Logarithms

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Pre-requisites

- Exponents and powers

1 Introduction

We all know how we represent an exponential equation:

$$x^a = y$$

In the above equation, x is the base, a is the exponent and y is the result.

We know that exponentiation is repeated multiplication and thus the above equation means that x is multiplied a times to obtain y. This is easy to imagine when a is a positive integer i.e. when $a \in \mathbb{Z}^+$, but what if a is actually a fraction. What is happening there? Let us try to get an intuitive idea. Just like exponentiation is repeated multiplication, similarly multiplication is repeated addition. Consider the following example:

$$16 \times \frac{1}{2} = x$$

How do we imagine this in terms of repeated addition?

$$2 \times x = 16$$

$$\Rightarrow x = 8$$

All we do is see that what would be added twice such that it would make 16. We can think about exponents in similar terms. For example,

$$16^{\frac{1}{2}}$$

Let this be equal to k. $\Rightarrow$ We need to multiply k twice to obtain 16. What should be multiplied twice to obtain 16. Simply, it is 4. So, you get the idea of fractional powers. So, rather that multiplying our base(x), exponent(a) times, we just find a number that when multiplied $\frac{1}{a}$ times, we get our base x. Simple right?

Now let us pose a different question, what if I want to know how many times do I need to multiply our base x to obtain the result y. In other words, we need to find a when x and y are given. This is where logarithms come in. Let us write the above equation in terms of logarithms.

$$x^a = y$$
$$\Rightarrow log_x y = a$$

The above two statements are equivalent. Here, x is the base. We say that a is the log of y to the base x. This means that we need to raise x to the power a to get y. Note the above two equations really well. We will be interchanging between the above equations a lot while proving the properties of logs. Logarithms are very useful in large calculations. Let us look at the properties of logarithms and relate them to the properties we looked in exponents and powers.

2 Properties of log

Property 2.1. $log_a a = 1$

Proof. Let $log_a a = k$
$\Rightarrow a^k = a$. [Refer to the two equations at the end of the previous page]
We know that, $a^1 = a \Rightarrow k = 1$. □

Property 2.2. $log_a m^n = nlog_a m$

Proof. Let $log_a m = x$
$\Rightarrow m = a^x \Rightarrow m^n = (a^x)^n \Rightarrow m^n = a^{xn} \Rightarrow log_a m^n = nx = nlog_a m$ □

Property 2.3. $log_a(mn) = log_a m + log_a n$

Proof. Let $x = log_a m, y = log_a n \Rightarrow m = a^x, n = a^y \Rightarrow mn = (a^x)(a^y) = a^{x+y} \Rightarrow log_a(mn) = x + y = log_a m + log_a n$ □

Property 2.4. $log_a(\frac{m}{n}) = log_a m - log_a n$

Proof. Left as an exercise. □

3 Example problem

Consider the following problem. Suppose you are given a number n. Now you need to calculate how many times do you need to divide it by 2 such that it becomes less than or equal to 1. So, we see that we divide it once, we get the number $\frac{n}{2}$. When we divide it twice we get the number $\frac{n}{4} = \frac{n}{2^2}$. When we divide it thrice, we get the number $\frac{n}{8} = \frac{n}{2^3}$ and so on. So let us say that we need to divide the number, x number of times such that it becomes less than or equal to 1. So, we can easily see that our equation becomes

$$\frac{n}{2^x} = 1$$
$$\Rightarrow n = 2^x$$
$$\Rightarrow log_2 n = log_2 2^x$$
$$\Rightarrow log_2 n = xlog_2 2$$
$$\Rightarrow log_2 n = x$$

See, how we use the properties of log to solve such problems.