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Instability in time-delayed switched systems induced by fast and random switching

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Abstract

In this paper, we consider a switched system comprising finitely or infinitely many subsystems described by linear time-delayed differential equations and a rule that orchestrates the system switching randomly among these subsystems, where the switching times are also randomly chosen. We first construct a counter-intuitive example where even though all the time-delayed subsystems are exponentially stable, the behaviors of the randomly switched system change from stable dynamics to unstable dynamics with a decrease of the dwell time. Then by using the theories of stochastic processes and delay differential equations, we present a general result on when this fast and random switching induced instability should occur and we extend this to the case of nonlinear time-delayed switched systems as well.

Keywords: Switched system; Random and fast switching; Time-delay; Stochastic instability; Dwell time

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1. Introduction

In natural and man-made systems, stochastic or periodic fluctuations due to environmental changes are unavoidable. Including the fluctuation influences in mathematical description of these systems usually invites modeling and exploration of switched systems. Mathematically, a switched system comprises a family of finitely or infinitely many subsystems and a rule that orchestrates the system switching among the subsystems. Switched systems along with switching rules can be largely divided into two types: one usually involves switching rules dependent on the loci of the system state variables [1–7], and the other includes rules crucially dependent on a set of switching time instants deterministically or randomly given [8–24].

In this paper, we discuss the stability problem for the second type of switched systems. In fact, many tools have been invented to solve the stability problem. For example, common quadratic Lyapunov function method [8,9] and switched quadratic Lyapunov function method [25,26] were proposed to deal with the stability problem for switched systems under arbitrary switching. A method of multiple Lyapunov functions [10–12] was developed for switched systems in which a concept of fixed switching duration, so-called dwell time, was introduced. It was shown in [13] that, when subsystems are all stable and the dwell time is sufficiently large, the switched system is exponentially stable for any switching law. Later, with an extension of the concept of dwell time to average dwell time, it has been shown that the switched system is still exponentially stable for sufficiently large average dwell time [14]. Aside from the case where all the subsystems are stable, the stability problem for large dwell time is also discussed for switched systems where subsystems could be either stable or unstable [15].

In addition, parallel works on stability have been done for switched systems when time-delay, an omnipresent phenomenon in real-world systems, is introduced. More concretely, it was shown that large dwell time could still guarantee the stability of systems with time-invariant or time-varying delays when all subsystems are stable [16] or when subsystems are either stable or unstable [17–19]. The requirement for sufficiently large values of average dwell time also could ensure the stability of time-delayed systems [20].

All the stability results mentioned above depend on a prerequisite of large dwell times or large average dwell times. Switched systems with large dwell times usually are regarded as slow switched systems; while, systems with small dwell times are regarded as fast switched systems. Also, all these stability results focus on the switching instants that are not stochastically but deterministically specified. However, as reported in the literature, fast and random switching usually brings some interesting dynamical behaviors contrary to the common knowledge. For example, this kind of switching is able to stabilize switched systems even though all the subsystems are unstable [21–23].

Thus, a question naturally arises: “Can fast and random switching bring dynamical behaviors different from those already reported in slow and deterministic time-delayed switched systems?” In this paper, we will fully address this question. More specifically, we will numerically and analytically show how fast and random switching is able to hamper the stability of time-delayed switched systems in spite that the corresponding slow switched systems with all stable subsystems are exponentially stable. This instability result just oppositely corresponds to the stability result induced by fast and random switching in systems with all unstable subsystems but without time-delay [23]. In real applications, addressing the problems of stability and instability is essential for understanding the mechanisms of dynamical behaviors in real systems. For instance, the coupled neurons in the brain area of hippocamp are observed to approach stable synchronization, which always triggers some mental disorders, such as epilepsy and Parkinson diseases

[27,28]. Therefore, using appropriate methods, such as fast and random switching, to suppress synchronization or destroy stable synchronization manifold becomes a topic of great physical and/or biological significance.

The organization of the remaining part of this paper is as follows. In Section 2, we provide an illustrative example to show that specific fast and random switching is able to destabilize time-delayed switched systems even with all stable subsystems. Also in this section, we expatiate intuitively on where this instability probably comes. Then, we extend the example to a more general case in Sections 3–5, where we formulate the model, present the main theorem, illustrate the example in Section 2 analytically, and perform the whole proof, respectively. In Section 6, we further extend our theoretical results to the case where fast and random switching induces instability in nonlinear time-delayed switched systems. Finally, we conclude this work by giving some remarks and perspectives.

2. Instability induced by fast and random switching: an example

In this section, we are to provide an example to show that fast and random switching could induce instability in switched systems even though all the subsystems are exponentially stable.

To begin with, we consider a model of 2-dimensional delay differential equations, which is given by $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + Bx(t-\tau)$, with $x = [x_1, x_2]^\top$ and an initial condition $x_0 = \phi \in \mathrm{C}([-\tau, 0], \mathbb{R}^2)$. Specifically, this time-delay model is supposed to switch with probability $\frac{1}{2}$ between the following two subsystems:

- Subsystem-1:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \begin{pmatrix} 5 & 10 \\ -10 & 5 \end{pmatrix} x(t) + \begin{pmatrix} -3 & -5 \\ 5 & -3 \end{pmatrix} x(t-0.1); \tag{1}$$

- Subsystem-2:

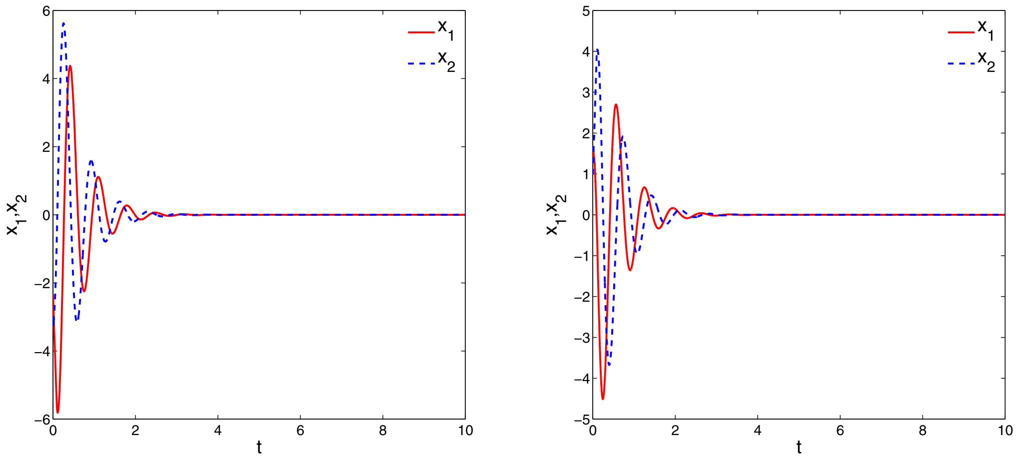
$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \begin{pmatrix} 5 & -10 \\ 10 & 5 \end{pmatrix} x(t) + \begin{pmatrix} -3 & 5 \\ -5 & -3 \end{pmatrix} x(t-0.1). \tag{2}$$

Without switching, each subsystem separately is exponentially stable, as numerically shown in Figs. 1(a) and 1(b). With a larger dwell time $T = 1$ for switching, i.e. slow switching, the randomly switched system above is still stable, as shown in Fig. 2(a). However, with decreasing the dwell time to a smaller value $T = 0.1$, i.e. fast switching, the randomly switched system becomes unstable, as surprisingly shown in Fig. 2(b).

To give an intuitive explanation on what kind of factor results in the instability for the fast and random switching, we first implement the transformation in complex variable: $z = x_1 + x_2\mathrm{i}$ to the switched system. Thus, both subsystems above could be written as:

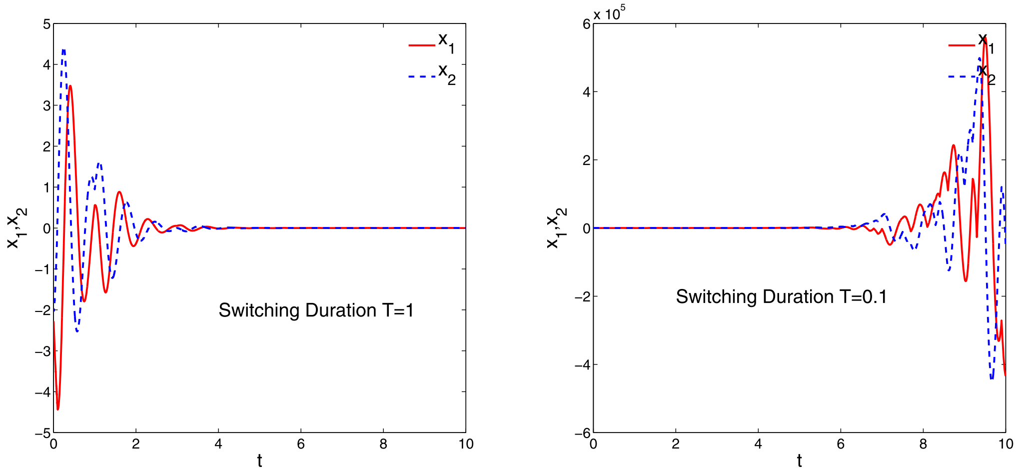
$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = \alpha z(t) + \beta z(t-\tau), \tag{3}$$

where $\alpha = \alpha_1 \triangleq 5 + 10\mathrm{i}$ and $\beta = \beta_1 \triangleq -3 - 5\mathrm{i}$ for Subsystem-1, and $\alpha = \alpha_2 \triangleq 5 - 10\mathrm{i}$ and $\beta = \beta_2 \triangleq -3 + 5\mathrm{i}$ for Subsystem-2. Thus, according to the necessary and sufficient conditions for stability of the complex-valued time-delay equation (3), which was established in [29], each subsystem separately is stable. More precisely, according to [29,30], the stability of Eq. (3) is



(a) (b)

Fig. 1. The convergent trajectories for Subsystem-1 (a), and for Subsystem-2 (b), respectively.



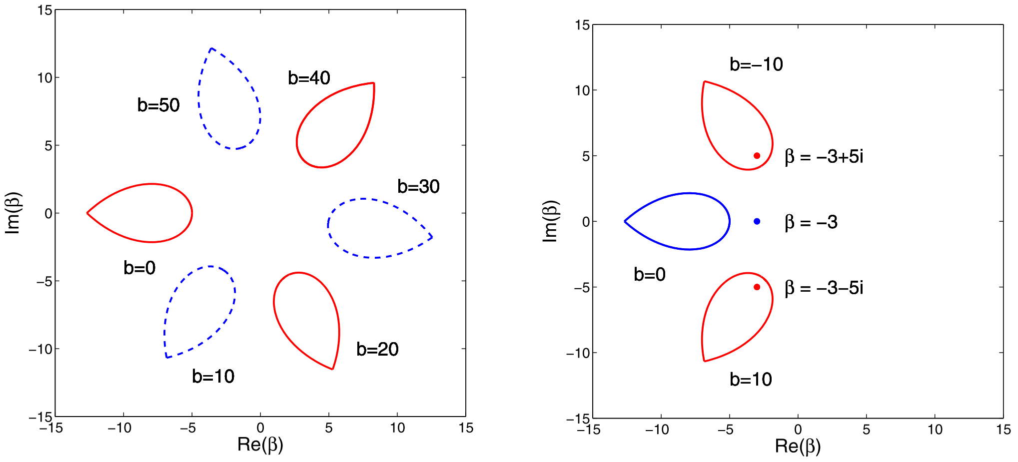
(a) (b)

Fig. 2. The convergent trajectories for the system randomly switching between Subsystem-1 and Subsystem-2 with probability $\frac{1}{2}$ and with a larger dwell time $T = 1$ (a). The divergent trajectories for this switched system when the dwell time decreases to $T = 0.1$ (b).

equivalent to the distribution of all the roots of the transcendental characteristic equation:

$$f_{\alpha,\beta}(\lambda) \triangleq \lambda - \alpha - \beta e^{-\lambda\tau} = 0 \tag{4}$$

being in the left-half complex plane, and hence equivalent to the proper choice of complex-valued β in some leaf-like region for given α and τ. As visually shown in Fig. 3(a), for $\alpha = 5 + bi$ and $\tau = 0.1$, the stability region for parameter β becomes only a leaf-like region in the complex



(a) (b)

Fig. 3. (a) The leaf-like region of parameter β for the stability of Eq. (3), rotating anti-clockwise around the coordinate center with increasing the value of b; (b) The leaf-like regions for stability correspond to the parameter sets in Subsystem-1, Subsystem-2, and the average system (5).

plane, which rotates anti-clockwise around the coordinate center with increasing the value of b. Notice that for $b = \pm 10$, the leaf-like regions for stability are symmetric with respect to the real axis and that $\beta_{1,2} = -3 \mp 5\mathrm{i}$ as defined above are located in these regions, as shown in Fig. 3(b). Therefore, each subsystem is stable, as concluded above.

