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# Instantaneous and non-zero tunneling time regimes


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We demonstrate how an operator-based theory of quantum time-of-arrival (TOA) reconciles the seemingly conflicting reports on the measured tunneling times. This is done by defining the barrier traversal time as the difference of the expectation values of the corresponding TOA-operators in the presence and absence of the barrier. We show that for an arbitrarily shaped potential barrier, there exists three traversal time regimes corresponding to *full-tunneling*, *partial-tunneling*, and *non-tunneling* processes, which are determined by the relation between the support of the incident wavepacket's momentum distribution $\tilde{\psi}(k)$, and shape of the barrier. The *full-tunneling* process occurs when the support of $\tilde{\psi}(k)$ is below the minimum height of the barrier, resulting to an instantaneous tunneling time. The *partial-tunneling* process occurs when the support or a segment of the support of $\tilde{\psi}(k)$ lies between the minimum and maximum height of the barrier. For this case, the particle does not "fully" tunnel through the entire barrier system resulting to a non-zero *traversal* time. The *non-tunneling* regime occurs when the support of $\tilde{\psi}(k)$ is above the maximum height of the barrier system, leading to a classical above-barrier traversal time. We argue that the zero and non-zero tunneling times measured in different attoclock experiments correspond to the full-tunneling and partial-tunneling processes, respectively.




The time it takes for a quantum particle to tunnel through a potential barrier has always eluded physicists since the advent of quantum mechanics [1, 2]. Numerous proposals to define a unique and simple expression for the tunneling time have been offered in the literature [3–13], e.g., Wigner phase time [3], Büttiker-Landauer time [4], Larmor time [5–7], Pollak-Miller time [8], dwell time [9], but a consensus has yet to be reached [14–21]. The development of ultraprecise techniques in strong-field physics has been expected to close the debate once and for all, but the existence of contradictory experimental results only further divided the physics community [22–29]. Specifically, some attoclock experiments reported instantaneous tunneling [22–27] while others reported non-zero tunneling times [28, 29]. This diversity posits a challenge to theoretical treatments that only predict either zero or non-zero tunneling times, imploring a formalism that could accommodate both seemingly contradictory results. In this Letter, we offer such a formalism using the theory of time-of-arrival (TOA) operators developed in Refs. [30, 31].

The initial application of the theory of TOA-operators on the tunneling time problem has been done by one of us for a square barrier in Ref. [32]. There, it is assumed that an initial wavepacket of the form $\psi(q) = e^{ip_o q/\hbar}\varphi(q)$ centered at $q = q_o$ with a mean momentum $p_o = \hbar k_o$, is prepared on the far left side of the square barrier $V(q) = V_o > 0$ placed at $-a < q < -b$. The arrival point is chosen to be the origin $q = 0$. The average barrier traversal time is deduced from the difference of the expectation values of the corresponding TOA-operators $\hat{\mathsf{T}}$ in the presence and absence of the barrier

$$\bar{\tau}_{\text{trav}} = \langle\psi|\hat{\mathsf{T}}_{\mathsf{F}}|\psi\rangle - \langle\psi|\hat{\mathsf{T}}_{\mathsf{B}}|\psi\rangle. \tag{1}$$

The subscripts $F$ and $B$ indicate the absence and presence of the barrier, respectively. It was shown that only the above-barrier energy components of the incident wavepacket's momentum distribution contribute to the measurable barrier traversal time while the contribution of the below-barrier components vanish, implying an instantaneous tunneling time. This phenomenon still persists in a recently proposed formalism on the construction of relativistic TOA-operators for spin-0 particles provided that the barrier height $V_o$ is less than the rest mass energy [33–35].

While these findings are expected to hold in general, a closed-form expression for the barrier traversal time for arbitrary potentials has not been derived. This issue becomes significant when comparing the results of the TOA-operator theory to attoclock tunneling ionization experiments which do not use a square barrier. In this Letter, we will show how the seemingly conflicting reports of instantaneous and non-zero tunneling times can be reconciled by the theory of TOA-operators, i.e., the measurable traversal time across the barrier depends on the support of the incident wavepacket's momentum distribution $\tilde{\psi}(k)$ in relation to the effective shape of the potential barrier. Identifying the relation between these two


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leads to a better understanding on the conditions resulting to either instantaneous or non-zero tunneling times. Specifically, this allows us to differentiate the traversal time defined by Eq. (1) from the tunneling time, and that these two 'times' are not necessarily equal.

