Dissertation Defense

Adviser: Eric A. Galapon, Ph.D.

Theory of quantized relativistic time-of-arrival operators for spin-0 particles and its application in the quantum tunneling time problem

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University of the Philippines Diliman, Quezon City

05 December 2022

Outline

1. **History and motivation**
2. **Relativistic free TOA-operator**
   Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. Physical Review A, 105(6), 062208.
3. **TOA-operators for the interacting case**
   Flores, P. C., & Galapon, E. A. (2022). Quantized relativistic time-of-arrival operators for spin-0 particles and the quantum tunneling time problem. arXiv preprint arXiv:2207.00343.
4. **Barrier traversal time**
   Flores, P. C., & Galapon, E. A. (2022). Instantaneous tunneling of relativistic massive spin-0 particles. arXiv preprint arXiv:2207.09040.
5. **Final Remarks**

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2. Relativistic free TOA-operator

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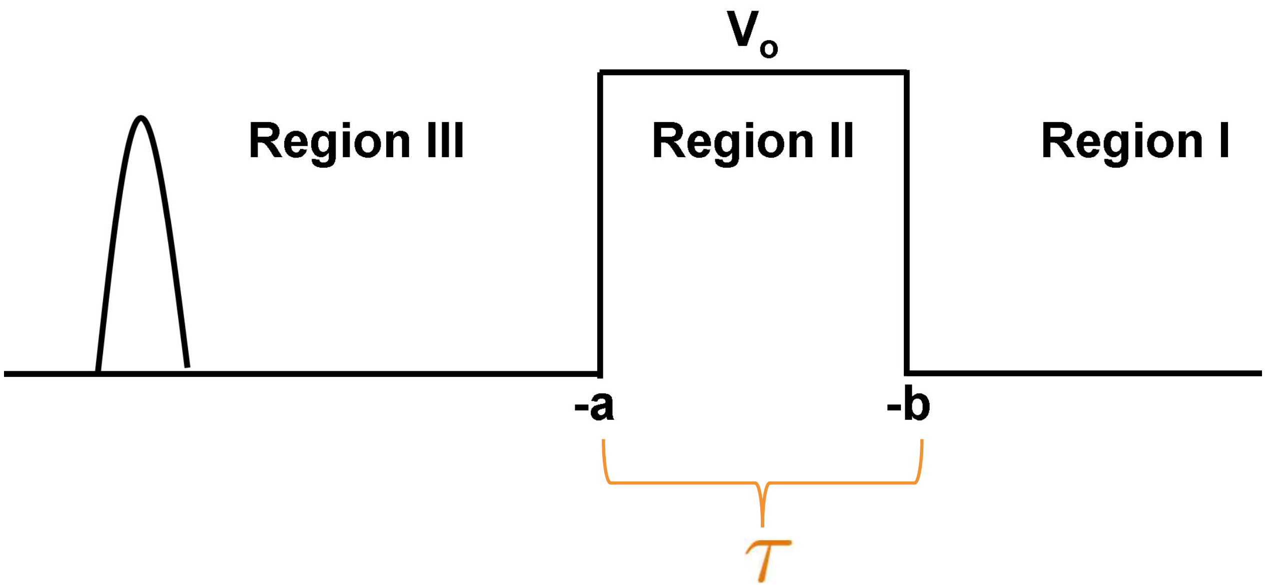
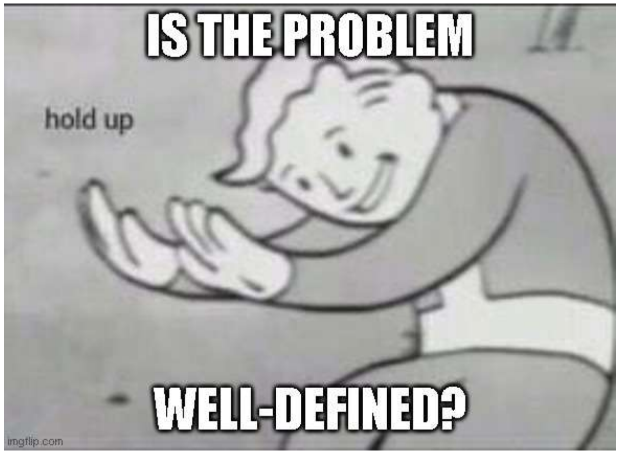
5. Final Remarks

The quantum tunneling time problem

How much time does the particle spend inside the barrier?





MacColl-Hartman effect:

The appearance of the transmitted peak on a ‘screen’ is independent of the barrier width.

MacColl, L. A. (1932). Note on the transmission and reflection of wave packets by potential barriers. *Physical Review*, *40*(4), 621
Hartman, T. E. (1962). Tunneling of a wave packet. *Journal of Applied Physics*, *33*(12), 3427-3433.

The quantum tunneling time problem

How much time does the particle spend inside the barrier?

The problem may be ill-defined because time is not an observable in standard quantum mechanics

Pauli's no-go theorem:

There exists no self-adjoint time operator canonically conjugate with the its corresponding system Hamiltonian.

Lower Limit for the Energy Derivative of the Scattering Phase Shift
Eugene P. Wigner
Phys. Rev. **98**, 145 – Published 1 April 1955

Lifetime Matrix in Collision Theory
Felix T. Smith
Phys. Rev. **118**, 349 – Published 1 April 1960; Erratum Phy

Traversal Time for Tunneling
M. Büttiker and R. Landauer
Phys. Rev. Lett. **49**, 1739 – Published 6 December 1982

Larmor precession and the traversal time for tunneling
M. Büttiker
Phys. Rev. B **27**, 6178 – Published 15 May 1983

New Physical Interpretation for Time in Scattering Theory
Eli Pollak and William H. Miller
Phys. Rev. Lett. **53**, 115 – Published 9 July 1984

Traversal time in quantum scattering
D. Sokolovski and L. M. Baskin
Phys. Rev. A **36**, 4604 – Published 1 November 1987

Dumont, R. S., Rivlin, T., & Pollak, E. (2020). The relativistic tunneling flight time may be superluminal, but it does not imply superluminal signaling. *New Journal of Physics*, *22*(9), 093060.
Landsman, A. S., & Keller, U. (2015). Attosecond science and the tunnelling time problem. *Physics Reports*, *547*, 1-24.
Landauer, R., & Martin, T. (1994). Barrier interaction time in tunneling. *Reviews of Modern Physics*, *66*(1), 217.
Hauge, E. H., & Støvneng, J. A. (1989). Tunneling times: a critical review. *Reviews of Modern Physics*, *61*(4), 917.
Pauli, W. (1933). Handbuch der physik. *Geiger and scheel*, *2*, 83-272.

The quantum tunneling time problem

Just perform an experiment and compare the measurements with the theoretical predictions

Measurement of the single-photon tunneling time

A. M. Steinberg, P. G. Kwiat, and R. Y. Chiao
Phys. Rev. Lett. **71**, 708 – Published 2 August 1993

Using a two-photon interferometer, we have measured the time delay for a photon to tunnel across a barrier consisting of a 1.1-μm-thick 1D photonic band-gap material. The peak of the photon wave packet appears on the far side of the barrier 1.47±0.21 fs *earlier* than it would if it were to travel at the vacuum speed of light *c*. Although the apparent tunneling velocity (1.7±0.2)*c* is superluminal, this is not a genuine signal velocity, and Einstein causality is not violated. The measured tunneling time is consistent with the group delay ("phase time"), but not with the semiclassical time.

Different experimental setups may measure different tunneling times

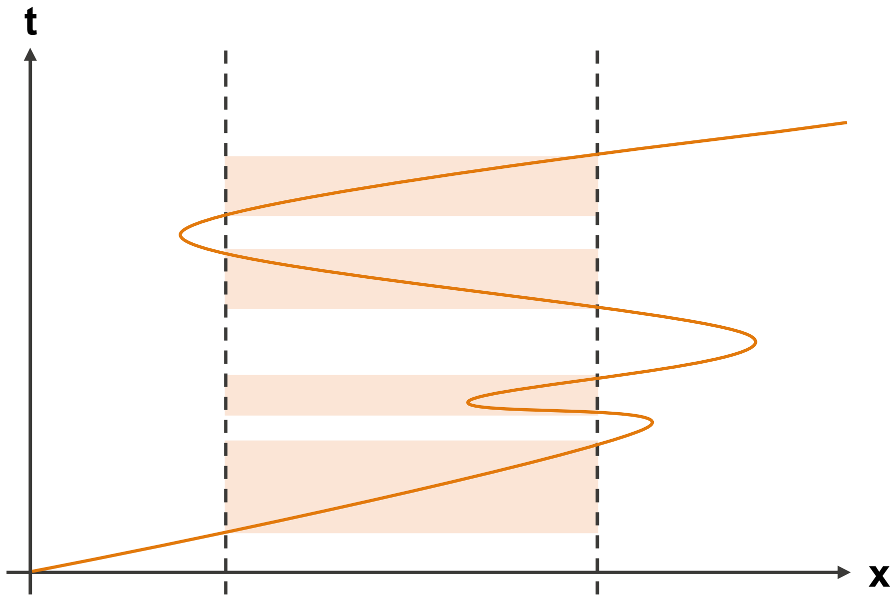
One must make a careful operational definition of exactly how the measurement of the tunneling time is performed.

Chiao, R. Y., & Steinberg, A. M. (1997). VI: Tunneling times and superluminality. In *Progress in Optics* (Vol. 37, pp. 345-405). Elsevier.
Spierings, D. C., & Steinberg, A. M. (2021). Observation of the Decrease of Larmor Tunneling Times with Lower Incident Energy. *Physical Review Letters*, *127*(13), 133001.

Tunneling time as an interaction time vs arrival time

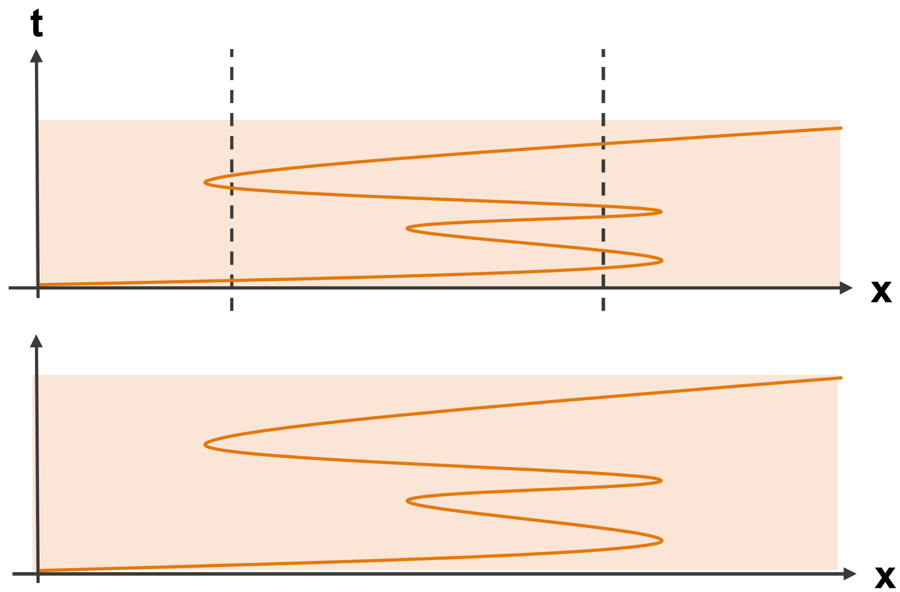
Interaction time:

- Total time the particle spent inside the barrier region



Arrival time:

- The moment at which a transmitted particle emerges on the far side



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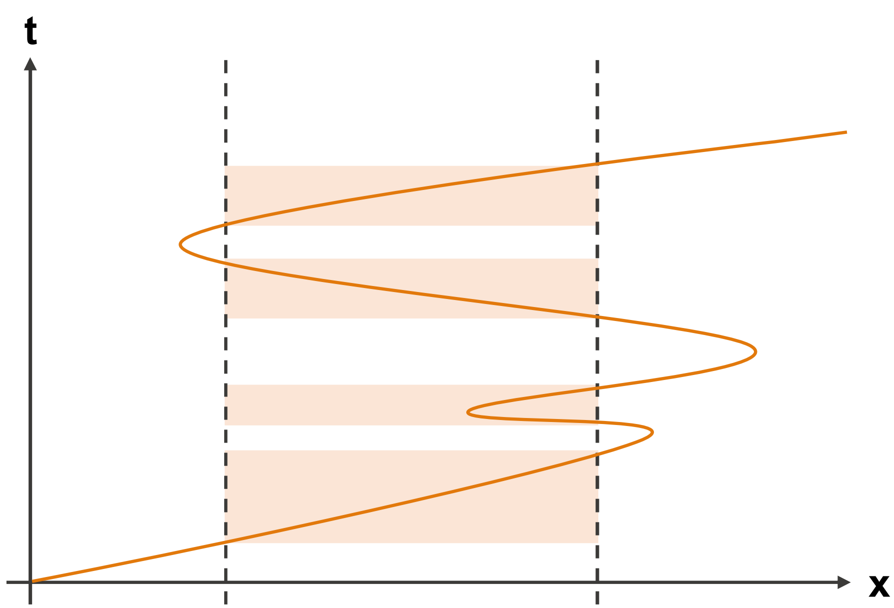
Yamada, N. (2004). Unified derivation of tunneling times from decoherence functionals. *Physical review letters*, *93*(17), 170401.

Tunneling time as an interaction time vs arrival time

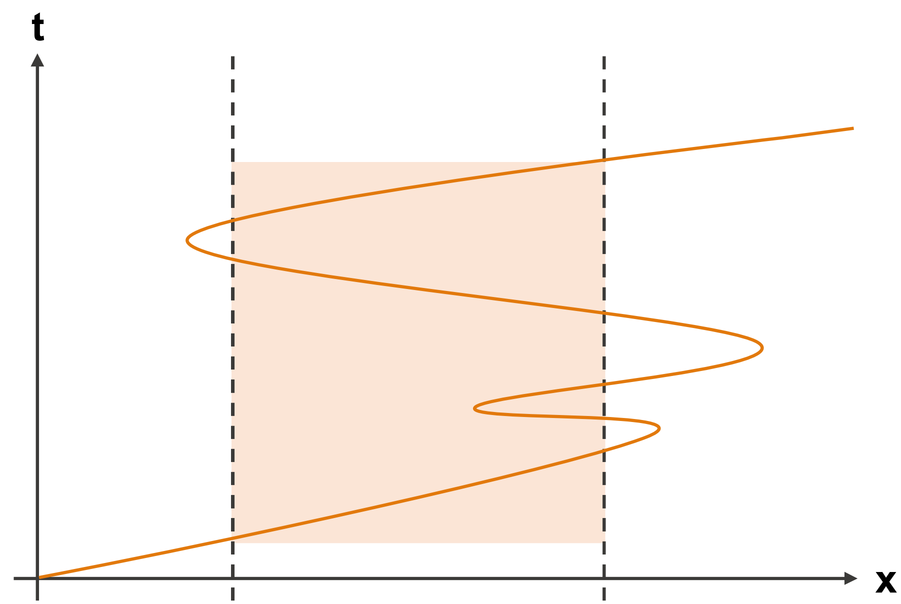
Interaction time:

- Total time the particle spent inside the barrier region



Passage time:

- Difference of time at first entry and last exit



Chiao, R. Y., & Steinberg, A. M. (1997). VI: Tunneling times and superluminality. In *Progress in Optics* (Vol. 37, pp. 345-405). Elsevier.
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Tunneling time as an interaction time vs arrival time

Interaction time:

- Total time the particle spent inside the barrier region
- Demonstrated by Larmor clock experiments
- Report finite tunneling times

Measurement of the time spent by a tunnelling atom within the barrier region

Ramón Ramos, David Spierings, Isabelle Racicot & Aephraim M. Steinberg

Nature **583**, 529–532 (2020) | Cite this article

Observation of the Decrease of Larmor Tunneling Times with Lower Incident Energy

David C. Spierings and Aephraim M. Steinberg

Phys. Rev. Lett. **127**, 133001 – Published 20 September 2021

Chiao, R. Y., & Steinberg, A. M. (1997). VI: Tunneling times and superluminality. In *Progress in Optics* (Vol. 37, pp. 345-405). Elsevier.
Spierings, D. C., & Steinberg, A. M. (2021). Observation of the Decrease of Larmor Tunneling Times with Lower Incident Energy. *Physical Review Letters*, *127*(13), 133001.
Yamada, N. (2004). Unified derivation of tunneling times from decoherence functionals. *Physical review letters*, *93*(17), 170401.