However, notice that the switched system switches between the two subsystems with probability $\frac{1}{2}$. Then, a direct calculation yields an average system as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} x(t) + \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} x(t-0.1), \tag{5}$$

or written in a complex-valued form: $\dot{z}(t) = \alpha_A z(t) + \beta_A z(t-0.1)$, where $\alpha_A = 5$ and $\beta_A = -3$. Still according to [29], β_A is located outside the leaf-like region for $\alpha = \alpha_A$, as shown in Fig. 3(b). Thus, the characteristic equation (4), $f_{\alpha_A,\beta_A}(\lambda) = 0$, has at least one root whose real part is positive. Therefore, the average system (5) becomes unstable.

It is noted that the instability of the average system (5) is not sufficient but essential for the instability of the switched system above. As shown in Fig. 2(a), the switched system with relatively slow switching is still stable. It loses stability when fast and random switching is implemented. In the following sections, we will give an analytical interpretation on how fast the random switching needs to be if the instability occurs and the average system is unstable.

3. Problem formulation and preliminaries

3.1. Model formulation

We consider the linear time-delay differential equation which reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = S_{\sigma(t)}x(t) + B_{\sigma(t)}x(t-\tau), \tag{6}$$

with an initial condition $x_0 = \phi \in \mathrm{C}([-\tau, 0], \mathbb{R}^m)$, where τ is the delay. Here $x(t) \in \mathbb{R}^m$ is a state variable, m could be regarded as the population dimension in applications. The function $\sigma : \mathbb{R}^+ \to \mathbb{N}$ is the switching rule, which is a right-sided continuous index function, and $S_{\sigma(t)}$ and $B_{\sigma(t)}$ are regarded as a connection matrix at time t. In this paper, we set $\sigma(t) \equiv k$ for all $t \in I_k = [t_{k-1}, t_k)$. Then S_k and B_k are the coefficient matrices of the subsystem, which are set to be randomly chosen throughout this paper. Here $\{t_k\}_{k\in\mathbb{N}}$ is a sequence of switching times. Denote by T_k, the duration of interval I_k, which means $T_k = t_k - t_{k-1}$. T_k are set to be randomly chosen in this paper, too. To our best knowledge, there is almost no result in the literature on discussing time-delayed switched systems with these settings of randomly chosen switching times. In addition, we will consider the following time-delayed switched nonlinear differential equation in this paper:

$$\frac{\mathrm{d}}{\mathrm{dt}}x(t) = f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t-\tau)), \tag{7}$$

where f and g are two nonlinear functions, and $\sigma(t)$ is defined as above.

3.2. Definitions of probability space and stochastic process

In the last subsection, we have supposed the switching rules for the switched system. Here, we need to specify the accurate definition of the corresponding probability space and stochastic process. First, let $T_1, T_2, \ldots$, the durations of the switching intervals, be independent and identically distributed random variables on a probability space $(\Omega^1, \mathscr{F}^1, \mathbb{P}^1)$. Denote $\mathscr{F}_k^1 = \sigma\{T_1, \ldots, T_k\}$, where $\sigma(\mathscr{A})$ stands for the σ-algebra generated by a set family $\mathscr{A}$. Then $\{T_k\}_{k\in\mathbb{N}}$ can be regarded as a stochastic process on the space $(\Omega^1, \mathscr{F}^1, \mathbb{P}^1)$, as well as $\{t_k\}_{k\in\mathbb{N}}$, with respect to the natural filtration $\{\mathscr{F}_k^1\}_{k\in\mathbb{N}}$. Second, we suppose the subsystem in the k-th time interval to be randomly chosen, which means S_k and B_k are random matrices defined on a probability space $(\Omega^2, \mathscr{F}^2, \mathbb{P}^2)$. Moreover, all the random matrices are mutually independent and identically distributed. Here the independence is only with respect to different k, which means we do not require the elements of the matrices to be independent with each other. And denote $\mathscr{F}_k^2 = \sigma\{S_1, \ldots, S_k, B_1, \ldots, B_k\}$. Then $\{(S_k, B_k)\}_{k\in\mathbb{N}}$ can be regarded as a stochastic process on $(\Omega^2, \mathscr{F}^2, \mathbb{P}^2)$ with respect to its natural filtration $\{\mathscr{F}_k^2\}_{k\in\mathbb{N}}$.

Finally, denote, respectively, $\Omega = \Omega^1 \times \Omega^2$, $\mathscr{F} = \mathscr{F}^1 \times \mathscr{F}^2$, $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^2$, and $\mathscr{F}_k = \mathscr{F}_k^1 \times \mathscr{F}_k^2 = \sigma(\mathscr{F}_k^1 \otimes \mathscr{F}_k^2)$. And let $\tilde{T}_k(\omega^1, \omega^2) = T_k(\omega^1)$, $\tilde{S}_k(\omega^1, \omega^2) = S_k(\omega^2)$ and $\tilde{B}_k(\omega^1, \omega^2) = B_k(\omega^2)$, for all $\omega = (\omega^1, \omega^2) \in \Omega$. For simplicity, in the following discussion, we still use T_k, S_k and B_k instead of $\tilde{T}_k$, $\tilde{S}_k$ and $\tilde{B}_k$, respectively. Thus $\{(T_k, S_k, B_k)\}_{k\in\mathbb{N}}$ as a whole can be regarded as a stochastic process on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with respect to the filtration $\{\mathscr{F}_k\}_{k\in\mathbb{N}}$. Hence the state variable can be written as $x(t) = x(t, \omega)$, and we get the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, which is the underlying space we study throughout this paper. Also for the nonlinear case, we define the probability space in an analogous way.

3.3. Notation

In this paper, for $x \in \mathbb{R}^m$, we use $\|x\|$ to denote the Euclidean norm as usual. For $\phi \in \mathrm{C}([-\tau, 0], \mathbb{R}^m)$, we use $\|\phi\|_{\mathrm{C}}$ to denote the usual norm on this continuous function space, which is given by $\|\phi\|_{\mathrm{C}} = \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\|$.

Since we have different probability spaces defined above, we need to use different notation for expectations. Denote by $\mathbb{E}^1$ the expectation on Ω^1, by $\mathbb{E}^2$ the expectation on Ω^2, and by $\mathbb{E}$ the expectation on Ω. However, according to the Fubini theorem for a nonnegative or integrable random variable ξ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, we have $\mathbb{E}[\xi] = \mathbb{E}^1\left[\mathbb{E}^2[\xi]\right] = \mathbb{E}^2\left[\mathbb{E}^1[\xi]\right]$.

3.4. Elementary definitions and lemmas

In this paper, we focus on the instability of switched systems, so we introduce its definition as follows.

Definition 3.1 *(Instability in the mean square sense).* The switched system (6) is said to be unstable in the mean square sense if, for given $\epsilon_0 > 0$ and any $\delta > 0$, there exists x_0 with $\|x_0\|_C < \delta$ such that, for $x(t)$ starting from x_0, we have $\mathbb{E}[\|x(t')\|^2] \geq \epsilon_0$ for some $t' > 0$.

Clearly, the solution of the switched system which satisfies $\lim_{t\to+\infty} \mathbb{E}[\|x(t)\|^2] = +\infty$ is unstable in the mean square sense.

Also, the following definitions for a stochastic process will be used in this paper.

Definition 3.2 *(Sub-martingale).* [31] A scalar stochastic process $\{\xi_k\}_{k\in\mathbb{N}}$ is said to be a sub-martingale, with respect to its natural filtration $\{\mathscr{F}_k\}_{k\in\mathbb{N}}$, provided that ξ_k is integrable and the conditional expectation inequality $\mathbb{E}\left[\xi_k|\mathscr{F}_{k-1}\right] \geq \xi_{k-1}$ $\mathbb{P}$-a.s. is satisfied for all $k \in \mathbb{N}$.

Definition 3.3 *(Stopping time and corresponding σ-algebra).* [31] A random variable $\kappa = \kappa(\omega)$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, that takes the values in $\mathbb{N}$, is called a stopping time with respect to the σ-algebra filtration $\{\mathscr{F}_k\}_{k\in\mathbb{N}}$, if the random variable $\mathbf{1}_{\kappa=k}(\omega)$ is $\mathscr{F}_k$-measurable. And the σ-algebra, corresponding to κ, is defined by

$$\mathscr{F}_\kappa = \left\{ A \in \mathscr{F} \,\middle|\, A \cap \{\kappa \leq k\} \in \mathscr{F}_k \right\}.$$

Furthermore, we say that a stopping time κ is finite, if $\mathbb{P}\{\kappa < \infty\} = 1$.

In this paper, the lemmas as introduced in the following will be useful.

Lemma 3.4. *[32] The matrix P is stable, which means all its eigenvalues have negative real parts, if and only if there exists a positive definite matrix Q such that*

$$\begin{cases} a\|x\|^2 \leq x^{\mathrm{T}} Q x \leq b\|x\|^2, \\ x^{\mathrm{T}}(QP + P^{\mathrm{T}} Q)x \leq -c\|x\|^2 \end{cases}$$

hold for any $x \in \mathbb{R}^m$. Here a, b and c are positive constants.

Lemma 3.5 *(Gronwall Inequality). [32] Let β and u be real-valued functions on the interval $I = [a, b]$. Assume that β and u are continuous, and α is a constant. If β is non-negative and if u satisfies the integrable inequality*

$$u(t) \leq \alpha + \int_a^t \beta(s)u(s)\mathrm{d}s, \quad \forall t \in I,$$

then we get

$$u(t) \leq \alpha \mathrm{e}^{\int_a^t \beta(s)\mathrm{d}s}, \quad \forall t \in I.$$

Lemma 3.6 *(Second Borel–Cantelli Lemma). [31] Suppose that the events A_n are independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = +\infty$. Then the probability that infinitely many of them occur is 1, that is,*

$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 1.$$

Lemma 3.7. *[31] Suppose that κ_1, κ_2 are two stopping times on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with respect to the natural filtration $\{\mathscr{F}_k\}_{k\in\mathbb{N}}$, and that $\mathscr{F}_{\kappa_1}$, $\mathscr{F}_{\kappa_2}$ are corresponding σ-algebras, respectively. If $\kappa_1 \leq \kappa_2$ $\mathbb{P}$-a.s., then we have $\mathscr{F}_{\kappa_1} \subset \mathscr{F}_{\kappa_2}$.*

Lemma 3.8 *(Doob's Optimal Stopping Theorem). [31] Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sub-martingale on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with respect to the natural filtration $\{\mathscr{F}_k\}_{k\in\mathbb{N}}$. And κ_1 and κ_2 are two finite stopping times, for which*

$$\begin{cases} \mathbb{E}[|\xi_{\kappa_i}|] < +\infty, \\ \lim\limits_{n\to\infty} \mathbb{E}\left[\xi_n; \kappa_i > n\right] = 0 \end{cases}$$

hold for $i = 1, 2$. If $\kappa_1 \leq \kappa_2$ holds $\mathbb{P}$-a.s., then $\mathbb{E}[X_{\kappa_2}|\mathscr{F}_{\kappa_1}] \geq X_{\kappa_1}$ $\mathbb{P}$-a.s. Furthermore, we have $\mathbb{E}[X_{\kappa_2}] \geq \mathbb{E}[X_{\kappa_1}]$.

4. Instability in time-delayed switched linear systems: the main theorem

In this section, we are to establish conditions under which the random and fast switching can destabilize the time-delayed switched linear equation (6).

