Ad Hoc-1

CC ICO Learning Program

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Outline I

Outline II

Brief Introduction

This may be repetitive for a few of you but it may be useful for others.

The course and lecture. This is the first lecture in the Ad-Hoc Advanced module of the ZIO training series as part of a wider ICO learning program run by Codechef and Unacademy. ZIO is the first stage of the Computing Olympiad and is a 4 problem(3 parts per problem) integer type exam where the algorithmic thinking and Informatics based intuition is tested and qualifiers get to appear for the INOI.

About me. I am a 12th grader from New Delhi. I have cleared INMO and am an IMOTCer and an APMO medallist. I have also cleared ZCO, ZIO in 2019. I am teaching the NT and Ad-Hoc advanced modules and hope that they will be helpful in your preparation and you will also enjoy this series..

Module Plan I

The module is divided across 3 lectures and 3 tests. The content appearing in the tests bears no direct relation to the content being covered in the module as these tests are designed to replicate a wide array of problems and we will be focusing on more specific ones in the lectures. With every lecture, I have planned a bonus thing we can discuss if time allows. Today's lecture will serve as advanced practice of recurrences. After the lecture, I will also try to send you a list of results that you should try to prove on your own(remind me!). Bonus: Josephus's Problem.

Module Plan II

The second lecture in the series will define basic terms and cover basics of graph theory. We will be covering trees, how trees are constructed, number of edges, connectedness, sum of degrees and Euler's formula for planar graphs.
The third lecture will be based on problem solving, we will discuss a few famous problems, find nice ideas and solve few ZIO problems using them as well!

Pre-Requisites

"It's not about how good you are at doing calculation, it's about how good you are at not doing calculation."

Unknown

I will be assuming that you all are comfortable with basic combinatarial, number theory and recurrence ideas covered in previous modules.

Derangements

Problem

You are given n letters and n boxes, such that every letter is associated with a fixed box. How many ways are there to put the letters in boxes such that no letter goes into the box associated with it?

We call this value as D_n.

Examples

Problem

What is D_3?

a) 4
b) 3
c) 2
d) 1

Problem

What is D_4?

a) 11
b) 10
c) 9
d) 8

As a recurrence I

The first few values of D_n are as follows-

n	1	2	3	4	5	6	7	8
D_n	0	1	2	9	44	265	1854	14833

Table 1: Derangements

A very interesting pattern seems to be appearing here so we conjecture that it always follows:

As a recurrence II

Conjecture

Prove that $D_n = nD_{n-1} + (-1)^n$

Proof. We prove by strong induction. The result can be directly proven for $1, 2$. Now, for larger n, we first try to place n. Observe that there are $n-1$ ways to do so. Let the position n goes to be m. Now, there are D_{n-1} cases where m doesn't go to position n and D_{n-2} cases where m goes to position n. Thus, we have

$$\begin{aligned} D_n &= n-1(D_{n-1} + D_{n-2}) = (n-1)D_{n-1} + (n-1)D_{n-2} \\ &= (n-1)D_{n-1} + D_{n-1} - (-1)^{n-1} = nD_{n-1} + (-1)^n \end{aligned}$$

Closed Form

We can infact now use our new found recurrence to show that-

Closed Form?

$$D_n = n!\left(\sum_{i=0}^{n} \frac{(-1)^i}{i!}\right)$$

You should now try to prove this on your own and I will not be showing the details.

ZIO 2016/3

Problem

How many ways are there to permute $\{1, 2, \cdots N\}$ *such that* $\forall k < N$*, the first* k *numbers are not just* $1, 2, \cdots k$ *when* $N =$

a) 5
b) 6
c) 7

Let us call the number of sequences that work as a_n. Now, we write first few terms of $\langle a_i \rangle$.

n	1	2	3	4
a_n	1	1	3	13

Table 2: First few terms of sequence

Now, there doesn't seem to be any apparent pattern but let's make some general observations. We can also perceive as a_i as the number of permutation σ of $\{1, 2, \cdots i\}$ such that the smallest number k for which $\{\sigma(1), \sigma(2), \cdots, \sigma(k)\} = \{1, 2, \cdots k\}$ is i.

