NT Final

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§1 Problem Set

Problem 1.1. The sequence of positive integers $a_1, a_2, \cdots$ has the property that $\gcd(a_m, a_n) > 1$ if and only if $|m - n| = 1$. Find the sum of the k smallest possible values of a_l when $(k, l) =$

a) $(2, 1)$

b) $(7, 3)$

c) $(20, 2020)$

Problem 1.2. How many ordered pairs of positive integers (x, y) satisfy $y \cdot x^y = y^k$ when $k =$

a) 4

b) 2020

c) 105106

Problem 1.3. The sequence of nonnegative integers $F_0, F_1, F_2, \ldots$ is defined recursively as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$. Let d be the largest positive integer such that, for all integers $n \geq 0$, d divides $F_{n+2k} - F_n$. Compute the remainder when d is divided by m for $(k, m) =$

a) $(2, 4)$

b) $(63, 11)$

c) $(2020, 1001)$

Problem 1.4. What is the largest integer $n \leq k$ such that for all integers $b > 1$, n has at least as many 1's in its base-a representation as it has in its base-b representation for $(a, k) =$

a) $(4, 10)$

b) $(8, 100)$

c) $(4, 2020)$

§2 Answers and Solutions

§2.1 Problem 1.1

1.1 Notice that $a_1 > 1$ as $\gcd(a_1, a_2) > 1$. Now, let the set of primes be $\{p_1, p_2, \cdots\}$. We can set $a_1 = p_1$ and $a_i = p_{i-1}p_i$ and this clearly works. So, for a_1 the minimum two possible values are $2, 3$ which sum to 5.
For, $i > 1$ a_i must have atleast two prime divisors as $\exists p \neq q$ such that $p|a_{i-1}, a_i$ and $q|a_i, a_{i+1}$.
Now, we claim that any number which has atleast two different prime factors can be a_i. For this, let us say we want $a_i = n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. And let the set of primes be $\{q_1, q_2, \cdots q_i, p_1, p_2, \cdots p_k, q_{i+2}q_{i+3}\cdots\}$ and let

$$a_j = \begin{cases} q_j q_{j+1} & \text{if } j < i-1 \text{ or } j > i+1 \\ q_{i-1}p_1 & \text{if } j = i-1 \\ p_k q_{i+2} & \text{if } j = i+1 \\ n & \text{if } j = i \end{cases} \tag{1}$$

Now, we can find the smallest 20 values which are not primes or prime-powers. These are

$$\{6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 30, 33, 34, 35, 36, 38, 39, 40\}$$

The smallest 7 sum to 95 and all 20 sum to 501.

a) 5

b) 95

c) 501

§2.2 Problem 1.2

1.2 Notice that no value of y can have multiple solutions. So, we just want to find y such that $\exists x \in \mathbb{N}$ such that $x^y = y^{k-1}$ or all y such that y^{k-1} is a y^{th} power. Observe that any y such that $y|k-1$ works. In fact, now if y is not a perfect power itself, we must have that $y|k-1$. Now, if y is a perfect power say $y = z^t$. We have $z^t|t(k-1)$. Now, you can argue via size that $z|k-1$. So, now in our options, we have the greatest prime power dividing $k-1$ as 2 and all factors of $k-1$ are odd so $t > 2$. So, if $t \geq 3$, we must have $p^{t-2}|t$ which is only possible for $(t, p) = (3, 3)$ but for $p = 3$ we have maximum power in all $k-1$ as 1 so we must have had $9|3$ which is impossible. So, only y that are factors of $k-1$ work. We can calculate by the formula for number of factors discussed earlier.
$4 - 1 = 3 = 3^1$ so there are 2 factors. $2020 - 1 = 2019 = 3^1 673^1$ so there are $(1+1)(1+1) = 4$ factors. $105106 - 1 = 105105 = 3^1 5^1 7^2 11^1 13^1$ so there are $(1+1)(1+1)(2+1)(1+1)(1+1) = 48$ factors.

a) 2

b) 4

c) 48

§2.3 Problem 1.3

1.3 Observe that $d|F_{n+2k} - F_n$ and $d|F_{n+2k+1} - F_{n+1} \implies d|F_{n+2k-1} - F_{n-1}$ and $d|F_{n+2k+2} - F_{n+2}$ so we have

$$\begin{aligned} d &= \gcd\left(F_{2k} + F_0, F_{2k+1} - F_1\right) \\ &= \gcd\left(F_{2k} + F_0, F_{2k-1} - F_1\right) \\ &= \gcd\left(F_{2k-2} + F_2, F_{2k-1} - F_1\right) \\ &= \gcd\left(F_{2k-2} + F_2, F_{2k-3} - F_3\right) \\ &\vdots \\ &= \gcd\left(F_{2k-k} + F_k, Fk + 1 - F_{k-1}\right) \\ &= \gcd\left(2F_k, F_k\right) \\ &= F_k \end{aligned}$$

So, we want $F_k \pmod{m}$. So, $F_2 \pmod{4} = 1$.
Observe that $\langle F_i \rangle \pmod{11}$ has period 10 so we have $F_{63} \equiv F_3 \equiv 2 \pmod{11}$.
Similarly, 7 has period 16 and 13 has period 28. So, we get that

$$\begin{aligned} F_{2020} &\equiv F_0 \equiv 0 \pmod{11} \\ F_{2020} &\equiv F_4 \equiv 3 \pmod{7} \\ F_{2020} &\equiv F_4 \equiv 3 \pmod{13} \end{aligned}$$

Now, by CRT, we get that $F_{2020} \equiv 913 \pmod{1001}$

a) 1

b) 2

c) 913

§2.4 Problem 1.4

1.4 Observe that $4, 8$ are both powers of 2. So, consider their binary representations. Now, if their base-a representation had a any digit other than $0, 1$ then it would add an extra 1 in their base 2 representations. So, the numbers must only have $0s$ and $1s$ in base a representations. So, we consider the largest numbers $< k$ with all $1s$ in their base a represntation. Observe that switching a 0 in the representation with a 1 is possible and only benefits us. So, we check these values and they work. So, a) is $4 + 1$, b) $64 + 8 + 1$ and c) is $1024 + 256 + 64 + 16 + 4 + 1$.

a) 5

b) 73

c) 1365