Traversal and tunneling time across barriers with jump discontinuities, and smooth barriers with compact support

Philip Caesar Flores, ^{1,*} Dean Alvin L. Pablico, ^{2,†} and Eric A. Galapon ^{2,‡}
¹Max-Born-Institute, Max-Born Straße 2A, 12489 Berlin, Germany
²Theoretical Physics Group, National Institute of Physics
University of the Philippines Diliman, 1101 Quezon City, Philippines
(Dated: September 25, 2023)

Using the theory of time-of-arrival operators, we introduce the concept of partial and full-tunneling to determine the traversal and tunneling time of an incident wavepacket $\psi(k)$ across barriers with jump discontinuities, and smooth barriers with compact support. These two 'times' are not necessarily equal. The full-tunneling process occurs when the support of $\tilde{\psi}(k)$ is below the minimum height of the barrier, resulting to an instantaneous tunneling time. Meanwhile, the partial-tunneling process occurs when the support or a segment of the support of $\tilde{\psi}(k)$ lies between the minimum and maximum height of the barrier. For this case, the particle does not "fully" tunnel through the entire barrier system resulting to a non-zero traversal time. We will show that potential barriers with jump discontinuities exhibit full tunneling and partial tunneling, while smooth barriers with compact support only exhibit partial tunneling. The implications of these results on the resolution of the tunneling time problem are then discussed.

The time it takes for a quantum particle to tunnel through a potential barrier has always eluded physicists since the advent of quantum mechanics [1, 2]. However, time is not a quantum observable [3], which has led to the development of several conflicting theories regarding the tunneling time [4–14]. The most prominent of which are the Büttiker-Landauer [4], Larmor [5], Pollak-Miller [6], and Eisenbud-Wigner times [7]. The problem was purely theoretical until the first tunneling time measurement done by Ref. [15] which compared the time-of-arrival of two entangled photons in the presence and absence of a potential barrier. It was shown that the presence of a barrier leads to earlier arrival times implying a non-zero tunneling time, but it did not resolve the tunneling time problem because it turns out that different experimental setups may measure different tunneling times. That is, it is highly dependent on the operational definition used to measure the tunneling time [16].

Tunneling times can be classified into two distinct categories [17], i.e., arrival time and interaction time. The former is concerned with the appearance of the tunneled particle at the far side of the barrier, and is demonstrated by attoclock experiments [18–25]. The first attoclock experiment was performed in Ref. [18] which reported an instantaneous tunneling time, and was supported by later studies [20–23]. However, some attoclock experiments involving multi-electron atoms have reported finite tunnelling times [24, 25]. Meanwhile, the interaction time determines the time duration spent inside the barrier, and is demonstrated by Larmor clock experiments [26]. A recent experiment using cold atoms also measured the

Larmor time and reported a non-zero tunneling time [26].

Comparing the results of the attoclock and Larmor clock to settle the debate on the tunneling time problem thus seems unfruitful as these experiments use different operational definitions of the tunneling time. Instead, we should compare the attoclock experiments [18, 20–23] with the observations of Ref. [15]. Specifically, we ask the following question: By treating tunneling time as an arrival time (at the single particle level), why does the measurement done by Ref. [15] yield a non-zero tunneling time while the attoclock experiments [18, 20–23] yield an instantaneous tunneling time?

In this Letter, we address this question by treating time-of-arrival (TOA) as a dynamical observable, and generalize the earlier results obtained by one of us in Ref. [27] to potential barriers with jump discontinuities, and smooth barriers with compact support. The theory assumes that an initial wavepacket of the form $\psi(q) = e^{ip_o q/\hbar}\varphi(q)$ centered at $q = q_o$ with a mean momentum $p_o = \hbar k_o$, is prepared on the far left side of the square barrier V(q) placed at -a < q < -b. The arrival point is chosen to be the origin q = 0. The average barrier traversal time is deduced from the difference of the expectation values of the corresponding TOA-operators \hat{T} in the presence and absence of the barrier

$$\Delta \tau = \langle \psi | \hat{\mathsf{T}}_{\mathsf{F}} | \psi \rangle - \langle \psi | \hat{\mathsf{T}}_{\mathsf{B}} | \psi \rangle. \tag{1}$$

The subscripts F and B indicate the absence and presence of the barrier, respectively. We will show that the measurable traversal time across the potential barrier depends on the relation between the support of the incident wavepacket's momentum distribution $\tilde{\psi}(k)$, and the effective shape of the potential barrier. This relation leads to a better understanding on the conditions resulting to either instantaneous or non-zero tunneling times. This allows us to differentiate the traversal and tunneling time,

^{*} flores@mbi-berlin.de

[†] dlpablico@up.edu.ph

[‡] eagalapon@up.edu.ph

and that these two 'times' are not necessarily equal.

