MAGIC SQUARES AND PYTHAGOREAN NUMBERS

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CRITICISMS AND DISCUSSIONS.

MAGIC SQUARES AND PYTHAGOREAN NUMBERS.

> "I have compiled this discourse, which asks for your consideration and pardon not only because the matter itself is by no means easy to be handled, but also because the doctrines herein contained are somewhat contrary to those held by most of the Platonic philosophers." *Plutarch.*

The fascinating series of articles upon "Magic Squares" by Mr. W. S. Andrews and the interesting "Reflections" upon the same by the Editor in recent numbers of *The Monist* have induced me to make a few comments upon a subject in which I have long been interested,—that of the relationship between magic squares and certain Pythagorean numbers.

The mysterious relationships of numbers have attracted the minds of men in all ages. The many-sided Franklin, whose 200th anniversary the philosophical, scientific, and literary worlds have recently celebrated, used to amuse himself with the construction of magic squares and in his memoirs has given an example of his skill in this direction, by showing a very complicated compound square with the comment that he believes the same to be the most magical magic square yet constructed by any magician. I would therefore attribute the discovery of compound magic squares to Franklin rather than to Professor Schubert as suggested by Mr. Andrews.

That magic squares have had in centuries past a deeper meaning for the minds of men than that of simple mathematical curios we may infer from the celebrated picture by Albert Dürer entitled "Melancolia," engraved in 1514. The symbolism of this engraving has interested to a marked degree almost every observer. The figure of the brooding genius sitting listless and dejected amid her uncompleted labors, the scattered tools, the swaying balance, the flow-

ing sands of the glass, and the magic square of 16 beneath the bell, —these and other details reveal an attitude of mind and a connection of thought, which the great artist never expressed in words, but left for every beholder to interpret for himself.

MELANCHOLY.

The discovery of the arrangement of numbers in the form of magic diagrams was undoubtedly known to the ancient Egyptians and this may have formed part of the knowledge which Pythagoras brought back from his foreign travels. We have no direct evidence

that the Pythagorean philosophers in their studies of the relationship of numbers ever combined them into harmonic figures, yet the supposition that they did so is not at all improbable. Such diagrams and their symbolic meanings may well have formed part of the arcana of the esoteric school of Pythagoras, for similar facts were accounted by ancient writers as constituting a part of the aporrheta of the order and the story is told of an unworthy disciple who revealed the secret of the construction of the dodecahedron inscribed within a sphere, this being a symbol of the universe.

Among the best expositions of the Pythagorean philosophy are sections of the "Timæus" and "Republic" of Plato. These dialogues were written after Plato's return from Magna Græcia, where from contact with Archytas of Tarentum and other philosophers, he imbibed so much of the Italian school that his whole system of philosophy became permeated with Pythagorean ideas. It is even suggested that he incorporated into these dialogues parts of the lost writings of Philolaus, whose works he is known to have purchased. No portions of the dialogues named have been more puzzling to commentators than the vague references to different numbers, such as the number 729, which is chosen to express the difference between the kingly man and the tyrant, or the so-called number of the State in the "Republic," or the harmonic number of the soul in the "Timæus" of which Plutarch said that "it would be an endless toil to recite the contentions and disputes that have from hence arisen among his interpreters." Either our text of these passages is corrupt or Plato is very obscure, throwing out indirect hints which would be intelligible only to those previously informed. Plato states himself in the "Phædrus" that "all writings are to be regarded purely as a means of recollection for him who already knows," and he, therefore, probably wrote more for the benefit of his hearers than for distant posterity.

It is upon the principle of a magic square that I wish to interpret the celebrated passage in the "Republic" referring to the number 729, proceeding from this to a discussion of certain other numbers of peculiar significance in the Pythagorean system. My efforts in this direction are to be regarded as purely fanciful; the same may be said, however, of the majority of other methods of interpretation.