Our analysis starts by considering the classical TOA for arrivals at the origin $q = 0$ given by

$$T(q,p) = -\mathrm{sgn}(p)\sqrt{\frac{\mu}{2}}\int_0^q dq' \frac{dq'}{\sqrt{H(q,p)-V(q')}}, \quad (2)$$

where $\mu$ is the mass of the particle and $\mathrm{sgn}(z)$ is the signum function. It is clear from Eq. (2) that $T(q,p)$ is only applicable for above-barrier traversal since it becomes complex-valued for the case of quantum tunneling. However, what we can quantize is its expansion around the free-TOA given by

$$\tau(q,p) = \sum_{j=0}^{\infty}\frac{(2j-1)!!}{j!}\frac{(-1)^{j+1}\mu^{j+1}}{p^{2j+1}} \times \int_0^q dq'(V(q)-V(q'))^j, \quad (3)$$

which we refer to as the local time-of-arrival (LTOA). Notice that the LTOA is now single and real-valued.

The main difference between the classical arrival time Eq. (2) and LTOA is the fact that the former holds in the entire region $\Omega = \Omega_q \times \Omega_p$, which may be complex-valued in the classically forbidden region. Meanwhile, the LTOA converges uniformly and absolutely to $T(q,p)$ only in some local neighborhood $\omega = \omega_q \times \omega_p$ determined by $|V(q) - V(q')| < p^2/2\mu$ for $p \neq 0$, where it is real-valued. This implies the inclusion $\tau(q,p) \subset T(q,p)$. This condition exactly describes above barrier traversal process. The LTOA diverges outside the region defined by $\omega$, physically signifying non-arrival at the arrival point. [36]. Hence, the LTOA correctly represents above and below barrier traversal times in the classical regime.

Now, for an arbitrary potential $V(q)$ that admits the expansion $V(q) = \sum_{n=0} a_n q^n$, the LTOA Eq. (3) can be rewritten as

$$\tau(q,p) = \sum_{j=0}^{\infty}(-1)^{j+1}\mu^{j+1}\frac{(2j-1)!!}{j!}\sum_{n=1}^{\infty}a_n^{(j)}\frac{q^n}{p^{2j+1}}, \quad (4)$$

which is now amenable to quantization. We perform canonical quantization by imposing Weyl-ordering to the monomial $q^n p^{-(2j+1)}$. In coordinate representation, the quantized TOA operator generally appear as

$$(\hat{\mathsf{T}}\phi)(q) = \int_{-\infty}^{\infty} dq' \frac{\mu}{i\hbar} T(q,q')\mathrm{sgn}(q-q')\phi(q'), \quad (5)$$

where $T(q,q')$ is referred to as the time kernel factor (TKF) [31], and is given by,

$$\tilde{T}(\eta,\zeta) = \frac{1}{2}\int_0^{\eta} {}_0F_1\left[;1;\frac{\mu}{2\hbar^2}(V(\eta)-V(s))\zeta^2\right], \quad (6)$$

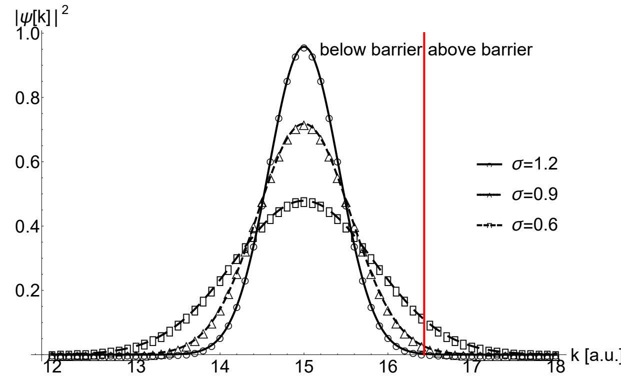


FIG. 1. Momentum density distribution of an incident Gaussian wavepacket initially centered at $q_o = -9$ with momentum $p_o = 15$ for the parameters $\mu = \hbar = 1$. The red line indicates the $\kappa_o$ for a square barrier of height $V_o = 135$.

where $\eta = (q+q')/2$, $\zeta = q - q'$, and ${}_0F_1(;a;z)$ is a particular hypergeometric function [31].