First, we present an argument on how to extend the interval of existence for any solution of Eq. (6) to the entire positive real line almost surely.

Lemma 4.1. *Suppose that $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of random variables, which are independent and identically distributed. Furthermore, suppose that the identical distribution is nonnegative and satisfies $\mathbb{P}^1 \left\{\omega^1 \in \Omega^1 \,\middle|\, \xi(\omega^1) > 0\right\} > 0$ where ξ is the random variable corresponding to the distribution. Then we have*

$$\mathbb{P}\left\{\omega = (\omega^1, \omega^2) \in \Omega \,\middle|\, \lim_{k\to\infty} t_k(\omega) = \lim_{k\to\infty} \sum_{i=1}^{k} T_i(\omega) = +\infty\right\} = 1. \tag{8}$$

Proof. Denote, respectively, $D_k(c) = \left\{\omega^1 \in \Omega^1 \,\middle|\, T_k(\omega^1) > c\right\}$ and $P_k(c) = \mathbb{P}^1(D_k(c))$. It follows from the assumption that for a positive constant $c > 0$, there exists a constant $\gamma > 0$ such that $P_k(c) \geq \gamma$ uniformly for all k. Thus, $\sum\limits_{k=1}^{\infty} P_k(c) = +\infty$. Now by the virtue of the independence of T_k and using the second Borel–Cantelli Lemma 3.6, we obtain $\mathbb{P}^1\left(\limsup\limits_{k\to\infty} D_k(c)\right) = 1$, so that

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} \sum_{i=1}^{k} T_i = +\infty \quad \mathbb{P}^1\text{-a.s.},$$

which, together with the definitions of the probability space in Subsection 3.2, implies a completion of the proof of (8). $\square$

Furthermore, we make the following assumption, which will be adopted in the following argument as well as in our main conclusion.

Assumption 4.2. Suppose that the random matrices S_k and B_k are bounded, that is, there exist positive constants M_1 and M_2 such that $\|S_k\| \leq M_1$ and $\|B_k\| \leq M_2$ for any $k \in \mathbb{N}$. Also suppose that the random variables T_k are bounded, that is, for some T, $T_k \leq T$ for all $k \in \mathbb{N}$.

Denote $F(t,\phi) = S_{\sigma(t)}\phi(0) + B_{\sigma(t)}\phi(-\tau)$. Then, from Assumption 4.2, we have

$$\|F(t,\phi)\| \leq \|S_{\sigma(t)}\| \cdot \|\phi(0)\| + \|B_{\sigma(t)}\| \cdot \|\phi(-\tau)\| \leq (M_1 + M_2)\|\phi\|_{\mathrm{C}},$$

which means F is a completely continuous function on the space $\mathbb{R}^+ \times \mathrm{C}([-\tau, 0], \mathbb{R}^m)$. Notice that F is a linear functional. Hence F is globally Lipschitz continuous with a global Lipschitz constant $L = M_1 + M_2$. This implies the existence and uniqueness of the solution of Eq. (6) on every interval $[t_{k-1}, t_k]$. Also, according to Lemma 4.1, $\lim\limits_{k\to\infty} t_k = +\infty$ $\mathbb{P}$-a.s., which means for any $t \in \mathbb{R}^+$, there are only finite such intervals $[t_{k-1}, t_k]$ on the time interval $[0, t]$ $\mathbb{P}$-a.s. Therefore, the solution exists uniquely on $[0, t]$ for any $t \in \mathbb{R}^+$ $\mathbb{P}$-a.s., which implies the existence and uniqueness of the solution of Eq. (6) on the entire positive real line $\mathbb{P}$-a.s.

In order to present the main theorem, we need the following assumptions and conclusion in the remark.

Assumption 4.3. Suppose that the expectation matrix of S_k, denoted by $\bar{S}$, is totally unstable, that is, all eigenvalues of matrix $\bar{S}$ have positive real parts.

Remark 4.4. According to Lemma 3.4, there exists a positive definite matrix Q of size $m \times m$, and positive numbers a, b, and c such that

$$\begin{cases} a\|x\|^2 \leq x^{\mathrm{T}} Q x \leq b\|x\|^2, \\ x^{\mathrm{T}}\left(Q\bar{S} + \bar{S}^{\mathrm{T}} Q\right) x \geq c\|x\|^2 \end{cases}$$

for any $x \in \mathbb{R}^m$.

Assumption 4.5. We suppose that the expectation matrix of B_k, denoted by $\bar{B}$, satisfies

$$\|Q\bar{B}\| \le \frac{c}{2} - \delta < \frac{c}{2},$$

where matrix Q is specified in Remark 4.4 and $\delta > 0$ is a positive constant.

Now, with the almost sure existence and uniqueness of the solution on the positive real line, we are in a position to present the main theorem on the instability induced by fast and random switching.

Theorem 4.6. *Suppose that for Eq. (6), Assumptions 4.2, 4.3 and 4.5 are all satisfied. Also suppose $V_0 = V(x_0) > 0$, where x_0 is the initial value for Eq. (6) and $V(\phi)$ is a functional satisfying:*

$$V(\phi) = \phi^{\mathrm{T}}(0)Q\phi(0) - \int_{-\tau}^{0} \phi^{\mathrm{T}}(s)P\phi(s)\mathrm{d}s. \tag{9}$$

Here, $\phi \in \mathrm{C}([-\tau, 0], \mathbb{R}^m)$, $P = \frac{1}{2}\left(Q\bar{S} + \bar{S}^{\mathrm{T}}Q\right)$, and Q is a matrix specified in Remark 4.4. If T, the upper bound of T_k, is sufficiently small such that both d_1 and d_2, specified later in (16) and dependent on T, are positive, then

$$\lim_{t\to+\infty} \mathbb{E}\left[\|x(t)\|^2\right] = +\infty,$$

which implies that Eq. (6) is unstable in the mean square sense.

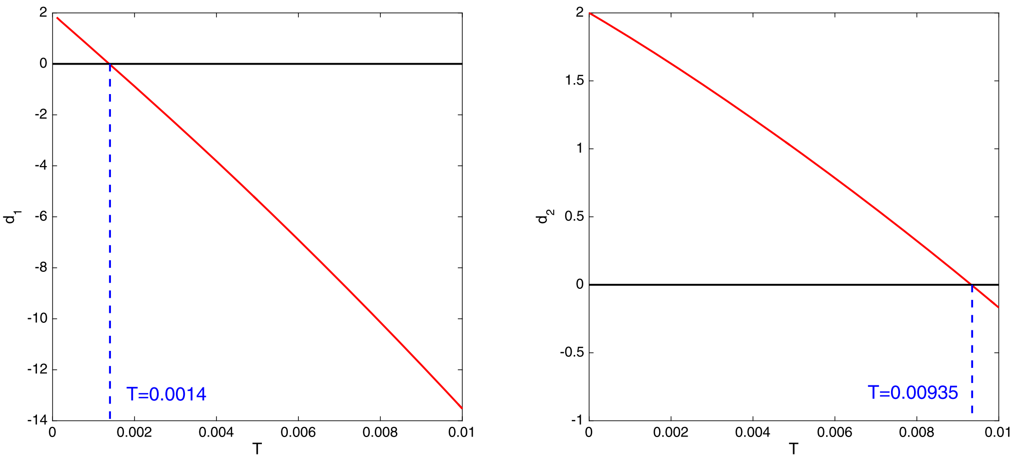
To illustrate the usefulness of the main theorem above, we continue to consider the example provided in Section 2, where the time-delayed switched system switches with a probability $\frac{1}{2}$ between Subsystem-1 and Subsystem-2. Also we consider the corresponding average system (5).

In light of the assumptions in Theorem 4.6 and Remark 4.4, we set $Q = \mathrm{I}$, so that $a = b = 5$, $c = 10$, and $\delta = 2$. Correspondingly, set $P = 5\mathrm{I}$. Hence the functional in (9) becomes $V(\phi) = \|\phi(0)\|^2 - 5\int_{-0.1}^{0} \|\phi(s)\|^2\mathrm{d}s$.

Now, according to Theorem 4.6, the switched system is unstable in the mean square sense, if the upper bound T for the switching duration satisfies $d_{1,2} > 0$, where $d_{1,2}$, according to (16), become

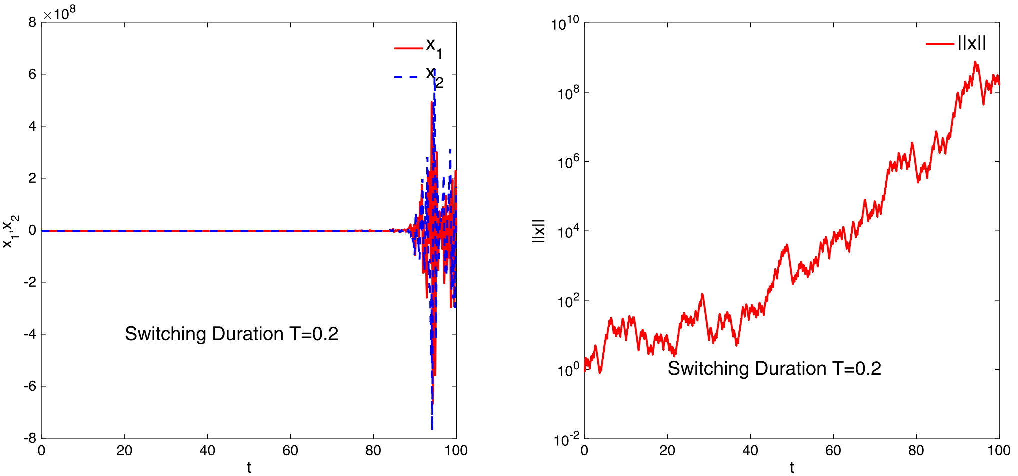
$$\begin{aligned}
d_1 &= 2 - 5T^{\frac{1}{2}} - \frac{2}{T}\left(\mathrm{e}^{5\sqrt{5}T} - 1\right)^2\left(5T^{\frac{1}{2}} + 5T\right) - \frac{5}{T}\left(\mathrm{e}^{10\sqrt{5}T} - 1 - 10\sqrt{5}T\right) > 0,\\
d_2 &= 2 - \mathrm{e}^{10\sqrt{5}T}68T\left(5T^{\frac{1}{2}} + 5T\right) - 170T > 0.
\end{aligned}$$

As shown in Figs. 4(a) and 4(b), the smallest critical point that satisfies $d_{1,2} = 0$ corresponds to the largest value of the upper bound T. Numerical computation yields $T < 0.0014$, which means that we need a very fast random switching to destabilize the time-delayed system even though the subsystems are both stable.



(a) (b)

Fig. 4. The plots of d_1 (a) and d_2 (b), respectively, with respect to the upper bound T for the random switching duration.



(a) (b)

Fig. 5. Instability in the switched time-delay system with an upper bound $T = 0.2$ for random switching duration. (a) The trajectories of the unstable solution. (b) The norm of the unstable solution.

Clearly, for getting instability, Theorem 4.6 establishes only a sufficient criterion for the upper bound of the switching duration. The largest T satisfying $d_{1,2} > 0$ is always smaller than the largest T that is computed directly from the instability appearing in a concrete time-delayed switched system, which still could be illustrated by the example above. As a matter of fact, numerical simulations shown in Figs. 5(a) and 5(b) manifest that $T \leq 0.2$ is sufficient for ensuring the appearance of instability in the example above.

5. Proof of the main theorem

In this section, we give the detailed proof of Theorem 4.6. To this end, we first establish the following lemmas on the estimations of the solutions of Eq. (6).

Lemma 5.1. *Suppose that Assumption 4.2 is satisfied. Then the trajectory $x(t)$ of Eq. (6) is uniformly bounded on any bounded interval $[0, t]$, that is,*

$$\|x(t)\| \le \mathrm{e}^{M_1(t+\tau)}(1 + M_2\tau)^{\frac{t+\tau}{\tau}} \|x_0\|_{\mathrm{C}}, \tag{10}$$

where x_0 is the initial value.

