A Simple and Robust Solution to the Minimal General Pose Estimation*

(Supplementary Material)

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In this annex we present the following supplementary material to the article "A Simple and Robust Solution to the Minimal General Pose Estimation": the *proof* of Theorem 1; the determination of the transformation parameters $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{t}}$; the determination of the coefficient parameters of the three polynomial equations $g_{i,j}(a,b,c)$; and the experimental results with noise.

I. Proof of the Theorem 1

Theorem 1: Consider a set of three points defined in the world coordinate system $\left\{\mathbf{p}_i^{(\mathscr{W})}\right\}$ and their correspondent lines in the camera coordinates system $\left\{\left(\mathbf{d}_i^{(\mathscr{C})},\mathbf{m}_i^{(\mathscr{C})}\right)\right\}$ for $i=1,2,3$. If the three points define a plane that does not pass through the origin, the dimension of the *column–space* of $\mathbf{M}$ (in the main article Equation (14)) will be $\mathscr{C}(\mathbf{M})=6$.

Proof: Using the definition of the *kronecker* product and rearranging rows and columns, we can get a matrix $\mathbf{M}^{(2)}$ such that $\mathscr{C}\left(\mathbf{M}^{(2)}\right)=\mathscr{C}(\mathbf{M})$ (note that the permutation of rows and columns does not change the *dimension of the column–space*) and

$$\mathbf{M}^{(2)}=\begin{pmatrix}\mathbf{0} & -\mathbf{D}^{(3)}\mathbf{P} & \mathbf{D}^{(2)}\mathbf{P}\\ \mathbf{D}^{(3)}\mathbf{P} & \mathbf{0} & -\mathbf{D}^{(1)}\mathbf{P}\\ -\mathbf{D}^{(2)}\mathbf{P} & \mathbf{D}^{(1)}\mathbf{P} & \mathbf{0}\end{pmatrix} \tag{I.1}$$

where

$$\mathbf{D}^{(i)}=\begin{pmatrix} d_i^{(1)} & 0 & 0\\ 0 & d_i^{(2)} & 0\\ 0 & 0 & d_i^{(3)}\end{pmatrix} \text{ and } \mathbf{D}^{(i)}=\begin{pmatrix}\mathbf{p}_1^{(\mathscr{W})\,T}\\ \mathbf{p}_2^{(\mathscr{W})\,T}\\ \mathbf{p}_3^{(\mathscr{W})\,T}\end{pmatrix}, \tag{I.2}$$

and $d_i^{(j)}$ is the i^{th} element of the vector $\mathbf{d}_j^{(\mathscr{C})}$.

On the other hand, matrix $\mathbf{M}^{(2)}$ can be decomposed as

$$\mathbf{M}^{(2)}=\underbrace{\begin{pmatrix}\mathbf{0} & -\mathbf{D}^{(3)} & \mathbf{D}^{(2)}\\ \mathbf{D}^{(3)} & \mathbf{0} & -\mathbf{D}^{(1)}\\ -\mathbf{D}^{(2)} & \mathbf{D}^{(1)} & \mathbf{0}\end{pmatrix}}_{\mathbf{Q}^{(1)}}\underbrace{\begin{pmatrix}\mathbf{P} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{P} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{P}\end{pmatrix}}_{\mathbf{Q}^{(2)}}. \tag{I.3}$$

Since we are considering that the points in the world $\left\{\mathbf{p}_i^{(\mathscr{W})}\right\}$ define a plane that does not pass through the origin, we conclude that $\mathscr{C}(\mathbf{P})=3$ and as a result $\mathscr{C}\left(\mathbf{Q}^{(2)}\right)=9$. Since the *dimension of the column–space* of $\mathbf{Q}^{(2)}$ is equal to the number of rows, one can conclude that $\mathscr{C}\left(\mathbf{M}^{(2)}\right)=\mathscr{C}\left(\mathbf{Q}^{(1)}\right)$.

