Relations and Functions

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Pre-requisites

- Set theory

1 Relations

Have you ever listened to music on FM radio? You must be knowing how there is a different channel on 91.1 MHz, a different one on 93.5 MHz etc. In fact, for every frequency you set, the corresponding channel starts playing. That means for every frequency, we have defined what channel (if any) is playing on it. We say that we have defined a relationship between the frequency values and the channels. e.g. - 91.1 MHz is related to ChannelA, 92.7 MHz is related to ChannelB. Now think about this, using the knowledge of sets. Can you say that the frequencies you use to change the channel is a ***set*** of values? Yes, ofcourse. It is a well-ordered collection of data. Similarly, the collection of all the channels is also a set of values. Thus, can we say that a relation is nothing but a **mapping** from one set to another.

Definition 1 *Relation is a mapping from one set to another.*

We can have a relation from any set A to any set B, where is not necessary that $A \neq B$. Now we know how a relation is but how do we denote a relation. It's time to move onto some notations.

2 Denoting a relation

Let us again consider the above example. Let us say frequency 91.1 MHz is related to ChannelA. This can be represented using the **ordered pair** $(91.1, ChannelA)$. Similarly, 92.7 is related to ChannelB and 93.5 is related to ChannelC. Thus, we have the ordered pairs $(92.7, ChannelB)$ and $(93.5, ChannelC)$.

So putting all of them together, we get the set $\{(91.1, ChannelA), (92.7, ChannelB), (93.5, ChannelC)\}$. This is nothing but a relation. All that is left to do is to identify this relation and to identify anything we need a name. So, let us name this relation and call it R.

$$R = \{(91.1, ChannelA), (92.7, ChannelB), (93.5, ChannelC)\}$$

Have you seen something similar? Well yes, this seems just like a cartesian product of two sets. If you do not know about cartesian product of two sets refer here. Relations are nothing but **subsets of cartesian products** of the two sets. In the above example let us define the set of frequencies as $P = \{91.1, 92.7, 93.5\}$ and the set of channels as $Q = \{ChannelA, ChannelB, ChannelC\}$.

$$\begin{aligned}
\Rightarrow P \times Q = \{&(91.1, ChannelA), (91.1, ChannelB), (91.1, ChannelC), \\
&(92.7, ChannelA), (92.7, ChannelB), (92.7, ChannelC), \\
&(93.5, ChannelA), (93.5, ChannelB), (93.5, ChannelC)\} \\
\Rightarrow R \subseteq P \times Q&
\end{aligned}$$

Notice how we use the term **ordered-pair** to define $(91.1, ChannelA)$. This is because the order of the pair is important.

$$\Rightarrow (91.1, ChannelA) \neq (ChannelA, 91.1)$$

And this is why, cartesian products are not commutative.
$\Rightarrow A \times B \neq B \times A$. This gives us an idea that there is a direction associated to a relation.
In other words, a relation from set A to set B is different from a relation from set B to set A. We denote this using the direction using the following notation:

$$R : A \rightarrow B$$

The set A is called **domain** and the set B is called **co-domain** of the relation.

Let us consider some more examples of relations. For defining the relations, we will use the following sets:

$$A = \{1, 2, 3, 4\}$$
$$B = \{3, 5, 7, 9\}$$

$$R_1 : A \rightarrow B$$
$$R_1 = \{(1, 3), (2, 5), (3, 7)\}$$

Notice, how we didn't use up all elements of the sets A and B. 4 is not mapped to any element from set B and also no element from set A is mapped to 9 from set B. Also, can you identify the domain and the co-domain of the relation?

$$R_2 : A \rightarrow B$$
$$R_2 = \{(1, 3), (1, 5), (2, 7), (3, 7)\}.$$

Notice, how 1 is related to two elements 3 and 5 and also that the 2 and 3 are related to the same element 7.

We are dealing with very small sets here. $\mid A \mid = 4$ and $\mid B \mid = 4$. But, what if we are dealing with infinite sets like $\mathbb{N}$ or $\mathbb{R}$. We need some notation to define relations in a less cumbersome manner. Can you think of any tool to do this? Ahah! We have the set builder notation. But, how can we use that to define relations? The relation we saw above are arbitrary relations but in reality most relations are well-defined. The elements of one set are related to another set by some mathematical expression. Let us consider an example. In this example we will use the sets A and B as defined above. Consider the following relation.

$$R_3 : A \rightarrow B$$
$$R_3 = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$$

We can define the above relation using the following the notation:

$$R_3 = \{(a, b) | a \in A, b \in B, b = 2a + 1\}$$

Isn't this much easier? Yeah, even I think so. The true power of this notation comes into play dealing with larger sets. Consider the following example.

$$R_4 = \{(a,b)|a,b \in \mathbb{N}; b = a^2\}$$
$$R_5 = \{(a,b)|a \in \mathbb{N}, b \in \mathbb{R}; b = a^{\frac{1}{2}}\}$$

See, how easy was it to define the above relations.

Now, we can conclude that any subset of the cartesian product is a relation. However, there are no constraints while selecting these subsets. What happens if we apply some constraints? What constraints can we apply? Let us see in the next section.