The passage from the "Republic" referred to (Book IX, § 587-8, Jowett's translation) reads as follows:

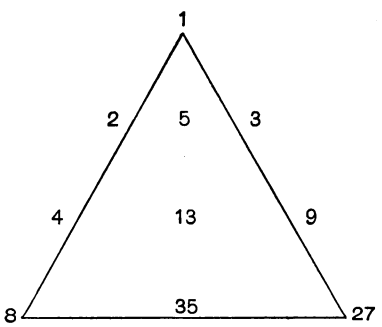
Socrates. "And if a person tells the measure of the interval which separates the king from the tyrant in truth of pleasure, he

will find him, when the multiplication is completed living 729 times more pleasantly, and the tyrant more painfully by this same interval."

Glancon. "What a wonderful calculation."

Socrates. "Yet a true calculation and a number which closely concerns human life, if human life is concerned with days and nights and months and years."

The number 729 is found to be of great importance all through the Pythagorean system. Plutarch states that this was the number belonging to the sun, just as 243 was ascribed to Venus, 81 to Mercury, 27 to the moon, 9 to the earth, and 3 to Antichthon (the earth opposite to ours). These and many similar numbers were derived from one of the progressions of the Tetractys,—1:2::4:8 and 1:3::9:27. The figures of the above proportions were combined by Plato into one series 1, 2, 3, 4, 9, 8, 27. ("Timæus, § 35). Plutarch in his "Procreation of the Soul," which is simply a commentary upon Plato's "Timæus," has represented the numbers in the form of a triangle; the interior numbers, 5, 13, and 35, representing the sums of the opposite pairs, were also of great importance.



The deep significance of the Tetractys in the system of Pythagoras may be inferred from a fragment of an oath contained in the "Golden Verses."

Ναὶ μὰ τὸν ἀμέτερον ψυχᾷ παραδόντα τετρακτὸν
Παγὰν, ἀενάου φύσεως ῥιζώματ' ἔχουσαν.

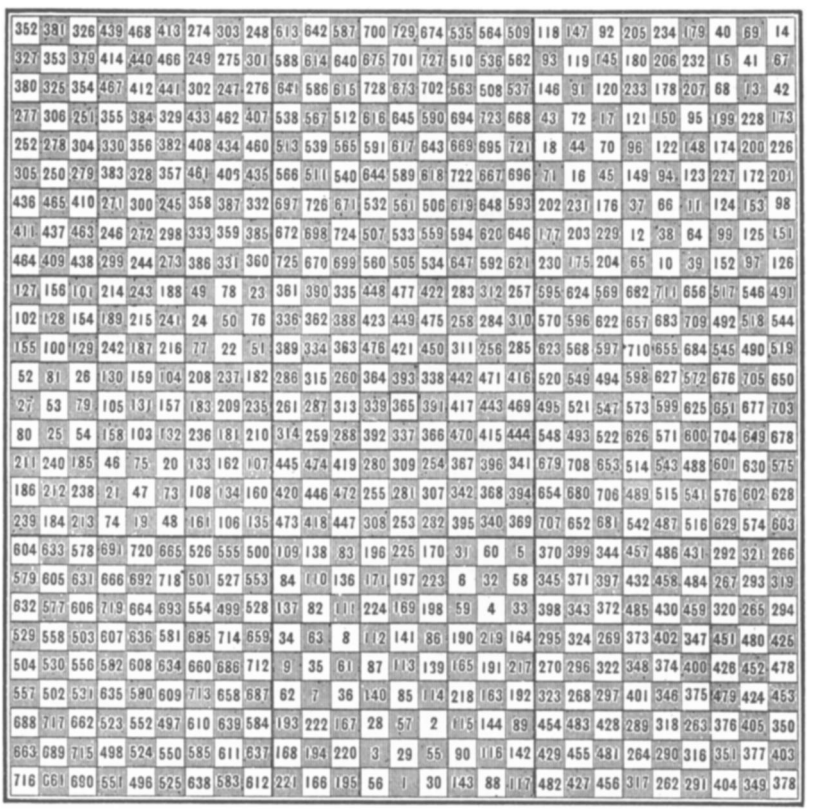
"Yea, by our Tetractys which giveth the soul the fount and source of ever flowing nature!"

