts of the path), Eq. (14) generalizes Eq. (12b) and is applicable to any situation. One sometimes considers the electromotive force between two points $A$ and $B$ in space. Then the integral in Eq. (15) is an open line integral between the points. In contrast to the electrostatic situation, the emf is in general path dependent. Care must be exercised in distinguishing between electrostatic potential differences and electromotive forces. With a closed path through which the magnetic flux is changing in time, parts of the path may be in regions where $\partial\mathbf{B}/\partial t=0$. Over such a segment the line integral of $\mathbf{E}$ can be interpreted as a potential difference without ambiguity, since $\nabla\times\mathbf{E}=\mathbf{0}$ there.

To connect Faraday's law, derived rather directly from the transient current flow in resistive circuits and embodied in Eq. (14), with the Maxwell equation, Eq. (11), one can consider how the magnetic flux $\Phi_M$, Eq. (13), changes in time if the circuit $C$ is in motion.

Whether $C$ is in motion or not, the flux changes if $\mathbf{B}$ changes in time. There will therefore be a contribution to $d\Phi_M/dt$ from $\mathbf{n}\cdot\partial\mathbf{B}/\partial t$ at each point of the surface $S$. If the circuit $C$ is in motion, there is an additional contribution whose nature can be seen from Fig. 1(a). In an infinitesimal time $\Delta t$ the path $C(t)$ changes into $C(t+\Delta t)$. The surface spanning $C(t+\Delta t)$ can be taken as the surface spanning $C(t)$, minus the infinitesimal area around the sides, defined by the directed element of area, $d\mathbf{l}\times\mathbf{v}\Delta t$. Per unit time, the total change in magnetic flux is therefore

$$\frac{d\Phi_M}{dt}=\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{n}\,da-\oint_C \mathbf{B}\cdot(d\mathbf{l}\times\mathbf{v}). \qquad (16)$$

Rearranging the triple scalar product and substituting Eqs. (15) and (16) into (14), one finds

$$\oint_C (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}=-\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{n}\,da + \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}. \qquad (17)$$

The contribution to the time rate of change of the flux from the motion of the path $C$ cancels the magnetic part of the electromotive force; Eq. (17) becomes Eq. (12a). We have thus returned to the third Maxwell equation, (11). It should be noted that these equations contain no reference to apparatus. Thus the path $C$ and its spanning surface $S$ can be viewed as mathematical abstractions. The integral and differential forms of Faraday's law in terms of $\mathbf{E}$ and $\mathbf{B}$ have a validity and generality far beyond their concrete origins.

A simple illustration of Faraday's law and its relation to other approaches is afforded by the situation sketched in Fig. 1(b). A conducting right circular cylinder of radius $a$ rotates about its axis at angular speed $\omega=2\pi f$. There is a uniform magnetic induction parallel to the axis. Consider the open radial path $OA$, which is of length $r$ and is rotating with the conductor. From Eq. (15), the emf from the axis at $O$ to the point $A$ is

$$\begin{aligned}\mathscr{E}_{OA}&=\int_O^A (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{r}=\tfrac{1}{2}\,\omega B r^2\\ &=f(\pi r^2)B\\ &=f\Phi_M(r).\end{aligned}$$

Here $\Phi_M(r)$ is the magnetic flux contained within the radius $r$. The emf $\mathscr{E}_{OA}$ is equal to the rate at which the flux is cut by the rotating radial line.

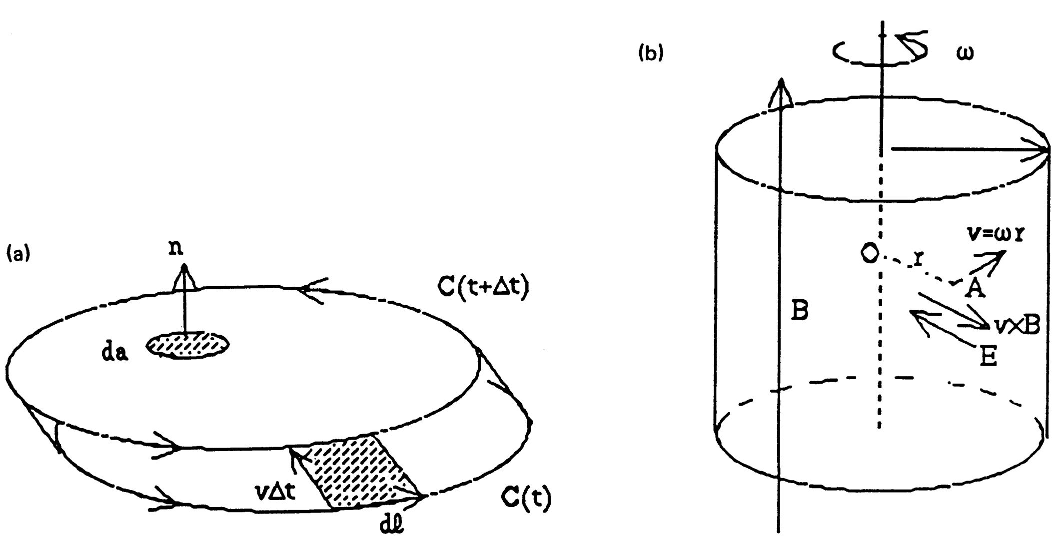


**FIG. 1.** (a) Closed circuit in Faraday's law. The closed path $C(t)$ changes in time $\Delta t$ to $C(t+\Delta t)$. The surface spanning $C(t+\Delta t)$, with directed area element $\mathbf{n}da$, can be taken as that spanning $C(t)$, minus the "sides" defined by the directed area elements $d\mathbf{l}\times\mathbf{v}\Delta t$ around the path. (b) A long conducting right circular cylinder of radius $a$ rotates about its axis at constant angular frequency $\omega$ in a uniform magnetic field $\mathbf{B}$ parallel to the axis.

An alternative way of approaching the problem is to consider Ohm's law in the rotating conductor. The current density is given by $\mathbf{J}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. In the steady state no current flows. Within the conductor, there is an electrostatic field developed to oppose the magnetic force. At the typical point $A$ (now considered as fixed in space, rather than rotating with the conductor), the electric field is radial and given by $E_r=-\omega Br$. There is a uniform volume charge density, $\rho=\epsilon_0\nabla\cdot\mathbf{E}=\epsilon_0(1/r)\partial(rE_r)/\partial r=-2\omega B\epsilon_0$. If the long cylinder is uncharged, there will be a surface charge density, $\sigma=\omega Ba\epsilon_0$. (There is a small magnetic field generated by the motion of these charge densities, but it is of higher order in smallness for surface speeds small compared to the speed of light.)

If sliding contacts fixed in space are made with the axis of the cylinder and its surface, a potential difference equal to the emf $\mathscr{E}_{OA}(a)$ or to the line integral of the electrostatic field,

$$V_{OA}=-\int_O^a E_r\,dr,$$

will be measured by a voltmeter. This example is qualitatively similar to one of Faraday's original experiments with a rotating copper disk.

### 2.4 Law of Magnetic Flux

The final Maxwell equation embodies the fact that, as far as is presently known, there are no magnetic charges or magnetic charge density. Its differential and integral forms parallel the first Maxwell equation, namely,

$$\nabla\cdot\mathbf{B}=0, \tag{18}$$

$$\int_S \mathbf{B}\cdot\mathbf{n}\,da=0. \tag{19}$$

Equation (19) is the integral magnetic flux theorem, stating that the total magnetic flux emerging from a closed surface $S$ vanishes.

### 2.5 Limitations

The equations listed above are the Maxwell equations of *classical* electrodynamics in vacuum, an accurate and wonderfully successful description of myriad phenomena at the classical level. At the atomic and subatomic levels, there are quantum modifications. For example, basic to the discussion so far is the assumption that the charge and current densities are continuous, corresponding to infinitely divisible charges. But Nature supplies charges in discrete units, integral multiples of the electronic charge ($e=-1.602\times10^{-19}$ C). In many practical macroscopic applications, this unit is so small that the discreteness of charge densities can be ignored. An important relativistic quantum effect is the modification of the inverse square law of electrostatics at short distances ($r<\hbar/m_ec$, the electronic Compton wavelength). This modification, causing the force to be stronger at short distances and resulting in a tiny shift of atomic energy levels, is the result of "vacuum polarization" (see QUANTUM ELECTRODYNAMICS). A discussion of the idealizations and limitations of classical electrodynamics can be found in the introductory chapter of Jackson (1975).

## 3. MATERIAL MEDIA, CONSTITUTIVE RELATIONS, BOUNDARY CONDITIONS, AND SIMPLE DIELECTRIC AND MAGNETIC PROPERTIES OF MATTER

### 3.1 Macroscopic Maxwell Equations

The basic equations of electromagnetism of the previous section can be viewed as a description at the microscopic level, with the charge and current densities representing all the sources, whether free or bound. At length and time scales large compared to the atomic ($L\gg10^{-10}$ m, $T\gg10^{-17}$ s), the rapid fluctuations of the microscopic fields are unobservable. Only suitable averages over microscopically large, but macroscopically small, regions of space and/or time are relevant. Such averaged sources and fields are called macroscopic fields and sources. They satisfy the *macroscopic Maxwell equations* given below, and reflect the response of the material media to applied fields and sources.

In the process of averaging the microscopic equations, a distinction is made between bound charges (and atomic circulating currents) and the free charges (conduction electrons) whose averages define the familiar macroscopic charge and current densities, $\rho(\mathbf{r},t)$ and $\mathbf{J}(r,t)$:

$$\rho(\mathbf{r},t)=\langle\rho_{\text{free}}(\mathbf{r},t)\rangle,\quad \mathbf{J}(\mathbf{r},t)=\langle\mathbf{J}_{\text{free}}(\mathbf{r},t)\rangle, \tag{20}$$

where the brackets ⟨ ⟩ indicate spatial averaging or its equivalent (Jackson, 1975, Sec. 6.7; Robinson, 1973, Chaps. 6–8). The contributions of the bound charges and currents are systematized in terms of the electric polarization $\mathbf{P}(\mathbf{r},t)$ [dominated by the averaged electric dipole moment density, but including in principle the electric quadrupole density $Q'_{jk}(\mathbf{r},t)$ and higher multipoles], and the magnetization $\mathbf{M}(\mathbf{r},t)$ (averaged magnetic dipole moment density, plus higher magnetic multipoles). These polarizations require the introduction of two new phenomenological fields $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$, called the electric displacement and magnetic field strength, respectively, in the equations involving sources [Eqs. (3) and (8)]. The local macroscopic Maxwell equations are

$$\nabla\cdot\mathbf{D}=\rho,$$

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t},$$

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0},$$

$$\nabla\cdot\mathbf{B}=0. \qquad (21)$$

Here the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ of the two homogeneous equations are just the averaged quantities of the corresponding microscopic equations. The new macroscopic fields that include the responses of the bound charges are

$$D_i=\epsilon_0 E_i+P_i \qquad (22)$$

and

$$H_i=\frac{1}{\mu_0}\,B_i-M_i. \qquad (23)$$

Since $\mathbf{B}$ is the fundamental magnetic field quantity and $\mathbf{H}$ is phenomenological, Eq. (23) is the rational way to express the relationship. The minus sign associated with the magnetization is a natural one. It is conventional, however, to rearrange Eq. (23) and write the connections between the four fields as

$$\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P},$$

$$\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}), \qquad (24)$$

even though the basic physics is thereby obscured.

Integral statements of the Maxwell equations can be written down from Eqs. (21) in complete analogy with the development in Sec. 2.

### 3.2 Constitutive Relations

The Maxwell equations (21) are eight coupled partial-differential equations in space and time for the twelve field quantities **E**, **B**, **D**, and **H**. The system cannot be solved without the *constitutive relations* relating **D** and **H** to **E** and **B**:

$$\mathbf{D}=\mathbf{D}[\mathbf{E},\mathbf{B}], \quad \mathbf{H}=\mathbf{H}[\mathbf{E},\mathbf{B}]. \qquad (25)$$

For conducting media, there is a generalized Ohm's law,

$$\mathbf{J}=\mathbf{J}[\mathbf{E},\mathbf{B}]. \qquad (26)$$

The square brackets in Eqs. (25) and (26) signify that the connection may be nonlinear, nonlocal, and dependent on past history (hysteresis). The connections (24) show that the electric polarization and the magnetization of matter determine the nature of the constitutive relations. In general, these relations are far from simple, especially in crystalline solids, with ferroelectric and ferromagnetic materials having nonzero **P** or **M** even in the absence of applied fields. The study and elucidation of these properties is the province of condensed-matter physics. Only the simplest aspects are discussed here (for more detail, see Kittel, 1986; OPTICAL PROPERTIES OF SOLIDS; DIELECTRIC PROPERTIES OF INSULATORS; MAGNETOSTATICS; MAGNETIC MATERIALS).

In substances other than ferroelectrics or ferromagnets, a weak applied electric or magnetic field induces a polarization proportional to the magnitude of the applied field. The response of the medium is then said to be linear, and the constitutive relations take the form

$$D_i=\sum_{k=1}^{3}\epsilon_{ik}E_k, \quad B_i=\sum_{k=1}^{3}\mu_{ik}H_k. \qquad (27)$$

The tensors $\epsilon_{ik}$ and $\mu_{ik}$ are called the electric permittivity or dielectric tensor and the magnetic permeability tensor, respectively. They are generally dependent on the frequency of the applied field, and on bulk properties of the material, such as density and temperature, as well as on details of molecular and perhaps crystalline structure. For simple materials (gases, liquids, some solids) the linear response is isotropic in space. Then $\epsilon_{ik}=\epsilon\delta_{ik}$ and $\mu_{ik}=\mu\delta_{ik}$; $\mathbf{D}=\epsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$; $\epsilon/\epsilon_0=\epsilon_r$ is called simply the (relative) dielectric "constant," and $\mu/\mu_0=\mu_r$ the (relative) magnetic permeability. Many crystals have anisotropies in their dielectric constant (or magnetic permeability). The elements of the tensor $\epsilon_{ik}$ form a nonsingular matrix with eigenvalues $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, and associated principal axes in the crystal (which may or may not be along the axes of symmetry of the crystal). When expressed with respect to these axes, the tensor $\epsilon_{ik}$ takes the diagonal form $\epsilon_{ik}=\epsilon_i\delta_{ik}$. Unless the eigenvalues are all equal, such crystals (e.g., calcite or mica) exhibit birefringence and double refraction. See Hecht and Zajac (1974), Sec. 8.2.

For orientation, we mention that at low frequencies ($\nu<10^6$ Hz), where all charges, regardless of their masses, respond to an applied electric field, solids have dielectric constants typically in the range $\epsilon_r=\langle\epsilon_{ii}\rangle/\epsilon_0\approx 2$–$20$, with larger values not uncommon. Systems with permanent molecular dipole moments can have much larger and temperature-sensitive dielectric constants. Distilled water, for example, has a static ($\nu=0$) dielectric constant of $\epsilon_r=80$ at 0 °C and $\epsilon_r=56$ at 100 °C. At optical frequencies only the electrons can respond significantly. The dielectric constants are frequency dependent in the range $\epsilon_r\approx 1.7$–$10$, with $\epsilon_r\approx 2$–$3$ for most solids. Water has $\epsilon_r\approx 1.77$–$1.80$ over the visible range, essentially independent of temperature from 0 to 100 °C.

Applied magnetic fields induce different responses, depending upon the properties of the individual atoms and also on their interactions. *Diamagnetic* materials consist of atoms or molecules with no net angular momentum. The response to an applied magnetic field is the creation of circulating atomic currents that produce a very small bulk magnetization opposed to the applied field ($\mu_{ii}<\mu_0$). Bismuth, the most diamagnetic substance known, has $1-\mu_r=1.8\times 10^{-4}$. Diamagnetism is clearly a very small effect. If the basic atomic unit of the material has a net angular momentum from an unpaired electron, the substance is *paramagnetic*. The magnetic moment of the odd electron is aligned parallel to the applied field and $\mu>\mu_0$. Typical values are in the range $\mu_r-1\approx 10^{-2}$–$10^{-4}$ at room temperature, decreasing at higher temperatures because of the randomizing effect of thermal excitations. (See MAGNETOSTATICS.)