Proof. First, consider the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = S_{\sigma(t)} x(t),$$

whose fundamental solution matrix is

$$\Phi(t,s) = \mathrm{e}^{S_{i+j}(t - t_{i+j})} \mathrm{e}^{S_{i+j-1}(t_{i+j} - t_{i+j-1})} \dots \mathrm{e}^{S_{i+1}(t_{i+2} - t_{i+1})} \mathrm{e}^{S_i(t_{i+1} - s)},$$

where $s \in [t_i, t_{i+1})$ and $t \in [t_{i+j}, t_{i+j+1})$. Furthermore, with the assumption $\|S_k\| \le M_1$, we have an estimation:

$$\|\Phi(t,s)\| \le \mathrm{e}^{M_1(t-s)}.$$

Then using the variation of constants formula, we get

$$x(t) = \Phi(t,s)x(s) + \int_s^t \Phi(t,\theta) B_k x(\theta - \tau) \mathrm{d}\theta \tag{11}$$

for any $t > s$. Further estimation gives:

$$\begin{aligned} \|x(t)\| &\le \mathrm{e}^{M_1(t-s)}\|x(s)\| + \int_s^t \mathrm{e}^{M_1(t-\theta)}\|B_k\| \cdot \|x(\theta-\tau)\| \mathrm{d}\theta \\ &\le \mathrm{e}^{M_1(t-s)}\|x(s)\| + \mathrm{e}^{M_1(t-s)} M_2(t-s) \sup_{\theta\in[s,t]} \|x(\theta-\tau)\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|x(t+s)\| &\le \mathrm{e}^{M_1 s}\|x(t)\| + \mathrm{e}^{M_1 s} M_2 s \sup_{\theta\in[t,t+s]} \|x(\theta-\tau)\| \\ &\le \mathrm{e}^{M_1\tau}(1 + M_2\tau) \sup_{\theta\in[t,t+\tau]} \|x(\theta-\tau)\| \\ &= \mathrm{e}^{M_1\tau}(1 + M_2\tau)\|x_t\|_{\mathrm{C}}, \end{aligned}$$

for any $t \in \mathbb{R}^+$ and $s \in [0, \tau]$. Hence we get

$$\|x_{t+\tau}\|_{\mathrm{C}} = \sup_{s\in[0,\tau]} \|x(t+s)\| \le \mathrm{e}^{M_1\tau}(1 + M_2\tau)\|x_t\|_{\mathrm{C}}$$

for any $t \in \mathbb{R}^+$. Accordingly, by induction on k, we get

$$\|x_{k\tau}\|_{\mathrm{C}} \le \mathrm{e}^{kM_1\tau}(1 + M_2\tau)^k\|x_0\|_{\mathrm{C}},$$

which yields:

$$\|x(t)\| \le \mathrm{e}^{kM_1\tau}(1 + M_2\tau)^k\|x_0\|_{\mathrm{C}} \le \mathrm{e}^{M_1(t+\tau)}(1 + M_2\tau)^{\frac{t+\tau}{\tau}}\|x_0\|_{\mathrm{C}}$$

for all $t \in \mathbb{R}^+$. This completes the proof. □

For simplicity of expression, denote $x_k = x(t_k)$ and $V_k = V(x_{t_k})$. Suppose that T is sufficiently small such that $T \le \tau$. Then, we have the following estimation.

Lemma 5.2. *For the solution $x(t)$ of Eq. (6), we have*

$$\|x(t) - x_{k-1}\|^2 \le c_1\|x_{k-1}\|^2 + c_2\int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2\mathrm{d}s \tag{12}$$

for any $t \in [t_{k-1}, t_k)$, where

$$\begin{cases} c_1 = c_1(\|S_k\|, T_k) \triangleq 2\left(\mathrm{e}^{\|S_k\|T_k} - 1\right)^2, \\ c_2 = c_2(\|S_k\|, \|B_k\|, T_k) \triangleq 2\mathrm{e}^{2\|S_k\|T_k}\|B_k\|^2T_k. \end{cases}$$

Proof. By using the variation of constants formula, we get

$$x(t) = \mathrm{e}^{S_k(t-t_{k-1})}x_{k-1} + \int_{t_{k-1}}^{t} \mathrm{e}^{S_k(t-s)}B_kx(s-\tau)\mathrm{d}s \tag{13}$$

for any $t \in [t_{k-1}, t_k]$. Then,

$$\begin{aligned} \|x(t) - x_{k-1}\| &\le \left(\mathrm{e}^{\|S_k\|T_k} - 1\right)\|x_{k-1}\| + \mathrm{e}^{\|S_k\|T_k}\|B_k\|\int_{t_{k-1}}^{t_k} \|x(s-\tau)\|\mathrm{d}s \\ &\le \left(\mathrm{e}^{\|S_k\|T_k} - 1\right)\|x_{k-1}\| + \mathrm{e}^{\|S_k\|T_k}\|B_k\|T_k^{\frac{1}{2}}\left(\int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2\mathrm{d}s\right)^{\frac{1}{2}}, \end{aligned}$$

where the second inequality is due to the Cauchy–Schwarz inequality. Therefore, we have

$$\|x(t)-x_{k-1}\|^2 \le 2\left(\mathrm{e}^{\|S_k\|T_k}-1\right)^2\|x_{k-1}\|^2+2\mathrm{e}^{2\|S_k\|T_k}\|B_k\|^2T_k\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s,$$

which verifies the estimation given in the lemma. □

Next, we need to prove the measurableness of the processes x_k and V_k.

Lemma 5.3. *Suppose $x(t)$ is a solution of Eq. (6). Then, both sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{V_k\}_{k\in\mathbb{N}}$ along the trajectory of the solution $x(t)$ can be regarded as a stochastic process on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ with respect to the natural filtration $\{\mathscr{F}_k\}_{k\in\mathbb{N}}$.*

Proof. Suppose that $x(t)$ is a $\mathscr{F}_{k-1}$-measurable variable for any $t\in[0,t_{k-1}]$. Then, it is also $\mathscr{F}_k$-measurable. Together with the variation of constants formula (13) and the $\mathscr{F}_k$-measurableness of B_k, S_k and $x(t)$ on $[0,t_{k-1}]$, $x(t)$ on $[t_{k-1},t_k]$ is $\mathscr{F}_k$-measurable and so is x_k. Hence, by induction on k, it is concluded that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is a stochastic process. Analogously, we can verify that $\{V_k\}_{k\in\mathbb{N}}$ is a stochastic process, too. □

Now, according to Lemmas 5.1 and 5.3, V_k is bounded, which implies that $\{V_k\}_{k\in\mathbb{N}}$ is an integrable stochastic process. Thus we are in a position to give an estimation on V_k by using the tools from the stochastic process in the literature.

Lemma 5.4. *Suppose $x(t)$ is a solution of Eq. (6) and consider the functional $V(\phi)$ specified in (9). Also suppose that Assumptions 4.2, 4.3 and 4.5 are all satisfied. Then, for sufficiently small T, the following inequality*

$$\mathbb{E}\left[V_k|\mathscr{F}_{k-1}\right]\ge V_{k-1}+d_1\cdot\mathbb{E}[T_k]\cdot\|x_{k-1}\|^2 \tag{14}$$

holds for all $k\in\mathbb{N}$, where d_1 is a positive constant specified in (16) in the following proof.

Proof. From the variation of constant formula (13), it follows that

$$\begin{aligned}
V_k-V_{k-1}&=x_k^{\mathrm{T}}Qx_k-x_{k-1}^{\mathrm{T}}Qx_{k-1}-\int_{t_{k-1}}^{t_k}x^{\mathrm{T}}(s)Px(s)\mathrm{d}s+\int_{t_{k-1}}^{t_k}x^{\mathrm{T}}(s-\tau)Px(s-\tau)\mathrm{d}s\\
&=2x_{k-1}^{\mathrm{T}}Q(x_k-x_{k-1})+(x_k^{\mathrm{T}}-x_{k-1}^{\mathrm{T}})Q(x_k-x_{k-1})\\
&\quad-\int_{t_{k-1}}^{t_k}x_{k-1}^{\mathrm{T}}Px_{k-1}\mathrm{d}s-2\int_{t_{k-1}}^{t_k}x_{k-1}^{\mathrm{T}}P(x(s)-x_{k-1})\mathrm{d}s\\
&\quad-\int_{t_{k-1}}^{t_k}(x(s)-x_{k-1}^{\mathrm{T}})P(x(s)-x_{k-1})\mathrm{d}s+\int_{t_{k-1}}^{t_k}x^{\mathrm{T}}(s-\tau)Px(s-\tau)\mathrm{d}s\\
&\ge 2x_{k-1}^{\mathrm{T}}Q(x_k-x_{k-1})-\int_{t_{k-1}}^{t_k}x_{k-1}^{\mathrm{T}}Px_{k-1}\mathrm{d}s+\int_{t_{k-1}}^{t_k}x^{\mathrm{T}}(s-\tau)Px(s-\tau)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&- 2\|P\|T_k \cdot \|x_{k-1}\| \max_{s\in[t_{k-1},t_k]} \|x(s) - x_{k-1}\| - \|P\|T_k \max_{s\in[t_{k-1},t_k]} \|x(s) - x_{k-1}\|^2 \\
\geq\, & 2x_{k-1}^{\mathrm{T}} Q(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} x_{k-1}^{\mathrm{T}} P x_{k-1} \mathrm{d}s + \int_{t_{k-1}}^{t_k} x^{\mathrm{T}}(s-\tau) P x(s-\tau) \mathrm{d}s \\
&- \|P\|T_k^{\frac{3}{2}} \cdot \|x_{k-1}\|^2 - (\|P\|T_k^{\frac{1}{2}} + \|P\|T_k) \max_{s\in[t_{k-1},t_k]} \|x(s) - x_{k-1}\|^2 \\
\geq\, & 2x_{k-1}^{\mathrm{T}} Q(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} x_{k-1}^{\mathrm{T}} P x_{k-1} \mathrm{d}s + \frac{c}{2} \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
&- \left[\|P\|T_k^{\frac{3}{2}} + c_1(\|P\|T_k^{\frac{1}{2}} + \|P\|T_k) \right] \|x_{k-1}\|^2 \\
&- c_2(\|P\|T_k^{\frac{1}{2}} + \|P\|T_k) \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
=\, & 2x_{k-1}^{\mathrm{T}} Q(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} x_{k-1}^{\mathrm{T}} P x_{k-1} \mathrm{d}s + \frac{c}{2} \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
&- c_3\|x_{k-1}\|^2 - c_4 \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{cases}
c_3 = c_3(\|S_k\|, T_k) \triangleq \|P\|T_k^{\frac{3}{2}} + c_1 \cdot \left(\|P\|T_k^{\frac{1}{2}} + \|P\|T_k \right), \\
c_4 = c_4(\|S_k\|, \|B_k\|, T_k) \triangleq c_2 \cdot \left(\|P\|T_k^{\frac{1}{2}} + \|P\|T_k \right).
\end{cases}
$$

