Complex Economics Dynamics 1

Homework assignment 2

Important

a) Submit the assignment as one .pdf file prepared with LaTeX or one of its variants, with a font size of at least 10pt.

b) **The first line of the .pdf document should contain the name or names of the submitters.**

c) You are allowed to code in `julia`, `matlab`, `python`, or `R`. If you want to use another programming language, please consult first with the tutorials teacher.

d) Submit any code as a `filename.txt` file: for instance, if you submit the file `foo.py`, rename it to `foo.py.txt`.

e) Clear, concise, well-documented answers and code gets more points.

f) In the course of the model description, more and more assumptions are introduced. These assumption can only be used when answering questions following their introduction, not earlier ones.

Consider a stock market, where investors can invest either in a riskless asset, with gross return $R > 1$, and a risky asset, which pays a stochastic dividend y_t each period.

The wealth of an investor of type h evolves according to

$$W_{h,t+1} = (1+r)W_{h,t} + (p_{t+1} + y_{t+1} - Rp_t)z_{h,t},$$

where $z_{h,t}$ is the number of shares of the risky asset the investor has bought.

Denote by $\mathbb{E}_{h,t}$ and $\mathbb{V}_{h,t}$ the beliefs of an investor of type h about mean and variance of a stochastic variable, based on the information available at the beginning of period t. Investors are mean-variance maximisers, that is, they choose $z_{h,t}$ in order to maximise the objective

$$\mathbb{E}_{h,t}W_{h,t+1} - \frac{a}{2}\mathbb{V}_{h,t}W_{t+1}.$$

> **Question 1.** Derive the demand $z_{h,t}$ of shares by an investor of type h that maximises the objective. (6 points)

In the following, we consider the case that the y_t are iid distributed with mean y^*. This is common knowledge among the investors. We introduce the fundamental price $p^* = y^*/(R-1)$.

Expectations. In the model, there will be two belief types. Both types have the same belief about variances:

$$\mathbb{V}_{h,t}(p_{t+1} + y_{t+1}) = \sigma^2, \qquad h = 1, 2.$$

Type 1 investors are 'fundamentalists', in that they believe that tomorrow's price tends towards the fundamental prices. They believe that

$$\mathbb{E}_{1,t}p_{t+1} = p^e_{1,t+1} = p^* + v(p_{t-1} - p^*),$$

where $0 < v < 1$ is a positive constant.

Type 2 investors are 'technical traders', which extrapolate price trends. They believe

$$\mathbb{E}_{2,t}p_{t+1} = p^e_{2,t+1} = p_{t-1} + g(p_{t-1} - p_{t-2}).$$

If $g > 0$, the investors follow a trend; if $g < 0$, they expect a reversal of the trend.

Market equilibrium. Let $n_{i,t}$ denote the fraction of investors using rule i at the beginning of period t. We assume that the total number of shares is 0: investors can trade by going 'long' — buying positive amounts — and 'short' — buying negative amounts — in the number of shares. Market equilibrium in period t reads as

$$Rp_t = \sum_{h=1}^{2} n_{h,t} p^e_{h,t+1} + y^*.$$

Fractions. After the market price p_t is determined, fractions are updated on the basis of the realised value of their objective in period t. These are given as

$$\pi_{h,t} = (p_t + y^* - Rp_{t-1}) z_{h,t-1} - \frac{a}{2}\sigma^2 z^2_{h,t-1}$$

> **Question 2.** Show that the fitness measure $\pi_{h,t}$ is equivalent to a type-independent term minus a constant times the squared prediction error. That is, show that $\pi_{h,t} = A_t - B_t(p^e_{h,t} - p_t)^2$ and determine A_t and B_t. (6 points)

The fractions $n_{h,t}$ of type h agents in period t are determined in two steps. In the first step, investors choose their belief type according to the discrete choice model

$$\hat{n}_{i,t} = \exp(\beta \pi_{i,t-1})/Z_t, \qquad Z_t = \sum_{i=1}^{N} \exp(\beta \pi_{i,t-1}),$$

where the normalisation Z_t ensures that the fractions add up to one. The parameter β is the intensity of choice.

In the second step, some of the investors that thought that technical trading is the best strategy have second thoughts if the price is too far away from the fundamental, and switch to the fundamentalist strategy. We model this by setting

$$\begin{aligned} n_{2,t} &= \hat{n}_{2,t}\, \mathrm{e}^{-(p_{t-1}-p^*)^2/\alpha}, \\ n_{1,t} &= 1 - n_{2,t}. \end{aligned}$$

Research questions. The assignment is centred around two questions:

(A) How does the two-type dynamics between fundamentalists and technical traders depend on the technical traders' extrapolation strength g?

(B) What kind of dynamical behaviour can occur in this market?

Question 3. Transformation of the model. (24 points)

a. Give the price evolution equation in full: in particular, write out terms like $p^e_{h,t+1}$, $n_{h,t}$ and $Z_{h,t}$.

b. Rewrite the model in terms of the deviation variable $x_t = p_t - p^*$.

c. Rewrite the model as a dynamical system $X_{t+1} = \Phi(X_t)$, where

$$X_t = (x_{t-1}, x_{t-2}, x_{t-3}, x_{t-4}).$$

d. Show that $X^* = (0, 0, 0, 0)$ is the unique steady state of the dynamical system.

Question 4. Local stability analysis. (24 points)

a. Find all values of (g, v) such that the steady state $X = (0, 0, 0)$ of the dynamical system Φ is hyperbolic and locally stable.

b. Find all values (g, v) such that the steady state $X = (0, 0, 0)$ satisfies the necessary conditions for one of the following bifurcations: saddle-node, transcritical, pitchfork, period-doubling, and Hopf.

c. Show that the dynamical system is symmetric, that is, show that $\Phi(-X) = -\Phi(X)$.

d. Investigate whether a symmetric 2-cycle $(X^*, -X^*)$ exists in the model.

Question 5. Dynamical numerical analysis. (24 points)

Set $\beta = 4$, $v = 0.6$, $R = 1.01$, $\alpha = 10$, $a = \sigma^2 = 1$.

a. Make a plot of the largest Lyapunov exponent of the dynamics as a function of g varying between 1 and at least 2.2. Let g increase in steps of at most 0.01.

b. Illustrate dynamics with a negative largest Lyapunov exponent by giving a (t, x_t) diagram as well as a (p_t, p_{t+1}) diagram.

c. Do the same for dynamics where the largest Lyapunov exponent equals 0.

d. Do the same when the largest Lyapunov exponent is positive.

Question 6. Describe your results in economic terms. (16 points)