Ferromagnetic materials are paramagnetic (unpaired electrons), but show drastically different behavior because of interactions among the atoms. Below the Curie temperature (1040 K for iron, 630 K for nickel), ferromagnetic substances show spontaneous magnetization; all the atomic magnetic moments within a microscopically large region called a domain are aligned. Neighboring domains have random directions of alignment, but the application of a very modest external magnetic field tends to cause domains to merge or to line up in the same direction, leading to the saturation of the bulk magnetization. Removal of the field leaves a considerable fraction of the moments still aligned. The residual "permanent" magnetization can be as large as $\mu_0|\mathbf{M}|\gtrsim 1$ T. (See MAGNETS; MAGNETIC MATERIALS.)

For data on dielectric and magnetic properties of materials, the reader may consult Weast, *Handbook of Chemistry and Physics*, 69th ed. (1989).

Materials that show a linear response in weak fields eventually show *nonlinear behavior* at high enough field strengths as the electronic or ionic oscillators are driven to large amplitudes. The linear relations (27) are modified to, for example,

$$D_i=\sum_k \epsilon_{ik}^{(1)}E_k+\sum_{k,m}\epsilon_{ikm}^{(2)}E_kE_m+\cdots. \tag{28}$$

For static fields the consequences are not dramatic, but for time-varying fields it is another matter. A large-amplitude wave of two frequencies $\omega_1$ and $\omega_2$ generates waves in the medium with frequencies 0, $2\omega_1$, $2\omega_2$, $\omega_1+\omega_2$, $\omega_1-\omega_2$, as well as the original $\omega_1$ and $\omega_2$. From cubic and higher nonlinear terms an even richer spectrum of signals can be generated. Lasers can produce peak electric fields $\gtrsim 10^{12}$ V/m, comparable to or larger than the internal electric fields in atoms ($e/4\pi\epsilon_0a_0^2\approx 5\times 10^{11}$ V/m). The regime of nonlinearity is readily accessible for research in the laboratory and provides a vast array of applications. (See LASER PHYSICS; OPTICS, NONLINEAR.) The treatment here is restricted to linear phenomena.

### 3.3 Boundary Conditions at Interfaces between Different Media

At interfaces between different media, with different constitutive relations between **E** and **D** and **B** and **H**, and perhaps with surface charge and current densities, the various fields obey boundary conditions that can be inferred from the integral forms of the four Maxwell equations. Application of the divergence theorem to the first and last equations in (21) yields the integral statements

$$\int_S \mathbf{D}\cdot\mathbf{n}\, da = \int_V \rho\, d^3x \tag{29a}$$

and

$$\int_S \mathbf{B}\cdot\mathbf{n}\, da = 0. \tag{29b}$$

Here $S$ is the closed surface bounding the volume $V$. Similarly, use of Stokes's theorem on the second and third equations in (21) yields the integral relations

$$\int_C \mathbf{H}\cdot d\mathbf{l} = \int_{S'} \mathbf{t}\cdot\left(\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}\right) da \tag{30}$$

and

$$\int_C \mathbf{E}\cdot d\mathbf{l} = -\int_{S'} \mathbf{t}\cdot\frac{\partial \mathbf{B}}{\partial t} da. \tag{31}$$

In (30) and (31) the surface $S'$ is an open surface bounded by the closed curve $C$, and **t** is a unit normal to the surface $S'$.

For an interface between different media the appropriate geometrical arrangement of volumes and surfaces is shown in Fig. 2. The Gaussian pillbox of volume $V$, with top and bottom surfaces just inside each medium, and normal **n** defined as going from medium 1 to medium 2, is suitable for Eqs. (29a) and (29b). In the limit of vanishing height of the pillbox, these equations yield relations between the normal components of **D** and **B** on either side of the boundary surface, namely,

$$(\mathbf{D}_2-\mathbf{D}_1)\cdot\mathbf{n}=\sigma, \quad (\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n}=0. \tag{32}$$

In words, the normal component of **B** is continuous, while the discontinuity of the normal component of **D** is equal to the surface charge density $\sigma$ at that point.

Applying Eqs. (30) and (31) to the infinitesimal Stokesian loop in Fig. 2, with sides of negligible length perpendicular to the surface, gives relations between the tangential components of **E** and **H** on either side of the interface,

$$\mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1)=\mathbf{0}, \quad \mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}. \tag{33}$$

The surface current **K** (with dimensions of current per unit length and units A/m) is assumed to have only components parallel to the interface. The tangential component of **E** is continuous across the interface, while the tangential component of **H** is discontinuous by an amount whose magnitude is equal to the magnitude of the surface current **K** and whose direction is parallel to $\mathbf{K}\times\mathbf{n}$. The discontinuity equations (32) and (33) for the normal and tangential components of the fields are useful in solving the Maxwell equations in different regions and then splicing them together to provide a solution throughout all space.

For completeness it can be noted that Eq. (9), the differential statement of charge–current-density conservation, leads to a corresponding boundary condition for the volume current density and the surface current and charge densities,

$$(\mathbf{J}_2-\mathbf{J}_1)\cdot\mathbf{n}+\nabla_{\parallel}\cdot\mathbf{K}+\frac{\partial\sigma}{\partial t}=0. \tag{34}$$

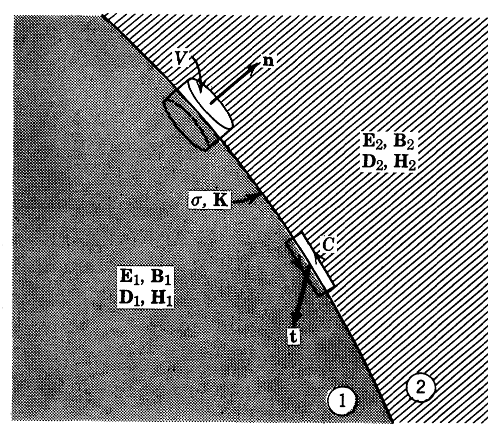


**FIG. 2.** Schematic diagram of boundary surface (heavy line) between different media. The boundary is assumed to carry idealized surface charge and current densities, $\sigma$ and **K**. The volume $V$ is a pillbox with its plane surfaces parallel to the interface. The normal **n** to the planes points from medium 1 to medium 2. The rectangular contour $C$, half in one medium and half in the other, is oriented so that the normal $t$ to its spanning surface is tangent to the interface. [From Jackson (1975), reproduced by permission of John Wiley & Sons, Inc.]

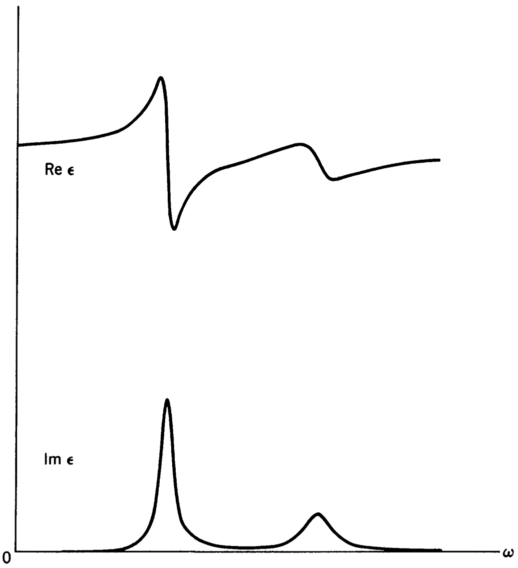


**FIG. 3.** Real and imaginary parts of the dielectric constant $\epsilon(\omega)$ as functions of frequency $\omega$ in the neighborhood of two resonances. [From Jackson (1975), reproduced by permission of John Wiley & Sons, Inc.]

### 3.4 Frequency Dependence of the Dielectric Constant

A completely correct description of the linear response of a material medium to applied electromagnetic fields requires the use of quantum mechanics, but the behavior of the dielectric constant can be inferred from largely classical considerations. Consider a model of the atom as a collection of $Z$ electrons, each of mass $m$ and charge $-e$, bound harmonically, with $f_k$ of them having circular binding frequency $\omega_k$ and damping constant $\gamma_k$ ($\Sigma_k f_k = Z$). In the low-density limit in which the applied fields can be assumed to be those acting on the individual atoms, each electron obeys an equation of motion of the form

$$m(\ddot{\mathbf{x}}+\gamma_k\dot{\mathbf{x}}+\omega_k^2\mathbf{x}) = -e\mathbf{E}(0,t),$$

where the amplitude of oscillation is assumed small enough that the applied electric field can be evaluated at the average position of the charge (dipole approximation). For sinusoidal time dependence at frequency $\omega$, the atomic dipole moment can be found as a sum over the contributions of all the electrons. With $N$ atoms per unit volume, the dielectric constant at frequency $\omega$ is

$$\epsilon(\omega) = \epsilon_0 + \frac{Ne^2}{m} \sum_k f_k(\omega_k^2 - \omega^2 - i\omega\gamma_k)^{-1}. \quad (35)$$

With suitable quantum-mechanical definitions of $f_k$ (called the oscillator strengths), $\gamma_k$, and $\omega_k$, Eq. (35) is an accurate description of the electronic contributions to the dielectric constant. The relation $\Sigma_k f_k = Z$ is called the oscillator strength sum rule. Classically, it merely records the fact that there are $Z$ electrons in the atom; quantum mechanically, it is a statement about the sum of the squares of the transitional dipole moments, weighted by the transitional frequencies, from a given quantum state to all others.

The damping constants $\gamma_k$ in Eq. (35) are generally very small compared to the resonance frequencies $\omega_k$. This means that $\epsilon(\omega)$ is approximately real for most frequencies, and, despite the fact that the overall trend with increasing frequency is downward, the most common behavior, called *normal dispersion,* is for $\mathrm{Re}[\epsilon(\omega)]$ to increase with frequency. The overall trend is produced by rather violent behavior in narrow regions around $\omega \approx \omega_k$, called regions of *anomalous dispersion,* in which $\mathrm{Re}[\epsilon(\omega)]$ decreases rapidly with frequency. A typical situation is illustrated in Fig. 3, which shows the behavior of the real and imaginary parts of $\epsilon(\omega)$ in the neighborhood of two resonances. In a region of anomalous dispersion, the imaginary part of $\epsilon(\omega)$ shows a typical resonant form. A significant $\mathrm{Im}[\epsilon(\omega)]$ represents dissipation of electromagnetic energy from the wave into the medium—such regions are called regions of resonant absorption. The fractional decrease in intensity $(\Delta I/I)$ of a monochromatic electromagnetic wave, per wavelength divided by $2\pi$ $(\Delta z = \lambda/2\pi)$, can be shown to be given by $\mathrm{Im}[\epsilon(\omega)]/\mathrm{Re}[\epsilon(\omega)]$.

Ohm's law is the statement that the electric field strength and the current density are related by $\mathbf{E} = \rho\mathbf{J}$, where the proportionality constant $\rho$, characteristic of the material, is called the electrical resistivity (units of ohms times meters, abbreviated as $\Omega$ m). Ohm's law holds in a wide range of practical situations, although it is clearly an approximation and sometimes a gross oversimplification. If it is inserted in the second Maxwell equation in Eq. (21) and the medium is linear, the right-hand side of the equation is linear in the electric field (and its time derivative). This suggests that electrical conductivity and dielectric response are actually two facets of the same phenomenon. Conduction is caused by the motion of quasifree electrons (and holes). Suppose a fraction $f_0$ of the "bound" electrons in the model for $\epsilon(\omega)$ are actually free, that is, they have no binding—their resonant frequency is zero. Their contribution to $\epsilon(\omega)$ in Eq. (35) can be separated off to give

$$\epsilon(\omega)=\epsilon_{\text{bound}}+i\frac{Ne^2f_0}{m\omega(\gamma_0-i\omega)}. \tag{36}$$

Because it is $-i\omega\epsilon(\omega)$ that enters the right-hand side of the Ampère–Maxwell equation, the singular behavior of the contribution from the free electrons represents a well-behaved resistivity,

$$\rho=m(\gamma_0-i\omega)/Ne^2f_0. \tag{37}$$

This expression is essentially the Drude (1900) expression for electrical resistivity, with $f_0N$ being the number of free electrons per unit volume. The damping constant $\gamma_0/f_0$ can be determined empirically. For copper, $N\approx 8\times 10^{22}$ atoms/cm$^3$, and at normal temperatures the low-frequency resistivity is $\rho\approx 1.8\times 10^{-8}$ $\Omega$ m. This gives $\gamma_0/f_0\approx 3\times 10^{13}$ s$^{-1}$. With the assumption that $f_0=O(1)$, Eq. (37) shows that up to frequencies well beyond the microwave region ($\omega\leqslant 10^{11}$ s$^{-1}$) resistivities of metals are essentially real and independent of frequency. At higher frequencies the simple model fails in detail, but still describes the qualitative behavior. The reader may consult Wilson (1953) or Beam (1965) for details on the role of lattice vibrations, lattice imperfections, and impurities in determining the temperature dependences and magnitudes of resistivities.

At frequencies far above the highest atomic resonance, the expression (35) for $\epsilon(\omega)$ simplifies to

$$\epsilon_r(\omega)\approx 1-\omega_p^2/\omega^2,$$

where

$$\omega_p^2=NZe^2/\epsilon_0 m. \tag{38a}$$

The frequency $\omega_p$, which depends only on the total number $NZ$ of electrons per unit volume, is called the *plasma frequency* of the medium. In a metal, if $\omega\gg\gamma_0$, the conduction electrons give a term like the second one in Eq. (38a), so that one has

$$\epsilon_r(\omega)\approx\epsilon_{r,\text{bound}}-\omega_p^2/\omega^2,$$

where

$$\omega_p^2=ne^2/\epsilon_0 m^*. \tag{38b}$$

The plasma frequency is now that of the conduction electrons only, with an effective mass $m^*$ to include effects of the binding (band structure). At frequencies such that $\omega\ll\omega_p$, the last term dominates; $\epsilon_r$ is negative; the index of refraction is purely imaginary; waves are reflected by the metal. When $\omega$ is sufficiently large that $\epsilon_r(\omega)>0$, waves can penetrate and propagate through the metal. Typically, this transition occurs in the ultraviolet and one speaks of the ultraviolet transparency of metals. Aluminum, for example, shows a sharp drop in reflectance for wavelengths shorter than about 80 nm, corresponding to a conduction-electron plasma energy of $\hbar\omega_p\approx 15$ eV compared to a theoretical value of 15.7 eV from Eq. (38b) with three conduction (valence) electrons per atom and $m^*=m$. For further discussion of this transition and other dielectric properties of metals in the optical and ultraviolet regions, see Pines (1963), Wooten (1972), and OPTICAL PROPERTIES OF SOLIDS.

We mentioned above that the free-electron contribution to the dielectric constant could be viewed alternatively as a representation of Ohm's law, while the bound electrons gave the usual dielectric constant. The dominance of one or the other contribution governs the behavior of fields in matter. Consider for simplicity a uniform isotropic linear medium with resistivity $\rho$, dielectric constant $\epsilon$, and magnetic permeability $\mu$, all real. For harmonic fields, the second and third Maxwell equations in Eq. (21), combined with the fourth and Ohm's law, lead to a Helmholtz wave equation for **H**,

$$\nabla^2\mathbf{H}+k^2\mathbf{H}=\mathbf{0},$$

where

$$k^2=i\omega\mu(\rho^{-1}-i\omega\epsilon)$$

is the square of the (complex) wave number. If the medium is poorly conducting, the Maxwell displacement current dominates and $k^2\approx\mu\epsilon\omega^2$. The wave vectors are approximately real, and the electric field is in time phase with the magnetic field. If, on the other hand, the resistivity is small enough (i.e., $\rho\omega\epsilon\ll 1$), the conduction current dominates and $k^2\approx i\omega\mu/\rho$. Then the wave numbers are complex,

$k \approx \pm[(1+i)/\sqrt{2}]\sqrt{\omega\mu/\rho}$,

with equal real and imaginary parts. The electric field is 135° out of phase with the magnetic field. For the plane surface of a good conductor, the field components vary with distance $x$ into the conductor as $e^{-x/\delta} \times \cos(x/\delta+\beta)$, where the skin depth $\delta$ is

$$\delta = \sqrt{2\rho/\omega\mu}. \tag{39}$$

The behavior in other geometries is similar—for cylinders, for example, the fields are proportional to Kelvin's ber and bei functions, Bessel functions of complex argument. For copper ($\rho \approx 1.7 \times 10^{-8}$ Ω m at 20 °C) the skin depth is numerically $\delta \approx 6.6/\sqrt{\nu(\text{Hz})}$ cm. At 10 GHz, the skin depth in copper is less than 1 μm. The concept of skin or penetration depth in conducting media applies in several domains, from quasistatic induction heating to microwave components.