For the first two terms after the last equality, we make the following estimation for its lower bound:

$$
\begin{aligned}
& 2x_{k-1}^{\mathrm{T}} Q(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} x_{k-1}^{\mathrm{T}} P x_{k-1} \mathrm{d}s \\
=\, & 2x_{k-1}^{\mathrm{T}} Q \left(S_k T_k + \frac{1}{2} S_k^2 T_k^2 + \ldots \right) x_{k-1} + 2x_{k-1}^{\mathrm{T}} Q \int_{t_{k-1}}^{t_k} \mathrm{e}^{S_k(t-s)} B_k x(s-\tau) \mathrm{d}s \\
&- \frac{1}{2} \int_{t_{k-1}}^{t_k} x_{k-1}^{\mathrm{T}} \left(Q\bar{S} + \bar{S}^{\mathrm{T}} Q \right) x_{k-1} \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= x_{k-1}^{\mathrm{T}}\left(QS_k + S_k^{\mathrm{T}}Q\right)T_k x_{k-1} - \frac{1}{2}x_{k-1}^{\mathrm{T}}\left(Q\bar{S}_k + \bar{S}_k^{\mathrm{T}}Q\right)T_k x_{k-1} \\
&\quad + 2x_{k-1}^{\mathrm{T}}Q\left(\frac{1}{2}S_k^2T_k^2 + \frac{1}{6}S_k^3T_k^3 + \dots\right)x_{k-1} + 2x_{k-1}^{\mathrm{T}}\int_{t_{k-1}}^{t_k} QB_k x(s-\tau)\mathrm{d}s \\
&\quad + 2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k}\left(S_k(t-s) + \frac{1}{2}S_k^2(t-s)^2 + \dots\right)B_k x(s-\tau)\mathrm{d}s \\
&\geq T_k x_{k-1}^{\mathrm{T}}\left(QS_k + S_k^{\mathrm{T}}Q\right)x_{k-1} - \frac{1}{2}x_{k-1}^{\mathrm{T}}\left(Q\bar{S}_k + \bar{S}_k^{\mathrm{T}}Q\right)T_k x_{k-1} \\
&\quad - 2b\left(\mathrm{e}^{\|S_k\|T_k} - 1 - \|S_k\|T_k\right)\|x_{k-1}\|^2 + 2x_{k-1}^{\mathrm{T}}\int_{t_{k-1}}^{t_k} QB_k x(s-\tau)\mathrm{d}s \\
&\quad - 2b\|B_k\|\left(\mathrm{e}^{\|S_k\|T_k} - 1\right)\|x_{k-1}\|\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|\mathrm{d}s \\
&\geq T_k x_{k-1}^{\mathrm{T}}\left(QS_k + S_k^{\mathrm{T}}Q\right)x_{k-1} - \frac{1}{2}x_{k-1}^{\mathrm{T}}\left(Q\bar{S}_k + \bar{S}_k^{\mathrm{T}}Q\right)T_k x_{k-1} \\
&\quad - b\left[2\left(\mathrm{e}^{\|S_k\|T_k} - 1 - \|S_k\|T_k\right) + \left(\mathrm{e}^{\|S_k\|T_k} - 1\right)^2\right]\|x_{k-1}\|^2 \\
&\quad - b\|B_k\|^2\left(\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|\mathrm{d}s\right)^2 + 2x_{k-1}^{\mathrm{T}}\int_{t_{k-1}}^{t_k} QB_k x(s-\tau)\mathrm{d}s \\
&\geq T_k x_{k-1}^{\mathrm{T}}\left(QS_k + S_k^{\mathrm{T}}Q\right)x_{k-1} - \frac{1}{2}x_{k-1}^{\mathrm{T}}\left(Q\bar{S}_k + \bar{S}_k^{\mathrm{T}}Q\right)T_k x_{k-1} \\
&\quad + 2x_{k-1}^{\mathrm{T}}\int_{t_{k-1}}^{t_k} QB_k x(s-\tau)\mathrm{d}s \\
&\quad - b\left(\mathrm{e}^{2\|S_k\|T_k} - 1 - 2\|S_k\|T_k\right)\|x_{k-1}\|^2 - b\|B_k\|^2T_k\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s \\
&= T_k x_{k-1}^{\mathrm{T}}\left(QS_k + S_k^{\mathrm{T}}Q\right)x_{k-1} - \frac{1}{2}x_{k-1}^{\mathrm{T}}\left(Q\bar{S}_k + \bar{S}_k^{\mathrm{T}}Q\right)T_k x_{k-1} \\
&\quad + 2x_{k-1}^{\mathrm{T}}\int_{t_{k-1}}^{t_k} QB_k x(s-\tau)\mathrm{d}s - c_5\|x_{k-1}\|^2 - c_6\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s,
\end{aligned}
$$

where

$$\begin{cases} c_5 = c_5(\|S_k\|, T_k) \triangleq b \cdot \left(\mathrm{e}^{2\|S_k\|T_k} - 1 - 2\|S_k\|T_k\right), \\ c_6 = c_6(\|B_k\|, T_k) \triangleq b \cdot \|B_k\|^2 \cdot T_k. \end{cases}$$

Furthermore, decompose, respectively, S_k and B_k into two parts as follows:

$$S_k = \bar{S} + \hat{S}_k, \quad B_k = \bar{B} + \hat{B}_k,$$

in which

$$\mathbb{E}[\hat{S}_k] = 0_{m\times m}, \quad \mathbb{E}[\hat{B}_k] = 0_{m\times m}.$$

From Remark 4.4 and Assumption 4.5, we have

$$\frac{1}{2}T_k x_{k-1}^{\mathrm{T}}\left(Q\bar{S} + \bar{S}^{\mathrm{T}}Q\right)x_{k-1} \geq \frac{c}{2}T_k \cdot \|x_{k-1}\|^2,$$

and

$$\begin{aligned} 2x_{k-1}^{\mathrm{T}}\int_{t_{k-1}}^{t_k} Q\bar{B}x(s-\tau)\mathrm{d}s &\geq -2\|x_{k-1}\|\int_{t_{k-1}}^{t_k} \|Q\bar{B}\|\cdot\|x(s-\tau)\|\mathrm{d}s \\ &\geq -\left(\frac{c}{2}-\delta\right)\int_{t_{k-1}}^{t_k}\left(\|x_{k-1}\|^2 + \|x(s-\tau)\|^2\right)\mathrm{d}s \\ &= -\left(\frac{c}{2}-\delta\right)T_k\|x_{k-1}\|^2 - \left(\frac{c}{2}-\delta\right)\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s. \end{aligned}$$

Accordingly, we obtain

$$\begin{aligned} T_k x_{k-1}^{\mathrm{T}}\left(QS_k + S_k^{\mathrm{T}}Q\right)x_{k-1} &= T_k x_{k-1}^{\mathrm{T}}\left(Q\bar{S} + \bar{S}^{\mathrm{T}}Q\right)x_{k-1} + T_k x_{k-1}^{\mathrm{T}}\left(Q\hat{S}_k + \hat{S}_k^{\mathrm{T}}Q\right)x_{k-1} \\ &\geq cT_k\cdot\|x_{k-1}\|^2 + T_k x_{k-1}^{\mathrm{T}}\left(Q\hat{S}_k + \hat{S}_k^{\mathrm{T}}Q\right)x_{k-1}, \end{aligned}$$

and

$$\begin{aligned} &2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k} B_k x(s-\tau)\mathrm{d}s = 2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k}\bar{B}x(s-\tau)\mathrm{d}s + 2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k}\hat{B}_k x(s-\tau)\mathrm{d}s \\ &\geq -\left(\frac{c}{2}-\delta\right)T_k\|x_{k-1}\|^2 - \left(\frac{c}{2}-\delta\right)\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s + 2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k}\hat{B}_k x(s-\tau)\mathrm{d}s. \end{aligned}$$

Hence a substitution of these estimations into the estimation obtained above for $V_k - V_{k-1}$ yields:

$$
\begin{aligned}
V_k - V_{k-1} \geq\ & \frac{c}{2} T_k \cdot \|x_{k-1}\|^2 + T_k x_{k-1}^{\mathrm{T}} \left(Q \hat{S}_k + \hat{S}_k^{\mathrm{T}} Q \right) x_{k-1} \\
& - \left(\frac{c}{2} - \delta \right) T_k \|x_{k-1}\|^2 - \left(\frac{c}{2} - \delta \right) \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
& + 2 x_{k-1}^{\mathrm{T}} Q \int_{t_{k-1}}^{t_k} \hat{B}_k x(s-\tau) \mathrm{d}s + \frac{c}{2} \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
& - c_5 \|x_{k-1}\|^2 - c_6 \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
& - c_3 \|x_{k-1}\|^2 - c_4 \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
\geq\ & T_k x_{k-1}^{\mathrm{T}} \left(Q \hat{S}_k + \hat{S}_k^{\mathrm{T}} Q \right) x_{k-1} + 2 x_{k-1}^{\mathrm{T}} Q \int_{t_{k-1}}^{t_k} \hat{B}_k x(s-\tau) \mathrm{d}s \\
& + T_k \left(\delta - \frac{c_3}{T_k} - \frac{c_5}{T_k} \right) \|x_{k-1}\|^2 + (\delta - c_4 - c_6) \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s
\end{aligned}
\tag{15}
$$

Notice that T_k is bounded by T. If T is sufficiently small, then c_i $(i = 1, \cdots, 6)$ have the following estimations:

$$
\begin{cases}
c_1 = O(T_k^2), \quad c_2 = O(T_k), \quad c_3 = O(T_k^{\frac{3}{2}}), \\
c_4 = O(T_k^{\frac{3}{2}}), \quad c_5 = O(T_k^2), \quad c_6 = O(T_k).
\end{cases}
$$

Thus, for sufficiently small T, we have

$$
d_1 \triangleq \inf \left\{ \omega \in \Omega \,\middle|\, \delta - \frac{c_3}{T_k} - \frac{c_5}{T_k} \right\} > 0, \quad d_2 \triangleq \inf \left\{ \omega \in \Omega \,\middle|\, \delta - c_4 - c_6 \right\} > 0,
$$

which, after tedious computations, yield:

$$
\begin{cases}
d_1 = \delta - \|P\| T^{\frac{1}{2}} - \dfrac{2}{T} \left(\mathrm{e}^{M_1 T} - 1 \right)^2 \left(\|P\| T^{\frac{1}{2}} + \|P\| T \right) - \dfrac{b}{T} \left(\mathrm{e}^{2 M_1 T} - 1 - 2 M_1 T \right), \\
d_2 = \delta - 2 \mathrm{e}^{2 M_1 T} M_2^2 T \left(\|P\| T^{\frac{1}{2}} + \|P\| T \right) - b M_2^2 T.
\end{cases}
\tag{16}
$$

Moreover, notice that $\mathbb{E}[\hat{S}_k] = \mathbb{E}[\hat{B}_k] = 0_{m\times m}$. Then, we have

$$\begin{aligned}\mathbb{E}\left[T_k x_{k-1}^{\mathrm{T}}\left(Q\hat{S}_k + \hat{S}_k^{\mathrm{T}} Q\right) x_{k-1}\middle|\mathscr{F}_{k-1}\right] &= x_{k-1}^{\mathrm{T}}\mathbb{E}\left[T_k\left(Q\hat{S}_k + \hat{S}_k^{\mathrm{T}} Q\right)\right] x_{k-1}\\ &= x_{k-1}^{\mathrm{T}}\mathbb{E}^1\left[\mathbb{E}^2\left[T_k\left(Q\hat{S}_k + \hat{S}_k^{\mathrm{T}} Q\right)\right]\right] x_{k-1}\\ &= x_{k-1}^{\mathrm{T}}\mathbb{E}^1\left[T_k\left(Q\mathbb{E}^2\left[\hat{S}_k\right] + \mathbb{E}^2\left[\hat{S}_k^{\mathrm{T}}\right] Q\right)\right] x_{k-1}\\ &= x_{k-1}^{\mathrm{T}}\mathbb{E}^1[T_k 0_{m\times m}]x_{k-1} = 0.\end{aligned} \tag{17}$$

Analogously, we can verify

$$\mathbb{E}\left[2x_{k-1}^{\mathrm{T}} Q\int_{t_{k-1}}^{t_k}\hat{B}_k x(s-\tau)\mathrm{d}s\middle|\mathscr{F}_{k-1}\right] = 0. \tag{18}$$

Consequently, substituting Eqs. (17) and (18) into the estimation (15) and then taking the conditional expectation on both sides of (15) result in the inequality (14), which eventually completes the proof. □

Finally, with all the preparations above, we can perform the proof for the main theorem.

