Slackish Business Cycles

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CHAPTER 4.

Matching function

In slackish markets, all trades are mediated by a matching function. These markets therefore differ from Walrasian markets, where all trades are mediated via an auction, and will have sharply different properties.

In this chapter we introduce the matching function. This will be a key tool we use to model slackish markets and study slack and unemployment. The matching function summarizes the complex process through which buyers find sellers: for instance, how workers searching for jobs meet firms searching for employees, and how firms searching for customers meet consumers searching for sellers.[1]

Pissarides (2000, pp. 3–4) explains the role of the matching function wonderfully in the case of the labor market, but this book's argument is that his description in fact applies to the vast majority of markets. Slightly generalizing his words:

> Trade in [any] market is a nontrivial economic activity because of the existence of heterogeneities . . . and information imperfections. If all [buyers and sellers] were identical to each other, and if there was perfect information about their [attributes], trade would be trivial. But without homogeneity on either side of the market and with costly acquisition of information, [buyers and sellers] find it necessary to spend resources to find [desirable] trades. . . . The matching function gives the outcome of the investment of resources by [buyers and sellers] in the trading process as a function of inputs. It is a modeling device

[1]See Petrongolo and Pissarides (2001, 425–427) for a detailed history of the matching function.

> that captures the implications of the costly trading process without the need to make ... the features that give rise to it explicit.

After having introduced a generic matching function and discussed its properties, we will cover four examples. We will start with the urn-ball matching function, which has a particularly simple microfoundation. We then discuss the constant-elasticity-of-substitution (CES) matching function, which is especially convenient for theoretical work. We next turn to the Cobb-Douglas matching function, which is widely used because it describes well the labor market, and because it is easy to manipulate. Finally, we introduce a penalized Cobb-Douglas matching function, which modifies slightly the regular Cobb-Douglas matching function to generate an isoelastic Beveridge curve.

4.1. Why do we need a matching function?

At any point in time, in almost any market, we have buyers and sellers who want to trade, but are not able to trade immediately. On the labor market, there are workers who want to sell their labor and firms who want to buy labor and yet, even though these two groups coexist, it takes time for workers to find a job and for firms to fill a vacant job. On the housing market, we have families who are trying to sell their houses and families who are looking for a house to buy—these two also always coexist, but it takes time for families to sell their homes and for other families to find a new home. On the car market, as we previously discussed, there are cars that sit in inventory and people who are looking to buy cars. It takes time for people to find a car, and it also takes time for the cars in inventory to be sold. On the service market, there may be babysitters who are looking for work, and at the same time, there are families looking to hire—it still takes time for them to find the right babysitter. There are restaurant tables that are empty and hungry people looking for a good restaurant to eat at.

The bottom line is that in almost all markets, buyers who want to buy and sellers who want to sell always coexist—it takes time for the transaction to occur. To model such markets, we cannot use a Walrasian market. In the Walrasian market, one can always buy and sell any quantity of a good at the market price, immediately and without constraints. So a Walrasian market ignores the fact that it takes time and effort to buy and sell in most real-world markets.

To model this complex trading process, we introduce a tool called the matching function. The matching function is an aggregate function that captures and summarizes, at the macro level, all the complexities of trading that happen at the micro level. It is very similar to the production function, which summarizes at the aggregate level all the production that happens at the micro level. It is a simple, well-behaved function that depends on only a few aggregate variables.

Most trading is quite complicated—it is hard for buyers and sellers to get together to execute a trade that both sides desire. For instance, on the labor market, a worker with specific skills would look for a job with specific requirements, whether it be a specific industry, specific location, or specific working conditions or responsibilities. Similarly, a firm advertising a vacant job would have specific needs and would be looking for a worker with the right skill set, the right experience, the right qualifications and character.

This complexity is not limited to the labor market. It would also occur when firms trade with other firms. When firms look for suppliers for a specific part in a product that they are building, they have a long list of requirements because that part is unique to their product. They also have additional constraints on the quality of the goods produced by the supplier, its reliability, its location and how it operates, and so on. The same is true on the supplier side as well.

This complexity exists in most markets, so the matching function can be applied to almost all markets in which trading is complex. The only exceptions are markets in which the goods that are traded are available in large quantity and entirely homogeneous. Then, there isn't this complexity of finding the right seller. An example is the market for common stocks, where all the shares are exactly the same. This is well-modeled by a Walrasian model. The market for common commodities such as gold or wheat is the same as well. But very few markets operate that way; in practice, the goods or services sold are all different and the buyers all have different specific needs. In fact many financial markets are not Walrasian but are better modeled using matching functions (Duffie 2012; Hugonnier, Lester, and Weill 2025).

Furthermore, the complexity of the trade process is quite visible by the effort that both buyers and sellers put into trying to trade. If we look at the labor market, firms that are trying to fill a vacant job have to spend a lot of time and effort recruiting. Similarly, workers spend a lot of time and effort trying to find a job, for example by browsing LinkedIn or Monster and sending out their CV. On the product market, a buyer spends time doing market research before purchasing a good or service, either by reading customer reviews or going on websites like Yelp or TripAdvisor. Similarly, sellers also spend time and money on marketing and advertising to try and find consumers. Again, this complexity of trading can be modeled with a matching function.

4.2. Theoretical properties of a matching function

The matching function is an aggregate function that summarizes all the complexities of the trading process at a micro level. The idea of summarizing the complex trading process by a generic function—without modeling the trading process explicitly—goes back to Pissarides (1979, 1984, 1985, 1986). Before that, people usually modeled specific trading processes that resulted in trading probabilities between 0 and 1—often from the urn-ball model that is commonly used in probability (covered in section 4.4).

Let us look at the standard theoretical properties of the function. In specific models, researchers usually assume specific functional forms for their matching functions, such as a Cobb-Douglas function (covered in section 4.6). But most matching functions share similar properties, reflecting natural properties of markets. We review these properties here, and discuss their implications for traders and market outcomes in the next section.

Consider a market for a specific good with $s > 0$ sellers and $b > 0$ buyers, open for one period. Each seller sells one good, and each buyer aims to purchase one good. The matching function m gives the number of trades in the period:

$$M = m(s, b). \tag{4.1}$$

The shape of the matching function is often characterized by the matching elasticity: the elasticity of the matching function with respect to the number of sellers, denoted η. Formally, the matching elasticity is defined by

$$\eta = \frac{s}{m(s, b)} \cdot \frac{\partial m(s, b)}{\partial s}.$$

As we will see, the matching elasticity may or may not be constant, depending on the type of matching function.

A few restrictions are typically imposed on the matching function, to be realistic as well as to simplify the analysis (Blanchard and Diamond 1989; Petrongolo and Pissarides 2001).

First, the matching function is assumed to be increasing in both arguments, s and b. This means that if there are more sellers in the market, or more buyers in the market, there will be more trades. This is a natural assumption. If more goods are available, chances are that a greater number of buyers will find an appropriate good to buy. Conversely if there are more buyers on the market, chances are that a greater number of sellers will find a buyer for their good.

A second important assumption is that the matching function has constant returns to scale:

$$m(\lambda s, \lambda b) = \lambda m(s, b) \tag{4.2}$$

for any $\lambda \geq 0$. This assumption critically simplifies the analysis, because it ensures that the trading probabilities can be expressed as functions of market tightness, as the next section shows. This assumption has also been tested extensively with labor market data and could not be rejected (Petrongolo and Pissarides 2001).

A third typical assumption is that there are no trades if there are no buyers or no sellers. This is also natural, since we need people on both sides of the market for trades to occur.

A fourth typical assumption is that the matching function is concave in both arguments, s and b.

A final assumption is that the number of trades that occur within the period considered is less than the numbers of sellers and buyers: $m(s,b) \leq \min(s,b)$. Indeed, there cannot be more goods sold than sellers, and more goods bought than buyers. This assumption is specific to discrete-time models, however. In a continuous-time model, the matching function gives the flow of trades occurring at any point in time, and there would be no restriction on the number of trades that are realized, so no constraint on the level of $m(s,b)$.

