Relativistic time of arrival

(Thesis proposal)

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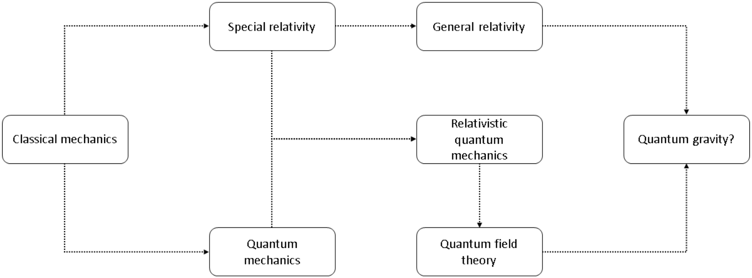
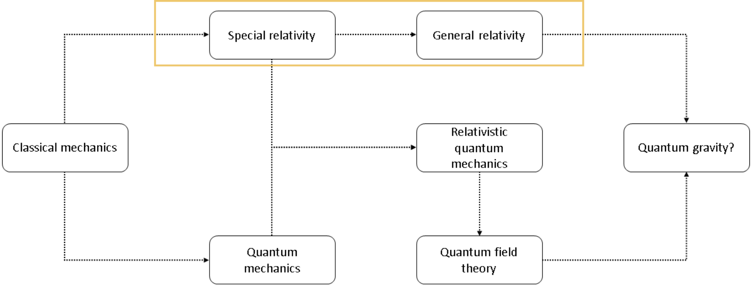
Current Physical Theories



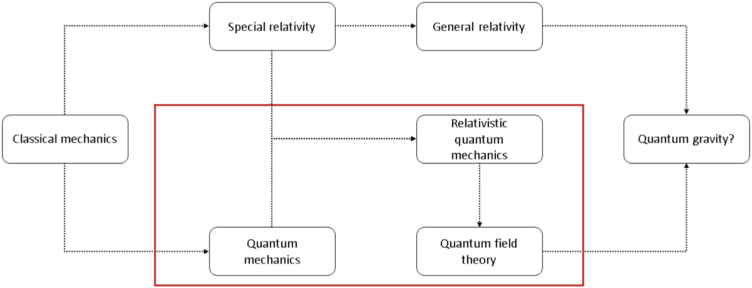
- Classical mechanics treats time only as a parameter:
  - Newton's equations
  - Maxwell's equations
- Time is absolute.

Current Physical Theories



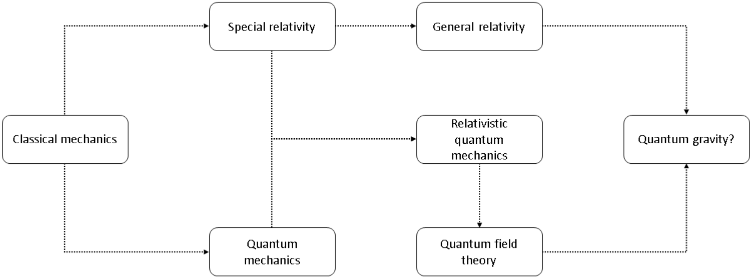
- Time in relativity is dynamical
  - SR treats time and space as components of a 4-vector
  - Time in GR is influenced by the geometry of spacetime

Current Physical Theories



- Quantum mechanics treats time as an external parameter which governs the evolution of the system
- Quantization schemes lack any fundamental notion of time

Current Physical Theories



- Different treatment of time then poses a problem in attempts to unify general relativity and quantum theory
- This constitutes one aspect of the problem of time in quantum gravity

Time of arrival in classical physics

Method 1:

We invert the classical equation of motion

$$x = q + \frac{p}{\mu} t \Rightarrow t = \mu \frac{(x - q)}{p} \tag{1}$$

where, μ is the mass of the particle and p is the initial momentum

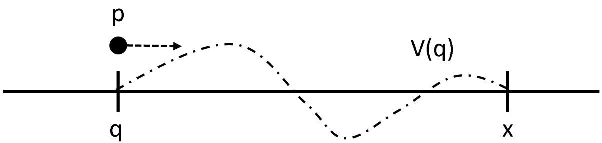
Time of arrival in classical physics

Method 2:

Use a detector to record the instant of time t_q at which the particle leaves q and another detector to record its arrival t_x at x.
The difference $\Delta t = t_x - t_q$ is the time of arrival.