Proof of Theorem 4.6. On the one hand, for $V_{k-1} \ge 0$, from the inequality (14) in Lemma 5.4, it follows that

$$\begin{aligned}\mathbb{E}[V_k|\mathscr{F}_{k-1}] &\ge V_{k-1} + d_1\mathbb{E}[T_k]\|x_{k-1}\|^2\\ &\ge V_{k-1} + \frac{d_1}{b}\mathbb{E}[T_k]x_{k-1}^{\mathrm{T}} Q x_{k-1}\\ &\ge V_{k-1} + \frac{d_1}{b}\mathbb{E}[T_k]V_{k-1} = \left(1+\frac{d_1}{b}\mathbb{E}[T_k]\right)V_{k-1}.\end{aligned}$$

On the other hand, for $V_{k-1} < 0$, we still have

$$\mathbb{E}[V_k|\mathscr{F}_{k-1}] \ge V_{k-1} \ge \left(1+\frac{d_1}{b}\mathbb{E}[T_k]\right)V_{k-1}.$$

Thus, we get

$$\begin{aligned}\mathbb{E}[V_k] = \mathbb{E}\big[\mathbb{E}[V_k|\mathscr{F}_{k-1}]\big] &\ge \left(1+\frac{d_1}{b}\mathbb{E}[T_k]\right)\mathbb{E}[V_{k-1}]\\ \ge\cdots\ge \prod_{i=1}^{k}\left(1+\frac{d_1}{b}\mathbb{E}[T_i]\right)\mathbb{E}[V_0] &= \left(1+\frac{d_1}{b}\mathbb{E}[T_1]\right)^k V_0.\end{aligned}$$

Also we have $\mathbb{E}[V_k|\mathscr{F}_{k-1}] \ge V_{k-1}$, which, according to Definition 3.2, implies that the sequence $\{V_k\}_{k\in\mathbb{N}}$ is a sub-martingale.

Denote $\tilde{V}_k = V(x_{t_k \wedge t})$, where $t_k \wedge t = \min\{t_k, t\}$ for any $t \in \mathbb{R}^+$. Although $\{V(x_t))\}_{t\in\mathbb{R}^+}$ is not a continuous sub-martingale with respect to t, we can prove that $\{\tilde{V}_k\}_{k\in\mathbb{N}}$ is also a sub-martingale sequence, similarly as proving $\{V_k\}_{k\in\mathbb{N}}$ being a sub-martingale. Denote, respectively,

$$\kappa_1 \triangleq \inf\left\{k \,\middle|\, t_k = \sum_{i=1}^{k} T_k \geq t - T\right\}, \quad \kappa_2 \triangleq \inf\left\{k \,\middle|\, t_k = \sum_{i=1}^{k} T_k \geq t\right\}.$$

Clearly, both $\kappa_{1,2}$ are stopping times. From (8) in Lemma 4.1, it follows that $0 \leq \left[\frac{t}{T}\right] - 1 \leq \kappa_1 < \kappa_2 < +\infty$ $\mathbb{P}$-a.s. for $t \geq T$. This means that both κ_1 and κ_2 are finite stopping times. Here, $[\cdot]$ stands for the rounding function.

Now, we are in a position to validate the two conditions used in Doob's Optimal Stopping Theorem (Theorem 3.8). Firstly, according to inequality (5.1), $x(s)$ is bounded by a constant M for all $s \in [-\tau, t]$, so is $V(x_s)$. This implies $\mathbb{E}[|V_{\kappa_i}|] \leq M < +\infty$ for $i = 1, 2$. Secondly, since $\kappa_{1,2}$ are two finite stopping times, we get

$$\lim_{n\to\infty} \mathbb{P}\big(\{\kappa_i > n\}\big) = 0, \quad i = 1, 2.$$

Then, combining with the boundedness of V_n on the set $\{\kappa_i > n\}$, we get

$$\left|\lim_{n\to\infty} \mathbb{E}\big[V_n; \kappa_i > n\big]\right| \leq M \cdot \lim_{n\to\infty} \mathbb{P}\big(\{\kappa_i > n\}\big) = 0$$

for $i = 1, 2$. The same conditions can be verified for the sub-martingale sequence $\left\{\tilde{V}_k\right\}_{k\in\mathbb{N}}$. Hence, by virtue of Doob's Optimal Stopping Theorem, we get

$$\mathbb{E}\left[V(x_t)\right] = \mathbb{E}\left[\tilde{V}_{\kappa_2}\right] \geq \mathbb{E}\left[\tilde{V}_{\kappa_1}\right] = \mathbb{E}[V_{\kappa_1}] \geq \mathbb{E}\left[V_{[\frac{t}{T}]-1}\right] \geq \left(1 + \frac{d_1}{b}\mathbb{E}\left[T_1\right]\right)^{[\frac{t}{T}]-1} V_0,$$

which yields:

$$\mathbb{E}\left[\|x(t)\|^2\right] \geq \frac{1}{b}\mathbb{E}\left[V(x_t)\right] \geq \frac{1}{b}\left(1 + \frac{d_1}{b}\mathbb{E}[T_1]\right)^{[\frac{t}{T}]-1} V_0$$

for any $t \geq T$. Therefore, we obtain $\lim_{t\to+\infty} \mathbb{E}[\|x(t)\|^2] = +\infty$, which completes the entire proof. □

6. Generalization from linear systems to nonlinear systems

In this section, we discuss the nonlinear time-delayed switched system (7). First, we decompose the nonlinear functions f_k and g_k into two parts as follows, respectively:

$$f_k(x) = L_k x + h_k(x), \quad g_k(x) = J_k x + l_k(x), \tag{19}$$

where

$$L_k \triangleq \left.\frac{\partial f_k}{\partial x}\right|_{x=0}, \quad J_k \triangleq \left.\frac{\partial g_k}{\partial x}\right|_{x=0},$$

are the Jacobian matrices of functions f_k and g_k at the point $x = 0$. Akin to the discussions performed above for Eq. (6), we also need some assumptions.

Assumption 6.1. Suppose that f_k and g_k are all globally Lipschitz continuous functions for any $k \in \mathbb{N}$. Also suppose that the random matrices L_k and J_k are bounded by constants M_1 and M_3, respectively, for any $k \in \mathbb{N}$. The random functions h_k and l_k are supposed to be linearly bounded by constants M_2 and M_4, respectively, which means that the inequalities $\|h_k(x)\| \le M_2\|x\|$ and $\|l_k(x)\| \le M_4\|x\|$ hold for any $k \in \mathbb{N}$. The random variables T_k are supposed to be bounded by T for any $k \in \mathbb{N}$.

Denote $G(t, \phi) = f_{\sigma(t)}(\phi(0)) + g_{\sigma(t)}(\phi(-\tau))$. Then, we have

$$\|G(t, \phi)\| \le \|f_{\sigma(t)}\| \cdot \|\phi(0)\| + \|g_{\sigma(t)}\| \cdot \|\phi(-\tau)\| \le (M_1 + M_2 + M_3 + M_4)\|\phi\|_{\mathrm{C}},$$

where the estimation is due to Assumption 6.1. Clearly, G is a completely continuous functional on the space $\mathbb{R}^+ \times \mathrm{C}([-\tau, 0], \mathbb{R}^m)$. Similar to the argument performed for the linear case, this complete continuity, combining with the fact that f_k and g_k are globally Lipschitz continuous, implies the existence and uniqueness of the solution of Eq. (7) on the entire positive real line $\mathbb{R}^+$ $\mathbb{P}$-a.s.

Assumption 6.2. Suppose that the expectation matrix $\bar{L}$ of L_k is totally unstable.

Similar to Remark 4.4, we conclude that there exists a positive definite matrix Q of size $m \times m$, and positive reals a, b, c, such that

$$\begin{cases} a\|x\|^2 \le x^{\mathrm{T}} Q x \le b\|x\|^2, \\ x^{\mathrm{T}} \left(Q\bar{L} + \bar{L}^{\mathrm{T}} Q\right) x \ge c\|x\|^2. \end{cases} \tag{20}$$

Assumption 6.3. We suppose that the functions $\bar{J}$, h_k and l_k satisfy

$$2\|Q\bar{J}\| + 2b\|h_k\| + 2b\|l_k\| \le 2\|Q\bar{J}\| + 2b(M_2 + M_4) \le c - \delta < c, \tag{21}$$

for some positive constant δ. Here, Q is a positive definite matrix specified in (20).

Lemma 6.4. *Suppose that Assumption 6.1 is satisfied. Then, the solution of Eq. (7) is uniformly bounded on any bounded interval* $[0, t]$*, more precisely,*

$$\|x(t)\| \le \|x_0\|_{\mathrm{C}} \cdot \mathrm{e}^{(M_1+M_3+M_4+M_2\mathrm{e}^{M_1 T})t},$$

where x_0 is the initial value.

Proof. With the decomposition of f_k and g_k in (19), Eq. (7) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = L_k x(t) + h_k(x(t)) + J_k x(t-\tau) + l_k(x(t-\tau))$$

for any $t \in [t_{k-1}, t_k)$. Using the variation of constants formula gives

$$x(t) = \mathrm{e}^{L_k(t-t_{k-1})}x_{k-1} + \int_{t_{k-1}}^{t} \mathrm{e}^{L_k(t-s)}(h_k(x(s)) + J_k x(s-\tau) + l_k(x(s-\tau)))\mathrm{d}s. \tag{22}$$

Then we make the following estimations:

$$\begin{aligned}
\|x(t)\| &\le \mathrm{e}^{\|L_k\|(t-t_{k-1})}\|x_{k-1}\| + \int_{t_{k-1}}^{t} \mathrm{e}^{\|L_k\|(t-s)}(\|h_k\| \cdot \|x(s)\| + (\|J_k\| + \|l_k\|)\|x(s-\tau)\|)\mathrm{d}s \\
&\le \mathrm{e}^{M_1(t-t_{k-1})}\left(\|x_{k-1}\| + \int_{t_{k-1}}^{t}(M_3+M_4)\|x(s-\tau)\|\mathrm{d}s + \int_{t_{k-1}}^{t} M_2\|x(s)\|\mathrm{d}s\right) \\
&\le \mathrm{e}^{M_1(t-t_{k-1})}\left((1 + M_3(t-t_{k-1}) + M_4(t-t_{k-1}))\|x_{t_{k-1}}\|_{\mathrm{C}} + \int_{t_{k-1}}^{t} M_2\|x(s)\|\mathrm{d}s\right),
\end{aligned}$$

where $\|h_k\| \triangleq \sup_{x\neq 0}\|h_k(x)\|/\|x\|$ and $\|l_k\| \triangleq \sup_{x\neq 0}\|l_k(x)\|/\|x\|$. Thus, according to the Gronwall Inequality (Lemma 3.5), we get

$$\begin{aligned}
\|x(t)\| &\le (1 + M_3(t-t_{k-1}) + M_4(t-t_{k-1}))\mathrm{e}^{M_1(t-t_{k-1})}\|x_{t_{k-1}}\|_{\mathrm{C}} \cdot \mathrm{e}^{\int_{t_{k-1}}^{t} M_2 \mathrm{e}^{M_1 T_k}\mathrm{d}s} \\
&\le \mathrm{e}^{(M_1+M_3+M_4)(t-t_{k-1})}\|x_{t_{k-1}}\|_{\mathrm{C}} \cdot \mathrm{e}^{M_2\mathrm{e}^{M_1 T}(t-t_{k-1})} \\
&\le \|x_{t_{k-1}}\|_{\mathrm{C}} \cdot \mathrm{e}^{\left(M_1+M_3+M_4+M_2\mathrm{e}^{M_1 T}\right)(t-t_{k-1})},
\end{aligned}$$

which implies that

$$\|x(t)\|_{\mathrm{C}} \le \|x_{t_{k-1}}\|_{\mathrm{C}} \cdot \mathrm{e}^{\left(M_1+M_3+M_4+M_2\mathrm{e}^{M_1 T}\right)(t-t_{k-1})},$$

if $T_k \le T \le \tau$. Consequently, by induction on k, we get

$$\|x(t)\|_{\mathrm{C}} \le \|x_0\|_{\mathrm{C}} \cdot \mathrm{e}^{\left(M_1+M_3+M_4+M_2\mathrm{e}^{M_1 T}\right)t}.$$

This therefore completes the proof. □

Lemma 6.5. *For a solution $x(t)$ of Eq. (7), we have*

$$\|x(t) - x_{k-1}\|^2 \le c_1\|x_{k-1}\|^2 + c_2\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s$$

for any $t \in [t_{k-1}, t_k]$*, where we denote, respectively,*

$$c_1 \triangleq 2\left(\frac{\mathrm{e}^{\|L_k\|T_k} - 1 + \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}{1 - \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}\right)^2, \quad c_2 \triangleq 2\left(\frac{\mathrm{e}^{\|L_k\|T_k}(\|J_k\| + \|l_k\|)}{1 - \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}\right)^2 T_k.$$