4.3. Market tightness and trading probabilities

The matching function tells us that, during a certain period, not all sellers are able to sell their good, and not all buyers are able to buy a good. Therefore, we need to figure out the probabilities that a buyer is able to buy and that a seller is able to sell. The two trading possibilities are the probability to sell f and the probability to buy q, given by

$$f = \frac{M}{s}, \quad q = \frac{M}{b}.$$

The assumptions we made on the matching function have clear implications for how these trading probabilities behave.

4.3.1. Market tightness

Before we start, we introduce a new fundamental variable, the market tightness:

$$\theta = \frac{b}{s}.$$

The tightness can be defined on any market with sellers and buyers.

Since both trading probabilities will only depend on market tightness, tightness will play a crucial role in the book—it is going to summarize the state of any market.

4.3.2. Selling probability

We then look at the selling probability:

$$f = \frac{m(s,b)}{s} = m\left(\frac{s}{s}, \frac{b}{s}\right) = m\left(1, \frac{b}{s}\right),$$

since the matching function is constant returns to scale. We can re-express our selling probability as a function of market tightness:

$$f(\theta) = m(1, \theta). \tag{4.3}$$

Since the matching function is increasing in its two arguments, we infer that the selling probability is increasing in tightness. The tighter the market, the more likely you are to sell. This makes sense because the tighter a market is, the more buyers there are for each seller. And since the matching function is concave in its two arguments, we infer that the selling probability is concave in tightness. This means that a higher tightness leads to a higher selling probability, but with diminishing returns.

We can also see that when tightness is 0, the selling probability must also be 0. This is because, in the matching function, when any of the two arguments are zero, there is no trade. This is intuitive because if there are no buyers, there is no chance of selling anything, so the probability of selling has to be 0.

Additionally, we know that the selling probability is always between 0 and 1 because we have assumed that the matching function is always less than the minimum of its two arguments. Since $m(s, b) \leq \min(s, b)$, then $m(s, b) \leq s$ and $f \leq 1$.

4.3.3. Buying probability

Now, we shift our attention to the buying probability:

$$q = \frac{m(s, b)}{b} = m\left(\frac{s}{b}, \frac{b}{b}\right) = m\left(\frac{s}{b}, 1\right),$$

since the matching function has constant returns to scale. Using market tightness, we can rewrite the probability as:

$$q(\theta) = m\left(\frac{1}{\theta}, 1\right), \tag{4.4}$$

so that the buying probability only depends on tightness.

We can go over the properties of the buying probability in the same way. When tightness increases, $1/\theta$ decreases. Because the matching function is increasing in both arguments, the buying probability is decreasing in tightness. This means that a buyer in a tight market is less likely to be able to buy the good they want, since there are very few sellers and a lot of buyers, increasing competition. This is true in any tight market: it is a good time to be a seller but a bad time to be a buyer.

Another thing that we see is when we take the limit, the probability to buy is 0 when tightness is infinite. This makes sense because when infinitely many buyers are competing to buy the goods they want, the probability to be able to buy that good is bound to go to 0.

Additionally, we know that the buying probability is always between 0 and 1 because we have assumed that the matching function is always less than the minimum of its two arguments. Since $m(s, b) \leq \min(s, b)$, then $m(s, b) \leq b$ and $q \leq 1$.

4.3.4. Relation between the trading probabilities

There is a simple relationship between the two trading probabilities. The number of trades is $f(\theta)s = q(\theta)b$, so $f(\theta)/q(\theta) = b/s = \theta$. That is, for any matching function:

$$f(\theta) = \theta q(\theta). \tag{4.5}$$

From this result we also see that the elasticities of the trading probabilities with respect to tightness are necessarily related:

$$\epsilon_{\theta}^{f} = 1 + \epsilon_{\theta}^{q}.$$

From (4.4), and the results in appendix A on the elasticity of composite functions, we see that the elasticity of the buying probability is directly related to the matching elasticity η:

$$\epsilon_{\theta}^{q} = -\eta. \tag{4.6}$$

Thus, the elasticity of the selling probability is also related to the matching elasticity:

$$\epsilon_{\theta}^{f} = 1 - \eta. \tag{4.7}$$

Once again, these elasticities might or might not be constant, depending on the matching function.

4.4. Urn-ball matching function

Let us now study the urn-ball matching function. The urn-ball matching function is interesting because it has a simple microfoundation, which draws on the urn-ball model in probability theory.

4.4.1. Foundation

Several researchers proposed an urn-ball foundation to explaining why trading probabilities might be less than 1 in a given period.

Butters (1977) used the urn-ball framework in the product market. In this model, firms advertise their products by dropping ads into customers' mailboxes. The probability to sell is less than 1 because several firms might drop their ad in the same mailbox, in which case the customer only buys from the cheapest firm. The probability to buy is also less than 1, because a customer might not receive any ad in their mailbox.

Hall (1979) used a similar setup in the labor market. In this model, firms simultaneously and randomly make job offers to job seekers. The probability to hire a job seeker is less than 1 because several firms might make a job offer to the same job seeker, who cannot accept more than one offer. The probability to find a job is also less than 1 because a job seeker might be unlucky and not receive any of the job offers.

Both of these situations correspond to an urn-ball setup. In Butters's example, the ads are the balls and the mailboxes are the urns. In Hall's example, the job offers are the balls and the job seekers are the urns.

In this section, we consider as an example the haircut market. There are b customers who are looking to get a haircut, and s hairdressers. Each hairdresser can only accommodate one customer at the time, so if two customers pick the same hairdresser, only one will get a haircut. We could add additional sources of randomness. For instance, hairdressers might only be open at certain hours and on certain days, which customers might not be aware of. If a customer arrives at a hair salon and it is closed, they would not be able to get a haircut. Another possible source of randomness is that each hairdresser specializes in a type of haircut (women, men, children, cheap, expensive, fashionable, professional, and so on), which again customers might not be aware of. Customers are looking for a specific type of haircut. If they arrive at a hair salon that does not offer the haircut that they are looking for, they would not be able to get a haircut. Customers might now know where hair salons are located, so they might pick a shopping center at random and not find a hair salon there. Of course, a hairdresser might not be able to sell a haircut if no interested customers show up during the hair salon's open hours.

We can see that the task of matching customers with hairdressers is complex and multidimensional. First, customers do not know where the others go, so they don't know which hairdresser already have a customer and which hairdressers are idle. Beside such coordination failure between customers, customers might not know the type and quality of haircuts offered by each hairdresser. Finally, customers might not know when hairdressers are operating or where hairdressers are located.

4.4.2. Expression

We now derive the number of haircuts sold in a simple situation: b customers in need of a haircut each go to one of s open hair salons. The customers pick their hair salons simultaneously and randomly. That day, customers can only make one trip to the hair salon. All hair salons offer the same haircut, which is the haircut that customers are looking for.

The probability that a specific customer goes to a specific hairdresser is $1/s$, and the probability that the customer does not go to the hairdresser is $1 - 1/s$. Accordingly, the probability that a specific hairdresser does not get a visit from any of the b customers is $(1 - 1/s)^b$. Conversely, the probability that a specific hairdresser gets at least one visit is $1 - (1 - 1/s)^b$. Since a hairdresser sells a haircut if at least one customer visits, this probability gives the probability to sell a haircut for a specific hairdresser, as well as the expected number of haircuts sold by the hairdresser (since the hairdresser sells either 0 or 1 haircut). Then, the expected number of haircuts sold by all hairdressers is just

$$M = s\left[1 - \left(1 - \frac{1}{s}\right)^b\right]. \tag{4.8}$$

This is the expected number of trades that are going to occur on the haircut market.