### 3.5 Causality and Analytic Properties of $\epsilon(\omega)$

The dielectric constant, Eq. (35), viewed as a function of complex $\omega$, has a pattern of singularities and domain of analyticity that are a consequence of the causal nature of the response of the charged atomic particles to applied fields. If a single-frequency component of the field is considered, **D** and **E** at a given point in space are related by

$$\mathbf{D}(\mathbf{r},\omega) = \epsilon(\omega)\mathbf{E}(\mathbf{r},\omega). \tag{40}$$

If use is made of the Fourier integrals in time and frequency [Eq. (1)], $\mathbf{D}(\mathbf{r},t)$ can be expressed nonlocally (in time) in terms of $\mathbf{E}(\mathbf{r},t)$ through

$$\mathbf{D}(\mathbf{r},t) = \epsilon_0\left(\mathbf{E}(\mathbf{r},t) + \int_{-\infty}^{\infty} G(\tau)\mathbf{E}(\mathbf{r},t-\tau)\,d\tau\right), \tag{41}$$

where $G(\tau)$ is the Fourier transform of $\epsilon_r(\omega) - 1$,

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\epsilon_r(\omega)-1]e^{-i\omega\tau}\,d\omega. \tag{42}$$

$G(\tau)$ is called the susceptibility kernel. A simple, single-oscillator model of $\epsilon(\omega)$ is sufficient to illustrate its general features. If $\epsilon_r(\omega)$ is given by

$$\epsilon_r(\omega) = 1 + \omega_p^2(\omega_0^2 - \omega^2 - i\omega\gamma)^{-1}, \tag{43}$$

then

$$G(\tau) = \omega_p^2 e^{-\gamma\tau/2}\,\frac{\sin \nu_0\tau}{\nu_0}\,\theta(\tau), \tag{44}$$

with

$$\nu_0 = \sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2},$$

where $\theta(\tau)$ is the unit step function [$\theta(x)=1$ for $x>0$; $\theta(x)=0$ if $x<0$]. Evidently the susceptibility kernel for Eq. (35) is just a linear superposition of terms like Eq. (44). There are three major features: (a) $G(\tau)$ is causal, that is, it vanishes for $\tau<0$. This is a general feature, not restricted to the particular model used here. (b) $G(\tau)$ is oscillatory, at a frequency that is closely the resonant frequency $\omega_0$ (since $\gamma \ll \omega_0$). (c) $G(\tau)$ is damped in time for $\tau>0$ with the damping constant of the electronic oscillator. These damping constants, quantum mechanically the widths in frequency of the relevant spectral lines, are typically $10^7$–$10^9$ s$^{-1}$. The nonlocality in time in Eq. (41) is therefore of the order of $10^{-7}$–$10^{-9}$ s. For frequencies above the microwave region many earlier cycles of the electric field oscillations contribute to the value of **D** at a given instant of time. [Actually, the response of the medium is nonlocal in space as well as time, but for insulators the spatial nonlocality is only of the order of atomic sizes. The concept of a dielectric constant $\epsilon(\omega)$ or index of refraction $n(\omega)$ therefore applies far beyond the visible into the soft-x-ray region. For metals, especially at low temperatures, where electrons have very long mean free paths, the nonlocal region can be macroscopic in size. See Pippard (1960).]

Viewed as a function of complex $\omega$, the dielectric constant, Eq. (35), or its special case, Eq. (43), has poles only in the lower half-plane. The absence of singularities above the real axis is directly responsible for the causal behavior of $G(\tau)$. The relation inverse to Eq. (42) expresses $\epsilon(\omega)$ in terms of $G(\tau)$ as

$$\epsilon_r(\omega) = 1 + \int_0^{\infty} G(\tau)e^{i\omega\tau}\,d\tau. \tag{45}$$

From the reality of $G(\tau)$ it can be inferred from Eq. (45) that (a) for complex $\omega$, $\epsilon(-\omega) = \epsilon^*(\omega^*)$; (b) $\epsilon(\omega)$ is an analytic function of $\omega$ in the upper half-plane and on the real axis ($\omega \neq 0$), provided $G(\tau)$ is finite for all $\tau$. At the origin it is necessary to invoke the requirement that $G(\tau)\rightarrow 0$ as $\tau\rightarrow\infty$ to assure that $\epsilon(\omega)$ is also analytic there. This is true for insulators, but not for conductors. For the latter, $G(\tau)\rightarrow$const as $\tau\rightarrow\infty$ and $\epsilon(\omega)$ has a pole at $\omega=0$, as already discussed in connection with Eq. (36).

The analyticity of $\epsilon(\omega)$ in the upper half-plane permits us to use Cauchy's theorem to deduce "dispersion relations" or Kramers–Kronig relations (1926–27), as they are called. [See Jackson (1975), p. 310ff, for a derivation.] These relations express the real part of $\epsilon_r(\omega)$ as an integral over the imaginary part, and vice versa. For insulators (without the pole at $\omega=0$), the Kramers–Kronig relations are

$$\mathrm{Re}\ \epsilon_r(\omega)=1+\frac{2}{\pi}\ \mathrm{P}\int_0^\infty \frac{\omega'\ \mathrm{Im}\ \epsilon_r(\omega')}{\omega'^2-\omega^2}\,d\omega',$$

$$\mathrm{Im}\ \epsilon_r(\omega)=-\frac{2}{\pi}\ \mathrm{P}\int_0^\infty \frac{[\mathrm{Re}\ \epsilon_r(\omega')-1]}{\omega'^2-\omega^2}\,d\omega'. \tag{46}$$

Here the symbol P denotes "principal part" of the integral, namely, the integral over $\omega'$, omitting the symmetric interval around $\omega'=\omega$ from $\omega'=\omega'-\epsilon$ to $\omega'=\omega+\epsilon$, in the limit $\epsilon\rightarrow 0$. The power of these relations lies in their generality. Experimental information on absorption [Im $\epsilon(\omega)$] permits calculation of Re $\epsilon(\omega)$, etc. The asymptotic behavior of $\epsilon(\omega)$ for large $\omega$, Eq. (38a), can be used as a definition of the plasma frequency, i.e., $\omega_p^2=\lim_{\omega\rightarrow\infty}\{\omega^2[1-\epsilon_r(\omega)]\}$. Application of this limit to the first equation in Eqs. (46) leads to a "sum rule" for the plasma frequency, equivalent to the sum rule for the oscillator strengths,

$$\omega_p^2=\frac{2}{\pi}\int_0^\infty \omega\ \mathrm{Im}\ \epsilon_r(\omega)d\omega. \tag{47}$$

This optical sum rule is one of many that can be developed (Altarelli *et al.*, 1972; Wooten, 1972).

The analyticity of $\epsilon(\omega)$ translates into analytic properties for its square root, the index of refraction, relevant for the propagation of electromagnetic waves in matter. In two historical papers in 1914, Sommerfeld and his student Brillouin (a) showed that no electromagnetic disturbance in matter can propagate faster than the speed of light in vacuum, despite the fact that the phase velocity of the wave in the material can exceed that speed; and (b) analyzed in detail the rather complex arrival of the signal at a distant point. See Sec. 7.11 of Jackson (1975) for a simplified account, including discussion of an experimental demonstration, and Brillouin (1960) for translations of the original papers and subsequent work.

## 4. ELECTROSTATIC AND MAGNETOSTATIC ENERGIES, CAPACITANCE, AND INDUCTANCE

(See also ELECTROSTATICS; MAGNETOSTATICS; CIRCUIT ELEMENTS; ELECTROSTATIC CAPACITIVE STORAGE OF ENERGY; INDUCTORS.)

### 4.1 Electrostatic Energy

For static fields Faraday's law states that the electric field has zero curl. It can therefore be written as the gradient of a scalar potential function $\phi(\mathbf{r})$, called the electric potential. Conventionally, the signs are chosen so that

$$\mathbf{E}=-\nabla\phi. \tag{48}$$

If an infinitesimal point charge $\delta q$ is moved from infinity to a point at $\mathbf{r}$ through a preexisting electric field $\mathbf{E}$ whose potential vanishes at infinity, the mechanical work done on the charge (the negative of the work done by the field) is $\delta W=\delta q\phi(r)$. This work represents the electrostatic energy of $\delta q$. Suppose now that the preexisting field is produced by a charge density $\rho(\mathbf{r})$ and that, instead of an infinitesimal point charge $\delta q$ being brought in from infinity, there is an infinitesimal change $\delta\rho$ in the charge density throughout space (think of all the little bits of charge making up $\delta\rho$ being brought into their places from infinity). The work done to accomplish the change is

$$\delta W=\int \delta\rho(\mathbf{r})\phi(\mathbf{r})d^3x, \tag{49}$$

where the integration is over all regions where $\delta\rho\neq 0$. Without any assumptions about linearity, uniformity, isotropy, etc., the infinitesimal change in the charge density $\delta\rho$ can be related to a change in the electric displacement $\delta\mathbf{D}$ through the first Maxwell equation in Eq. (21):

$$\delta\rho=\nabla\cdot(\delta\mathbf{D}).$$

An integration by parts then leads to

$$\delta W = \int \mathbf{E} \cdot \delta\mathbf{D} \, d^3x,$$

where it has been assumed that $\rho(\mathbf{r})$ is a localized charge distribution. The total electrostatic energy (total work done in assembling the charge density) can be written formally as

$$W = \int d^3x \int_0^D \mathbf{E} \cdot \delta\mathbf{D}[\mathbf{E}]. \qquad (50)$$

The symbolic integration over $\delta\mathbf{D}$ from zero to a final value contains all the information in the constitutive relations. For linear media, $2\mathbf{E}\cdot\delta\mathbf{D} = \delta(\mathbf{E}\cdot\mathbf{D})$. The electrostatic energy for linear media is thus

$$W = \frac{1}{2} \int \mathbf{E}\cdot\mathbf{D} \, d^3x, \qquad (51)$$

where the integration is over all space. The integrand in Eq. (51) may be identified as an electrostatic energy density distributed in space. Use of Eq. (48) and the first equation in (21) will transform Eq. (51) into

$$W = \frac{1}{2} \int_V \rho(\mathbf{r})\phi(\mathbf{r}) d^3x - \int_S \phi(\mathbf{r})\mathbf{D}\cdot\mathbf{n} \, da. \qquad (52)$$

Here a formal device of finite volume $V$ and surrounding surface $S$ has been used to show explicitly the integration by parts. If the charge density is localized, the volume $V$ can be expanded to infinity. The potential decreases inversely or faster with distance and the field $\mathbf{D}$ falls off with one higher power. The surface integral therefore manifestly vanishes as $S$ goes to infinity. [It must vanish as soon as $S$ is large enough to contain all of $\rho(\mathbf{r})$, but that is not immediately obvious.]

The last expression for the energy is what occurs for the microscopic Maxwell equations, where all the charges, both bound and free, are included in the computation of the work. In macroscopic electromagnetism, it is a valid result only for linear media. In general, Eq. (50) must be used.

## 4.2 Capacitance and Coefficients of Capacitance

A set of $n$ conductors is embedded in a linear (but not necessarily isotropic or uniform) medium, with each conductor having a total charge $Q_j$ and a corresponding potential $V_j$ ($j = 1,2,\ldots,n$). The electrostatic energy of the system can be expressed in terms of the potentials $V_j$ alone and certain geometrical quantities called coefficients of capacitance. The simplest example is an isolated metal sphere of radius $R$ with a total charge $Q$ in an isotropic, uniform medium with dielectric constant $\epsilon$. The electrostatic potential at the surface of the sphere is $V = Q/4\pi\epsilon R$ and the energy is $W = QV/2$. Elimination of $Q$ gives $W = CV^2/2$, where the coefficient $C = 4\pi\epsilon R$ is called the capacitance of a sphere. This example illustrates the geometrical aspect of capacitance. Apart from the conventional factor $4\pi\epsilon$, the dimensions of capacitance are those of length. For the isolated sphere, capacitance is defined as the ratio of charge to potential, $C = Q/V$. A closely parallel definition applies for each set of $n$ conductors, as is immediately apparent.

For a given configuration of conductors in a linear medium, the linear connection of the fields and potential with the charge density implies that the potential of the $j$th conductor is related linearly to the charges on all the conductors, that is,

$$V_j = \sum_{k=1}^{n} p_{jk} Q_k,$$

where the coefficients $p_{jk}$ depend on the geometry of the conductors and the medium (if nonisotropic and/or nonuniform). These $n$ equations can be inverted to yield the charge on the $j$th conductor in terms of all the potentials, with coefficients that are purely geometrical and independent of field strengths,

$$Q_j = \sum_{k=1}^{n} C_{jk} V_k, \quad j = 1,2,\ldots,n. \qquad (53)$$

The diagonal coefficients $C_{jj}$ are called *capacitances*, while the $C_{jk}$ ($j \neq k$) are called coefficients of capacitance. From Eq. (53) it is evident that the capacitance of a conductor is the total charge on the conductor when it is maintained at unit potential, all other conductors being held at zero potential.

From Eqs. (52) and (53) it is seen that the electrostatic energy of the system of $n$ conductors is

$$W = \frac{1}{2} \sum_{j=1}^{n} Q_j V_j = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} V_j C_{jk} V_k. \qquad (54)$$

This expression for the energy, equal to Eq. (51), can be used to find capacitances of specific configurations, either exactly or approximately (by variational methods, for example). The coefficients of capacitance are symmetric, i.e., $C_{jk}=C_{kj}$, as can be proven by means of Green's reciprocation theorem (Sec. 4.5).

Some common example of capacitances (with the surrounding medium isotropic, linear, and uniform) are

1. Isolated conducting sphere of radius $R$:

   $C=4\pi\epsilon R$.

2. Isolated thin flat circular conducting disk of radius $R$:

   $C=8\epsilon R$.

3. Two equal flat parallel conducting sheets of area $A$ located one above the other with spacing $d$ (with the smallest dimension across any part of the sheets large compared to $d$), as shown in Fig. 4, upper left:

   $$C=A\epsilon/d \quad +\text{fringe-field corrections } O(\epsilon d).$$

   To the neglect of the fringing field, the same capacitance applies for a flat sheet of area $A$ positioned parallel to and a distance $d$ away from any larger parallel flat sheet of conductor that extends beyond the projection of the smaller one.

4. Two coaxial cylindrical surfaces, the inner one of radius $a$, the outer of radius $b>a$, and length $l$ ($l\gg b$), as shown in Fig. 4, upper right, with the potential difference between them:

   $$C=\frac{2\pi\epsilon l}{\ln(b/a)} \quad +\text{end-field corrections } O(\epsilon(b-a)).$$

5. Two long straight parallel right circular cylinder wires of radii $a_1$ and $a_2$ and common length $l$, whose central axes are separated by a distance $d$ as shown in Fig. 4, lower left:

   $$C=2\pi\epsilon l\left[\cosh^{-1}\left(\frac{d^2-a_1^2-a_2^2}{2a_1a_2}\right)\right]^{-1} \approx\frac{\pi\epsilon l}{\ln(d/\sqrt{a_1a_2})}.$$

   The approximate result applies when $d\gg a_1,a_2$. End-field corrections are of the relative order $d/l$ or less.

For conducting surfaces of not too severe shapes, capacitances can be estimated by comparison with known results. It can be proved that the capacitance of a closed surface $S$ is bounded between those of the surfaces $S_<$ and $S_>$, where $S_<$ ($S_>$) lies totally inside (outside) $S$, as sketched in Fig. 4, lower right. For example, the capacitance of a conducting oblate spheroid with axes $a=b=R$ and $c<R$ has a capacitance between those of a sphere and a disk of radii $R$. The capacitance of a cube with sides of length $a$ lies between those of the inscribed and circumscribed spheres, $C_<=0.5(4\pi\epsilon a)$ and $C_>=(\sqrt{3}/2)(4\pi\epsilon a)$. The arithmetic average of these bounds, $0.683(4\pi\epsilon a)$, is close to the numerically determined result, $0.655(4\pi\epsilon a)$.

