Syllabus: Interest rate modelling

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Syllabus outline

Financial derivatives are instruments that derive their value from some underlying asset, think for example of forwards, futures or options. If the underlying asset is the interest rate (IR), we speak of interest rate derivatives. This class of financial securities has by far the highest trading volume on the bilateral market. This is because IR derivatives are used to hedge interest rate risk, to which any corporate that needs to attract funding is exposed. Interest rate modelling constitutes the foundation to pricing, hedging and risk-managing IR derivatives. A good understanding of this topic is therefore highly relevant for many financial institutions.

This syllabus will provide a compact introduction to the basic concepts, definitions and prerequisites that are fundamental to IR modelling. Most of the content that is presented here is derived from [2], [6] and [3]. We highly recommend these books for more information on this field. The syllabus will touch upon the following topics:

1. **Mathematical prerequisites:** A brief introduction to the mathematical notation and techniques required in the subsequent sections. It treats Brownian motion and Itô calculus.

2. **Modelling the financial market:** A brief introduction to modelling the financial market with Itô processes. We treat important results, including risk-neutral pricing and changing the numéraire.

3. **Interest rate definitions and terminology:** The field of IR modelling entails certain specific definitions and jargon. Here we will treat the most important ones and introduce some common interest rate derivatives.

4. **Modelling the short-rate:** There exist many different models to simulate interest rate dynamics. Here we will introduce the Hull-White model, which is popular amongst practitioners in risk-management.

1 Mathematical prerequisites

In this chapter we will provide a brief summary of the relevant mathematical notions and propositions.

1.1 Brownian motion and Itô calculus

Brownian motions can be considered as the continuous-time equivalent of a random-walk. We will consider stochastic processes to model quantities that are exposed to randomness. The Brownian motion is a continuous process with convenient properties, that is often used to introduce "randomness" to a model. We will start with a formal definition.

Definition 1. ***One-dimensional Brownian motion:*** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. Suppose that* $W : \Omega \times [0, \infty) \to \mathbb{R}$ *is a continuous stochastic process such that* $\forall_{\omega \in \Omega} : W(0) = 0$. *Then* $W(t)$ *is a Brownian motion if for all* $0 = t_0 < \ldots < t_m$ *the increments* $W(t_1) - W(t_0), \ldots, W(t_m) - W(t_{m-1})$ *are independent and normally distributed such that*

$$W(t_i) - W(t_{i-1}) \sim N(0, t_i - t_{i-1})$$

The Brownian motion restricted to a finite time-horizon $[0, T]$ is a text-book example of a Martingale.

Definition 2. ***Martingale:*** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* T *a fixed positive constant and* $\mathcal{F}_t$ *a filtration of* $\mathcal{F}$ *for* $0 \leq t \leq T$. *Let* $M(t)$ *be an adapted stochastic process relative to* $\mathcal{F}_t$ *for* $0 \leq t \leq T$. *Then* $M(t)$ *is a Martingale if for all* $0 \leq s \leq t \leq T$ *we have that*

$$\mathbb{E}\left(M(t) \middle| \mathcal{F}_s \right) = M(s) \quad a.s.$$

The stochastic processes that are typically considered in financial models, take the form of an Itô process $X(t)$ adapted to the filtration $\mathcal{F}_t$. This means that X can be written as

$$X(t) = X(0) + \int_0^t \Theta(u) du + \int_0^t \Delta(u) dW(u)$$

where $\Theta(t)$ and $\Delta(t)$ are adapted, integrable processes. Often the dynamics of a process X are expressed in its differential form, given by

$$dX(t) = \Theta(t) dt + \Delta(t) dW(t)$$

The term $\int_0^t \Delta(u) dW(u)$ is known as an Itô integral. The properties of an Itô integral are listed in the following proposition.

Proposition 3. ***Properties of the Itô-integral:*** *Let* T *be a positive constant,* $t \in [0, T]$ *and* $\Delta(t)$ *some stochastic process adapted to* $\mathcal{F}_t$. *Assume that*

$$\mathbb{E}\left(\int_0^T \Delta^2(t) dt \right) < \infty$$

Then the Itô-integral $I(t) := \int_0^t \Delta(s) dW(s)$ *has the following properties:*

1. **(Continuity)** $I(t)$ has continuous paths w.r.t. the variable t.

2. **(Adaptivity)** $I(t)$ is $\mathcal{F}_t$-measurable.

3. **(Martingale)** $I(t)$ is a Martingale.

4. **(Itô-isometry)** $\mathbb{E}\left(I^2(t)\right) = \mathbb{E}\left(\int_0^t \Delta^2(s)ds\right)$

For a proof we refer to [6]. If $\Delta(t)$ is a deterministic process, we have additionally the following result.

Proposition 4. ***Itô-integral of a deterministic integrand:*** *Let $\Delta(t)$ be a deterministic function of time. Then for all $t \geq 0$*

$$\int_0^t \Delta(u)dW(u) \sim N\left(0, \int_0^t \Delta^2(u)du\right).$$

See [6] for a proof. An important general result, which can be used to evaluate stochastic differential equations (SDEs), is known as Itô's lemma or the Itô-Doeblin formula. Below we state the one-dimensional version of this proposition. Generalisations to $d > 1$ dimensions exist, but are not treated here.

Proposition 5. ***Itô's lemma:*** *Let $f(t,x)$ denote a continuous function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ with well-defined, continuous partial derivatives f_t, f_x and f_{xx}. Let $X(t)$ denote an Itô processes on the space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying $dX(t) = \Theta(t, X(t))\,dt + \Delta(t, X(t))\,dW(t)$. Then $Y(t) = f(t, X(t))$ defines an Itô process that satisfies the SDE*

$$\begin{aligned} dY(t) = &\left(f_t(t, X(t)) + f_x(t, X(t))\,\Theta(t, X(t)) + \frac{1}{2} f_{xx}(t, X(t))\,\Delta^2(t, X(t))\right) dt \\ &+ f_x(t, X(t))\,\Delta(t, X(t))\,dW(t) \end{aligned}$$

A proof can be found in [6].

Definition 6. ***Equivalent measures:*** *Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathbb{P}$, $\tilde{\mathbb{P}}$ denote two probability measures on this space. Then $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent if they agree on all null-sets, i.e. for all $A \in \mathcal{F}$ we have*

$$\mathbb{P}(A) = 0 \Leftrightarrow \tilde{\mathbb{P}}(A) = 0$$

Two equivalent measures are related through a unique random variable, typically referred to as the Radon-Nikodym derivative. The following theorem is an important result, which allows us to construct equivalent measures on a common measurable space.