Our analysis starts by considering the classical TOA for arrivals at the origin given by

$$T(q,p) = -\operatorname{sgn}(p)\sqrt{\frac{\mu}{2}} \int_0^q dq' \frac{dq'}{\sqrt{H(q,p) - V(q')}},$$
 (2)

where $H(q,p) = p^2/2\mu + V(q)$ is the Hamiltonian, μ the mass of the particle and $\operatorname{sgn}(z)$ the signum function. It is clear from Eq. (2) that T(q,p) is only applicable for above-barrier traversal since it becomes complex-valued for the case of quantum tunneling. However, what we can quantize is its expansion around the free-TOA [28] given by

$$\tau(q,p) = \sum_{j=0}^{\infty} \frac{(2j-1)!!}{j!} \frac{(-\mu)^{j+1}}{p^{2j+1}} \int_0^q dq' (V(q) - V(q'))^j,$$
(3)

which we refer to as the local time-of-arrival (LTOA). The LTOA is now single and real-valued, and is amenable to quantization.

The main difference between the classical arrival time Eq. (2) and LTOA Eq. (3) is the fact that the former holds in the entire region $\Omega = \Omega_q \times \Omega_p$, which may be complex-valued in the classically forbidden region. Meanwhile, the LTOA converges uniformly and absolutely to T(q,p) only in some local neighborhood $\omega = \omega_q \times \omega_p$ determined by $|V(q) - V(q')| < p^2/2\mu$ for $p \neq 0$, where it is real-valued. This implies the inclusion $\tau(q,p) \subset T(q,p)$, which exactly describes above barrier traversal process. The LTOA diverges outside the region defined by ω , physically signifying non-arrival at q=0 [29]. Hence, the LTOA correctly represents above and below-barrier traversal times in the classical regime.

Now, for an arbitrary potential V(q) that admits the expansion $V(q) = \sum_{n=0} a_n q^n$, the LTOA Eq. (3) can be rewritten as

$$\tau(q,p) = \sum_{j=0}^{\infty} (-1)^{j+1} \mu^{j+1} \frac{(2j-1)!!}{j!} \sum_{n=1}^{\infty} a_n^{(j)} \frac{q^n}{p^{2j+1}}. \quad (4)$$

which is canonically quantized by imposing Weylordering to the monomial $q^n p^{-(2j+1)}$. In coordinate representation, the TOA-operator has the general form

$$(\hat{\mathsf{T}}\phi)(q) = \int_{-\infty}^{\infty} dq' \frac{\mu}{i\hbar} T(q, q') \operatorname{sgn}(q - q') \phi(q'), \quad (5)$$

where T(q, q') is referred to as the time kernel factor (TKF) [28], and is given by,

$$\tilde{T}(\eta,\zeta) = \frac{1}{2} \int_{0}^{\eta} {}_{0}F_{1}\left[;1;\frac{\mu}{2\hbar^{2}}(V(\eta) - V(s))\zeta^{2}\right],$$
 (6)

where $\eta = (q + q')/2$, $\zeta = q - q'$, and ${}_{0}F_{1}(;a;z)$ is a particular hypergeometric function [28].

In the absence of the barrier, the corresponding TKF is easily obtained by substituting V(q) = 0 into Eq. (6)

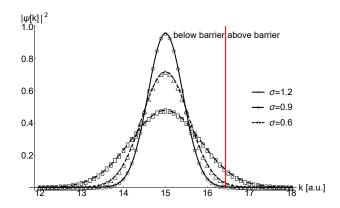


FIG. 1. Momentum density distribution of an incident Gaussian wavepacket initially centered at $q_o = -9$ with momentum $p_o = 15$ for the parameters $\mu = \hbar = 1$. The red line indicates the κ_o for a square barrier of height $V_o = 135$.

