In Response to 'Advice on Comparing Two Independent Samples of Circular Data in Biology'

Patrick Mellady

May 13, 2024

Summary

Methodology and Goals

- In this presentation, we will explore statistical analysis of directional data, in particular directional data in two dimensions.
- After developing the necessary fundamentals, we will state some distributional results that are useful for testing purposes.
- We will finish by examining why the implementation of the P statistic by the authors in [1] is incorrect, and see a correct application of the P statistic to data presented in [1].

Directional Data

- Directional data is a collection of *random directions*, which are random vectors on the sphere
- Circular data is when the random directions come from the unit circle in $\mathbb{R}^2$
- We will assume a Langevin-von Mises-Fisher distribution on which has density

$$f(\mathbf{u}; \boldsymbol{\mu}, \kappa) = \frac{(\kappa/2)^{d/2-1}}{\Gamma(d/2) I_{d/2-1}(\kappa)} e^{\kappa \boldsymbol{\mu}^T \mathbf{u}}, \quad \mathbf{u} \in S^{d-1} \tag{1}$$

Fundamental Quantities

Given a random sample of directions $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, we define

- Resultant vector: $\sum_{i=1}^{n} \mathbf{u}_i$
- Sample mean vector: $\overline{\mathbf{u}} = n^{-1} \sum_{i=1}^{n} \mathbf{u}_i$
- Resultant length: $R = \left\| \sum_{i=1}^{n} \mathbf{u}_i \right\|$
- Sample mean resultant length: $\overline{R} = n^{-1} R$
- Sample mean direction vector: $(\overline{R})^{-1} \overline{\mathbf{u}}$

A natural question to ask is "why so many quantities?"

Fundamental Quantities Cont'd

- Note that, $\|\overline{\mathbf{u}}\| \leq 1$. So we must standardize our resultant vectors in order to have statistics that remain on the sphere.
- If $\overline{R} \approx 1$, then we see that our data is highly concentrated. Conversely, if $\overline{R} \approx 0$, we see that our data is highly dispersed.
- These quantities have population level analogues: $\boldsymbol{\mu} = \frac{E(\mathbf{u})}{\|E(\mathbf{u})\|}$ is the mean direction vector and $\rho = \|E(\mathbf{u})\|$ is the mean resultant length
- For $\mathbf{u}$ following a Langevin-von Mises-Fisher distribution, $\rho = A_d(\kappa) = \frac{I_{d/2}(\kappa)}{I_{d/2-1}(\kappa)}$

Tests of Multiple Samples

Possible goals for comparing two independent samples of directional data include:

1. Testing for equality of concentration
2. Testing for equality of mean direction
3. Testing for equality of distribution

The Test

Let $\mathbf{u}_{11}, \cdots, \mathbf{u}_{1n_1}, \mathbf{u}_{21}, \cdots, \mathbf{u}_{2n_2}, \cdots, \mathbf{u}_{k1}, \cdots, \mathbf{u}_{kn_k}$ be k random samples of size n_i for $i = 1, 2, 3, \cdots, k$, possibly from different distributions described in (1) but all with the same concentration. We wish to test:

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \cdots = \boldsymbol{\mu}_k \tag{2}$$

$$H_a : \boldsymbol{\mu}_i \neq \boldsymbol{\mu}_j \text{ for at least one (i,j) pair}$$

Likelihood Ratio Test

Let $n = \sum_{i=1}^{k} n_i$ and $U = (\mathbf{u}_{11}, \mathbf{u}_{12}, \cdots, \mathbf{u}_{k,n_k})$ be a matrix whose columns are the observed directions, then the likelihood is given by:

$$L(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \cdots, \boldsymbol{\mu}_k, \kappa : U) = (\frac{(\kappa/2)^{d/2-1}}{\Gamma(d/2) I_{d/2-1}(\kappa)})^n e^{\sum_{i=1}^{k} \sum_{j=1}^{n_i} \kappa \boldsymbol{\mu}_i^T \mathbf{u}_{ij}}$$

which gives the following MLEs:

- Under H_0, $\hat{\boldsymbol{\mu}_0} = (\overline{R})^{-1}\overline{\mathbf{u}}, \quad \hat{\kappa_0} = A_d^{-1}(\overline{R})$
- Under H_a, $\hat{\boldsymbol{\mu}}_{ai} = (\overline{R}_i)^{-1}\overline{\mathbf{u}}_i, \quad \hat{\kappa_a} = A_d^{-1}(\tilde{R})$, where $\tilde{R} = n^{-1} \sum_{i=1}^{k} R_i$

Using all the above, we have:

$$\Lambda = \frac{L(\hat{\boldsymbol{\mu}}_0, \hat{\kappa}_0; U)}{L(\hat{\boldsymbol{\mu}}_{a1}, \cdots, \hat{\boldsymbol{\mu}}_{ak}, \hat{\kappa}_a; U)}$$

And $G = -2\ln(\Lambda)$ has an approximate χ^2 distribution.