Proposition 7. ***(Radon-Nikodym)*** *Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathbb{P}$, $\tilde{\mathbb{P}}$ denote two probability measures on this space. Then there exists an a.s. positive random variable Z, such that $\mathbb{E}Z = 1$ and for every $A \in \mathcal{F}$*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega)$$

We refer to [6] for a proof. The variable Z is called the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. By convention we usually write $Z \equiv \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$. Related to Z is the Radon-Nikodym derivative process $Z(t)$, which is defined as

$$Z(t) = \mathbb{E}\left(Z | \mathcal{F}_t\right)$$

By the tower-property of conditional expectation, it can easily be shown that $Z(t)$ defines a Martingale.

2 Modelling the financial market

This section serves as an overview of the relevant financial terminology and concepts. The first part considers the concepts of no-arbitrage and risk-neutral pricing. The subsequent part concerns the change of numéraire technique. This technique is required to perform certain pricing routines, which would otherwise be difficult to compute.

2.1 A stochastic economy

We start with a brief, but general description of a no-arbitrage economy. We largely follow the set-up described in [3] and [2]. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ representing all possible states of the economy and let $T > 0$ be some date in the future defining a finite time horizon $[0, T]$. Let $\{\mathcal{F}_t | \, 0 \leq t \leq T\}$ be a filtration on $\mathcal{F}$, such that $\mathcal{F}_T = \mathcal{F}$. Intuitively, $\mathcal{F}_t$ can be considered all the information generated by the economy until time$-t$. Defined on this probability space we consider a multi-dimensional financial market, that consists of a finite number m of risky assets (stocks, bonds, commodities, etc.). We assume that the price of these assets can be captured by stochastic Itô processes, which we will denote as $X_1(t), \dots, X_m(t)$. We assume these assets to be continuously tradable on the market and let their price-process satisfy the following SDE

$$dX_i(t) = \mu_i \left(X_i(t), t\right) dt + \sigma_i \left(X_i(t), t\right) dW_i(t), \quad i = 1, \dots, m$$

Both $\mu_i(x, t)$ (the "drift") and $\sigma_i(x, t)$ (the "volatility") denote continuous functions which are adapted to $\mathcal{F}_t$. With W_i we denote standard one-dimensional Brownian motions, which are typically correlated to one another. On top of these assets, we consider a *money-market account* B which satisfies

$$\frac{dB(t)}{B(t)} = r(t)dt, \quad B(0) = 1$$

The process B is often also referred to as *the bank account.*

Definition 8. ***Bank account process*** $B(t)$***:*** *the value of a unit of currency at time$-t$ if it were invested in the money market at time zero, where the value accumulates according to the continuously compounded, short-term interest rate $r(t)$.*

An expression for B is given by:

$$B(t) = e^{\int_0^t r(s)ds}$$

The process r is usually referred to as the *short-rate.* Closely related to the bank account is the *discount process.*

Definition 9. ***Discount process*** $D(t, T)$***:*** *the amount of currency at time$-t$ that will yield one unit of currency at time$-T$ if it were invested in the money-market.*

An expression for D is given by:

$$D(t, T) := \frac{B(t)}{B(T)} = e^{-\int_t^T r(s)ds}$$

As we will be considering securities directly related to interest rates, we will consider a stochastic interest rate economy and a related interest rate model. We will thus assume that r is a stochastic process, adapted to the filtration $\mathcal{F}_t$.

2.2 Risk-neutral pricing

First consider the definition of a risk-neutral measure.

Definition 10. ***Risk-neutral measure:*** *Let $\mathbb{Q}$ be a probability measure defined on the measurable space $(\Omega, \mathcal{F})$. Then we call $\mathbb{Q}$* risk-neutral *if:*

i) $\mathbb{P}$ and $\mathbb{Q}$ are equivalent (i.e. $\forall_{A \in \mathcal{F}} : \mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0$).

ii) For each $i = 0, \ldots, m$, the process $D(0,t)X_i(t)$ is a Martingale under $\mathbb{Q}$.

With this definition in mind we, we continue to consider the concept of *hedging*. Each asset X_i in our economy is assumed to be continuously tradable on the market. Hence a potential investor is free to compose a portfolio that contains any number of assets. Let for $\phi_i(t)$ denote the quantity of asset X_i the investor holds in his portfolio for $i \in \{1, \ldots, m\}$ and $\phi_0(t)$ the amount of currency that he invested in the money-market at time t. Given that each function ϕ_i is bounded and predictable on $[0, T]$, then the $(m+1)$-dimensional process

$$\phi(t) = (\phi_0(t), \ldots, \phi_m(t)), \quad 0 \leq t \leq T$$

defines a *trading strategy*. This represents a portfolio Π, which at time t corresponds to a portfolio-value of

$$\Pi(t) = \sum_{i=0}^{m} \phi_i(t) X_i(t)$$

The strategy ϕ is additionally called a *self-financing* strategy if any change in $V(\phi, t)$ over time is solely induced by changes in the asset-values. In other words, once the portfolio is composed, no cash is added or extracted from it until time T.

Let us now consider a derivative security V, which matures at time T and has payoff $V(T)$, such that $V(T)$ is square-integrable and non-negative. Also consider an agent that traded this derivative and wishes to hedge it. The agent would then want to set up a portfolio Π according to a self-financing trading strategy ϕ, such that at time T it is guaranteed that

$$\Pi(T) = V(T)$$

Now assume that such a strategy exists, i.e. assume that derivative V can indeed be hedged. In that case we call the security an *attainable contingent claim*.

Definition 11. ***Attainable contingent claim:*** *A contingent claim is a square-integrable, non-negative random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A contingent claim V is called attainable if there exists a self-financing trading strategy ϕ, such that the associated portfolio at time T yields $\Pi(T) = V$ almost surely (i.e. with probability 1).*

Since each ϕ_i of the trading strategy ϕ is predictable and $D(0,t)X_i(t)$ is a Martingale under $\mathbb{Q}$ by definition, it follows that

$$D(0,t)\Pi(t) = \sum_{i=0}^{m} \phi_i(t) D(0,t) X_i(t)$$

is a Martingale under $\mathbb{Q}$. The amount of cash required to set up the hedge for security V at time t is represented by $\Pi(t)$. Therefore, the fair value $V(t)$ of the derivative at time t should be equal to $\Pi(t)$ as otherwise an arbitrage opportunity would occur. Hence, using the Martingale property, the following relation must hold

$$D(0,t)V\left(t\right) = D(0,t)\Pi\left(t\right) = \mathbb{E}^{Q}\left(D(0,T)\Pi(T)|\,\mathcal{F}_t\right) = \mathbb{E}^{Q}\left(D(0,T)V(T)|\,\mathcal{F}_t\right)$$

Rewriting the equality above yields

$$V\left(t\right) = \frac{1}{D(0,t)}\mathbb{E}^{Q}\left(D(0,T)V(T)|\,\mathcal{F}_t\right) = \mathbb{E}^{Q}\left(D(t,T)V(T)|\,\mathcal{F}_t\right)$$