Odd numbers were especially favored by the Pythagoreans and of these certain ones such as 3 and its higher powers were considered to have a higher significance than others and in this way, perhaps, arose the distinction between expressible and inexpressible or ineffable numbers (*ἀριθμοὶ ῥητοὶ καὶ ἄῤῥητοι*). Numbers which expressed some astronomical fact also held high places of honor, as may be seen from a statement by Plutarch (*loc. cit.*) in reference to the Tetractys. "Now the final member of the series, which is 27, has this peculiarity, that it is equal to the sum of the preceding numbers (1+2+3+4+9+8); it also represents the periodical num-

ber of days in which the moon completes her monthly course; the Pythagoreans have made it the tone of all their harmonic intervals."

This passage indicates sufficiently the supreme importance of the number *27*.

If we construct a magic square 27×27 upon the plan of a



checker-board—arranging the numbers 1 to 729 first in numerical order, then shifting the 9 largest squares (9×9) into the positions indicated in the familiar 3×3 square, repeating the process with the subdivisions of the 9×9 squares and so on down—we will arrive at the following combination.[1]

It will be noted that we have 365 white squares or days and

[1] This method of constructing compound magic squares is, so far as I know, original with the writer. It bears some resemblance to the method of Schubert (*Monist*, XV, p. 566); the numbers of each square, however, increase in periods of threes instead of by sequence.

364 dark squares or nights—a veritable "checkerboard of nights and days." The number 365, the days of the solar year, very appropriately occupies the centre of the system. The columns, horizontals, and diagonals of the central square 3×3 foot up 1095, or the days of a 3 year period, those of the larger center square 9×9 foot up 3285 the days of a 9 year period, while those of the entire combination 27×27 foot up 9855,[2] the days of a 27 year period,—in other words, periods of years corresponding to the Tetractys 1, 3, 9, 27. We may with safety borrow the language of Plato and say that the above arrangement of numbers "is concerned with days and nights and months and years."

The interpretation of the other passage referred to in the "Republic"—the finding of the number of the State—(Book VIII, § 546) has been a subject of the greatest speculation and by consulting the various editions of Plato it will be found that scarcely any two critics agree upon a solution.[3] As Jowett remarks, it is a puzzle almost as great as that of the Beast in the Book of Revelation. Unfortunately we have no starting-point from which to begin our calculations; this and the very uncertain meanings of many of the Greek terms have caused many commentators to give up the solution of the problem in sheer despair. Aristotle, who was a hearer of Plato's, writes as if having a full knowledge of the mystery; Cicero, however, was unable to solve the riddle and his sentiment became voiced in the proverb *numeris Platonicis nihil obscurius.*

By taking a hint from our magic square and starting with the number 27, I believe we may arrive at as good a solution of the problem as any that I have seen suggested. The following interpretation of the Greek terms is offered.

αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι	the square of the number times its root,	$27^2 \times \sqrt[3]{27} = 2187$
ῥεῖς ἀποστάσεις	increased by thrice the first terms (of the Tetractys)	$(1+2+3+4+9)\times 3 = 57$
τέτταρας δὲ ὅρους λαβοῦσαι	and four times the whole series	$(1+2+3+4+9+8+27)\times 4 = 216$

[2] Not only the perpendiculars, horizontals, and diagonals of this large square foot up 9855, but there are an almost indefinite number of zig-zag lines, which give the same footing.

[3] Schleiermacher, Donaldson, and Schneider suggest 216, and much may be said in favor of this number. Jowett gives 8000 as the possible solution. Others suggest 951, 5040, 17,500, 1728, 10,000, etc.