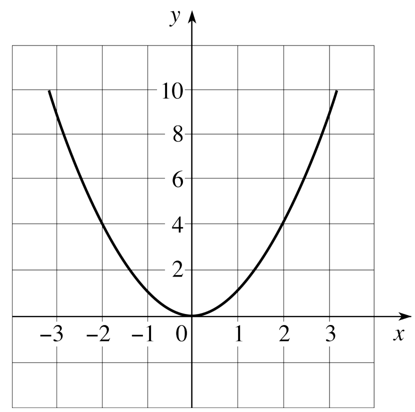
3 Functions

Functions. A word you use much too often in every day life. The function of a headphones is to convert electrical signals into sound; the function of a keyboard to convert the key pressed into some sort of signal that can be identified the computer; the function of a fan is to convert electrical energy into rotational energy; the function of a gas stove is to convert take in LPG gas and release it in specific amounts to produce flame. Do you notice a similarity in all this? Think hard. Well, we can see that there is an input for each of the function and it produces some kind of input to some meaningful and useful output. In other words, each input is mapped to some output. Thus, if we combine all the inputs into a set and all the outputs into another set, a function will be nothing but a relation from one set to another. But the obvious question comes to mind? What is the difference between a relation and a function. Hmmm.. That seems to be hard to answer. But, did you notice that when we defined relation R_2 above, how 1 from set A is mapped to 2 elements 3 and 5. Do you think there exists a function which produces different outputs when given the same input. NO! So we found our difference. The function is a relation where an element of the input set (domain) is mapped to only one element of the output set (co-domain). However, it is noteworthy that two elements of the input set can map to the same element of the output set. Phew, enough of theory. Let's look at some examples now.

Since functions are relations only, let us start with relations only and see whether they are functions too. Let us consider the relation R_4 defined above. Do you think it is a function too? Well, the square of every number is different and no two numbers can have the same square. Thus, no element from the domain will be mapped to more than one element of the co-domain. So yeah, it is a function. But how do we write it? Since, functions are relations only we write them the same way. However, there is another notation that we can use. Since, every element of the input maps to a specific output. We can write the output in terms of the input in the following manner.

$$\begin{aligned} b =& f(a) = a^2; a \in \mathbb{N} \\ & OR \\ y =& f(x) = x^2 \end{aligned}$$

Oh! Did something click? Yeah, I saw x and y there. Could this mean they represent the (x, y) on the cartesian plane? (Cartesian plane is the rectangular plane with X and Y axes.) Oh yes, they certainly do. Here is where the great contribution of Rene Descartes comes into picture. (Rene Descartes introduced the concept of cartesian plane and hence graphs.) So, that means we can actually represent a function as a graph on the cartesian plane where x and y are mapped according to a function. What will the graph to above function look like? Well, we can see that in the following figure.



Cool, right? Nice. Now, let us consolidate our understanding. Consider the relation R_5 defined above. Is it a function? Yes? No? Well, if you think hard, you will notice that the square root of 4 is both 2 and -2 and similarly the square root of 9 is 3 and -3. Thus, one element from the domain is actually mapped to more than one elements of the co-domain. Hence, this is not a function. Good Work!

4 Dependent and independent variables

Let us consider a small problem. You are in a transportation company and you need to transport bags of sand(Odd work right?). Now, you have a lot of trucks for the job. One truck can carry 10 bags of sand. You need to calculate the minimum number of trucks required if you need transport x bags of sand. That seems to be a pretty simple. All you need to do is calculate $\frac{x}{10}$ and your answer will be the next smallest integer, e.g. if $x = 23$, $\frac{x}{10} = 2.3$, the next smallest integer is 3. So the answer becomes 3. Now let us think in terms of functions. What you need to do is calculate the number of trucks required. Let us denote that by y. Now, is the value of y dependent on something, or you can give it an arbitrary value? It is dependent on the number of bags of sand i.e x! Here, the value of x is not dependent on anything, we just get an arbitrary value for it, we say x is an independent variable whereas y is a dependent variable dependent on x by the function:

$$y = f(x) = (\frac{x}{10})$$

where () denote the next smallest integer.

We say that y is a function of x.

5 Domain and Range

We have already discussed what domain is in terms of relations. However, let us consolidate that definition for functions. **Domain** is the set of values that the **independent variable** can take. Similarly, **Range** is the set of values that the **dependent variable** takes. It is worthy to note that we can decide our own domain (since we can give any value that we want to our independent variable) but the range depends on the function. However, sometimes we have to be clever while choosing the domain, because at some values, our function ceases to exist. Consider the following example:

$$y = f(x) = \frac{1}{x}$$

Over here, if we say that our domain is the set $A = \{1, 2, 3\}$. Then our range becomes the set $B = \{1, 0.5, 0.333...\}$. However, if we say that our domain is the set $P = \{0, 1, 2, 3\}$. The function ceases to exist at the point where $x = 0$.

Now let us do some practice. For the following functions, choose a domain(try to make it as large as possible) and find the corresponding range. The first one has been done for you.

1. $y = f(x) = x^2$
   - Domain = $\mathbb{R}$
   - Range = $\mathbb{R}^+$
2. $y = f(x) = x^2 - 2x + 10$
3. $y = f(x) = x^{\frac{1}{2}}$
4. $y = f(x) = \frac{1}{x^2+2x}$