We refer to this expression as the *risk-neutral pricing formula.*

Proposition 12. ***The risk-neutral pricing formula:*** *Let $\mathbb{Q}$ denote a risk-neutral probability measure on the measurable space $(\Omega, \mathcal{F})$ and $V(T)$ an attainable contingent claim. Then for any time $t \leq T$, the fair value or* risk-neutral price *$V(t)$ associated with V is given by*

$$V(t) = \mathbb{E}^{Q}\left(D(t,T)V(T)|\,\mathcal{F}_t\right)$$

A market model is called *complete* if for each derivative security, a self-financing strategy ϕ can be determined to hedge it.
Finally we can state the two fundamental theorems of asset pricing. Formal statement and proofs can for example be found in [6]. The first one concerns the absence of arbitrage:

Proposition 13. *If a market model admits a risk-neutral measure $\mathbb{Q}$, then there doesn't exist a strategy ϕ that imposes an opportunity of arbitrage.*

This results indicates that it is sufficient to mathematically show the existence of a risk-neutral measure. Once this is done, we immediately satisfy an important condition for a realistic model, namely that it doesn't admit arbitrage. The second theorem concerns risk-neutral pricing:

Proposition 14. *Assume that a market model admits at least one risk-neutral measure. Then this measure is unique if and only if the the model is complete.*

From this theorem we can conclude that the fair value, which we compute through the risk-neutral pricing formula, is in fact unique. This is an important result for pricing derivative securities. Given such an instrument there exists exactly one price for which an agent is able to hedge it.

2.3 Changing the numéraire

We have seen that an asset denominated by the bank-account ${}^{X(t)}/_{B(t)}$ is a Martingale under the risk-neutral measure. This property can be generalized if we introduce the notion of a *numéraire.*

Definition 15. ***numéraire:*** *A continuously tradable, positively priced asset that is free of transaction costs and dividend payments.*

So far we have used the bank-account as numéraire and used the Martingale property to construct the risk-neutral pricing formula. However, pricing a derivative or portfolio can sometimes be more convenient under a different numéraire. It appears that this is possible by application of a measure change. A portfolio process denominated by any numéraire N defined in the market model is still a Martingale under the risk-neutral measure $\mathbb{Q}^N$ associated to that numéraire.

To see why such a measure indeed always exists, we follow the arguments of [4]. First of all let N be any numéraire. Secondly, consider the random variable Z, which is given by

$$Z = \frac{N(T)}{N(0) \cdot B(T)}$$

By their definition we have that $N(T)$, $N(0)$ and $B(T)$ are non-negative random variables and $\mathbb{E}^Q \left(\frac{N(T)}{N(0) \cdot B(T)} \right) = \frac{N(0)}{N(0) \cdot B(0)} = 1$. For any $A \in \mathcal{F}$ it therefore follows that the measure

$$\mathbb{Q}^N (\omega) \equiv \int_A Z(\omega) d\mathbb{Q}(\omega)$$

defines a probability measure on the measurable space $(\Omega, \mathcal{F})$. Furthermore, it follows directly from its definition that $\mathbb{Q}^N$ is equivalent to $\mathbb{Q}$, from which we can conclude that Z is in fact the Radon-Nikodym derivative $\frac{d\mathbb{Q}^N}{d\mathbb{Q}}$.

Now, we only need to show that the price of any security X_i denominated by the numéraire N is in fact a Martingale under $\mathbb{Q}^N$. To do so, consider the Radon-Nikodym derivative process given by $Z(T) := Z$ and for $0 \leq t < T$

$$Z(t) = \mathbb{E}^Q \left(\left. \frac{d\mathbb{Q}^N}{d\mathbb{Q}} \right| \mathcal{F}_t \right)$$

Then according to [6], lemma 5.2.2, we have for any $\mathcal{F}$-measurable random variable Y and $0 \leq s \leq t \leq T$ the following relation:

$$\mathbb{E}^{Q^N} \left(Y | \mathcal{F}_s \right) = \frac{1}{Z(s)} \mathbb{E}^Q \left(Y \cdot Z(t) | \mathcal{F}_s \right) \tag{2.1}$$

Since both X_i and N are adapted to $\mathcal{F}_t$, we know that $\frac{X_i(t)}{N(t)}$ is $\mathcal{F}_t$-measurable. Furthermore, by the definition of the risk-neutral measure, the processes $\frac{N(t)}{B(t)}$ and $\frac{X_i(t)}{B(t)}$ are Martingales under $\mathbb{Q}$. Therefore we have

$$Z(s) = \mathbb{E}^Q \left(\left. \frac{d\mathbb{Q}^N}{d\mathbb{Q}} \right| \mathcal{F}_s \right) = \frac{1}{N(0)} \mathbb{E}^Q \left(\left. \frac{N(T)}{B(T)} \right| \mathcal{F}_s \right) = \frac{N(s)}{N(0) \cdot B(s)}$$

and

$$\mathbb{E}^Q \left(\left. \frac{X_i(t)}{N(t)} \cdot Z(t) \right| \mathcal{F}_s \right) = \mathbb{E}^Q \left(\left. \frac{X_i(t)}{N(t)} \cdot \frac{N(t)}{N(0) \cdot B(t)} \right| \mathcal{F}_s \right) = \frac{X_i(s)}{N(0) \cdot B(s)}$$

Using these two results and setting $Y = \frac{X_i(t)}{N(t)}$ in equation (2.1), we find that

$$\mathbb{E}^{Q^N} \left(\left. \frac{X_i(t)}{N(t)} \right| \mathcal{F}_s \right) = \frac{1}{\frac{N(s)}{N(0) \cdot B(s)}} \cdot \frac{X_i(s)}{N(0) \cdot B(s)} = \frac{X_i(s)}{N(s)}$$

from which we conclude that X_i discounted by N is indeed a Martingale under $\mathbb{Q}^N$. We summarize this result in the proposition below, as also formulated in [4].