Other examples of capacitance can be found in Sec. 5.b of Gray, *American Institute of Physics Handbook* (1972) and in ELECTROSTATIC CAPACITIVE STORAGE OF ENERGY.

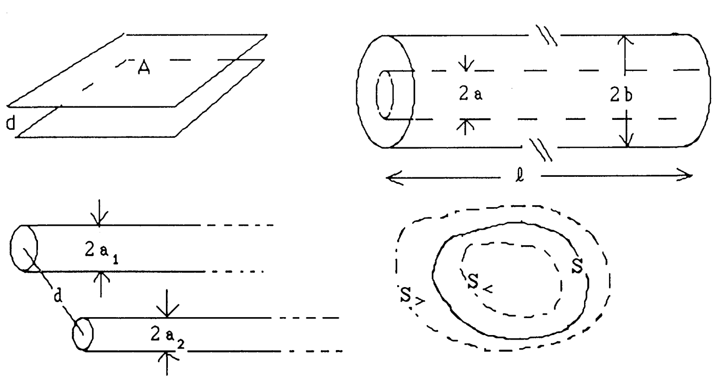


**FIG. 4.** Some capacitance geometries: parallel plates (upper left), coaxial cylinders (upper right), parallel circular wires (lower left), and closed conducting surface $S$, with inscribed ($S_<$) and circumscribed ($S_>$) surfaces (lower right).

### 4.3 Magnetostatic Energy

In deriving the electrostatic energy it is not necessary to consider time variation of the fields explicitly. Charges can be assumed to be moved arbitrarily slowly from infinity to their final locations. For magnetic fields, however, the variation in time of the fields as they are built up to their final magnetostatic configuration (defined by $\nabla\cdot\mathbf{J}=0$, or equivalently, $\partial\rho/\partial t=0$) produces electromotive forces (Faraday's law) that do work on the charges and currents. A treatment paralleling that of Sec. 4.1 [see Jackson (1975), Sec. 6.2] leads to a result for the infinitesimal change in magnetic energy caused by an infinitesimal change in the magnetic flux density of the form

$$\delta W=\int \mathbf{H}\cdot\delta\mathbf{B}\, d^3x. \tag{55}$$

This expression is the magnetic equivalent to the equation above Eq. (50). The constitutive relation between **B** and **H** is necessary to complete the determination of the total magnetostatic energy. For linear media, the energy is

$$W=\frac{1}{2}\int \mathbf{H}\cdot\mathbf{B}\, d^3x, \tag{56}$$

where the integration is over all space. In analogy with electrostatics, the integrand can be thought of as a magnetic energy density.

An expression for the magnetic energy analogous to Eq. (52) can be written by introduction of the vector potential **A**, an auxiliary vector field that takes into account the fourth Maxwell equation, $\nabla\cdot\mathbf{B}=0$. Any divergenceless vector field can be written as the curl of another vector field. The magnetic flux density can therefore be defined as

$$\mathbf{B}=\nabla\times\mathbf{A}. \tag{57}$$

Substitution of this definition into Eq. (56), followed by an integration by parts and use of the Ampère–Maxwell equation, leads to the magnetic energy of a linear medium as

$$W=\frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}\, d^3x. \tag{58}$$

The sources **J** are assumed to be localized in space; otherwise, there would be an additional surface integral at infinity. [For a localized steady current source, the vector potential decreases asymptotically inversely with distance and the fields decrease as the cube of the distance. The integrand in Eq. (56) therefore vanishes rapidly enough to make (56) equal (58) without a surface term.]

### 4.4 Inductance

A common situation in magnetostatics is a set of $n$ currents $I_j$ flowing in conductors $C_j$ embedded in a nonconducting, magnetically linear (although not necessarily isotropic or uniform) medium. The conductors, which are also taken to be magnetically linear, are not assumed to be infinitesimally thin wires, although this may be a permissible approximation in some computations. The magnetic energy of the configuration can be expressed as a quadratic form in the currents, with geometrical coefficients of inductance, just as with the electrostatic energy and the coefficient of capacitance. The starting point is Eq. (58), which can be written as

$$W=\frac{1}{2}\sum_{j=1}^{n}\int_{V_j}\mathbf{J}_j\cdot\mathbf{A}_j\, d^3x + \sum_{j=1}^{n}\sum_{k>j}^{n}\int_{V_j}\mathbf{J}_j\cdot\mathbf{A}_k\, d^3x, \tag{59}$$

where $\mathbf{J}_j$ is the current density in the $j$th conductor (proportional to the total current $I_j$) and $\mathbf{A}_j$ is the vector potential resulting from that current density. The integrations are over the volumes $V_j$ of the set of conductors. With a linear medium, the magnetic fields $\mathbf{B}_j$ and $\mathbf{H}_j$ and also the vector potential $\mathbf{A}_j$ throughout space are linear in the current $I_j$. Equation (59) may therefore be expressed in the form

$$W=\frac{1}{2}\sum_{j=1}^{n} L_j I_j^2 + \sum_{j=1}^{n}\sum_{k>j}^{n} M_{jk} I_j I_k, \tag{60}$$

where the coefficients $L_j$ and $M_{jk}$ are purely geometrical quantities, dependent upon the configuration of conductors and intervening medium and how the current flows through the conductors, but not dependent on the magnitudes of the currents $I_j$. The quantities $L_j$ ($M_{jk}$) are called coefficients of self- (mutual) inductance. Like the coefficients of capacitance, the mutual inductances are symmetric, i.e., $M_{jk}=M_{kj}$.

The approximation of infinitesimally thin wires can sometimes be used in the calculation of mutual inductances (but not for self-inductances, where an infinite value is the result). Consider such a conducting wire of circuit $C_1$ in the presence of another such circuit $C_2$. From Eq. (58) the mutual energy is

$$W_{12}=I_1 \oint_{C_1} \mathbf{A}_2 \cdot d\mathbf{l}_1. \tag{61}$$

Stokes's theorem, together with Eq. (57) connecting **B** and **A**, can be used to transform $W_{12}$ into

$$W_{12}=I_1 \int_{S_1} \mathbf{B}_2 \cdot \mathbf{n}\, da_1.$$

The open surface $S_1$ spans the circuit $C_1$. The integral is the magnetic flux $\Phi_{12}$ (caused by the second circuit with its current $I_2$) through the first circuit $C_1$. The mutual inductance $M_{12}$ is therefore $M_{12}=\mathscr{F}_{12}$, where $\mathscr{F}_{12}$ is the magnetic flux through circuit $C_1$ caused by unit current flowing in circuit $C_2$. For wire circuits in a uniform, isotropic medium of permeability $\mu$, it can be shown, from Eq. (61) and the expression for the vector potential in terms of the current, that $M_{12}$ can be written in this approximation in the purely geometric form

$$M_{12}=\frac{\mu}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r_{12}}. \tag{62}$$

Clearly such an expression is meaningless when applied to self-inductance.

Equation (56) or (58) for the energy can be used together with Eq. (60) to calculate the self-inductances of specific configurations of currents. For mutual inductances, the same approach can be employed, although in some circumstances the linked flux approach or Eq. (62) can be substituted.

Some examples of self-inductances (circuits are in vacuum and at low frequency unless stated otherwise) are the following.

1. A straight right circular wire of radius $a$, length $l$ ($l \gg a$), and relative permeability $\mu_r$:

$$L \approx \frac{\mu_0}{2\pi} l \left[ \ln\left(\frac{2l}{a}\right) - 1 + \frac{\mu_r}{4} \right].$$

The contribution $\mu_r/4$ comes from the interior of the wire. At very high frequencies, when the skin depth (39) $\delta \ll a$, this interior contribution is absent. In actuality, the argument of the logarithm is uncertain, or rather, it depends upon the actual way in which the wire of length $l$ and radius $a$ is connected into a complete circuit.

2. Two parallel long straight circular nonpermeable wires of length $l$ with radii $a_1$ and $a_2$ and a separation $d$ between their axes, with current flowing down one wire and back the other, as shown in Fig. 5, upper left:

$$L=\frac{\mu_0 l}{\pi} \left[ \ln\left(\frac{d}{\sqrt{a_1 a_2}}\right) + \frac{1}{4} \right].$$

At very high frequencies the "interior" contribution, $\frac{1}{4}$, is absent.

3. Two coaxial conducting right circular cylinders of length $l$ with current flow in opposite directions, the inner one solid with radius $a$ and relative permeability $\mu_1$, the outer one hollow, with inner radius $b$, thickness $t$ ($t \ll b$), and relative permeability $\mu_3$, and the space between the cylinders having relative permeability $\mu_2$, as sketched in Fig. 5, upper right:

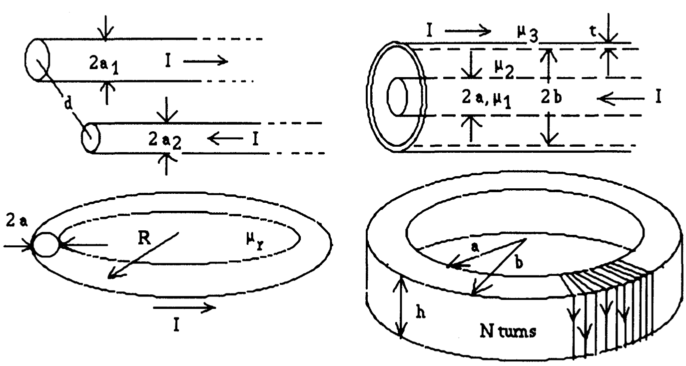


**FIG. 5.** Some inductance geometries: parallel circular wire transmission line (upper left), coaxial cylinder transmission line (upper right), circular loop of wire (lower left), circular toroid of rectangular cross section (lower right).

$$L \approx \frac{\mu_0}{2\pi}\, l \left[\frac{\mu_1}{4} + \mu_2 \ln\left(\frac{b}{a}\right) + \frac{\mu_3 t}{3b}\right].$$

At very high frequencies the terms in $\mu_1$ and $\mu_3$ are absent, as they would be if the inner and outer cylinders were both very thin cylindrical shells.

**4.** A circular wire of radius $a$ and relative permeability $\mu_r$ bent into a circle of mean radius $R$ ($R \gg a$), as shown in Fig. 5, lower left:

$$L \approx \mu_0 R \left[\ln\left(\frac{8R}{a}\right) - 2 + \frac{\mu_r}{4}\right].$$

There are corrections of order $a^2/R^2$ and $(a^2/R^2)\ln(R/a)$, and the same comment as above applies about the absence of the interior contribution at high enough frequencies. For a coil of $N$ turns wound closely together such that the bundle has an approximately circular cross section of radius $a$, the self-inductance is approximately $N^2$ times that of the single turn, provided the mean radius is used for $R$.

**5.** A toroid of rectangular cross section, with inner (outer) radius $a$ ($b$) and height $h$, and permeability $\mu$, wound uniformly with $N$ turns, as sketched in Fig. 5, lower right:

$$L \approx \frac{\mu N^2 h}{2\pi} \ln\left(\frac{b}{a}\right).$$

It is assumed that the wire size is very small compared to the dimensions of the toroid and that the number of turns is large enough that there is negligible spacing between turns.

**6.** A toroid of circular cross section of radius $a$ whose center is a radius $R$ ($R > a$) from the axis of the toroid, with permeability $\mu$, wound uniformly with $N$ turns:

$$L \approx \mu N^2 (R - \sqrt{R^2 - a^2}).$$

The same assumptions about wire size, etc. apply here as in example **4.**

For examples **4–6**, approximate mutual inductances can be obtained for coils wound closely on top of each other by replacing $N^2$ by $N_1 N_2$ and using mean dimensions. With the idealized toroids of examples **5** and **6**, the second coil of $N_2$ turns need not extend around the complete toroid because the magnetic flux is contained entirely within the toroid.

Further examples and more precise expressions for self- and mutual inductances can be found in Sec. 5.b of Gray, *American Institute of Physics Handbook* (1972), and in INDUCTORS.

### 4.5 Symmetry and Reciprocity Properties

There are useful symmetries and reciprocity relations based on Green's reciprocation theorem of electrostatics and its generalization to magnetostatics. Consider a volume $V$ defined by a set of conducting boundary surfaces collectively labeled $S$ (one of which may be at infinity). Let $\phi(\mathbf{r})$ be the electrostatic potential produced by a volume charge density $\rho$ and a surface charge density $\sigma$ on the surface(s) $S$, and let $\phi'(\mathbf{r})$ be the potential produced by another charge distribution, $\rho'$ and $\sigma'$. Green's reciprocation theorem states that

$$\int_V \rho\phi'\, d^3x + \int_S \sigma\phi'\, da = \int_V \rho'\,\phi\, d^3x + \int_S \sigma'\phi\, da. \tag{63}$$

Physically, the two sides of Eq. (63) are just different expressions for the mutual electrostatic potential energy of one charge distribution in the presence of the electric potential caused by the other. The equality can be traced back to Newton's third law. For a set of conducting surfaces raised to potentials $V_j$ ($V_j'$) by the placement of charges $Q_j$ ($Q_j'$) on them, Eq. (63) becomes

$$\sum_j Q_j' V_j = \sum_k Q_k V_k'. \tag{64}$$

Suppose that in one situation $Q_1 = q$, but all other $Q_j = 0$, while in the other, $Q_2' = q$ and all other $Q_j' = 0$. Equation (64) shows that the uncharged surface $S_2$ is raised by the presence of $q$ on $S_1$ to the same potential as is the uncharged surface $S_1$ when the same charge $q$ is placed on $S_2$. In another application, the symmetry of the coefficients of capacitance $C_{ij}$ follows from Eq. (64) if Eq. (53) is used to express the charges in terms of the potentials on both sides of the equation.

The reciprocation theorem can be used to answer questions about induced charges. Consider, for example, a point $P_1$ a distance $r$ from the center of a conducting sphere of radius $a$

on which a charge $q$ has been placed. The potential at $P_1$ is $V_1=q/4\pi\epsilon r$. If, instead, a point charge $q$ is placed at $P_1$, the uncharged sphere will be raised to the same potential, $V_2'=V_1$. With the point charge still at $P_1$, suppose that the sphere is now grounded. Enough charge $q'$ flows in from (out to) infinity to make the sphere's potential zero, i.e., $q'/4\pi\epsilon a+V_2'=q'/4\pi\epsilon a+q/4\pi\epsilon r=0$, or $q'=-aq/r$, the well-known result for the charge induced on a grounded sphere by a nearby point charge.

In magnetostatics there is a corresponding, although less familiar, reciprocation theorem involving currents in a set of circuits and the resulting magnetic fluxes through them.

## 5. POTENTIALS, GAUGE TRANSFORMATIONS, AND WAVE EQUATIONS

### 5.1 Scalar and Vector Potentials and Gauge Transformations

In Sec. 4 the scalar potential $\phi$ and the vector potential $\mathbf{A}$ were introduced in the context of time-independent fields. For time-dependent phenomena they are equally useful. The definition, Eq. (57), of the magnetic flux density in terms of the vector potential is applicable to the more general situation because it stems from the fourth Maxwell equation, $\nabla\cdot\mathbf{B}=0$, but the connection of the electric field to the potentials, Eq. (48), must be generalized. Recall that the curl of a gradient vanishes. In contrast, Faraday's law shows that

$$\nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=\mathbf{0}.$$

With time-dependent fields, it is not $\mathbf{E}$, but $\mathbf{E}+\partial\mathbf{A}/\partial t$ that can be expressed as a gradient of the scalar function. The definitions of the fields $\mathbf{E}$ and $\mathbf{B}$ in terms of the potentials $\phi$ and $\mathbf{A}$ are thus

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t},$$

$$\mathbf{B}=\nabla\times\mathbf{A}. \tag{65}$$

With these definitions, the two homogeneous Maxwell equations are automatically satisfied.