Tunneling time as an interaction time vs arrival time

Attosecond Ionization and Tunneling Delay Time Measurements in Helium

P. ECKLE, A. N. PFEIFFER, C. CIRELLI, A. STAUDTE, R. DÖRNER, H. G. MULLER, M. BÜTTIKER, AND U. KELLER Authors Info & Affiliations

SCIENCE • 5 Dec 2008 • Vol 322, Issue 5907 • pp. 1525-1529 • DOI: 10.1126/science.1163439

Attosecond angular streaking and tunnelling time in atomic hydrogen

U. Satya Sainadh, Han Xu, Xiaoshan Wang, A. Atia-Tul-Noor, William C. Wallace, Nicolas Douguet, Alexander Bray, Igor Ivanov, Klaus Bartschat, Anatoli Kheifets, R. T. Sang & I. V. Litvinyuk

Nature 568, 75–77 (2019) | Cite this article

Attoclock reveals natural coordinates of the laser-induced tunnelling current flow in atoms

Adrian N. Pfeiffer, Claudio Cirelli, Mathias Smolarski, Darko Dimitrovski, Mahmoud Abu-samha, Lars Bojer Madsen & Ursula Keller

Nature Physics 8, 76–80 (2012) | Cite this article

Experimental Evidence for Quantum Tunneling Time

Nicolas Camus, Enderalp Yakaboylu, Lutz Fechner, Michael Klaiber, Martin Laux, Yonghao Mi, Karen Z. Hatsagortsyan, Thomas Pfeifer, Christoph H. Keitel, and Robert Moshammer
Phys. Rev. Lett. **119**, 023201 – Published 14 July 2017

Interpreting attoclock measurements of tunnelling times

Lisa Torlina, Felipe Morales, Jivesh Kaushal, Igor Ivanov, Anatoli Kheifets, Alejandro Zielinski, Armin Scrinzi, Harm Geert Muller, Suren Sukiasyan, Misha Ivanov & Olga Smirnova

Nature Physics 11, 503–508 (2015) | Cite this article

Arrival time:

- The moment at which a transmitted particle emerges on the far side
- Demonstrated by attoclock experiments
- Early measurements report instantaneous tunneling
- Recent measurements report non-zero values

Is tunneling instantaneous?

Only Above Barrier Energy Components Contribute to Barrier Traversal Time

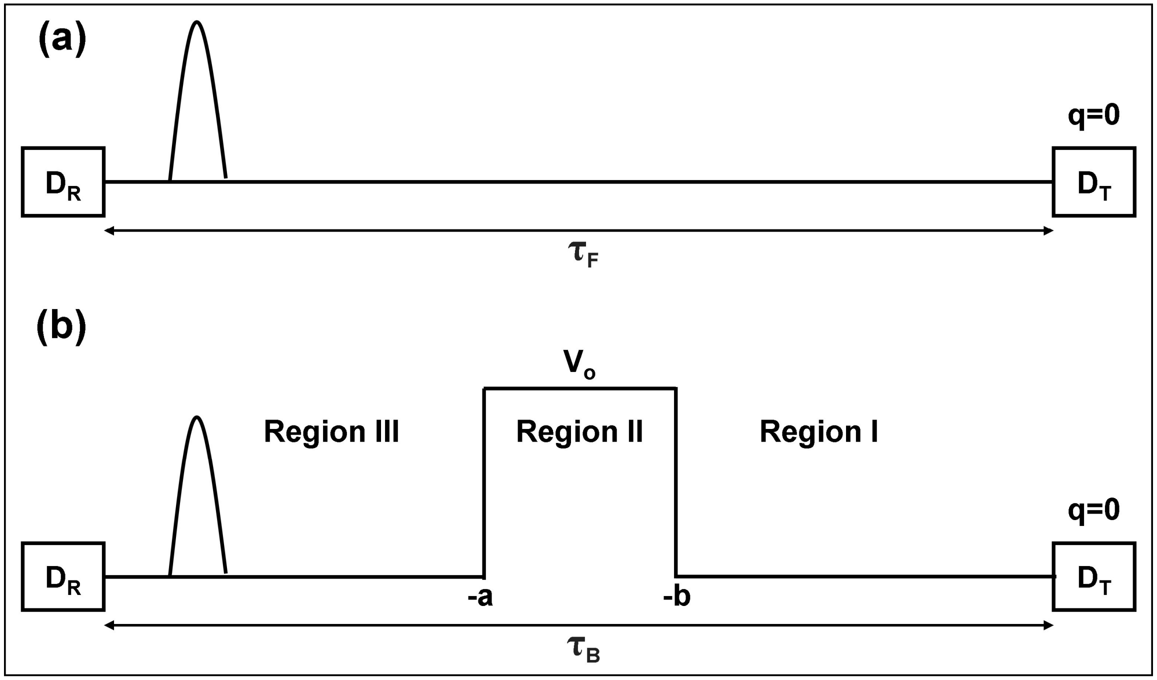
Eric A. Galapon
Phys. Rev. Lett. **108**, 170402 – Published 24 April 2012

- ✓ **Tunneling time is a dynamical observable**

$$\Delta\tau = \tau_F - \tau_B$$

$$\tau_F = \langle\psi|\hat{\mathrm{T}}_{\mathrm{F}}|\psi\rangle$$

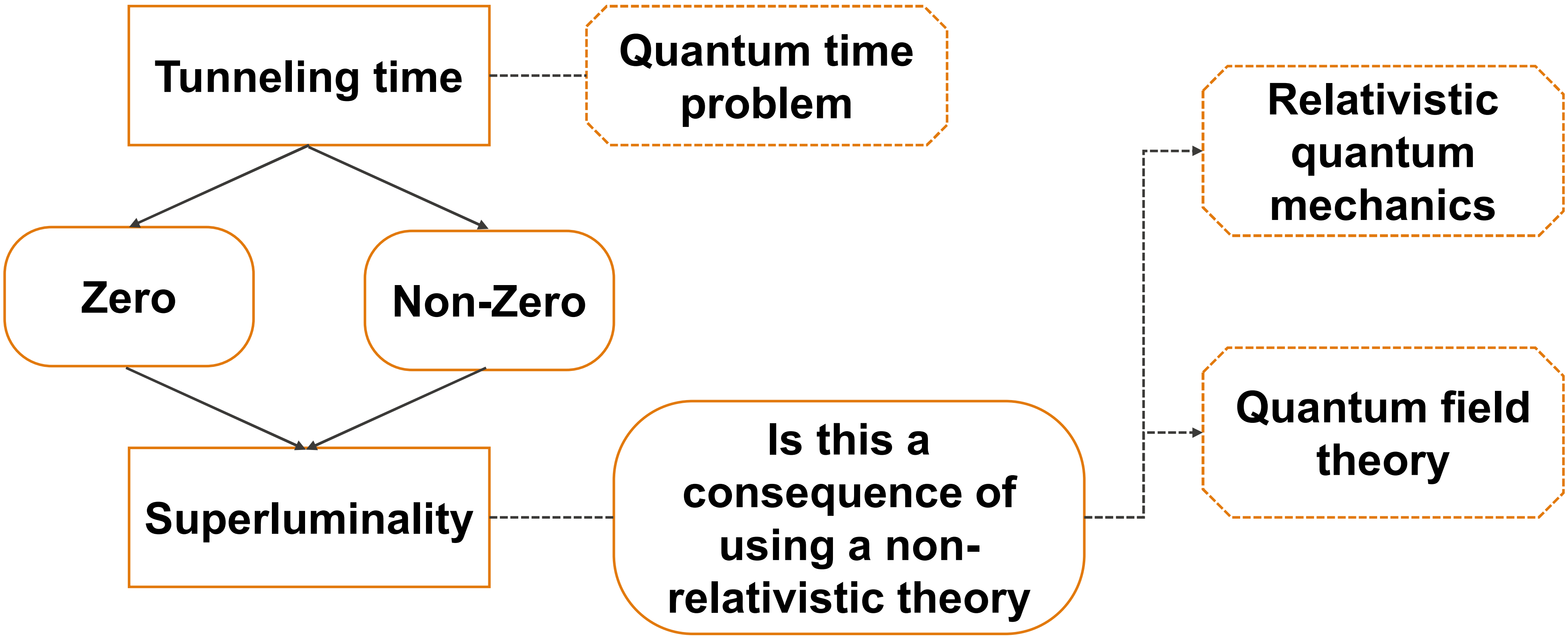
$$\tau_B = \langle\psi|\hat{\mathrm{T}}_{\mathrm{B}}|\psi\rangle$$



- ✓ **Tunneling is instantaneous**

Galapon, E. (2002). Pauli's theorem and quantum canonical pairs: the consistency of a bounded, self–adjoint time operator canonically conjugate to a Hamiltonian with non–empty point spectrum. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, *458*(2018), 451-472.
Galapon, E. A. (2002). Self–adjoint time operator is the rule for discrete semi–bounded Hamiltonians. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, *458*(2027), 2671-2689.

Maybe non-relativistic quantum mechanics is not enough



Maybe non-relativistic quantum mechanics is not enough

Relativistic quantum mechanics

Superluminal behavior is still present!

The relativistic tunneling flight time may be superluminal, but it does not imply superluminal signaling

Randall S Dumont[1], Tom Rivlin[2] and Eli Pollak[3,2]

Published 18 September 2020 • © 2020 The Author(s). Published by IOP Publishing Ltd on behalf of the Institute of Physics and Deutsche Physikalische Gesellschaft

New Journal of Physics, Volume 22, September 2020

Citation Randall S Dumont *et al* 2020 *New J. Phys.* **22** 093060

Dirac equation studies in the tunneling energy zone

S. De Leo & P.P. Rotelli

The European Physical Journal C **51**, 241–247 (2007) | Cite this article

A study of transit times in Dirac tunneling

Stefano De Leo[1]

Published 3 April 2013 • © 2013 IOP Publishing Ltd

Journal of Physics A: Mathematical and Theoretical, Volume 46, Number 15

Citation Stefano De Leo 2013 *J. Phys. A: Math. Theor.* **46** 155306

Relativistic analysis of a wave packet interacting with a quantum-mechanical barrier

Vittoria Petrillo and Davide Janner
Phys. Rev. A **67**, 012110 – Published 30 January 2003

Effects of relativity on the time-resolved tunneling of electron wave packets

P. Krekora, Q. Su, and R. Grobe
Phys. Rev. A **63**, 032107 – Published 14 February 2001

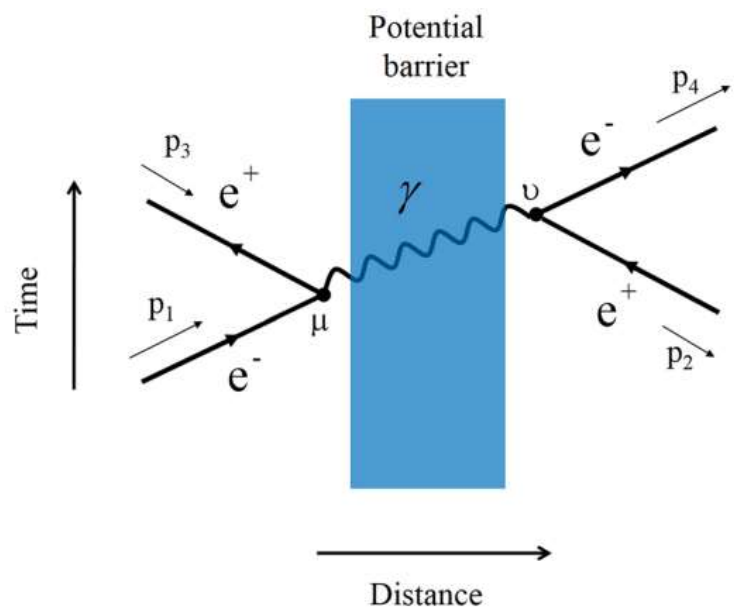
Larmor-clock transmission times for resonant double barriers

C. R. Leavens and G. C. Aers
Phys. Rev. B **40**, 5387 – Published 15 September 1989

Maybe non-relativistic quantum mechanics is not enough

Quantum field theory

Instantaneous tunneling is allowed for virtual particles in spacelike transitions



Martín-Palma, R. J. (2020). Quantum tunneling in low-dimensional semiconductors mediated by virtual photons. *AIP Advances*, *10*(1), 015145.

Our approach

Tunneling as a time of arrival problem

Quantization

Relativistic TOA-operators

Relativistic QM is not a well-defined one-particle theory

Careful with the operational definition of the tunneling time

The identity of the particle should be the same throughout the process

$$V_o < \mu_o c^2$$

Spin-0

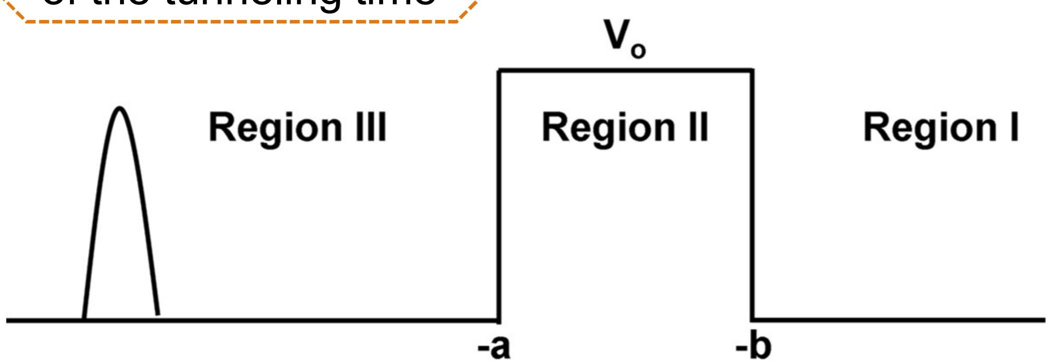
~~Spin-1/2~~

Simplest case

Vacuum polarization

Free case

Interacting case



Galapon, E. A. (2012). Only above barrier energy components contribute to barrier traversal time. *Physical review letters*, *108*(17), 170402.

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5. Final Remarks

Razavi's TOA-operator

❑ There have been attempts to construct a corresponding TOA-operator

No consensus on the free case

No position operator in RQM

Measurement of the position leads to pair-production

$$\hat{\mathsf{T}}_{\mathsf{Ra}} = -\frac{\mu}{2}\left(\sqrt{1+\frac{\hat{\mathsf{p}}^2}{\boxed{\mu}^2\boxed{c}^2}}\,\hat{\mathsf{p}}^{-1}\boxed{\hat{\mathsf{q}}} + \hat{\mathsf{q}}\sqrt{1+\frac{\hat{\mathsf{p}}^2}{\mu^2c^2}}\,\boxed{\hat{\mathsf{p}}}^{-1}\right)$$

rest mass — speed of light — non-relativistic position operator — non-relativistic momentum operator

$$\lim_{c\to\infty}\hat{\mathsf{T}}_{\mathsf{Ra}} = \hat{\mathsf{T}}_{\mathsf{AB}} = -\frac{\mu}{2}\left(\hat{\mathsf{p}}^{-1}\hat{\mathsf{q}} + \hat{\mathsf{q}}\hat{\mathsf{p}}^{-1}\right)$$

Detailed study of using its position-space representation within the rigged Hilbert space formulation of quantum mechanics

Aharonov, Y., & Bohm, D. (1961). Time in the quantum theory and the uncertainty relation for time and energy. *Physical Review*, *122*(5), 1649.
Razavy, M. (1969). Quantum-mechanical conjugate of the hamiltonian operator. *Il Nuovo Cimento B (1965-1970)*, *63*(1), 271-308.
Newton, T. D., & Wigner, E. P. (1949). Localized states for elementary systems. *Reviews of Modern Physics*, *21*(3), 400.
León, J. (1997). Time-of-arrival formalism for the relativistic particle. *Journal of Physics A: Mathematical and General*, *30*(13), 4791.

Razavi's TOA-operator

Why the Rigged Hilbert space formulation?