Proposition 16. *Let $\mathbb{Q}$ be the risk-neutral measure, such that $\frac{X_i(t)}{B(t)}$ is a Martingale for all $i \in \{1, \ldots, m\}$ under $\mathbb{Q}$ and let N be any numéraire. Then there exists a probability measure $\mathbb{Q}^N$ equivalent to $\mathbb{Q}$ such that $\frac{X_i(t)}{N(t)}$ is a Martingale under $\mathbb{Q}^N$. The Radon-Nikodym derivative is given by $\frac{d\mathbb{Q}^N}{d\mathbb{Q}} = \frac{N(T)}{N(0) \cdot B(T)}$.*

A direct consequence of the proposition is that it expands our toolbox in pricing derivative securities. This follows from the Martingale property of a portfolio process that is denominated by the numéraire. Let N be a numéraire, $\mathbb{Q}^N$ the associated measure and $V(t)$ the value process of an attainable contingent claim maturing at T. Then we have the following relations

$$\begin{aligned} \frac{V(t)}{B(t)} &= \mathbb{E}^Q\left(\left.\frac{V(T)}{B(T)}\right| \mathcal{F}_t\right) \\ \frac{V(t)}{N(t)} &= \mathbb{E}^N\left(\left.\frac{V(T)}{N(T)}\right| \mathcal{F}_t\right) \end{aligned}$$

Hence, it follows that the fair price of a derivative is invariant to the underlying numéraire N and can equivalently be computed according to the formula

$$V(t) = N(t) \cdot \mathbb{E}^N\left(\left.\frac{V(T)}{N(T)}\right| \mathcal{F}_t\right)$$

3 Interest rate definitions and terminology

Here we will introduce standard terminology related to interest rate modeling and define some basic interest rate derivatives. Additionally we will briefly discuss the risk-neutral price of some of these instruments. We do so by applying the risk-neutral pricing formula. Most of the terminology that is introduced in this section follows the formulation presented in [2].

Definition 17. ***Zero-coupon bond:*** *A contract that guarantees the buyer 1 unit of currency at maturity T.*

We denote the value of a zero-coupon bond at time $t < T$ by $P(t,T)$. By its definition, we have $P(T,T) = 1$. For $t < T$ we have that

$$P(t,T) = \mathbb{E}^Q \left(D(t,T)P(T,T) \middle| \mathcal{F}_t \right) = \mathbb{E}^Q \left(e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t \right)$$

The zero-coupon bond maturing at T is a common alternative to the bank-account as a choice in numéraire when pricing instruments. This is due to the convenient characteristic that $P(T,T) = 1$. In the risk-neutral pricing formula this implies

$$\begin{aligned} V(t) &:= B(t) \cdot \mathbb{E}^Q \left(\frac{V(T)}{B(T)} \middle| \mathcal{F}_t \right) \\ &= P(t,T) \cdot \mathbb{E}^T \left(\frac{V(T)}{P(T,T)} \middle| \mathcal{F}_t \right) \\ &= P(t,T) \cdot \mathbb{E}^T \left(V(T) \middle| \mathcal{F}_t \right) \end{aligned}$$

We refer to the measure associated with a zero-coupon bond numéraire as the T-forward measure.

Definition 18. ***T-forward measure:*** *The probability measure $\mathbb{Q}^T$ under which any asset-price process denominated by the zero-coupon bond price process $t \mapsto P(t,T)$ becomes a Martingale.*

From the definition of a zero-coupon bond we can easily move to interest rates. We distinguish two types of interest rates: the continuously compounded and the simply compounded. In practice the latter is most common.

Definition 19. ***Continuously compounded interest rate:*** *The constant rate $R(t,T)$ prevailing at time t at which an investment of $P(t,T)$ units of currency is required to grow if one is to obtain 1 unit of currency at time T, given that the investment accrues continuously.*

Let Δt denote the year-fraction between time t and T. Then by its definition, the continuously-compounded rate $R(t,T)$ can be written as

$$e^{R(t,T)\Delta t} P(t,T) = 1$$

Or equivalently

$$R(t,T) = -\frac{\log \left(P(t,T) \right)}{\Delta t}$$

Definition 20. ***Simply compounded interest rate:*** *The constant rate $L(t,T)$ prevailing at time t at which an investment of $P(t,T)$ units of currency is required to grow if one is to obtain 1 unit of currency at time T, given that the accruing is proportional to the investment time.*

A well known example of a simply-compounded rate is the LIBOR (the London InterBank Offered Rate), hence the notation L. For multiple maturity dates T, the LIBOR is quoted in the market and updated on a daily basis. The LIBOR is inferred from a collection of interest rates, which are quoted by a panel of prominent banks. Let Δt denote the year-fraction between time t and T. Then by its definition, the simply-compounded rate $L(t,T)$ can be written as

$$(1 + L(t,T)) \cdot \Delta t \cdot P(t,T) = 1$$

Or equivalently

$$L(t,T) = \frac{1 - P(t,T)}{\Delta t \cdot P(t,T)}$$

LIBOR rates over future time intervals, say $[S,T]$ for $S > t$, are not known today. Consider an investor that would like to fix the LIBOR today for a future time instant. Hypothetically, this could be done through a contract, called a prototypical *forward rate agreement* (FRA). This is a contract in which a fixed interest rate K is settled today (time t) over some notional amount N. Subsequently, at maturity T (when the LIBOR $L(S,T)$ is already known) the two parties exchange $N \cdot \Delta t \cdot L(S,T)$ and $N \cdot \Delta t \cdot K$. The rate K that at time t sets the risk-neutral price of a FRA to zero is called the *forward rate*. In reality, only a generalization of the FRA is traded in the market, namely the interest rate swap (which will be treated later on).

Definition 21. ***Simply compounded forward rate:*** *The fixed rate $F(t,S,T)$ for which a prototypical FRA expiring at time S and maturing at time T has risk-neutral price 0 at inception date t.*

The forward rate can be considered the current expectation of the future realization of the LIBOR. We can derive a formula for the forward rate in terms of zero-coupon bond prices, by application of a measure change. According to the definition, $F(t,S,T)$ should satisfy the relation

$$\begin{aligned} \mathbb{E}^Q\left(D(t,T)\left(L(S,T) - F(t,S,T)\right) \middle| \mathcal{F}_t\right) &= 0 \\ \Rightarrow \qquad \frac{\mathbb{E}^Q\left(D(t,T) L(S,T) \middle| \mathcal{F}_t\right)}{\mathbb{E}^Q\left(D(t,T) \middle| \mathcal{F}_t\right)} &= F(t,S,T) \end{aligned}$$

The denominator is by definition equal to $P(t,T)$. The numerator can be computed by changing to the T-forward measure.

$$\begin{aligned} \mathbb{E}^Q\left(D(t,T) L(S,T) \middle| \mathcal{F}_t\right) &= P(t,T) \cdot \mathbb{E}^T\left(L(S,T) \middle| \mathcal{F}_t\right) \\ &= P(t,T) \cdot \mathbb{E}^T\left(\frac{P(S,S) - P(S,T)}{\Delta t \cdot P(S,T)} \middle| \mathcal{F}_t\right) \\ &= P(t,T) \cdot \frac{P(t,S) - P(t,T)}{\Delta t \cdot P(t,T)} \end{aligned}$$

The last equality follows from the fact that $P(t,S)$ and $P(t,T)$ are both tradable assets, implying that denominated by the numéraire $P(S,T)$, they are Martingales under $\mathbb{Q}^T$. Therefore we find that in terms of zero-coupon bonds the value of the forward rate can be written as

$$F(t,S,T) = \frac{P(t,T) \cdot \frac{P(t,S) - P(t,T)}{\Delta t \cdot P(t,T)}}{P(t,T)} = \frac{1}{\Delta t}\left(\frac{P(t,S)}{P(t,T)} - 1 \right)$$