The matching function given by (4.8) is a little cumbersome, but it can be greatly simplified when the number of hairdressers is large enough. To simplify, we use a linear approximation of the logarithm: $\ln(1-x) \approx -x$ when x is small. This approximation allows us to rewrite the probability that a hairdresser gets no visit at all:

$$\left(1 - \frac{1}{s}\right)^b = \exp\left(b \cdot \ln\left(1 - \frac{1}{s}\right)\right) \approx \exp\left(b \cdot \frac{-1}{s}\right) = \exp\left(-\frac{b}{s}\right).$$

We can then rewrite our matching function, which is the expected number of haircuts sold:

$$m(s, b) = s\left[1 - \exp\left(-\frac{b}{s}\right)\right]. \tag{4.9}$$

This is the urn-ball matching function, which gives the number of trades for a given number of sellers, s, and buyers, b.

4.4.3. Properties

By looking at the matching function (4.9), we can verify that it satisfies the general properties mentioned in section 4.2. Firstly, we see that it has constant returns to scale:

$$m(\lambda s, \lambda b) = \lambda s\left[1 - \exp\left(-\frac{\lambda b}{\lambda s}\right)\right] = \lambda m(s, b),$$

since the λs in the numerator and denominator of the fraction cancel each other out.

Secondly, we can check that the matching function is increasing in both its arguments. We can immediately see from (4.9) that if the number of buyers goes up, the number of haircuts will also increase. Formally, the partial derivative of the matching function with respect to the number of buyers is simply:

$$\frac{\partial m}{\partial b} = \exp\left(-\frac{b}{s}\right) > 0.$$

Showing that the number of haircuts increases with the number of hairdressers is a little bit more tricky since s has two opposite influences on the matching function. We must calculate the derivative formally to check that it is positive. After simplifying, we get:

$$\frac{\partial m}{\partial s} = 1 - \left(1 + \frac{b}{s}\right)\exp\left(-\frac{b}{s}\right). \tag{4.10}$$

By using the fact that $\exp(x) \geq 1 + x$ for all x, we infer that $1 \geq (1 + x)\exp(-x)$ for all x, so we verify that $\partial m/\partial s \geq 0$. From this we conclude that the matching function is increasing in s. Thus, we confirm that the matching function is increasing in its two arguments.

Thirdly, we can confirm that if the number of sellers or buyers is zero, the matching function is going to be equal to zero: $m(0, b) = m(s, 0) = 0$.

Fourthly, we can verify that the matching function is concave in both arguments. Clearly $\partial m/\partial b$ is decreasing in b, so $\partial^2 m/\partial b^2 < 0$, which tells us the matching function is concave in the number of buyers. The second derivative of the matching function with respect to the number of sellers is

$$\frac{\partial^2 m}{\partial s^2} = -\frac{b^2}{s^3}\exp\left(-\frac{b}{s}\right);$$

since $\partial^2 m/\partial s^2 < 0$, the function is also concave with respect to the number of hairdressers.

Finally, we see that the matching function is less than its two arguments. Since $\exp(-x) > 0$, it is clear that $m(s, b) \leq s$. Using again $\exp(x) \geq 1 + x$, we see that

$$m(s, b) \leq s\left[1 - \left(1 - \frac{b}{s}\right)\right] = b.$$

So overall $m(s, b) \leq \min(s, b)$.

The trading probabilities given by the urn-ball matching function are illustrated in figure 4.1.

4.4.4. Matching elasticity

A key statistic to describe the shape of the matching function is the matching elasticity: the elasticity of the matching function with respect to the number of sellers. We compute the matching elasticity to better understand how the urn-ball matching function behaves.

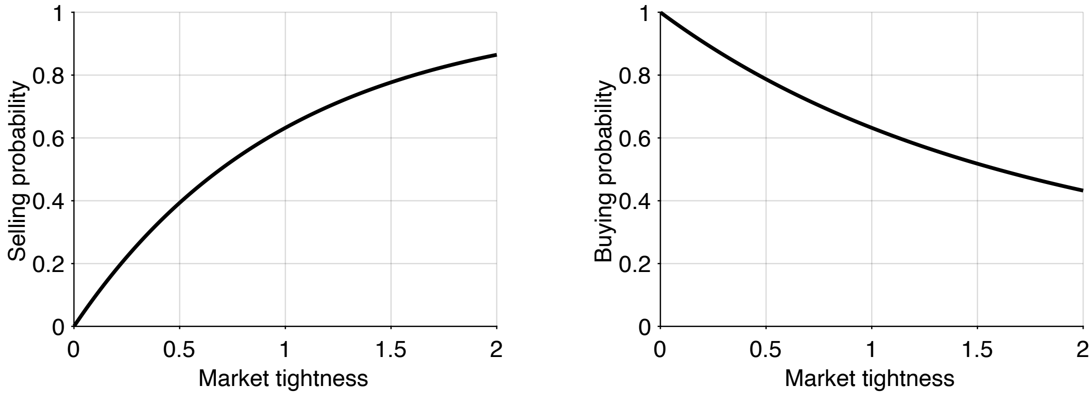


FIGURE 4.1. Urn-ball matching function

The urn-ball matching function is given by (4.9). The selling probability is obtained from the matching function with (4.3). The buying probability is obtained from the matching function with (4.4).

We start from the definition of the matching elasticity:

$$\eta = \frac{s}{m(s,b)} \cdot \frac{\partial m}{\partial s}.$$

Using the partial derivative (4.10), we can write the matching elasticity as

$$\eta = \frac{s}{m(s,b)} \cdot \left[1 - \exp\left(\frac{-b}{s}\right)\right] - \frac{b}{m(s,b)} \cdot \exp\left(\frac{-b}{s}\right).$$

Given that the matching function $m(s,b)$ is given by (4.9), we can easily express the matching elasticity as a function of the tightness of the market:

$$\eta(\theta) = 1 - \frac{\theta}{\exp(\theta) - 1}. \tag{4.11}$$

The matching elasticity is an increasing function of tightness, growing from 0 when tightness is 0, to $1 - 1/e$ when tightness is 1, to 1 when tightness is infinite.

The property of the matching elasticity can be showed by using the definition of the exponential function via Taylor series:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

so

$$\exp(x) - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!},$$

and

$$\frac{\exp(x) - 1}{x} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{(n+1)!},$$

All the terms in the sum are increasing in $x \geq 0$, so the function $[\exp(x) - 1]/x$ is increasing, which tells use that $h(\theta)$ is increasing in θ. All the terms in the sum are 0 when $x = 0$, which tells us that $\eta(0) = 1$. And all the terms in the sum grow to ∞ when $x \to \infty$, which tells us that $\lim_{\theta\to\infty} \eta(\theta) = 1$.

The behavior of the elasticity means that when there are no buyers on the market ($\theta = 0$), adding more sellers does not lead to any more trades. The intuition is that the market is already extremely congested with sellers, so more sellers do not bring more trade: only more buyers would generate more trades. On the other hand, when there are almost no sellers ($\theta \to \infty$), adding 1% more sellers generates 1% more trades, even with a fixed number of buyers. In that case the number of trades is almost solely determined by the number of sellers, so numbers of sellers and trades grow proportionally.

4.5. Constant-elasticity-of-substitution matching function

Despite its simple microfoundation, the urn-ball matching function is not used often in theoretical work, because it is a little cumbersome. Instead, it is preferable to use the constant-elasticity-of-substitution (CES) matching function, as it is much more tractable. While it does not have a microfoundation as simple as the urn-ball process, it is possible to obtain a CES matching function from a Poisson queuing process in which sellers call buyers to advertise their goods (Stevens 2007), and from an Erdos-Renyi network in which sellers and buyers are connected (Angelis and Bramoulle 2023). Another drawback to the urn-ball matching function compared to the other matching functions that we will see in this chapter—including the CES matching function—is that it does not include any parameters, so in quantitative work it cannot be calibrated to match features of real-world data, which is a disadvantage.