The electric and magnetic fields are fundamental; the potentials are useful auxiliary fields, but they are not unique. The magnetic flux density is unchanged if the vector potential is modified by adding the gradient of some scalar function $\chi(\mathbf{r},t)$ to it. To keep the electric field unchanged, the scalar potential must also be modified. The simultaneous change of the two potentials,

$$\mathbf{A}\to\mathbf{A}'=\mathbf{A}+\nabla\chi,$$

$$\phi\to\phi'=\phi-\frac{\partial\chi}{\partial t} \tag{66}$$

is called a gauge transformation. It preserves the electric and magnetic fields, but can modify the equations satisfied by the potentials.

Consider the Maxwell equations in free space. If $\mathbf{E}$ and $\mathbf{B}$ are eliminated by means of Eq. (65) from the inhomogeneous pair of the Maxwell equations, they become equations for the potentials,

$$\nabla^2\phi+\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A})=\frac{1}{\epsilon_0}\rho, \tag{67}$$

$$\nabla^2\mathbf{A}-\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right)=-\mu_0\mathbf{J}. \tag{68}$$

The four coupled first-order partial-differential equations of Maxwell have been replaced by two coupled second-order partial-differential equations for the potentials. The freedom to make gauge transformations without affecting the fields means that Eqs. (67) and (68) can be decoupled or otherwise put into forms that are simpler or more convenient. For example, potentials can be chosen such that they satisfy the Lorentz condition,

$$\nabla\cdot\mathbf{A}+\mu_0\epsilon_0\frac{\partial\phi}{\partial t}=0. \tag{69}$$

Then the potentials satisfy the standard second-order wave equation (with $\mu_0\epsilon_0=1/c^2$)

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\frac{1}{\epsilon_0}\rho, \tag{70}$$

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}. \tag{71}$$

Potentials that satisfy Eq. (69) are said to be in the class of Lorentz gauge potentials. If the original set did not satisfy Eq. (69), one can make a gauge transformation, Eq. (66), with $\chi$ a solution of

$$\nabla^2\chi-\frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2}=-\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right). \qquad (72)$$

The new potentials will satisfy the Lorentz condition. Since the solution of Eq. (72) is not unique (any solution of the homogeneous wave equation can be added), it is clear that there is an infinite set of potentials in the Lorentz gauge class. Potentials in the Lorentz gauge are convenient because the equations for $\phi$ and $\mathbf{A}$ are simple decoupled wave equations with source terms given by the charge density $\rho$ and the current density $\mathbf{J}$, respectively.

Another useful choice of gauge is the gauge in which $\nabla\cdot\mathbf{A}=0$, called the Coulomb, transverse, or radiation gauge. Equation (73) for the scalar potential becomes Poisson's equation, with the solution

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,d^3x'. \qquad (73)$$

The scalar potential is just the *instantaneous* Coulomb potential caused by the charge density $\rho(\mathbf{r},t)$. The vector potential satisfies an inhomogeneous wave equation,

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}_t,$$

where

$$\mathbf{J}_t=\mathbf{J}-\epsilon_0\nabla\frac{\partial\phi}{\partial t} \qquad (74)$$

is the transverse current density, so named because it is the part of the current that has zero divergence, $\nabla\cdot J_t=0$. The name "transverse gauge" comes from this circumstance.

The Coulomb gauge is often used when no sources are present. The scalar potential can be taken to be zero, while $\mathbf{A}$ satisfies the homogeneous wave equation and $\nabla\cdot\mathbf{A}=0$. Then the fields are given solely in terms of $\mathbf{A}$ by Eq. (65) with $\phi=0$. These fields propagate with the speed of light in vacuum, $c$. A plane wave, harmonic in time with frequency $\omega$ and wave vector $\mathbf{k}$, can be represented by a vector potential, $\mathbf{A}=A_0\boldsymbol{\epsilon}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$, where $|\mathbf{k}|=\omega/c$ and $\mathbf{k}\cdot\boldsymbol{\epsilon}=0$. From Eq. (65), the fields are $\mathbf{E}=i\omega\mathbf{A}$ and $\mathbf{B}=i\mathbf{k}\times\mathbf{A}$. The transversality of the fields ($\mathbf{k}\cdot\mathbf{E}=\mathbf{k}\cdot\mathbf{B}=0$) is manifest, as well as the orthogonality of $\mathbf{E}$ and $\mathbf{B}$. If $\boldsymbol{\epsilon}$ is a real vector, the plane wave is said to be plane polarized, with the directions of $\mathbf{E}$ and $\mathbf{B}$ fixed in space. If $\boldsymbol{\epsilon}$ is complex, e.g., $\boldsymbol{\epsilon}=(\mathbf{e}_1+i\mathbf{e}_2)/\sqrt{2}$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are real unit vectors, perpendicular to each other and to $\mathbf{k}$, then the wave is elliptically polarized. (The example is a right circularly polarized or positive helicity wave. See OPTICS, LINEAR.)

Since the electric field strength $\mathbf{E}$ has the dimensions of electric potential per unit length (units of V/m) while the magnetic field strength $\mathbf{H}$ has dimensions of current per unit length (units of A/m), their ratio has the dimensions of impedance (V/A or $\Omega$). For a plane wave in free space this ratio is $Z_0=\mu_0\omega/k=\sqrt{\mu_0/\epsilon_0}$, called the impedance of free space, and is equal to 376.730 $\Omega$. Analogous ratios of electric and magnetic fields in transmission lines and wave guides permit the use of lumped-circuit concepts in systems large compared to the free-space wavelength $\lambda=2\pi c/\omega$. (See ELECTROMAGNETIC WAVE PROPAGATION.)

### 5.2 Green's Functions for the Wave Equation

The first step in solving the wave equation, Eq. (70) or (71) or (74), is to find a Green's function, that is, a solution of the scalar wave equation with a unit impulse source in space and time. If the source point is at $(\mathbf{r}',t')$, the Green's function $G(\mathbf{r},t;\mathbf{r}',t')$ satisfies

$$\nabla^2G(\mathbf{r},t;\mathbf{r},'t')-\frac{1}{c^2}\frac{\partial^2G(\mathbf{r},t;\mathbf{r}',t')}{\partial t^2}$$
$$=-\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \qquad (75)$$

If there are no boundary surfaces at finite distances in space and time, the Green's function can only be a function of the relative coordinates, $\mathbf{R}=\mathbf{r}-\mathbf{r}'$ and $\tau=t-t'$. Furthermore, the source is spherically symmetric. The Green's function can be a function only of $R=|\mathbf{r}-\mathbf{r}'|$ and $\tau$. Now introduce a frequency-domain Green's function through the Fourier transforms

$$\delta(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega\tau}\,d\omega \qquad (76)$$

and

$$G(R,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(R,\omega)e^{-i\omega\tau}\,d\omega. \tag{77}$$

The equation obeyed by $G(R,\omega)$ is the inhomogeneous Helmholtz wave equation,

$$\nabla_{\mathbf{R}}^2 G(R,\omega)+k^2G(R,\omega)=-\delta(\mathbf{R}), \tag{78}$$

where $k^2=\omega^2/c^2$ is the square of the wave number associated with the frequency $\omega$. Because of the spherical symmetry, only the radial part of the Laplacian gives a nonvanishing contribution. For all points with $R\neq 0$, the right-hand side is zero and the equation reads

$$\frac{d^2}{dR^2}RG(R,\omega)+k^2RG(R,\omega)=0$$

with the solution

$$RG(R,\omega)=Ae^{ikR}+Be^{-ikR}.$$

Furthermore, the $\delta$ function in Eq. (78) influences the solution only at the origin, where the second term on the left-hand side is negligible compared with the first. In these circumstances, Eq. (78) becomes Laplace's equation of electrostatics; $G$ must reduce to the potential for a point charge and therefore

$$\lim_{kR\to 0} RG(R,\omega)=1/4\pi.$$

The coefficients $A$ and $B$ above must satisfy $4\pi(A+B)=1$.

It is useful to define the so-called retarded and advanced Green's functions, $G_\pm(R,\omega)$, to be

$$G_\pm(R,\omega)=\frac{1}{4\pi}\frac{e^{\pm ikR}}{R}. \tag{79}$$

Given the convention of $e^{-i\omega t}$ for time dependence, $G_+$ ($G_-$) represents a spherically diverging (converging) wave propagating outwards from (inwards towards) the origin. To understand the designations, retarded and advanced, it is only necessary to insert $G_\pm(R,\omega)$ into Eq. (77) to determine the time-domain Green's function,

$$G_\pm(R,\tau)=\frac{1}{4\pi R}\,\delta\!\left(\tau\mp\frac{R}{c}\right)$$
$$=\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\delta\!\left(t'-\left[t\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right). \tag{80}$$

Evidently the retarded Green's function, $G_+$, represents an effect at $(\mathbf{r},t)$ caused by the action of an impulsive source a distance $R$ away acting at an earlier time $t'=t-R/c$. The time difference $R/c$ is just the time of propagation of the disturbance from one point to the other. Similarly, the advanced Green's function $G_-$ represents sources acting at *later* times.

Because causality is observed in natural phenomena, it is customary to express the action of sources only in terms of the retarded Green's function, but that is not necessarily appropriate. Let $\Psi(\mathbf{r},t)$ and $f(\mathbf{r},t)$ represent $\phi$ (or a component of $\mathbf{A}$) and $\rho/\epsilon_0$ (or a component of $\mu_0\mathbf{J}$), respectively, in the wave equations, Eqs. (70) and (71). It is assumed that $f(\mathbf{r},t)$ is localized in space and time. At remote times, $t\to\pm\infty$, the wave amplitude $\Psi$ is a solution of the homogeneous wave equation. If $\Psi$ is specified to be equal to a given initial wave, $\Psi_{\text{in}}(\mathbf{r},t)$, at $t\to-\infty$, the solution at all times is given in terms of the retarded Green's function,

$$\Psi(\mathbf{r},t)=\Psi_{\text{in}}(\mathbf{r},t)+\int\int G_+(\mathbf{r},t;\mathbf{r}',t')f(\mathbf{r}',t')d^3x'\,dt. \tag{81}$$

With the sources localized in time, it is clear that at early enough times the integral will be zero; the wave will be given by its initial free-wave form. At late times the sources will have acted; $\Psi$ will consist of the initial wave $\Psi_{\text{in}}$ plus waves causally radiated outwards from the sources.

If, on the other hand, the wave is specified to be $\Psi_{\text{out}}(\mathbf{r},t)$ for $t\to+\infty$, the solution is in terms of the advanced Green's function, that is, Eq. (81) with "in" replaced by "out" and $G_+$ replaced by $G_-$. The radiation field $\Psi_{\text{rad}}$ can be defined formally as the difference, $\Psi_{\text{out}}-\Psi_{\text{in}}$:

$$\Psi_{\text{rad}}(\mathbf{r},t)=\int\int [G_+(\mathbf{r},t;\mathbf{r}',t')-G_-(\mathbf{r},t;\mathbf{r}',t')]f(\mathbf{r}',t')d^3x'\,dt. \tag{82}$$

For the usual situation in which the radiation fields are considered only after the sources have acted, $G_+$ alone is effective; the familiar retarded solution for the radiation fields emerges.

### 5.3 Harmonic Fields and Potentials in Uniform, Linear, Isotropic, Dispersive Media

The dispersive nature of dielectric constants and the resulting nonlocality in time for the relation between $\mathbf{D}$ and $\mathbf{E}$ has been discussed in Secs. 3.4 and 3.5. It is therefore only approximate to generalize the free-space wave equations such as Eqs. (70) and (71) merely by the replacement of $\mu_0$ and $\epsilon_0$ by $\mu$ and $\epsilon$. Only if $\epsilon(\omega)$ and $\mu(\omega)$ are sensibly constant over the range of frequencies of interest can one make such a simple substitution. Instead, one must consider one frequency component of the fields at a time. Assume that all the fields and sources oscillate in time according to $e^{-i\omega t}$ and write $\mathbf{E}(\mathbf{r},t)=\mathbf{E}(\mathbf{r})\,e^{-i\omega t}$, etc., where $\mathbf{E}(\mathbf{r})$ is a complex vector field, and similarly for the other fields and sources. With the generally dispersive connections, $\mathbf{D}=\epsilon(\omega)\mathbf{E}$, $\mathbf{B}=\mu(\omega)\mathbf{H}$, appropriate to a linear, isotropic, uniform medium, the Maxwell equations (21) become

$$\begin{aligned}&\nabla\cdot\mathbf{E}=\rho/\epsilon,\\&\nabla\times\mathbf{H}+i\epsilon\omega\mathbf{E}=\mathbf{J},\\&\nabla\times\mathbf{E}-i\omega\mu\mathbf{H}=\mathbf{0},\\&\nabla\cdot\mathbf{H}=0.\end{aligned}\tag{83}$$

The continuity equation for charge and current densities now reads $\nabla\cdot\mathbf{J}-i\omega\rho=0$. If the potentials are similarly decomposed into spatial factors times $e^{-i\omega t}$, the fields are related to them via $\mathbf{E}=-\nabla\phi+i\omega\mathbf{A}$ and $\mu\mathbf{H}=\nabla\times\mathbf{A}$. The Lorentz condition, Eq. (70), is generalized for harmonic time dependence, but dispersive media, to $\nabla\cdot\mathbf{A}=i\mu\epsilon\omega\phi$. In the Lorentz class of gauges, the scalar potential is not an independent quantity. Once $\mathbf{A}$ is known, $\phi$ can be computed, but it is not needed—fields can be expressed in terms of $\mathbf{A}$ alone. The equation satisfied by the vector potential is an inhomogeneous Helmholtz wave equation,

$$\nabla^2\mathbf{A}+\mu\epsilon\omega^2\mathbf{A}=-\mu\mathbf{J}.\tag{84}$$

The solution of Eq. (84) can be written down in terms of the frequency-domain retarded Green's function $G_+(R,\omega)$, Eq. (79):

$$\mathbf{A}(\mathbf{r})=\mathbf{A}_{\mathrm{in}}(\mathbf{r})+\mu\int\frac{e^{ikR}}{4\pi R}\,\mathbf{J}(\mathbf{r}')d^3x',\tag{85}$$

where $k=\omega\sqrt{\mu\epsilon}=\omega\sqrt{\mu_r\epsilon_r}/c$ is the wave number at frequency $\omega$ in the dispersive medium. This formula can be used to compute the radiation from harmonically driven sources such as antennas. (See ELECTROMAGNETIC RADIATION.)

If there are no external sources of charge and current, but the medium is conducting, the Ohm's-law contribution ($\mathbf{J}=\mathbf{E}/\rho$) can be written explicitly or included in a complex dielectric constant $\epsilon(\omega)$, as described in Sec. 3.4. With the latter choice, the wave equation (84) becomes a homogeneous equation with a complex wave number.

Auxiliary potentials, called electric and magnetic Hertz vectors, $\mathbf{Z}_e$ and $\mathbf{Z}_m$, are sometimes introduced. For harmonic fields and with any Ohm's-law contribution included in $\epsilon(\omega)$, the electric Hertz vector $\mathbf{Z}_e$ is just proportional to $\mathbf{A}$. The connections to $\phi$ and $\mathbf{A}$ are

$$\phi=-\nabla\cdot\mathbf{Z}_e,\quad \mathbf{A}=-i\omega\mu\epsilon\mathbf{Z}_e.\tag{86}$$

It may be seen that these definitions yield potentials in the Lorentz class of gauges. In the absence of external sources, $\mathbf{Z}_e$ satisfies the homogeneous wave equation and the fields are given by

$$\begin{aligned}&\nabla^2\mathbf{Z}_e+\mu\epsilon\omega^2\mathbf{Z}_e=0,\\&\mathbf{E}=\nabla\times\nabla\times\mathbf{Z}_e,\\&\mathbf{H}=-i\omega\epsilon\nabla\times\mathbf{Z}_e.\end{aligned}\tag{87}$$

Since the Maxwell equations (83) transform into themselves under the simultaneous substitutions $\mu\leftrightarrow-\epsilon$, $\mathbf{E}\leftrightarrow\mathbf{H}$, in the absence of sources, the same transformation can be made in Eq. (87), leading to the magnetic Hertz vector $\mathbf{Z}_m$, which obeys the same wave equation and has the fields given by

$$\begin{aligned}&\mathbf{E}=i\omega\mu\nabla\times\mathbf{Z}_m,\\&\mathbf{H}=\nabla\times\nabla\times\mathbf{Z}_m.\end{aligned}\tag{88}$$

Hertz originally introduced these "superpotentials" in a treatment of the macroscopic Maxwell equations that kept explicit the polarization and magnetization densities, $\mathbf{P}$ and $\mathbf{M}$, treating them as given external sources. The two Hertz vectors then satisfied inhomogeneous Helmholtz wave equations with $\mathbf{P}$ and $\mathbf{M}$ on the right-hand sides, with solutions like

Eq. (85), but with **J** replaced by **P** or **M**. In all but a few circumstances, it is simpler to deal with the usual scalar and vector potentials.