In the absence of the barrier, the corresponding TKF is easily obtained by substituting $V(q) = 0$ into Eq. (6) yielding $\tilde{T}_F(\eta) = \eta/2$. The integral form of the TKF in Eq. (6) allow us to consider not only analytic continuous potentials but also piecewise potentials such as a square barrier system. In the presence of a square potential barrier, the TKFs are constructed by mapping the potential $V(q)$ to the $\eta-$coordinate wherein $V(\eta) = V_o$ for $-a < \eta < -b$. Straightforward calculations yield the following TKFs

$$\begin{aligned}
\tilde{T}_I(\eta,\zeta) &= \frac{\eta}{2} && ,\eta \geq -b\\
\tilde{T}_{II}(\eta,\zeta) &= \frac{\eta+b}{2} - \frac{b}{2}I_0(\kappa_o|\zeta|) && ,-a < \eta < -b\\
\tilde{T}_{III}(\eta,\zeta) &= \frac{\eta+L}{2} - \frac{L}{2}J_0(\kappa_o|\zeta|) && ,\eta \leq -a
\end{aligned}$$

where $\kappa_o = \sqrt{2\mu V_o}/\hbar$, while $I_0(z)$ and $J_0(z)$ are specific Bessel functions. The same TKFs can be obtained using Born-Jordan and simple symmetric ordering [31]. The equality of the TKFs despite using different ordering rules rests on the linearity of the potential on $q$ for each region in the barrier system.

We assume that the incident wavepacket $\psi(q)$ is prepared on the far left of the square potential barrier, so that it does not initially 'leak' inside the barrier region. This assumption suggests that the relevant TKF of our barrier TOA operator is $\tilde{T}_{III}(\eta,\zeta)$ alone. In accordance with Eq. (1), it is easy to show that the resulting traversal time across the square barrier potential is

$$\bar{\tau}_{\mathrm{trav}} = \frac{L}{\nu_o}R \quad (7)$$

wherein $\nu_o = p_o/\mu$ is the group velocity. The factor $R$ is interpreted as the effective index of refraction (IOR) of the square barrier and has the form $R = (R_+ - R_-)$

where

$$R_{\pm} = k_o \int_{\kappa_o}^{\infty} dk \frac{|\tilde{\psi}(\pm k)|^2}{\sqrt{k^2 - \kappa_o^2}}, \tag{8}$$

and $\tilde{\psi}(k)$ is the Fourier transform of the incident wavepacket [32]. The terms $R_+$ and $R_-$ correspond to the contributions of the positive and negative momentum components of the incident wavepacket, respectively. For arrivals at the transmission channel, only $R_+$ is physically relevant.

It can be immediately seen from Eq. (8) that the components below $\kappa_o$ do not contribute to the barrier traversal time, and the IOR vanishes if all the energy components of the incident wavepacket is below $\kappa_0$, implying an instantaneous tunneling time. This happens when the incident wavepacket is (i) sufficiently spatially wide such that all the energy components are below the barrier, and (ii) sufficiently far from the barrier so that it has zero probability of being in the barrier region at the initial time $t = 0$. This is illustrated in Fig. 1 for an incident Gaussian wavepacket $\psi(q) = (\sigma\sqrt{2\pi})^{-1/2} e^{-(q-q_o)^2/4\sigma^2} e^{ip_o q/\hbar}$. It can be seen that if the incident wavepacket is spatially narrow, then the momentum density distribution has components above the barrier. This results to a non-zero traversal time but this is not the tunneling time since this 'time' comes from the above-barrier components of $\tilde{\psi}(k)$ which do not tunnel through the barrier. Meanwhile, a spatially wide incident wavepacket with only below-barrier components yields a zero traversal time which equivalently means an instantaneous tunneling time.

The corresponding IOR of a square barrier for different heights $V_o$ is shown in Fig. 2. The IOR vanishes as the barrier height increases when all the momentum components are below the barrier. Moreover, the barrier traversal time has superluminal and subluminal regions where $R < 1$ and $R > 1$, respectively, but this does not mean that the particle traverses the barrier at superluminal speeds when it has components that are above and below the barrier. This can be seen by rewriting the barrier traversal time as the weighted average of the classical above-barrier traversal time

$$\bar{\tau}_{\text{trav}} = \int_{\kappa_o}^{\infty} dk |\tilde{\psi}(\pm k)|^2 \tau_{\text{top}}(k) \tag{9}$$

$$\tau_{\text{top}}(k) = \frac{\mu L}{\hbar} \frac{1}{\sqrt{k^2 - \kappa_o^2}}, \tag{10}$$

with weights $|\tilde{\psi}(\pm k)|^2$. This implies that the above-barrier components still traverse the barrier at subluminal speeds but the instantaneous tunneling of the below-barrier components effectively lowers the average to superluminal traversal times.