4.5.1. Expression

The CES matching function takes the following form:

$$m(s, b) = \omega \cdot \left(s^{-\gamma} + b^{-\gamma}\right)^{-1/\gamma}. \tag{4.12}$$

The parameter $\omega \in (0, 1]$ governs the efficacy of the matching process. The parameter $\gamma > 0$ governs the elasticity of substitution between s and b.

4.5.2. Properties

We can again check that all the general properties of a matching function are satisfied here. First, we see that the CES matching function is increasing in both its arguments. To check that the function is concave in both arguments, we compute its derivatives. For instance, the partial derivative with respect to b is

$$\frac{\partial m}{\partial b} = \omega \cdot \left[1 + \left(\frac{b}{s}\right)^{\gamma}\right]^{-\frac{1+\gamma}{\gamma}}.$$

We see that the partial derivative is not only positive but also decreasing in b, so $\partial^2 m/\partial b^2 < 0$. From this we infer that the CES matching function is concave in b. The matching function is completely symmetric in b and s, so it is also concave in s.

We also see that the CES function has constant returns to scale:

$$m(\lambda s, \lambda b) = \omega \cdot \left[(\lambda s)^{-\gamma} + (\lambda b)^{-\gamma}\right]^{-1/\gamma} = \lambda m(s, b). \tag{4.13}$$

We can also easily see that when there are no buyers or no sellers, the number of matches is zero: $m(0, b) = m(s, 0) = 0$.

Lastly, we verify that the CES matching function is less than the minimum of its two arguments. We can first note that since $b^{-\gamma} > 0$, then $s^{-\gamma} + b^{-\gamma} > s^{-\gamma}$. Moreover, $\omega < 1$. Hence,

$$m(s, b) = \omega \cdot \left(s^{-\gamma} + b^{-\gamma}\right)^{-1/\gamma} < \left(s^{-\gamma}\right)^{-1/\gamma} = s. \tag{4.14}$$

Similarly, we find $m(s, b) < b$, which overall tells us that $m(s, b) \leq \min(s, b)$.

The trading probabilities given by the CES matching function are illustrated in figure 4.2.

4.5.3. Matching elasticity

The matching elasticity η is the elasticity of the matching function with respect to the number of sellers. With the CES matching function (4.12), the elasticity varies with the tightness of the market.

We compute the matching elasticity using the elasticity results from appendix A. We can express the matching elasticity as a function of market tightness:

$$\eta(\theta) = \frac{-1}{\gamma} \cdot \frac{s^{-\gamma}}{s^{-\gamma} + b^{-\gamma}} \cdot (-\gamma)$$

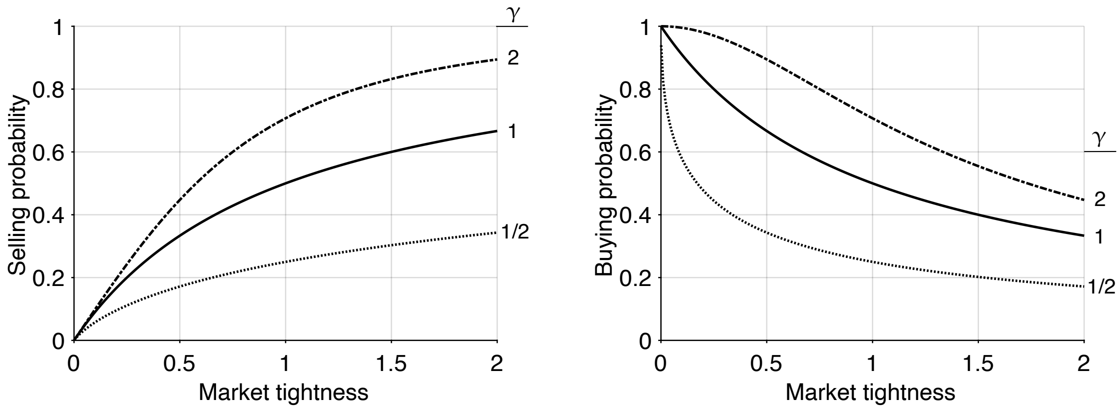


FIGURE 4.2. CES matching function for different elasticities γ

The CES matching function is given by (4.9). The selling probability is obtained from the matching function with (4.3). The buying probability is obtained from the matching function with (4.4). In the figure, the matching function is calibrated with $\omega = 1$.

so that

$$\eta(\theta) = \frac{1}{1+\theta^{-\gamma}}. \tag{4.15}$$

The matching elasticity is an increasing function of tightness, growing from 0 when tightness is 0, to 1/2 when tightness is 1, to 1 when tightness is infinite. Just like in the case of the urn-ball matching function, the behavior of the matching elasticity implies that when there are no buyers on the market, adding more sellers does not lead to any more trades, while when there are almost no sellers, numbers of sellers and trades grow proportionally.

With the CES matching function, it is actually easy to see what happens when there are very few sellers and buyers. We can rewrite the matching function (4.12) as

$$m(s,b) = \omega \cdot s \cdot \left[1 + \left(\frac{s}{b}\right)^{\gamma}\right]^{-1/\gamma}.$$

When the number of sellers is small relative to the number of buyers, $(s/b)^{\gamma}$ is negligible compared to 1, so the matching function is approximately linear in the number of sellers: $m(s,b) \approx \omega \cdot s$. From this approximation we see that when the number of sellers is small, the elasticity of the matching function with respect to s is 1, so numbers of sellers and trades grow proportionally.

We can apply the same logic when there are very few buyers relative to the number of sellers. In that case the matching function is approximately linear in the number of buyers: $m(s,b) \approx \omega \cdot b$. Then the elasticity of the matching function with respect to s is 0, so the number of sellers has no effect on the number of trades. Instead, the number of trades grows proportionally with the number of buyers.

4.6. Cobb-Douglas matching function

Maybe the most popular of all matching functions is the Cobb-Douglas matching function. Its appeal comes from the facts that it is that even easier to manipulate than the CES matching function, and that it describes the US labor market well (Blanchard and Diamond 1989; Petrongolo and Pissarides 2001). There are no standard microfoundations for the Cobb-Douglas matching function, but just like the CES matching function, it is possible to obtain it from a Poisson queuing process or an Erdos-Renyi network (Stevens 2007; Angelis and Bramoulle 2023). In theoretical work, however, it is sometimes preferrable to stick with the CES matching function because the Cobb-Douglas matching function does not guarantee that $m(s, b) \leq \min(s, b)$, so if it is not truncated it can produce trading probabilities that are greater than 1.

4.6.1. Expression

The Cobb-Douglas matching function takes the following form:

$$m(s, b) = \omega \cdot s^{\eta} \cdot b^{1-\eta}.$$

The parameter $\omega > 0$ governs the efficacy of the matching process. The parameter $\eta \in (0, 1)$ is the matching elasticity, which is constant here.

4.6.2. Properties

It is easy to check that the Cobb-Douglas function satisfies the typical properties of a matching function. First, the function is clearly increasing and concave in both arguments—since any function of the form $x \mapsto ax^{b}$ with $a > 0$ and $b \in (0, 1)$ is increasing and concave in x. Second, we see that the matching function has constant returns to scale:

$$m(\lambda s, \lambda b) = \omega(\lambda s)^{\eta}(\lambda b)^{1-\eta} = \lambda m(s, b).$$

Last, we can easily see that if there are no sellers or no buyers, there are no trades: $m(0, b) = m(s, 0) = 0$.