The expression above is standard when working with the forward rate. Note that by the definition of the T-forward measure, the forward rate itself

$$F(t,S,T) = \frac{1}{\Delta t}\left(\frac{P(t,S)}{P(t,T)} - 1\right) = \frac{P(t,S) - P(t,T)}{\Delta t \cdot P(t,T)}$$

must be a Martingale under $\mathbb{Q}^T$. If we choose $t = S$, then it follows by the fact that $P(S,S) = 1$ that we can write

$$F(t,S,T) = \mathbb{E}^T\left(\left.\frac{P(S,S) - P(S,T)}{\Delta t \cdot P(S,T)}\right| \mathcal{F}_t\right) = \mathbb{E}^T\left(L(S,T)\middle|\, \mathcal{F}_t\right)$$

And thus we conclude that the expected value of the LIBOR under the T-forward measure in fact yields the forward rate.
Considering the forward rate for infinitesimal accrual periods $[T, T + dt]$, one arrives at the *instantaneous forward rate*.

Definition 22. ***Instantaneous forward rate:*** $f(t,T) = \lim_{S\to T} F(t,S,T)$

Computation of the limit yields.

$$\begin{aligned} f(t,T) &= \lim_{S\to T}\frac{1}{T-S}\left(\frac{P(t,S)}{P(t,T)} - 1\right) \\ &= \frac{1}{P(t,T)}\lim_{S\to T}\left(\frac{P(t,S) - P(t,T)}{T-S}\right) \\ &= -\frac{\partial}{\partial T}\log\left(P(t,T)\right) \end{aligned}$$

Finally, if we let the maturity of the instantaneous forward rate approach today, we obtain the instantaneous spot rate.

Definition 23. ***Instantaneous spot rate:*** $r(t) = \lim_{T\to t} f(t,T)$

This quantity is often called the *short-rate* and corresponds to the drift of the money-market account.

3.1 Interest rate derivatives

Here we introduce the two popular interest rate derivatives, namely the interest rate swap and the swaption. We treat the set-up of the instruments and illustrate how their risk-neutral price is evaluated. The definitions we present here are largely based on [2].

3.1.1 The interest rate swap

Recall that a FRA is a contract that allows the buyer to fix a future interest rate. A generalization of such an instrument is the interest rate swap (IRS). It is one of the most commonly traded OTC derivatives and there exist multiple variations to it. We discuss here the floating-fix version of the IRS.
An IRS is a contract that settles a sequence of cash exchanges between two parties. The instrument specifies several properties.

- A set $\mathcal{T}$ of future dates $t < T_0 < T_1 < \ldots < T_m$. The time-instants are usually equidistant, meaning that $\Delta t_i = T_i - T_{i-1}$ is equal for each $i \in \{1, \ldots, m\}$. Typical accrual periods Δt are 1 month, 3 months, 6 months or 1 year.

- A notional amount N.
- A fixed rate K.

One sequence of payments is referred to as the fixed leg as it is associated with the fixed rate K. At each date $T_i \in \{T_1, \dots, T_m\}$, this leg pays out an amount

$$N \cdot K \cdot \Delta t_i$$

In return the floating leg, associated to the floating LIBOR, pays out

$$N \cdot L\left(T_{i-1}, T_i\right) \cdot \Delta t_i$$

We assume that the fixed and floating cashflows are each time exchanged at the same date. Note that the interest rate $L\left(T_{i-1}, T_i\right)$ is fixed at date T_{i-1}, which corresponds to the end-date of the previous coupon, but is only paid out at T_i, the end date of the current coupon.

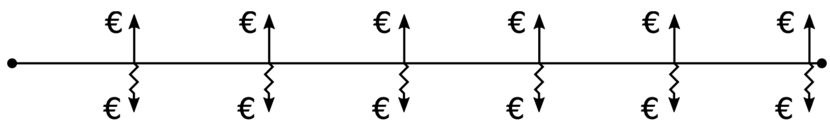

Figure 3.1: Graphic illustration of the cashflows of a floating-fix IRS.

Since the IRS is a derivative security with multiple pay-offs, we will treat it as a sequence of single payments trades. This allows us to calculate its fair price by the risk-neutral pricing formula. Denote the value of the swap at time $t < T_0$ by $V(t)$, then we have

$$\begin{aligned} V(t) &= \mathbb{E}^Q\left(\sum_{i=1}^m D\left(t, T_i\right) N \Delta t_i\left(L\left(T_{i-1}, T_i\right) - K\right) \middle| \mathcal{F}_t\right) \\ &= N \sum_{i=1}^m \mathbb{E}^Q\left(D\left(t, T_i\right) \Delta t_i\left(\frac{1 - P\left(T_{i-1}, T_i\right)}{\Delta t_i \cdot P\left(T_{i-1}, T_i\right)} - K\right) \middle| \mathcal{F}_t\right) \end{aligned}$$

Switch to the T_i-forward measure, recall that $P\left(T_{i-1}, T_{i-1}\right) = 1$ and use the Martingale property of assets denominated by $P\left(T_{i-1}, T_i\right)$ under $\mathbb{Q}^{T_i}$ to see

$$\begin{aligned} V(t) &= N \sum_{i=1}^m P\left(t, T_i\right) \mathbb{E}^{Q^{T_i}}\left(\Delta t_i\left(\frac{P\left(T_{i-1}, T_{i-1}\right) - P\left(T_{i-1}, T_i\right)}{\Delta t_i \cdot P\left(T_{i-1}, T_i\right)} - K\right) \middle| \mathcal{F}_t\right) \\ &= N \sum_{i=1}^m P\left(t, T_i\right) \Delta t_i\left(\frac{P\left(t, T_{i-1}\right) - P\left(t, T_i\right)}{\Delta t_i \cdot P\left(t, T_i\right)} - K\right) \\ &= N \sum_{i=1}^m \left(P\left(t, T_{i-1}\right) - P\left(t, T_i\right) - P\left(t, T_i\right) \Delta t_i K\right) \\ &= N\left(P\left(t, T_0\right) - P\left(t, T_m\right) - K \sum_{i=1}^m P\left(t, T_i\right) \Delta t_i\right) \end{aligned}$$

It is common practice to enter an IRS deal *at par*. This means that at inception of the trade, the fixed rate is chosen such that the contract has zero value at that time. This particular rate

K is called the *swap rate*. Hence, by its definition, at any time t the swap rate $S_{0,m}(t)$ is defined by the following equation:

$$\begin{aligned} N\left(P\left(t,T_0\right)-P\left(t,T_m\right)-S_{0,m}(t)\sum_{i=1}^{m}P\left(t,T_i\right)\Delta t_i\right) &= 0 \\ \Rightarrow \qquad \frac{P\left(t,T_0\right)-P\left(t,T_m\right)}{\sum_{i=1}^{m}P\left(t,T_i\right)\Delta t_i} &= S_{0,m}(t) \end{aligned}$$

3.1.2 The swaption

Second we consider an interest rate derivative called the swaption. A swaption is a contract that gives the holder the right to enter an IRS at a future time instant for a specific fixed rate K (physical settlement) or to receive the cash value of the IRS (cash settlement). A swaption thus defines a European option written on a swap. Consider for the underlying IRS the following properties

- A set $\mathcal{T}$ of future dates $t < T_0 < T_1 < \ldots < T_m$, with equidistant accrual periods, denoted by $\Delta t_i = T_i - T_{i-1}$ for each $i \in \{1, \ldots, m\}$.
- A notional amount N.
- A fixed rate K.