yielding $T_F(\eta) = \eta/2$. The integral form of the TKF in Eq. (6) allow us to consider not only analytic continuous potentials but also piecewise potentials such as a square barrier system. In the presence of a square potential barrier, the TKFs are constructed by mapping the potential V(q) to the η -coordinate wherein $V(\eta) = V_o$ for $-a < \eta < -b$. It is straightforward to show that the TKFs have the form

$$\tilde{T}_{I}(\eta,\zeta) = \frac{\eta}{2} \qquad , \eta \ge -b$$

$$\tilde{T}_{II}(\eta,\zeta) = \frac{\eta+b}{2} - \frac{b}{2}I_{0}(\kappa_{o}|\zeta|) \qquad , -a < \eta < -b$$

$$\tilde{T}_{III}(\eta,\zeta) = \frac{\eta+L}{2} - \frac{L}{2}J_{0}(\kappa_{o}|\zeta|) \qquad , \eta \le -a$$

where $\kappa_o = \sqrt{2\mu V_o}/\hbar$, while $I_0(z)$ and $J_0(z)$ are specific Bessel functions. The same TKFs can be obtained using Born-Jordan and simple symmetric ordering [28]. The equality of the TKFs despite using different ordering rules rests on the linearity of the potential on q for each region in the barrier system.

We assume that the incident wavepacket $\psi(q)$ is prepared on the far left of the square potential barrier, so that it does not initially 'leak' inside the barrier region. This assumption suggests that the relevant TKF of our barrier TOA operator is $\tilde{T}_{III}(\eta,\zeta)$ alone. In accordance with Eq. (1), it is easy to show that the resulting traversal time across the square barrier potential is

$$\bar{\tau}_{\text{trav}} = \frac{L}{\nu_o} R \tag{7}$$

wherein $\nu_o = p_o/\mu$ is the group velocity. The factor R is interpreted as the effective index of refraction (IOR) of the square barrier and has the form $R = (R_+ - R_-)$, where

$$R_{\pm} = k_o \int_{\kappa_o}^{\infty} dk \frac{|\tilde{\psi}(\pm k)|^2}{\sqrt{k^2 - \kappa_o^2}},$$
 (8)

and $\tilde{\psi}(k)$ is the Fourier transform of the incident wavepacket [27]. The terms R_+ and R_- correspond to the contributions of the positive and negative momentum components of the incident wavepacket, respectively. For arrivals at the transmission channel, only R_+ is physically relevant.

It can be immediately seen from Eq. (8) that the components below κ_o do not contribute to the barrier traversal time, and the IOR vanishes if all the energy components of the incident wavepacket is below κ_0 , implying an instantaneous tunneling time. This happens when the incident wavepacket is (i) sufficiently spatially wide such that all the energy components are below the barrier, and (ii) sufficiently far from the barrier so that it has zero probability of being in the barrier region at the initial time t = 0. This is illustrated in Fig. 1 for an incident Gaussian wavepacket $\psi(q) =$ $(\sigma\sqrt{2\pi})^{-1/2}e^{-(q-q_o)^2/4\sigma^2}e^{ip_oq/\hbar}$. It can be seen that if the incident wavepacket is spatially narrow, then the momentum density distribution has components above the barrier. This results to a non-zero traversal time but this is not the tunneling time since this 'time' comes from the above-barrier components of $\psi(k)$ which do not tunnel through the barrier. Meanwhile, a spatially wide incident wavepacket with only below-barrier components yields a zero traversal time which equivalently means an instantaneous tunneling time.

The corresponding IOR of a square barrier for different heights V_o is shown in Fig. 2. The IOR vanishes as the barrier height increases, that is, all the momentum components are below the barrier. Moreover, the barrier traversal time has superluminal (R < 1) and subluminal (R > 1) regions, but this does not mean that the particle traverses the barrier at superluminal speeds when it has above $(k > \kappa_o)$ and below-barrier $(k < \kappa_o)$ momentum components. This can be seen by rewriting the barrier traversal time as the weighted average of the classical above-barrier traversal time

$$\bar{\tau}_{\text{trav}} = \int_{\kappa}^{\infty} dk |\tilde{\psi}(\pm k)|^2 \tau_{\text{top}}(k)$$
 (9)

$$\tau_{\rm top}(k) = \frac{\mu L}{\hbar} \frac{1}{\sqrt{k^2 - \kappa_o^2}},\tag{10}$$

with weights $|\tilde{\psi}(\pm k)|^2$. This implies that the above-barrier components still traverse the barrier at subluminal speeds but the instantaneous tunneling of the below-barrier components effectively lowers the average to superluminal traversal times.