- Methods 1 and 2 will always yield the same result.

Time of arrival in classical physics



Interacting case:

Given the Hamiltonian $H(q,p) = \dfrac{p^2}{2\mu} + V(q)$, the time of arrival is

$$T_x(q,p) = -\mathrm{sgn}(p)\sqrt{\frac{\mu}{2}} \int_x^q \frac{dq'}{\sqrt{H(q,p) - V(q')}} \tag{2}$$

Time of arrival in quantum physics

We cannot use method 1

In the Heisenberg picture, the equation of motion of the position operator is

$$\frac{d\hat{x}}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{x}\right] \tag{3}$$

- We can quantize the classical expression of the time of arrival but this leads to ordering ambiguity.
- The classical TOA may be complex and/or multiple valued.

Time of arrival in quantum physics

We cannot use method 2

The first detector will collapse the wavefunction and the state of the particle upon arrival at x is no longer causally related to the state during preparation.

- The standard formulation of quantum mechanics has no standard solution to the time of arrival of a particle.
- It is non-sensical to ask the time of arrival of the particle because time is not an observable in quantum mechanics.

Time operators

Pauli's theorem:[1]

There is no self-adjoint time operator that is canonically conjugate with its corresponding bounded system Hamiltonian .

- This has lead to a diverse treatment of time within and beyond the standard formulation of quantum mechanics to bypass Pauli's theorem[2].
- Classical mechanis is always a fundamental reference for all approaches.

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[1] Wolfgang Pauli et al. "Handbuch der physik". In: *Geiger and scheel* 2 (1933), pp. 83–272.

[2] Juan Gonzalo Muga and C Richard Leavens. "Arrival time in quantum mechanics". In: *Physics Reports* 338.4 (2000), pp. 353–438; Iñigo L. Egusquiza J. Gonzalo Muga Rafael Sala Mayato. *Time in quantum mechanics Volume 1*. 2nd ed. Lecture Notes in Physics 734. Springer-Verlag Berlin Heidelberg, 2007. ISBN: 9783540734727; 3540734724; Adolfo Campo Gonzalo Muga Andreas Ruschhaupt. *Time in Quantum Mechanics - Vol. 2*. 1st ed. Lecture Notes in Physics 789. Springer-Verlag Berlin Heidelberg, 2009. ISBN: 3642031730; 9783642031731.

Time operators

- Pauli's proof was only formal, without regard to the domains of the operators involved and to the validity of the operations leading to his conclusion[3].
- Several studies on time of arrival operators have also been made.

[3] Eric Galapon. "Pauli's theorem and quantum canonical pairs: the consistency of a bounded, self–adjoint time operator canonically conjugate to a Hamiltonian with non–empty point spectrum". In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 458.2018 (2002), pp. 451–472.

Free time of arrival operator

Aharonov-Bohm operator[4]

Symmetric quantization of the classical TOA yields

$$\hat{T}_{AB} = -\frac{\mu}{2}\left(\hat{p}^{-1}\hat{q} + \hat{q}\hat{p}^{-1}\right) \tag{4}$$

- This is a maximally symmetric operator.
- The eigenfunctions are complete but non-orthogonal[5]

$$\langle p|t,\pm\rangle = \sqrt{\frac{|p|}{\mu\hbar}}\exp\left(\frac{i}{\hbar}\frac{p^2}{2\mu}t\right)\Theta(\pm p) \tag{5}$$

[4] Yakir Aharonov and David Bohm. "Time in the quantum theory and the uncertainty relation for time and energy". In: *Physical Review* 122.5 (1961), p. 1649.

[5] JG Muga, CR Leavens, and JP Palao. "Space-time properties of free-motion time-of-arrival eigenfunctions". In: *Physical Review A* 58.6 (1998), p. 4336.