In addition, we can decompose $\mathbf{Q}^{(1)}$ into $\mathbf{Q}^{(1)}=\mathbf{S}^{(1)}+\mathbf{S}^{(2)}+\mathbf{S}^{(3)}$ as it is shown in Equation (I.4). Since matrices $\mathbf{S}^{(i)}$ (for $i=1,2,3$) represent orthogonal subspaces [2], we can write $\mathbf{S}^{(i)}\cap\mathbf{S}^{(j)}=\{0\}$ for $i\neq j$ and, as a result,

$$\mathscr{C}\left(\mathbf{Q}^{(1)}\right)=\sum_{i=1}^{3}\mathscr{C}\left(\mathbf{S}^{(i)}\right). \tag{I.5}$$

Moreover and eliminating zero rows and columns of each matrix $\mathbf{S}^{(i)}$, we can see that $\mathscr{C}\left(\mathbf{S}^{(i)}\right)=\mathscr{C}\left(\widehat{\mathbf{d}}_i^{(\mathscr{C})}\right)$[1], and as a result

$$\mathscr{C}\left(\mathbf{Q}^{(1)}\right)=\sum_{i=1}^{3}\mathscr{C}\left(\widehat{\mathbf{d}}_i^{(\mathscr{C})}\right). \tag{I.6}$$

Since $\mathbf{d}_i^{(\mathscr{C})}$ are non–zero vectors, we can conclude that $\mathscr{C}\left(\widehat{\mathbf{d}}_i^{(\mathscr{C})}\right)=2$ and as a result $\mathscr{C}\left(\mathbf{Q}^{(1)}\right)=6$.

To conclude, $\mathscr{C}\left(\mathbf{Q}^{(1)}\right)=6$ implies $\mathscr{C}\left(\mathbf{M}^{(2)}\right)=6$ and as a result $\mathscr{C}(\mathbf{M})=6$, which proves the theorem. ■

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[1]The anti-symmetric matrix represented by a hat $\widehat{\mathbf{a}}\subset\mathbb{R}^{3\times 3}$ is the linear representation of the exterior product of three dimensional vector, such that $\mathbf{a}\times\mathbf{b}\doteq\widehat{\mathbf{a}}\mathbf{b}$.

$$
\mathbf{Q}^{(1)} = \underbrace{\begin{pmatrix}
0 & 0 & 0 & -d_3^{(1)} & 0 & 0 & d_2^{(1)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_3^{(1)} & 0 & 0 & 0 & 0 & 0 & -d_1^{(1)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-d_2^{(1)} & 0 & 0 & d_1^{(1)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}}_{\mathbf{S}^{(1)}} + \underbrace{\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -d_3^{(2)} & 0 & 0 & d_2^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_3^{(2)} & 0 & 0 & 0 & 0 & 0 & -d_1^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -d_2^{(2)} & 0 & 0 & d_1^{(2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}}_{\mathbf{S}^{(2)}} + \underbrace{\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -d_3^{(3)} & 0 & 0 & -d_2^{(3)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_3^{(3)} & 0 & 0 & 0 & 0 & 0 & -d_1^{(3)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -d_2^{(3)} & 0 & 0 & d_1^{(3)} & 0 & 0 & 0
\end{pmatrix}}_{\mathbf{S}^{(3)}} \tag{I.4}
$$