Proof. From (22), it follows that

$$\begin{aligned}
&\max_{t\in[t_{k-1},t_k]} \|x(t) - x_{k-1}\| \\
&\le \left(\mathrm{e}^{\|L_k\|T_k} - 1\right)\|x_{k-1}\| + \mathrm{e}^{\|L_k\|T_k}\int_{t_{k-1}}^{t_k}\Big[\|h_k\|\cdot\|x(s)\| + (\|J_k\| + \|l_k\|)\|x(s-\tau)\|\Big]\mathrm{d}s \\
&\le \left(\mathrm{e}^{\|L_k\|T_k} - 1\right)\|x_{k-1}\| + \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k\cdot\left(\|x_{k-1}\| + \max_{t\in[t_{k-1},t_k]}\|x(t) - x_{k-1}\|\right) \\
&\quad + \mathrm{e}^{\|L_k\|T_k}(\|J_k\| + \|l_k\|)\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|\mathrm{d}s \\
&\le \left(\mathrm{e}^{\|L_k\|T_k} - 1 + \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k\right)\|x_{k-1}\| + \mathrm{e}^{\|L_k\|T_k}(\|J_k\| + \|l_k\|)\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|\mathrm{d}s \\
&\quad + \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k\cdot\max_{t\in[t_{k-1},t_k]}\|x(t) - x_{k-1}\|.
\end{aligned}$$

This implies

$$\begin{aligned}
&\max_{t\in[t_{k-1},t_k]} \|x(t) - x_{k-1}\| \\
&\le \frac{\mathrm{e}^{\|L_k\|T_k} - 1 + \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}{1 - \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}\|x_{k-1}\| + \frac{\mathrm{e}^{\|L_k\|T_k}(\|J_k\| + \|l_k\|)}{1 - \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|\mathrm{d}s.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\|x(t) - x_{k-1}\|^2 \le\ & 2\left(\frac{\mathrm{e}^{\|L_k\|T_k} - 1 + \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}{1 - \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}\right)^2\|x_{k-1}\|^2 \\
& + 2\left(\frac{\mathrm{e}^{\|L_k\|T_k}(\|J_k\| + \|l_k\|)}{1 - \mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k}\right)^2 T_k\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s.
\end{aligned}$$

Consequently, this gives the estimation in the lemma. □

Lemma 6.6. *Suppose that Assumptions 6.1, 6.2 and 6.3 are all satisfied. Then, for sufficiently small $T > 0$, the following inequality for the solution of Eq. (7):*

$$\mathbb{E}[V_k|\mathscr{F}_{k-1}] \geq V_{k-1} + d_1\mathbb{E}[T_k]\cdot\|x_{k-1}\|^2$$

holds for all $k \in \mathbb{N}$, where $V_k = V(x_{t_k})$ and the functional V is specified in (9) but with different $P = \alpha\cdot\left(Q\bar{L} + \bar{L}^{\mathrm{T}}Q\right)$, where

$$\alpha = \frac{\|Q\bar{J}\| + bM_4}{2\|Q\bar{J}\| + 2bM_2 + 2bM_4}.$$

Here, $\bar{L}$ is the expectation matrix of L_k, and Q is a positive definite matrix specified in (20). Moreover, d_1 is a positive constant specified in (24) in the following proof.

Proof. Similar to the proof for the linear case in the last section, we consider the functional V specified in this lemma. Then, we have

$$\begin{aligned} V_k - V_{k-1} \geq\; & 2x_{k-1}^{\mathrm{T}}Q(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} x_{k-1}^{\mathrm{T}}Px_{k-1}\mathrm{d}s + \alpha c\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s \\ & - c_3\|x_{k-1}\|^2 - c_4\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s, \end{aligned}$$

where

$$c_3 \triangleq \|P\|T_k^{\frac{3}{2}} + c_1(\|P\|T_k^{\frac{1}{2}} + \|P\|T_k),\quad c_4 \triangleq c_2(\|P\|T_k^{\frac{1}{2}} + \|P\|T_k).$$

Next, estimating on the first two terms in the above estimation after the last inequality symbol gives:

$$\begin{aligned} & 2x_{k-1}^{\mathrm{T}}Q(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} x_{k-1}^{\mathrm{T}}Px_{k-1}\mathrm{d}s \\ &= 2x_{k-1}^{\mathrm{T}}Q\left(L_kT_k + \frac{1}{2}L_k^2T_k^2 + \dots\right)x_{k-1} + 2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k}\mathrm{e}^{L_k(t-s)}J_kx(s-\tau)\mathrm{d}s \\ &\quad + 2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k}\mathrm{e}^{L_k(t-s)}h_k(x(s))\mathrm{d}s + 2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k}\mathrm{e}^{L_k(t-s)}l_k(x(s-\tau))\mathrm{d}s \\ &\quad - \alpha T_k\cdot x_{k-1}^{\mathrm{T}}\left(Q\bar{L} + \bar{L}^{\mathrm{T}}Q\right)x_{k-1} \end{aligned}$$

$$
\begin{aligned}
&\geq x_{k-1}^{\mathrm{T}}\left(QL_k+L_k^{\mathrm{T}}Q\right)x_{k-1}T_k+2x_{k-1}^{\mathrm{T}}Q\int\limits_{t_{k-1}}^{t_k}J_kx(s-\tau)\mathrm{d}s\\
&\quad-2b\left(\mathrm{e}^{\|L_k\|T_k}-1-\|L_k\|T_k\right)\|x_{k-1}\|^2\\
&\quad-2b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|x_{k-1}\|\int\limits_{t_{k-1}}^{t_k}\|J_k\|\cdot\|x(s-\tau)\|\mathrm{d}s\\
&\quad-2b\mathrm{e}^{\|L_k\|T_k}\|x_{k-1}\|\int\limits_{t_{k-1}}^{t_k}\|h_k\|\cdot\|x(s)\|\mathrm{d}s\\
&\quad-2b\mathrm{e}^{\|L_k\|T_k}\|x_{k-1}\|\int\limits_{t_{k-1}}^{t_k}\|l_k\|\cdot\|x(s-\tau)\|\mathrm{d}s\\
&\quad-\alpha T_k\cdot x_{k-1}^{\mathrm{T}}\left(Q\bar{L}+\bar{L}^{\mathrm{T}}Q\right)x_{k-1},
\end{aligned}
$$

in which

$$
\begin{aligned}
&2b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|x_{k-1}\|\int\limits_{t_{k-1}}^{t_k}\|J_k\|\cdot\|x(s-\tau)\|\mathrm{d}s\\
&\leq b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|J_k\|\int\limits_{t_{k-1}}^{t_k}\left(\|x_{k-1}\|^2+\|x(s-\tau)\|^2\right)\mathrm{d}s\\
&\leq b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|J_k\|T_k\cdot\|x_{k-1}\|^2+b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|J_k\|\int\limits_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
&2b\mathrm{e}^{\|L_k\|T_k}\|x_{k-1}\|\int\limits_{t_{k-1}}^{t_k}\|h_k\|\cdot\|x(s)\|\mathrm{d}s\\
&\leq 2b\mathrm{e}^{\|L_k\|T_k}\|h_k\|\cdot\|x_{k-1}\|\int\limits_{t_{k-1}}^{t_k}\left(\left\|x(s)-x_{k-1}\right\|+\|x_{k-1}\|\right)\mathrm{d}s\\
&\leq 2b\mathrm{e}^{\|L_k\|T_k}\|h_k\|\cdot\|x_{k-1}\|\int\limits_{t_{k-1}}^{t_k}\left(\left(1+\sqrt{\frac{c_1}{2}}\right)\|x_{k-1}\|+\sqrt{\frac{c_2}{2T_k}}\int\limits_{t_{k-1}}^{t_k}\|x(\theta-\tau)\|\mathrm{d}\theta\right)\mathrm{d}s\\
&\leq 2b\mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k\left(1+\sqrt{\frac{c_1}{2}}\right)\cdot\|x_{k-1}\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2b\mathrm{e}^{\|L_k\|T_k}\|h_k\|T_k\sqrt{\frac{c_2}{2T_k}}\|x_{k-1}\|\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|\mathrm{d}s \\
&\le b\|h_k\|T_k\left(2+2\left(\mathrm{e}^{\|L_k\|T_k}-1\right)+2\mathrm{e}^{\|L_k\|T_k}\sqrt{\frac{c_1}{2}}+\mathrm{e}^{\|L_k\|T_k}\sqrt{\frac{c_2}{2}}\right)\cdot\|x_{k-1}\|^2 \\
&+b\|h_k\|T_k\mathrm{e}^{\|L_k\|T_k}\sqrt{\frac{c_2}{2}}\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
&2b\mathrm{e}^{\|L_k\|T_k}\|x_{k-1}\|\int_{t_{k-1}}^{t_k}\|l_k\|\cdot\|x(s-\tau)\|\mathrm{d}s \\
&\le b\mathrm{e}^{\|L_k\|T_k}\|l_k\|\int_{t_{k-1}}^{t_k}\left(\|x_{k-1}\|^2+\|x(s-\tau)\|^2\right)\mathrm{d}s \\
&= b\mathrm{e}^{\|L_k\|T_k}\|l_k\|T_k\cdot\|x_{k-1}\|^2+b\mathrm{e}^{\|L_k\|T_k}\|l_k\|\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s.
\end{aligned}
$$

Therefore, we further obtain

$$
\begin{aligned}
&2x_{k-1}^{\mathrm{T}}Q(x_k-x_{k-1})-\int_{t_{k-1}}^{t_k}x_{k-1}^{\mathrm{T}}Px_{k-1}\mathrm{d}s \\
&\ge x_{k-1}^{\mathrm{T}}\left(QL_k+L_k^{\mathrm{T}}Q\right)x_{k-1}T_k+2x_{k-1}^{\mathrm{T}}\int_{t_{k-1}}^{t_k}QJ_kx(s-\tau)\mathrm{d}s \\
&-\left((2b\|h_k\|+b\|l_k\|)T_k+c_5\right)\cdot\|x_{k-1}\|^2-\left(b\|l_k\|+c_6\right)\int_{t_{k-1}}^{t_k}\|x(s-\tau)\|^2\mathrm{d}s \\
&-\alpha T_k\cdot x_{k-1}^{\mathrm{T}}\left(Q\bar{L}+\bar{L}^{\mathrm{T}}Q\right)x_{k-1},
\end{aligned}
$$

where $c_{5,6}$ are denoted by

$$
\begin{aligned}
c_5 &\triangleq 2b\left(\mathrm{e}^{\|L_k\|T_k}-1-\|L_k\|T_k\right)+b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|J_k\|T_k+2b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|h_k\|T_k \\
&+b\mathrm{e}^{\|L_k\|T_k}\left(2\sqrt{\frac{c_1}{2}}+\sqrt{\frac{c_2}{2}}\right)\|h_k\|T_k+b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|l_k\|T_k, \\
c_6 &\triangleq b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|J_k\|+b\mathrm{e}^{\|L_k\|T_k}\sqrt{\frac{c_2}{2}}\cdot\|h_k\|T_k+b\left(\mathrm{e}^{\|L_k\|T_k}-1\right)\|l_k\|.
\end{aligned}
$$