ὁμοιούντων τε καὶ ἀνομοιουντων καὶ ἀυξόντων καὶ φθινότνων	of numbers unlike yet bearing the same ratio whether increasing or decreasing	[carried over from last page]	2460
	(i. e. 1 : 2 : : 4 : 8 or 8 : 4 : : 2 : 1 It may also refer to the ascending and descending figures of the triangle. 8, 4, 2, 1, 3, 9, 27)		
πάντα προσήγορα καὶ ῥητὰ πρὸς ἀλληλα ἀπέφηναν	makes the sum commensurable and expressible in all its parts.		sum= 2460
	(i. e 2460 is easily divisible by 1, 2, 3, 4, 5, 6, 10, 12, etc.)		
ὢν ἐπιτριτος πυθμὴν,	this sum increased by ⅓		2460×1⅓= 3280
πεμπάδι συζυγείς	and adding 5		3280+5= 3285
τρὶς αὐξηθείς	is multiplied by 3		3285×3= 9855

This solution of the problem, 9855, it will be noted, brings us again but by a different route to the magic number of our large square. The second part of the passage contains a description of the number by which the above calculation may be verified.

δύο ἁρμονίας παρέχεται	(the number) yields two harmonic parts,		
τὴν μὲν ἴσην ἰσάκις,	one of which is a square	3×3= 9	
ἑκατὸν τοσαυτάκις,	multiplied by 100:		9×100= 900
τὴν δὲ ἰσομήκη μὲν,	the other has one side equal to the square		3
τῇ προμήκει δὲ,	and the other oblong		3×2985= 8955
			sum= 9855

The remainder of the passage describes the length of the oblong which we have shown above to be 2985:

ἑκατὸν μὲν ἀριθμῶν ἀπὸ διαμέτρων πεμπάδος,	(the oblong) is 100 times the side of a rectangle having diagonals of 5.	100×3= 300
	(i. e having sides of 3 and 4.)	
οητῶν δεομένων ἑνὸς ἑκάστων,	less of one each of the expressible parts, i. e. 4 and 5	

ἀῤῥήτων δὲ δυεῖν,	and 2 of the inexpressible	300—(5+4+3+3)= 285
ἑκατὸν δὲ κύβων τριάδος	plus 100 times the cube of 3	$(3)^3 \times 100 = 2700$
		sum= 2985

Plato states that the number of the State "represents a geometrical figure which has control over the good and evil of births. For when your guardians are ignorant of the right seasons and unite bride and bridegroom out of due time, the children will not be goodly and happy." The number 9855, expressing a period of 27 years, might thus represent the dividing line between the ages when men and women should begin to bear children to the State,—20-27 years for women, 27-34 years for men. (See also "Republic," Book V, § 460). Aristotle in his "Politics" (V, 12. 8) says in reference to the number of the State that when the progression of number is increased by $^1/_3$ and 5 is added, 2 harmonies are produced giving a solid diagram. This, as may be seen from our analysis of the first part of the passage, may have reference to the number 3285, which, being represented by $3^2 \times 365$, may be said to have the dimensions of a solid.

In the January number of *The Monist* the Editor gave some very striking examples of the relationship between magic squares and the musical figures of Chladni. I would like to touch before concluding upon a closely related subject and show certain connections which exist between the magic square, which we have constructed, and the numbers of the Pythagorean harmonic scale. This scale had, however, more than a musical significance among the Greek philosophers; it was extended to comprehend the harmony of planetary movements and above all else to represent the manner in which the "soul of the universe" was composed. It is especially in the latter sense that Plato employs the scale in his "Timæus."

In a treatise by Timæus the Locrian upon the "Soul of the World and Nature," we find the following passage: "Now all these proportions are combined harmonically according to numbers, which proportions the demiurge has divided according to a scale scientifically, so that a person is not ignorant of what things and by what means the soul is combined; which the deity has not ranked after the substance of the body...., but he made it older by taking the *first of unities* which is 384. Now of these the first being assumed it is easy to reckon the double and triple; and all the terms, with

their complements and eights must amount to 114,695." (Translation by Burge.)