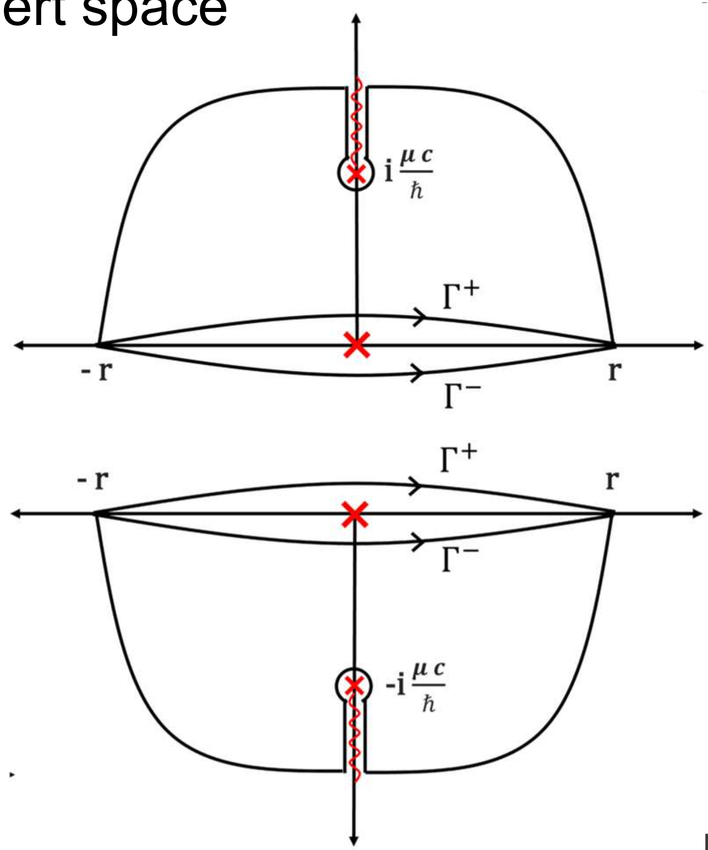
To accommodate non-square integrable functions that are outside the Hilbert space

❑ In coordinate representation

$$(\hat{\mathsf{T}}_{\mathsf{Ra}}\varphi)(q) = \int_{-\infty}^{\infty} dq' \boxed{\langle q|\hat{\mathsf{T}}_{\mathsf{Ra}}|q'\rangle} \varphi(q')$$

$$\boxed{-\mu\frac{q+q'}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\exp\left[\frac{i}{\hbar}(q-q')p\right]\frac{1}{p}\left(\sqrt{1+\frac{p^2}{\mu^2c^2}}\right)}$$

this integral is equal to the Cauchy Principal Value



Galapon, E. A. (2016). The Cauchy principal value and the Hadamard finite part integral as values of absolutely convergent integrals. *Journal of Mathematical Physics*, *57*(3), 033502.

Razavi's TOA-operator

- The Rigged Hilbert space extension of Razavi's TOA-operator is now

$$(\hat{\mathsf{T}}_{\mathsf{Ra}}\varphi)(q) = \int_{-\infty}^{\infty} dq' \langle q|\hat{\mathsf{T}}_{\mathsf{Ra}}|q'\rangle \varphi(q')$$

$$\langle q|\hat{\mathsf{T}}_{\mathsf{Ra}}|q'\rangle = \frac{\mu}{i\hbar}\left(\frac{q+q'}{4}\right) T_c(q,q')\mathrm{sgn}(q-q')$$

0

$$T_c(q,q') = 1 + \frac{2}{\pi}\int_1^{\infty} dz \exp\left[-\frac{\mu c}{\hbar}|q-q'|z\right]\frac{\sqrt{z^2-1}}{z}$$

Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. *Physical Review A*, *105*(6), 062208.

Dynamics of the TOA eigenfunctions in position space

- Legitimate TOA operator has eigenfunctions that exhibit unitary arrival
- Evolve through time to localize at the intended arrival point at their corresponding eigenvalues

$$\tilde{\phi}_\tau(q,t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} e^{ipq/\hbar} e^{-iE_p t/\hbar} \phi_\tau(p)$$

$$E_p = \sqrt{p^2c^2 + \mu^2c^4}$$

- We compare the eigenfunctions solved by Razavi with that of the Rigged Hilbert space extension

Galapon, E. A., Caballar, R. F., & Bahague Jr, R. T. (2004). Confined quantum time of arrivals. *Physical review letters*, *93*(18), 180406.
Galapon, E. A., Caballar, R. F., & Bahague, R. (2005). Confined quantum time of arrival for the vanishing potential. *Physical Review A*, *72*(6), 062107.
Galapon, E. A., & Magadan, J. J. P. (2018). Quantizations of the classical time of arrival and their dynamics. *Annals of Physics*, *397*, 278-302.
Sombillo, D. L. B., & Galapon, E. A. (2016). Particle detection and non-detection in a quantum time of arrival measurement. *Annals of Physics*, *364*, 261-273.

Dynamics of the TOA eigenfunctions in position space

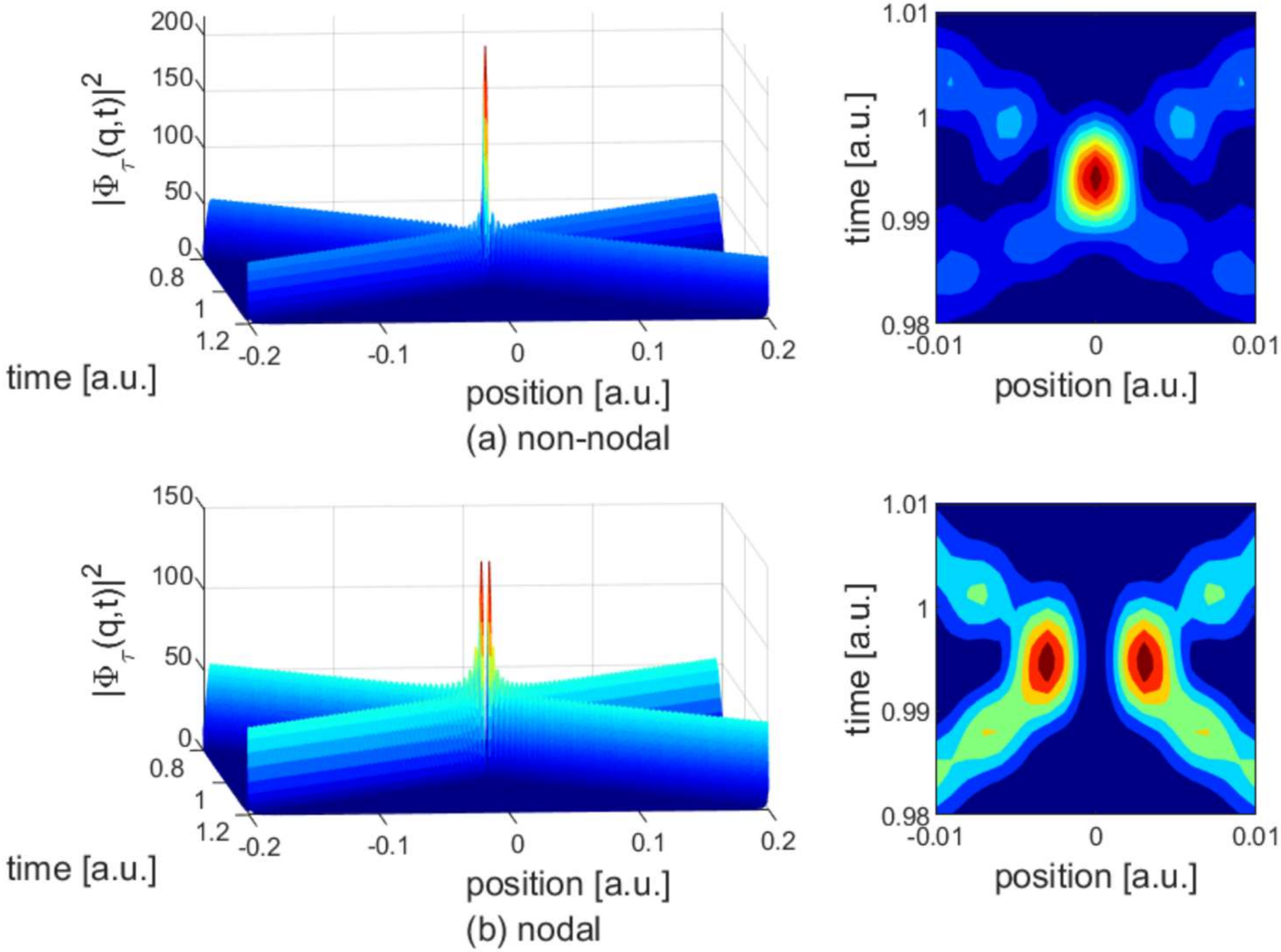
Eigenfunctions of the RHS extension

$$\tau\tilde{\Phi}_\tau(q) = \int_{-\infty}^{\infty} dq' \left[1 + \frac{2}{\pi}\int_1^\infty dz \exp(-\frac{\mu c}{\hbar}|q-q'|z)\frac{\sqrt{z^2-1}}{z}\right] \times \frac{\mu}{i\hbar}\left(\frac{q+q'}{4}\right)\mathrm{sgn}(q-q')\tilde{\Phi}_\tau(q')$$

Nystrom method:

$$\tau\tilde{\Phi}(q_k) \approx \sum_{l=1}^{2n+1} w_l \langle q_k|\hat{\mathsf{T}}_{\mathsf{Ra}}|q_l\rangle\tilde{\Phi}(q_l)$$

Dynamics of the TOA eigenfunctions in position space



Exhibits non-nodal and nodal eigenfunctions similar to the non-relativistic case

Dynamics of the TOA eigenfunctions in position space

Razavi's eigenfunction

$$\hat{\mathsf{T}}_{\mathsf{Ra}}\phi_\tau(p) = -\frac{i\hbar}{c^2}\left[\frac{E_p}{p}\frac{\partial}{\partial p} - \frac{1}{2}\frac{\mu^2 c^4}{p^2 E_p}\right]\phi_\tau(p) = \tau\phi_\tau(p)$$

$$\phi_\tau(p) = N\sqrt{\frac{|p|c}{E_p}}\exp\left[\frac{i}{\hbar}E_p\boxed{\tau}\right]$$

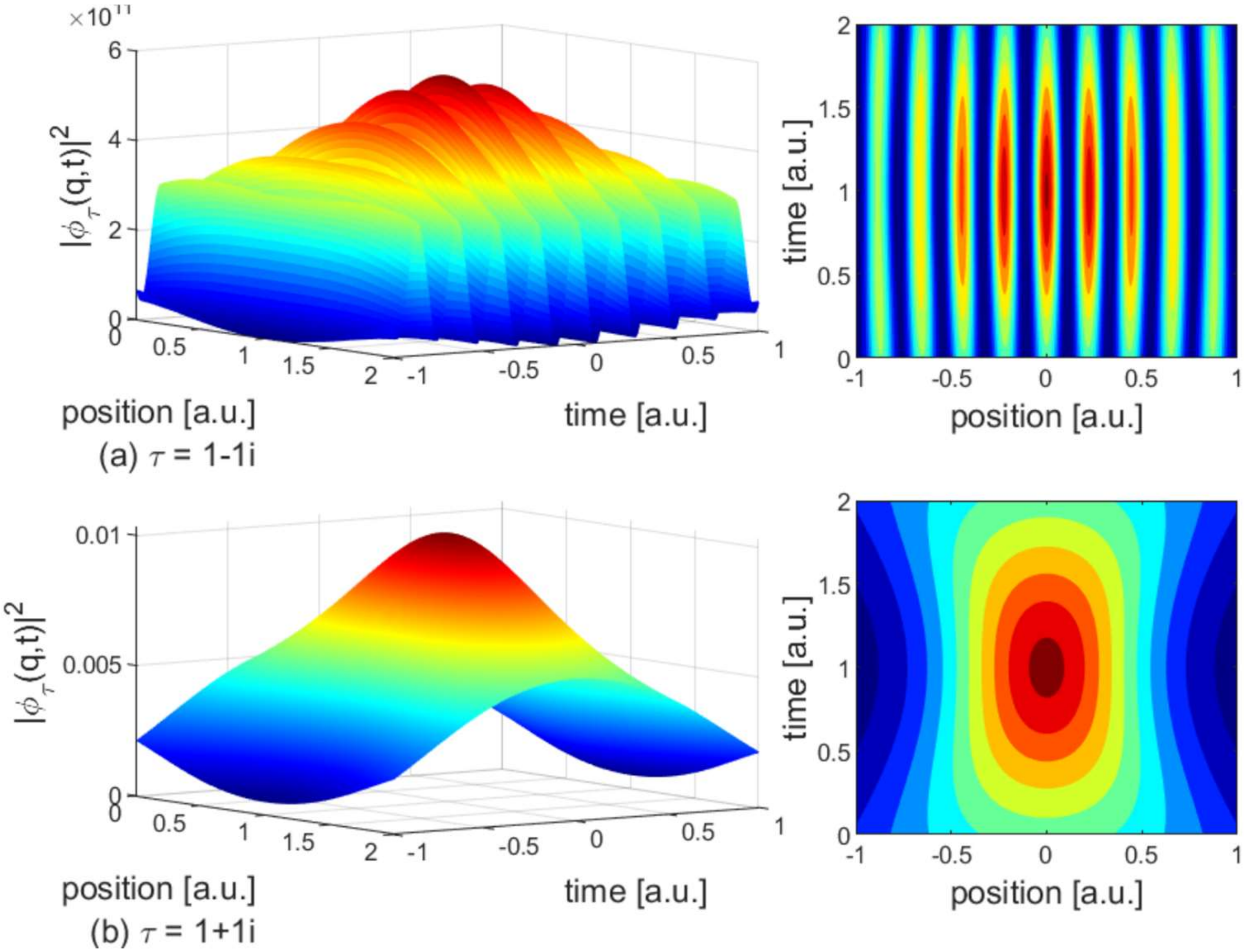
Complex-valued

$$\phi_\tau^{(Re)}(\epsilon,p) = \sqrt{\frac{\hbar}{\pi\boxed{\epsilon}}\frac{\sin(\epsilon E_p/\hbar)}{E_p}}\sqrt{\frac{|p|c}{E_p}}\exp\left[\frac{i}{\hbar}E_p\boxed{\tau}\right]$$

Goes to zero

Real-valued

Dynamics of the TOA eigenfunctions in position space

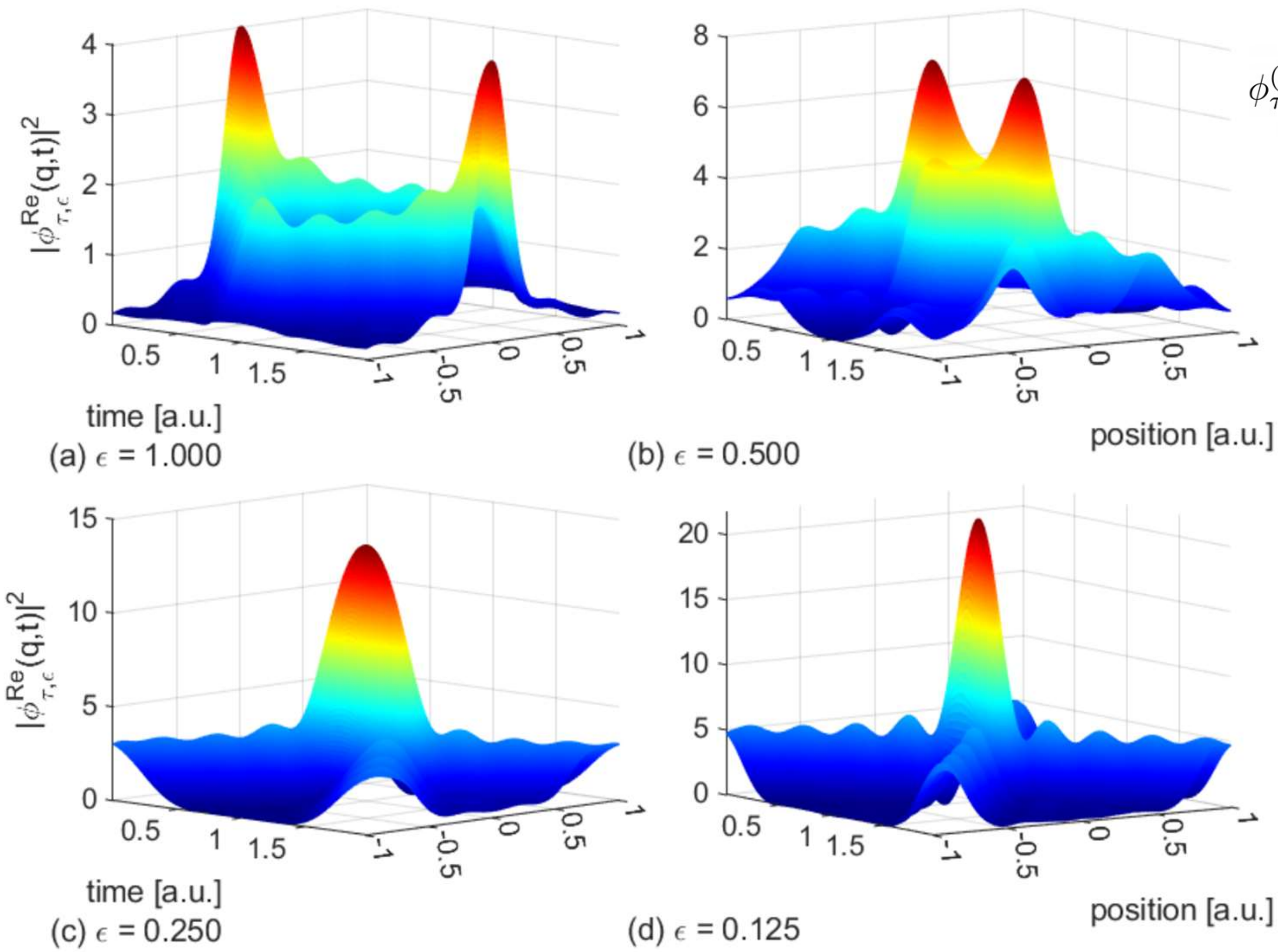


$$\phi_\tau(p) = N\sqrt{\frac{|p|c}{E_p}}\exp\left[\frac{i}{\hbar}E_p\tau\right]$$