The Cobb-Douglas matching function has one main limitation. There is no guarantee that it is less than the minimum of its two arguments: $m(s, b) > \min(s, b)$ if s or b is large. In a continuous-time model, where the matching function describes the flow of trades at any point in time, this is not an issue. But in a discrete-time model, where the matching function describes the number of trades in a period, this could be an issue

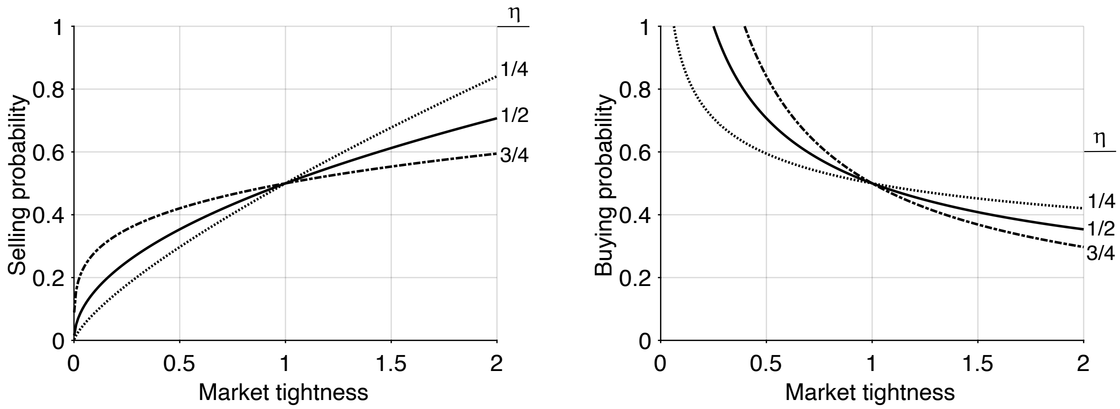


FIGURE 4.3. Cobb-Douglas matching function for different elasticities η

The Cobb-Douglas matching function is given by (4.9). The selling probability is obtained from the matching function with (4.3). The buying probability is obtained from the matching function with (4.4). In the figure, the matching function is calibrated with $\omega = 0.5$.

because it might lead to trading probabilities that are greater than 1. To avoid trading probabilities that are greater than 1, it is required to truncate the matching function by imposing $m(s,b) = \min\left(\omega \cdot s^{\eta} \cdot b^{1-\eta}, s, b\right)$. The truncation in turn might create difficulties in numerical and theoretical work. So if it is important to guarantee that the matching function remains less than the minimum of its two arguments, it is more convenient to use the CES matching function (den Haan, Ramey, and Watson 2000).

The trading probabilities given by the Cobb-Douglas matching function are illustrated in figure 4.3.

4.6.3. Matching elasticity

A reason why the Cobb-Douglas matching function is particularly convenient is that its matching elasticity is constant, equal to the parameter η, unlike in the case of the CES matching function. This property often greatly simplifies the analysis.

Relatedly, the trading probabilities that it produces are really easy to deal with. We can obtain them directly using (4.3) and (4.4):

$$f(\theta) = m(1,\theta) = \omega\theta^{1-\eta}, \quad q(\theta) = m\left(\frac{1}{\theta}, 1\right) = \omega\theta^{-\eta}.$$

In particular, the trading probabilities have constant elasticity with respect to tightness: $\epsilon^{f}_{\theta} = 1 - \eta$ and $\epsilon^{q}_{\theta} = -\eta$. This property also often simplifies analytical work.

4.7. Penalized Cobb-Douglas matching function

The Cobb-Douglas matching function is very popular, but it does not produce an isoelastic Beveridge curve. This property is inconsistent with the US Beveridge curve, which appears to be isoelastic: it seems to have a constant elasticity (chapter 3). The non-isoelastic Beveridge curve also complicates some of the welfare and policy analysis (Michaillat and Saez 2021).

However, it is easy to generate a Beveridge curve of constant elasticity by adding a small penalty to the Cobb-Douglas function, as proposed by Michaillat and Saez (2024). It is designed to be used in continuous-time dynamic models to produces a Beveridge curve with constant elasticity (chapter 8).

4.7.1. Expression

The penalized Cobb-Douglas matching function takes the following form:

$$m(s, b) = \omega \cdot s^{\eta} \cdot b^{1-\eta} - \sigma \cdot s. \tag{4.16}$$

The novelty with the standard Cobb-Douglas matching function is the small penalty $-\sigma \cdot s$ added to the function. The parameter $\sigma > 0$ dictates the size of the penalty; it is small relative to ω in practice.

4.7.2. Properties

The penalized Cobb-Douglas function satisfies the standard properties of a matching function, but requires that the arguments remain within a slightly reduced range compared to the Cobb-Douglas function.

First, the matching function has constant returns to scale:

$$m(\lambda s, \lambda b) = \omega(\lambda s)^{\eta}(\lambda b)^{1-\eta} - \sigma \cdot (\lambda s) = \lambda m(s, b).$$

Second, the function is increasing and concave in b, just like the standard Cobb-Douglas function.

Third, the function is 0 when $s = 0$.

Fourth, the function is concave in s, as it is the sum of two functions that are themselves concave in s: the Cobb-Douglas function $s \mapsto \omega s^{\eta} b^{1-\eta}$ and the linear function $s \mapsto -\sigma s$.

Fifth, the function is increasing in s, as long as tightness $\theta = b/s$ is not too low. Taking the partial derivative of the matching function, we see that the partial derivative is actually a function of tightness only:

$$\frac{\partial m}{\partial s} = \omega\eta\theta^{1-\eta} - \sigma.$$

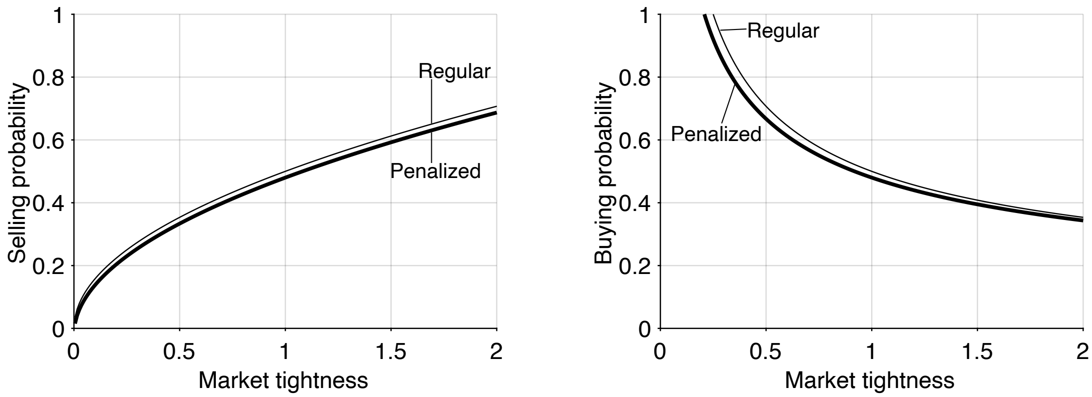


FIGURE 4.4. Penalized Cobb-Douglas matching function

The penalized Cobb-Douglas matching function is given by (4.9). The selling probability is obtained from the matching function with (4.3). The buying probability is obtained from the matching function with (4.4). In the figure, the matching function is calibrated with $\omega = 0.5$, $\eta = 0.5$ and $\sigma = 0.05$. The figure also compares the penalized Cobb-Douglas matching function to the regular Cobb-Douglas matching function calibrated with $\omega = 0.5$ and $\eta = 0.5$.

Hence the derivative is positive for any tightness above the lower bound $\underline{\theta}$, given by

$$\underline{\theta} = \left(\frac{\sigma}{\omega\eta}\right)^{\frac{1}{1-\eta}}. \tag{4.17}$$

Sixth, the function is positive, as long as tightness is not too low. Dividing (4.16) by s, we see that the function is positive as long as $\omega\theta^{1-\eta} - \sigma \geq 0$, which requires tightness to be high enough:

$$\theta \geq \left(\frac{\sigma}{\omega}\right)^{\frac{1}{1-\eta}}.$$

This bound on tightness is lower than $\underline{\theta}$, since $\eta < 1$.

Overall, for any tightness in $(\underline{\theta}, \infty)$, the penalized Cobb-Douglas matching function is positive, has constant returns to scale, is increasing and concave in both arguments. The penaly parameter σ is generally much lower than the efficacy parameter ω, so the lower bound on tightness $\underline{\theta}$ is close to 0.