The distinction between the traversal time Eq. (1) and tunneling time is now clear, i.e., the traversal time arises from the segment of the support of $\tilde{\psi}(k)$ that lies above $\kappa_o$ while the tunneling time arises from the segment of the support of $\tilde{\psi}(k)$ that lies below $\kappa_o$. These two 'times' are only equal if the support of $\tilde{\psi}(k)$ lies below $\kappa_o$ such that the traversal time identically vanishes and is equivalent to an instantaneous tunneling time.

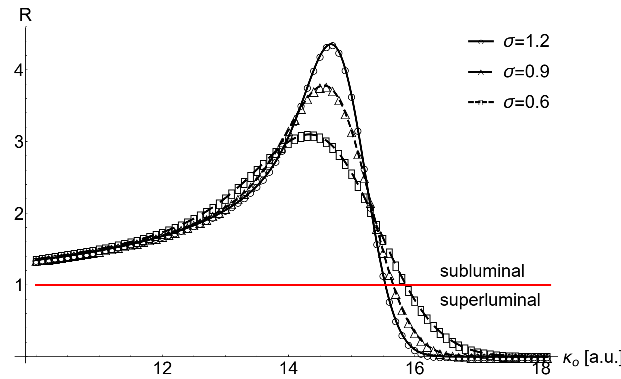


FIG. 2. The effective IOR of a square barrier as the height $V_o$ increases for the Gaussian wavepacket in Fig. 1.

We now extend the same analysis to a system of two square barriers with $V(q) = V_1$ for $-a < q < -l$ and $V(q) = V_2$ for $-l < q < -b$, where $V_1 < V_2$. A similar barrier system was also considered in Ref. [37] in the investigation of the generalized Hartman effect. It can be shown that the barrier traversal time of an incident wavepacket with group velocity $\nu_o$ is simply $\tau_{\text{trav}} = \sum_{n=1}^{2} (w_n/\nu_o) R_n$, where $w_n$ is the width of each barrier and $R_n$ is the corresponding IOR.

The results of the single square barrier case implies that when the support of $\tilde{\psi}(k)$ lies between $0 < k < \kappa_1 = \sqrt{2\mu V_1}/\hbar$, then it follows that $\tau_{trav} = 0$ implying instantaneous tunneling time. We shall describe this as the *full-tunneling regime*. However, if the support or a segment of the support of $\tilde{\psi}(k)$ lies between $\kappa_1 < k < \kappa_2 = \sqrt{2\mu V_2}/\hbar$, then $\tau_{trav} \neq 0$ and we can rewrite the traversal time as

$$\bar{\tau}_{\text{trav}} = \frac{L}{\nu_o} (R_{\text{abo}}^{(2)} + R_{\text{bel}}^{(2)}) \tag{11}$$

wherein,

$$R_{\text{abo}}^{(2)} = \frac{w_1 k_o}{L} \int_{\kappa_2}^{\infty} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa_1^2}} + \frac{w_2 k_o}{L} \int_{\kappa_2}^{\infty} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa_2^2}} \tag{12}$$

$$R_{\text{bel}}^{(2)} = \frac{w_1 k_o}{L} \int_{\kappa_1}^{\kappa_2} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa_1^2}}. \tag{13}$$

We describe this as the *partial-tunneling regime* since this is equivalent to a particle that classically traverses above the first barrier and tunnels across the second barrier. In other words, the particle does not tunnel through the entire barrier system, hence the name. The quantity $\bar{\tau}_{trav} = (L/\nu_o) R_{\text{bel}}^{(2)}$ can thus be interpreted as a traversal time, and likely seems to be the origin of the reported non-zero tunneling times in the discussion to