Free time of arrival operator

Grot-Rovelli-Tate operator[6]

Regularize $\hat{T}_{AB}$ at the point $p = 0$ which yields

$$\hat{T}_{GRT} = -\mu\sqrt{f_\epsilon(\hat{p})}\hat{q}\sqrt{f_\epsilon(\hat{p})} \tag{6}$$

$$f_\epsilon(\hat{p}) = \begin{cases} \hat{p}^{-1}, & |p| > \epsilon \\ \epsilon^2 \hat{p}, & |p| < \epsilon \end{cases} \tag{7}$$

- This is a self-adjoint operator.
- Regularization introduces negative energies to bypasses Pauli's theorem.

[6] Norbert Grot, Carlo Rovelli, and Ranjeet S Tate. "Time of arrival in quantum mechanics". In: *Physical Review A* 54.6 (1996), p. 4676.

Free time of arrival operator

Grot-Rovelli-Tate operator[6]

Regularize $\hat{T}_{AB}$ at the point $p = 0$ which yields

$$\hat{T}_{GRT} = -\mu\sqrt{f_\epsilon(\hat{p})}\hat{q}\sqrt{f_\epsilon(\hat{p})} \tag{6}$$

$$f_\epsilon(\hat{p}) = \begin{cases} \hat{p}^{-1}, & |p| > \epsilon \\ \epsilon^2\hat{p}, & |p| < \epsilon \end{cases} \tag{7}$$

- The eigenfunctions are

$$\langle p|t,\pm\rangle_\epsilon = \sqrt{\frac{1}{\mu\hbar f_\epsilon(p)}}\exp\left(\frac{i}{\hbar}\frac{t}{\mu}\int_{\pm\epsilon}^{p}\frac{dp'}{f_\epsilon(p')}\right)\Theta(\pm p) \tag{8}$$

[6] Norbert Grot, Carlo Rovelli, and Ranjeet S Tate. “Time of arrival in quantum mechanics”. In: *Physical Review A* 54.6 (1996), p. 4676.

Free time of arrival operator

Kijowski-Delgado-Muga operator[7]

Self-adjoint extension of $\hat{T}_{AB}$ is obtained using the combination

$$\hat{T}_{KDM} = \hat{T}_{AB}\Theta(\hat{p}) - \hat{T}_{AB}\Theta(\hat{-p}) \tag{9}$$

- Bypasses Pauli's theorem because $\hat{T}_{KDM}$ is not canonically conjugate to $\hat{H}$, instead

$$\left[\mathrm{sgn}(\hat{p})\hat{H}, \hat{T}_{KDM}\right] = i\hbar \tag{10}$$

[7] Jerzy Kijowski. "On the time operator in quantum mechanics and the Heisenberg uncertainty relation for energy and time". In: *Reports on Mathematical Physics* 6.3 (1974), pp. 361–386; V Delgado and JG Muga. "Arrival time in quantum mechanics". In: *Physical Review A* 56.5 (1997), p. 3425.

Free time of arrival operator

Confined time of arrival operator[8]

The operator in position space is

$$\hat{T}_\gamma \varphi(q) = \int_{-l}^{l} dq' \, T_\gamma(q,q')\varphi(q') \tag{11}$$

$$T_\gamma(q,q') = -\mu \frac{(q+q')}{4\hbar \sin\gamma} \left(e^{i\gamma}\Theta(q-q') + e^{-i\gamma}\Theta(q'-q) \right) \tag{12}$$

- Self adjointness was addressed by spatial confinement and imposing non-vanishing boundary conditions $\varphi(-l) = e^{-2i\gamma}\varphi(l)$ in the system Hilbert space.

[8] Eric A Galapon, Roland F Caballar, and Ricardo T Bahague Jr. "Confined quantum time of arrivals". In: *Physical review letters* 93.18 (2004), p. 180406; Eric A Galapon, Roland F Caballar, and Ricardo Bahague. "Confined quantum time of arrival for the vanishing potential". In: *Physical Review A* 72.6 (2005), p. 062107.