The trading probabilities given by the penalized Cobb-Douglas matching function are illustrated in figure 4.4. As we can see on the figure, the trading probabilities are very close to those for a regular Cobb-Douglas matching function when σ/ω is small (as an illustration, we set $\sigma/\omega = 0.1$ in the figure).

4.8. Empirical properties of the matching function

To conclude this section, we briefly review the empirical properties of the matching function to convince ourselves that the theoretical assumptions that we made give us a matching function that describes the real world well. When matching functions were first developed, researchers examined the data to ensure that they built functions that were descriptive of the real world (Blanchard and Diamond 1989; Petrongolo and Pissarides 2001). Here we review modern US data to do the same.

Data on sellers and buyers are most available on the labor market, where we have good counts of who sells labor (job seekers) and who buys labor (vacant jobs). Thus we focus on the matching function for the labor market.

A central assumption is that the matching function has constant returns to scale. The main implication is that the market tightness determines the rates at which sellers and buyers trade on the market.

On the labor market, this means that labor market tightness θ determines the rate at which job seekers find jobs—the job-finding rate f—and the rate at which vacant jobs are filled—the job-filling rate q. Even on the labor market, we need to make some assumption to be able to compute the trading rates and tightness. We take the perspective that only unemployed workers are actively looking for jobs and filling vacancies, and that workers neither enter nor exit the labor force but simply transit between employment and unemployment. In reality, there are employment-to-employment transitions as well as flows between employment and unemployment and outside of the labor force (Davis, Faberman, and Haltiwanger 2006, figure 1). But we ignore that complication here. Our results will be valid as long as the flows of unemployed workers who exit the labor force is offset by the flow of people who enter the pool of unemployed from outside of the labor force, and as long as a fixed share of vacancies is devoted to the unemployed workers.

For reference, we plot the US labor market tightness between 1948 and 2024. Labor market tightness is computed as the number of job vacancies per job seeker, or equivalently, the ratio of vacancy rate to unemployment rate. We start in 1948 because that is when the BLS started to collect the data that we need to compute the job-finding rate (CPS data). We see that labor market tightness is sharply procyclical. It averages 0.69 between 1948 and 2024. Over that period, tightness reached its highest level, 1.98, during the recovery from the pandemic (2022:Q2) and its lowest level, 0.16, during the Great Depression (2009:Q3).

We start with the rate at which jobseekers find jobs, and correlate this rate with labor market tightness. To compute the job-finding rate, we follow Shimer (2012). In month t, $u(t)$ workers are looking for jobs. Some of them find a job in the month, and some do not. Those who do not find a job remain unemployed, so we could think that $u(t) - u(t+1)$ indicates the number of workers who have been able to find a job. There is a complication, however. Some workers who were employed lose their jobs in month t and join the pool of unemployed workers. We need to account for those when we compute the number of job seekers who found a job. We denote by $u^s(t+1)$ the number of workers who were previously employed and who have joined unemployment between months t and $t+1$. The

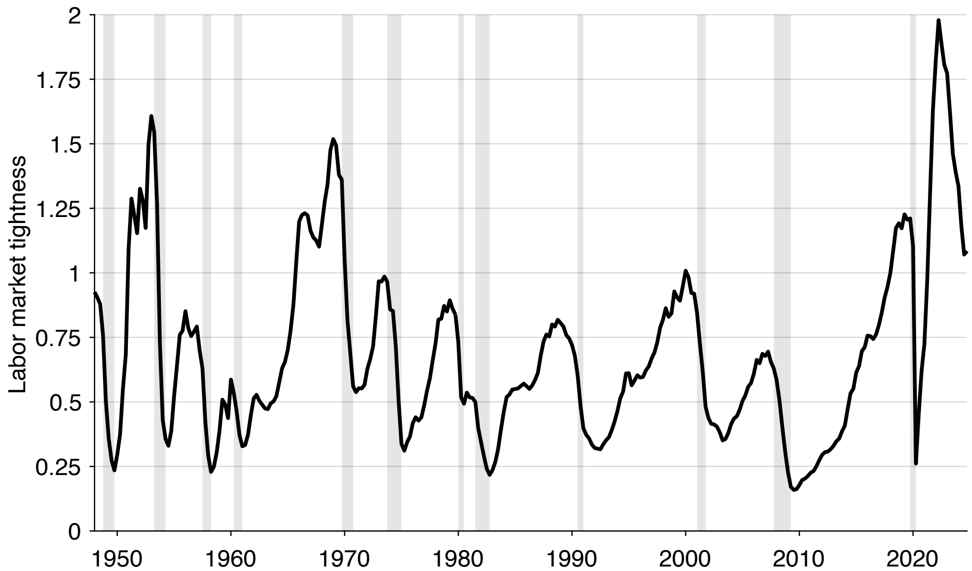


FIGURE 4.5. Labor market tightness in the United States, 1948–2024

Labor market tightness is the ratio of vacancy rate to unemployment rate. The unemployment rate comes from figure 3.1 while the vacancy rate come from figure 3.5.

actual number of workers who have found a job within month t is $u(t)-[u(t+1)-u^{s}(t+1)]$. Dividing this number of job finders by the number of job seekers in month t, we obtain the probability of finding a job in month t:

$$F(t) = 1 - \frac{u(t+1) - u^{s}(t+1)}{u(t)}. \tag{4.18}$$

To calculate $F(t)$, we measure $u(t)$ as the number of unemployed persons in month t computed by the BLS (2025b), and $u^{s}(t)$ as the number of persons who have been unemployed for less than 5 weeks in month t computed by the BLS (2025a).[2]

Assuming that unemployed workers find a job according to a Poisson process with monthly arrival rate $f(t)$, then the probability that they have not found a job at after a month is $\exp(-f(t))$, so the probability that they have found a job within one month is $F(t) = 1 - \exp(-f(t))$. We can thus infer the job-finding rate from the job-finding probability:

$$f(t) = -\ln(1 - F(t)). \tag{4.19}$$

We thus obtain the monthly job-finding rate in the United States (figure 4.6A). Over

[2]Following Shimer (2012, appendix A), we multiply the series for $u^{s}(t)$ by 1.1 after January 1994 to correct for a change in the design of the CPS.

1948–2024, the job-finding rate is highly procyclical and averages 0.56 per month. This means that on average, a US job seeker takes $1/0.56 = 1.8$ months to find a job.

Our objective was to assess how whether the assumption that labor market tightness determines the job-finding rate was accurate. Just like in figure 3.8, we focus on the 1948–2019 period, which is when the Beveridge curve and therefore the matching function was stable. To uncover the relationship between tightness and job-finding rate, we plot the log of the job-finding rate against the log of tightness. To remove slow changes in the matching function, we detrend both series using an HP filter with smoothing parameter 10,000. We find that log job-finding rate is largely determined by log labor market tightness, supporting the assumption of constant returns to scale (figure 4.6B). The least-squares regression gives $R^2 = 0.89$, with a coefficient of 0.40, which indicates that the job-finding rate is approximately an isoelastic function of tightness, with an elasticity of 0.40.

