NT Introductory

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§1 Problem Set

Problem 1.1. Find

a) Number of factors of 297000.

b) Number of numbers $n < 297000$ co-prime to 297000.

c) Sum of divisors of 297000.

Problem 1.2. Master Oogway recently declared Po as the Dragon Warrior, against the judgement of Master Shifu. To make his point, Master Shifu challenges Po to a puzzle. He writes down five consecutive integers and then erases one of them. The four remaining integers sum to k. Help Po compute the integer that Master Shifu erased and earn his place as the true Dragon Warrior for $k =$

a) 23

b) 100

c) 57321

Problem 1.3. As Hobbes is getting bored, so he writes all rational numbers r, $0 < r < 1$ on a blackboard in their lowest terms such that the numerator and denominator sum to n. Calvin keeps losing count whenever he tries to find the number of numbers Hobbes writes. Help Calvin find the number of numbers on the board without looking at the board when $n =$

a) 10

b) 87

c) 17017

Problem 1.4. Katie has a fair k-sided die with sides labeled $1, 2, \cdots, k$. After each roll, she replaces her n-sided die with an $(n+1)$-sided die having the n sides of her previous die and an additional side with the number she just rolled. The probability that Katie's a^{th} roll is b is $\frac{x}{y}$ where x and y are coprime and non-negative(y is positive). Find $x + y$ for $(k, a, b) =$

a) $(2, 3, 1)$

b) $(3, 2, 5)$

c) $(2020, 47, 2020)$

§2 Answers with Solutions

§2.1 Standard Results

1.1 Note that $297000 = 2^3 \cdot 3^3 \cdot 5^3 \cdot 11$.

a) **Answer.** 128
The number of factors of a number n is usually donated by $\sigma(n)$ and we have that when $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then $\sigma(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$. Applying the formula here, we get $\sigma(2^3 \cdot 3^3 \cdot 5^3 \cdot 11) = 4 \cdot 4 \cdot 4 \cdot 2 = 128$.

b) **Answer.** 72000.
The number of numbers co-prime to n and less than n is usually denoted by $\varphi(n)$ and we have that when $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $\alpha_i \geq 1$ then $\varphi(n) = p_1^{\alpha_1 - 1}(p_1 - 1) p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_k^{\alpha_k - 1}(p_k - 1)$. Applying the formula here, we get $\varphi(2^3 \cdot 3^3 \cdot 5^3 \cdot 11) = 4 \cdot 1 \cdot 9 \cdot 2 \cdot 25 \cdot 4 \cdot 1 \cdot 10 = 72000$.

c) **Answer.** 1123200
The sum of divisors of a number n is denoted by $\tau(n)$ and we have that when $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then $\tau(n) = \dfrac{p_1^{\alpha+1} - 1}{p_1 - 1} \cdot \dfrac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \cdots \dfrac{p_k^{\alpha_k+1} - 1}{p_k - 1}$. Applying the formula here, we get the answer as $\tau(2^3 \cdot 3^3 \cdot 5^3 \cdot 11) = 15 \cdot 40 \cdot 156 \cdot 12 = 1123200$.

§2.2 Shifu writes numbers! (Rephrased from OMO)

1.2 Assume the numbers he wrote were $n, n+1, n+2, n+3, n+4$. Then, after erasing the number left would be one of $\{4n+6, 4n+7, \cdots 4n+10\}$. Considering this $\pmod 4$, we get that there is a unique n for numbers of the form $4n+7, 4n+8, 4n+9$. So we find n for each of the given values and thus the corresponding number to be deleted.

a) **Answer.** 7
23 is of the form $4n+7$ and $n = 4$. So, number deleted is $n + 3 = 7$

b) **Answer.** 25
100 is of the form $4n+8$ for $n = 23$. So, number deleted is $n + 2 = 25$.

c) **Answer.** 14329
57321 is of the form $4n+9$ for $n = 14328$. So, number deleted is $n + 1 = 14329$.

§2.3 So many rationals(Rephrased from AIME)

1.3 Suppose $\frac{a}{b}$ is a number Hobbes can write. Then, we have three conditions $\gcd(a, b) = 1$, $a < b$ and $a + b = k$. Now, we know that $1 = \gcd(a, b) = \gcd(a, a+b) = \gcd(a, k)$. So, we have only 2 conditions on a, $\gcd(a, k) = 1$ and $a < \frac{k}{2}$.
Observe that $(a, k) = (b, k) = 1$ now. So, for every a that satisfies above conditions, we get a corresponding b. So, our answer is $\frac{\phi(k)}{2}$ where $\phi(k)$ represents the number of numbers less than k co-prime to k.

a) **Answer.** 2
$\phi(10) = 4$ so answer is 2.

b) **Answer.** 28
$\phi(87) = \phi(3) \times \phi(29) = 2 \cdot 28 = 56$ so answer is 28.

c) **Answer.** 5760
$\phi(17017) = \phi(7) \cdot \phi(11) \cdot \phi(13) \cdot \phi(17) = 6 \cdot 10 \cdot 12 \cdot 16 = 11520$. So, answer is $\frac{11520}{2} = 5760$.

§2.4 Katie rolls die (Rephrased from HMMT)

1.4 Observe that Katie can never roll a number larger than k. So, if $b > k$ then $\frac{x}{y} = \frac{0}{1}$ and $x + y = 1$. In case, $b \in \{1, 2, \cdots k\}$ then observe that the die is symmetric for $1, 2, \cdots k$ so each has equal probability so $\frac{x}{y} = \frac{1}{k}$ so answer is $k + 1$.

a) **Answer.** 3
$k = 2$ and $b < k$ so answer is $k + 1$.

b) **Answer.** 1
$5 = b > k = 3$ so answer is 1.

c) **Answer.** 2021
$k = 2020 \geq 2020 = b$ so answer is $k + 1 = 2021$.