Confined time of arrival operator

- In the limit of large confining length, the eigenfunctions become two-fold degenerate and are related to the Aharonov-Bohm TOA eigenfunctions as

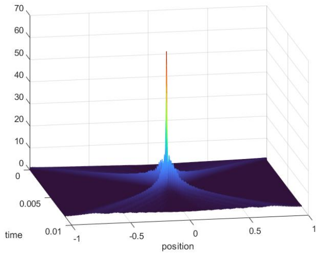
$$\langle p|t\rangle_{non} = \frac{1}{\sqrt{2}}\left(\langle p|t,+\rangle - \langle p|t,-\rangle\right) \tag{13}$$

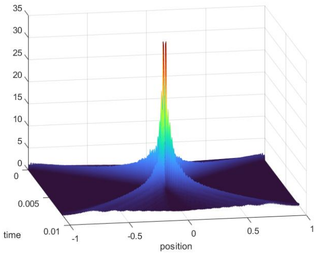
$$\langle p|t\rangle_{nod} = \frac{1}{\sqrt{2}}\left(\langle p|t,+\rangle + \langle p|t,-\rangle\right) \tag{14}$$

- Non-nodal: arrival with detection[9]
- Nodal: arrival with non-detection

[9] Denny Lane B Sombillo and Eric A Galapon. "Particle detection and non-detection in a quantum time of arrival measurement". In: *Annals of Physics* 364 (2016), pp. 261–273.

Confined time of arrival operator





- The eigenfunctions exhibit unitary collapse[10]
- The peaks occur at at time equal to the CTOA eigenvalue where the position uncertainty is minimum.

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[10] Eric A Galapon. "Theory of quantum arrival and spatial wave function collapse on the appearance of particle". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 465.2101 (2009), pp. 71–86.

TOA operators in the interacting case

- Quantizing the classical TOA

$$T_x(q,p) = -\mathrm{sgn}(p)\sqrt{\frac{\mu}{2}}\int_x^q \frac{dq'}{\sqrt{H(q,p)-V(q')}}$$

  has been deemed not meaningful because it is not generally real and single valued everywhere[11] but can be addressed on physical grounds[12]
- The TOA of a quantum particle is always real and single valued because
  - it can tunnel through the classically forbidden region
  - measuring the first TOA will collapse the wavefunction

[11] Asher Peres. *Quantum theory: concepts and methods*. Vol. 57. Springer Science & Business Media, 2006.

[12] Eric A Galapon and John Jaykel P Magadan. "Quantizations of the classical time of arrival and their dynamics". In: *Annals of Physics* 397 (2018), pp. 278–302.

Quantized time of arrival operators

- Sum of the Bender-Dunne basis operators[13]

$$\hat{T}_{m,n} = \frac{1}{\sum_{k=0}^{n} a_k^{(n)}} \sum_{k=0}^{n} a_k^{(n)} \hat{q}^k \hat{p}^m \hat{q}^{n-k} \tag{15}$$

$$= \frac{1}{\sum_{j=0}^{m} a_j^{(m)}} \sum_{j=0}^{n} a_j^{(m)} \hat{p}^j \hat{q}^n \hat{p}^{m-j} \tag{16}$$

$$a_k^{(n)} = \begin{cases} \binom{n}{k}, & \text{Weyl ordering} \\ 1, & \text{Born-Jordan ordering} \\ \delta_{n,0} + \delta_{n,n}, & \text{simple symmetric ordering} \end{cases} \tag{17}$$

[13] Carl M. Bender and Gerald V. Dunne. "Polynomials and Operator Orderings". In: *Journal of Mathematical Physics* 29.8 (Aug. 1988), pp. 1727–1731. ISSN: 0022-2488, 1089-7658. DOI: 10.1063/1.527869; Carl M. Bender and Gerald V. Dunne. "Exact Solutions to Operator Differential Equations". In: *Physical Review D* 40.8 (Oct. 15, 1989), pp. 2739–2742. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.40.2739; Carl M. Bender and Gerald V. Dunne. "Integration of Operator Differential Equations". In: *Physical Review D* 40.10 (Nov. 15, 1989), pp. 3504–3511. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.40.3504.