- ❑ probability density oscillates
- ❑ single peak gathers at the arrival point

no sharp localization

Dynamics of the TOA eigenfunctions in position space



$$\phi_\tau^{(Re)}(\epsilon,p) = \sqrt{\frac{\hbar}{\pi\epsilon}\frac{\sin(\epsilon E_p/\hbar)}{E_p}}\sqrt{\frac{|p|c}{E_p}}\exp\left[\frac{i}{\hbar}E_p\tau\right]$$

Only captures the dynamics of the non-nodal eigenfunctions

Recovering nodal and non-nodal from Razavi's eigenfunctions

$$\hat{\mathrm{T}}_{\mathrm{Ra}}\phi_\tau(p) = -\frac{i\hbar}{c^2}\left[\frac{E_p}{p}\frac{\partial}{\partial p} - \frac{1}{2}\frac{\mu^2 c^4}{p^2 E_p}\right]\phi_\tau(p) = \tau\phi_\tau(p)$$

Solve for p>0 and p<0

$$\phi_\tau^{(\pm)}(p) = \sqrt{\frac{c}{4\pi\hbar}}\sqrt{\frac{|p|c}{E_p}}\exp\left[\frac{i}{\hbar}E_p\tau\right]\underbrace{\Theta(\pm p)}_{\text{Heaviside function}}$$

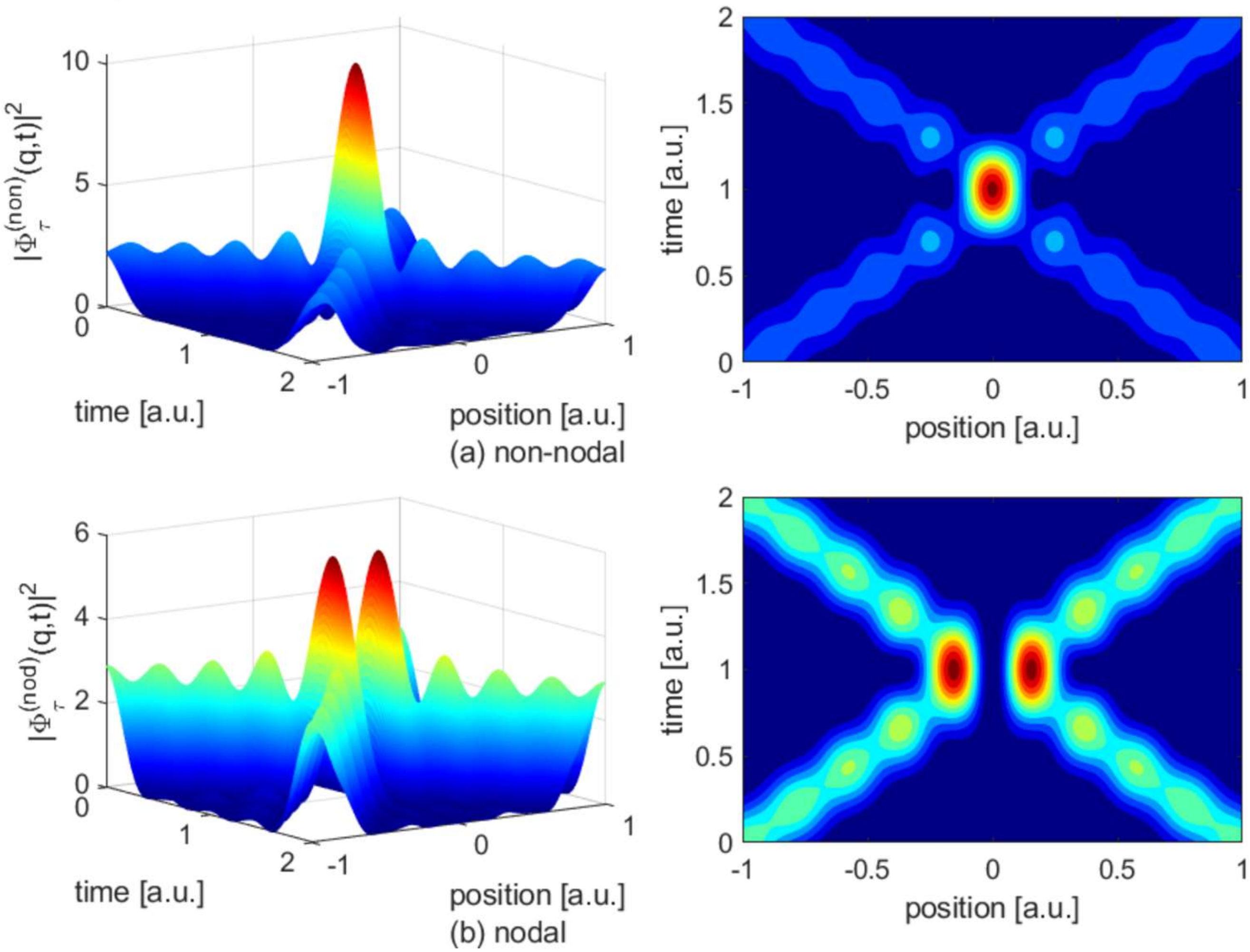
Recovering nodal and non-nodal from Razavi's eigenfunctions

$$\hat{\mathsf{T}}_{\mathsf{Ra}}\phi_\tau(p) = -\frac{i\hbar}{c^2}\left[\frac{E_p}{p}\frac{\partial}{\partial p} - \frac{1}{2}\frac{\mu^2 c^4}{p^2 E_p}\right]\phi_\tau(p) = \tau\phi_\tau(p)$$

$$\Phi_\tau^{(non)}(p) = \sqrt{\frac{c}{4\pi\hbar}}\sqrt{\frac{|p|c}{E_p}}\exp\left[\frac{i}{\hbar}E_p\tau\right]$$

$$\Phi_\tau^{(nod)}(p) = \sqrt{\frac{c}{4\pi\hbar}}\sqrt{\frac{|p|c}{E_p}}\exp\left[\frac{i}{\hbar}E_p\tau\right]\operatorname{sgn}(p)$$

Dynamics of the TOA eigenfunctions in position space

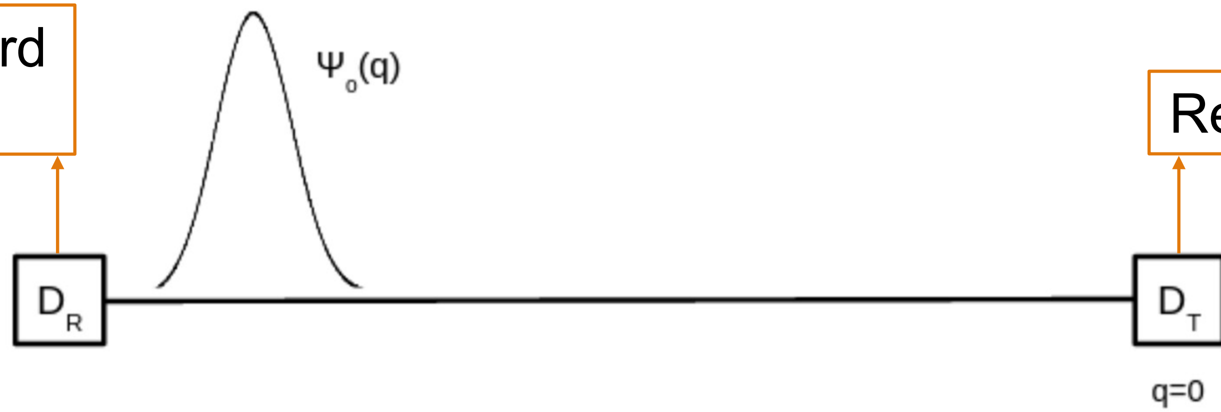


The same dynamics are observed

Expected quantum TOA

Does not record the TOA

Records the TOA



provides an indirect but accurately realistic way of obtaining the TOA of the particle at the origin

- ❑ The expected TOA is given by the expectation value of the TOA-operator

$$\tau = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dq' \psi^*(q) \langle q | \hat{\mathrm{T}}_{\mathrm{Ra}} | q' \rangle \boxed{\psi(q')} \longrightarrow \boxed{e^{ipq'/\hbar} \varphi(q')}$$

Galapon, E. A. (2012). Only above barrier energy components contribute to barrier traversal time. *Physical review letters*, *108*(17), 170402.
Sombillo, D. L., & Galapon, E. A. (2014). Quantum traversal time through a double barrier. *Physical Review A*, *90*(3), 032115.
Sombillo, D. L. B., & Galapon, E. A. (2018). Barrier-traversal-time operator and the time-energy uncertainty relation. *Physical Review A*, *97*(6), 062127.
Pablico, D. A. L., & Galapon, E. A. (2020). Quantum traversal time across a potential well. *Physical Review A*, *101*(2), 022103.

Expected quantum TOA

- The asymptotic expansion of the expected TOA is now

$$\tau = -\mu \sum_{n=0}^{\infty} \frac{\hbar^{2n}}{p^{2n+1}} \boxed{\gamma_c^{(2n)}} \boxed{\chi_1^{(n)}} + \mu \sum_{n=0}^{\infty} \frac{(-1)^n \hbar^{2n+1}}{p^{2n+2}} \boxed{\gamma_c^{(2n+1)}} \boxed{\chi_2^{(n)}}$$

$$\boxed{\gamma_c^{(n)}(p) = 1 + \frac{2}{\pi} \int_1^{\infty} dz \frac{\sqrt{z^2-1}}{z} \left(\frac{p}{p^2 + \mu^2 c^2 z^2} \right)^{n+1} \mathrm{Re}\left[(p + i\mu c z)^{n+1} \right]}$$

$$\boxed{\chi_1^{(n)} = \int_{-\infty}^{\infty} dq |\varphi^{(n)}(q)|^2 q}$$

$$\boxed{\chi_2^{(n)} = \int_{-\infty}^{\infty} dq \mathrm{Im}\left[\varphi^*(q) \varphi^{(2n+1)}(q) \right] q}$$

Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. *Physical Review A*, *105*(6), 062208.
Galapon, E. A. (2009). Quantum wave-packet size effects on neutron time-of-flight spectroscopy. *Physical Review A*, *80*(3), 030102.
Flores, P. C. M., Caballar, R. C. F., & Galapon, E. A. (2016). Synchronizing quantum and classical clocks made of quantum particles. *Physical Review A*, *94*(3), 032123.

Expected quantum TOA

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$$\tau = -\mu \underbrace{\sum_{n=0}^{\infty} \frac{\hbar^{2n}}{p^{2n+1}} \boxed{\gamma_c^{(2n)}} \boxed{\chi_1^{(n)}}}_{} + \mu \sum_{n=0}^{\infty} \frac{(-1)^n \hbar^{2n+1}}{p^{2n+2}} \gamma_c^{(2n+1)} \chi_2^{(n)}$$

n=0: classical term

$$\Rightarrow t = -\frac{\mu q_0}{p}\sqrt{1+\frac{p^2}{\mu^2 c^2}}$$

$$\chi_1^{(0)} = \int_{-\infty}^{\infty} dq |\varphi(q)|^2 q = q_o$$

$$\gamma_c^{(0)}(p) = 1 + \frac{2}{\pi}\int_1^{\infty} dz \frac{p^2}{(p^2+\mu^2 c^2 z^2)} \frac{\sqrt{z^2-1}}{z} = \sqrt{1+\frac{p^2}{\mu^2 c^2}}$$

Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. *Physical Review A*, *105*(6), 062208.
Galapon, E. A. (2009). Quantum wave-packet size effects on neutron time-of-flight spectroscopy. *Physical Review A*, *80*(3), 030102.
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Example: Quantum correction for Gaussian wavepackets

- The asymptotic expansion of the expected TOA is now

$$\tau = -\mu \sum_{n=0}^{\infty} \frac{\hbar^{2n}}{p^{2n+1}} \boxed{\gamma_c^{(2n)}} \boxed{\chi_1^{(n)}} + \cancel{\mu \sum_{n=0}^{\infty} \frac{(-1)^n \hbar^{2n+1}}{p^{2n+2}} \gamma_c^{(2n+1)}} \boxed{\chi_2^{(n)}} \rightarrow 0$$

$$\boxed{\gamma_c^{(n)}(p) = 1 + \frac{2}{\pi} \int_1^{\infty} dz \frac{\sqrt{z^2-1}}{z} \left(\frac{p}{p^2 + \mu^2 c^2 z^2} \right)^{n+1} \mathrm{Re}\left[(p + i\mu c z)^{n+1} \right]}$$

$$\boxed{\chi_1^{(n)} = \int_{-\infty}^{\infty} dq |\varphi^{(n)}(q)|^2 q} \quad \Rightarrow \chi_1^{(n)} = q_0 \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2\sigma^2)^n}$$

$$\boxed{\chi_2^{(n)} = \int_{-\infty}^{\infty} dq \mathrm{Im}\left[\varphi^*(q) \varphi^{(2n+1)}(q) \right] q} \qquad \varphi(q) = \frac{1}{\sqrt{\sigma\sqrt{2\pi}}} \exp\left[-\frac{(q-q_0)^2}{4\sigma^2} \right]$$

Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. *Physical Review A*, *105*(6), 062208.
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Example: Quantum correction for Gaussian wavepackets

- The asymptotic expansion can be re-summed as

$$\tau = \left(-\frac{\mu q_o}{p}\sqrt{1+\frac{p^2}{\mu^2c^2}}\right)\boxed{Q_c(\mu,p,\sigma)} \rightarrow \boxed{= \left(1+\frac{p^2}{\mu^2c^2}\right)^{-1/2}\left(Q_c^{(1)}+Q_c^{(2)}\right)}$$

$$Q_c^{(1)} = \frac{1}{\sqrt{\pi}}\mathrm{P.V.}\int_0^\infty e^{-s}s^{-1/2}$$

$$Q_c^{(2)} = \frac{2}{\pi^{3/2}}\int_1^\infty dz\frac{\sqrt{z^2-1}}{z}\int_0^\infty dse^{-s}s^{-1/2}$$
$$\mathrm{Re}\left[\frac{1}{1-i\frac{\mu c}{p}z}\left(1-\frac{1}{(1-i\frac{\mu c}{p}z)^2}\frac{\hbar^2}{p^2}\frac{s}{2\sigma^2}\right)^{-1}\right]$$

Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. *Physical Review A*, *105*(6), 062208.
Galapon, E. A. (2009). Quantum wave-packet size effects on neutron time-of-flight spectroscopy. *Physical Review A*, *80*(3), 030102.
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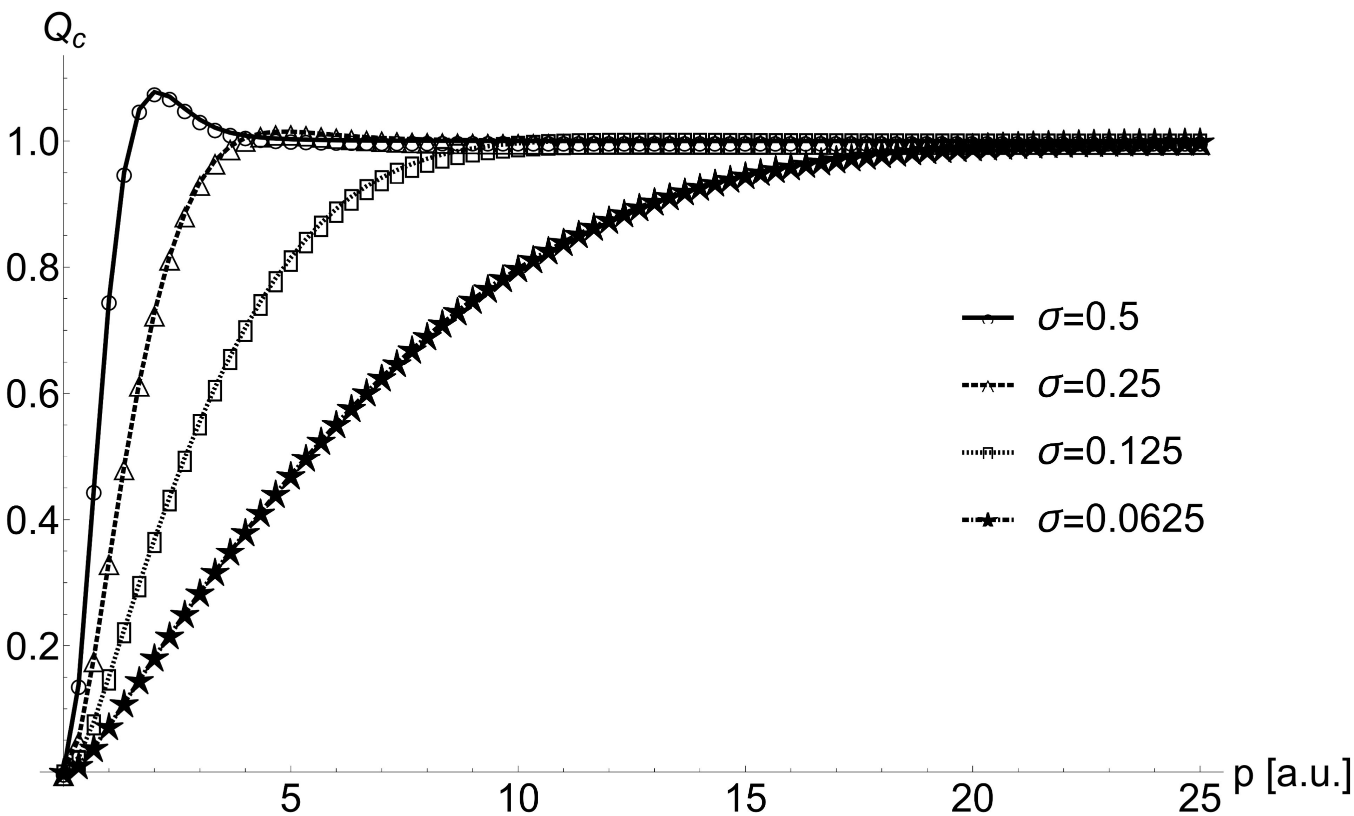
Example: Quantum correction for Gaussian wavepackets



- ❑ The correction goes to unity as the momentum increases
- ❑ The correction vanishes as the momentum vanishes

↓

Expected TOA does not vanish

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Annals of Physics

Volume 353, February 2015, Pages 83-106

A one-particle time of arrival operator for a free relativistic spin-0 charged particle in $(1+1)$ dimensions

Joseph Bunao, Eric A. Galapon

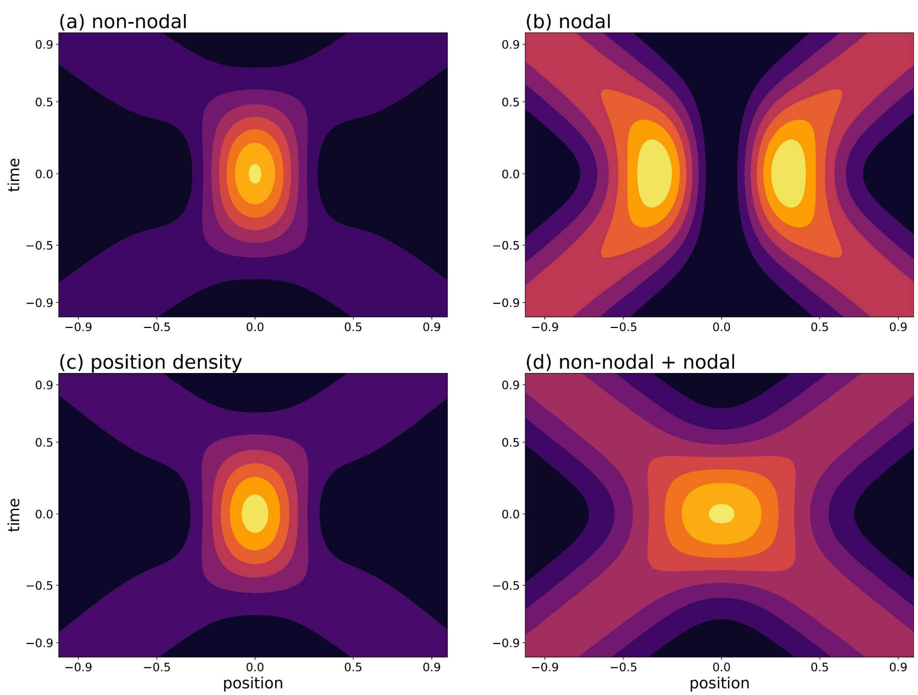
Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. *Physical Review A*, *105*(6), 062208.
Galapon, E. A. (2009). Quantum wave-packet size effects on neutron time-of-flight spectroscopy. *Physical Review A*, *80*(3), 030102.
Flores, P. C. M., Caballar, R. C. F., & Galapon, E. A. (2016). Synchronizing quantum and classical clocks made of quantum particles. *Physical Review A*, *94*(3), 032123.

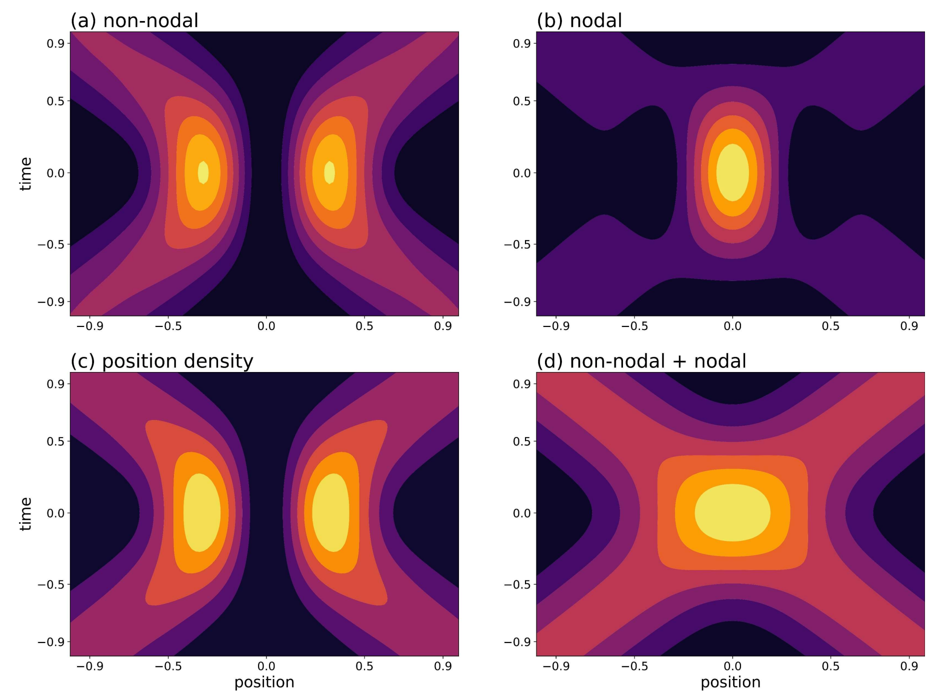
Example: TOA distribution for Gaussian wavepackets

- The asymptotic expansion can be re-summed as

$$\Pi_{\psi}(\tau) = \left| \int_{-\infty}^{\infty} dp \tilde{\psi}^*(p) \boxed{\Phi_{\tau}^{(non)}(p)} \right|^2$$

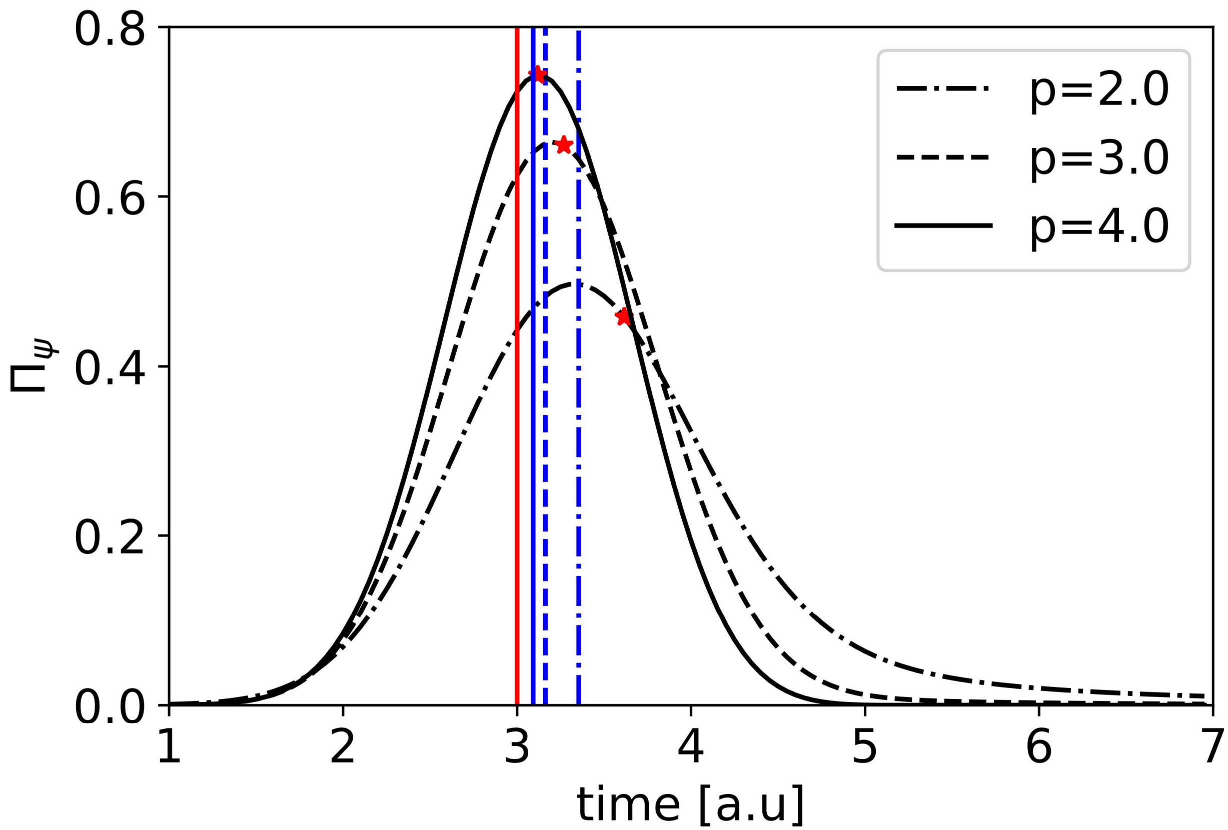
Particle arrival with detection





Sombillo, D. L. B., & Galapon, E. A. (2016). Particle detection and non-detection in a quantum time of arrival measurement. *Annals of Physics*, *364*, 261-273.

Example: TOA distribution for Gaussian wavepackets



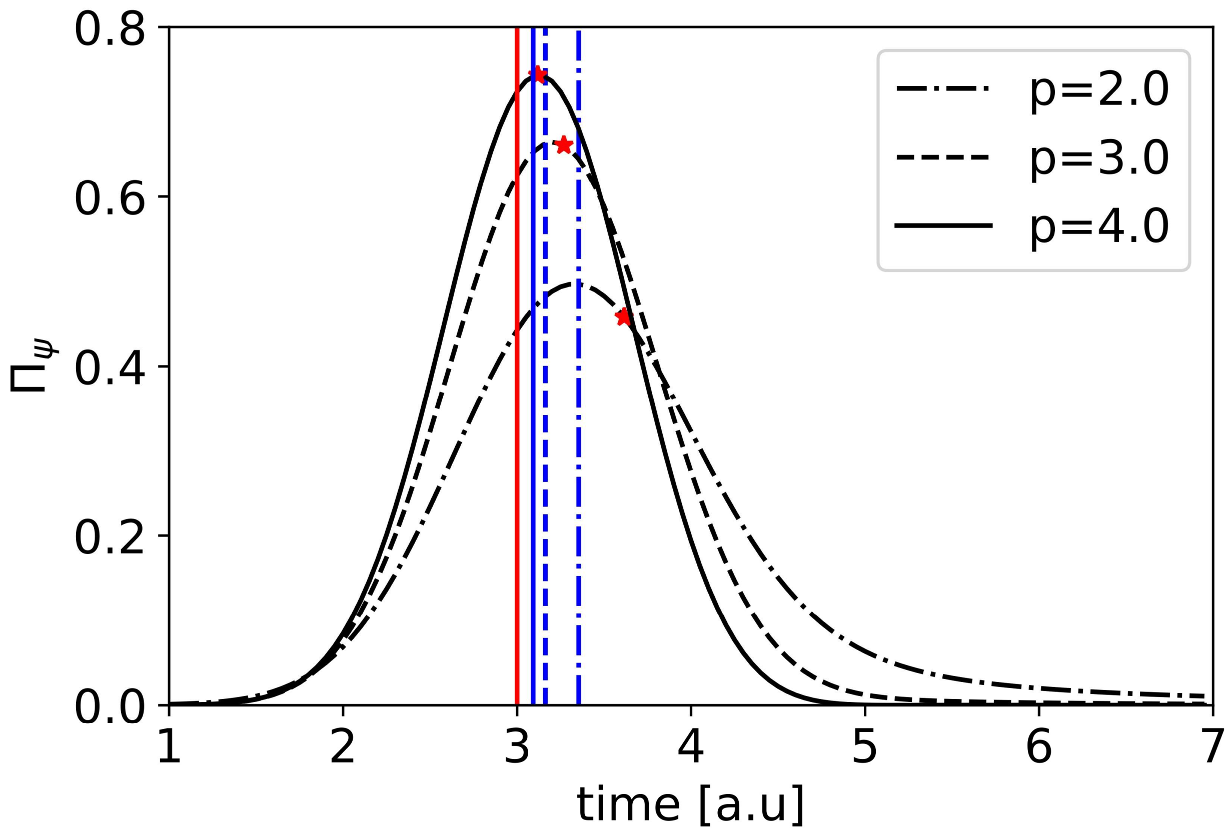
- ❑ the peak of the TOA distributions, and expectation values are greater than the TOA for a photon

↓

particles will, on average, always arrive later than a photon

Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. *Physical Review A*, *105*(6), 062208

Example: TOA distribution for Gaussian wavepackets



- ❑ TOA distributions can spread to values less than the TOA for a photon

↓

Non-zero probability of superluminal TOA

Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. *Physical Review A*, *105*(6), 062208

1. History and motivation

2. Relativistic free TOA-operator

Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. Physical Review A, 105(6), 062208.