II. Parameters of the Transformation $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{t}}$

In the main article we consider a transformation $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{t}}$ such that, applied to the set world points $\left\{\mathbf{p}_i^{(\mathscr{W})}\right\}$ and plane coordinates $\Pi^{(\mathscr{W})} \doteq \left(\zeta^{(\mathscr{W})}, \pi^{(\mathscr{W})}\right)$,

$$
\widetilde{\mathbf{p}}_i^{(\mathscr{W})} = \widetilde{\mathbf{R}}\mathbf{p}_i^{(\mathscr{W})} + \widetilde{\mathbf{t}}, \quad \forall i \tag{II.1}
$$

$$
\left(\widetilde{\zeta}^{(\mathscr{W})}, \widetilde{\pi}^{(\mathscr{W})}\right) = \left(\zeta^{(\mathscr{W})} - \widetilde{\mathbf{t}}^T \widetilde{\mathbf{R}}\pi^{(\mathscr{W})}, \widetilde{\mathbf{R}}\pi\right), \tag{II.2}
$$

we obtain $\widetilde{\pi}$ parallel to the z–axis. To get the transformation parameters $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{t}}$ for general points we considered the following method: The rotation

$$
\widetilde{\mathbf{R}} = \begin{pmatrix} \mathbf{d}_4 & \mathbf{d}_5 & \mathbf{d}_6 \end{pmatrix}^T \tag{II.3}
$$

where

$$
\mathbf{d}_6 = \pi. \tag{II.4}
$$

To compute the vector $\mathbf{d}_4$ we consider two vectors $\mathbf{n}_-$ and $\mathbf{n}_+$ such that $\mathbf{n}_- = \mathbf{d}_6 \times (0,1,0)$ and $\mathbf{n}_+ = \mathbf{d}_6 \times (1,0,0)$. We define $\mathbf{d}_4 = \frac{\mathbf{n}}{|\mathbf{n}|}$, such that $\mathbf{n}$ is equal to the vector of largest magnitude out of $\mathbf{n}_-$ and $\mathbf{n}_+$. To conclude,

$$
\mathbf{d}_5 = \mathbf{d}_6 \times \mathbf{d}_4. \tag{II.5}
$$

To define a specific depth for the plane, we can use

$$
\widetilde{\mathbf{t}} = \widetilde{\mathbf{R}}\overline{\mathbf{p}}^{(\mathscr{W})} - \begin{pmatrix} 0 & 0 & \widetilde{\zeta} \end{pmatrix}^T. \tag{II.6}
$$

III. Coefficients of the Polynomial Equations $g_{i,j}(a,b,c)$

Let us consider the two first columns of the matrix $\mathbf{H}$ represented as a function of the unknown parameters α_1, α_2 and α_3 as

$$
\mathbf{h}_1 \doteq \alpha_1\mathbf{f}_1^{(1)} + \alpha_2\mathbf{f}_1^{(2)} + \alpha_3\mathbf{f}_1^{(3)} + \mathbf{f}_1^{(4)} \tag{III.1}
$$

$$
\mathbf{h}_2 \doteq \alpha_1\mathbf{f}_2^{(1)} + \alpha_2\mathbf{f}_2^{(2)} + \alpha_3\mathbf{f}_2^{(3)} + \mathbf{f}_2^{(4)}. \tag{III.2}
$$

For more information see the main article.

TABLE I

In this table, we show the coefficient parameters $\kappa_n^{(i,j)}$ of the polynomial equations $g_{i,j}(a,b,c)$.