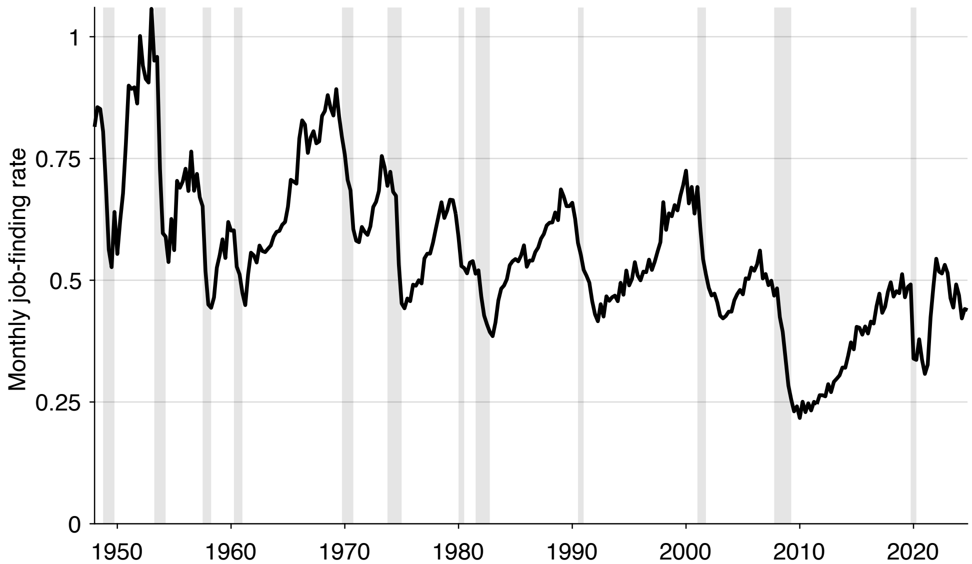
Under the assumption that matching on the labor market only involves vacancies and unemployed workers, we know from (4.5) that the job-filling and job-finding rates are related by $q = f/\theta$. This means that we can easily construct the US job-filling rate between 1948 and 2024 as

$$q(t) = \frac{f(t)}{\theta(t)}. \tag{4.20}$$

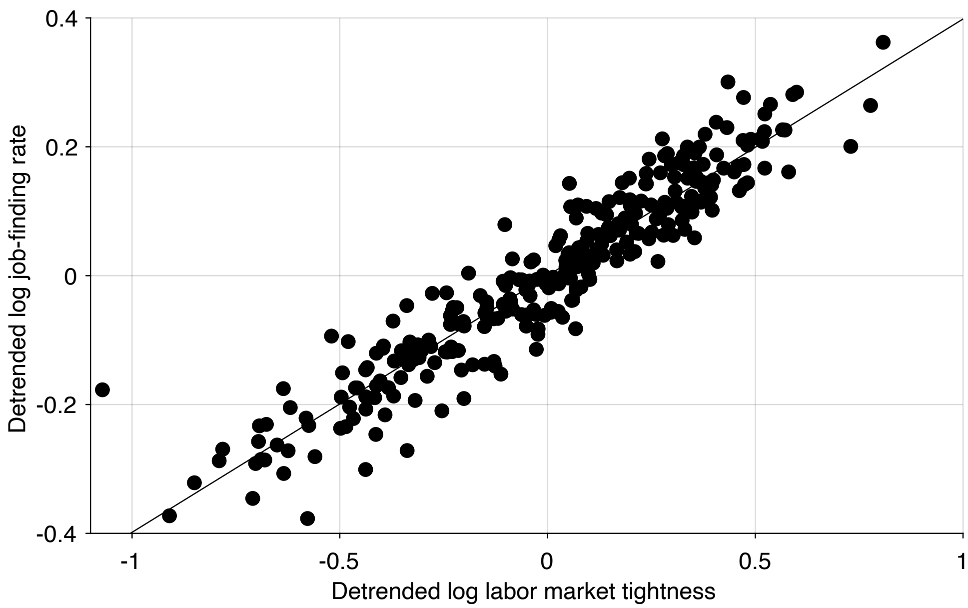
We see that the job-filling rate is countercyclical and averages 0.94 per month (figure 4.7A). Given that the job-finding rate is roughly an isoelastic function of tightness, it is expected that the job-filling rate also appears to be an isoelastic function of tightness (figure 4.7B). An least-squares regression of log job-filling rate on log tightness—both detrended by HP filter with smoothing parameter 10,000—gives a great fit, $R^2 = 0.95$, with a coefficient of -0.60.

Overall, we find that the job-finding and job-filling rates are largely determined by labor market tightness, which supports the assumption that the matching function has constant returns to scale. In addition, we have seen that the elasticity of the job-finding and job-filling rates with respect to tightness appears constant, which suggests that the matching elasticity is constant. This finding suggests that a Cobb-Douglas matching function, which has a constant matching elasticity, describes the US labor market better than the urn-ball or CES matching functions, which have matching elasticities that are highly responsive to tightness. Furthermore, the matching elasticity that appears from the regressions is $\eta = 0.6$, using (4.7) and (4.6).

The empirical findings in this section are in line with findings in the literature on the matching function. In their survey, Petrongolo and Pissarides (2001) conclude that aggregate studies of the US labor market found that the matching function involved the stock of unemployed workers and job vacancies with constant returns to scale. The studies converged on a Cobb-Douglas form with a matching elasticity η between 0.5 and 0.7. More



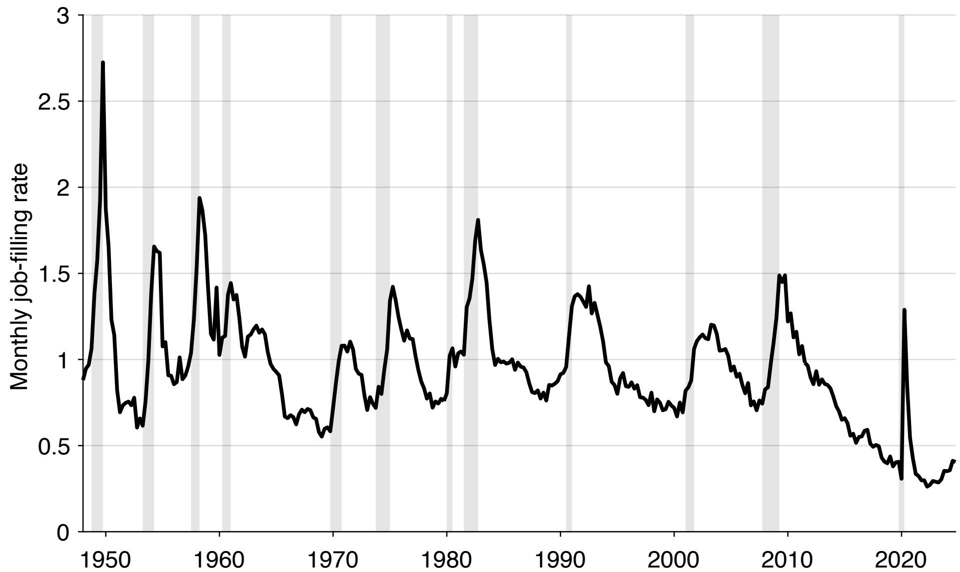
A. Monthly job-finding rate, 1948–2024



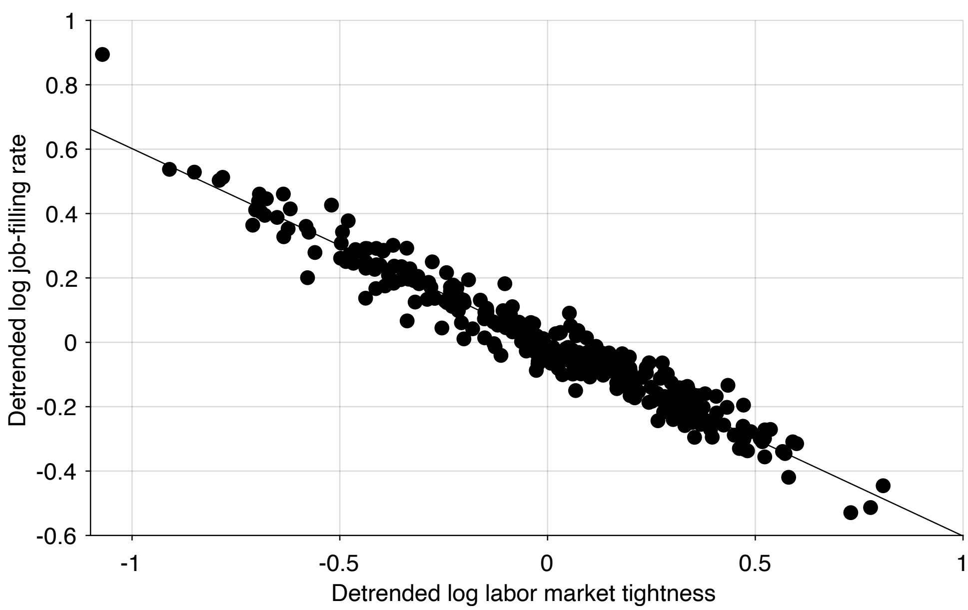
B. Detrended log tightness and job-finding rate, 1948–2019

FIGURE 4.6. Job-finding rate approximates an isoelastic function of labor market tightness in the United States

The monthly job-finding rate is computed from (4.19). Labor market tightness comes from figure 4.5. In panel B, the series are detrended by applying a HP filter with smoothing parameter of 10,000.



