EEG Feature Calculation Algorithms

Power Spectral Density Estimation:

$M = number\ of\ samples$

$x[n] = raw\ signal\ at\ sample\ n$

$\hat{R}_{xx}[k] = autocorrelation\ function\ at\ sample\ lag\ k$

$PSD = power\ spectral\ density$

$HFFT\{\cdot\} = Hermitian\ Fast\ Fourier\ Transform$

1. Estimate autocorrelation function:

$$\hat{R}_{xx}[k] \ = \frac{1}{M}\sum_{n=k}^{M} x[n]x[n-k], \qquad for\ 0 \leq k \leq M$$

2. Calculate (unnormalized) Fourier transform of autocorrelation function:

$$PSD = HFFT\left\{\hat{R}_{xx}[k]\right\}$$

3. Normalize Fourier transform:

$$PSD \ := \frac{1}{M} PSD$$

Algorithm 1: Frequency Bin-Average PSD

$$PSD[f] = power\ spectral\ density\ at\ frequency\ f$$

$$bins[i,0] = lower\ frequency\ of\ bin\ i,\ \ bins[i,1] = upper\ frequency\ of\ bin\ i$$

$$PSD_{avg}[i] = average\ PSD\ value\ for\ bin\ i$$

1. Calculate average PSD values in selected frequency bins:

$$PSD_{avg}[i] = mean\big(PSD\big[bins[i,0] : bins[i,1]\big]\big)$$

$$PSD_{avg}[i] = \frac{1}{bins[i,1] - bins[i,0]} \sum_{f=bins[i,0]}^{bins[i,1]} PSD[f]\,, \qquad if\ f \in \{0, 1, 2, \dots\}$$

Algorithm 2: PCA on PSD

$selected\ frequencies\!:\{f_1,\ f_2,\dots,f_L\}$

$L = number\ of\ frequencies$

$PSD_n^{(p)}[f] = power\ spectral\ density\ of\ example\ p, channel\ n, frequency\ f$

$P = number\ of\ examples$

$N = number\ of\ channels$

$K_n[i,j] = statistical\ matrix\ of\ channel\ n\ at\ row\ i, column\ j$

$v_{k,n}[i] = ith\ element\ of\ kth\ eigenvector\ of\ K_n$

$v_{k,n}\ is\ indexed\ starting\ at\ 1$

$W_n^{(p)}[k] = projection\ weight\ of\ log -$
$normalized\ PSD_n^{(p)}\ onto\ kth\ principal\ component\ of\ K_n$

$\varepsilon = small\ positive\ parameter\ to\ prevent\ \ln(0)$

$Q = number\ of\ principal\ components\ to\ keep\ (Q \le L)$

1. Log-normalize PSD values across examples:

$$PSD_n^{(p)}[f] \;:=\; \ln\left(PSD_n^{(p)}[f] + \varepsilon\right) - \;\ln\left(\frac{1}{P}\sum_{p=1}^{P} PSD_n^{(p)}[f] + \varepsilon\right),$$

$$for\ p \in \{1,\dots,P\}\ \ and\ \ n \in \{1,\dots,N\}\ \ and\ \ f \in \{f_1,\dots,f_L\}$$

2. *3 different variations (see 2 pages ahead).
3. Determine eigenvectors/eigenvalues of each statistical matrix:

$$v_{k,n} = kth\ eigenvector\ of\ K_n, \qquad for\ n \in \{1,\dots,N\}\ \ and\ \ k \in \{1,\dots,L\}$$

$$\lambda_{k,n} = kth\ eigenvalue\ of\ K_n, \qquad for\ n \in \{1,\dots,N\}\ \ and\ \ k \in \{1,\dots,L\}$$

4. Sort eigenvectors (principal components) of each statistical matrix in decreasing order of the magnitude of their corresponding eigenvalues:

$$\{v_{1,n},\dots,v_{L,n}\} \;:=\; sort(\{v_{1,n},\dots,v_{L,n}\}), \qquad for\ n \in \{1,\dots,N\}$$