(i,j)	$(1,2)$	$(1,1)$	$(2,2)$
$\kappa_1^{(i,j)}$	$\mathbf{f}_1^{(1)} \cdot \mathbf{f}_2^{(1)}$	$\mathbf{f}_1^{(1)} \cdot \mathbf{f}_1^{(1)}$	$\mathbf{f}_2^{(1)} \cdot \mathbf{f}_2^{(1)}$
$\kappa_2^{(i,j)}$	$\mathbf{f}_1^{(2)} \cdot \mathbf{f}_2^{(2)}$	$\mathbf{f}_1^{(2)} \cdot \mathbf{f}_1^{(2)}$	$\mathbf{f}_2^{(2)} \cdot \mathbf{f}_2^{(2)}$
$\kappa_3^{(i,j)}$	$\mathbf{f}_1^{(3)} \cdot \mathbf{f}_2^{(3)}$	$\mathbf{f}_1^{(3)} \cdot \mathbf{f}_1^{(3)}$	$\mathbf{f}_2^{(3)} \cdot \mathbf{f}_2^{(3)}$
$\kappa_4^{(i,j)}$	$\mathbf{f}_1^{(1)} \cdot \mathbf{f}_2^{(2)} + \mathbf{f}_1^{(2)} \cdot \mathbf{f}_2^{(1)}$	$2\mathbf{f}_1^{(1)} \cdot \mathbf{f}_1^{(2)}$	$2\mathbf{f}_2^{(1)} \cdot \mathbf{f}_2^{(2)}$
$\kappa_5^{(i,j)}$	$\mathbf{f}_1^{(1)} \cdot \mathbf{f}_2^{(3)} + \mathbf{f}_1^{(3)} \cdot \mathbf{f}_2^{(1)}$	$2\mathbf{f}_1^{(1)} \cdot \mathbf{f}_1^{(3)}$	$2\mathbf{f}_2^{(1)} \cdot \mathbf{f}_2^{(3)}$
$\kappa_6^{(i,j)}$	$\mathbf{f}_1^{(2)} \cdot \mathbf{f}_2^{(3)} + \mathbf{f}_1^{(3)} \cdot \mathbf{f}_2^{(2)}$	$2\mathbf{f}_1^{(2)} \cdot \mathbf{f}_1^{(3)}$	$2\mathbf{f}_2^{(2)} \cdot \mathbf{f}_2^{(3)}$
$\kappa_7^{(i,j)}$	$\mathbf{f}_1^{(1)} \cdot \mathbf{f}_2^{(4)} + \mathbf{f}_1^{(4)} \cdot \mathbf{f}_2^{(1)}$	$2\mathbf{f}_1^{(1)} \cdot \mathbf{f}_1^{(4)}$	$2\mathbf{f}_2^{(1)} \cdot \mathbf{f}_2^{(4)}$
$\kappa_8^{(i,j)}$	$\mathbf{f}_1^{(2)} \cdot \mathbf{f}_2^{(4)} + \mathbf{f}_1^{(4)} \cdot \mathbf{f}_2^{(2)}$	$2\mathbf{f}_1^{(2)} \cdot \mathbf{f}_1^{(4)}$	$2\mathbf{f}_2^{(2)} \cdot \mathbf{f}_2^{(4)}$
$\kappa_9^{(i,j)}$	$\mathbf{f}_1^{(3)} \cdot \mathbf{f}_2^{(4)} + \mathbf{f}_1^{(4)} \cdot \mathbf{f}_2^{(3)}$	$2\mathbf{f}_1^{(3)} \cdot \mathbf{f}_1^{(4)}$	$2\mathbf{f}_2^{(3)} \cdot \mathbf{f}_2^{(4)}$
$\kappa_{10}^{(i,j)}$	$\mathbf{f}_1^{(4)} \cdot \mathbf{f}_2^{(4)}$	$\mathbf{f}_1^{(4)} \cdot \mathbf{f}_1^{(4)} - 1$	$\mathbf{f}_2^{(4)} \cdot \mathbf{f}_2^{(4)} - 1$