3. TOA-operators for the interacting case

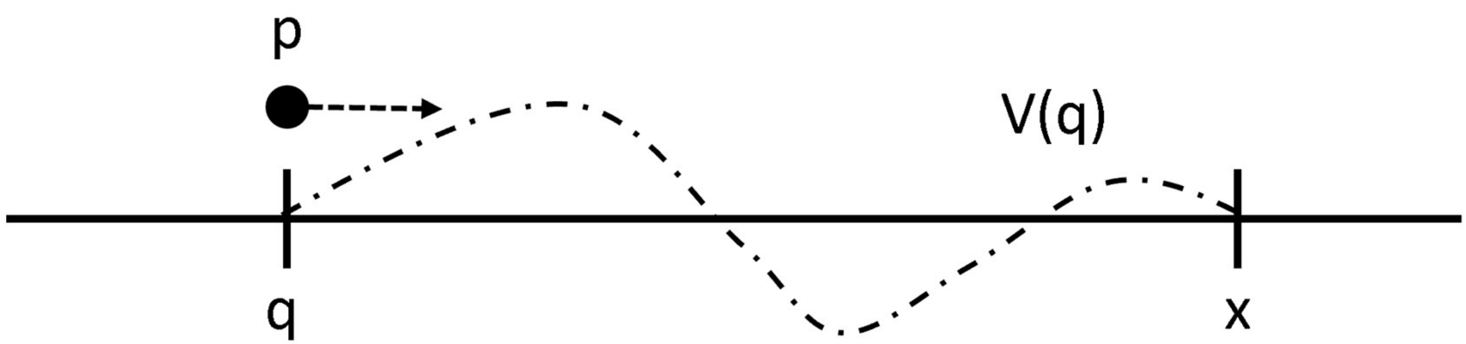
Flores, P. C., & Galapon, E. A. (2022). Quantized relativistic time-of-arrival operators for spin-0 particles and the quantum tunneling time problem. arXiv preprint arXiv:2207.00343.

4. Barrier traversal time

Flores, P. C., & Galapon, E. A. (2022). Instantaneous tunneling of relativistic massive spin-0 particles. arXiv preprint arXiv:2207.09040.

5. Final Remarks

- The classical TOA is given as



mass   initial position

$$T(q,p) = -\boxed{\mathrm{sgn}(p)}\sqrt{\frac{\boxed{\mu}}{2}}\int_{\boxed{x}}^{\boxed{q}} \frac{dq'}{\sqrt{\boxed{H(q,p)} - \boxed{V(q')}}}$$

interaction potential

signum function   arrival point   Hamiltonian

Galapon, E. A., & Magadan, J. J. P. (2018). Quantizations of the classical time of arrival and their dynamics. Annals of Physics, 397, 278-302.

Review: Quantization prescription for the non-relativistic case

- The classical TOA is given as

$$T(q,p) = -\mathrm{sgn}(p)\sqrt{\frac{\mu}{2}}\int_x^q \frac{dq'}{\boxed{\sqrt{H(q,p)-V(q')}}}$$

  - can be complex and/or multiple valued
  - quantization is not well-defined in the entire phase space

> There is still no consensus on how to construct TOA-operators in the presence of an interaction potential

Galapon, E. A., & Magadan, J. J. P. (2018). Quantizations of the classical time of arrival and their dynamics. Annals of Physics, 397, 278-302.
León, J., Julve, J., Pitanga, P., & De Urríes, F. J. (2000). Time of arrival in the presence of interactions. *Physical Review A*, *61*(6), 062101.
Peres, A. (1997). *Quantum theory: concepts and methods*. Dordrecht: Kluwer Academic.

Review: Quantization prescription for the non-relativistic case

- ❑ The classical TOA is given as

$$T(q,p) = -\mathrm{sgn}(p)\sqrt{\frac{\mu}{2}}\int_x^q \frac{dq'}{\sqrt{H(q,p)-V(q')}}$$

- ❑ Physical arguments to justify quantization
  - Quantum TOA is always single-valued (wavefunction collapse)
  - Quantum TOA is always real-valued (tunneling)

We must quantize the local time-of-arrival (LTOA) which is the expansion around the free TOA

Galapon, E. A., & Magadan, J. J. P. (2018). Quantizations of the classical time of arrival and their dynamics. Annals of Physics, 397, 278-302.
León, J., Julve, J., Pitanga, P., & De Urríes, F. J. (2000). Time of arrival in the presence of interactions. *Physical Review A*, *61*(6), 062101.
Peres, A. (1997). *Quantum theory: concepts and methods*. Dordrecht: Kluwer Academic.

Review: Quantization prescription for the non-relativistic case

- ❑ The classical TOA is given as

$$T(q,p) = -\mathrm{sgn}(p)\sqrt{\frac{\mu}{2}}\int_x^q \frac{dq'}{\sqrt{H(q,p)-V(q')}}$$

- ❑ The quantization prescription is as follows

$$\boxed{\begin{array}{l}\text{Step 1:}\\ \text{Obtain the LTOA}\end{array}} \longrightarrow \boxed{\begin{array}{l}\text{Step 2:}\\ \text{Impose an ordering rule}\end{array}} \longrightarrow \boxed{\begin{array}{l}\text{Step 3:}\\ \text{Take the coordinate-representation}\end{array}}$$

Galapon, E. A., & Magadan, J. J. P. (2018). Quantizations of the classical time of arrival and their dynamics. Annals of Physics, 397, 278-302.
León, J., Julve, J., Pitanga, P., & De Urríes, F. J. (2000). Time of arrival in the presence of interactions. *Physical Review A*, *61*(6), 062101.
Peres, A. (1997). *Quantum theory: concepts and methods*. Dordrecht: Kluwer Academic.

Quantization prescription for the relativistic case

Step 1:
Obtain the LTOA

$$H(q,p) = \sqrt{p^2c^2 + \mu_o^2 c^4} + V(q)$$

- The classical TOA is given as

$$t(q,p) = -\mathrm{sgn}p \int_{\boxed{0}}^{q} \frac{dq'}{c}\left(1 - \frac{\mu_o^2 c^4}{(H(q,p) - V(q'))^2}\right)^{-1/2}$$

Let the arrival point be x=0

$$t(q,p) = -\mu_o \sum_{j=0}^{\infty}\sum_{k=0}^{j} \binom{-\frac{1}{2}}{j}\binom{j}{k} \frac{(2\mu_o)^j}{(2\mu_o c^2)^{j-k}} \sum_{n=1}^{\infty} a_n^{(2j-k)} \frac{\gamma_p^{k+1}}{p^{2j+1}} q^n$$

$$\gamma_p = \sqrt{1 + \frac{p^2}{\mu_o^2 c^2}}$$

Assume that the potential is analytic at the origin

Goldstein, H., Poole, C., & Safko, J. (2002). Classical mechanics.
Greiner, W. (2000). *Relativistic quantum mechanics* (Vol. 2). springer, Berlin.

Quantization prescription for the relativistic case

Step 1:
Obtain the LTOA

Step 2:
Impose an ordering rule

❑ The relativistic LTOA is given as

$$t(q,p) = -\mu_o \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{-\frac{1}{2}}{j} \binom{j}{k} \frac{(2\mu_o)^j}{(2\mu_o c^2)^{j-k}} \sum_{n=1}^{\infty} a_n^{(2j-k)} \boxed{\frac{\gamma_p^{k+1}}{p^{2j+1}} q^n}$$

❑ We now quantize the functions $Q[q^n \gamma_p^{k+1} p^{-2j-1}] \Rightarrow Q[q^n h(p)]$

$$Q[q^n h(p)] = \frac{\sum_{k=0}^{n} \boxed{\alpha_k^{(n)}} \hat{\mathsf{q}}^k h(\hat{\mathsf{p}}) \hat{\mathsf{q}}^{n-k}}{\sum_{k=0}^{n} \boxed{\alpha_k^{(n)}}}$$

$$\boxed{\alpha_k^{(n)} = \begin{cases} \dfrac{n!}{k!(n-k)!} & , \quad \text{Weyl} \\ 1 & , \quad \text{Born-Jordan} \\ \delta_{k,0} + \delta_{k,n} & , \quad \text{simple-symmetric} \end{cases}}$$

Galapon, E. A., & Magadan, J. J. P. (2018). Quantizations of the classical time of arrival and their dynamics. Annals of Physics, 397, 278-302.
Bender, C. M., & Dunne, G. V. (1988). Polynomials and operator orderings. *Journal of mathematical physics*, *29*(8), 1727-1731.

Quantization prescription for the relativistic case

Step 1:
Obtain the LTOA

Step 2:
Impose an ordering rule

Step 3:
Take the coordinate rep.

- We see that the relevant value to compute is

$$\langle q|\hat{\mathsf{p}}^{-2j-1}\gamma_{\hat{\mathsf{p}}}^{\mathsf{k}+1}|q'\rangle = \int_{-\infty}^{\infty}\frac{dp}{2\pi\hbar}\exp\left[\frac{i}{\hbar}(q-q')p\right]\boxed{\frac{1}{p^{2j+1}}}\boxed{\left(\sqrt{1+\frac{p^2}{\mu_o^2c^2}}\right)^{k+1}}$$

  - Diverges because of the pole
  - Also has branch points

Evaluated as a distributional Fourier transform which coincides with the Hadamard finite-part

Galapon, E. A. (2016). The Cauchy principal value and the Hadamard finite part integral as values of absolutely convergent integrals. *Journal of Mathematical Physics*, *57*(3), 033502.

Quantization prescription for the relativistic case

Step 1: Obtain the LTOA	**Step 2:** Impose an ordering rule	**Step 3:** Take the coordinate rep.

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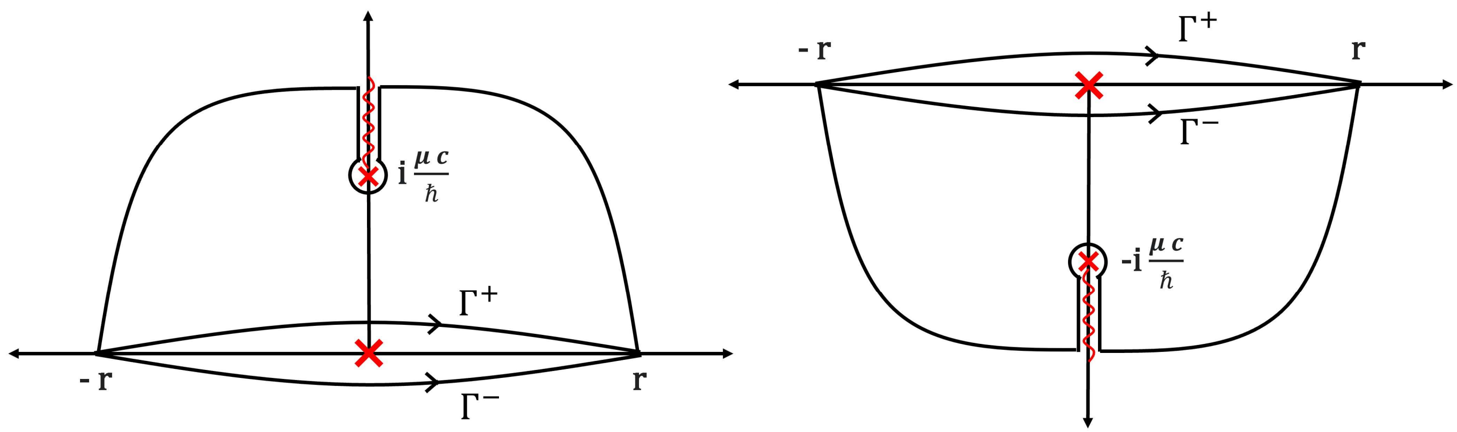

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Quantization prescription for the relativistic case

Step 1:
Obtain the LTOA

Step 2:
Impose an ordering rule

Step 3:
Take the coordinate rep.

- We see that the relevant value to compute is

$$\langle q|\hat{\mathsf{p}}^{-2j-1}\gamma_{\hat{\mathsf{p}}}^{k+1}|q'\rangle = -\frac{1}{2i\hbar}\left(\boxed{f_{j,k}(q,q')} + \boxed{g_{j,k}(q,q')}\right)\operatorname{sgn}(q-q')$$

$$\boxed{f_{j,k}(q,q') = \frac{1}{(2j)!}\left(\frac{i}{\hbar}(q-q')\right)^{2j}\int_0^\infty dy e^{-y}\oint_R \frac{dz}{2\pi i}\frac{1}{z}\sqrt{1+\frac{z^2}{\mu_o^2 c^2}}^{\,k+1}\left(1 - i\frac{\hbar}{q-q'}\frac{y}{z}\right)^{2j}}$$

$$\boxed{g_{j,k}(q,q') = \frac{(-1)^j i^k}{(\mu_o c)^{2j}}\left(\frac{1-(-1)^{k+1}}{2}\right)\frac{2}{\pi}\int_1^\infty dy \exp\left[-\frac{\mu_o c}{\hbar}|q-q'|y\right]\frac{\sqrt{y^2-1}^{\,k+1}}{y^{2j+1}}}$$

Galapon, E. A. (2016). The Cauchy principal value and the Hadamard finite part integral as values of absolutely convergent integrals. *Journal of Mathematical Physics*, *57*(3), 033502.

Quantization prescription for the relativistic case

- The TOA-operator now has the form $(\hat{\mathrm{T}}\varphi)(q) = \int_{-\infty}^{\infty} dq' \frac{\mu}{i\hbar} \boxed{T(q,q')} \mathrm{sgn}(q-q')\varphi(q')$

$$T(q,q') = \frac{1}{2}\int_0^{\frac{q+q'}{2}} ds \left[\boxed{\mathcal{W}_s^{(1)}(q,q')} + \frac{2}{\pi}\int_1^{\infty} dz \exp[-\frac{\mu c}{\hbar}|q-q'|z] \frac{\sqrt{z^2-1}}{z} \mathcal{W}_{s,z}^{(2)}(q,q') \right]$$

$$\mathcal{W}_s^{(1)}(q,q') = \int_0^{\infty} dy e^{-y} \oint_R \frac{dz}{2\pi i} \frac{1}{z} \sqrt{1 + \frac{z^2}{\mu^2 c^2}}$$

$${}_0F_1\left[;1;\frac{\mu V_s^{(W)}(q,q')}{2\hbar^2}\left((q-q') - i\hbar\frac{y}{z}\right)^2 \mathcal{P}_{\mathcal{W}}(s,z,q,q')\right]$$

$$\mathcal{P}_{\mathcal{W}}(s,z,q,q') = \left(\sqrt{1+\frac{z^2}{\mu^2 c^2}} + \frac{V_s^{(W)}(q,q')}{2\mu c^2}\right)$$

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$$T(q,q') = \frac{1}{2}\int_0^{\frac{q+q'}{2}} ds \left[\mathcal{W}_s^{(1)}(q,q') + \frac{2}{\pi}\int_1^{\infty} dz \exp[-\frac{\mu c}{\hbar}|q-q'|z] \frac{\sqrt{z^2-1}}{z} \boxed{\mathcal{W}_{s,z}^{(2)}(q,q')} \right]$$

$$\mathcal{W}_{s,z}^{(2)}(q,q') = \frac{1}{2}\left[1 - \frac{1}{z^2}\left(\frac{V_s^{(W)}(q,q')}{\mu c^2}\right)^2 + 2i\frac{\sqrt{z^2-1}}{z^2}\left(\frac{V_s^{(W)}(q,q')}{\mu c^2}\right)\right]^{-1/2} + c.c.$$

$$V_s^{(W)}(q,q') = V\left(\frac{q+q'}{2}\right) - V(s)$$

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Quantization prescription for the relativistic case

- The TOA-operator now has the form $(\hat{\mathrm{T}}\varphi)(q) = \int_{-\infty}^{\infty} dq' \frac{\mu}{i\hbar} T(q,q')\mathrm{sgn}(q-q')\varphi(q')$

$$T(q,q') = \frac{1}{2}\int_0^{\frac{q+q'}{2}} ds \left[\mathcal{W}_s^{(1)}(q,q') + \frac{2}{\pi}\int_1^{\infty} dz \exp[-\frac{\mu c}{\hbar}|q-q'|z]\frac{\sqrt{z^2-1}}{z}\mathcal{W}_{s,z}^{(2)}(q,q')\right]$$

$$\mathcal{W}_s^{(1)}(q,q') = \int_0^{\infty} dy e^{-y} \oint_R \frac{dz}{2\pi i}\frac{1}{z}\sqrt{1+\frac{z^2}{\mu^2c^2}}$$

$$ {}_0F_1\left[;1;\frac{\mu V_s^{(W)}(q,q')}{2\hbar^2}\left((q-q') - i\hbar\frac{y}{z}\right)^2 \mathcal{P}_{\mathcal{W}}(s,z,q,q')\right]$$

$$\mathcal{W}_{s,z}^{(2)}(q,q') = \frac{1}{2}\left[1 - \frac{1}{z^2}\left(\frac{V_s^{(W)}(q,q')}{\mu c^2}\right)^2 + 2i\frac{\sqrt{z^2-1}}{z^2}\left(\frac{V_s^{(W)}(q,q')}{\mu c^2}\right)\right]^{-1/2} + c.c.$$

1. History and motivation

2. Relativistic free TOA-operator

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4. Barrier traversal time

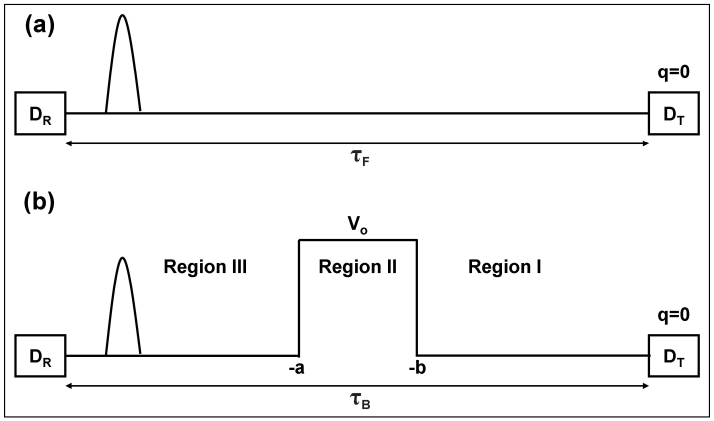
Flores, P. C., & Galapon, E. A. (2022). Instantaneous tunneling of relativistic massive spin-0 particles. arXiv preprint arXiv:2207.09040.