Now, let us try to count the number of sequences of length n where the smallest number k for which $\{\sigma(1), \sigma(2), \cdots, \sigma(k)\} = \{1, 2, \cdots k\}$ is i, then there are $a_i \cdot (n-i)!$ such permutations. Thus, we get the very nice result that $\sum_{i=1}^{n} a_i(n-i)! = n!$. Thus, we get that

$$a_n = n! - \sum_{i=1}^{n-1} a_i(n-i)!$$

Now, using this we compute further a_i

n	**Calculation**	a_n
1	1	1
2	$2! - 1$	1
3	$3! - 2! - 1!$	3
4	$4! - 3 \cdot 1! - 2! - 3!$	13
5	$120 - 13 - 6 - 6 - 24$	71
6	$720 - 71 - 26 - 18 - 24 - 120$	461
7	$5040 - 461 - 142 - 78 - 72 - 120 - 720$	3447

Table 3: ZIO 2016/3

Grid-Walk

This was one of the problems given in practice test 1 and is relatively much more direct and you can find the recursion directly.

Problem

You are standing on a lattice at the point $(0, 0)$. *From any point* (x, y), *you can go to either* $(x + 1, y)$ *or* $(x, y + 1)$ *or* $(x + 1, y + 1)$. *How many distinct paths are there from* $(0, 0)$ *to* (m, n) *when* $(m, n) =$

a) $(2, 1)$
b) $(3, 3)$
c) $(4, 5)$

Solution

Let us build a 2 dimensional recurrence, so let the number of paths to reach (m, n) from $(0, 0)$ be $p[m, n]$. Then, we find that $p[m, n] = p[m-1, n] + p[n-1, m] + p[m-1, n-1]$ when $mn > 0$ and $p[m, n] = 1$ when $mn = 0$. Thus, we get the following grid and our answers-

5	1	11	61	231	681	1683
4	1	9	41	129	321	681
3	1	7	25	63	129	231
2	1	5	13	25	41	61
1	1	3	5	7	9	11
0	1	1	1	1	1	1
	0	1	2	3	4	5

Table 4: Values of p[m,n]

The problem statement

Josephus's statement

There are n people standing around a circle and in every move, the kth person in the circle is executed and removed from the circle, the process continues with the $n - 1$ people with the first person as the person in front of the executed person. Find the position in which the survivor is standing.

We will first try to figure out the case in which $k = 2$ and then you can try find a general recurrence for all k.

k=2

Let us check a few small values and then conjecture on the basis of that.

Problem

Which place is the survivor when there are 3 people?

a) 1
b) 2
c) 3

Problem

Which place is the survivor when there are 10 people?

a) 1
b) 5
c) 8
d) 10

First few values

We now see the first few values in the following table(Call the survivor position J_n):

n	1	2	3	4	5	6	7	8
J_n	1	1	3	1	3	5	7	1

Table 5: Josephus's k=2

Clearly, a very nice pattern seems to be emerging. We now make the brave claim that:

Conjecture

If $2^k \leq n < 2^{k+1}$, then $J_n = 2n - 2^{k+1} + 1$.

Sketch of proof

Sketch. Try to show that in the case that $n = 2^k$ then winner is always position 1. Now, finish by induction on n.
Now, that you have this, try to find a recurrence for the general case!

Suggestions

There is not much to cover in basic recurrences other than the things that have been covered in the lectures. So, I would only recommend going through some easier Codeforces DP problems, Atcoder's educational DP contest, and past ZIO papers. In case you are interested in learning stuff beyond ZIO, I suggest reading about generating functions.

The End

Thanks for Attending
Discord Username: rg230403#8247 and @Rohan Goyal on the server.
Please also fill the Feedback Form.