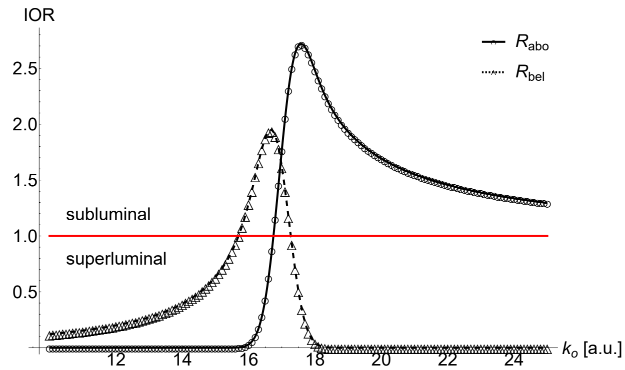


FIG. 3. Contributions of $R_{\rm abo}$ and $R_{\rm bel}$ for the "deformed" square barrier of height $V_o = 145$ and $A = 0.2$, as the momentum of the incident Gaussian wavepacket increases, wherein $q_o = -9$ and $\sigma = 1.2$ for the parameters $\mu = \hbar = 1$.

follow. Last, the *non-tunneling regime* occurs when the support of $\tilde{\psi}(k)$ lies above $k > \kappa_2$, and is equivalent to a classical above-barrier traversal time.

We repeat the same analysis to an arbitrary potential barrier $V(q)$ and model the system as a sum of square barriers with varying heights and width, i.e., $V(q) = \sum_{n=1}^{\infty} V_n$ each having a width $w_n$. In the continuous limit, $w_n \to 0$, we find the effective IOR has the form $R = R_{\rm abo} + R_{\rm bel}$, where

$$R_{\rm abo} = \frac{k_o}{L}\int_b^a dx \int_{\kappa_{\rm max}}^{\infty} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa(x)^2}}, \tag{14}$$

$$R_{\rm bel} = \frac{k_o}{L}\int_b^a dx \int_{\kappa(x)}^{\kappa_{\rm max}} dk \frac{|\tilde{\psi}(k)|^2}{\sqrt{k^2 - \kappa(x)^2}}, \tag{15}$$

in which $\kappa_{\rm max} = \sqrt{2\mu V_{\rm max}}/\hbar$ indicates the maximum value of the the barrier and $\kappa(x) = \sqrt{2\mu V(x)/\hbar^2}$. Hence, an arbitrary potential barrier has a traversal time contribution $\bar{\tau}_{\rm trav}^{\rm bel} = (L/\nu_o)R_{\rm bel}$ which can be mistakenly identified as a tunneling time. Morover, $\bar{\tau}_{\rm trav}^{\rm bel}$ arises from the "deformation" of the square barrier, and it follows that if we choose the potential to be a square barrier $V(x) = V_0$, then $\bar{\tau}_{\rm trav}^{\rm bel}$ vanishes, which is consistent with the predictions of [32] that tunneling is instantaneous.

The IOR contribution $R_{\rm bel}$ also indicates that if a potential barrier $V(q)$ has no sharp discontinuity along the edges, then the traversal time $\bar{\tau}_{\rm trav}^{\rm bel}$ will always be non-zero since the minimum value of $\kappa(x) = 0$, i.e., $\bar{\tau}_{\rm trav}^{\rm bel} \neq 0$ for smooth barriers. This also implies that there are regions in which $\bar{\tau}_{\rm trav}^{\rm bel}$ may indicate either subluminal or superluminal traversal times when $R_{\rm bel} > 1$ or $R_{\rm bel} < 1$, respectively, but this does not mean that the particle traverses the barrier at superluminal speeds. This can be seen by rewriting $\bar{\tau}_{\rm trav}^{\rm bel}$ as

$$\bar{\tau}_{\rm trav}^{\rm bel} = \int_a^b dx \int_{\kappa(x)}^{\kappa_{\rm max}} dk |\tilde{\psi}(\pm k)|^2 \tau_{\rm bel}(k,x) \tag{16}$$

$$\tau_{\rm bel}(k,x) = \frac{\mu}{\hbar}\frac{1}{\sqrt{k^2 - \kappa(x))^2}}. \tag{17}$$

Eq. (16) indicates that the traversal time $\bar{\tau}_{\rm trav}^{\rm bel}$ is equal to the sum of the weighted average of the above barrier traversal time of each energy component $\kappa_d$ that is below $\kappa_{\rm max}$ with weights $|\tilde{\psi}(\pm k)|^2$. Thus, the superluminal traversal time $\bar{\tau}_{\rm trav}^{\rm bel}$ is also due to the effective lowering of the average traversal time due to the instantaneous tunneling of the components below $\kappa_d$. This can be illustrated using a "deformed" square barrier

$$V(q) = V_o \exp\left[\frac{4A^2}{(a-b)^2}\right]\exp\left[\frac{A^2}{(q+a)(q+b)}\right] \tag{18}$$

for $-a \leq q \leq -b$ and zero everywhere else, in which $A < 1$. The contributions of $R_{\rm bel}$ and $R_{abo}$ are shown in Fig. 3. Since the barrier has no sharp discontinuity at the edges, it can be seen that the traversal time $\bar{\tau}_{\rm trav}^{\rm bel}$ is always non-zero as long as the incident wavepacket has components below the maximum height of this barrier.

