Complex Economics Dynamics 1

Homework assignment 1

Important

a) Submit the assignment as one .pdf file prepared with LaTeX or one of its variants, with a font size of at least 10pt.

b) **The first line of the .pdf document should contain the name or names of the submitters.**

c) You are allowed to code in `julia`, `matlab`, `python`, or `R`. If you want to use another programming language, please consult first with the tutorials teacher.

d) Submit any code as a `filename.txt` file: for instance, if you submit the file `foo.py`, rename it to `foo.py.txt`.

e) Clear, concise, well-documented code gets more points.

Consider the following model of stock market trading. There are $2N$ investors, of which n_+ are optimists and n_- are pessimists, $n_- + n_+ = 2N$. We introduce $n = \frac{1}{2}(n_+ - n_-)$ and $x = n/N$ as the 'average opinion': note that $x \in [-1, 1]$.

Pessimistic traders become optimistic at a rate p_{+-}; optimistic traders become pessimistic at a rate p_{-+}. We assume that these rates only depend on the average opinion x.

Then we obtain that $\dot{n}_+ = n_- p_{+-} - n_+ p_{-+}$, a similar expression for $\dot{n}_-$, and consequently an evolution equation for the average opinion

$$\dot{x} = (1-x)p_{+-}(x) - (1+x)p_{-+}(x). \tag{1}$$

Question 1. Make the derivation of this evolution equation precise.

We use the following model for the switching rates:

$$p_{+-}(x) = v \exp(ax), \qquad p_{-+}(x) = v \exp(-ax);$$

the parameter a measures the strength of herding behaviour, and v is the speed of change.

Question 2. a. Show that for this specification of the switching rates, the system (1) has for $0 \le a < 1$ a unique stable steady state at $x = 0$

b. Show that for $a > 1$, the steady state $x = 0$ is unstable, and there are two additional stable steady states. (Hint: it is possible to compute for most values of x the value of a such that x is a steady state for that value of a. The identity $\frac{1}{2}\log\frac{1+x}{1-x} = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}$, valid for $|x| < 1$, may be useful.).

c. Compute and plot the bifurcation diagram. Show families of stable steady states as solid curves, and families of unstable steady states as dashed curves.

d. Which bifurcation occurs at $a = 1$?

To model trading, we assume that every optimistic trader offers to buy, and every pessimistic trader offers to sell, a fixed quantity t_N of the stock. This yields the net excess demand $D_N = n_+ t_N - n_- t_N = 2n t_N$. Recalling that $n = Nx$ and introducing the trading volume $T_N = 2N t_n$, we find for the net demand

$$D_N = T_N x.$$

To allow the traders to actually trade, we introduce a second group of traders into this model. The excess demand of this group, which we shall call fundamentalists, depends on the difference between the fundamental price p_f and the actual price p of the assets. We take a simple, linear, specification

$$D_F = T_F(p_f - p);$$

here T_F is a measure of the trading volume of the fundamentalists. The market is then modelled using a 'market-maker' that matches supply and demand at any instance in time, and lets prices adjust in the direction of zero excess demand. This is captured in the following dynamic law

$$\dot{p} = \beta(D_N + D_F) = \beta\left(T_N x + T_F(p_f - p)\right), \tag{2}$$

with β the speed of price adjustment.

To model the influence of the price dynamics on the opinion dynamics, we change the model of the switching rates to

$$p_{+-}(x,p) = v\exp(a_1\dot{p}/v + a_2 x), \qquad p_{-+}(x,p) = v\exp(-a_1\dot{p}/v - a_2 x);$$

here a_1 describes how much information traders draw from price changes, whereas a_2 models, as a before, the strength of herding behaviour; finally, $\dot{p}$ is given by (2).

We obtain the following two-dimensional system for the opinion–price dynamics

$$\begin{aligned} \dot{x} &= (1-x)p_{+-}(x,p) - (1+x)p_{-+}(x,p), \\ \dot{p} &= \beta\left(T_N x + T_F(p_f - p)\right). \end{aligned}$$

Question 3. a. Show that for $0 \le a_2 \le 1$ this system has a unique steady state $E_0 = (0, p_f)$.

b. Show that for $a_2 > 1$, the system has two additional steady states E_- and E_+.

c. Show that if $a_2 > 1$, the steady state E_0 is unstable.

d. Show that if $0 \le a_2 < 1$, the steady state E_0 may be stable or unstable: derive a stability condition in terms of the parameters a_1, a_2, β, v, T_N and T_F.

e. What kind of bifurcation destabilises E_0?

f. Find parameter values such that the system has periodic solutions and compute these numerically. Give the code. Plot the solution in the (x,p)-plane. Also give $(t, x(t))$ and $(t, p(t))$ plots for this case.