The plane waves described at the end of Sec. 5.1 have their counterparts in uniform, isotropic, linear media. It is only necessary to replace $\mu_0$ and $\epsilon_0$ everywhere by $\mu(\omega)$ and $\epsilon(\omega)$, respectively. The wave number becomes $k=n(\omega)\omega/c$, where $n(\omega)=\sqrt{\mu_r\epsilon_r}$ is the index of refraction of optics.

## 6. CONSERVATION LAWS FOR ENERGY AND MOMENTUM

### 6.1 Poynting's Theorem of Conservation of Electromagnetic Energy

An important theorem of Poynting (1884) concerns the continuity equation for the flux of electromagnetic energy and the energy content in the fields. If an infinitesimal point charge $\delta q$ is acted on by electric and magnetic fields, **E** and **B**, the rate at which the fields do work on the charge is $\delta W=\delta q\ \mathbf{v}\cdot\mathbf{E}$, where **v** is the velocity of the charge. (The magnetic field does no work because the magnetic force is perpendicular to the velocity.) If there is a continuous distribution of charge and current, the total rate of doing work by the fields in a finite volume $V$ is

$$\int_V \mathbf{J}\cdot\mathbf{E}\ d^3x.$$

This power represents a conversion of electromagnetic energy into mechanical or thermal energy and must result in a decrease in the total electromagnetic energy within $V$. The necessary conservation law is the content of Poynting's theorem. To find the explicit statement, one substitutes for **J** from the second line in Eq. (21):

$$\int_V \mathbf{J}\cdot\mathbf{E}\ d^3x=\int_V \left(\mathbf{E}\cdot(\nabla\times\mathbf{H})-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right)d^3x. \tag{89}$$

The vector identity

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{H})$$

may be used, together with Faraday's law [third line in Eq. (21)], to transform the right-hand side of Eq. (89) and obtain

$$\int_V \mathbf{J}\cdot\mathbf{E}\ d^3x=-\int_V \left(\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)d^3x. \tag{90}$$

Three assumptions are needed to proceed further, namely, that the medium is linear, that it is nondispersive, and that the sum of Eqs. (51) and (56), derived in stationary situations, gives the total electromagnetic energy when the fields are time dependent. With these assumptions, the second and third terms on the right-hand side of Eq. (90) represent the time derivative of the total electromagnetic energy density $u$, where

$$u=\tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})=\tfrac{1}{2}(\epsilon E^2+\mu B^2). \tag{91a}$$

If the medium is dispersive then **D** and **B** are nonlocal in time and the conversion of the last two terms in Eq. (90) is more complicated. As first shown by Brillouin in 1921 (Brillouin, 1960; Landau *et al.*, 1984, Sec. 80), if the fields are concentrated in frequency around $\omega$, with slow modulation in time, the last two terms in Eq. (90) correspond to the steady time derivative of

$$\bar{u}=\frac{1}{2}\left(\frac{d(\omega\epsilon)}{d\omega}\overline{E^2}+\frac{d(\omega\mu)}{d\omega}\overline{B^2}\right), \tag{91b}$$

where bars denote time averages over the rapid oscillations. If the medium is nondispersive, Eq. (91a) is recovered.

The first term on the right-hand side of Eq. (90) can be converted to a surface integral by means of the divergence theorem. It evidently represents a flow of energy out through the surface $S$ bounding the volume $V$. It is therefore natural to define an electromagnetic energy flux density, called the *Poynting vector*,

$$\mathbf{S}=\mathbf{E}\times\mathbf{H} \tag{92}$$

with dimensions of energy per unit area per unit time. Equation (90) can then be written as a differential or integral statement of conservation of electromagnetic energy,

$$\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}+\mathbf{J}\cdot\mathbf{E}=0,$$

$$\int_V \frac{\partial u}{\partial t}\, d^3x+\int_S \mathbf{n}\cdot\mathbf{S}\, da+\int_V \mathbf{J}\cdot\mathbf{E}\, d^3x=0. \tag{93}$$

The first two terms correspond to the rate of change of electromagnetic energy, either from doing work on charges and currents or from outward flow, while the third is the energy gained by the charges and currents from the fields. Equation (93) is Poynting's theorem.

### 6.2 Poynting's Theorem for Harmonic Fields and Field Definitions of Impedance and Admittance

A useful form of Poynting's theorem is that of time-averaged quantities with harmonic fields of a single frequency. For such fields the Maxwell equations take the form

$$\nabla\cdot\mathbf{B}=0,$$

$$\nabla\cdot\mathbf{D}=\rho,$$

$$\nabla\times\mathbf{E}-i\omega\mathbf{B}=\mathbf{0},$$

$$\nabla\times\mathbf{H}+i\omega\mathbf{D}=\mathbf{J}, \tag{94}$$

where all the quantities are complex functions of $\mathbf{r}$ with time dependence, $e^{-i\omega t}$, understood. In keeping with the conventions of Sec. 1 concerning time averages, the time-average rate of doing work on the charge and current distribution is

$$\frac{1}{2}\mathrm{Re}\left(\int_V \mathbf{J}^*\cdot\mathbf{E}\, d^3x\right).$$

If the steps from Eqs. (89)–(93) are repeated with obvious changes, the result is the pair of complex equations,

$$2i\omega(w_e-w_m)+\nabla\cdot\mathbf{S}+\tfrac{1}{2}\,\mathbf{J}^*\cdot\mathbf{E}=0,$$

$$2i\omega\int_V (w_e-w_m)d^3x+\int_S \mathbf{n}\cdot\mathbf{S}\, da +\frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}\, d^3x=0, \tag{95}$$

where the complex Poynting vector and harmonic electric and magnetic energy densities are

$$\mathbf{S}=\tfrac{1}{2}\,\mathbf{E}\times\mathbf{H}^*,\quad w_e=\tfrac{1}{4}\,\mathbf{E}\cdot\mathbf{D}^*,\quad w_m=\tfrac{1}{4}\,\mathbf{B}\cdot\mathbf{H}^*. \tag{96}$$

Equations (95) and (96) are the harmonic analog of Eq. (93). The real part of Eq. (95) is the statement of conservation of energy for the time-averaged quantities. Its imaginary part relates the reactive or stored energy to its alternating flow.

The complex form of Poynting's theorem, Eq. (95), can be exploited to define the input (junction) impedance of a general, two-terminal, linear, passive, electromagnetic system. Imagine the isolated system within a volume $V$ with boundary surface $S$, as shown in Fig. 6. Only the two input–output terminals are protruding. Let the complex harmonic current and voltage at the terminals be $I_i$ and $V_i$. Then the complex power input is $I_i^* V_i/2$. This power is assumed to flow *into* the volume $V$ through the surface $S_i$, e.g., the cross section of the coaxial line shown in the bottom half of Fig. 6. In terms of the Poynting vector, one has

$$\frac{1}{2}\, I_i^* V_i=-\oint_{S_i} \mathbf{S}\cdot\mathbf{n}\, da,$$

where $\mathbf{n}$ is the outwardly directed normal to $S$, as indicated in Fig. 6. Now applying Eq. (95) to the volume $V$ and total surface $S$, one obtains

$$2i\omega\int_V (w_e-w_m)d^3x+\int_{S-S_i} \mathbf{n}\cdot\mathbf{S}\, da +\frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}\, d^3x=\frac{1}{2}\, I_i^* V_i. \tag{97}$$

The surface integral on the left represents the energy flow out of the volume $V$. If the surface $S-S_i$ is taken to infinity, this integral is real and describes escaping radiation. At low frequencies radiation is generally negligible; no distinction need be made between $S$ and $S_i$; the upper diagram in Fig. 6 is adequate.

The input impedance $Z=R-iX$ (engineers, please recall that the conventions of Sec. 1 imply $j=-i$) of the system follows from Eq. (97) with its definition, $V_i=ZI_i$. The right-hand side of Eq. (97) is $Z|I_i|^2/2$.

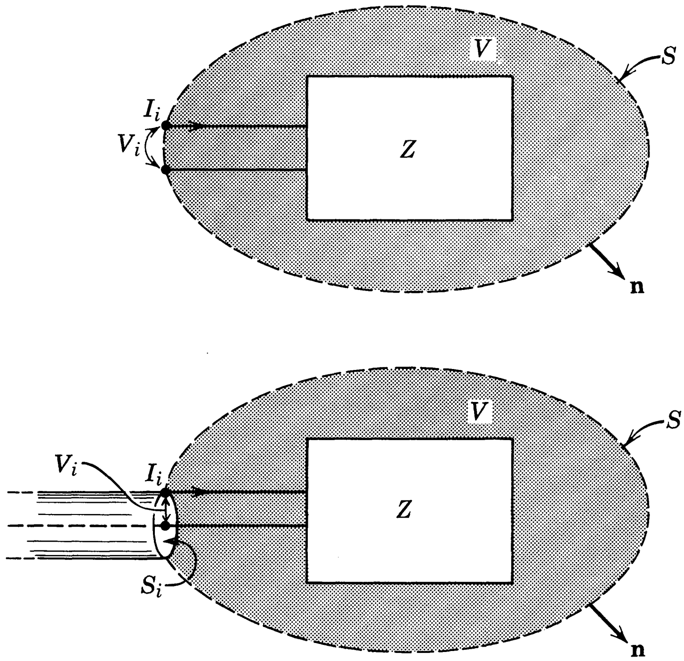


**FIG. 6.** Schematic diagrams of arbitrary, two-terminal, linear, passive electromagnetic systems. The surface $S$ completely surrounds the system; only the input terminals protrude. At the terminals, the input current and voltage are $I_i$ and $V_i$, with the input impedance defined through $V_i = ZI_i$. The upper diagram pertains to low frequencies, the lower, with its coaxial feed, to high frequencies where radiation resistance may enter. [From Jackson (1975), reproduced by permission of John Wiley & Sons, Inc.]

The real and imaginary parts of $Z$ are

$$R = \frac{1}{|I_i|^2}\left[\operatorname{Re}\left(\int_V \mathbf{J}^* \cdot \mathbf{E}\, d^3x\right) + 4\omega \operatorname{Im}\left(\int_V (w_m - w_e)\, d^3x\right) + 2\int_{S-S_i} \mathbf{n}\cdot\mathbf{S}\, da\right],$$

$$X = \frac{1}{|I_i|^2}\left[4\omega \operatorname{Re}\left(\int_V (w_m - w_e)\, d^3x\right) - \operatorname{Im}\left(\int_V \mathbf{J}^* \cdot \mathbf{E}\, d^3x\right)\right]. \qquad (98)$$

Here it has been assumed that the power flow out through $S - S_i$ is real. The last term in the expression for $R$ is the *radiation resistance*, important at high frequencies. At low frequencies, the Ohmic losses are generally the only dissipation ($\mathbf{J}$ in phase with $\mathbf{E}$); the electric and magnetic energies densities are real throughout $V$. Then the expressions simplify to

$$R \approx \frac{1}{|I_i|^2}\int_V \rho |\mathbf{J}|^2\, d^3x,$$

$$X \approx \frac{4\omega}{|I_i|^2}\int_V (w_m - w_e)\, d^3x. \qquad (99)$$

Here $\rho$ is the real (low-frequency) resistivity. The expression for the resistance $R$ exhibits the Ohmic heat loss throughout the system. Similarly, the expression for the reactance $X$ shows the expected behavior. If stored magnetic energy dominates, as for a lumped inductance, the reactance is positive, etc. The different dependences on frequency of the reactance for inductances ($X = \omega L$) and capacitances ($X = -1/\omega C$) can be traced to the definition of $L$ in terms of current and voltage ($V = L\, dI/dt$) on the one hand, and of $C$ in terms of charge and voltage ($V = Q/C$) on the other.

The admittance $Y = G - iB$, where $G$ is the conductance and $B$ the susceptance, can be defined via Eq. (97) using $I_i = YV_i$. The low-frequency forms analogous to Eq. (99) are

$$G \approx \frac{1}{|V_i|^2}\int_V \frac{1}{\rho} |\mathbf{E}|^2\, d^3x,$$

$$B \approx -\frac{4\omega}{|V_i|^2}\int_V (w_m - w_e)\, d^3x. \qquad (100)$$

It should be noted that the situation depicted in Fig. 6 and the above definitions of impedance or admittance are not immediately applicable to circumstances in which energy input is distributed in space, as, for example, in the exchange of energy between an electron beam and the cavity structure in a magnetron or free-electron laser. Poynting's theorem is applicable and circuit concepts can be used, but care must be taken in the definitions. For example, the voltage $V_i$ in the definition of admittance is replaced by an appropriate line integral of the electric field. For discussions in the microwave domain, see Chap. IX of Slater (1950) or Chap. 2 of Collins (1948).

### 6.3 Conservation of Momentum, Maxwell Stress Tensor, and Radiation Pressure

The conservation of momentum can be similarly addressed. A particle with infinitesimal charge $\delta q$ in the presence of fields $\mathbf{E}$ and $\mathbf{B}$ experiences a force, called the Lorentz force,

$$\delta\mathbf{F}=\delta q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

If $\mathbf{P}_{\text{mech}}$ is the sum of the mechanical momenta of all the particles within a volume $V$, Newton's second law can be expressed as

$$\frac{d\mathbf{P}_{\text{mech}}}{dt}=\int_V(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})d^3x, \tag{101}$$

where a discrete sum over charged particles is formally replaced by integration over the charge and current densities times the appropriate fields. For simplicity it is assumed that the charges described by $\rho$ and $\mathbf{J}$ are in otherwise free space, i.e., $\mathbf{D}=\epsilon_0\mathbf{E}$ and $\mathbf{B}=\mu_0\mathbf{H}$, and that the volume $V$ is such that there is no flow of particles into or out of it. If the first two of the Maxwell equations in Eq. (21) are used to eliminate $\rho$ and $\mathbf{J}$ from Eq. (101), the integrand can be cast in the form

$$\begin{aligned}\rho\mathbf{E}+\mathbf{v}\times\mathbf{B}=\epsilon_0[&\mathbf{E}(\nabla\cdot\mathbf{E})-\mathbf{E}\times(\nabla\times\mathbf{E})]\\&+\mu_0[\mathbf{H}(\nabla\cdot\mathbf{H})-\mathbf{H}\times(\nabla\times\mathbf{H})]\\&-\mu_0\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}).\end{aligned} \tag{102}$$

In Cartesian components, the square brackets have the form

$$[\mathbf{E}(\nabla\cdot\mathbf{E})-\mathbf{E}\times(\nabla\times\mathbf{E})]_j=\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(E_jE_k-\tfrac{1}{2}\,\delta_{jk}\mathbf{E}\cdot\mathbf{E}). \tag{103}$$

and similarly for the second one. The last term in Eq. (102) has the form of the time derivative of a momentum density ($\mu_0\epsilon_0\,\mathbf{E}\times\mathbf{H}=\mathbf{S}/c^2$). The total electromagnetic momentum within $V$ is thus defined as

$$\mathbf{P}_{\text{em}}=\mu_0\epsilon_0\int_V\mathbf{E}\times\mathbf{H}\,d^3x. \tag{104}$$

Equation (101) can then be written in component form to express conservation of the sum of mechanical and electromagnetic momentum,

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}}+\mathbf{P}_{\text{em}})_j=\sum_{k=1}^{3}\int_V\frac{\partial}{\partial x_k}T_{jk}\,d^3x, \tag{105}$$

where the *Maxwell stress tensor* $T_{jk}$ is

$$T_{jk}=\epsilon_0E_jE_k+\mu_0H_jH_k-\tfrac{1}{2}\delta_{jk}(\epsilon_0\mathbf{E}\cdot\mathbf{E}+\mu_0\mathbf{H}\cdot\mathbf{H}). \tag{106}$$

Application of the divergence theorem to the volume integral in Eq. (105) gives Newton's second law for the combined system of particles and fields within the volume $V$,

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}}+\mathbf{P}_{\text{em}})_j=-\int_S\sum_k T_{jk}n_k\,da, \tag{107}$$

where $\mathbf{n}$ is the *inward*-directed normal to the closed surface $S$ surrounding $V$. The integrand on the right-hand side of Eq. (107) is evidently the force per unit area transmitted across an element of the surface $S$ and acting on the combined system inside $V$.