Quantized time of arrival operators

In position space, the TOA operators are integral operators[14]

$$(\hat{T}\phi)(q) = \frac{\mu}{i\hbar}\int_{-\infty}^{\infty} dq'\, T(q,q')\mathrm{sgn}(q-q')\phi(q') \tag{18}$$

where,

$$T_W(q,q') = \frac{1}{2}\int_0^{\frac{q+q'}{2}} ds\, {}_0F_1\left[;1;\frac{\mu}{2\hbar^2}(q-q')^2 V\left(\frac{q+q'}{2},s\right)\right]$$

[14] Eric A Galapon and John Jaykel P Magadan. "Quantizations of the classical time of arrival and their dynamics". In: *Annals of Physics* 397 (2018), pp. 278–302.

$$
\begin{aligned}
T_{BJ}(q,q') =& \frac{1}{2(q-q')}\int_0^q ds \int_0^s du \\
& \times {}_0F_1\left[;1;\frac{\mu}{2\hbar^2}(q-q')^2 V(s,u)\right] \\
& - \frac{1}{2(q-q')}\int_0^{q'} ds \int_0^s du \\
& \times {}_0F_1\left[;1;\frac{\mu}{2\hbar^2}(q-q')^2 V(s,u)\right]
\end{aligned}
$$

$$
\begin{aligned}
T_{SS}(q,q') =& \frac{1}{4}\int_0^q ds {}_0F_1\left[;1;\frac{\mu}{2\hbar^2}(q-q')^2 V(q,s)\right] \\
& - \frac{1}{4}\int_0^{q'} ds {}_0F_1\left[;1;\frac{\mu}{2\hbar^2}(q-q')^2 V(q',s)\right]
\end{aligned}
$$

where, $V(x,y) = V(x) - V(y)$.

Supraquantized time of arrival operators

Supraquantization

Method of constructing quantum observables without quantization.

- Process of quantization is circular

$$\begin{tikzcd}
\text{Classical observable} \arrow[r, bend left, "\text{ordering rule}"] & \text{Quantum operator} \arrow[l, bend left, "\text{correspondence principle}"]
\end{tikzcd}$$

- No quantization exists such that for all classical observables f and g, the Dirac condition is satisfied

$$\{f, g\} \Rightarrow \left[\hat{Q}_f, \hat{Q}_g\right] = i\hbar$$

Supraquantized kernel[15]

The time kernel function must satisfy the PDE

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial q^2}-\frac{\partial^2}{\partial q'^2}\right)T(q,q')+\left(V(q)-V(q')\right)T(q,q')=0 \tag{19}$$

with the boundary conditions $T(q,q)=\frac{q}{2}$ and $T(q,-q)=0$

- The supraquantized kernel is unique, in contrast to the quantized kernel.
- Supraquantization uses the classical observable as a boundary condition.

[15] Eric A Galapon. "Shouldn't there be an antithesis to quantization?" In: *Journal of mathematical physics* 45.8 (2004), pp. 3180–3215.

Supraquantized time of arrival operators

Linear systems

For potentials of the form $V(q) = aq^2 + bq + c$, the supraquantized kernel is equal to the Weyl-ordered kernel

Non-linear systems

The supraquantized kernel is equal to the Weyl-ordered kernel plus corrections terms which arise due to the obstruction to quantization.

Why relativistic TOA operators?

- If position can be treated as an observable then it is natural to promote time, e.g. TOA, as an observable with a corresponding operator[16].
- There are ‘negative energies’and ‘negative probabilities’

Additional problem

Although the presence of anti-particles naturally bypasses Pauli’s theorem, we are now faced with the problem that we do not know if the particle that arrived is the same particle we started with.

[16] Joseph Bunao and Eric A Galapon. “A one-particle time of arrival operator for a free relativistic spin-0 charged particle in (1+ 1) dimensions”. In: *Annals of Physics* 353 (2015), pp. 83–106.

Razavi TOA operator

- Earlieast construction of a relativistic free TOA operator was done by Razavi[17]

Construction

The classical TOA is first calculated

$$t = \left\{ \int dq \left(\left. \frac{\partial H}{\partial p} \right|_{p=p(q,H)} \right)^{-1} \right\}_{H=H(p,q)} \tag{20}$$

then quantized using simple-symmetric quantization.

[17] M Razavy. "Quantum-mechanical conjugate of the hamiltonian operator". In: *Il Nuovo Cimento B (1965-1970)* 63.1 (1969), pp. 271–308.

