Ad-Hoc-2

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Time: 180 minutes

§1 Problem Set

Problem 1.1. A candy store has c_i pieces of candy to give away. When you get to the store, there are k people in front of you, numbered from 1 to k. The ith person in line considers the set of positive integers congruent to i modulo k which are at most the number of pieces of candy remaining. If this set is empty, then they take no candy. Otherwise they pick an element of this set and take that many pieces of candy. For example, the first person in line will pick an integer from the set $\{1, k+1, 2k+1, \cdots\}$ and take that many pieces of candy. How many ways can the first five people take their share of candy so that after they are done there are at least c_e pieces of candy remaining when $(c_i, c_e, k) =$

a) $(50, 1, 7)$

b) $(100, 34, 5)$

c) $(5025, 2020, 3)$

Problem 1.2. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i, k), (i, l), (j, k), (j, l)$ for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively, or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid when $(m, n) =$

1. $(1, 7)$
2. $(2, 3)$
3. $(4, 13)$

Problem 1.3. Find the number of functions $f : \mathbb{N} \mapsto \mathbb{N}$ such that $f(f(n)) = n + k$ $\forall n \in \mathbb{N}$ when $k =$

1. 4
2. 8
3. 2021

Problem 1.4. Uncle Shiva has a plate containing $3c$ circular crumbling mooncakes, arranged as follows:

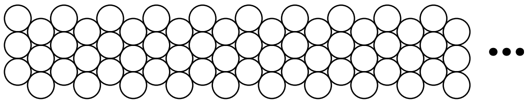

Figure 1: Circular Crumbling Mooncakes

(This continues for c total columns). Nikhil wants to pick some of the mooncakes to eat, however whenever she takes a mooncake all adjacent mooncakes will be destroyed and cannot be eaten. Let M be the maximal number of mooncakes she can eat, and let n be the number of ways she can pick M mooncakes to eat (Note: the order in which she picks mooncakes does not matter). Let the remainder when n is divided by 1001 be R. Compute $1001M + R$ when $c =$

a) 6

b) 31

c) 100

§2 Answers and Solutions

§2.1 Cute Combi

1.1

Answer. $\binom{\left\lfloor \frac{c_e - c_i}{k} + \frac{k-1}{2} \right\rfloor}{k}$

Solution. We only care about $c = c_e - c_i$ so the actual values of each are irrelevant. Now, we present two approaches are the same but you can show that in essence they are exactly same.

Approach 1. We consider the generating function $f(x) = (x + x^{k+1} + x^{2k+1} + \cdots)(x^2 + x^{k+2} + \cdots) \cdots (x^k + x^{2k} + \cdots)$. Observe that we want to find the sum of coefficients of x^i where $i \le c$. Now, this is essentially, $x^{\frac{k^2+k}{2}}(1 + x^k + x^{2k} + \cdots)^k$ and we want to find sum of coefficients of x^i where $i \le k$. Now, we can remove $x^{\frac{k^2+k}{2}}$. Thus, we want to find sum of terms with power $i \le d = c - \frac{k^2+k}{2}$. So, the answer becomes $\binom{k-1}{k-1} + \binom{k}{k-1} + \binom{k+2}{k-1} + \cdots \binom{\lfloor \frac{d}{k} \rfloor + k - 1 = \lfloor \frac{c}{k} + \frac{k-3}{2} \rfloor}{k-1} = \binom{\lfloor \frac{c}{k} + \frac{k-1}{2} \rfloor}{k}$.

Approach 2. Let the number of candies each kid picks be $a_1, a_2, \cdots a_k$ and let $a_i = b_i \cdot k + i$. Now, we have that $\sum k \cdot b_i \le c - \frac{k^2+k}{2}$ and $b_i \ge 0$. So, using the Ball and sticks formula, we get the number of ways of assigning b_i as $\binom{k-1}{k-1} + \binom{k}{k-1} + \cdots \binom{\left\lfloor \frac{c - \frac{k^2+k}{2}}{k} + k - 1 \right\rfloor}{k-1} = \binom{\lfloor \frac{c}{k} + \frac{k-1}{2} \rfloor}{k}$. □

Values.

a) 120

b) 3003

c) 167167000

§2.2 Rocks and Squares

Answer. $\binom{m+n-1}{m}^2$

Proof by CantonMathsGuy on AoPS: The answer is $\binom{m+n-1}{m}^2$. Let $\mathcal{A}$ be the multiset of the stone rows; similarly define $\mathcal{B}$ for columns.