The force per unit area or pressure, $-\Sigma_kT_{jk}n_k$, can be used to calculate forces on material objects such as conductors by integration over an appropriate surface. A simple example of the ponderomotive nature of electromagnetic radiation is *radiation pressure.* A plane wave with wave vector $\mathbf{k}$ has both $\mathbf{E}$ and $\mathbf{H}$ perpendicular to $\mathbf{k}$, in the plane of constant phase, with $|\mathbf{E}|=\sqrt{\mu_0/\epsilon_0}|\mathbf{H}|$. The force per unit area in the direction of $\mathbf{k}$ then comes only from the $\delta_{jk}$ terms in Eq. (106). The radiation pressure in the direction of propagation is therefore

$$P_{\text{rad}}=-\sum_{i,j}\hat{k}_iT_{ij}\hat{k}_j=\frac{1}{c}|\mathbf{S}|=u. \tag{108}$$

Here $u$ is the energy density and $\mathbf{S}$ the Poynting vector. The pressure (flux of momentum) has the relation to the Poynting vector (flux of energy) expected for relativistically moving (massless) particles (photons). If a plane wave in vacuum is incident and reflected at angle $\theta$ to the normal by a perfectly reflecting plane, the radiation pressure on the plane is easily found to be $P_{\text{rad}}=2\cos^2\theta|\mathbf{S}_{\text{inc}}|/c$, where $\mathbf{S}_{\text{inc}}$ is the Poynting vector of the incident wave.

Radiation pressure is the reason why most comets' tails point away from the Sun at perihelion—the dust particles that make up the tail are pushed away from the Sun by the radiation pressure of the solar radiation [a few microjoules per cubic meter or $10^{-11}$ atmospheres at the radius of the Earth and increasing as $(R_E/R)^2$ at smaller $R$].

For material media, care must be exercised in defining the purely electromagnetic energy density $u$, energy flow $\mathbf{S}$, momentum flow, and stress tensor $T_{jk}$ because what is considered electromagnetic and what is mechanical is to some extent arbitrary. After all, the mechanical properties of matter are governed by electromagnetic forces, even if the mass is not. It is generally agreed that in matter the energy flow is still given by the $\mathbf{S}$ of Eq. (92) and that the momentum density is $\mathbf{S}/c^2$. For detailed discussions, see deGroot (1969), and also Landau *et al.* (1984) or Robinson (1973).

## 7. SPECIAL RELATIVITY AND CLASSICAL FIELD THEORY

### 7.1 Summary of Results in Special Relativity

The special theory of relativity is based on two postulates:

1. There exists a multiply infinite set of equivalent Euclidean reference frames or coordinate systems moving with constant velocities in rectilinear paths relative to one another in which all physical phenomena occur in an identical matter. For brevity, these frames of reference are called inertial frames. This principle of relativity is very ancient, dating from Copernicus's time or earlier. It conforms with all known observations of ordinary mechanical systems, as well as the Michelson–Morley experiment (failure to detect motion through the putative aether).
2. There is a limiting speed of an object, regardless of the inertial frame from which it is observed. Experimentally, the speed is the speed of light *in vacuo*.

This second postulate, simplicity itself, has radical consequences, changing the concept of time as something absolute and putting it on an equal footing with the spatial coordinates. In high-energy physics, special relativity is verified routinely in very practical ways in the operation of accelerators, in the transport of beams of unstable particles over long distances, etc. See Sec. 11.2 of Jackson (1975) for a discussion of some of the experimental tests.

It is convenient to introduce the time coordinate $x_0=ct$ to give time the same dimensions as the spatial coordinates, $x_1=x$, $x_2=y$, $x_3=z$. Let the unprimed coordinates $(x_0,x_1,x_2,x_3)$ be the coordinates of a point $P$ in space-time with respect to axes fixed in an inertial frame $K$, and let the primed coordinates $(x_0',x_1',x_2',x_3')$ be the coordinates of $P$ with respect to axes fixed in the inertial frame $K'$, moving with speed $\beta c$ parallel to the $x_1$ axis of $K$. The axes in the two frames are defined to be parallel to each other and coincident at $x_0=x_0'=0$. The second postulate is equivalent to the invariance of the quantity $x_0^2-x_1^2-x_2^2-x_3^2$. The first postulate implies the isotropy and homogeneity of space-time. Together they lead to the *Lorentz transformation* among the coordinates in $K$ and $K'$,

$$x_0'=\gamma(x_0-\beta x_1),\quad x_1'=\gamma(x_1-\beta x_0),$$
$$x_2'=x_2,\quad x_3'=x_3. \tag{109a}$$

The Lorentz factor $\gamma$ is

$$\gamma=1/\sqrt{1-\beta^2}. \tag{109b}$$

If the coordinate axes in $K$ and $K'$ remain parallel, but the velocity $\mathbf{v}=\boldsymbol{\beta}c$ of frame $K'$ with respect to frame $K$ is in an arbitrary direction, the Lorentz transformation (109a) has the generalization

$$x_0'=\gamma(x_0-\boldsymbol{\beta}\cdot\mathbf{x}),$$
$$\mathbf{x}'=\mathbf{x}+\frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta}-\gamma\boldsymbol{\beta}x_0. \tag{110}$$

Energy and momentum of a particle transform in the same manner as the time and space coordinates. If $E$ is the particle's total energy (including its rest energy, $mc^2$) and $\mathbf{p}$ its momentum, then $(p_0=E/c,p_1,p_2,p_3)$ transform in the same way as $(x_0,x_1,x_2,x_3)$. One calls the four quantities a four-vector and sometimes writes $x^\mu=(x_0;\mathbf{x})$. Similarly, $p^\mu=(p_0;\mathbf{p})$ is the energy–momentum four-vector. For electromagnetic radiation, $k_0=\omega/c$ and $\mathbf{k}$ form the frequency–wave vector four-vector. The four quantities $G^\mu$ $(\mu=0,1,2,3)$ form a four-vector if they transform under the above Lorentz transformation in the same way as the space and time coordinates, namely,

$$G_0'=\gamma(G_0-\beta G_1),\quad G_1'=\gamma(G_1-\beta G_0),$$
$$G_2'=G_2,\quad G_3'=G_3, \tag{111}$$

or the relations equivalent to Eq. (110) for a general velocity $\mathbf{v}$ of $K'$ in $K$. The inverse transformation is obtained by interchanging primed and unprimed coordinates and changing the sign of $\beta$ or $\boldsymbol{\beta}$.

Application of Eq. (111) to frequency and wave number yields the *relativistic Doppler formulas,*

$$\omega'=\gamma\omega(1-\beta\cos\theta),$$
$$\tan\theta'=\sin\theta/\gamma(\cos\theta-\beta), \tag{112}$$

where $\theta$ and $\theta'$ are the angles of $\mathbf{k}$ and $\mathbf{k}'$ relative to the direction of $\mathbf{v}$, the relative velocity of the inertial frame $K'$ with respect to $K$. The inverse relations can be obtained by interchanging primed and unprimed quantities and reversing the sign of $\beta$. The more familiar nonrelativistic Doppler shift has $\gamma\approx 1$ in Eq. (112). The presence of $\gamma\geqslant 1$ shows that there is a frequency shift even for radiation emitted perpendicular to the velocity. This relativistic effect has been detected for rapidly rotating sources (but still $\beta\ll 1$) by means of the Mössbauer effect.

The relativistic law of transformation for velocity is not that of a four-vector. Let $\mathbf{u}'$ be a velocity in the frame $K'$, and let the velocity of $K'$ with respect to $K$ be $\mathbf{v}$. Then the velocity $\mathbf{u}$ in $K$ has components parallel $(u_\parallel)$ and perpendicular $(\mathbf{u}_\perp)$ to $\mathbf{v}$ given by

$$u_\parallel=\frac{u_\parallel'+v}{1+(\mathbf{v}\cdot\mathbf{u}')/c^2},\quad \mathbf{u}_\perp=\frac{\mathbf{u}_\perp'}{\gamma_v[1+(\mathbf{v}\cdot\mathbf{u}')/c^2]}. \tag{113}$$

Here $\gamma_v$ is the Lorentz factor associated with the velocity $\mathbf{v}$. The inverse relation is obtained by interchanging primed and unprimed quantities and changing the sign of $\mathbf{v}$. For parallel velocities, the first equation in Eq. (113) shows that, while the intuitive sum of velocities holds for nonrelativistic speeds, the factor in the denominator assures that the second postulate is not violated. In particular, if $u'=c(1-\epsilon)$ and $v=c(1-\epsilon_v)$, then $u\approx c(1-\epsilon\epsilon_v)$, correct to second order in small quantities.

While velocity does not transform in the same way as energy and momentum, a closely related four-vector, $U^\mu$, the four-velocity, can be defined through the ratio of a particle's four-momentum to its mass, i.e., $U^\mu=(\gamma c;\gamma\mathbf{u})$. The four-vector law of transformation, Eq. (111) can be applied to $U^\mu$ and then expressions for the parallel and perpendicular components of $\mathbf{u}$ in terms of $\mathbf{u}'$ and $\mathbf{v}$ can be found; Eq. (113) results.

Because of the possibility of interpreting the invariant quantity, $s^2=x_0^2-x_1^2-x_2^2-x_3^2$, as the square of a "length" in a non-Euclidean space, it is useful to introduce the ideas of upper and lower indices, of a metric tensor, and the Einstein summation convention of implicit summation over any repeated index, one lower and one upper. Thus the familiar coordinates $(ct,x,y,z)$ form a contravariant four-vector $x^\mu=(x^0=ct,\ x^1=x,\ x^2=y,\ x^3=z)$. The covariant four-vector $x_\mu=(x_0=ct,\ x_1=-x,\ x_2=-y,\ x^3=-z)$ is obtained from $x^\mu$ through multiplication by the metric tensor $g^{\mu\nu}$. Thus $x_\mu=g_{\mu\nu}x^\nu$, where $g^{\mu\nu}=g_{\mu\nu}$ and the only nonvanishing elements are

$$g^{00}=1,\quad g^{11}=g^{22}=g^{33}=-1. \tag{114}$$

The invariant "length" can then be written $s^2=x_\mu x^\mu$. The differential invariant length element in the space described by the metric tensor, Eq. (114) (called Minkowski space), is $(ds)^2=g_{\alpha\beta}\,dx^\alpha\,dx^\beta=dx_\beta\,dx^\beta=dx^\alpha\,dx_\alpha$. The scalar product of two four-vectors is defined as $A\cdot B=A_\mu B^\mu=A^0B^0-\mathbf{A}\cdot\mathbf{B}$, where boldface type means an ordinary vector in three-space. It should be noted that the Jacobian

$$J(x,x')=\frac{\partial(x^0,x^1,x^2,x^3)}{\partial(x'^0,x'^1,x'^2,x'^3)}$$

connecting the four-dimensional volume element $d^4x\equiv dx^0dx^1dx^2dx^3$ and $d^4x'\equiv dx'^0\,dx'^1dx'^2dx'^3$ is equal to unity. The four-volume element is therefore "invariant."

In Minkowski space, differential operators are defined appropriately as

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\nabla\right),$$

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right),$$

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{02}} - \nabla^2. \quad (115)$$

The four-divergence of a four-vector $A^\mu$ is the invariant quantity

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \mathbf{A}, \quad (116)$$

an equation familiar from the continuity of charge and current densities $\rho$ and $\mathbf{J}$, and the Lorentz condition for the potentials $\phi$ and $\mathbf{A}$. (Here is the first suggestion of specific Lorentz transformation properties of electromagnetic quantities.) The second line in Eq. (115) shows that the wave-equation operator is a Lorentz-invariant four-dimensional Laplacian (denoted by $\Box$).

### 7.2 Relativistic Formulation of Electrodynamics

If $c\rho$ and $\mathbf{J}$ are interpreted as the time and space components of a four-vector density $J^\mu$, the continuity equation for charge and current, Eq. (9), can be expressed in the invariant divergence form of Eq. (116):

$$\partial_\mu J^\mu = 0. \quad (117)$$

The validity of such an identification can be inferred from the fact that electric charge is known experimentally to be a Lorentz-invariant quantity. [See Sec. 11.9 of Jackson (1975) for a brief discussion.] Since $\rho\, dx\, dy\, dz = \rho\, dx^1\, dx^2\, dx^3$ is such a quantity of charge and since the four-dimensional volume element $d^4x$ is invariant, one can infer that the charge density $\rho$ transforms as $dx^0$, that is, as the time component of a four-vector. The requirement that the continuity equation must be true in all inertial frames then forces $\mathbf{J}$ to be the space part of the same four-vector $J^\mu = (c\rho; \mathbf{J})$.

The Lorentz condition, Eq. (69), takes the invariant form of Eq. (116) if the potentials $\phi/c$ and $\mathbf{A}$ are identified as the components of a four-vector potential $A^\mu = (\phi/c; \mathbf{A})$. The wave equations, Eqs. (70) and (71), are seen to involve the invariant four-dimensional Laplacian operator, with components of the four-vector potential on one side and the four-current density on the other. They therefore provide a properly Lorentz-covariant description summarized in the wave equation

$$\Box A^\mu = \mu_0 J^\mu \quad (118)$$

with the invariant subsidiary condition,

$$\partial_\nu A^\nu = 0. \quad (119)$$

The definitions of the electric and magnetic fields in terms of the potentials [Eq. (65)] can be shown to imply that the field strengths $(E_x, E_y, E_z, B_x, B_y, B_z)$ are the elements of a second-rank, antisymmetric, field-strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The explicit form as a matrix array is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \quad (120)$$

The Maxwell equations can be written in covariant form as

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta,$$

$$\partial^\alpha F^{\beta\delta} + \partial^\beta F^{\delta\alpha} + \partial^\delta F^{\alpha\beta} = 0. \quad (121)$$

The first equation embodies Coulomb's law ($\beta = 0$) and the three Ampère–Maxwell equations ($\beta = 1,2,3$); the second, in which $\alpha$, $\beta$, and $\delta$ are any three of (0,1,2,3), the four source-free equations. Note that current conservation is implicit in the first equation in (121). If the four-divergence is taken of both sides, the antisymmetric nature of $F^{\alpha\beta}$ means that $\partial_\beta \partial_\alpha F^{\alpha\beta} = 0$ and Eq. (117) is recovered. The completely antisymmetric unit tensor $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$ (defined to be zero if two or more indices are the same, and equal to $\pm 1$ depending on the even or odd number of permutations of the indices needed to go from $\epsilon_{0123} = +1$) can be used to define the dual field-strength tensor, $\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$, which has the general appearance of Eq. (120), but with the roles of $\mathbf{E}$ and $\mathbf{B}$ interchanged (and some signs different). The contraction of $F^{\alpha\beta}$ with itself and with its dual tensor yield two Lorentz-invariant combinations, quadratic in the field strengths:

$$F_{\alpha\beta}F^{\alpha\beta} = -2(\mu_0\epsilon_0 \mathbf{E}\cdot\mathbf{E} - \mathbf{B}\cdot\mathbf{B}),$$

$$F^{\alpha\beta}\mathcal{F}_{\alpha\beta} = -4\sqrt{\mu_0\epsilon_0}\,\mathbf{E}\cdot\mathbf{B}. \qquad (122)$$

The Lorentz transformation properties of the field strengths as a second-rank tensor can be expressed in vectorial form as

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \frac{\gamma^2}{\gamma+1}\,\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}),$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}/c) - \frac{\gamma^2}{\gamma+1}\,\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B}). \qquad (123)$$

The fields $\mathbf{E}'$ and $\mathbf{B}'$ are those in the inertial frame $K'$, which is moving with velocity $\mathbf{v} = \boldsymbol{\beta}c$ with respect to the frame $K$. The inverse transformation is obtained, as usual, by interchanging primed and unprimed quantities and changing the sign of $\mathbf{v}$ (and $\boldsymbol{\beta}$).