Time T_0 then denotes the expiry date of the swaption, meaning that the holder will exercise the option if it has positive value at that time. We know that the value of an IRS at time T_0 is given by

$$\begin{aligned} V\left(T_0\right) &= N\left(P\left(T_0,T_0\right)-P\left(T_0,T_m\right)-K\sum_{i=1}^{m}P\left(T_0,T_i\right)\Delta t_i\right) \\ &= N\left(\sum_{i=1}^{m}P\left(T_0,T_i\right)\Delta t_i\right)\cdot\left(\frac{P\left(T_0,T_0\right)-P\left(T_0,T_m\right)}{\sum_{i=1}^{m}P\left(T_0,T_i\right)\Delta t_i}-K\right) \end{aligned}$$

Recall that the swap rate at time T_0 was defined as

$$S_{0,m}\left(T_0\right)=\frac{P\left(T_0,T_0\right)-P\left(T_0,T_m\right)}{\sum_{i=1}^{m}P\left(T_0,T_i\right)\Delta t_i}$$

To obtain a more convenient expression for the value of a swaption, we introduce the notion of an *annuity*. We will denote an annuity corresponding to the time schedule $\mathcal{T}$ as $A_{0,m}\left(t\right)$. It is defined as

$$A_{0,m}(t)=\sum_{i=1}^{m}P\left(t,T_i\right)\Delta t_i$$

Note that an annuity is nothing more that the weighted sum of a finite set of zero-coupon bonds. As zero-coupon bonds are freely tradable, positively priced assets, so are annuities. Therefore, $A_{0,m}(t)$ is in fact a well-defined numéraire. Using this notation, the value of a swaption at expiry can be written as

$$V\left(T_0\right)=N\cdot A_{0,m}\left(T_0\right)\cdot\left(S_{0,m}\left(T_0\right)-K\right)^{+}$$

Now we are interested in the fair price of a swaption at some time $t < T_0$. Naturally this is given by the risk-neutral pricing formula, which yields

$$\begin{aligned} V(t) &= \mathbb{E}^{Q}\left(D(t, T_0) V(T_0) \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}^{Q}\left(D(t, T_0) \cdot N \cdot A_{0,m}(T_0) \cdot (S_{0,m}(T_0) - K)^{+} \middle| \mathcal{F}_t \right) \end{aligned}$$

Since the annuity $A_{0,m}(t)$ is a numéraire, we know that there exist a measure $\mathbb{Q}^{0,m}$ such that any asset denominated by $A_{0,m}(t)$ becomes a Martingale under this measure.

Definition 24. ***Annuity measure:*** *The probability measure $\mathbb{Q}^{0,m}$ under which any asset-price process denominated by the annuity process $t \mapsto A_{0,m}(t) := \sum_{i=1}^{m} P(t, T_i)\Delta t_i$ becomes a Martingale.*

By changing to the annuity-measure, we see that we can therefore rewrite

$$\begin{aligned} V(t) &= B(t)\,\mathbb{E}^{0,m}\left(\frac{N \cdot A_{0,m}(T_0) \cdot (S_{0,m}(T_0) - K)^{+}}{B(T_0)} \middle| \mathcal{F}_t \right) \\ &= A_{0,m}(t)\,\mathbb{E}^{0,m}\left(\frac{N \cdot A_{0,m}(T_0) \cdot (S_{0,m}(T_0) - K)^{+}}{A_{0,m}(T_0)} \middle| \mathcal{F}_t \right) \\ &= N \cdot A_{0,m}(t)\,\mathbb{E}^{0,m}\left((S_{0,m}(T_0) - K)^{+} \middle| \mathcal{F}_t \right) \end{aligned}$$

Note that this expression is now only dependent on the stochastic behavior of the swap rate. We can therefore evaluate the swaption price by assuming appropriate dynamics on the swap rate process. The assumption that $S_{0,m}$ follows a Gaussian process would imply that we can price a swaption using Bachelier's formula. Likewise, the assumption that $S_{0,m}$ follows a geometric Brownian motion would imply that we can use Black's formula.

4 Modelling the short-rate

Interest rate models are widely used to price interest rate related derivatives and financial risk-management in general. In the past decades many different models have been treated in the academic literature. Here we will treat the one-factor model proposed by John Hull and Alan White in 1990 [5]. The model is an extension of the 1977 Vasicek model and allows for a perfect fit of today's term-structure of interest rates. Due to the Gaussian character of the Hull-White state variables we can derive explicit pricing formulas for a large range of interest rate derivatives.

4.1 The one-factor Hull-White model

A common approach to interest rate modeling is the simulation of a mathematical variable $r(t)$, which we call the short-rate. This one-dimensional instantaneous spot rate is in reality is not observed in the market. This variable corresponds to the drift-term of the money-market account. In order to price and model interest rate derivatives, the consideration of a suitable interest rate model is key. We will introduce the one-factor Hull-White short-rate model, which can be categorized as an affine term-structure model. The main characteristic of such a model is that the continuously compounded spot interest rate $R(t,T)$ is an affine function (linear term plus constant) of the short-rate $r(t)$ [2]:

$$R(t,T) = \alpha(t,T) + \beta(t,T)r(t)$$

An important implication of this definition is that a zero-coupon bond price can be written in the following form:

$$P\left(t,T\right) = A\left(t,T\right)e^{-B(t,T)r(t)}$$

Recall that the continuously compounded interest rate is defined through the relation

$$R\left(t,T\right) = -\frac{\log\left(P\left(t,T\right)\right)}{\Delta t}$$

We hence arrive at the above relation for P if we set

$$A\left(t,T\right) = e^{-\alpha(t,T)\Delta t}, \quad B\left(t,T\right) = \beta\left(t,T\right)\Delta t$$

For the pricing procedure of interest rate derivatives, this is a convenient property. We have seen that the model-independent definition of the zero-coupon bond price is given by

$$P(t,T) = \mathbb{E}^Q\left(e^{-\int_t^T r(s)ds}\middle|\,\mathcal{F}_t\right)$$

Depending on which model you use, the above expression can be difficult to compute. Since the Hull-White model belongs to the class of affine-term structure models, we will see that a convenient closed-form formula can be derived for $P\left(t,T\right)$. Zero-bonds are often considered the fundamental quantities in interest rate modeling. The risk-neutral value of many interest rate derivatives can be expressed in terms of $P\left(t,T\right)$. This implies that the model offers efficient pricing routines for these instrument, which is particularly important for calibration purposes. Additionally, its implementation is easy and efficient in comparison to other interest rate models. All together, the Hull-White model is still a popular tool in risk-management for financial institutions.