The distinction between the traversal time and tunneling time is now clear, i.e., the traversal time arises from the segment of the support of $\tilde{\psi}(k)$ that lies above κ_o while the tunneling time arises from the segment of the support of $\tilde{\psi}(k)$ that lies below κ_o . These two 'times' are only equal if the support of $\tilde{\psi}(k)$ lies below κ_o such that the traversal time identically vanishes and is equivalent to an instantaneous tunneling time.

We now extend the same analysis to a system of two

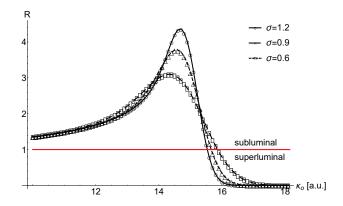


FIG. 2. The effective IOR of a square barrier as the height V_o increases for the Gaussian wavepacket in Fig. 1.

square barriers with $V(q) = V_1$ for -a < q < -land $V(q) = V_2$ for -l < q < -b, where $V_1 < V_2$. A similar barrier system was also considered in Ref. [30] in the investigation of the generalized Hartman effect. It can be shown that the barrier traversal time of an incident wavepacket with group velocity ν_o is simply $\tau_{\text{trav}} = \sum_{n=1}^{2} (w_n/\nu_o) R_n$, where w_n is the width of each barrier and $\sum_{n=1}^{2} w_n = L$, while R_n is the corresponding IOR. The results of the single square barrier case implies that when the support of $\psi(k)$ lies between $0 < k < \kappa_1 = \sqrt{2\mu V_1}/\hbar$, then it follows that $\tau_{trav} = 0$ implying instantaneous tunneling time. We shall describe this as the full-tunneling regime. It easily follows that if the support of $\psi(k)$ lies above $k > \kappa_2 = \sqrt{2\mu V_2}/\hbar$, then the particle does not tunnel through the barrier system and is equivalent to a classical above-barrier traversal time.

A special case happens when the support or segment of the support of $\tilde{\psi}(k)$ lies between $\kappa_1 < k < \kappa_2$, and is equivalent to a particle that classically traverses above the barrier V_1 and tunnels across the barrier V_2 . Hence, the particle does not tunnel through the entire barrier system, and shall be referred to as the *partial-tunneling regime*. It will be convenient to rewrite the traversal time for the double square barrier as

$$\bar{\tau}_{\text{trav}} = \frac{L}{\nu_{\text{o}}} (R_{\text{abo}}^{(2)} + R_{\text{bel}}^{(2)})$$
 (11)

wherein,

$$R_{\rm abo}^{(2)} = \frac{w_1 k_o}{L} \int_{\kappa_2}^{\infty} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa_1^2}} + \frac{w_2 k_o}{L} \int_{\kappa_2}^{\infty} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa_2^2}}$$
(12)

$$R_{\text{bel}}^{(2)} = \frac{w_1 k_o}{L} \int_{\kappa_1}^{\kappa_2} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa_1^2}}.$$
 (13)

It is easy to see that

$$\lim_{V_1 \to V_2} R_{\text{bel}}^{(2)} = 0, \quad \lim_{V_1 \to V_2} R_{\text{abo}}^{(2)} = R_+, \tag{14}$$

in which we recover the results of the single barrier case discussed previously. The quantity $\bar{\tau}_{\rm bel}^{(2)} = (L/\nu_o) R_{\rm bel}^{(2)}$ may be mistakenly identified as a tunneling time because it can be interpreted as the below-barrier traversal across V_2 , but this is not the case. The factor $\sqrt{k^2 - \kappa_1^2}$ in the integrand of $R_{\rm bel}^{(2)}$ makes it clear that it should be interpreted as the above-barrier traversal across V_1 , i.e., $\bar{\tau}_{\rm bel}^{(2)}$ is a traversal time and not a tunneling time.

