Equation 13 from Article 2: $I(R,\theta) = N Z^2 I_{inc} \frac{\omega^4}{\omega_0^4} \frac{r_0^2}{R^2} \sin^2\theta$

Oscillating dipole fields are given by: $\vec{E}(\vec{r},t) = \frac{\mu_0}{4\pi r} \hat{r} \times \left[\hat{r} \times \left(\frac{d^2\vec{p}}{dt^2}\right)_{t-(r/c)}\right]$

where $\vec{p}$ is the induced dipole moment.

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{r} \times \vec{E}(\vec{r},t)$$

Through a substitution of $\vec{p} = q\vec{s}$

and $\vec{a} = \ddot{\vec{s}} = (1/q)\, d^2\vec{p}/dt^2$ and some geometrical arguments,

we find: $\vec{E}(\vec{r},t) = -\frac{q}{4\pi\varepsilon_0 c^2} \frac{\vec{a}_\perp(t_{ret})}{r}$.

with the magnitude becoming: $E(r,\theta;t) = \frac{q}{4\pi\varepsilon_0 c^2} \frac{a_{ret}\sin\theta}{r}$

We construct a differential equation for undamped incident light:

$$m\ddot{z} = q E_0 \cos(\omega t) - m\omega_0^2 z$$

The solution is quickly obtained as: $z = \frac{q E_0 \cos(\omega t)}{m(\omega_0^2 - \omega^2)}$

We find, by substitution into our equation for a_{ret}:

$$a_{ret} = -\frac{q}{m} \frac{\omega^2}{\omega_0^2} E_0 \cos(\omega t_{ret})$$

Which further can be substituted as: $E(r,\theta;t) = -\frac{\omega^2}{\omega_0^2} \frac{r_0}{r} \sin\theta\, E_0 \cos(\omega t_{ret})$

with $r_0 = q^2/4\pi\varepsilon_0 m c^2$

From our definition of intensity we find: (with Z being the atomic # of an atom)

$$I(\omega,r,\theta) = \varepsilon_0 c \langle E^2 \rangle = I_{inc} Z^2 \frac{\omega^4}{\omega_0^4} \frac{r_0^2}{r^2} \sin^2\theta.$$

Finally, with N dipoles in the solution we have:

$$I(R,\theta) = N Z^2 I_{inc} \frac{\omega^4}{\omega_0^4} \frac{r_0^2}{R^2} \sin^2\theta$$

Scattering of Light
Laboratory Exercises

Patrick McMillin
Phys 465 Dr. Choudhary
8 October 2017

Equation 20 from article 1:

$$\vec{E}_s = \sum_{n=1}^{\infty} E_n\left(ia_n\vec{N}^{(3)}_{e1n} - b_n\vec{M}^{(3)}_{o1n}\right)$$

$$\vec{H}_s = \frac{k}{\omega\mu}\sum_{n=1}^{\infty} E_n\left(ib_n\vec{N}^{(3)}_{o1n} + a_n\vec{M}^{(3)}_{e1n}\right)$$

We can begin by defining some vector fields:

$$\vec{E}_i = E_0 e^{(i(\vec{k}\cdot\vec{x}-\omega t))} \; ; \; \vec{H}_i = H_0 e^{(i(\vec{k}\cdot\vec{x}-\omega t))}$$

which describe the electric and magnetic fields.

These fields must satisfy Maxwell's equations: $\nabla\cdot\vec{E}=0, \nabla\cdot\vec{H}=0; \nabla\times\vec{E}=i\omega\mu\vec{H}$ and $\nabla\times\vec{H}=-i\omega\varepsilon\vec{E}$

where our values of ε and μ are the premitivity and permeability in a material. ω is an angular frequency, k is the wave vector, and x and t are spatial and temporal components.

Through a quick rearrangement of these equations, we find:

$$\nabla^2\vec{E}+k^2E=0 \; ; \text{ and } \nabla^2\vec{H}+k^2\vec{H}=0 \quad \text{with } k=\omega^2\varepsilon\mu$$

Which are vector wave equations with vector solutions $\vec{M}\equiv\nabla\times(\vec{r}\psi)$ and $\vec{N}\equiv\frac{\nabla\times\vec{M}}{k}$. Since our problem is based in spherical coordinates, through separation of variables and using our definition of the Laplacian in spherical coordinates, we obtain:

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0 \; ; \; \frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right]+\left(n(n+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0 \; ; \; \frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(k^2r^2-n(n+1)\right)R=0$$

If we change variables, it is easy to describe our coordinates in terms of special functions: $\Phi=\{\cos(m\phi),\sin(m\phi)\}$; $\Theta=\{P_n^m(\cos(\theta))\}$; $R=\{j_n(\rho), y_n(\rho)\}$

Where P_n^m is a first kind Legendre function of degree n and order m, and j_n and y_n are Spherical Bessel functions of order n of the first and second kind.

The special functions define a basis on which we can construct the vector spherical harmonics for $\vec{M}$ and $\vec{N}$.

These harmonics are very messy, but when we consider the plane waves of the $\vec{E}$, we find $\vec{E}_i = E_0\sum_{n=1}^{\infty} i^n\frac{2n+1}{n(n+1)}\left(\vec{M}^{(1)}_{o1n}-i\vec{N}^{(1)}_{e1n}\right)$ where $\vec{M}_{o1n}$ is the 1st order spherical harmonic for $\vec{m}$ with radial dependence, similarly for $\vec{N}$.

By utilizing boundary conditions, and switching to Henkel functions, we eventually find:

$$\vec{E}_s = \sum_{n=1}^{\infty} E_n\left(ia_n\vec{N}^{(3)}_{e1n} - b_n\vec{M}^{(3)}_{o1n}\right)$$

$$\vec{H}_s = \frac{k}{\omega\mu}\sum_{n=1}^{\infty} E_n\left(ib_nN^{(3)}_{o1n} + a_n\vec{M}^{(3)}_{o1n}\right)$$