4.2 The dynamics of the short-rate

Hull and White have examined several variations of the short-rate process. Here we will treat the extension of the Vasicek model that considers the following short-rate dynamics under the risk-neutral measure $\mathbb{Q}$:

$$dr(t) = (\theta(t) - a \cdot r(t))\, dt + \sigma dW(t)$$

Here a, the mean-reversion rate, and σ, the volatility are deterministic scalars. W is a standard one-dimensional Brownian motion under $\mathbb{Q}$. $\theta(t)$ is a deterministic function of time, which is calibrated such that the corresponding yield curve matches the currently observed term-structure of interest rates in the market. The dynamics of $r(t)$ follow an Ornstein-Uhlenbeck process. This yields a mean-reverting character, which is a desired property as this behavior is also observed in reality. As the short-rate is driven by a single one-dimensional Brownian motion, this model is referred to as the *one-factor* Hull-White model.

Given its dynamics, an expression for $r(t)$ can be derived by an application of Itô's lemma. Let $f(t, r) = r \cdot e^{a \cdot t}$, it then follows that

$$\begin{aligned} df\,(t, r(t)) &= ar(t)e^{at}dt + e^{at}dr(t) \\ &= ar(t)e^{at}dt + e^{at}\left[(\theta(t) - a \cdot r(t))\, dt + \sigma dW(t)\right] \\ &= e^{at}\theta(t)dt + e^{at}\sigma dW(t) \end{aligned}$$

Hence we see that for any $0 \leq s \leq t$, we have

$$r(t) \cdot e^{at} = r(s) \cdot e^{as} + \int_s^t e^{au}\theta(u)du + \int_s^t e^{au}\sigma dW(u)$$

which can equivalently be rewritten as

$$r(t) = r(s)e^{a(s-t)} + \int_s^t e^{a(u-t)}\theta(u)du + \int_s^t e^{a(u-t)}\sigma dW(u)$$

What we know from the properties of an Itô integral with deterministic integrand that $\int_s^t e^{a(u-t)}\sigma dW(u)$ is Gaussian with mean zero. Its variance is computed by the application of Itô isometry. Due to the deterministic nature of θ and $r(s)$ conditioned on $\mathcal{F}_s$, we find that $r(t)$ is normally distributed with moments:

$$\begin{aligned} \mathbb{E}^Q\left(r(t)|\,\mathcal{F}_s\right) &= r(s)e^{a(s-t)} + \int_s^t e^{a(u-t)}\theta(u)du \\ \mathrm{Var}\left(r(t)|\,\mathcal{F}_s\right) &= \int_s^t \left(e^{a(u-t)}\sigma\right)^2 du = \frac{\sigma^2}{2a}\left(1 - e^{2a(s-t)}\right) \end{aligned}$$

4.3 Pricing a zero-coupon bond

The Gaussian nature of the short-rate has as a convenient consequence that not only $r(t)$ itself, but also $\int_s^t r(u)du$ conditioned on $\mathcal{F}_s$ is normally distributed. Considering the integrated short-rate we find that

$$\begin{aligned} \int_s^t r(u)du &= \int_s^t r(s)e^{a(s-u)}du + \int_s^t \int_s^u e^{a(v-u)}\theta(v)dvdu \\ &\quad + \int_s^t \int_s^u e^{a(v-u)}\sigma dW(v)du \end{aligned}$$

By the positive integrability of $e^{a(v-u)}\sigma$ and $\theta(v)$ on $[0,T]$, we can apply Fubini's theorem to change the order of integration of the last two terms (see [1]), yielding

$$\begin{aligned}\int_s^t r(u)du &= \int_s^t r(s)e^{a(s-u)}du + \int_s^t \int_v^t e^{a(v-u)}\theta(v)dudv \\ &\quad + \int_s^t \int_v^t e^{a(v-u)}\sigma dudW(v) \\ &= \frac{r(s)}{a}\left(1-e^{a(s-t)}\right) + \int_s^t \frac{1}{a}\left(1-e^{a(v-t)}\right)\theta(v)dv \\ &\quad + \int_s^t \frac{1}{a}\left(1-e^{a(v-t)}\right)\sigma dW(v)\end{aligned}$$

As a result we find that $\int_s^t r(u)du$ is indeed Gaussian. To shorten the notation, define $B(S,T) = \frac{1}{a}\left(1-e^{a(S-T)}\right)$. The moments of the integrated short-rate can be obtained by application of Itô isometry, so that

$$\begin{aligned}\mathbb{E}^Q\left(\int_s^t r(u)du\middle|\mathcal{F}_s\right) &= r(s)B(s,t) + \int_s^t B(v,t)\theta(v)dv \\ \mathrm{Var}\left(\int_s^t r(u)du\middle|\mathcal{F}_s\right) &= \int_s^t \left(\frac{1}{a}\left(1-e^{a(v-t)}\right)\sigma\right)^2 dv \\ &= \frac{\sigma^2}{a^2}\left(t-s+\frac{2}{a}e^{a(s-t)} - \frac{1}{2a}e^{2a(s-t)} - \frac{3}{2a}\right)\end{aligned}$$

Now we have come to a point where we can write an expression for a zero-coupon bond price. Recall that its fair value under the risk-neutral measure is defined by $\mathbb{E}^Q\left(e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right)$. We now know that $\int_t^T r(s)ds$ is normally distributed. As a consequence, $e^{-\int_t^T r(s)ds}$ is a lognormal random variable. The moments of the lognormal distribution are known. Let X be a Gaussian random random variable with parameters μ and σ^2. Then the mean of $Y = \exp\{X\}$ is given by $\mathbb{E}(Y) = \exp\{\mu + \sigma^2/2\}$. Using this given property lets us compute

$$\begin{aligned}P(s,t) &= \mathbb{E}^Q\left(e^{-\int_s^t r(u)du}\middle|\mathcal{F}_s\right) \\ &= \exp\left\{-\mathbb{E}^Q\left(\int_s^t r(u)du\middle|\mathcal{F}_s\right) - \frac{1}{2}\mathrm{Var}\left(\int_s^t r(u)du\middle|\mathcal{F}_s\right)\right\} \\ &= A(s,t)\,e^{-B(s,t)r(s)}\end{aligned}$$

Where A and B are both deterministic functions of time, defined as

$$\begin{aligned}A(s,t) &= \exp\left\{-\int_s^t B(u,t)\theta(u)du \right.\\ &\quad \left. -\frac{\sigma^2}{2a^2}\left(t-s+\frac{2}{a}e^{a(s-t)} - \frac{1}{2a}e^{2a(s-t)} - \frac{3}{2a}\right)\right\} \\ B(s,t) &= \frac{1}{a}\left(1-e^{a(s-t)}\right)\end{aligned}$$

As a result we have derived a tractable expression for a zero-coupon bond price, being a deterministic function of time and $r(s)$. What remains to be done, is to compute $\theta(t)$. We do so by a fitting procedure, which we will treat in the following paragraph.