The same analysis can be extended to an arbitrary potential barrier V(q), by discretizing the system as a sum of square barriers with varying heights and width, i.e., $V(q) = \sum_{n=1}^{\infty} V_n$ each having a width w_n . In the continuous limit, $w_n \to 0$, we find the effective IOR has the form $R = R_{\text{abo}} + R_{\text{bel}}$, where

$$R_{\rm abo} = \frac{k_o}{L} \int_b^a dx \int_{\kappa_{\rm max}}^{\infty} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa(x)^2}},\tag{15}$$

$$R_{\text{bel}} = \frac{k_o}{L} \int_b^a dx \int_{\kappa(x)}^{\kappa_{\text{max}}} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa(x)^2}}, \tag{16}$$

in which $\kappa_{\rm max} = \sqrt{2\mu V_{\rm max}}/\hbar$ indicates the maximum value of the the barrier and $\kappa(x) = \sqrt{2\mu V(x)/\hbar^2}$. Hence, an arbitrary potential barrier has a traversal time contribution $\bar{\tau}_{\rm trav}^{\rm bel} = (L/\nu_o)R_{\rm bel}$. Moreover, $\bar{\tau}_{\rm trav}^{\rm bel}$ arises from the "deformation" of the square barrier, and it follows that if we choose the potential to be a square barrier $V(x) = V_0$, then $\bar{\tau}_{\rm trav}^{\rm bel}$ vanishes, which is consistent with the predictions of [27] that tunneling time is instantaneous.

The IOR contribution $R_{\rm bel}$ also indicates that if a potential barrier V(q) has no sharp discontinuity along the edges, then the traversal time $\bar{\tau}_{\rm trav}^{\rm bel}$ will always be nonzero since the minimum value of $\kappa(x)=0$, i.e., $\bar{\tau}_{\rm trav}^{\rm bel}\neq 0$ for smooth barriers with compact support. This also implies that there are regions in which $\bar{\tau}_{\rm trav}^{\rm bel}$ may indicate either subluminal or superluminal traversal times, but this does not mean that the particle traverses the barrier at superluminal speeds. This can be seen by rewriting $\bar{\tau}_{\rm trav}^{\rm bel}$ as

$$\bar{\tau}_{\text{trav}}^{\text{bel}} = \int_{a}^{b} dx \int_{\kappa(x)}^{\kappa_{\text{max}}} dk |\tilde{\psi}(\pm k)|^{2} \tau_{\text{bel}}(k, x)$$
 (17)

$$\tau_{\rm bel}(k,x) = \frac{\mu}{\hbar} \frac{1}{\sqrt{k^2 - \kappa(x))^2}}.$$
(18)

Eq. (17) indicates that the traversal time $\bar{\tau}_{\text{trav}}^{\text{bel}}$ is equal to the sum of the weighted average of the above barrier traversal time of each energy component κ_d that is below κ_{max} with weights $|\tilde{\psi}(\pm k)|^2$. Thus, the superluminal traversal time $\bar{\tau}_{\text{trav}}^{\text{bel}}$ is also due to the effective lowering of the average traversal time due to the instantaneous tunneling of the components below κ_d . This can be illustrated using a "deformed" square barrier

$$V(q) = V_o \exp\left[\frac{4A^2}{(a-b)^2}\right] \exp\left[\frac{A^2}{(q+a)(q+b)}\right]$$
 (19)

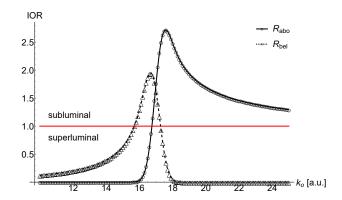


FIG. 3. Contributions of $R_{\rm abo}$ and $R_{\rm bel}$ for the "deformed" square barrier of height $V_o=145$ and A=0.2, as the momentum of the incident Gaussian wavepacket increases, wherein $q_o=-9$ and $\sigma=1.2$ for the parameters $\mu=\hbar=1$.

for $-a \leq q \leq -b$ and zero everywhere else, in which A < 1. The contributions of $R_{\rm bel}$ and R_{abo} are shown in Fig. 3. Since the barrier has no sharp discontinuity at the edges, it can be seen that the traversal time $\bar{\tau}_{\rm trav}^{\rm bel}$ is always non-zero as long as the incident wavepacket has components below the maximum height of this barrier.