Razavi TOA operator

Spin-0

The positive-energy solution of the Klein-Gordon equation is generated by $H = \sqrt{p^2c^2 + \mu^2c^4}$ which leads to the operator

$$\hat{T}_{\mathrm{Ra}}^{(0)} = \frac{1}{2}\left(\frac{\sqrt{\hat{p}^2c^2+\mu^2c^4}}{\hat{p}c^2}\hat{q} + \hat{q}\frac{\sqrt{\hat{p}^2c^2+\mu^2c^4}}{\hat{p}c^2}\right) \tag{21}$$

- This looks like the relativistic version of the Aharonov-Bohm TOA operator

Razavi TOA operator

Spin-0

The eigenfunction is given as

$$\varphi_\tau(p) = N\sqrt{\frac{|p|c}{\sqrt{p^2c^2+\mu^2c^4}}}\exp\left(\frac{i}{\hbar}\tau\sqrt{p^2c^2+\mu^2c^4}\right) \tag{22}$$

- Not square integrable unless τ is complex where $\mathrm{Im}(\tau) < 0$
- If τ is real then the eigenfunction becomes

$$\varphi_\tau(p,\epsilon) = \int_{\tau-\epsilon}^{\tau+\epsilon} d\tau' \varphi_{\tau'}(p) \tag{23}$$

Razavi TOA operator

Spin-1/2

Apply a Foldy-Wouthuysen transformation to $H = \alpha \cdot \hat{p} + \beta\mu c^2$ where

$$H \to \hat{U} H \hat{U}^{-1} = \beta\sqrt{p^2 c^2 + \mu^2 c^4} \tag{24}$$

where, $\hat{U}^{\pm 1} = \cos\theta \pm \beta\alpha \cdot \hat{p} \sin\theta$ such that $\tan 2\theta = \frac{|p|}{\mu c}$. Since the unitary transform preservers the commutation relation, then

$$\hat{T}_{\text{Ra}}^{(1/2)} = \hat{U} \hat{T}_{\text{Ra}}^{(0)} \hat{U}^{-1} \tag{25}$$

León TOA operator

- Extended the Grot-Rovelli-Tate TOA operator to the positive energy solutions of the Klein-Gordon equation[18]

Spin-0

The operator is given as

$$\hat{T}_{\mathrm{Le}} = -\frac{\sqrt{E_p}}{c}\hat{p}^{-1/2}\hat{Q}_{NW}\hat{p}^{1/2}\frac{\sqrt{E_p}}{c} \tag{26}$$

where $E_p = \sqrt{p^2c^2 + \mu^2 c^4}$ and $\hat{Q}_{NW}$ is the Newton-Wigner position operator[19]

[18] Juan León. "Time-of-arrival formalism for the relativistic particle". In: *Journal of Physics A: Mathematical and General* 30.13 (1997), p. 4791.

[19] Theodore Duddell Newton and Eugene P Wigner. "Localized states for elementary systems". In: *Reviews of Modern Physics* 21.3 (1949), p. 400.

León TOA operator

- The Newton-Wigner position operator is

$$\hat{Q}_{NW} = \hat{x} - i\hbar \frac{pc^2}{2E_p^2} \tag{27}$$

- By construction, its eigenfunctions are not Lorentz covariant
- The localized state in position space is not a Dirac delta

Wang-Xiong TOA operator

- Direct extension of the non-relativistic TOA operator[20]

$$t = -\mu \frac{q}{p} \Rightarrow -\frac{E_p}{c^2}\frac{q}{p} \tag{28}$$

Spin-1/2

Total symmetrization of the TOA yields the operator

$$\hat{T}_{WX} = -\frac{1}{4c^2}(\hat{H}\hat{T}_{AB} + \hat{T}_{AB}\hat{H}) = -\alpha_1 \frac{\hat{q}}{c^2} + \beta \hat{T}_{AB} \tag{29}$$

where $\hat{H} = \alpha_1 \hat{p} + \beta \mu c^2$

[20] Zhi-Yong Wang and Cai-Dong Xiong. "Relativistic free-motion time-of-arrival". In: *Journal of Physics A: Mathematical and Theoretical* 40.8 (2007), p. 1897.