Transforming the Likelihood Ratio Test Statistic

A transformation of G, similar to the sums of squares ratio in classical ANOVA, gives us the following statistic:

$$P = \frac{n-k}{k-1}(\Lambda^{-2/n(d-1)} - 1)$$

Additionally, by implementing a Maclaurin series, we see that

$$P = \frac{G}{(k-1)(d-1)} + O_p(n^{-1})$$

Asymptotic Considerations

The test above has two asymptotic considerations: large sample and high concentration

- For large sample, fix $\kappa > 0$ and assume $\min(n_1, n_2, \cdots, n_k) \to \infty$, in which case, the LRT statistic is distributed

$$\chi^2((k-1)(d-2))$$

- In the high concentration case, we let κ become large, in which case, the P statistic is distributed

$$F((k-1)(d-1), (n-k)(d-1))$$

Why the P Statistic?

- In the above, the scaled χ^2 distribution is the large sample limit of the F distribution
- Thus, the F approximation (via the P statistic) is valid in both high concentration and large sample settings
- Other famous tests for (2) are only applicable in the high concentration setting
- As shown, the test based on P outperforms all other tests for equality of mean directions in both asymptotic settings

The Error in Landler et al.

- In the supplemental material for [1], the authors used the following *approximation* to the P statistic

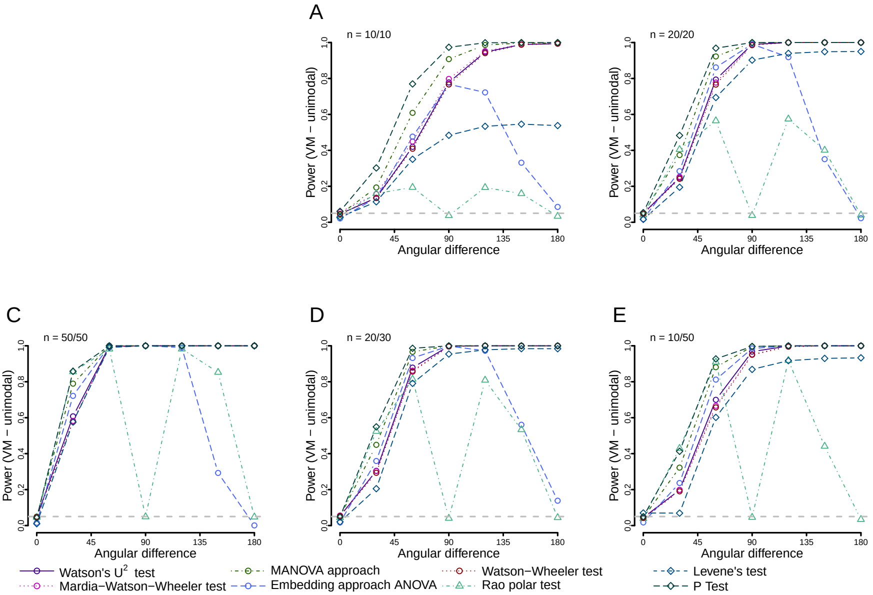
$$P = W + \frac{n-k}{k-1}\left(\frac{1-(d-2)^2}{2(d-1)^2}\right)\frac{(\tilde{R}-\overline{R})(1-\overline{R})}{(1-\tilde{R})} + O_p(\kappa^{-2})$$

- The choice to use an approximation of P does not accurately reflect the efficacy of the P test
- The corresponding recommendations that exclude the use of the P test are unfounded

Power

- We will analyze how the difference in angle between the two mean directions affects the rejection rate of the test
- We will use differences of $0, 30, 60, 90, 120, 150, 180$ degrees between the two mean direction vectors
- Performing the tests exactly as done in [1] reveals that the test based on the P statistic outperforms all other tests for testing equality of mean directions
- The power of the test based on the P statistic for different sample sizes and mean directions is shown below:

Power

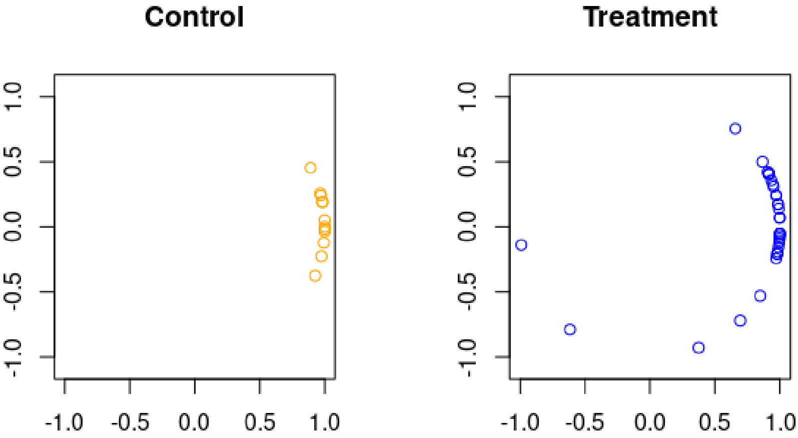

A
n = 10/10
n = 20/20
C
n = 50/50
D
n = 20/30
E
n = 10/50
Power (VM – unimodal)
Angular difference
0
45
90
135
180
0.0
0.2
0.4
0.6
0.8
1.0
Watson's U^2 test
Mardia–Watson–Wheeler test
MANOVA approach
Embedding approach ANOVA
Watson–Wheeler test
Rao polar test
Levene's test
P Test

Introduction to the Real Data Problems

- The authors of [1] apply methods for testing equality of mean direction to biological data for bats, and ants.
- Due to the misapplication of the P statistic, they do not use the P statistic to test for equality of mean direction as they claim it does not perform well
- This claim is based on the incorrect implementation of the P statistic, so, in this work, we will test the equality of mean direction in these three cases using the correct P statistic

Ant Data

Researchers interested in whether or not ants could transfer visual information from one eye to another. It was suspected that there should be no difference in mean direction or concentration. The data is shown below:



Ant Data Cont'd

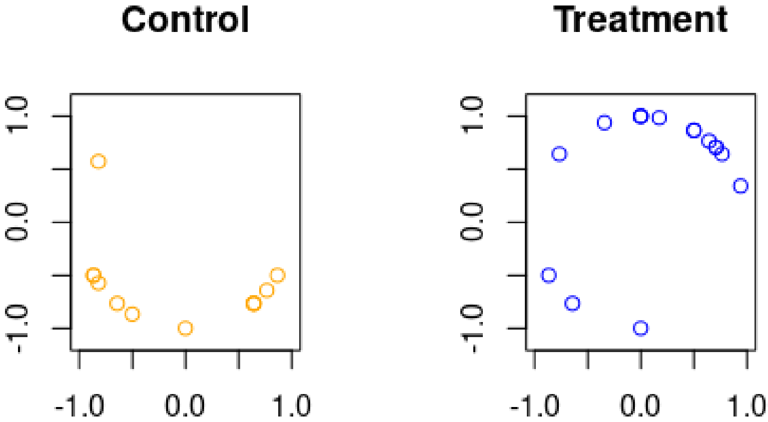
The test based on the P stat recognizes that the mean directions are not significantly different. We obtain the following R output from the data:

- H_0 :Equal Mean Directions
- P.stat=0.016442
- pval=0.8985968

So we fail to reject the null hypothesis at any reasonable significance level.

Bat Data

Data on the direction of bats were collected to see if a difference in mean direction existed between bats that were exposed to the sun's position at dusk and those that were not. The data is plotted below.



Bat Data Cont'd

The test based on the P stat recognizes that the mean directions are significantly different. We obtain the following R output from the data:

- H_0 :Equal Mean Directions
- P.stat=109.212
- pval=2.064711×10^{-10}

So we reject the null hypothesis at the $\alpha = 0.05$ level.

Conclusion

- There are many tests available for statistical analysis of directional data for testing different flavors of hypotheses
- The test based on the P statistic presented in [2] is specifically for testing equality of mean direction while concentrations are held equal and it outperforms other tests for this kind of hypothesis

Conclusion

- The implementation and corresponding recommendations about the P statistic presented in [1] fail to accurately represent this test
- This work was meant to elaborate on the shortcomings of the implementation in [1] and correct any misconceptions it imparted
- It is maintained, as stated in [2], that the test based on the P statistic has no reason to not be adopted in preference to the other tests for comparing mean directions

References

1. Landler, Lukas, et al. “Advice on comparing two independent samples of circular data in Biology.” Scientific Reports, vol. 11, no. 1, 13 Oct. 2021, https://doi.org/10.1038/s41598-021-99299-5.
2. Rumcheva, Pavlina, and Brett Presnell. “An improved test of equality of mean directions for the Langevin-von Mises-Fisher Distribution.” Australian amp; New Zealand Journal of Statistics, vol. 59, no. 1, Mar. 2017, pp. 119–135, https://doi.org/10.1111/anzs.12183.