The main advantage of the traversal $\bar{\tau}_{\rm trav}^{\rm bel}$ in comparison with other tunneling time definitions is its simplicity. Specifically, other tunneling time definitions such as the Büttiker-Landauer time, Larmor time, and Pollak-Miller time involves calculating the transmission amplitude for propagating through the barrier, which will require solving the Schödinger equation. However, $\bar{\tau}_{\rm trav}^{\rm bel}$ only requires information on the incident wavepacket and the interaction potential, which allows it to be applicable in any system without the need of any further calculations, as long as the incident wavepacket does not initially 'leak' into the barrier.

In conclusion, we explicitly showed how the momentum distribution of the incident wave packet dictates the measurable traversal time across the barrier region for arbitrary potential $V(q)$. Our results do not only answer the time it takes for a quantum particle with some specific incident energy to traverse a potential barrier, but also touches the question on what really constitutes a tunneling particle. Before we can close the debate on quantum tunneling time, one must first recognize that there are, in fact, two possible tunneling processes, i.e., full tunneling and partial tunneling.

Textbook quantum mechanics only considers an incident plane wave with incident energy $E_o < V_o$ to illustrate quantum tunneling. The same set-up was also used by MacColl and Hartman to investigate the quantum tunneling time which sparked this entire field of study. However, such a setup only considers a single energy component which entails that the location of the incident particle is ill-defined and consequently makes the instance at which it emerges from the barrier also ill-defined [38]. Using an incident wavepacket with multiple energy components is able to address this problem since both the position and momentum density distribution are now well-defined. This allows for a better understanding on which components tunnel through the barrier, and traverses above the barrier. Thus, there is a clear distinction between the traversal and tunneling times, and that these two 'times' are not necessarily equal.

Now, we argue that $\bar{\tau}_{\rm trav}^{\rm bel}$, Eq. (16), arising from the *partial tunneling regime* may explain the reported non-zero tunneling times in attoclock experiments [28, 29]. Although our theory does not exactly model the tunnel ionization of electrons in the attoclock, wherein the electron starts as a bound state and tunnels through the potential to become a scattering state, it still follows that potential $V(q) = -Z_{eff}/q - Fq$ used in the attoclock will yield $\bar{\tau}_{\text{trav}}^{\text{bel}} \neq 0$ as long as it is confined in a finite region of length $L$. This now raises the question on whether the reported non-zero tunneling times in Refs. [28, 29] are actually tunneling times. Using $\bar{\tau}_{\text{trav}}^{\text{bel}}$ in attoclock experiments may thus settle the conflicting reports on tunneling time. Specifically, the main challenge in attoclock experiments is extracting the tunneling time from the measured offset angle, and that the theoretical offset angle is first calculated by assuming an instantaneous tunneling time [39]. However, the offset angle is highly dependent on the Coulomb potential which also leads to model-dependent conclusions regarding the tunneling time based on the removal of Coulomb effects. The traversal time $\bar{\tau}_{\text{trav}}^{\text{bel}}$ can circumvent the initial assumption of instantaneous tunneling for the theoretical calculation of the offset angle by imposing that this is a *partial tunneling process*, which may prove to be useful in calibrating the attoclock.