Wang-Xiong TOA operator

- The proper-time eigenfunctions are solved by

$$\hat{T}_{WX}\varphi(p) = t\varphi(p) \tag{30}$$

$$t = -\lambda \frac{E_p}{pc^2} q \quad \text{and} \quad \tau = \mu \frac{q}{p} \tag{31}$$

which yields

$$\varphi(p) = N \left(\frac{q^2/c^2}{q^2/c^2 + \tau^2} \right)^{1/4} \xi(x) e^{-ipx/\hbar} \tag{32}$$

$$\xi(p) = \sqrt{\frac{\tau + bt_x}{2bt_x}} \begin{bmatrix} \eta_s \\ \dfrac{\sigma_1 x/c^2}{\tau + bt_x} \eta_s \end{bmatrix} \tag{33}$$

where $b = \pm 1$ and $t_x = \sqrt{x^2/c^2 + \tau^2}$

Wang-Xiong TOA operator

- Dual relations
  - $t^2 = q^2/c^2 + \tau^2 \leftrightarrow E_p^2 = p^2c^2 + \mu^2c^4$
  - $\hat{T}_{WX} = -\alpha_1\hat{q}/c^2 + \beta\hat{T}_{AB} \leftrightarrow \hat{H} = \alpha_1\hat{p} + \beta\mu c^2$
  - $-i\hbar\partial_E\varphi = \hat{T}_{WX}\varphi \leftrightarrow i\hbar\partial_t\psi = \hat{H}\psi$
- They have also applied this operator to quantum field theory[21]

[21] ZY Wang, B Chen, and CD Xiong. "Time in quantum mechanics and quantum field theory". In: *Journal of Physics A: Mathematical and General* 36.18 (2003), p. 5135; Zhi-Yong Wang, Cai-Dong Xiong, and Bing He. "Arrival time in quantum field theory". In: *Physics Letters B* 666.4 (2008), pp. 382–385; Zhi-Yong Wang, Qi Qiu, and Cai-Dong Xiong. "Time operator in QFT with Virasoro constraints". In: *Physics Letters B* 718.4-5 (2013), pp. 1515–1518.

Bauer-Aguillon TOA operator

- Applies Lorentz and Born's reciprocity[22,23] invariance to the canonical quantization of special relativity
  - $p_\mu p^\mu = p_o^2 - \vec{p}^2 = (m_o c)^2$ leads to
  $$\left(\hat{p}_\mu \hat{p}^\mu - (m_o c)^2\right)|\psi\rangle = 0 \Rightarrow c\hat{p}_o |\psi\rangle = \left(c\alpha\cdot\hat{p} + \beta m_o c^2\right)|\psi\rangle$$
  provided $[\hat{p}_\mu, \hat{p}_\nu] = 0$
  - $x_\mu x^\mu = x_o^2 - \vec{r}^2 = s_o^2$ leads to
  $$\left(\hat{x}_\mu \hat{x}^\mu - s_o^2\right)|\psi\rangle = 0 \Rightarrow \frac{\hat{x}_o}{c}|\psi\rangle = \left(\frac{\alpha\cdot\hat{r}}{c} + \beta\tau_o\right)|\psi\rangle$$
  provided $[\hat{x}_\mu, \hat{x}_\nu] = 0$
  - $\mathcal{O}^- = x_\mu p^\mu - p^\mu x_\mu \rightarrow [\hat{x}_\mu, \hat{p}_\nu] = i\hbar\eta_{\mu\nu}\infty$

[22] Max Born. "A suggestion for unifying quantum theory and relativity". In: *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 165.921 (1938), pp. 291–303; Max Born. "Reciprocity theory of elementary particles". In: *Reviews of Modern Physics* 21.3 (1949), p. 463.

[23] The transformation $x \rightarrow p$ and $p \rightarrow -x$ leaves the Hamilton equations invariant

Bauer-Aguillon TOA operator

Spin-1/2

The time operator is

$$\hat{T}_{BA} = \frac{\alpha \cdot \hat{r}}{c} + \beta \tau_o \tag{34}$$

where τ_o is a real-valued constant[24]

- This time operator is not canonically conjugate to the Hamiltonian.

