**Oscillatory Motion and Chaos**

In introductory physics, we have all seen the simple pendulum as an approximate simple harmonic oscillator with the equation of motion:



The solution to the simple harmonic oscillator problem is well known. We would like to study the problem of a driven nonlinear pendulum – where the oscillation amplitudes get large enough that the mass can swing all the way around the pivot point (replacing the string with a rigid massless rod). We will include the effects of friction (possible sources include the effective bearing where the pendulum connects to the support, air resistance, etc.) which we will take to be proportional to the velocity. By driven, we mean an external driving force that acts on the pendulum. A convenient choice is to assume a sinusoidal driving force (which may arise if the pendulum mass is charged and we apply an oscillating electric field) with amplitude *FD* and frequency Ω*D*. The equation of motion becomes



where *q* is a parameter that is a measure of the strength of the damping due to friction. This model contains some very rich and interesting behavior but we will only touch upon a few of them. Denote the angular velocity by . In all of the following questions, we will use the initial conditions  and . Use the parameters ,  and .

Questions:

1. First, solve the problem of the simple harmonic oscillator by solving the differential equation numerically. Plot as function of time for up to 60 s. Compare with the exact results.
2. Next, solve the driven nonlinear pendulum problem for the three values of driving amplitude: .

The results for the driven nonlinear pendulum shows three distinct behaviors – the first is damped motion and has no driving force; the second displays oscillatory behavior; the third has no discernable pattern. The third one actually displays chaotic behavior and by this, we mean that although the equation of motion is deterministic, it is difficult to predict the future behavior because the solution has a very sensitive dependence on the initial conditions. To ensure that the differences are not an artifact of the instability in the algorithm that was used to solve the differential equation, we will investigate the following.

1. Imagine we have two identical pendulums and start one off with the initial conditions above and the other with the slightly different initial conditions:  and . With , calculate  and  – the amplitudes of the two pendula respectively. Now plot  versus time on a semilog plot for up to 50 s.

You should find a series of sharp dips that occur approximately every 3 s. They occur when one of the pendulums reaches a turning point.  vanishes near each tuning point since the trajectories  and  cross each other. The plateau values of  away from these dips exhibit a steady and fairly rapid decrease with time. This means that the motion of the two pendula become more and more similar and the difference between the two angles approaches zero as the motion proceeds. This in turn means that the motion is predictable and small differences in initial conditions will all evolve to the same final motion. Hence we would not call such motion chaotic. It also shows us that the MATLAB differential equation solver(s) is stable.

1. Now repeat (3) with  but now plot till *t* = 150 s.

You should find that  increases rapidly and irregularly with time and we say the two trajectories diverge from one another. You should notice that the trend of the divergence is linear on the semilog plot at short times.  stops changing at long time periods because it has reached a value of order 2*π* and simply cannot get any larger. The irregular variation of  cannot be described by any simple function. However, if we were to repeat this calculation for a range of different initial values of *θ*1 (keeping (0) fixed at 0.001) and average the results, we will find a much smoother linear increase in  on the semilog plot. This means that the divergence is exponential and is characterized by an exponent called the Lyapunov exponent. Lyapunov exponents are ubiquitous in chaotic dynamics.