Plato's account of the combination of the soul is very similar to the above, though he seems to have selected 192, (384/2) for the first number. Plutarch in his commentary makes no mention of Timæus, but states that Crantor[4] was the first to select 384, for the reason that it represented the product of $8^2 \times 6$, and is the lowest number which can be taken for the increase by eighths without leaving fractions. Another very possible reason, which I have not seen mentioned, is that 384 is the harmonic ratio of $27^2/2$ or 364.5, a number which expresses very closely the days of the year.

$$243 : 256 :: 364.5 : 384.$$

The proportion $243 : 256 (3^5 : 4^4)$ was employed by the Pythagoreans to mark the ratio[5] which two unequal semitones of the harmonic scale bear to one another.

Batteux has calculated the 36 terms of the Pythagorean scale starting with 384 and his series must be considered correct, for it fulfils the conditions specified by Timæus,—the numbers all footing up 114,695: A few of the numbers of this harmonic scale marking the "first unity" and several of the semitones will be given.

1st octave	E	384
	C	486
	F	729
2nd octave	C	972
	F	1458
3rd octave	C	1944
	B-flat	2187
4th octave	B-flat	4374

(For Batteux' full series and method of calculation the reader is referred to Burge's translation of Plato Vol. VI. p. 171).

By referring to our magic square it will be noted that the *first* of *unities,*" 384, constitutes the magic nmber of the small 3×3 square beginning with the number 100. If we arrange the magic numbers of the 81 squares (3×3) in the order of their magnitudes we find that they fall into 9 series of 9 numbers, each series beginning as follows:

I	II	III	IV	V	VI	VII	VIII	IX
87	330	573	816	1059	1302	1545	1788	2031

[4] Crantor lived nearly 100 years after Timæus the Locrian. The treatise upon the "Soul of the World and Nature," which bears the latter's name probably belongs to a much later period.

[5] for further references to this ratio see Plato's "Timæus," § 36, and Plutarch's "Procreation of the Soul," § 18.

The intervals between these series are worthy of note.

INTERVALS.

Between I and II	243	the first member of the ratio 243:256.	
" I " III	486	C of the 1st octave	
" I " IV	729	F " " 1st "	
" I " V	972	C " " 2nd "	
" I " VII	1458	F " " 2nd "	
" I " IX	1944	C " " 3rd "	

If we arrange the magic numbers of the large squares (9×9) in the same way, it will be found that they fall into 3 series of 3 numbers, each series beginning

I	II	III
1017	3204	5391

Interval between I and II = 2187 B-flat of the 3rd octave.
" " I " III = 4374 B-flat " " 4th "

Numerous other instances might be given of the very intimate connection between magic squares and various Pythagorean numbers, but these must be left for the curious-minded to develop for themselves. Such connections as we have noted are no doubt in some respects purely accidental, being due to the *intrinsic harmony of numbers* and therefore not implying a knowledge by the ancients of magic squares as we now know them. The harmonic arrangement by the Greeks of numbers in geometrical forms both plane and solid may, however, be accepted, and Plato's descriptions of various numbers obscure and meaningless as they were to succeeding generations, may have been easily comprehended by his hearers when illustrated by a mathematical diagram or model.[6]

Differences between the methods of notation in ancient and modern times have necessarily produced differences in the conception of numerical relations. The expression of numbers among the Greeks by letters of the alphabet was what led to the idea that every name must have a numerical attribute, but the connection of the letters of the name was in many cases lost, the number being regarded as a pure attribute of the object itself. A similar confusion of symbols arose in the representation of various concepts by geometrical forms, such as the five letters of ΥΓΕΙΑ and the symboliza-

[6] The description of the number of the State in the "Republic" and that of the Soul in the "Timæus" render such a mode of representation almost necessary. Plutarch ("Procreation of Soul," § 12) gives an illustration of an harmonic diagram 5×7 containing 35 small squares "which comprehends in its subdivisions all the proportions of the first concords of music."

tion of health by the Pythagoreans under the form of the pentalpha or five-pointed star.