5. Final Remarks

Time kernel factors

- Recall that the TOA-operators have the form

$$(\hat{\mathrm{T}}\varphi)(q) = \int_{-\infty}^{\infty} dq' \frac{\mu}{i\hbar} \boxed{T(q,q')} \mathrm{sgn}(q-q')\varphi(q')$$



$$(q,q') \to (\eta,\zeta)$$

$$\zeta = q - q'$$

$$\eta = \frac{q+q'}{2}$$

Free case:

$$\tilde{T}_F(\eta,\zeta) = \frac{\eta}{2}\mathcal{T}_F(\zeta)$$

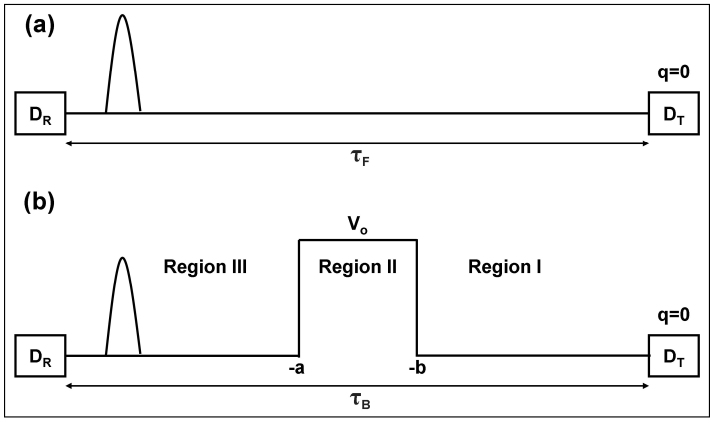
$$\mathcal{T}_F(\zeta) = 1 + \frac{2}{\pi}\int_1^{\infty} dz \frac{\sqrt{z^2-1}}{z}\exp[-\frac{\mu c}{\hbar}|\zeta|z]$$

Flores, P. C., & Galapon, E. A. (2022). Instantaneous tunneling of relativistic massive spin-0 particles. *arXiv preprint arXiv:2207.09040*.

Time kernel factors

- Recall that the TOA-operators have the form

$$(\hat{\mathrm{T}}\varphi)(q) = \int_{-\infty}^{\infty} dq' \frac{\mu}{i\hbar} \boxed{T(q,q')} \operatorname{sgn}(q-q')\varphi(q')$$



Barrier case:

$$\begin{cases} \frac{\eta}{2}\mathcal{T}_F(\zeta) & , \quad \eta \geq -b \\ \frac{\eta+b}{2}\mathcal{T}_F(\zeta) - \frac{b}{2}\boxed{\mathcal{T}_B(V_o,\zeta)} & , \quad -a < \eta < -b \\ \frac{\eta+L}{2}\mathcal{T}_F(\zeta) - \frac{L}{2}\boxed{\mathcal{T}_B(-V_o,\zeta)} & , \quad \eta \leq -a \end{cases}$$

$$\mathcal{T}_B(V_o,\zeta) = \mathcal{F}_B(V_o,\zeta) + \frac{2}{\pi}\int_1^{\infty} dy \frac{\sqrt{y^2-1}}{y} \exp[-\frac{\mu c}{\hbar}|\zeta|y]\mathcal{G}_B(V_o,y)$$

Flores, P. C., & Galapon, E. A. (2022). Instantaneous tunneling of relativistic massive spin-0 particles. *arXiv preprint arXiv:2207.09040*.

Time kernel factors

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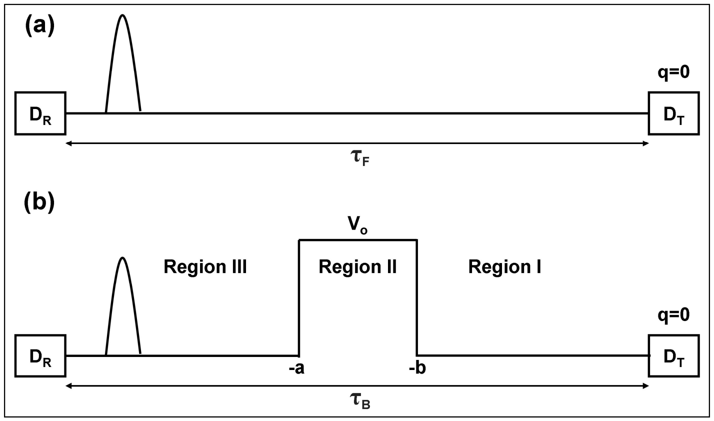


Barrier case:

$$\mathcal{T}_B(V_o,\zeta) = \boxed{\mathcal{F}_B(V_o,\zeta)} + \frac{2}{\pi}\int_1^{\infty} dy \frac{\sqrt{y^2-1}}{y} \exp[-\frac{\mu c}{\hbar}|\zeta|y]\mathcal{G}_B(V_o,y)$$

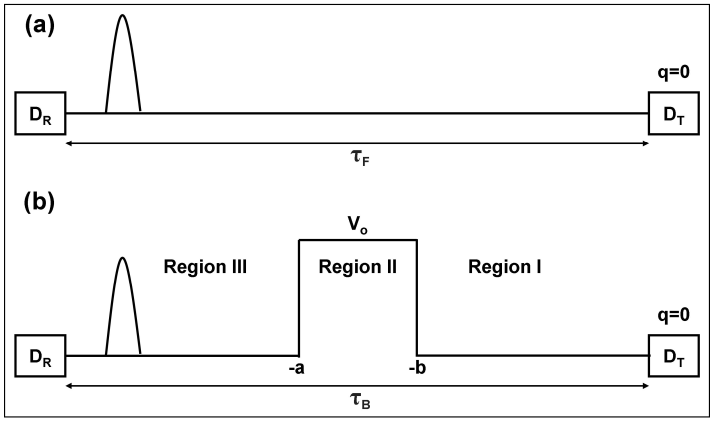
$$\boxed{\mathcal{F}_B(V_o,\zeta) = \int_0^{\infty} dy e^{-y} \oint_R \frac{dz}{2\pi i}\frac{1}{z}\sqrt{1+\frac{z^2}{\mu^2c^2}} {}_0F_1\left[;1;\frac{\mu V_o}{2\hbar^2}\left(\zeta - i\hbar\frac{y}{z}\right)^2 \mathcal{P}_B(V_o,z)\right]}$$

$$\mathcal{P}_B(V_o,z) = \sqrt{1+z^2/(\mu c)^2} + V_o/(2\mu c^2)$$

Flores, P. C., & Galapon, E. A. (2022). Instantaneous tunneling of relativistic massive spin-0 particles. *arXiv preprint arXiv:2207.09040*.

Time kernel factors

- Recall that the TOA-operators have the form



$$(\hat{\mathrm{T}}\varphi)(q) = \int_{-\infty}^{\infty} dq' \frac{\mu}{i\hbar} \boxed{T(q,q')} \mathrm{sgn}(q-q')\varphi(q')$$

Barrier case:

$$\mathcal{T}_B(V_o,\zeta) = \mathcal{F}_B(V_o,\zeta) + \frac{2}{\pi}\int_1^{\infty} dy \frac{\sqrt{y^2-1}}{y} \exp[-\frac{\mu c}{\hbar}|\zeta|y] \boxed{\mathcal{G}_B(V_o,y)}$$

$$\boxed{\mathcal{G}_B(V_o,y) = \frac{1}{2}\left(1 - \left(\frac{1}{y}\frac{V_o}{\mu c^2}\right)^2 + 2i\frac{\sqrt{y^2-1}}{y^2}\left(\frac{V_o}{\mu c^2}\right)\right)^{-1/2} + g_{i\rightarrow -i}}$$

Flores, P. C., & Galapon, E. A. (2022). Instantaneous tunneling of relativistic massive spin-0 particles. *arXiv preprint arXiv:2207.09040*.

Classical limits of the time kernel factors

- To check for consistency, we verify that the TOA-operators corresponding to the TKF reduce to the classical value

Weyl-Wigner transform:

$$\tilde{t}(q_o, p_o) = \frac{\mu}{i\hbar} \int_{-\infty}^{\infty} d\zeta e^{-ip_o\zeta/\hbar} \tilde{T}(q_o, \zeta) \mathrm{sgn}(\zeta)$$

Classical value:

$$\tilde{t}(q_o, p_o) = -\mathrm{sgn}(p_o) \int_0^{q_o} \frac{dq'}{c} \left(1 - \frac{\mu^2 c^4}{(H(q_o, p_o) - V(q'))^2}\right)^{-1/2}$$

Classical limits of the time kernel factors

Weyl-Wigner transform: $$\tilde{t}(q_o, p_o) = \frac{\mu}{i\hbar}\int_{-\infty}^{\infty} d\zeta e^{-ip_o\zeta/\hbar}\tilde{T}(q_o,\zeta)\mathrm{sgn}(\zeta)$$

Free case

$$\tilde{t}_F = \frac{\mu}{i\hbar}\frac{q_o}{2}\int_{-\infty}^{\infty} d\zeta e^{-ip_o\zeta/\hbar}\mathrm{sgn}(\zeta)$$

$$+\frac{\mu}{i\hbar}\frac{q_o}{2}\frac{2}{\pi}\int_{1}^{\infty} dz\frac{\sqrt{z^2-1}}{z}\int_{-\infty}^{\infty} d\zeta\,\exp[-\frac{\mu c}{\hbar}|\zeta|z]e^{-ip_o\zeta/\hbar}\mathrm{sgn}(\zeta)$$

$$= -\frac{\mu q_o}{p_o}\sqrt{1+\frac{po^2}{\mu^2 c^2}}$$

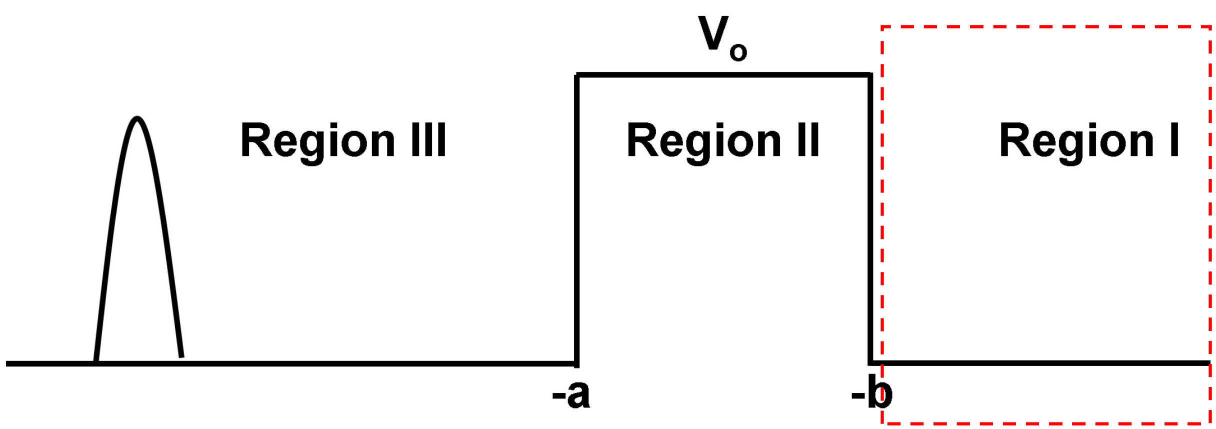
Classical limits of the time kernel factors

Weyl-Wigner transform: $$\tilde{t}(q_o, p_o) = \frac{\mu}{i\hbar} \int_{-\infty}^{\infty} d\zeta e^{-ip_o\zeta/\hbar} \tilde{T}(q_o, \zeta) \mathrm{sgn}(\zeta)$$

Barrier case

$$\tilde{t}_B^{(I)} = \boxed{-\frac{\mu q_o}{p_o}\sqrt{1 + \frac{po^2}{\mu^2 c^2}}}$$

The particle is free in this region



Classical limits of the time kernel factors

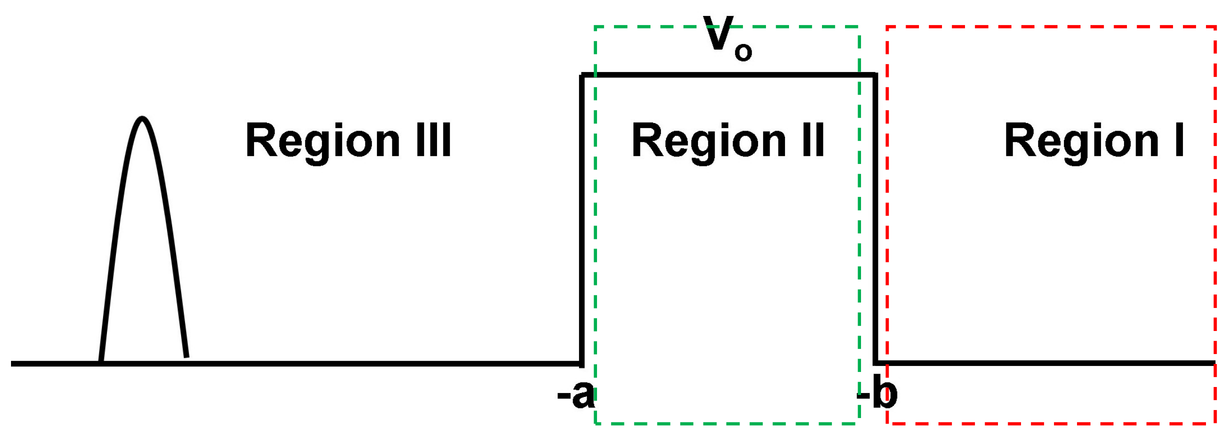
Weyl-Wigner transform: $$\tilde{t}(q_o,p_o)=\frac{\mu}{i\hbar}\int_{-\infty}^{\infty}d\zeta e^{-ip_o\zeta/\hbar}\tilde{T}(q_o,\zeta)\mathrm{sgn}(\zeta)$$

Barrier case

$$\tilde{t}_B^{(II)}=\boxed{-\frac{\mu(q_o+b)}{p_o}\sqrt{1+\frac{p_o^2}{\mu^2c^2}}}+\boxed{\frac{b}{c}\sqrt{\frac{1+\frac{p_o^2}{\mu^2c^2}}{\left(\sqrt{1+\frac{p_o^2}{\mu^2c^2}}+\frac{V_o}{\mu c^2}\right)^2-1}}}$$

TOA from the edge of the barrier

traversal time on top of the barrier



Classical limits of the time kernel factors

Weyl-Wigner transform:

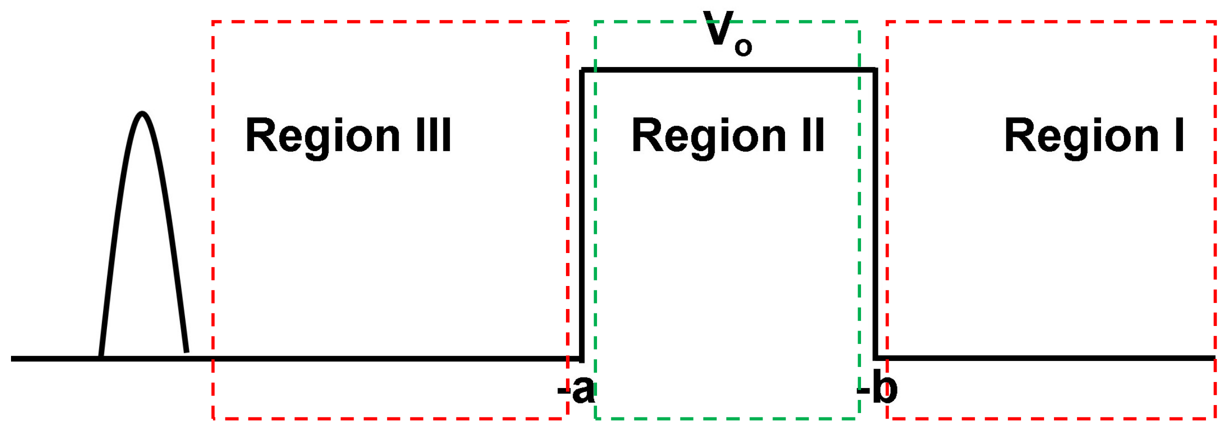
$$\tilde{t}(q_o,p_o) = \frac{\mu}{i\hbar}\int_{-\infty}^{\infty} d\zeta e^{-ip_o\zeta/\hbar}\tilde{T}(q_o,\zeta)\mathrm{sgn}(\zeta)$$

Barrier case

$$\tilde{t}_B^{(III)} = -\frac{\mu(q_o+L)}{p_o}\sqrt{1+\frac{p_o^2}{\mu^2c^2}} + \frac{L}{c}\sqrt{\frac{1+\frac{p_o^2}{\mu^2c^2}}{\left(\sqrt{1+\frac{p_o^2}{\mu^2c^2}}-\frac{V_o}{\mu c^2}\right)^2-1}}$$

traversal time in the free region

traversal time across the barrier



Expected barrier traversal time

❑ The expectation value is explicitly given as

$$\bar{\tau} = \int_{-\infty}^{\infty} dq \psi^*(q) \int_{-\infty}^{\infty} dq' \frac{\mu}{i\hbar} T(q,q') \mathrm{sgn}(q-q') \boxed{\psi(q')} \longrightarrow \boxed{e^{ip_o q'/\hbar} \varphi(q')}$$

$$(q,q') \rightarrow (\eta,\zeta) \qquad \zeta = q - q' \qquad \eta = \frac{q+q'}{2}$$

$$\bar{\tau}^* = -\frac{2\mu}{\hbar} \int_{-\infty}^{\infty} d\eta \int_0^{\infty} d\zeta e^{ik_o\zeta} \tilde{T}(\eta,\zeta) \varphi^* \left(\eta - \frac{\zeta}{2}\right) \varphi \left(\eta + \frac{\zeta}{2}\right)$$

$$\Delta\bar{\tau} = \mathrm{Im}[\Delta\bar{\tau}^*] = \mathrm{Im}[\bar{\tau}_F^* - \bar{\tau}_B^*]$$

Expected barrier traversal time

- The barrier traversal time has the form

$$\Delta\bar{\tau}^* = \frac{\mu L}{p_o}(Q_c^* - R_c^*)$$

$$Q_c^* = k_o \int_0^\infty d\zeta e^{ik_o\zeta} \mathcal{T}_F(\zeta)\Phi(\zeta)$$

$$R_c^* = k_o \int_0^\infty d\zeta e^{ik_o\zeta} \mathcal{T}_B(-V_0, \zeta)\Phi(\zeta)$$

$$\Phi(\zeta) = \int_{-\infty}^\infty d\eta \varphi^* \left(\eta - \frac{\zeta}{2}\right) \varphi \left(\eta + \frac{\zeta}{2}\right)$$

Expected barrier traversal time

- The barrier traversal time has the form $$\Delta\bar{\tau}^* = \frac{\mu L}{p_o}(Q_c^* - R_c^*)$$

$$Q_c^* = k_o \int_0^\infty d\zeta e^{ik_o\zeta} \boxed{\mathcal{T}_F(\zeta)} \Phi(\zeta)$$

$$\boxed{\mathcal{T}_F(\zeta) = 1 + \frac{2}{\pi}\int_1^\infty dz \frac{\sqrt{z^2-1}}{z}\exp[-\frac{\mu c}{\hbar}|\zeta|z]}$$

$$Q_c \sim \sqrt{1 + \frac{p_o^2}{\mu^2 c^2}}$$

relativistic correction to the non-relativistic free TOA

Expected barrier traversal time

- The barrier traversal time has the form $\Delta\bar{\tau}^* = \frac{\mu L}{p_o}(Q_c^* - R_c^*)$

$$R_c^* = k_o \int_0^\infty d\zeta e^{ik_o\zeta} \boxed{\mathcal{T}_B(-V_0,\zeta)} \Phi(\zeta)$$

$$\boxed{\mathcal{T}_B(V_o,\zeta) = \boxed{\mathcal{F}_B(V_o,\zeta)} + \frac{2}{\pi}\int_1^\infty dy \frac{\sqrt{y^2-1}}{y} \exp[-\frac{\mu c}{\hbar}|\zeta|y] \boxed{\mathcal{G}_B(V_o,y)}}$$

$$\boxed{\mathcal{F}_B(V_o,\zeta) = \int_0^\infty dy e^{-y} \oint_R \frac{dz}{2\pi i}\frac{1}{z}\sqrt{1+\frac{z^2}{\mu^2c^2}}\, {}_0F_1\left[;1;\frac{\mu V_o}{2\hbar^2}\left(\zeta - i\hbar\frac{y}{z}\right)^2 \mathcal{P}_B(V_o,z)\right]}$$

$$\boxed{\mathcal{G}_B(V_o,y) = \frac{1}{2}\left(1 - \left(\frac{1}{y}\frac{V_o}{\mu c^2}\right)^2 + 2i\frac{\sqrt{y^2-1}}{y^2}\left(\frac{V_o}{\mu c^2}\right)\right)^{-1/2} + g_{i\to -i}}$$

Expected barrier traversal time

- The barrier traversal time has the form $\Delta\bar{\tau}^* = \dfrac{\mu L}{p_o}(Q_c^* - R_c^*)$

$$R_c^* = k_o \int_0^\infty d\zeta e^{ik_o\zeta} \mathcal{T}_B(-V_0, \zeta)\Phi(\zeta)$$

$$R_c \sim \frac{p_o}{\mu c}\sqrt{\frac{p^2c^2 + \mu^2c^4}{(\sqrt{p^2c^2 + \mu^2c^4} - V_o)^2 - \mu^2c^4}}$$

effective index of refraction

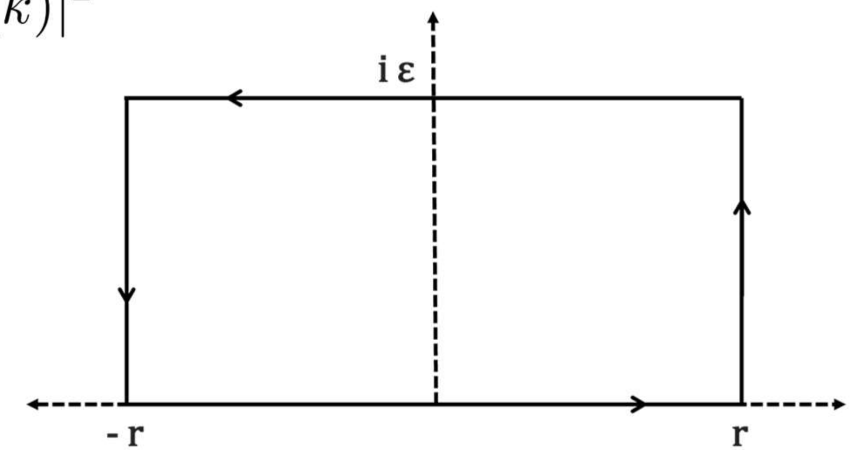
The effective index of refraction

- Explicit form of the IOR $$R_c^* = k_o \int_0^\infty d\zeta e^{ik_o\zeta}\mathcal{T}_B(-V_0,\zeta)\Phi(\zeta)$$

$$R_c^* = k_o \boxed{\int_0^\infty d\zeta}\,\mathcal{T}_B(-V_0,\zeta)\boxed{\int_{-\infty}^{\infty} dk}\,e^{ik\zeta}|\tilde{\psi}(k)|^2$$

Interchanging the orders will lead to divergent integrals

we can now perform a term-by-term integration



$$\int_{-\infty}^{\infty} e^{ix\zeta}p(x) = \int_{-\infty}^{\infty} dx e^{-(\epsilon-ix)\zeta}p(x+i\epsilon) \quad ,p(z) = |\tilde{\psi}(z)|^2$$

Pablico, D. A. L., & Galapon, E. A. (2020). Quantum traversal time across a potential well. *Physical Review A*, *101*(2), 022103.
Tica, C. D., & Galapon, E. A. (2019). Finite-part integration of the generalized Stieltjes transform and its dominant asymptotic behavior for small values of the parameter. II. Non-integer orders. *Journal of Mathematical Physics*, *60*(1), 013502.
Galapon, E. A. (2017). The problem of missing terms in term by term integration involving divergent integrals. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *473*(2197), 20160567.

The effective index of refraction

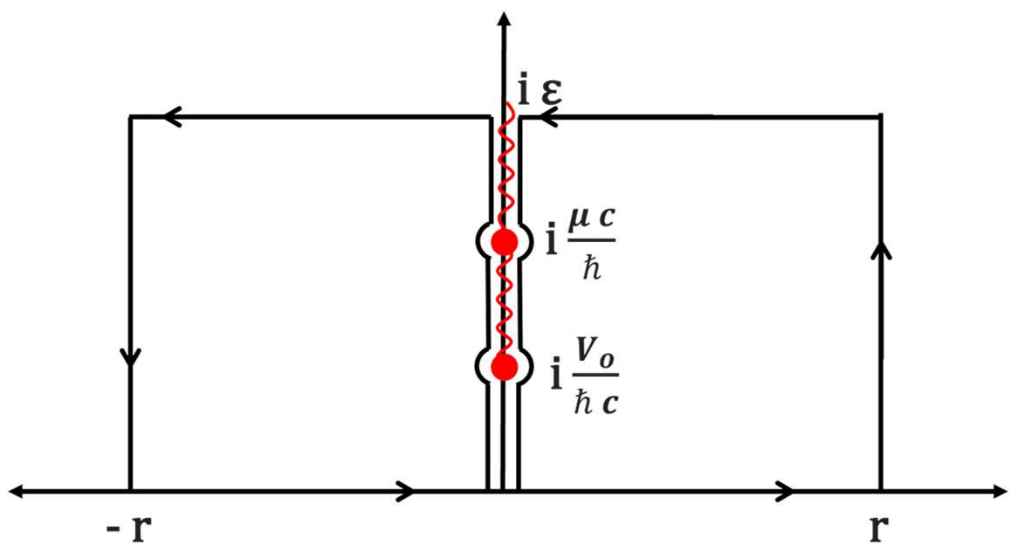
- Explicit form of the IOR $R_c^* = k_o \int_0^\infty d\zeta e^{ik_o\zeta} \mathcal{T}_B(-V_0,\zeta)\Phi(\zeta)$

$$\frac{R_c^*}{k_o} = i\sum_{n=0}^{\infty} \frac{(2n)!}{(1)_n n!}\left(\frac{\mu_o V_o}{2\hbar^2}\right)^n \boxed{\int_{-\infty}^{\infty} dk p(k+i\epsilon)\,\mathrm{csgn}(k+i\epsilon)\left((k+i\epsilon)^2+\frac{V_o^2}{\hbar^2 c^2}\right)^{-n-\frac{1}{2}}\left(\sqrt{1+\frac{\hbar^2(k+i\epsilon)^2}{\mu_o^2 c^2}}\right)^{n+1}}$$

$$-2\int_1^\infty dy \frac{\sqrt{y^2-1}}{y}\mathcal{G}_B(V_o,z)p\left(i\frac{\mu_o c}{\hbar}y\right)+\frac{2}{\pi}\int_1^\infty dy\frac{\sqrt{y^2-1}}{y}\mathcal{G}_B(V_o,z)\int_{-\infty}^{\infty} dk \frac{\frac{\mu_o c}{\hbar}y}{k^2+\frac{\mu_o^2 c^2}{\hbar^2}y^2}p(k)$$

$$z=\left\{\pm i\frac{\mu_o c}{\hbar},\pm i\frac{V_o}{\hbar c}\right\}$$

$$V_o < \mu_o c^2 \Rightarrow \frac{V_o}{\hbar c} < \frac{\mu_o c}{\hbar}$$



The effective index of refraction

- Explicit form of the IOR

$$R_c^* = k_o \int_0^\infty d\zeta e^{ik_o\zeta} \mathcal{T}_B(-V_0, \zeta)\Phi(\zeta)$$

$$R_c = \mathrm{Im}[R_c^*] = \frac{\hbar k_o}{\mu_o c}\tilde{R}_c = \frac{\hbar k_o}{\mu_o c} \boxed{\mathrm{Re}\left\{\int_0^\infty dk \left(|\tilde{\psi}(k)|^2 - |\tilde{\psi}(-k)|^2\right)\sqrt{\frac{\tilde{E}_k^2}{(\tilde{E}_k - V_o)^2 - \mu_o^2 c^4}}\right\}}$$

$$\tilde{E}_k = \sqrt{\hbar^2 k^2 c^2 + \mu_o^2 c^4}$$

$$\boxed{|k| > \kappa_c = \sqrt{\frac{2\mu_o V_o}{\hbar^2}\left(1 + \frac{V_o}{2\mu_o c^2}\right)}}$$

$$\boxed{R_c = \frac{\hbar k_o}{\mu_o c}\tilde{R}_c = \frac{\hbar k_o}{\mu_o c}\int_{\kappa_c}^\infty dk \left(|\tilde{\psi}(k)|^2 - |\tilde{\psi}(-k)|^2\right)\sqrt{\frac{\tilde{E}_k^2}{(\tilde{E}_k - V_o)^2 - \mu_o^2 c^4}}}$$

The barrier traversal time

- ❑ Thus, we finally have

$$\bar{\tau}_{\mathrm{trav}} = \frac{\mu L}{p_o}\frac{\hbar k_o}{\mu c}\tilde{R}_c = \frac{L}{c}\tilde{R}_c = \frac{L}{c}\int_{\kappa_c}^{\infty} dk\left(|\tilde{\psi}(k)|^2 - |\tilde{\psi}(-k)|^2\right)\sqrt{\frac{\tilde{E}_k^2}{(\tilde{E}_k - V_o)^2 - \mu^2 c^4}}$$

✓ Only above barrier components contribute to the traversal time

✓ Tunneling is instantaneous

The incident wavepacket must be spatially wide

The incident wavepacket must be very far from the barrier

1. **History and motivation**

2. **Relativistic free TOA-operator**

   Flores, P. C., & Galapon, E. A. (2022). Relativistic free-motion time-of-arrival operator for massive spin-0 particles with positive energy. Physical Review A, 105(6), 062208.

3. **TOA-operators for the interacting case**

   Flores, P. C., & Galapon, E. A. (2022). Quantized relativistic time-of-arrival operators for spin-0 particles and the quantum tunneling time problem. arXiv preprint arXiv:2207.00343.

4. **Barrier traversal time**

   Flores, P. C., & Galapon, E. A. (2022). Instantaneous tunneling of relativistic massive spin-0 particles. arXiv preprint arXiv:2207.09040.

5. **Final Remarks**

- The results of this Dissertation implies that instantaneous tunneling is an inherent quantum effect in the context of arrival times

  - The cases when the barrier height is greater than the rest mass energy is expected to have non-zero tunneling times

- It is enough to use the non-relativistic TOA-operators
    - The square barrier constitutes a non-linear system, the correction terms to the TOA-operator may be able to yield a non-zero tunneling time
    - The shape of the barrier might play a crucial role in instantaneous tunneling

Pablico, D. A. L., & Galapon, E. A. (2022). Quantum corrections to the Weyl quantization of the classical time of arrival. *arXiv preprint arXiv:2205.08694*