[1] L. MacColl, Physical Review **40**, 621 (1932).
[2] T. E. Hartman, Journal of Applied Physics **33**, 3427 (1962).
[3] E. P. Wigner, Physical Review **98**, 145 (1955).
[4] M. Büttiker and R. Landauer, Physical Review Letters **49**, 1739 (1982).
[5] A. Baz, Yadern. Fiz. **4** (1966).
[6] V. Rybachenko, Sov. J. Nucl. Phys. **5**, 635 (1967).
[7] M. Büttiker, Physical Review B **27**, 6178 (1983).
[8] E. Pollak and W. H. Miller, Physical review letters **53**, 115 (1984).
[9] F. T. Smith, Physical Review **118**, 349 (1960).
[10] J. Petersen and E. Pollak, The Journal of Physical Chemistry A **122**, 3563 (2018).
[11] D. Sokolovski and L. Baskin, Physical Review A **36**, 4604 (1987).
[12] N. Yamada, Physical review letters **93**, 170401 (2004).
[13] S. Brouard, R. Sala, and J. Muga, Physical Review A **49**, 4312 (1994).
[14] S. Yusofsani and M. Kolesik, Phys. Rev. A **101**, 052121 (2020).
[15] C. A. de Carvalho and H. M. Nussenzveig, Physics Reports **364**, 83 (2002).
[16] H. G. Winful, Physics Reports **436**, 1 (2006).
[17] K. Imafuku, I. Ohba, and Y. Yamanaka, Physical Review A **56**, 1142 (1997).
[18] W. Jaworski and D. M. Wardlaw, Physical Review A **38**, 5404 (1988).
[19] C. Leavens and G. Aers, Physical Review B **39**, 1202 (1989).
[20] E. Hauge, J. Falck, and T. Fjeldly, Physical Review B **36**, 4203 (1987).
[21] E. Hauge and J. Støvneng, Reviews of Modern Physics **61**, 917 (1989).
[22] L. Torlina, F. Morales, J. Kaushal, I. Ivanov, A. Kheifets, A. Zielinski, A. Scrinzi, H. G. Muller, S. Sukiasyan, M. Ivanov, *et al.*, Nature Physics **11**, 503 (2015).
[23] P. Eckle, M. Smolarski, P. Schlup, J. Biegert, A. Staudte, M. Schöffler, H. G. Muller, R. Dörner, and U. Keller, Nature Physics **4**, 565 (2008).
[24] P. Eckle, A. Pfeiffer, C. Cirelli, A. Staudte, R. Dorner, H. Muller, M. Buttiker, and U. Keller, science **322**, 1525 (2008).
[25] A. N. Pfeiffer, C. Cirelli, M. Smolarski, D. Dimitrovski, M. Abu-Samha, L. B. Madsen, and U. Keller, Nature Physics **8**, 76 (2012).
[26] A. N. Pfeiffer, C. Cirelli, M. Smolarski, and U. Keller, Chemical Physics **414**, 84 (2013).
[27] U. S. Sainadh, H. Xu, X. Wang, A. Atia-Tul-Noor, W. C. Wallace, N. Douguet, A. Bray, I. Ivanov, K. Bartschat, A. Kheifets, *et al.*, Nature **568**, 75 (2019).
[28] A. S. Landsman, M. Weger, J. Maurer, R. Boge, A. Ludwig, S. Heuser, C. Cirelli, L. Gallmann, and U. Keller, Optica **1**, 343 (2014).
[29] N. Camus, E. Yakaboylu, L. Fechner, M. Klaiber, M. Laux, Y. Mi, K. Z. Hatsagortsyan, T. Pfeifer, C. H. Keitel, and R. Moshammer, Physical review letters **119**, 023201 (2017).
[30] E. A. Galapon, Journal of mathematical physics **45**, 3180 (2004).
[31] E. A. Galapon and J. J. P. Magadan, Annals of Physics **397**, 278 (2018).
[32] E. A. Galapon, Phys. Rev. Lett. **108**, 170402 (2012).
[33] P. C. Flores and E. A. Galapon, Physical Review A **105**, 062208 (2022).
[34] P. C. Flores and E. A. Galapon, Europhysics Letters **141**, 10001 (2023).
[35] P. C. M. Flores and E. A. Galapon, The European Physical Journal Plus **138**, 375 (2023).
[36] D. A. L. Pablico and E. A. Galapon, The European Physical Journal Plus **138**, 1 (2023).
[37] D. L. Sombillo and E. A. Galapon, Physical Review A **90**, 032115 (2014).
[38] K. L. Jensen, J. Riga, J. L. Lebowitz, R. Seviour, and D. A. Shiffler, Journal of Applied Physics **132**, 124303 (2022).
[39] M. Yu, K. Liu, M. Li, J. Yan, C. Cao, J. Tan, J. Liang, K. Guo, W. Cao, P. Lan, *et al.*, Light: Science & Applications **11**, 215 (2022).