A. Monthly job-filling rate, 1948–2024



B. Detrended log tightness and job-filling rate, 1948–2019

FIGURE 4.7. Job-filling rate approximates an isoelastic function of labor market tightness in the United States

The monthly job-filling rate is computed from (4.20). Labor market tightness comes from figure 4.5. In panel B, the series are detrended by applying a HP filter with smoothing parameter of 10,000.

recent studies obtain comparable results. Shimer (2005, table 2) estimates the matching elasticity at 0.72; Rogerson and Shimer (2011, p. 638) obtain an estimate of 0.58; and Borowczyk-Martins, Jolivet, and Postel-Vinay (2013, table 1) report a lower estimate of 0.30.

4.9. Summary

In this chapter we have introduced the matching function as a key tool for modeling slackish markets, where trade is not instantaneous and requires time and effort from both buyers and sellers. Unlike Walrasian markets where trades are seamless, real-world markets are characterized by complexities that make it difficult for buyers and sellers to find each other. The matching function is an aggregate tool, similar to a production function, that captures the outcome of this complex trading process without modeling the micro-level details explicitly.

A matching function, $M = m(s, b)$, gives the total number of trades (M) as a function of the number of sellers (s) and buyers (b). It is generally assumed to have constant returns to scale, which implies that the probability of selling, $f(\theta)$, and the probability of buying, $q(\theta)$, depend only on the market's tightness $\theta = b/s$. A tighter market—more buyers per seller—increases the selling probability but decreases the buying probability.

The chapter presents four specific types of matching functions: urn-ball, CES (constant elasticity of substitution), Cobb-Douglas, and penalized Cobb-Douglas, which each have their advantages and drawbacks. We discussed how to choose which matching function in different situations. For instance, the CES function is useful in theoretical work, while the penalized Cobb-Douglas function is useful to produce an isoelastic Beveridge curve.

Finally, the chapter reviews empirical evidence on matching from the US labor market, which strongly supports the use of a matching function with constant returns to scale. The data show that the job-finding rate is a stable, increasing function of labor market tightness, while the job-filling rate is a stable, decreasing function of labor market tightness. This evidence suggests a Cobb-Douglas function is a good approximation for the US labor market, because the matching elasticity appears roughly constant.

Bibliography

Angelis, Georgios and Yann Bramoulle. 2023. "The Matching Function: a Unified Look into the Black Box." CEPR Discussion Paper 18318. https://ideas.repec.org/p/cpr/ceprdp/18318.html.

Blanchard, Olivier J. and Peter Diamond. 1989. "The Beveridge Curve." *Brookings Papers on Economic Activity* 20 (1): 1–76. https://doi.org/10.2307/2534495.

BLS. 2025a. "Number Unemployed for Less Than 5 Weeks." FRED, Federal Reserve Bank of St. Louis. https://fred.stlouisfed.org/series/UEMPLT5.

BLS. 2025b. "Unemployment Level." FRED, Federal Reserve Bank of St. Louis. https://fred.stlouisfed.org/series/UNEMPLOY.

Borowczyk-Martins, Daniel, Gregory Jolivet, and Fabien Postel-Vinay. 2013. "Accounting for Endogeneity in Matching Function Estimation." *Review of Economic Dynamics* 16 (3): 440–451. https://doi.org/10.1016/j.red.2012.07.001.

Butters, Gerard R. 1977. "Equilibrium Distributions of Sales and Advertising Prices." *Review of Economic Studies* 44 (3): 465–491. https://doi.org/10.2307/2296902.

Davis, Steven J., R. Jason Faberman, and John Haltiwanger. 2006. "The Flow Approach to Labor Markets: New Data Sources and Micro-Macro Links." *Journal of Economic Perspectives* 20 (3): 3–26. https://doi.org/10.1257/jep.20.3.3.

den Haan, Wouter J., Garey Ramey, and Joel Watson. 2000. "Job Destruction and Propagation of Shocks." *American Economic Review* 90 (3): 482–498. https://doi.org/10.1257/aer.90.3.482.

Duffie, Darrell. 2012. *Dark Markets: Asset Pricing and Information Transmission in Over-the-Counter Markets*. Princeton, NJ: Princeton University Press. https://muse.jhu.edu/book/31191.

Hall, Robert E. 1979. "A Theory of the Natural Unemployment Rate and the Duration of Employment." *Journal of Monetary Economics* 5 (2): 153–169. https://doi.org/10.1016/0304-3932(79)90001-1.

Hugonnier, Julien, Benjamin Lester, and Pierre-Olivier Weill. 2025. *The Economics of Over-the-Counter Markets: A Toolkit for the Analysis of Decentralized Exchange*. Princeton, NJ: Princeton University Press. https://muse.jhu.edu/book/136051.

Michaillat, Pascal and Emmanuel Saez. 2021. "Beveridgean Unemployment Gap." *Journal of Public Economics Plus* 2: 100009. https://doi.org/10.1016/j.pubecp.2021.100009.

Michaillat, Pascal and Emmanuel Saez. 2024. "Beveridgean Phillips Curve." arXiv:2401.12475. `https://doi.org/10.48550/arXiv.2401.12475`.

Petrongolo, Barbara and Christopher A. Pissarides. 2001. "Looking into the Black Box: A Survey of the Matching Function." *Journal of Economic Literature* 39 (2): 390–431. `https://doi.org/10.1257/jel.39.2.390`.

Pissarides, Christopher A. 1979. "Job Matchings with State Employment Agencies and Random Search." *Economic Journal* 89 (356): 818–833. `https://doi.org/10.2307/2231501`.

Pissarides, Christopher A. 1984. "Efficient Job Rejection." *Economic Journal* 94 (Supplement): 97–108. `https://doi.org/10.2307/2232658`.

Pissarides, Christopher A. 1985. "Short-Run Equilibrium Dynamics of Unemployment, Vacancies, and Real Wages." *American Economic Review* 75 (4): 676–690. `https://www.jstor.org/stable/1821347`.

Pissarides, Christopher A. 1986. "Unemployment and Vacancies in Britain." *Economic Policy* 1 (3): 500–559. `https://doi.org/10.2307/1344583`.

Pissarides, Christopher A. 2000. *Equilibrium Unemployment Theory*. Cambridge, MA: MIT Press, 2nd ed.

Rogerson, Richard and Robert Shimer. 2011. "Search in Macroeconomic Models of the Labor Market." In *Handbook of Labor Economics*, vol. 4A, edited by Orley C. Ashenfelter and David Card, chap. 7. Amsterdam: Elsevier. `https://doi.org/10.1016/S0169-7218(11)00413-8`.

Shimer, Robert. 2005. "The Cyclical Behavior of Equilibrium Unemployment and Vacancies." *American Economic Review* 95 (1): 25–49. `10.1257/0002828053828572`.

Shimer, Robert. 2012. "Reassessing the Ins and Outs of Unemployment." *Review of Economic Dynamics* 15 (2): 127–148. `https://doi.org/10.1016/j.red.2012.02.001`.

Stevens, Margaret. 2007. "New Microfoundations for the Aggregate Matching Function." *International Economic Review* 48 (3): 847–868. `https://doi.org/10.1111/j.1468-2354.2007.00447.x`.