Convexity

$f : \mathbb{R}^n \to \mathbb{R}^n$

f is convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$

$\forall x, y \in \mathbb{R}^n$ & $\forall\, 0 \leq \lambda \leq 1$

Differentiation Equivalence $f \in C^2$

TFAE : 1. f is convex.

2. $f(y) \geq f(x) + \nabla^T f(x)(y-x) \quad \forall x, y \in \mathbb{R}^n$

3. $F(x) \succeq 0 \quad \forall x \in \mathbb{R}^n$

Pf. $1 \to 2$

$$f(\lambda y + (1-\lambda)x) \leq \lambda f(y) + (1-\lambda) f(x)$$

$$f(x + \lambda(y-x)) \leq f(x) + \lambda(f(y) - f(x))$$

$$f(y) - f(x) \geq \frac{f(x + \lambda(y-x)) - f(x)}{\lambda}$$

$\lim \lambda \downarrow 0$

$$= \nabla^T f(x)(y-x)$$

$2 \to 1$

Let $x, y \in \mathbb{R}^n$, $z := \lambda x + (1-\lambda)y$

so $z \in \mathbb{R}^n$

$$\left(f(x) \geq f(z) + \nabla f^T(z)(x-z)\right) \lambda$$

$$\left[f(y) \geq f(z) + \nabla f^T(z)(y-z)\right](1-\lambda)$$

$$\lambda f(x) + (1-\lambda) f(y) \geq f(z) + \cancelto{0}{\nabla f^T(z)}(\underbrace{\lambda x + (1-\lambda)y}_{z} - z)$$

$$= f(z)$$

$$= f(\lambda x + (1-\lambda)y)$$