The main advantage of the traversal time $\bar{\tau}_{\text{trav}}^{\text{bel}}$ in comparison with other tunneling time definitions is its simplicity. Specifically, other tunneling time definitions such as the Büttiker-Landauer time, Larmor time, and Pollak-Miller time involves calculating the transmission amplitude for propagating through the barrier, which will require solving the Schödinger equation. However, $\bar{\tau}_{\mathrm{trav}}^{\mathrm{bel}}$ only requires information on the incident wavepacket and the interaction potential, which allows it to be applicable in any system without the need of any further calculations, as long as the incident wavepacket does not initially 'leak' into the barrier. It is also clear that potential barriers with jump discontinuities exhibit full tunneling and partial tunneling which can lead to either instantaneous tunneling time or non-zero traversal time, respectively. Meanwhile, smooth barriers with compact support will only exhibit partial tunneling leading to a non-zero traversal time.

Our results imply that the 'time' reported in Ref. [15] is a traversal time, and not a tunneling time. Specifically, the tunnel barrier used was a multilayer dielectric mirror that has an $(HL)^5H$ structure, where H represents titanium oxide with a refractive index of $n_H = 2.22$ while L represents fused silica with a refractive index of $n_L = 1.41$. This setup is an optical analogue of a system of contiguous square barriers with heights V_H and V_L , where $V_L < V_H$, and can be considered as a potential barrier with jump discontinuties. Since there was no way that the momentum distribution of the incident photon can be controlled such that all the momentum components are below V_L , then the photon exhibits partial tunneling resulting to a non-zero traversal time. Moreover, our results imply that the superluminality observed by Ref. [15] is due to the effective lowering of the traversal

time because of the instantaneous tunneling of the momentum components $k < \sqrt{2\mu V_L}/\hbar$. It is thus possible to observe a subluminal traversal time by lowering V_L .

We also argue that $\bar{\tau}_{\rm trav}^{\rm bel}$ may explain any upper bound in the tunneling time of attoclook experiments which reported instantaneous tunneling time [18, 20–23]. Although our theory does not exactly model the tunnel ionization of electrons in the attoclock, wherein the electron starts as a bound state and tunnels through the potential to become a scattering state, it still follows that potential $V_a(q) = -Z_{eff}/q - Fq$ used in the attoclock will yield $\bar{\tau}_{\rm trav}^{\rm bel} \neq 0$ as long as it is confined in a finite region of length L, i.e., partial tunneling. As demonstrated in Fig. 3, $\bar{\tau}_{\rm trav}^{\rm bel}$ is always non-zero as long as a segment of the support of $\tilde{\psi}(k)$ lies below the maximum height of $V_a(q)$, and can be very small when the support is below $V_a(q)$.

In conclusion, we explicitly showed how the momentum distribution of the incident wave packet dictates the measurable traversal time across the barrier region for arbitrary potential V(q). Textbook quantum mechanics only considers an incident plane wave with incident energy $E_o < V_o$ to illustrate quantum tunneling. The same set-up was also used by MacColl and Hartman to investigate the quantum tunneling time which sparked this entire field of study. However, such a setup only considers a single energy component which entails that the location of the incident particle is ill-defined and conse-

quently makes the instance at which it emerges from the barrier also ill-defined [31]. Using an incident wavepacket with multiple energy components is able to address this problem since both the position and momentum density distribution are now well-defined. This allows for a better understanding on which components tunnel through the barrier, and traverses above the barrier. Thus, there is a clear distinction between the traversal and tunneling times, and that these two 'times' are not necessarily equal. Before we can close the debate on quantum tunneling time, one must first recognize that there are, in fact, two possible tunneling processes, i.e., full tunneling and partial tunneling.

Our results do not only answer the time it takes for a quantum particle with some specific incident energy to traverse a potential barrier, but also touches the question on what really constitutes a tunneling particle. Moreover, we are able to address the apparent superluminal traversal times, i.e., the above-barrier components still traverse the barrier at subluminal speeds but the instantaneous tunneling of the below-barrier components effectively lowers to average to superluminal speeds. The origin of the instantaneous tunneling time of the belowbarrier components for the process still remains unknown and even persists in a recently proposed formalism of relativistic TOA-operators for spin-0 particles provided that the barrier height V_o is less than the rest mass energy [32–34].

