Analysis

Assume $\|x^*\| \le B$, $\quad x^1 = 0$

$$f(\bar{x}) - f(x^*) = f\left(\frac{1}{T}\sum x^t\right) - f(x^*)$$

$$\overset{\text{Jenson's}}{\le} \frac{1}{T}\sum f(x^t) - f(x^*)$$

$$= \frac{1}{T}\sum \left(f(x^t) - f(x^*)\right) \qquad v_t \in \partial f(x^t)$$

$$\le (v_t)^T (x^t - x^*)$$

$$\le \frac{1}{T}\sum \underbrace{(v_t)^T (x^t - x^*)}$$

$$\frac{1}{\eta}\underbrace{(\eta v_t)^T}_{a}\underbrace{(x^t - x^*)}_{b}$$

Aside

$\|a-b\|^2 = \|a\|^2 + \|b\|^2 - 2ab$

$ab = \dfrac{\|a\|^2 + \|b\|^2 - \|a-b\|^2}{2}$

$$\frac{1}{2\eta}\left(-\|\underbrace{x^t - x^* - \eta v_t}_{x^{t+1}}\|^2 + \|x^t - x^*\|^2 + \eta^2\|v_t\|^2\right)$$

$$\frac{1}{2\eta}\left(-\|x^{t+1} - x^*\|^2 + \|x^t - x^*\|^2\right) + \frac{\eta}{2}\|v_t\|^2$$

$$\frac{1}{2\eta T}\underbrace{\sum\left(-\|x^{t+1} - x^*\|^2 + \|x^t - x^*\|^2\right)}_{\|x^1 - x^*\|^2 - \|x^{T+1} - x^*\|^2} + \frac{\eta}{T2}\sum \|v_t\|^2$$

$$\frac{1}{2\eta T}\left(\|\overset{0}{\cancel{x^1}} - x^*\|^2 - \|x^{T+1} - x^*\|^2\right) + \frac{\eta}{T2}\sum \|v_t\|^2$$

$$\le \frac{1}{2\eta T}\|x^*\|^2 + \frac{\eta}{T2}\sum \|v_t\|^2$$