Ad-Hoc 1

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Time: 180 minutes

§1 Problem Set

Problem 1.1. Let the number of permutations σ of $[n]$ such that $|\sigma(i) - i| \leq 1$ be a_n. Find a_n when $n =$

a) 5

b) 13

c) 19

Problem 1.2. You are standing on a lattice at the point $(0, 0)$. From any point (x, y), you can go to either $(x + 1, y)$ or $(x, y + 1)$ or $(x + 1, y + 1)$. How many distinct paths are there from $(0, 0)$ to (m, n) when $(m, n) =$

a) $(2, 1)$

b) $(3, 3)$

c) $(4, 5)$

Problem 1.3. You are given 8 distinct points on a circle and all pairs of points are joined by a chord. We know that no three chords are concurrent at any point inside the circle. Find the number of:

a) Different Intersection points

b) Triangles(all three sides are given by edges and vertices by intersection points or the n points on the circle.)

c) Pairs of adjacent points(i.e. points that are on the same chord and don't have any other intersections between them. Note that you can consider points on the circle as well).

Problem 1.4. The wheel shown below consists of two circles and five spokes, with a label at each point where a spoke meets a circle. A bug walks along the wheel, starting at point A. At every step of the process, the bug walks from one labeled point to an adjacent labeled point. Along the inner circle the bug only walks in a counterclockwise direction, and along the outer circle the bug only walks in a clockwise direction. For example, the bug could travel along the path $AJABCHCHIJA$, which has 10 steps. Let n be the number of paths with k steps that begin and end at point A. Find n when $k =$

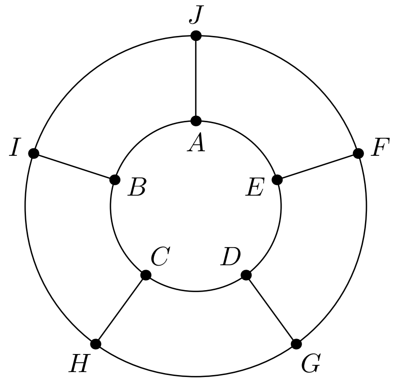

Figure 1: Bug path

a) 4

b) 9

c) 15

§2 Answers and Solutions

§2.1 Permutation Count

Solution. 1.1 Observe that if $\sigma(n) = n$, then the number of such sequences is a_{n-1}. If we get that $\sigma(n) = n-1$ then we must have $\sigma(n-1) = n$. So, we get a_{n-2} sequences. Thus, $a_n = a_{n-1} + a_{n-2}$. Clearly, $a_1 = 1, a_2 = 2$. So, we $a_n = F_{n+1}$. Now, we can just find F_{n+1}.

Answer 2.1. a) 8

b) 377

c) 6765

□

§2.2 Grid-Walk

We call the number of paths to reach (m, n) $p[m, n]$. Now, we clearly have $p[m, n] = p[m-1, n-1] + p[m-1, n] + p[m, n-1]$. So, we construct a grid as:

1	11	61	231	681	1683
1	9	41	129	321	681
1	7	25	63	129	231
1	5	13	25	41	61
1	3	5	7	9	11
1	1	1	1	1	1

Table 1: Values of p[m,n]

Thus, we get

a) 5

b) 63

c) 681

§2.3 Cute Counting idea

Assume for now that there are n points on the circle. Consider any 4 points on the circle v_i, v_j, v_k, v_l and observe that they give us a unique intersection point and every intersection point can be attributed to a set of 4 points by the edges it comes from intersecting. So, the number of intersection points is $\binom{n}{4}$.

Similarly, we can get the result for triangles as $\binom{n}{3} + 4\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}$. You can refer to Stack-Exchange.

For edges, you can do a degree sum and get the answer as $\binom{n}{2} + 2\binom{n}{4}$. This actually leads to a more interesting result given by the number of regions in a circle given by arcs and such chords. We may discuss this result in the class but you may read up on it at StackExchange or Wikipedia. So, the answers are

a) 70

b) 644

c) 168

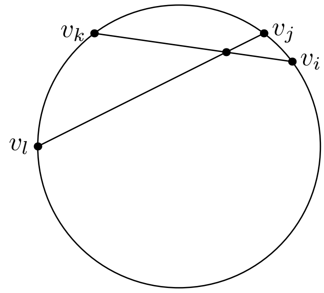

Figure 2: Intersections

§2.4 AIME 2018/10

From djmathman on AoPS. Note that the set of sequences of moves the bug makes is in bijective correspondence with the set of strings of Xs and Ys of length k, where X denotes a move which is either counterclockwise or inward along a spoke and Y denotes a move which is either clockwise or outward along a spoke. (The proof of this basically boils down to the fact that which one depends on whether the bug is on the inner wheel or the outer wheel.) Now the condition that the bug ends at A implies that the difference between the number of Xs and the number of Ys is a multiple of 5 and last move must be an X. So, for first case the only way is 2Ys in remaining 3 moves so there are $\binom{3}{2}$ ways. In second case we have the number of Y can be 2 or 7 in the remaining cases so the answer is $\binom{8}{7} + \binom{8}{2} = 8 + 28 = 36$ and in the last case, we can have $0, 5$ or 10Ys so the answer is $\binom{14}{0} + \binom{14}{5} + \binom{14}{10} = 3004$. Please refer to AoPS for more approaches and explanations.

a) 3

b) 36

c) 3004