5. Calculate projection weights of log-normalized PSD values onto principal components (PC’s):

$$W_n^{(p)}[k] \;=\; \langle PSD_n^{(p)}, v_{k,n}\rangle \;=\; \sum_{i=1}^{L} PSD_n^{(P)}[f_i] * v_{k,n}[i],$$

$$for\ n \in \{1,\dots,N\}\ and\ p \in \{1,\dots,P\}\ and\ \ k \in \{1,\dots,Q\}$$

5.1. Matrix representation:

$$Let\ the\ PSD\ matrix\ for\ channel\ n\ be:\ PSD_n = \left[PSD_n^{(1)}, \dots, PSD_n^{(P)}\right]^T \in \mathbb{R}^{P\ x\ L},$$

$$for\ n \in \{1, \dots, N\},$$

$$where\ PSD_n^{(p)} = row\ p\ of\ PSD_n = PSD\ values\ of\ example\ p, channel\ n$$

$$Let\ the\ principal\ component\ matrix\ for\ channel\ n\ be:\ V_n = \left[v_{1.n}, \dots, v_{Q,n}\right] \in \mathbb{R}^{L\ x\ Q},$$

$$for\ n \in \{1, \dots, N\}$$

$$Projection\ weights\ of\ log - normalized\ PSD\ values\ onto\ principal\ components:$$

$$W_n = PSD_n * V_n \in \mathbb{R}^{P\ x\ Q}, \qquad for\ n \in \{1, \dots, N\},$$

$$where\ row\ p\ of\ W_n = projection\ weights\ of\ example\ p\ onto\ Q\ PC's,$$

$$and\ W_n[p,k] = \langle PSD_n^{(p)}, v_{k,n} \rangle$$

6. Size of feature vector for each example p: $N * Q$

*Variations for step 2:

1. Variation 1: Calculate channel-specific autocorrelation matrices across examples:

$$K_n[i,j] \;=\; \frac{1}{P}\sum_{p=1}^{P} PSD_n^{(p)}[f_i] * PSD_n^{(p)}[f_j], \qquad for\ n \in \{1, \dots, N\}\ and\ i,j \in \{1, \dots, L\}$$

2. Variation 2: Calculate channel-specific autocovariance matrices across examples:

$$K_n[i,j] \;=\; \frac{1}{P}\sum_{p=1}^{P} \left(PSD_n^{(p)}[f_i] - \overline{PSD_n[f_i]}\right) * \left(PSD_n^{(p)}[f_j] - \overline{PSD_n[f_j]}\right),$$

$$for\ n \in \{1, \dots, N\}\ and\ i,j \in \{1, \dots, L\}$$

   2.1. Mean across examples:

$$\overline{PSD_n[f_i]} \;=\; \frac{1}{P}\sum_{p=1}^{P} PSD_n^{(p)}[f_i]$$

3. Variation 3: Calculate Pearson (correlation coefficient) channel-specific autocovariance matrices across examples:

$$K_n[i,j] \;=\; \frac{1}{\sigma_i * \sigma_j}\sum_{p=1}^{P} \left(PSD_n^{(p)}[f_i] - \overline{PSD_n[f_i]}\right) * \left(PSD_n^{(p)}[f_j] - \overline{PSD_n[f_j]}\right),$$

$$for\ n \in \{1, \dots, N\}\ and\ i,j \in \{1, \dots, L\}$$

   3.1. Unnormalized standard deviation across examples:

$$\sigma_i \;=\; \sqrt{\sum_{p=1}^{P} \left(PSD_n^{(p)}[f_i] - \overline{PSD_n[f_i]}\right)^2}$$

   3.2. Mean across examples:

$$\overline{PSD_n[f_i]} \;=\; \frac{1}{P}\sum_{p=1}^{P} PSD_n^{(p)}[f_i]$$