Next, we decompose L_k and J_k into two parts as follows:

$$L_k = \bar{L} + \hat{L}_k, \quad J_k = \bar{J} + \hat{J}_k,$$

where $\mathbb{E}[\hat{L}_k] = 0_{m\times m}$ and $\mathbb{E}[\hat{J}_k] = 0_{m\times m}$. Further computation gives

$$\begin{aligned}
V_k - V_{k-1} \geq{}& T_k x_{k-1}^{\mathrm{T}} \left(Q\hat{L}_k + \hat{L}_k^{\mathrm{T}} Q\right) x_{k-1} + 2x_{k-1}^{\mathrm{T}} Q \int_{t_{k-1}}^{t_k} \hat{J}_k x(s-\tau)\mathrm{d}s \\
&+ (1-\alpha) T_k \cdot x_{k-1}^{\mathrm{T}} \left(Q\bar{L} + \bar{L}^{\mathrm{T}} Q\right) x_{k-1} + \alpha c \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
&- \|Q\bar{J}\| \left(T_k \cdot \|x_{k-1}\|^2 + \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \right) \\
&- \Big((2b\|h_k\| + b\|l_k\|) T_k + c_3 + c_5 \Big) \cdot \|x_{k-1}\|^2 \\
&- \Big(b\|l_k\| + c_4 + c_6 \Big) \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s \\
\geq{}& T_k x_{k-1}^{\mathrm{T}} \left(Q\hat{L}_k + \hat{L}_k^{\mathrm{T}} Q\right) x_{k-1} + 2x_{k-1}^{\mathrm{T}} Q \int_{t_{k-1}}^{t_k} \hat{J}_k x(s-\tau)\mathrm{d}s \\
&+ \left((1-\alpha)c - \|Q\bar{J}\| - 2b\|h_k\| - b\|l_k\| - \frac{1}{T_k}(c_3 + c_5) \right) T_k \cdot \|x_{k-1}\|^2 \\
&+ \left(\alpha c - \|Q\bar{J}\| - b\|l_k\| - (c_4 + c_6) \right) \int_{t_{k-1}}^{t_k} \|x(s-\tau)\|^2 \mathrm{d}s.
\end{aligned} \tag{23}$$

According to Assumption 6.3, we obtain

$$(1-\alpha)c - \|Q\bar{J}\| - 2b\|h_k\| - b\|l_k\| \geq (1-\alpha)\delta$$

and

$$\alpha c - \|Q\bar{J}\| - b\|l_k\| \geq \alpha\delta.$$

Notice that T_k is bounded by T. Then, for sufficiently small T, we have the following estimations:

$$c_1 = O(T_k^2), \quad c_2 = O(T_k), \quad c_3 = O(T_k^{\frac{3}{2}}),$$
$$c_4 = O(T_k^{\frac{3}{2}}), \quad c_5 = O(T_k^{\frac{3}{2}}), \quad c_6 = O(T_k),$$

and we get

$$d_1 \triangleq \inf\left\{\omega\in\Omega \,\middle|\, (1-\alpha)\delta - \frac{c_3}{T_k} - \frac{c_5}{T_k}\right\} > 0,\quad d_2 \triangleq \inf\left\{\omega\in\Omega \,\middle|\, \alpha\delta - c_4 - c_6\right\} > 0.$$

After some computation, we get

$$\begin{cases} d_1 = (1-\alpha)\delta - \|P\|T^{\frac{1}{2}} - \dfrac{2}{T}\left(\dfrac{\mathrm{e}^{M_1T}-1+\mathrm{e}^{M_1T}M_2T}{1-\mathrm{e}^{M_1T}M_2T}\right)^2\left(\|P\|T^{\frac{1}{2}}+\|P\|T\right) \\ \qquad -\dfrac{b}{T}\Bigg[2\mathrm{e}^{M_1T}-2-2M_1T+\left(\mathrm{e}^{M_1T}-1\right)(2M_2T+M_3T+M_4T) \\ \qquad\qquad + \mathrm{e}^{M_1T}\left(2\dfrac{\mathrm{e}^{M_1T}-1+\mathrm{e}^{M_1T}M_2T}{1-\mathrm{e}^{M_1T}M_2T}+\dfrac{\mathrm{e}^{M_1T}(M_3+M_4)}{1-\mathrm{e}^{M_1T}M_2T}T^{\frac{1}{2}}\right)M_2T\Bigg], \\ d_2 = \alpha\delta - 2\left(\dfrac{\mathrm{e}^{M_1T}(M_3+M_4)}{1-\mathrm{e}^{M_1T}M_2T}\right)^2\left(\|P\|T^{\frac{1}{2}}+\|P\|T\right)T \\ \qquad - b\left[\left(\mathrm{e}^{M_1T}-1\right)(M_3+M_4)+\mathrm{e}^{M_1T}\dfrac{\mathrm{e}^{M_1T}(M_3+M_4)}{1-\mathrm{e}^{M_1T}M_2T}M_2T^{\frac{3}{2}}\right] \end{cases} \tag{24}$$

which are more complicated than those specified in (16), due to the fact that we are taking into account the nonlinear case.

Moreover, from $\mathbb{E}[\hat{L}_k] = \mathbb{E}[\hat{J}_k] = 0_{m\times m}$, it follows that

$$\mathbb{E}\left[\frac{1}{2}T_k x_{k-1}^{\mathrm{T}}\left(Q\hat{L}_k+\hat{L}_k^{\mathrm{T}}Q\right)x_{k-1}\,\middle|\,\mathscr{F}_{k-1}\right]=0,\quad \mathbb{E}\left[2x_{k-1}^{\mathrm{T}}Q\int_{t_{k-1}}^{t_k}\hat{B}_k x(s-\tau)\mathrm{d}s\,\middle|\,\mathscr{F}_{k-1}\right]=0,$$

which, together with the estimations in (23), gives

$$\mathbb{E}[V_k|\mathscr{F}_{k-1}] \geq V_{k-1} + d_1\cdot\mathbb{E}[T_k]\cdot\|x_{k-1}\|^2.$$

This completes the entire proof. □

Finally, akin to the argument performed for the linear case, we present the instability result for the nonlinear Eq. (7) without showing the detailed proof.

Theorem 6.7. *Suppose that Assumptions 6.1, 6.2, and 6.3 are all satisfied. Also suppose that $V_0 = V(x_0) > 0$ where V is specified in Lemma 6.6. If T is sufficiently small such that $d_{1,2} > 0$, where $d_{1,2}$ are given in (24), then the solution of the system (7) satisfies*

$$\lim_{t\to+\infty}\mathbb{E}[\|x(t)\|^2] = +\infty,$$

which implies the instability of the system (7) in the mean square sense.

7. Concluding remarks

By using the theory of stochastic process, especially the theory of martingales, we have shown that fast and random switching is able to destabilize time-delayed switched systems even with all stable subsystems. However, without time delay, fast and random switching cannot easily destabilize the switched system, mainly because the eigenvalue distribution of the characteristic equation for high-dimensional time-delayed systems, as discussed in Section 2 as well as in [29], is far more complicated than that for systems without time delay. Also, we have generalized the results for the linear case to the nonlinear one, which makes the results we established in this paper more applicable to intervention between synchronization and desynchronization in complex networks.

It is valuable to point out that the techniques we have developed in this paper could be used to deal with switched systems with various types of time-delays, including switched systems with more than one delay and switched systems with distributed delays. Also it should be pointed out that our discussion in this paper focuses only on the instability defined in the mean square sense. However, showing the instability in a sense of probability one awaits further exploration, because this kind of instability is of more physical significance. All these constitute our present and future research work.

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References

[1] M.R. Jeffrey, S.J. Hogan, The geometry of generic sliding bifurcations, SIAM Rev. 53 (3) (2011) 505–525.
[2] M.R. Jeffrey, Dynamics at a switching intersection: hierarchy, isonomy, and multiple sliding, SIAM J. Appl. Dyn. Syst. 13 (3) (2014) 1082–1105.
[3] M.R. Jeffrey, Hidden dynamics in models of discontinuity and switching, Phys. D 273–274 (15) (2014) 34–45.
[4] M. Guardia, T.M. Seara, M.A. Teixeira, Generic bifurcations of low codimension of planar Filippov systems, J. Differential Equations 250 (2011) 1967–2023.
[5] Y. Chitour, M. Gaye, P. Mason, Geometric and asymptotic properties associated with linear switched systems, J. Differential Equations 259 (2015) 5582–5616.
[6] X. Chen, W. Zhang, Isochronicity of centers in a switching Bautin system, J. Differential Equations 252 (2012) 2877–2899.
[7] Y. Tian, P. Yu, Center conditions in a switching Bautin system, J. Differential Equations 259 (2015) 1203–1226.
[8] R. Shorten, O. Mason, F.O. Cairbre, P. Curran, A unifying framework for the SISO circle criterion and other quadratic stability criteria, Internat. J. Control 77 (1) (2004) 1–8.
[9] R. Shorten, K. Narendra, O. Mason, A result on common quadratic Lyapunov functions, IEEE Trans. Automat. Control 48 (1) (2003) 110–113.
[10] R.A. Decarlo, M.S. Branicky, S. Pettersson, B. Lennartson, Perspectives and results on the stability and stabilizability of hybrid systems, in: Special Issue: Hybrid Systems, Proc. IEEE 88 (2000) 1069–1082.
[11] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag. 19 (5) (1999) 59–70.
[12] A.N. Michel, Recent trends in the stability analysis of hybrid dynamical systems, IEEE Trans. Circuits Syst. I 46 (1) (1999) 120–134.

[13] A.S. Morse, Supervisory control of families of linear set-point controllers – part 1: exact matching, IEEE Trans. Automat. Control 41 (1996) 1413–1431.

[14] J.P. Hespanha, A.S. Morse, Stability of switched systems with average dwell-time, in: Proc. 38th IEEE Conf. Decision Control, 1999, pp. 2655–2660.

[15] G. Zhai, B. Hu, K. Yasuda, A.N. Michel, Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach, Internat. J. Systems Sci. 32 (2001) 1055–1061.

[16] P. Yan, H. Özbay, Stability analysis of switched time-delay systems, SIAM J. Control Optim. 47 (2008) 936–949.

[17] M.S. Alwan, X. Liu, On stability of linear and weakly nonlinear switched systems with time delay, Math. Comput. Modelling 48 (2008) 1150–1157.

[18] X. Sun, J. Zhao, D.J. Hill, Stability and l_2-gain analysis for switched delay systems, Automatica 42 (2006) 1769–1774.

[19] S. Kim, S.A. Campbell, X. Liu, Delay independent stability of linear switching systems with time delay, J. Math. Anal. Appl. 339 (2008) 785–801.

[20] Y. Zhang, X. Liu, X. Shen, Stability of switched systems with time-delay, Nonlinear Anal. Hybrid Syst. 1 (2007) 44–58.

[21] D.J. Stilwell, E.M. Bollt, D.G. Roberson, Sufficient conditions for fast switching synchronization in time-varying network topologies, in: Proc. 2006 American Control Conference, 2006.

[22] W. Wang, D. Nešič, Input-to-state stability and averaging of linear fast switching systems, IEEE Trans. Automat. Control 55 (5) (2010) 1274–1279.

[23] Y. Guo, W. Lin, M.A.F. Sanjuán, The efficiency of a random and fast switch in complex dynamical systems, New J. Phys. 14 (2012) 083022.

[24] Y. Guo, W. Lin, D.W.C. Ho, Discrete-time systems with random switches: from systems stability to networks synchronization, Chaos 3 (2016) 033113.

[25] J. Daafouz, P. Riedinger, C. Iung, Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach, IEEE Trans. Automat. Control 47 (11) (2002) 1883–1887.

[26] L. Fang, H. Lin, P.J. Antsaklis, Stabilization and performance analysis for a class of switched systems, in: Proc. 43rd IEEE Conf. Decision Control, 2004, pp. 3265–3270.

[27] P.A. Tass, A model of desynchronizing deep brain stimulation with a demand-controlled coordinated reset of neural subpopulations, Biol. Cybernet. 89 (2) (2003) 81–88.

[28] P.A. Tass, D. Smirnov, A. Karavaev, U. Barnikol, T. Barnikol, I. Adamchic, C. Hauptmann, N. Pawelcyzk, M. Maarouf, V. Sturm, H.-J. Freund, B. Bezruchko, The causal relationship between subcortical local field potential oscillations and parkinsonian resting tremor, J. Neural Eng. 7 (1) (2010) 016009.

[29] W. Lin, Y. Pu, Y. Guo, J. Kurths, Oscillation suppression and synchronization: frequencies determine the role of control with time delays, Europhys. Lett. 102 (2013) 20003.

[30] J.K. Hale, Theory of Functional Differential Equations, 1st edition, Springer, 1977.

[31] A.N. Shiryaev, Probability, 2nd edition, Springer, 1995.

[32] C. Robinson, Dynamical Systems – Stability, Symbolic Dynamics and Chaos, 2nd edition, CRC Press, 1998.