Every move preserves $\mathcal{A}$ and $\mathcal{B}$. More strongly, letting the m stones have row coordinates $a_1, \ldots, a_m$ and column coordinates $b_1, \ldots, b_m$, any stone move corresponds to a swap in the b_is.

Since any permutation is obtained by swaps, equivalence classes are characterized completely by the signature $(\mathcal{A}, \mathcal{B})$; by stars and bars there are $\binom{m+n-1}{m}^2$ possible signatures.

To finish we only need to show that every signature is represented. Write any sequences $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$, and place stones at each (a_i, b_i)!

For more detailed answers, refer to Aops

Answer 2.1. a) 49

b) 36

c) 3312400

§2.3 IMO 1987/4 General

Claim 2.2 — f is injective.

Proof. Let m, n be such that $f(m) = f(n)$. Then, we have $f(f(m)) = f(f(n)) \implies m + k = n + k \implies m = n$. □

Definition 2.3 (Cute numbers and meows)**.** Call a number n **cute** if $\exists m$ such that $f(m) = n$. And call m, the **meow** of n and observe that if m exists then it is unique as f is injective and call it **very cute** if $\exists m$ such that $f(f(m)) = n$.

Claim 2.4 — n is *very cute* iff $n > k$.

Proof. $n = f(f(m)) = m + k \implies n > k$ and $n > k \implies n = f(f(n-k))$. □

Claim 2.5 — If $i \in \{1, 2, \cdots k\}$ then either i is *cute* and the *meow* of i is $\leq k$ or $f(i) \leq k$.

Proof. If i is not cute then we have that $f(i)$ is *cute* but not *very cute* so it must be $\leq k$. If i is *cute* then its *meow* is not cute as otherwise i would be *very cute* so the *meow* of i is not *cute* and consequently $< n$. □

Claim 2.6 — $\text{Fix}(f) = \phi$.

Proof. If $n \in \text{Fix}(f)$ then $f(n) = n \implies f(f(n)) = n \implies n + k = n \implies k = 0$. Contradiction! □

Now, we have that the numbers $\leq k$ can be divided into pairs of not *cute* numbers and their images. So, if such an f exists then k must be even. If k is odd, any set of $\frac{k}{2}$ ordered pairs gives a unique f. So, answer is $\dfrac{k!}{\frac{k}{2}!}$

a) 12

b) 1680

c) 0

§2.4 Crumbling Mooncakes

Solution. The value of M is simply $2\lceil \frac{c}{2} \rceil$ as you can pick upto 2 from two consecutive rows and that's always possible. In odd number of rows, only way to achieve this is to pick 2 from each odd row. So, $n = r = 1$. Thus, answer would be $1001(c+1) + 1 = 1001c + 1002$.

In case c is even, check P8 solution from CMIMC 2020 C-CS test. Thus, we want to find in case of evens, $F_{c+4} + \binom{\frac{c}{2}}{2} - 2 \pmod{1001} + 1001c$. Thus, the answer when $c = 6$, $6006 + (F_{10} + 3 - 2 \pmod{1001}) = 6006 + 56 = 6062$. $c = 31$, the answer is 32033.

For, $F_{104} \equiv F_8 \equiv 0 \pmod 7, F_{104} \equiv F_4 \equiv 3 \pmod{11}$ and $F_{104} \equiv F_{20} \equiv -F_6 \pmod 8 \equiv 5 \pmod{13} \implies F_{104} \equiv 707 \pmod{1001}$. Thus, our final answer is $100100 + (707 + 222) = 101029$.

$\square$

Answer. a) 6062

b) 32033

c) 101029