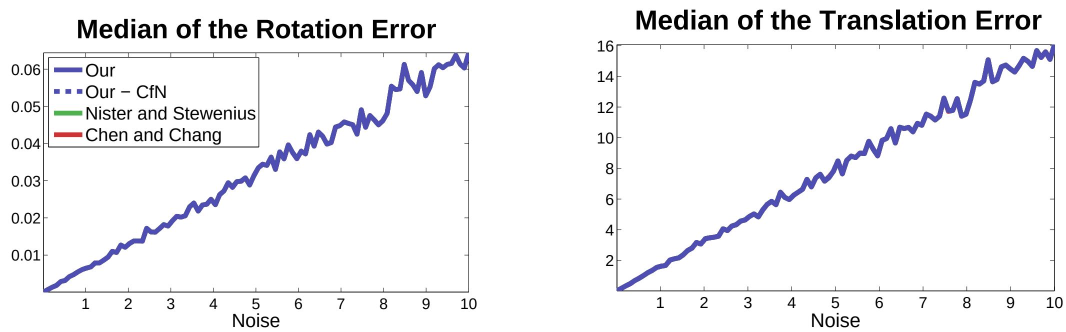

Fig. 1. In this figure we show the analysis of the errors for data with noise. We consider the `Noise` variable as the standard deviation of three random noise vector $\mathbf{n}_i^{(\mathscr{W})}$ and compute the pose using the data-set $\left\{\mathbf{l}_i^{(\mathscr{C})}, \breve{\mathbf{p}}_i^{(\mathscr{W})}\right\}$, where $\breve{\mathbf{p}}_i^{(\mathscr{W})} = \mathbf{p}_i^{(\mathscr{W})} + \mathbf{n}_i^{(\mathscr{W})}$. For each value of the `Noise`, we use 10^3 trials and the errors are computed as the median of all the errors. As it can be seen from the figures, the four plots overlap, which means that the errors are modeled only by the noise.

In the main article, we define three constraints such that

$$\mathbf{h}_1 \cdot \mathbf{h}_2 = 0, \ \ \mathbf{h}_1 \cdot \mathbf{h}_1 = 1 \ \text{ and } \ \mathbf{h}_2 \cdot \mathbf{h}_2 = 1. \tag{III.3}$$

where the operator $(\cdot)$ represents the inner product. Substituting the vectors $\mathbf{h}_i$ on Equation (III.3) by the space of solutions defined at Equations (III.1) and (III.2), we derive three quadric equations

$$g_{1,2}\left(\alpha_1, \alpha_2, \alpha_3\right) = 0, \ \ g_{1,1}\left(\alpha_1, \alpha_2, \alpha_3\right) = 0 \ \text{ and } \ g_{2,2}\left(\alpha_1, \alpha_2, \alpha_3\right) = 0 \tag{III.4}$$

where each function $g_{i,j}\left(\alpha_1, \alpha_2, \alpha_3\right)$ represents a degree-two polynomial equation with three unknowns

$$g_{i,j}\left(\alpha_1, \alpha_2, \alpha_3\right) = \kappa_1^{(i,j)} \alpha_1^2 + \kappa_2^{(i,j)} \alpha_2^2 + \kappa_3^{(i,j)} \alpha_3^2 + \kappa_4^{(i,j)} \alpha_1 \alpha_2 + \kappa_5^{(i,j)} \alpha_1 \alpha_3 + \kappa_6^{(i,j)} \alpha_2 \alpha_3 + \kappa_7^{(i,j)} \alpha_1 + \kappa_8^{(i,j)} \alpha_2 + \kappa_9^{(i,j)} \alpha_3 + \kappa_{10}^{(i,j)}. \tag{III.5}$$

The coefficient parameters $k_n^{(i,j)}$ are shown in the Table I.

IV. Experiments with Noise

As was emphasized and discussed by Nistér and Stewénius [1], for experiments with noise, solutions of the minimal problems yield similar results which means that these tests are not relevant for comparison. If both the problem formulation and minimal solver are right, the solutions must be the same (in number and value). If we add noise to the data, and since the solutions must be the same, errors in rotation and translation estimates must be the same. If we plot the errors as a function of noise, the plots for the four methods should overlap. However, to see the effects of the noise on the general minimal absolute pose, we add noise to the 3D world points and plot the errors. The results were obtained for the general (non-central) case. The results of the errors for rotation are shown in Figure 1.

For experiments with real data, the same analysis can be made.

References

[1] D. Nistér and H. Stewénius. A Minimal Solution to the Generalized 3–Point Pose Problem. J. Math. Imaging Vis, 2007.

[2] G. Strang. Linear Algebra and its Applications. Academic Press, 1980.