It was the great defect of the Greek schools that in their search for truth, methods of experimental research were not cultivated. Plato in his "Republic" (Book VII, § 530-531) ridicules the empiricists, who sought knowledge by studying the stars or by comparing the sounds of musical strings, and insists that no value is to be placed upon the testimony of the senses. "Let the heavens alone and train the intellect" is his constant advice.

If the examples set by Pythagoras in acoustics and by Archimedes in statics had been generally followed by the Greek philosophers, our knowledge of natural phenomena might have been advanced a thousand years. But as it happened there came to prevail but one idea intensified by both Plato and Aristotle, and handed down through the scholastics even to the present time, that knowledge was to be sought for only from within. Hence came the flood of idle speculations which characterized the later Pythagorean and Platonic schools and which eventually undermined the structure of ancient philosophy. But beneath the abstractions of these schools one can discover a strong undercurrent of truth. Many Pythagoreans understood by number that which is now termed natural law. Such undoubtedly was the meaning of Philolaus when he wrote "Number is the bond of the eternal continuance of things," a sentiment which the modern physicist could not express more fittingly.

As the first study of importance for the youth of his "Republic" Plato selected the science of numbers; he chose as the second geometry and as the third astronomy, but the point which he emphasized above all was that these and all other sciences should be studied in their "mutual relationships that we may learn the nature of the bond which unites them." "For only then," he states, "will a pursuit of them have a value for our object, and the labor, which might otherwise prove fruitless, be well bestowed." Noble utterance! and how much greater need of this at the present day with our complexity of sciences and tendency towards narrow specialization.

In the spirit of the great master whom we have just quoted we may compare the physical universe to an immense magic square. Isolated investigators in different areas have discovered here and there a few seemingly restricted laws, and paying no regard to the territory beyond their confines, are as yet oblivious of the great pervading and unifying Bond which connects the scattered parts

and binds them into one harmonious system. Omar, the astronomer-poet, may have had such a thought in mind, when he wrote:

> "Yes; and a single Alif were the clue—
> Could you but find it—to the treasure-house
> And peradventure to the Master too;
> . . . . . .
> Whose secret presence, through creation's veins
> Running quicksilverlike eludes your pains;" etc.

When Plato's advice is followed and the "mutual relationships between our sciences" are understood we may perchance find this clue, and having found it be surprised to discover as great a simplicity underlying the whole fabric of natural phenomena as exists in the construction of a magic square.

C. A. BROWNE, JR.

NEW ORLEANS, LA.

THOUGHTS ON TIME, SPACE, AND EXISTENCE.*

I.

All existence is one. Every thing that is, is part of the All. That all of the parts are adjusted to each other, and work in harmony, proves the relationship of all the parts to the whole, and the unity of the All.

Existence extends, and existence endures. While it endures there is an unending and unceasing succession of events occurring.

When we mentally think away all of the features of existence except the feature of extension, that feature which remains when all the rest are thought away, forms, in consciousness, the conception we term space. Space, then, is abstracted mentally from reality, and in so far is a mental existence. But in reality itself, space, or the property of extension can not be separated from existence. We think of it as a thing in itself, or by itself, but this is not the

* A great many of the ideas expressed in this article have been obtained by a study of the philosophical works of Dr. Paul Carus, and I wish to give proper credit for numerous expressions and quotations which I can not well avoid using. Prior to my study of his writings, my philosophical studies had been confined mostly to the works of Herbert Spencer. While I had in a certain degree noticed that he overlooked the importance of the formal and the subjective features of existence, my ideas along these lines were very hazy and undeveloped. I supposed, that in his philosophy, finality had indeed been reached. However, when I began to study the works of Dr. Carus, a new world was opened to my view.