4.4 Fitting to the current market

A strong improvement of the Hull-White model compared to the Vasicek model, is that the time-dependence of $\theta(t)$ allows for a perfect fit of the model to the currently observed term-structure of zero-coupon bonds. Our objective is therefore to find an expression for θ, such that that our modeled zero-coupon prices $P(0,T)$ match the bond prices observed in the market $P^M(0,T)$. In other words: $\forall_{T>0}\ P^M(0,T) = P(0,T)$. From a computational stand-point, the calibration procedure it is easier if we fit θ to the term-structure of instantaneous forward rates $f^M(0,T)$. Recall that an expression for f is given by

$$f(t,T) = -\frac{\partial \log(P(t,T))}{\partial T}$$

With the model consistent expression for a zero-coupon bond from the previous section, we can substitute P in the relation above. We write $f^M(0,T)$ for the instantaneous forward that would correspond to the current market. Substitution of P yields

$$\begin{aligned}
f^M(0,T) &= \frac{-\partial\left(\log(A(0,T)) - B(0,T)\, r(0)\right)}{\partial T} \\
&= \frac{\partial}{\partial T}\left(\frac{1}{a}\int_0^T \left(1-e^{a(u-T)}\right)\theta(u)du\right) \\
&\quad +\frac{\partial}{\partial T}\frac{\sigma^2}{2a^2}\left(T + \frac{2}{a}e^{-aT} - \frac{1}{2a}e^{-2aT} - \frac{3}{2a}\right) \\
&\quad -\frac{\partial}{\partial T}\frac{1}{a}\left(1-e^{-aT}\right)r(0) \\
&= e^{-aT}\int_0^T e^{au}\theta(u)du - \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 + e^{-aT}r(0)
\end{aligned}$$

In order to isolate $\theta(T)$ from the expression, we differentiate f^M a second time, so that we obtain

$$\begin{aligned}
\frac{\partial f^M(0,T)}{\partial T} &= -ae^{-aT}\int_0^T \theta(u)e^{au}du + \theta(T) \\
&\quad -\frac{\sigma^2}{a}e^{-aT} + \frac{\sigma^2}{a}e^{-2aT} - ae^{-aT}r(0)
\end{aligned}$$

Note that we have the relation $-ae^{-aT}\int_0^T \theta(u)e^{au}du - ae^{-aT}r(0) = -a\cdot f^M(0,T) - \frac{\sigma^2}{2a}\left(1-e^{-aT}\right)^2$. We substitute the relation into the expression above and do some rewriting. By doing so we end up with a compact expression for θ.

$$\begin{aligned}
\frac{\partial f^M(0,T)}{\partial T} &= -a\cdot f^M(0,T) - \frac{\sigma^2}{2a}\left(1-e^{-aT}\right)^2 + \theta(T) - \frac{\sigma^2}{a}e^{-aT} + \frac{\sigma^2}{a}e^{-2aT} \\
\Rightarrow \quad \theta(T) &= \frac{\partial f^M(0,T)}{\partial T} + a\cdot f^M(0,T) + \frac{\sigma^2}{2a}\left(1-e^{-2aT}\right)
\end{aligned}$$

If we substitute this result into our earlier derived formula, we arrive at a closed-form expression for the zero-coupon bond price.

Proposition 25. ***Zero-coupon bond price:*** *Under the Hull-White short-rate model, the zero-*

coupon bond price $P(s,t)$ admits a closed-form expression. It is given by

$$
\begin{aligned}
P(s,t) &= A(s,t)e^{-B(s,t)r(s)} \\
A(s,t) &= \frac{P^M(0,t)}{P^M(0,s)} \exp\left\{B(s,t)f^M(0,s) - \frac{\sigma^2}{4a}B^2(s,t)\left(1-e^{-2as}\right)\right\} \\
B(s,t) &= \frac{1}{a}\left(1-e^{a(s-t)}\right)
\end{aligned}
$$

4.5 The shifted short-rate process

Note that our formula for pricing a zero-coupon bond requires the instantaneous forward rate f^M as an input (next to $P^M(0,s)$ and $P^M(0,t)$). Although this parameter is mathematically well-defined, it can in reality not be observed in the market. Therefore it would be convenient to remove this term from the expression. It appears we can do so if we consider a related zero-mean process $x(t)$, of which the dynamics are defined as

$$dx(t) = -a \cdot x(t)dt + \sigma dW(t), \quad x(0) = 0$$

By a simple application of Itô's lemma on the function $f(t,x) = x \cdot e^{at}$, it can be shown that an expression for $x(t)$, conditioned on $\mathcal{F}_s$ is given by

$$x(t) = x(s)e^{a(s-t)} + \int_s^t e^{a(u-t)}\sigma dW(u)$$

If we compare this formula to that of $r(t)$, we see that for each $t > 0$, $r(t)$ can be constructed from the shifted short-rate process $x(t)$ through the relation $r(t) = x(t) + \alpha(t)$, where $\alpha(t) = f^M(0,t) + \frac{\sigma^2}{a^2}\left(1-e^{-at}\right)^2$. If we substitute $x(t)+\alpha(t)$ in the formula for a zero-coupon bond, we find that

$$
\begin{aligned}
P(s,t) &= \frac{P(0,t)}{P(0,s)} \exp\left\{B(s,t)f^M(0,s) - \frac{\sigma^2}{4a}B^2(s,t)\left(1-e^{-2as}\right) - B(s,t)\left(x(s)+\alpha(s)\right)\right\} \\
&= \frac{P(0,t)}{P(0,s)} \exp\left\{-\frac{\sigma^2}{4a}B^2(s,t)\left(1-e^{-2s}\right) - \frac{\sigma^2}{a^2}B(s,t)\left(1-e^{-as}\right)^2 - B(s,t)x(s)\right\}
\end{aligned}
$$

For simulation purposes this is an important result. For the computation of a zero-coupon bond, all we have to consider are the shifted short-rate process $x(t)$ and the current term-structure of bonds.

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