[24] Mariano Bauer. "A dynamical time operator in Dirac's relativistic quantum mechanics". In: *International Journal of Modern Physics A* 29.06 (2014), p. 1450036; M Bauer. "A time operator in the simulations of the Dirac equation". In: *International Journal of Modern Physics A* 34.22 (2019), p. 1950114; CA Aguillón, M Bauer, and GE García. "Time and energy operators in the canonical quantization of special relativity". In: *European Journal of Physics* 41.3 (2020), p. 035601; M Bauer, CA Aguillón, and GE García. "Conditional interpretation of time in quantum gravity and a time operator in relativistic quantum mechanics". In: *International Journal of Modern Physics A* 35.21 (2020), p. 2050114.

Bunao-Galapon one-particle TOA operator

- A TOA operator was constructed by solving the commutation relation $\left[\hat{H}, \hat{T}\right] = i\hbar$
- The one-particle TOA operator was constructed by taking transform $\hat{T}_{\Phi} = \hat{U}\hat{T}\hat{U}^{-1}$ where[25]

$$\hat{U}^{\pm} = \frac{(\mu c^2 + E_p)\hat{\sigma}_0 \mp (\mu c^2 - E_p)\hat{\sigma}_1}{\sqrt{4\mu c^2 E_p}} \tag{35}$$

[25] Herman Feshbach and Felix Villars. "Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles". In: *Reviews of Modern Physics* 30.1 (1958), p. 24; Walter Greiner et al. *Relativistic quantum mechanics*. Vol. 2. Springer, 2000.

Bunao-Galapon one-particle TOA operator

Spin-0

The one particle operator is[26]

$$\hat{T}_{BG}^{(\Phi)} = \hat{\sigma}_3 \left[-\frac{1}{2E_p} \left(\frac{\hat{p}\hat{q} + \hat{q}\hat{p}}{2} \right) - \left(\frac{\mu^2 c^2}{E_p} + \frac{p^2}{2E_p} \right) \left(\frac{\hat{p}^{-1}\hat{q} + \hat{q}\hat{p}^{-1}}{2} \right) \right] \tag{36}$$

- Reduces to the $\hat{\sigma}_3 \hat{T}_{AB}$ as $c \to \infty$
- Not self-adjoint but maximally symmetric

[26] Joseph Bunao and Eric A Galapon. “A one-particle time of arrival operator for a free relativistic spin-0 charged particle in (1+ 1) dimensions”. In: *Annals of Physics* 353 (2015), pp. 83–106.

Bunao-Galapon one-particle TOA operator

Spin-1/2

The one particle operator is[27] the same as that for the spin-0

- There is an extra term but it was dropped to impose parity inversion symmetry to the operator
- The extra term commutes with the Hamiltonian

[27] Joseph Bunao and Eric A Galapon. "A relativistic one-particle Time of Arrival operator for a free spin-1/2 particle in (1+ 1) dimensions". In: *Annals of Physics* 356 (2015), pp. 369–382.

Comments

- There is still no consensus on the status of time in non-relativistic quantum mechanics
- Additional problems arise because of special relativity
  - Lorentz invariance: Hamiltonian formulation of only works in some specific Lorentz frame and the TOA is also measured in that frame[28]
  - Identity of the particle upon arrival
  - Klein-Gordon and Dirac equation cannot provide a well-defined local probability distribution[29]

[28] Herbert Goldstein, Charles Poole, and John Safko. *Classical mechanics*. 2002.

[29] AJ Kálnay. *The localization problem. in "Problems in the Foundations of Physics"(M. Bunpe. Ed.). Vol. 4. pp. 93-J 10*. 1971; Donald Reed. "Lawrence P. Horwitz: Relativistic Quantum Mechanics". In: *Foundations of Physics* 47.11 (2017), pp. 1498–1502; Theodore Duddell Newton and Eugene P Wigner. "Localized states for elementary systems". In: *Reviews of Modern Physics* 21.3 (1949), p. 400.

Problem

- Given a relativistic particle with a certain initial state, what is the TOA probability distribution?
- Does this TOA distribution have an underlying ideal distribution generated by a corresponding TOA operator?
- How do we construct the TOA operator?

Thank You