Consider a point charge $q$ at rest in $K'$, with electric field $\mathbf{E}' = q\mathbf{r}'/4\pi\epsilon_0 r'^3$ and $\mathbf{B}' = \mathbf{0}$ in that frame. In the frame $K$ the charge is moving with velocity $\mathbf{v}$. The inverse of Eq. (123) shows that, to first order in $\beta$ (i.e., the nonrelativistic approximation), the electric field in $K$ is the same as before, but there is now a magnetic field,

$$\mathbf{B} \approx \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\times\mathbf{r}}{r^3}, \qquad (124)$$

which is just the Ampère–Biot–Savart expression for the magnetic field of a moving charge.

The above considerations are restricted to the Maxwell equations in vacuum, apart from the source four-vector, $J^{\mu}$. For a description of the relativistic formulation of moving media, where $(\mathbf{E}/c, \mathbf{B})$, $(c\mathbf{P}, \mathbf{M})$, and $(c\mathbf{D}, \mathbf{H})$ are separately second-rank tensors (with the electric and magnetic properties of the medium defined in its rest frame), see Pauli (1958) or Penfield and Haus (1967).

### 7.3 Lagrangian Formulation of Electrodynamics

The powerful methods of Lagrangians, familiar in the classical mechanics of systems with a finite number of degrees of freedom, can be applied to continuous systems (with an infinite number of degrees of freedom). The finite number of coordinates and velocities, $q_i(t)$, $\dot{q}_i(t)$, $i = 1,2,3,\ldots,n$, are replaced by continuous fields and their derivatives. The discrete index $i$ is replaced by the continuous space-time coordinate $x^{\mu}$ and perhaps by a finite number of discrete values at each $x^{\mu}$. The correspondences are $q_i(t) \to \phi_k(x)$, $\dot{q}_i(t) \to \partial^{\beta}\phi_k(x)$. Here $x$ stands for the space and time coordinates, and there are $N$ different fields $\phi_k$, $k = 1,2,3,\ldots,N$, defined at each point $x$. The dynamics is defined by a Lagrangian density $\mathcal{L}$ whose integral over all three-space is the Lagrangian $L$. The Euler–Lagrange equations follow from the stationary property of the action integral (of $L$ over time) with respect to variations $\delta\phi_k$ and $\delta(\partial^{\beta}\phi_k)$ around the physical values. There are thus the following correspondences:

$$i \to x^{\mu}, k,$$

$$q_i \to \phi_k(x),$$

$$\dot{q}_i \to \partial^{\beta}\phi_k(x),$$

$$L = \sum_i L_i(q_i, \dot{q}_i) \to \int \mathcal{L}(\phi_k, \partial^{\beta}\phi_k)\, d^3x,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \to \partial^{\beta}\frac{\partial\mathcal{L}}{\partial(\partial^{\beta}\phi_k)} = \frac{\partial\mathcal{L}}{\partial\phi_k}. \qquad (125)$$

The action integral $S$ takes the form

$$S = \int dt \int d^3x\, \mathcal{L} = \int \mathcal{L}\, d^4x. \qquad (126)$$

The necessary Lorentz-invariant nature of the action is preserved provided the Lagrangian density $\mathcal{L}$ is a Lorentz scalar. By analogy with discrete-particle dynamics it is expected that the kinetic energy part of the Lagrangian density is quadratic in the velocities, i.e., in $\partial^{\beta}\phi_k$. The combination, $\partial^{\beta}\phi_k\partial_{\beta}\phi_k$, for each $k$ has the required invariance under Lorentz transformations. Additional terms in $\mathcal{L}$ depend on the physical properties of the fields and their interactions.

For the electromagnetic field, the four-vector potential $A^{\beta}$ plays the role of the set of $\phi_k$. The Lagrangian density is expected to be quadratic in the derivatives $\partial^{\alpha}A^{\beta}$ or equivalently in $F^{\alpha\beta}$. Equation (122) displays the only Lorentz-invariant quadratics in $F^{\alpha\beta}$. Restriction of $\mathcal{L}$ to a scalar under spatial inversions eliminates the second form and leaves a multiple of $F^{\alpha\beta}F_{\alpha\beta}$ as the only possibility. If one recalls that the Hamiltonian (energy) density involves subtraction of $\mathcal{L}$ and compares Eq. (91) with the first term in Eq. (122), it is apparent that the Lagrangian density must be

$$\mathscr{L}=-\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}-J_\alpha A^\alpha. \tag{127}$$

The interaction of the potentials with the external sources $J^\beta$ has also been included. In order to find the Euler–Lagrange equations of motion [last line in Eq. (125)], the Lagrangian density must be written in terms of the potentials:

$$\mathscr{L}=-\frac{1}{4\mu_0}g_{\lambda\mu}\,g_{\nu\sigma}(\partial^\mu A^\sigma-\partial^\sigma A^\mu)(\partial^\lambda A^\nu-\partial^\nu A^\lambda)$$
$$-J_\alpha A^\alpha. \tag{128}$$

The ingredients of the Euler–Lagrange equation of motion are

$$\frac{\partial\mathscr{L}}{\partial(\partial^\beta A^\alpha)}=-\frac{1}{\mu_0}F_{\beta\alpha}=\frac{1}{\mu_0}F_{\alpha\beta},\quad \frac{\partial\mathscr{L}}{\partial A^\alpha}=-J_\alpha$$

with the equation of motion taking the form

$$\partial^\beta F_{\beta\alpha}=\mu_0 J_\alpha, \tag{129}$$

equivalent to the first equation in Eq. (121). The homogeneous Maxwell equations do not emerge as Euler–Lagrange equations of motion—they are already implicit in the definition of the fields in terms of the potentials (See Sec. 5.1).

### 7.4 Further Ramifications

The reader who wishes to explore further ramifications of the relativistic formulation of classical field theory may consult Landau and Lifshitz (1975), Rohrlich (1990), or Soper (1976). Construction of the relativistic generalization of the Hamiltonian density (called the stress–energy–momentum tensor) and its connection to a unified description of the conservation of energy and momentum are discussed briefly in Jackson (1975), Sec. 12.10, and in the just cited references.

## GLOSSARY

**Admittance:** Symbol, $Y=G-iB$ (engineers: read $Y=G+jB$); reciprocal of impedance ($Z$).

**Ampere:** The SI unit of electric current, abbreviated A, equal to one coulomb per second.

**Capacitance:** Amount of charge carried by a conductor raised to a potential difference of one volt.

**Capacitance, Coefficients of:** Generalization of capacitance; geometrical coefficients for a given configuration of conducting surfaces and linear dielectric media between, which permit the total electrostatic energy to be expressed in terms of the potentials of the conductors—see Eq. (54).

**Causality:** Loosely, cause precedes effect; the response of a system to an external stimulus always follows in time the application of the stimulus. In relativity, it also involves the concept that signals do not travel faster than the speed of light in vacuum.

**Conductivity:** Symbol, $\sigma$; local property of conducting media, reciprocal of resistivity.

**Constitutive Relations:** Relations specifying the dielectric and magnetic properties of a medium.

**Coulomb:** The SI unit of electric charge, abbreviated C. The magnitude of the charge on the electron or the proton in this unit is $e=1.6022\times10^{-19}$ C.

**Dielectric Constant:** Symbol, $\epsilon(\omega)$ [or $\epsilon_r(\omega)$, for $\epsilon(\omega)/\epsilon_0$]; frequency-dependent proportionality between electric displacement and electric field in simple linear media. More generally a tensor, $\epsilon_{jk}(\omega)$.

**Dispersion:** Property of dielectrics or magnetic media that the product of the wavelength and the frequency of electromagnetic waves depends on the frequency (in vacuum, $\lambda\nu=c$).

**Dispersion Relations:** Relations between the real and imaginary parts of the dielectric constant; consequences of causality—see Eq. (46). Have applications in other fields.

**Displacement, Electric:** Symbol **D**; field variable in dielectrics, related to electric field and electric polarization—see Eq. (24). The SI unit is coulomb per square meter.

**Electric Field:** Symbol, **E**; a vectorial quantity, defined at a given space-time point as the mechanical force per unit charge experienced by an infinitesimal test charge placed at the point. The SI unit is volts per meter.

**Electromotive Force:** Line integral of the electric field around a closed path in space; accompanies a magnetic flux density changing in time.

**Farad:** The SI unit of capacitance, abbreviated F; one coulomb per volt.

**Gauge, Gauge Transformation:** Particular choice of scalar and vector potentials; modification of choice of potentials, without change in the physically observable quantities.

**Henry:** The SI unit of inductance, abbreviated H. One henry equals one volt–second per ampere.

**Hertz Vectors:** Auxiliary potentials used in solution of the wave equation—see Sec. 5.3.

**Impedance:** Symbol, $Z=R-iX$ (engineers: read $R+jX$). Complex quantity with dimensions of resistance (ohms) that describes the magnitude and phase of the voltage across the terminals of an electrical device with respect to an input current.

**Impedance of Free Space:** Constant ($\sqrt{\mu_0/\epsilon_0}$) with dimensions of resistance; ratio of electric to magnetic field strengths in a plane electromagnetic wave in vacuum. Numerically, $\sqrt{\mu_0/\epsilon_0}=376.730\ \Omega$.

**Inductance:** Symbol, $L$ (self-inductance) or $M_{12}$ (mutual inductance); geometrical property of a coil of wire or other circuit describing its ability to develop a voltage difference across it when the current through it (or a neighboring circuit) changes with time—a manifestation of Faraday's law. Symbolically, $V=L\ dI/dt$. The SI unit is the henry.

**Inductance, Coefficients of:** Totality of self- and mutual inductances of a configuration of circuits, permitting the total energy of the configuration to be expressed in terms of the currents in the circuits—see Eq. (60).

**Kramers–Kronig Relations:** Another name for dispersion relations.

**Linear Media:** Substances whose induced electric or magnetic polarizations are linear in externally applied fields.

**Lorentz Transformation:** Linear transformation in special relativity of space-time coordinates from one coordinate frame to another moving at uniform velocity with respect to the first. More generally, the transformation of four-vectors and higher-rank four-tensors.

**Magnetic Field:** Symbol, **H**; field variable related to the magnetic flux density and magnetization—see Eq. (24). The SI unit is ampere-turn per meter.

**Magnetic Flux Density:** Symbol, **B**; analog for currents of electric field. Defined in magnitude as force per unit current, but directions of current flow, **B**, and force are not parallel—see discussion above Eq. (6). The SI unit is the tesla (sometimes, weber per square meter).

**Magnetization:** Symbol, **M**; macroscopic magnetic property of a substance, the average molecular or atomic magnetic dipole moment density—see Sec. 3.1. The SI unit is ampere per meter.

**Maxwell Stress Tensor:** Symbol, $T_{jk}$; tensor bilinear in the electric and magnetic field strengths, used to calculate electromagnetic forces—see Eq. (106).

**Minkowski Space:** Non-Euclidean space-time—see Eq. (114).

**Ohm:** The SI unit of resistance, abbreviated $\Omega$; one ohm is equal to one volt per ampere.

**Ohm's Law:** Law obeyed by many conducting substances: current density locally is proportional to electric field. Generally, there is a tensor of proportionality; often it is a scalar quantity.

**Plasma Frequency:** Symbol, $\omega_p$; material property governing the high-frequency behavior of dielectric constants—see Eq. (38); depends only on density of electrons (not details of atomic structure).

**Polarization, Electric:** Symbol, **P**; macroscopic electric property of a substance, the average molecular or atomic electric dipole moment density—see Sec. 3.1. The SI unit is coulomb per square meter.

**Pressure, Radiation:** Electromagnetic momentum per unit area per unit time.

**Resistance:** Symbol, $R$; property of conductor or circuit element to limit the flow of current for a given applied voltage. Symbolically, $V=IR$. The SI unit is the ohm.

**Resistance, Radiation:** Effective resistance of a radiating element; ratio of radiated power to the square of the driving current.

**Resistivity:** Symbol, $\rho$; local property of a conducting substance; enters Ohm's law: $\mathbf{E}=\rho\mathbf{J}$.

**Permeability of Free Space:** Constant ($\mu_0$) entering the relation between electric current and magnetic flux density [see Eq. (8)]. Defined to be $\mu_0=4\pi\times10^{-7}$ henrys per meter.

**Permittivity of Free Space:** Constant ($\epsilon_0$) entering the relation between charge and electric field [see Eq. (3)]. Its inverse, divided by $4\pi$, is a 17-digit number related to the speed of light in vacuum. Numerically, $\epsilon_0=8.854\ 19\ldots\times10^{-12}$ farads per meter.

**Potential, Four-Vector:** Symbol, $A^{\mu}$; set of four potentials, $\phi$ and the three components of **A**, which transform relativistically in the same way as $x^{\mu}=(x^0,\mathbf{x})$.

**Potential, Scalar:** Symbol, $\phi$; in static situations, the difference $\phi(\mathbf{r}_1)-\phi(\mathbf{r}_2)$ represents the line integral of the electric field from the point $\mathbf{r}_1$ to the point $\mathbf{r}_2$.

**Potential, Vector:** Symbol, **A**; an auxiliary field related to the magnetic flux density through $\mathbf{B}=\nabla\times\mathbf{A}$.

**Skin Depth:** Symbol, $\delta$; depth of penetration of fields into a good conductor—see Eq. (39).

**Système Internationale (SI):** An internationally agreed system of weights and measures; roughly, the metric system.

**Tesla:** The SI unit of magnetic flux density, abbreviated T; one tesla is equal to one newton per ampere–meter.

**Volt:** The SI unit of electric potential difference, abbreviated V; one volt equals one joule per coulomb.

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## Further Reading

Electromagnetism is part and parcel of almost everything that exists or happens. Its applications are everywhere. Recommendations for further reading can only touch on a few areas or subjects, largely reflecting the writer's knowledge and prejudices.

For more leisurely or elementary accounts of electricity and magnetism, consult the physics books,


Griffths, D. J. (1989), *Introduction to Classical Electrodynamics*, 2nd ed., New York: Prentice-Hall.

Purcell, E. M. (1985), *Electricity and Magnetism*, 2nd ed., New York: McGraw-Hill.


For a more applied, but still elementary, account, see


Kraus, J. D. (1984), *Electromagnetics*, 3rd ed., New York: McGraw-Hill.


For applications in engineering, particularly transmission and radiation, see


Collin, R. E. (1966), *Field Theory of Guided Waves*, New York: McGraw-Hill.

Collin, R. E. (1985), *Antennas and Radiowave Propagation*, New York: McGraw-Hill.

Ramo, S., Whinnery, J. R., Van Duzer, T. (1984), *Fields and Waves in Communication Electronics*, 2nd ed., New York: Wiley.


The subjects of radiation, scattering, and diffraction are treated in


Jackson, J. D. (1975), *Classical Electrodynamics*, 2nd ed., New York: Wiley, Chaps. 9 and 16.

Landau, L. D., Lifshitz, E. M. (1975), *Classical Theory of Fields*, 4th rev. Eng. ed., Oxford: Pergamon.

Special relativity is a topic far broader than electromagnetism. For discussion of its place in mechanics, see

Møller, C. (1972), *The Theory of Relativity*, 2nd ed., Oxford: Clarendon.

Sard, R. D. (1970), *Relativistic Mechanics*, New York: Benjamin.

For the relativistic theory of classical charged particles and fields, consult

Rohrlich, F. (1990), *Classical Charged Particles*, 2nd ed., Redwood City, CA: Addison-Wesley.

Soper, D. E. (1976), *Classical Field Theory*, New York: Wiley.

Discussion of relativistic radiation phenomena, including synchrotron radiation, transition radiation, classical and semiclassical bremsstrahlung, etc., may be found in

Jackson, J. D. (1975), *Classical Electrodynamics*, 2nd ed., New York: Wiley, Chaps. 14 and 15.

# ELECTRODYNAMICS, QUANTUM

See QUANTUM ELECTRODYNAMICS