- [1] L. MacColl, Physical Review 40, 621 (1932).
- [2] T. E. Hartman, Journal of Applied Physics 33, 3427 (1962).
- [3] W. Pauli et al., Geiger and scheel 2, 83 (1933).
- [4] M. Büttiker and R. Landauer, Physical Review Letters 49, 1739 (1982).
- [5] M. Büttiker, Physical Review B 27, 6178 (1983).
- [6] E. Pollak and W. H. Miller, Physical review letters 53, 115 (1984).
- [7] E. P. Wigner, Physical Review 98, 145 (1955).
- [8] A. Baz, Yadern. Fiz. 4 (1966).
- [9] V. Rybachenko, Sov. J. Nucl. Phys. 5, 635 (1967).
- [10] F. T. Smith, Physical Review 118, 349 (1960).
- [11] J. Petersen and E. Pollak, The Journal of Physical Chemistry A 122, 3563 (2018).
- [12] S. Brouard, R. Sala, and J. Muga, Physical Review A 49, 4312 (1994).
- [13] D. Sokolovski and L. Baskin, Physical Review A 36, 4604 (1987).
- [14] N. Yamada, Physical review letters **93**, 170401 (2004).
- [15] A. M. Steinberg, P. G. Kwiat, and R. Y. Chiao, Physical Review Letters 71, 708 (1993).
- [16] R. Y. Chiao and A. M. Steinberg, in *Progress in Optics*, Vol. 37 (Elsevier, 1997) pp. 345–405.
- [17] D. C. Spierings and A. M. Steinberg, Phys. Rev. Lett. 127, 133001 (2021).
- [18] P. Eckle, A. Pfeiffer, C. Cirelli, A. Staudte, R. Dorner, H. Muller, M. Buttiker, and U. Keller, science 322, 1525 (2008).

- [19] P. Eckle, M. Smolarski, P. Schlup, J. Biegert, A. Staudte, M. Schöffler, H. G. Muller, R. Dörner, and U. Keller, Nature Physics 4, 565 (2008).
- [20] A. N. Pfeiffer, C. Cirelli, M. Smolarski, D. Dimitrovski, M. Abu-Samha, L. B. Madsen, and U. Keller, Nature Physics 8, 76 (2012).
- [21] A. N. Pfeiffer, C. Cirelli, M. Smolarski, and U. Keller, Chemical Physics 414, 84 (2013).
- [22] U. S. Sainadh, H. Xu, X. Wang, A. Atia-Tul-Noor, W. C. Wallace, N. Douguet, A. Bray, I. Ivanov, K. Bartschat, A. Kheifets, et al., Nature 568, 75 (2019).
- [23] L. Torlina, F. Morales, J. Kaushal, I. Ivanov, A. Kheifets, A. Zielinski, A. Scrinzi, H. G. Muller, S. Sukiasyan, M. Ivanov, et al., Nature Physics 11, 503 (2015).
- [24] A. S. Landsman, M. Weger, J. Maurer, R. Boge, A. Ludwig, S. Heuser, C. Cirelli, L. Gallmann, and U. Keller, Optica 1, 343 (2014).
- [25] N. Camus, E. Yakaboylu, L. Fechner, M. Klaiber, M. Laux, Y. Mi, K. Z. Hatsagortsyan, T. Pfeifer, C. H. Keitel, and R. Moshammer, Physical review letters 119, 023201 (2017).
- [26] R. Ramos, D. Spierings, I. Racicot, and A. M. Steinberg, Nature 583, 529 (2020).
- [27] E. A. Galapon, Phys. Rev. Lett. 108, 170402 (2012).
- [28] E. A. Galapon and J. J. P. Magadan, Annals of Physics 397, 278 (2018).
- [29] D. A. L. Pablico and E. A. Galapon, The European Physical Journal Plus 138, 1 (2023).
- [30] D. L. Sombillo and E. A. Galapon, Physical Review A

- **90**, 032115 (2014).
- [31] K. L. Jensen, J. Riga, J. L. Lebowitz, R. Seviour, and D. A. Shiffler, Journal of Applied Physics 132, 124303 (2022).
- [32] P. C. Flores and E. A. Galapon, Physical Review A **105**, 062208 (2022).
- $[33]\,$ P. C. Flores and E. A. Galapon, Europhysics Letters ${\bf 141},$ 10001 (2023).
- [34] P. C. M. Flores and E. A. Galapon, The European Physical Journal Plus 138, 375 (2023).